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Constrained Mean Field Games Equilibria as Fixed Point of Random Lifting of Set-Valued Maps

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Abstract: We introduce an abstract framework for the study of general mean field game and mean field control problems. Given a multiagent system, its macroscopic description is provided by a time-depending probability measure, where at every instant of time the measure of a set represents the fraction of (microscopic) agents contained in it. The trajectories available to each of the microscopic agents are affected also by the overall state of the system. By using a suitable concept of random lift of set-valued maps, together with fixed point arguments, we are able to derive properties of the macroscopic description of the system from properties of the set-valued map expressing the admissible trajectories for the microscopical agents. We apply the results in the case in which the admissible trajectories of the agents are the minimizers of a suitable integral functional depending also from the macroscopic evolution of the system.

Keywords: Multi-agents system, Differential or dynamic games, Set-valued map, Mean Field Games, state constraints.

1. INTRODUCTION

The mathematical analysis of complex systems with multiple interacting agents has recently attracted increasing attention from researchers in the area of applied mathematics. Social dynamics (e.g., pedestrian dynamics, social network models, opinion formation, infrastructure planning, financial markets, big data analysis, life sciences are only some examples of fields where the mathematical modeling of multi-agent systems has been successfully applied.

The core of the analysis is the fact that the collective (macroscopic) behavior is deeply influenced by complex interactions that usually arise among the subjects (agents like e.g., cell populations, fish swarms, insect colonies, human crowds, bird flocks). The interactions between the agents, which may range from the simplest, e.g., avoiding collision, or attraction/repulsion effects, to more complex ones, involving also penalization of overcrowding/dispersion, or further state constraints on the density of the agents. As a consequence, in general the macroscopic description of the system cannot be reduced to the simple superposition of the microscopic description of each individual.

The reference framework for many mathematical models of multi-agent systems is provided by Mean Field Games (MFG) theory.

The concept of Mean Field Games (MFGs) was first introduced around 2006 by two independent groups, P. E. Caines, M. Huang, and R. P. Malhamé (Huang et al., 2007, 2006), and J.-M. Larsy and P.-L. Lions (Lasry and Lions, 2006a,b, 2007), motivated by problems in economics and engineering and building upon previous works on games with infinitely many agents such as (Aumann and Shapley, 1974; Jovanovic and Rosenthal, 1988). Roughly speaking, MFGs are game models with a continuum of indistinguishable, rational agents influenced only by the average behavior of other agents, and the typical goal of their analysis is to characterize their equilibria.

In general, the equilibria are defined through the system of PDEs, known as MFG system, involving two unknown functions: the value function of the optimal control problem that a typical agent seeks to solve, and the time-dependent density of the population of agents. This interpretation fails in presence of state constraints, i.e., when agents are confined in the closure of a bounded domain. This case has been studied in (Cannarsa and Capuani, 2018; Cannarsa et al., 2019, 2021), where the authors used a different approach to attack the problem. They define the equilibrium for the constrained problem replacing probability measures with measures on arcs, the so-called Lagrangian Formulation.

Recently, a deep comparison between the Lagrangian formulation and other formulation for mean-field problems has been developed in Cavagnari et al. (2020). Basing on the possibility to represent the evolution of the system as the evolution of L^2-random variables in a suitable probability space, it is possible to give an alternative formulation of many classes of problems in terms of L^2-random variables, and the authors establish a comparison between the original problem, the Lagrangian approach, and the L^2-random variable approach.

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In this paper, we focus on constrained MFG equilibria, introduced in (Cannarsa and Capuani, 2018, Definition 3.1), proving that they can be seen as a special case of a more general construction, that we called *random lift*, whose aim is to directly transfer properties from set-valued map describing the microscopical behaviour of the agent, to the macroscopical description of the system. A similar setting was investigated also in Cavagnari et al. (2018), but only when the trajectories of microscopic agents obey to a differential inclusion and there were no interactions between them.

The definition of nonlocal dynamics in Wasserstein space, not necessarily related to control problem, has been studied in Piccoli (2019), Camilli et al. (2021), Cavagnari et al. (2021), also providing numerical schemes for the approximation of the trajectories. In Bonnet and Frankowska (2021), it is developed a full theory for differential inclusions in Wasserstein space, but under severe restrictions on the allowed concept of solution in the space of measures. We emphasize that the approach outlined here and applied to constrained mean field games can be extended to cover cases where less regularity both of the microscopic trajectories and of the macroscopic evolution are considered. From a macroscopical point of view, the control of continuity equation has been studied in Pogodaev (2016) and in Pogodaev and Staritsyn (2020), where an extension to impulsive control systems is presented.

