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RESEARCH ARTICLE

Strategy-proofness implies minimal participation under single-peakedness

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Abstract
We study a model in which agents with single-peaked preferences can participate in a costly voting procedure to determine the value of a one-dimensional variable. We show that, for all positive participation costs and all profiles of individual preferences, there exists a unique equilibrium outcome with one single participant whenever the voting rule is strategy-proof, anonymous, and responsive in the sense that the outcome reacts to a unanimous move of the votes of all agents in the same direction; moreover, the single participant is always one of the 'extremist' voters, i.e. either one with the lowest or one with the highest peak. While this uncovers a strong tension between strategy-proofness and participation for all deterministic voting rules on the single-peaked domain (just as in the case of an unrestricted domain), there are simple probabilistic and strategy-proof voting rules that induce full participation in equilibrium.

Keywords Costly voting · Strategy-proofness · Generalized medians · Participation

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1 Introduction

The celebrated Gibbard-Satterthwaite Theorem (Gibbard 1973; Satterthwaite 1975) states that, under very general circumstances, every strategy-proof social choice function on an unrestricted preference domain is dictatorial. In particular, if voting comes at some cost all voters except one will prefer not to participate in the election process. One interpretation of the Gibbard-Satterthwaite Theorem is thus that two fundamental incentive properties are incompatible on an unrestricted domain: strategy-proofness, i.e. the incentive to reveal preferences truthfully, and participation, i.e. the incentive to invest the cost of casting one's vote.

In this paper, we show that a similar conclusion holds if all voters' preferences are single-peaked, which is one of the paradigmatic cases in which there exist non-dictatorial and strategy-proof voting rules (Black 1948; Moulin 1980). Concretely, the following is our main result (Theorem 1 in Section 3). Suppose that a group of agents collectively decides on the level of a one-dimensional variable and that all agents have single-peaked preferences over that level. Also assume that each agent prefers not to participate whenever such unilateral abstention does not change the outcome of the election. Moreover, suppose that the voting rule employed is strategy-proof, anonymous and responsive in the sense that the outcome reacts to any unanimous shift of all peaks in one direction. Then, there exists a unique equilibrium in which every participant uses her (weakly) dominant strategy; in this equilibrium there is a single participant which is either the agent with the lowest peak, or the agent with the highest peak.[1]

Our analysis brings together two different strands of the literature. On the one hand, we rely on Moulin's path-breaking characterization of all strategy-proof social choice functions on the single-peaked domain as the *generalized medians*, i.e. medians with a set of 'phantom voters' (Moulin 1980); on the other hand, we employ a version of the *pivotal voter model* (Downs 1957; Palfrey and Rosenthal 1983) which posits that rational agents will engage in a voting procedure if and only if the expected benefit of doing so exceeds its costs. The main departure from the bulk of the literature on the pivotal voter model is that we assume a one-dimensional space of alternatives with single-peaked preferences in a complete information environment. Our model thus applies to decisions in small committees rather than to large elections.

In our setting agents take two decisions: whether or not to participate in the voting procedure, and if so, which vote to cast. We consider two versions of this general set-up: a sequential model and a simultaneous model. The sequential model has two stages: in the first stage agents simultaneously decide whether or not to participate in a committee, and in the second stage a simultaneous vote is cast by the committee members on the level of a one-dimensional variable. In the simultaneous model, the participation and voting decisions are made simultaneously by all agents. In either model, we assume that agents incur positive cost if and only if they in fact cast a vote. We shall see that while the two models may lead to different predictions in general, for our main result the timing of decisions is in fact not relevant.

[1] More precisely, the equilibrium *outcome* is unique since different agents could have the same peaks in which case there is a coordination issue of who among them votes and who abstains.

As a simple example, think of a faculty meeting on a Friday afternoon at which the yearly expenditure shares, say, for research and teaching are to be determined (given a fixed budget). Each faculty member deliberates about whether or not to participate in the voting procedure. In the sequential model, one can think of the participation decision as being taken before the actual meeting. In the simultaneous model, there could be an announcement during the meeting that a vote would be taken after extensive discussion, and committee members may decide to leave early thereby abstaining from the collective decision. The assumption of complete information is strong but does not seem to be unrealistic in such a scenario; in fact, since all strategy-proof voting rules (for a fixed set of participants) only depend on the top alternative of each voter, it is sufficient to know each colleagues' top choice.

The conclusion of our main result—that under strategy-proofness, anonymity and responsiveness a single individual casts her vote in equilibrium—stands in stark contrast to other voting rules that are not strategy-proof. For instance, if the collective outcome is determined to be the *average* of the individual votes (Renault and Trannoy 2005) on a cardinal scale, every voter can shift the outcome by a positive amount and full participation is indeed an equilibrium if participation costs are sufficiently small. But while anonymous and responsive, taking the average vote does evidently not define a strategy-proof rule.

The intuition behind the single participation equilibrium under strategy-proofness can be explained by looking at the case of two voters. Strategy-proofness and the responsiveness condition jointly imply that in the case of two voters the outcome must coincide with one of the two voters' top alternative; anonymity implies that it cannot depend on the identity of the voter, hence it must be either the lower or the higher top alternative. In the first case, if (one of) the agent(s) with the lowest top alternative participates, no other agent has an incentive to participate, since no other agent can change the result by unilateral participation; in the latter case, the same holds if (one of) the agent(s) with the highest top alternative participates. The proof that there are no other equilibria is more involved (see Section 3). Ultimately then, our 'impossibility' result can be traced back to the fact that strategy-proofness necessitates the use of an asymmetric tie-breaking rule in the case of an even number of participants; and the counterfactual situation of an even number of participants is of course also relevant for equilibrium even if the potential and actual number of voters is odd. This observation suggests that the incompatibility of strategy-proofness and participation may vanish under a *probabilistic* and symmetric tie-breaking rule in the case of an even number of participants, and we show by means of a simple example that this is indeed true.

A full-fledged analysis of the probabilistic case is beyond the scope of the present paper. But we investigate if the standard *symmetric median*, i.e. the average of the two middle tops in case of an even number of voters, can solve the participation problem in the deterministic case at the expense of loosing the strategy-proofness property for an even number of voters. We answer this in the negative and provide a complete characterization of the subgame perfect equilibria of the sequential game if participation costs are sufficiently small but positive (Theorem 2 in Section 4). Clearly, the very definition of the symmetric median requires a cardinal scale. Taking without much loss of generality the unit interval as the outcome space, we show that there are only two types of equilibria: the single participation of the voter whose peak is closest to the midpoint 0.5, or (almost) full participation with the fixed outcome 0.5. In the latter case, if there is a voter with top alternative at 0.5 that voter will be the only one to abstain.

Relation to the Literature

The question of participation when voting is costly has been extensively discussed in the literature ever since the first formal formulation of the pivotal voter model by Palfrey and Rosenthal (1983). The vast majority of the contributions in the literature studies the case of majority voting among two alternatives. Under complete information, a key issue is to analyze the equilibrium consequences of the fact that a large fraction of voters shares the same preferences. The resulting coordination problem is typically addressed by an analysis of mixed equilibria, see Nöldeke and Peña (2016); Mavridis and Serena (2018) for recent contributions. The focus of the present paper is different due to the assumption of a one-dimensional outcome space with single-peaked preferences. For instance, note that in the case of a rich (e.g. continuous) space of alternatives, different individuals generically have different preference peaks.

