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Prescribed-time control for a class of semilinear hyperbolic PDE-ODE systems

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Abstract: A prediction-based controller is shown to achieve prescribed-time stabilization of a nonlinear infinite-dimensional system, which consists of a general boundary controlled first-order semilinear hyperbolic PDE that is bidirectionally interconnected with nonlinear ODEs at its unactuated boundary. The approach uses a coordinate transformation to map the nonlinear system into a form suitable for control. In particular, this transformation is based on predictions of system trajectories, which can be obtained by solving a general nonlinear Volterra integro-differential equation. Then, a prediction-based controller is designed to stabilize the system in prescribed-time. Numerical simulations illustrate the performance of both the prescribed-time controller and an asymptotically stabilizing one, which follows as a special case.

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1. INTRODUCTION

Several technological processes can be modeled by hyperbolic partial differential equations (PDEs) that may also be coupled with ordinary differential equations (ODEs): for instance, drilling systems (Saldivar et al. (2016)), hydraulic networks (Bastin and Coron (2016)), road traffic networks (Lattanzio et al. (2011)), communication networks (Espitia et al. (2017)). The stabilization of these infinite dimensional systems has been studied intensively in the literature and continues to be challenging. For boundary control, backstepping designs (Krstic and Smyshlyaev (2008)) and Lyapunov techniques (Bastin and Coron (2016)) are the most commonly used. These methods have been widely applied to hyperbolic PDE-ODE systems (see, e.g., Krstic (2009); Bekiaris-Liberis and Krstic (2012); Bresch-Pietri and Krstic (2014); Mazenc et al. (2014); Bekiaris-Liberis and Krstic (2018); Karafyllis and Krstic (2019); Deutscher et al. (2019); Redaud et al. (2021))

Nevertheless, most of the existing results on stabilization are based on asymptotic or exponential guarantees; though in many cases, transient processes must not exceed a given finite time. The need to meet time constraints and increase temporal performance has motivated methods for achieving so-called *non-asymptotic stability*. For instance, finite-time stability means that the solutions of a stable system converge to the equilibrium in a finite time which depends on the initial conditions (ICs). If the settling time is uniformly bounded by a parameter independent of the ICs, the system is said to be fixed-time stable (Polyakov (2012)). Such concepts have been extensively studied within the framework of linear and nonlinear ODEs (see, e.g., Bhat and Bernstein (2000); Hong (2002); Lopez-Ramirez et al. (2018); Polyakov et al. (2015)).

Efforts have been made to extend these notions to the infinite-dimensional setting. Finite-time stabilization and null-controllability of linear parabolic PDEs are studied in Coron and Nguyen (2017) using time-varying backstepping approaches. Similar results are obtained for hyperbolic PDEs in, e.g., Coron et al. (2013); Auriol and Di Meglio (2016); Coron et al. (2017); Perrollaz and Rosier (2014); Deutscher and Gabriel (2020); Coron and Nguyen (2021); Strecker et al. (2022).

More recently, the prescribed-time stabilization concept has arisen, which allows the terminal time to be prescribed independently of ICs and parameters. It was originally introduced in Song et al. (2017) and has been the basis of several contributions not only for finite-dimensional systems (see, e.g., Holloway and Krstic (2019); Krishnamurthy et al. (2020); Krishnamurthy and Khorrami (2020); Tran and Yucelen (2020); Zhou (2020); Chitour et al. (2020); Irscheid et al. (2021a)) but also in the infinite dimensional case (see, e.g., Espitia et al. (2019); Steeves and Krstic (2021); Espitia and Perruquetti (2021)). However, finite-/fixed-/prescribed-time concepts for hyperbolic PDE-ODE systems remain sparse and constitute a challenging topic; especially for nonlinear dynamics.

The present work proposes a prescribed-time prediction-based controller to stabilize a general boundary controlled first-order semilinear hyperbolic PDE that is intercon-

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nected with nonlinear ODEs at its unactuated boundary. The interconnection is bidirectional in the sense that the PDE boundary drives the ODE subsystem which, in turn, excites the PDE through an in-domain coupling. Building upon Irscheid et al. (2021b), a coordinate transformation is used to map the system into a form suitable for control. Then, the controller is designed to achieve prescribed-time stability. For that, predictions of the system trajectories are required, which are obtained from the solution of a general nonlinear Volterra integro-differential equation. To the best of the authors' knowledge, this is the first work for this system class not only on prescribed-time but also on asymptotic stabilization, which follows as a special case.

The paper is organized as follows. Section 2 presents the considered nonlinear bidirectionally coupled PDE-ODE system and states the control objective. The system equations are solved by the method of characteristics in Section 3 to simplify the subsequent control design in Section 4. Therein, a general prediction-based approach is used to derive the prescribed-time controller. It uses a state transformation to map the PDE-ODE system into a suitable target system, on which the stability analysis is performed. Numerical simulations illustrate the performance of the prescribed-time controller in Section 5, where an asymptotically stabilizing controller is used for reference. Finally, concluding remarks are given in Section 6.

Notation: Define $\|u\|_\infty = \sup_{z\in[0,1]} |u(z)|$ for $u : [0,1] \ni z \mapsto \mathbb{R}$. Let u_z and u_t stand for the partial derivatives of $u : (z,t) \mapsto u(z,t)$ w.r.t. z and t, respectively. The set of nonnegative real numbers is denoted by $\mathbb{R}_0^+$. A continuous function $\alpha : [0,a) \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is a class $\mathcal{K}_\infty$ function if, additionally, $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_0^+$, the mapping $\mathbb{R}_0^+ \ni r \mapsto \beta(r,t)$ belongs to class $\mathcal{K}_\infty$ w.r.t. r and, for each fixed $r \in \mathbb{R}_0^+$, the mapping $\mathbb{R}_0^+ \ni t \mapsto \beta(r,t)$ is decreasing w.r.t. t and $\beta(r,t) \to 0$ as $t \to \infty$.

