A Direct Construction of Even Length ZCPs with Large ZCZ Ratio

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Abstract

This paper presents a direct construction of aperiodic $\boldsymbol{q}$-ary ($\boldsymbol{q}$ is a positive even integer) even length Z-complementary pairs (ZCPs) with large zero-correlation zone (ZCZ) width using generalised Boolean functions (GBFs). The applicability of ZCPs increases with the increasing value of ZCZ width, which plays a significant role in reducing interference in a communication system with asynchronous surroundings. For $\boldsymbol{q = 2}$, the proposed ZCPs reduce to even length binary ZCPs (EB-ZCPs). However, to the best of the authors' knowledge, the highest ZCZ ratio for even length ZCPs which are directly constructed to date using GBFs is $\mathbf{3/4}$. In the proposed construction, we provide even length ZCPs with ZCZ ratios $\mathbf{5/6}$ and $\mathbf{6/7}$, which are the largest ZCZ ratios achieved to date through direct construction.

Keywords: Even length binary Z-complementary pairs (EB-ZCPs), generalised Boolean functions (GBFs), Golay complementary pair (GCP), zero-correlation-zone (ZCZ).

1 Introduction

The Golay complementary pairs (GCPs) were first introduced by Golay [1]. The aperiodic auto-correlation sum (AACS) of a GCP diminishes to zero for all time shifts not equal to zero. GCPs have found many engineering applications including channel estimation [2], radar [3] and peak power control in orthogonal frequency division multiplexing (OFDM)[4] etc. But the lengths of GCPs are limited to the form $2^{\alpha}10^{\beta}26^{\gamma}$, where α, β and γ are non-negative integers [5, 6]. As a generalization of GCPs, Fan *et al.* introduced the idea of Z-complementary pairs (ZCPs) [7]. The AACS of the two sequences in a ZCP is zero within a certain zone that is known as the zero-correlation zone (ZCZ). Unlike GCPs, ZCPs exist for arbitrary lengths with several ZCZ widths.

In [8], a construction of GCPs based on generalised Boolean functions (GBFs) over $\mathbb{Z}_q$ is provided by Davis and Jedwab, where $q{=}2^h$, for some positive integer h. Later, Paterson expanded Davis and Jedwab's results to q-ary GCPs for even q [9]. Such constructed GCPs based on the GBFs given in [8] and [9] are called the Golay-Davis-Jedwab (GDJ) pairs [10–12] (or called the standard GCPs [13]).

ZCPs can be used as spreading sequences in quasi-synchronous code-division multiple access (CDMA) systems to reduce inter-symbol interference (ISI) and multiple access interference (MAI) [14]. As introduced in [7], there are two types of ZCPs, namely even length binary ZCPs (EB-ZCPs) and odd length binary ZCPs (OB-ZCPs). Furthermore, it was conjectured in [7] that, the maximum ZCZ width for a binary ZCP of length N is given by $(N+1)/2$ when N is odd, and it is $N-2$ when N is even.

Several direct constructions of ZCPs of lengths in the form of a power of two can be found in [15–17]. A direct construction of q-ary ZCPs based on GBF having larger ZCZ width is introduced by Chen [18]. For $q = 2$, this direct construction produces binary ZCPs with a ZCZ ratio 2/3. For non-negative integers α, β and γ, ZCPs of length $2^{\alpha+2}10^{\beta}26^{\gamma}+2$ and ZCZ width $3\times2^{\alpha}10^{\beta}26^{\gamma}+1$, with asymptotic ZCZ ratio 3/4 are systematically constructed using insertion [19]. A direct construction of q-ary even length ZCPs through GBFs was provided in [20]. The length and ZCZ width of the constructed ZCP are $2^{m-1}+2,\ m \geq 3$, and $2^{m-2}+2^{\pi(m-3)}+1$, respectively, where π is a permutation over $m-2$ symbols. When $\pi(m-3) = m-3$, the asymptotic ZCZ ratio becomes 3/4 which is the largest ZCZ ratio available in the existing literature to date. Recently, direct GBF constructions of ZCPs of non-power-of-two lengths have been reported with maximum ZCZ ratio of 2/3 in [21] and 4/7 in [22]. In [23], different construction methods for ZCPs have been discussed and provided their upper bounds of peak to mean envelope power ratio (PMEPR). A direct construction of Z-complementary code sets (ZCCSs) has been reported in [24] which produces non-power-of-two lengths ZCPs with a maximum ZCZ ratio of 3/4. The authors in [20, 24] also introduced an open problem of having higher ZCZ ratio of ZCPs. The authors in [25] provided a systematic construction of q-ary even length ZCPs for ZCZ ratio 6/7 using the concatenation of GCPs and their complementary mates, where GCPs and their mates are used as seed sequences. However, they did not provide a direct construction and thus introduced it is an open problem for future work.

