Bound states in bent soft waveguides

Pavel Exner, Semjon Vugalter

KARLSRUHE INSTITUTE OF TECHNOLOGY

CRC 1173

Participating universities

Universität Stuttgart

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ABSTRACT. The aim of this paper is to show that a two-dimensional
Schrödinger operator with the potential in the form of a 'ditch' of
a fixed profile can have a geometrically induced discrete spectrum;
this happens if such a potential channel has a single or multiple
bends being straight outside a compact. Moreover, under stronger
geometric restrictions the claim remains true in the presence of a
potential bias at one of the channel 'banks'.

1. INTRODUCTION

Behavior of quantum particles confined to tubular regions attracted
a lot of attention in the last decades with the motivation coming from
two sources. On the physics side it was the possibility to use such
models to describe a guided dynamics in various condensed matter
systems. At the same time, this appeared to be a source of interesting
mathematical problems, in particular, those concerning spectral effects
coming from the geometry of the confinement; for an introduction to
the topic and a bibliography we refer to the book [EK15].

There are different ways how to localize a particle in the configuration
space. One possibility is a hard confinement where the Hamiltonian is
typically the Dirichlet Laplacian associated with a tube in $\mathbb{R}^d$ (or more
complicated regions such as layers, networks, etc.). From the point of
view of application to objects like semiconductor wires such a model
has a drawback; it does not take into account the tunneling between
different parts the waveguide. This fact motivated investigation of the
'leaky' confinement in which the Hamiltonian is instead Schrödinger
operator with an attractive singular interaction supported by a curve
(or a surface, metric graph, etc.); to have it well defined, the codimen-
sion of the interaction support must not exceed three.

If we stay for simplicity in the two dimensional situation, both mod-
els exhibit *curvature-induced bound states*: whenever the strip, or the
curve supporting the δ interaction, is non-straight but asymptotically
straight, the corresponding Hamiltonian has a non-void discrete spec-
trum; this claim is valid universally modulo technical requirements on
the regularity and asymptotic behavior.

Leaky guide model has another drawback in assuming that the inter-
action support has zero width. This motivated recently investigation

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of a more realistic situation when the potential in the Schrödinger operator is regular in the form of a channel of a fixed profile [Ex20]. The term coined was *soft waveguides*; the analogous problem was studied in three dimensions [Ex22] as well as for soft layers [EKP20, KK22]. One has to add that such operators were considered before [EI01, WT14], however, the focus was then on the limit in which the potential shrinks transversally to a manifold; in the physics literature the idea of determining the right 'quantization' on a manifold through such a limit was examined a long time ago [KJ71, To88].

Not very surprisingly, soft waveguides were already shown to share properties with their hard and leaky counterparts, an example is the ground state optimization in a loop-shaped geometry [EL21]. Some results have also been obtained concerning the problem we are interested in here, the existence of curvature-induced bound states, however, so far they lack the universal character indicated above. In [Ex20] Birman-Schwinger principle was used to derive a sufficient condition under which the discrete spectrum is nonempty, expressed in terms of of positivity of a certain integral which, in general, is not easy to evaluate. An alternative is to apply the variational method; in this way the existence was established in the example of a particular geometry, often referred to as a 'bookcover' [KKK21]. We note in passing that it is paradoxically easier to establish the existence in conic-shaped soft layers, where the discrete spectrum is infinite [EKP20, KK22].

The trouble with the variational approach is that it is not easy, beyond the simple example mentioned, to find a suitable trial function. The aim of this paper is to extend the existence result using a variational method to a much wider, even if still not optimal class of soft waveguides. The main restrictions in our analysis are the limitation of the curved part into a bounded region, a compact support of the potential defining the channel profile, and the requirement of the profile symmetry. The latter restriction can be relaxed in some situations, in particular, if the profile potential is sign-changing and the transverse part of operator, the operator (2.2) below, has zero-energy resonance.

We will also consider the situation when the system has a constant positive potential bias in one of the regions separated by the profile potential support. In this case we have a stronger geometry restriction: we have to assume that one of the two regions is *convex*. If the bias potential is supported in it, we can again prove the existence of a discrete spectrum, even without the symmetry assumption. If the bias is supported in the opposite region, we have the existence again, however, except in the situation when the operator (2.2) has a zero-energy resonance; this is in agreement with the result of [EV16] where we treated a system which can be regarded as a singular version of the present system. Let us stress that the convexity makes it also possible

to prove the existence in the absence of the bias and the symmetry restriction, provided that operator (2.2) has a negative eigenvalue.

In the following section we will state the problem in proper terms and present the main results. The rest of the paper is devoted to the proofs. The next two sections deal with case without the bias; in Sec. 3 we prove part (a) of Theorem 2.2 which concerns the situation when the operator (2.2) has a zero-energy resonance, Sec. 4 provides the proof of part (a) of Theorem 2.4 which addresses the case when the operator has a negative eigenvalue and the channel profile is symmetric. Finally, in Sec. 5 we prove parts (b) of the two theorems which establish the existence results in the situation when one of the two regions to which the potential channel, not necessarily symmetric, divides the plane is convex, even in the absence of the bias, except in the zero-energy resonance case.

2. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let us now state the problem described in the introduction. We begin with the assumptions which are split into two groups; the first one concerns the support of the potential, the other the channel profile. The former is a strip built around a curve Γ, understood as the graph of a function $\Gamma : \mathbb{R} \to \mathbb{R}^2$ such that $|\dot{\Gamma}(s)| = 1$. Without repeating it at every occasion we always exclude the trivial situation when Γ is a straight line; in addition to that we suppose:

(s1) Γ is C^3-smooth, non-straight but straight outside a compact; its curved part consists of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved,

(s2) $|\Gamma(s_+) - \Gamma(s_-)| \to \infty$ as $s_\pm \to \pm\infty$, in other words, the two straight parts of Γ are either not parallel, or if they are, they point in the opposite directions,

(s3) the strip neighborhood $\Omega^a := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < a\}$ of Γ with a halfwidth $a > 0$ does not intersect itself.

Assumption (s3) has various equivalent expressions: one can say, for instance, that the function $\operatorname{dist}(x, \Gamma(\cdot))$ has for any fixed $x \in \Omega^a$ a unique minimum, or that the map

$$x(s,t) \mapsto \left(\Gamma_1(s) - t\Gamma_2'(s), \Gamma_2(s) + t\Gamma_1'(s) \right) \tag{2.1}$$

from the straight strip $\Omega_0^a := \mathbb{R} \times (-a, a)$ to $\mathbb{R}^2$ is a bijection, in fact, a diffeomorphism; $\vec{n}(s) = (-\Gamma_2'(s), \Gamma_1'(s))$ is, of course, the (inward) normal to the curve at the point $\Gamma(s)$. Under assumption (s1), the signed curvature $\kappa : \kappa(s) = (\Gamma_2'\Gamma_1'' - \Gamma_1'\Gamma_2'')(s)$ is smooth and compactly supported function; a necessary, but in general not sufficient condition for (s3) to hold is $a\|\kappa\|_\infty < 1$ which ensures the local injectivity of the map. The curve divides the plane into open regions which we denote
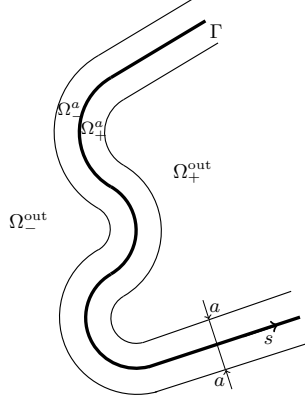
FIGURE 1. Scheme of the waveguide

as $\Omega_\pm$; for the sake of definiteness we assume that Ω_+ is *at the left side*
when one looks in the direction of the increasing arc length variable s.

We also introduce $\Omega^a_\pm := \Omega_\pm \cap \Omega^a$ so that we have $\Omega^a = \Omega^a_+ \cup \Gamma \cup \Omega^a_-$;
given our choice of the normal orintation, the labels correspond to the
sign of the transversal variable t. Finally, we will use a natural symbol
for the complement of the strip, namely $\Omega_{\mathrm{out}} := \mathbb{R}^2 \setminus \Omega^a$, and its one-
sided components will be denoted as $\Omega^{\mathrm{out}}_\pm := \Omega_\pm \setminus \Omega^a$ – cf. Fig. 1.

The second group of assumptions concerns the potential. Its profile
is determined by a function $v \colon \mathbb{R} \to \mathbb{R}$ of which we assume

(p1) $v \in L^2(\mathbb{R})$ and $\operatorname{supp} v \subset [-a, a]$;

in some situations, specifically in part (a) of Theorem 2.4 below, we
will require it additionally to be mirror-symmetric,

(p2) $v(t) = v(-t)$ for $t \in [-a, a]$.

In addition to the potential defining the channel we are going to con-
sider, in general, also a one-sided potential bias of the system. To this
aim, we introduce the one-dimensional Schrödinger operator

$$h := -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0 \chi_{[a,\infty)}(t), \quad V_0 \geq 0. \qquad (2.2)$$

The crucial role will be played by the spectral bottom of this operator,
specifically we will be concerned with the following two possibilities:

(p3) $\inf \sigma(h)$ is a negative (ground state) eigenvalue μ associated with
a real-valued eigenfunction ϕ_0 which we may without loss of gen-
erality normalize by the requirement $\phi_0(-a) = 1$,

(p4) operator h has a zero-energy resonance, meaning that $h \geq 0$ and
$-(1-\varepsilon)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + v(t) + V_0 \chi_{[a,\infty)}(t)$ has a negative eigenvalue for any
$\varepsilon > 0$. In this case, the equation $h\phi = 0$ has a real-valued solution
$\phi_0 \in H^2_{\mathrm{loc}}(\mathbb{R})$ not increasing at infinity; it will be again supposed
to satisfy the normalization condition $\phi_0(-a) = 1$.

