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Labor supply when productivity keeps growing

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ABSTRACT

We examine the intensive and extensive margins of labor supply in an incomplete-markets framework where productivity keeps growing. What are, in particular, the long-run implications for who will work how much, and how the distribution of economic welfare among households will change? We insist the relative strengths of income and substitution effects to be such as to match historical and cross-country observations. That is, hours will fall toward zero as productivity and income rise, while wages per hour will keep rising and be consistent with stable income shares for labor and capital. Despite this rather drastic path toward zero hours worked, we find that few features of the distribution of outcomes in the population are affected much at all by productivity growth. In particular, the relative distribution of hours worked and of consumption will look very similar to the case without productivity growth.

1. Introduction

Modern macroeconomic models all depart from the basic, frictionless neoclassical framework with dynastic households who make optimizing saving and labor-supply decisions in an environment with labor-augmenting technology growth. The early literature built models of this sort with representative agents. The go-to reference for how to operationalize studies relying on these frameworks is Cooley and Prescott (1995); there, the authors showed in particular how to discipline the model quantitatively.

Beginning with the papers by Imrohoroğlu (1989), Huggett (1993), and Aiyagari (1994), a large literature on models with incomplete markets and idiosyncratic risk followed. These papers considered a version of the core representative-agent framework without elastic labor supply but with idiosyncratic productivity shocks for households and a complete absence of insurance against these shocks, except by saving in a riskless asset. Such a model typically delivers long-run joint distributions of consumption and assets that are independent of initial conditions and hence can be compared to data. An active labor-supply channel was later added, and the resulting framework has since established itself as a core setting for studying aggregate fluctuations, including versions with additional frictions such as price and wage stickiness (also referred to as Aiyagali or HANK settings).[1]

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[1] Early papers with elastic labor supply include Krusell and Smith (1998), Pijoan-Mas (2006) and Chang and Kim (2006); for settings with aggregate risk, see Krusell and Smith (1998) and Chang and Kim (2007) and, with nominal stickiness added, see Oh and Reis (2012), Gornemann et al. (2016), McKay et al. (2016), and Kaplan et al. (2018). See Chang et al. (2019) for an incomplete market model with both the intensive and extensive margin of labor supply.

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Motivated, as explained in Cooley and Prescott (1995), by the striking constancy of hours worked in the postwar U.S. period, all the above-mentioned settings use utility functions defined over consumption and leisure (or hours worked) as given in King et al. (1988) and typically labeled KPR. The KPR class is a formulation that makes income and substitution effects cancel exactly along a balanced growth path, such that we obtain the sought constancy of hours, despite significant growth in labor productivity over time. A broader class of utility functions, labeled BK, was then proposed in Boppart and Krusell (2020), allowing income effects to possibly exceed substitution effects along a balanced path. This case was argued to be relevant since both across countries and within countries over time, it appears that higher labor productivity is associated with lower hours worked.[2] However, the adoption of BK preferences raises the question of how hours will develop over time under incomplete markets: will hours go to zero, and how will different households be affected by labor productivity growth in their hours outcomes? How will aggregate quantities and prices evolve? We address these questions in the present paper.

We first present a core framework in Section 2 with the purpose of introducing the reader, by means of an equilibrium definition, to the dynamic incomplete-markets model with hours choice. In Section 3, we then study a static version of this model. This section, which occupies a significant part of the paper, serves to introduce BK preferences and discuss the determination of individual labor supply in significant detail. The version of preferences that we use here, which is due to MaCurdy (1981) and nowadays constitute the main functional form used in applied work, admits both the KPR and BK cases through the choice of one parameter: utility's curvature in consumption. This convenient functional form does not, however, admit aggregation in wealth (either in the KPR or the BK case), so one issue we first look at is the quantitative effect of this departure from aggregation. We find it to be small, given the observed heterogeneity in wealth. We also show that the BK (but not KPR) version of MaCurdy preferences delivers the textbook "backward-bending labor supply function".

A second part of Section 2 looks at extensive-margin labor supply, still in a static model. This analysis is important background for the dynamic model, where a central issue, given growing productivity and strong income effects, is the push toward lower hours worked and, possibly, an increasing degree to which the extensive margin binds. We model the extensive margin simply by requiring hours to be chosen from the non-convex set $\{0\} \cup [\underline{h}, 1]$, where $\underline{h} \in (0, 1)$. Given a non-convex choice set, the Frisch elasticity of the household's labor supply, which is constant in an interior (intensive-margin) solution with the MaCurdy formulation, is nontrivially determined. We illustrate this feature and also compute an aggregate Frisch elasticity given a realistic wealth distribution.

Section 4 introduces the dynamic model and long-run productivity growth. We first show that, in the special case where individual shocks are shut down—thus, in the absence of any frictions—labor supply goes to zero at a constant rate (under BK preferences). This was shown for the intensive-margin case in Boppart and Krusell (2020); here we show that under the non-convex hours choice set, there is also an exact balanced growth path where the *fraction* of people working goes to zero at a constant rate. In the extensive-margin case, aggregate labor hours fall somewhat faster than in the intensive-margin case. This is because the extensive-margin case can be thought of as one with linear (not convex) utility costs of working, so that the household does not consider the smoothing of hours over time to be of relevance; in fact, the household is indifferent as to the timing of work, subject to bringing in a determinate present-value labor income.

We then introduce incomplete markets and show that, in the intensive-margin case, both the KPR and BK case are consistent with exact balanced growth paths where the relative distribution of wealth, consumption, and hours worked across agents is unchanged over time, while wealth and consumption grow as hours shrink. We look at the relevant aggregates and inequality measures and conclude that all the different cases look very similar—comparing BK to KPR and comparing growth to no growth. Thus, growth, along with strong income effects, does not lead to a very different economy from the one we live in today, though of course we consume increasing amounts and enjoy more and more leisure and hours therefore need to be detrended (see Gali (2005)).

We also show that an extensive-margin case where the positive lower bound on hours ($\underline{h}$) is allowed to fall (exogenously) at an appropriate rate also delivers an exact balanced growth path. On this balanced path, all growth rates are identical to those found under the intensive margin; thus, the two cases are very similar. The exogenously falling $\underline{h}$ is, however, challenging to motivate; consumers would surely want $\underline{h}$ to fall at the assumed rate but it is not clear how it can be made technologically feasible. Thus, we also discuss the case where $\underline{h}$ remains constant over time. This case turns out to be interesting in that we have been unable to solve the model, and it is not even clear that an equilibrium exists. One possibility is that it does but that, in the limit, the relative wealth, consumption, and hours distributions explode; agents with low idiosyncratic productivity values do not work and run down their assets but, when they receive a good shock they obtain an "infinite" amount of new income from choosing to work. We leave this extensive-margin case as an unresolved, curious case. Section 5 concludes.

2. A core quantitative model

In this section, we describe a simple version of the model we will later analyze and we indicate how we will develop it further. We do not consider aggregate shocks, but we study aggregate dynamics, i.e., transitions. We abstract from growth

[2] Thus, the postwar period in the U.S. case is an exception; this was a period of high growth in women's labor-force participation, while men's labor supply fell.

in productivity here, since it requires further discussion and we cover this in detail later in the paper. We do allow an extensive-margin labor supply and consider a general utility function at this point. The steady state corresponding to the model described here is the standard Aiyagari (1994) setting, augmented to allow for endogenous labor supply (as in Krusell and Smith (1998) and Pijoan-Mas (2006)). The main modification compared to the appendix of Krusell and Smith (1998) is that we model explicitly an extensive margin of labor supply as studied in Chang and Kim (2006) and Chang and Kim (2007).

2.1. The benchmark model

There is a unit mass of households, each with some asset level a and some idiosyncratic productivity state ω. We denote the joint distribution of assets and productivity across people by Γ. The remainder of the variables will be described as the definition of equilibrium is laid out. The benchmark model—defined as a recursive competitive equilibrium (RCE)—can thus be described as follows.[3]

Definition 1. A RCE consists of pricing functions r_k and w, a value function V, decision rules f^a and f^h, an aggregate labor supply function H^n, and a law of motion for the distribution, H^k, such that:

1. V solves the household's problem: for all (a, ω, Γ),
$$V(a,\omega,\Gamma) = \max_{a',h}\left\{u(a(1-\delta+r_k(\Gamma)) + h\omega w(\Gamma) - a', h) + \beta E[V(a',\omega',H^k(\Gamma))|\omega]\right\}$$
s.t. $a' \geq \underline{a}, h \in \mathcal{H} \subseteq [0,\infty)$.
2. $f^a(a,\omega,\Gamma)$ and $f^h(a,\omega,\Gamma)$ solve the maximization problem on the right-hand side of the dynamic-programming equation above for all (a,ω,Γ).
3. r_k and w satisfy $r_k(\Gamma) = F_1(\bar{k},\bar{h})$ and $w(\Gamma) = F_2(\bar{k},\bar{h})$, where
$$\bar{k} \equiv \sum_{\omega}\int_a a\Gamma(da,\omega)$$
and
$$\bar{h} = H^n(\Gamma).$$
4. H^n satisfies
$$H^n(\Gamma) = \sum_{\omega}\int_a \omega f^h(a,\omega,\Gamma)\Gamma(da,\omega)$$
for all (Γ).
5. H^k satisfies
$$H^k(\Gamma)(B,\omega) = \sum_{\hat{\omega}}\pi_{\omega|\hat{\omega}}\int_{a:f^a(a,\hat{\omega},\Gamma)\in B}\Gamma(da,\hat{\omega})$$
for all (Γ), all Borel sets B, and all ω.

The individual productivity process we assume is exogenous and discrete, i.e., $\omega \in \{\omega_1,\omega_2,\ldots,\omega_I\}$ and $\pi_{\omega|\hat{\omega}}$ captures the entries of the transition matrix. We assume a standard constant-returns-to-scale production function F and a time-independent borrowing constraint. The model described here is rather standard and is, when $\mathcal{H} = [0,\infty)$, identical to that discussed in the appendix of Krusell and Smith (1998), where $u(c,h)$ was assumed to be a power function of a Cobb-Douglas aggregate of c and h. The set $\mathcal{H}$ allows a focus on the extensive margin, e.g., if it equals $\{0,\underline{h}\}$ for some strictly positive $\underline{h}$.

2.2. The models in the remainder of the paper

Our aim is to introduce productivity growth into this model, including extensive-margin examples. Because our focus is on the long run, we will allow our utility function to feature income effects that—on a balanced growth path—exceed substitution effects, as in Boppart and Krusell (2020). To begin with, we will therefore discuss some features of such a preference relation in detail; we will focus on a static model (and thus there are no shocks to discuss). This material will

[3] The definition is a straightforward extension of that in Cooley and Prescott (1995) pages 9–10.

be covered in Section 3. The feature we focus on first is the nature of the Marshallian demand for hours; in particular we will illustrate that, under some conditions, our simple utility function—that due to MaCurdy (1981)—implies a "backward-bending" demand curve: hours worked are initially increasing, but eventually decreasing, in the wage. This is motivated by the fact that we will look at productivity (i.e., wage) growth later. Second, we will illustrate the departure from (Gorman) aggregation and we will do this with a quantitative example. This discussion is motivated by our heterogeneous-agent perspective: even with complete markets Gorman aggregation fails in this economy.

In Section 4.2 we then introduce incomplete markets and connect with the setup described just above. In that section, the focus is on productivity growth with and without non-convexities in $\mathcal{H}$.

3. The static version of the model

The dynamic model just described has been studied in the literature for special cases. Some features of the model, however, have not been highlighted so we begin with a short discussion of these. We particularly look for the determinants of aggregate consumption and hours worked and the distribution of consumption, hours, and wealth across people.

3.1. Restriction to a class of utility functions

We first of all restrict attention to utility functions that are consistent with balanced growth. In Boppart and Krusell (2020) it is shown that an hours path with constant negative hours growth is consistent with a balanced growth path, where the other main economic aggregates—output, consumption, investment, and the stock of capital—all grow at constant rates, if and only if the per-period preferences fall into the "BK class", i.e., if the utility function is of the form:

$$u(c,h)=\frac{\left(c\cdot v\left(hc^{\frac{\nu}{1-\nu}}\right)\right)^{1-\sigma}-1}{1-\sigma} \tag{1}$$

for $\sigma\neq 1$, or

$$u(c,h)=\log(c)+\log\left(v\left(hc^{\frac{\nu}{1-\nu}}\right)\right) \tag{2}$$

where v is an arbitrary, twice continuously differentiable function. The parameter ν is key in that it regulates the relative strength of the income and substitution effects (of wages on labor supply) along a balanced growth path; when $\nu>0(<0)$, the former (latter) is stronger. The formulation nests the classic balanced-growth utility function with zero growth in hours as proposed by King et al. (1988): by setting $\nu=0$ we obtain the standard "KPR class". In this case the utility function takes the form:

$$u(c,h)=\frac{(c\cdot v\,(h))^{1-\sigma}-1}{1-\sigma} \tag{3}$$

for $\sigma\neq 1$, or

$$u(c,h)=\log(c)+\log\left(v\,(h)\right) \tag{4}$$

again with v being an arbitrary, twice continuously differentiable function.

In this paper we will restrict attention to the familiar MaCurdy (1981) formulation,

$$\frac{c^{1-\sigma}-1}{1-\sigma}-\psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \tag{5}$$

which is a special case of this class.[4]

Note that a KPR function is allowed as a special case: $\sigma=1$, so that the function of consumption is logarithmic. The MaCurdy function (5) has the convenient feature that the Frisch elasticity of labor supply, i.e., the percentage change in hours as a result of a one-percent change in the wage while keeping the marginal utility of wealth constant, is constant and equal to θ.

