PULSATING WAVES IN NONLINEAR MAGNETOCONVECTION

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Numerical experiments on compressible magnetoconvection reveal a new type of periodic oscillation, associated with alternating streaming motion. Analogous behaviour in a Boussinesq fluid is constrained by extra symmetry. A low-order model confirms that these pulsating waves appear via a pitchfork-Hopf-gluing bifurcation sequence from the steady state.

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1. Symmetry-breaking in compressible magnetoconvection

A nonlinear system may possess steady solutions that share a spatial symmetry of the system. This symmetry may be broken at a pitchfork bifurcation, giving rise to solutions related by the broken symmetry, or at a Hopf bifurcation, giving rise to oscillations with a spatiotemporal symmetry corresponding to an advance of half a period in time followed by the broken symmetry operation. Systems with $O(2)$ symmetry possess Z_2-symmetric steady solutions. Breaking this reflection symmetry at a pitchfork bifurcation leads to a pair of travelling waves [1]. If it is broken at a Hopf bifurcation there is a pulsating wave solution with a spatiotemporal symmetry which ensures that there is no net drift over a full period P of the oscillation [2]; such solutions have also been described as direction-reversing travelling waves [1] or sloshing [3, 4]. Interactions between vortices and shear flows in fluid dynamics provide many examples of these symmetry-breaking bifurcations.

We consider two-dimensional convection in a plane layer with periodic lateral boundary conditions and stress-free horizontal boundaries, in the presence of an imposed magnetic field. The system has $O(2)$ horizontal symmetry but owing to compressibility there is no up-down symmetry. For a given reference atmosphere the configuration is described by a Rayleigh number R (a dimensionless measure of the rate of heating from below), the Chandrasekhar number Q (a measure of the field strength) and the aspect ratio λ [5]. Fig. 1 shows velocity streaklines for solutions obtained with a horizontal magnetic field [6]. The steady convection rolls in fig. 1a are mirror-symmetric about vertical planes separated by $\frac{1}{2}\lambda$. After a pitchfork bifurcation there are asymmetric waves that travel leftward or rightward without change of form, as in fig. 1b. Alternatively, there may be a Hopf bifurcation leading to pulsating waves such that solutions separated by half a period in time are mirror images, as shown in fig. 1c. These oscillations are stationary with respect to the frame in which there is no net horizontal momentum. Travelling waves and pulsating waves have also been found in a related study using the anelastic approximation [3].

Numerical experiments on convection in a vertical magnetic field reveal a new and more dramatic type of pulsating wave [7, 8], which possesses the same symmetry but is reached via a different bifurcation sequence. We choose parameters so that the first bifurcation from the trivial solution leads to steady convection, as illustrated in fig. 2a. This solution again has symmetry under reflection in a vertical plane. As R is increased, a symmetry-breaking pitchfork bifurcation occurs, leading to a travelling wave (TW), shown in fig. 2b. The bifurcation introduces a sheared horizontal flow, streaming in the prograde direction at the top of the layer and in the retrograde direction at the bottom. The shearing flow causes the rolls to appear tilted, with one significantly larger than the other. At a higher Rayleigh number, a Hopf bifurcation occurs, leading to a modulated wave (MW), which is periodic in a uniformly moving frame. Global quantities, such as the Nusselt number N (a dimensionless measure of total heat transport) vary periodically, with period $\bar{P}$, as shown in fig. 2c, while the velocity pattern drifts, as shown in fig. 2d. The amplitude of the convection and strength of the shearing flow oscillate as the wave travels but the motion is dominated by a single roll, in the same sense as the shear, which does not reverse direction. Similar TW, MW and aperiodically modulated waves also appear in the absence of a magnetic field [3, 4, 9, 10].

As R is increased further the amplitude and period of the oscillation increase until a gluing bifurcation occurs. Near this global bifurcation, which involves homoclinic connections to the circle of unstable fixed points representing steady convection, solutions exhibit complicated time-dependent behaviour [11]. After the bifurcation we find pulsating waves (PW) of the type illustrated in fig. 3. Once again, motion is dominated by a single prominent eddy but now the eddy, and with it the the sheared streaming flow, reverse direction so that the spatiotemporal symmetry is preserved. At yet higher values of R this symmetry is broken, yielding modulated waves, which subsequently become chaotic.

