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Hyperspaces, shape theory and computational topology

(Hiperespacios, teoría de la forma y topología computacional)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Diego Mondéjar Ruiz

Director

Manuel Alonso Morón

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Memoria para optar al título de Doctor

Diego Mondéjar Ruiz

Director: Manuel Alonso Morón

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A mi Compi

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Every problem is an opportunity.
The bigger the problem,
the bigger the opportunity
— Somewhere in Stanford

Contents

Resumen

Introducción

Esta tesis trata sobre aproximaciones de espacios métricos compactos. La aproximación y reconstrucción de espacios topológicos mediante otros más sencillos es un tema antigüo en topología geométrica. La idea es construir un espacio muy sencillo lo más parecido posible al espacio original. Como es muy difícil (o incluso no tiene sentido) intentar obtener una copia homeomorfa, el objetivo será encontrar un espacio que preserve algunas propriedades topológicas (algebraicas o no) como compacidad, conexión, axiomas de separación, tipo de homotopía, grupos de homotopía y homología, etc.

Los primeros candidatos como espacios sencillos con propiedades del espacio original son los poliedros. Ver el artículo [45] para los resultados principales. En el germen de esta idea, destacamos los estudios de Alexandroff en los años 20, relacionando la dimensión del compacto métrico con la dimensión de ciertos poliedros a través de aplicaciones con imágenes o preimágenes controladas (en términos de distancias). En un contexto más moderno, la idea de aproximación puede ser realizada construyendo un complejo simplicial basado en el espacio original, como el complejo de Vietoris-Rips o el complejo de Čech y comparar su realización con él. En este sentido, tenemos el clásico lema del nervio [12, 21] el cual establece que para un recubrimiento por abiertos "suficientemente bueno" del espacio (es decir, un recubrimiento con miembros e intersecciones contractibles o vacías), el nervio del recubrimiento tiene el tipo de homotopía del espacio original. El problema es encontrar estos recubrimientos (si es que existen). Para variedades Riemannianas, existen algunos resultados en este sentido, utilizando los complejos de Vietoris-Rips. Hausmann demostró [35] que la realización del complejo de Vietoris-Rips de la variedad, para valores suficientemente bajos del parámetro, tiene el tipo de homotopía de dicha variedad. En [40], Latschev demostró una conjetura establecida por Hausmann: El tipo de homotopía de la variedad se puede recuperar utilizando un conjunto finito de puntos (suficientemente denso) para el complejo de Vietoris-Rips. Los resultados de Petersen [58], comparando la distancia Gromov-Hausdorff de los compactos métricos con su tipo de homotopía, son también interesantes. Aquí, los poliedros salen a relucir en las demostraciones, no en los resultados.

Otro punto importante en este tema son los espacios topológicos finitos. Se podría pensar que los espacios topológicos finitos son demasiado sencillos para detectar propiedades topológicas complejas, pero esto no es así y corresponde a la errónea identificación de los espacios finitos como discretos. Otro inconveniente al uso de espacios finitos es que tienen unas propiedades de separación muy deficientes. Cualquier espacio topológico finito que tenga la propiedad T_1 es directamente un espacio discreto. Como los espacios no-Hausdorff parecen poco manejables, los espacios finitos podrían presentar más dificultades en sí mismos que los espacios que queremos aproximar. Los artículos de Stong [64] y McCord [50] significaron un gran avance en el estudio de los espacios topológicos finitos. Stong estudió los tipos de homotopía y homeomórficos de los espacios finitos. Entre otros resultados, demostró que los tipos homeomórficos están en correspondencia biyectiva con ciertas clases de matrices y que todo espacio finito tiene un *núcleo(core)*, con el mismo tipo de homotopía. McCord definió un functor de los espacios finitos T_0 a los poliedros que conserva los grupos de homotopía y homología (mediante una equivalencia débil de homotopía entre ellos). Este es un resultado de gran relevancia ya que podemos representar todos los grupos de homotopía y homología de un poliedro compacto como los grupos de un espacio finito. La propiedad esencial de los espacios finitos, que hace posible este resultado, es que la intersección arbitraria de abiertos es abierta (los espacios con esta propiedad son llamados Alexandroff) y, por lo tanto, tienen una base minimal. Si el espacio finito es T_0, la base minimal propociona una estructura de conjunto parcialmente ordenado (este hecho es observado en [2] por primera vez) y este hecho se usa extensivamente en la prueba del resultado. Ambos artículos fueron recuperados en una serie de notas de May [49, 48] muy instructivas donde estos resultados son puestos en valor. Basándose en los resultados probados en esos artículos, Barmak y Minian [10, 11, 8] introdujeron recientemente una teoría de topología algebraica para espacios topológicos finitos.

También podemos hacer uso de la construcción del límite inverso. Si no es posible obtener la aproximación buscada usando un único espacio, podemos intentar alcanzarla mediante el límite de un proceso de refinamiento por espacios con buenas propiedades. Esta idea se puede llevar a cabo utilizando los límites inversos. Podemos pensar en una aproximación similar a la que obtenemos mediante las series de Tailor para las funciones. El origen del uso de los límites inversos para la aproximación de espacios compactos nos remonta de nuevo a los trabajos de Alexandroff [1], donde demuestra que para todo espacio métrico compacto hay una sucesión inversa de espacios finitos T_0 tal que hay un subespacio del límite inverso homeomorfo al espacio original. También debemos citar el trabajo de Freudenthal, que demostró [31] que todo métrico compacto es el límite inverso de una sucesión inversa de poliedros. Más recientes son los trabajos de Kopperman y sus colaboradores [37, 38] que demuestran que todo compacto Hausdorff *reflexión Hausdorff* del límite inverso de una sucesión inversa de espacios finitos y T_0. Ellos, definen y usan el concepto *aplicación calmante (calming map)* para demostrar que si las aplicaciones de esta sucesión inversa son calmantes, entonces se puede asociar una sucesión inversa de poliedros cuyo límite es homeomorfo al espacio original. Estos resultados son muy interesantes desde un punto de vista teórico, pero las nociones de reflexión Hausdorff y aplicación calmante hacen que el cálculo efectivo pueda resultar difícil (o imposible). Otro resultado importante es el de Clader [19], que demuestra que todo poliedro compacto tiene el tipo de homotopía del límite inverso de una sucesión inversa de espacios finitos T_0.

La teoría de la forma explota esta idea de aproximación de los límites inversos. Esta teoría nació en 1968 con el artículo de Borsuk [13]. Es una teoría desarrollada para extender la teoría de homotopía a espacios donde no funciona bien, debido a sus patologías (por ejemplo, malas propiedades locales).

Aunque la teoría de la forma definida por Borsuk no hace uso explícito de los límites inversos, es claro que están presentes en la base de dicha teoría. La idea de Borsuk fue extender el conjunto de morfismos entre compactos métricos sumergiéndolos en el cubo de Hilbert y definiendo morfismos entre los abiertos de las copias de los espacios. Más tarde Mardesic y Segal iniciaron en [46] el uso de sistemas inversos para la teoría de la forma. Aquí, el sentido aproximativo de esta teoría está claro: Todo espacio compacto Hausdorff se puede escribir como un límite inverso de un sistema inverso de ANR's compactos, que actúan como espacios sencillos. Así, los nuevos morfismos se pueden definir esencialmente como aplicaciones entre los sistemas inversos. Este punto de vista para la teoría de la forma es desarrollado extensamente en [47] para espacios topológicos más generales y para ellos se introducen nuevos conceptos, como las *expansiones* y las *resoluciones*, para generalizar el concepto de límite inverso cuestiones técnicas, aunque la idea es la misma. Es evidente que la aproximación de espacios topológicos mediante límites inversos está estrechamente relacionada con la teoría de la forma. Existen varios invariantes para la forma de un espacio. Destaca, entre otros, la homología de Čech que se puede definir como el límite inverso de los grupos de homología singular y de homomorfismos inducidos del sistema inverso que define la forma del espacio.

En los últimos años ha habido un gran interés en la aproximación y reconstrucción de espacios topológicos, en parte por el desarrollo de la topología computacional y concretamente el análisis topológico de datos (leer el excelente artículo de Carlsson [17] como introducción a este tema). La idea es recuperar las propiedades topológicas de algún espacio usando solo una información parcial o defectuosa (también llamada ruidosa). Normalmente, se conoce solo un conjunto finito de puntos y las distancias entre ellos (esto es conocido como *nube de puntos*), que constituye una muestra de un espacio topológico desconocido, y el objetivo es reconstruir la topología del espacio o, al menos, encontrar algunas propiedades topológicas suyas. Además de los ya mencionados complejos de Vietoris-Rips y Čech, se pueden definir otros muchos, como los complejos *testigo (witness)*, *Delaunay* o las *alfa formas (alpha shapes)* [26]. En este contexto, destacan los resultados de Niyogi et al [57, 56], en las que se establecen condiciones para reconstruir el tipo de homotopía y la homología de una variedad con un conjunto finito de puntos (posiblemente con ruido) pertenecientes a una subvariedad de un espacio euclídeo. Para estos resultados se usan distribuciones de probabilidad. Hay una ran cantidad de artículos recientes dedicados a este problema de reconstrucción. Por ejemplo, Attali et al [7], desde una perspectiva más computacional, establecen condiciones para las que el complejo de Vietoris-Rips de una nube de puntos en un espacio euclídeo detecta el tipo de homotopía del espacio del cual son muestra. De entre todos los procedimientos, hemos de destacar *homología persistente*. La idea es fácil y muy efectiva: En lugar de considerar un único poliedro basado en la nube de puntas para representar la topología del espacio desconocido, consideramos toda una familia de poliedros construidos mediante los datos y las aplicaciones naturales inducidas por la inclusión entre ellos. Así, no seleccionamos una resolución concreta para analizar la nube de puntos, sino que consideramos todos los posibles valores del parámetro y sus conexiones a la vez y los usamos conjuntamente para determinar la evolución de la topología según la variación de dicho parámetro.

El vínculo entre la teoría de la forma y la homología persistente fue señalado por primera vez en 1999 por Vanessa Robins [60]. En este artículo, ella propone utilizar la teoría de la forma para aproximar compactos métricos utilizando tan solo un cojnunto finito de datos. Introdujo el concepto *número de Betti persistente*, como la evolución de los números de Betti en la sucesión inversa de poliedros en diferentes escalas (o resoluciones) de la aproximacion. Su propuesta es la siguiente: Dada una muestra (conjunto finito de puntos, posiblemente con ruido) de un espacio topológico desconocido, construir un sistema inverso de ε-entornos del conjunto finito de puntos junto con las correspondientes inclusiones. Hecho esto, triangular los ε-entornos utilizando α-formas y así obtenemos una sucesión inversa de poliedros basados en la muestra. Podemos entonces considerar la evolución de los numeros de Betti a lo largo de este sistema inverso. En el caso de algunos ejemplos concretos (con origen en sistemas dinámicos) determina cotas para la evolución de los números de Betti, cuando la resolución crece y tiende a infinito y por tanto la muestra es más ajustada. Robins predice que cuanto más ajustada sea la muestra, más exacta será la reconstrucción y es en este punto donde sugiere la teoría de la forma como una teoría que de soporte teórico a este y otros métodos similares.

Con esta intención, Morón et al [3] definen lo que llaman la *construcción principal*[1]. Esta consiste en una sucesión inversa de espacios topológicos finitos construidos a partir de aproximaciones finitas cada vez más densas del compacto métrico. Los espacios finitos no son exactamente las aproximaciones sino subespacios del hiperespacio de cada aproximación con la topología semifinita superior. Este paso técnico es necesario para poder definir aplicaciones continuas entre las aproximaciones. Estas aplicaciones están definidas en términos de proximidad entre los puntos de aproximaciones consecutivas. Por tanto, no son inclusiones (porque los espacios finitos no están necesariamente anidados). Entonces, se aplica la *correspondencia de Alexandroff-McCord*, el functor que asigna un poliedro a cada espacio finito T_0. La functorialidad sirve para poder definir aplicaciones continuas entre los poliedros inducidos y así obtenemos una sucesión inversa de poliedros. El proceso por el que esta sucesión inversa está definida, utilizando aproximaciones finitas, les induce a conjeturar que su límite inverso está, de alguna forma, relacionado con la topología del compacto métrico original. Esta conjetura se establece como el *principio general*, en el que se propone esta sucesión para detectar las propiedades shape (que conciernen a la teoría de la forma) como, por ejemplo y en especial, la homología de Čech. Este trabajo comienza aquí, comprendiendo y profundizando en las propiedades de la construcción principal.

Objetivos

La intención inicial de este trabajo era demostrar que la construcción principal de [3] es un proceso adecuado para determinar la topología de cualquier compacto métrico. En particular, los objetivos planteados son:

1. Determinar qué propiedades o invariantes shape es posible recuperar mediante la sucesión inversa de poliedros definida en la construcción principal de [3]. Demostrar o negar el principio general.

2. Estudiar el límite inverso de la sucesión inversa de espacios finitos T_0 y encontrar la información topológica disponible en el. Relacionar las dos

[1]Este no fue el primer artículo de este grupo de investigación en estas cuestiones. También es tratado, desde otra perspectiva, en [34].

sucesiones.

3. Considerar posibles modificaciones o particularizaciones de la construcción para obtener más propiedades sobre el espacio original, añadiendo si es necesario, condiciones topológicas sobre el.

4. Construir ejemplos de la construcción principal explícitamente en los que se traten los problemas de aproximación y reconstrucción planteados. Adaptar este método en contextos de problemas con datos reales.

5. Generalizar el contexto donde se realizan estas construcciones y determinar algunas propiedades de la topología semifinita superior en hiperespacios con la topología discreta, inmersiones de espacios finitos o el cálculo de la homología de Čech de compactos métricos utilizando sucesiones inversas de poliedros determinados por subespacios finitosde dicho hiperespacio.

Resultados

En el primer capítulo se redactan la teoría y resultados necesarios para la comprensión y seguimiento del resto del texto. Por tanto, los resultados de este capítulo no son originales.

En el capítulo dos comenzamos demostrando el principio general. La sucesión inversa de poliedros definida en [3] es una HPol expansión del compacto métrico de la cual es construida. Por tanto, esta sucesión representa el shape del espacio y, por tanto, el límite inverso tiene el shape del espacio original. Además, constrimos más sucesiones inversas de poliedros, todas ellas inducidas por la sucesión inversa de finitos y utilizando diferentes poliedros basados en las aproximaciones finitas: Čech, Witness y Dowker. Probamos que todas estas sucesiones son HPol expansiones del espacio. Definimos algunos tipos de error en las sucesiones inversas de grupos inducidas en homología (de hecho, definidos para cualquier sucesión inversa de grupos abelianos) para medir lo apropiado que son los trozos finitos de las sucesiones inversas para determinar la homología de Čech del espacio y lo relacionamos con la movilidad del espacio.