[4] It is obtained by the choice of a particular functional form for v: $v(x)=\left(1-\frac{\psi(1-\sigma)}{1+\frac{1}{\theta}}x^{1+\frac{1}{\theta}}\right)^{\frac{1}{1-\sigma}}$, with $x\equiv hc^{\frac{\nu}{1-\nu}}$, and the following parameter restriction:

$$\nu=\frac{\sigma-1}{\sigma+\frac{1}{\theta}}. \tag{6}$$

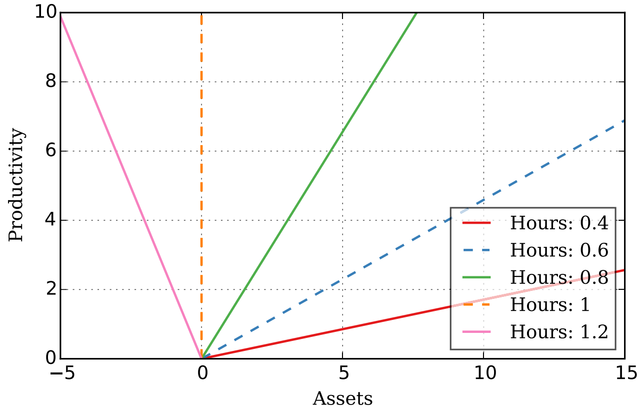


Fig. 1. Hours choice with $\sigma = 1$. Illustration of the combinations of assets (x-axis) and productivity (y-axis) that yield the same hours choice. Each line represents an hours isoquant. $\psi = 1$ and $\theta = 0.5$.

3.2. Features of the static model: the intensive margin

We start by assuming that households can choose labor freely along the intensive margin. In other words: there are no restrictions on how much or little the household can choose to work, and we can really think of hours as "effort" in this sense rather than a time restriction. When interpreting h as time there is a natural upper bound. Such an interpretation is possible too as long as this upper bound does not bind and the choice is interior.

First, we analyze the household problem in some detail, and contrast KPR preferences with the more general BK preference formulation. Thereafter, we will turn to the question about aggregation.

3.2.1. The household problem

The household maximizes $u(c, h)$ by choice of c and h subject to a budget

$$c = \omega h + a,$$

where ω is the household's productivity (wage) and a is the asset (wealth) level.

3.2.2. The effect of wages on hours worked

Under KPR preferences, the optimal amount of hours worked does not change if the wage changes, as long as the wealth of the household changes proportionally. On the other hand, if wealth is not scaled with the wage change, wages affect hours: with constant, positive (negative) wealth, hours worked are increasing (decreasing) in the wage. Fig. 1 illustrates by plotting isohours curves (based on the KPR version of the MaCurdy function (5)). At zero wealth, the isocurve is vertical: a given level of hours is consistent with any productivity level. For a positive (negative) wealth level, a productivity increase increases (decreases) hours worked; the linearity reflects the cancellation of income and substitution effect defining KPR preferences.

When we go outside the KPR class to the more general BK class, these statements need to be altered. Using the version of the MaCurdy function (5) where the income effect exceeds the substitution effect (along a balanced growth path), so that hours fall when a country gets richer, we need to have $\sigma > 1$. In contrast, the case of $\sigma = 1$ corresponds to KPR preferences.

In parallel with the previous figure, now for the case with $\sigma > 1$, Fig. 2 illustrates how, for a given asset level, the optimal choice of hours depends on the productivity level in a non-monotonic way (the example has $\sigma = 2.5$). Given a positive level of wealth, and for sufficiently low wage levels, hours increase as wages increase, just like under KPR preferences. Intuitively, the wage bill is small enough here that wage changes do not affect the household's wealth much and hence the substitution effect dominates. As wages grow further, however, eventually their effect on work is always negative; now wages dominate the household's income and the income effect then dominates in this preference class. Under KPR and a fixed positive wealth level, a wage increase raises hours by less and less as the wage grows; under BK and $\sigma > 1$, it goes further and eventually leads to falling hours.

Putting these illustrations together in order to directly show (Marshallian) labor supply, we have Fig. 3. Here we see that under KPR preferences, for a given positive (negative) level of asset wealth, hours worked increase (decrease) as wages grow but that under BK preferences with $\sigma > 1$, wages first increase and then decrease in hours. In fact, the second panel of the figure illustrates—for the cases with positive asset wealth—the *backward-bending labor supply curve*.[5]

[5] To literally see something backward-bending, you do need to turn your head so as to see hours on the x-axis and wages on the y-axis. Since hours are endogenous here and wages exogenous, we find our figure more intuitive.

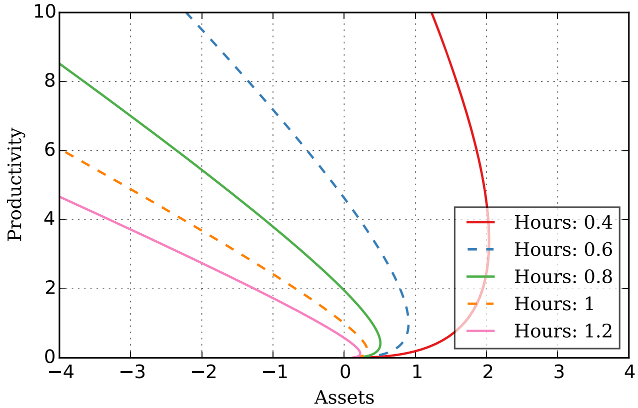
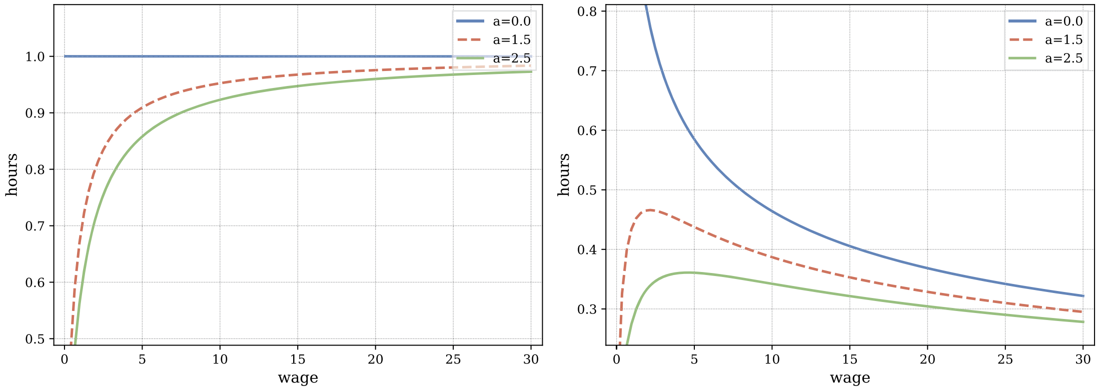


Fig. 2. Hours choice with $\sigma = 2.5$. Illustration of the combinations of assets (x-axis) and productivity (y-axis) that yield the same hours choice. Each line represents an hours isoquant. $\psi = 1$ and $\theta = 0.5$.



(a) KPR preferences. (b) BK preferences.

Fig. 3. Hours as a function of assets and wage.

The non-monotonicity feature of BK preferences will be important for understanding hours and participation in the long run, which we study later in the paper.

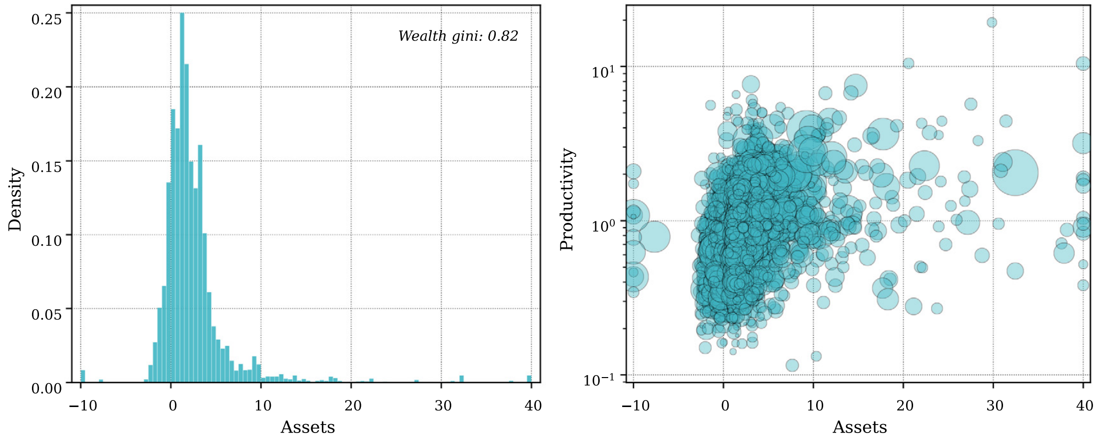
3.2.3. Quantitative departures from aggregation

The preference class we study here does not admit Gorman aggregation. This is a general feature of most applied aggregate models of endogenous labor supply, and most of these models are representative-agent models.[6] The question addressed in what follows is how much of a difference this non-aggregation makes in practice. To give a rough quantitative answer to this question we will use the static model just studied in a version that corresponds to a snapshot from a general-equilibrium model with exact balanced growth, but with a nontrivial distribution of agents over wealth and productivity levels. In particular, we will impose an exogenous bivariate distribution over assets and productivity that is similar to that observed in the data. Fig. 4 shows both the asset distribution and the distribution of asset vs. productivity, using data from the PSID from the years 1998 to 2008.[7]

In the incomplete-markets models that are the ultimate aim of this study, the wealth-productivity distribution is endogenous. In the present section, it is not: the distribution can be seen as a "parameter" that we are free to choose; hence we choose it to approximately mimic the data.

[6] The MaCurdy function (5) is, for example, very often used in newkeynesian models.

[7] Assets are the sum of cash, bonds, stocks, business assets, pension assets and real estate, net of mortgages and other debt and is a snapshot from 2008. For productivity, we use the average of the observed wages for the last 10 years in the case of the individual having at least one recorded wage during this time period. To calculate hourly wage we take annual labor income (sum of regular labor income, labor income from business, and farm income equally split between husband and wife) and divide by annual hours. Wage observations below half the state minimum wage for that particular year are set to that number. If the individual has no wage observations in the period 1998–2008, we impute a productivity based on observables (age, race, education, marital status, presence of children in the household, supporting children living outside the home).



(a) Asset distribution.

(b) Bivariate distribution of assets and productivity. Size of marker indicates weight of cell.

Fig. 4. Distribution of assets and productivity used for quantitative examples in this section. Sample consists of 4,461 households. Source: PSID.

In our general-equilibrium model here, we assume a unit mass of households, each with an individual productivity ω_i and an individual asset holdings a_i. The households rent out their labor services and capital services to a production sector that produces with a standard Cobb-Douglas technology.

Hence, the household wants to maximize utility, subject to the budget constraint (omitting the i subscripts for readability), which reads

$$\max_{c,h} u(c,h) \quad \text{s.t. } c = h\omega w + (1 + r - \gamma)a \tag{7}$$

where r is the rental rate of capital and w is the economy-wide wage rate, in equilibrium given by $F_1(K,L) - \delta$ and $F_2(K,L)$ respectively. The gross growth rate γ appears in the budget because there is growth at that rate; hence the equation is expressed in terms of variables that are transformed into stationary form. The resulting static model is thus our way of capturing the economy's behavior along a balanced growth path, where per-capita consumption and capital grow at rate γ.

Along a balanced growth path the interest rate is given by $r = \gamma/\beta - 1$, and we assume $\gamma = 1.02$ and $\beta = 0.98$. The capital share in the production function is assumed to be one third and the depreciation rate is 5% yearly.

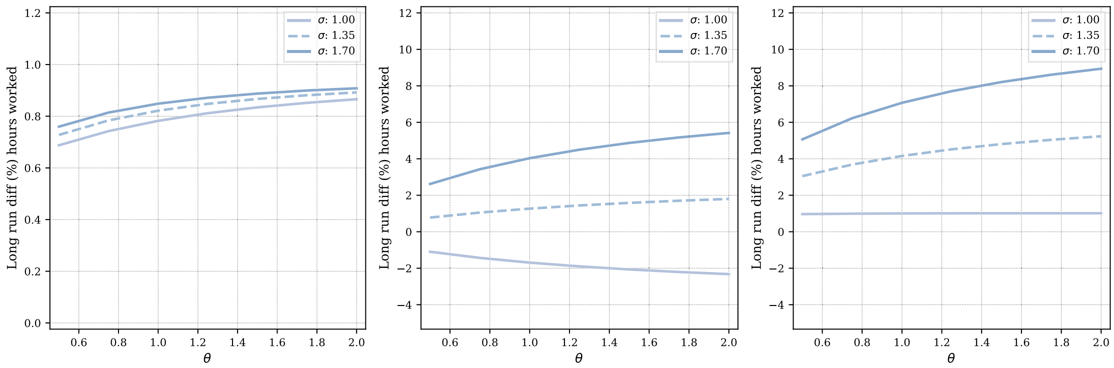
3.2.3.1. Heterogeneity in assets only First, we ignore the heterogeneity in productivity and assume that everyone in the economy has equal productivity, $\omega_i = 1\ \forall i$. Asset holdings are heterogeneous and correspond to the asset distribution we observe in the data. With heterogeneous assets and preferences that are not in the Gorman class, the marginal propensity to decrease hours worked out of wealth differs between rich and poor households and therefore the wealth distribution affects total hours worked. Hence, we should expect a difference between the heterogeneous-agent case and the representative agent case; the question is just how large it is.

We select ψ (this parameter guides the level of labor supplied by the households) so that the labor market and the capital market clear at the balanced growth interest rate level. Thereafter we compare the outcome to one where the assets are redistributed so that all households have the same amount of assets. In other words, we give each household the average asset holdings, and thereafter find the new corresponding equilibrium. Thus, in this "representative-agent" economy individual households are literally homogeneous in every respect: preferences, productivity, and assets. We then measure the difference in outcomes between those two economies: how much more (or less) labor is supplied in the economy with heterogeneous asset holdings?