The two contributions in the literature closest to ours are Osborne et al. (2000) and Cohensius et al. (2017). These authors also study costly voting in committees in a complete information environment with individuals that have single-peaked preferences. Osborne et al. (2000) *assume* that agents vote truthfully and show that ‘extreme’ voters are more likely to participate than ‘moderate’ voters. However, in the relevant results of Osborne et al. (2000) the outcome with an even number of participants is determined by the symmetric median in which case sincere voting does not generally constitute an equilibrium. While our main result does not contradict the intuition put forward by Osborne et al. (2000), it qualifies it in an important way. Under the responsiveness condition, the single participant is indeed always an ‘extreme’ voter: either the agent with the lowest or the agent with the highest top alternative. But as explained above, the rationale is not that the moderate voters cancel each other out. More importantly, without the responsiveness condition, equilibria can occur in which only ‘moderate’ agents participate, see Section 3.1 below.

In independent work, Cohensius et al. (2017) observe that participation by a single voter is the only equilibrium in the special case of the ‘lower median’ voting rule (and by symmetry also for the ‘upper median’ rule), but they do not offer a general impossibility result for the class of all strategy-proof voting rules akin to the main result provided here.

2 The model

We denote the set of agents by $N = \{1, 2, ..., n\}$, and by $(X, >)$ a linearly ordered set of alternatives; moreover, let $\underline{x} := \inf X$, $\overline{x} := \sup X$, and $\overline{X} := X \cup \{\underline{x}, \overline{x}\}$. Our lead example will be the case in which X is the unit interval $[0, 1] \subseteq \mathbb{R}$; in this case, a possible interpretation is that each $0 \leq x \leq 1$ represents the expenditure share for a public project. But the analysis is completely general; for instance, the elements of

X could represent abstract positions on a discrete political spectrum. Each agent i is characterized by a single-peaked (ordinal) preference relation $\succcurlyeq_i$ over X. Single-peakedness means that each agent i has a unique top alternative $p_i \in X$ (the 'peak') such that, for all $x, y \in X$, we have $x \succ_i y$ whenever $y < x \leq p_i$ or $p_i \leq x < y$.[2]

Agents decide whether or not to participate in a committee that decides on the level of the 'one-dimensional' variable $x \in X$ by a voting procedure. Each agent i faces a positive participation cost $c_i > 0$. This cost may vary from agent to agent, it may depend on the finally chosen outcome, and even on the set of the other participating agents; in fact, all what matters for our purpose is that these costs are strictly positive for all agents. In particular, we could allow the cost to be private information. For each possible non-empty set $K \subseteq N$ of participants, there is a social choice function $F^K(\succcurlyeq_1, ..., \succcurlyeq_{\#K}) \in X$ that maps every profile of preferences of the agents in K to an outcome in X. The collection $F = \{F^K\}_{\emptyset \neq K \subseteq N}$ of these social choice functions is referred to as a *voting rule*. The employed voting rule is common knowledge among the agents.

In our model with endogenous participation, we need to specify agents' preferences $\widehat{\succcurlyeq}_i$ over pairs (x, K) of outcomes and sets of participants K who actually cast a vote. We denote by $x_0 \notin X$ the (exogenously determined) outcome if nobody participates in the voting process, and will make the following assumptions. For all $i \in N$,

(i) the outcome x_0 is strictly worse than the most preferred outcome with single own participation, i.e.

$$(p_i, \{i\}) \mathrel{\widehat{\succ}_i} (x_0, \emptyset),$$

(ii) for every fixed set $K \neq \emptyset$ the preference over outcomes is given by $\succcurlyeq_i$, i.e.

$$(x, K) \mathrel{\widehat{\succcurlyeq}_i} (y, K) \iff x \succcurlyeq_i y,$$

(iii) for every fixed $x \in X$, non-participation is strictly preferred to participation (and indifference with respect to the composition of the set of participants otherwise), i.e. for all $K, K' \neq \emptyset$,

$$\{i \notin K \quad \text{or } i \in K'\} \Longrightarrow (x, K) \mathrel{\widehat{\succcurlyeq}_i} (x, K'),$$
$$\{i \notin K \text{ and } i \in K'\} \Longrightarrow (x, K) \mathrel{\widehat{\succ}_i} (x, K').$$

Condition (i) is very weak and says that voting costs are not prohibitively high so that agents have a general incentive to participate. Except for this 'non-triviality' condition, no assumptions are made about how agents compare an outcome without participation to a *strictly better* outcome with own participation; indeed, such trade-offs would determine the particular magnitude of participation cost which plays no role in our present analysis. Condition (ii) simply says that the preference for outcomes does not depend on the set of voters who determine it, and condition (iii) says that

[2] Note that we do not need to make any assumptions about the comparisons of alternatives on different sides of the peak; in fact, the preference relation may even be incomplete and refrain from making such comparisons.

agents would rather abstain provided that they are not pivotal. The latter condition has been studied by Desmedt and Elkind (2010); Elkind et al. (2015) under the name of *lazy bias*.[3]

We will consider two variants of the model, a simultaneous and a sequential version. In the sequential version, agents first simultaneously decide whether or not to participate and vote simultaneously in a second stage after having observed who the other participants are. By contrast, in the simultaneous version, both the voting and participation decisions are made simultaneously. While the equilibria in general differ in the two models (see the discussion section below), the main conclusion of the present paper is robust with respect to the timing of decision.

In our main result, Theorem 1 in Section 3 below, we will require the voting rule to be anonymous and strategy-proof. The *anonymity* condition has two components: first, for each given set of participants K, the outcome under F^K is invariant with respect to permutations of the agents in K; secondly, the employed social choice function F^K should depend only on the number of agents in K. Using the latter condition, we can write F^k for all social choice functions F^K with $\#K = k$, and describe the voting rule $F = \{F^k\}_{1 \leq k \leq n}$ in terms of n social choice functions, one for each number of participants.

Strategy-proofness requires that truth-telling be a (weakly) dominant strategy for all participating agents: for all $K, i \in K, \succcurlyeq_i, \succcurlyeq'_i, \succcurlyeq_{K-i}$,

$$F^k(\succcurlyeq_i, \succcurlyeq_{K-i}) \succcurlyeq_i F^k(\succcurlyeq'_i, \succcurlyeq_{K-i}),$$

where $k = \#K$ and $\succcurlyeq_{K-i}$ denotes any profile of preferences for the agents in K other than i.

By a well-known result of Moulin (1980), the conditions of anonymity and strategy-proofness jointly imply that all social choice functions F^k are 'generalized medians' with $k+1$ so-called 'phantom voters.' Specifically, for each $k \in \{2, ..., n\}$, F^k only depends on the individual peaks, i.e. for some function $f^k : X^k \to X$

$$F^k(\succcurlyeq_1, ..., \succcurlyeq_k) = f^k(p_1, ..., p_k),$$

and there exist fixed values $\alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k \in \overline{X}$ such that

$$f^k(p_1, ..., p_k) = med\{p_1, ..., p_k, \alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k\}, \tag{1}$$

where med denotes the usual median operator and the p_i are the peaks of $\succcurlyeq_i$ for each participating agent i; note that there are $2k+1$, i.e. an odd number, of values in (1). An important example is the standard median rule with an odd number of participants; in this case, half of the phantom voters are placed at $\underline{x}$ and half are placed at $\overline{x}$.[4] We emphasize that strategy-proofness in the sense considered in this paper is a condition

[3] These studies find that in the context of plurality voting, equilibria with lazy-biased voters often (but not always) involve participation of a single individual, see also Meir (2018) for a recent survey.

