# Power spectral density and rms-amplitude

Lecture notes on „Physics of Seismic Instruments“
Thomas Forbriger (BFO, GPI, Karlsruhe Institute of Technology KIT)
Thomas.Forbriger@kit.edu


## 1 Stochastic, stationary noise

*Power spectral density* (PSD) is a quantity to specify the signal level of *stochastic, stationary noise*.

The word 'noise' frequently is used to summarize all unwanted components in a seismic record. This comprises drifts, spikes, glitches, steps, and stationary stochastic signals and more. It then serves as a 'garbage can' to summarize all signal components which appear annoying because they might hide signals of interest, distort the signal, are of unknown origin, or simply lack a proper quantitative model to describe the signal in a deterministic way.

Here, in the context of power spectral density, 'noise' refers to a signal with the following properties:

- The signal appears non-deterministic.
- The signal's phase varies randomly with time.
- Only stochastic properties of the signal can be specified.

If the stochastic properties remain constant with time the signal is called *stationary*. Properties in that sense may be the mean amplitude, the average power, or the frequency distribution of sample values.

Seismological signals to which this concept at least partly applies are the marine microseisms, high-frequency vibrations caused by forces excerted by turbulent wind at the surface, and electronic noise. In practice these signals might not exactly match the definition given above. They might by partly correlated and not completely random. Their properties like signal power depending on frequency might slowly (sometimes quickly) change with time, such that they are not exactly stationary. We have to keep these limitation in mind. Nevertheless it is appropriate to apply the concept of power spectral density at least to finite sections of the recordings.

## 2 Signal power and rms-amplitude

### 2.1 Energy and power in physics

If a current $I(t)$ flows through a resistor of resistance $R$, there appears a voltage drop of

$$U(t) = R\,I(t) \tag{1}$$

across the resistor. Eq. (1) is called 'Ohm's law'. Then

$$P(t) = U(t)\,I(t) = R\,I^2(t) \tag{2}$$

is the power, which is dissipated at the resistor at any instance of time $t$.

In the time from $t = T_1$ to $t = T_2$ a total of electric energy

$$E = \int_{T_1}^{T_2} P(t)\,\mathrm{d}t = \int_{T_1}^{T_2} R\,I^2(t)\,\mathrm{d}t \tag{3}$$

is transformed to thermal energy at the resistor. The resistor is heated, it becomes warm.

Power is energy per unit of time. The average power of heating is

$$\overline{P} = \frac{E}{\Delta T} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} R\,I^2(t)\,\mathrm{d}t. \tag{4}$$

If $I(t)$ is constant or if

$$I(t) = I\,\sin(\omega t) \tag{5}$$

oscillates with constant amplitude, the dissipated energy will on average increase at constant rate. The average power $\overline{P}$ converges to a constant value for large time windows $\Delta T = T_2 - T_1$.

The physical unit of energy is the joule (J) and the unit of power is the watt (W).

### 2.2 Energy and power in signal theory

In signal theory an abstraction of energy and power is used to characterize signals. If $a(t)$ is a signal of some physical quantity, then

$$E = \int_{T_1}^{T_2} a^2(t)\,\mathrm{d}t \tag{6}$$

is the energy of this signal in the time window from $T_1$ to $T_2$. Likewise

$$\overline{P} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} a^2(t)\,\mathrm{d}t \tag{7}$$

is the average power of the signal. Neither is the joule the unit of signal energy nor is the watt the unit of signal power. The unit of signal energy

$$[E] = [a(t)]^2 \text{ s} \tag{8}$$

is the square of the unit of the signal times the unit of time. The unit of signal power

$$[P] = [a(t)]^2 \tag{9}$$

is the square of the unit of the signal.

### 2.2.1 Signals of finite energy

A transient signal has an onset at a given time and decays to vanishing amplitude after a given time. It is confined to a limited time window, just like an earthquake signal. For such a signal the total signal energy

$$E = \int_{-\infty}^{+\infty} a^2(t)\,\mathrm{d}t \tag{10}$$

is finite. The signal in jargon sometimes is called an 'energy signal' for this reason. One essential condition for the existence of a Fourier transform is that the value of eq. (10) is finite.

### 2.2.2 Signals of finite power

If the signal lasts forever (like the tides or continuously ongoing noise) the total energy as defined by eq. (10) becomes infinite. The Fourier transformation no longer is applicable. However, if the signal is well behaved and its amplitude remains within finite limits, its average power

$$\overline{P} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} a^2(t)\,\mathrm{d}t \tag{11}$$

remains finite. The signal in jargon sometimes is called a 'power signal' for this reason.

## 2.3 Root-mean-square (rms) amplitude

A proper definition of average amplitude of a stationary random signal is the 'root-mean-square' (rms) amplitude

$$a_{\text{rms}} = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} a^2(t)\,\mathrm{d}t} = \sqrt{\overline{P}}. \tag{12}$$

**Table 1:** Properties for a time continuous signal $a(t)$ in correspondence to a discrete time series of $N$ samples $a_l$ with sampling interval $\Delta t$.

| | discrete | continuous |
|---|---|---|
| value | $a_l$ | $a(t)$ |
| instantaneous power | $a_l^2$ | $a^2(t)$ |
| total energy | $\sum_{l=1}^{N} a_l^2\,\Delta t$ | $\int_{-T}^{+T} a^2(t)\,\mathrm{d}t$ |
| average power ('mean square') | $\frac{1}{N}\sum_{l=1}^{N} a_l^2$ | $\frac{1}{2T}\int_{-T}^{+T} a^2(t)\,\mathrm{d}t$ |
| rms-amplitude | $\sqrt{\frac{1}{N}\sum_{l=1}^{N} a_l^2}$ | $\sqrt{\frac{1}{2T}\int_{-T}^{+T} a^2(t)\,\mathrm{d}t}$ |

which equals the square root of average power

$$\overline{P} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} a^2(t)\,\mathrm{d}t. \tag{13}$$

The reason for the term 'root-mean-square' becomes obvious from the expression for a discrete time series as given in Table 1 in correspondence to the continuous time expressions. If $a_l$ is a series with vanishing average

$$\sum_{l=1}^{N} a_l = 0 \tag{14}$$

then the expression for the mean squares equals the variance of the sequence and the root-mean-square equals the expression for the standard deviation for large $N$.