The main object of our interest is the Schrödinger operator

$$H_{\Gamma,V} = -\Delta + V(x) \qquad (2.3a)$$

on $L^2(\mathbb{R}^2)$ with the potential defined using the locally orthogonal coordinates (s,t) appearing in (2.1) as

$$V(x) = \begin{cases} v(t) & \text{if } x \in \Omega^a \\ V_0 & \text{if } x \in \Omega_+ \setminus \Omega^a \\ 0 & \text{otherwise} \end{cases} \qquad (2.3b)$$

We will often drop the subscript of $H_{\Gamma,V}$ if it is clear from the context.

Proposition 2.1. *Under the assumptions (s1)–(s3), (p1) and (p3), the operator* (2.3) *is self-adjoint, $D(H_{\Gamma,V}) = H^2(\mathbb{R}^2)$, and $\sigma_{\text{ess}}(H_{\Gamma,V}) = [\mu, \infty)$. If $h \geq 0$, the same is true with $\mu = 0$.*

Proof. The self-adjointness is easy to check; it is sufficient to ascertain that the potential (2.3b) is infinitely small with respect to $-\Delta$, that is, to any $a > 0$ there is a $b > 0$ such that $\|V\psi\| \leq a\|\Delta\psi\| + b\|\psi\|$ holds for all $\psi \in H^2(\mathbb{R}^2)$. Suppose first that $V_0 = 0$. In view of assumption (p1) we can use Kato-Rellich theorem [RS, Sec. X.1]. We decompose any given ψ into a sum $\psi = \psi_+ + \psi_0 + \psi_-$ of H^2 functions such that $\text{supp}\,\psi_\pm \restriction_{\Omega^a}$ lies in the straight parts of the strip and $\text{supp}\,\psi_0 \restriction_{\Omega^a}$ contains the curved part. To the latter the theorem applies directly since ψ_0 is essentially bounded with bounded support so that $V \restriction_{\text{supp}\,\psi_0} \in L^2$. In the straight parts we get first using (p1) the one-dimensional version of the inequality in the transverse variable, then we lift it to two dimensions using the fact that $\|\partial_t^2\psi\| \leq \|\Delta\psi\|$. The constants b_j, $j = 0, \pm$, in the obtained inequalities are in general different; we put $b := \sqrt{3}\max\{b_+, b_0, b_-\}$. Using then the triangle and Schwarz inequalities which, in particular, give $\|\psi\| \leq \sqrt{3}\sum_j \|\psi_j\|$, we arrive at the desired conclusion. Finally, the self-adjointness is not affected by adding the bounded potential $V_0\chi_{\Omega_+^{\text{out}}}$.

The identification of the essential spectrum of $H_{\Gamma,V}$ with the interval $[\mu, \infty)$, where $\mu = \inf\sigma(h)$, was established in [Ex20, Proposition 3.1] under slightly different assumptions. The argument can be easily modified for our present purpose; the requirement on the smoothness of Γ we made is stronger than there, and neither the substitution of a bounded negative v by a possibly sign indefinite square integrable one, nor the addition of a potential bias alters the conclusion. $\quad\square$

Note also that the above Hamiltonian can be investigated using the associate quadratic form $Q_{\Gamma,V}$, mostly written without the indices specifying the curve and the potential, and defined by

$$Q[\psi] = \|\nabla\psi\|^2 + \int_{\Omega^a} V(x)|\psi(x)|^2\,\mathrm{d}x, \quad D(Q) = H^1(\mathbb{R}^2); \qquad (2.3c)$$

in the presence of a potential bias we have instead formula (**??**) below.

Now we are in position to state our main results. The assumptions may appear in various combinations; we group them according the according to the spectral threshold μ starting with the situation when operator (2.2) has a zero-energy resonance:

Theorem 2.2 (threshold resonance case). *Assume (s1)–(s3), (p1) and (p4); then the following claims are valid:*
(a) If the bias is absent, $V_0 = 0$, and

$$[\phi_0(a)^2 - \phi_0(-a)^2] \int_{\mathbb{R}} \kappa(s)\,\mathrm{d}s \leq 0 \qquad (2.4)$$

holds, then $H_{\Gamma,V}$ has at least one negative eigenvalue.
(b) The same is true if $V_0 > 0$ and Ω_+ is convex.

Remark 2.3. Recall that κ does not vanish identically. The condition (2.4) is naturally satisfied if $\phi_0(a) = \phi_0(-a)$, in particular, under the mirror-symmetry assumption (p2). Consider further the asymmetric situation, $\phi_0(a) \neq \phi_0(-a)$, and recall that the integral in (2.4) equals $\pi - 2\theta$ where 2θ is the angle between the asymptotes. Consequently, at least one bound state exists then in the zero-energy resonance case if the asymptotes of Γ are parallel and pointing in the opposite directions, $\theta = \frac{1}{2}\pi$, or if they are not parallel and the resonance solution ϕ_0 is larger at the 'outer' side of the strip Ω^a.

If h has negative eigenvalues so that $\mu < 0$, the situation is more complicated and we have to make stronger restrictions on the profile or the shape of the waveguide:

Theorem 2.4 (eigenvalue case). *Assume (s1)–(s3) together with (p1) and (p3). Then $\sigma_{\mathrm{disc}}(H_{\Gamma,V})$ is nonempty under any of the following conditions:*
(a) $V_0 = 0$ and assumption (p2) is satisfied.
(b) $V_0 \geq 0$ and one of the regions $\Omega_\pm$ is convex.

3. Proof of Theorem 2.2 – the first part

With the later purpose in mind we will formulate the argument first in the general situation which involves both the bound-state and zero-energy-resonance cases as well as the possible potential bias. In view of Proposition 2.1, it is sufficient to construct a trial function $\psi \in H^1(\mathbb{R}^2)$ such that $Q[\psi] < \mu\|\psi\|^2$. Let us first fix the geometry. If the two straight parts of Γ are not parallel – cf. Fig. 1 – their line extensions intersect at a point which we choose as the origin O, and use polar coordinates with this center, in which the two halflines correspond to the angles $\pm\theta_0$ for the appropriate $\theta_0 \in (0, \frac{1}{2}\pi)$. Furthermore, we fix the point $s = 0$ in such a way that for large $|s|$ the points with the coordinates $\pm s$ have the same Euclidean distance from O.

If the asymptotes are parallel (and pointing in the opposite directions according to (s2)), we choose the origin as the point with equal distance from the endpoints of the two halflines. The point with $s = 0$ on the curve is likewise chosen so that those with the coordinates $\pm s$ have the same Euclidean distance from the origin; in both cases one can check easily that such a choice is unique.

3.1. **Trial function inside the strip.** For fixed values s_0, such that the points with coordinates $\pm s_0$ lay outside the curved part of Γ, and $s^* > s_0$, to be chosen later, we define

$$
\chi_{\text{in}}(s) := \begin{cases} 1 & \text{if } |s| < s_0 \\ \ln \frac{s^*}{|s|} \left(\ln \frac{s^*}{s_0} \right)^{-1} & \text{if } s_0 \leq |s| \leq s^* \\ 0 & \text{if } |s| > s^* \end{cases} \qquad (3.1)
$$

Recalling that ϕ_0 is the ground-state eigenfunction or the zero-energy solution normalized by $\phi_0(-a) = 1$, we put

$$
\psi(s,t) = \phi_0(t)\chi_{\text{in}}(s) + \nu g(s,t), \quad |t| \leq a, \qquad (3.2)
$$

where the parameter ν and the function g, compactly supported within $(-s_0, s_0) \times (-a, a)$, will be chosen later. We denote by $Q_{\text{int}}[\psi]$ the contribution to the shifted quadratic form, $Q[\psi] - \mu\|\psi\|^2$, coming from the strip Ω^a, which can be using the parallel coordinates expressed as

$$
Q_{\text{int}}[\psi] = \int_{|t| \leq a} \left\{ \left(\frac{\partial \psi}{\partial s} \right)^2 (1 - \kappa(s)t)^{-1} + \left(\frac{\partial \psi}{\partial t} \right)^2 (1 - \kappa(s)t) \right.
$$
$$
\left. + (v(t) - \mu)|\psi|^2 (1 - \kappa(s)t) \right\} \mathrm{d}s\mathrm{d}t.
$$

The first term on the right-hand side can be estimated as

$$
\int_{|t| \leq a} \left(\frac{\partial \psi}{\partial s} \right)^2 (1 - \kappa(s)t)^{-1} \, \mathrm{d}s\mathrm{d}t \leq 2\tau_0^{-1} \|\phi_0 \restriction_{[-a,a]} \|^2 \|\chi_{\text{in}}'\|^2 + C\nu^2,
$$

where the norm refers to $L^2(\mathbb{R})$, $\tau_0 := 1 - a\|\kappa\|_\infty$ is positive by (s3), and C depends on g only; we will use the same letter for generic constants in the following. Note that choosing the parameter s^* in (3.1) large one can make the norm $\|\chi_{\text{in}}'\| = \left(\ln \frac{s^*}{s_0} \right)^{-1} \left(\frac{1}{s_0} - \frac{1}{s^*} \right)^{1/2}$ small. As for the

other two terms, we have

$$\int_{|t|\le a} \left\{ \left(\frac{\partial\psi}{\partial t}\right)^2 (1-\kappa(s)t) + (v(t)-\mu)|\psi|^2(1-\kappa(s)t) \right\} \mathrm{d}s\mathrm{d}t$$

$$= \int_{|t|\le a} \left\{ (\phi_0'(t))^2 + (v(t)-\mu)|\phi_0(t)|^2 \right\} \chi_{\mathrm{in}}^2(s)(1-\kappa(s)t)\, \mathrm{d}s\mathrm{d}t$$

$$+ 2\nu \int_{|t|\le a} \left\{ \phi_0'\frac{\partial g}{\partial t} + (v(t)-\mu)\phi_0 g \right\} \chi_{\mathrm{in}}(s)(1-\kappa(s)t)\, \mathrm{d}s\mathrm{d}t$$

$$+ \nu^2 \int_{|t|\le a} \left\{ \left(\frac{\partial g}{\partial t}\right)^2 + (v(t)-\mu)|g|^2 \right\}(1-\kappa(s)t)\, \mathrm{d}s\mathrm{d}t, \qquad (3.3)$$

where the last term on right-hand side can be again estimated by $C\nu^2$ with a C depending on the function g only. Furthermore, integrating the middle term by parts with respect to t, we get

$$2\nu \int_{|t|\le a} \left[-\phi_0'' + (v(t)-\mu)\phi_0 \right] \chi_{\mathrm{in}}(s) g(s,t)(1-\kappa(s)t)\, \mathrm{d}s\mathrm{d}t$$

$$- 2\nu \int_{|t|\le a} \phi_0'(t)\chi_{\mathrm{in}}(s) g(s,t)\kappa(s)\, \mathrm{d}s\mathrm{d}t, \qquad (3.4)$$

where the square bracket in the first integral is zero by assumption.