Finalmente, exhibimos ejemplos concretos en los que hacemos la construcción (a mano) para compactos métricos paradigmáticos en teoría de la forma: El círculo polaco y el anillo hawaiano.

El capítulo tres contiene el resultado más importante y sorprendente (para nosotros) de este trabajo. El límite inverso de toda sucesión inversa de espacios finitos definida por la construcción principal tiene el tipo de homotopía del espacio original y contiene una copia homeomorfa del espacio original como subespacio. Además, identificamos explícitamente este subespacio. Posteriormente, analizamos algunas propiedades de la construcción principal y el resultado de aplicar la construcción principal a algunas clases particulares de espacios como en espacios densos, numerables y ultramétricos. En el caso de estos últimos, encontramos que podemos hacer la construcción de modo que el límite inverso del sistema inverso de finitos es homeomorfo al espacio ultramétrico. Comparamos nuestros resultados con los de Clader y Kopperman y colaboradores previamente citados. Podemos obtener el resultado de Clader como un corolario de nuestro teorema principal. En el otro caso, observamos que sus resultados de aproximaciones son para espacios compactos Hausdorff y nosotros no alcanzamos ese nivel de generalización. Sin embargo, para compactos métricos, obtenemos consecuencias similares y obtenemos además que todo compacto métrico tiene el tipo de homotopía de un límite inverso de espacios finitos T_0, el cual, parece ser un resultado novedoso. Además, demostramos que la reflexión Hausdorff preserva el shape y, por lo tanto, de los resultados de Kopperman et al se deduce que todo compacto Hausdorff tiene el mismo shape que un límite inverso de espacios finitos T_0. Finalmente, generalizamos el resultado principal de esta sección para el hiperespacio del compacto métrico original con la topología semifinita superior (el cual no es un espacio métrico) demostrando que es, salvo tipo de homotopía, el límite inverso de espacios finitos T_0 (realmente, hiperespacios de espacios finitos con la topología discreta).

El capítulo cuarto está dedicado al estudio y desarrollo del uso de hiperespacios con la topología semifinita superior, especialmente de espacios con la topología discreta. Primero, probamos algunas propiedades básicas de estos espacios. Después, demostramos que son universales (en términos de inmersión) para espacios de Alexandorff T_0. Finalmente, la simple observación de que un complejo simplicial se puede interpretar como un abierto de un hiperespacio de un espacio discreto con la topología semifinita superior que contiene a la copia canónica del espacio, construimos la categoría de entornos simpliciales, como un nuevo punto de vista para tratar con complejos simplicales. Esta perspectiva nos permite demostrar que ciertos hiperespacios pueden actuar como contenedores universales para todas las homologías de Čech de todas las posibles métricas que hacen a un conjunto ser un espacio compacto métrico.

Conclusiones

El estudio en profundidad de la construcción principal revela que es un proceso constructivo que recupera toda la información topológica de un compacto métrico. Esto significa que las aproximaciones y las aplicaciones construidas están definidas de manera coherente con la topología del espacio. Además, la topología semifinita superior para los hiperespacios resulta fácilmente manipulable para tratar espacios no-Hausdorff con cierta comodidad.

El capítulo cinco contiene solo resultados parciales o direcciones y observaciones para un trabajo futuro. Se propone la construcción principal como el origen de una nueva perspectiva para un futuro trabajo. Establecemos las bases para la implementación de estos resultados en un contexto más práctico. La construcción principal nos permite esbozar un algoritmo para obtener módulos persistentes como sucesiones finitas extraídas de una sucesión inversa de poliedros. Estos módulos persistentes se obtienen de un modo esencialmente distinto de los habituales por loque bautizamos este punto de vista como persistencia inversa. Proponemos la implementación de este proceso y la comparación con el habitual en términos de estabilidad. Finalizamos definiendo algunos conceptos relacionados con la estabilidad de las sucesiones inversas de poliedros obtenidos en el capítulo dos. Definimos el concepto de ser homotópicamente (o shape) estable y exhibimos algún ejemplo de espacios que satisfacen esta propiedad. Dejamos algunas conjeturas y preguntas abiertas relacionadas con estos conceptos.

Summary

Introduction

This thesis is about approximations of metric compacta. The approximation and reconstruction of topological spaces using simpler ones is an old theme in geometric topology. One would like to construct a very simple space as similar as possible to the original space. Since it is very difficult (or does not make sense) to obtain a homeomorphic copy, the goal will be to find an space preserving some (algebraic) topological properties such as compactness, connectedness, separation axioms, homotopy type, homotopy and homology groups, etc.

The first candidates to act as the simple spaces reproducing some properties of the original space are polyhedra. See the survey [45] for the main results. In the very beginnings of this idea, we must recall the studies of Alexandroff around 1920, relating the dimension of compact metric spaces with dimension of polyhedra by means of maps with controlled (in terms of distance) images or preimages. In a more modern framework, the idea of approximation can be carried out constructing a simplicial complex, based on our space, such as the Vietoris-Rips complex or the Čech complex, and compare its realization with it. In this direction, for example, we find the classical Nerve Lemma [12, 21] which claims that for a "good enough" open cover of the space (meaning an open covering with contractible or empty members and intersections), the nerve of the cover has the homotopy type of our original space. The problem is to find those good covers (if they exist). For Riemannian manifolds, there are some results concerning its approximation by means of the Vietoris-Rips complex. Hausmann showed [35] that the realization of the Vietoris-Rips complex of the manifold, for a small enough parameter choice, has the homotopy type of the manifold. In [40], Latschev proved a conjecture made by Hausmann: The homotopy type of the manifold can be recovered using only a (dense enough) finite set of points of it, for the Vietoris-Rips complex. The results of Petersen [58], comparing the Gromov-Hausdorff distance of metric compacta with their homotopy types, are also interesting. Here, polyhedra are just used in the proofs, not in the results.

Another important point, concerning this topic, are finite topological spaces. It could be expected that finite topological spaces are too simple to capture any topological property, but this is far from reality and comes from thinking about finite spaces as discrete ones. Another obstruction to the use of finite spaces is that a very basic observation reveals that they have very poor separation properties. Any finite topological space satisfying just the T_1 axiom of separation is really a discrete space. Since non-Hausdorff spaces seem to be less manageable, finite spaces could represent themselves a more difficult problem to study that the spaces we want to approximate with. There were two papers of Stong [64] and McCord [50] that were a breakthrough in finite topological spaces. Stong studied the homeomorphism and homotopy type of finite spaces. Among other results, he showed that the homeomorphism types are in bijective correspondence with certain equivalence classes of matrices and that every finite space has a *core*, which is homotopy equivalent to it. McCord defined a functor from finite T_0 spaces to polyhedra preserving the homotopy and homology groups (defining a weak homotopy equivalence between them). This is a very important result, since we obtain that the homotopy and homology groups of every compact polyhedron can be obtained as the groups of a finite space. The essential property of finite spaces, making possible this result, is that the arbitrary intersection of open sets is open (every space satisfying this property is called Alexandroff space) and hence they have minimal basis. If the finite space is T_0, the minimal basis gives the space a structure of a poset (first noticed in [2]) which is used in the cited result. Both papers were retrieved in a series of very instructive notes by May [49, 48], where these results are adequately valued. Based on the theorems and relations proved in those papers, Barmak and Minian [10, 11, 8] introduced a whole algebraic topology theory over finite spaces.

One step further is to make use of the inverse limit construction. If we cannot obtain the desired approximation using only one simple space, we can try to obtain it as some kind of limit of an infinite process of refinement by good spaces. That idea is accomplished by the notion of inverse limit. It is similar in spirit to the use of the Taylor series to approximate a function. For the origins of using inverse limits to approximate compacta, we should go back, again, to the work of Alexandroff [1], where it is shown that every compact metric space has an associate inverse sequence of finite T_0 spaces such that there is a subspace of the inverse limit homeomorphic to the original one. We also have to mention Freudenthal, who showed [31] that every compact metric space is the inverse limit of an inverse sequence of polyhedra. More recent results were obtained by Kopperman et al [37, 38]. They showed that every compact Hausdorff space is the *Hausdorff reflection* of the inverse limit of an inverse sequence of finite T_0 spaces. Also they define the concept of *calming map* and show that if the maps in this sequence are calming, then an inverse sequence of polyhedra can be associated and its limit is homeomorphic to the original space. Those are good results, although the technical concepts of Hausdorff reflection and calming map, make its real computation hard to achieve. Another important result is the one obtained by Clader [19], who proved that every compact polyhedron has the homotopy type of the inverse limit of an inverse sequence of T_0 finite spaces.

Shape theory makes use of this notion of approximation by inverse limits. This theory was founded in 1968 with Borsuk's paper [13]. It is a theory developed to extend homotopy theory for spaces where it does not work well, because of its pathologies (for example, bad local properties). Although Borsuk's original approach does not make explicit use of inverse limits, they are in the underlying machinery. The idea of Borsuk was to enlarge the set of morphisms between metric compacta by embedding the spaces into the Hilbert cube and define some kind of morphisms between the open neighborhoods of those embedded spaces. Later, Mardesic and Segal initiated in [46] the inverse system approach to Shape Theory. Here, the approximative sense of Shape Theory is clear: Every compact Hausdorff space can be written as the inverse limit of an inverse system (or an inverse sequence if the space is metric) of compact ANR's, which act as the good spaces. Then, the new morphisms are essentially defined as maps between the systems. Shape theory, in its invese system approach, is then defined and developed [47] for more general topological spaces and new concepts, as *expansions* and *resolutions*, have to take the role of the inverse limit for technical reasons, but the point of view is similar. It is evident that the inverse limit approximation point of view for spaces is closely related with Shape Theory. There are several shape invariants. Among others, we have the Čech homology, which is the inverse limit of the singular homology groups and the induced maps in homology of the inverse system defining the shape of the space.

In the last years, there has been a renewed interest in the approximation and reconstruction of topological spaces, in part because the development of the Computational Topology and more concretely the Topological Data Analysis (read the excellent survey of Carlsson [17] as an introduction for this topic). Here, the idea is to recapture the topological properties of some space using partial or defective (sometimes called noisy) information about it. Usually we only know a finite set of points and the distances between them (this is known as *point cloud*) which is a sample of an unknown topological space, and the goal is to reconstruct the topology of the space or, at least, be able to detect some topological properties. Besides the classical Vietoris-Rips and Čech complexes, several other complexes (as the *witness*, *Delaunay* complexes or the *alpha shapes* [26]) are defined with this purpose. Some important results in this setting were obtained by Niyogi et al [57, 56], where they give conditions to reconstruct the homotopy type and the homology of the manifold when only a finite set of points (possibly with noise) lying in a submanifold of some euclidean space, is known. They also use probability distributions in their results. There are a large amount of recent papers devoted to this kind of reconstructions. For instance, Attali et al [7], in a more computational approach, give conditions in which a Vietoris-Rips complex of a point cloud in an euclidean space recovers the homotopy type of the sampled space. Among other techniques, we have to highlight the *persistent homology*. The idea here is as easy as effective: Instead of considering only one polyhedron based on the point cloud to recover the topology of the hidden space, consider a family of polyhedra constructed from the data and natural maps induced by the inclusion connecting them. Then, we do not choose one concrete resolution to analyze the point cloud, but we consider all possible values of the parameter and their connections at once and use them together to determine the evolution of the topology of the point cloud along the parameter changes.

The first link between Shape Theory and Persistent Homology was made in 1999 by Vanessa Robins [60]. There, she propose to use the machinery of Shape Theory to approximate compact metric spaces from finite data sets. She introduced the concept of *persistent Betti number*, which is the evolution of the Betti numbers in the inverse sequence of polyhedra at different scales (or resolution) of approximation. Her approach is the following: Given a sample (finite set of points, possibly with noise) of an unknown topological space, construct an inverse system of ε-neighborhoods of the finite set and inclusion maps. Then, triangulate the ε-neighborhoods using the α-shapes and we obtain an inverse system of polyhedra based on the sample. Then, track the Betti numbers over this system. For some examples arising from dynamical systems, she is able to give bounds for the behavior of the Betti numbers, when the resolution parameter tends to infinity, and hence the sample is more accurate. Her guess is that the more accurate the sample is, the more exactness in the prediction can be made, and is here where shape theory is proposed as a theory to support this and other similar methods.

In this direction, Morón et al [3] introduced what they called the *main construction*[2]. This is an inverse sequence of finite topological spaces constructed from more and more tight approximations of a given compact metric space. The finite spaces are not exactly the approximations but some subspaces of the hyperspace of the approximations with the upper semifite topology. This is necessary in order to define continuous maps between these approximations. These maps are defined in terms of proximity between points of consecutive approximations. Hence, they are not the inclusion (because the finite spaces are not necesarilly nested). At this point, they make use of the so called *Alexandroff-McCord correspondence*, which is the functor assigning a polyhedron to every T_0 finite space,

[2]This was not the first paper of this research group in this topic. From another point of view, this theme is treated in [34].

mentioned above. The functoriality is used to define maps between the induced polyhedra and hence we obtain an inverse sequence of polyhedra. The way that this sequence is constructed, using finite approximations, induces them to conjeture that the inverse limit of the inverse sequence of polyhedra is somehow related with the topology of the original compact metric space. This conjecture is stated as the *general principle*, proposing this sequence to detect the shape properties of the space such as the Čech homology. Our work is placed here, understanding and expanding the properties of the main construction.

Objectives

The aim of this work was to show that the setting of the main construction, defined in [3], is a good framework to determine the topology of any compact metric space. In particular, the goals raised can be enumerated as follows:

1. Determine what shape properties or invariants are recovered by the inverse sequence of polyhedra defined in the main construction of [3]. Prove (or disprove) the general principle.

2. Find what information about the original space is contained in the inverse limit of finite T_0 spaces defined in the main construction and relate the two sequences.

3. Study suitable modifications of the construction to obtain more properties about the original space, adding if necessary, topological conditions over it.

4. Since the main construction is really computable at hand (or by a computer), construct explicit examples in which the reconstruction and the approximation problem is treated. Try to adapt this method for computational purposes in real data problems.

5. Generalize the framework where these constructions are defined and determine some properties of the upper semifinite topology of hyperspaces with the discrete topology, embeddability of finite spaces or the computation of Čech homology of compact metric spaces using inverse sequences of polyhedra determined by finite subspaces of this hyperspace.

Results

Chapter 1 contains the necessary theory and results to follow the rest of the text. So, the results contained there are not original.

In chapter 2 we begin by showing the general principle. The inverse sequence of polyhedra defined in [3] is a HPol expansion of the compact metric space over they are constructed. Hence this sequence represents the shape type of the space and, hence, the inverse limit of the sequence has the shape type of the original space. Moreover, we construct more induced sequences of polyhedra, all of them based on the inverse sequence of finite spaces, using different simplicial complexes based on the finite approximations: Čech, Witness and Dowker. We prove that all of these sequences are HPol expansions of the space. We define some kind of errors in the induced homology inverse sequences of these sequences of polyhedra (actually, they are defined for every inverse sequence of abelian groups) to measure the suitability of finite portions of the inverse sequences to determine the Čech homology of the space and we relate it with the movability of the space. Finally, we show some explicit and constructible (by hand) examples of how this main construction can be carried out in some metric compacta intimately related with shape theory: The Warsaw circle and the Hawaiian earring.