2. PRELIMINARIES

Let (X, d_X) be a separable metric space. We denote by $\mathscr{P}(X)$ the set of Borel probability measures on X endowed with the weak* topology induced by the duality with the Banach space $C^0_b(X)$ of the real-valued continuous bounded functions on X with the uniform convergence norm. The support of $\mu \in \mathscr{P}(X)$, $\mathrm{supp}(\mu)$, is the closed set defined by

$$\mathrm{supp}(\mu)=\Big\{x \in X : \mu(V) > 0 \text{ for any open } V \text{ s.t. } x \in V\Big\}.$$

For any $p \geq 1$, we set the space of Borel probability measures with finite *p-moment* as

$$\mathscr{P}_p(X) = \Big\{\mu \in \mathscr{P}(X) : \int_X d^p_X(x, \bar{x})\, d\mu(x) < +\infty \text{ for some } \bar{x} \in X\Big\}.$$

Given complete separable metric spaces (X, d_X), (Y, d_Y), for any Borel map $r : X \to Y$ and $\mu \in \mathscr{P}(X)$, we define the *push forward measure* $r\sharp\mu \in \mathscr{P}(Y)$ by setting $r\sharp\mu(B) = \mu(r^{-1}(B))$ for any Borel set B of Y.

Definition 2.1. (Transport plans and Wasserstein distance). Let X be a complete separable metric space, $\mu_1, \mu_2 \in \mathscr{P}(X)$. We define the set of *admissible transport plans* between μ_1 and μ_2 by setting

$$\Pi(\mu_1, \mu_2) = \{\boldsymbol{\pi} \in \mathscr{P}(X \times X) : \mathrm{pr}_i\sharp\boldsymbol{\pi} = \mu_i,\, i = 1, 2\},$$

where for $i = 1, 2$, we defined $\mathrm{pr}_i : X \times X \to X$ by $\mathrm{pr}_i(x_1, x_2) = x_i$. The *inverse* $\boldsymbol{\pi}^{-1}$ of a transport plan $\boldsymbol{\pi} \in \Pi(\mu, \nu)$ is defind by $\boldsymbol{\pi}^{-1} = i\sharp\boldsymbol{\pi} \in \Pi(\nu, \mu)$, where $i(x, y) = (y, x)$ for all $x, y \in X$. The *p-Wasserstein distance* between μ_1 and μ_2 is

$$W^p_p(\mu_1, \mu_2) = \inf_{\boldsymbol{\pi}\in\Pi(\mu_1,\mu_2)} \int_{X\times X} d^p_X(x_1, x_2)\, d\boldsymbol{\pi}(x_1, x_2).$$

If $\mu_1, \mu_2 \in \mathscr{P}_p(X)$ then the above infimum is actually a minimum, and we define

$$\Pi^p_o(\mu_1, \mu_2) = \Big\{\boldsymbol{\pi} \in \Pi(\mu_1, \mu_2) : W^p_p(\mu_1, \mu_2) = \int_{X\times X} d^p_X(x_1, x_2)\, d\boldsymbol{\pi}(x_1, x_2)\Big\}.$$

The space $\mathscr{P}_p(X)$ endowed with the W_p-Wasserstein distance is a complete separable metric space, moreover for all $\mu \in \mathscr{P}_p(X)$ there exists a sequence $\{\mu^N\}_{N\in\mathbb{N}} \subseteq \mathrm{co}\{\delta_x : x \in \mathrm{supp}\,(\mu)\}$ such that $W_p(\mu^N, \mu) \to 0$ as $N \to +\infty$.

Fact 1. When X is compact then for all $p \geq 1$ the p-Wasserstein distances are all equivalent.

Definition 2.2. (Set-valued maps). Let X, Y be sets. A *set-valued* map F from X to Y is a map associating to each $x \in X$ a (possible empty) subset $F(x)$ of Y. We will write $F : X \rightrightarrows Y$ to denote a set-valued map from X to Y. The *graph* of a set-valued map F is

$$\mathrm{graph}\, F := \{(x, y) \in X \times Y : y \in F(x)\} \subseteq X \times Y,$$

while the *domain* of F is

$$\mathrm{dom}\, F := \{x \in X : F(x) \neq \emptyset\} \subseteq X.$$

Given $A \subseteq X$, we set

$$\mathrm{graph}(F_{|A}) := \mathrm{graph}\, F \cap (A \times Y) = \{(x, y) \in A \times Y : y \in F(x)\}.$$

A *selection* of F is a map $f : \mathrm{dom}\, F \to Y$ such that $f(x) \in F(x)$ for all $x \in \mathrm{dom}\, F$.

