On Watanabe's characterisation and change of intensity à la Girsanov for Cox processes

On Watanabe's characterisation and change of intensity à la Girsanov for Cox processes

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Abstract

We consider Girsanov changes of measures that alter the intensity of point processes in a suitable way, e.g., for the probability reference approach in filtering. Such change of measure exists if a suitably related stochastic exponential involving the intended new target intensity process is a proper martingale. We show that the stochastic exponential is always a martingale when the point process is a Cox process. In doing so, we show how Watanabe's characterisation for conditional Poisson processes in terms of local martingales can be formulated as a necessary and sufficient condition.

Keywords: Change of intensity by change of measure, conditional Poisson process, Cox process, Girsanov theorem for point processes, Watanabe's characterisation.

1 Introduction

We show that the stochastic exponential

$$Z_t = \exp\left(-\int_0^t (Y_s - 1) X_s \,\mathrm{d}s \right) \prod_{s \in]0,t]} \big(1 + (Y_s - 1)\,\Delta N_s \big) \quad \text{for } \; t \geq 0$$

is a martingale in the underlying filtration $(\mathcal{F}_t)_{t\geq 0}$ when N is a conditional Poisson process with $\mathcal{F}_0$-measurable intensity X (under $\mathbb{P}$) and Y is a non-negative $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_0$-measurable process (intended as a modifier to the intensity under another changed measure, to be constructed) such that

$$\mathbb{P}\left[\int_0^t Y_s X_s \,\mathrm{d}s < \infty \quad \forall\, t \geq 0 \right] = 1.$$

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If the stochastic exponential process Z is not just a local martingale but a martingale, we can use it as a Radon-Nikodym density process to define a change to a probability measure under which the intensity of the conditional Poisson process is changed from X to YX. For example, that the stochastic exponential Z is a martingale is a crucial assumption in the reference probability approach to filtering for point processes, see Brémaud (1981, Chapter VI). Here, the goal is to extract information about X from the observation N. That might be easy or difficult depending on the complexity of the joint distribution of X and N. Now, the reference probability approach introduces a change of measure that decouples X from N such that the two become independent, thereby simplifying their joint distribution. Then, further properties like convenient marginal distributions enable computations of conditional expectations of X given N by the Kallianpur–Striebel formula (see Zakai (1969) or Bain & Crisan (2009, Chapter 3) for the approach in a diffusion setting).

Whether a stochastic exponential is a martingale or only a local martingale is of mathematical interest on its own. We refer the reader to Mijatović & Urusov (2012) or Ruf (2013) and the references therein for literature on this topic in the diffusion setting. The known conditions in the point process setting typically impose bounds on the integrands involved (and work irrespective of any class of point processes) are presented to in Boel et al. (1975, Section 3.3), Hansen & Sokol (2015, Theorem 2.4), or Brémaud (2020, Remark 5.5.2 and subsequent examples).

We state our results for conditional Poisson processes, which are point processes indexed by time $t \geq 0$. Their defining feature is that the conditional distribution of any increment follows the law of a Poisson distribution with stochastic intensity X (see Definition 2.1). Most notably, conditional Poisson processes are defined with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$. They are related to Cox processes. However, Cox processes are defined on general measurable spaces (not necessarily representing time) without reference to any filtration, see Last & Penrose (2017, Chapter 13). Nevertheless, any Cox process indexed by time with an absolutely continuous intensity measure is a conditional Poisson process with respect to the filtration $(\mathcal{F}_t^N \vee \sigma(X))_{t\geq 0}$.

Even though we state our result for unmarked point processes for simplicity, the extension to marked point processes is immediate.

Whether or not our particular stochastic exponential Z is a martingale is related to an integrability issue that appears in the characterisation of conditional Poisson processes in the spirit of Watanabe (1964, Remark to Theorem 2.3). Watanabe famously proved that a simple point process N is a Poisson process with parameter $\lambda > 0$ if and only if $t \mapsto N_t - \lambda t$ is a martingale. Grigelionis (1975, Theorem 4.1) tried to generalise that famous result by claiming that a simple point process N is a conditional Poisson process with intensity X if and only if $t \mapsto N_t - \int_0^t X_s \,\mathrm{d}s$ is a local martingale. He showed only one of the directions for this equivalence and stated that the local martingale property "follows from [the] definition [...]". Since then, other authors have been more cautious at the expense of more restrictive assumptions. More precisely, if N is a conditional Poisson process, then it holds for $t \geq r \geq 0$ that

