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Mathematica Bohemica, Vol. 148 (2023), No. 3, 303–327

Persistent URL: `http://dml.cz/dmlcz/151762`

BOUNDEDNESS CRITERIA FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DELAY

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Received November 3, 2021. Published online August 3, 2022.
Communicated by Leonid Berezansky

Abstract. We consider certain class of second order nonlinear nonautonomous delay differential equations of the form

$$a(t)x'' + b(t)g(x,x') + c(t)h(x(t-r))m(x') = p(t,x,x')$$

and

$$(a(t)x')' + b(t)g(x,x') + c(t)h(x(t-r))m(x') = p(t,x,x'),$$

where a, b, c, g, h, m and p are real valued functions which depend at most on the arguments displayed explicitly and r is a positive constant. Different forms of the integral inequality method were used to investigate the boundedness of all solutions and their derivatives. Here, we do not require construction of the Lyapunov-Krasovskiĭ functional to establish our results. This work extends and improve on some results in the literature.

Keywords: boundedness; nonlinear; differential equation of third order; integral inequality

MSC 2020: 34C11, 34C12, 34K12

1. Introduction

The qualitative behaviour of solutions of differential equations with and without delay has been extensively studied by many researchers. The surge interest in this area in over five decades is evidenced by numerous research papers on the subject and lots of published books. For example, we refer to the works of Bellman and Cooke [6], Driver [14], Èl'sgol'ts [16], Èl'sgol'ts and Norkin [17], Gopalsamy [19], Hale [22], Kolmanovskii and Myshkis [25], Krasovskiĭ [27]. The expositions of Adams and Olutimo [1], Ademola et al. [2], Burton and Hatvani [11], Burton and Hering [12], Afuwape and Omeike [3], Gabsi et al. [18], Mahmoud and Tunç [30], Olutimo and

DOI: 10.21136/MB.2022.0166-21

Adams [34], Omeike [35], Omeike et al. [36], Remili and Beldjerd [40], [41], Tunç [43], [44], [46], [47], [49]–[51], Tunç and Erdur [52], Tunç and Tunç [53]–[55], Yao and Wang [60] among others are important contributions in this regard.

It is important to note that differential equations with delay play significant role in the research field of various applied sciences such as control theory, electrical networks, population dynamics, environmental science, biology and life science (see [15]). Several authors have considered work on second order nonlinear delay differential equations. For instance, in [63], Zhang considered the retarded Liènard equation

$$x'' + f(x)x' + g(x(t-h)) = 0$$

in which h is a nonnegative constant and g, f are continuous with $f(x) > 0$ for all $x \in \mathbb{R}$. The author obtained conditions for the boundedness and global asymptotic stability results. Furthermore, in [64], Zhang examined the same equation and gave results on the uniform boundedness, uniform ultimate boundedness and oscillation of solutions. Moreover, Peng (see [38]) studied the second order nonlinear system with delay

$$x''(t) + f(x(t), x'(t)) + g(x(t), x'(t))\psi(x(t-\tau)) = p(t),$$

where f, g, p are continuous functions, ψ is a differentiable function, τ is a positive constant, and gave four theorems on the stability of zero solution, the boundedness of solutions, the existence of periodic solutions, and the existence and uniqueness of stationary oscillation. Also, in [48], Tunç established some results for the stability and the boundedness of solutions of nonautonomous differential equations of second order with a variable deviating argument of the form

$$x''(t) + f(t, x(t), x'(t))x'(t) + b(t)g(x(t-\tau(t))) = q(t),$$

where $\tau(t)$ is variable deviating argument; f, b, g and q are continuous functions in their arguments on $\mathbb{R}^3$, $\mathbb{R}$, $\mathbb{R}$ and $[0, \infty)$, respectively.

Ogundare et al. in [32] considered second order nonlinear differential equations of the form

$$x'' + a(t)f(x, x') + g(x(t-\tau)) = p(t, x, x'),$$

where a, f, g and p are continuous functions that depend (at most) only on the arguments displayed explicitly and $\tau \in [0, h]$, $\tau > 0$. The authors obtained results for the global asymptotic stability, boundedness and ultimate boundedness of the solutions, respectively.

However, Athanassov (see [5]) considered the boundedness of all solutions and their first derivatives for the second order nonlinear differential equations

$$(a(t)x')' + b(t)f(x, x') + c(t)g(x)h(x') = p(t, x, x')$$

and

$$a(t)x'' + b(t)f(x,x') + c(t)g(x)h(x') = p(t,x,x'),$$

where a, b, c, f, g, h, and p are real valued functions which depend at most on the arguments displayed explicitly. These forms of nonlinear differential equations have been extensively investigated by several authors. Majority of the results obtained require the use of energy (Lyapunov) functions while few considered the use of integral inequalities as a viable tool in establishing boundedness results.

The motivation for this work comes from the work by Athanassov (see [5]), where the author made use of two forms of the second mean value theorem for integrals and Stieltjes integral inequalities to investigate the boundedness of a class of second order nonlinear nonautonomous differential equations. It can be observed that the investigation of qualitative behaviour in nonlinear differential equations with delay has been carried out by several authors (see Yao and Wang [60]) who made use of Lyapunov-Krasovskiĭ functionals to obtain their results. The challenge now is using the notions employed by Athanassov (see [5]) in the investigation of a certain class of second order nonlinear nonautonomous differential equations with delay in which to the best of our knowledge this approach is new.

In this paper, we consider the second order nonlinear nonautonomous differential equations with delay

$$a(t)x'' + b(t)g(x,x') + c(t)h(x(t-r))m(x') = p(t,x,x'), \tag{1}$$

$$(a(t)x')' + b(t)g(x,x') + c(t)h(x(t-r))m(x') = p(t,x,x'), \tag{2}$$

where a, b, c, g, h, m and p are real valued functions which depend at most on the arguments displayed explicitly and r is a positive constant. It is assumed that solutions of the class of delay differential equations being considered exist, a, b, c, g, h, m and p are continuous in their respective arguments (see Rao [39]). This work extends and improves on the paper by Athanassov (see [5]) and some references listed below. It also gives an alternative approach to the study of qualitative behaviour of solutions for a certain class of second order nonlinear delay differential equations.

2. Notation and preliminaries

First, we give some basic information for a general nonautonomous differential system with delay (see Burton [9], Burton and Markay [13], Tunç [45], see also Kolmanovskii and Myshkis [25], Kolmanovskii and Nosov [26], Krasovskiĭ [27] and Yoshizawa [61]).

Consider the general nonautonomous differential system with delay

$$z'(t) = f(t, z_t), \quad z_t(s) = z(t+s), \quad -r \leqslant s \leqslant 0, \quad t \geqslant 0, \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}, \tag{3}$$

where $f\colon [0,\infty) \times C_H \to \mathbb{R}^n$ is continuous and maps bounded sets into bounded sets, and $f(t,0) = 0$. Here, $(C, \|\cdot\|)$ is the Banach space of continuous functions $\varphi\colon [-r,0] \to \mathbb{R}^n$ with supremum norm, σ is a nonnegative constant, C_H is the open H-ball in $C_H := \{\varphi \in (C[-r,0], \mathbb{R}^n)\colon \|\varphi\| < H\}$.

