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EXISTENCE OF RENORMALIZED SOLUTIONS FOR SOME  
DEGENERATE AND NON-COERCIVE ELLIPTIC EQUATIONS

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*Abstract.* This paper is devoted to the study of some nonlinear degenerated elliptic equations, whose prototype is given by

$$\begin{aligned} -\operatorname{div}(b(|u|)|\nabla u|^{p-2}\nabla u) + d(|u|)|\nabla u|^p &= f - \operatorname{div}(c(x)|u|^\alpha) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^N$  ( $N \geq 2$ ) with  $1 < p < N$  and  $f \in L^1(\Omega)$ , under some growth conditions on the function  $b(\cdot)$  and  $d(\cdot)$ , where  $c(\cdot)$  is assumed to be in  $L^{N/(p-1)}(\Omega)$ . We show the existence of renormalized solutions for this non-coercive elliptic equation, also, some regularity results will be concluded.

*Keywords:* renormalized solution; nonlinear elliptic equation; non-coercive problem

*MSC 2020:* 35J60, 46E30, 46E35

## 1. INTRODUCTION

In [7], Boccardo et al. have studied the quasilinear elliptic problem with degenerate coercivity

$$(1.1) \quad \begin{cases} -\operatorname{div}(A(x, u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the data  $f$  is assumed to be in  $L^m(\Omega)$  with  $m \geq 1$ . They have proved the existence and some regularity results; we refer the reader to [2], [9], and also [16] for the case of measure data.

Alvino et al. have considered in [1] the nonlinear degenerated elliptic problem of the form

$$(1.2) \quad \begin{cases} -\operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}$$

they have proved the existence of solutions and some regularity results for  $f$  a measurable function in  $L^m(\Omega)$  with  $m \geq 1$ .

In [19], Murat has proved the existence of renormalized solutions for the quasilinear elliptic problem

$$(1.3) \quad \begin{cases} \lambda u - \operatorname{div}(A(x)\nabla u + \phi(u)) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $f \in L^1(\Omega)$  and  $\lambda > 0$ . The uniqueness of the solution was concluded under some locally Lipschitz continuous conditions on the vector field  $\phi(\cdot)$ . We refer also to [10], where Del Vecchio et al. have proved the existence of weak solutions for the non-coercive problem by using the symmetrization method.

In [13], Droniou has studied the nonlinear non-coercive elliptic problems

$$(1.4) \quad \begin{cases} Au - \operatorname{div}\phi(x, u) = f(x) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with  $Au = -\operatorname{div}a(x, u, \nabla u)$  being a Leray-Lions operator on  $W_0^{1,p}(\Omega)$ , and  $\Phi(x, s)$  being convection term with growth properties, where  $f \in W^{-1,p'}(\Omega)$  and  $\mu \in \mathcal{M}(\Omega)$ . He has proved the existence and some regularity results. Also, the author has proved in [12] the existence and uniqueness of solutions for some elliptic problems.

In [5], Bensoussan, Boccardo and Murat have studied the nonlinear elliptic problem

$$Au + g(x, u, \nabla u) = f \quad \text{in } \Omega,$$

where  $A$  is a Leray-Lions operator acted from  $W_0^{1,p}(\Omega)$  into  $W^{-1,p'}(\Omega)$ , where  $g$  is a Carathéodory function satisfying the sign and growth conditions, the data  $f$  belongs to  $W^{-1,p'}(\Omega)$ . They proved the existence of the solution in the sense of distributions  $u \in W_0^{1,p}(\Omega)$  such that  $g(x, u, \nabla u) \in L^1(\Omega)$  and  $g(x, u, \nabla u)u \in L^1(\Omega)$ .

In the case of  $f \in L^1(\Omega)$ , Boccardo and Gallouet (see [8]) have proved the existence of solutions  $u \in W_0^{1,p}(\Omega)$  with  $g(x, u, \nabla u) \in L^1(\Omega)$  under the additional assumption:

$$\text{There exist } \sigma > 0, \gamma > 0 \text{ such that } |g(x, s, \xi)| \geq \gamma|\xi|^p \text{ for } |s| \geq \sigma.$$

In [3], Ben Cheikh Ali and Guibé have studied some quasilinear elliptic equations of the type

$$(1.5) \quad \begin{cases} \lambda(x, u) - \operatorname{div}(a(x, \nabla u) + \Phi(x, u)) = f & \text{in } \Omega, \\ (a(x, \nabla u) + \Phi(x, u)) \cdot n = 0 & \text{on } \Gamma_n, \\ u = 0 & \text{on } \Gamma_d, \end{cases}$$

where  $Au = -\operatorname{div} a(x, \nabla u)$  is a Leray-Lions type operator and the Carathéodory functions  $\lambda(x, s): \Omega \times \mathbb{R} \mapsto \mathbb{R}$  and  $\Phi(x, s): \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$  satisfy only some growth conditions. They have proved the existence of renormalized solutions for this equation. Moreover, the uniqueness of solution is obtained under some additional conditions on the function  $\Phi(x, s)$ ; we refer the reader to [11], [18], [22], [23].

In [15], Guibé et al. have studied a class of nonlinear elliptic problems whose prototype is

$$(1.6) \quad \begin{cases} -\Delta_p u - \operatorname{div}(c(x)|u|^\gamma) + b(x)|\nabla u|^\lambda = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Delta_p$  is the  $p$ -Laplace operator  $1 < p < N$ , and  $\mu$  is a Radon measure with bounded variation on  $\Omega$ . They have proved the existence of renormalized solutions in the case of  $0 \leq \gamma \leq p - 1$  and  $0 \leq \lambda \leq p - 1$  (see also [14]).

In this paper, we are interested in proving the existence of renormalized solutions to the following nonlinear elliptic problem having a degenerate coercivity:

$$(1.7) \quad \begin{cases} Au + g(x, u, \nabla u) = f - \operatorname{div} \phi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^N$  ( $N \geq 2$ ) with  $1 < p < N$  and  $Au = -\operatorname{div} a(x, u, \nabla u)$  is a non-coercive Leray-Lions operator acting from  $W_0^{1,p}(\Omega)$  into its dual  $W^{-1,p'}(\Omega)$ , the Carathéodory functions  $g(x, s, \xi)$  and  $\phi(x, s)$  verify only some growth conditions, and the data  $f$  is assumed to belong to  $L^1(\Omega)$ . Under such assumptions, the solution  $u$  may not be finite in general. This means that, at least for solutions obtained through approximation, such solutions may reach the values  $\infty$  and  $-\infty$ . For more details we refer the reader to [6].

The novelty of this work is the fact of overcoming several difficulties at the same time. We prove the existence of renormalized solutions for the strongly nonlinear and non-coercive elliptic problem (1.7), the existence result is obtained by using an approximation procedure and some a priori estimate. The functions test used in this work are essentially inspired from the standard analysis; we refer the reader for example to [1], [3], [13], [20], [21].

This paper is organized as follows: In Section 2, we present some non-standard assumptions on the Carathéodory functions  $a(x, s, \xi)$ ,  $g(x, s, \xi)$  and  $\phi(x, s)$  for which our nonlinear elliptic problem (1.7) has a renormalized solution. In Section 3, we will state the main results. Section 4 is devoted entirely to prove the existence of renormalized solutions for our nonlinear elliptic equation, also, some regularity results will be proved. Finally in Section 5, we will prove Proposition 4.1.

## 2. ESSENTIAL ASSUMPTIONS

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^N$  ( $N \geq 2$ ) and let  $1 < p < N$ . We consider a Leray-Lions operator  $A$  from  $W_0^{1,p}(\Omega)$  into its dual  $W^{-1,p'}(\Omega)$ , defined by the formula

$$(2.1) \quad Au = -\operatorname{div} a(x, u, \nabla u),$$

where  $a(x, s, \xi): \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}^N$  is a Carathéodory function (i.e., measurable with respect to  $x$  in  $\Omega$  for every  $(s, \xi)$  in  $\mathbb{R} \times \mathbb{R}^N$  and continuous with respect to  $(s, \xi)$  in  $\mathbb{R} \times \mathbb{R}^N$  for almost every  $x$  in  $\Omega$ ) and verifies the following conditions:

$$(2.2) \quad |a(x, s, \xi)| \leq \beta(a_0(x) + |s|^{p-1} + |\xi|^{p-1})$$

for a positive function  $a_0(x) \in L^{p'}(\Omega)$ , and  $\beta > 0$ ;

$$(2.3) \quad (a(x, s, \xi) - a(x, s, \eta)) \cdot (\xi - \eta) > 0 \quad \text{for any } \xi \neq \eta.$$

There exists a positive decreasing function  $b: ]0, \infty[ \rightarrow ]0, \infty[$ , and two constants  $b_0, s_0 > 0$  such that

$$(2.4) \quad a(x, s, \xi) \cdot \xi \geq b(|s|)|\xi|^p \quad \text{with } b(|s|) \geq \frac{b_0}{(1 + |s|)^\lambda} \text{ for all } |s| > s_0,$$

for a.e.  $x \in \Omega$  and all  $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ , where  $0 \leq \lambda \leq p - 1$ . As a consequence of (2.4) and the continuity of the function  $a(x, s, \cdot)$  with respect to  $\xi$ , we have

$$a(x, s, 0) = 0.$$

The lower order term  $g(x, s, \xi): \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function which satisfies only the growth condition

$$(2.5) \quad |g(x, s, \xi)| \leq f_0(x) + d(|s|)|\xi|^p,$$

where  $f_0(x)$  is assumed to be a positive measurable function in  $L^1(\Omega)$ , and  $d(\cdot): \mathbb{R} \mapsto \mathbb{R}^+$  is a continuous decreasing function such that  $d(|\cdot|)/b(|\cdot|)$  is decreasing and belongs to  $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ .

The Carathéodory function  $\phi(\cdot, \cdot): \Omega \times \mathbb{R} \mapsto \mathbb{R}^N$  satisfies the growth condition

$$(2.6) \quad |\phi(x, s)| \leq c(x)(1 + |s|)^\alpha,$$

where  $0 \leq \alpha \leq p - 1 - \lambda$  and  $c(x)$  is a positive function in  $L^{N/(p-1)}(\Omega)$ .

We consider the strongly nonlinear and non-coercive elliptic Dirichlet problem

$$(2.7) \quad \begin{cases} Au + g(x, u, \nabla u) = f - \operatorname{div} \phi(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the data  $f$  is assumed to be in  $L^1(\Omega)$ .

**Definition 2.1.** Let  $k > 0$ , the truncation function  $T_k(\cdot): \mathbb{R} \mapsto \mathbb{R}$  is given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \frac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,p}(\Omega) := \{u: \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,p}(\Omega) \text{ for any } k > 0\}.$$

**Proposition 2.1** (cf. [4]). *Let  $u \in \mathcal{T}_0^{1,p}(\Omega)$ . There exists a unique measurable function  $v: \Omega \mapsto \mathbb{R}^N$  such that*

$$\nabla T_k(u) = v \chi_{\{|u| < k\}} \quad \text{a.e. in } \Omega \text{ for any } k > 0,$$

where  $\chi_A$  denotes the characteristic function of a measurable set  $A$ . The function  $v$  is called the weak gradient of  $u$  and is still denoted by  $\nabla u$ . Moreover, if  $u$  belongs to  $W_0^{1,1}(\Omega)$ , then  $v$  coincides with the gradient of  $u$ , that is  $v = \nabla u$ .

### 3. MAIN RESULT

We begin by introducing the definition of renormalized solutions for the elliptic equation (2.7):

**Definition 3.1.** A measurable function  $u$  is called a renormalized solution of the strongly nonlinear elliptic problem (2.7) if  $u \in \mathcal{T}_0^{1,p}(\Omega)$ ,  $g(x, u, \nabla u) \in L^1(\Omega)$ , and

$$(3.1) \quad \lim_{h \rightarrow \infty} \frac{1}{h} \int_{\{|u| \leq h\}} a(x, u, \nabla u) \nabla u \, dx = 0$$

such that  $u$  satisfies the equality

$$(3.2) \quad \int_{\Omega} a(x, u, \nabla u) \cdot (S'(u)\varphi \nabla u + S(u)\nabla \varphi) \, dx + \int_{\Omega} g(x, u, \nabla u) S(u)\varphi \, dx \\ = \int_{\Omega} f S(u)\varphi \, dx + \int_{\Omega} \phi(x, u) \cdot (S'(u)\varphi \nabla u + S(u)\nabla \varphi) \, dx$$

for every  $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  and for any smooth function  $S(\cdot) \in W^{1,\infty}(\mathbb{R})$  with a compact support.

The goal of the present paper is to prove the following existence result:

**Theorem 3.1.** *Let  $f \in L^1(\Omega)$ , assuming that conditions (2.2)–(2.6) hold true, and one of the following additional hypothesis holds true:*

- (i) *There exist two positive constants  $s_0$  and  $d_0$  such that  $d(|s|) \leq d_0/(1 + |s|)^p$  for any  $s \geq s_0$ .*
- (ii)  $0 \leq \alpha < p - 1 - \lambda$ .
- (iii)  $\|c(x)\|_{L^{N/(p-1)}(\Omega)} \leq c_0$ , where  $c_0$  is small enough.

*Then there exists at least one renormalized solution for the strongly nonlinear and non-coercive elliptic problem (2.7).*

**Remark 3.1.** Note that, under the assumption of Theorem 3.1, the renormalized solution of problem (2.7) belongs to  $u \in \mathcal{T}_0^{1,p}(\Omega)$  so that  $d(|u|)^{1/p}u$  belongs to  $L^p(\Omega)$ .

In all remaining parts of this paper, we will denote by  $C_p$  the constant of Poincaré's inequality, and by  $C_s$  the constant of Sobolev's inequality. The real constants  $C_i$  for  $i = 0, 1, \dots$  are different in each step of the proof of Theorem 3.1.

#### 4. PROOF OF THEOREM 3.1

The proof will be divided into several steps.

**Step 1: Approximate problems.** We consider a sequence of smooth functions  $(f_n)_{n \in \mathbb{N}^*}$  in  $W^{-1,p'}(\Omega) \cap L^1(\Omega)$  that converges strongly to  $f$  in  $L^1(\Omega)$ , such that  $|f_n| \leq |f|$  (for example  $f_n = T_n(f)$ ).

For any  $n \in \mathbb{N}^*$  we consider the approximate problem

$$(4.1) \quad \begin{cases} A_n u_n + g_n(x, u_n, \nabla u_n) = f_n - \operatorname{div} \phi_n(x, u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with  $\phi_n(x, s) = \phi(x, T_n(s))$  and  $g_n(x, s, \xi) = T_n(g(x, s, \xi))$ , where the operator  $A_n$  is given by

$$A_n v = -\operatorname{div} a(x, T_n(v), \nabla v) \quad \text{for any } v \in W_0^{1,p}(\Omega).$$

Note that the operator  $A_n$  is coercive and satisfies the classical Leray-Lions conditions.

Indeed, by using condition (2.4) we have

$$\langle A_n v, v \rangle = \int_{\Omega} a(x, T_n(v), \nabla v) \cdot \nabla v \, dx \geq b(n) \int_{\Omega} |\nabla v|^p \, dx$$

for all  $v \in W_0^{1,p}(\Omega)$  with  $b(n) > 0$ .

In view of the classical results of Leray-Lions (see [17]), there exists at least one weak solution  $u_n \in W_0^{1,p}(\Omega)$  for the approximate problem (4.1); we refer the reader also to [15] for more details.

**Proposition 4.1.** *Assume that conditions (2.2)–(2.6) hold true, and let  $u_n$  be a weak solution of the approximate problem (4.1). If one of assumptions (i)–(iii) in Theorem 3.1 is satisfied, then for any  $n \in \mathbb{N}^*$ , the weak solution of approximate problem (4.1) verifies*

$$(4.2) \quad \lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{meas}\{|u_n| > k\} \rightarrow 0.$$

The proof of Proposition 4.1 is in Appendix.

**Step 2: Weak convergence of truncations.** Let  $k \geq 1$ , we set  $B(s) = T_k(s)(1 + |T_k(s)|)^\lambda$  and  $H(s) = 2 \int_0^s d(|\tau|)/b(|\tau|) \, d\tau$ .

We have  $d(|\cdot|)/b(|\cdot|) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and  $0 \leq H(\infty) := 2 \int_0^\infty d(|\tau|)/b(|\tau|) \, d\tau$  is a finite real number. Then  $B(u_n)e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$ .

By taking  $B(u_n)e^{H(|u_n|)}$  as a test function for the approximate problem (4.1), we have

$$\begin{aligned} & \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u_n) B'(u_n) e^{H(|u_n|)} \, dx \\ & \quad + 2 \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} |B(u_n)| e^{H(|u_n|)} \, dx \\ & \quad + \int_{\Omega} g_n(x, u_n, \nabla u_n) B(u_n) e^{H(|u_n|)} \, dx \\ & = \int_{\Omega} f_n B(u_n) e^{H(|u_n|)} \, dx + \int_{\Omega} \phi(x, T_n(u_n)) \cdot \nabla T_k(u_n) B'(u_n) e^{H(|u_n|)} \, dx \\ & \quad + 2 \int_{\Omega} \phi(x, T_n(u_n)) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} |B(u_n)| e^{H(|u_n|)} \, dx. \end{aligned}$$



In view of (2.4), (2.5) and (2.6), and since  $(1 + |T_k(s)|)^\lambda \leq B'(s) \leq (\lambda + 1) \times (1 + |T_k(s)|)^\lambda$  for  $|s| < k$ , we conclude that

$$\begin{aligned}
 (4.3) \quad & \int_{\{|u_n| \leq k\}} b(|u_n|) |\nabla T_k(u_n)|^p (1 + |u_n|)^\lambda \, dx + 2 \int_{\Omega} d(|u_n|) |\nabla u_n|^p |B(u_n)| e^{H(|u_n|)} \, dx \\
 & \leq (1 + k)^{\lambda+1} e^{H(\infty)} \int_{\Omega} (|f_n| + |f_0|) \, dx + \int_{\Omega} d(|u_n|) |\nabla u_n|^p |B(u_n)| e^{H(|u_n|)} \, dx \\
 & \quad + (\lambda + 1) e^{H(\infty)} \int_{\{|u_n| \leq k\}} c(x) (1 + |T_k(u_n)|)^\alpha |\nabla T_k(u_n)| (1 + |T_k(u_n)|)^\lambda \, dx \\
 & \quad + 2 \int_{\Omega} c(x) (1 + |u_n|)^\alpha |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} |B(u_n)| e^{H(|u_n|)} \, dx.
 \end{aligned}$$