$$\mathbb{E}\big[N_t - N_r \,\big|\, \mathcal{F}_r\big] = \int_r^t X_s \,\mathrm{d}s,$$

which implies the martingale property when either side of the equation is integrable. For example, Brémaud (1981, Chapter II.2) assumes $\mathbb{E}[N_t] < \infty$ and Grandell (1991, Proposition 18) assumes $\mathbb{E}[\int_0^t X_s \,\mathrm{d}s] < \infty$. However, if neither side is integrable, we need to localise with a sequence of stopping times $(T_n)_{n=0}^{\infty}$ to make either side integrable. Thus, recognising that the stopping times might not be $\mathcal{F}_r$-measurable, we need to know that

$$\mathbb{E}\big[N_{t\wedge T_n} - N_{r\wedge T_n} \,\big|\, \mathcal{F}_r\big] = \mathbb{E}\bigg[\int_{r\wedge T_n}^{t\wedge T_n} X_s \,\mathrm{d}s \,\bigg|\, \mathcal{F}_r\bigg]$$

to deduce the local martingale property. If we cannot show that, we must use other approaches or weaken the statements. For example, Grigelionis (1998, Theorem 1) uses a different method to show a related result instead of relying on his characterisation (see also Grandell (1997, Proposition 6.1)), and Brémaud (2020, Exercise 5.9.9) drops the word "characterisation" from the theorem.

We show that Z is a martingale (see Theorem 3.4) and that the generalised Watanabe characterisation for conditional Poisson processes holds (see Theorem 3.2) by using the following observation. The distribution of X fixes the joint distribution of N and X. Thus, we should be able to write down any conditional expectation for N and X given the information X. Because N is a point process, we can reduce any conditional expectation to the conditional distributions of arrival times $(T_n)_{n=0}^{\infty}$ of N. After deriving the explicit formulas for the conditional expectations of $(T_n)_{n=0}^{\infty}$ (see Lemma 3.1), we can check any martingale property by direct computation.

Surprisingly, the literature has missed that despite Chou & Meyer (1975) (see also Brémaud (2020, Theorem 5.2.2)) proved statements about the conditional distributions of interarrival times $(T_{n+1} - T_n)_{n=0}^{\infty}$ to derive formulas for the compensators of general point processes.

We have organised the remainder of our note as follows. Section 2 contains the precise definition of a conditional Poisson process. Section 3 contains our statements and proofs. In particular, Lemma 3.1 shows that we can write down any conditional expectation relevant to our context. Theorem 3.2 shows that $t \mapsto N_t - \int_0^t X_s \,\mathrm{d}s$ is a local martingale if N is a conditional Poisson process with intensity X. And Theorem 3.4 establishes that our stochastic exponential Z is a martingale. Section 4 finishes our note with two examples demonstrating how to use the reference probability approach to filtering.

2 Setting

If not otherwise stated, we make the following assumptions for the remainder of the paper. There is a sequence of ordered stopping times $(T_n)_{n=0}^{\infty}$ with values in $[0,\infty]$ on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ such that

$$\mathbb{P}\big[T_0 = 0\big] = 1 \quad \text{and} \quad \mathbb{P}\big[T_{n+1} < \infty \;\Rightarrow\; T_n < T_{n+1}\big] = 1 \quad \text{for } n \in \mathbb{N}_0.$$

The process N is the simple point process corresponding to $(T_n)_{n=0}^{\infty}$, i.e.

$$N_t = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{T_n \leq t\}} \quad \text{for } t \geq 0.$$

There is a non-negative $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_0$-measurable process X that is locally integrable, i.e.

$$\mathbb{P}\bigg[\int_0^t X_s \,\mathrm{d}s < \infty \quad \forall\, t \geq 0\bigg] = 1.$$

And N and X fulfil the following definition:

Definition 2.1. N is a conditional Poisson process with $\mathcal{F}_0$-measurable intensity X if

$$\mathbb{P}\big[N_t - N_r = n \,\big|\, \mathcal{F}_r\big] = \mathrm{e}^{-\int_r^t X_s \,\mathrm{d}s} \frac{1}{n!} \bigg(\int_r^t X_s \,\mathrm{d}s \bigg)^n \quad \mathbb{P}\text{-a.s.} \quad \text{for } t \geq r \geq 0,\ n \in \mathbb{N}_0.$$

It is easy to see from the local integrability of X that $\mathbb{P}[N_t < \infty \,|\, \mathcal{F}_0] = 1$. In particular, the process N and $t \mapsto \int_0^t X_s \,\mathrm{d}s$ are finite outside a set of probability measure zero. Because our filtration is complete, we can and do assume that both processes N and X are finite in the remainder of the paper.