The results for a number of parameter combinations—close to calibrations considered in the literature—in the utility function (adjusting σ and θ) are shown in Fig. 5a. As can be seen, the departure from aggregation depends on the parameters in the utility function, but is limited to around 1%. As previously pointed out, there are departures from aggregation also for utility functions within the KPR class (the case of $\nu = 0$).

Finding 1. *The departures from aggregation in total hours worked due to wealth heterogeneity alone are between 0 and 1% for the parameter constellations considered, including KPR preferences. Work is higher under wealth heterogeneity.*

3.2.3.2. Heterogeneity in productivity only Next, we assume that everyone in the economy has an equal amount of assets but that productivity is heterogeneous and corresponds to what we observe in the data. That this economy does not aggregate



(a) Asset heterogeneity (b) Productivity heterogeneity (c) Asset and productivity het.

Fig. 5. Difference in total hours worked between model with heterogeneity (using actual productivity and/or asset distribution) and the corresponding representative agent solution. The x-axis indicates different levels of θ. The lines indicate different values of σ.

is also clear; heterogeneous productivity can be viewed as heterogeneity in the relative price between consumption and leisure such that preferences do not even aggregate if they are part of the Gorman class.

The experiment is exactly the same as in the previous section: for a number of different configurations of parameters in the utility function we first solve for the equilibrium in the economy with heterogeneous productivity, and thereafter compare to an economy where productivity is distributed evenly ($\omega_i = 1\ \forall i$). The results are shown in Fig. 5b. The deviation between the representative-agent economy and the heterogeneous-productivity economy now ranges between hours worked being 6% more to being 2% less in the case of heterogeneity in productivity.

Finding 2. *The departures from aggregation in total hours worked due to productivity heterogeneity alone are between -2% and 6% for the parameter constellations considered.*

3.2.3.3. Heterogeneity in both assets and productivity Finally, we introduce heterogeneity in both the asset and productivity dimensions, with a correlation structure like that in the data. Again, we compare the labor supply in the economy with heterogeneity to the labor supply in the representative-agent economy. The results are shown in Fig. 5c and, as can be seen, the difference between hours worked in the economy with heterogeneous agents and those in the representative-agent economy is now between 1 and 9%. Thus, the interaction of heterogeneity in wealth and productivity raises hours worked.

Finding 3. *The departures from aggregation in total hours worked due to wealth and productivity heterogeneity together are up to 9% for the parameter constellations considered. Work is higher under heterogeneity.*

3.3. The static model with an extensive (and intensive) margin

So far we have only dealt with labor choice along the intensive margin. However, if we look at people's working hours, many people work zero hours. In the U.S., the civilian labor force participation in the beginning of 2019 was 63%. Even in the age group 25–54, i.e., among prime-aged individuals, 17% of the population is out of the labor force. It thus seems hard to ignore the extensive margin in a model of labor supply.

As argued by Hansen (1985) and Rogerson (1988), models with an extensive-margin labor choice can behave very differently than models with an intensive-margin choice only. Their settings, however, are frameworks with full insurance and, in essence, the representative-agent feature is then kept to a large extent. In contrast, in incomplete-markets economies households are not fully insured and as a result can be in very different situations. Therefore, their responses to shocks can differ very widely; this point is also clear from previous work, such as Chang and Kim (2007) and Krusell et al. (2008) in the context of a steady state, and Krusell et al. (2017) in the context of aggregate fluctuations. What we point out here is that some of these heterogeneous effects can be studied also under complete markets: i.e., we simply explore how agents with different wealth and productivity positions differ in their hours worked and in their responses to shocks.

In this section, we will thus look at a framework where there is only an extensive margin but we will also consider the case with both an extensive and an intensive margin. We introduce a tractable way of modelling these features and discuss the implications for aggregation and for the aggregate Frisch elasticity.

3.3.1. The household problem with both margins

To introduce an extensive margin into the labor supply choice we assume that the hours choice is constrained as follows:

$$h_i \in \{0\} \cup [\underline{h}, 1],\ \forall i. \tag{8}$$

If $\underline{h} = 0$, the problem is the same as in the previous section, with intensive margin choice only (we impose an upper bound for hours worked of 1 here for simplicity only; in general we can use any $\tilde{h}$). When $\underline{h} > 0$ this is a non-convex set; if it equals 1, the constraint implies that the household can only choose between working and not working, i.e., only has an extensive margin choice. We do not consider the deeper sources of the non-convexity, which could be various forms of fixed costs, such as commuting costs or the need to purchase specific clothes for work, or involve ways in which productivity per time unit falls when less time is spent on the job. The main advantage of our formulation is its simplicity.[8]

We will now look at the household problem in detail in the presence of this non-convexity before turning to the implications for the household's labor-supply elasticity.

3.3.1.1. The household problem and basic results We start by writing down the household problem to gain some intuition about the non-convex hours choice set. Consider a static problem where an agent is endowed with a certain asset level, and chooses labor and consumption. Hence, the agent's problem is to maximize the one-period utility $u(c,h)$ with respect to c and h given the budget constraint $c = \omega h + a$. We use the MaCurdy formulation for utility:

$$u(c,h) = \frac{c^{1-\sigma}-1}{1-\sigma} - \psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}.$$

Taking the first-order conditions we obtain the familiar relationship for marginal utility of consumption and marginal disutility of work:

$$c^{-\sigma} = \frac{\psi h^{\frac{1}{\theta}}}{\omega},$$

which, combined with the budget constraint, gives us the preferred hours and consumption choice as functions of the asset level a and the productivity level ω. We call these preferred choices from the unconstrained problem $h^*(a,\omega)$ and $c^*(a,\omega)$ respectively.

Now assume that the hours choice is constrained as described above: $h \in \{0\} \cup [\underline{h}, 1]$. If the household's preferred hours choice from the unconstrained problem, $h^*(a,\omega)$, falls within $[\underline{h}, 1]$, the solution to the constrained problem is the same as in the unconstrained problem. But what if $h^*(a,\omega) \in (0, \underline{h})$? Then the choice of working $\underline{h}$ or 0 is determined by whether or not $u(a+\omega\underline{h}, \underline{h})$ exceeds $u(a,0)$. With our choice of utility function, the decision to work $\underline{h}$ or not work at all is hence given by

$$\frac{(a+\underline{h}\omega)^{1-\sigma}}{1-\sigma} - \psi \frac{\underline{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} \gtrless \frac{a^{1-\sigma}}{1-\sigma}.$$

We call the choices from a restricted problem $h^R(a,\omega)$ and $c^R(a,\omega)$. For a given productivity level the household will choose to work if the asset level is low. However, if the asset level is large enough, the household will switch to not working. This is a manifestation of a simple income effect on hour worked (as leisure is a normal good).

3.3.1.2. Introducing lotteries Given the presence of the non-convexity in the choice set for h, in some cases households would, if they could, choose to randomize between 0 and $\underline{h}$. We will consider randomization in some of our analysis below. When we do, we thus assume that a household can assign a probability $e \in [0,1]$ to be employed and work. Hence, the household's problem is now given by:

$$\max_{c,h,e} \left[\frac{c^{1-\sigma}-1}{1-\sigma} - e\psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} \right] \tag{9}$$

subject to

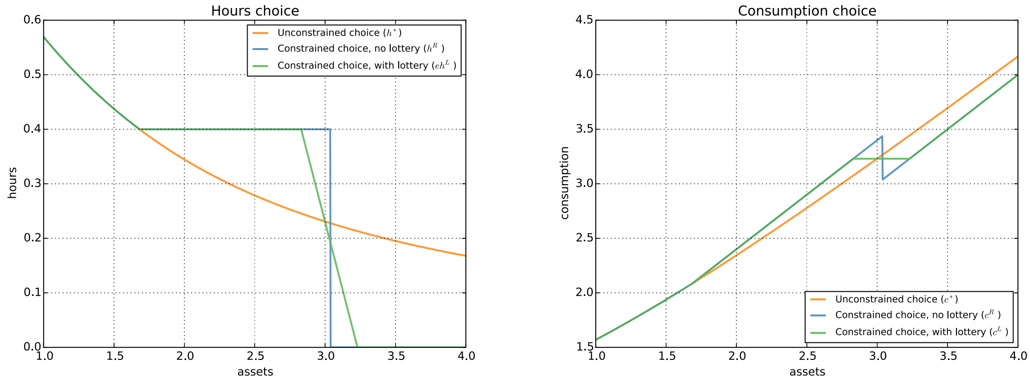
$$c = e\omega h + a \tag{10}$$

$$e \in [0,1] \tag{11}$$

$$h \in [\underline{h}, 1]. \tag{12}$$

We will assume for now, just to simplify the notation and discussion, that the combination of assets, productivity, and preference parameters is such that the constraint $h \leq 1$ is not binding. We then have the following first-order conditions:

[8] Alternative formulations could be to use a part-time penalty, as in, e.g., French (2005), or a combination of fixed cost and non-linear earnings, as in Erosa et al. (2016).



(a) Choice of hours (b) Choice of consumption

Fig. 6. Illustration of the choice of hours and consumption as a function of assets, in the unconstrained problem compared to the constrained problem, with and without lotteries. For the lottery problem, the hours choice is defined as $eh^L(a)$. Productivity is fixed, $\omega = 1$ and we have $\underline{h} = 0.4$.

$$\begin{aligned}
&c^{-\sigma} - \mu_1 = 0 \\
&-e\psi h^{\frac{1}{\theta}} + \mu_1 e\omega + \mu_4 = 0 \\
&-\psi \frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} + \mu_1 \omega h + \mu_2 + \mu_3 = 0 \\
&\mu_2 e = 0 \\
&\mu_3 (1-e) = 0 \\
&\mu_4 (h - \underline{h}) = 0,
\end{aligned}$$

where μ_1 denotes Lagrangian multiplier and $\mu_2, \mu_3, \mu_4 \geq 0$ the Kuhn-Tucker multipliers. If the solution to the unconstrained problem for a given asset level and productivity level, $h^*(a, \omega)$, is larger than $\underline{h}$, the solution to this problem with lotteries, denoted by the superscript L, is simply given by $h^L(a, \omega) = h^*(a, \omega)$, $c^L(a, \omega) = c^*(a, \omega)$ and $e^L(a, \omega) = 1$.

When a household chooses to randomize, i.e., $e^L(a, \omega) \in (0, 1)$ we obtain the following expression for consumption:

$$c^{-\sigma} = \frac{1}{\omega \underline{h}} \cdot \psi \frac{\underline{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}.$$

Hence, only at this consumption level is the marginal benefit of consumption equal to the marginal disutility of increasing the fraction working. If this consumption level is between a (which is the consumption the household achieves by not working) and $\omega \underline{h} + a$ (which is the consumption the household achieves by not using the lottery and working $\underline{h}$), the household will choose to randomize.

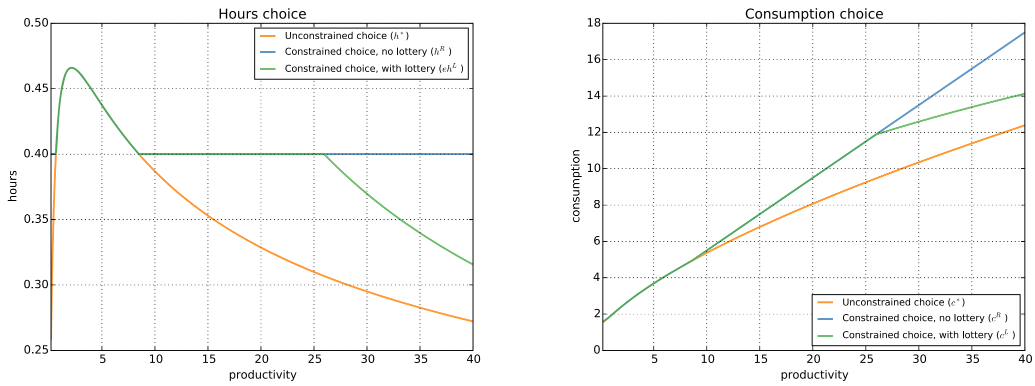
3.3.1.3. Illustrations It is informative to compare the household choices for three cases: the unconstrained choice ($\underline{h} = 0$), the constrained choice ($\underline{h} > 0$), and the constrained choice where we allow for lotteries.

Fig. 6 contrasts the optimal choice of hours and consumption for these three cases, and how the choice depends on asset holdings. Hours is (weakly) decreasing for all cases: the richer the household, the less it will choose to work. For the unconstrained problem, the hours will gradually fall towards (but never reach) 0. For the constrained problem, hours will gradually fall until they hit the $\underline{h}$ level. Then for a wealth range hours will be constant at this level, until the household is rich enough to choose $h = 0$. The convexification of this problem makes the drop from $h = \underline{h}$ to $h = 0$ go via assigning less and less probability to work until the probability is 0.

In contrast, for optimal consumption, the non-convexity creates a non-monotonic jump at the point where the household is rich enough to decide to withdraw completely from the labor market. The convexification of the problem, by allowing the household to choose a fraction working, removes the non-monotonicity in the consumption choice. In the region where the household is using lotteries, the consumption is flat.

When the household becomes richer in terms of assets, the constraint in the hours choice becomes less and less binding. Asymptotically, when wealth goes to infinity, the hours choice in the unconstrained problem approaches zero and the constrained and the unconstrained solutions coincide.

Next, Fig. 7 shows how the optimal choices of hours and consumption depend on productivity. Since we assume $\sigma > 1$, i.e., we work outside the KPR class and assume that the (balanced-growth) income effect of a wage increase on hours



(a) Hours choice. (b) Consumption choice.