Chapter 3 contains the more surprising (for us) and more important result of this work. The inverse limit of every inverse sequence of finite spaces defined by the main construction has the homotopy type of the original space, and it contains an homeomorphic copy of the original space as a subspace. We identify explicitly this subspace. After that, we study some properties of the main construction and the result of performing the main construction to some specific classes of spaces as dense subspaces, countable and ultrametric spaces. For the last, we obtain that in this case we can choose a suitable construction such that the inverse limit of the finite spaces is homeomorphic to the ultrametric space. We compare our results with that of Clader and Kopperman et al (previously cited). We can obtain Clader's result as a corollary of our main theorem. In the other case, their approximations are made for Hausdorff compact spaces, and we do not obtain their generality. In contrast, for the case of metric compacta, we obtain the same consecuences, and we also deduce that every compact metric space has the homotopy type of an inverse sequence of finite T_0 spaces, which seems to be an unknown result until now. Also, we show that the Hausdorff reflection preserves the shape type and hence the results of Kopperman et al implies that every Hausdorff compact space has the same shape as an inverse sequence of finite T_0 spaces. Finally, we generalize the main result of this section for the hyperspace of the compact metric space with the upper semifinite topology (which is not a metric space) proving that it is the inverse limit of finite T_0 spaces (actually, hyperspaces of finite spaces with the discrete topology), up to homotopy type.

The fourth chapter is devoted to the study and development of the use of the hyperspaces with the upper semifinite topology, specially of spaces with the discrete topology. First, we prove some basic properties of these spaces. Next, we show that they are universal spaces (in terms of embeddability) for T_0 Alexandroff spaces. Finally, under the observation that every simplicial complex is just an open subset of some hyperspace of a discrete space with the upper semifinite topology containing the canonical copy of the space, we construct the simplicial neighborhood category, as a new point of view to deal with simplicial complexes. This perspective allows us to show that certain hyperspaces acts as universal containers for all the Čech homologies corresponding to all the possible metrics making a set a compact metric space.

Conclusions

The deep study of the main construction reveals that it is a constructive process that is able to recover the whole topological information about a compact metric space. That means that the approximations and the maps constructed are defined coherently with the topology of the space. Also, the upper semifinite topology for the hyperspaces is very tractable and enables to deal with non-Hausdorff spaces with some convenience.

Chapter five contains only partial results or just new directions or observations for future work. We propose the main construction as the origin of a new perspective for future work. There, we set the basis for the implementation of these results in a more practical framework. The main construction allows us to outline an algorithm to obtain persistence modules as finite sequences extracted from an inverse sequence of polyhedra. These persistence modules are obtained in a different way from the usual ones so we call this new point of view *inverse persistence*. We propose the implementation of this process and its comparison with the usual in terms of stability. We finish by defining some concepts regarding the stability of the inverse sequences of polyhedra obtained in chapter 2. We define the concept of being homotopically (or shape) stable and show some examples of known spaces satisfying this property. Some results concerning this properties are posed as conjectures and open questions.

Chapter 1

Preliminaries

1.1 Hyperspaces

This is an old theme in topology. It is a natural way of constructing a new space from a topological space, and use the properties of the original space to deduce some of the hyperspace. These relations will depend on the topology given to the hyperspace. As a general reference for Hyperspaces, we recommend the paper [51] and the book [55].

Given a topological space X we define the *hyperspace* of X as the set of its non-empty closed subsets

$$2^X = \{C \subset X : C \text{ is closed }\}.$$

We can endow 2^X with several topologies. Before that, we can consider two distinguished elements of 2^X. The subset X is always a closed subspace of X, so it is a point of 2^X that will be called *the fat point*. If X is T_1, then every point is closed, so we can consider every singleton $\{x\}$, with $x \in X$, as a point of 2^X. The subset

$$\{\{x\} : x \in X\} \subset 2^X,$$

is the *canonical copy* of X in 2^X.

If (X,d) is a compact metric space (one of the best situations we can have), we can define a metric in the hyperspace which is the most used. For two points

C, D of 2^X, the *Hausdorff distance* of C and D is

$$d_H(C,D) = \inf\{\varepsilon > 0 : C \subset D_\varepsilon, D \subset C_\varepsilon\},$$

where

$$C_\varepsilon = \{x \in X : \mathrm{d}(x, C) < \varepsilon\}$$

is the *generalized ball* of radius ε. With this metric, $2^X_H = (2^X, \mathsf{d}_H)$ is a compact metric space. Moreover, it is shown that the inclusion map

$$\begin{aligned} \phi : X &\longrightarrow 2^X_H \\ x &\longmapsto \{x\}, \end{aligned}$$

with image the canonical copy of X in 2^X, is an isometry. That means, in particular, that the canonical copy $\phi(X)$ is homeomorphic to the original space X, which seems to be a very desirable feature. In other words, X is embedded in 2^X_H. More results about hyperspaces with the Hausdorff metric and its relations with the base space can be seen in [5].

1.1.1 Upper semifinite topology

We next define a topology for hyperspaces that will be used widely along the text. The advantage of using it is that it has a very easy handling, with the cost that the hyperspace has very poor topological properties. The general references for hyperspaces contain the definition and some properties for this topology. We add two more references [4, 6] about this topology and some of its properties, that will be used here. In general, this is a non-Hausdorff topology.

Let X be a topological space. For every open set $U \subset X$ define

$$B(U) = \left\{C \in 2^X : C \subset U\right\} \subset 2^X.$$

The family

$$B = \{B(U) : U \subset X \text{ open}\}$$

is a base for the *upper semifinite topology* for the hyperspace 2^X_u. The closure operator of this topology is very easy to describe. Given a T_1 space X, and

$C \in 2^X$, then, the closure of the set constisting of just this point is

$$\overline{\{C\}} = \left\{ D \in 2^X : C \subset D \right\}.$$

We have the following properties from [4]

Proposition 1. *Let X, Y be Tychonoff spaces. We have the following.*

i) The set X is the unique closed point in 2^X_u.
ii) The space 2^X_u is a compact connected space.
iii) X is homeomorphic to Y if and only if 2^X_u is homeomorphic to 2^Y_u.
iv) If X is non-degenerate[1], 2^X_u is a T_0 but not T_1 space.

In this context, we also have that, if X is a T_1 space the inclusion map

$$\begin{aligned} \phi : X &\longrightarrow 2^X_u \\ x &\longmapsto \{x\}, \end{aligned}$$

is a topological embedding.

In the case of metric compacta, we have some extra properties. Let (X,d) be a compact metric space. Consider for every $\varepsilon > 0$ the subspace of 2^X consisting of the closed subsets of X

$$U_\varepsilon = \left\{ C \in 2^X : \operatorname{diam}(C) < \varepsilon \right\}.$$

The following result is key in the use of the upper semifinite topology for hyperspaces in this text.

Proposition 2. *[2][6]] The family $U = \{U_\varepsilon\}_{\varepsilon>0}$ is a base of open neighborhoods of the canonical copy $\phi(X)$ inside 2^X_u.*

Remark 1. Note that if we consider any decreasing and tending to zero sequence of positive real numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$, we have that $\{U_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is a nested countable base of 2^X_u.

[1]Actually, X just need to be a non-degenerate T_1 space to satisfy this property.
[2]This result is also shown [5] for the hyperspace 2^X_H with the Hausdorff metric.

Now consider we have a continuous map of compact metric spaces $f : X \to Y$. We define the *elevation induced* by f as the function $2^f : 2^X_u \to 2^Y_u$ defined in the natural way: For $C \in 2^X_u$, $2^f(C) = \bigcup_{c\in C} f(c)$. This is a continuous[3] map. Moreover, for every map from a topological space to a hyperspace (of the same space or a different one), we can consider an extension to the whole hyperspace. Let X, Y be compact metric spaces. If $f : X \to 2^Y_u$ is a continuous map, its *extension* is the function $F : 2^X_u \to 2^Y_u$, given by

$$F(C) = \bigcup_{x\in C} f(x).$$

It is an extension in the sense that we can consider that f is actually a continuous map from the canonical copy of X in 2^X_u. That is, strictly speaking, F would be the extension of the map $f^* : \phi(X) \to 2^Y_u$, with $f^*(\{x\}) = f(x)$, which is continuous because f is. This is Lemma 3 in [6]

Lemma 1 (Continuity of the extension map)**.** *The extension of every continuous map $f : X \to 2^Y_u$ is well defined and continuous.*

1.2 Polyhedra

Polyhedra are topological spaces that can be triangulated. As a consecuence, they are well behaved in terms of homotopy theory. They play a very important role in shape theory and as approximations of metric compacta. We recommend [63, 48] and appendix 1 of [47].

1.2.1 Abstract and geometric simplicial complexes

Before defining polyhedra, we define the abstract and geometric concepts of triangulation. An *abstract simplicial complex* K is a set of *vertices* $V(K)$ and a set K of non-empty finite subsets of $V(K)$, called *simplices*, satisfying this condition: if $\sigma \in K$ and $\tau \subset \sigma$, then $\tau \in K$. In this case, we say that τ is a *face* of σ. We will denote the simplices as $\sigma = \langle v_0, \ldots v_s\rangle$, sometimes. The abstract simplicial complex K is said to be *finite* if so is $V(K)$. The *dimension* of a

[3]In weaker topological assumptions for the spaces X and Y, this is not always true.

simplex $\sigma = \{x_0, \dots, x_s\}$ is s. A *simplicial map* $g : K \to L$ of abstract simplicial complexes is a function $g : V(K) \to V(L)$ sending simplices to simplices. We say K is a *subcomplex* of L if every vertex and simplex of K is in L. Moreover, it is called a *full subcomplex*, if every simplex of L with vertices in K is indeed a simplex of K.

Now, we turn into the geometric translation of this concept. We say that a set of points $\{v_0, \dots, v_n\} \subset \mathbb{R}^n$ is *geometrically independent* if the vectors $v_i - v_0$, with $1 \leqslant i \leqslant n$, are linearly independent. We define an *n-simplex* σ spanned by $\{v_0, \dots, v_n\}$ as the set of points

$$\left\{ x \in \mathbb{R}^n : x = \sum_{i=0}^{n} t_i v_i, 0 \leqslant t_i \leqslant 1, \sum_{i=0}^{n} t_i = 1 \right\}.$$

The numbers t_i are the *barycentric coordinates* of the point x. In the case $t_i = \frac{1}{n+1}$ for every $i = 0, \dots, n$, the point x is called the *barycenter* of σ. A *(proper) face* of σ is a simplex spanned by a (proper) subset of the vertices of σ. The n-simplex Δ_n spanned by the standard basis of $\mathbb{R}^n$ is called the *standard n-simplex*. A *geometric simplicial complex* K is a collection of simplices in $\mathbb{R}^N$ (for some finite or infinite cardinal N), such that, every face of a simplex in K is a simplex and the intersection of two simplices of K is a simplex. The notions of *vertices*, *(full) subcomplex* and *map of simplicial complexes* are straightforward. Note that we do not require the whole set of vertices to be geometrically independent.

It is evident that every abstract simplicial complex K gives us a geometric simplicial complex (and vice versa), that we will denote also as K, considering any bijection between the vertices $V(K)$ and a geometrically independent subset of points of $\mathbb{R}^N$, for some N (for instance, we can use the standard basis of $\mathbb{R}^N$, where N is the number of vertices $V(K)$). Then, a geometric simplex is spanned if it is the image under the bijection of a simplex of K. Also, a simplicial map determines a map of simplicial complexes. With this correspondence, we will not distinguish between abstract or geometric complexes unless necessary.

Some simplicial complexes

Given any topological space X and a covering of it $U = \{U_\alpha\}_{\alpha \in A}$, we can construct the *nerve* of the covering $\mathcal{N}_U(X)$, whose vertices are the elements of the covering and a finite set of members of the covering $\{U_0, \dots, U_s\}$ is a s–simplex if $U_0 \cap \dots U_s \neq \emptyset$. If we consider the case where X is a metric space, and the covering $B_\varepsilon = \{\mathrm{B}(x, \varepsilon) : x \in X\}$, for $\varepsilon > 0$, the nerve $\check{C}_\varepsilon(X)$ of this covering is sometimes called the *Čech* complex.

Another well known simplicial complex is the following. Given a metric space X, we define the *Vietoris* (or *Rips*) complex $\mathcal{V}_\varepsilon(X)$, for $\varepsilon > 0$, as the simplicial complex having as vertices the points of X and as simplices the finite sets $\{x_0, \dots, x_s\}$ such that $\operatorname{diam}\{x_0, \dots, x_s\} < \varepsilon$.

1.2.2 Geometric realizations

Given a simplicial complex K, its *geometric realization* $|K|$ is the union of simplices of K, as a subspace of $\mathbb{R}^N$, and topologized defining as closed sets, the sets meeting each simplex in a closed subset. If K is finite, then this topology is inherited as a subspace of $\mathbb{R}^N$ and, in this case, $|K|$ becomes a compact metric space. A topological space X is a called a *polyhedron* if there exists a simplicial complex K such that $X = |K|$. If we have a simplicial map $g : K \to L$, the *realization of the map* g is the continuous map $|g| : |K| \to |L|$ defined sending $\sum t_i v_i$ to $\sum t_i g(v_i)$. If g is an isomorphism (that is, a bijection on vertices and simplices) then $|g|$ is an homeomorphism. One important result concerning simplicial maps and realizations is that we have a combinatorial way of showing if two maps are homotopic. We say that two continuous maps $f, g : X \to |K|$ are *contiguous* if, for every $x \in X$, $f(x) \cup g(x)$ belongs to the closure of a simplex σ of K. The claimed result is the following.

Proposition 3. *Contiguous maps are homotopic.*

1.2.3 Subdivisions

A *subdivision* L of a simplicial complex K is a simplicial complex such that every simplex of L is a subset of a simplex of K and every simplex of K is the union of finitely many simplices of L (we can think of it as a kind of refinement). It can be shown that this new simplicial complex does not change the corresponding topological space. That is, if L is a subdivision of K, then $|K| = |L|$. Among subdivisions, there is an outstanding one. The *barycentric subdivision* of a simplicial complex K is the simplicial complex whose vertices are the simplices of K and its simplices are finite chains of simplices $\{\sigma_0, \ldots, \sigma_s\}$ satisfying $\sigma_0 \subset \ldots \subset \sigma_s$. It is clear that we can repeat this process sequentially, say n times, obtaining the corresponding n-th barycentric subdivision $K^{(n)}$ of K. Concerning barycentric subdivisions, we have the following two results.