In our presentation, (in)equalities and properties stated for random variables are meant to hold $\mathbb{P}$ almost surely ($\mathbb{P}$-a.s.). We omit writing $\mathbb{P}$-a.s. by common convention.

3 Statements and Proofs

Lemma 3.1 (Conditional Density). Let Y be a $\mathcal{B}([0,\infty]) \otimes \mathcal{F}_r$-measurable non-negative process for given $r \geq 0$ and $n \in \mathbb{N}_0$. Then

$$\mathbb{E}\big[Y_{T_{n+1}} \,\big|\, \mathcal{F}_r\big] = \begin{cases} \int_r^\infty Y_t \Psi_t^{n,r} \mathrm{d}t + Y_\infty \Psi_\infty^{n,r} & \text{on } \{N_r \leq n\} \\ Y_{T_{n+1}} & \text{on } \{N_r > n\} \end{cases}$$

with conditional density $\Psi^{n,r} \colon [0,\infty] \times \Omega \to [0,\infty)$ given by

$$\Psi_t^{n,r} = \begin{cases} X_t \,\mathrm{e}^{-\int_r^t X_s \,\mathrm{d}s} \dfrac{1}{(n-N_r)!} \bigg(\displaystyle\int_r^t X_s \,\mathrm{d}s \bigg)^{n-N_r} \mathbb{1}_{\{N_r \leq n\}} & \text{if } t < \infty \\ \mathrm{e}^{-\int_r^\infty X_s \,\mathrm{d}s} \displaystyle\sum_{k=0}^{n-N_r} \frac{1}{k!} \bigg(\int_r^\infty X_s \,\mathrm{d}s \bigg)^k \mathbb{1}_{\{N_r \leq n\}} & \text{if } t = \infty \end{cases}$$

using the conventions $0^0 = 1$ and $\mathrm{e}^{-\infty}\infty = 0$.

Proof: We prove the statement for the special case $Y_s = \mathbb{1}_{\{s>t\}}$ for given finite $t \geq r \geq 0$. The general result follows then from measurability and a monotone class argument.

Because N is a conditional Poisson process with intensity X, we have

$$\begin{aligned} \mathbb{P}\big[T_{n+1} > t \,\big|\, \mathcal{F}_r\big] &= \mathbb{P}\big[N_t \leq n \,\big|\, \mathcal{F}_r\big] = \mathbb{P}\big[N_t - N_r \leq n - N_r \,\big|\, \mathcal{F}_r\big] \\ &= \sum_{k=0}^{n-N_r} \mathbb{P}\big[N_t - N_r = k \,\big|\, \mathcal{F}_r\big] \\ &= \mathrm{e}^{-\int_r^t X_s \,\mathrm{d}s} \sum_{k=0}^{n-N_r} \frac{1}{k!} \bigg(\int_r^t X_s \,\mathrm{d}s \bigg)^k \mathbb{1}_{\{N_r \leq n\}}. \end{aligned}$$

The above process is absolutely continuous with respect to t because sums, products, and compositions (with increasing absolutely continuous functions) of absolutely continuous functions are again absolutely continuous. In particular, there is a derivative $\Psi^{n,r}$ such that for all finite $v \geq t \geq r$

$$\int_t^v \Psi_s^{n,r} \,\mathrm{d}s = \mathbb{P}\big[T_{n+1} > t \,\big|\, \mathcal{F}_r\big] - \mathbb{P}\big[T_{n+1} > v \,\big|\, \mathcal{F}_r\big] \quad \text{and} \tag{1}$$

$$\Psi_t^{n,r} = X_t \,\mathrm{e}^{-\int_r^t X_s \,\mathrm{d}s} \frac{1}{(n-N_r)!} \bigg(\int_r^t X_s \,\mathrm{d}s \bigg)^{n-N_r} \mathbb{1}_{\{N_r \leq n\}}. \tag{2}$$