The physical mechanisms responsible for this behaviour are straightforward. The symmetric steady (SS) solution is unstable to perturbations that tilt the rolls, thereby transporting horizontal momentum towards the boundaries so as to create a sheared flow that increases the tilt [12]. The sheared flow enhances the vorticity of rolls with the same sense of motion and suppresses rolls whose vorticity has the opposite sign. (This process has been referred to as 'peeling of convection cells' [13, 14] – an apt description of the kinematics that misses the essential dynamics.) The

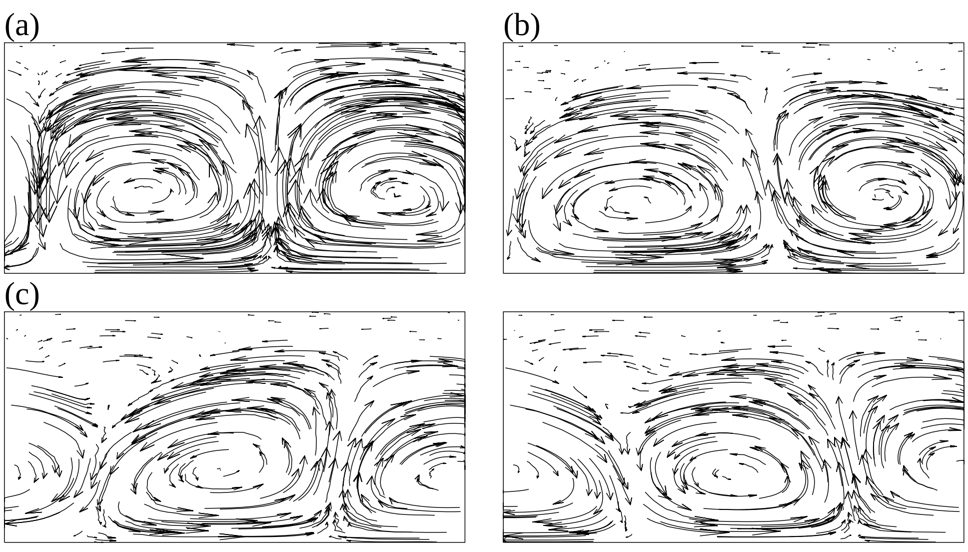

Figure 1. *Compressible convection in a horizontal magnetic field. The relevant parameters for this problem are the aspect ratio ($\lambda = 2$) and plasma beta ($\beta = 128R/9Q$ for this atmosphere). (a) Symmetric steady (SS) convection: $R = 32{,}000$, $\beta = 512$. (b) Asymmetric rightward-travelling wave (TW): $R = 32{,}000$, $\beta = 128$. (c) Pulsating wave (PW): $R = 128{,}000$, $\beta = 128$; the two frames are separated by $\frac{1}{2}P$.*