2.1 Mean-Field setting

Let $T > 0$, $\Omega \subset \mathbb{R}^d$ be a bounded open domain with boundary of class $C^{1,1}$, and set $I = [0, T]$. Endowed $C^0(I; \mathbb{R}^d)$ with the uniform convergence metric, let Γ be defined by

$$\Gamma = \Big\{\gamma \in AC(I; \mathbb{R}^d) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in I\Big\} \subset C^0(I; \mathbb{R}^d).$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}.$$

For any $t \in [0, T]$, we denote by $e_t : \Gamma \to \overline{\Omega}$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \qquad \forall \gamma \in \Gamma. \tag{1}$$

Let $f : \mathscr{P}(\overline{\Omega}) \times \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfied the following conditions.

(L1) There exists a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ such that
$$|f(\theta_1, x, v) - f(\theta_2, x, v)| \leq \omega(d(\theta_1, \theta_2))$$
for all $\theta_1, \theta_2 \in \mathscr{P}(\overline{\Omega})$, $x \in \overline{\Omega}$, $v \in \mathbb{R}^d$, and d denotes any p-Wasserstein distance.

(L2) $\theta \mapsto f(\theta, x, v)$ is locally Lipschitz for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^d$ and for a.e. $(\theta, x, v) \in \mathscr{P}(\overline{\Omega}) \times \overline{\Omega} \times \mathbb{R}^d$,
$$|D_x f(\theta, x, v)| \leq C(1 + |v|^2),$$
$$|D_v f(\theta, x, v)| \leq C(1 + |v|),$$
for some constant $C > 0$.

(L3) There exist constants $c_1, c_0 > 0$ such that
$$f(\theta, x, v) \geq c_1|v|^2 - c_0,$$
for all $(\theta, x, v) \in \mathscr{P}(\overline{\Omega}) \times \overline{\Omega} \times \mathbb{R}^d$.

(L4) $v \longmapsto f(\theta, x, v)$ is convex for all $(\theta, x) \in \mathscr{P}(\overline{\Omega}) \times \overline{\Omega}$.

Let $g : \mathscr{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}$ be a uniformly continuous function on $\mathscr{P}(\overline{\Omega}) \times \overline{\Omega}$, that is, there exists a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ such that

$$|g(\theta_1, x) - g(\theta_2, x)| \leq \omega(d(\theta_1, \theta_2))$$

for all $\theta_1, \theta_2 \in \mathscr{P}(\overline{\Omega})$ and $x \in \overline{\Omega}$.

Throughout the paper we suppose that f and g satisfies the above assumptions.

For all $\boldsymbol{\eta} \in \mathscr{P}(\Gamma)$, we define

$$J^{\boldsymbol{\eta}}[\gamma] = \int_0^T f(e_t\sharp\boldsymbol{\eta}, \gamma(t), \dot{\gamma}(t))\, dt + g(e_T\sharp\boldsymbol{\eta}, \gamma(T)), \quad (2)$$

for any $\gamma \in \overline{\Omega}$.

Fact 2. If f has the special form $f(e_t\sharp\boldsymbol{\eta}, x, v) = L(x, v) + F(x, e_t\sharp\boldsymbol{\eta})$, then L and F satisfy the assumptions in (Cannarsa and Capuani, 2018, Section 3.2), see for more details (Cannarsa and Capuani, 2018, Remark 3.3).

Definition 2.3. (**Constrained equilibrium**). Let $m_0 \in \mathscr{P}(\overline{\Omega})$. We say that $\boldsymbol{\eta} \in \mathscr{P}(\Gamma)$ is a constrained MFG equilibrium for m_0 if

- $e_0\sharp\boldsymbol{\eta} = m_0$
- $\mathrm{supp}(\boldsymbol{\eta}) \subseteq \bigcup_{x \in \overline{\Omega}} \left\{ \gamma \in \Gamma[x] : J^{\boldsymbol{\eta}}[\gamma] = \min_{\Gamma[x]} J^{\boldsymbol{\eta}} \right\}$.

In other words, $\boldsymbol{\eta}$ is a constrained MFG equilibrium for m_0 if for $\boldsymbol{\eta}$-a.e. $\overline{\gamma} \in \Gamma$ we have that

$$J^{\boldsymbol{\eta}}[\overline{\gamma}] \leq J^{\boldsymbol{\eta}}[\gamma], \quad \forall \gamma \in \Gamma[\overline{\gamma}(0)].$$

Fact 3. Under our assumptions, there exists at least one constrained MFG equilibrium, see (Cannarsa and Capuani, 2018, Theorem 3.1).