# 3 Spectral density

## 3.1 Energy density

The signal energy for a transient signal $a(t)$ is

$$E = \int_{-\infty}^{+\infty} a^2(t)\,\mathrm{d}t. \tag{15}$$

The Fourier transform of $a(t)$ is

$$\tilde{a}(\omega) = \int_{-\infty}^{+\infty} a(t)\,e^{-i\omega t}\,\mathrm{d}t \tag{16}$$

and because of Parceval's theorem[1]

$$E = \int_{-\infty}^{+\infty} \left|\tilde{a}(\omega)\right|^2 \frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} a^2(t)\,\mathrm{d}t. \tag{17}$$

Hence the signal's energy in the frequency band from $\omega_1$ to $\omega_2$ is

$$E_{\omega_1,\omega_2} = 2 \int_{\omega_1}^{\omega_2} \left|\tilde{a}(\omega)\right|^2 \frac{\mathrm{d}\omega}{2\pi}. \tag{18}$$

That is why we call $\left|\tilde{a}(\omega)\right|^2$ the *energy density* of the signal. The factor 2 in eq. (18) accounts for the contribution at negative frequency, because $\left|\tilde{a}(\omega)\right|$ is an even function of frequency. This is a consequence of the symmetry condition for the Fourier transform of a signal $a(t)$ of real values.

### 3.2 Power spectral density

The normalized auto correlation function for a stationary stochastic signal $a(t)$ is

$$P(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} a(t)\,a(t+\tau)\,\mathrm{d}t. \tag{19}$$

Its Fourier transform

$$\tilde{P}(\omega) = \int_{-\infty}^{+\infty} P(\tau)\,e^{-i\omega\tau}\,\mathrm{d}\tau \tag{20}$$

is called the *power spectral density*. Why?

The average signal power is

$$\overline{P} = P(\tau=0) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} a^2(t)\,\mathrm{d}t. \tag{21}$$

With eq. (20) the normalized auto-correlation function

$$P(\tau) = \int_{-\infty}^{+\infty} \tilde{P}(\omega)\,e^{i\omega\tau}\,\frac{\mathrm{d}\omega}{2\pi} \tag{22}$$

can be expressed by Fourier expansion. The average total power as defined in eq. (21) then is computed from $\tilde{P}(\omega)$ by

$$\overline{P} = P(\tau=0) = \int_{-\infty}^{+\infty} \tilde{P}(\omega)\,e^{i\omega\cdot 0}\,\frac{\mathrm{d}\omega}{2\pi} = \int_{-\infty}^{+\infty} \tilde{P}(\omega)\,\frac{\mathrm{d}\omega}{2\pi}. \tag{23}$$

Hence the total average signal power is

$$\overline{P} = \int_{-\infty}^{+\infty} \tilde{P}(\omega)\,\frac{\mathrm{d}\omega}{2\pi} \tag{24}$$

and analogous to the definition of energy density. Because of its even symmetry with $\tilde{P}(-\omega) = \tilde{P}(\omega)$ the components at negative frequency do not contribute additional information. Hence the average power in the frequency band from $\omega_1$ to $\omega_2$ is

$$\overline{P}_{\omega_1,\omega_2} = 2 \int_{\omega_1}^{\omega_2} \tilde{P}(\omega)\,\frac{\mathrm{d}\omega}{2\pi}. \tag{25}$$

That is why we call $\tilde{P}(\omega)$ the *power spectral density* (PSD) of the signal.

### 3.3 One-sided PSD

In engineering $2\tilde{P}(\omega)$ is frequently used and called the *one-sided* PSD because it allows the computation of total power by integration over positive frequency only (in contrast to the *two-sided* PSD $\tilde{P}(\omega)$). Also in seismology it is quite common to specify the *one-sided PSD* as a result of signal analysis. When analyzing signals it is essential to check whether the used computer function returns one-sided PSD ($2\tilde{P}(\omega)$) or twosided PSD ($\tilde{P}(\omega)$) to make diagrams consistent. As well the caption of the diagram should specify whether one-sided or two-sided PSD is displayed.

### 3.4 Total average power and bandwidth

Obviously we require $P(\tau)$ for all lag-times $\tau$ in order to obtain a frequency dependent power spectral density $\tilde{P}(\omega)$ with eq. (20). However, with eq. (24) and eq. (22) it is obvious that the total average power is only present in the value of $P(\tau)$ for $\tau = 0$. When computing the total average power $\overline{P}$, we extract one scalar value from the information contained either in $P(\tau)$ or $\tilde{P}(\omega)$. We do this by taking $\overline{P} = P(0)$ or by integration in eq. (24).

This is similar to the Fourier transform of signal $a(t)$ in eq. (16). The value of $a(t)$ at $t = 0$ is given by

$$a(t=0) = \int_{-\infty}^{+\infty} \tilde{a}(\omega)\,\frac{\mathrm{d}\omega}{2\pi}. \tag{26}$$

As a result we may draw the conclusion that a signal of infinite bandwidth must contain infinite power. Signals of finite power must be band-limited, always.

## 4 Amplitude in a finite frequency band

$\tilde{P}(\omega)$ as defined in eq. (20) specifies power spectral *density*. The total power (or variance) in the frequency band from $f_1$

[1]Jenkins and Watts (1968, eq. A2.1.4) specify the theorem in its most general form for the cross-correlogram of two signals and the integral over the normalized cross-spectrum.

to $f_2$ then is

$$P_{f_1,f_2} = 2 \int_{f_1}^{f_2} \tilde{P}(2\pi f)\,\mathrm{d}f. \tag{27}$$

The rms-amplitude of the band-pass filtered time series consequently is

$$a_{\mathrm{rms}\,f_1,f_2} = \sqrt{P_{f_1,f_2}}. \tag{28}$$

This provides a means to specify average signal amplitude of a random noise signal as a function of frequency. If the frequency interval $\Delta f = f_2 - f_1$ is sufficiently small and $\tilde{P}(2\pi f)$ is well behaved, power spectral density at the center frequency can be approximated by

$$\tilde{P}(2\pi f_c) \approx \frac{P_{f_1,f_2}}{2\Delta f}, \tag{29}$$

where power spectral density explicitly is signal power over bandwidth. rms-amplitude as a function of frequency then would be

$$a_{\mathrm{rms}}(f_c;\Delta f) \approx \sqrt{\tilde{P}(2\pi f_c)\,2\Delta f}. \tag{30}$$

Notice that **a value of rms-amplitude without a specification of bandwidth is meaningless**.