Furthermore, consider $\Psi^{n,r}_\infty = \mathbb{P}[T_{n+1} = \infty \,|\, \mathcal{F}_r]$, then the monotone convergence theorem and $x^k \mathrm{e}^{-x} \to 0$ when $x \uparrow \infty$ for any $k \in \mathbb{N}_0$ shows that

$$\Psi^{n,r}_\infty = \lim_{v\to\infty} \mathbb{P}\big[T_{n+1} > v \,\big|\, \mathcal{F}_r\big] = \mathrm{e}^{-\int_r^\infty X_s\,\mathrm{d}s} \sum_{k=0}^{n-N_r} \frac{1}{k!}\bigg(\int_r^\infty X_s\,\mathrm{d}s\bigg)^k \mathbb{1}_{\{N_r\le n\}}. \tag{3}$$

Combining equations (1), (2), (3) yields the required result

$$\mathbb{P}\big[T_{n+1} > t \,\big|\, \mathcal{F}_r\big] = \int_t^\infty \Psi^{n,r}_s\,\mathrm{d}s + \Psi^{n,r}_\infty = \int_r^\infty Y_s\,\Psi^{n,r}_s\,\mathrm{d}s + Y_\infty\,\Psi^{n,r}_\infty.$$

□

Theorem 3.2 (Characterisation à la Watanabe (1964) and Grigelionis (1975)). Let N be a simple point process corresponding to $(T_n)_{n=0}^\infty$. Then, the following statements are equivalent

(i) N is a conditional Poisson process with intensity X (see Definition 2.1),

(ii) $\widetilde{N}_t = N_t - \int_0^t X_s\,\mathrm{d}s$ for $t \ge 0$, called compensated point process, is a local martingale,

(iii) $\mathbb{E}[\int_0^\infty \varphi_s\,\mathrm{d}N_s] = \mathbb{E}[\int_0^\infty \varphi_s X_s\,\mathrm{d}s]$ holds for any non-negative predictable process φ.

Proof: The implications (ii) ⇒ (iii) and (iii) ⇒ (i) are well-known, see for example Brémaud (1981, Chapter II.2). Thus, we only show the implication (i) ⇒ (ii).

Using the notation of Lemma 3.1, we have for finite $t \ge r \ge 0$ that

$$\begin{aligned}\mathbb{E}\bigg[\int_{r\wedge T_n}^{t\wedge T_n} X_s\,\mathrm{d}s \,\bigg|\, \mathcal{F}_r\bigg] &= \mathbb{E}\bigg[\mathbb{1}_{\{T_n>t\}}\int_r^t X_s\,\mathrm{d}s + \mathbb{1}_{\{t\ge T_n>r\}}\int_r^{T_n} X_s\,\mathrm{d}s \,\bigg|\, \mathcal{F}_r\bigg] \\ &= \mathbb{P}\big[T_n > t \,\big|\, \mathcal{F}_r\big]\int_r^t X_s\,\mathrm{d}s + \int_r^t \Psi^{n-1,r}_u \int_r^u X_s\,\mathrm{d}s\,\mathrm{d}u. \end{aligned}\tag{4}$$

Using the special form of Ψ in particular $\Psi^{n-1,r}_u \int_r^u X_s\,\mathrm{d}s = (n-N_r)\Psi^{n,r}_u$, it holds that

$$\int_r^t \Psi^{n-1,r}_u \int_r^u X_s\,\mathrm{d}s\,\mathrm{d}u = (n-N_r)\int_r^t \Psi^{n,r}_u\,\mathrm{d}u = (n-N_r)\,\mathbb{P}\big[t \ge T_{n+1} > r \,\big|\, \mathcal{F}_r\big]. \tag{5}$$

By the definition of a conditional Poisson process, we have

$$\begin{aligned}&\mathbb{P}\big[T_n > t \,\big|\, \mathcal{F}_r\big]\int_r^t X_s\,\mathrm{d}s = \mathbb{P}\big[N_t < n \,\big|\, \mathcal{F}_r\big]\int_r^t X_s\,\mathrm{d}s \\ &= \mathbb{P}\big[N_t - N_r < n - N_r \,\big|\, \mathcal{F}_r\big]\int_r^t X_s\,\mathrm{d}s = \mathrm{e}^{-\int_r^t X_s\,\mathrm{d}s}\sum_{k=0}^{n-N_r} k\,\frac{1}{k!}\bigg(\int_r^t X_s\,\mathrm{d}s\bigg)^k \\ &= \sum_{k=0}^{n-N_r} k\,\mathbb{P}\big[N_t - N_r = k \,\big|\, \mathcal{F}_r\big] = \mathbb{E}\big[(N_t - N_r)\,\mathbb{1}_{\{T_{n+1}>t\}} \,\big|\, \mathcal{F}_r\big]. \end{aligned}\tag{6}$$