3. RANDOM-LIFTING OF SET-VALUED MAPS

In this section, we will introduce a more general notion of random lift of set-valued maps then the one introduced in (Capuani et al., 2022, Definition 3).

Let X, Y, Z be complete separable metric spaces, and $S : Z \times X \rightrightarrows Y$ be a set valued map. Given $z \in Z$, we denote by $S^z : X \rightrightarrows Y$ the set valued map $x \mapsto S(z, x)$. In particular, $(z, x, y) \in \mathrm{graph}\, S$ if and only if $(x, y) \in \mathrm{graph}\, S^z$.

Definition 3.1. (Random lift of a general set-valued maps). The set-valued map $\boldsymbol{\Xi} : Z \times \mathscr{P}(X) \rightrightarrows \mathscr{P}(X \times Y)$ defined as

$$\boldsymbol{\Xi}(z, \mu) := \{\boldsymbol{\eta} \in \mathscr{P}(X \times Y) : \ \mathrm{supp}\,(\boldsymbol{\eta}) \subseteq \mathrm{graph}\, S^z \text{ and } \mathrm{pr}_1\sharp\boldsymbol{\eta} = \mu\},$$

where $\mathrm{pr}_1(x, y) = x$ for all $(x, y) \in X \times Y$, will be called the *random lift* of S. Given $z \in Z$, we denote by $\boldsymbol{\Xi}^z : \mathscr{P}(X) \rightrightarrows \mathscr{P}(X \times Y)$ the set valued map $\mu \mapsto \boldsymbol{\Xi}(z, \mu)$, which will be called the *random lift* of $S^z(\cdot)$.

Definition 3.2. (Evaluation for random lift). Given $p \geq 1$, a map $\mathscr{E}_p : \mathscr{P}_p(X \times Y) \to Z$ will be called an *evaluation* for the random lift $\boldsymbol{\Xi}(\cdot)$. We say that an evaluation $\mathscr{E}_p$ is *Borel* (resp. *continuous*, *Lipschitz*) if $\mathscr{E}_p : \mathscr{P}_p(X \times Y) \to Z$ is a *Borel* (resp. *continuous*, *Lipschitz*) map. When Z is a convex subset of a linear space, we say that $\mathscr{E}_p$ is *affine* if for all $\lambda \in [0, 1]$, $\mu \in X$, and $\boldsymbol{\eta}^{(i)} \in \boldsymbol{\Xi}^z(\mu) \cap \mathscr{P}_p(X \times Y)$, $i = 1, 2$, it holds

$$\mathscr{E}_p(\lambda\boldsymbol{\eta}^{(1)} + (1 - \lambda)\boldsymbol{\eta}^{(2)}) = \lambda\mathscr{E}_p(\boldsymbol{\eta}^{(1)}) + (1 - \lambda)\mathscr{E}_p(\boldsymbol{\eta}^{(2)}).$$

In the following definition we characterize the set of fixed-points associated to the random lift $\boldsymbol{\Xi}$ and its evaluation map.

Definition 3.3. Given an evaluation $\mathscr{E}_p$ for the random lift $\boldsymbol{\Xi}(\cdot)$, we define the set valued map $\Upsilon_{\mathscr{E}_p} : Z \times \mathscr{P}_p(X) \rightrightarrows Z$ by setting for all $z \in Z$ and $\mu \in \mathscr{P}_p(X)$

$$\Upsilon_{\mathscr{E}_p}(z, \mu) := \mathscr{E}_p(\boldsymbol{\Xi}(z, \mu) \cap \mathscr{P}_p(X \times Y)) \subseteq Z, \qquad (3)$$

and the set valued map $\mathscr{A} : \mathscr{P}_p(X) \rightrightarrows Z$ by

$$\mathscr{A}(\mu) := \left\{ z \in Z : \ z \in \Upsilon_{\mathscr{E}_p}(z, \mu) \right\}.$$

Given $z \in Z$, we denote by $\Upsilon^z_{\mathscr{E}_p} : \mathscr{P}_p(X) \rightrightarrows Z$ the set valued map $\mu \mapsto \Upsilon(z, \mu)$.

Proposition 4. Let X, Y, Z be complete separable metric spaces. Let $S : Z \times X \rightrightarrows Y$ be a set valued map with random lift $\boldsymbol{\Xi}(\cdot)$.