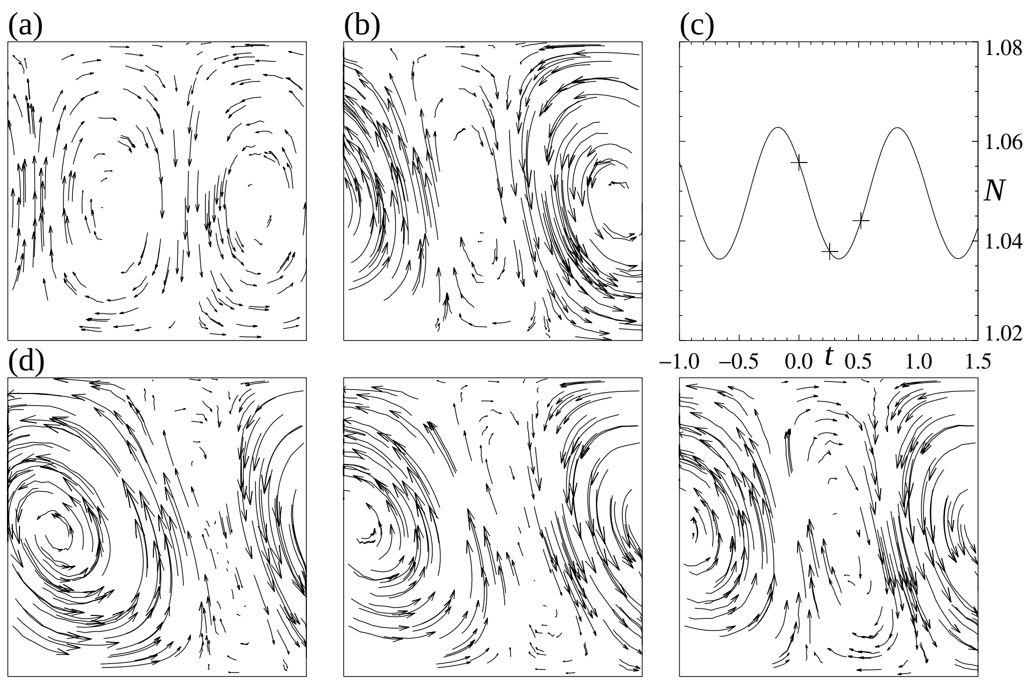

Figure 2. *Compressible convection in a vertical magnetic field, with* $\lambda = 1$ *and* $Q = 111.125$. *(a) Symmetric steady (SS) convection:* $R = 4950$; *(b) leftward-travelling wave (TW):* $R = 5000$. *Modulated wave (MW) at* $R = 5250$: *(c) periodic variation of the Nusselt number* N *(with time in units of* $\bar{P}$*); (d) velocity vectors at times* $t = 0, 0.26, 0.52\bar{P}$, *which are represented by crosses in (c).*

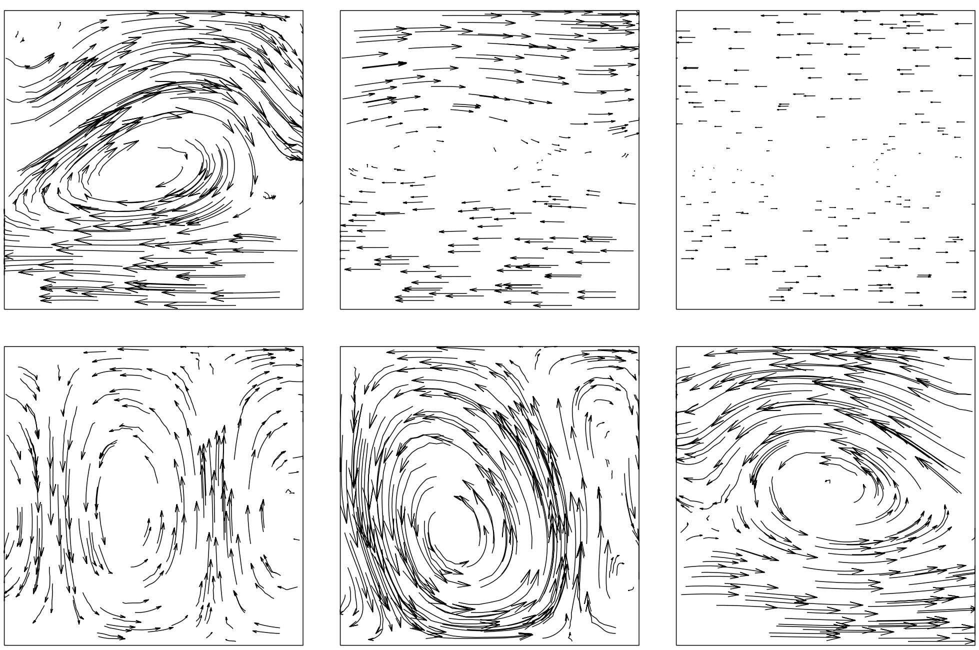

Figure 3. *Compressible convection in a vertical magnetic field: a pulsating wave (PW), with* $\lambda = 1$, $\beta = 4096$ *and* $R = 8000$. *Reading from left to right, the frames are at* $t = 0, 0.06, 0.24, 0.43, 0.47, 0.50P$: *note that the time intervals were chosen to be unequal to illustrate the rapid development of shear. The system begins with a single prominent eddy, with strong horizontal shear. The eddy is suppressed by the shear, which is itself eventually reversed by magnetic forces. Short-lived convection rolls develop, one of which grows at the expense of the other, and the system finishes (after* $\frac{1}{2}P$*) in a state that is the mirror image of the first frame.*