Thanks to (2.4), we have  $b(|u_n|)(1 + |u_n|)^\lambda \geq b_0$  for  $|u_n| \geq s_0$ . We set

$$b_1 = \min \left\{ b_0, \inf_{|s| \leq k} b(|s|)(1 + |s|)^\lambda \right\},$$

then we get

$$\begin{aligned}
 b_1 \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p \, dx + \int_{\Omega} d(|u_n|) |\nabla u_n|^p |B(u_n)| e^{H(|u_n|)} \, dx \\
 \leq (1 + k)^{\lambda+1} e^{H(\infty)} (\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) \\
 \quad + (\lambda + 1) e^{H(\infty)} \int_{\{|u_n| \leq k\}} c(x) |\nabla T_k(u_n)| (1 + |T_k(u_n)|)^{\alpha+\lambda} \, dx \\
 \quad + 2 e^{H(\infty)} \int_{\Omega} c(x) (1 + |u_n|)^\alpha |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} |B(u_n)| \, dx.
 \end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}
 \frac{b_1}{2} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p \, dx + \frac{1}{2} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda \, dx \\
 \leq C_0 (1 + k)^{\lambda+1} + C_1 \int_{\{|u_n| \leq k\}} |c(x)|^{p'} (1 + |T_k(u_n)|)^{(\alpha+\lambda)p'} \, dx \\
 \quad + C_2 \int_{\Omega} |c(x)|^{p'} (1 + |u_n|)^{\alpha p'} \frac{d(|u_n|)}{b(|u_n|)^{p'}} |B(u_n)| \, dx.
 \end{aligned}$$

Let  $R \geq 1$ , since  $0 \leq \alpha + \lambda \leq p - 1$ , having in mind (2.4), it follows that

$$\begin{aligned}
 (4.4) \quad & \frac{b_1}{2} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p \, dx + \frac{1}{2} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda \, dx \\
 & \leq C_0 (1 + k)^{\lambda+1} + C_1 \int_{\{|u_n| \leq k\}} |c(x)|^{p'} (1 + |T_k(u_n)|)^{(\alpha+\lambda)p'} \, dx \\
 & \quad + C_3 \int_{\Omega} |c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} d(|u_n|) |B(u_n)| \, dx
 \end{aligned}$$

$$\begin{aligned} &\leq C_0(1+k)^{\lambda+1} + C_4 \int_{\{R < |u_n| \leq k\}} |c(x)|^{p'} |T_k(u_n)|^p dx \\ &\quad + C_5 \int_{\{R < |u_n|\}} |c(x)|^{p'} |u_n|^p d(|u_n|) |B(u_n)| dx + C_5. \end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned} (4.5) \quad &\frac{b_1}{2} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p dx + \frac{1}{2} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda dx \\ &\leq C_0(1+k)^{\lambda+1} + C_4 \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \|T_k(u_n)\|_{L^{p^*}(\Omega)}^p \\ &\quad + C_5 \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \|u_n d(|u_n|)^{1/p} |B(u_n)|^{1/p}\|_{L^{p^*}(\Omega)}^p + C_6, \end{aligned}$$

where  $C_6$  is a constant depending on  $R$ ,  $p$  and  $\|c(\cdot)\|_{L^{p'}(\Omega)}$ .

Recall that if  $g(s)$  is positive and decreasing function on the set  $[0, r]$ , then  $rg(r) \leq \int_0^r g(s) ds$ .

We have that  $d(|s|)$  is a decreasing function, and  $B(|s|)$  is a constant function on the set  $\{|s| > k\}$ . Therefore, the function  $d(|s|)B(|s|)$  is a decreasing function for  $\{|s| > k\}$ , and we obtain

$$|u_n| d(|u_n|)^{1/p} |B(u_n)|^{1/p} \leq \int_0^{|u_n|} d(|\tau|)^{1/p} |B(\tau)|^{1/p} d\tau + C_7 k^{(p+\lambda+1)/p} \quad \text{a.e. in } \Omega.$$

Thanks to Sobolev inequality and (4.2), we conclude that

$$\begin{aligned} (4.6) \quad &\frac{b_1}{2} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u_n|^p d(|u_n|) |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda dx \\ &\leq C_9(1+k)^{p+\lambda+1} + C_4 \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \|T_k(u_n)\|_{L^{p^*}(\Omega)}^p \\ &\quad + C_8 \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \left\| \int_0^{|u_n|} d(|\tau|)^{1/p} |B(\tau)|^{1/p} d\tau \right\|_{L^{p^*}(\Omega)}^p \\ &\leq C_9(1+k)^{p+\lambda+1} + C_4 C_S^p \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \|\nabla T_k(u_n)\|_{L^p(\Omega)}^p \\ &\quad + C_8 C_S^p \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \|d(|u_n|)^{1/p} |B(u_n)|^{1/p} \nabla u_n\|_{L^p(\Omega)}^p. \end{aligned}$$

Since  $\text{meas}\{|u_n| > R\} \rightarrow 0$  as  $R$  tends to  $\infty$ , we can choose  $R \geq 1$  large enough such that

$$C_4 C_S^p \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \leq \frac{b_1}{4} \quad \text{and} \quad C_8 C_S^p \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \leq \frac{1}{4}.$$

We obtain

$$\begin{aligned}
 (4.7) \quad & \frac{b_1}{2} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p dx + \frac{1}{2} \int_{\Omega} |\nabla u_n|^p d(|u_n|) |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda dx \\
 & \leq C_9 (1+k)^{p+\lambda+1} + \frac{b_1}{4} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p dx \\
 & \quad + \frac{1}{4} \int_{\Omega} |\nabla u_n|^p d(|u_n|) |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda dx.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 (4.8) \quad & \frac{b_1}{4} \int_{\{|u_n| \leq k\}} |\nabla T_k(u_n)|^p dx + \frac{1}{4} \int_{\Omega} |\nabla u_n|^p d(|u_n|) |T_k(u_n)| (1 + |T_k(u_n)|)^\lambda dx \\
 & \leq C_{10} k^{p+\lambda+1}.
 \end{aligned}$$

Then the sequence  $(T_k(u_n))_n$  is bounded in  $W_0^{1,p}(\Omega)$ , and there exists a measurable function  $\eta_k \in W_0^{1,p}(\Omega)$  such that

$$(4.9) \quad \begin{cases} T_k(u_n) \rightharpoonup \eta_k & \text{weakly in } W_0^{1,p}(\Omega), \\ T_k(u_n) \rightarrow \eta_k & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

In view of (4.2) and (4.9), and following the same approach as in [4], we conclude that the sequence of weak solutions  $(u_n)_n$  converges almost everywhere to a measurable function  $u$ , and thanks to (4.9), we obtain

$$(4.10) \quad \begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega), \\ T_k(u_n) \rightarrow T_k(u) & \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{cases}$$

Moreover, taking  $k = 1$  in inequality (4.8) and since  $d(\cdot)$  is a decreasing function,

$$(4.11) \quad \int_{\Omega} d(|u_n|) |u_n|^p dx \leq \int_{\Omega} \left| \int_0^{|u_n|} d(s)^{1/p} ds \right|^p dx \leq C_p^p \int_{\Omega} d(|u_n|) |\nabla u_n|^p dx \leq C_{11},$$

where  $C_{11}$  is a constant that does not depend on  $n$ . Then the sequence  $(d(|u_n|))^{1/p} \times |u_n|_n$  is uniformly bounded in  $L^p(\Omega)$  and it follows that

$$(4.12) \quad d(|u_n|)^{1/p} |u_n| \rightharpoonup d(|u|)^{1/p} |u| \quad \text{weakly in } L^p(\Omega).$$

**Step 3: Some regularity results.** In this step, we denote by  $\varepsilon_i(n)$ , for  $i = 1, 2, \dots$ , some real-valued functions that converge to 0 as  $n$  tends to infinity. Similarly, we define  $\varepsilon_i(h)$  and  $\varepsilon_i(n, h)$ .

In this step, we will show the following estimate:

$$(4.13) \quad \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = 0.$$

Indeed, by taking  $h^{-1}T_h(u_n)e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$  as a test function in the approximate problem (4.1), we have

$$\begin{aligned} & \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{H(|u_n|)} \, dx \\ & \quad + \frac{1}{h} \int_{\Omega} g_n(x, u_n, \nabla u_n) T_h(u_n) e^{H(|u_n|)} \, dx \\ & \quad + \frac{2}{h} \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n |T_h(u_n)| \frac{d(|u_n|)}{b(|u_n|)} e^{H(|u_n|)} \, dx \\ & = \frac{1}{h} \int_{\Omega} f_n T_h(u_n) e^{H(|u_n|)} \, dx + \frac{1}{h} \int_{\{|u_n| \leq h\}} \phi(x, T_n(u_n)) \cdot \nabla u_n e^{H(|u_n|)} \, dx \\ & \quad + \frac{2}{h} \int_{\Omega} \phi(x, T_n(u_n)) \cdot \nabla u_n |T_h(u_n)| \frac{d(|u_n|)}{b(|u_n|)} e^{H(|u_n|)} \, dx. \end{aligned}$$

In view of (2.5) and (2.6), we conclude that

$$\begin{aligned} & \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{H(|u_n|)} \, dx \\ & \quad + \frac{2}{h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| e^{H(|u_n|)} \, dx \\ & \leq \int_{\Omega} (|f_n| + |f_0|) \frac{|T_h(u_n)|}{h} e^{H(|u_n|)} \, dx \\ & \quad + \frac{1}{h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| e^{H(|u_n|)} \, dx \\ & \quad + \frac{1}{h} \int_{\{|u_n| \leq h\}} c(x) (1 + |T_n(u_n)|)^\alpha |\nabla u_n| e^{H(|u_n|)} \, dx \\ & \quad + \frac{2}{h} \int_{\Omega} c(x) (1 + |T_n(u_n)|)^\alpha |\nabla u_n| |T_h(u_n)| \frac{d(|u_n|)}{b(|u_n|)} e^{H(|u_n|)} \, dx. \end{aligned}$$