(1) $\boldsymbol{\Xi}(\cdot)$ has always convex images w.r.t. the linear structure of $(C^0_b(X \times Y))'$, even if $S(\cdot)$ has not convex images. Moreover, if Z is a convex subset of a linear space and $\mathscr{E}_p$ is an affine evaluation, we have that $\Upsilon_{\mathscr{E}_p}(\cdot)$ has convex or empty images.

(2) If $S(\cdot)$ has closed graph then $\boldsymbol{\Xi}(\cdot)$ has closed graph.

(3) Given $z \in Z$, suppose that for every compact $K \subseteq X$ the set $\mathrm{graph}\left(S^z_{|K}\right)$ is compact in $X \times Y$. Then for every relative compact $\mathscr{K} \subseteq \mathscr{P}(X)$, the set $\boldsymbol{\Xi}^z(\mathscr{K}) := \bigcup_{\mu \in \mathscr{K}} \boldsymbol{\Xi}^z(\mu)$ is relative compact.

Proof.

(1) Given $(z, \mu) \in Z \times \mathscr{P}(X)$, $\boldsymbol{\eta_i} \in \boldsymbol{\Xi}(z, \mu)$, $i = 0, 1$, and $\lambda \in [0, 1]$, set $\boldsymbol{\eta}_\lambda = \lambda\boldsymbol{\eta_1} + (1 - \lambda)\boldsymbol{\eta_0}$, and notice that

$$\mathrm{supp}\,(\boldsymbol{\eta}_\lambda) \subseteq \mathrm{supp}\,(\boldsymbol{\eta_0}) \cup \mathrm{supp}\,(\boldsymbol{\eta_1}) \subseteq \mathrm{graph}\, S^z,$$

$$\begin{aligned}\boldsymbol{\eta}_\lambda\left(\mathrm{pr}_1^{-1}(A)\right) &= \lambda\boldsymbol{\eta_1}\left(\mathrm{pr}_1^{-1}(A)\right) + (1 - \lambda)\boldsymbol{\eta_2}\left(\mathrm{pr}_1^{-1}(A)\right) \\ &= \lambda\mu(A) + (1 - \lambda)\mu(A) = \mu(A),\end{aligned}$$

for all Borel set $A \subseteq X$, and so $\mathrm{pr}_1\sharp\boldsymbol{\eta_\lambda} = \mu$.

The last assertion follows from the fact that, by definition, an affine evaluation sends convex sets to convex sets, and $\boldsymbol{\Xi}^z(\mu) \cap \mathscr{P}_p(X \times Y)$ is either empty or convex.

(2) Suppose that S has closed graph. Let $(z, \mu, \boldsymbol{\eta}) \in \overline{\mathrm{graph}\,\boldsymbol{\Xi}}$ and $(x, y) \in \mathrm{supp}\,(\boldsymbol{\eta})$. First we prove that $\mathrm{pr}_1\sharp\boldsymbol{\eta} = \mu$ and $(z, x, y) \in \overline{\mathrm{graph}\, S} = \mathrm{graph}\, S$. Indeed, let $\{(z_n, \mu_n, \boldsymbol{\eta_n})\}_{n\in\mathbb{N}} \subseteq \mathrm{graph}\,\boldsymbol{\Xi}$ be a sequence converging to $(z, \mu, \boldsymbol{\eta}) \in Z \times \mathscr{P}(X) \times \mathscr{P}(X \times Y)$. Since pr_1 is continuous, we have that $\{\mathrm{pr}_1\sharp\boldsymbol{\eta_n}\}_{n\in\mathbb{N}}$ narrowly converges to $\mathrm{pr}_1\sharp\boldsymbol{\eta}$, and therefore, since $\mathrm{pr}_1\sharp\boldsymbol{\eta_n} = \mu_n$, for all $\in \mathbb{N}$, by passing to the limit we get $\mathrm{pr}_1\sharp\boldsymbol{\eta} = \mu$. On the other hand, recalling Proposition 5.1.8 in Ambrosio et al. (2008), for every $(x, y) \in \mathrm{supp}\,(\boldsymbol{\eta})$ there exists a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converging to (x, y) and with $(x_n, y_n) \in \mathrm{supp}\,(\boldsymbol{\eta_n}) \subseteq \mathrm{graph}\, S^{z_n}$ for all $n \in \mathbb{N}$. In particular, we have that $\overline{(z_n, x_n, y_n)} \to (z, x, y)$ in $Z \times X \times Y$, so $(z, x, y) \in \overline{\mathrm{graph}\, S}$. This implies $(x, y) \in \mathrm{graph}\, S^z$, so $\mathrm{supp}\,(\boldsymbol{\eta}) \subseteq S^z$. Thus $(z, \mu, \boldsymbol{\eta}) \in \mathrm{graph}\,\boldsymbol{\Xi}$, and therefore $\mathrm{graph}\,\boldsymbol{\Xi}$ is closed.