2. PROBLEM STATEMENT

Consider the nonlinear infinite-dimensional system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), u(0,t)) \tag{1a}$$

$$u_t(z,t) = \lambda u_z(z,t) + g(z, \boldsymbol{x}(t), u[t](z)) \tag{1b}$$

$$u(1,t) = U(t) \tag{1c}$$

with the ODE state $\boldsymbol{x}(t) \in \mathbb{R}^n$, the PDE state $u(z,t) \in \mathbb{R}$ defined for $t \geq 0$ and $z \in [0,1]$ and the input $U(t) \in \mathbb{R}$. In addition, $\lambda > 0$ is the transport velocity, $u[t](z)$ denotes the profile $u(\zeta,t)$ of the distributed state for $\zeta \in [0,z]$ at time $t \geq 0$ and the general source term g is of the form [1]

$$g(z, \boldsymbol{x}(t), u[t](z)) = g_0(z, \boldsymbol{x}(t), u(0,t)) + g_1(z, \boldsymbol{x}(t), u(z,t)). \tag{2}$$

To ensure that the origin $\boldsymbol{x}(t) = \boldsymbol{0}$ and $u(z,t) = 0$ is a steady-state solution of (1) for $U(t) = 0$, let $\boldsymbol{f}(\boldsymbol{0},0) = \boldsymbol{0}$ and $g(z,\boldsymbol{0},0) = 0$, $\forall z \in [0,1]$. The ICs are $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ and $u(z,0) = u_0(z) \in \mathbb{R}$ piecewise continuous.

[1] The notation $u[t](z)$ is only used here to simplify the presentation. It also allows for direct generalizations in future works.

Note that (1b) is a first-order semilinear hyperbolic PDE, which is actuated at the boundary $z = 1$. It is also excited by $\boldsymbol{x}(t)$ and $u(0,t)$ through the nonlinear source term g. At the unactuated boundary, the ODE (1a) is excited by the boundary value $u(0,t)$ of the PDE state $u(z,t)$. Hence, the PDE and ODE subsystems are bidirectionally coupled.

The goal is to design a controller stabilizing the origin of the nonlinear infinite-dimensional plant (1) in prescribed time $\bar{T}$ such that $\lim_{t\to\bar{T}^-} \|\boldsymbol{x}(t)\| = 0$ and $\lim_{t\to\bar{T}^-} \|u(\cdot,t)\|_\infty = 0$. Due to the presence of a transport phenomenon, the action of the input U affects the ODE (1a) only after a time delay $\Delta = \lambda^{-1}$. Therefore, the prescribed time $\bar{T}$ has to exceed Δ, which is satisfied by setting $\bar{T} = T + \Delta$ with $T > 0$. The following assumptions are imposed to guarantee the existence of such controller.

Assumption 1. Both the vector field $\boldsymbol{f}$ and the function g are globally Lipschitz continuous w.r.t. $\boldsymbol{x}(t)$, $u(0,t)$ as well as $u(z,t)$ and g is also continuous w.r.t. z.

The Lipschitz continuity ensures global existence of solutions $t \mapsto \boldsymbol{x}(t)$ and $(z,t) \mapsto u(z,t)$ of (1) for all bounded ICs and every measurable locally essentially bounded input U. This excludes finite escape time especially for that part of the solution which is determined by the ICs only.

Remark 2. Note that the global Lipschitz continuity of $\boldsymbol{f}$ and g is only sufficient for global existence of solutions. Hence, Assumption 1 can be replaced with weaker conditions (such as, e.g., forward completeness) without substantially changing the results of this paper.

The proposed prescribed-time controller for (1), as detailed in Section 4, uses results from finite-dimensional theory (see, e.g., Krishnamurthy et al. (2020) for nonlinear systems in (generalized) feedback forms or Irscheid et al. (2021a) for nonlinear flat systems) by considering $u(0,t) = \kappa(t, \boldsymbol{x}(t))$ in (1a). Existence of such a feedback is formulated in the next assumption.

Assumption 3. There exists a function $\kappa(t, \boldsymbol{x}(t))$ such that

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \kappa(t, \boldsymbol{x}(t))), \qquad t > 0 \tag{3}$$

is prescribed-time stable for all ICs $\boldsymbol{x}(0) \in \mathbb{R}^n$ with convergence time $T > 0$. Additionally, κ is bounded in the sense that for $t \in [0,T]$ and $\boldsymbol{x}(t)$ satisfying (3) there exist a constant $M > 0$ with $|\kappa(t, \boldsymbol{x}(t))| < M$ and $\lim_{t\to T^-} \kappa(t, \boldsymbol{x}(t)) = 0$. Also, for convenience, define

$$\kappa(t,\cdot) = 0, \qquad t \leq 0 \tag{4a}$$

$$\kappa(t,\cdot) = 0, \qquad t \geq T. \tag{4b}$$

The notion of prescribed-time stability is clarified next.

Definition 4. The equilibrium $\boldsymbol{x}(t) = \boldsymbol{0}$ of (3) is said to be (uniformly) stable in prescribed-time $T > 0$ if there exist a class $\mathcal{KL}$ function β, a positive constant c, and a function $\mu : [0,T) \to \mathbb{R}_0^+$ with $\mu \to \infty$ for $t \to T^-$ such that $\|\boldsymbol{x}(t)\| \leq \beta(\|\boldsymbol{x}(0)\|, \mu(t))$, for all $t \in [0,T)$ and $\|\boldsymbol{x}(0)\| < c$. The corresponding notions are global if $c \to \infty$.