Fig. 7. Illustration of the choice of hours and consumption as a function of productivity, in the unconstrained problem compared to the constrained problem, with and without lotteries. For the lottery problem, the hours choice is defined as $eh^L(a, \omega)$. Assets are fixed, $a = 1.5$ and we have $\underline{h} = 0.4$.

exceeds that of the substitution effect, optimal hours is not a monotonic function; this is entirely in line with the findings in Section 3.2.1, but now the focus is on how the extensive margin affects household outcomes. Recalling the key intuition, for a very low household productivity, there is little point in working, since the wage is just too low. At the same time, because the wage is low in this region relative to assets, the effect of a wage increase is that the substitution effect dominates, so that hours worked rise. For a middle productivity level, the choice of hours is at its maximum. For high productivity, the income effect dominates, and the optimal hours is decreasing in productivity.

In this case, the higher is productivity, the more binding is the hours constraint. When productivity approaches infinity, an unconstrained household would choose to work an infinitesimal amount of hours (still delivering substantial income), while the constrained household has to keep on working at $\underline{h}$. The household allowed to randomize, on the other hand, approaches the unconstrained solution as productivity increases.

3.3.1.4. The Frisch elasticity of the individual household One implication of using the MaCurdy preferences is, as previously mentioned, that the Frisch elasticity is constant and equal to θ. With non-convexity in the choice set for hours, this is no longer the case. Again, we can gain insight by looking at Fig. 6. For the asset region where the choices of hours coincide for the unconstrained and the constrained problem, the Frisch elasticities also coincide. However, for the asset region where the household is choosing $\underline{h}$, the lower bound of working, the Frisch elasticity is effectively zero. For the asset region where the household continuously shifts from working to not working, i.e., is randomizing, the objective function becomes

$$u(c,h) = \frac{c^{1-\sigma}-1}{1-\sigma} - e\psi\frac{\underline{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}, \tag{13}$$

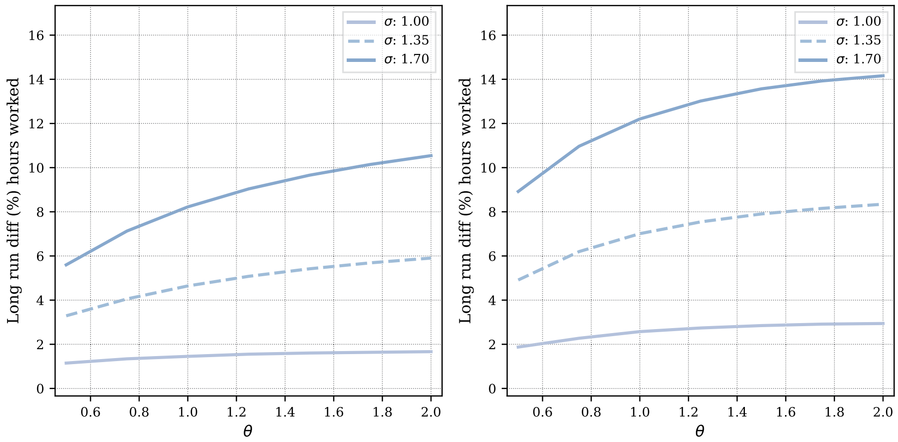
which is linear in the choice variable e, and consequently the Frisch elasticity tends to infinity. For higher asset levels, where the household works zero hours, the Frisch elasticity is effectively zero again. These feature that the elasticity is infinity over a range is of course well known from Hansen (1985) and Rogerson (1988) but here the focus is on the regions where it is, instead, zero. How households are placed across 0, θ, and infinity Frisch elasticity regions is crucial in models with significant heterogeneity across households, whether or not markets are incomplete. We turn to this issue next.

3.3.2. Aggregation and the aggregate Frisch elasticity

Just as the simpler economy with only intensive-margin hours choice, the economy described in this section, with non-convexities in the hours choice, does not admit a representative agent. By restricting the choice set of hours worked with a positive $\underline{h}$ some agents are potentially constrained or forced to randomize between 0 and $\underline{h}$. This introduces yet another reason why the overall distribution starts to matter for the aggregates: it determines the fraction of constrained households, on top of the reasons mentioned earlier (the fact that the MaCurdy preferences are not in the Gorman class and that we allow for heterogeneity in productivity).

In this case, the difference between the heterogeneous-agent economy and the economy with a representative agent who can choose any number of hours becomes larger. Quantitatively, the difference depends on how many individuals are restricted. Fig. 8 shows the difference in total hours worked for two cases: one where the lower limit for work, $\underline{h}$, is set to 0.4 and one where it is set to 0.44; the difference here is thus small.

As can be seen from the figure, the difference between the aggregate hours worked in the heterogeneous-agent economy now differs substantially from the aggregate hours worked in the representative agent economy and can easily amount to



(a) $\underline{h} = 0.40$ (b) $\underline{h} = 0.44$

Fig. 8. Difference in total hours worked between model with heterogeneity in assets and productivity and the corresponding representative agent solution. A positive number indicates higher number of hours worked in the heterogeneous agent economy.

more than 10%. Thus, heterogeneity per se—under the preferences adopted here and when there is an extensive-margin labor choice—can matter significantly for hours worked in the aggregate economy.[9] Thus, in our incomplete-markets economy studied later, this force will be present as well.

Finding 4. *Aggregate hours worked is quite sensitive to the precise constraints on hours choice on the individual level for realistic wealth-productivity distributions.*

We now turn to the Frisch elasticity. As discussed above, the Frisch elasticity for the individual household can be θ, 0, or ∞. Since the fraction of households falling into each category depends on household heterogeneity, the aggregate Frisch elasticity is a non-trivial object.

To make an assessment, we solve for equilibrium in the heterogeneous agent economy in the same way as before. We then imagine a wage increase by 1%, and take note of how much the labor supply would increase for each household. For a household with a Frisch elasticity of θ, the labor supply would increase accordingly, for a household which is currently randomizing between 0 and $\underline{h}$, the labor supply would increase to $\underline{h}$, and for a household with a Frisch elasticity of 0 there would be no effect. We then sum up the total increase in hours worked and compare that to the previous hours worked in this economy. The result we report as the average Frisch elasticity in the economy. As can be seen in the results reported in Fig. 9, the average Frisch elasticity can be both higher and lower in the heterogeneous-agent economy (note that the Frisch elasticity of the representative agent trivially is θ, indicated by the identity line, since we are using MaCurdy preferences (5)).

The discussion based on the static model makes clear that the aggregate Frisch elasticity does not have to remain constant over time, if the distribution of households changes relative to its cutoffs in the decision rule. Over time, there can be significant such changes. In particular, it should be noted that under some conditions, all the households either randomize or quit work altogether as time goes to infinity. Therefore, we may expect the Frisch elasticity of labor supply to change over time and, in general, increase, under BK preferences with income effects exceeding substitution effects.

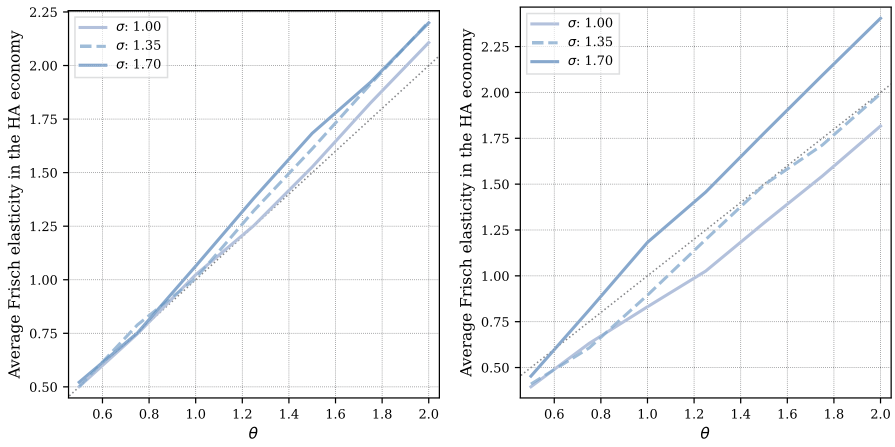
4. The long run

The present section looks at long-run work determination. The focus is on the case where income effects exceed substitution effects along a balanced path. We begin by briefly discussing a benchmark, namely the case without shocks (and unrestricted borrowing and lending).

4.1. No shocks and complete markets

In Boppart and Krusell (2020) it is shown that BK preferences, along with a standard neoclassical technology with labor-augmenting productivity growth at gross rate γ, deliver a balanced growth path where hours worked—modeled as an intensive-margin choice—fall over time at a constant rate $\gamma^{-\nu}$, where again the preference parameter ν describes the

9 See Attanasio et al. (2018) for an empirical study of the heterogeneity of labor supply elasticities that is underlying the aggregate number.



(a) $\underline{h} = 0.40$ (b) $\underline{h} = 0.44$

Fig. 9. The aggregate Frisch elasticity. For description of calculation, please see text.

extent to which the income effect exceeds the substitution effect. This finding presumes a representative agent, the absence of risk, and complete markets, and our purpose below is of course to explore heterogeneity in the context of incomplete markets and endogenous wealth inequality.

The result that hours fall toward zero at a constant rate, so long as income effects exceed substitution effects and the hours choice occurs along the intensive margin only, is rather abstract: people work a zero fraction of their (daily or weekly) time in the limit but their income is still to a large extent labor income, since wages per hour are infinitely large. We would still defend the framework with an intensive margin as a useful model: over a finite time horizon going forward, i.e., over the "foreseeable future", until an extensive-margin constraint of some kind would bind, it can be used to describe hours choices.

Given that, over the very long run, it nevertheless seems relevant to consider the extensive margin—in particular a minimum working time per time period—we need to discuss this case. We begin with a social planner problem and then discuss the decentralized equilibrium.

4.1.1. The extensive margin: the planner's choice

For simplicity, we consider all agents to have the same productivity and, moreover, that they have the same utility weight in the planning problem. We also consider the simplest version of our extensive-margin constraint in (8), where hours on the individual level is simply 0 or $\underline{h}$. Given that we use MaCurdy utility, it follows straightforwardly that all agents' consumption levels will coincide and that the period utility function will simplify to

$$\frac{c_t^{1-\sigma}}{1-\sigma} - \lambda_t \psi \frac{\underline{h}^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}},$$

where λ_t is the fraction of agents assigned to work in period t. In the resource constraint at t, we have labor input equal to $\lambda_t \underline{h}$. Clearly, this planning problem gives rise to a balanced growth path (asymptotically or at all t for the appropriate initial condition on capital). The growth rate of hours on this path will occur at the rate $\gamma^{-\nu}$, where ν is given by equation (6): $\nu = \frac{\sigma-1}{\sigma+\frac{1}{\theta}}$, where θ now equals ∞, so that $\nu = \frac{\sigma-1}{\sigma}$. Thus, we see that hours fall at a higher rate with an extensive-margin labor choice; correspondingly, consumption grows at a slower rate. Intuitively, with an extensive margin, changing hours over time is less costly.

4.1.2. The extensive margin: market equilibrium

Are there balanced-growth market equilibria for our economy in case of an extensive-margin labor choice? The section above suggests an affirmative answer, but perhaps only in case consumers have access to lotteries—so as to implement the planning allocation. However, lotteries are not required. Let us now briefly look at how consumers choose hours in a competitive equilibrium with an extensive margin.

Let us consider the household's problem on a balanced path. It reads:

$$\max_{\{c_t, h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \psi h_t \right]$$

subject to

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = a_0 + \sum_{t=0}^{\infty} \frac{w_t h_t}{(1+r)^t},$$

where $w_t = w_0 \gamma^t$, and the constraint that $h_t \in \{0, 1\}$ for all t.[10]

The standard Euler equation implies along a balanced growth path that consumption will grow at a gross rate $(\beta(1+r))^{\frac{1}{\sigma}} \equiv g_c$. We take as given that $1 + r = \gamma/\beta$ on a balanced path; hence, $g_c = \gamma^{\frac{1}{\sigma}}$. We can then write the left-hand side of the budget constraint as

$$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = c_0 \sum_{t=0}^{\infty} \left(\frac{g_c}{1+r}\right)^t = c_0 \sum_{t=0}^{\infty} \left(\beta\gamma^{\frac{1-\sigma}{\sigma}}\right)^t.$$

The right-hand side of the budget constraint can be written as

$$a_0 + \sum_{t=0}^{\infty} \frac{w_t h_t}{(1+r)^t} = a_0 + w_0 \sum_{t=0}^{\infty} h_t \left(\frac{\gamma}{1+r}\right)^t.$$

We are not sure which periods the household will work, but let us define the following:

$$\sum_{t=0}^{\infty} h_t \left(\frac{\gamma}{1+r}\right)^t = \sum_{t=0}^{\infty} h_t \beta^t = \lambda,$$

with $\lambda \in [0, \frac{1}{1-\beta}]$, where the lower limit corresponds to never working and the upper limit represents working in every period for the rest of eternity. Thus, we have recast the problem in terms of the total remaining time, appropriately discounted, that the household chooses to work. It is also clear that the household will be indifferent as to the timing here, so long as the total satisfies the chosen λ.

We can now write the problem as follows:

$$\max_{c_0,\lambda} \left\{ \sum_{t=0}^{\infty} \beta^t \frac{\left(c_0 \gamma^{\frac{t}{\sigma}}\right)^{1-\sigma}}{1-\sigma} - \psi\lambda \right\}$$

subject to

$$\frac{c_0}{1-\beta\gamma^{\frac{1-\sigma}{\sigma}}} = a_0 + w_0\lambda.$$

Inserting the budget constraint into the maximization problem gives a static maximization problem

$$\max_{\lambda} \left\{ \frac{\left(a_0 + w_0\lambda\right)^{1-\sigma}}{(1-\sigma)\left(1-\beta\gamma^{\frac{1-\sigma}{\sigma}}\right)^{\sigma}} - \psi\lambda \right\}.$$

The first-order condition with respect to λ (assuming an interior solution) reads

$$\frac{w_0(a_0 + w_0\lambda)^{-\sigma}}{x} = \psi,$$

where we define $x = (1-\beta\gamma^{\frac{1-\sigma}{\sigma}})^{\sigma}$. We then see that if assets are large enough, there is no interior solution to the problem, but the household will choose to work zero periods, i.e., $\lambda = 0$. If the assets are low enough, the household will have to work every period, i.e., $\lambda = \frac{1}{1-\beta}$.