Using Young's inequality, we deduce that

$$\begin{aligned} & \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n e^{H(|u_n|)} \, dx + \frac{1}{h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| e^{H(|u_n|)} \, dx \\ & \leq e^{H(\infty)} \int_{\Omega} (|f_n| + |f_0|) \frac{|T_h(u_n)|}{h} \, dx + \frac{1}{2h} \int_{\{|u_n| \leq h\}} b(|u_n|) |\nabla u_n|^p e^{H(|u_n|)} \, dx \\ & \quad + \frac{C_0}{h} \int_{\{|u_n| \leq h\}} |c(x)|^{p'} \frac{(1 + |u_n|)^{\alpha p'}}{b(|u_n|)^{p'/p}} \, dx + \frac{1}{2h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| e^{H(|u_n|)} \, dx \\ & \quad + \frac{C_0}{h} \int_{\Omega} |c(x)|^{p'} (1 + |u_n|)^{\alpha p'} |T_h(u_n)| \frac{d(|u_n|)}{b(|u_n|)^{p'}} e^{H(|u_n|)} \, dx. \end{aligned}$$

In view of (4.2), we have that  $\text{meas}\{|u_n| > h\} \rightarrow 0$  as  $h$  tends to infinity, thus  $|T_h(u_n)|/h \rightarrow 0$  weak-\* in  $L^\infty(\Omega)$ . Thanks to the Lebesgue dominated convergence theorem, we obtain

$$(4.14) \quad \varepsilon_1(h) = \lim_{h \rightarrow \infty} e^{H(\infty)} \int_{\Omega} (|f_n| + |f_0|) \frac{|T_h(u_n)|}{h} dx \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Let  $R \geq 1$  and since  $0 \leq (\alpha + \lambda)p' \leq p$ , thanks to (2.4) and (4.2) and Hölder's inequality, we conclude that

$$(4.15) \quad \begin{aligned} & \frac{1}{2h} \int_{\{|u_n| \leq h\}} a(x, T_h(u_n), \nabla u_n) \cdot \nabla u_n dx + \frac{1}{2h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| dx \\ & \leq \varepsilon_1(h) + \frac{C_1}{h} \int_{\{|u_n| \leq h\}} |c(x)|^{p'} (1 + |u_n|)^{(\alpha + \lambda)p'} b(|u_n|) dx \\ & \quad + \frac{C_1}{h} \int_{\Omega} |c(x)|^{p'} (1 + |u_n|)^{(\alpha + \lambda)p'} |T_h(u_n)| d(|u_n|) dx \\ & \leq \varepsilon_2(h) + \frac{C_2}{h} \int_{\{|u_n| \leq h\}} |c(x)|^{p'} |u_n|^p b(|u_n|) dx \\ & \quad + \frac{C_2}{h} \int_{\Omega} |c(x)|^{p'} |u_n|^p |T_h(u_n)| d(|u_n|) dx \\ & \leq \varepsilon_2(h) + \frac{C_2}{h} \int_{\{|u_n| \leq R\}} |c(x)|^{p'} |u_n|^p b(|u_n|) dx \\ & \quad + \frac{C_2}{h} \int_{\{|u_n| \leq R\}} |c(x)|^{p'} |u_n|^p |T_h(u_n)| d(|u_n|) dx \\ & \quad + \frac{C_2}{h} \int_{\{R < |u_n| \leq h\}} |c(x)|^{p'} |u_n|^p b(|u_n|) dx \\ & \quad + \frac{C_2}{h} \int_{\{R < |u_n|\}} |c(x)|^{p'} |u_n|^p |T_h(u_n)| d(|u_n|) dx \\ & \leq \varepsilon_2(h) + \varepsilon_3(h) + \frac{C_2}{h} \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \\ & \quad \times (\|u_n b(|u_n|)^{1/p}\|_{L^{p^*}(\{|u_n| \leq h\})}^p + \|u_n |T_h(u_n)|^{1/p} d(|u_n|)^{1/p}\|_{L^{p^*}(\Omega)}^p), \end{aligned}$$

where

$$\begin{aligned} \varepsilon_2(h) &= \varepsilon_1(h) + \frac{2^{p-1}C_1}{h} \left( \int_{\{|u_n| \leq h\}} |c(x)|^{p'} b(|u_n|) dx + \int_{\Omega} |c(x)|^{p'} |T_h(u_n)| d(|u_n|) dx \right) \\ &= \varepsilon_1(h) + \frac{2^{p-1}C_1}{h} \left( \|b(\cdot)\|_{L^\infty(\mathbb{R})} \int_{\{|u_n| \leq h\}} |c(x)|^{p'} dx \right. \\ & \quad \left. + \|d(\cdot)\|_{L^\infty(\mathbb{R})} \int_{\Omega} |c(x)|^{p'} |T_h(u_n)| dx \right) \rightarrow 0 \quad \text{as } h \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned}
\varepsilon_3(h) &= \frac{C_2}{h} \int_{\{|u_n| \leq R\}} |c(x)|^{p'} |u_n|^p b(|u_n|) dx \\
&\quad + \frac{C_2}{h} \int_{\{|u_n| \leq R\}} |c(x)|^{p'} |u_n|^p |T_h(u_n)| d(|u_n|) dx \\
&= \frac{C_2}{h} (R^p \|b(\cdot)\|_{L^\infty(\mathbb{R})} + R^{p+1} \|b(\cdot)\|_{L^\infty(\mathbb{R})}) \int_{\Omega} |c(x)|^{p'} dx \rightarrow 0 \quad \text{as } h \rightarrow \infty.
\end{aligned}$$

We have that  $b(|s|)$  and  $d(|s|)$  are two decreasing functions, and having in mind that  $d(|\cdot|)/b(|\cdot|)$  is a decreasing function and belongs to  $L^1(\mathbb{R})$ , then there exist two positive constants  $\mu$  and  $s_0$  such that  $d(|s|)/b(|s|) \leq \mu/|s|$  for any  $|s| \geq s_0$ . Then  $|T_h(s)|^{1/p} d(|s|)^{1/p} \leq \mu^{1/p} b(|s|)^{1/p} \in L^1(\mathbb{R})$  and we obtain

$$\begin{aligned}
&|s| |T_h(s)|^{1/p} d(|s|)^{1/p} \cdot \chi_{\{|s| > s_0\}} \\
&\leq \mu^{1/p} |s| b(|s|)^{1/p} \cdot \chi_{\{|s| \leq h\}} + |s| |T_h(s)|^{1/p} d(|s|)^{1/p} \cdot \chi_{\{|s| > h\}} \\
&\leq \mu^{1/p} \int_0^{|T_h(s)|} b(\tau)^{1/p} d\tau + \int_0^{|s|} |T_h(s)|^{1/p} d(|s|)^{1/p} \cdot \chi_{\{|s| > h\}} d\tau.
\end{aligned}$$

This follows that there exists a positive constant  $C_3$  such that

(4.16)

$$|s| |T_h(s)|^{1/p} d(|s|)^{1/p} \leq C_3 + \mu^{1/p} \int_0^{|T_h(s)|} b(\tau)^{1/p} d\tau + \int_0^{|s|} |T_h(\tau)|^{1/p} d(\tau)^{1/p} d\tau.$$

Thanks to (4.15), we conclude that

$$\begin{aligned}
(4.17) \quad &\frac{1}{2h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx + \frac{1}{2h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| dx \\
&\leq \varepsilon_4(h) + \frac{C_4}{h} \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \left( \left\| \int_0^{|u_n|} b(\tau)^{1/p} d\tau \right\|_{L^{p^*}(\{|u_n| \leq h\})}^p \right. \\
&\quad \left. + \left\| \int_0^{|u_n|} |T_h(\tau)|^{1/p} d(\tau)^{1/p} d\tau \right\|_{L^{p^*}(\Omega)}^p + 1 \right) \\
&\leq \varepsilon_5(h) + \frac{C_s^p C_4}{h} \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \\
&\quad \times \left( \left\| \nabla \int_0^{|u_n|} b(\tau)^{1/p} d\tau \right\|_{L^p(\{|u_n| \leq h\})}^p + \left\| \nabla \int_0^{|u_n|} |T_h(\tau)|^{1/p} d(\tau)^{1/p} d\tau \right\|_{L^p(\Omega)}^p \right) \\
&= \varepsilon_5(h) + \frac{C_s^p C_4}{h} \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \\
&\quad \times \left( \int_{\Omega} b(|u_n|) |\nabla T_h(u_n)|^p dx + \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| dx \right).
\end{aligned}$$

By taking  $R$  large enough such that  $C_s^p C_4 \|c(x)\|_{L^{N/(p-1)}(\{R < |u_n|\})}^{p'} \leq \frac{1}{4}$ , we conclude that

$$(4.18) \quad \frac{1}{4h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx + \frac{1}{4h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| \, dx \leq \varepsilon_5(h)$$

for any  $n \in \mathbb{N}^*$ . Thus, by letting  $h$  tend to infinity in (4.18), we conclude that

$$(4.19) \quad \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = 0.$$

Moreover, we have

$$(4.20) \quad \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\Omega} d(|u_n|) |\nabla u_n|^p |T_h(u_n)| \, dx = 0,$$

then

$$(4.21) \quad \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{|u_n| > h\}} d(|u_n|) |\nabla u_n|^p \, dx = 0.$$

**Step 4: Convergence of the gradients.** Let  $h \geq k \geq 1$ , we set

$$(4.22) \quad S_h(s) = 1 - \frac{|T_{2h}(s) - T_h(s)|}{h} \quad \text{and} \quad H(s) = 2 \int_0^s \frac{d(|\tau|)}{b(|\tau|)} \, d\tau.$$