Remark 5. The design of κ, determination of functions β and μ as well as known robustness issues are inherited from the finite-dimensional problem. With this in mind, the present results similarly apply to an asymptotically stabilizing feedback or tracking controller κ by (essentially) replacing T with ∞. Moreover, with further modifications, the design of finite/fixed-time stabilizing controllers is possible. However, this is out of the scope of this paper.

3. WELL-POSEDNESS OF THE CAUCHY PROBLEM

This section is devoted to obtaining the solution $\tau \mapsto \boldsymbol{x}(\tau)$ and $(z, \tau) \mapsto u(z, \tau)$ of the plant (1) on the domain $\Omega_t = \{(z, \tau) : z \in [0, 1], \tau \in [t, t + \Delta - \frac{z}{\lambda}]\}$ on the basis of the (initial) data $\boldsymbol{x}(t)$ and $u(z, t)$. For that, the method of characteristics is used to construct the solution from integral curves of the vector fields $\boldsymbol{f}$ and g.

3.1 The method of characteristics

The characteristic projections of the PDE (1b) can be parameterized by the curves $(z_{\mathrm{p}}, t_{\mathrm{p}})$ with

$$z_{\mathrm{p}}(\tau; z) = z - \lambda\tau, \qquad t_{\mathrm{p}}(\tau; t) = t + \tau \tag{5}$$

for the fixed pair $(z, t) \in [0, 1] \times \mathbb{R}_0^+$ describing a point on a characteristic line with the curve parameter τ (see the inclined lines in Fig. 1). For constructing the solution on Ω_t, consider $\tau \in [0, \frac{z}{\lambda}]$, i.e. each line starting from the point (z, t) and ending at $(0, t + \frac{z}{\lambda})$. Then, introduce

$$\boldsymbol{x}_{\mathrm{p}}(\tau; t) = \boldsymbol{x}(t_{\mathrm{p}}(\tau; t)) \tag{6a}$$
$$u_{\mathrm{p}}(\tau; (z, t)) = u(z_{\mathrm{p}}(\tau; z), t_{\mathrm{p}}(\tau; t)), \tag{6b}$$

which are the restrictions of the ODE and PDE states $\boldsymbol{x}$ and u on the characteristic line starting from the point (z, t). Hence, for each curve, z and t are fixed parameters.

Fig. 1 illustrates the characteristic projections on the rectangular domain $\mathcal{R} = \{(z, t) : z \in [0, 1], t \in [0, \bar{T}]\}$. Note that $\mathcal{R}$ can be split into the lower left triangular domain $\Omega_0 = \{(z, t) : z \in [0, 1], t \in [0, \Delta - \frac{z}{\lambda}]\}$, on which the solution is solely determined by the ICs $\boldsymbol{x}_0$ and $u_0(z)$, and the complementary domain $\Omega_0^c = \mathcal{R} \backslash \Omega_0$, on which the state of the PDE-ODE system (1) is excited by the boundary condition (1c). The latter is also split into the upper right triangular domain $\mathcal{D}_u$, which only contains characteristic lines that do not reach $z = 0$, and the remaining domain $\mathcal{D}_{\boldsymbol{x}}$, where all characteristic lines end at $z = 0$. The red, blue and cyan domains in Fig. 1 correspond to Ω_0, $\mathcal{D}_{\boldsymbol{x}}$ and $\mathcal{D}_u$, respectively.

It now remains to determine $\boldsymbol{x}_{\mathrm{p}}(\tau; t)$ and $u_{\mathrm{p}}(\tau; (z, t))$ on the basis of the initial data $\boldsymbol{x}_{\mathrm{p}}(0; t) = \boldsymbol{x}(t)$ and $u_{\mathrm{p}}(0; (z, t)) = u(z, t)$ for $z \in [0, 1]$ (cf. (5)–(6)). By using the method of characteristics, one obtains the nonlinear initial value problem (IVP)

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{x}_{\mathrm{p}}(\tau; t) = \boldsymbol{f}(\boldsymbol{x}_{\mathrm{p}}(\tau; t), u_{\mathrm{p}}(\tau; (\lambda\tau, t))) \tag{7a}$$
$$\boldsymbol{x}_{\mathrm{p}}(0; t) = \boldsymbol{x}(t) \tag{7b}$$

coupled with

$$\tfrac{\mathrm{d}}{\mathrm{d}s}u_{\mathrm{p}}(s; (\lambda\tau, t)) = g_0(z_{\mathrm{p}}(s; \lambda\tau), \boldsymbol{x}_{\mathrm{p}}(s; t), u_{\mathrm{p}}(s; (\lambda s, t))) + g_1(z_{\mathrm{p}}(s; \lambda\tau), \boldsymbol{x}_{\mathrm{p}}(s; t), u_{\mathrm{p}}(s; (\lambda\tau, t))) \tag{7c}$$
$$u_{\mathrm{p}}(0; (\lambda\tau, t)) = u(\lambda\tau, t) \tag{7d}$$

for $\tau \in [0, \Delta]$, $s \in [0, \tau]$ and a fixed $t \geq 0$.

Lemma 6. There exists a unique solution of the nonlinear IVP (7) for $\tau \in [0, \Delta]$ and $s \in [0, \tau]$.