If assets are somewhere in between, the interior solution for how much to work is given by

$$\lambda = \frac{1}{w_0}\left(\left(\frac{\psi x}{w_0}\right)^{-\frac{1}{\sigma}} - a_0\right).$$

Clearly, consumers with different wealth will choose different remaining workloads. Thus, the model with an extensive margin is consistent with balanced growth—at the rates of the different aggregates solved for in the planning problem—even with heterogeneity in initial asset levels. To complete the construction of an equilibrium—in particular to ensure market clearing for the labor input—one would need to specify who works at which point in time. That can be accomplished with many degrees of freedom since consumers are indifferent as to the timing of their work (if their solution is interior).

[10] Note that here we normalized $\underline{h} = 1$.

4.2. Incomplete markets

In the incomplete-markets model as outlined in Section 2.1 households can benefit from working longer hours when their productivity is high, and less hours when their productivity is low, but they cannot do this as effectively here as there is a lower bound on asset holdings. The incomplete insurance, moreover, implies that interest rate is lower than under complete markets, making richer households—who are relatively well insured—less willing to save. As a result of these features, in the event of a sequence of many bad labor productivity shocks, households are forced to work longer hours despite their current low productivity, since they already have drawn down their assets and are borrowing-constrained. This is an argument for average productivity in the economy with incomplete markets to be lower than in the corresponding complete markets economy.[11] On the other hand, total hours may be higher if the insurance motive for working is strong. All these issues will be discussed below.

We first formulate the model in the presence of growth, and with intensive-margin labor choice only. Thereafter, we turn the to case of extensive-margin labor choice. After that, we present our quantitative results and interpret them in light of the complete-markets analysis.

4.2.1. The intensive margin

We begin by defining a balanced-growth equilibrium in levels and then show how it can be transformed, focusing first on the case with an intensive-margin choice. Growth is labor-augmenting at gross rate γ. This means, given the more general preference class in Boppart and Krusell (2020), that consumption (and assets) can grow at a different rate (let us call its gross rate g) than labor productivity, while hours grow at the gross rate g_h. The discrepancy is regulated by the parameter ν (i.e., the KPR formulation is the special case obtaining for $\nu = 0$).

Definition 2. A balanced-growth equilibrium consists of growth rates g and g_h, prices r_k and w_t, a value function V_t, decision rules f_t^k and f_t^h, and distributions Γ_t such that, for all t,

1. $g = \gamma g_h = \gamma^{1-\nu}$.
2. V_t solves the household's problem: for all (a, ω),

$$V_t(a,\omega) = \max_{a',h} u(a(1-\delta+r_k) + h\omega w_t - a', h) + \beta E[V_{t+1}(a',\omega')|\epsilon]$$

   s.t. $a' \geq \underline{a} g^{t+1}, h \in [0,\infty)$. Notice, here, that the borrowing constraint changes over time (unless $\underline{a} = 0$) and gets more and more stringent with $\underline{a} < 0$.
3. $f_t^a(a,\omega)$ and $f_t^h(a,\omega)$ solve the maximization problem on the right-hand side of the dynamic-programming problem above for all (a,ω).
4. r_k and w satisfy $r_k = F_1(\bar{k}_t, \gamma^t \bar{h}_t)$ and $w_t = \gamma^t F_2(\bar{k}_t, \gamma^t \bar{h}_t)$, where $\bar{k}_t \equiv \sum_\omega \int_a a\Gamma_t(da,\omega)$ and $\bar{h}_t \equiv \sum_\omega \int_a \omega f_t^h(a,\omega) \times \Gamma_t(da,\omega)$.
5. $\Gamma_{t+1}(B,\omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{a: f_t^a(a,\hat{\omega}) \in B} \Gamma_t(da,\hat{\omega})$ for all Borel sets B and for all ω.
6. $f_t^a(ag^t,\omega) = g^t f_0^a(a,\omega)$, $f_t^h(ag^t,\omega) = g_h^t f_0^h(a,\omega)$, and $\Gamma_t(Bg^t,\omega) = \Gamma_0(B,\omega)$ for all a, B, and ω.

Note that due to growth, the distribution over a will not be stationary. However, as we will show below, once a is detrended by the appropriate growth rate we obtain a stationary distribution.

The level-based definition just defined can be stated in stationary form as follows:

Claim 1. *The balanced-growth equilibrium defined above is equivalent to a stationary equilibrium defined by prices r_k and w, a value function V, decision rules f^a and f^h, and a distribution Γ such that*

1. *V solves*

$$V(a,\omega) = \max_{a',h} u(a(1-\delta+r_k) + h\omega w - a'g, h) + \beta g^{1-\sigma} E[V(a',\omega')|\omega]$$

$$s.t.\ a' \geq \underline{a}, h \in [0,\infty).$$

2. *$f^a(a,\omega)$ and $f^h(a,\omega)$ solve the maximization problem on the right-hand side of the dynamic-programming problem above for all (a,ω).*
3. *$r_k = F_1(\bar{k},\bar{h})$ and $w = F_2(\bar{k},\bar{h})$, where $\bar{k} \equiv \sum_\omega \int_a a\Gamma(da,\omega)$ and $\bar{h} \equiv \sum_\omega \int_a \omega f^h(a,\omega)\Gamma(da,\omega)$.*
4. *$\Gamma(B,\omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{a: gf^a(a,\hat{\omega}) \in B} \Gamma(da,\hat{\omega})$ for all Borel sets B and for all ω.*

[11] See Pijoan-Mas (2006) for a full discussion about this topic.

Note that w, $\bar{k}$, $\bar{h}$, a, and h in this Claim 1 should be understood as detrended objects (detrended at their respective long-run growth rates). We now prove the claim.

Proof. Using the last condition of the balanced-growth equilibrium, note that in the third condition we can write $\bar{k}_t = (\sum_\omega \int_a a\Gamma_0(\frac{da}{g^t}, \omega))$, which is equivalent to $\tilde{\bar{k}}_t \equiv \frac{\bar{k}_t}{g^t} = (\sum_\omega \int_{\tilde{a}} \tilde{a}\Gamma_0(d\tilde{a}, \omega))$, where we have defined $\tilde{a} = a/g^t$. Notice also that $\tilde{\bar{k}}_t = \tilde{\bar{k}}$, i.e., a constant, in a balanced-growth equilibrium.

Similarly, we obtain $\bar{h}_t = \sum_\omega \int_a \omega g_h^t f_0^h(\frac{a}{g^t}, \omega)\Gamma_0(\frac{da}{g^t}, \omega)$, implying that

$$\tilde{\bar{h}}_t \equiv \frac{\bar{h}_t}{g_h^t} = \sum_\omega \int_{\tilde{a}} \omega f_0^h(\tilde{a}, \omega)\Gamma_0(d\tilde{a}, \omega),$$

which also is constant under balanced growth: $\tilde{\bar{h}}_t = \tilde{\bar{h}}$.

Given $g = \gamma g_h$ and that F_1 and F_2 are both homogeneous of degree 0, we now see that the two firm first-order conditions can be expressed as

$$r_k = F_1(\tilde{\bar{k}}, \tilde{\bar{h}}) \quad \text{and} \quad w_0 = F_2(\tilde{\bar{k}}, \tilde{\bar{h}}). \tag{14}$$

Turning to the fourth equilibrium condition, using the (very) last condition stating that the distribution is (in an appropriate sense) constant on the balanced growth path, we obtain

$$\Gamma_0(B/g^{t+1}, \omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{\tilde{a}: f_0^a(\tilde{a},\hat{\omega})g^t \in B} \Gamma_0(d\tilde{a}, \hat{\omega}),$$

where we used the definition of $\tilde{a}$. Defining $\tilde{B} = B/g^t$ for any Borel set B, we obtain

$$\Gamma_0(\tilde{B}/g, \omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{\tilde{a}: f_0^a(\tilde{a},\hat{\omega}) \in \tilde{B}} \Gamma_0(d\tilde{a}, \hat{\omega}),$$

which can equivalently be stated as

$$\Gamma_0(\tilde{B}, \omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{\tilde{a}: g f_0^a(\tilde{a},\hat{\omega}) \in \tilde{B}} \Gamma_0(d\tilde{a}, \hat{\omega}). \tag{15}$$

Looking at consumer optimization under balanced growth, finally, we obtain (after using the same kinds of definitions as above),

$$V_t(\tilde{a}(1-\delta+r_k)g^t, \omega) = \max_{\tilde{a}', \tilde{h}} u(\tilde{a}g^t + \tilde{h}g_h^t \omega w_0 \gamma^t - \tilde{k}' g^{t+1}, \tilde{h}g_h^t) + \beta E[V_{t+1}(\tilde{a}' g^{t+1}, \omega')|\omega]$$

s.t. $\tilde{a}' g^{t+1} \geq \underline{a} g^{t+1}, \tilde{h}g_h^t \in [0, \infty)$.

Now consider our instantaneous utility function as formulated in equation (1) for u and let $g_h = \gamma^{-\nu}$ and $g = \gamma^{1-\nu}$. Then $g^{t(1-\sigma)}$ can be factorized out from u. Dividing both sides of the equation by this quantity and defining $V_t(\tilde{a}g^t, \omega) \equiv g^{t(1-\sigma)}\tilde{V}(\tilde{a}, \omega)$, we can write

$$\tilde{V}(\tilde{a}, \omega) = \max_{\tilde{a}', \tilde{h}} u(\tilde{a}(1-\delta+r_k) + \tilde{h}\omega w_0 - \tilde{k}' g, \tilde{h}) + \beta g^{1-\sigma} E[\tilde{V}(\tilde{a}', \omega')|\omega] \tag{16}$$

s.t. $\tilde{a}' \geq \underline{a}, \tilde{h} \in [0, \infty)$, with associated policy functions $\tilde{f}_t^a(\tilde{a}, \omega)$ and $\tilde{f}_t^h(\tilde{a}, \omega)$.

Now r_k, w_0, $\tilde{V}$, $\tilde{f}^a$, $\tilde{f}^h$, and Γ_0, determined by equations (14), (15), and (16), define a stationary equilibrium for detrended objects $\tilde{\bar{k}}$, $\tilde{\bar{h}}$, $\tilde{a}$ and $\tilde{h}$. To ease notation we removed the tildes on these objects in Claim 1. □

Four items differ compared to the formulation of our quantitative baseline model in Definition 1 in Section 2.1. First, we restrict attention to balanced paths here. Second, the discount factor in the consumer's problem is multiplied by $g^{1-\sigma}$. Third, an additional gross "cost" of saving, g, appears in the consumer's budget. Fourth, and finally, g appears in the updating on the right-hand side of the equation determining the stationary distribution.

4.2.2. The extensive margin: version I

When the labor choice is in a two-point set $\{0, \underline{h}\}$, the equilibrium definition needs to be altered only slightly: $f^h(a,\omega)$ is now an indicator function, taking on the value 1 if the agent chooses to work and 0 if the agent chooses not to work. For convenience, we restrict attention here to utility functions u that are additively separable in consumption and leisure (the MaCurdy family (5)).

A complication that arises when there is an extensive-margin choice is that in the case where $\nu > 0$—the case where ongoing labor-productivity growth would call for lower and lower hours worked—workers cannot choose to work less and less per time unit, since the set of available hours choices only contains 0 and $\underline{h}$. Instead, they would work more and more rarely, as detailed below in the complete-markets formulation in Section 4.2.5: there would be a continually decreasing participation rate. Individual choice would entail an f^h that would be equal to 1 on an increasingly small (relative) part of its domain for cash on hand ω as wages keep rising; when the worker works one period there is instead a very large addition to assets. These features may or may not be consistent with an exact balanced-growth equilibrium; we discuss this important issue further in Section 4.2.5 below. Because of this challenge, let us for the time being consider a different model formulation: our "version 1". This extensive-margin model is designed to generate balanced growth by allowing the constraint itself change with time.

Thus, consider a two-point set for hours worked that changes with productivity: if productivity grows, the working option involves less hours (in the case with stronger income effects than for KPR). This formulation is motivated by what workers would like, given their preferences: it would allow working regularly while at the same time enjoying more and more leisure. In particular, assume that the labor choice set is $\{0, \underline{h}\gamma^{-\nu t}\}$. The implied setup is consistent with a (transformed) stationary equilibrium.

Claim 2. *The balanced-growth equilibrium with a labor choice set $\{0, \underline{h}\gamma^{-\nu t}\}$ is equivalent to a stationary equilibrium defined by prices r and w, a value function V, decision rules f^a and f^h, and a distribution Γ such that*

1. *V solves*

$$V(a,\omega) = \max_{a',h} u(a(1-\delta+r) + h\underline{h}\omega w - a'g, h) + \beta g^{1-\sigma} E[V(a',\omega')|\epsilon] \tag{17}$$

s.t. $a' \geq \underline{a}$, $h \in \{0,1\}$.

2. *$f^a(a,\omega)$ and $f^h(a,\omega)$ solve the maximization problem on the right-hand side of the dynamic-programming problem for all (a,ω).*
3. *r and w satisfy $r = F_1(\bar{k},\bar{h})$ and $w = F_2(\bar{k},\bar{h})$, where $\bar{k} \equiv (\sum_\omega \int_a a\Gamma(da,\omega))$ and $\bar{h} \equiv \sum_\omega \int_a \omega\underline{h} f^h(a,\omega)\Gamma(da,\omega)$.*
4. *$\Gamma(B,\omega) = \sum_{\hat{\omega}} \pi_{\omega|\hat{\omega}} \int_{a: g f^a(a,\hat{\omega})\in B} \Gamma(da,\hat{\omega})$ for all Borel sets B and for all ω.*

The proof of this claim follows the proof of the previous claim line by line.