## 4.1 Relative bandwidth

If $a_{\mathrm{rms}}(f_c;\Delta f)$ is to be given for a large frequency range, the specification of a constant bandwidth $\Delta f$ is not appropriate. $\Delta f = 1\,\mathrm{Hz}$ might be a reasonable choice for $a_{\mathrm{rms}}(f_c;\Delta f)$ being given in the range from 10 Hz to 100 Hz but not for values in the frequency range of 1 mHz. For this reason we prefer to specify the bandwidth

$$\Delta f = f_2 - f_1 \tag{31}$$

proportional to the center frequency

$$f_c = \sqrt{f_1\,f_2} \tag{32}$$

such that

$$\Delta f = f_c\,R_{\mathrm{BW}}, \tag{33}$$

where $R_{\mathrm{BW}}$ is called the *bandwidth factor*. It is common to specify the bandwidth in

$$n = \log_2 \frac{f_2}{f_1} \qquad \text{octaves} \tag{34}$$

or

$$m = \log_{10} \frac{f_2}{f_1} \qquad \text{decades.} \tag{35}$$

The bandwidth factor then is

$$R_{\mathrm{BW}} = \frac{\Delta f}{f_c} = \frac{f_2 - f_1}{f_c} = \frac{2^n - 1}{2^{n/2}} = \frac{10^m - 1}{10^{m/2}} \tag{36}$$

**Table 2:** Values of bandwidth factor for different value of relative bandwidth.

| bandwidth | $R_{\mathrm{BW}}$ |
|---|---|
| 1 decade | 2,846 |
| 1/2 decade | 1,215 |
| 1 octave | 0,707 |
| 1/6 decade | 0,386 |
| 1/2 octave | 0,348 |
| 1/3 octave | 0,232 |

and rms-amplitude as a function of frequency is specified by

$$\boxed{a_{\mathrm{rms}}(f_c;R_{\mathrm{BW}}\,f_c) = \sqrt{2\tilde{P}(2\pi\,f_c)\,f_c\,R_{\mathrm{BW}}}.} \tag{37}$$

Values of $R_{\mathrm{BW}}$ for commonly used values of bandwidth are given in Table 2.

## 4.2 Average peak amplitude

Average peak amplitude can be defined by the mean value of the envelope of narrow-band filtered, Gaussian distributed stochastic noise (Rice 1945, section 3.7). By coincidence the value of average peak amplitudes in 1/3 octave equals the rms amplitude in 1/6 decade. The exact factor is (Peterson 1993, page 36)

$$\begin{aligned} a_{\mathrm{peak}}(f_c;1/3\,\mathrm{octave}) &= \sqrt{\frac{\pi}{2}}\,a_{\mathrm{peak}}(f_c;1/3\,\mathrm{octave}) \\ &= \sqrt{\frac{\pi}{2}\,\frac{0.232}{0.386}}\,a_{\mathrm{rms}}(f_c;1/6\,\mathrm{decade}) \\ &\approx 0.972\,a_{\mathrm{rms}}(f_c;1/6\,\mathrm{decade}), \end{aligned} \tag{38}$$

where $R_{\mathrm{BW}} = 0.232$ for 1/3 octave and $R_{\mathrm{BW}} = 0.386$ for 1/6 decade.

# 5 Recipe for practical computation

In practical application to time series data the signal power in fact is computed from

$$\text{power} = \frac{\text{energy}}{\text{time}}$$

i. e. signal energy being present in a finite time window over the length of the time window. The signal then is available in terms of $N$ samples $a_l$ over a finite time window $T = N\Delta t$, where $\Delta t$ is the sampling interval. All computation in the frequency domain is based on the Fourier coefficients

$$\tilde{a}_k = \sum_{l=1}^{N} a_l\,e^{-2i\pi\,\frac{(l-1)\,(k-1)}{N}}\,\Delta t, \tag{39}$$

which usually are obtained by an appropriately scaled algorithm of *Fast-Fourier-Transformation* (FFT). One-sided power spectral density at frequency $f_k = k/T$ then is

$$2\tilde{P}_k = 2\frac{|\tilde{a}_k|^2}{T} \tag{40}$$

such that

$$a_{\text{rms}\,k} = \sqrt{2\tilde{P}_k\, f_k\, R_{\text{BW}}} \tag{41}$$

is the corresponding value for rms-amplitude in a bandwidth of $\Delta f = f_k\, R_{\text{BW}}$.

### 5.1 Application of a taper

The time series $a_l$ represents an ever lasting random noise signal. As a non-transient signal it is not naturally confined to a finite time window. Prior to application of the discrete Fourier transformation as defined in eq. (39) a taper function should be applied in order to reduce spectral side-lobes. This taper function must be appropriately scaled such that it does not alter the signals energy in the analyzed time window.

### 5.2 Necessity to take the average

The values computed by eqs. (40) and (41) will strongly fluctuate due to the stochastic nature of the signal's phase. The computed value for this reason depends on the actual choice of time window. To obtain a proper value which would represent the average property of a stationary signal, values of $2\tilde{P}_k$ as computed for several different time windows must be averaged or the average over adjacent index $k$ must be computed similar to eq. (29).

## 6 The decibel scale

Because of the large dynamic range of signal levels to be graphed over a large interval of frequency, it is common practice to display the curves on log-log-scales. Frequency or signal period simply is given on a logarithmic axis of abscissae, *power-spectral density* (PSD) is given in units of decibels, which is logarithmic.

Values given in decibel always express a ratio with respect to a reference value. It is absolutely necessary to include the reference value in the diagram (ideally in the labeling of the ordinate).

The decibel scale is a logarithmic scale for signal level (in terms of signal power or amplitude). A logarithmic scale is preferable if levels cover several orders of magnitude and in particular, when multiplicative operations (like amplification) are involved. A level is called to be *larger by one order of magnitude* when it is larger by a factor of ten. The decibel scale is a relative scale. It refers signal level of one signal to that of another. Examples can be a) an output signal of an amplifier referred to the input, b) a signal level referred to a reference level, or c) the level of a disturbance referred to the total signal level.

The decibel scale is defined for signal power. Signal power scales with the square of amplitude. Details of the definition of signal power will be given elsewhere together with the definition of power spectral density.