Proof. By Assumption 1, (7c)–(7d) has a unique solution

$$u_{\mathrm{p}}(s; (\lambda\tau, t)) = u(\lambda\tau, t) + \int_0^s g_0(\lambda(\tau - \sigma), \boldsymbol{x}_{\mathrm{p}}(\sigma; t), u_{\mathrm{p}}(\sigma; (\lambda\sigma, t)))\,\mathrm{d}\sigma + \int_0^s g_1(\lambda(\tau - \sigma), \boldsymbol{x}_{\mathrm{p}}(\sigma; t), u_{\mathrm{p}}(\sigma; (\lambda\tau, t)))\,\mathrm{d}\sigma \tag{8}$$

for $s \in [0, \tau]$, which is obtained through formal integration. Therefore, the IVP (7a)–(7b), after inserting $u_{\mathrm{p}}(\tau; (\lambda\tau, t))$ from (8) at $s = \tau$, is a general nonlinear Volterra integro-differential equation for $\boldsymbol{x}_{\mathrm{p}}(\tau; t)$. The existence of a unique solution $[0, \Delta] \ni \tau \mapsto \boldsymbol{x}_{\mathrm{p}}(\tau; t)$ is inferred from Assumption 1 (see Feldstein and Sopka (1974) for details). □

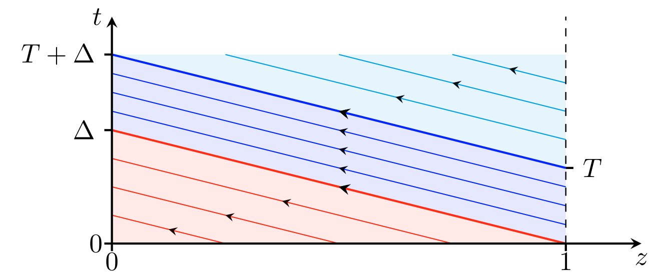

Fig. 1. The characteristic projections of the PDE (1b) on the rectangular domain $\mathcal{R} = \Omega_0 \cup \mathcal{D}_{\boldsymbol{x}} \cup \mathcal{D}_u$.

Note that (7) has to be solved successively in the sense that obtaining the solution of (7a)–(7b) for each τ requires solving (7c)–(7d) for $s \in [0, \tau]$ first. In fact, (7c)–(7d) can be replaced by the nonlinear integral equation (8).

3.2 Constructing the solution

The following observations form the basis of the prediction-based control design in Section 4. In particular, the controller uses those future values of the ODE and PDE states that are uniquely determined by the current system state only. This is clarified in the next theorem.

Theorem 7. The solution $\tau \mapsto \boldsymbol{x}(\tau)$ and $(\bar{z}, \tau) \mapsto u(\bar{z}, \tau)$ of the PDE-ODE system (1) exists on the domain Ω_t and is uniquely determined by $\boldsymbol{x}(t)$ and $u(z, t)$ at each time t.

Proof. Lemma 6 and the definition (6) of the restrictions $\boldsymbol{x}_{\mathrm{p}}$ and u_{p} of $\boldsymbol{x}$ and u on the curves (5) imply that $\boldsymbol{x}(\tau)$ and $u(\bar{z}, \tau)$ on Ω_t can be obtained by solving the nonlinear IVP (7) for $\tau \in [0, \Delta]$ and fixed t. □

A key component of the control design is the coordinate transformation in Section 4.1. The next lemma guarantees that this transformation is well defined.

Lemma 8. The function

$$G(s; (z, t)) = \int_0^s g_0(z - \lambda\sigma, \boldsymbol{x}(t + \sigma), u(0, t + \sigma))\,\mathrm{d}\sigma + \int_0^s g_1(z - \lambda\sigma, \boldsymbol{x}(t + \sigma), u(z - \lambda\sigma, t + \sigma))\,\mathrm{d}\sigma \tag{9}$$

is well-defined and bounded for all $s \in [0, \frac{z}{\lambda}]$ and $(z, t) \in \mathcal{R}$.

Proof. The substitution $\lambda\tau \mapsto z$ in (8) yields

$$u(z - \lambda s, t + s) = u(z, t) + G(s; (z, t)) \tag{10}$$

in light of (5)–(6) and (9). The existence of the (unique) solution $\Omega_t \ni (\bar{z}, \tau) \mapsto u(\bar{z}, \tau)$ (cf. Theorem 7) implies that $G(s; (z, t))$ is bounded for $s \in [0, \frac{z}{\lambda}]$. □

4. CONTROL DESIGN

The proposed prescribed-time controller makes use of future values of the ODE and PDE states $\boldsymbol{x}$ and u, respectively, on $\Omega_t \cap \mathcal{R}$. These values, which can be obtained by solving an IVP on the basis of the state at time t (as detailed in Section 3), are then used in a transformation of the PDE state u to map the PDE-ODE system (1) into a form that is suitable for stability analysis. This approach is motivated by the prediction-based design presented in Irscheid et al. (2021b) for 2×2 hyperbolic PDE-ODE systems. Moreover, there is an obvious connection to the framework of nonlinear prediction-based backstepping transformations pursued in (Krstic, 2009, Chapter 11) for stabilizing nonlinear ODEs with input delays.