4.2.3. Results

We begin with the calibration and then outline how we use numerical methods to find equilibria; these methods go beyond what is used in the literature in that they also offer insights into convergence to balanced growth paths. We then go over the nature of the policy rules, with maintained emphasis on comparisons with the static/complete-markets models discussed earlier. After this, we discuss the aggregate results, first focusing on interest rates and aggregate hours worked and then on various measures of inequality as well as efficiency.

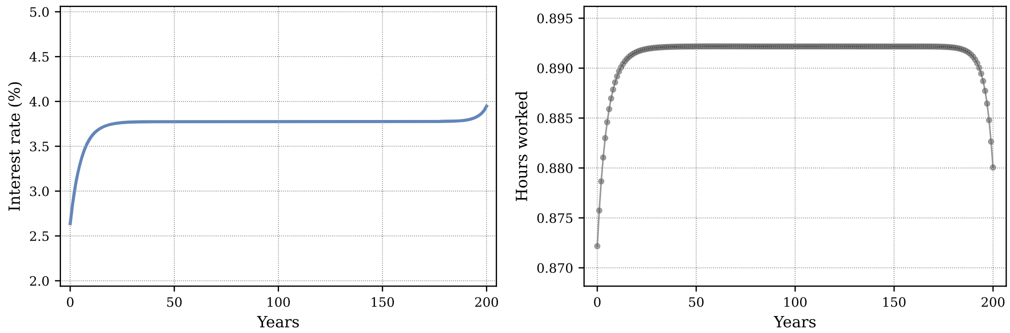
4.2.3.1. Calibration For the purpose of this section we again use the MaCurdy utility function:

$$u(c,h) = \frac{c^{1-\sigma}-1}{1-\sigma} - \psi\frac{h^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}. \tag{18}$$

We will both use the KPR version, $\sigma = 1$, and a BK version, where we set $\sigma = 1.7$, a calibration that is also entertained in Boppart and Krusell (2020); with this value follows a value for θ that is equal to 1.5, so that hours fall at roughly the rate observed in data in a cross-section of countries, a rate that corresponds to $\gamma^{-\nu}$ in the model, where $\nu = (\sigma-1)/(\sigma+\frac{1}{\theta})$. Different values of ψ are then considered, including $\psi = 0$ (no valued leisure), delivering different levels of aggregate hours. We examine both a model with an intensive margin only and a model with an extensive margin only.[12]

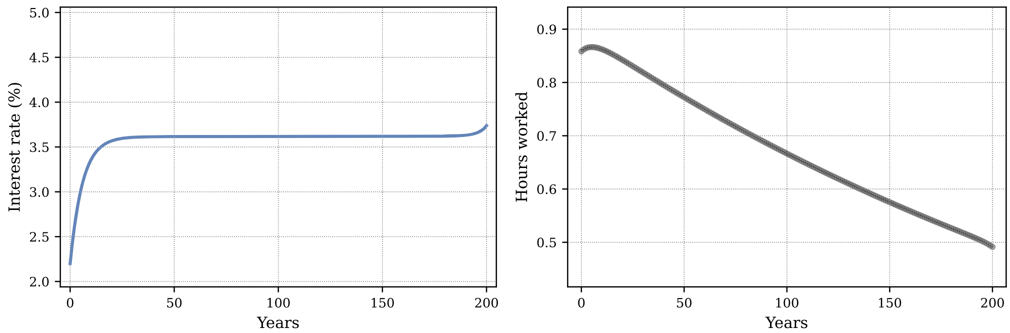
For all models we set the yearly growth rate $\gamma = 1.01$ and we use a(n annual) discount factor $\beta = 0.96$. The rest of the model is parameterized in a standard way, with idiosyncratic productivity shocks following an AR(1) process in logs with persistence 0.9 and conditional standard deviation of 0.1744, discretized into a 7-state Markov chain. The firm side produces with Cobb-Douglas technology with capital share of 1/3. Depreciation, δ, is assumed to be 10%.

[12] The case with both an intensive and an extensive margin would also be interesting to study; here, the two extreme cases are focused on for ease of comparison.



(a) Interest rate path (b) Path of average hours worked

Fig. 10. Results from the turnpike approach. Model with KPR preferences and intensive margin labor choice, $\psi = 1.0$.



(a) Interest rate path (b) Path of average hours worked

Fig. 11. Results from the turnpike approach. Model with BK preferences and intensive margin labor choice, $\psi = 1.0$.

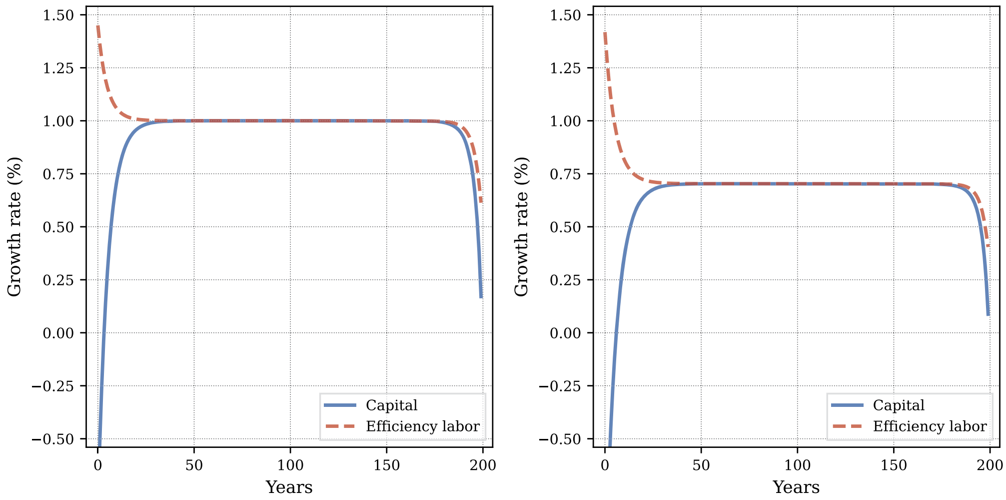
4.2.3.2. Finding a steady state: the use of turnpikes The existence of steady states in the Aiyagari model has been established in the literature; see, e.g., Acemoglu and Jensen (2015). There are results on uniqueness—see, e.g., Light (2018)—but there are also examples of multiplicity of steady states. We know of no proofs of global convergence to steady state for this class of models. For this reason, caution has to be exercised when trying to find a steady state with numerical methods.

We follow two procedures here. One is standard: the model is transformed into stationary form—by use of the growth rates that we conjecture will characterize a balanced path; see the previous sections—and then an iterative procedure is used to find the steady state. This iteration is particularly simple here because it only involves one equilibrium price: the real interest rate.[13] So one guesses on the interest rate, solves the dynamic programming problem of the household given this interest rate, which implies choices for hours and capital savings, and then finds the implied stationary distribution of these variables given the stochastic process assumed. That stationary distribution in turn gives the economy-wide capital-hours ratio, which returns the real interest rate. We find this iterative procedure to be fast and stable.

The second procedure is to instead solve for a long transition path for the untransformed model. That is, one fixes a final-period interest rate—arbitrarily chosen—and then solves for a path of equilibrium interest rates. This numerical task is more challenging, because it involves guessing and iterating on a whole sequence of interest rates. However, we find that also this procedure is stable and fast. We use 200 time periods and our end-period guess is an interest rate that is a steady state for the equivalent economy without further growth. I.e., we know that the guess is wrong, and in some cases the guess is quite far from the steady state sought. The key now is that once an equilibrium is reached here, one can assess whether it appears to converge to a steady state, and whether that steady state coincides with the steady state found using the first procedure.

We find that this "turnpike" approach works very well and strongly indicates global convergence. Fig. 10 shows the resulting interest rate path and average hours worked for the KPR model with labor choice on the intensive margin. As can be seen, the results confirm the results from the transformed stationary equilibrium: in the long run, the real interest rate as well as hours worked settle down to constant after roughly 20 years and then remain at those constants until about 10 years from the endpoint. I.e., the economy gets on the turnpike rather quickly, stays on it for a long time, and then exits at the end, just like a car would.

[13] The interest rate maps uniquely into the aggregate wage rate, given a constant-returns-to-scale production function.



(a) KPR preferences (b) BK preferences

Fig. 12. Growth of capital and labor measured in efficiency units. Results from the model with intensive margin labor supply, solved with the turnpike approach.

Fig. 11 shows the corresponding figures for the model with BK preferences. Here, as can be seen, average hours are falling toward zero at a constant rate in the long run, as expected.

Fig. 12 illustrates the resulting growth rates for capital and labor, measured in efficiency units from the turnpike model (for KPR preferences and BK preferences, respectively). As can be seen, with KPR preferences, the growth rate converges to γ, while in the model with BK preferences, the growth rate converges to $\gamma^{1-\nu}$. Very similar results are found for the extensive-margin versions of the model. We omit them for the sake of brevity.

In sum, our balanced growth paths are straightforwardly computed and there are strong indications of global convergence to these paths. The long-run growth rates coincide with those of the corresponding complete-markets economies.

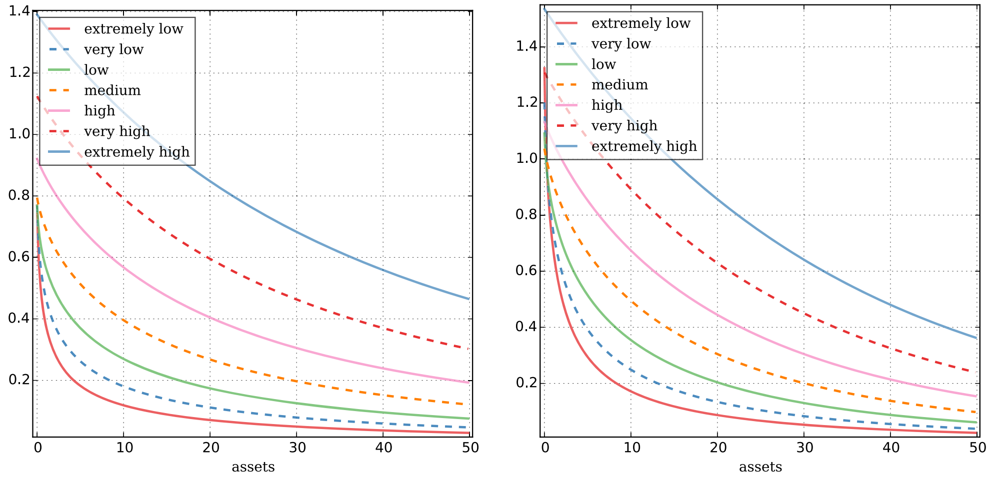
4.2.3.3. Policy rules We now discuss the numerically computed policy rules for our economies and compare them to corresponding rules under complete markets. We begin by showing the decision rules in the stationary state and then show how growth in wages affect these graphs. Throughout we focus on labor supply, i.e., we omit consumption decision rules, as hours worked are our focus here.

First, Fig. 13 shows policy functions for labor from the stationary growth model with an intensive-margin labor choice. The figure illustrates two important features. One is rather obvious: for a given productivity level (on a given colored line in the graph), higher assets lowers labor supply. The second is a feature of intertemporal substitution in the incomplete-markets economy: the higher the productivity, the more the agent chooses to work, given a fixed asset level. That is, the agent takes the opportunity to work when the wage is high and, at that point, saves a large portion of the earnings for the future. The intertemporal-substitution mechanism is a strong force and influences, e.g., hours-productivity correlations in the cross-sectional data significantly.

Second, and relatedly, it is instructive to think back to the simpler static setting discussed in Section 3.2. To facilitate, Fig. 14 shows the hours choice in a static model (repeating exactly the same information as Fig. 1 and Fig. 2).[14] In this figure, we contrast hours isoquants in two stylized static models: one with KPR preferences and one with BK preferences. For the household, a productivity increase is perceived as an increase in wages. In the (static) KPR model we see that, given a level of positive assets, an increase in wages always leads to higher optimal hours. However, in the case of BK preferences, increased productivity can actually lead to lower optimal hours. As for the static model, the reason is that, when the wage is low relative to assets, the substitution effect dominates, and we find an effect similar to that in the KPR case, whereas when the wage is high relative to assets, the income effect will dominate under BK and, hence, hours fall when productivity rises. We thus see, in our figure, that the non-monotonicity appears sooner for low asset levels.

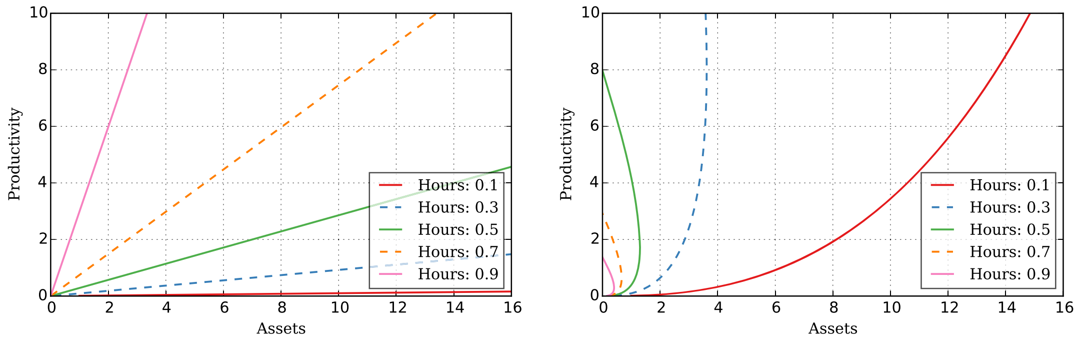
Third, we now turn to how decisions change over time when wages increase. So consider Fig. 15, which results from the intensive-margin model. For three points in time, it shows the point in the asset/productivity space where the household chooses to work exactly 0.5. As can be seen from the graph in the left panel of the figure, in the model with KPR preferences, the isohours line shifts out monotonically to the right. I.e., as aggregate wages grow, it takes higher and higher asset levels to reproduce the same hours choice: wages induce higher work, and the income effect of higher assets are needed to balance the wage increase.