(3) By Prokhorov's theorem (see e.g. Theorem 5.1.3 in Ambrosio et al. (2008)), for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq X$ such that $\mu(X \setminus K_\varepsilon) \leq \varepsilon$ for all $\mu \in \mathscr{K}$. By (2), for all $\boldsymbol{\eta} \in \boldsymbol{\Xi}^z(\mathscr{K})$ it holds

$$\boldsymbol{\eta}\Big((X \times Y) \setminus \mathrm{graph}\,\left(S^z_{|K_\varepsilon}\right)\Big) = \mu(X \setminus K_\varepsilon) \leq \varepsilon.$$

Since by assumption graph $\left(S_{|K_\varepsilon}\right)$ is compact in $X \times Y$, we have that $\boldsymbol{\Xi}^z(\mathscr{K})$ is relatively compact in $\mathscr{P}(X \times Y)$ again by Prokhorov's theorem.

Theorem 5. (Fixed point). Let X be a compact metric space. Let $p \geq 1$, and Z be a subset of a linear space. Suppose that

(i) $(z, x) \mapsto S^z(x)$ is upper semicontinuous with nonempty compact images;
(ii) $\mathscr{E}_p$ is a continuous affine evaluation;
(iii) there exists a compact convex set $K \subseteq Z$ such that $\Upsilon^z_{\mathscr{E}_p}(\mu) \subseteq K$ for all $z \in K$, $\mu \in \mathscr{P}_p(X)$;
(iv) if we endow K with the topology inherited by Z, we have that
 · every point of K has a base of convex (not necessarily open) neighborhoods,
 · the map $K \times K \times [0,1] \to K$ defined by $(k_1, k_2, \lambda) \mapsto \lambda k_1 + (1-\lambda)k_2$ is continuous.

Then $\mathscr{A}(\mu) \neq \emptyset$ for all $\mu \in \mathscr{P}_p(X)$ and $\mathscr{A}$ is upper semicontinuous.

Proof. Under the assumptions on K, according to Lawson (1976) and Roberts (1978), there exists a locally convex topological vector space $\mathscr{L}$, a subset $\mathscr{L}_K \subseteq \mathscr{L}$ and an homeomorphism $h : K \to \mathscr{L}_K$ satisfying $h(\lambda z_1 + (1-\lambda)z_2) = \lambda h(z_1) + (1-\lambda)h(z_2)$ for all $z_1, z_2 \in K$, $\lambda \in [0,1]$, where $\mathscr{L}_K$ will be endowed with the topology inherited by $\mathscr{L}$. Set $\mathscr{L}_K := h(K)$. Notice that $\mathscr{L}_K$ is compact since K is compact and h is continuous.

Given $\mu \in \mathscr{P}_p(X)$, the set-valued map $\ell \mapsto h(\Upsilon^{h^{-1}(\ell)}_{\mathscr{E}_p}(\mu))$ from $\mathscr{L}_K \rightrightarrows \mathscr{L}_K$ has non-empty compact images and closed graph since $h(\cdot)$ is an homeomorphism and $\Upsilon_{\mathscr{E}_p}$ has closed graph and compact images by our assumptions on X and $\mathscr{E}_p$. Moreover, it has convex images since $h(\cdot)$ sends convex sets to convex sets, and the images are all contained in a common compact set. By Kakutani-Fan-Glicksberg fixed point theorem (see e.g. Theorem 13.1 in Pata (2019)), this set-valued map admits a fixed point, i.e., there exists $\bar{\ell} \in \mathscr{L}_K$ such that $\bar{\ell} \in h(\Upsilon^{h^{-1}(\bar{\ell})}_{\mathscr{E}_p}(\mu))$.

Set $\bar{z} = h^{-1}(\bar{\ell}) \in K$, we have $\bar{z} \in \Upsilon^{\bar{z}}_{\mathscr{E}_p}(\mu)$, thus $\bar{z} \in \mathscr{A}(\mu)$, which concludes the proof.

In Theorem 5, we note that the assumption (iv) can be dropped if Z is a locally convex topological vector space.