4.1 Coordinate transformation

The transformation

$$\omega(z,t) = u(z,t) - k(z,t) \tag{11}$$

of the PDE state u with

$$k(z,t) = \kappa(t + \tfrac{z-1}{\lambda}, \boldsymbol{x}(t + \tfrac{z}{\lambda})) - \int_t^{\min(t+\frac{z}{\lambda},\bar{T})} g(z + \lambda(t-s), \boldsymbol{x}(s), u[s](z+\lambda(t-s)))\,\mathrm{d}s \tag{12}$$

leads to the equivalent description

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \omega(0,t) + \kappa(t-\Delta, \boldsymbol{x}(t))) \tag{13a}$$

$$\omega_t(z,t) = \lambda\omega_z(z,t) \tag{13b}$$

$$\omega(1,t) = U(t) - k(1,t) \tag{13c}$$

of the plant (1) for $(z,t) \in \mathcal{R}$. This is a result of k satisfying

$$k_t(z,t) = \lambda k_z(z,t) + g(z, \boldsymbol{x}(t), u[t](z)) \tag{14}$$

in spite of the minimum condition $\min(t + \frac{z}{\lambda}, \bar{T})$ in the upper integration limit in (12). Note that the integral in (12) is well-defined (cf. Lemma 8 with an appropriate substitution of the integration variable), but it remains to prove that $\kappa(t + \frac{z-1}{\lambda}, \boldsymbol{x}(t+\frac{z}{\lambda}))$ is finite in order for (11) to be a (bounded) transformation. This is shown as part of the stability analysis in Section 4.2.

Since κ vanishes when its first argument is non-positive (cf. (4a)), one obtains the IC

$$\omega(z,0) = u_0(z) + \int_0^{\frac{z}{\lambda}} g(z - \lambda s, \boldsymbol{x}(s), u[s](z-\lambda s))\,\mathrm{d}s \tag{15}$$

for $z \in [0,1]$ from (11) and the IC $u(z,0) = u_0(z)$. In particular, (15) implies that the IC of the transformed PDE state ω depends on the ODE state $\boldsymbol{x}(t)$ and the original PDE state $u(z,t)$ on the triangular domain Ω_0 (see Fig. 1). In fact, this corresponds to the solution of the plant (1) on Ω_0 being determined by the initial conditions $\boldsymbol{x}_0$ and u_0 only (as discussed in Section 3). Thereby, Theorem 7 implies that the IC (15) is well-defined, however, it is not required explicitly for the following considerations.

Fig. 1 shows that the prescribed-time stabilization of the ODE subsystem at $z = 0$ has to be realized by an appropriate control action at $z = 1$ for $t \in [0, T]$. Therefore, one expects κ to affect the transformation (11) inside the domain $\mathcal{D}_{\boldsymbol{x}}$ only. This is clarified in next lemma.

Lemma 9. The function k in (12) and, thus, the transformation (11) are independent of κ on the domain $\mathcal{D}_u$ with

$$k(z,t) = -G(\min(z, \lambda(\bar{T} - t)); (z,t)) \tag{16}$$

for all $(z,t) \in \mathcal{D}_u$, where G is defined in (9).

Proof. The statement of Lemma 9 follows immediately from (4b), (9) and (12) after a substitution of the integration variables. □

4.2 Target System and Stability Analysis

The subsequent analysis proves the prescribed-time stability of the origin of the proposed target system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \kappa(t-\Delta, \boldsymbol{x}(t)) + \omega(0,t)) \tag{17a}$$

$$\omega_t(z,t) = \lambda\omega_z(z,t) \tag{17b}$$

$$\omega(1,t) = 0 \tag{17c}$$

with ICs $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ and $\omega(z,0) = \omega_0(z) \in \mathbb{R}$ for $z \in [0,1]$ (cf. (15)). Note that (17) follows from (13) with

$$U(t) = k(1,t). \tag{18}$$

First, observe that

$$\omega(z,t) = \begin{cases} \omega_0(z + \lambda t), & (z,t) \in \Omega_0 \\ 0, & (z,t) \in \Omega_0^c \end{cases} \tag{19}$$

is the unique solution of the PDE (17b) with the boundary condition (17c) and the IC $\omega_0(z)$. This means that the PDE state $\omega(z,t)$ vanishes everywhere in Ω_0^c and, thus, is stabilized in the PDE's inherent finite time Δ (cf. Fig. 1). In light of (17a) with (4a) and (19), it remains to analyze

$$\dot{\boldsymbol{x}}(t) = \begin{cases} \boldsymbol{f}(\boldsymbol{x}(t), \omega_0(\lambda t)), & t \in (0, \Delta] \\ \boldsymbol{f}(\boldsymbol{x}(t), \kappa(t-\Delta, \boldsymbol{x}(t))), & t > \Delta. \end{cases} \tag{20}$$

Lemma 10. The origin $\boldsymbol{x}(t) = \boldsymbol{0}$ of (20) is prescribed-time stable with convergence time $\bar{T} = T + \Delta$. Additionally, $\kappa(t - \Delta, \boldsymbol{x}(t))$ is bounded for $t \geq 0$.

Proof. The solution $t \mapsto \boldsymbol{x}(t)$ of (20) exists on the interval $\mathcal{I}_0 = (0, \Delta]$ (by Assumption 1) and is determined by the ICs $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\omega_0(\lambda t)$ for $t \in \mathcal{I}_0$. In particular, let

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}_0(t; \boldsymbol{x}_0, \omega_0), \qquad t \in \mathcal{I}_0 \tag{21}$$

denote the unique solution of (20) on $\mathcal{I}_0$. In order to simplify the following reasoning concerning the solution $t \mapsto \boldsymbol{x}(t)$ for $t > \Delta$, define the auxiliary variable

$$\boldsymbol{\chi}(t) = \boldsymbol{x}(t + \Delta), \qquad t \geq 0. \tag{22}$$

Then, (20), (21) and (22) yield the IVP

$$\dot{\boldsymbol{\chi}}(t) = \boldsymbol{f}(\boldsymbol{\chi}(t), \kappa(t, \boldsymbol{\chi}(t))), \qquad t > 0 \tag{23a}$$

$$\boldsymbol{\chi}(0) = \boldsymbol{\Phi}_0(\Delta; \boldsymbol{x}_0, u_0). \tag{23b}$$

By Assumption 3, the origin of (23) is prescribed-time stable with convergence time T and $\kappa(t, \boldsymbol{\chi}(t))$ is bounded for all $t \geq 0$. This concludes the proof in light of (22). □

As previously mentioned, the prescribed-time stable target system (17) is obtained from (13) with the controller (18). However, this control law uses predictions of the ODE and PDE states $\boldsymbol{x}$ and u (cf. (12)).