By motivation of the above open problems, in this work we present a direct construction of aperiodic q-ary even length ZCPs for ZCZ ratio 5/6 and 6/7. Like the direct construction proposed in [18] and [20], the proposed construction also does not require any special seed sequences initially. In addition, the proposed q-ary even length ZCPs have a larger ZCZ ratio than the ZCPs reported in [20], and they also provide the largest ZCZ width. For q=2, the proposed construction reduces to EB-ZCP.

The remaining paper is organized as follows. Basic notations and definitions are provided in Section 2. In Section 3, the proposed construction of q-ary even length ZCPs is discussed. Finally, concluding remarks are provided in Section 4.

2 Notations and Definitions

The following notations will be followed throughout this paper: $\bar{x} = 1 - x$, where $x \in \{0, 1\}$, $\lfloor x \rfloor$ and $\lceil x \rceil$ represents the floor and ceiling function of x respectively, for $n > 1$, $\mathbf{0}_n$ and $\mathbf{1}_n$ represents n consecutive 0's and 1's, respectively.

Definition 1 Let $\mathbf{a_1} = (a_{1,0}, a_{1,1}, \dots, a_{1,N-1})$ and $\mathbf{a_2} = (a_{2,0}, a_{2,1}, \dots, a_{2,N-1})$ be two q-ary sequences over $\mathbb{Z}_q$. The aperiodic cross-correlation function (ACCF) at a shift τ is defined by

$$\rho_{\mathbf{a_1},\mathbf{a_2}}(\tau) = \begin{cases} \sum_{k=0}^{N-1-\tau} \omega^{a_{1,k}-a_{2,k+\tau}}, & 0 \leq \tau \leq N-1, \\ \sum_{k=0}^{N-1-\tau} \omega^{a_{1,k+\tau}-a_{2,k}}, & -N+1 \leq \tau \leq -1, \\ 0, & |\tau| \geq N, \end{cases}$$

where q (≥ 2) is an integer and $\omega = \exp(2\pi\sqrt{-1}/q)$. When $\mathbf{a_1} = \mathbf{a_2}$, $\rho_{\mathbf{a_1},\mathbf{a_2}}(\tau)$ is called aperiodic auto-correlation function (AACF) of $\mathbf{a_1}$ and it is denoted by $\rho_{\mathbf{a_1}}(\tau)$.

Definition 2 A pair of sequences $\mathbf{a}$ and $\mathbf{b}$ of length N is called a GCP if their AACS is zero for all non-zero time shifts, i.e.,

$$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0, \quad \text{for } \tau \neq 0. \tag{1}$$

Definition 3 A GCP $(\mathbf{c}, \mathbf{d})$ of length N is called a complementary mate of a GCP $(\mathbf{a}, \mathbf{b})$ of length N, if

$$\rho_{\mathbf{a},\mathbf{c}}(\tau) + \rho_{\mathbf{b},\mathbf{d}}(\tau) = 0, \quad \text{for } 0 \leq \tau \leq N-1. \tag{2}$$

Definition 4 A pair of length N sequences $\mathbf{a_1}$ and $\mathbf{a_2}$ is called a ZCP with ZCZ width Z, if and only if

$$\rho_{\mathbf{a_1}}(\tau) + \rho_{\mathbf{a_2}}(\tau) = 0, \quad \text{for } 1 \leq \tau \leq Z-1. \tag{3}$$

We call $(\mathbf{a_1}, \mathbf{a_2})$ an (N, Z)-ZCP. When $Z = N$, then the pair $(\mathbf{a_1}, \mathbf{a_2})$ becomes a GCP. The sequence pair is called an EB-ZCP if the sequences $\mathbf{a_1}, \mathbf{a_2}$ are binary and length of the sequence is even.

Definition 5 ([26]) If $(\mathbf{a_1}, \mathbf{a_2})$ is a ZCP of even length N with ZCZ width Z, then the ZCZ ratio is defined as

$$\text{ZCZ ratio} = Z/N. \tag{4}$$

2.1 Generalised Boolean Function

A GBF f in m binary variables $x_0, x_1, \ldots, x_{m-1}$ is a function from $\{0,1\}^m$ to $\mathbb{Z}_q$. A monomial of degree r is defined as the product of any r variables among $x_0, x_1, \ldots, x_{m-1}$. So there are $\sum_{r=0}^{m}\binom{m}{r}=2^m$ monomials, namely $1, x_0, x_1, \ldots, x_{m-1}, x_0x_1, x_0x_2, \ldots, x_{m-2}x_{m-1}, \ldots, x_0x_1\cdots x_{m-1}$. With the linear combinations of these 2^m monomials by taking coefficients from $\mathbb{Z}_q$, a GBF can be expressed uniquely. In the expression of a GBF of order r, there exists at least one highest-degree monomial of order r with a non-zero coefficient. The complex-valued sequence corresponding to GBF f of m variables $x_0, x_1, \ldots, x_{m-1}$ is expressed as