Standard existence (see Burton [9]) shows that if $\varphi \in C_H$ and $t \geqslant 0$, then there is at least one continuous solution $z(t, t_0, \varphi)$ on $[t_0, t_0+\alpha)$ satisfying (3) for $t > t_0$, $z_t(t_0, \varphi) = \varphi$ and α being some positive constant; if there is a closed subset $B \subset C_H$ such that the solution remains in B, then $\alpha = \infty$. In addition, $|\cdot|$ denotes the norm in $\mathbb{R}^n$ with $|z| = \max\limits_{1 \leqslant i \leqslant n} |z_i|$.

However, in this paper we use the following notation: $\mathbb{R}$ is the real line, $\mathbb{R}$ and I are intervals $(0,\infty)$ and $[0,\infty)$, respectively, and $|\cdot|$ is the absolute value. Moreover, $C(X,\mathbb{R})$ and $C'(X,\mathbb{R})$ denote the sets of $\mathbb{R}$-valued functions defined on the set X that are continuous and continuously differentiable with respect to each variable, respectively. $L^1(X)$ is the set of Lebesgue integrable functions on X. It is assumed that all solutions of (1) and (2) are continuous (for instance, see [5], [57]).

The following lemmas are two forms of the second mean value theorem for integrals which will be useful in the proofs. For example, one can refer to Hildebrandt [23] and Athanassov [5].

Lemma 1. *If $u \in L^1[\alpha_1, \beta_1]$ and v is a positive, bounded and nonincreasing function on $[\alpha_1, \beta_1]$, then there is a number $\delta \in [\alpha_1, \beta_1]$ such that*

$$\int_{\alpha_1}^{\beta_1} u(\eta)v(\eta)d\eta = v(\alpha_1 + 0)\int_{\alpha_1}^{\delta} u(\eta)\,\mathrm{d}\eta.$$

Lemma 2. *If $u \in L^1[\alpha_1, \beta_1]$ and v is a positive, bounded and nondecreasing function on $[\alpha_1, \beta_1]$, then there is a number $\delta \in [\alpha_1, \beta_1]$ such that*

$$\int_{\alpha_1}^{\beta_1} u(\eta)v(\eta)\,\mathrm{d}\eta = v(\beta_1 - 0)\int_{\delta}^{\beta_1} u(\eta)\,\mathrm{d}\eta.$$

The following generalization of Gronwall's inequality for Riemann-Stieltjes integrals is a modification of a result given by Jones [24], see also Athanassov [5].

Lemma 3. *Let u, v, w be real valued functions defined and continuous on $[\alpha_1, \beta_1]$. Let u, v be nonnegative and let w be nondecreasing on $[\alpha_1, \beta_1]$. If*

$$u(t) \leqslant c + \int_{\alpha_1}^{t} u(\eta)v(\eta)\,\mathrm{d}w(\eta)$$

for some positive constant c, then

$$u(t) \leqslant c \exp\left(\int_{\alpha_1}^{t} v(\eta)\,\mathrm{d}w(\eta)\right) \quad \text{for all } t \in [\alpha_1, \beta_1].$$

Lemma 4. *If, for $t \in [\alpha_1, \beta_1]$, $u(t)$ is real valued, continuous, of bounded variation, and if $u(t) > 0$, then*

$$\int_{\alpha_1}^{\beta_1} \frac{1}{u(\eta)}\,\mathrm{d}u(\eta) = \log u(\beta_1) - \log u(\alpha_1).$$

3. Main results

The following are the basic assumptions used to formulate the results:

(i) $a, b, c \in C(I, \mathbb{R}^+)$;

(ii) $g \in C(\mathbb{R}^2, \mathbb{R})$, $h \in C(\mathbb{R}, \mathbb{R})$, $m \in C(\mathbb{R}, \mathbb{R}^+)$, $p \in C(I \times \mathbb{R}^2, \mathbb{R})$;

(iii) $g(x, y)y > 0$ for all $(x, y) \in \mathbb{R}^2$, $y \neq 0$;

(iv) $H(x) \to \infty$ as $|x| \to \infty$, where $H(x) = \int_0^x h(\tau)\,\mathrm{d}\tau \geqslant 0$;

(v) $M(y) \to \infty$ as $|y| \to \infty$, where $M(y) = \int_0^y (\tau/m(\tau))\,\mathrm{d}\tau$;

(vi) $\int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \beta$ for all $(x, y) \in \mathbb{R}^2$ and r, β are constants;

(vii) There is a nonnegative function $e(t) \in L^1(I)$ such that $|p(t, x, y)y| \leqslant e(t)m(y)$ for all $(t, x, y) \in I \times \mathbb{R}^2$.

(viii) There are positive constants α and k such that $y^2/m(y) \leqslant \alpha M(y)$ for all $|y| \geqslant k$ and a nonnegative function $e_1(t) \in L^1(I)$ such that $|p(t, x, y)| \leqslant e_1(t)$ for all $(t, x, y) \in I \times \mathbb{R}^2$.

Remark 5. The assumptions (i) and (ii) guarantee the local existence of solutions of (1) and (2), (iii) is standard in the case when $a(t) = b(t) = c(t) \equiv 1$, $m(x') \equiv 1$ and $h(x(t - r)) \equiv h(x)$, $t > r$ (see [4], [5], [8], [56]). The assumptions (iv) and (v) have been used by a number of papers to establish boundedness and continuability theorems (see [5], [10], [20], [28], [59]). In (vi), the double integrals are bounded by a constant. Moreover, the condition (vii) is a generalization of a condition by Legatos (see [5], [29]). The first part of (viii) is less restrictive than bounding m from above and below or asking $y^2/m(y) \leqslant \alpha M(y)$ for all y (see [5], [10], [33], [37]) and this does not contradict the assumption (v). The second part of (viii) generalizes a condition by Tejumola (see [5], [42]).

Now, we state and prove theorems on the boundedness of solutions for the nonlinear delay differential equation

$$x'' + b(t)g(x, x') + c(t)h(x(t-r))m(x') = p(t, x, x') \tag{4}$$

which is a special case of the equation (1) when $a(t) \equiv 1$. This can be re-written as: let $x' = y$, then

$$y' + b(t)g(x, y) + c(t)h(x)m(y) - c(t)m(y) \int_{t-r}^{t} h'(x(s))y(s)\,\mathrm{d}s = p(t, x, y). \tag{5}$$

The results obtained are then related to the equivalent system for the equation (1) as corollaries.

Theorem 6. *Suppose that the conditions* (i)–(vi) hold and $c(t)$ is nondecreasing on I. Then any solution $x(t)$ of (5) is bounded. If, in addition, $c(t)$ is bounded from above on I, then $y(t)$ is also bounded.