horizontal shear eventually inhibits convection, leaving the streaming to decay on a viscous timescale until convective instability recurs. In magnetoconvection the Lorentz force acts so as to oppose the shear flow: the elasticity of the magnetic field causes a reversal of the shear flow and leads to a pulsating wave. In the absence of a magnetic field we find modulated waves, with a preferred sense of shear and no reversals of the flow, at parameter values similar to those in fig. 3.

2. Incompressible magnetoconvection

With an incompressible (Boussinesq) fluid the system has an additional up-down symmetry with respect to reflection about the horizontal mid-plane. As a result the SS solutions have point symmetry as well as reflection symmetry, so their symmetry group is D_2 rather than Z_2 [2]. Fig. 4a shows streamlines for steady convection in rolls with point symmetry about their axes; note that, despite appearances, the streamlines are not (quite) reflection-symmetric about the vertical planes that pass through the centres of the rolls ($x = \frac{1}{4}\lambda, \frac{3}{4}\lambda$). In cells that are not too wide there is a sequence of transitions that are qualitatively similar to those described above [15, 16], with one important difference. The solutions all retain point symmetry, which ensures that the oppositely directed horizontal velocities are exactly equal and so prevents the waves from travelling. After the secondary pitchfork bifurcation to shear there are therefore asymmetric tilted rolls that do not travel; this stationary tilted convection is illustrated in fig. 4b. Thus the analogue of TW in the compressible case is steady tilted convection (STC) in the incompressible case. This solution then undergoes a Hopf bifurcation, leading to oscillatory tilted convection (OTC), as shown in fig. 4c and d, which corresponds to MW in the compressible case. Finally, there is a gluing bifurcation, followed by the pulsating waves shown in fig. 5. Note that the point symmetry fixes the positions of the eddies in all these solutions. For some extreme parameter values this point symmetry is broken, allowing waves to travel.

Related bifurcation sequences were first described by Howard and Krishnamurti [12], who developed a sixth-order truncated model to describe the development of streaming instabilities in experiments on Rayleigh–Bénard convection. Their approach can be extended to include vertical [15, 16] or horizontal [4] magnetic fields. With a horizontal field, the resulting ninth-order system exhibits a bifurcation structure similar to that described above [4]. With a vertical field, these models can be further simplified by taking the limit of narrow convection rolls [17], leading to the fifth-order system [15]

$$\begin{aligned}
\dot{A} &= \mu A + AE - BC, \\
\dot{B} &= \frac{1}{4}[\frac{3(1+\sigma)}{\sigma}AC - \sigma(B + \frac{Q}{\pi^2}G)], \\
\dot{C} &= [\mu - \frac{9\sigma}{4(1+\sigma)}]C + AB, \\
\dot{E} &= -E - A^2, \\
\dot{G} &= \frac{1}{4}\zeta(B - G).
\end{aligned} \tag{1}$$

Here A, B and C represent contributions to the stream function, while E and G represent the perturbed mean temperature and magnetic field. Symmetrical steady convection is represented by A and E only, the sheared velocity and magnetic field are given by B and G, and C describes the tilting forced by nonlinear interactions. The relevant parameters are μ (a measure of the supercritical Rayleigh number) and Q, together with the ratios σ and ζ of the viscous and magnetic diffusivities to the thermal diffusivity.

We expect that this model system will track the behaviour of the full PDEs in the limit of narrow rolls, with σ, ζ and Q of order unity, as indicated by comparison with numerical solutions of the PDEs [16]. A partial unfolding diagram for the system (1) is shown in fig. 6. If μ is increased for Q of order unity, we recover the same sequence of bifurcations as in the full Boussinesq problem. This is represented by the bifurcation diagram in fig. 7: the initial pitchfork (PF) to one or other of a pair of (SS) fixed points at $\mu = 0$ is followed by a secondary pitchfork to a symmetrical pair of fixed points (STC), then a Hopf bifurcation to an asymmetric periodic orbit (OTC), followed by a gluing