Notice next that ϕ_0' cannot vanish identically in the interval $[-a, a]$. Indeed, it is continuous in $\mathbb{R}$ and we have $v(t) = 0$ for $|t| > a$, hence should the derivative ϕ_0' be zero in $[-a, a]$, the function must have been a constant one, however, that is impossible for an eigenfunction or a zero-energy resonance solution. This observation allows us to choose the function g in such a way that the last integral is positive; it suffices to have it supported in a region where both ϕ_0' and κ do not change sign and to pick the sign of $g(s,t)$ accordingly. With such a choice the expression (3.4) will be smaller than $\delta\nu$ with some $\delta > 0$; for small ν this linear term will dominate over those estimated by $C\nu^2$.

It remains to deal with the first term on the right-hand side of (3.3). To simplify the notation, we introduce the following symbols,

$$\phi_+ = \phi_0(a), \quad \xi_+ = -\sqrt{|\mu|+V_0}, \quad \xi_- = \sqrt{|\mu|}. \qquad (3.5)$$

which allows us to write $\phi_0'(a) = \xi_+\phi_+$ and $\phi'(-a) = \xi_-$; recall that $\phi_0(-a) = 1$ holds by assumption. The expression in question then can

be rewritten using integration by parts as follows:

$$\int_{|t|\leq a}\big\{(\phi_0'(t))^2+(v(t)-\mu)|\phi_0(t)|^2\big\}\chi_{\mathrm{in}}^2(s)(1-\kappa(s)t)\,\mathrm{d}s\mathrm{d}t$$

$$=\int_{\mathbb{R}}\big[\xi_+\phi_+^2(1-\kappa(s)a)-\xi_-(1+\kappa(s)a)\big]\chi_{\mathrm{in}}^2(s)\,\mathrm{d}s$$

$$+\int_{|t|\leq a}\big\{-\phi_0''(t)+(v(t)-\mu)\phi_0(t)\big\}\phi_0(t)(1-\kappa(s)t)\chi_{\mathrm{in}}^2(s)\,\mathrm{d}s\mathrm{d}t$$

$$+\int_{|t|\leq a}\big(-\phi_0'(t)\phi_0(t)\big)(\kappa(s))\chi_{\mathrm{in}}^2(s)\,\mathrm{d}s\mathrm{d}t$$

$$=\big[\xi_+\phi_+^2-\xi_-\big]\|\chi_{\mathrm{in}}\|^2-\big[\xi_+\phi_+^2+\xi_-\big]a\int_{\mathbb{R}}\kappa(s)\chi_{\mathrm{in}}^2(s)\,\mathrm{d}s$$

$$+\frac{1}{2}(\phi_+^2-1)\int_{\mathbb{R}}\kappa(s)\chi_{\mathrm{in}}^2(s)\,\mathrm{d}s, \tag{3.6}$$

where the norm in the last expression refers to $L^2(\mathbb{R})$ and we have used the identity $\phi_0'\phi_0=\frac{1}{2}(\phi_0^2)'$. Since κ has a compact support and $\chi_{\mathrm{in}}^2(s)=1$ holds on it by (3.1), we can replace the integrals in the last part of (3.6) by $\int_{\mathbb{R}}\kappa(s)\,\mathrm{d}s$. Summarizing the estimate, we have obtained for all sufficiently small ν the inequality

$$Q_{\mathrm{int}}[\psi]\leq-\frac{1}{2}\delta\nu+\big[\xi_+\phi_+^2-\xi_-\big]\|\chi_{\mathrm{in}}\|^2-\big[\xi_+\phi_+^2+\xi_-\big]a\int_{\mathbb{R}}\kappa(s)\,\mathrm{d}s$$

$$+\frac{1}{2}(\phi_+^2-1)\int_{\mathbb{R}}\kappa(s)\,\mathrm{d}s+\tau_0^{-1}\|\phi_0\!\restriction_{[-a,a]}\|^2\|\chi_{\mathrm{in}}'\|^2. \tag{3.7}$$

Choosing then the coordinates $s^*\gg s_0$ at the right-hand side of (3.1), one can achieve that the last term in (3.7) will be smaller than $\frac{1}{4}\delta\nu$.

Now we finally use assumptions of Theorem 2.2. First of all, in the zero-energy resonance situation we have $\xi_\pm=0$, so that the second and the third term on the right-hand side vanish, and since the fourth one is supposed to be nonpositive, the estimate reduces to

$$Q_{\mathrm{int}}[\psi]\leq-\frac{1}{4}\delta\nu-2|\mu|^{1/2}\|\chi_{\mathrm{in}}\|^2. \tag{3.8}$$

At the same time, we have $\mu=0$ in this case, hence (3.8) simply becomes $Q_{\mathrm{int}}[\psi]\leq-\frac{1}{4}\delta\nu$ and to conclude the proof we have to choose the outer part of trial function in such a way that its contribution to the quadratic form can be made smaller than any fixed positive number.

3.2. **Trial function outside the strip.** In the zero-energy resonance case, $\mu=0$, the absence of the bias V_0 means that $\phi_0(t)=\mathrm{const}$ holds for $|t|\geq a$. To construct a suitable mollifier χ_{out} we require the following properties:

(i) in $\mathbb{R}^2\setminus\Omega^a$ the function depends on $\rho=\mathrm{dist}(x,O)$ only,

(ii) we have continuity at the boundary: at the points $x(s, \pm a)$ the relation $\chi_{\text{out}}(x) = \chi_{\text{in}}(s)$ holds.

Let us consider the situation where the extensions of the asymptotes of Γ cross; the case of parallel asymptotes pointing in the opposite directions can dealt with analogously. We again choose s_0 in such a way that the points $\Gamma(\pm s_0)$ belong to the straight parts of the curve, then $\text{dist}(\Gamma(s), O) = \rho_s := (|s| - s_0) + d_0$, where $d_0 = \text{dist}(\Gamma(s_0), O)$ (recall that $\Gamma(-s_0) = \Gamma(s_0)$ holds by assumption).

Given that the distance of the points $x(s, \pm a)$ from the origin is $\sqrt{\rho_s^2 + a^2}$, in accordance with the requirements (i), (ii) we put

$$
\chi_{\text{out}}(\rho) := \begin{cases} \chi_{\text{in}}(\sqrt{\rho^2 - a^2} - d_0 + s_0) & \text{if } \sqrt{\rho^2 - a^2} \geq d_0 \\ 1 & \text{if } \rho \leq \sqrt{d_0^2 + a^2} \end{cases}
$$

This, in particular means, that χ_{out} vanishes if its argument exceeds s^*, in other words, for $\rho > \sqrt{(s^* - s_0 + d_0)^2 + a^2}$. The external trial function is then just the appropriate restriction mollifier χ_{out} itself:

$$
\psi_{\text{out}}(x) := \chi_{\text{out}}(x) \quad \text{if } x \in \Omega_\pm \setminus \Omega_\pm^a. \tag{3.9}
$$

Since $\mu = 0$ by assumption and the potential is zero away from Ω^a, the quantity to be estimated is the kinetic energy contribution to the form (2.3c) from the outer part of the trial function,

$$
\int_{\Omega \setminus \Omega^a} |\nabla \psi_{\text{out}}(x)|^2 dx \tag{3.10}
$$

$$
= 2\pi \int_{\sqrt{d_0^2 + a^2}}^{\sqrt{(s^* - s_0 + d_0)^2 + a^2}} \left| \frac{d}{d\rho} \chi_{\text{in}}(\sqrt{\rho^2 - a^2} - d_0 + s_0) \right|^2 \rho \, d\rho.
$$

Relation (3.8) tells us that one can choose parameters δ and ν for which the inner contribution to the form is negative (using a sufficiently large s^*), hence to prove the claim it is enough to show that the integral on the right-hand side of (3.10) vanishes if $s_0, d_0 \to \infty$ with the difference $s_0 - d_0$ bounded and $\frac{s^*}{s_0} \to \infty$. The values of the integrated function on the support of $\nabla \psi_{\text{out}}$ can be expressed using (3.1) to be

$$
\left| \frac{d}{d\rho} \chi_{\text{in}}(\sqrt{\rho^2 - a^2} - d_0 + s) \right|^2 = \left| \left(\ln \frac{s^*}{s_0} \right)^{-1} \frac{1}{\sqrt{\rho^2 - a^2} - d_0 + s_0} \frac{\partial s}{\partial \rho} \right|^2
$$

$$
= \left(\ln \frac{s^*}{s_0} \right)^{-2} \left(\sqrt{\rho^2 - a^2} - d_0 + s_0 \right)^{-2},
$$

because $\frac{\partial s}{\partial \rho} = 1$ in the considered region. Substituting from here to (3.9) we get

$$\int_{\Omega \setminus \Omega^a} |\nabla \psi_{\text{out}}(x)|^2 \mathrm{d}x$$

$$= 2\pi \left(\ln \frac{s^*}{s_0} \right)^{-2} \int_{\sqrt{d_0^2+a^2}}^{\sqrt{(s^*-s_0+d_0)^2+a^2}} \frac{\rho \mathrm{d}\rho}{(\sqrt{\rho^2 - a^2} - d_0 + s_0)^2}$$

Since $s_0 - d_0$ is bounded and a is fixed, we can choose a sufficiently large $\rho \geq d_0$ in such a way that

$$\sqrt{\rho^2 - a^2} - d_0 + s_0 \geq \frac{1}{2}\rho$$

in which case we have

$$\int_{\Omega \setminus \Omega^a} |\nabla \psi_{\text{out}}(x)|^2 \mathrm{d}x \leq 8\pi \left(\ln \frac{s^*}{s_0} \right)^{-2} \ln \rho \, \Big|_{\sqrt{d_0^2+a^2}}^{\sqrt{(s^*-s_0+d_0)^2+a^2}}. \qquad (3.11)$$

Using again the fact that $s^*, d_0 \to \infty$ while a is fixed and $s_0 - d_0$ bounded, we see that the parameters can be chosen so that

$$\ln \frac{\sqrt{(s^* - s_0 + d_0)^2 + a^2}}{\sqrt{d_0^2 + a^2}} \leq \ln \frac{2s^*}{\frac{1}{2}s_0} = \ln \frac{s^*}{s_0} + 2\ln 2, \qquad (3.12)$$

and substituting from (3.12) into (3.11), we get the needed result; this concludes the proof of part (a) of Theorem 2.2.