$$\Psi(f) = \left(\omega^{f_0}, \omega^{f_1}, \ldots, \omega^{f_{2^m-1}}\right), \tag{5}$$

where $f_i = f(i_0, i_1, \ldots, i_{m-1})$, $\omega = \exp\left(2\pi\sqrt{-1}/q\right)$, and $(i_0, i_1, \ldots, i_{m-1})$ is the binary vector representation of i, where as in the remainder of this paper, q is an even integer not less than 2. Corresponding to a GBF f with m variables the sequence $\Psi(f)$ is of length 2^m. In this paper, we focus on q-ary (N,Z)-ZCPs, where $N \neq 2^m$. Hence, we define the truncated complex-valued sequence $\Psi_L(f)$ corresponding to the GBF f by deleting the last L elements of the sequence $\Psi(f)$.

Lemma 1 ([27]) *Let $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ be a GBF defined by*

$$f = \frac{q}{2}\sum_{k=0}^{m-2} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=0}^{m-1} g_k x_k + g', \tag{6}$$

where π is a permutation of $\{0,1,2,\ldots,m-1\}$ and $g', g_k \in \mathbb{Z}_q$. Then for any $c' \in \mathbb{Z}_q$ $(\mathbf{a_1}, \mathbf{a_2}) = \left(\Psi(f), \Psi\left(f + \frac{q}{2}x_{\pi(0)} + c'\right)\right)$ is a GCP and $(\mathbf{b_1}, \mathbf{b_2}) = \left(\Psi\left(f + \frac{q}{2}x_{\pi(m-1)}\right), \Psi(f + \frac{q}{2}\left(x_{\pi(0)} + x_{\pi(m-1)}\right) + c'\right)$ is a complementary mate of $(\mathbf{a_1}, \mathbf{a_2})$.

3 Proposed Construction

For proving our main result, we define the following function which will be used throughout this section.

For any integer $m \geq 5$, let π be a permutation of $\{0,1,2,\ldots,m-5\}$. Let the GBF $f : \mathbb{Z}_2^{m-4} \rightarrow \mathbb{Z}_q$ be defined as

$$f = \frac{q}{2}\sum_{i=0}^{m-6} x_{\pi(i)}x_{\pi(i+1)} + \sum_{i=0}^{m-5} c_i x_i. \tag{7}$$

Define the functions $\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3$ as follows:

$$\mathrm{f}_1 = f + \frac{q}{2}x_{\pi(0)}; \quad \mathrm{f}_2 = f + \frac{q}{2}x_{\pi(m-5)}; \quad \mathrm{f}_3 = \mathrm{f}_2 + \frac{q}{2}x_{\pi(0)}. \tag{8}$$

Theorem 1 *Let the GBF* $g : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ *be defined as*

$$\begin{aligned} g = f + \frac{q}{2}\left(\bar{x}_{m-1}x_{m-2}\left(\bar{x}_{m-3} + x_{m-3}x_{m-4}\right) + x_{m-1}\bar{x}_{m-2}\bar{x}_{m-3}x_{m-4}\right) \\ + \frac{q}{2}x_{\pi(m-5)}\left(\bar{x}_{m-1}\bar{x}_{m-3}\left(x_{m-2} + x_{m-4}\bar{x}_{m-2}\right) + x_{m-1}\bar{x}_{m-2}\left(\bar{x}_{m-3} + x_{m-3}x_{m-4}\right)\right), \end{aligned} \tag{9}$$

where f *is defined in (7),* q *is an even integer and* $c_i \in \mathbb{Z}_q$*. Then for any choice of* $c' \in \mathbb{Z}_q$*,* $(\mathbf{a}, \mathbf{b}) = \left(\Psi_{2^{m-2}}(g), \Psi_{2^{m-2}}(h = g + \frac{q}{2}x_{\pi(0)} + c')\right)$ *forms a* $\left(2^{m-1} + 2^{m-2}, 2^{m-1} + 2^{m-3}\right)$*-ZCP.*

Proof For $0 < \tau < 2^{m-1} + 2^{m-3}$, we have from definition

$$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = \sum_{i=0}^{2^{m-1}+2^{m-2}-1-\tau} \left(\omega^{g_i - g_{i+\tau}} + \omega^{h_i - h_{i+\tau}}\right). \tag{10}$$

For any integer i, let $j = i + \tau$ and $(i_0, i_1, \ldots, i_{m-1})$ and $(j_0, j_1, \ldots, j_{m-1})$ is the binary vector representation of i and j respectively. The proof is split into four cases.