### 6.1 Definition

The definition is based on the definition of the unit bel (symbol B). If the power of signal 1 is $P_1$ and that of signal 2 is $P_2$ then their ratio $P_2/P_1$ is the power of signal 2 referred to that of signal 1 is given by

$$V = \log_{10}\frac{P_2}{P_1}\,\text{B} \tag{42}$$

in the units of 1 B.

Because the power scales with the square of the amplitude, the ratio can be computed from signal amplitudes $A_1$ and $A_2$ by

$$V = \log_{10}\frac{P_2}{P_1}\,\text{B} = \log_{10}\left(\frac{A_2}{A_1}\right)^2\,\text{B} \tag{43}$$

$$= 2\,\log_{10}\frac{A_2}{A_1}\,\text{B} \tag{44}$$

in the units of 1 B. The decibel simply is one tenth of a bel, such that

$$1\,\text{dB} = 0.1\,\text{B}. \tag{45}$$

This way

$$\boxed{V = 10\,\log_{10}\frac{P_2}{P_1}\,\text{dB} = 20\,\log_{10}\frac{A_2}{A_1}\,\text{dB}} \tag{46}$$

specifies the ratio $V$ in units of 1 dB. Eq. (46) is the commonly used form.

### 6.2 Specifying signal level

If a signal level $V_{\text{signal}}$ is given in units of 1 dB, then a reference level must be specified along with this value (ideally this is done in the labeling of the ordinate). Unfortunately authors sometimes forget to do so and assume a standard reference level implicitly. In seismology it then is very likely that the author specifies the signal level with respect to acceleration amplitude of $1\,\text{m}\,\text{s}^{-2}$. In electronic and communications engineering there exist signal units like 1 dBm based on the decibel scale, which implicitly refer to a specific reference level.

Such units do not exist in seismology. **Missing to specify a reference level must be regarded as a fault.**

Values of PSD then are

$$V_{\text{PSD}}(\omega) = 10\,\log_{10}\left(\frac{2\tilde{P}(\omega)}{P_{\text{ref}}}\right)\,\text{dB} \qquad (47)$$

if PSD is given as one-sided PSD $2\tilde{P}(\omega)$ with respect to a reference value $P_{\text{ref}}$ or

$$V_{\text{PSD}}(\omega) = 20\,\log_{10}\left(\frac{a_{\text{rms}}(f_c; R_{\text{BW}}\,f_c)}{A_{\text{ref}}}\right)\,\text{dB} \qquad (48)$$

if PSD is given by rms-amplitude $a_{\text{rms}}(f_c; R_{\text{BW}}\,f_c)$ with respect to a reference value $A_{\text{ref}}$. Common choices for the reference naturally are $P_{\text{ref}} = 1\,\text{m}^2\,\text{s}^{-4}\,\text{Hz}^{-1}$ and $A_{\text{ref}} = 1\,\text{m}\,\text{s}^{-2}$. if PSD levels are specified for acceleration.

If the level of a signal is x dB higher than that of another signal (where x may be negative if the level is lower actually), then this statement applies regardless of whether the level difference was calculated from the power spectral density or from the rms amplitude (x is identical in both cases). This is a nice feature of the dB-scale.

### 6.3 Examples

- Dynamic range as the ratio of the largest signal with respect to the smallest detectable signal level appropriately is specified in units of 1 dB. A dynamic range of 140 dB implies that the largest non-clipping signal has an amplitude of 7 orders of magnitude (factor 10 millions) larger than that of the detection level.
- If an amplifier amplifies the signal by 6 dB the output amplitude is about twice the input amplitude.
- If a signal level is 100 dB larger than the reference, the signal power is ten orders of magnitude (factor $10^{10}$) larger than the reference and the amplitude is five orders of magnitude (factor $10^5$) larger than the reference.
- If the level of non-linear distortion is 90 dB below the total signal level and the total signal level $V_{\text{signal}}$ is given with respect to a reference level, then the distortion level with respect to the reference level is $V_{\text{distortion}} = V_{\text{signal}} - 90\,\text{dB}$.
- The signal-to-noise ratio can be specified in units of 1 dB. If signal to noise ratio is 40 dB than the signal amplitude is by a factor of 100 larger than the noise amplitude.

### 6.4 Literature

The decibel is not an official SI-unit, however is accepted for use with the SI, and units based on fundamental constants (Taylor 2008, section 4.1). Bormann and Wielandt (2012b, section 4.2.3.6) introduce the decibel scale very briefly and refer to concepts like power spectral density. Bormann and Wielandt (2012a) provide exercises on conversion of power spectral density, which include computations based on the decibel scale. Havskov and Alguacil (2016) introduce the decibel scale in appendix A.3.1. Scherbaum (1994) uses the decibel scale in section 2.6 to specify filter roll-off and in section 6.1 to specify dynamic range. Parker (2010) introduces signal-to-noise-ratio for quantization noise in units of 1 dB in section 3.2. Aki and Richards (2002, box 12.1) give an introduction to the application of the decibel scale in seismology. However, all these reference do not significantly go beyond the statements given above.

## 7 Caveats

### 7.1 Representation of signal level

In literature the quantities of

$$\tilde{P}(\omega), 2\tilde{P}(\omega), \sqrt{\tilde{P}(\omega)}, \sqrt{2\tilde{P}(\omega)}, \text{ and } \sqrt{2\tilde{P}(\omega)\,\Delta f}$$

all occasionally are called 'power spectral density' by the authors. This can cause confusion. Actually only the first two are 'power spectral density', where the second should be called 'one-sided power spectral density'. They are given in units

$$\left[\tilde{P}(\omega)\right] = \frac{[a(t)]^2}{\text{Hz}}.$$

A multiplication of $2\tilde{P}(\omega)$ with a value of bandwidth $\Delta f$ results in the total average power in this bandwidth, with

$$\left[\tilde{P}(\omega)\,\Delta f\right] = [a(t)]^2\,.$$

The square root of twice this value is rms-amplitude in units

$$\left[\sqrt{2\tilde{P}(\omega)\,\Delta f}\right] = [a(t)]$$

of signal amplitude. Hence values of power spectral density can easily be converted to total power or rms-amplitude in a given bandwidth.