Proposition 4. *There exists a simplicial map $\xi : K' \to K$ such that its realization is contiguous (hence homotopic) to the identity.*

Proof. Any simplicial map sending σ to any vertex of σ satisfies it ✓

Proposition 5. *Any simplicial map $g : K \to L$ induces a subdivided simplicial map $g' : K' \to L'$ whose realization is contiguous (hence homotopic) to $|g|$.*

Proof. Define $g'(\sigma) = g(\sigma)$ ✓

1.2.4 The homotopy type of polyhedra

We recall here two important and useful theorems concerning the homotopy type of polyhedra. The first says that, homotopically, polyhedra are the same thing as ANRs.

Theorem 1 (West [66], Mardešić [42]). *For every topological space it is equivalent to have the homotopy type of a polyhedron or an ANR. Moreover, every CW-complex has the homotopy type of a polyhedron.*

The second one is about the reconstruction of topological spaces in terms of the nerves of their coverings.

Theorem 2 (Nerve Lemma [12, 21]). *Let X be a topological space and $U = \{U_n\}_{n\in\mathbb{N}}$ a numerable open covering. Suppose that, for every $S \subset \mathbb{N}$, we have that $\bigcap_{n\in S} U_n$ is empty or contractible. Then, the realization of the nerve $|\mathcal{N}_U(X)|$ is homotopically equivalent to X.*

1.3 Shape theory

1.3.1 Origins of shape

Shape theory is a suitable extension of homotopy theory for topological spaces with bad local properties, where this theory does not give any information about the space. The paradigmatic example is the Warsaw circle $\mathcal{W}$: It is the graph of the function $\sin\left(\frac{1}{x}\right)$ in the interval $(0, \frac{2}{\pi}]$ adding its closure (that is, the segment joining $(0, -1)$ and $(0, 1)$) and closing the space by any simple (not intersecting itself or the rest of the space) arc joining the points $(0, -1)$ and $(\frac{\pi}{2}, 1)$. See figure 1.1. It is readily seen that the fundamental group of $\mathcal{W}$ is trivial. Moreover, so are all its homology and homotopy groups. But it is also easy to see that $\mathcal{W}$ has not the homotopy type of a point (for example, it decomposes the plane in two connected components), so it has some homotopy type information that the homotopy and homology groups are not able to capture. It is then evident that homotopy theory does not work well for $\mathcal{W}$. Shape theory was initiated by Karol Borsuk in 1968 to overcome these limitations, defining a new category, containing the same information about well behaved topological spaces, but giving some information about spaces with bad local properties. The idea is that, no matter how bad the space is, its neighborhoods when it is embedded into a larger space (for example the Hilbert cube Q) are not too bad. In our example, it is easy to see that the neighborhoods of $\mathcal{W}$ are annuli, having then the homotopy type of $\mathbb{S}^1$. The space $\mathcal{W}$ share some global properties with $\mathbb{S}^1$. There are no non-trivial maps from $\mathbb{S}^1$ to $\mathcal{W}$, so the method will be to compare them in terms of maps between its neighborhoods.

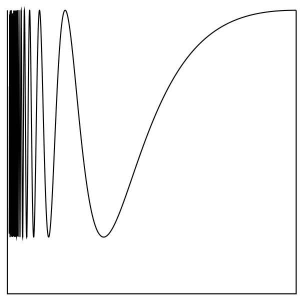


Figure 1.1: The Warsaw circle.

Specifically, Borsuk defined a new class of morphism between metric compacta embedded in the Hilbert cube, called fundamental sequences, as sequences of continuous maps $f_n : Q \to Q$ satisfying some homotopy conditions on the neighborhoods of the spaces embedded in the Hilbert cube. He introduced a notion of homotopy among fundamental sequences, setting the shape category of metric compacta as the homotopy classes for this homotopy relation. It is shown that the new category differs only formally from the homotopy category when the space under consideration is an ANR. For the details, see the original source [13], or the books [15, 14].

After Borsuk's description of the shape category for metric compacta, there was a lot of work in shape theory, such as different descriptions of shape, extensions to more general spaces (for instance, Fox's extension of shape for metric spaces [30]), classifications of shape types or shape invariants. As general references, we recommend the books [15, 14, 47, 24] and the surveys [43, 44].

1.3.2 Inverse system approach to Shape

In this text, we will use the inverse system approach to shape theory, initiated by Mardesic and Segal for compact Hausdorff spaces in [46], and further developed by them and some other authors. The best reference for this approach, is the book by the same authors [47], where all the details and proofs of this section can be found.

Inverse systems and expansions

In this section, we recall inverse systems and expansions, the main technical tools for the inverse system approach to shape theory. We will use generic categories and, later, we will focus in our concrete case.

Let $\mathcal{C}$ be any category and Λ be a directed set (called the *index set*). An *inverse system* in $\mathcal{C}$ consists of a triple $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, where X_λ is an object (*term*) of $\mathcal{C}$, for every $\lambda \in \Lambda$, and $p_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$ is a morphism (*bonding morphisms or maps*) of $\mathcal{C}$, for every pair $\lambda \leqslant \lambda'$ of indices, satisfiying $p_{\lambda\lambda} = id_{X_\lambda}$ and $p_{\lambda\lambda'}p_{\lambda'\lambda''} = p_{\lambda\lambda''}$, for every triple $\lambda \leqslant \lambda' \leqslant \lambda''$. If the index set of an inverse system is $\Lambda = \mathbb{N}$, then it is called *inverse sequence*, and it is written $\mathbf{X} = (X_n, p_{nn+1})$, since the rest of bonding maps are determined by the composition of those. Given two inverse systems $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$, a *morphism of inverse systems* $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ is a function $\phi : M \to \Lambda$ and a collection of morphisms of $\mathcal{C}$, for every $\mu \in M$, $f_\mu : X_{\phi(\mu)} \to Y_\mu$, such that, for every pair $\mu \leqslant \mu'$, there exists a $\lambda \in \Lambda$ satisfying $\lambda \geqslant \phi(\mu), \phi(\mu')$ for which the following diagram[4] is commutative.

$$\begin{array}{ccccc} X_{\phi(\mu)} & \longleftarrow & X_\lambda & \longrightarrow & X_{\phi(\mu')} \\ {\scriptstyle f_\mu}\downarrow & & & & \downarrow{\scriptstyle f_{\mu'}} \\ Y_\mu & & \longleftarrow & & Y_{\mu'} \end{array}$$

This morphism of systems will be called a *level morphism of systems* if $\Lambda = M$, $\phi = id_\Lambda$ and, for every $\lambda \leqslant \lambda'$, the following diagram is commutative.

$$\begin{array}{ccc} X_\lambda & \longleftarrow & X_{\lambda'} \\ {\scriptstyle f_\lambda}\downarrow & & \downarrow{\scriptstyle f_{\lambda'}} \\ Y_\lambda & \longleftarrow & Y_{\lambda'} \end{array}$$

The composition of these morphisms is defined straightforward. Then, an equivalence relation $\sim$ between morphisms $(f_\mu, \phi), (f'_\mu, \phi') : \mathbf{X} \to \mathbf{Y}$ is defined as follows: $(f_\mu, \phi) \sim (f'_\mu, \phi')$ if and only if, for every $\mu \in M$, there is a $\lambda \in \Lambda$, with $\lambda \geqslant \phi(\mu), \phi'(\mu)$ such that the following diagram is commutative.

$$\begin{array}{ccccc} X_{\phi(\mu)} & \longleftarrow & X_\lambda & \longrightarrow & X_{\phi(\mu')} \\ & {\scriptstyle f_\mu}\searrow & & \swarrow{\scriptstyle f_{\mu'}} & \\ & & Y_\mu & & \end{array}$$

Define the *category pro-$\mathcal{C}$* to be the category with objects inverse systems $\mathbf{X}$ (over all directed sets) in $\mathcal{C}$ and morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, equivalence classes of morphisms of systems under the relation $\sim$. Next, we state a very useful characterization about isomorphisms in pro-$\mathcal{C}$.

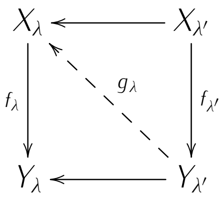
Theorem 3 (Morita's lemma [52]). *A level morphism of systems*

$$\mathbf{f} : \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda) \longrightarrow \mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$$

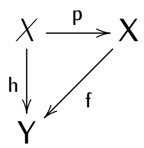
in pro-$\mathcal{C}$, is an isomorphism if and only if every $\lambda \in \Lambda$ admits a $\lambda' \geqslant \lambda$ and a

[4]We do not write the morphisms p and q, since its subindices are evident.

morphism $g_\lambda : Y_{\lambda'} \to X_\lambda$ *in* $\mathcal{C}$ *making the following diagram commutative.*



Let $\mathcal{T}$ be a category and $\mathcal{P}$ a subcategory of $\mathcal{T}$. Let X be an object of $\mathcal{T}$, a $\mathcal{T}$*-expansion* of X is a morphism in pro-$\mathcal{T}$ $\mathbf{p} : X \to \mathbf{X}$ (consider X as an inverse system in which every term is X and the bonding maps are the identity) to an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{T}$ satisfying the following universal condition: For every inverse system $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in $\mathcal{P}$ and any morphism $\mathbf{h} : X \to \mathbf{Y}$ in pro-$\mathcal{T}$, there exists a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{T}$ closing the following diagram.



Moreover, $\mathbf{p}$ is called a $\mathcal{P}$*-expansion* of X whenever $\mathbf{X}$ and $\mathbf{f}$ are in pro-$\mathcal{P}$. It is straightforward to show that two expansions of the same space are isomorphic. Moreover, the isomorphism is unique. Also, a composition of an expansion with an isomophism is again an expansion.

Given a category $\mathcal{T}$ and a subcategory $\mathcal{P}$, we say that $\mathcal{P}$ is *dense* in $\mathcal{T}$ if every object of $\mathcal{T}$ admits a $\mathcal{P}$-expansion.

The Shape category

Let $\mathcal{T}$ be a category and $\mathcal{P}$ a dense subcategory. Consider two objects X, Y of $\mathcal{T}$ and two $\mathcal{P}$-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p}' : X \to \mathbf{X}'$ of X and another two $\mathbf{q} : Y \to \mathbf{Y}$ and $\mathbf{q}' : Y \to \mathbf{Y}'$ for Y. Let us set that two morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{f}' : \mathbf{X}' \to \mathbf{Y}'$ in pro-$\mathcal{P}$ are equivalent, denoted $\mathbf{f} \sim \mathbf{f}$, when the following diagram[5]

[5] The horizontal arrows stand for the unique isomorphisms quoted before.

is commutative in pro-$\mathcal{P}$.

$$\begin{array}{ccc} \mathsf{X} & \longrightarrow & \mathsf{X}' \\ {\scriptstyle \mathsf{f}}\downarrow & & \downarrow{\scriptstyle \mathsf{f}'} \\ \mathsf{Y} & \longrightarrow & \mathsf{Y}' \end{array}$$

Define the *shape category* Sh for $(\mathcal{T}, \mathcal{P})$ as the one having as objects the objects of $\mathcal{T}$, and, for $X, Y \in \mathcal{T}$, the morphisms $X \to Y$ are the equivalence classes for $\sim$ of morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{P}$.

Usually, the shape category is used for $(\mathcal{T} = HTop, \mathcal{P} = HPol)$. The term $HTop$ stands for the homotopy category of topological spaces: Objects are homotopy classes of topological spaces and morphisms are homotopy classes of maps between topological spaces, called Hmaps. So, in this category, two homotopically equivalent topological spaces are considered isomorphic and two homotopic maps are considered the same map. Similarly, $HPol$ is the homotopy category of polyhedra (with similar considerations). We will call these spaces and maps *up to homotopy* and there are two reasons of using this condition. On one hand, because of technical reasons, there are some diagrams that need to be commutative up to homotopy. On the other hand, shape is an extension of homotopy, so two spaces homotopically equivalent must have the same shape. Two isomorphic spaces X, Y in Sh are said to have the *same shape (type)*, written $Sh(X) = Sh(Y)$.

Polyhedra is then considered as the "good" spaces to be used for the expansions. Theorem 1, shows that we can use indistinctly polyhedra, CW-complexes or ANRs for it.

We have an extension of the homotopy category, enlarging the set of morphisms. Thus not every shape morphism is represented by a continuous function, but we have that every continuous function induces a shape morphism. From [41], we have the following useful characterization for a function to induce an isomorphism in the shape category.

Theorem 4. *Let X and Y be topological spaces and $f : X \to Y$ a continuous map. Then f is a shape equivalence (that is, the shape morphism induced by f is an isomorphism in the shape category) if and only if, for every CW-complex*

P, the function[6]

$$f : [Y, P] \longrightarrow [X, P]$$
$$[h] \longmapsto [h \cdot f]$$

is a bijection.

Čech and Vietoris expansions

We describe two concrete $HPol$ expansions that can be defined for every topological space X, that will be used later. First, we define the *Čech system*, $\check{\mathbf{C}}(X) = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$. The indexes Λ are all the normal coverings of X ordered by refinement (that is, $\lambda \leqslant \lambda'$ if λ' refines λ. The term X_λ is the nerve of the covering λ. The bonding morphism $p_{\lambda\lambda'}$ is the homotopy class of any simplicial projection: Every vertex V' of the covering λ' is sent to a vertex V of λ satisfying $V' \subset V$. This projections are not well defined maps, but they are well defined (and uniquely determined) Hmaps. It can be shown that there are canonical maps producing a unique homotpy class $p_\lambda : X \to X_\lambda$ in such a way that the morphism $\mathbf{p} : X \to \check{\mathbf{C}}(X)$ is an $HPol$ expansion, the *Čech expansion* of X.

Similarly, we have the *Vietoris expansion* $\mathbf{q} : X \to \mathbf{V}(X) = (K_\lambda, q_{\lambda\lambda'}, \Lambda)$, where Λ is exactly the same as in the Čech system. The polyhedron K_λ is the realization of the following simplicial complex: $\{x_0, \ldots, x_s\}$ is a simplex in K_λ if there is a member U of the covering λ containing $\{x_0, \ldots, x_s\}$. The bonding morphism $q_{\lambda\lambda'}$ is the homotopy class of the realization of the simplicial map $K_{\lambda'} \to K_\lambda$ defined by the identity in the vertex set of K_λ. It was shown in [22] that, for every topological space X, the Čech and Vietoris systems are isomorphic in pro-$HPol$, so the *Vietoris system* $\mathbf{V}(X)$ and $\mathbf{q}$ form an $HPol$ expansion of X.

Inverse limits and shape

Inverse limits of inverse systems are a good source of expansions. Actually, the beginning of the inverse system approach of shape for compact Hausdorff spaces was to use inverse systems of ANRs [46] instead the more general concept of

[6]Notation: For topological spaces Z, R, $[Z, R]$ is the set of homotopy classes of continuous functions from Z to R. For a map $h : Z \to R$, we represent by $[h]$ its homotopy class.

expansion.