P r o o f. Let $x(t)$ and $y(t)$ be solutions defined on $[0, t]$, respectively. Multiplying (5) by $y(t)/(c(t)m(y(t)))$ and integrating both the sides of the resulting equation from 0 to t, we have

$$\begin{aligned} \int_0^t \frac{y(\tau)y'(\tau)}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{b(\tau)g(x(\tau), y(\tau))y(\tau)}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau. \end{aligned}$$

The integral in the second term on the left is nonnegative because of (i)–(iii) and by Lemma 1 it follows that there is $\delta \in [0, t]$ such that

$$\begin{aligned} \frac{1}{c(0)} \int_0^\delta \frac{y(\tau)}{m(y(\tau))y'(\tau)}\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\ \leqslant \frac{1}{c(0)} \int_0^\delta \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau. \end{aligned}$$

Applying (iv)–(vii), then

$$\frac{1}{c(0)}(M(y(\delta)) - M(y(0))) + H(x(t)) - H(x(0)) - \beta \leqslant \frac{1}{c(0)} \int_0^\infty e(\tau)\,\mathrm{d}\tau.$$

Moreover, (i) leads to the estimate

$$H(x(t)) \leqslant H(x(t)) + \frac{1}{c(0)}M(y(\delta)) \leqslant H(x(0)) + \beta + \frac{1}{c(0)}\Big(M(y(0)) + \int_0^\infty e(\tau)\,\mathrm{d}\tau\Big).$$

The right-hand side of the last inequality is a constant independent of t, say K, and therefore (iv) implies that $x(t)$ is bounded on I.

We now suppose that $c(t) \leqslant c_0$ on I. Substitute $x(t)$ and $y(t)$ into (5), multiply both the sides by $y(t)/m(y(t))$ and integrate from 0 to t. By (i)–(iii), we obtain

$$\int_0^t \frac{y(\tau)y'(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{b(\tau)g(x(\tau),y(\tau))y(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

The integral in the second term on the left is nonnegative because of (i)–(iii) and by Lemma 2 there exists $\delta \in [0,t]$ such that

$$\int_0^t \frac{y(\tau)}{m(y(\tau))y'(\tau)}\,\mathrm{d}\tau + c(t)\int_\delta^t h(x(\tau))y(\tau)\,\mathrm{d}\tau - c(t)\int_\delta^t\int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\ \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

Applying (iv)–(vii), then

$$M(y(t)) - M(y(0)) + c(t)(H(x(t)) - H(x(\delta))) - c(t)\beta \leqslant \int_0^\infty e(\tau)\,\mathrm{d}\tau.$$

Since $c(t)H(x(t))$ is nonnegative on I, we have

$$\begin{aligned} M(y(t)) &\leqslant M(y(t)) + c(t)H(x(t)) \\ &\leqslant M(y(0)) + c(t)H(x(\delta)) + c(t)\beta + \int_0^\infty e(\tau)\,\mathrm{d}\tau \\ &\leqslant M(y(0)) + c_0(K+\beta) + \int_0^\infty e(\tau)\,\mathrm{d}\tau = L, \end{aligned}$$

a constant independent of t. Hence, (v) implies that $y(t)$ is bounded on I. □

Remark 7. The equation (4) considered above improved on the equation considered by Athanassov (see [5]) for the case in which $a(t) \equiv 1$ and $h(x(t-\tau)) = h(x)$, where $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $t > \tau$, and $\tau > 0$ is a constant.

As a consequence of Theorem 6 we have the following result.

An equivalent system of (1) becomes $x' = y$, then

$$a(t)y' + b(t)g(x,y) + c(t)h(x)m(y) - c(t)m(y)\int_{t-r}^t h'(x(s))y(s)\,\mathrm{d}s = p(t,x,y). \tag{6}$$

Corollary 8. *Suppose that the conditions* (i)–(vii) *hold and* $a(t)$ *is bounded away from zero on* I. *If the quotient* $c(t)/a(t)$ *is nondecreasing on* I, *then any solution* $x(t)$ *of* (6) *is bounded. If, in addition,* $c(t)/a(t)$ *is bounded from above on* I *then* $y(t)$ *is also bounded.*

P r o o f. Multiply both the sides of (6) by $a^{-1}(t)y(t)/(c(t)m(y(t)))$, and then by $a^{-1}(t)y(t)/m(y(t))$, respectively. So, the conclusion follows by Theorem 6. We have that

$$H(x(\delta)) \leqslant H(x(0)) + \frac{a(0)}{c(0)} M(y(0)) + \beta + \frac{1}{c(0)} \int_0^\infty e(\tau)\,\mathrm{d}\tau = W_1,$$

where W_1 is a constant independent of t and by (iv) this implies that $x(t)$ is bounded on I. Further,

$$M(y(t)) \leqslant M(y(0)) + \frac{c_0}{a_0}(W_1 + \beta) + \frac{1}{a_0} \int_0^\infty e(\tau)\,\mathrm{d}\tau = W_2,$$

where W_2 is a constant independent of t. Hence, (v) implies that $y(t)$ is bounded on I. □

Theorem 9. *Suppose that the conditions* (i)–(vi) *and* (viii) *hold and let* $c(t)$ *be nonincreasing on* I. *Then, for any solution* $x(t)$ *of* (5), $y(t)$ *is bounded. If, in addition,* $c(t)$ *is bounded away from zero on* I, *then* $x(t)$ *is also bounded.*

P r o o f. Multiplying (5) by $y(t)/m(y(t))$, integrating both the sides of the resulting equation from 0 to t, and by the assumptions (i)–(iii), we obtain

$$\int_0^t \frac{y(\tau)y'(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\ \leqslant \int_0^t \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

By Lemma 1 it follows that there is $\delta \in [0, t]$ such that

$$\int_0^t \frac{y(\tau)}{m(y(\tau))y'(\tau)}\,\mathrm{d}\tau + c(0)\int_0^\delta h(x(\tau))y(\tau)\,\mathrm{d}\tau - c(0)\int_0^\delta \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\ \leqslant \int_0^t \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

If $|y| \leqslant \max\{a, k\}$, $y^2/m(y) \leqslant d_1$ for some $d_1 > 0$, so $y^2/m(y) \leqslant d_1 + \alpha M(y)$ for all y. Also, for $|y| \leqslant \max\{1, k\}$, $|y|/m(y) \leqslant d_2$, $d_2 > 0$ and for $|y| \geqslant \max\{1, k\}$, $|y|/m(y) \leqslant y^2/m(y)$, so $|y|/m(y) \leqslant d_2 \pm y^2/m(y) \leqslant d_1 + d_2 + \alpha M(y) = D + \alpha M(y)$ for all y.

Thus, using (vii), (viii) and Lemma 1, there exists $\delta \in [0,t]$ such that

$$\begin{aligned}
M(y(t)) - M(y(0)) + c(0)H(x(\delta)) - c(0)H(x(0)) - c(0)\beta \\
\leqslant (D + \alpha M(y)) \int_0^\infty e_1(\tau)\,\mathrm{d}\tau, \\
M(y(t)) \leqslant M(y(t)) + c(0)H(x(\delta)) \\
\leqslant M(y(0)) + c(0)H(x(0)) + c(0)\beta \\
+ D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau + \alpha \int_0^\infty e_1(\tau)M(y(\tau))\,\mathrm{d}\tau, \\
M(y(t)) \leqslant K_1 + \alpha \int_0^\infty e_1(\tau)M(y(\tau))\,\mathrm{d}\tau,
\end{aligned}$$

where $K_1 = M(y(0)) + c(0)H(x(0)) + c(0)\beta + D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau$ is a nonnegative constant. By Gronwall's inequality, it follows that

$$M(y(t)) \leqslant K_1 \exp\Big(\alpha \int_0^\infty e_1(\tau)\,\mathrm{d}\tau\Big) = L_1,$$

a constant independent of t. Thus the condition (v) implies that $y(t)$ is bounded on I.