Moreover, the following equation holds,

$$N_{t\wedge T_n} = N_{r\wedge T_n} + (n-N_r)\,\mathbb{1}_{\{t\ge T_{n+1}>r\}} + (N_t - N_r)\,\mathbb{1}_{\{T_{n+1}>t\}}. \tag{7}$$

Now, combining equations (4), (5), (6), (7) and noting that all following terms are quasi-integrable (possibly $-\infty$ but never ∞) or bounded yields

$$\mathbb{E}\left[N_{t\wedge T_n} - \int_0^{t\wedge T_n} X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_r\right] = N_{r\wedge T_n} - \int_0^{r\wedge T_n} X_s \,\mathrm{d}s$$
$$+ \mathbb{E}\left[(n - N_r)\, \mathbb{1}_{\{t\geq T_{n+1}>r\}} + (N_t - N_r)\, \mathbb{1}_{\{T_{n+1}>t\}} - \int_{r\wedge T_n}^{t\wedge T_n} X_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_r\right]$$
$$= N_{r\wedge T_n} - \int_0^{r\wedge T_n} X_s \,\mathrm{d}s.$$

Thus, the process $t \mapsto N_t - \int_0^t X_s \,\mathrm{d}s$ stopped with T_n fulfils the martingale property; it is integrable (which follows from the special case $r = 0$); and it is obviously adapted to $(\mathcal{F}_t)_{t\geq 0}$; i.e. it is a martingale.

Note that $T_n \uparrow \infty$ for $n \uparrow \infty$ because $\mathbb{P}[N_t < \infty \,|\, \mathcal{F}_0] = 1$, see Section 2, i.e. $\mathbb{P}[N_t = \infty \,|\, \mathcal{F}_0] = 0$, for all $t \geq 0$. In particular, $(T_n)_{n=0}^{\infty}$ is a suitable localising sequence. □

Remark 3.3. Applying the optional stopping theorem to Lemma 3.1 shows that for any $\mathcal{B}([0,\infty]) \otimes \mathcal{F}_{T_n}$-measurable non-negative process W for given $n \in \mathbb{N}_0$

$$\mathbb{E}\big[W_{T_{n+1}} \,\big|\, \mathcal{F}_{T_n}\big] = \int_{T_n}^{\infty} W_s \,\Phi_s^n \,\mathrm{d}s + W_\infty \,\Phi_\infty^n \tag{8}$$

with $\Phi^n \colon [0,\infty] \times \Omega \to [0,\infty)$ given by

$$\Phi_s^n = \begin{cases} X_s \,\mathrm{e}^{-\int_{T_n}^s X_r \,\mathrm{d}r} & \text{if } s < \infty \\ \mathrm{e}^{-\int_{T_n}^\infty X_r \,\mathrm{d}r} & \text{if } s = \infty \end{cases}.$$

Now, we can also show the implication (i) ⇒ (ii) from Theorem 3.2 by applying (8) to general point process theory. Chou & Meyer (1975) (see also Brémaud (2020, Theorem 5.2.2)) developed the idea of computing the compensator of the entire point process from the compensators of every single jump. More precisely, in our context,

$$\lambda_t = \sum_{n=0}^{\infty} \frac{\Phi_t^n}{1 - \int_{T_n}^t \Phi_s^n \,\mathrm{d}s} \,\mathbb{1}_{\{T_{n+1}>t\geq T_n\}} \quad \text{for } t \geq 0$$

has the property that $t \mapsto N_t - \int_0^t \lambda_s \,\mathrm{d}s$ is a local martingale. Plugging Φ^n in the above equation reveals $\lambda_t = X_t$.