We can define inverse limits for inverse systems in every category by an universal property that reminds us the definition of expansion. But we will only introduce the equivalent definition for topological spaces, because it will be the one used. Let $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of topological spaces (i.e. in the Top category). Let $\prod X_\lambda$ be the topological product of all terms, and consider the projection onto the term X_λ,

$$\pi_\lambda : \prod X_\lambda \longrightarrow X_\lambda.$$

The *inverse limit* of $\mathbf{X}$ is the subspace

$$X = \lim_{\leftarrow} \mathbf{X} = \left\{ x \in \prod X_\lambda : \pi_\lambda(x) = p_{\lambda\lambda'}\pi_{\lambda'}(x), \ \ \forall \lambda \leqslant \lambda' \right\},$$

together with the projections $p_\lambda = \pi_\lambda \mid X$, for every $\lambda \in \Lambda$. Our spaces, can be always obtained as inverse limits:

Theorem 5. *Every compact Hausdorff (metric) space is the inverse limit of an inverse system (sequence) of compact polyhedra and PL-bonding maps.*

In order to obtain expansions from inverse limits, we need to define the *homotopy functor* $H : Top \to Top$, that keeps the objects fixed and sends every map f to its homotopy class $Hf = [f]$. Obviously, H assigns to every pro-Top system a system in pro-$HTop$. Then, we have that expansions are obtained applying the homotopy functor to inverse limits.

Theorem 6. *Let* $\mathbf{X}$ *be an inverse system of compact ANRs and suppose* $\mathbf{p} : X \to \mathbf{X}$ *is an inverse limit of* X. *Then* $H\mathbf{p} : X \to H\mathbf{X}$ *is an* $HPol$*-expansion of* X.

In the case of the Warsaw circle, it could be written as the infinite intersection a decreasing sequence $\{W_n\}$ of nested annulus containing $\mathcal{W}$. That is, it is the inverse limit of that sequence of annulus with the inclusion as bonding maps. But that inverse sequence gives us, using the previous theorem, an $HPol$-expansion which is also an $HPol$-expansion for a circle $\mathbb{S}^1$, so $Sh(\mathcal{W}) = Sh(\mathbb{S}^1)$, as was wanted, because they share their global properties.

Shape invariants

In the same way homology and homotopy groups are homotopy invariants, we will define some shape invariants. Let X be any topological space and $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ an $HPol$ expansion of X. For every abelian group G, we can consider the k-th homology group $H_k(X_\lambda; G)$ of each term and the induced homology maps $H_k(p_{\lambda\lambda'}; G)$ of the bonding maps. Then we obtain an inverse system of abelian groups

$$H_k(\mathbf{X}; G) = (H_k(X_\lambda; G), H_k(p_{\lambda\lambda'}; G), \Lambda) \,,$$

called the *k-th homology pro-group* of X. We define the *k-th Čech homology group* of X as the inverse limit of this inverse system of groups,

$$\check{H}_k(X) = \lim_{\leftarrow} H_k(\mathbf{X}; G).$$

Similarly, we can take the k-th homotopy group[7] of each term and the induced homotopy maps to obtain an inverse system of groups

$$\check{\pi}_k(X) = \lim_{\leftarrow} \pi_k(\mathbf{X}),$$

(the *k-th homotopy pro-group*) whose inverse limit is called the *K-th shape group* of X,

$$\check{\pi}_k(X) = \lim_{\leftarrow} \pi_k(\mathbf{X}).$$

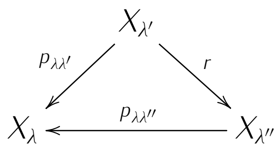
It is shown that the Čech homology and shape groups are well defined, that is, they do not depend on the $HPol$ expansion we use to compute them. Moreover, they are shape invariants.

Theorem 7. *Let X, Y be topological spaces and G an abelian group. If $Sh(X) = Sh(Y)$, then $\check{H}_k(X; G) \approx \check{H}_k(Y; G)$ and $\check{\pi}_k(X) \approx \check{\pi}_k(Y)$.*

The last shape invariant we want to recall is movability, introduced by Borsuk for metric compacta, trying to generalize the concept of spaces having the shape of ANRs. This property allows us to use the inverse limit instead of the whole inverse system to proof some theorems. We can define movability for arbitrary inverse systems. An inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in pro-$\mathcal{C}$ is *movable* provided

[7] With the corresponding considerations about the base point that we do not include here.

every $\lambda \in \Lambda$ admits $\lambda' \geqslant \lambda$, called *movability index* of λ, such that, for every $\lambda'' \geqslant \lambda$, there exists a morphism $r : X_{\lambda'} \to X_{\lambda''}$ of $\mathcal{C}$ making the following diagram commutative.



Movability is well defined, since, if **X** and **Y** are isomorphic in pro-$\mathcal{C}$, then **X** is movable if and only if so is **Y**. A topological space X is *movable* if it has movable $HPol$-expansions. It is a shape invariant property.

An inverse system **X** in pro-$\mathcal{C}$ is *stable* provided it is isomorphic in pro-$\mathcal{C}$ to an object $X \in \mathcal{C}$. It is evident that if **X** is stable, then it is movable. A topological space is said to be *stable* provided its $Hpol$-expansions are stable. It is equivalent to have the same shape as a polyhedron (or ANR) and it is obviously a shape invariant property. Then, an stable space is movable, but the converse is not always true. An example of a movable but not stable space is the Hawaiian earring, another important space in shape theory. It is an infinite union of circles in $\mathbb{R}^2$ intersecting only in the point $(0, 0)$. Specifically, it is the subspace of $\mathbb{R}^2$,

$$\bigcup_{n\in\mathbb{N}\cup\{0\}} \mathrm{S}\left(\left(\frac{1}{2^n}, 0\right), \frac{1}{2^n}\right),$$

where $\mathrm{S}(a, b)$ stands for the 1-sphere of center a and radius b in $\mathbb{R}^2$. See figure 1.2. The Hawaiian earring can be described as an inverse limit of the inverse sequence (C_n, r_{nn+1}), where $C_n = \bigcup_0^n \mathrm{S}\left(\left(\frac{1}{2^n}, 0\right), \frac{1}{2^n}\right)$ and the bonding maps are the retractions $r_{nn+1} : C_{n+1} \to C_n$ sending $\mathrm{S}\left(\left(\frac{1}{2^{n+1}}, 0\right), \frac{1}{2^{n+1}}\right)$ to $(0, 0)$ and being the identity elsewhere. This inverse limit gives us an $HPol$-expansion of $\mathcal{H}$ which is movable. For instance, its first homology pro-group has the Mittag-Leffler property with index 1 as can be easily computed. This space has not the shape of any polyhedron, so it is not stable.

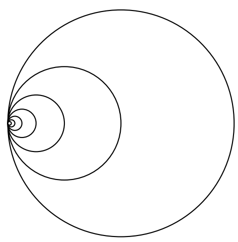


Figure 1.2: The Hawaiian earring.

We will introduce one more concerning movability. An inverse system of groups $\mathbf{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ has the *Mittag-Leffler* (ML) property if every $\lambda \in \Lambda$ admits a $\lambda' \geqslant \lambda$ (called an *ML index* for λ) such that, for every $\lambda'' \geqslant \lambda'$, we have $p_{\lambda\lambda''}(X_{\lambda''}) = p_{\lambda\lambda'}(X_\lambda)$. Using that every movable inverse system of groups has the Mittag-Leffler property and that every functor preserves the property of being movable, we have that the homology and homotopy pro-groups $H_k(\mathbf{X}; G)$, $\pi_k(\mathbf{X})$ are movable and hence have the Mittag-Leffler property.

The last example of this section is the dyadic solenoid $\mathcal{S}$. It is a very suitable space for shape theory, because it is defined as an inverse limit. There is a more geometric definition as the intersection of an infinite sequence of nested solid tori, but we will use the following[8]. Consider

$$\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$$

as the unit circle in the complex plane, and define a map $p : \mathbb{S}^1 \to \mathbb{S}^1$ sending an element $z = e^{i\theta}$ to $p(z) = e^{2i\theta}$. It is a fairly complicated space, arising in some dynamical systems as an atractor. It is a non-movable metric continuum, since its induced first homology pro-group

$$\mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow \dots,$$

where the bonding maps are the multiplication by 2, has not the Mittag-Leffler property. Moreover, the first Čech homology group is the trivial one.

1.3.3 Multivalued maps and hyperspaces

In this section, we will recall the multivalued theory of shape for metric compacta, initiated by Sanjurjo in [61] and the reinterpretation of this theory in terms of hyperspaces with the upper semifinite topology.

The key and acute idea of multivalued shape theory is to replace the shape morphisms by sequences of multivalued maps with decreasing diameters of their images, which is, in some sense, a very natural way of defining them, but hard to formalize. By defining a non-trivial sort of homotopic classes in this maps, it is possible to establish a category isomorphic to the shape category of metric

[8]The equivalence between these two definitions of the dyadic solenoid can be found in [36].

compacta. We describe the multivalued theory here, but it is recommended to read the original source. Let X, Y be metric compacta. A *multivalued function* $F : X \to Y$ is a function that sends each $x \in X$ to a closed subset $F(x) \subset Y$. It is said to be *upper semicontinuous* if, for every $x \in X$ and every open neighborhood V of $F(x)$ in Y, there is an open neighborhood of $x \in U$ such that $F(U) \subset V$. Moreover, a multivalued map $F : X \to Y$ is *ε-small* if, for every $x \in X$, $\operatorname{diam}(F(x)) < \varepsilon$. Now, given two ε-small upper semicontinuous multivalued functions $F, G : X \to Y$, they are said to be *ε-multihomotopic*, written $F \simeq_\varepsilon G$, if there is an ε-small upper semicontinuous multivalued function $H : X \times I \to Y$ with $H(x, 0) = F(x)$ and $H(x, 1) = G(x)$, for $x \in X$. Now, we define a *multi-net* from X to Y as a sequence of upper semicontinuous multivalued functions $\widehat{F} = \{F_n : X \to Y\}_{n\in\mathbb{N}}$ such that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $F_n \simeq_\varepsilon F_{n+1}$ for every $n \geqslant n_0$. Finally, two multi-nets $\widehat{F}, \widehat{G}$ are *homotopic*, written $\widehat{F} \simeq \widehat{G}$ if, given $\varepsilon > 0$, there is an index $n_0 \in \mathbb{N}$ such that $F_n \simeq_\varepsilon G_n$ for every $n \geqslant n_0$. Then, it is defined a notion of composition for multinets, and it is possible to prove the following

Theorem 8 (Sanjurjo [61]). *The class of metric compacta with homotopy classes of multi-nets (with the quoted composition) is a category isomorphic with the shape category of metric compacta.*

The importance of this theory lies on the fact that it is internal. That is, we do not make use of external elements (such as the Hilbert cube or polyhedra) to describe the morphisms, as in other shape theories. We just use maps between the metric compacta to define the morphisms.

This multivalued theory of shape was reinterpreted later by Alonso-Morón and González Gómez in [6]. It is based on the observation that multivalued functions are just maps into hyperspaces. Moreover, the upper semifinite topology in the hyperspace is equivalent to the upper semicontinuity in the multivalued maps. We need to define two concepts here.

Definition 1. Let X and Y be two compact metric spaces. Consider 2^Y_u the hyperspace of Y with the upper semifinite topology. An *approximative map* from X to Y is a sequence of continuous maps $\widehat{f} = \{f_n\}_{n\in\mathbb{N}}$, with $f_n : X \to 2^Y_U$, such that, for every neighborhood U of the canonical copy of Y in 2^Y_u, there exists

$n_0 \in \mathbb{N}$ such that f_n is homotopic to f_{n+1} in U (written $f_n \simeq_U g_n$, meaning there exists a homotopy $H : X \times I \to U \subset 2^Y_u$ between f_n and f_{n+1}) for every $n > n_0$.

Definition 2. We say that two approximative maps $\widehat{f} = \{f_n\}_{n\in\mathbb{N}}$ and $\widehat{g} = \{g_n\}_{n\in\mathbb{N}}$ from X to Y are *homotopic* $\widehat{f} \simeq \widehat{g}$ when, for each open neighborhood U of the canonial copy of Y in 2^Y_u, there exists $n_0 \in \mathbb{N}$ such that f_n is homotopic to g_n in U for every $n > n_0$.

These concepts are related in a very simple way. The following statements are proved in [6].

Proposition 6. *Let X, Y be metric compacta. A sequence $\widehat{F} = \{F_n\}_{n\in\mathbb{N}}$ is a multi-net from X to Y if and only if the sequence $\widehat{f} = \{f_n\}_{n\in\mathbb{N}}$, given by $f_n(x) = F_n(x)$ for every $n \in \mathbb{N}$ and $x \in X$, is an approximative map. Moreover, given two multi-nets $\widehat{F}, \widehat{G}$ and two approximative maps $\widehat{f}, \widehat{g}$ such that $f_n(x) = F_n(x)$ and $g_n(x) = G_n(x)$, $\widehat{F}$ and $\widehat{G}$ are homotopic if and only if $\widehat{f}$ and $\widehat{g}$ are homotopic.*

The multivalued theory of shape can be reformulated as follows:

Corollary 1. *Let X, Y be metric compacta. There is a bijective correspondence between the set of homotopy classes of approximative maps from X to Y and the set of homotopy classes of multi-nets from X to Y. Hence there is also a bijection with the set of shape morphisms from X to Y.*

1.4 Alexandroff spaces

Alexandroff spaces are topological spaces satisfying a topological condition that makes them very special spaces. The notion was introduced by Alexandroff [2]. We wil use them along the text because of its simplicity. Many of the hyperspaces considered will be Alexandroff. A good reference for Alexandroff and finite topological spaces are the notes of May [49, 48]. We also recommend two papers about Alexandroff and finite spaces [64, 50] that were essential in its development. Finite topological spaces have captured a lot of attention in the last years because of the developments of digital and computational topology. In a series of papers, Barmak and Minian have shown very interesting theorems about the algebraic topology of finite topological spaces (for example, generalizating notions such as collapsibility and simple homotopy type to finite topological spaces). See, for instance, [10, 9, 11] or Barmak's book [8]. A topological space X is said to be *Alexandroff* provided arbitrary intersections of open sets are open. A special case of Alexandroff spaces are the *finite* topological spaces. One could have the intuition that a topological space with a finite set of points cannot contain a deep geometric information, but this will be shown to be not the case. Concerning Alexandroff spaces, is good to have in mind finite topological spaces, for simplicity. We can not require too strong separation properties to Alexandroff spaces, because they will turn trivial: An Alexandroff T_1 space is discrete. But, on the other hand, finite T_0 spaces have some geometric interest, since they have, at least, one closed point. Moreover, in terms of algebraic topology, we can consider only Alexandroff T_0 spaces because of the following theorem.