We now suppose that $c(t) \geqslant c_0 > 0$ on I. Multiply (5) by $y(t)/(c(t)m(y(t)))$ and integrate from 0 to t. By (i)–(iii) and Lemma 2, it follows that there exists $\delta \in [0,t]$ such that

$$\begin{aligned}
\frac{1}{c(t)} \int_\delta^t \frac{y(\tau)}{m(y(\tau))y'(\tau)}\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\
\leqslant \frac{1}{c(t)} \int_\delta^t \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.
\end{aligned}$$

Applying (iv)–(vi) and (viii), then

$$\begin{aligned}
\frac{M(y(t)) - M(y(\delta))}{c(t)} + H(x(t)) - H(x(0)) - \beta \leqslant \frac{D + \alpha M(y)}{c(t)} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau, \\
H(x(t)) \leqslant \frac{M(y(t))}{c_0} + H(x(t)) \leqslant \frac{M(y(\delta))}{c_0} + H(x(0)) + \beta + \frac{\alpha L_1 + D}{c_0} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Since $1/c_0 M(y(t))$ is nonnegative on I, then

$$H(x(t)) \leqslant H(x(0)) + \frac{1}{c_0}\bigg(L_1 + \beta + (\alpha L_1 + D)\int_0^\infty e_1(\tau)\,\mathrm{d}\tau\bigg) = L_2,$$

a constant independent of t. Hence, (iv) implies that $x(t)$ is bounded on I. $\square$

Corollary 10. *Suppose that the conditions* (i)–(vi) *and* (viii) *hold and let $a(t)$ be bounded away from zero on I. If the quotient $c(t)/a(t)$ is nonincreasing on I, then for any solution $x(t)$ of* (6), *$y(t)$ is bounded. If, in addition, $c(t)/a(t)$ is bounded from below on I, then $x(t)$ is bounded.*

Proof. Multiply both the sides of (6) by $a^{-1}(t)y(t)/m(y(t))$, and then by $a^{-1}(t)y(t)/c(t)m(y(t))$, respectively. By the proof of Theorem 9, the following results can be obtained:

$$M(y(t)) \leqslant W_3 \exp\biggl(\alpha \int_0^\infty e_1(\tau)\,\mathrm{d}\tau\biggr) = W_4,$$

a constant independent of t, where

$$W_3 = M(y(0)) + \frac{c(0)}{a(0)}(H(x(0)) + \beta) + \frac{D}{a(0)} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau$$

is a nonnegative contant. Thus, the condition (v) implies that $y(t)$ is bounded on I. Now, suppose that $a(t)c(t) \geqslant u_0 > 0$, then for the boundedness of $x(t)$, by the assumption (iv), we have

$$H(x(t)) \leqslant H(x(\delta)) + \frac{a_0}{u_0}\biggl(W_4 + \beta + (\alpha W_4 + D) \int_0^\infty e_1(\tau)\,\mathrm{d}\tau\biggr) = W_5,$$

a constant independent of t. $\square$

We now consider the boundedness of solutions for the equation (2). We construct the equivalent system as: let $x' = y$,

$$\begin{aligned} a'(t)y + a(t)y' + b(t)g(x,y) + c(t)h(x)m(y)& \\ - c(t)m(y) \int_{t-r}^t h'(x(s))y(s)\,\mathrm{d}s &= p(t,x,y). \end{aligned} \tag{7}$$

The proof of the following theorem resembles that of Theorem 9.

Theorem 11. *Suppose that the assumptions* (i)–(vi) *and* (viii) *hold and let $c(t)$ be nonincreasing on I. If $a(t) \in C^1(I,\mathbb{R}^+)$, $a'(t) \leqslant 0$, and if $a(t)$ is bounded away from zero on I, then for any solutions $x(t)$ of* (7), *$y(t)$ is bounded. If, in addition, $c(t)$ is bounded away from zero on I, then $x(t)$ is bounded.*

Proof. Multiplying (7) by $y(t)/m(y(t))$ and integrating from 0 to t, we have

$$\begin{aligned} \int_0^t \frac{a'(\tau)(y(\tau))^2}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{a(\tau)y(\tau)y'(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{b(\tau)g(x(\tau),y(\tau))y(\tau)}{m(y(\tau))}\,\mathrm{d}\tau& \\ + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau& \\ \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.& \end{aligned}$$

Considering the assumptions (i)–(iii), we have

$$\int_0^t \frac{a'(\tau)(y(\tau))^2}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{a(\tau)y(\tau)y'(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

As in the proof of Theorem 9, there are $d_1 > 0$ and $d_2 > 0$ such that $y^2/m(y) \leqslant d_1 + \alpha M(y)$ and $|y|/m(y) \leqslant d_1 + d_2 + \alpha M(y)$ for all y. Thus, using the supposition that $a'(t) \leqslant 0$ and (viii), we obtain

$$(d_1 + \alpha M(y)) \int_0^t a'(\tau)\,\mathrm{d}\tau + \int_0^t a(\tau)(M(y(\tau))'\,\mathrm{d}\tau + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant (D + \alpha M(y)) \int_0^\infty e_1(\tau)\,\mathrm{d}\tau. \tag{8}$$

Applying integration by parts on the second integral and using Lemma 1 on the third and fourth integrals of the inequality (8), we obtain

$$d_1 \int_0^t a'(\tau)\,\mathrm{d}\tau + \alpha \int_0^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + a(t)M(y(t)) - a(0)M(y(0)) \\ - \int_0^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + c(0)\int_0^\delta h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - c(0)\int_0^\delta \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\ \leqslant D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau + \alpha \int_0^\infty e_1(\tau)M(y(\tau))\,\mathrm{d}\tau.$$

Then, by the assumption (vi), we have

$$d_1 a(t) - d_1 a(0) + \alpha \int_0^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + a(t)M(y(t)) - a(0)M(y(0)) \\ - \int_0^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + c(0)H(x(\delta)) - c(0)H(x(0)) - c(0)\beta \\ \leqslant D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau + \alpha \int_0^\infty e_1(\tau)M(y(\tau))\,\mathrm{d}\tau.$$

So

$$a(t)M(y(t)) \leqslant a(t)M(y(t)) + c(0)H(x(\delta)) + d_1 a(t) \\ \leqslant R + (1-\alpha)\int_0^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + \alpha \int_0^\infty e_1(\tau)M(y(\tau))\,\mathrm{d}\tau,$$

where

$$R = d_1 a(0) + a(0)M(y(0)) + c(0)H(x(0)) + c(0)\beta + D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau.$$

If $\alpha \leqslant 1$ then by Gronwall's inequality we obtain

$$M(y(t)) \leqslant \frac{R}{a_0} \exp\left(\frac{\alpha}{a_0} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau\right) = V_1$$

and if $\alpha \geqslant 1$ (which implies $1-\alpha \leqslant 0$), this gives

$$M(y(t)) \leqslant \frac{R}{a_0} \exp \frac{(\alpha-1)a(0)}{a_0} \exp\left(\frac{\alpha}{a_0} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau\right) = V_2,$$

where a_0 is the lower bound (positive) of $a(t)$ on I. Thus (v) implies that $y(t)$ is bounded on I.