Theorem 3.4 (Changing intensity by change of measure). Let Y be a non-negative $\mathcal{B}([0,\infty)) \otimes \mathcal{F}_0$-measurable process such that

$$\mathbb{P}\left[\int_0^t Y_s \,X_s \,\mathrm{d}s < \infty \quad \forall\, t \geq 0\right] = 1.$$

Consider the stochastic exponential $Z = \mathcal{E}(\int_0^\cdot Y_s - 1 \,\mathrm{d}\widetilde{N}_s)$ with compensated process $\widetilde{N}$ from Theorem 3.2(ii), i.e.

$$Z_t = \exp\left(-\int_0^t (Y_s - 1)\, X_s \,\mathrm{d}s\right) \prod_{s\in]0,t]} \big(1 + (Y_s - 1)\Delta N_s\big) \quad \text{for } t \geq 0$$

with the convention $\prod_\varnothing = 1$. Then, the following statements are true:

(i) Z is a non-negative right-continuous martingale with $Z_0 = 1$.

(ii) The probability measure induced by Z and a bounded $\mathcal{F}_0$-measurable random variable $S \geq 0$, i.e.

$$\mathbb{Q}(A) = \mathbb{E}\big[\mathbb{1}_A Z_S\big] \quad \text{for } A \in \mathcal{F},$$

is such that $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_0$ and the process $\widetilde{N}^{\mathbb{Q}}_{t\wedge S} = N_t - \int_0^{t\wedge S} Y_s X_s \,\mathrm{d}s$ for $t \geq 0$ is a local martingale.

In particular, Theorem 3.2 implies that N is a conditional Poisson process with intensity YX up to time S under $\mathbb{Q}$.

Proof: We show that $\mathbb{E}[Z_t] = 1$ for finite $t \geq 0$. Then, the statements (i) and (ii) follow from well-known results, see for example Brémaud (1981, Ch VI.2).

We show by induction over $j \in \mathbb{N}_0$ with $n \geq j \geq 0$ and the convention $0^0 = 1$ that

$$\mathbb{E}\Big[\prod_{k=1}^{n} Y_{T_k} \mathbb{1}_{\{N_t = n\}} \,\Big|\, \mathcal{F}_{T_{n-j}}\Big] = \mathrm{e}^{-\int_{T_{n-j}}^t X_s \,\mathrm{d}s} \frac{1}{j!} \Big(\int_{T_{n-j}}^t Y_s X_s \,\mathrm{d}s\Big)^j \mathbb{1}_{\{t \geq T_{n-j}\}} \prod_{k=1}^{n-j} Y_{T_k}. \tag{9}$$

The base case $j = 0$ follows from (8) in Remark 3.3 for the specific choice $W_s = \mathbb{1}_{\{s > t \geq T_n\}}$ together with the identity $\int_{T_n}^\infty \Phi_s^n \,\mathrm{d}s + \Phi_\infty^n = \exp(-\int_{T_n}^t X_s \,\mathrm{d}s)$ on $\{t \geq T_n\}$. More precisely, we have

$$\begin{aligned}\mathbb{E}\Big[\prod_{k=1}^{n} Y_{T_k} \mathbb{1}_{N_t = n} \,\Big|\, \mathcal{F}_{T_n}\Big] &= \mathbb{P}\big[T_{n+1} > t \geq T_n \,|\, \mathcal{F}_{T_n}\big] \prod_{k=1}^{n} Y_{T_k} \\ &= \mathrm{e}^{-\int_{T_n}^t X_s \,\mathrm{d}s} \mathbb{1}_{\{t \geq T_n\}} \prod_{k=1}^{n} Y_{T_k}.\end{aligned}$$

Let us continue with the induction step; we assume that (9) holds for $j-1$;