[4] See Jennings et al. (2020) for alternative characterizations of the class of all strategy-proof social choice functions on the domain of single-peaked preferences on an arbitrary linearly ordered set.

imposed on a social choice function for any *fixed* set of participants; it does not refer to a property of the overall mechanism (which includes the participation decision).

We will say that F^k, respectively f^k, satisfies *responsiveness* if, for all $p_1, ..., p_k$, $p'_1, ..., p'_k$,

$$p'_i > p_i \text{ for all } i \in K \Rightarrow f^k(p'_1, ..., p'_k) \neq f^k(p_1, ..., p_k).$$

Responsiveness can be viewed as a condition of 'local non-imposition:' if *every* agent desires a strictly higher (lower) outcome, the chosen alternative should move at least minimally.[5] While arguably a weak and plausible condition, responsiveness does restrict the set of admissible voting rules, as follows.

Observation. *The generalized median functions f^k in (1) satisfy responsiveness if and only if all 'phantom voters' are either at $\underline{x}$ or at $\overline{x}$, i.e. for all $j = 1, ..., k+1$, $\alpha^k_j \in \{\underline{x}, \overline{x}\}$, and neither are all phantom voters located at $\underline{x}$ nor all at $\overline{x}$. In particular, in this case the generalized median always coincides with one of the peaks of the agents and the corresponding voting rule is efficient.*

Proof It is easily verified that the stated condition is sufficient for responsiveness. To show its necessity suppose that, for some k and $j_0 \in \{1, ..., k+1\}$, one has $\underline{x} < \alpha^k_{j_0} < \overline{x}$. (Note that the statement evidently holds in the case of two alternatives.) Suppose that there are $\ell \in \{1, ..., k+1\}$ phantom voters with exactly the same location at $\alpha^k_{j_0}$, and $h \in \{0, ..., k\}$ phantom voters with a location strictly smaller than $\alpha^k_{j_0}$. Consider a profile $(p_1, ..., p_k)$ that puts the peaks of $k-h$ voters strictly below $\alpha^k_{j_0}$ and the peaks of h voters at $\alpha^k_{j_0}$. Then, exactly k elements of $\{p_1, ..., p_k, \alpha^k_1, ..., \alpha^k_{k+1}\}$ are strictly below $\alpha^k_{j_0}$, and hence $med\{p_1, ..., p_k, \alpha^k_1, ..., \alpha^k_{k+1}\} = \alpha^k_{j_0}$. If we now move the peaks of the $k-h$ voters strictly below $\alpha^k_{j_0}$ to $\alpha^k_{j_0}$, and the peaks of the h voters at $\alpha^k_{j_0}$ strictly above $\alpha^k_{j_0}$, we obtain a profile $(p'_1, ..., p'_n)$ at which exactly $h \leq k$ elements of $\{p'_1, ..., p'_k, \alpha^k_1, ..., \alpha^k_{k+1}\}$ are strictly below $\alpha^k_{j_0}$, and $k+\ell \geq k+1$ are below or equal to $\alpha^k_{j_0}$; hence $med\{p'_1, ..., p'_k, \alpha^k_1, ..., \alpha^k_{k+1}\} = \alpha^k_{j_0}$ in violation of responsiveness. □

It is well-known (Moulin 1980) that under efficiency, the generalized median functions f^k in (1) can be assumed to have $k-1$ instead of $k+1$ phantom voters. Generalized medians for which all $k-1$ phantom voters are at one of the two extremes are also known as the *order statistics*. Specifically, the choice of the i-th lowest value of the $\{p_1, ..., p_k\}$ is referred to as the i-th order statistic, and corresponds to the generalized median in which $k-i$ phantom voters are at $\underline{x}$ and $i-1$ phantom voters are at $\overline{x}$, see Caragiannis et al. (2016) for further discussion.

If all social choice functions F^k are strategy-proof, voting truthfully is the unique (weakly) dominant strategy for every participant in the simultaneous game, and in every second-stage voting subgame of the sequential game. We will therefore assume that all participants who actually cast a vote submit their true peak. This assumption

5 Intuitively, it should clearly move in the *same* direction; in the present context, this slightly stronger requirement is redundant because it follows from responsiveness plus strategy-proofness.

could be further justified by an appeal to equilibrium refinement concepts such as perfectness (Selten 1975), or strong equilibrium (Aumann 1959).[6]

3 Main result

The following is our main result. Let us call a Nash equilibrium in which each participant chooses her unique (weakly) dominant strategy, a *truthful Nash equilibrium.*

Theorem 1 *Suppose that the voting rule is anonymous, strategy-proof and responsive, and that all individuals' voting costs satisfy conditions (i)–(iii) above.*

a) In every truthful Nash equilibrium of the simultaneous game exactly one agent participates.
b) In every truthful subgame perfect Nash equilibrium of the sequential game exactly one agent participates.

In either model, the participating agent is either (one of) the individual(s) with the highest peak, or (one of) the individual(s) with the lowest peak.

Proof The assumptions on the voting rule imply that, for all non-empty sets $K \subseteq N$ of participating agents, the outcome is determined by a generalized median with $\#K - 1$ phantom voters. Moreover, by anonymity, the set of phantom voters only depends on $k = \#K$. Observe in particular that all social choice functions functions f^k are efficient, i.e. for all $p_1, ..., p_k$, we have

$$\min_{j=1,...,k} p_j \leq f^k(p_1, ..., p_k) \leq \max_{j=1,...,k} p_j. \tag{2}$$

First, we show that there exists a truthful Nash equilibrium with a single participant in the simultaneous game. Assume without loss of generality that $p_1 \leq p_2 \leq ... \leq p_n$. For $\#K = 2$, there is one phantom voter α_1^2, and by the Observation in the previous section, we have either $\alpha_1^2 = \underline{x}$, or $\alpha_1^2 = \overline{x}$. Suppose the former, then the single participation of any agent with peak p_1 (who reports truthfully) constitutes a truthful Nash equilibrium. Indeed, the outcome then is p_1 which by assumption is preferred by any such agent to x_0 (the outcome if nobody participates). Every other agent has a (weakly) higher peak and can thus not unilaterally change the outcome because $\alpha_1^2 = \underline{x}$; hence, by the participation condition (iii), each other agent prefers not to participate. The argument is completely symmetric if $\alpha_1^2 = \overline{x}$, in which case single participation of any agent with peak p_n (who reports truthfully) constitutes a truthful Nash equilibrium. Clearly, single participation of an agent j with $p_1 < p_j < p_n$ cannot be a truthful Nash equilibrium since either agent 1 or agent n would have an incentive to participate. Note that the same argument shows that single participation

[6] In the implementation literature, there has been some discussion on the fact that the median rule (as well as generalized medians) may have other Nash equilibria, in which (some) agents do not follow their unique weakly dominant strategy; see, e.g., Yamamura and Kawasaki (2013). For instance, if $k \geq 3$ and all agents cast exactly the same (non-truthful) vote nobody is pivotal, hence any such unanimous vote profile constitutes a Nash equilibrium. However, such equilibria are evidently neither robust against trembles, nor against deviations by coalitions of agents.

of either one of the agents with the lowest peak, or single participation of one of the agents with the highest peak constitutes a truthful subgame perfect Nash equilibrium in the sequential game. (Of course, the complete strategy in the sequential game also specifies, for each non-participant, truth-telling in all counterfactual participation situations.)