4.3 Prediction-based control design

The future values $\boldsymbol{x}(\tau)$ and $u(z,\tau)$ on $\Omega_t \cap \mathcal{R}$ can be obtained by solving the nonlinear IVP (7). Consequently, the controller (18) can be implemented by substituting the future values by their corresponding predictions according to (6). The following theorem presents the main result.

Theorem 11. Let Assumptions 1 and 3 hold. The controller consisting of

$$U(t) = \kappa(t, \boldsymbol{x}_\mathrm{p}(\Delta; t)) - \int_0^{\min(\Delta, \bar{T}-t)} g_0(\lambda(\Delta - s), \boldsymbol{x}_\mathrm{p}(s;t), u_\mathrm{p}(s;(\lambda s, t)))\,\mathrm{d}s - \int_0^{\min(\Delta,\bar{T}-t)} g_1(\lambda(\Delta - s), \boldsymbol{x}_\mathrm{p}(s;t), u_\mathrm{p}(s;(1,t)))\,\mathrm{d}s \tag{24}$$

and the IVP (7) stabilizes the nonlinear PDE-ODE system (1) in prescribed-time $\bar{T} > \Delta$ such that $\lim_{t\to\bar{T}^-}\|\boldsymbol{x}(t)\| = 0$ and $\lim_{t\to\bar{T}^-}\|u(\cdot,t)\|_\infty = 0$ for all ICs $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{R}^n$ and $u(z,0) = u_0(z) \in \mathbb{R}$ piecewise continuous.

Proof. The IVP (7) has a unique solution (cf. Lemma 6). By (5)–(6), the controller (24) is in fact equivalent to (18), which is obvious after a substitution of integration variables. Therefore, the transformed system (13) with the controller (24) is equivalent to the target system (17). Since the latter is prescribed-time stable with convergence time $\bar{T}$ (see (19) and Lemma 10), the same holds true for the transformed system (13) in closed loop. Hence, $\lim_{t\to\bar{T}^-}\|\boldsymbol{x}(t)\| = 0$ and $\omega(z,t) = 0$ for $(z,t) \in \Omega_0^c$. Whereas the prescribed-time stability of $\boldsymbol{x}(t) = \boldsymbol{0}$ is thereby shown, it remains to analyze the stability of the original PDE subsystem. In order to infer the stability properties of the plant (1) from those of the transformed system (13) with the feedback (24), the (bounded) invertibility of the coordinate transformation (11) has to be shown. However, this is equivalent to showing that $k(z,t)$ is bounded (cf. (11)). In fact, this is a direct implication of Lemma 8 and of the second part of Lemma 10. The inverse relation $u(z,t) = \omega(z,t) + k(z,t)$ implies $u(z,t) = k(z,t)$ pointwise in space for $(z,t) \in \Omega_0^c$ since $\omega(z,t)$ vanishes on that domain. Furthermore, in light of Lemma 9,

$$u(z,t) = -G(\min(z,\lambda(\bar{T}-t));(z,t)), \quad (z,t) \in \mathcal{D}_u, \quad (25)$$

which is bounded and well-defined (cf. Lemma 8). By taking the limit of the latter for $t \to \bar{T}^-$ one obtains $\lim_{t\to\bar{T}^-} u(z,t) = 0$ pointwise in space, which implies convergence in the supremum norm. □

5. NUMERICAL EXAMPLE

Consider the plant (1) with

$$\boldsymbol{f}(\boldsymbol{x}(t),u(0,t)) = \begin{pmatrix} x_2(t) - x_2^2(t)u(0,t) \\ u(0,t) \end{pmatrix}, \quad (26)$$

the transport velocity $\lambda = 1$ and the source term

$$g(z,\boldsymbol{x}(t),u[t](z)) = \sin(x_1(t) - x_2(t)) + \tfrac{z}{2}u(0,t)\tanh(x_1(t)) + (1-z^2)(\cos^3(u(z,t)) - 1). \quad (27)$$

The prescribed time is $T = 5$ and, thus, $\bar{T} = 6$. Two scenarios are presented: a prescribed-time stabilization using $\kappa = \kappa_\mathrm{F}$ and an asymptotic one using $\kappa = \kappa_\mathrm{A}$ with

$$\kappa_\mathrm{F}(t,\boldsymbol{x}(t)) = \begin{pmatrix} \frac{-c_1c_2}{(T-t)^2} & \frac{1-c_1-c_2}{T-t} \end{pmatrix} \begin{pmatrix} x_1(t) + \frac{1}{3}x_2^3(t) \\ x_2(t) \end{pmatrix} \quad (28a)$$

$$\kappa_\mathrm{A}(t,\boldsymbol{x}(t)) = \begin{pmatrix} -c_1 & -c_2 \end{pmatrix} \begin{pmatrix} x_1(t) + \frac{1}{3}x_2^3(t) \\ x_2(t) \end{pmatrix} \quad (28b)$$

as well as $c_1 = 7$ and $c_2 = 8$. Note that Assumptions 1 and 3 hold (cf. (Krstic, 2009, Chapter 12.1) and Irscheid et al. (2021a)). Introduce the parameter $C \in \{1, 10\}$ to account for two different families of ICs according to

$$u(z,0) = \tfrac{2}{5}C\sin^3(4\pi z), \qquad \boldsymbol{x}(0) = \tfrac{C}{10}(3,-1)^\mathrm{T}. \quad (29)$$