Let  $\varphi(s) = s \exp(\frac{1}{2} \gamma^2 s^2)$ , where  $\gamma = \frac{7}{2} \|d(|\cdot|)/b(|\cdot|)\|_{L^\infty(\mathbb{R})}$ . It is obvious that

$$\varphi'(s) - \gamma |\varphi(s)| \geq \frac{1}{2} \quad \text{for all } s \in \mathbb{R}.$$

By taking  $\varphi(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$  as a test function in the approximate problem (4.1), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla (\varphi(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)}) \, dx \\ & \quad + \int_{\Omega} g_n(x, u_n, \nabla u_n) (\varphi(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)}) \, dx \\ & = \int_{\Omega} f_n(\varphi(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)}) \, dx \\ & \quad + \int_{\Omega} \phi(x, u_n) \cdot \nabla (\varphi(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)}) \, dx. \end{aligned}$$

We have  $S_h(u_n) = 1$  on the set  $\{|u_n| \leq h\}$ , and since  $\varphi(T_k(u_n) - T_k(u))$  has the same sign as  $u_n$  on the set  $\{|u_n| \geq k\}$ , in view of assumptions (2.4)–(2.6), we conclude that

$$\begin{aligned}
(4.23) \quad & \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \varphi'(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} dx \\
& - 2 \int_{\{|u_n| \leq k\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} \\
& \times |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
& + 2 \int_{\{|u_n| > k\}} d(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
\leq & e^{H(\infty)} \int_{\Omega} (|f_0| + |f_n|) |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) dx \\
& + \int_{\Omega} d(u_n) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
& + \frac{1}{h} \int_{\{h < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n |\varphi(T_k(u_n) - T_k(u))| e^{H(|u_n|)} dx \\
& - \frac{1}{h} \int_{\{h < |u_n| \leq 2h\}} c(x) (1 + |u_n|)^{\alpha} |\nabla u_n| |\varphi(T_k(u_n) - T_k(u))| e^{H(|u_n|)} dx \\
& + \int_{\Omega} c(x) (1 + |u_n|)^{\alpha} |\nabla T_k(u_n) - \nabla T_k(u)| \varphi'(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} dx \\
& + 2 \int_{\{|u_n| \leq 2h\}} c(x) (1 + |u_n|)^{\alpha} |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} \\
& \times |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx.
\end{aligned}$$

For the first term on the right-hand side of (4.23) we have  $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$  weak-\* in  $L^{\infty}(\Omega)$ , and since  $f_n$  converges strongly to  $f$  in  $L^1(\Omega)$  as  $n$  goes to infinity, we obtain

$$\begin{aligned}
(4.24) \quad \varepsilon_1(n) &= \left| \int_{\Omega} (|f_0| + |f_n|) |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) dx \right| \\
&\leq \int_{\Omega} (|f_0| + |f_n|) |\varphi(T_k(u_n) - T_k(u))| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Concerning the third and fourth terms on the right-hand side of (4.23), in view of (4.19), we have

$$\begin{aligned}
(4.25) \quad \varepsilon_2(n) &= \frac{1}{h} \int_{\{h < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n |\varphi(T_k(u_n) - T_k(u))| e^{H(|u_n|)} dx \\
&\leq \frac{\varphi(2k) e^{H(\infty)}}{h} \int_{\{h < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\end{aligned}$$



and thanks to Young's inequality, since  $b(s)$  is a decreasing function, and similarly as in (4.17), we can show that

$$\begin{aligned}
(4.26) \quad \varepsilon_3(h) &= \left| \frac{1}{h} \int_{\{h < |u_n| \leq 2h\}} c(x)(1 + |u_n|)^\alpha |\nabla u_n| |\varphi(T_k(u_n) - T_k(u))| e^{H(|u_n|)} dx \right| \\
&\leq \frac{e^{H(\infty)}}{h} \int_{\{h < |u_n| \leq 2h\}} |c(x)|^{p'} \frac{(1 + |u_n|)^{\alpha p'}}{b(|u_n|)^{p'-1}} |\varphi(T_k(u_n) - T_k(u))| dx \\
&\quad + \frac{e^{H(\infty)}}{h} \int_{\{h < |u_n| \leq 2h\}} b(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| dx \\
&\leq \frac{\varphi(2k)e^{H(\infty)}}{h} \int_{\{h < |u_n| \leq 2h\}} |c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} b(|u_n|) dx \\
&\quad + \frac{\varphi(2k)e^{H(\infty)}}{h} \int_{\{h < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n dx \\
&\leq \frac{2^{p-1} \varphi(2k) k e^{H(\infty)}}{h} \int_{\{|u_n| \leq 2h\}} |c(x)|^{p'} |u_n|^p b(|u_n|) dx + \varepsilon_4(h) \\
&\leq \frac{2^{p-1} \varphi(2k) k e^{H(\infty)}}{h} \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \\
&\quad \times \int_{\Omega} b(|u_n|) |\nabla T_{2h}(u_n)|^p dx + \varepsilon_4(h) \rightarrow 0 \quad \text{as } h \rightarrow \infty.
\end{aligned}$$

For the fifth term on the right-hand side of (4.23), since  $1 \leq \varphi'(T_k(u_n) - T_k(u)) \leq \varphi'(2k)$ , and we have that  $(1 + |T_{2h}(u_n)|)^\alpha$  converges strongly to  $(1 + |T_{2h}(u)|)^\alpha$  in  $L^{Np'/(N-p)}(\Omega)$  and  $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$  in  $L^p(\Omega)$ , it follows that

$$\begin{aligned}
(4.27) \quad \varepsilon_5(n) &= \left| \int_{\Omega} c(x)(1 + |u_n|)^\alpha |\nabla T_k(u_n) - \nabla T_k(u)| \right. \\
&\quad \left. \times \varphi'(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} dx \right| \\
&\leq e^{H(\infty)} \varphi'(2k) \int_{\Omega} |c(x)| (1 + |T_{2h}(u_n)|)^\alpha \\
&\quad \times |\nabla T_k(u_n) - \nabla T_k(u)| dx \rightarrow 0 \quad \text{as } h \rightarrow \infty.
\end{aligned}$$

Concerning the last term on the right-hand side of (4.23), we have

$$\begin{aligned}
(4.28) \quad &\int_{\{|u_n| \leq 2h\}} c(x)(1 + |u_n|)^\alpha |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
&\leq \frac{1}{4} \int_{\{|u_n| \leq 2h\}} d(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
&\quad + \int_{\{|u_n| \leq 2h\}} |c(x)|^{p'} (1 + |u_n|)^{\alpha p'} \frac{d(|u_n|)}{b(|u_n|)^{p'}} \\
&\quad \times |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4} \int_{\{|u_n| \leq 2h\}} d(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
&\quad + e^{H(\infty)} \int_{\{|u_n| \leq 2h\}} |c(x)|^{p'} (1 + |T_{2h}(u_n)|)^p d(|u_n|) |\varphi(T_k(u_n) - T_k(u))| dx \\
&\leq \frac{1}{4} \int_{\{|u_n| \leq 2h\}} d(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx + \varepsilon_6(n)
\end{aligned}$$

with

$$\varepsilon_6(n) = e^{H(\infty)} \int_{\{|u_n| \leq 2h\}} |c(x)|^{p'} (1 + |T_{2h}(u_n)|)^p d(|u_n|) |\varphi(T_k(u_n) - T_k(u))| dx \rightarrow 0$$

as  $n \rightarrow \infty$ . By combining (4.23) and (4.24)–(4.28), we conclude that

$$\begin{aligned}
(4.29) \quad &\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \varphi'(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} dx \\
&\quad + \frac{1}{2} \int_{\{|u_n| > k\}} d(|u_n|) |\nabla u_n|^p |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
&\quad - \frac{7}{2} \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\
&\leq \varepsilon_7(n, h).
\end{aligned}$$

For the first term on the left-hand side of (4.29), we have  $a(x, T_k(u_n), \nabla T_k(u_n)) = 0$  on the set  $\{k < |u_n|\}$ . Then

$$\begin{aligned}
(4.30) \quad &\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \varphi'(T_k(u_n) - T_k(u)) S_h(u_n) e^{H(|u_n|)} dx \\
&= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
&\quad \times \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \\
&\quad - \int_{\{k < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \\
&= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
&\quad \times \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \\
&\quad + \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
&\quad \times \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \\
&\quad - \int_{\{k < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx.
\end{aligned}$$

For the second term on the right-hand side of (4.30), we have  $a(x, T_k(u_n), \nabla T_k(u)) \rightarrow a(x, T_k(u), \nabla T_k(u))$  strongly in  $(L^{p'}(\Omega))^N$ , and since  $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$  weakly in  $(L^p(\Omega))^N$ ,

$$(4.31) \quad \begin{aligned} \varepsilon_8(n) &= \left| \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \right. \\ &\quad \left. \times \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \right| \\ &\leq e^{H(\infty)} \varphi'(2k) \int_{\Omega} |a(x, T_k(u_n), \nabla T_k(u))| \\ &\quad \times |\nabla T_k(u_n) - \nabla T_k(u)| dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Concerning the last term on the right-hand side of (4.30), we have that  $(|a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))|)_n$  is bounded in  $L^{p'}(\Omega)$ . Then there exists a function  $\psi_{2h} \in L^{p'}(\Omega)$  such that  $|a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| \rightharpoonup \psi_{2h}$  weakly in  $L^{p'}(\Omega)$ , which yields that

$$(4.32) \quad \begin{aligned} \varepsilon_8(n) &= \left| \int_{\{k < |u_n| \leq 2h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u) \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \right| \\ &\leq e^{H(\infty)} \varphi'(2k) \int_{\{k < |u_n| \leq 2h\}} |a(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| |\nabla T_k(u)| dx \\ &\rightarrow e^{H(\infty)} \varphi'(2k) \int_{\{k < |u| \leq 2h\}} \psi_{2h} |\nabla T_k(u)| dx = 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