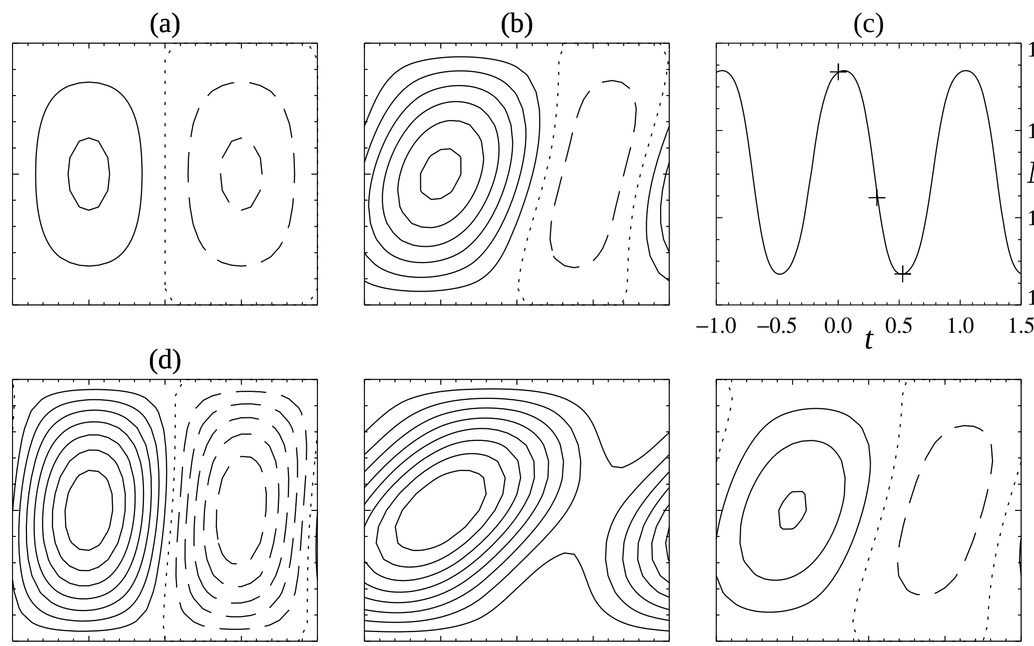

Figure 4. *Incompressible convection in a vertical magnetic field. The spacing between the streamlines is uniform and is the same in each figure. The zero streamline is dotted and negative streamlines are dashed. Note that the system retains point symmetry throughout. The parameters are* $\sigma = 0.5$, $\zeta = 0.2$, $\lambda = 0.756$ *and* $Q = 63.2$. *(a) Symmetric steady (SS) convection:* $R = 8193$. *(b) Steady tilted convection (STC):* $R = 9262$. *Oscillatory tilted convection (OTC) at* $R = 10{,}331$*: (c) periodic variation of the Nusselt number* N *(with time in units of* P*); (d) streamlines at times* $t = 0, 0.32, 0.53P$*, which are represented by crosses in (c).*