Case-1: $i_{\pi(0)} \neq j_{\pi(0)}$ and $\tau \neq C_{m-4}2^{m-4} + C_{m-3}2^{m-3} + C_{m-2}2^{m-2} + C_{m-1}2^{m-1}$, where $C_{m-\alpha} \in \{0, 1\}$ for $\alpha = 1, 2, 3, 4$. Then we have

$$g_i - g_j - \left(h_i - h_j\right) = \frac{q}{2}\left(j_{\pi(0)} - i_{\pi(0)}\right) = \frac{q}{2} \pmod{q}.$$

Therefore, $\omega^{g_i - g_j}/\omega^{h_i - h_j} = \omega^{q/2} = -1$, which implies $\omega^{g_i - g_j} + \omega^{h_i - h_j} = 0$.

Case-2: $i_{\pi(0)} = j_{\pi(0)}$ and $\tau \neq C_{m-4}2^{m-4} + C_{m-3}2^{m-3} + C_{m-2}2^{m-2} + C_{m-1}2^{m-1}$, where $C_{m-\alpha} \in \{0, 1\}$ for $\alpha = 1, 2, 3, 4$.

Let t be the smallest integer such that $i_{\pi(t)} \neq j_{\pi(t)}$. Since $\tau \neq C_{m-4}2^{m-4} + C_{m-3}2^{m-3} + C_{m-2}2^{m-2} + C_{m-1}2^{m-1}$, there exists at least one $C_k \neq 0$, $1 \leq k \leq m-5$, such that for $C_k \in \{0, 1\}$, we can write: $\tau = C_0 2^0 + C_1 2^1 + \ldots, C_{m-5}2^{m-5} + C_{m-4}2^{m-4} + \ldots + C_{m-1}2^{m-1}$. Since $j - i = \tau$, we have $t \leq m - 5$. Let i' and j' differ from i and j at only one position $\pi_{(t-1)}$, i.e., $i'_{\pi(t-1)} = 1 - i_{\pi(t-1)}$ and $j'_{\pi(t-1)} = 1 - j_{\pi(t-1)}$ respectively, such that $j' = i' + \tau$. Then we have

$$g_{i'} - g_i = \frac{q}{2}\left(i_{\pi(t-2)} + i_{\pi(t)}\right) + c_{\pi(t-1)}\left(1 - 2i_{\pi(t-1)}\right), \tag{11}$$

in case of $t = 1$, we will just delete the terms involving $i_{\pi(t-2)}$. Since $i_{\pi(t-1)} = j_{\pi(t-1)}$ and $i_{\pi(t-2)} = j_{\pi(t-2)}$, we have

$$g_i - g_j - g_{i'} + g_{j'} = \frac{q}{2}\left(j_{\pi(t)} - i_{\pi(t)}\right) = \frac{q}{2} \pmod{q},$$

which implies that $\omega^{g_i - g_j} + \omega^{g_{i'} - g_{j'}} = 0$. Therefore,

$$\omega^{g_i - g_j} + \omega^{g_{i'} - g_{j'}} + \omega^{h_i - h_j} + \omega^{h_{i'} - h_{j'}} = 0. \tag{12}$$

So from *Case-1* and *Case-2*, it is shown that the AACS of **a** and **b** is zero when τ is not an integer multiple of 2^{m-4} , i.e., $\tau \neq n \times 2^{m-4}$, $1 \leq n \leq 10$. Since the ZCZ width of the constructed ZCP is $Z = 10\times 2^{m-4}$, so the AACS at $\tau = 10\times 2^{m-4}$ is not zero. So, it will be sufficient to prove that the AACS is zero when $\tau = n\times 2^{m-4}$, $1 \leq n \leq 9$, which has been proved below, by considering odd and even cases of n in *Case-3* and *Case-4* respectively. For *Case 3, Case 4* $f, \mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3$ are as defined in (7) and (8) respectively.

Case-3: $\tau = n \times 2^{m-4}$, n is odd and $n \leq 9$.

First the calculation is done for $n = 3$, and later generalised for all odd integer $n \leq 9$.

$$\begin{aligned}\rho_{\mathbf{a}}\left(3 \times 2^{m-4}\right) = &\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - f_k} - \sum_{k=0}^{2^{m-4}} \omega^{(\mathrm{f}_2)_k - (\mathrm{f}_2)_k} \\ &+ \sum_{k=0}^{2^{m-4}} \omega^{(\mathrm{f}_2)_k - f_k} - 2\sum_{k=0}^{2^{m-4}} \omega^{f_k - (\mathrm{f}_2)_k}.\end{aligned}$$