Since there is $c_0 > 0$ such that $c(t) \geqslant c_0$ on I. Multiplying (7) by $y(t)/(c(t)m(y(t)))$, integrating from 0 to t, using the assumptions (i)–(iii) and (viii), and applying Lemma 2, we have

$$\begin{aligned}\int_0^t \frac{a'(\tau)(y(\tau))^2}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{a(\tau)y(\tau)y'(\tau)}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau.\end{aligned}$$

This implies

$$\begin{aligned}\frac{d_1+\alpha M(y)}{c(t)} \int_\delta^t a'(\tau)\,\mathrm{d}\tau + \frac{1}{c(t)} \int_\delta^t a(\tau)(M(y(\tau)))'\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \frac{D+\alpha M(y)}{c(t)} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Applying integration by parts on the second integral and the conditions (iv)–(vi) on the above inequality, we have

$$\begin{aligned}\frac{d_1}{c(t)} \int_\delta^t a'(\tau)\,\mathrm{d}\tau + \frac{\alpha}{c(t)} \int_\delta^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + \frac{a(t)}{c(t)}M(y(t)) - \frac{a(\delta)}{c(t)}M(y(\delta)) \\ - \frac{1}{c(t)} \int_\delta^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + H(x(t)) - H(x(0)) - \beta \\ \leqslant \frac{D+\alpha M(y)}{c(t)} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Thus,

$$\begin{aligned}H(x(t)) \leqslant H(x(0)) - \frac{a(t)}{c(t)}M(y(t)) + \frac{a(\delta)}{c(t)}M(y(\delta)) - \frac{d_1}{c(t)} \int_\delta^t a'(\tau)\,\mathrm{d}\tau \\ - \frac{\alpha}{c(t)} \int_\delta^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau + \frac{1}{c(t)} \int_\delta^t a'(\tau)M(y(\tau))\,\mathrm{d}\tau \\ + \beta + \frac{D+\alpha M(y)}{c(t)} \int_0^\infty e_1(\tau)\,\mathrm{d}\tau,\end{aligned}$$

where $\delta \in [0,t]$. Put $V = \max(V_1, V_2)$ and then, since $M(y(t)) \leqslant V$ on I, we obtain

$$H(x(t)) \leqslant H(x(0)) - \frac{1}{c(t)}(d_1 + \alpha V - V)\int_\delta^t a'(\tau)\,\mathrm{d}\tau + \beta + \frac{D + \alpha V}{c(t)}\int_0^\infty e_1(\tau)\,\mathrm{d}\tau.$$

Hence, with $c(t) \geqslant c_0$ such that $c_0 > 0$, then

$$H(x(t)) \leqslant H(x(0)) + V\frac{a(\delta)}{c_0}\Big(\frac{d_1}{V} + \alpha - 1\Big) + \beta + \frac{1}{c_0}(D + \alpha V)\int_0^\infty e_1(\tau)\,\mathrm{d}\tau.$$

Therefore, $H(x(t))$ is bounded on I and (iv) implies that $x(t)$ is bounded. This completes the proof. □

In the following two theorems, by using Riemann-Stieltjes integrals and examining the quotients $a(t)/c(t)$ and $c(t)/a(t)$, the boundedness results for the solutions of (7) are obtained as follows.

Theorem 12. *Suppose that the conditions* (i)–(vii) *hold. If* $a(t) \in C^1(I, \mathbb{R}^+)$, $a'(t) \geqslant 0$, *and the quotient* $a(t)/c(t)$ *is nondecreasing and bounded from above on* I, *then any solution* $x(t)$ *of* (7), *along with its derivative* $y(t)$, *is bounded for all* $t \in I$.

Proof. Multiplying (7) by $y(t)/(c(t)m(y(t)))$, integrating both the sides of the equation from zero to some $t \geqslant 0$, and using (i)–(iii), we have

$$\begin{aligned}\int_0^t \frac{a'(\tau)(y(\tau))^2}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{a(\tau)y(\tau)y'(\tau)}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau, x(\tau), y(\tau))y(\tau)|}{c(\tau)m(y(\tau))}\,\mathrm{d}\tau.\end{aligned}$$

Applying (iv), (vii), and the fact that $a'(t) \geqslant 0$, then

$$\begin{aligned}\int_0^t \frac{a(\tau)}{c(\tau)}(M(y(\tau)))'\,\mathrm{d}\tau + \int_0^t h(x(\tau))y(\tau)\,\mathrm{d}\tau \\ - \int_0^t \int_{t-r}^t h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \leqslant \int_0^t \frac{e(\tau)}{c(\tau)}\,\mathrm{d}\tau.\end{aligned} \tag{9}$$

By the assumed monotonicity of the quotient $a(t)/c(t)$ we can conclude that $a(t)/c(t)$ is of bounded variation on $[0,t]$. Thus, by the theorem of reduction of a Riemann-Stieltjes integral to a Riemann integral for the first term in the inequality (9), and applying the assumptions (iv) to (vii) and Lemma 1 where necessary, we obtain

$$\int_0^t \frac{a(\tau)}{c(\tau)}\,\mathrm{d}M(y(\tau)) + H(x(t)) - H(x(0)) - \beta \leqslant \frac{1}{c(0)}\int_0^\infty e(\tau)\,\mathrm{d}\tau.$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\frac{a(t)}{c(t)}M(y(t))-\frac{a(0)}{c(0)}M(y(0))-\int_0^t M(y(\tau))\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big)+H(x(t))-H(x(0))-\beta \leqslant \frac{1}{c(0)}\int_0^\infty e(\tau)\,\mathrm{d}\tau.$$

Then the inequality above becomes

$$\frac{a(t)}{c(t)}M(y(t))+H(x(t))\leqslant N+\int_0^t M(y(\tau))\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big), \tag{10}$$

where $N=H(x(0))+(a(0)/c(0))M(y(0))+\beta+(1/c(0))\int_0^\infty e(\tau)\,\mathrm{d}\tau$.

Since $H(x(t))\geqslant 0$ on I, we have

$$\frac{a(t)}{c(t)}M(y(t))\leqslant N+\int_0^t \frac{a(\tau)}{c(\tau)}\frac{c(\tau)}{a(\tau)}M(y(\tau))\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big).$$

By Lemma 4, we obtain

$$\frac{a(t)}{c(t)}M(y(t))\leqslant N\exp\Big(\int_0^t \frac{1}{(a(\tau)/c(\tau))}\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big)\Big)\leqslant N\exp\Big(\log\frac{a(t)}{c(t)}-\log\frac{a(0)}{c(0)}\Big)$$
$$\leqslant N\exp\Big(\ln\frac{(a(t)/c(t))}{a(0)/c(0)}\Big)$$

which implies that

$$\frac{a(t)}{c(t)}M(y(t))\leqslant N\frac{a(t)/c(t)}{a(0)/c(0)}.$$

Hence,

$$M(y(t))\leqslant N\frac{c(0)}{a(0)} \tag{11}$$

and by assumption (v) this implies that $y(t)$ is bounded on I. Since $M(y(t))\geqslant 0$ and $a(t)/c(t)$ is bounded from above on I, we have from (10) that

$$H(x(t))\leqslant N+\int_0^t M(y(\tau))\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big). \tag{12}$$

Now, substituting the inequality (11) in the integral in (12), we have

$$H(x(t))\leqslant N+\int_0^t N\frac{c(0)}{a(0)}\,\mathrm{d}\Big(\frac{a(\tau)}{c(\tau)}\Big)\leqslant N+N\frac{c(0)}{a(0)}\Big(\frac{a(t)}{c(t)}-\frac{a(0)}{c(0)}\Big).$$

By assumption, there exists $r_0>0$ such that $a(t)/c(t)\leqslant r_0$ for all $t\in I$. Then $1/c(t)\leqslant r$ on I, where $r=r_0/a(0)$. We have

$$H(x(t))\leqslant Nr_0\frac{c(0)}{a(0)}.$$

Therefore, (iv) implies that $x(t)$ is bounded on I and this completes the proof. □

Theorem 13. *Suppose that the conditions* (i)–(vii) *hold. If* $a(t)\in C'(I,\mathbb{R}^+)$, $a'(t)\geqslant 0$, *and the quotient* $c(t)/a(t)$ *is nondecreasing and bounded above on* I, *then any solution* $x(t)$ *of* (7) *along with its derivative* $y(t)$ *is bounded for all* $t\in I$.