Theorem 9 (McCord [50]). *Let X be an Alexandroff space. There exists a quotient T_0 space $q_X : X \to X_0$ homotopically equivalent to X (q is a homotopy equivalence). Moreover, for every map between Alexandroff spaces, $f : X \to Y$ there is a unique map $f_0 : X_0 \to Y_0$, between T_0 Alexandroff spaces, such that $q_Y f = f_0 q_X$.*

1.4.1 Alexandroff spaces and posets

The most important property of an Alexandroff space X is that it has a distinguished basis. For every $x \in X$, we can consider the intersection

$$B_x = \bigcap_{x \in U \text{ open}} U$$

of all the open sets containing x, which is open and it is called the *minimal neighborhood* of x, because, by definition, it is contained in every open set containing x. It can be shown, that the set of minimal neighborhoods, $\{B_x : x \in X\}$ is a base for the topology of X, called the *minimal basis* of X. This minimal basis defines a reflexive and transitive relation on the space X. For $x, y \in X$, say $x \leqslant y$ if $B_x \subset B_y$. This relation is a partial order if and only if X is T_0. On the other hand, every reflexive and transitive relation on a set X determines an

Alexandroff topology, with basis the sets $U_x = \{y \in X : y \leqslant x\}$. So, we have the following correspondence.

Proposition 7. *For every set, its Alexandroff topologies are in bijective correspondence with its reflexive and transitive relations. The topology is T_0 if and only if the relation is a partial order.*

We call a set with a partial order a *poset*. Last proposition tells us that Alexandroff T_0 spaces (sometimes called A-spaces) and posets are the same thing. In what follows we will use both points of view without distinction. With this notation, continuous maps are easily characterized. A function $f : X \to Y$ of Alexandroff spaces is continuous if and only if is order preserving, that is, $x \leqslant y$ implies $f(x) \leqslant f(y)$.

1.4.2 Alexandroff-McCord correspondence

We recall the correspondence proved by McCord [50] (we call it the Alexandroff-McCord correspondence because it was Alexandroff who first worked in it) in which simplicial complexes are related with Alexandroff T_0 spaces. Given an A-space space X, define $\mathcal{K}(X)$ as the abstract simplicial complex having has vertex set X and as simplices the finite totally ordered subsets $x_0 \leqslant \ldots \leqslant x_s$ of the poset X. A continuous map $f : X \to Y$ of A-spaces defines a simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$, since it is order preserving. Now, we can define the following map $\psi = \psi_X : |\mathcal{K}(X)| \to X$ as follows. Every point $z \in |\mathcal{K}(X)|$ is contained in the interior of a unique simplex σ spanned by a strictly increasing finite sequence $x_0 < x_1 < \ldots < x_s$ of points of X. We define $\psi(z) = x_0$, and the following theorem holds.

Theorem 10 (McCord [50])**.** *The map ψ_X is a weak homotopy equivalence. Moreover, given a map $f : X \to Y$ of A-spaces, the induced simplicial map $\mathcal{K}(f)$ makes the following diagram commutative.*

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ \uparrow{\scriptstyle \psi_X} & & \uparrow{\scriptstyle \psi_Y} \\ |\mathcal{K}(X)| & \xrightarrow[|\mathcal{K}(f)|]{} & |\mathcal{K}(Y)| \end{array}$$

Example 1. Consider the finite space $X = \{a, b, c, d\}$ with proper open sets

$$\tau = \{\{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}\,.$$

Its minimal basis is

$$\{B_a = \{a\}, B_b = \{a, b, c\}, B_c = \{c\}, B_d = \{a, c, d\}\}\,.$$

Hence, X is a poset with $a \leqslant b, d$, $c \leqslant b, d$. The corresponding simplicial complex $\mathcal{K}(X)$ has vertices a, b, c, d and simplices $\langle a, b\rangle, \langle a, d\rangle, \langle c, b\rangle, \langle c, d\rangle$, whose realization is homeomorphic to a sphere $\mathbb{S}^1$. Hence X has the homotopy and singular homology groups of $\mathbb{S}^1$.

On the other direction, given a simplicial complex K, we can define an A-space $\mathcal{X}(K)$ whose points are the simplices of K and the relation is given as $\sigma \leqslant \tau$ if and only if $\sigma \subset \tau$ as simplices. Also, from any simplicial map $g : K \to L$ it is evident that we obtain a continuous map $\mathcal{X}(g) : \mathcal{K} \to \mathcal{L}$ of A-spaces. Now, since $\mathcal{X}(K)$ in an A-space, we can apply the previous theorem to obtain the simplicial complex $\mathcal{K}(\mathcal{X}(K)) = K'$ and the weak homotopy equivalence

$$\phi_K = \psi_{\mathcal{X}(K)} : |K| = |K'| = |\mathcal{K}(\mathcal{X}(K))| \longrightarrow \mathcal{X}(K).$$

Again, for every simplicial map $g : X \to Y$ we have that the following diagram commutes up to homotopy.

$$\begin{array}{ccc} |K| & \xrightarrow{\quad g \quad} & |L| \\ \downarrow{\scriptstyle \phi_K} & & \downarrow{\scriptstyle \phi_L} \\ \mathcal{X}(K) & \xrightarrow[\mathcal{X}(g)]{} & \mathcal{X}(L) \end{array}$$

So, there is a mutual correspondence of simplicial complexes and A-spaces (or posets) preserving homotopy and singular homology groups. Note that this means that there are A-spaces with the same homotopy and singular homology groups as every possible simplicial complex. Concretely, there are finite T_0 spaces with the same homotopy and singular homology groups as any compact polyhedron.

Note that given a simplicial complex K, we can apply the correspondences sequentially to obtain $\mathcal{K}(\mathcal{X}(\cdot\overset{n}{\cdot\cdot}\cdot\mathcal{K}(\mathcal{X}(K)))) = K^{(n)}$ the n-th barycentric subdivision of K. Similarly, given any A-space X, we can apply the correspondences n times to obtain what we will call the *n-th barycentric subdivision* $X^{(n)} = \mathcal{K}(\mathcal{X}(\cdot\overset{n}{\cdot\cdot}\cdot\mathcal{K}(\mathcal{X}(K))))$ of the A-space X.

1.5 Persistent homology

In the recent years, the fields of Computational Topology and Applied Algebraic Topology have had a great and successful development. The deep and abstract mathematical concepts and theorems of (Algebraic) Topology have been shown as a very useful tool in real world problems, so the interest of other areas of science in them, is becoming bigger and bigger. As general references for these topics we give the books, [68, 26, 33]. We are interested in the more specific field of Topological Data Analysis. This consists of the study and management of (maybe belonging to real world) data sets using topological constructions and techniques. The excellent surveys [17] by Carlsson and [32] by R. Ghrist are strongly recommended for this topic.

In particular, we recall the powefull tool of persistent homology. Persistence is an algebraic topological tool used to detect topological features in contexts where we have not all the information about the space or the information we have is somehow noisy. We recommend, besides the general references quoted, the surveys [25, 65]. It is usually agreed that the concept of persistence born in three different ways: Frosini and Ferri's group, studying the persistence of 0-dimensional homology of functions (using the concept os size function) [29], Vanessa Robins introducing the concept of persistent Betti numbers in a shape theory context to understand the evolution of homology in fractals [60] and Edelsbrunner group [27].

1.5.1 The idea of persistence

We illustrate the notion of persistent homology through a very schematic example. Consider we have a finite set of points $\mathbb{X}$ (and we know the distances between them), possibly as a noisy sample of an unknown topological space X. If we want to detect some topological properties of X from $\mathbb{X}$, one way could be to construct a simplicial complex based on this set of points and study its topologicalproperties. For example, in figure 1.3, we have the Vietoris-Rips

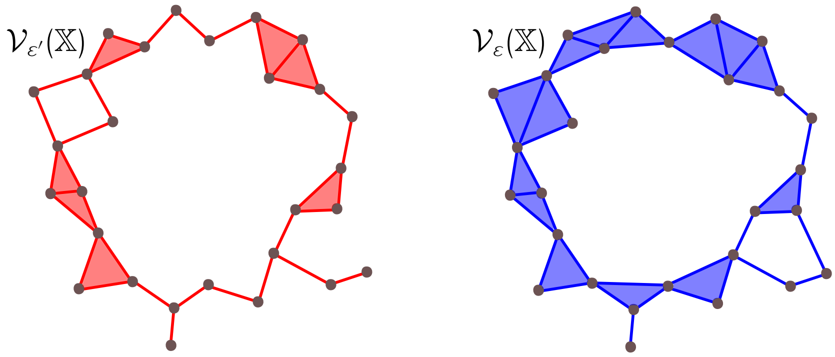


Figure 1.3: The Vietoris-Rips complexes of a point cloud with two parameters.

complexes of a finite set of points $\mathbb{X}$, which is a noisy sample of an underlying space $X = \mathbb{S}^1$, with two different real parameters $0 < \varepsilon' < \varepsilon$. Both detect the main feature of X, the central hole or 1-cycle. But we have that none of them really determine the first homology group of X, because

$$H_1(\mathcal{V}_{\varepsilon'}(\mathbb{X}); \mathbb{Z}) = H_1(\mathcal{V}_{\varepsilon}(\mathbb{X}); \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \ncong \mathbb{Z} \cong H_1(X; \mathbb{Z}).$$

The persistent homology idea is just to consider the inclusion $\mathcal{V}_{\varepsilon'}(\mathbb{X}) \hookrightarrow \mathcal{V}_{\varepsilon}(\mathbb{X})$ and the image of the induced maps on the first homology groups, that is,

$$\mathrm{Im}\big(H_1(\mathcal{V}_{\varepsilon'}(\mathbb{X}); \mathbb{Z}) \hookrightarrow H_1(\mathcal{V}_{\varepsilon}(\mathbb{X}); \mathbb{Z})\big) \cong \mathbb{Z} = H_1(X; \mathbb{Z})$$

which really captures the desired feature.

1.5.2 Filtrations

In general, suppose we have a *filtration*, i.e., a finite sequence of nested simplicial complexes

$$\emptyset = K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_s.$$

We are interested in the topological evolution of the sequence of the homology groups, so, for every $p \in \mathbb{N}$ and every abelian group G, we can consider the induced p-th homology finite sequence

$$\{0\} = H_p(K_0; G) \hookrightarrow H_p(K_1; G) \hookrightarrow \ldots \hookrightarrow H_p(K_s; G).$$

As we move forward in the sequence, new homology classes can appear and some could merge or vanish. We collect the homology classes as follows. The *p-th persistent homology groups* are the images of the homomorphisms induced by inclusion

$$H_p^{ij} = \mathrm{Im}\big(H_p(K_i; G) \hookrightarrow H_p(K_j; G)\big)$$

for $0 \leqslant i < j \leqslant s$. Similarly, the *p-th persistent Betti numbers* are the ranks of these groups $\beta_p^{ij} = \mathrm{rk} H_p^{ij}$. We can do the same definitions with reduced homology. The collection of persistent Betti numbers can be visualized in a *persistence diagram*. Given a filtration of simplicial complexes, there are several algorithms determining these numbers and the evolution of the homology classes. See references for more details.

There are several ways of arriving to a filtration of simplicial complexes. We mention the main two of them.

- A finite set of points and its distances. Given any finite metric space $\mathbb{X}$ (as in the previous example), called a *point cloud*, we can produce filtrations of simplicial complexes taking the Vietoris-Rips, Čech or other complexes of $\mathbb{X}$ for every $\varepsilon > 0$. There will be only a finite number of different complexes since $\mathbb{X}$ is finite, so we obtain a filtration of simplicial complexes.

- Consider a simplicial complex K and a real valued function $f : X \to \mathbb{R}$ which is monotonic (meaning that if τ is a face of σ, then $f(\tau) \leqslant f(\sigma)$. Then, supposing the different values of the function are $-\infty = a_0 < a_1 < \ldots < a_s$, if we set $K_i = f^{-1}(-\infty, a_i]$ for $i = 0, 1, \ldots, s$, we have that K_i are subcomplexes of K, K_i is a subcomplex of K_{i+1}, for every $i = 0, 1, \ldots, s-1$, and $K_a = K$. Thus we have a filtration called the *filtration of the function* f.

1.5.3 Structure of persistence

One step further in the study of persistence is to find some structure in the evolution of the homology classes in a given filtration. In this direction, we recall the Structure Theorem by Carlsson and Zomorodian [67]. For the algebraic definitions see the cited article or the book [23]. Let F be a field. We define a *persistence module* $\mathcal{M}$ as a family of vector spaces [9] M_i over F and homomorphisms $\varphi_i : M_i \to M_{i+1}$, for $i \in \mathbb{N}$. For example, the induced homology finite sequence of a filtration, where the maps φ send a homology class to the one containing it. We will say that $\mathcal{M}$ is of finite type if M_i is a finitely generated R-module and there exists an integer m such that φ_i is an isomorphism for $i \geqslant m$. Now we define the elements for the classification which, in some sense, represents the beginning and end of an homology class. A *persistence interval* is an ordered pair (i, j), with $0 \leqslant i < j$, $i, j \in \mathbb{Z} \cup \{+\infty\}$. A finite set of persistence intervals is called a *barcode*. The following correspondence is stablished.

Theorem 11 (Correspondence)**.** *The isomorphism classes of persistence modules of finite type over a field are in bijective correspondence with barcodes.*

The proof of this theorem uses some advanced algebra, including the structure theorem of finitely generated modules and graded modules over PIDs, which we do not include here for simplicity. For the algebraic machinery used in the proof, see [23]. The importance of this result, which gives a structure to the persistence modules, is that we know that the barcodes, a very intuitive way of representing the evolution of the homology classes, really determines the persistence module, up to isomorphism. So they are a good way to represent persistence. On the other hand, this result enables to modify the standard reduction algorithm for homology using the properties of the persistence module to derive a rather simple algorithm to compute the barcodes. This is implemented in the Matlab routine Plex.

[9] The definition still holds if we replace F by a commutative ring with unity, obtaining then R modules M_i, but we need this stronger condition for the structure theorem.

Chapter 2

Shape approximations of compacta

In this chapter, we recall the construction over compact metric spaces, done in [3], to obtain an inverse sequence of finite approximations, closer to our space in each step. We can define some sequences of polyhedra associated with it. This construction is based in the multivalued shape theory and we will show how it describes the shape of the original space.

2.1 Main construction

We begin by recalling the main construction done in section 6 of [3]. There, given a compact metric space, it is obtained an inverse sequence of finite approximations of our space and some sequences of real numbers that allow us to define continuous maps between the approximations. Since the space is compact, we can find finite approximations as small as wanted. The naive idea would be to connect them in terms of proximity. That is, we would send a point in one approximation to its closest point in the previous one. The problem is that it is possible for one point of one approximation to be exactly at the same distance from two points of another approximation. Hence, we would not have a well defined map and, even if we have it, the approximations are just discrete spaces and the map is trivial. By making use of hyperspaces and the upper semifinite topology, we can define more suitable finite topological spaces and continuous maps between them. Moreover, this inverse sequence will lead to inverse sequences of polyhedra, which will be shown to recover the shape of the original space.