Similarly, by calculating for each odd integer $n \leq 9$ we get

$$\begin{aligned}\rho_{\mathbf{a}}(\tau) = &\sum_{k=0}^{2^{m-4}-1} C_1^n + (-1)^{\lfloor \frac{n}{2} \rfloor} C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - (\mathrm{f}_2)_k}\right) \\ &+ \sum_{k=0}^{2^{m-4}-1} C_2^n + (-1)^{\lfloor \frac{n}{3} \rfloor + 1} C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k}\right),\end{aligned}$$

where

$$C_1^n = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \left(\left\lfloor \frac{n+1}{4} \right\rfloor \pmod 2\right), C_2^n = -C_1^n, \tag{13}$$

$$C_3^n = (n-1) \pmod 4 + (-1)^{\lfloor \frac{n}{2} \rfloor} \left(\left\lfloor \frac{n}{4} \right\rfloor \pmod 2\right), \tag{14}$$

$$C_4^n = \left(\left\lfloor \frac{n}{4} \right\rfloor + 1\right) \pmod 2. \tag{15}$$

Replacing f, f_2 with $\mathrm{f}_1 + c', \mathrm{f}_3 + c'$ respectively in $\rho_{\mathbf{a}}(\tau)$, we get

$$\begin{aligned}\rho_{\mathbf{b}}(\tau) = &(-1)^{\lfloor \frac{n}{2} \rfloor} C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_1 + c')_k - (\mathrm{f}_3 + c')_k}\right) \\ &+ (-1)^{\lfloor \frac{n}{3} \rfloor + 1} C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_3 + c')_k - (\mathrm{f}_1 + c')_k}\right),\end{aligned}$$

so from *Lemma* 1, we get $\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0$.

Case-4: $\tau = n \times 2^{m-4}$, n is even and $n \leq 9$.

First the calculation is done for $n = 2$, and later generalised for all even integer $n \leq 9$.

$$\begin{aligned}\rho_{\mathbf{a}}\left(2^{m-3}\right) = &\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - f_k} - \sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - (\mathrm{f}_2)_k} \\ &+ 2\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k} = 2\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k}.\end{aligned}$$

Similarly, by calculating for each even integer $n \le 8$, we get

$$\begin{aligned}\rho_{\mathbf{a}}(\tau) &= (-1)^{\lfloor \frac{n}{4} \rfloor} (n+2 \pmod 4)) \left(\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - (\mathrm{f}_2)_k} \right) \\ &+ 2^{m-4} - 2^{m-4} + (-1)^{\lfloor \frac{n}{4} \rfloor} (n \pmod 4)) \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k} \right).\end{aligned} \tag{16}$$

Replacing f, f_2 with $\mathrm{f}_1 + c', \mathrm{f}_3 + c'$ respectively in $\rho_{\mathbf{a}}(\tau)$, we get

$$\begin{aligned}\rho_{\mathbf{b}}(\tau) &= (-1)^{\lfloor \frac{n}{4} \rfloor} (n \pmod 4)) \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_3+c')_k - (\mathrm{f}_1+c')_k} \right) \\ &+ (-1)^{\lfloor \frac{n}{4} \rfloor} (n+2 \pmod 4)) \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_1+c')_k - (\mathrm{f}_3+c')_k} \right),\end{aligned}$$

so from *Lemma* 1, we get $\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0$. Hence, $(\mathbf{a}, \mathbf{b})$ is a ZCP with ZCZ width $Z = 2^{m-1} + 2^{m-3}$. So the ZCZ ratio$= 5/6$. □

Example 1 Let us consider $q = 4$, $m = 6$ and $\pi(0) = 0, \pi(1) = 1$, and the Boolean function g is given by

$$\begin{aligned}g &= 2x_0x_1 + 2x_0 + 3x_1 + 2\left(\bar{x}_5x_4\left(\bar{x}_3 + x_3x_2\right) + x_5\bar{x}_4\bar{x}_3x_2\right) \\ &+ 2x_1\left(\bar{x}_5\bar{x}_3\left(x_4 + x_2\bar{x}_4\right) + x_5\bar{x}_4\left(\bar{x}_3 + x_3x_2\right)\right), \text{ and} \\ h &= g + 2x_0 + 1.\end{aligned}$$

Then from *Theorem* 1, the pair $(\mathbf{a}, \mathbf{b}) = (\Psi_{16}(g), \Psi_{16}(h))$ gives a length 48 quaternary ZCP, with ZCZ width 40. The sequence pair $(\mathbf{e}, \mathbf{f})$ is given below explicitly, where i represents ω^i.