4. Proof of Theorem 2.4 – the first part

Let us pass to the situation where there is again *no bias*, $V_0 = 0$, the channel profile is *symmetric*, and the transverse operator (2.2) is *subcritical*, $\mu < 0$. The most difficult part is now to construct the exterior part of the trial function, for the interior we can use the result of Sec. 3.1 noting that in view of the assumption (p2) we have $\phi_+ = 1$ and $\xi_+ = -\xi_-$ which means that the inequality (3.8) is still valid.

4.1. **Curves with a piecewise constant curvature.** We divide the construction into two parts, considering first a particular class of the generating curves assuming additionally that

(s4) the curved part of Γ is piecewise C^∞-smooth consisting of a *finite array of circular arcs*; at its endpoints it is C^1-smoothly connected to the halflines

Consequently, the signed curvature $\kappa(\cdot)$ of such a curve is a step function. To begin with, we define in Ω_{out} function ϕ by

$$\phi(x) := \exp\{-\xi(\text{dist}(x, \Gamma) - a)\}, \quad x \in \mathbb{R}^2 \setminus \Omega^a, \qquad (4.1)$$

where $\xi := \xi_- = -\xi_+ = |\mu|^{1/2}$. The sought trial function will be then of the form $\psi_{\text{out}} = \phi \chi_{\text{out}}$ with the mollifier χ_{out} to be specified below.

As before, we will focus on the situation where the asymptotes of Γ are not parallel, the case with $\theta_0 = \frac{\pi}{2}$ can be treated in a similar way.

Since $\theta_0 > 0$ by assumption, we can choose conical neighborhoods of the asymptotes which do not intersect, that is, to pick $\Delta\theta_0$ sufficiently small so that $[-\theta_0 - \Delta\theta_0, -\theta_0 + \Delta\theta_0] \cap [\theta_0 - \Delta\theta_0, \theta_0 + \Delta\theta_0] = \emptyset$. Furthermore, we pick an $r_0 > 0$ large enough to ensure that the curved part of Γ is contained in the disk of one half that radius, $B_{\frac{1}{2}r_0}(O)$, centered at the coordinate origin O. At the points of the corresponding conical sectors, $x = (\rho, \theta) \in \mathbb{R}^2 \setminus B_{r_0}(O)$ with $\theta \in [\theta_0 - \Delta\theta_0, \theta_0 + \Delta\theta_0]$ or $\theta \in [-\theta_0 - \Delta\theta_0, -\theta_0 + \Delta\theta_0]$ we can use the (s,t) coordinates and define the mollifier χ_{out} depending on the longitudinal variable only,

$$\chi_{\mathrm{out}}(s,t) = \chi_{\mathrm{in}}(s),$$

where the right-hand side is given by (3.1). Furthermore, at the points $x \in B_{r_0}(O) \setminus \Omega^a$ we put $\chi_{\mathrm{out}}(x) = 1$, and finally, in the remaining part of the plane we choose χ_{out} independent of θ, in other words, as a function of the distance ρ from the origin O only, and such that χ_{out} is continuous in Ω_{out}. It is clear that the radial decay of such an external mollifier is determined by the behavior of the function (3.1).

Since the potential is supported in Ω^a, the contribution to the quadratic form (2.3c) in the exterior region comes from the kinetic term only. The trial function factorizes into a product and our first goal is to show that the cross-term containing the integral of $2\nabla\phi \cdot \nabla\chi_{\mathrm{out}}$ is small for large r_0, in particular, that one can make it smaller than $\frac{1}{16}\delta\nu$ with respect to the quantities appearing in (3.8).

Lemma 4.1. *We have*

$$\int_{\Omega_{\mathrm{out}}} |\nabla\psi_{\mathrm{out}}(x)|^2 \mathrm{d}x \leq \int_{\Omega_{\mathrm{out}}} |\nabla\phi(x)|^2 \chi_{\mathrm{out}}^2(x)\,\mathrm{d}x$$

$$+ \int_{\Omega_{\mathrm{out}}} |\phi(x)|^2 |\nabla\chi_{\mathrm{out}}(x)|^2 \mathrm{d}x + \mathcal{O}(r_0^{-1}) \quad as \ r_0 \to \infty. \qquad (4.2)$$

Proof. Since $\psi_{\mathrm{out}} = \phi\chi_{\mathrm{out}}$ for $x \in \Omega_{\mathrm{out}}$, we have to estimate the integral $\int_{\Omega_{\mathrm{out}}} |\nabla\phi(x) \cdot \nabla\chi_{\mathrm{out}}(x)|\,\mathrm{d}x$ to deal with the cross-term. To this aim we first note that $\chi_{\mathrm{out}} = 1$ holds inside $B_{r_0}(O)$ so we have to consider only the complement of the disk. In the conical sectors of $\mathbb{R}^2 \setminus B_{r_0}(O)$ with $[\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0]$ the point nearest to (x,θ) lies on the straight part of Γ, as the distance to it is at most $\rho\Delta\theta < \frac{1}{2}\rho$ while that to the curved part is at least $\rho - \frac{1}{2}r_0 > \frac{1}{2}\rho$. This implies that $\nabla\phi$ is perpendicular to $\nabla\chi_{\mathrm{out}}$, and the corresponding contribution to the integral vanishes too. Finally, in view of our definition of χ_{out} in combination with (3.1) we see that $\nabla\chi_{\mathrm{out}}$ is bounded outside $B_{r_0}(O)$ and the two sectors, and furthermore, we have $|\nabla\phi(x)| \leq \xi\phi(x) \leq C\,\mathrm{e}^{-\rho/2}$ which yields

$$\int_{\Omega_{\mathrm{out}}} |\nabla\phi(x) \cdot \nabla\chi_{\mathrm{out}}(x)|\,\mathrm{d}x \leq C' \int_{\Omega_{\mathrm{out}}} |\nabla\phi(x)|\,\mathrm{d}x = \mathcal{O}(r_0^{-1})$$

as $r_0 \to \infty$ which we set out to prove. $\qquad\square$

Let us turn to the second term on the right-hand side of (4.2).

Lemma 4.2. *We have*

$$\int_{\Omega_{\text{out}}} |\phi(x)|^2 |\nabla\chi_{\text{out}}(x)|^2 \mathrm{d}x = \mathcal{O}(r_0^{-1}) \quad as \ r_0 \to \infty. \qquad (4.3)$$

Proof. The integral over the disk is again zero and using an argument analogous to that of the previous proof, one can check that the integral over the region outside the conical sectors is $\mathcal{O}(r_0^{-1})$ as $r_0 \to \infty$. Inside the sectors we have

$$|\phi(x)|^2 |\nabla\chi_{\text{out}}(x)|^2 = |\phi_0(t)|^2 |\chi'_{\text{in}}(s)|^2 \le |\chi'_{\text{in}}(s)|^2$$

with χ_{in} given by (3.1); recall that outside Ω^a the function ϕ_0 decays exponentially with the distance from Γ and $\phi_0(\pm a) = 1$ holds by assumption. Hence the integral in (4.3) can be estimated by the squared norm of χ'_{in}, and since to a given r_0 one can choose $s^* = s^*(r_0)$ in such a way that $\ln\frac{s^*}{s_0} > Cr_0$ for some $C > 0$, the claim follows. $\qquad\square$

Combining the two lemmata, we see that choosing r_0 sufficiently large one can achieve that the outer contribution to the first term of (2.3c) can be estimated by the first expression on right-hand side of (4.2) with a small error, say

$$\int_{\Omega_{\text{out}}} |\nabla\psi_{\text{out}}(x)|^2 \mathrm{d}x \le \int_{\Omega_{\text{out}}} |\nabla\phi(x)|^2 \chi^2_{\text{out}}(x) \, \mathrm{d}x + \frac{|\mu|}{8}\delta\nu. \qquad (4.4)$$

Next we note that in part (a) of Theorem 2.4 the bias is absent, $V_0 = 0$, which means that the function (4.1) satisfies

$$|\nabla\phi|^2 - \mu|\phi|^2 = 2|\nabla\phi|^2$$

almost everywhere in Ω_{out}. This means that we can estimate the whole exterior contribution to the form $Q[\psi] - \mu\|\psi\|^2$ by doubling the kinetic term and neglecting the one containing the eigenvalue μ; in combination with (3.8) this tells us that in order to prove the theorem it is sufficient to check that

$$2\int_{\Omega_{\text{out}}} |\nabla\phi(x)|^2 \chi^2_{\text{out}}(x) \, \mathrm{d}x \le 2|\mu|^{1/2}\|\chi_{\text{in}}\|^2 + \frac{|\mu|}{8}\delta\nu,$$

and that is in view of $|\nabla\phi|^2 = -\mu|\phi|^2$ further equivalent to

$$\int_{\Omega_{\text{out}}} |\phi(x)\chi_{\text{out}}(x)|^2 \, \mathrm{d}x \le |\mu|^{-1/2}\|\chi_{\text{in}}\|^2 + \frac{1}{16}\delta\nu. \qquad (4.5)$$