[14] To facilitate intuition, we omit the case of negative wealth here.



(a) KPR preferences (b) BK preferences

Fig. 13. Policy functions for work from a stationary growth model with intensive margin labor choice. Each line represents a productivity level. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)



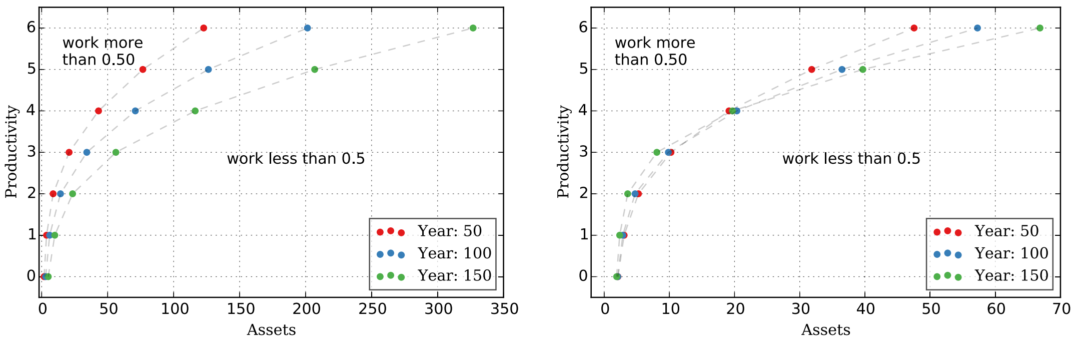
(a) KPR preferences (b) BK preferences

Fig. 14. Hours choice in a static model. Illustration of the combination of assets (x-axis) and productivity (y-axis) that yield the same hours choice. Each line represents an hours isoquant.

With BK preferences, depicted in the graph on the right-hand panel of the figure, the same monotonicity is not present. At low asset levels, the indifference curve is moving left over time, as aggregate wages increase, but at high levels it is moving right. This feature, to which we will return when we discuss the extensive margin in the BK model, derives from the features uncovered in the static model and just discussed above, namely the backward-bending nature of isohours curves in asset-productivity space under BK preferences. At low asset levels, higher wages, or productivity, will make hours worked decrease in the BK model: here, earnings are high relative to other wealth, and the income effect dominates the substitution effect. Hence, for a given, low asset level (say, 5) in the graph, the green-dotted isohours curve (where wages are the highest) is above the other lines, indicating a smaller area of high work effort. At high asset levels, in contrast, the substitution effect dominates and higher wages increase the area of high work effort.

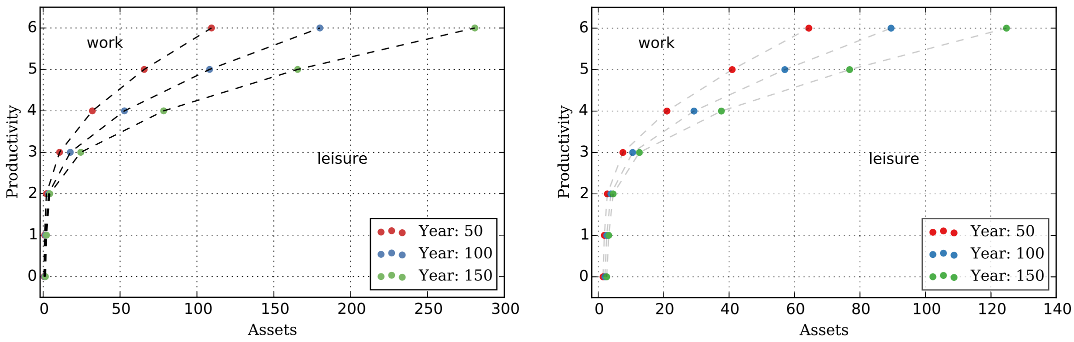
The exact same patterns as just discussed emerge when we consider the extensive-margin models. The policy functions for work/leisure at three points in time for the model with KPR preferences are shown in Fig. 16a. If a household is in the south-east region of this graph, the household has a relatively low productivity and is rich enough to be able to afford leisure and postpone the working decision. As the figure shows, the breakpoints are shifting out over time: as the economy grows, the household needs more assets to consider itself rich enough to enjoy leisure, given a fixed productivity level.

Fig. 16b shows the same type of information as Fig. 16a, but for the model with BK preferences and a bound for the feasible hours choice set that falls at the appropriate rate ($\gamma^{-\nu}$). As can be seen, as for the KPR case, the breakpoints in asset space for the indifference between hours and leisure move monotonically. However, this monotonic movement is due



(a) KPR preferences (b) BK preferences

Fig. 15. Isohours lines at three points in time, where later years imply higher wage level. Results for a model with intensive margin labor choice. The line for each year indicates where the household chooses to work exactly 0.5. The y-axis depicts indices on the discretized productivity grid (note that the corresponding productivity levels are not equidistant). The model is solved with the turnpike approach.



(a) KPR preferences (b) BK preferences

Fig. 16. Policy functions for work decisions at three points in time, where later years imply higher wage level. Results from the model with extensive margin labor choice (with a falling upper bound on labor supply in the case of BK preferences). The y-axis depicts indices on the discretized productivity grid (note that the corresponding productivity levels are not equidistant). The model is solved with the turnpike approach.

to the fact that the interpretation of the indifference curves changes as wages grow: they depict indifference between not working and working at a level of hours that falls over time.[15]

Armed with these insights about how agents behave, in the next section we turn to the equilibrium determination of interest rate and aggregate hours worked. There, we will only look at variables transformed by their respective growth rates, i.e., all variables will be stationary. Interest rates do not have a trend, so do not need to be transformed, but in this model hours need to be transformed to the extent preferences depart from KPR. Hence, aggregate hours will be reported relative to a trend that equals $\gamma^{-\nu t}$ (where $\nu = 0$ is the KPR case). Hence, a high value should be reported as high relative to this trend.

4.2.3.4. Interest rates and aggregate hours Table 1 summarizes our main results for the aggregates. As is well known, the interest rate in the incomplete-markets model is lower than in the complete-markets counterpart, due to the precautionary savings motive. This is illustrated by the interest rates in the first rows of the table, corresponding to the complete-markets case, being significantly higher than the values in the rows below, which all report results for incomplete-markets economies.[16]

[15] As we shall see below, monotonicity appears to fail if $\underline{h}$ is fixed.

[16] Note that in the complete-markets case the interest rate is increasing in σ for the case of no valued leisure or intensive margin labor choice. However, in the case of extensive margin, the interest rate in the complete-market model with KPR preferences and BK preferences coincide. In a complete markets economy with preferences given by (18), the long run interest rate is given by $(1+r) = g^{\sigma}/\beta$, with g, the growth rate of consumption, given by $g = \gamma^{1-\nu}$ (with ν given by (6)). In the case of KPR preferences, $\nu = 0$. In the case of BK preferences and an extensive margin labor choice, we have effectively a $\theta = \infty$, which gives $g = \gamma^{1/\sigma}$. Hence, the interest rate in a complete-markets setting with extensive margin labor choice with KPR preferences and BK preferences coincide.

Table 1
Equilibrium interest rates and aggregate hours (in parentheses) for stationary growth models. Aggregate hours are expressed as average hours worked for the intensive margin labor supply models, and as labor force participation for the extensive margin labor supply models. For the model with BK preferences and intensive margin labor choice, hours are reported relative to trend.

	KPR preferences $\sigma = 1.0$	BK preferences $\sigma = 1.7$
Complete markets		
No valued leisure	5.21%	5.94%
Valued leisure, $h \in [0, \infty)$	5.21%	5.41%
Valued leisure, $h \in \{0, \underline{h}\gamma^{-\nu t}\}$	5.21%	5.21%
Incomplete markets		
No valued leisure	4.39%	3.89%
	(100%)	(100%)
Valued leisure, $h \in [0, \infty)$		
$\psi = 1.0$	3.77%	3.61%
	(0.89)	(0.90)
Valued leisure, $h \in \{0, \underline{h}\gamma^{-\nu t}\}$		
$\psi = 0.8$	3.84%	3.26%
	(89%)	(87%)
$\psi = 1.0$	3.30%	2.97%
	(78%)	(78%)

The model without valued leisure ($\psi = 0$) is represented by numbers in the second row. Here, the numbers in parenthesis refer to hours worked and we see that they are 100%. We see that the interest rate in equilibrium is lower in the BK economy; the higher consumption curvature in utility for the BK case induces a stronger precautionary savings motive.

We then introduce valued leisure ($\psi = 1$) and an intensive-margin labor choice: $h \in [0, \infty)$. As can be seen in the table, the interest rate now falls further, and rather significantly. There are, in principle, two effects here. With an active labor-supply channel, the de-facto insurance is higher, and households with a bad past string of productivity realizations can improve their asset position, and protect against further bad luck, by working harder. One might then expect that the effects of the frictions would be lower, and hence that interest rates would be higher. They are, however, lower. It turns out, namely, that another channel dominates, and it precisely builds on the fact that households now are able to intertemporally substitute and work more in times of high productivity and work less in times of low productivity. The mechanism works as follows: households now shift more assets from good periods of high productivity, because they now work harder too, than under fixed hours. On average, this increases the amount of capital households carry, because of the asymmetry given by the lower bound on assets: they increase assets more in good times than they decrease it in bad times. As a result of these mechanisms, there is a higher aggregate capital stock under valued leisure the interest rate has to fall in order to clear the capital market.

We see that the interest rate, as expected is lower under BK than under KPR, since there is a higher consumption curvature/risk aversion in the former case, inducing more saving. We also see that hours rise somewhat as under BK preferences relative to KPR, but the rise is small: from 0.89 to 0.90. We interpret this as an effect of higher saving under higher consumption curvature/risk aversion, implying lower consumption and hence higher hours worked.[17]

Next, we turn to the case of a labor supply choice on the extensive margin: $h \in \{0, 1\}$. We contrast two models, one with a lower disutility of work than the other ($\psi = 0.8$ vs. $\psi = 1$). Under the former, i.e., the lower disutility of work, the KPR preference case gives a labor force participation of 89%. The interest rate in this model is 3.86%, which is higher than in the intensive margin model. However, when the disutility is increased to 1.0 (still maintaining KPR preferences), labor force participation falls to 78% and the interest rate falls to 3.30%. The reason is that with lower labor force participation, there are more periods when the household has no income, and therefore the need to shift assets to extended periods of no working increases. Thus, the interest rate needs to fall to clear the capital market.

In the extensive margin labor supply choice model with BK preferences, we observe the same pattern, but generally the interest rate is slightly lower than in the KPR case, again due to the higher risk aversion. In this model, which is the "version I" case of the extensive margin (where the bound on hours is falling over time), the labor force participation does not fall when productivity grows: instead, the amount of hours conditional on working falls.

4.2.3.5. Inequality Table 2 displays information about various inequality measures in the different models considered. We look at Gini coefficients for asset inequality, consumption inequality (within parentheses), inequality in hours worked (within brackets), and inequality for earnings (within double brackets). As robustness Table 4 in the Appendix provides the coefficient of variation in these objects as an alternative measure of inequality.

[17] Keeping consumption and wages fixed, the direct effect of an increase in σ on hours, i.e., going from KPR to BK, is negative, and hence goes the other way.

Table 2
Inequality. Resulting Gini coefficient for wealth, consumption (in parentheses), hours [in brackets], and earnings [[in double-brackets]].

		KPR preferences $\sigma = 1.0$	BK preferences $\sigma = 1.7$
No valued leisure	*wealth*	0.58	0.56
	consumption	(0.19)	(0.18)
	hours	[−]	[−]
	earnings	[[0.22]]	[[0.22]]
Valued leisure, $h \in [0, \infty)$			
$\psi = 1.0$	*wealth*	0.65	0.63
	consumption	(0.19)	(0.14)
	hours	[0.12]	[0.14]
	earnings	[[0.31]]	[[0.27]]
Valued leisure, $h \in \{0, \underline{h}\gamma^{-\nu t}\}$			
$\psi = 0.8$	*wealth*	0.67	0.65
	consumption	(0.17)	(0.14)
	hours	[0.12]	[0.14]
	earnings	[[0.31]]	[[0.33]]
$\psi = 1.0$	*wealth*	0.64	0.66
	consumption	(0.17)	(0.13)
	hours	[0.22]	[0.22]
	earnings	[[0.40]]	[[0.39]]

Before commenting on how different specific models compare, let us make some overall observations about the extent of inequality in our benchmark models with an active labor-supply channel. We see that wealth inequality is high, not as high as in the data (it is around 0.8), but it is much higher than it is in the baseline model (Aiyagari, 1994). This is noteworthy, given the discussion in Hubmer et al. (2021), which various extensions to the baseline model aimed at raising wealth inequality: heterogeneity in discount factors, heterogeneity in returns, and superstar earnings processes.[18] Apparently, the model with an active labor-supply channel we study here helps significantly in making wealth inequality match data better. We also observe that earnings inequality is high, though not as high as in the data (in the data it is a little over 0.4). Again, an active labor-supply channel helps: it makes hours go up when the wage is high, reflecting intertemporal substitution. Consumption inequality has the lowest Gini, reflecting a strong need to smooth over time and across states of nature.