$\mathbf{a} =$ (023302110233023320332033023320110211203302330211)
$\mathbf{b} =$ (110211201102110233023302110233201120330211021120)

Theorem 2 *Let the GBF $g : \mathbb{Z}_2^m \to \mathbb{Z}_q$ be defined as*

$$\begin{aligned}g = f &+ \frac{q}{2}\left(\bar{x}_{m-1}x_{m-2}x_{m-4} + x_{m-1}\bar{x}_{m-3}\left(\bar{x}_{m-2}\bar{x}_{m-4} + x_{m-2}\right)\right) + \frac{q}{2}x_{\pi(m-5)} \\ &\left(\bar{x}_{m-1}\left(\bar{x}_{m-3}x_{m-4} + x_{m-2}\bar{x}_{m-4}\right) + x_{m-1}\left(\bar{x}_{m-2}x_{m-3} + x_{m-2}\bar{x}_{m-3}x_{m-4}\right)\right),\end{aligned} \tag{17}$$

where f is defined in (7), q is an even integer and $c_i \in \mathbb{Z}_q$. Then for any choice of $c' \in \mathbb{Z}_q$, $(\mathbf{a}, \mathbf{b}) = \left(\Psi_{2^{m-3}}(g), \Psi_{2^{m-3}}(h = g + \frac{q}{2}x_{\pi(0)} + c')\right)$ forms a $\left(2^{m-1} + 2^{m-2} + 2^{m-3},\ 2^{m-1} + 2^{m-2}\right)$-ZCP.

Proof For $0 < \tau < 2^{m-1} + 2^{m-2}$, we have from definition

$$\rho_{\mathbf{a}}(\tau) = \sum_{k=0}^{2^{m-1}+2^{m-2}+2^{m-3}-1-\tau} \omega^{g_k - g_{k+\tau}}. \tag{18}$$

The proof is split into four cases. *Case-1* and *Case-2* are same as in proof of *Theorem* 1 and can be proved similarly. For *Case-3* and *Case-4* the function $f, \mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3$ are as defined in (7) and (8) respectively.

Case-3: $\tau = n \times 2^{m-4}$, n is odd and $n \leq 11$.

$$\rho_{\mathbf{a}}(\tau) = \sum_{k=0}^{2^{m-4}-1} C_1^n + (-1)^{\left(\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{9} \rfloor\right)} C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - (\mathrm{f}_2)_k} \right) + \sum_{k=0}^{2^{m-4}-1} C_2^n + (-1)^{\left(\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{9} \rfloor + 1\right)} C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k} \right),$$

where

$$C_1^n = (-1)^{\lfloor \frac{n}{3} \rfloor} \left(\left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + 1 \pmod 2 \right), C_2^n = -C_1^n, \tag{19}$$

$$C_3^n = \left\lceil \frac{8}{n+1} \right\rceil \left(\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{7} \right\rfloor + 1 \pmod 2 \right), \tag{20}$$

$$C_4^n = \left(\left\lfloor \frac{n}{6} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + 1 \pmod 2 \right). \tag{21}$$

Similarly,

$$\rho_{\mathbf{b}}(\tau) = (-1)^{\left(\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{9} \rfloor\right)} C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_1 + c')_k - (\mathrm{f}_3 + c')_k} \right) + (-1)^{\left(\lfloor \frac{n}{3} \rfloor + \lfloor \frac{n}{9} \rfloor + 1\right)} C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_3 + c')_k - (\mathrm{f}_1 + c')_k} \right),$$

so from *Lemma* 1, we get $\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0$.

Case-4: $\tau = n \times 2^{m-4}$, n is even and $n \leq 11$.

$$\rho_{\mathbf{a}}(\tau) = \sum_{k=0}^{2^{m-4}-1} C_1^n + 2C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{f_k - (\mathrm{f}_2)_k} \right) + \sum_{k=0}^{2^{m-4}-1} C_2^n - 2C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_2)_k - f_k} \right),$$

where

$$C_1^n = -\left\lfloor \frac{n}{8} \right\rfloor \text{ and } C_2^n = \left\lfloor \frac{n}{8} \right\rfloor, \tag{22}$$

$$C_3^n = \left\lfloor \frac{n}{5} \right\rfloor \pmod 2, \tag{23}$$

$$C_4^n = \left\lfloor \frac{n}{4} \right\rfloor \pmod 2. \tag{24}$$

Similarly,

$$\rho_{\mathbf{b}}(\tau) = 2C_3^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_1 + c')_k - (\mathrm{f}_3 + c')_k} \right) - 2C_4^n \left(\sum_{k=0}^{2^{m-4}-1} \omega^{(\mathrm{f}_3 + c')_k - (\mathrm{f}_1 + c')_k} \right),$$

so from *Lemma* 1, we get $\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0$. Hence, $(\mathbf{a}, \mathbf{b})$ is a ZCP with ZCZ width $Z = 2^{m-1} + 2^{m-2}$. So the ZCZ ratio= 6/7. □