It remains to show that there are no other truthful equilibria. By contradiction, suppose we have a truthful equilibrium in which the set of participants is $K \subseteq N$ where $\#K = k > 1$. By the Observation above, there exists $j \in K$ such that $f^k(p_1, ..., p_k) = p_j$. First assume that $j = 1$, i.e. that one of the voters with the lowest peak gets her most preferred alternative. Then, voter k (the one with the highest peak among the participants) has an incentive to abstain; indeed, by the efficiency of f^{k-1} the outcome without voter k cannot be smaller than p_1. By a similar argument, one can show that $j \neq k$. Note, that if $p_1 = p_k$ every participant would prefer to (unilaterally) abstain as the outcome would remain unchanged.

Thus, we must have that $1 < j < k$ for the individual j who receives her peak p_j. In this case, the assumed optimality of participation by agent 1 implies that

$$f^{k-1}(p_2, ..., p_k) \; = \; med\{p_2, ..., p_k, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} > p_j, \tag{3}$$

since otherwise agent 1 would prefer not to participate by the participation condition (iii). Similarly, the assumed participation of agent k implies that

$$f^{k-1}(p_1, ..., p_{k-1}) \; = \; med\{p_1, ..., p_{k-1}, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} < p_j. \tag{4}$$

Without agent 1 there are at least $j - 1$ peaks that are below or equal to p_j. By (3), the generalized median $f^{k-1}(p_2, ..., p_k)$ with $k - 1$ participants (i.e. agents 2 to k) is strictly above p_j; this implies that at most $(k - 1 - j)$ of the $k - 2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at $\underline{x}$. Similarly, without agent k there are at least $k - j$ peaks above or equal to p_j. By (4), the generalized median $f^{k-1}(p_1, ..., p_{k-1})$ with $k - 1$ participants (i.e. agents 1 to $k - 1$) is strictly below p_j; this implies that at most $j - 2$ of the $k - 2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at $\overline{x}$. By the responsiveness, all of the $k - 2$ phantom voters have to be located either at $\underline{x}$ or at $\overline{x}$. But we have just shown that under conditions (3) and (4) this is not possible since

$$(k - 1 - j) \; + \; (j - 2) = k - 3 \; < \; k - 2.$$

Thus, there can be no Nash, or subgame-perfect Nash equilibrium in which more than one agent participates and all participants vote truthfully. This completes the proof of Theorem 1. □

3.1 Discussion

In order to assess the scope and the robustness of Theorem 1, we now consider each of its assumptions. We explain why they are necessary for the conclusion and we discuss what happens if they were dropped.

Truthful voting

The requirement that all participants vote according to their (weakly) dominant strategy (i.e. truthfully) can neither be dropped in the simultaneous nor in the sequential game. This can be demonstrated by the following two examples.

Suppose that there are three agents 1, 2, 3, and for simplicity assume that $X = [0, 1]$. Suppose that with three participants the two phantom voters are both located at 0, and with two participants the single phantom voter is located at 1. Suppose that the peaks of the three agents are distributed as follows: $p_1 = 0.2$, $p_2 = 0.3$ and $p_3 = 0.4$; moreover, assume that all agents preferences are symmetrically single-peaked, i.e. representable by the negative Euclidean distance to the peak. As can be verified, the vote profile (i.e. the reported 'peaks') $(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (0.2, 0.8, 0.5)$ constitutes a non-truthful Nash equilibrium in the simultaneous game with full participation. Observe, however, that this equilibrium is not stable against 'small' perturbations: for instance, if agent 1 abstains with positive probability, agent 2 does not respond optimally.

Next, consider the sequential game. Again, let there be three individuals 1, 2, 3 and two alternatives $x, y \in X$ such that $p_1 < p_2 < p_3 < x < y$. Suppose all three individuals use the following strategy: choose $\tilde{p}_i = x$ if the number of participants is $k = 3$, $\tilde{p}_i = y$ if the number of participants is $k = 2$, and $\tilde{p}_i = p_i$ under single participation. Then, full participation is a non-truthful subgame-perfect Nash equilibrium for the rule that chooses the median of the votes if $k = 3$, the upper median if $k = 2$, and the unique vote if $k = 1$. Again, this equilibrium is not stable, since deviations by coalitions of agents can be profitable in some subgames.

Anonymity

Our anonymity condition has two components. First, it requires the voting rule not to depend on the 'names' of voters *for any given set of participants*; secondly, it requires that the same voting rule is employed for all subsets with the *same number* of participants. Arguably, both conditions are natural in the present context. The first part is a standard assumption in voting theory, and in fact Moulin's characterization of all strategy-proof rules for single-peaked preferences in terms of phantom voters needs this assumption. In our present variable electorate context the second part also appears to be highly plausible. Importantly, it also guarantees the existence of an equilibrium in pure strategies (and its outcome uniqueness). We show this by means of two simple examples, as follows.

Assume that all conditions of Theorem 1 are satisfied except the second part of the anonymity condition, and consider the following examples with $N = \{1, 2, 3\}$. Suppose that if the set of participants consists of agents 1 and 2, the outcome function $f^{\{1,2\}}$ chooses the higher peak, i.e. we have $\alpha_1^{\{1,2\}} = \overline{x}$ for the corresponding phantom voter; if the set of participants consists of agents 1 and 3, the outcome function $f^{\{1,3\}}$ chooses the lower peak, i.e. $\alpha_1^{\{1,3\}} = \underline{x}$ for the corresponding phantom voter; and finally, if the set of participants consists of agents 2 and 3, the outcome function $f^{\{2,3\}}$ chooses again the higher peak, i.e. $\alpha_1^{\{2,3\}} = \overline{x}$ for the corresponding phantom voter. Evidently, this specification violates the (second part of the) anonymity condition.

Suppose that agents are ordered so that $p_1 < p_2 < p_3$. For no agent single participation is an equilibrium: if agent 1 is the single voter, agent 2 has an incentive to join; if agent 2 is the single voter, agent 3 has an incentive to join; and if 3 is the single voter, agent 1 has an incentive to join. A situation with two participants cannot be an equilibrium either because, by the responsiveness condition, one of the two gets her peak in which case the other has an incentive to abstain and save the voting costs. Finally, full participation cannot be an equilibrium either. Indeed, suppose that all agents participate; again by the responsiveness condition, one of the agents must receive her peak. If agent 1 receives her peak, agent 2 has an incentive to abstain, because this would not change the outcome and agent 2 would save the participation cost; similarly, if agent 2 receives her peak, agent 3 has an incentive to abstain, and if agent 3 receives her peak, agent 1 has an incentive to abstain. Hence, in this example there is no equilibrium in pure strategies.