The simulation uses an explicit Euler discretization with an equidistant temporal step size of 10^{-2}. Fig. 2 illustrates the total error $\|\boldsymbol{x}(t)\| + \|u(\cdot,t)\|_\infty$ (in logarithmic scale) of the closed-loop system (1) with (24) for both cases $\kappa = \kappa_\mathrm{F}$ (prescribed-time) and $\kappa = \kappa_\mathrm{A}$ (asymptotic), where each case is simulated with ICs (29) for both $C = 1$ and $C = 10$. The ODE state $\boldsymbol{x}(t)$ and the PDE state $u(z,t)$ in closed-loop are depicted in Fig. 3 for the case $\kappa = \kappa_\mathrm{F}$ and $C = 10$.

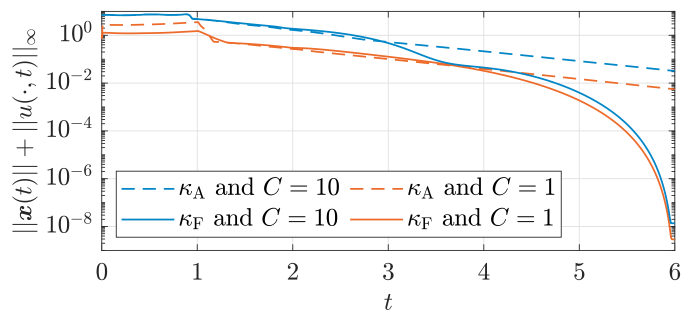

Fig. 2. Evolution of $\|\boldsymbol{x}(t)\| + \|u(\cdot,t)\|_\infty$ (logarithmic scale) in closed loop with the controller (24) for all possible combinations of $\kappa = \kappa_\mathrm{F}, \kappa_\mathrm{A}$ and $C = 1, 10$.

6. CONCLUDING REMARKS

It seems promising to use the presented ideas for an observer design to estimate the state in prescribed-time. Future research is concerned with extensions to finite/fixed-time approaches as well as to a more general class of infinite-dimensional systems, where the source term g in (1b) can depend on integrals of the PDE state.

REFERENCES

Auriol, J. and Di Meglio, F. (2016). Minimum time control of heterodirectional linear coupled hyperbolic PDEs. *Automatica*, 71, 300–307.

Bastin, G. and Coron, J.M. (2016). *Stability and Boundary Stabilization of 1-D Hyperbolic Systems*. Birkhäuser.

Bekiaris-Liberis, N. and Krstic, M. (2012). Compensation of state-dependent input delay for nonlinear systems. *IEEE Trans. Autom. Control*, 58, 275–289.

Bekiaris-Liberis, N. and Krstic, M. (2018). Compensation of actuator dynamics governed by quasilinear hyperbolic PDEs. *Automatica*, 92, 29–40.

Bhat, S. and Bernstein, D. (2000). Finite time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38, 751–766.

Bresch-Pietri, D. and Krstic, M. (2014). Delay-adaptive control for nonlinear systems. *IEEE Trans. Autom. Control*, 59, 1203–1218.

Chitour, Y., Ushirobira, R., and Bouhemou, H. (2020). Stabilization for a perturbed chain of integrators in prescribed time. *SIAM J. Control Optim.*, 52, 1022–1048.

Coron, J.M., Hu, L., and Olive, G. (2017). Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Automatica*, 84, 95–100.

Coron, J.M. and Nguyen, H.M. (2017). Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225, 993–1023.

Coron, J.M. and Nguyen, H.M. (2021). Null-controllability of linear hyperbolic systems in one dimensional space. *Syst. Control Lett.*, 148, 104851.

Coron, J.M., Vazquez, R., Krstic, M., and Bastin, G. (2013). Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM J. Control Optim.*, 51, 2005–2035.

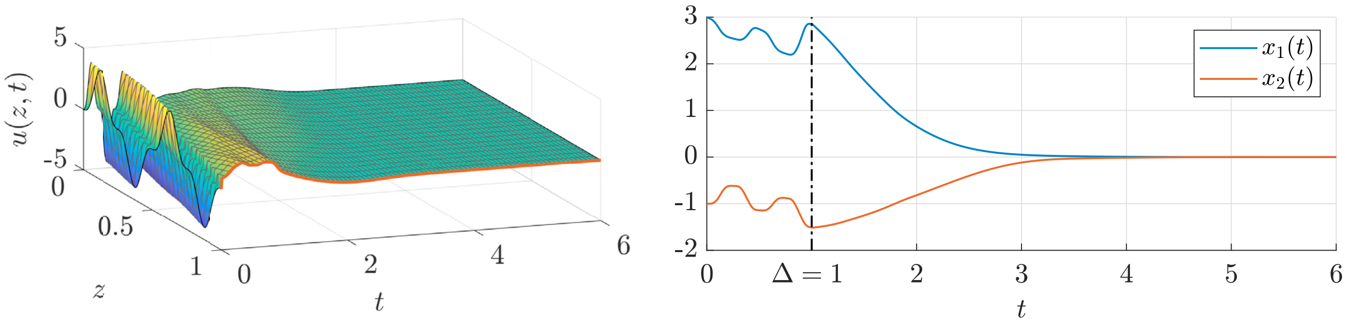

Fig. 3. Simulation results for $\kappa = \kappa_{\mathrm{F}}$ and $C = 10$. On the left: PDE state $u(z,t)$ on the domain $\mathcal{R}$, where the input $U(t) = u(1,t)$ is highlighted in red. On the right: the components of the ODE state $\boldsymbol{x}(t)$ on the domain $[0, \bar{T}]$.