The rest of the proof consists of verification of the inequality (4.5). To begin with, we estimate the contribution to its left-hand side from the parts of the plane adjacent to the straight parts of the waveguide; we choose them as conical sectors similar to those used in the proof of Lemma 4.1. We recall that for $x = (\rho, \theta)$ with $\rho \ge r_0$ and $\theta \in [\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0]$ we can use the (s, t) coordinates simultaneously

with the polar ones. We choose an $\hat{s} \geq r_0$ so that the parts of Γ with $|s| \geq \hat{s}$ lay outside $B_{r_0}(O)$, and at the same time we choose s_0 of (3.1) is such a way that $s_0 > \hat{s}$. Then we define

$$K_\pm := \left\{ x : \ |s| \geq \hat{s}, \ |t| \geq a, \ \theta \in [\pm\theta_0 - \Delta\theta_0, \pm\theta_0 + \Delta\theta_0] \right\} \qquad (4.6)$$

Within these sets, the closest points of Γ are those on the straight parts of the curve with the same coordinate s. Then it is easy to see that

$$\int_{\Omega_{\text{out}} \cap \{K_+ \cup K_-\}} |\phi(x)\chi_{\text{out}}(x)|^2 \, \mathrm{d}x \leq |\mu|^{-1/2} \|\chi_{\text{in}}\|^2_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))} \quad (4.7)$$

It remains to integrate the function $|\phi\chi_{\text{out}}|^2$ over $\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}$. Obviously, the integral will increase if we replace χ_{out} by one, hence to complete the proof, it is in view of (3.8) and (4.7) enough to check that

$$\int_{\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}} |\phi(x)|^2 \, \mathrm{d}x \leq 2\hat{s} \, |\mu|^{-1/2} + \frac{1}{16}\delta\nu \, ; \qquad (4.8)$$

we have used here the fact that $\|\chi_{\text{in}}\|^2_{L^2((-\hat{s},\hat{s}))} = 2\hat{s}$.

To estimate the indicated integral we employ the additional assumption (s4); the two part of Γ corresponding to $|s| > \hat{s}$ will be considered as arcs of zero curvature, cf. Remark 4.4 below. First of all, we note that the function $d_x : \mathbb{R} \to \mathbb{R}_+$, defined by $d_x(s) := \text{dist}(x, \Gamma(s))$, is C^1 smooth for any $x \in \mathbb{R}^2$, and under the the assumption (s4) it is piecewise monotonous because on each arc it can have at most one extremum. At the same time, $d_x(s) \to \infty$ holds as $s \to \pm\infty$, hence the function has a global minimum, positive as long as x does not lie on the curve, and in view of its continuity it may also have a finite number of local extrema which come in pairs, a minimum adjacent to a maximum. Let s_x^0 be the coordinate of the global minimum and denote by s_x^i with i running over an appropriate finite set of integers the coordinates of all the extrema; we introduce the symbol $M_x^\uparrow$ for the subset referring to the local maxima and $M_x^\downarrow$ for the set of coordinates of the minima. Then it obviously holds

$$\exp\{-2\xi(d_x(s_x^0) - a)\} \leq - \sum_{s_x^i \in M_x^\uparrow} \exp\{-2\xi(d_x(s_x^i) - a)\}$$

$$+ \sum_{s_x^i \in M_x^\downarrow} \exp\{-2\xi(d_x(s_x^i) - a)\} \qquad (4.9)$$

for all $x \in \Omega_{\text{out}}$. To estimate the integral in (4.8), we have to integrate the right-hand side of (4.9) over $\Omega_{\text{out}} \setminus \{K_+ \cup K_-\}$. To this aim, let us first collect several simple geometric statements easy to check:

Proposition 4.3. *Let Γ_j be one the arcs of Γ and denote by $\omega_{1j}, \omega_{2j}, \omega_{3j}$ and Ω_j^a the open regions shown in Fig. 2. Then the following holds true:*

(i) If $x \in \omega_{1j} \cup \omega_{2j}$, then $d_x(\cdot)$ has a minimum in the interior of Γ_j.
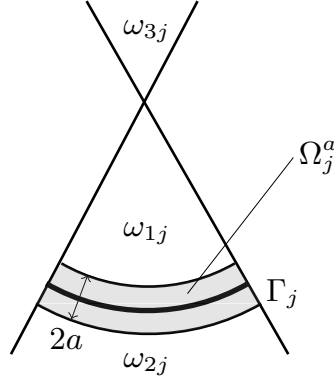*(ii) If $x \in \omega_{3j}$, then $d_x(\cdot)$ has a maximum in the interior of Γ_j.*

FIGURE 2. The regions used in Proposition 4.3

(iii) $x \notin \bar{\omega}_{1j} \cup \bar{\omega}_{2j} \cup \bar{\omega}_{3j} \cup \bar{\Omega}_j^a$, *then $d_x(\cdot)$ has no extremum on Γ_j.*
(iv) $d_x(\cdot)$ *cannot have more than one critical point in the interior of Γ_j.*
(v) *If $x \in \omega_{kj}$ for any of $k = 1, 2, 3$, then the one-sided derivative $d_x'(s) \neq 0$ at the endpoints of Γ_j.*

Remark 4.4. With an abuse of terminology we include into (s4) also situations when a Γ_j is a straight segment, that is, $\kappa(s) = 0$ holds on Γ_j. In that case the wedge-shaped regions ω_{1j} and ω_{2j} become semi-infinite strips and ω_{3j} does not exist. This concerns, in particular, the two straight parts of Γ corresponding to $|s| > \hat{s}$.

Within the regions we introduced the minimal and maximal distances are easily expressed; we have

$$
\begin{aligned}
d_x(s_x^i) &= \mathrm{dist}(x, \Gamma_j) \quad \text{if } s_x^i \in \Gamma_j \cap M_x^{\downarrow}, \\
d_x(s_x^i) &= |\kappa_j|^{-1} + \mathrm{dist}(x, O_j) \quad \text{if } s_x^i \in \Gamma_j \cap M_x^{\uparrow},
\end{aligned}
\tag{4.10}
$$

where O_j is the center of the corresponding circular arc.

Let $\iota_j^{1,2}$ and ι_j^3 be the characteristic functions of the sets $\omega_{1j} \cup \omega_{2j}$ and ω_{3j}, respectively. In view of of Proposition 4.3 and relations (4.10), we can replace the first term at the right-hand side of (4.9), everywhere except the zero measure set referring to the boundaries of the regions ω_{kj}, $k = 1, 2, 3$, with

$$
-\sum_j \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\iota_j^3(x)
\tag{4.11}
$$

and the second term similarly by

$$
\sum_j \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\}\iota_j^{1,2}(x)
\tag{4.12}
$$

Integrating now (4.11) and (4.12) over $\Omega_{\mathrm{out}} \backslash \{K_+ \cup K_-\}$ and exchanging the order of integration over x with summation over j, we can using

(4.9) estimate $\int_{\Omega_{\text{out}}\backslash\{K_+\cup K_-\}} \exp\{-2\xi(d_x(s_x^0)-a)\}\,\mathrm{d}x$ from above by

$$\sum_j \int_{(\omega_{1j}\cup\omega_{2j})\cap\{\Omega_{\text{out}}\backslash\{K_+\cup K_-\}\}} \exp\{-2\xi(\operatorname{dist}(x,\Gamma_j)-a)\}\,\mathrm{d}x \qquad (4.13)$$

$$-\sum_j \int_{\omega_{3j}\cap\{\Omega_{\text{out}}\backslash\{K_+\cup K_-\}\}} \exp\{-2\xi(|\kappa_j|^{-1}+\operatorname{dist}(x,O_j)-a)\}\,\mathrm{d}x,$$

where the sums run over all the indices of Γ_j including those of the straight segments of the curve with $|s| > \hat{s}$. Note that this estimate includes in general a double counting since the same x may belong to different ω_{kj}; this does not matter as long as we consider the contributions referring of a given Γ_j together.

Our next goal is to show that the expression (4.13) cannot decrease if we replace the integration domains by $(\omega_{1j}\cup\omega_{2j})\backslash\{K_+\cup K_-\}$ and $\omega_{3j}\backslash\{K_+\cup K_-\}$, respectively. To this aim, consider a fixed arc Γ_{j_0} and the respective segment $\Omega_{j_0}^a$ of the strip Ω^a as indicated in Fig. 2. For a point $x \in \Omega_{j_0}^a$ the function $d_x(\cdot)$ has the global minimum on Γ_{j_0} with a coordinate s_x^0 and all the local extrema, if they exist, come in pairs situated outside Γ_{j_0}. This yields the estimate

$$0 \leq -\sum_{s_x^i\in M_x^\uparrow} \exp\{-2\xi(d_x(s_x^i)-a)\} + \sum_{\substack{s_x^i\in M_x^\downarrow \\ s_x^i\neq s_x^0}} \exp\{-2\xi(d_x(s_x^i)-a)\}$$