By combining (4.30)–(4.32), we conclude that

$$(4.33) \quad \begin{aligned} &\int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) S_h(u_n) \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx \\ &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u))) \\ &\quad \times (\nabla T_k(u_n) - \nabla T_k(u)) \varphi'(T_k(u_n) - T_k(u)) e^{H(|u_n|)} dx + \varepsilon_9(n). \end{aligned}$$

Similarly, we can show that

$$(4.34) \quad \begin{aligned} &\int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n) \cdot \nabla u_n \frac{d(|u_n|)}{b(|u_n|)} |\varphi(T_k(u_n) - T_k(u))| S_h(u_n) e^{H(|u_n|)} dx \\ &\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\ &\quad \times \left\| \frac{d(\cdot)}{b(\cdot)} \right\|_{L^\infty(\mathbb{R})} |\varphi(T_k(u_n) - T_k(u))| e^{H(|u_n|)} dx + \varepsilon_{10}(n). \end{aligned}$$

We have  $\varphi'(s) - \gamma|\varphi(s)| \geq \frac{1}{2}$  for any  $s \in \mathbb{R}$ , thus, by combining (4.29), (4.33) and (4.34) we conclude that

$$\begin{aligned}
 (4.35) \quad 0 &\leq \frac{1}{2} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \\
 &\leq \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \\
 &\quad \times \left( \varphi'(T_k(u_n) - T_k(u)) - \frac{7}{2} \left\| \frac{d(\cdot)}{b(\cdot)} \right\|_{L^\infty(\mathbb{R})} |\varphi(T_k(u_n) - T_k(u))| \right) e^{H(|u_n|)} \, dx \\
 &\quad + \varepsilon_{11}(n) \\
 &\leq \varepsilon_7(n, h).
 \end{aligned}$$

By letting  $n$  and  $h$  tend to infinity, we obtain

$$(4.36) \quad \lim_{n \rightarrow \infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \cdot (\nabla T_k(u_n) - \nabla T_k(u)) \, dx = 0.$$

Under assumptions (2.2)–(2.4), it is well known that this implies

$$(4.37) \quad T_k(u_n) \rightarrow T_k(u) \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad \nabla u_n \rightarrow \nabla u \text{ a.e. in } \Omega.$$

Moreover, since  $a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n$  tends to  $a(x, u, \nabla u) \cdot \nabla u$  almost everywhere in  $\Omega$ , and in view of Fatou's lemma and (4.19), we conclude that

$$\begin{aligned}
 (4.38) \quad &\lim_{h \rightarrow \infty} \frac{1}{h} \int_{\{|u| \leq h\}} a(x, u, \nabla u) \cdot \nabla u \, dx \\
 &\leq \lim_{h \rightarrow \infty} \liminf_{n \rightarrow \infty} \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \\
 &\leq \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \frac{1}{h} \int_{\{|u_n| \leq h\}} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx = 0,
 \end{aligned}$$

which proves (3.1).

**Step 5: The equi-integrability of  $g_n(x, u_n, \nabla u_n)$ .** To prove that

$$g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega),$$

using Vitali's theorem, it is sufficient to show that the sequence  $(g_n(x, u_n, \nabla u_n))_n$  is uniformly equi-integrable. Indeed, thanks to (4.21), we have

$$(4.39) \quad \lim_{h \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{\{|u_n| > h\}} d(|u_n|) |\nabla u_n|^p \, dx = 0.$$

Having in mind that  $f_0 \in L^1(\Omega)$ , we conclude that

$$\int_{\{|u_n|>h\}} |g(x, u_n, \nabla u_n)| \, dx \leq \int_{\{|u_n|>h\}} |f_0| \, dx + \int_{\{|u_n|>h\}} d(|u_n|) |\nabla u_n|^p \, dx \rightarrow 0$$

as  $h \rightarrow \infty$ , thus, for all  $\varepsilon > 0$ , there exists  $h_0(\varepsilon) > 0$  such that

$$(4.40) \quad \int_{\{|u_n|>h\}} |g(x, u_n, \nabla u_n)| \, dx \leq \frac{\varepsilon}{2} \quad \text{for any } h \geq h_0(\varepsilon).$$

On the other hand, for any measurable subset  $E \subset \Omega$  we have

$$(4.41) \quad \int_E |g_n(x, u_n, \nabla u_n)| \, dx \leq \int_E |g_n(x, T_h(u_n), \nabla T_h(u_n))| \, dx \\ + \int_{\{|u_n|>h\}} |g(x, u_n, \nabla u_n)| \, dx.$$

Thanks to (4.37), there exists  $\beta(\varepsilon) > 0$  small enough such that

$$(4.42) \quad \int_E |g_n(x, T_h(u_n), \nabla T_h(u_n))| \, dx \leq \int_E |f_0(x)| \, dx + \int_E d(|T_h(u_n)|) |\nabla T_h(u_n)|^p \, dx \leq \frac{\varepsilon}{2}.$$

By combining (4.40), (4.41) and (4.42), we deduce that for any  $\varepsilon > 0$  there exists  $\beta(\varepsilon) > 0$  such that

$$(4.43) \quad \int_E |g_n(x, u_n, \nabla u_n)| \, dx \leq \varepsilon \quad \text{with } E \subseteq \Omega \text{ such that } \text{meas}(E) \leq \beta(\varepsilon).$$

We conclude that the sequence  $(g_n(x, u_n, \nabla u_n))_n$  is uniformly equi-integrable, and thanks to (4.37), we have

$$(4.44) \quad g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \quad \text{a.e. in } \Omega.$$

Thus, in view of Vitali's theorem, we obtain

$$(4.45) \quad g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega).$$

**Step 6: Passage to the limit.** Let  $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ , and let  $S(\cdot)$  be a smooth function in  $W^{1,\infty}(\mathbb{R})$  such that  $\text{supp}(S(\cdot)) \subseteq [-M, M]$  for some  $M \geq 0$ .

By choosing  $S(u_n)\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$  as a test function in the approximate problem (4.1), we obtain

$$(4.46) \quad \int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi) \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) S(u_n)\varphi \, dx \\ = \int_\Omega f_n S(u_n)\varphi \, dx + \int_\Omega \phi(x, T_n(u_n)) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi) \, dx.$$

We begin by the first term on the left-hand side of (4.46). We have

$$\begin{aligned} & \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi) \, dx \\ &= \int_{\Omega} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (S'(u_n)\varphi \nabla T_M(u_n) + S(T_M(u_n))\nabla\varphi) \, dx. \end{aligned}$$

In view of (2.3), we have that  $(a(x, T_M(u_n), \nabla T_M(u_n)))_n$  is bounded in  $(L^{p'}(\Omega))^N$ , and since  $a(x, T_M(u_n), \nabla T_M(u_n))$  tends to  $a(x, T_M(u), \nabla T_M(u))$  almost everywhere in  $\Omega$ , it follows that

$$a(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a(x, T_M(u), \nabla T_M(u)) \quad \text{in } (L^{p'}(\Omega))^N,$$

and since  $S'(u_n)\varphi \nabla T_M(u_n) + S(T_M(u_n))\nabla\varphi$  tends strongly to  $S'(u)\varphi \nabla T_M(u) + S(T_M(u))\nabla\varphi$  in  $(L^p(\Omega))^N$ , we deduce that

$$\begin{aligned} (4.47) \quad & \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot (\nabla u_n S'(u_n)\varphi + S(u_n)\nabla\varphi) \, dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_M(u_n), \nabla T_M(u_n)) \cdot (\nabla T_M(u_n) S'(T_M(u_n))\varphi + S(T_M(u_n))\nabla\varphi) \, dx \\ &= \int_{\Omega} a(x, T_M(u), \nabla T_M(u)) \cdot (\nabla T_M(u) S'(T_M(u))\varphi + S(T_M(u))\nabla\varphi) \, dx \\ &= \int_{\Omega} a(x, u, \nabla u) \cdot (\nabla u S'(u)\varphi + S(u)\nabla\varphi) \, dx. \end{aligned}$$

Concerning the second term on the right-hand side of (4.46), we have  $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$  weak-\* in  $L^\infty(\Omega)$ , and thanks to (4.45), we have that  $g_n(x, u_n, \nabla u_n) \rightarrow g(x, u, \nabla u)$  strongly in  $L^1(\Omega)$ , which yields that

$$\begin{aligned} (4.48) \quad & \lim_{n \rightarrow \infty} \int_{\Omega} g_n(x, u_n, \nabla u_n) S(T_M(u_n))\varphi \, dx = \int_{\Omega} g(x, u, \nabla u) S(T_M(u))\varphi \, dx \\ &= \int_{\Omega} g(x, u, \nabla u) S(u)\varphi \, dx. \end{aligned}$$

Similarly, we have  $f_n \rightarrow f$  strongly in  $L^1(\Omega)$ , then

$$(4.49) \quad \lim_{n \rightarrow \infty} \int_{\Omega} f_n S(T_M(u_n))\varphi \, dx = \int_{\Omega} f S(T_M(u))\varphi \, dx = \int_{\Omega} f S(u)\varphi \, dx.$$

For the last term on the right-hand side of (4.46), we have  $\phi_n(x, u_n)S(u_n) = \phi(x, T_M(u_n))S(T_M(u_n))$  for  $n$  large enough (for example  $n \geq M$ ), and since

$\phi(x, T_M(u_n)) \rightarrow \phi(x, T_M(u))$  strongly in  $(L^{p'}(\Omega))^N$ , we obtain  
(4.50)

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} \phi_n(x, u_n) \cdot (\nabla u_n S'(u_n) \varphi + S(u_n) \nabla \varphi) \, dx \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \phi(x, T_M(u_n)) \cdot (\nabla T_M(u_n) S'(T_M(u_n)) \varphi + S(T_M(u_n)) \nabla \varphi) \, dx \\ &= \int_{\Omega} \phi(x, T_M(u)) \cdot (\nabla T_M(u) S'(T_M(u)) \varphi + S(T_M(u)) \nabla \varphi) \, dx \\ &= \int_{\Omega} \phi(x, u) \cdot (\nabla u S'(u) \varphi + S(u) \nabla \varphi) \, dx. \end{aligned}$$

By combining (4.46)–(4.50), we conclude that

$$\begin{aligned} (4.51) \quad & \int_{\Omega} a(x, u, \nabla u) \cdot (S'(u) \varphi \nabla u + S(u) \nabla \varphi) \, dx + \int_{\Omega} g(x, u, \nabla u) S(u) \varphi \, dx \\ &= \int_{\Omega} f S(u) \varphi \, dx + \int_{\Omega} \phi(x, u) \cdot (S'(u) \varphi \cdot \nabla u + S(u) \nabla \varphi) \, dx, \end{aligned}$$

which completes the proof of Theorem 3.1.