Here is an example in which there are several equilibria, including one with *full participation*. If all agents participate, the social choice function $f^{\{1,2,3\}}$ chooses the standard median, in other words, the corresponding phantom voters are given by $\alpha_1^{\{1,2,3\}} = \underline{x}$ and $\alpha_2^{\{1,2,3\}} = \overline{x}$; if agents 1 and 2 participate, the social choice function $f^{\{1,2\}}$ chooses the lower peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{1,2\}} = \underline{x}$; if agents 2 and 3 participate, the social choice function $f^{\{2,3\}}$ chooses the higher peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{2,3\}} = \overline{x}$. No matter how we specify the outcome in the case in which the set of participants consists of agents 1 and 3, this already implies that for sufficiently small participation costs full participation is an equilibrium. Indeed, if all agents participate the agent with the median peak gets her peak and has no incentive to abstain if her participation costs are sufficiently small; for either of the other two agents, unilateral non-participation would move the outcome further away from their respective peak, so neither of them has an incentive to abstain as well. There also exists an additional single participation equilibrium. Indeed, for the set of participants $\{1, 3\}$ we either have $\alpha_1^{\{1,3\}} = \underline{x}$ or $\alpha_1^{\{1,3\}} = \overline{x}$. In the first case, single participation of the agent with the lowest peak is an equilibrium (since none of the other two agents can unilaterally change the outcome); in the second case, single participation of the agent with the highest peak is an equilibrium.

Responsiveness

Above, we have justified the responsiveness condition by an appeal to a 'local non-imposition' property: if all agents uniformly move in one direction, the outcome should not remain unchanged. We have also shown that this condition is equivalent to the property that all phantom voters should be at the two extreme points $\underline{x}$ or $\overline{x}$. There may exist an even deeper justification for the responsiveness condition in purely ordinal contexts. Indeed, if the set of alternatives is linearly ordered but in a purely ordinal way, any specific location of a phantom voter in the interior of X seems arbitrary. On the other hand, if cardinal information is available, say $X = [0, 1]$ such as in the example of the dividing a fixed budget, phantoms may be placed in the midpoint (at

0.5), or distributed uniformly in [0, 1]; the latter specification is also known as the 'uniform' or 'linear' median, see Jennings et al. (2020). In any case, Theorem 1 fails without the responsiveness condition. As a simple example, consider the case of an even number of agents $N = \{1, 2, ..., 2m\}$, and suppose that the phantom voters are located as follows: $m - 1$ phantom voters are at $\underline{x}$, $m - 1$ phantom voters are at $\overline{x}$, and one phantom voter is located in the interior, say at x with $\underline{x} < x < \overline{x}$. Also, suppose that for any set of $2m - 1$ participants the standard median is chosen as the outcome. Consider any distribution such that m agents have their peak below x and m agents have their peak above x. Then, if all individuals' costs are sufficiently small, full participation is an equilibrium. Indeed, the outcome under full participation is x, and for any agent unilateral abstention moves the outcome further away from her peak.

But even without the responsiveness condition, there often also exist profiles of peaks such that only one single agent participates in equilibrium. Specifically, let $p_1 < p_2 < \cdots < p_N$ be such that (a) $\{\alpha_j^k\}_{j=1,...,k+1}^{k=1,...,N} \cap [p_1, p_N] = \emptyset$, i.e. all phantom voters are either below the smallest peak or above the highest peak, and (b) for all $k = 1, ..., N$, $\min_j\{\alpha_j^k\} < p_1$ and $p_N < \max_j\{\alpha_j^k\}$. (Note that condition (b) is implied by efficiency of the voting rule.) Then, we obtain single participation as the unique equilibrium by the same logic as in the proof of Theorem 1.

Strategy-proofness

The assumption of strategy-proofness (for any fixed set of participants) is essential for the conclusion of Theorem 1. Strategy-proofness guarantees that truthful voting is a weakly dominant strategy for every participant, and as shown above, Theorem 1 hinges on the use of that strategy. Not surprisingly then, Theorem 1 fails without the strategy-proofness property.

To illustrate this, suppose we can use cardinal information and employ the following *symmetric* version of the median rule. Let $X = [0, 1]$, and assume that the peaks of the participants be ordered such that $p_1 < p_2 < \cdots < p_k$; for an odd number $k = 2m - 1$ the outcome is the standard median p_m, and for an even number $k = 2m$ the outcome is the midpoint between the two middle peaks $(p_m + p_{m+1})/2$. This rule is not strategy-proof, and therefore we cannot assume that the reported peaks coincide with the true peaks.[7] Consider the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$. Let $\tilde{p}_i$ be the reported peak by agent i, and assume that the participation costs of the two extreme agents 1 and 3 are small but positive. Then, the equilibrium depends, among other things, on the magnitude of the participation cost (i.e. the precise shape of the preferences) of the median agent. If the participation cost of agent 2 is sufficiently small, full participation and truth-telling is an equilibrium in the simultaneous game; on the other hand, if agent 2 prefers the outcome 0.5 without own participation to the outcome 0.45 while participating, $\tilde{p}_1 = 0$ and $\tilde{p}_3 = 1$ is a (non-truthful) equilibrium.

Moreover, it is because of truthful voting in equilibrium that the conclusion of Theorem 1 is robust with respect to the timing of decisions. Indeed, in the sequential

[7] It is well-known that there exist no strategy-proof, anonymous and symmetric ('neutral') social choice functions on the domain of single-peaked preferences for an even number of individuals, see Moulin (1980, 1988).

model participants' voting strategy may depend on the set of other participants which becomes common knowledge after all agents have made their participation decision. To illustrate this point, consider again the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$ and the symmetric median rule, but now assume that agents move sequentially. In this case, full participation is no longer an equilibrium; indeed, if agent 3 does not participate, agents 1's and 2's optimal votes are $\tilde{p}_1 = 0$ and $\tilde{p}_2 = 0.9$, respectively, with the outcome 0.45. Since this is the same outcome as under full participation, agent 3 prefers to abstain whenever she has positive participation costs. However, participation of agents 1 and 2 with outcome 0.45 can also not be an equilibrium since then agent 1 has an incentive to abstain. In this example, if all agents have sufficiently small (but strictly positive) participation cost, the unique subgame perfect equilibrium of the sequential voting game is given by participation of agents 2 and 3 with votes $\tilde{p}_2 = 0$ and $\tilde{p}_3 = 1$, resulting in the outcome 0.5. A complete characterization of the equilibria under the symmetric median rule in the sequential model is provided in the next section under the assumption that in the second stage a strong Nash equilibrium is played.

4 Probabilistic versus deterministic voting rules

One may interpret Theorem 1 as saying that, if voting is costly, no anonymous and deterministic voting rule can implement the median if all participants vote according to their unique dominant strategy. Specifically, we have the following corollary of Theorem 1.