$$\begin{aligned}
&\mathbb{E}\Big[\prod_{k=1}^{n} Y_{T_k} \mathbb{1}_{\{N_t = n\}} \,\Big|\, \mathcal{F}_{T_{n-j}}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\prod_{k=1}^{n} Y_{T_k} \mathbb{1}_{\{N_t = n\}} \,\Big|\, \mathcal{F}_{T_{n-(j-1)}}\Big] \,\Big|\, \mathcal{F}_{T_{n-j}}\Big] \\
&\overset{(9)}{=} \mathbb{E}\Big[Y_{T_{n-(j-1)}} \mathrm{e}^{-\int_{T_{n-(j-1)}}^t X_s \,\mathrm{d}s} \frac{1}{(j-1)!} \Big(\int_{T_{n-(j-1)}}^t Y_s X_s \,\mathrm{d}s\Big)^{j-1} \mathbb{1}_{\{t \geq T_{n-(j-1)}\}} \,\Big|\, \mathcal{F}_{T_{n-j}}\Big] \prod_{k=1}^{n-j} Y_{T_k} \\
&\overset{(8)}{=} \int_{T_{n-j}}^t Y_r X_r \mathrm{e}^{-\int_{T_{n-j}}^r X_s \,\mathrm{d}s} \mathrm{e}^{-\int_r^t X_s \,\mathrm{d}s} \frac{1}{(j-1)!} \Big(\int_r^t Y_s X_s \,\mathrm{d}s\Big)^{j-1} \mathbb{1}_{t \geq T_{n-j}} \,\mathrm{d}r \prod_{k=1}^{n-j} Y_{T_k} \\
&= \mathrm{e}^{-\int_{]T_{n-j},t]} X_s \,\mathrm{d}s} \int_{]T_{n-j},t]} Y_r X_r \frac{1}{(j-1)!} \Big(\int_{]r,t]} Y_s X_s \,\mathrm{d}s\Big)^{j-1} \mathrm{d}r \;\mathbb{1}_{t \geq T_{n-j}} \prod_{k=1}^{n-j} Y_{T_k} \\
&= \mathrm{e}^{-\int_{T_{n-j}}^t X_s \,\mathrm{d}s} \frac{1}{j!} \Big(\int_{T_{n-j}}^t Y_s X_s \,\mathrm{d}s\Big)^j \mathbb{1}_{t \geq T_{n-j}} \prod_{k=1}^{n-j} Y_{T_k},
\end{aligned}$$

which concludes the induction proof.

Now, taking $j = n$ in (9) and summing over n yields

$$\sum_{n=0}^{\infty} \mathbb{E}\Big[\prod_{k=1}^{n} Y_{T_k} \mathbb{1}_{N_t = n} \,\Big|\, \mathcal{F}_0\Big] = \mathrm{e}^{-\int_0^t X_s \,\mathrm{d}s} \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\int_0^t Y_s X_s \,\mathrm{d}s\Big)^n$$

$$= \exp\Big(\int_0^t (Y_s - 1)\, X_s \,\mathrm{d}s\Big).$$

Hence, after applying tower property and monotone convergence

$$\mathbb{E}\big[Z_t\big] = \mathbb{E}\bigg[\exp\Big(-\int_0^t (Y_s - 1) X_s \,\mathrm{d}s\Big) \sum_{n=0}^{\infty} \mathbb{E}\bigg[\prod_{k=1}^{n} Y_{T_k}\, \mathbb{1}_{N_t = n} \,\bigg|\, \mathcal{F}_0\bigg]\bigg]$$
$$= \mathbb{E}\bigg[\exp\Big(-\int_0^t (Y_s - 1)\, X_s \,\mathrm{d}s\Big)\, \exp\Big(\int_0^t (Y_s - 1)\, X_s \,\mathrm{d}s\Big)\bigg] = 1.$$

□

4 Filtering Examples

As mentioned in the introduction, Theorem 3.4 is a crucial stepping stone in the reference probability approach to filtering with conditional Poisson processes. Here, we consider a conditional Poisson process N with intensity X, and we want to derive explicit formulas for conditional expectations of X given the information on N.

Zakai (1969) suggested deriving such formulas through a change of measure that makes the processes X and N independent. Brémaud (1981) observed that the stochastic exponential Z from Theorem 3.4 with $Y = 1/X$ (if it is a martingale) yields a new measure $\mathbb{Q}$ under which the conditional Poisson process N has constant intensity 1 on a given interval $[0, T]$. Hence, N is a Poisson process on $[0, T]$ under $\mathbb{Q}$ according to the characterisation of Poisson processes by Watanabe (1964). In particular, its increments are independent of past information. Because we consider the information on X to be part of the initial information $\mathcal{F}_0$ in the context of conditional Poisson processes, any increment of N and therefore N itself on $[0, T]$ is independent of X under $\mathbb{Q}$. Now, assuming that $1/X$ is well-defined, then Theorem 3.4 guarantees that Brémaud's assumptions hold and Zakai's change of measure exists on $[0, T]$.