Proof. Multiplying (7) by $y(t)/(a(t)m(y(t)))$, integrating both sides of the equation from zero to some $t\geqslant 0$, using (i)–(iii) as in the proof of Theorem 12, and since $a'(t)\geqslant 0$, we have

$$
\begin{aligned}
(13)\quad \int_0^t \frac{y(\tau)y'(\tau)}{m(y(\tau))}\,\mathrm{d}\tau + \int_0^t \frac{c(\tau)}{a(\tau)} h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t\int_{t-r}^t \frac{c(\tau)}{a(\tau)} h'(x(s))y(s)y(\tau)\,\mathrm{d}s\,\mathrm{d}\tau \\
\leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{a(\tau)m(y(\tau))}\,\mathrm{d}\tau.
\end{aligned}
$$

Applying the assumptions (iv) to (vii) and Lemma 1 where necessary in inequality (13), we obtain

$$
(14)\qquad M(y(t))-M(y(0))+\int_0^t \frac{c(\tau)}{a(\tau)}\,\mathrm{d}H(x(\tau)) - \frac{c(0)}{a(0)}\beta \leqslant \frac{1}{a(0)}\int_0^t e(\tau)\,\mathrm{d}\tau.
$$

Following an argument similar to that used in the proof of Theorem 12, we have that $c(\tau)/a(\tau)$ is Riemann-Stieltjes integrable with respect to $H(x(\tau))$ on [0,t]. That is,

$$
\int_0^t \frac{c(\tau)}{a(\tau)} h(x(\tau))y(\tau)\,\mathrm{d}\tau = \int_0^t \frac{c(\tau)}{a(\tau)}\,\mathrm{d}H(x(\tau)).
$$

Hence, the integral on the left above is a Riemann-Stieltjes integral. As a consequence of this, using the integration by parts formula for the Riemann-Stieltjes integral, the inequality (14) becomes

$$
\begin{aligned}
M(y(t))-M(y(0))+\frac{c(t)}{a(t)}H(x(t)) - \frac{c(0)}{a(0)}H(x(0)) \\
-\int_0^t H(x(\tau))\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big) - \frac{c(0)}{a(0)}\beta \leqslant \frac{1}{a(0)}\int_0^t e(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Now

$$
\frac{c(t)}{a(t)}H(x(t)) \leqslant M(y(t)) + \frac{c(t)}{a(t)}H(x(t)) \leqslant Q_1 + \int_0^t H(x(\tau))\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big),
$$

where $Q_1 = M(y(0)) + (c(0)/a(0))H(x(0)) + (c(0)/a(0))\beta + (1/a(0))\int_0^\infty e(\tau)\,\mathrm{d}\tau$.

It then follows that

$$
\frac{c(t)}{a(t)}H(x(t)) \leqslant Q_1 + \int_0^t \frac{a(\tau)}{c(\tau)}\frac{c(\tau)}{a(\tau)} H(x(\tau))\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big).
$$

By Lemma 3, we have

$$\frac{c(t)}{a(t)}H(x(t)) \leqslant Q_1 \exp\left(\int_0^t \frac{a(\tau)}{c(\tau)}\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big)\right)$$

and by Lemma 4, we obtain

$$\begin{aligned}\frac{c(t)}{a(t)}H(x(t)) &\leqslant Q_1 \exp\left(\int_0^t \frac{1}{c(\tau)/a(\tau)}\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big)\right) \leqslant Q_1 \exp\Big(\log\frac{c(t)/a(t)}{c(0)/a(0)}\Big),\\ H(x(t)) &\leqslant Q_1\frac{a(0)}{c(0)}\end{aligned}$$

on I. Then by (iv), this implies the boundedness of $x(t)$.

By hypothesis, $(c(t)/a(t)) \leqslant \varrho$ for some $\varrho > 0$ and for all $t \in I$, then

$$\begin{aligned}M(y(t)) &\leqslant Q_1 + \int_0^t H(x(\tau))\,\mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big) \leqslant Q_1 + Q_1\frac{a(0)}{c(0)}\int_0^t \mathrm{d}\Big(\frac{c(\tau)}{a(\tau)}\Big)\\ &\leqslant Q_1 + Q_1\frac{a(0)}{c(0)}\Big(\frac{c(t)}{a(t)} - \frac{c(0)}{a(0)}\Big).\end{aligned}$$

Thus,

$$M(y(t)) \leqslant Q_1\varrho\frac{a(0)}{c(0)}.$$

□

Remark 14. A number of papers have dealt with the boundedness of solutions of the form (7) for the case $h(x(t-r)) = h(x)$, where $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $t > r$, $r > 0$ is a constant. For instance see Graef and Spikes [21], Nápoles [31] and Zarghamee and Mehri [62]. Theorem 12 and 13 generalized the corresponding results in Athanassov [5].

Throughout the remainder of this study, we now replace the monoticity conditions of $a(t)$ and $c(t)$ by integral conditions on the derivatives $a'(t)$, $c'(t)$ and $(c(t)/a(t))'$.

Theorem 15. *Let assumptions* (i)–(vii) *hold and let* $c(t) \in C'(I, \mathbb{R}^+)$ *be bounded away from zero and* $|c'(t)| \in L^1(I)$. *Then both the solution* $x(t)$ *of* (5) *and its derivative* $y(t)$ *are bounded.*

Proof. Multiplying (5) by $y(t)/m(y(t))$, integrating from 0 to t and using (i)–(vii) with the application of Lemma 2 where necessary, we have

$$M(y(t)) - M(y(0)) + \int_0^t c(\tau)\,\mathrm{d}H(x(\tau)) - c(0)\beta \leqslant \int_0^t e(\tau)\,\mathrm{d}\tau. \tag{15}$$

Using integration by parts at the third term of inequality (15), we obtain

$$
\begin{aligned}
(16) \qquad & M(y(t)) - M(y(0)) + c(t)H(x(t)) - c(0)H(x(0)) \\
& \quad - \int_0^t H(x(\tau))\,\mathrm{d}c(\tau) - c(0)\beta \leqslant \int_0^t e(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Then

$$
\begin{aligned}
(17) \qquad c(t)H(x(t)) &\leqslant M(y(t)) + c(t)H(x(t)) \\
&\leqslant M(y(0)) + c(0)H(x(0)) + c(0)\beta \\
&\quad + \int_0^\infty e(\tau)\,\mathrm{d}\tau + \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau \\
&\leqslant Q_2 + \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau,
\end{aligned}
$$

where $Q_2 = M(y(0)) + c(0)H(x(0)) + c(0)\beta + \int_0^\infty e(\tau)\,\mathrm{d}\tau$. Therefore,

$$
H(x(t)) \leqslant \frac{Q_2}{c(t)} + \frac{1}{c(t)} \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau.
$$

By the hypothesis $c(t) \geqslant c_0$ for some $c_0 > 0$ and for all $t \in I$, and by Gronwall's inequality, we obtain

$$
H(x(t)) \leqslant \frac{Q_2}{c_0} \exp\Bigl(\int_0^\infty |c'(\tau)| \frac{\mathrm{d}\tau}{c_0}\Bigr) = \Phi.
$$

So, by assumption (iv) it implies that $x(t)$ is bounded.