Example 2 Let us consider q=2, m=6 and $\pi(0) = 1, \pi(1) = 0$, and let the Boolean function g is given by

$$\begin{aligned} g &= x_0x_1 + x_0 + \bar{x}_5x_4x_2 + x_5\bar{x}_3\left(\bar{x}_4\bar{x}_2 + x_4\right) \\ &+ x_0\left(\bar{x}_5\left(\bar{x}_3x_2 + x_4\bar{x}_2\right) + x_5\left(\bar{x}_4x_3 + x_4\bar{x}_3x_2\right)\right), \text{ and} \\ h &= g + x_1 + 1. \end{aligned}$$

Then from *Theorem* 2, the pair $(\mathbf{a}, \mathbf{b}) = (\Psi_8(g), \Psi_8(h))$ gives a length 56 binary ZCP, with ZCZ width 48. The sequence pair $(\mathbf{e}, \mathbf{f})$ is given below explicitly, where $0, 1$ represents ω^0, ω^1 respectively.

$$\mathbf{a} = (01\mathbf{0}_5101\mathbf{0}_31\mathbf{0}_5\mathbf{1}_4\mathbf{0}_4\mathbf{1}_20\mathbf{1}_30\mathbf{1}_201\mathbf{0}_51\mathbf{0}_3\mathbf{1}_20\mathbf{1}_50)$$

$$\mathbf{b} = (1\mathbf{0}_3\mathbf{1}_20\mathbf{1}_2\mathbf{0}_31\mathbf{0}_3\mathbf{1}_201\mathbf{0}_2101\mathbf{1}_20101\mathbf{1}_30\mathbf{1}_4\mathbf{0}_3\mathbf{1}_20\mathbf{1}_3010\mathbf{1}_3\mathbf{0}_210)$$

In Table 1, we have compared the method of construction, length and ZCZ ratio of the proposed construction with [18–22, 25].

Remark 1 We have proposed a construction of ZCPs with ZCZ ratios 5/6 and 6/7 based on GBFs. Although, ZCPs with same ratio can be constructed through an indirect method in [25], the proposed method generated different set of ZCPs than [25]. The ZCPs with ZCZ ratio 6/7 from *Theorem* 2 are different from those in [25]. In fact the ZCP generated above in *Example* 2 can't be generated from [25].

Table 1 Comparison of the proposed construction with [18–22, 25].

Construction	Method	Length	Constraint	ZCZ Ratio
[18]	Direct	$2^{m-1} + 2^{\nu}$	$m \geq 2$, $\nu \leq m-2$	$= 2/3$
[19]	Indirect	$2^{\alpha+2}10^{\beta}26^{\gamma} + 2$	$\alpha, \beta, \gamma \in \mathbb{N}$	$\approx 3/4$
[20]	Direct	$2^{m-1} + 2$	$m \geq 4$	$\approx 3/4$
[21]	Direct	$N \in \mathbb{N}$, $N \neq 2^m$	$m \geq 0$	$= 2/3$
[22, Th. 1]	Direct	$2^{m+3} + 2^{m+2} + 2^{m+1}$	$m \geq 0$	$= 4/7$
[22, Th. 2]	Indirect	$14 \times 2^{\alpha}10^{\beta}26^{\gamma}$	$\alpha, \beta, \gamma \in \mathbb{N}$	$= 6/7$
[25]	Indirect	$14 \times 2^{\alpha}10^{\beta}26^{\gamma}$	$\alpha, \beta, \gamma \in \mathbb{N}$	$= 6/7$
Theorem 1	Direct	$2^{m-1} + 2^{m-2}$	$m \geq 5$	$= 5/6$
Theorem 2	Direct	$2^{m-1} + 2^{m-2} + 2^{m-3}$	$m \geq 5$	$= 6/7$

4 Conclusion

The proposed direct construction generates aperiodic q-ary even length ZCPs using GBFs. Our proposed method based on GBFs facilitates the construction of q-ary even length ZCPs of ZCZ ratio 5/6 and 6/7 which are the largest ZCZ ratio, i.e., the largest ZCZ width achieved to date using GBFs. For $q = 2$, the proposed construction reduces to EB-ZCP.

References

[1] Golay, M.: Complementary series. IRE Trans. Inf. Theory **7**(2), 82–87 (1961). https://doi.org/10.1109/TIT.1961.1057620

[2] Spasojevic, P., Georghiades, C.N.: Complementary sequences for ISI channel estimation. IEEE Trans. Inf. Theory **47**(3), 1145–1152 (2001). https://doi.org/10.1109/18.915670

[3] G.Welti: Quaternary codes for pulsed radar. IRE Trans. Inf. Theory **IT-6**(3), 400–408 (1960)

[4] Boyd, S.: Multitone signals with low crest factor. IEEE Trans. Circuits Syst. **CAS-33**(10), 1018–1022 (1986)

[5] Fan, P., Darnell, M.: Sequence Design for Communications Applications. Wiley, New York (1996)