4. APPLICATION TO CONSTRAINED MEAN FIELD GAMES PROBLEM

In this section we provide an application of previously described construction. More precisely, we show that the constrained MFG equilibria are fixed-points of a suitable set-valued map built in terms of random lift. Choosing $Z = \mathscr{P}(\Gamma)$, $X = \overline{\Omega}$ (which is compact) and $Y = \Gamma$ in the Defintion 3.1, we can rewrite S and the random lift $\boldsymbol{\Xi}$ of S as $S : \mathscr{P}(\Gamma) \times \overline{\Omega} \rightrightarrows \Gamma$ and $\boldsymbol{\Xi} : \mathscr{P}(\Gamma) \times \mathscr{P}(\overline{\Omega}) \rightrightarrows \mathscr{P}(\overline{\Omega} \times \Gamma)$, respectively. Taking into account the construction of constrained MFG equilbria in Section 2.1, we have to define $S^{\cdot}$, in terms of a set of minimizers of the functional $J^{\cdot}$, defined in (2). In this framework, fixed $\boldsymbol{\theta} \in \mathscr{P}(\Gamma)$ the set-valued map $S^{J,\boldsymbol{\theta}}_I : \overline{\Omega} \rightrightarrows \Gamma$ and its lift $\boldsymbol{\Xi}^{J,\boldsymbol{\theta}}_I : \mathscr{P}(\overline{\Omega}) \rightrightarrows \mathscr{P}(\overline{\Omega} \times \Gamma)$ are defined as follows

- for all $x \in \overline{\Omega}$
$$S^{J,\boldsymbol{\theta}}_I(x) = \left\{\gamma \in \Gamma[x] : J^{\boldsymbol{\theta}}[\gamma] = \min_{\Gamma[x]} J^{\boldsymbol{\theta}}\right\},$$
where $J^{\boldsymbol{\theta}}$ is defined in (2);
- for all $\mu \in \mathscr{P}(\overline{\Omega})$
$$\boldsymbol{\Xi}^{J,\boldsymbol{\theta}}_I(\mu) = \Big\{\boldsymbol{\eta} \in \mathscr{P}(\overline{\Omega} \times \Gamma) : \mathrm{supp}\,(\boldsymbol{\eta}) \subseteq \mathrm{graph}\, S^{J,\boldsymbol{\theta}}_I \text{ and } (e_0 \circ \mathrm{pr}_2)\sharp\boldsymbol{\eta} = \mu\Big\},$$
where e_0 is given by (1) and we define $\mathrm{pr}_2 : \overline{\Omega} \times \Gamma \to \Gamma$ by $\mathrm{pr}_2(x, \gamma) = \gamma$.

Moreover, for any $p \geq 1$ the evaluation map $\mathscr{E}_p : \mathscr{P}_p(\overline{\Omega} \times \Gamma) \rightrightarrows \mathscr{P}(\Gamma)$ for the random lift $\boldsymbol{\Xi}$ is defined by $\mathcal{E}_p(\boldsymbol{\eta}) = \mathrm{pr}_2\sharp\boldsymbol{\eta}$. Therefore, we get
$$\mathscr{E}_p(\boldsymbol{\Xi}^{J,\boldsymbol{\theta}}_I(\mu)) = \Upsilon^{J,\boldsymbol{\theta}}_I(\mu) \subseteq \mathscr{P}(\Gamma), \tag{4}$$
where $\Upsilon^{J,\boldsymbol{\theta}}_I : \mathscr{P}_p(\overline{\Omega}) \rightrightarrows \mathscr{P}(\Gamma)$. Finally, we define $\mathscr{A}^{J,\boldsymbol{\theta}}_I : \mathscr{P}_p(\overline{\Omega}) \rightrightarrows \mathscr{P}(\Gamma)$ to be the set-valued map given by
$$\mathscr{A}^{J,\boldsymbol{\theta}}_I(\mu) := \{\boldsymbol{\theta} \in \mathscr{P}(\Gamma) : \boldsymbol{\theta} \in \Upsilon^{J,\boldsymbol{\theta}}_I(\mu)\},$$
i.e., $\mathscr{A}^{J,\boldsymbol{\theta}}_I(\mu)$ is the set of fixed points of $\boldsymbol{\theta} \mapsto \Upsilon^{J,\boldsymbol{\theta}}_I(\mu)$.

Theorem 6. Let $\mu \in \mathcal{P}(\overline{\Omega})$. There exists at least one $\boldsymbol{\theta} \in \mathscr{P}(\Gamma)$ such that $\boldsymbol{\theta} \in \mathscr{A}^{J,\boldsymbol{\theta}}_I(\mu)$. Moreover, the fixed-point $\boldsymbol{\theta} \in \mathscr{A}^{J,\boldsymbol{\theta}}_I(\mu)$ is a constrained MFG equilibrium for μ.