Values of $\sqrt{\tilde{P}(\omega)}$ or $\sqrt{2\tilde{P}(\omega)}$ sometimes are considered by the authors to be closer to amplitude values because the square root already is taken. This however is not true. Their units are as strange as

$$\left[\sqrt{\tilde{P}(\omega)}\right] = \frac{[a(t)]}{\sqrt{\text{Hz}}}.$$

The computation of rms-amplitude from these values is not straightforward, because they do not represent a spectral density. Taking the product of these values with a value of bandwidth is meaningless. The product with the square-root of bandwidth provides values of average amplitude.

## 7.2 Necessity to specify bandwidth

The value of rms-amplitude $\sqrt{2\tilde{P}(\omega)\,\Delta f}$ always must be accompanied by a specification of bandwidth $\Delta f$. Unfortunately there are examples of diagrams of amplitude over frequency in the literature, without a specification of bandwidth. In the worst case they are useless, because there is no way to deduce their actual meaning.

## 7.3 Smoothing and averaging

Due to the stochastic nature of the analyzed signal, appropriate averaging must be applied as discussed in section 5.2. Likewise the resulting curves of PSD commonly are smoothed by a moving average along the frequency axis. This provides appropriate average values of PSD, while values PSD obtained from discrete spectral analysis may strongly fluctuate from frequency to frequency due to the stochastic phase of the signal. If signal level is displayed on a logarithmic scale (values given in decibels) averaging and smoothing must take place for PSD, not for the logarithmic value. Smoothing the linear value is similar to finding the upper envelope of the logarithmic value. Smoothing or averaging the logarithmic value will systematically bias the results to smaller values.

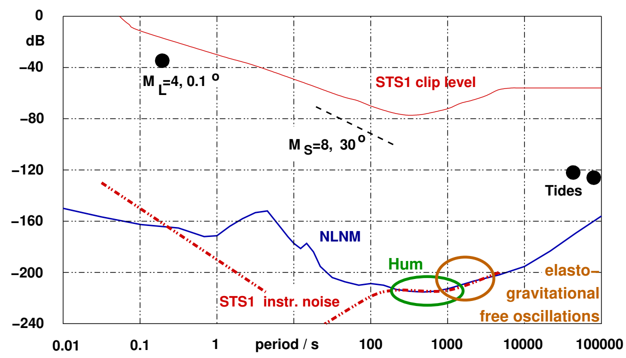
## 7.4 Transient signals

The recording of a seismometer may contain a transient signal (e. g. earthquake) together with some stochastic background. The level of the earthquake signal is not appropriately represented by PSD, because it is not stationary. The larger the time window which is analyzed, the less the earthquake signal (of finite energy) contributes to average power.

## 7.5 Harmonic signals

The recording of a seismometer may contain a harmonic signals (e. g. tidal signals or free oscillations) together with some stochastic background. The level of harmonic signals is not appropriately represented by PSD. They are of finite (and maybe stationary) amplitude and average signal power. Harmonic signals are phase-coherent over large time intervals, in contrast to stochastic signals. Because they exist at a single frequency, their power spectral *density* theoretically is infinite at this frequency. The PSD value obtained by the analysis of a time series of finite length will thus strongly depend on the length of the analyzed recording. The larger the time window, the larger the resulting apparent PSD.

This property of harmonic signals consequently results in a signal-to-noise ratio with respect to a stochastic background




(Courtesy of E. Wielandt, 2002, New Manual of Seismological Observatory Practice, ed. P. Bormann)

**Figure 1:** Comparison of different seismological signals, the dynamic range of the STS-1 and the New Low Noise Model (NLNM) by Peterson (1993). The values given are for a local earthquake ($M_\mathrm{L} = 4$, $\Delta = 10\,\mathrm{km}$), a teleseismic earthquake ($M_\mathrm{S} = 8$, $\Delta = 30°$) and the semi-diurnal and diurnal tides. The so-called *hum* names the *background free oscillations*, a resonant response of the Earth's body to broadband forcing due to the turbulent atmosphere and oceans at its surface. The signal levels are given in decibels relative to $1\,\mathrm{m\,s^{-2}}$. They can be understood as rms values in a bandwidth of 1/6 decade or as mean peak values in a bandwidth of 1/3 octave.

signal, which depends on the length of the available recording interval. The longer the time series, the larger the chance to detect small amplitude harmonics. Zürn (1974) demonstrated that even harmonic signals with an amplitude smaller than the least-significant bit of the digitizer (i. e. signals with amplitude smaller than quantization noise) might be detected if the time series is sufficiently long.

## 7.6 Graphing clip level

Notwithstanding what was discussed above in sections 7.4 and 7.5, transient signals, harmonic signals, and levels of stochastic noise are charted together in Figure 1. This is correct because signal level is not specified in terms of PSD but in terms of a time-domain signal amplitude in a finite, well specified bandwidth. This may be a reason to prefer diagrams of rms-amplitude.

The only disputable element in the diagram is the 'STS1 clip level'. This curve specifies the amplitude level of a harmonic signal at which the instrument would saturate is exceeded. The curve displayed (rather than a single data point like in the case of tides) might suggest that this level can be applied to a broadband signal. This however is not correct. Saturation takes place in the time domain. What limits the instrument is a maximum amplitude of the instantaneous time domain signal. Consider a stochastic signal of constant and finite PSD $2\tilde{P}(\omega) = P_\mathrm{const}$. If not band-limited its total time domain

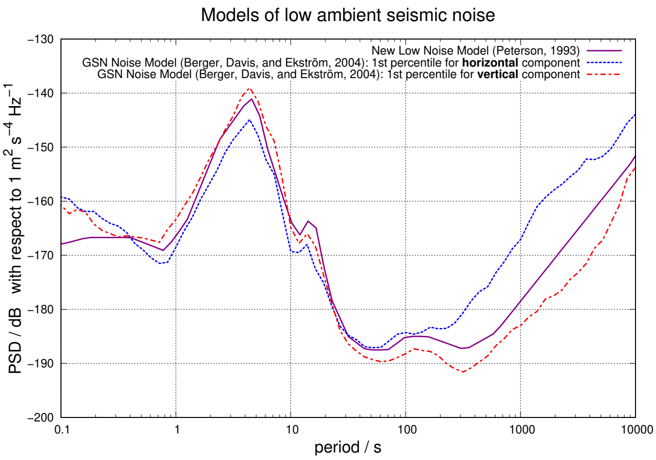



**Figure 2:** Power spectral density as defined by the low-noise models by Peterson (1993) and Berger et al. (2004). The specified values of the one-sided spectral power density are given in decibels with respect to $1\,\mathrm{m}^2\,\mathrm{s}^{-4}\,\mathrm{Hz}^{-1}$. All values are based on observations from the Global Seismographic Network (GSN). Berger et al. (2004) distinguish between signals of the horizontal components on the one hand and vertical components on the other hand.