Furthermore, following from the inequality (16), it is clear that

$$
M(y(t)) \leqslant Q_2 + \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau \leqslant Q_2 + \Phi \int_0^t |c'(\tau)|\,\mathrm{d}\tau
$$

and by (v), this implies that $y(t)$ is bounded. □

Corollary 16. *Suppose that assumptions* (i)–(vii) *hold. If* $a(t)$ *and* $c(t)/a(t) \in C'(I, \mathbb{R}^+)$ *are bounded away from zero and* $|(c(t)/a(t))'| \in L^1(I)$*, then any solution* $x(t)$ *of* (5) *along with its derivative* $y(t)$ *is bounded.*

Theorem 17. *Let the conditions* (i)–(vii) *hold and let* $a(t)$ *be nonincreasing on* I*. If* $c(t) \in C'(I, \mathbb{R}^+)$ *is bounded away from zero and* $|c'(t)| \in L^1(I)$*, then any solution* $x(t)$ *of* (6) *is bounded. If, in addition,* $a(t)$ *is bounded away from zero, then* $y(t)$ *is also bounded.*

P r o o f. Multiply (6) by $y(t)/m(y(t))$ and integrate from 0 to t. Then, we have

$$\int_0^t \frac{a(\tau)y(\tau)y'}{m(x'(\tau))}\,\mathrm{d}\tau + \int_0^t c(\tau)h(x(\tau))y(\tau)\,\mathrm{d}\tau - \int_0^t \int_{t-r}^t c(\tau)h'(x(s))y(s)y(\tau)\,\mathrm{d}\tau \leqslant \int_0^t \frac{|p(\tau,x(\tau),y(\tau))y(\tau)|}{m(y(\tau))}\,\mathrm{d}\tau.$$

Applying the assumptions (i) to (iii), (vi), (vii) and using Lemma 1 where necessary, we have

$$a(0)M(y(\delta)) - a(0)M(y(0)) + \int_0^t c(\tau)\,\mathrm{d}H(x(\tau)) - c(0)\beta \leqslant \int_0^t e(\tau)\,\mathrm{d}\tau.$$

Employing integration by parts at the third term of the above inequality, then

$$a(0)M(y(\delta)) + c(t)H(x(t)) \leqslant Q_3 + \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau,$$

where $Q_3 = a(0)M(y(0)) + c(0)H(x(0)) + c(0)\beta + \int_0^\infty e(\tau)\,\mathrm{d}\tau$ and $\delta = [0,t]$.

Now

$$c(t)H(x(t)) \leqslant c(t)H(x(t)) + a(0)M(y(\delta)) \leqslant Q_3 + \int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau.$$

Since $c(t) \geqslant c_0$ for some $c_0 > 0$ and all $t \in I$, we have

$$H(x(t)) \leqslant \frac{Q_3}{c_0} + \frac{1}{c_0}\int_0^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau.$$

By Gronwall's inequality and the assumption (iv), it follows that $x(t)$ is bounded on I.

Suppose now that $a(t) \geqslant a_0$ for some $a_0 > 0$ and all $t \in I$. Multiplying (6) by $y(t)/m(y(t))$, integrating from 0 to t and applying assumptions (i) to (vii) including Lemma 2, we have

$$M(y(t)) - M(y(0)) + \frac{1}{a(t)}\int_\delta^t c(\tau)\,\mathrm{d}H(x(\tau)) - \frac{1}{a(t)}\int_\delta^t c(\tau)\beta\,\mathrm{d}\tau \leqslant \frac{1}{a(t)}\int_0^t e(\tau)\,\mathrm{d}\tau,$$

where $\delta \in [0,t]$.

Using integration by parts at the first integral above, we have

$$M(y(t)) - M(y(0)) + \frac{c(t)}{a(t)}H(x(t)) - \frac{c(\delta)}{a(t)}H(x(\delta)) + \frac{c(t)}{a(t)}\beta - \frac{c(\delta)}{a(\delta)}\beta \leqslant \frac{1}{a(t)}\int_\delta^t |c'(\tau)|H(x(\tau))\,\mathrm{d}\tau + \frac{1}{a(t)}\int_0^t e(\tau)\,\mathrm{d}\tau.$$

From the condition on $c(t)$ we have that $c(t)$ tends to a positive limit as $t \to \infty$ and then, $c(\delta)$ is bounded from above, say by c_1. Thus, from the above inequality, we have the estimate

$$\begin{aligned} M(y(t)) &\leqslant M(y(t)) + \frac{c(t)}{a(t)} H(x(t)) + \frac{c(t)}{a(t)} \beta \\ &\leqslant M(y(0)) + \frac{c(t)}{a(t)} \Phi + \frac{c_1}{a_0} \beta + \frac{\Phi}{a_0} \int_\delta^t |c'(\tau)| \,\mathrm{d}\tau + \frac{1}{a(t)} \int_0^t e(\tau) \,\mathrm{d}\tau, \end{aligned}$$

where

$$\Phi = \frac{Q_2}{c_0} \exp\Bigl(\int_0^\infty |c'(\tau)| \frac{\mathrm{d}\tau}{c_0} \Bigr).$$

Therefore, (v) implies the boundedness of $y(t)$ and this completes the proof. $\square$

Remark 18. In the relationship with some of the results on the boundedness behaviour of solutions for equations the (1) and (2), it can be observed in Theorem 11 that $c(t)$ is nonincreasing on I, and $a(t) \in C'(I, \mathbb{R}^+)$ and $a(t)$ bounded away from zero while in Theorem 17 $a(t)$ is nonincreasing on I and $c(t) \in C'(I, \mathbb{R}^+)$ and is bounded away from zero. Then it was satisfied that (iv) and (v) imply that $x(t)$ and $y(t)$ are bounded, respectively, in the theorems.