Corollary 1 *Suppose that the number of agents is odd. There does not exist an anonymous and strategy-proof voting rule that yields the median peak for all distributions of individual peaks if voting is costly and all actual participants vote according to their unique dominant strategy.*

To verify this simply note that by the remarks in Section 3.1, even without the responsiveness condition there always exist (generic) peak distributions for which the unique dominant strategy equilibrium involves the single participation either of the agent with the highest peak, or the agent with the lowest peak.[8]

An important implicit assumption of our analysis is that we require voting rules to be *deterministic*. One may argue that this is a strong assumption; and in fact, together with anonymity and strategy-proofness it forces an asymmetric tie-breaking rule in the case of an even number of participants. Consider the following probabilistic voting rule: for an odd number of participants choose the median peak, and for an even number choose a fair lottery that yields the two middle peaks with equal probability of $1/2$. Moreover, suppose that all voters have expected utility preferences with a single-peaked von-Neumann-Morgenstern utility function. Then, for any fixed number of participants, it is the unique (weakly) dominant strategy to submit one's true preference peak as in the deterministic case. But unlike in the deterministic case, participation and truthful

[8] In independent work, Cohensius et al. (2017) have shown that participation of a single agent is the unique equilibrium under the 'lower median' rule, i.e. under the rule that chooses the median if the number of participants is odd and the lower of the two middle votes if the number of participants is even.

reporting now always changes the outcome to one's benefit at profiles with distinct peaks.[9] In particular, at every profile with pairwise distinct peaks full participation with truthful reporting is an equilibrium if voting costs are sufficiently small.

A detailed analysis of costly voting in the probabilistic case is beyond the scope of the present paper, but we note the following corollary to Theorem 1. Suppose that we extend the class of voting rules to include all probabilistic rules, and generalize the notions of anonymity, responsiveness and strategy-proofness in the natural way[10]; then, we have:

Corollary 2 *Let $\{F^k\}_{1 \leq k \leq n}$ be a probabilistic voting rule that is anonymous, responsive and strategy-proof. Suppose that for some $h \in \{2, ..., n\}$, both F^h and F^{h-1} are in fact deterministic social choice functions. If voting is costly (in the sense described in Section 2 above), then there does not exist a truthful equilibrium with h participants.*

This follows by exactly the same logic as in the proof of Theorem 1 above. By Corollary 2, randomization must occur for sufficiently many sizes of the participant sets in order to induce participation when costs are small but positive. For instance, taking $h = n$ in Corollary 2 we can deduce that full participation requires that either F^n or F^{n-1} must be non-deterministic.

If $X = [0, 1]$, which we assume for the remainder of this section, the fair lottery over the two middle peaks in the case of an even number of votes yields the symmetric median in expectation. One may wonder whether the deterministic symmetric median[11] can 'mimic' the randomized median rule and induce full participation (perhaps under additional assumptions) if costs are sufficiently small. The following result provides a complete characterization of all subgame-perfect equilibria of the sequential game if costs are sufficiently small and answers this question to the negative. Note that in the case of an even number of participants truth-telling no longer constitutes a dominant strategy (cf. Section 3.1 above). To avoid the discussion of an artificial multiplicity of equilibria, we assume that in the second stage a strong Nash equilibrium is played (i.e. an equilibrium such that no subgroup of agents can profitably deviate); for every fixed number k of participants, a strong equilibrium exists (see Appendix). Say that a participant with peak $p_i \neq 0.5$ exhibits *extreme reporting* if she reports 0 if $p_i < 0.5$ and 1 if $p_i > 1$.

Theorem 2 *Consider the symmetric median rule and assume that individuals have sufficiently small but positive costs of participation in the sense of conditions (i), (ii) and (iii) in Section 2 above. Then all subgame-perfect equilibria such that in the second stage a strong Nash equilibrium is played are of the following four types:*

Case 1. *There exists at least one agent with peak 0.5. Then, the single participation of one of these agents (who reports truthfully her peak 0.5) is an equilibrium.*
Case 2. *The peaks of all agents are on the same side of 0.5, i.e. either $p_i \geq 0.5$ or $p_i \leq 0.5$ for all $i \in N$. Then, the single participation of (one of) the individual(s) whose peak is closest to 0.5 (who reports truthfully) is an equilibrium.*

[9] We are grateful to Klaus Nehring and Martin Osborne who independently pointed this out to us.

[10] For the relevant probabilistic version of strategy-proofness, see, e.g., Ehlers et al. (2002).

[11] Recall that the 'symmetric median' selects the standard median in the case of an odd number of votes, and the average of the two middle votes in the case of an even number of votes.

Case 3. *The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them having a peak strictly above* 0.5*. Then, full participation with extreme reporting is an equilibrium and the outcome is* 0.5*.*
Case 4. *The number of potential voters n is odd, there is an odd number of agents (smaller than n) with peak 0.5 and the peaks of the other agents are evenly split to both sides of* 0.5*. Then, it is an equilibrium that all agents with peak different from 0.5 participate with extreme reporting; the resulting outcome is* 0.5*.*

In all other cases, there does not exist a pure strategy equilibrium such that in the second stage a strong Nash equilibrium is played.

(Proof in Appendix)

Note that there are cases in which the equilibrium outcome of the symmetric median rule is very far from the median peak. For instance, in Case 2 the median peak could be close to 0 while the equilibrium outcome is the highest peak which could be even at 0.5. The intuition behind this equilibrium is as follows. Suppose that only the agent with the highest peak participates and votes truthfully; clearly, this gives the best outcome for that agent (given her participation). If any of the remaining agents decides to participate, the voter with the highest peak can adjust her voting behavior in the sequential model and receive her peak again by optimally responding to the vote of the other participant. Thus, the outcome does not change hence none of the remaining agents has an incentive to participate. In Case 3, the outcome is always 0.5 while the symmetric median (i.e. the midpoint between the two middle peaks) could be arbitrarily close to 0.25 (resp. 0.75).

Finally, let us compare the findings of Theorem 2 with the intuition put forward by Osborne et al. (2000) that more extreme voters are more likely to participate. Cases 1, 3 and 4 do not confirm this intuition: in Case 1 the single participant is not determined by her being ‘moderate’ or ‘extreme,’ but simply by the fact that her peak is at 0.5; in Cases 3 and 4 we have (almost) full participation (albeit with extreme reporting). On the other hand, Case 2 comes closer to the intuition since the single participant is either the voter with the left-most or the right-most peak; however, within the spectrum of *conceivable* positions, this is also the voter with the most ‘moderate’ view among all potential voters since, by assumption, all of them are on the same side of 0.5.

5 Conclusion

Our main result reveals a strong tension between two kinds of incentive properties if preferences are single-peaked and voting is costly: participation and truthful reporting. If a voting rule is anonymous, strategy-proof and responsive the only equilibrium in which all participants follow their unique dominant strategy consists of the single participation of either one of the agents with the highest, or one of the agents with the lowest peak. In particular, there is no anonymous and strategy-proof deterministic voting rule that yields the median peak if participants vote according to their unique dominant strategy. While in this result the need for a tie-breaking rule in the case of an even number of actual participants plays an important role, it is remarkable that the result holds for any number of potential voters.

A possible way out of the problem is to consider probabilistic rules, and indeed we have shown by means of example that a simple and natural strategy-proof probabilistic rule implements the median with full participation if voting costs are sufficiently small. A full fledged analysis of probabilistic rules in the context of costly voting in committees appears to be a worthwhile subject for future work.

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Appendix: Proof of Theorem 2

We split the proof of Theorem 2 in two parts. Proposition 1 deals with Cases 1 and 2, while Proposition 2 proves the claim for Cases 3 and 4.

The following lemma ensures that in all subgames on the second stage of the sequential voting game there exists a strong Nash equilibrium and additionally, that if several strong Nash equilibria exist, they necessarily result in the same outcome.