rms-amplitude is

$$a_{\mathrm{rms}} = \lim_{f_1=0\,\mathrm{Hz}, f_2\to\infty} \sqrt{P_{f_1,f_2}}$$

$$= \lim_{f_1=0\,\mathrm{Hz}, f_2\to\infty} \sqrt{\int_{f_1}^{f_2} P_{\mathrm{const}}\,\mathrm{d}f.} \to \infty \qquad (49)$$

infinite, despite its finite (and arbitrary small) level $P_{\mathrm{const}}$ of PSD. Remember what has been said in section 3.4. A signal may have a finite PSD, which certainly always goes along with a finite level of rms-amplitude in a finite bandwidth. However total signal power and total rms-amplitude will only be finite if the signal is band-limited.

A signal may saturate the instrument (i. e. exceed the clip level) even if the amplitude of a band-limited part of the signal is below clip level at each frequency. Only harmonic signals are reliably band-limited without further specification.

# 8 Low-noise models

So-called 'low-noise models' specify PSD as the lower envelope of a large collection of recorded signals. They do not represent the signal level of a single time series. Well established low-noise models for the Global Seismographic Network (GSN) are those published by Peterson (1993, so-called New Low Noise Model, NLNM) and a newer one by Berger et al. (2004, so-called GSN noise model). Figure 2 displays both of them. They represent the lowest signal level at each frequency observed in a well-defined subset of all recordings in the GSN. Both differ in the approach of statistical analysis applied to the signals.

Although the PSD levels specified by the models do not represent the PSD of single recording, they are commonly used as reference values. A seismometer with a self-noise level below the low-noise curve would qualify to be able to detect the signals of smallest amplitude at any GSN-station. If a seismometer station produces signals with PSD above the low-noise models it is not able to compete with the best stations in the GSN. Low-noise models thus also serve as a quality standard for instruments and seismometer sites. This does mean that the signal level specified by the low-noise model represents a smallest level of ground motion originating from the Earth's interior in all cases (see below in section 8.3). Future observational techniques shall be able to detect signals below the level of the low-noise models and in fact gravimeters with appropriate signal correction are already able to do so as demonstrated by Zürn and Widmer (1995, Figure 1) and Rosat and Hinderer (2011, Figure 1) for observations at frequencies below $1\,\mathrm{mHz}$.

## 8.1 The low-noise model by Peterson (1993)

A classic and still frequently used low-noise model is the so-called *New Low Noise Model* (NLNM) by Peterson (1993). He examined the data from 75 stations of the GSN and two additional stations. For each station he selected three quiet time windows, each 24 hours long. For each component of the seismometer, Peterson calculated the values of the power spectral density in these time windows and averaged them. Values of PSD are given for the one-sided spectral power density (Bormann and Wielandt 2012b, Section 4.4.4) as defined in section 3.3 above. The NLNM forms the lower envelope of the family of curves for all results (for all stations and all components), which was found by graphically fitting linear segments in decibels (Peterson 1993, Fig. 13).

For comparison with amplitudes in a finite frequency band, Peterson (1993, Fig. 18) additionally gives average peak-to-peak amplitudes in 1/3 octave. He defines this as twice the mean values of the envelope of narrow-band filtered, normally distributed noise signals (Rice 1945, Section 3.7). The calculation shows that the mean peak values defined in this way in 1/3 octave are approximately equal (factor 0.972) to the rms values in 1/6 decade (see section 4.2 above). Peterson's definition of the amplitude values seems unnecessarily complicated.

Peterson misses to explicitly specify signal level being the onesided PSD. A quantitative conversion of PSD (Peterson 1993, Fig. 15) into average peak-to-peak values (Peterson 1993, Fig. 18) however, leaves no room for doubt. The average peak-to-peak amplitude in 1/3 octave is

$$a_{\mathrm{PP}}(f_c; 1/3\,\mathrm{octave})$$

$$= 2\sqrt{\frac{\pi}{2}\,\frac{0.232}{0.386}}\,a_{\mathrm{rms}}(f_c; 1/6\,\mathrm{decade})$$

$$= \sqrt{2\pi}\,\sqrt{2\tilde{P}(2\pi\,f_c)\,f_c\,0.232}. \qquad (50)$$

If levels are given in decibels for one-sided PSD with respect to $1\,\mathrm{m}^2\,\mathrm{s}^{-4}\,\mathrm{Hz}^{-1}$,

$$V = 10\,\log_{10}\left(\frac{2\pi\,f_c\,0.232}{1\,\mathrm{Hz}}\right)\,\mathrm{dB} \qquad (51)$$

must be added to obtain the value for average peak-to-peak amplitude in 1/3 octave specified in decibels with respect to $1\,\mathrm{m}\,\mathrm{s}^{-2}$.

## 8.2 The low-noise model by Berger et al. (2004)

Berger et al. (2004) created an updated low-noise model for the GSN. They used the records of 118 stations covering a whole year (July 2001 up to and including June 2002). They split data into time windows, where the length of the individual window (between one hour and one day) depends on the sampling rate of the data stream. Windows with obvious data problems (gaps, malfunctioning instruments, etc) were discarded. Appropriately scaled Fourier amplitude spectra after smoothing in the frequency domain then provide one-sided PSD for each time window. Berger et al. (2004) in their Figs. 6 and 7 present the frequency distribution of PSD values for the entire data set at each frequency for the vertical components and the horizontal components, respectively. The authors then take quantiles of the PSD values at each given frequency. In the supplement they provide the levels of 1st percentile, 5th percentile, 1st quartile, and median in each recorded channel individually. Fig. 4 of Berger et al. (2004) summarizes these levels for all channels in comparison to the NLNM. Their Fig. 9 sorts the values by instrument type, clearly indicating that the seismometers of type STS-1 provide the lowest level of detection threshold at frequencies below $10\,\mathrm{mHz}$. These instruments therefore define the low noise model for vertical components in this frequency band.