In terms of model comparisons, the table shows, first of all, that wealth inequality is higher as a result of the active labor-supply channel: the associated Gini goes up by 7 percentage points (both for KPR and BK preferences) when we compare $\psi = 0$ to $\psi = 1$. The interpretation here is that intertemporal substitution makes hours comove with wages, hence increasing earnings inequality. As a result, savings rise overall, but not sufficiently to outweigh the direct earnings-based effect. We also see that whether we look at a model with an intensive or an extensive margin makes only a small difference for wealth Ginis.

Overall, consumption inequality is much lower than wealth inequality. This is a well known, but of course important, point: wealth inequality in large part is a result of consumption smoothing and hence higher wealth inequality helps keep consumption inequality low. As for how consumption inequality differs across models, the most striking feature is that BK preferences give lower Ginis. This is, again, not surprising, since higher consumption curvature means that it is costlier to accept higher consumption variation, and hence consumers make sure to decrease it. Consumption inequality differs only slightly depending on whether labor supply is active or not.

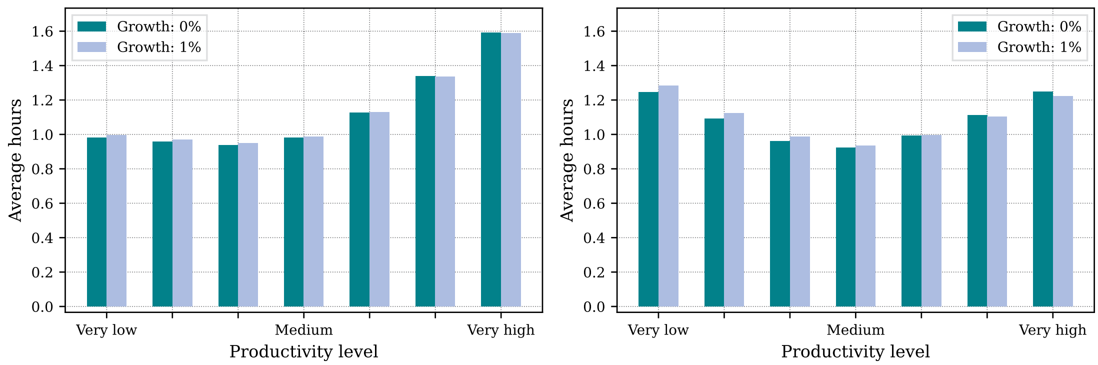
Inequality in hours worked is also very low, of course reflecting the fact that it is costly to vary hours: $\theta = 1.5$ is a high enough Frisch elasticity to prevent much movement in hours, given the calibrated process for individual wages. We also note, conditional on an extensive-margin choice, that hours inequality increases significantly if the cost of working rises (ψ goes from 0.8 to 1): fewer people now work and, hence, inequality rises.

Let us, finally, briefly comment on another feature of our equilibria: the “efficiency” with which people are allocated to work. In particular, one can look at whether agents with high labor productivities work more on average (than do agents with low productivity). In separate work (Boppart et al., 2023) we address this question more generally in a similar model to those considered here. In particular, in a social planning problem with utilitarian weights, there would be a tight connection between work hours and productivity. Thus, it is relevant to ask to what extent the market equilibria with incomplete markets mimic this benchmark. It turns out that for KPR utility, a strong positive correlation is borne out: it is 0.53 for the intensive-margin model and about half that for the extensive-margin model. For the BK model, however, the correlation is positive but closer to zero; one reason for this is the stronger income effect, making higher wages also lower hours worked. We also note that assets and earnings are positively correlated. Relatedly, a decentralization of the utilitarian planning solution would require that the higher productivity of the agent, the lower the agent's assets. This is because

[18] The key feature of the superstar earnings process is not only that extremely high productivity can occur but also that extremely large drops in productivity can occur from the highest productivity levels.

Table 3
Key results from a model with and without growth. The average productivity is measured as the sum of efficiency units of labor divided by the total hours worked in the economy. The labor productivity is normalized so that average productivity would be equal to 1 if hours worked were equal across productivity states.

	KPR preferences $\sigma = 1.0$		BK preferences $\sigma = 1.7$	
	$g = 0\%$	$g = 1\%$	$g = 0\%$	$g = 1\%$
Interest rate	2.82%	3.77%	2.50%	3.61%
Wealth Gini	0.64	0.65	0.62	0.63
Hours Gini	0.13	0.12	0.14	0.14
Average productivity	1.052	1.050	1.012	1.006



(a) KPR preferences (b) BK preferences

Fig. 17. Average hours worked by productivity level with and without growth.

with strong income effects, high-productivity agents would not work more for a given level of assets, so lower assets are needed to make them work more in the decentralization delivering strongly positive hours-productivity correlations. In the incomplete-markets model, a strong negative correlation is not borne out, in line with the data. The model rather predicts a positive correlation here—in stark contrast to the utilitarian social planner solution.

4.2.4. Growth vs. no growth

Table 3 shows key statistics from the different models considered, evaluating the impact of growth in the economy. We focus on the case of intensive margin choice of labor in this section, and the table compares key outcomes under the assumption of 1% growth (the premise used above) vs. zero growth.

As expected, the interest rate is lower and thus the capital-to-labor ratio higher in the no-growth scenario, both considering the model with KPR preferences and BK preferences. However, the distribution of wealth does not change much; the Gini coefficient for wealth hardly moves across the two scenarios. The same is true for the distribution of hours: the Gini coefficient hardly moves as growth changes.

As Table 3 shows, average labor productivity does fall slightly when the economy grows, both in the case with KPR preferences and with BK preferences. In other words: when the growth rate increases, there is a relative shift of hours from more highly productive to less productive agents in the economy. The main driver is the higher interest rate with growth, which increases non-labor income relatively more for the rich (who on average also are more high-productive). In the KPR case, the fall is −0.20%, while it is slightly more severe in the BK case: −0.54%.

Fig. 17 shows the average hours worked by productivity level for the four models considered. As can be seen, the hours distribution shifts slightly across productivity states depending on the assumption of growth, but the assumption of the strength of the income effect in preferences is far more important in shaping the hours distribution.

4.2.5. The extensive margin: version II

Let us now return to the model with an extensive margin that does not change with productivity. Recall that, with complete markets and a fixed extensive margin (where thus $\underline{h} > 0$ and does not change with time), we showed in Sections 4.1.1–4.1.2 above that there is an exact balanced growth path. This path features a constantly (in percentage terms) declining set of people working, but indifference as to the timing of work for each household.[19] With incomplete markets and the same extensive margin, we discovered a challenge. The purpose here is to discuss this challenge: it is not clear to us how to characterize equilibria with an extensive-margin model where the lower bound on hours is not falling as

[19] Moreover, recall that total hours decline somewhat faster in this case than under intensive-margin labor supply.

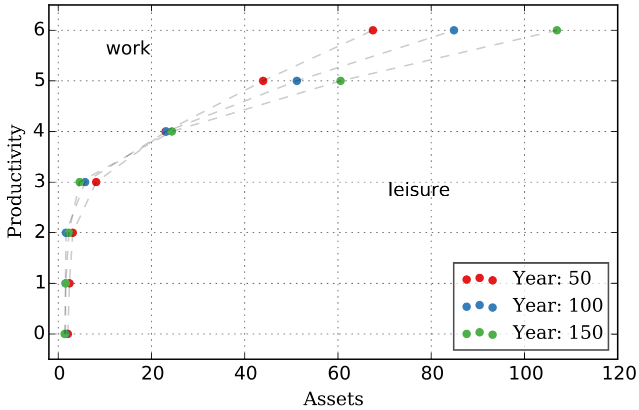


Fig. 18. Policy functions for work decisions at three points in time. Results from a model with BK preferences and extensive margin labor choice, with a time-invariant upper bound on labor supply. The y-axis depicts indices on the discretized productivity grid (note that the corresponding productivity levels are not equidistant). The model is solved with the turnpike approach.

productivity rises, but is constant. We will present a conjecture as to the qualitative asymptotic behavior of this economy, but it is only a conjecture and it is not even clear that an equilibrium exists.

First, let us simply note that the kind of transformation used for the economy with an intensive margin does not work if $\underline{h}$ does not shrink over time. Second, we noted that individuals will want to work less and less in this economy but that it is not obvious whether an exact balanced growth path exists. Under complete markets, as seen above, such a balanced path exists, and it is associated with a total commitment to work going forward that is shrinking at a constant rate. It is also, however, associated with indifference as to the timing of the work, and under incomplete markets shocks and the inability to fully transfer resources over time will in general prevent this.

One possibility is that the asymptotic path has all agents, or all agents except a vanishingly small set of agents, effectively fully insured due to individual capital accumulation. However, what is key in this economy is assets relative to an appropriate transformation of wages rather than the absolute asset level and if people keep withdrawing from the labor force, wages will grow faster than will output, by implication, and assets. This would contradict effectively full insurance.

Given incomplete insurance, we know, from the individual's Euler equation, that the real interest rate will be somewhat depressed. In particular, if an individual would save in a permanent-income manner, thus consuming the return and saving the rest, assets would grow less fast than the growth rate of consumption and output. As a result, assets would decline relative to consumption so long as the consumer does not work. On the other hand, any consumer who works in the current period would, in the limit, accumulate an enormous amount of assets. Thus, the domain for assets will contain the whole real line and not lend itself easily to a transformation: to the right of the asset level at which the consumer is indifferent between working or not assets will jump further and further to the right; to the left they will shrink.

Fig. 18 illustrates the lack of monotonicity: here, we display decision rules based on aggregate wages growing over time—not equilibrium wages, since we have not been able to solve the model. As can be seen, as for the intensive choice, the breakpoints in asset space for the indifference between hours and leisure move to the left, as wages grow, if assets are low, but to the right, if assets are high. This is intuitive and, again, follows the logic emphasized in the static analysis: when assets are low enough, raised wages lead to lower desired work hours, since the income effect dominates when the wage is high relative to assets. This qualitative feature flips over as assets increase and wage changes mainly generate substitution effects.

The non-monotonicity is difficult to manage numerically and it also seems that the entire asset domain will expand as wages grow: for low assets, working becomes more and more rare as wages grow, and hence assets fall further in the absence of earnings income. At the same time higher and higher asset holdings will also materialize when assets are high enough. Our conjecture, still, is that there will be an asymptotic steady state with a stable interest rate. Preliminary calculations indicate that this interest rate will be quite low, but at the very least the details are still quite open. It also seems that, if an asymptotic path exists, it may have wealth and consumption inequality. Thus, the question of work hours are distributed under extensive-margin constraints and ever-rising productivity does warrant more work.

4.2.6. Summary of results

This section briefly summarizes the results from the different experiments. We have looked at models that vary in a number of dimensions. One is the strength of income vs. substitution effects. Another is complete vs. incomplete markets. A third is growth vs. no growth. And, finally, a fourth is the intensive vs. the extensive margin. If we limit attention to the case with growth and the intensive margin—or an extensive margin that admits a shrinking lower bound for hours—then the most important quantitative factor behind *differences* in outcomes derives from market incompleteness. Clearly, market incompleteness drives inequality in wealth (and other individual variables) and lowers the real interest rate, but in addition market incompleteness is a key determinant behind the correlation between hours and wages in the population.

In particular, despite strong income effects, this correlation is positive, reflecting intertemporal substitution. Perhaps even more strikingly, incomplete markets make the extensive-margin case very different in nature than its complete-markets counterpart: we find that no exact balanced path exists and we are not even sure that an asymptotically balanced path exists. Overall, the strength of income vs. substitution effects, while important in the qualitative sense in the longer run as it guides whether hours stay stationary or fall toward zero, does not affect aggregates (such as the relative importance of wage vs. capital incomes or the interest rate) or inequality measures more than rather marginally. The other main noteworthy feature, touched on briefly above, is that in an economy with incomplete markets, an extensive margin with a positive lower bound on hours that is fixed over time, and income effects that exceed substitution effects, we have not been able to compute an equilibrium.

5. Concluding comments

In this paper we have analyzed incomplete-markets economies with growing labor productivity and strong income effects that call for less and less hours worked over time. We have found these economies to behave like models without growth in many respects: with an appropriate transformation of variables, we obtain exact steady state with a determinate distribution of consumption, wealth, and hours worked, while of course the levels of consumption and wealth grow at constant rates while hours shrink at a constant rate. The comparative statics of these transformed steady-state economies do not differ greatly as a function of the growth rate of labor productivity, and the strength of income affects equilibrium variables just like they do in complete-markets economies. We have, finally, failed to characterize the case where hours are constrained by a fixed, positive lower bound, i.e., when the extensive margin binds, even though the complete-markets version of this case is well behaved (an exact balanced growth path exists). The extensive-margin, incomplete-markets case is thus challenging but must be left for future work.

Data availability

Replication package has already been approved by Christian Zimmermann.

Appendix A. Additional table

Table 4
Inequality. Resulting coefficient of variation for wealth, consumption (in parentheses), hours [in brackets], and earnings [[in double-brackets]].

		KPR preferences $\sigma = 1.0$	BK preferences $\sigma = 1.7$
No valued leisure	*wealth*	1.16	1.10
	consumption	(0.34)	(0.33)
	hours	[–]	[–]
	earnings	[[0.41]]	[[0.41]]
Valued leisure, $h \in [0, \infty)$			
$\psi = 1.0$	*wealth*	1.36	1.28
	consumption	(0.34)	(0.25)
	hours	[0.23]	[0.24]
	earnings	[[0.61]]	[[0.52]]
Valued leisure, $h \in \{0, \underline{h}\gamma^{-\nu t}\}$			
$\psi = 0.8$	*wealth*	1.44	1.40
	consumption	(0.30)	(0.26)
	hours	[0.37]	[0.39]
	earnings	[[0.57]]	[[0.59]]
$\psi = 1.0$	*wealth*	1.34	1.41
	consumption	(0.30)	(0.24)
	hours	[0.53]	[0.52]
	earnings	[[0.71]]	[[0.71]]

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