The Kallianpur–Striebel formula (see Bain & Crisan (2009, Proposition 3.16)) allows us to rewrite any conditional expectation under $\mathbb{P}$ in terms of the conditional expectations under $\mathbb{Q}$. More precisely, let f be a positive Borel measurable function and $T > t \geq 0$, and note that Theorem 3.4 implies $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}|_{\mathcal{F}_t} = Z_t$, then

$$\mathbb{E}[f(X_t) \,|\, \mathcal{F}_t^N] = \frac{\mathbb{E}_{\mathbb{Q}}[f(X_t)\,\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} \,|\, \mathcal{F}_t^N]}{\mathbb{E}_{\mathbb{Q}}[\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q} \,|\, \mathcal{F}_t^N]} = \frac{\mathbb{E}_{\mathbb{Q}}[f(X_t)/Z_t \,|\, \mathcal{F}_t^N]}{\mathbb{E}_{\mathbb{Q}}[1/Z_t \,|\, \mathcal{F}_t^N]}.$$

Observe that $f(X_t)/Z_t$ and $1/Z_t$ are explicit functions of N and X only (see Theorem 3.4 with $Y = 1/X$). In particular, we can compute the conditional expectations on the right side of the above equation by integrating out X while treating N as a known fixed parameter because X and N are independent under $\mathbb{Q}$.

Moreover, integrating out X under $\mathbb{Q}$ is as easy as integrating out X under $\mathbb{P}$ because $\mathbb{Q} = \mathbb{P}$ on $\mathcal{F}_0$, which includes the information of X, by Theorem 3.4. In particular, we can express $\mathbb{E}[f(X_t) \,|\, \mathcal{F}_t^N]$ as a ratio of two explicit integrals with respect to the distribution of X in which N plays the role of a fixed parameter only.

The following two examples show what those ratios of explicit integrals look like.

Example 4.1 ($\mathcal{F}^N$-intensity). Let N be a conditional Poisson process with positive intensity X. Let $t \geq 0$ be a time where X is right-continuous and $\mathbb{E}[X_t] < \infty$.

It can be shown that

$$\mathbb{E}\big[X_t \,\big|\, \mathcal{F}_t^N\big] = \frac{\mathbb{E}_{\mathbb{Q}}[X_t \exp(\int_0^t 1 - X_s \,\mathrm{d}s) \prod_{s\in]0,t]}(1 + (X_s - 1)\Delta N_s) \,|\, \mathcal{F}_t^N]}{\mathbb{E}_{\mathbb{Q}}[\exp(\int_0^t (1 - X_s)\,\mathrm{d}s) \prod_{s\in]0,t]}(1 + (X_s - 1)\,\Delta N_s) \,|\, \mathcal{F}_t^N]}$$

$$= -\partial_{t+} \log\left(\mathbb{E}_{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^t X_s \,\mathrm{d}s} \prod_{s\in]0,t]} \big(1 + (X_s - 1)\,\Delta N_s\big) \,\middle|\, \mathcal{F}_t^N\right]\right),$$

in which ∂_{t+} is the right-derivative at t, and $\mathbb{Q}$ is such that N and X are independent and the distribution of X is the same as under $\mathbb{P}$.

Example 4.2 (Filtering compound Poisson processes from observations about the jump times). Let X be a compound Poisson process initially in $x_0 > 0$ with independent Poisson process M and positive jumps $(\xi_n)_{n=1}^{\infty}$, i.e. $X_t = x_0 + \sum_{k=0}^{M_t} \xi_k$ for $t \geq 0$.

We want to describe the distribution of X_t given the observation $\mathcal{F}^N$ of the conditional Poisson process N with intensity X and given the additional knowledge $\mathcal{F}^M$ of the jump times using the Laplace transform, i.e. $\mathbb{E}[\exp(\alpha X_t) \,|\, \mathcal{F}_t^N \vee \mathcal{F}_t^M]$ for $\alpha \in \mathbb{R}$.

It can be shown that

$$\mathbb{E}\Big[\exp\big(\alpha X_t\big) \,\Big|\, \mathcal{F}_t^N \vee \mathcal{F}_t^M\Big] = \frac{\Lambda_t^\alpha}{\Lambda_t^0} \quad \text{with}$$

$$\Lambda_t^\alpha = \int_{\mathbb{R}^m} \exp\big(\alpha X_t^j\big) \exp\left(\int_0^t (1 - X_s^j)\,\mathrm{d}s\right) \prod_{s\in]0,t]} \big(1 + (X_s^j - 1)\,\Delta N_s\big)\,\mathrm{d}\Gamma_m(j)\Big|_{m=M_t},$$

in which Γ_m is the joint distribution of $(\xi_k)_{k=1}^m$, and $X_s^j = x_0 + \sum_{k=1}^{M_s} j_k$.

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