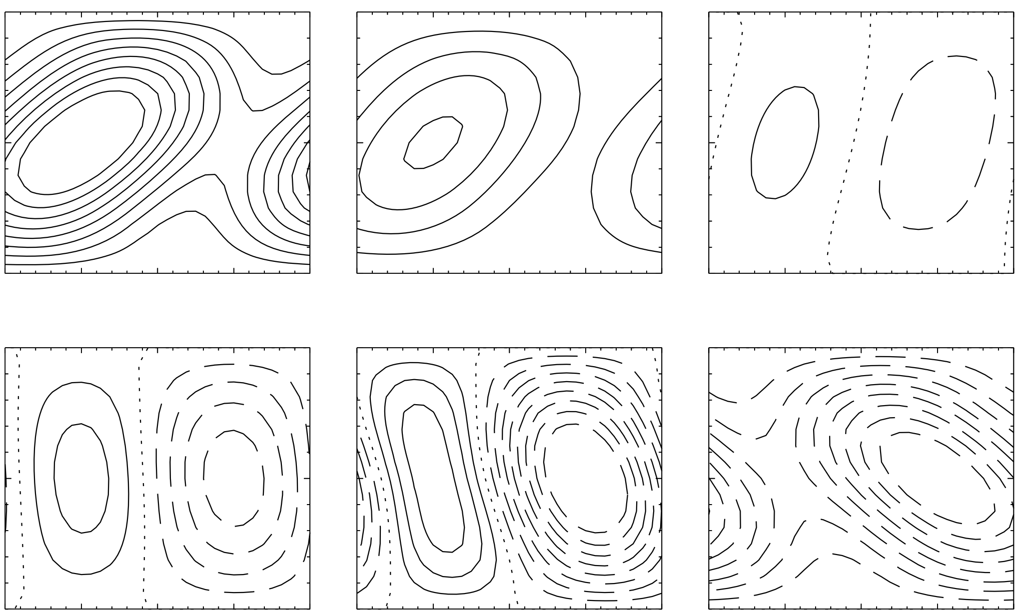

Figure 5. *Incompressible convection in a vertical magnetic field: as in fig. 4 but with* $R = 10{,}687$. *Pulsating wave (PW) at times* $t = 0, 0.08, 0.14, 0.26, 0.41, 0.50P$. *The behaviour is qualitatively the same as in fig. 3, but the solution remains point-symmetric, so the wave never drifts.*

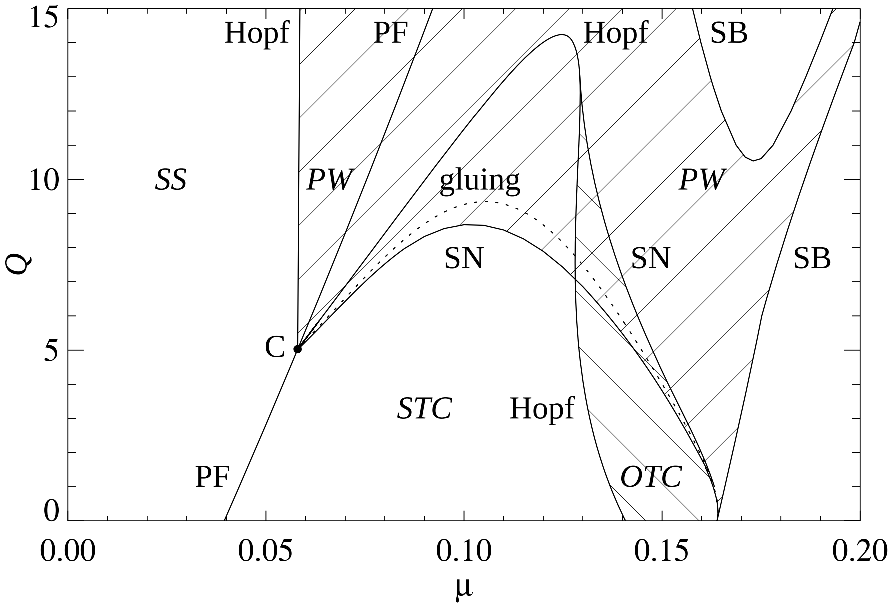

Figure 6. *Partial unfolding diagram for the model equations (1) with $\sigma = 0.5$ and $\zeta = 0.2$. The attracting solutions in different regions of the (μ, Q)-plane are indicated; OTC and PW occur in the regions hatched slanting down to the right and left, respectively. There are parameter values for which two attractors coexist. Only the first few bifurcations are shown, and the gluing bifurcation is indicated by a dotted line. The PW are destroyed in a saddle-node (SN) bifurcation as Q decreases, or lose stability in a (spatio-temporal) symmetry-breaking (SB) bifurcation as μ increases.*

bifurcation to a symmetric orbit in which the shear reverses (PW). Close inspection shows that this gluing bifurcation involves homoclinic connections to the fixed point corresponding to SS convection when $Q \neq 0$. For larger Q, however, there is a Hopf bifurcation directly from SS convection to PW. This transition occurs for a wide range of parameter values in the model.