Let us start with the kind of approximations that we will use.

Definition 3. Let (X, d) be a compact metric space and $\varepsilon > 0$ a real number. A finite subset $A \subset X$ is said to be an ε–approximation of X if, for every $x \in X$, there is at least one point $a \in A$ such that $\mathrm{d}(x, a) < \varepsilon$.

Remark 2. It is straighforward to see that, for a compact metric space, there are ε–approximations for every $\varepsilon > 0$.

Given a non-empty finite subset $A \subset X$ of a compact metric space (X, d), we consider, for each point $x \in X$, the set of closest points of A as

$$A(x) = \{a \in A : \mathrm{d}(x, a) = \mathrm{d}(x, A)\}.$$

It is natural, then, to define a function from the space to its closest sets. We will call the *nearby* map from X to A to the function

$$q_A : X \to 2^A \subset 2^X_u,$$

defined by $q_A(x) = A(x)$. The extension of the nearby map will be usually written as

$$r_A : 2^X_u \to 2^X_u.$$

Moreover, we can define the *distance* map $d_A : X \to \mathbb{R}$, with $d_A(x) = d(x, A)$. Both will be shown to be continuous maps because of the following lemma.

Lemma 2. *Let (X, d) be a compact metric space and $A \subset X$ a finite subset. For every $x \in X$ there exists $\delta > 0$ such that, for every $y \in B(x, \delta)$, $A(y) \subset A(x)$.*

Proof. Let $x \in X$ and consider the distances $\delta^- = \mathrm{d}(x, A) \geqslant 0$ and $\delta^+ = \mathrm{d}(x, A \setminus A(x)) > 0$ (if $A \setminus A(x) = \emptyset$, then $A(x) = A$, so we will assume that it is not empty). Now, fix

$$\delta = \frac{\delta^+ - \delta^-}{2} > 0.$$

If $a \in A(x)$ and $b \in A \setminus A(x)$, we see that, for every $y \in \mathrm{B}(x, \delta)$,

$$\mathrm{d}(y, a) \leqslant \mathrm{d}(y, x) + \mathrm{d}(x, a) < \delta + \delta^- = \frac{\delta^+ + \delta^-}{2},$$
$$\delta^+ \leqslant \mathrm{d}(x, b) \leqslant \mathrm{d}(x, y) + \mathrm{d}(y, b) < \delta + \mathrm{d}(y, b).$$

Whence

$$\mathrm{d}(y,b) > \frac{\delta^+ + \delta^-}{2} > \mathrm{d}(y,a),$$

so $A(y) \subset A(x)$ ✓

As an immediate corollary, we obtain the continuity of the nearby map.

Corollary 2. *Let (X, d) be a compact metric space and $A \subset X$ a finite subset. The nearby map $q_A : X \to 2^X_u$ is continuous. Hence its extension r_A is also continuous.*

Proof. The map q_A satisfies that, for every $x \in X$, there exists $\delta > 0$ such that

$$q_A(\mathrm{B}(x,\delta)) \subset q_A(x),$$

hence q_A is continuous ✓

Remark 3. If A is a finite ε-approximation of a compact metric space (X, d) , the images of the points $x \in X$ are sent to the subespace $U_{2\varepsilon}(A)$ because of the triangle inequality. That is, the nearby map is $q_A : X \to U_{2\varepsilon}(A)$.

This result is the more important one concerning the main construction. It says that given an approximation, we always can find a more accurate approximation and define finite spaces based on them and connected by a nearby map.

Lemma 3. *Let (X, d) be a compact metric space and consider a real number $\varepsilon > 0$ and a finite ε-approximation A of X. There exists $0 < \varepsilon' < \varepsilon$ such that, for every finite ε'-approximation A', the map $p : U_{2\varepsilon'}(A') \to U_{2\varepsilon}(A)$, defined by $p(C) = r_A(C)$, is well defined and continuous. Moreover, we can select $\varepsilon' < \frac{\varepsilon-\gamma}{2}$ where $\gamma \geqslant d(x, A)$, for every $x \in X$.*

Proof. Since the map $r_A : 2^X_u \to 2^A \subset 2^X_u$ is continuous and $\{U_\alpha\}_{\alpha>0}$ is a base of open neighborhoods of the canonical copy of X in 2^X_u, we have that there exist $\delta > 0$ such that $r_A(U_\delta(X)) \subset U_{2\varepsilon}(A)$. In words, two points of X that are δ-close, are sent by q_A to subsets of A whose points are ε-close. Now, pick a real number $0 < \varepsilon' < \frac{\delta}{2}$ and any ε'-approximation A' of X. Then, $U_{2\varepsilon'}(A) \subset U_\delta(X)$, so $r_A(U_{2\varepsilon'}(A)) \subset U_{2\varepsilon}(A)$. Hence, the map $p : U_{2\varepsilon'}(A') \to U_{2\varepsilon}(A)$, defined as the restriction of r_A to $U_{2\varepsilon'}(A')$ (that is, $p(C) = r_A(C)$, for every $C \in U_{2\varepsilon'}(A')$), is well defined and continuous.

For the second part, let us consider the distance funcion to A, $d_A : X \to \mathbb{R}$ which is a continuous map. Since A is an ε-approximation, $\mathrm{d}(x, A) < \varepsilon$ for every $x \in X$. Moreover, for being X compact, there exists a supremum

$$\gamma = \sup \{\mathrm{d}(x, A) : x \in X\} < \varepsilon,$$

so it is enough to select

$$0 < \varepsilon' < \min \left\{ \frac{\varepsilon - \gamma}{2}, \frac{\delta}{2} \right\}$$

and we are done ✓

The reason of the second part is that we want the described approximation to be tight enough. It will be seen to be useful to derive some properties in what follows.

Definition 4. Let (X, d) be a compact metric space. Given two real numbers $0 < \varepsilon' < \varepsilon$, two finite subsets $A, A' \subset X$, ε and ε'-approximations respectively, we will say that A' is *adjusted* to A if ε' and ε satisfy the conditions of the previous result.

Remark 4. In these terms, lemma 3 simply says that for every approximation of a compact metric space, there exists another adjusted to it.

Finally, by induction, we can repeat the process indefinitely, to obtain sequences of approximations.

Proposition 8 (Main construction)**.** *For every compact metric space (X, d) , there exists a decreasing sequence of positive real numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ tending to zero, and a sequence $\{A_n\}_{n\in\mathbb{N}}$ of finite ε_n-approximations of X, such that A_{n+1} is adjusted to A_n, for every $n \in \mathbb{N}$.*

Proof. Let (X, d) be a compact metric space. The construction is done inductively. Consider the diameter $\mathrm{diam}\,(X) = M$ and any $\varepsilon_1 > M$. Consider the subset A_1

consisting of one point of X. It is clear that A_1 is an ε_1 approximation of X. In the next step, we consider

$$0 < \varepsilon_2 < \min\left\{\frac{\varepsilon_1 - M}{2}, \frac{M}{2}\right\}$$

and a finite ε_2-approximation A_2. The finite subsets $A_1, A_2 \subset X$ inherit the metric. Consider the finite spaces $U_{2\varepsilon_1}(A_1), U_{2\varepsilon_2}(A_2) \subset 2^X_u$ with the subsepace topology and the constant (and hence continuous) map $p_{1,2} : U_{2\varepsilon_2}(A_2) \to U_{2\varepsilon_1}(A_1)$. For the next step, apply lemma 3 to the ε_2-approximation A_2 to obtain an adjusted ε_3-approximation. In general, apply lemma 3 to the ε_n-approximation A_n to obtain a ε_{n+1}-approximation A_{n+1}, adjusted to A_n ✓

Remark 5. Note that, given a compact metric space, this process is completely constructive. We can compute all the real numbers and select finite approximations that satisfy the quoted properties. It is an inductive process, so we compute the numbers and approximations in this strictly necessary order:

$$M, \varepsilon_1, A_1, \varepsilon_2, A_2, \gamma_2, \delta_2, \ldots, \varepsilon_n, A_n, \gamma_n, \delta_n, \varepsilon_{n+1}, A_{n+1}, \gamma_{n+1}, \delta_{n+1}, \ldots$$

Given a compact metric space (X, d) , from this construction we obtain a sequence of finite spaces $\{U_{2\varepsilon_n}(A_n)\}_{n\in\mathbb{N}}$ and continuous maps $p_{n,n+1} : U_{2\varepsilon_{n+1}}(A_{n+1}) \to U_{2\varepsilon_n}(A_n)$, for every $n \in \mathbb{N}$. We thus obtain an inverse sequence of finite spaces

$$U_{2\varepsilon_1}(A_1) \xleftarrow{p_{1,2}} U_{2\varepsilon_2}(A_2) \xleftarrow{p_{2,3}} \ldots \xleftarrow{p_{n-1,n}} U_{2\varepsilon_n}(A_n) \xleftarrow{p_{n,n+1}} U_{2\varepsilon_{n+1}}(A_{n+1}) \xleftarrow{p_{n+1,n+2}} \ldots$$

We will give a name to every inverse sequence of finite spaces obtained in this way.

Definition 5. Let (X, d) be a compact metric space. An inverse sequence of finite spaces

$$\{U_{2\varepsilon_n}(A_n), p_{n,n+1}\}$$

obtained as indicated from a sequence of adjusted ε_n approximations $\{A_n\}_{n\in\mathbb{N}}$, where $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is decreasing and tending to zero, is said to be a *finite approximative sequence (usually written FAS)* of X.

Remark 6. Strictly speaking, a FAS will be the inverse sequence of finite spaces quoted above. But we will use FAS to make reference also to the approximations and the numbers obtained, $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$, because they determine uniquely the finite spaces and maps.

Remark 7. Theorem 8 implies that every compact metric space has a FAS. In general, FASs are not unique.

Moreover, we can use the Alexandroff-McCord correspondence (Theorem 10) to obtain a sequence of polyhedra. For every $n \in \mathbb{N}$ and finite T_0 space $U_{2\varepsilon_n}(A_n)$, there exists a simplicial complex $\mathcal{K}(U_{2\varepsilon_n}(A_n))$ with vertex set the points $D \in U_{2\varepsilon_n}(A_n)$ and simplexes $\langle D_0, D_1, \dots, D_s\rangle$ with $D_0 \subset D_1 \subset \dots \subset D_s$ such that there is a weak homotopy equivalence between the finite space and the geometric realization of the simplicial complex

$$f_n : |\mathcal{K}(U_{2\varepsilon_n}(A_n))| \longrightarrow U_{2\varepsilon_n}(A_n),$$

defined as follows. Every point $x \in |\mathcal{K}(U_{2\varepsilon_n}(A_n))|$ is contained in the interior of a unique simplex $\sigma = \langle D_0, D_1, \dots, D_s\rangle$ and $f_n(x) = D_0$.

We also have simplicial maps[1] between the polyhedra, defined on the vertices and extended as usual to simplices:

$$\begin{aligned} p_{n,n+1} : \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) &\longrightarrow \mathcal{K}(U_{2\varepsilon_n}(A_n)) \\ D &\longmapsto p_{n,n+1}(D) \\ \langle D_0, D_1, \dots, D_s\rangle &\longmapsto \langle p_{n,n+1}(D_0), p_{n,n+1}(D_1), \dots, p_{n,n+1}(D_s)\rangle \end{aligned}$$

where, if

$$D_0 \subset D_1 \subset \dots \subset D_s,$$

then

$$p_{n,n+1}(D_0) \subset p_{n,n+1}(D_1) \subset \dots \subset p_{n,n+1}(D_s).$$

The realizations of these simplicial maps satisfy that, for every $n \in \mathbb{N}$, the

[1] Following McCords paper's notation we should write $\mathcal{K}(p_{n,n+1})$ for the simplicial maps but we will omit this notation, using the same as for the maps between the finite spaces, $p_{n,n+1}$, for the sake of simplicity.

diagram

$$\begin{array}{ccc} |\mathcal{K}(U_{2\varepsilon_n}(A_n))| & \xleftarrow{|p_{n,n+1}|} & |\mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1}))| \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ U_{2\varepsilon_n}(A_n) & \xleftarrow[p_{n,n+1}]{} & U_{2\varepsilon_{n+1}}(A_{n+1}) \end{array}$$

commutes. So we obtain an inverse sequence of polyhedra and a map (actually a level map) between the inverse sequences of finite spaces and polyhedra.

$$\begin{array}{ccccccccccc} |\mathcal{K}(U_{2\varepsilon_1}(A_1))| & \xleftarrow{|p_{1,2}|} & |\mathcal{K}(U_{2\varepsilon_2}(A_2))| & \longleftarrow & \dots & \longleftarrow & |\mathcal{K}(U_{2\varepsilon_n}(A_n))| & \xleftarrow{|p_{n,n+1}|} & |\mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1}))| & \longleftarrow & \dots \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \\ U_{2\varepsilon_1}(A_1) & \xleftarrow[p_{1,2}]{} & U_{2\varepsilon_2}(A_2) & \longleftarrow & \dots & \longleftarrow & U_{2\varepsilon_n}(A_n) & \xleftarrow[p_{n,n+1}]{} & U_{2\varepsilon_{n+1}}(A_{n+1}) & \longleftarrow & \dots \end{array}$$

The analysis of these two inverse sequences, their limits and their relations with the original space, will play a fundamental role in the following.

2.2 Approximative maps

In this section we will analyze the shape properties of the main construction in terms of the shape theory for compact metric spaces with multivalued maps (see section 1.3.3). We will prove some propositions concerning this relationship as well as two results proposed in [3]. The purpose of this relationship is to reflect that the main construction captures the shape properties of the space in which is done.

We will need two technical lemmas about homotopies in hyperspaces with the upper semifinite topology in order to prove some results. The upper semifinite topology is shown here to be very useful, because it easily gives us homotopies between those kind of maps.