Using again Proposition 4.3, we get for any $x \in \Omega_{j_0}^a$ the inequality

$$0 \leq -\sum_{j\neq j_0} \exp\{-2\xi(|\kappa_j|^{-1}+\operatorname{dist}(x,O_j)-a)\}\iota_j^3(x)$$

$$+\sum_{j\neq j_0} \exp\{-2\xi(\operatorname{dist}(x,\Gamma_j)-a)\})\iota_j^{1,2}(x)\,; \qquad (4.14)$$

since $x \in \Omega_{j_0}^a$, we are able to replace the indicator functions in this expression by their restriction $\iota_j^3\upharpoonright\Omega_{j_0}^a$ and $\iota_j^{1,2}\upharpoonright\Omega_{j_0}^a$, respectively. Noting further that $\omega_{kj_0}\cap\Omega_{j_0}^a=\emptyset$ holds for $k=1,2,3$, we see that (4.14) remains valid if the summation is taken over all the j's. Integrating then the right-hand side over $x \in \Omega^a$ we arrive at the inequality

$$0 \leq -\sum_j \int_{\omega_{3j}\cap\Omega_{j_0}^a} \exp\{-2\xi(|\kappa_j|^{-1}+\operatorname{dist}(x,O_j)-a)\}\,\mathrm{d}x$$

$$+\sum_j \int_{(\omega_{1j}\cup\omega_{2j})\cap\Omega_{j_0}^a} \exp\{-2\xi(\operatorname{dist}(x,\Gamma_j)-a)\}\,\mathrm{d}x,$$

and summing this result over j_0 we get

$$0 \leq -\sum_j \int_{\omega_{3j} \cap \Omega^a} \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\} \, \mathrm{d}x$$

$$+ \sum_j \int_{(\omega_{1j} \cup \omega_{2j}) \cap \Omega^a} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\} \, \mathrm{d}x. \qquad (4.15)$$

Combining now (4.13) and (4.15), we obtain

$$\int_{\Omega_{\mathrm{out}} \backslash \{K_+ \cup K_-\}} |\phi(x)|^2 \, \mathrm{d}x \qquad (4.16)$$

$$\leq \sum_j \int_{(\omega_{1j} \cup \omega_{2j}) \backslash \{K_+ \cup K_-\}} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\} \, \mathrm{d}x$$

$$- \sum_j \int_{\omega_{3j} \backslash \{K_+ \cup K_-\}} \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\} \, \mathrm{d}x.$$

The summation in (4.16) runs over all the curve segments including the straight ones. Let us first estimate the contribution of these infinite 'arcs' to the positive part of (4.16) having in mind that in accordance with Remark 4.4 the segments with $\kappa = 0$ do not contribute to the negative one. We denote by Γ_+ the segment with $s > \hat{s}$ and by ω_{1+}, ω_{2+} the corresponding semi-infinite strips, then we have

$$\int_{(\omega_{1+} \cup \omega_{2+}) \backslash \{K_+ \cup K_-\}} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_+) - a)\} \, \mathrm{d}x \qquad (4.17)$$

$$\leq 2 \int_{\omega_{1+} \backslash K_+} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_+) - a)\} \, \mathrm{d}x$$

$$= 2 \int_{\rho(\hat{s}) \cos \Delta\theta}^{\infty} \int_{s \sin \Delta\theta_0}^{\infty} \exp\{-2\xi(t - a)\} \, \mathrm{d}t \mathrm{d}s$$

$$= \frac{\mathrm{e}^{2\xi a}}{4\xi^2 \sin \Delta\theta_0} \mathrm{e}^{-\xi \sin 2\Delta_0 \theta \cdot \rho(\hat{s})},$$

where $\rho(\hat{s})$ is the distance of the point $\Gamma(\hat{s})$ to the origin. In view of our choice of $\hat{s}$ we have $\rho(\hat{s}) \geq r_0$ and the integral at the right-hand side of (4.17) can be made arbitrarily small by choosing r_0 large enough. An analogous argument applies to the segment of Γ with $s < -\hat{s}$.

Denote now by $\sum_j^*$ the sum over all the Γ_j except of $\Gamma_{\pm}$. The conclusion just made allows us to replace the sum $\sum_j$ in (4.17) by $\sum_j^*$ with an error which can be made arbitrarily small by choosing an appropriately large r_0. Furthermore, we note that the positive part of (4.13) cannot decrease if we enlarge the integration domain in all the integrals there replacing $(\omega_{1j} \cup \omega_{2j}) \backslash \{K_+ \cup K_-\}$ by $\omega_{1j} \cup \omega_{2j}$.

Our next goal is to argue that we can do the same in the negative part of (4.13) replacing $\omega_{3j} \backslash \{K_+ \cup K_-\}$ by ω_{3j}. In such a case, of

course, the corresponding change of the integrals goes in the wrong way; our aim is to show that it again produces an error which can be made small if r_0 is large. Indeed, regions ω_{3j} exist only for the curved segments of Γ and those are by assumption inside $B_{\frac{1}{2}r_0}(O)$, while the regions $K_\pm$ are outside $B_{r_0}(O)$. Consequently, the contributions from the extended integration domains are

$$\int_{\omega_{3j}\cap\{K_+\cup K_-\}} \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\,\mathrm{d}x \qquad (4.18)$$

$$\leq \mathrm{e}^{2\xi a}\,|\Gamma_j|\int_{\rho\geq\sqrt{3}r_0/2}^{\infty} \mathrm{e}^{-\sqrt{3}\xi\rho}\,\rho\,\mathrm{d}\rho = |\Gamma_j|\,\mathcal{O}(\mathrm{e}^{-3\xi r_0/2})$$

uniformly in j, and since the length of the curved part is finite, the error coming from the extension of the integration domain is $\mathcal{O}(\mathrm{e}^{-3\xi r_0/2})$. Combining (4.18) with (4.16) we get

$$\int_{\Omega_{\mathrm{out}}\backslash\{K_+\cup K_-\}} |\phi(x)|^2\,\mathrm{d}x \qquad (4.19)$$

$$\leq \sum_{j}^{*} \int_{\omega_{1j}\cup\omega_{2j}} \exp\{-2\xi(\mathrm{dist}(x, \Gamma_j) - a)\}\,\mathrm{d}x$$

$$- \sum_{j}^{*} \int_{\omega_{3j}} \exp\{-2\xi(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\}\,\mathrm{d}x + \mathcal{O}(\mathrm{e}^{-3\xi r_0/2}).$$

It is not difficult to evaluate the integrals appearing at the right-hand side of (4.19): we have

$$\int_{\omega_{2j}} \exp\{-2\xi(\mathrm{dist}(x, O_j) - a)\}\,\mathrm{d}x = \left(\frac{1}{2\xi} + \frac{a|\kappa_j|}{2\xi} + \frac{|\kappa_j|}{4\xi^2}\right)|\Gamma_j|$$

$$= \frac{|\Gamma_j|}{2\xi} + \frac{a}{2\xi}\int_{\Gamma_j} |\kappa(s)|\,\mathrm{d}s + \frac{1}{4\xi^2}\int_{\Gamma_j} |\kappa(s)|\,\mathrm{d}s \qquad (4.20)$$

and

$$\int_{\omega_{1j}} \exp\{-2\xi(\mathrm{dist}(x, O_j) - a)\}\,\mathrm{d}x = \frac{|\Gamma_j|}{2\xi} - \frac{a}{2\xi}\int_{\Gamma_j} |\kappa(s)|\,\mathrm{d}s$$

$$- \frac{1}{4\xi^2}\int_{\Gamma_j} |\kappa(s)|\,\mathrm{d}s + \frac{1}{4\xi^2}\int_{\Gamma_j} \mathrm{e}^{-2\xi(|\kappa(s)|^{-1}-a)}|\kappa(s)|\,\mathrm{d}s.$$

for the positive part of the estimate, while in the negative one we use

$$\int_{\omega_{3j}} \exp\{-2\xi(\kappa_j^{-1} + \mathrm{dist}(x, O_j))\}\,\mathrm{d}x = \frac{1}{4\xi^2}\int_{\Gamma_j} \mathrm{e}^{-2\xi(|\kappa(s)|^{-1}-a)}|\kappa(s)|\,\mathrm{d}s.$$

Summing finally the contributions from given Γ_j we get $|\Gamma_j|\xi^{-1}$, hence the expression (4.13) is smaller that $2|\mu|^{-1/2}\hat{s} + o(r_0)$ which according to inequality (4.8) proves part (a) of Theorem 2.4 under the additional assumption (s4).

4.2. **Completing the proof.** We will use the same trial function as before, in particular, its outer part will be again of the form $\psi_{\text{out}} = \phi\chi_{\text{out}}$ with ϕ given (4.1). We have to show that (4.8) remains to be valid without the assumption (s4). The idea is to approximate the curve Γ satisfying (s1) by curves with a piecewise constant curvature, the same length and the same halfline asymptotes. Specifically, we are going to use the following result:

Theorem 4.5 (Sabitov-Slovesnov [SS10])**.** *Let Γ be a C^3-smooth curve consisting of a finite number of segments such that on each of them the monotonicity character of the signed curvature $\kappa(\cdot)$ of Γ and its sign are preserved. Then Γ can be approximated by a C^1-smooth function $\hat{\Gamma}$ of the same length, the curvature of which is piecewise constant having jumps at the points $s_1 < s_2 < \cdots < s_N$, in the sense that the estimates*

$$\|\Gamma^{(m)} - \hat{\Gamma}^{(m)}\|_\infty \leq C \max_{1 \leq k \leq N-1} (s_{k+1} - s_k)^{3-m}, \quad m = 0, 1, 2, \quad (4.21)$$

hold with some $C > 0$ for the function Γ and its two first derivatives.

The approximation bears a local character so we can refine it at a fixed part of Γ without changing anything at the rest. The length here means the arc length distance between a fixed pair of points, and naturally, the second derivative $\hat{\Gamma}^{(2)}$ does not exist in general at the points $\{s_k\}$. Note also that the result does not require Γ to be a unit-speed curve. It is obvious that the hypotheses of Theorem 4.5 are satisfied under our assumption (s1).

Following the construction of [SS10], we can to any given $\varepsilon > 0$ divide Γ into a finite union of segments such that on each of the corresponding intervals of the arc-length coordinate s we approximate it by a pair of circular arcs with the following properties:

  (i) the distance between Γ and the arcs does not exceed ε,
 (ii) The curvature of the arcs is in the interval $[\kappa_-, \kappa_+]$, where $\kappa_\pm$ is, respectively, the maximum and minimuum of $|\kappa(s)|$ over the interval in question,
(iii) the arcs are C^1-smoothly connected mutually and the the rest of the curve corresponding to s outside the interval,
(iv) the sum of arc length is the same as that of the approximated segment of Γ.