Deutscher, J. and Gabriel, J. (2020). Minimum time output regulation for general linear heterodirectional hyperbolic systems. *Int. J. Contr.*, 93, 1826–1838.

Deutscher, J., Gehring, N., and Kern, R. (2019). Output feedback control of general linear heterodirectional hyperbolic PDE-ODE systems with spatially-varying coefficients. *Int. J. Control*, 92, 2274–2290.

Espitia, N., Girard, A., Marchand, N., and Prieur, C. (2017). Dynamic boundary control synthesis of coupled PDE-ODEs for communication networks under fluid flow modeling. In *Proc. IEEE Conf. Decis. Control*, 1260–1265.

Espitia, N. and Perruquetti, W. (2021). Predictor-feedback prescribed-time stabilization of LTI systems with input delay. *IEEE Trans. Autom. Control*.

Espitia, N., Polyakov, A., Efimov, D., and Perruquetti, W. (2019). Boundary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction-diffusion systems. *Automatica*, 103, 398–407.

Feldstein, A. and Sopka, J. (1974). Numerical methods for nonlinear Volterra integro-differential equations. *SIAM J. Numer. Anal.*, 11, 826–846.

Holloway, J. and Krstic, M. (2019). Prescribed-time output feedback for linear systems in controllable canonical form. *Automatica*, 107, 77–85.

Hong, Y. (2002). Finite-time stabilization and stabilizability of a class of controllable systems. *Syst. Control Lett.*, 46, 231–236.

Irscheid, A., Bleymehl, S., and Rudolph, J. (2021a). Prescribed finite-time stabilization for flat systems (in German). *at – Automatisierungstechnik*, 69, 585–596.

Irscheid, A., Gehring, N., and Rudolph, J. (2021b). Trajectory tracking control for a class of 2 × 2 hyperbolic PDE-ODE systems. *IFAC-PapersOnLine*, 54, 416–421.

Karafyllis, I. and Krstic, M. (2019). *Input-to-State Stability for PDEs*. Springer.

Krishnamurthy, P. and Khorrami, F. (2020). Prescribed-time output-feedback stabilization of uncertain nonlinear systems with unknown time delays. In *Proc. Am. Control Conf.*, 2705–2710.

Krishnamurthy, P., Khorrami, F., and Krstic, M. (2020). A dynamic high-gain design for prescribed-time regulation of nonlinear systems. *Automatica*, 115, 108860.

Krstic, M. (2009). *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhäuser.

Krstic, M. and Smyshlyaev, A. (2008). *Boundary control of PDEs: A course on backstepping designs*. SIAM.

Lattanzio, C., Maurizi, A., and Piccoli, B. (2011). Moving bottlenecks in car traffic flow: A PDE-ODE coupled model. *SIAM J. Math. Anal.*, 43, 50–67.

Lopez-Ramirez, F., Polyakov, A., Efimov, D., and Perruquetti, W. (2018). Finite-time and fixed-time observer design: Implicit lyapunov function approach. *Automatica*, 87, 52–60.

Mazenc, F., Malisoff, M., and Niculescu, S.I. (2014). Reduction model approach for linear time-varying systems with delays. *IEEE Trans. Autom. Control*, 59, 2068–2082.

Perrollaz, V. and Rosier, L. (2014). Finite-time stabilization of 2×2 hyperbolic systems on tree-shaped networks. *SIAM J. Control Optim.*, 52, 143–163.

Polyakov, A. (2012). Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Trans. Autom. Control*, 57, 2106–2110.

Polyakov, A., Efimov, D., and Perruquetti, W. (2015). Finite-time and fixed-time stabilization: Implicit Lyapunov function approach. *Automatica*, 51, 332–340.

Redaud, J., Auriol, J., and Niculescu, S.I. (2021). Output-feedback control of an underactuated network of interconnected hyperbolic PDE–ODE systems. *Syst. Control Lett.*, 154, 104984.

Saldivar, B., Mondié, S., and Ávila Vilchis, J. (2016). The control of drilling vibrations: A coupled PDE-ODE modeling approach. *Int. J. Appl. Math. Comput. Sci.*, 26, 335–349.

Song, Y.D., Wang, Y.J., Holloway, J.C., and Krstic, M. (2017). Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*, 83, 243–251.

Steeves, D. and Krstic, M. (2021). Prescribed-time stabilization of ODEs with diffusive actuator dynamics. *IFAC-PapersOnLine*, 54, 434–439.

Strecker, T., Aamo, O., and Cantoni, M. (2022). Boundary feedback control of 2 × 2 quasilinear hyperbolic systems: Predictive synthesis and robustness analysis. *IEEE Trans. Autom. Control*, 67, 1397–1413.

Tran, D. and Yucelen, T. (2020). Finite-time control of perturbed dynamical systems based on a generalized time transformation approach. *Syst. Control Lett.*, 136, 104605.

Zhou, B. (2020). Finite-time stabilization of linear systems by bounded linear time-varying feedback. *Automatica*, 113, 108760.