Lemma 1 *Consider the symmetric median rule with a fixed number of participants k. There always exists a strong Nash equilibrium. Moreover,*

- *if k is odd, the outcome is the median of the peaks in all strong Nash equilibria.*
- *if k is even, the outcome is the median of $\{p_{\frac{k}{2}}, 0.5, p_{\frac{k}{2}+1}\}$ in all strong Nash equilibria, given that the peaks $p_1, \ldots, p_k$ are ordered.*

Proof Let k be odd. Thus, the symmetric median rule coincides with the median rule which is well-known to be strategy-proof. Hence, there exists a strong Nash equilibrium, and in all strong equilibria the outcome is the median of the peaks of the participants.

Next, let k be even.

Case 1: $p_{\frac{k}{2}} < 0.5 < p_{\frac{k}{2}+1}$.

In this case, it is easily verified that there is a unique strong Nash equilibrium: the $k/2$ agents with peak lower than 0.5 vote for 0, while the $k/2$ agents with peak larger than 0.5 vote for 1, resulting in the outcome 0.5.

Case 2: $p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1} \leq 0.5$.

In this case strong Nash equilibrium is not unique (unless $p_{\frac{k}{2}+1} = 0.5$) but all strong equilibria in fact result in the same outcome. If $p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$, then in all strong equilibria the $k/2$ agents with the lowest peaks vote for 0, agent $\frac{k}{2}+1$ votes for $2 \cdot p_{\frac{k}{2}+1}$ (≤ 1), and all other agents submit a vote between $2 \cdot p_{\frac{k}{2}+1}$ and 1, resulting in the outcome $p_{\frac{k}{2}+1}$. If $p_{\frac{k}{2}} = p_{\frac{k}{2}+1}$, then there exist at least two individuals with peak

$p_{\frac{k}{2}+1}$. In this case there exist further strong equilibria in which all those individuals coordinate their votes such that the outcome corresponds to their common peak $p_{\frac{k}{2}+1}$.
Case 3: $0.5 \leq p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1}$.
Analogously to Case 2, in all strong equilibria the $k/2$ individuals with the highest peaks vote for 1, individual $k/2$ votes for $2 \cdot p_{\frac{k}{2}} - 1$ (≥ 0), and all other individuals submit a vote between 0 and $2 \cdot p_{\frac{k}{2}} - 1$ resulting in the outcome $p_{\frac{k}{2}}$. Again, there might exist further equilibria, but in all of those the outcome is identical to the case described here. □

As the symmetric median rule differentiates between odd and even number of participants, so do we in our equilibrium analysis, starting with an odd number of participants.

Lemma 2 *Consider the symmetric median rule. There are no subgame-perfect equilibria in which a strong Nash equilibrium is played in the second stage with an odd number of participants greater than* 1.

Proof By contradiction, let $k > 1$ be the number of participants and let k be odd. Then, in every (strong) Nash equilibrium, the median participant $i = \frac{k+1}{2}$ determines the outcome by truthfully revealing her peak $p_{\frac{k+1}{2}}$.
Case 1: $p_{\frac{k+1}{2}} = 0.5$.
Then, there are as many participants (at least $\frac{k+1}{2}$) with a peak at or below 0.5 than there are participants with a peak at or above 0.5. Hence, if agent $\frac{k+1}{2}$ abstains, the outcome will be the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+3}{2}}$ by Lemma 1, that is the outcome will be 0.5. Thus, since the outcome would not change, agent $\frac{k+1}{2}$ has an incentive to abstain.
Case 2: $p_{\frac{k+1}{2}} < 0.5$.
If an agent $i > \frac{k+1}{2}$ (i.e. an agent with peak at or above the outcome) abstains, the outcome becomes the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+1}{2}}$ by Lemma 1. But since $p_{\frac{k-1}{2}} \leq p_{\frac{k+1}{2}} < 0.5$, this means that the outcome does not change; hence, agent i will rather abstain.
Case 3: $p_{\frac{k+1}{2}} > 0.5$.
This case is symmetric to Case 2. □

With the help of Lemma 2, we can characterize all equilibria with an odd number of participants.

Proposition 1 *Consider the symmetric median rule. There exists a single participation equilibrium (with a strong Nash equilibrium played in the second stage) for all positive participation costs if and only if one of the following two conditions holds:*

1. *There exists at least one individual $j \in N$ with peak $p_j = 0.5$.*
2. *The peaks of all individuals are on the same side of* 0.5, *i.e. either $p_i \geq 0.5$ for all $i \in N$ or $p_i \leq 0.5$ for all $i \in N$.*

In Case 1 the single participant is (one of) the individual(s) with peak $p_j = 0.5$. In Case 2 the single participant is an individual whose peak is the closest to 0.5.

Proof Case 1: It is easy to see that the existence of an individual j with peak $p_j = 0.5$ leads to a single participation equilibrium with this individual being the single participant. If j is the single participant, the outcome is $p_j = 0.5$. Whenever one individual joins, individual j will adapt her vote such that the outcome remains 0.5. Hence, no individual (apart from j) has an incentive to participate. Moreover, there is no other single participant equilibrium with an individual with a peak different from 0.5 for all positive participation costs since individual j could move the outcome to 0.5 by joining which is profitable for her for sufficiently small participation costs. Note that if there are several individuals with peak 0.5, then obviously single-participation of any one of them is an equilibrium.

Case 2: Assume that all peaks are on the same side of 0.5, w.l.o.g. assume $p_i \geq 0.5$ for all $i \in N$. Then by a similar argument one can show that there is only one type of a single participation equilibrium in which individual 1 (or another individual with the same peak as individual 1) is the only participant. As before, if another individual decides to join her, she can adapt her vote such that the outcome doesn't change. Thus, no individual has an incentive to join. Moreover, individual 1 has an incentive to join given that there is a single participant with a different peak as she can move the outcome to her peak which again is profitable for sufficiently small costs of participation. The case of all peaks below 0.5 can be dealt with by replacing individual 1 by individual n (and individuals with the same peak as individual n, respectively), who in this case is the individual with peak closest to 0.5.

It remains to show that in all other cases there exists no single participation equilibrium for all positive participation costs. As neither Case 1 nor Case 2 apply, there is no individual with a peak of 0.5 and given ordered peaks we have $p_1 < 0.5 < p_n$, i.e. at least one individual with a peak strictly below 0.5 and at least one individual with a peak strictly above 0.5. Assume that there is a single participation equilibrium for all positive participation costs with an individual with peak below 0.5. Then individual n has an incentive to join, as she could move the outcome to 0.5 by participating, which is profitable for sufficiently small participation costs. Analogously, if there were a single participation equilibrium with an individual with peak above 0.5, then individual 1 has an incentive to join for sufficiently small costs of participation. Hence, there exists no single participation equilibrium for all positive participation costs if neither Case 1 nor Case 2 applies. □

One can easily see that if there exists a single-participation equilibrium for some participation cost, then this equilibrium remains an equilibrium if participation costs increase. Even with the lower costs of participation all non-participants preferred to abstain and the single-participant will never want to abstain. Thus, we obtain that there are no other single-participation equilibria for small but strictly positive costs of participation, than the ones identified in Proposition 1.