We use therefore a family $\{\Gamma_n\}$ of such arcwise curves approximating the non-straight part of Γ which corresponds to a decreasing sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$; in view of Theorem 4.5 the corresponding sequence of partitions, $\{s_k^{(n)}\}$, must be refining in the parts of the curve where its curvature is non-constant, so that $s_{k+1}^{(n)} - s_k^{(n)} \to 0$ as $n \to \infty$; without loss of generality we may suppose that $\{s_k^{(n)}\} \subset \{s_k^{(n+1)}\}$. If $\kappa(\cdot)$ is non-constant in the vicinity of a point s, we can thus find a subsequence $\{s_{k_n}^{(n)}\}$ such $s_{k_n}^{(n)} \to s-$ as $n \to \infty$ and $s_{k+n+1}^{(n)} \to s+$, by

the C^3-smoothness of Γ and the property (ii) above we then infer that $\kappa_{k_n}^{(n)}$ and $\kappa_{k_n+1}^{(n)}$, the curvatures of the arcs with the endpoints at $\{s_{k_n}^{(n)}\}$ and $s_{k_n+1}^{(n)}$, respectively, converge to $\kappa(s)$ when $n \to \infty$.

Denoting then the piecewise constant curvature of the approximating curve Γ_n as κ_n and putting

$$\delta_n := 2 \int_{|t| \leq a} \phi_0'(t) \chi_{\mathrm{in}}(s) g(s,t) \kappa_n(s) \,\mathrm{d}s\mathrm{d}t,$$

we see that $\delta_n \to \delta$, two times the integral in the second term at the right-hand of (3.4), as $n \to \infty$. Keeping the same parameter ν, we have also $\delta_n \nu \to \delta \nu$ which means that the estimate (3.8) holds uniformly for all the approximating curves with n large enough, with the first term on its right-hand side replaced, say, by $-\frac{1}{5}\delta\nu$.

Note further that the trial function we have constructed is supported in a disk of radius R_{supp} which depends on the value of $\delta\nu$ – it must be large enough to ensure that the estimate (4.8) is still valid – and on the radius $\frac{1}{2}r_0$ of the disk containing the curved part of Γ and its approximants $\tilde\Gamma_n$. Since $\|\Gamma - \Gamma_n\|_\infty \to 0$ as $n \to \infty$ by (4.21), we can choose R_{supp} satisfying the requirements for all n large enough. Moreover, by construction

$$\mathrm{dist}(x,\Gamma) \geq \mathrm{dist}(x,\Gamma_n) - \varepsilon_n$$

holds for $x \in \Omega_{\mathrm{out}}$, and consequently,

$$\int_{\tilde\Omega_{\mathrm{out}}} \exp\{-2\xi(\mathrm{dist}(x,\Gamma_n) - a)\} \,\mathrm{d}x$$

$$\leq \mathrm{e}^{2\xi\varepsilon_n} \pi R_{\mathrm{supp}}^2 \int_{\tilde\Omega_{\mathrm{out}}} \exp\{-2\xi(\mathrm{dist}(x,\Gamma_n) - a)\} \,\mathrm{d}x.$$

Applying now the estimates of the previous section to the curves Γ_n which satisfy assumption (s4) and taking the limit $\varepsilon_n \to 0$, we get the sought estimate for the curve Γ which concludes the proof. $\qquad\square$

5. Concluding proofs of Theorems 2.2 and Theorem 2.4

It remains to establish parts (b) of both the main results. Let us begin with Theorem 2.4 and prove it in the situation when Ω_+ is *convex*; by assumption (p3) we have $\mu < 0$. Inside Ω^a we choose the trial function as in the previous proofs so that inequality (3.7) is valid for any $V_0 \geq 0$; recall that we derived it assuming the presence of a bias. Moreover, picking a suitable coordinate $s^* \gg s_0$ at the right-hand side of (3.1), the last term in (3.7) can be made as before smaller than $\frac{1}{4}\delta\nu$. Outside the strip Ω^a we set

$$\phi(x) := \phi_\pm \exp\{-|\xi_\pm|(\mathrm{dist}(x,\Gamma) - a)\} \quad \text{if } x \in \Omega_\pm^{\mathrm{out}}, \qquad (5.1)$$

recalling that $\phi_- = 1$, which is a natural generalization of (4.1), and we employ the same mollifier χ_{out} as before, cf. Sec. 4.1. Repeating the

argument of this section, we arrive at the inequality (4.4), however, now with the function ϕ given by (5.1). Let us split the outer contribution to the quadratic form into two parts referring, respectively, to $\Omega_\pm^{\text{out}}$, for which we have

$$Q_{\text{out}}^{(+)}[\psi_{\text{out}}] = \int_{\Omega_+^{\text{out}}} |\nabla\psi_{\text{out}}(x)|^2\,\mathrm{d}x + \int_{\Omega_+^{\text{out}}} (V_0 - \mu)|\psi_{\text{out}}(x)|^2\,\mathrm{d}x$$

$$\leq \int_{\Omega_+^{\text{out}}} \left\{|\nabla\phi(x)|^2 + (V_0 - \mu)|\phi(x)|^2\right\}\chi_{\text{out}}(x)^2\,\mathrm{d}x + \frac{1}{16}\delta\nu, \quad (5.2\text{a})$$

$$Q_{\text{out}}^{(-)}[\psi_{\text{out}}] \leq \int_{\Omega_-^{\text{out}}} \left\{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\right\}\chi_{\text{out}}(x)^2\,\mathrm{d}x + \frac{1}{16}\delta\nu \quad (5.2\text{b})$$

in view Lemmata 4.1 and 4.2 provided that r_0 is chosen large enough. As in the proof of the first part of Theorem 2.2 we choose an $\hat{s} \in [r_0, s_0)$ for which the parts of Γ with $|s| \geq \hat{s}$ are outside $B_{r_0}(O)$ and use the regions $K_\pm$ defined by (4.6). By the definition (5.1) we have

$$|\nabla\phi|^2 = \xi_\pm^2|\phi|^2 \quad \text{for } x \in \Omega_\pm^{\text{out}}.$$

Within $\Omega_{\text{out}} \cap \{K_+ \cup K_-\}$ we may use the (s,t) coordinates, and noting the ϕ is independent of s there, and as a function of t it coincides with the eigenfunction ϕ_0 of h, cf. (2.2), we get

$$\int_{\Omega_+^{\text{out}} \cap \{K_+ \cup K_-\}} \left\{|\nabla\phi(x)|^2 + (V_0 - \mu)|\phi(x)|^2\right\}\chi_{\text{out}}(x)^2\,\mathrm{d}x$$

$$\leq |\xi_+|\,\phi_+^2\,\|\chi_{\text{in}}\|_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))}^2 \quad (5.3\text{a})$$

and

$$\int_{\Omega_-^{\text{out}} \cap \{K_+ \cup K_-\}} \left\{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\right\}\chi_{\text{out}}(x)^2\,\mathrm{d}x$$

$$\leq \xi_-\|\chi_{\text{in}}\|_{L^2((-\infty,-\hat{s}]\cup[\hat{s},\infty))}^2. \quad (5.3\text{b})$$

So far we have not employed the convexity of Ω_+; we will need it from now on to estimate the integrals (5.3). As before we will first prove the second claim of Theorem 2.2 under the additional assumption (s4) using again the notation introduced in Fig. 2.

The part $\Omega_{\text{out}}^{(-)}$ consists then of a finite number of sectors ω_{2j} which in view of the convexity assumption do not overlap mutually. Let $\Gamma_\pm$ and $\omega_{k\pm}$, $k = 1, 2$, be the same as in part (a) of Theorem 2.4. By the same reasoning as in the proof of the latter, cf. (4.17), one can check that the contribution of the regions $\omega_{k\pm} \setminus \{K_+ \cup K_-\}$ to the integrals (5.3) can be made arbitrarily small by choosing r_0 sufficiently large. Using

further the fact that $|\chi_{\text{out}}| \leq 1$ in combination with (4.20), we get

$$\int_{\Omega_{\text{out}}^{(-)}\setminus\{K_+\cup K_-\}} \{|\nabla\phi(x)|^2 - \mu|\phi(x)|^2\}\chi_{\text{out}}(x)^2\, \mathrm{d}x \tag{5.4}$$

$$\leq 2|\mu|\Big[\frac{2\hat{s}}{2\xi_-} + \frac{a}{2\xi_-}\int_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}s + \frac{1}{4\xi_-^2}\int_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}s\Big] + \mathcal{O}(\mathrm{e}^{-cr_0})$$

$$= 2\xi_-\hat{s} + a\xi_-\int_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}s + \frac{1}{2}\int_{-\hat{s}}^{\hat{s}}\kappa(s)\,\mathrm{d}s + \mathcal{O}(\mathrm{e}^{-cr_0})$$

for some $c > 0$, and since $\kappa(s) = 0$ for $|s| > \hat{s}$ we can let the variable s in the above integrals run over the whole $\mathbb{R}$. Comparing now the right-hand side of (5.4) with that of (3.7), we see that the terms containing ξ_- in the latter have their counterparts here with the opposite sign, hence they cancel mutually.