## 5. APPENDIX: PROOF OF PROPOSITION 4.1

*Case 1.* Assuming that there exist two positive constants  $d_0$  and  $s_0$  such that

$$(5.1) \quad d(|s|) \leq \frac{d_0}{(1 + |s|)^p} \quad \text{for any } s \geq s_0,$$

we have

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{meas}(\{|u_n| > k\}) = 0.$$

**Proof of Case 1.** Let  $k \geq 1$  and  $n$  large enough. We consider the function  $\psi(\cdot)$  defined by

$$(5.2) \quad \psi(s) = \frac{1}{p - \lambda - 1} \left( 1 - \frac{1}{(1 + |T_k(s)|)^{p - \lambda - 1}} \right) \text{sign}(s),$$

where  $\lambda < p - 1$  and  $H(s) = 2 \int_0^s d(|\tau|) / b(|\tau|) \, d\tau$ . By taking  $\psi(u_n) e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$  as a test function in the approximate problem (4.1), we have

$$\begin{aligned} (5.3) \quad & \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla (\psi(u_n) e^{H(|u_n|)}) \, dx + \int_{\Omega} g_n(x, u_n, \nabla u_n) \psi(u_n) e^{H(|u_n|)} \, dx \\ &= \int_{\Omega} f_n \psi(u_n) e^{H(|u_n|)} \, dx + \int_{\Omega} \phi(x, T_n(u_n)) \cdot \nabla (\psi(u_n) e^{H(|u_n|)}) \, dx. \end{aligned}$$

In view of (2.4), (2.5) and (2.6), we conclude that

$$\begin{aligned}
(5.4) \quad & \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx + 2 \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| e^{H(|u_n|)} dx \\
& \leq \int_{\{|u_n| \leq k\}} \frac{a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx \\
& \quad + 2 \int_{\Omega} a(x, T_n(u_n), \nabla u_n) \cdot \nabla u_n |\psi(u_n)| \frac{d(|u_n|)}{b(|u_n|)} e^{H(|u_n|)} dx \\
& \leq \int_{\Omega} (|f_n(x)| + |f_0(x)|) |\psi(u_n)| e^{H(|u_n|)} dx \\
& \quad + \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| e^{H(|u_n|)} dx \\
& \quad + \int_{\{|u_n| \leq k\}} \frac{c(x)(1 + |u_n|)^\alpha |\nabla T_k(u_n)|}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx \\
& \quad + 2 \int_{\Omega} c(x)(1 + |u_n|)^\alpha |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} |\psi(u_n)| e^{H(|u_n|)} dx.
\end{aligned}$$

Thanks to Young's inequality, we obtain

$$\begin{aligned}
(5.5) \quad & \int_{\{|u_n| \leq k\}} \frac{c(x)(1 + |u_n|)^\alpha |\nabla T_k(u_n)|}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx \\
& \leq \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla T_k(u_n)|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx \\
& \quad + C_0 \int_{\{|u_n| \leq k\}} \frac{|c(x)|^{p'} (1 + |u_n|)^{\alpha p'}}{(1 + |T_k(u_n)|)^{p-\lambda} b(|u_n|)^{p'-1}} e^{H(|u_n|)} dx \\
& \leq \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla T_k(u_n)|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} e^{H(|u_n|)} dx \\
& \quad + C_1 \int_{\{|u_n| \leq k\}} \frac{|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'-\lambda}}{(1 + |u_n|)^{p-\lambda}} e^{H(|u_n|)} dx,
\end{aligned}$$

and thanks to (2.4), we have

$$\begin{aligned}
(5.6) \quad & \int_{\Omega} c(x)(1 + |u_n|)^\alpha |\nabla u_n| \frac{d(|u_n|)}{b(|u_n|)} |\psi(u_n)| e^{H(|u_n|)} dx \\
& \leq C_2 \int_{\Omega} |c(x)|^{p'} (1 + |u_n|)^{\alpha p'} \frac{d(|u_n|)}{b(|u_n|)^{p'}} |\psi(u_n)| e^{H(|u_n|)} dx \\
& \quad + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| e^{H(|u_n|)} dx \\
& \leq C_3 \int_{\Omega} d(|u_n|) |c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'-\lambda} |\psi(u_n)| e^{H(|u_n|)} dx \\
& \quad + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| e^{H(|u_n|)} dx.
\end{aligned}$$



By combining (5.4), (5.5) and (5.6), we conclude that

$$\begin{aligned}
(5.7) \quad & \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \\
& \leq C_4(\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) \\
& \quad + C_1 e^{H(\infty)} \int_{\{|u_n| \leq k\}} \frac{|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p' - \lambda}}{(1 + |u_n|)^{p-\lambda}} dx \\
& \quad + C_3 e^{H(\infty)} \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx \\
& \leq C_4(\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)}) + C_1 e^{H(\infty)} \int_{\{|u_n| \leq k\}} |c(x)|^{p'} dx \\
& \quad + C_3 e^{H(\infty)} \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx \\
& \leq C_5 + C_3 e^{H(\infty)} \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx.
\end{aligned}$$

In view of (5.1) we have  $d(|s|)(1 + |s|)^p \leq d_0$  for any  $s \geq s_0$ , which yields that

$$\begin{aligned}
(5.8) \quad & \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \\
& \leq C_5 + C_3 d_0 e^{H(\infty)} \int_{\Omega} |c(x)|^{p'} dx \leq C_6.
\end{aligned}$$

In view of (2.4), we conclude that

$$(5.9) \quad b_0 \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^p}{(1 + |T_k(u_n)|)^p} dx \leq \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx \leq C_7.$$

Using the Poincaré inequality, we get

$$\begin{aligned}
|\log(1+k)|^p \text{meas}(\{|u_n| > k\}) &= \int_{\{|u_n| > k\}} |\log(1 + |T_k(u_n)|)|^p dx \\
&\leq \int_{\Omega} |\log(1 + |T_k(u_n)|)|^p dx \\
&\leq C_p^p \int_{\Omega} |\nabla \log(1 + |T_k(u_n)|)|^p dx \\
&= C_p^p \int_{\Omega} \frac{|\nabla T_k(u_n)|^p}{(1 + |T_k(u_n)|)^p} dx \\
&\leq C_8
\end{aligned}$$

with  $C_8$  being a positive constant that does not depend on  $n$  and  $k$ . Thus,

$$(5.10) \quad \limsup_{n \rightarrow \infty} \text{meas}(\{|u_n| > k\}) \leq \frac{C_8}{|\log(1+k)|^p} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which concludes the proof of Case 1. □

Case 2. We assume that  $0 \leq \alpha < p - 1 - \lambda$ . Then

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{meas}(\{|u_n| > k\}) = 0.$$

Proof of Case 2. By taking  $\psi(u_n)e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$  as a test function in the approximate problem (4.1) and similarly as in (5.7) we can prove that

$$(5.11) \quad \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \\ \leq C_5 + C_3 e^{H(\infty)} \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx.$$

We have that  $d(\cdot)$  is a decreasing function, then there exists a positive constant  $r_0 > 0$  such that  $1/r_0 \leq |\psi(u_n)| \leq 1/(p - \lambda - 1)$  on the set  $\{|u_n| \geq 1\}$ , which yields that

$$|u_n|d(|u_n|)^{1/p} |\psi(u_n)|^{1/p} \leq \frac{|u_n|d(|u_n|)^{1/p}}{(p - \lambda - 1)^{1/p}} \leq \frac{1}{(p - \lambda - 1)^{1/p}} \int_0^{|u_n|} d(s)^{1/p} ds \\ \leq \frac{1}{(p - \lambda - 1)^{1/p}} \int_0^1 d(s)^{1/p} ds \\ + \frac{r_0^{1/p}}{(p - \lambda - 1)^{1/p}} \int_1^{|u_n|} d(s)^{1/p} |\psi(u_n)|^{1/p} ds. \\ = C_9 + C_{10} \int_0^{|u_n|} d(s)^{1/p} |\psi(u_n)|^{1/p} ds.$$