[6] Jedwab, J., Parker, M.G.: Golay complementary array pairs. Des. Codes Cryptogr. **44**, 209–216 (2007)

[7] Fan, P., Yuan, W., Tu, Y.: Z-complementary binary sequences. IEEE Signal Process. Lett. **14**(8), 509–512 (2007). https://doi.org/10.1109/LSP.2007.891834

[8] Davis, J.A., Jedwab, J.: Peak-to-mean power control in OFDM, Golay complementary sequences, and Reed-Muller codes. IEEE Trans. Inf. Theory **45**(7), 2397–2417 (1999)

[9] Paterson, K.G.: Generalized Reed-Muller codes and power control in OFDM modulation. IEEE Trans. Inf. Theory **46**(1), 104–120 (2000). https://doi.org/10.1109/18.817512

[10] Li, Y., Chu, W.B.: More Golay sequences. IEEE Trans. Inf. Theory **51**(3), 1141–1145 (2005)

[11] Liu, Z., Parampalli, U., Guan, Y.L.: On even-period binary Z-complementary pairs with large ZCZs. IEEE Signal Process. Lett. **21**(3), 284–287 (2014)

[12] Liu, Z., Parampalli, U., Guan, Y.L.: Optimal odd-length binary Z-complementary pairs. IEEE Trans. Inf. Theory **60**(9), 5768–5781 (2014)

[13] Fiedler, F., Jedwab, J., Parker, M.G.: A framework for the construction of Golay sequences. IEEE Trans. Inf. Theory **54**(7), 3114–3129 (2008)

[14] Tang, X., Mow, W.H.: Design of spreading codes for quasi-synchronous CDMA with intercell interference. IEEE J. Sel. Areas Commun. **24**(1),

84–93 (2006)

[15] Sarkar, P., Majhi, S.: A direct construction of optimal ZCCS with maximum column sequence PMEPR two for MC-CDMA system. IEEE Commun. Lett. **25**(2), 337–341 (2021)

[16] Sarkar, P., Majhi, S., Liu, Z.: Optimal Z-complementary code set from generalized Reed-Muller codes. IEEE Trans. Commun. **67**(3), 1783–1796 (2019). https://doi.org/10.1109/TCOMM.2018.2883469

[17] Sarkar, P., Majhi, S., Vettikalladi, H., Mahajumi, A.S.: A direct construction of inter-group complementary code set. IEEE Access **6**, 42047–42056 (2018). https://doi.org/10.1109/ACCESS.2018.2856878

[18] Chen, C.-Y.: A novel construction of Z-complementary pairs based on generalized Boolean functions. IEEE Signal Process. Lett. **24**(7), 284–287 (2017)

[19] Adhikary, A.R., Majhi, S., Liu, Z., Guan, Y.L.: New sets of even-length binary Z-complementary pairs with asymptotic ZCZ ratio of 3/4. IEEE Signal Process. Lett. **25**(7), 970–973 (2018). https://doi.org/10.1109/LSP.2018.2834143

[20] Adhikary, A.R., Sarkar, P., Majhi, S.: A direct construction of q-ary even length Z-complementary pairs using generalized Boolean functions. IEEE Signal Process. Lett. **27**, 146–150 (2020)

[21] Pai, C., Wu, S., Chen, C.: Z-complementary pairs with flexible lengths from generalized Boolean function. IEEE Commun. Lett. **24**(6), 1183–1187 (2020)

[22] Xie, C., Sun, Y.: Constructions of even-period binary Z-complementary pairs with large ZCZs. IEEE Signal Process. Lett. **25**(8), 1141–1145 (2018)

[23] Chen, C.-Y., Pai, C.-Y.: Binary Z-complementary pairs with bounded peak-to-mean envelope power ratios. IEEE Commun. Lett. **23**(11), 1899–1903 (2019). https://doi.org/10.1109/LCOMM.2019.2934692

[24] Sarkar, P., Roy, A., Majhi, S.: Construction of Z-complementary code sets with non-power-of-two lengths based on generalized Boolean functions. IEEE Commun. Lett. **24**(8), 1607–1611 (2020)

[25] Yu, T., Du, X., Li, L., Yang, Y.: Constructions of even-length Z-complementary pairs with large zero correlation zones. IEEE Signal Process. Lett. **28**, 828–831 (2021). https://doi.org/10.1109/LSP.2021.3055482

[26] Liu, Z., Parampalli, U., Guan, Y.L.: On even-period binary Z-complementary pairs with large ZCZs. IEEE Signal Process. Lett. **21**(3), 284–287 (2014)

[27] Wu, S.-W., Chen, C.-Y.: Optimal Z-complementary sequence sets with good peak-to-average power-ratio property. IEEE Signal Process. Lett. **25**(10), 1500–1504 (2018). https://doi.org/10.1109/LSP.2018.2864705