We now turn to the analysis of an even number of participants, starting with a lemma stating that in a strong Nash equilibrium with a fixed even number of participants the outcome is 0.5. Moreover the peaks of the participants need to differ from 0.5 and need to be evenly distributed to both sides of 0.5.

Lemma 3 *Consider an equilibrium of the sequential participation game under the symmetric median rule with an even number of participants. Given that in the second* *stage participants play a strong Nash equilibrium, the outcome is* 0.5. *Moreover, in all such equilibria half of the participants have a peak strictly above and half of the participants have a peak strictly below 0.5.*

Proof Assume, by way of contradiction, that there exists an equilibrium of the required sort with an even number of participants k and an outcome that is different from 0.5. We will show that this is not possible. By Lemma 1, the outcome is the median of $p_{\frac{k}{2}}$, 0.5 and $p_{\frac{k}{2}+1}$. As the outcome is assumed to be different from 0.5, we must have either $p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1} < 0.5$ or $0.5 < p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1}$.
Case 1: $p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1} < 0.5$.
In this case, Lemma 1 implies that the outcome is $p_{\frac{k}{2}+1}$. If an agent $i < \frac{k}{2}+1$ (i.e. with a peak at or below $p_{\frac{k}{2}+1}$) abstains, then there are as many participants (at least $\frac{k}{2}+1$) with a peak at or below $p_{\frac{k}{2}+1}$ as there are participants with a peak above $p_{\frac{k}{2}+1}$. Hence, $j = \frac{k}{2}+1$ is the median participant and the outcome is $p_{\frac{k}{2}+1}$. Since the outcome is unchanged, agent i has an incentive to abstain whenever her participation costs are positive.
Case 2: $0.5 < p_{\frac{k}{2}} \leq p_{\frac{k}{2}+1}$.
By a completely symmetric argument, one shows that in this case every agent $i > \frac{k}{2}$ (i.e. with a peak at or above $p_{\frac{n}{2}}$) has an incentive to abstain since this would again not change the outcome.

Hence, we have shown that the outcome in equilibrium is 0.5. This implies directly that the number of participants whose peak is below 0.5 needs to be same as the number of those participants with a peak above 0.5. Furthermore, there cannot be individuals with peak 0.5. Otherwise, some individual will find it profitable to abstain since the outcome will not change as 0.5 will be the median vote after the abstention. □

With the result of Lemma 3, we obtain an equilibrium characterization result for an even number of participants. We find full participation equilibria when the number of individuals is even, and equilibria in which all but those individuals with peak at 0.5 participate, when the number of individuals is odd. We say that a participant with peak $p_i \neq 0.5$ exhibits *extreme reporting* if she reports 0 if $p_i < 0.5$ and 1 if $p_i > 1$. In both types of equilibria we find that all participants exhibit extreme reporting.

Proposition 2 *Consider the symmetric median rule. The only subgame-perfect equilibria such that a strong Nash equilibrium is played in the second stage with an even number of participants for small but positive participation costs are of one of the following types:*

- *The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them a peak strictly above* 0.5. *Then, full participation with extreme reporting is an equilibrium and the outcome is* 0.5.
- *The number of potential voters n is odd and there exists a set $M \subset N$ of individuals with $\#M = m$ such that m is odd, $m < n$, $p_j = 0.5$ exactly for all $j \in M$, and the peaks of the other $n - m$ individuals are evenly split on both sides of* 0.5. *Then, it is an equilibrium that exactly the individuals in $N \setminus M$ participate with extreme reporting; the resulting outcome is* 0.5.

In all other cases there does not exist an equilibrium of the specified form (with an even number of participants).

Proof Let the peaks $p_1, \ldots, p_n$ be ordered. Independently of the number of individuals, we always have $\#\{i \in N : p_i < 0.5\} = \#\{i \in N : p_i > 0.5\}$, which by Lemma 2 and Lemma 3 is a necessary condition for the existence of an equilibrium with more than one participant.
Case 1: n is even.
By Lemma 3 we obtain that the outcome if all individuals participate is 0.5. If an individual i with peak below 0.5 abstains, the outcome shifts to $p_{(n/2)+1}$ which is worse for individual i for sufficiently small participation costs. Similarly, if an individual i' with peak above 0.5 abstains, the outcome changes to $p_{n/2}$ which again is worse for individual i' for sufficiently small participation costs. Hence full participation is an equilibrium given this distribution of peaks.
Case 2: n is odd.
By Lemma 3 we obtain that the outcome if exactly the individuals in $N \setminus M$ participate is 0.5. Evidently no participant has an incentive to abstain for small but positive participation costs. As the outcome corresponds already to their common peak, there is no incentive for individuals in M to participate, hence this constitutes an equilibrium.

It remains to show that the existence of these types of equilibria implies the even distribution of the peaks required in the proposition. Assume that there exists an equilibrium with full participation if n is even, and with an odd number of abstentions (not exceeding the number of individuals) if n is odd. Since the number of participants is even in both cases, we know by Lemma 3 that there need to be as many individuals with a peak strictly below 0.5 as there are with peak strictly above 0.5. If n is odd, there exists at least one individual who did not participate and even for small costs has no incentive to participate. This implies that the peak of all of those individuals have to correspond to the outcome, which is 0.5.

It finally remains to show that there are no other equilibria with an even number of participants, i.e. no other equilibria in which an individual in $N \setminus M$ abstains. As we have an even number of participants, we know by Lemma 3 that the outcome is 0.5. If an individual $j \in N \setminus M$ abstained, then she could shift the outcome closer to her peak by participating. For (sufficiently) small participation costs this is profitable for her. Hence this cannot constitute an equilibrium. □

Theorem 2 follows from combining Propositions 1 and 2. We conclude with two examples. The first demonstrates that there could be several equilibria of the sort required in Theorem 2 in the sequential model, the second shows that there could exist no equilibria in pure strategies.

Example 1 Let $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.7$. Then there exists an equilibrium of the sort required in Theorem 2 with one participant, and another one with an even number of participants and outcome 0.5. Indeed, if agent 2 is the only participant, the outcome is 0.5, and neither agent 1 nor agent 3 has an incentive to participate since the outcome would not change. If, on the other hand, agents 1 and 3 participate the outcome is again 0.5, hence agent 2 has no incentive to participate. If cost are sufficiently small, agents 1 and 3 indeed prefer to participate, since otherwise the outcome changes to 0.1 or 0.7, respectively.

Example 2 Let $p_1 = 0.1$, $p_2 = 0.8$ and $p_3 = 0.9$. Then there is no equilibrium of the sort required in Theorem 2 (provided that costs of participation are small). This can be seen as follows. With full participation the outcome would be the median of the votes cast, that is: 0.8. In that case agent 1 has an incentive to abstain, since without her the outcome remains unchanged: indeed, agent 2 would vote $\tilde{p}_2 = 0.6$ and agent 3 would vote of $\tilde{p}_3 = 1$ in the equilibrium of the subgame. But this situation cannot constitute an equilibrium either, since agent 3 would rather abstain. If agents 1 and 2, or agents 1 and 3 are the participants, the outcome is 0.5. But for small participation costs, the respective abstainer would prefer to join and change the result to 0.8. Finally, all single participation cases do not constitute an equilibrium since then there always exists a non-participant who could profitably change the outcome to 0.5 (provided that participation costs are sufficiently small).

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