Theorem 19. *Let the conditions* (i)–(vi) *and* (viii) *hold, and let $a(t)$ and $c(t)$ be bounded away from zero. If $a(t), c(t) \in C'(I, \mathbb{R}^+)$ and $|a'(t)|, |c'(t)| \in L^1(I)$, then any solution $x(t)$ of* (6) *along with its derivative $y(t)$ is bounded on I.*

Proof. Multiplying (6) by $y(t)/m(y(t))$, integrating from 0 to t and using (i) to (vi) and (viii), and also from the proof of Theorem 9, we get:

$$\begin{aligned} \int_0^t a(\tau)(M(y(\tau)))' \,\mathrm{d}\tau + \int_0^t c(\tau)(H(x(\tau)))' \,\mathrm{d}\tau + \int_0^t c(\tau)\beta \\ \leqslant D \int_0^t e_1(\tau) \,\mathrm{d}\tau + \alpha \int_0^t e_1(\tau) M(y(\tau)) \,\mathrm{d}\tau. \end{aligned}$$

Then

$$\begin{aligned} a(t)M(y(t)) - a(0)M(y(0)) - \int_0^t |a'(\tau)| M(y(\tau)) \,\mathrm{d}\tau + c(t)H(x(t)) \\ - c(0)H(x(0)) - \int_0^t |c'(\tau)| H(x(\tau)) \,\mathrm{d}\tau + c(t)\beta - c(0)\beta \\ \leqslant D \int_0^t e_1(\tau) \,\mathrm{d}\tau + \alpha \int_0^t e_1(\tau) M(y(\tau)) \,\mathrm{d}\tau. \end{aligned}$$

This implies

$$
\begin{aligned}
&a(t)M(y(t)) + c(t)H(x(t)) + c(t)\beta \\
&\qquad \leqslant Q_4 + \int_0^t (|a'(\tau)| + |c'(\tau)| + \alpha e_1(\tau))(M(y(\tau)) + H(x(\tau)))\,\mathrm{d}\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&j(M(y(t)) + H(x(t)) + \beta) \\
&\qquad \leqslant Q_4 + \int_0^t (|a'(\tau)| + |c'(\tau)| + \alpha e_1(\tau))(M(y(\tau)) + H(x(\tau)))\,\mathrm{d}\tau,
\end{aligned}
$$

where $j = \min(a_0, c_0)$, a_0 and c_0 being the lowest bounds of $a(t)$ and $c(t)$, respectively, and

$$Q_4 = a(0)M(y(0)) + c(0)H(x(0)) + c(0)\beta + D\int_0^\infty e_1(\tau)\,\mathrm{d}\tau.$$

By Gronwall's inequality, it follows that $M(y(t)) + H(x(t)) + \beta$ is bounded and then the conditions (iv) to (vi) imply that $x(t)$ and $y(t)$ are bounded on I. The proof is now completed. □

Remark 20. Theorem 19 generalizes Theorem 7 of Bihari [7], Theorem 4 of Wong [58] and Theorem 11 of Athanassov [5].

4. Examples

First, consider a special case of (1) which is the second order nonlinear delay differential equation

$$x'' + (1+t)(2+\sin x)x' + \frac{2(1+t)}{1+x'^2}x(t-r) = \frac{1}{2}\sin t. \tag{18}$$

The equivalent system of (18) is given as follows: let $x' = y$ then

$$y' + (1+t)(2+\sin x)y + \frac{2(1+t)x}{1+y^2} - \frac{2(1+t)}{1+y^2}\int_{t-r}^t \mathrm{d}(x(\theta)) = \frac{1}{2}\sin t. \tag{19}$$

Take $r = \frac{2}{25}$ and the initial conditions $x(0) = 1$, $y(0) = 1$.

The following basic conditions are satisfied for the equation (19) which can be related to the assumptions (i) to (v) of the main results.

(1) $a(t) = 1 > 0$, $b(t) = (1+t) > 0$, $c(t) = 2(1+t) > 0$;
(2) $g(x,y) = (2+\sin x) \geqslant 0$, $h(x) = x \geqslant 0$, $m(y) = 1/(1+y^2) > 0$, $p(t,x,y) = \frac{1}{2}\sin t \geqslant 0$;
(3) $g(x,y)y = (2+\sin x)y^2 > 0$ for all $(x,y) \in \mathbb{R}^2$, $y \neq 0$;
(4) $H(x) = \int_0^x \tau\,\mathrm{d}\tau = \frac{1}{2}x^2 \to \infty$ as $x \to \infty$ where $\int_0^x h(\tau)\,\mathrm{d}\tau = \int_0^x \tau\,\mathrm{d}\tau \geqslant 0$ and
(5) $\int_0^y \tau/m(\tau)\,\mathrm{d}\tau = \int_0^y \tau(1+\tau^2)\,\mathrm{d}\tau = \frac{1}{2}y^2 + \frac{1}{4}y^4 \to \infty$ as $y \to \infty$.

Consequently, we consider a special case of the equation (2) which is the second order nonlinear delay differential equation

$$(5\exp(-t)x')' + (1+t)(2+\sin x)x' + \frac{2(1+t)}{1+x'^2}x(t-r) = \frac{1}{2}\sin t. \tag{20}$$

The equation (20) is re-written as

$$(5\exp(-t)y)' + (1+t)(2+\sin x)y + \frac{2(1+t)x}{1+y^2} - \frac{2(1+t)}{1+y^2}\int_{t-r}^{t} \mathrm{d}(x(\theta)) = \frac{1}{2}\sin t. \tag{21}$$

Following the basic condtions (1) to (5) above for the equation (19), and taking $a(t) = 5\exp(t) > 0$ along with conditions (2) to (5) for the equation (21), we now have graphically the boundedness of solution satisfying equations (19) and (21), respectively, see Figures 1 and 2.

5. Conclusion

It is interesting to note that several authors have studied the boundedness of solutions of autonomous and nonautonomous delay differential equations by using the Lyapunov-Krasovskiĭ method to investigate the property. However, in this work, we have successfully investigated the boundedness of all solutions and their derivatives by applying different forms of the integral inequality method. We have applied Gronwall's inequality, two forms of the second mean value theorem for integrals and Riemann-Stieltjes integrals to nonlinear differential equations with delay to establish our boundedness results. Examples to corroborate the established results for the equations (1) and (2) are also included with graphical representations by using Maple 2015.

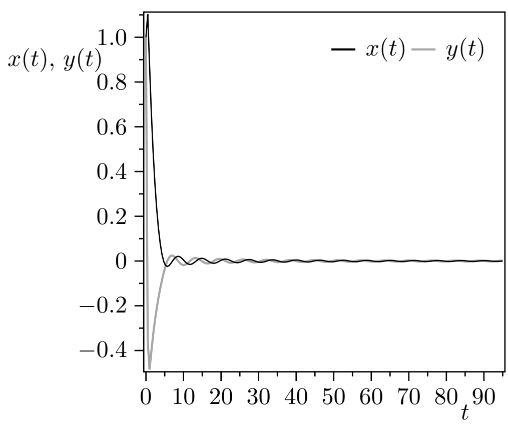

Figure 1. The graph showing the boundedness of solution for the equation (19). Hence, the solution of the nonlinear delay differential equation (19) is bounded.

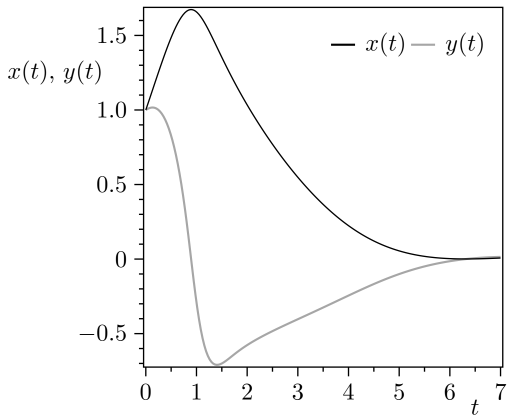

Figure 2. The graph showing the boundedness of solution for the equation (21). Thus, the solution of the nonlinear differential equation (21) with delay is also bounded.

Acknowledgment. The authors would like to appreciate the anonymous reviewer for his/her comments and useful suggestions which have improved this work. Thank you for your effort.

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