Table 2 in the paper specifies the final *GSN noise model* in terms of the 1st percentile for the vertical components and the horizontal components of the entire data set separately. Figure 2 in the current lecture notes displays the GSN noise model in comparison with the NLNM.

The noise power of the horizontal components is about $10\,\mathrm{dB}$ larger than that of the vertical components at signal periods longer than $300\,\mathrm{s}$. This is due to tilting of the instruments caused by crustal deformation as a consequence of loading forces on the Earth's surface due to fluctuating air-pressure. Tilting of the instrument changes the component of gravity being coupled into the sensitive axis of the seismometer. This effect is linear with respect to tilt angle in horizontal components and quadratic (and therefore negligible at small angles) for vertical components.

## 8.3 Common properties of low-noise models

The processes which control the level of low-noise models are largely understood but different in different frequency ranges. In the order of decreasing frequency they are the following.

At frequencies above $1\,\mathrm{Hz}$ ground vibrations dominate the background level. They typically originate from local sources, which can be man-made (traffic, industry, etc) or natural (rivers, dynamic forces of wind on the Earth's surface, etc). Force-balance feedback seismometers are not able to resolve the smallest levels of ground vibrations at frequencies larger than about $10\,\mathrm{Hz}$. This is because this type of instruments needs detecting mass displacement, which becomes smaller and smaller with increasing frequency. At these large frequencies passive seismometers with electrodynamic (velocity-)transducer provide a lower level of self-noise and are able to resolve ground vibrations of smallest amplitude.

A signal produced by fluctuating pressure at the sea bottom in the oceans can be observed throughout the continents in the frequency band between $0.02\,\mathrm{Hz}$ and $1\,\mathrm{Hz}$. This signal is called the *marine microseisms* and exact frequencies depend on the size of the ocean producing the signal. Typically we see amplified signal levels at $0.07\,\mathrm{Hz}$ (the peak of the so-called *primary marine microseisms*) and at $0.14\,\mathrm{Hz}$ (the larger peak of the so-called *secondary marine microseisms*). Webb (1992, 1998, 2002) describes the oceanographic processes driving this signal which even elevate the curves of low-noise models. Longuet-Higgins (1950) contributes the theory for the coupling mechanism, which allows oceans to act as very effective seismic sources.

At frequencies smaller then $10\,\mathrm{mHz}$ other forces than those of inertial acceleration start to contribute. Contributions of inertial acceleration decrease with decreasing frequency. A major source of background noise at these small frequencies is the Earth's atmosphere with ever fluctuating mass density, resulting in fluctuating gravitational attraction and fluctuating deformation of Earth's crust due to surface loading by air pressure. There are clear indications that for the vertical component recordings the atmospheric effects cancel near $3\,\mathrm{mHz}$ as motivated by Zürn and Wielandt (2007) and demonstrated by Zürn and Meurers (2009). As a consequence vertical component background noise level at frequencies between about $2\,\mathrm{mHz}$ and $10\,\mathrm{mHz}$ is controlled by the so-called background free oscillations (the so-called *hum*) as demonstrated by Berger et al. (2004, Fig. 8). Benioff et al. (1959) first proposed the existence of this signal which represents the resonant response of the Earth to broadband, random forcing at the surface. With modern instruments the hum is clearly identified in the vertical component (Kobayashi and Nishida 1998; Nawa et al. 1998; Suda et al. 1998) as well as in horizontal component (Kurrle and Widmer-Schnidrig 2008) recordings.

Zürn and Widmer (1995, Fig. 1) first demonstrated that the signal level of the NLNM below $1\,\mathrm{mHz}$ does not represent the smallest signals originating from Earth's interior. It is widely accepted that detection level of gravimeters can be lowered below the level of NLNM by an appropriate correction for signals originating in the atmosphere (Rosat and Hinderer 2011). Zürn and Wielandt (2007) have proposed physical models of the coupling mechanism, which can successfully be used for signal correction in many cases. It is also accepted that the level of the horizontal component low-noise model at frequencies below $1\,\mathrm{mHz}$ is controlled by tilting of the seismometers (Wielandt 2012; Zürn et al. 2007). This is caused by pressure fluctuations at the surface of Earth which cause a deformation of the subsurface and tilting of the instrument and thus couple a component of gravity into the horizontal components. Whether the level of the vertical component low-noise models at frequencies below $1\,\mathrm{mHz}$ represents the smallest signal level originating from the atmosphere or whether it represents the instrumental noise of the seismometers of type STS-1 in the GSN (Berger et al. 2004, Fig. 9) currently is under debate and is subject of ongoing research.

# 9 Further reading

The classic text on the estimation of power spectral density was written by Blackman and Tukey (1958). Welch (1967) demonstrated the equivalence of averaging over time windows on the one hand and averaging over adjacent index $k$ on the other hand. Jenkins and Watts (1968) further develop concepts of random signals with a sound background of statistical theory. They further develop the theory of Parceval's theorem to the quantity of *cross-power spectral density*, which allows the computation of signal *coherence*. The application of power spectral density to signals in seismology is discussed by Bormann and Wielandt (2012b).

# References


Aki, K. and P. G. Richards (2002). *Quantitative Seismology*. 2nd ed. Sausalito: University Science Books. URL: `https://services.bibliothek.kit.edu/primo/start.php?recordid=KITSRC104011955` (visited on Oct. 19, 2018).

Benioff, H., J. C. Harrison, L. LaCoste, W. H. Munk, and L. B. Slichter (1959). "Searching for the Earth's free oscillations". In: *J. Geophys. Res.* 64.9, pp. 1334–1337. DOI: `10.1029/JZ064i009p01334`.

Berger, J., P. Davis, and G. Ekström (2004). "Ambient Earth noise: A survey of the Global Seismographic Network". In: *J. Geophys. Res.* 109.B11. B11307, B11307. ISSN: 2156-2202. DOI: `10.1029/2004JB003408`.

Blackman, R. B. and J. W. Tukey (1958). *The measurement of power spectra*. New York: Dover Publications. URL: `https://services.bibliothek.kit.edu/primo/start.php?recordid=KITSRC064221814` (visited on Oct. 19, 2018).

Bormann, P. and E. Wielandt (2012a). "Bandwidth-dependent transformation of noise data from frequency into time domain and vice versa". In: *New Manual of Seismological Observatory Practice 2 (NMSOP-2)*. Ed. by P. Bormann. Exercise 4.1. Potsdam, Germany: GeoForschungsZentrum GFZ, pp. 1–8. DOI: `10.2312/GFZ.NMSOP-2_EX_4.1`.