In the non-magnetic case, the gluing bifurcation involves heteroclinic connections between three fixed points. Without a magnetic field to encourage the shear to change direction, the system may never develop pulsating waves, in which the shear changes sign after half a period. The transition from SS to STC to OTC (with no transition to PW) occurs in a truncated model [12]. For the same model with other parameter values, PW appear after a global bifurcation [18]; this seems to require a large value of σ. STC and OTC have also been observed [19, 20] for the full partial differential equations with stress-free boundaries, and we have found both routes to pulsating waves: PW are created in a Hopf bifurcation from SS (with $\sigma = 10$) and in a global bifurcation. The direct Hopf bifurcation to PW has also been reported for convection between rigid plates [21].

We have shown how pulsating waves with reversals of the flow arise for compressible and incompressible magnetoconvection and for a truncated model. In each case we find the same qualitative behaviour: PW can develop gradually after a Hopf bifurcation from steady convection or appear suddenly with large amplitude after a gluing bifurcation. In three-dimensional magnetoconvection, pulsating waves become more complicated, since shear does not affect convection in rolls whose axes are aligned with the streaming flow. As a result, rolls develop, are suppressed by the resulting shear, and are replaced by orthogonal rolls, producing an elaborate alternating pattern [11, 22].

Figure 7. *Schematic bifurcation diagram, with $Q = 4$ and μ increasing, taken below the Takens–Bogdanov point labelled C in fig. 6. For these parameter values, the gluing bifurcation occurs on an unstable part of the OTC/PW branch.*

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References

[1] A.S. Landsberg and E. Knobloch, Phys. Lett. A A 159 (1991) 17–20.

[2] M.R.E. Proctor and N.O. Weiss, Geophys. Astrophys. Fluid Dynamics 70 (1993) 137–160.

[3] S.R. Lantz and R.N. Sudan, *Astrophys. J.,* submitted.

[4] S.R. Lantz, *Astrophys. J.,* submitted.

[5] N.E. Hurlburt, M.R.E. Proctor, N.O. Weiss and D.P. Brownjohn, J. Fluid Mech. 207 (1989) 587–628.

[6] D.P. Brownjohn, N.E. Hurlburt, M.R.E. Proctor and N.O. Weiss, in preparation.

[7] N.O. Weiss, in: Solar and Stellar Granulation, eds. R.J. Rutten and G. Severino (Kluwer, Dordrecht, 1989) pp. 471–480.

[8] M.R.E. Proctor, N.O. Weiss, D.P. Brownjohn and N.E. Hurlburt, *J. Fluid Mech.*, submitted.

[9] N.E. Hurlburt, J. Toomre and J.M. Massaguer, Astrophys. J. 282 (1984) 557–573.

[10] G.P. Ginet and R.N. Sudan, Phys. Fluids 30 (1987) 1667–1677.

[11] P.C. Matthews, M.R.E. Proctor, A.M. Rucklidge and N.O. Weiss, in preparation.

[12] L.N. Howard and R. Krishnamurti, J. Fluid Mech. 170 (1986) 385–410.

[13] J.F. Drake *et al.*, Phys. Fluids B 4 (1992) 488–491.

[14] J.M. Finn, J.F. Drake and P.N. Guzdar, Phys. Fluids B 4 (1992) 2758–2768.

[15] A.M. Rucklidge and P.C. Matthews, in: Solar and Planetary Dynamos, eds. M.R.E. Proctor, P.C. Matthews and A.M. Rucklidge (Cambridge University Press, Cambridge, 1993) pp. 257–264.

[16] A.M. Rucklidge and P.C. Matthews, in preparation.

[17] D.W. Hughes and M.R.E. Proctor, Nonlinearity 3 (1990) 127–153.

[18] K.A. Julien, N. Brummell and J. Hart, preprint.

[19] J.W. Deardorff and G.E. Willis, J. Fluid Mech. 23 (1965) 337–353.

[20] J.M. Finn, Phys. Fluids B 5 (1993) 415–432.

[21] J. Prat, J.M. Massaguer and I. Mercader, in: Mixing in Geophysics, eds. J.M. Redondo and O. Metais (Diseño Gráfico), in press.

[22] P.C. Matthews, M.R.E. Proctor, A.M. Rucklidge and N.O. Weiss, in: Solar Magnetic Fields, eds. M. Schüssler and W. Schmidt (Cambridge University Press), in press.