Next we estimate the contribution to (5.3a) coming from $\Omega_{\text{out}}^{(+)}$. We note that $|\nabla\phi|^2 = (-\mu+V_0)|\phi|^2 = |\xi_+|^2|\phi(x)|^2$ holds almost everywhere in Ω_+^{out} which means that the integral at the right-hand side of (5.2a) can be rewritten as $2|\xi_+|^2\int_{\Omega_+^{\text{out}}}|\phi(x)|^2\chi_{\text{out}}(x)^2\,\mathrm{d}x$. In analogy with (4.9) we can estimate the function ϕ using local extrema of the distance function, namely

$$|\phi(x)|^2 = \phi_+^2\,\exp\{-2|\xi_+|(d_x(s_x^0) - a)\} \tag{5.5}$$

$$\leq \phi_+^2\Big[-\sum_{s_x^i\in M_x^\uparrow}\exp\{-2|\xi_+|(d_x(s_x^i)-a)\} + \sum_{s_x^i\in M_x^\downarrow}\exp\{-2|\xi_+|(d_x(s_x^i)-a)\}\Big].$$

As in part (a) of Theorem 2.4, we want to replace the integral of the expression at the right-hand side of (5.5) over $\Omega_{\text{out}}^{(+)}\setminus\{K_+\cup K_-\}$ by the sum of the integrals over the regions ω_{1j} and ω_{3j} corresponding to the partition of the curve segment with $s\in[-\hat{s},\hat{s}]$ into circular arcs. In analogy with relation (4.13) we get

$$\int_{\Omega_{\text{out}}^{(+)}\setminus\{K_+\cup K_-\}}|\phi(x)|^2\,\mathrm{d}x \tag{5.6}$$

$$\leq \phi_+^2\sum_j\Big\{\int_{\omega_{1j}\cap\{\Omega_{\text{out}}^{(+)}\setminus\{K_+\cup K_-\}\}}\exp\{-2|\xi_+|(\text{dist}(x,\Gamma_j)-a)\}\,\mathrm{d}x$$

$$-\int_{\omega_{3j}\cap\{\Omega_{\text{out}}^{(+)}\setminus\{K_+\cup K_-\}\}}\exp\{-2|\xi_+|(|\kappa_j|^{-1}+\text{dist}(x,O_j)-a)\}\,\mathrm{d}x\Big\},$$

where in contrast to (4.13) the right-hand side (5.6) does not involve integrals over ω_{2j} because in view of the convexity assumption we have $\Omega_{\text{out}}^{(+)}\cap\omega_{2j} = \emptyset$ holds for any j.

Following the strategy used in the proof of part (a) of Theorem 2.4, we want to replace integrals over $\omega_{kj}\cap\{\Omega_{\text{out}}^{(+)}\setminus\{K_+\cup K_-\}\}$, $k = 1, 3$, with those over the extended regions $\omega_{kj}\setminus\{K_+\cup K_-\}$, respectively. To this aim, we employ the following simple geometric result:

Lemma 5.1. *Suppose that $x \in \Omega_-$ does not belong to the boundaries of ω_{kj}, $k = 1, 2, 3$, for any j. Let further the distance function $d_x(s)$ reach a minimum which is not global at a point of the curve belonging to an arc Γ_{j*}, then we have $x \in \omega_{1j*}$.*

The lemma in fact says that if Ω_+ is convex, it cannot happen that $x \in \omega_{2j*}$, which is obviously equivalent to the following claim:

Lemma 5.1'. *Let $x \in \Omega_-$. For any distance function extremum, except the global minimum, the segment L_x^i connecting the points x and $\Gamma(s_x^i)$ approaches the curve from the side of Ω_+.*

Proof. The point of global minimum is obviously approached for the region where x lies, that is, from Ω_-. The next two extrema on both sides of s_x^0, provided they exist, are necessarily maxima, and in view of the assumed convexity of Ω_+ the segments L_x^i cannot approach $\Gamma(s_x^i)$ from the side of Ω_-. We denote by $L(s)$ the segment connecting the point x with $\Gamma(s)$. The side from which $L(s)$ approaches the curve can change only at the points where the angle $\beta(s)$ between the segment $L(s)$ and L_x^0 corresponding to the global minimum of $d_x(\cdot)$ has, as a function of s, a local maximum or minimum. Since the curve Γ is by assumption C^1-smooth, and so is $\beta(\cdot)$, the lines connecting such points with x are tangent to Ω, however, a convex region cannot cross its own tangent, hence the extrema of the function $\beta(\cdot)$ are global, one maximum and one minimum. The corresponding points s_x^i, provided both of them exist, lie on both sides of s_x^0 because a convex region can have only two tangents passing through an exterior point x and the point $\Gamma(s_x^i)$ lies between the two tangent points on the boundary of Ω_+. The same tangent argument shows that once the $L(s)$ switches the side from which it approached Γ it can never come back. $\qquad\square$

As before all the local extrema of $d_x(\cdot)$ for $x \in \Omega_-$ except the global minimum come in pairs, so in analogy with (4.13) we are able to estimate the expression $\phi_+^2 \int_{\Omega_- \setminus \{K_+ \cup K_-\}} \exp\{-2|\xi_+|(d_x(s_x^0) - a)\} \, \mathrm{d}x$ from above by

$$\phi_+^2 \sum_j \left\{ \int_{\omega_{1j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(\mathrm{dist}(x, \Gamma_j) - a)\} \, \mathrm{d}x \quad (5.7) \right.$$

$$\left. - \int_{\omega_{3j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}} \exp\{-2|\xi_+|(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\} \, \mathrm{d}x \right\},$$

where in view of Lemma 5.1 the first part does not include integration over $\omega_{2j} \cap \{\Omega_- \setminus \{K_+ \cup K_-\}\}$. Adding (5.7) to (5.6), we get

$$\int_{\Omega_{\text{out}}^{(+)} \setminus \{K_+ \cup K_-\}} |\phi(x)|^2 \, \mathrm{d}x \tag{5.8}$$

$$\leq \phi_+^2 \sum_j \left\{ \int_{\omega_{1j} \cap \tilde{\Omega}} \exp\{-2|\xi_+|(\mathrm{dist}(x, \Gamma_j) - a)\} \, \mathrm{d}x \right.$$

$$\left. - \int_{\omega_{3j} \cap \tilde{\Omega}} \exp\{-2|\xi_+|(|\kappa_j|^{-1} + \mathrm{dist}(x, O_j) - a)\} \, \mathrm{d}x \right\},$$

where $\tilde{\Omega} := \Omega_- \cup \{\Omega_{\text{out}}^{(+)} \setminus \{K_+ \cup K_-\}\}$. Moreover, applying again the argument that lead to (4.18) we infer that one can replace $\omega_{1j} \cap \tilde{\Omega}$ and $\omega_{3j} \cap \tilde{\Omega}$ in (5.8) by $\omega_{1j} \setminus \{K_+ \cup K_-\}$ and $\omega_{3j} \setminus \{K_+ \cup K_-\}$, respectively, with an error which can be made arbitrarily small by choosing r_0 large enough. The rest of the proof of part (b) of Theorem 2.4 for a convex Ω_+ repeats the corresponding part of the proof of the part (a); in the final step we take into account that a convex Γ can be approximated by convex curves of piecewise constant curvature.

To complete the proof of part (b) of Theorem 2.4, assume next that Ω_+ is *concave*. This case is already easy given the fact that in the first part of the proof we have not used the difference between $|\xi_+|$ and $|\xi_-|$, or between ϕ_+ and ϕ_-; the latter was set to one for convenience only. The role of the convexity was just to help us to distinguish the extrema of the distance function referring to the two outer parts of the trial function; if Ω_- is convex, we can repeat the argument step by step interchanging the roles of Ω_- and Ω_+ arriving thus at the sought claim.

It remains to prove part (b) of Theorem 2.2 where we have $\mu = 0$ by assumption and Ω_+ is again convex. Since $V_0 > 0$, the equation $h\phi = 0$ has a resonance solution ϕ_0 which is constant for $t \leq -a$ and decays exponentially for $t > a$; as before we normalize it putting $\phi_- = 1$. We have to construct a trial function $\psi \in H^2(\mathbb{R}^2)$ which makes the quadratic form (2.3c), now containing the potential bias, negative. We use elements of the previous proofs. In particular, inside Ω^a the function will be given by (3.2) and (3.1). Outside Ω^a the trial function in Ω_- will be the same as in the proof of part (a) of Theorem 2.2, cf. (3.9), while in Ω_+ we choose it as in the of part (b) of Theorem 2.4 discussed above, putting there $\mu = 0$, in other words, as (5.1) in which in view of (3.5) we set $\xi_+ = -\sqrt{V_0}$. Repeating then the estimates used to prove part (a) Theorem 2.2 in Ω_- and part (a) of Theorem 2.4 in Ω_+, we obtain

$$Q[\psi] = -\frac{1}{8}\delta\nu - \int_{\mathbb{R}} \kappa(s) \, \mathrm{d}s + o(\psi), \tag{5.9}$$

where the error term can be made arbitrarily small by choosing large r_0 and s^* in (3.1). In view of the assumed convexity of Ω_+ we have

$\int_{\mathbb{R}} \kappa(s) \, \mathrm{d}s > 0$, hence choosing the parameters properly we can make the form negative; this concludes the proof of part (b) of Theorem 2.2.

Remark 5.2. As we have noted in the introduction, the 'two-sided' validity of part (b) Theorem 2.4 does not extend to the zero-energy resonance case. The above proof indicates the source of this difference. While for $\mu < 0$ we can use the trial function from the proof of part (b) of Theorem 2.2 and simply switch the roles of Ω_+ and Ω_-, a similar interchange does not work if $\mu = 0$ because it leads to the sign change of the second term on the right-hand side of (5.9) and we are obviously not free to choose $\delta\nu$ to compensate this positive number.

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DOPPLER INSTITUTE FOR MATHEMATICAL PHYSICS AND APPLIED MATHEMATICS, CZECH TECHNICAL UNIVERSITY, BŘEHOVÁ 7, 11519 PRAGUE, CZECHIA, AND DEPARTMENT OF THEORETICAL PHYSICS, NPI, ACADEMY OF SCIENCES, 25068 ŘEŽ NEAR PRAGUE, CZECHIA
  Email address: exner@ujf.cas.cz
  URL: http://gemma.ujf.cas.cz/~exner/

KARLSRUHE INSTITUTE OF TECHNOLOGY
  Email address: semjon.wugalter@kit.edu