— (2012b). "Seismic Signals and Noise". In: *New Manual of Seismological Observatory Practice 2 (NMSOP2)*. Ed. by P. Bormann. 2nd ed. online resource. Potsdam, Germany: GeoForschungsZentrum. Chap. 4. DOI: `10.2312/GFZ.NMSOP-2_ch4`.

Havskov, J. and G. Alguacil (2016). *Instrumentation in Earthquake Seismology*. 2nd ed. online resource. Cham: Springer. ISBN: 978-3-319-21314-9. DOI: `10.1007/978-3-319-21314-9`.

Jenkins, G. M. and D. G. Watts (1968). *Spectral analysis and its applications*. San Francisco: Holden-Day.

Kobayashi, N. and K. Nishida (1998). "Continuous excitation of planetary free oscillations by atmospheric disturbances". In: 395.6700, pp. 357–360. DOI: `10.1038/26427`.

Kurrle, D. and R. Widmer-Schnidrig (2008). "The horizontal hum of the Earth: A global background of spheroidal and toroidal modes". In: *Geophys. Res. Lett.* 35.6. DOI: `10.1029/2007GL033125`.

Longuet-Higgins, M. S. (1950). "A theory of the origin of microseisms". In: *Phil. Trans. Roy. Soc. London, Ser. A* 243, pp. 1–35. DOI: `10.1098/rsta.1950.0012`.

Nawa, K., N. Suda, Y. Fukao, T. Sato, Y. Aoyama, and K. Shibuya (1998). "Incessant excitation of the Earth's free oscillations". In: *Earth Planets and Space* 50.1, pp. 3–8. DOI: `10.1186/BF03352080`.

Parker, M. A. (2010). *Digital signal processing: everything you need to know to get started*. eBook. online resource. Amsterdam: Newnes. ISBN: 978-1-282-73831-7. URL: `http://lib.myilibrary.com/detail.asp?id=273831`.

Peterson, J. (1993). *Observations and modelling of background seismic noise*. Open-file report 93-322. online resource. Albuquerque, New Mexico: U.S. Geological Survey. DOI: `10.3133/ofr93322`.

Rice, S. O. (Jan. 1945). "Mathematical analysis of random noise". In: *The Bell System Technical Journal* 24.1, pp. 46–156. ISSN: 0005-8580. DOI: `10.1002/j.1538-7305.1945.tb00453.x`.

Rosat, S. and J. Hinderer (2011). "Noise Levels of Superconducting Gravimeters: Updated Comparison and Time Stability". In: *Bull. Seism. Soc. Am.* 101.3, pp. 1233–1241. DOI: `10.1785/0120100217`.

Scherbaum, F. (1994). *Basic concepts in digital signal processing for seismologists*. Lecture notes in earth sciences; 53. online resource. Berlin: Springer. ISBN: 0-387-57973-7. DOI: `10.1007/BFb0017794`.

Suda, N., K. Nawa, and Y. Fukao (1998). "Earth's background free oscillations". In: 279.5359, pp. 2089–2091. DOI: 10.1126/science.279.5359.2089.

Taylor, B. N. (2008). *The International System of Units (SI)*. 2008 Edition. 330. online resource. National Institute of Standards and Technology (NIST). DOI: 10.6028/NIST.SP.330e2008.

Webb, S. C. (1992). "The equilibrium oceanic microseism spectrum". In: *J. Acoust. Soc. Am.* 92.4, pp. 2141–2157. DOI: 10.1121/1.405226.

— (1998). "Broadband seismology and noise under the ocean". In: *Rev. Geophys.* 36.1, pp. 105–142. DOI: 10.1029/97RG02287.

— (2002). "Seismic Noise on Land and on the Sea Floor". In: *International Handbook of Earthquake and Engineering Seismology*. Ed. by W. H. K. Lee, H. Kanamori, P. C. Jennings, and C. Kisslinger. Vol. Volume 81A. International Geophysics Series. London: Academic Press. Chap. 19, pp. 305–318. URL: https://services.bibliothek.kit.edu/primo/start.php?recordid=KITSRC103629610 (visited on Oct. 8, 2020).

Welch, P. D. (1967). "The Use of Fast Fourier Transform for the Estimation of Power Spectra: A Method Based on Time Averaging Over Short, Modified Periodograms". In: *IEEE Trans. Audio Electroacoust.* 15.2, pp. 70–73. DOI: 10.1109/TAU.1967.1161901.

Wielandt, E. (2000). *Seismometry*. preprint of 'Seismometry' by Erhard Wielandt (2002). Institut für Geophysik, Universität Stuttgart. URL: http://www.software-for-seismometry.de/textfiles/Seismometry/handbook.ps (visited on Feb. 4, 2019).

— (2012). "Seismic Sensors and their Calibration". In: *New Manual of Seismological Observatory Practice*. Ed. by P. Bormann. 2nd ed. online resource. Potsdam, Germany: GeoForschungsZentrum. Chap. 5. DOI: 10.2312/GFZ.NMSOP-2_ch5.

Zürn, W. (1974). "Detectability of Small Harmonic Signals in Digitized Records". In: *J. Geophys. Res.* 79.29, pp. 4433–4438. DOI: 10.1029/JB079i029p04433.

Zürn, W., J. Exß, S. H., C. Kroner, T. Jahr, and M. Westerhaus (2007). "On reduction of long-period horizontal seismic noise using local barometric pressure". In: *Geophys. J. Int.* 171, pp. 780–796. DOI: 10.1111/j.1365-246X.2007.03553.x.

Zürn, W. and B. Meurers (2009). "Clear evidence for the sign-reversal of the pressure admittance to gravity near 3 mHz". In: *J. Geodyn.* 48, pp. 371–377. DOI: 10.1016/j.jog.2009.09.040.

Zürn, W. and R. Widmer (1995). "On noise reduction in vertical seismic records below 2 mHz using local barometric pressure". In: *Geophys. Res. Lett.* 22.24, pp. 3537–3540. DOI: 10.1029/95GL03369.

Zürn, W. and E. Wielandt (2007). "On the minimum of vertical seismic noise near 3 mHz". In: *Geophys. J. Int.* 168, pp. 647–658. DOI: 10.1111/j.1365-246X.2006.03189.x.