Proof. In order to apply Theorem 5, we have to prove that $(i) - (iv)$ hold. First of all, using (Cannarsa and Capuani, 2018, Lemma 3.3) and (Cannarsa and Capuani, 2018, Lemma 3.4) we have that for all $x \in \overline{\Omega}$ and for all $\boldsymbol{\theta} \in \mathscr{P}(\Gamma)$ the set-valued map $S^{J,\boldsymbol{\theta}}_I$ has closed graph and nonempty images. Moreover, all minimizers $\gamma \in S^{J,\boldsymbol{\theta}}_I(x)$ are $\frac{1}{2}$-Holder continuous of constant $M > 0$. Indeed, for all $\gamma \in S^{J,\boldsymbol{\theta}}_I(x)$ we have by (L3)
$$c_1 \int_0^T |\dot{\gamma}(t)|^2\,dt - Tc_0 \leq J^{\boldsymbol{\theta}}[\gamma] \leq J^{\boldsymbol{\theta}}[\hat{\gamma}],$$
where $\hat{\gamma}(t) = x$ for all $t \in I$, noticing that $\hat{\gamma} \in \Gamma[x]$. This yields $||\dot{\gamma}||^2_{L^2} \leq M$, where M is defined as
$$M^2 = \frac{T}{c_1}\left[\max_{\substack{\zeta \in \mathscr{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} f(\zeta, x, 0) + c_0\right] + \max_{\substack{\zeta \in \mathscr{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} |g(\zeta, x)|$$
where c_0 and c_1 are the constants in (L3). Therefore, we have that
$$S^{J,\boldsymbol{\theta}}_I(x) \subset K := \overline{\{\gamma \in C^{0,1/2}(I; \mathbb{R}^d) : ||\gamma||_{C^{0,1/2}} \leq M + R\}},$$
where R is the minimum radius of the ball B of center 0 that contains $\bar{\Omega}$ and the norm $||\gamma||_{C^{0,1/2}}$ is defined as
$$||\gamma||_{C^{0,1/2}} = |\gamma(0)| + \sup_{\substack{t,s \in I \\ t \neq s}} \frac{|\gamma(t) - \gamma(s)|}{|t-s|^{1/2}}.$$

By Ascoli-Arzelà's Theorem, K is a compact set in $C^0(I; \mathbb{R}^d)$. By (Aubin and Frankowska, 2009, Proposition 1.4.8) $(\boldsymbol{\theta}, x) \mapsto S^{J,\boldsymbol{\theta}}_I(x)$ is upper semicontinuous with nonempty compact images. Since $\mathscr{E}_p$ is the projection on the second component, the assumptions of Theorem 5 are satisfied. Therefore, for any $\mu \in \mathcal{P}(\overline{\Omega})$ there exists at least one $\boldsymbol{\theta} \in \mathscr{P}(\Gamma)$ such that $\boldsymbol{\theta} \in \mathscr{A}_I^{J,\boldsymbol{\theta}}(\mu)$.

Disintegrating $\boldsymbol{\theta}$ w.r.t. the map $\gamma \mapsto \gamma(a)$ from Γ to $\overline{\Omega}$ yields $\boldsymbol{\theta} = \mu \otimes \theta_x$, where $\{\theta_x\}_{x\in\overline{\Omega}}$ is a Borel family of probability measures, uniquely defined μ-a.e. and for μ-a.e. $x \in \overline{\Omega}$ and θ_x-a.e. $\gamma \in \Gamma$, we have that $\gamma(a) = x$. Thus γ is a minimizer of $J^{\boldsymbol{\theta}}$, in particular $(x, \gamma) \in \operatorname{graph} S_I^{J,\boldsymbol{\theta}}$. This completes the proof.

5. CONCLUSION

The proposed technique to lift set-valued map can be generalized to the case of noncompact state space. To this aim, more careful a priori estimates on the minimizers are needed, and the choice of a particular Wasserstein distance becomes relevant.

Beside the lift of set-valued maps associating to a point the minimizing curves of an integral functional starting from that point, it can be taken into account also the lift of the solution map for a differential inclusion satifying certain structural assumptions, or generalized characteristics of some classes of Hamiltonian PDE, with potential applications to gradient flow of certain classes of functional defined in the Wasserstein space.

The state constraints considered in the last section are a special class of state constraints defined in the underlying finite-dimensional space on the trajectory of each agent. The treatment of more general state constraints (of nonlocal type) penalizing for instance concentrations or rarefaction seems to be quite complicated, due to their low regularity property.

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