Having in mind that  $0 \leq \alpha < p - 1 - \lambda$  let  $\varepsilon > 0$  and by using Young's and Sobolev inequalities, we conclude that

$$(5.12) \quad \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \\ \leq C_5 + C_{11} \int_{\Omega} d(|u_n|)|c(x)|^{p'} dx + \varepsilon \int_{\Omega} d(|u_n|)|c(x)|^{p'} |u_n|^p |\psi(u_n)| dx \\ \leq C_{12} + 2^{p-1} C_{10}^p \varepsilon \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \left\| \int_0^{|u_n|} d(s)^{1/p} |\psi(s)|^{1/p} ds \right\|_{L^{p^*}(\Omega)}^p \\ \leq C_{12} + 2^{p-1} C_{10}^p \varepsilon C_s^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \left\| \nabla \int_0^{|u_n|} d(s)^{1/p} |\psi(s)|^{1/p} ds \right\|_{L^p(\Omega)}^p \\ = C_{12} + 2^{p-1} C_{10}^p \varepsilon C_s^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx.$$

By taking  $\varepsilon > 0$  small enough such that  $2^{p-1} C_{10}^p \varepsilon C_s^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \leq \frac{1}{4}$ , we obtain

$$(5.13) \quad \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{4} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \leq C_{12}.$$

In view of (2.4), we conclude that

$$(5.14) \quad b_0 \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^p}{(1 + |T_k(u_n)|)^p} dx \leq \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx \leq C_{13}.$$

Following the same steps as in the previous case, we conclude that

$$(5.15) \quad \limsup_n \text{meas}(\{|u_n| > k\}) \leq \frac{C_{14}}{|\log(1+k)|^p} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which concludes the proof of Case 2.  $\square$

*Case 3.* We assume that  $\|c(x)\|_{L^{N/(p-1)}(\Omega)}$  is small enough. Then

$$\lim_{k \rightarrow \infty} \limsup_{n \rightarrow \infty} \text{meas}(\{|u_n| > k\}) = 0.$$

**Proof of Case 3.** By taking  $\psi(u_n)e^{H(|u_n|)} \in W_0^{1,p}(\Omega)$  as a test function in the approximate problem (4.1) and similarly as in (5.7) we can prove that

$$(5.16) \quad \begin{aligned} & \frac{1}{2} \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx + \frac{1}{2} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \\ & \leq C_5 + C_3 e^{H(\infty)} \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx \end{aligned}$$

with  $C_3 = 2^{p'-1}/(p'p^{p'-1})$ . For the last term on the right-hand side of (5.16), we have

$$(5.17) \quad \begin{aligned} & \int_{\Omega} d(|u_n|)|c(x)|^{p'} (1 + |u_n|)^{(\alpha+\lambda)p'} |\psi(u_n)| dx \\ & \leq 2^{p-1} \int_{\Omega} d(|u_n|)|c(x)|^{p'} |\psi(u_n)| dx + 2^{p-1} \int_{\Omega} d(|u_n|)|c(x)|^{p'} |u_n|^p |\psi(u_n)| dx \\ & \leq C_{15} + 2^{p-1} C_{10}^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \left\| \int_0^{|u_n|} d(s)^{1/p} |\psi(s)|^{1/p} ds \right\|_{L^{p^*}(\Omega)}^p \\ & \leq C_{15} + 2^{p-1} C_{10}^p C_s^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \left\| \nabla \int_0^{|u_n|} d(s)^{1/p} |\psi(s)|^{1/p} ds \right\|_{L^p(\Omega)}^p \\ & = C_{15} + 2^{p-1} C_{10}^p C_s^p \|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \int_{\Omega} d(|u_n|)|\nabla u_n|^p |\psi(u_n)| dx \end{aligned}$$

with  $C_{10}^p = r_0/(p - \lambda - 1)$  and  $C_s$  being the constant of the Sobolev inequality. Choosing the measurable function  $c(x)$  such that  $\|c(x)\|_{L^{N/(p-1)}(\Omega)}$  is small enough, for example

$$\|c(x)\|_{L^{N/(p-1)}(\Omega)}^{p'} \leq \frac{1}{2^{p+1}C_{10}^p C_s^p C_3 e^{H(\infty)}},$$

it follows that

$$(5.18) \quad b_0 \int_{\{|u_n| \leq k\}} \frac{|\nabla u_n|^p}{(1 + |T_k(u_n)|)^p} dx \leq \int_{\{|u_n| \leq k\}} \frac{b(|u_n|)|\nabla u_n|^p}{(1 + |T_k(u_n)|)^{p-\lambda}} dx \leq C_{16}.$$

Using similar process as in the first case, we deduce that

$$(5.19) \quad \limsup_{n \rightarrow \infty} \text{meas}(\{|u_n| > k\}) \leq \frac{C_{17}}{|\log(1+k)|^p} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which proves Case 3. Thus, the proof of Proposition 4.1 is complete.  $\square$

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#### References

- [1] *A. Alvino, L. Boccardo, V. Ferone, L. Orsina, G. Trombetti:* Existence results for non-linear elliptic equations with degenerate coercivity. *Ann. Mat. Pura Appl., IV. Ser.* 182 (2003), 53–79. [zbl](#) [MR](#) [doi](#)
- [2] *A. Alvino, V. Ferone, G. Trombetti:* A priori estimates for a class of nonuniformly elliptic equations. *Atti Semin. Mat. Fis. Univ. Modena* 46 (1998), 381–391. [zbl](#) [MR](#)
- [3] *M. Ben Cheikh Ali, O. Guibé:* Nonlinear and non-coercive elliptic problems with integrable data. *Adv. Math. Sci. Appl.* 16 (2006), 275–297. [zbl](#) [MR](#)
- [4] *P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vazquez:* An  $L^1$ -theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.* 22 (1995), 241–273. [zbl](#) [MR](#)
- [5] *A. Bensoussan, L. Boccardo, F. Murat:* On a nonlinear partial differential equation having natural growth terms and unbounded solution. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 5 (1988), 347–364. [zbl](#) [MR](#) [doi](#)
- [6] *D. Blanchard, O. Guibé:* Infinite valued solutions of non-uniformly elliptic problems. *Anal. Appl., Singap.* 2 (2004), 227–246. [zbl](#) [MR](#) [doi](#)
- [7] *L. Boccardo, A. Dall’Aglio, L. Orsina:* Existence and regularity results for some elliptic equations with degenerate coercivity. *Atti Semin. Mat. Fis. Univ. Modena* 46 (1998), 51–81. [zbl](#) [MR](#)
- [8] *L. Boccardo, T. Gallouët:* Strongly nonlinear elliptic equations having natural growth terms and  $L^1$  data. *Nonlinear Anal., Theory Methods Appl.* 19 (1992), 573–579. [zbl](#) [MR](#) [doi](#)
- [9] *G. Croce:* The regularizing effects of some lower order terms in an elliptic equation with degenerate coercivity. *Rend. Mat. Appl., VII. Ser.* 27 (2007), 299–314. [zbl](#) [MR](#)
- [10] *T. Del Vecchio, M. R. Posteraro:* Existence and regularity results for nonlinear elliptic equations with measure data. *Adv. Differ. Equ.* 1 (1996), 899–917. [zbl](#) [MR](#)
- [11] *F. Della Pietra:* Existence results for non-uniformly elliptic equations with general growth in the gradient. *Differ. Integral Equ.* 21 (2008), 821–836. [zbl](#) [MR](#)

- [12] *J. Droniou*: Non-coercive linear elliptic problems. *Potential Anal.* *17* (2002), 181–203. [zbl](#) [MR](#) [doi](#)
- [13] *J. Droniou*: Global and local estimates for nonlinear noncoercive elliptic equations with measure data. *Commun. Partial Differ. Equations* *28* (2003), 129–153. [zbl](#) [MR](#) [doi](#)
- [14] *O. Guibé, A. Mercaldo*: Existence and stability results for renormalized solutions to noncoercive nonlinear elliptic equations with measure data. *Potential Anal.* *25* (2006), 223–258. [zbl](#) [MR](#) [doi](#)
- [15] *O. Guibé, A. Mercaldo*: Existence of renormalized solutions to nonlinear elliptic equations with two lower order terms and measure data. *Trans. Am. Math. Soc.* *360* (2008), 643–669. [zbl](#) [MR](#) [doi](#)
- [16] *C. Leone, A. Porretta*: Entropy solutions for nonlinear elliptic equations in  $L^1$ . *Nonlinear Anal., Theory Methods Appl.* *32* (1998), 325–334. [zbl](#) [MR](#) [doi](#)
- [17] *J. L. Lions*: Quelques méthodes de résolution des problèmes aux limites non linéaires. *Etudes mathématiques*. Dunod, Gauthier-Villars, Paris, 1969. (In French.) [zbl](#) [MR](#)
- [18] *C. Maderna, C. D. Pagani, S. Salsa*: Quasilinear elliptic equations with quadratic growth in the gradient. *J. Differ. Equations* *97* (1992), 54–70. [zbl](#) [MR](#) [doi](#)
- [19] *F. Murat*: Soluciones renormalizadas de EDP elípticas no lineales. Technical Report R93023, Laboratoire d’Analyse Numérique, Paris, 1993. (In French.)
- [20] *A. Porretta*: Nonlinear equations with natural growth terms and measure data. *Electron. J. Differ. Equ. Conf.* *09* (2002), 183–202. [zbl](#) [MR](#)
- [21] *S. Segura de León*: Existence and uniqueness for  $L^1$  data of some elliptic equations with natural growth. *Adv. Differ. Equ.* *8* (2003), 1377–1408. [zbl](#) [MR](#)
- [22] *C. Trombetti*: Non-uniformly elliptic equations with natural growth in the gradient. *Potential Anal.* *18* (2003), 391–404. [zbl](#) [MR](#) [doi](#)
- [23] *W. Zou*: Existence of solutions for a class of porous medium type equations with a lower order terms. *J. Inequal. Appl.* *2015* (2015), Article ID 294, 23 pages. [zbl](#) [MR](#) [doi](#)

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