Lemma 4. *Let X, Y be metric compacta and $f, g : X \to 2^Y_u$ two continuous maps. The map $f \cup g : X \to 2^Y_u$, defined by $(f \cup g)(x) = f(x) \cup g(x)$, is continuous.*

Proof. For every $x \in X$, the application $f \cup g$ is well defined because $f(x)$ and $g(x)$ are closed subsets of Y, and so is $f(x) \cup g(x)$. Let us take a neighborhood $B(V)$ of $f(x) \cup g(x)$ in 2^Y_u, where $B(V) = \{C \in 2^Y_u : C \subset V\}$ with V an open subset of Y which contains $f(x) \cup g(x)$. The applications f and g are continuous, so there are two neighborhoods of x, namely U_1 and U_2, such that $f(U_1), g(U_2) \subset V$. Then $U = U_1 \cap U_2$ is a neighborhood of x such that $f \cup g(U) = f(U) \cup g(U) \subset f(U_1) \cup g(U_2) \subset V$, so $f \cup g(U) \subset B(V)$, hence $f \cup g$ is continuous at x ✓

Lemma 5. *Consider two compact metric spaces X, Y. Let $f, g, h : X \to 2^Y_u$ be continuous maps such that, for every $x \in X$, $f(x), g(x) \subset h(x) \subset 2^Y_u$. Then, f and g are homotopic.*

Proof. The map

$$H : X \times I \longrightarrow 2^X_u$$

defined by

$$H(x,t) = \begin{cases} f(x) & \text{if } t \in [0, \frac{1}{2}), \\ h(x) & \text{if } t = \frac{1}{2}, \\ g(x) & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

is continuous and hence a homotopy between the two maps. Indeed, H is obviously continuous in every point (x,t) with $t \neq \frac{1}{2}$. Consider $(x, \frac{1}{2}) \in X \times \frac{1}{2}$ and an open neighborhood $B(V)$ of it. Because of the continuity of h we have that there is an open neighborhood U of x such that $h(U) \subset B(V)$. But $f(U), g(U) \subset h(U)$, so $U \times I$ is an open neighborhood of $(x, \frac{1}{2})$ such that $H(U \times I) \subset B(V)$, and the continuity is proved ✓

Remark 8. From the previous two lemmas we can derive the following result: For X, Y compact metric spaces, every two maps $f, g : X \to 2^Y_u$ are homotopic. Despite it seems to be a disappointing result, we will be usually interested not in homotopies in the whole space 2^Y_u but in subespaces of it. Note that we are considering sequences of this kind of map with smaller and smaller diameters.

We can show now, the following result, proposed in [3](Proposition 21), relating the main construction with the multivalued shape theory.

Proposition 9. *Let X be a compact metric space. Consider we obtain the sequences $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$ by performing the main construction to X. The sequence of maps $\{q_{A_n}\}_{n\in\mathbb{N}}$ is an approximative map representing the identity shape morphism on X.*

Proof. Let us first prove that $\{q_{A_n}\}_{n\in\mathbb{N}}$ is indeed an approximative map. For each $n \in \mathbb{N}$ the map $q_{A_n} : X \to 2^Y_u$ is continuous, because of lemma 2. The family $\{U_\varepsilon\}$ is a base of open neighborhoods of the canonical copy of X inside 2^X_u, so there exists an $\varepsilon > 0$ such that $X \subset U_\varepsilon \subset U$. Recall that $\{\varepsilon_n\}$ is a decreasing sequence of positive real numbers tending to cero, so we can choose n_0 such that $2\varepsilon_{n_0} < \varepsilon$. We claim that, for every $n \geqslant n_0$, the map $H : X \times I \to U \subset 2^X_u$ defined by

$$H(x,t) = \begin{cases} q_{A_n}(x) & \text{if } t \in [0, \frac{1}{2}), \\ q_{A_n}(x) \cup q_{A_{n+1}}(x) & \text{if } t = \frac{1}{2}, \\ q_{A_{n+1}}(x) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is an homotopy between q_{A_n} and $q_{A_{n+1}}$ in U. The map is continuous and well defined because, for every $x \in X$,

$$\begin{aligned} q_{A_{n+1}}(x) &\subset \mathrm{B}(x, \varepsilon_{n+1}), \\ q_{A_n}(x) &\subset \mathrm{B}(x, \varepsilon_n), \end{aligned}$$

and then

$$\operatorname{diam}\left(q_{A_n}(x) \cup q_{A_{n+1}}(x)\right) < 2\varepsilon_n < \varepsilon,$$

so the images of the applications are in $U_\varepsilon \subset U$

It is clear that the approximative map $\mathrm{id} : X \to 2^X_u$, with $\mathrm{id}(x) = \{x\}$, corresponds to the identity shape morphism of X. To prove that $\{q_{A_n}\}_{n\in\mathbb{N}}$ it is homotopic to the identity, we just choose n_0 such that $2\varepsilon_{n_0} < \varepsilon$, and use the homotopy $H : X \times I \to U \subset 2^X_u$ defined by

$$H(x,t) = \begin{cases} q_{A_n}(x) & \text{if } t \in [0, \frac{1}{2}), \\ q_{A_n}(x) \cup \{x\} & \text{if } t = \frac{1}{2}, \\ \{x\} & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \quad \checkmark$$

Note that, in the previous result, we obtain an equivalent approximative map with finite images. We can generalize this to any class of approximative maps in order to describe a shape theory for metric compacta in simpler terms.

Definition 6. Let X, Y be compact metric spaces. An approximative map $\{f_n\}_{n\in\mathbb{N}}$ from X to Y is said to be of *finite type* if, for every $n \in \mathbb{N}$ and every $x \in X$, the image $f_n(x)$ is a finite set.

Now, we prove that for every homotopy class of approximative maps, we can allways find a representative of finite type.

Proposition 10. *Let X, Y be compact metric spaces and $\{f_n\}_{n\in\mathbb{N}}$ an approximative map from X to Y. There exists an approximative map $\{\mathfrak{f}_n\}_{n\in\mathbb{N}}$ of finite type from X to Y homotopic to $\{f_n\}_{n\in\mathbb{N}}$.*

Proof. Consider a decreasing sequence of positive real numbers $\{\beta_n\}_{n\in\mathbb{N}}$ converging to zero and a sequence of finite β_n-approximations[2] B_n of Y. Define, for every $n \in \mathbb{N}$, the map $\mathfrak{f}_n = r_{B_n} \circ f_n$, where $r_{B_n} : 2^Y_u \to 2^Y_u$, is the extension of the map $q_{B_n} : Y \to 2^Y_u$. It is clear that $\mathfrak{f}$ is continuous. Now, we need to show that $\{\mathfrak{f}_n\}_{n\in\mathbb{N}}$ is an approximative map and it is homotopic to $\{f_n\}_{n\in\mathbb{N}}$. We are going to prove both statements as consecuences of the following claim: For every open set $U \subset 2^Y_u$ containing the canonical copy of Y, there exists n_0 such that, for every $n \geqslant n_0$, $f_n \simeq_U g_n$. Indeed, let $U \subset 2^Y_u$ such an open set. Consider, for every $n \in \mathbb{N}$, the diameter D_n of the map f_n. Since f_n is an approximative map, it is clear that the sequence $\{D_n\}_{n\in\mathbb{N}}$ converges to zero. The diameter of g_n depends on D_n, for each $n \in \mathbb{N}$. For every $x \in X$, and for every two points $y_1, y_2 \in f_n(x)$, consider $z_1 \in B_n(y_1), z_2 \in B_n(y_2)$. Then, we have

$$\mathrm{d}(z_1, z_2) \leqslant \mathrm{d}(z_1, y_1) + \mathrm{d}(y_1, y_2) + \mathrm{d}(y_2, z_2) < 2\beta_n + D_n,$$

hence $\mathrm{diam}\,(g_n) < \beta_n + D_n$. Now, let $\varepsilon > 0$ be a real number such that $U_\varepsilon \subset U$, and select n_0 such that $2\beta_n + D_n < \varepsilon$ for every $n \geqslant n_0$. Then, the map $H : X \times I \to U$, defined by

$$H(x,t) = \begin{cases} f_n(x) & \text{if } t \in [0, \frac{1}{2}), \\ f_n(x) \cup g_n(x) & \text{if } t = \frac{1}{2}, \\ g_n(x) & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is continuous by lemma 5, and hence a homotopy between f_n and g_n in U. Moreover, being $\{f_n\}_{n\in\mathbb{N}}$ an approximative map, there exists m_0 such that, for every $n \geqslant m_0$, f_n is homotopic to f_{n+1} in U. Finally, for $n > \max n_0, m_0$, we have

$$g_n \simeq_U f_n \simeq_U f_{n+1} \simeq_U g_{n+1},$$

[2] In this case, we do not need the sequences to be as in the main construction, with the quoted properties is enough.

which shows the two statements which finish the proof ✓

We finish this section by showing this result, also proposed in [3] (Proposition 20), which establishes a deeper connection of the main construction with the shape of the space, because it takes into account the maps $p_{n,n+1}$ between the finite spaces.

Proposition 11. *Let X be a compact metric space and consider the sequences obtained with the main construction over X. The following diagram is commutative, up to homotopy, for every $n \in \mathbb{N}$:*

$$\begin{array}{ccc} X & \xrightarrow{\quad id \quad} & X \\ \Big\downarrow{\scriptstyle q_{A_{n+1}}} & & \Big\downarrow{\scriptstyle q_{A_n}} \\ U_{2\varepsilon_{n+1}}(A_{n+1}) & \xrightarrow[p_{n+1,n}]{} & U_{2\varepsilon_n}(A_n). \end{array}$$

Proof. To prove this commutativity we need a homotopy between the maps $p_{n,n+1} \circ q_{A_{n+1}}$ and q_{A_n}. Such a homotopy is given by $H : X \times I \to U_{2\varepsilon_n}(A_n)$, with

$$H(x,t) = \begin{cases} q_{A_n}(x) & \text{if } t \in [0, \frac{1}{2}), \\ q_{A_n}(x) \cup p_{n,n+1} \circ q_{A_{n+1}}(x) & \text{if } t = \frac{1}{2}, \\ p_{n,n+1} \circ q_{A_{n+1}}(x) & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

This is a homotopy because of Lemmas 4 and 5, and the following fact: If $x \in X$, $y \in q_{A_n}(x)$ and $z \in p_{n,n+1}(y)$ we have that

$$\begin{aligned} \mathrm{d}(x,z) &\leqslant \mathrm{d}(x,y) + \mathrm{d}(y,z) < \\ &< 2\gamma_{n+1} + \gamma_n < \frac{\varepsilon_n - \gamma_n}{2} + \gamma_n < \varepsilon_n. \end{aligned}$$

Then

$$\begin{aligned} p_{n,n+1} \circ q_{A_{n+1}}(x) &\subset \mathrm{B}(x, \varepsilon_n), \\ q_{A_n}(x) &\subset \mathrm{B}(x, \varepsilon_n), \end{aligned}$$

so

$$\operatorname{diam}\left(q_{A_n}(x) \cup p_{n,n+1} \circ q_{A_{n+1}}(x)\right) < 2\varepsilon_n$$

and the homotopy is well defined and continuous ✓

2.3 Polyhedral approximative sequences and the General Principle

From the main construction we obtain a sequence of finite spaces and maps between them related with the notion of proximity between points of contiguous approximations. This reminds the concept of shape theory of an expansion associated to a space X. In this section we would construct several sequences of polyhedra, all of them based on the main construction. They will provide all the shape information of the space. We will call any of the sequences of this section a *polyhedral approximative sequence* of X.

2.3.1 The Alexandroff-McCord approximative sequence

We begin by considering the inverse sequence of polyhedra we mentioned above,

$$A\mathcal{M}(X) = \{|\mathcal{K}(U_{2\varepsilon_n}(A_n))|, |p_{n,n+1}|, \mathbb{N}\}.$$

We will call this inverse sequence, the *Alexandroff-McCord approximative sequence*. The polyhedra involved in this sequence are actually realizations of another well known simplicial complex.

Definition 7. Let (X, d) a metric space, and consider a real number $\varepsilon > 0$, The Vietoris-Rips complex $\mathcal{R}_\varepsilon(X)$ is the simplicial complex with vertex set X and a q-simplex is a subset $\{x_0, \dots, x_q\} \subset X$ such that $\mathrm{diam}\{x_0, \dots, x_q\} < \varepsilon$.

The relation between the Vietoris-Rips complexes and the McCord complexes asociated to our finite spaces is stated in corollary 7 of [3]. Basically, the McCord complex is the barycentric subdivision of the Vietoris Rips complex.

Proposition 12. *Let (A, d) a finite metric space and consider $\varepsilon > 0$. Then*

$$\mathcal{K}(U_\varepsilon(A)) = \mathcal{R}'_\varepsilon(A).$$

The main result we want to prove here is the so called "general principle" of [3]. It says that this sequence reconstructs the shape properties of the space X. Namely, we are going to prove

Theorem 12 (General principle). *The inverse system $A\mathcal{M}(X)$ is an HPol-expansion of X.*

Proof. We are going to see that $A\mathcal{M}(X)$ is isomorphic to the Vietoris system

$$\mathcal{V}(X) = \{|\mathcal{R}_\varepsilon(X)|, i_{\varepsilon\varepsilon'}, \varepsilon > 0\},$$

where, for every $\varepsilon \leqslant \varepsilon'$,

$$i_{\varepsilon\varepsilon'} : |\mathcal{R}_{\varepsilon'}(X)| \longrightarrow |\mathcal{R}_\varepsilon(X)|$$

is just the inclusion induced by the simplicial inclusion. This is a well known an HPol-expansion of X (see [47]). The main differences between the two systems, that make harder their comparison, are that they are defined over different index sets, and that the polyhedra of the former are the barycentric subdivisions of the polyhedra of the latter. So, we are going to see the isomorphism with a chain of isomorpisms between the two sequences.

First of all, we can consider the sequence

$$\mathcal{V}^*(X) = \{|\mathcal{R}_{2\varepsilon_n}(X)|, i_{\varepsilon_n\varepsilon_{n+1}}, \mathbb{N}\}$$

which is cofinal with $\mathcal{V}(X)$ (and then, isomorphic), because $\{\varepsilon_n\}$ is decreasing and tending to zero.

Now, the system $A\mathcal{M}(X)$ induces a sequence with maps defined over the Vietoris Rips complexes as follows: We define, for all $n \in \mathbb{N}$, the simplicial map

$$\begin{array}{rcl} p^*_{n,n+1} : \mathcal{R}_{2\varepsilon_n}(A_{n+1}) & \longrightarrow & \mathcal{R}_{2\varepsilon_n}(A_n) \\ a & \longmapsto & b \in q_{A_n}(a) = p_{n,n+1}(\{a\}). \end{array}$$

We have to see that the realization of this map on the corresponding polyhedra is well defined up to homotopy type[3]: If $b, b' \in q_{A_n}(a)$, then

$$\mathrm{d}(b, b') \leqslant \mathrm{d}(b, a) + \mathrm{d}(a, b') < 2\varepsilon_n,$$

so the two posible definitions of the map, b and b', are homotopic. And it is

[3]Meaning that, although we do not define a map, we define a homotopy class of maps

simplicial: If the simplex $\langle a_0, a_1, \ldots, a_s \rangle$, where $\mathrm{diam}(\{a_0, a_1, \ldots, a_s\}) \leqslant 2\varepsilon_{n+1}$, the image $\langle b_0, b_1, \ldots, b_s \rangle$, with $b_i \in q_{A_n}(a_i)$ is a simplex of $\mathcal{R}_{2\varepsilon_n}(A_n)$ because

$$\begin{aligned} \mathrm{d}(b_i, b_j) &\leqslant \mathrm{d}(b_i, a_i) + \mathrm{d}(a_i, a_j) + \mathrm{d}(a_j, b_j) < \\ &< \gamma_n + 2\varepsilon_{n+1} + \gamma_n < 2\gamma_n + 2\frac{\varepsilon_n - \gamma_n}{2} = \gamma_n + \varepsilon_n < 2\varepsilon_n. \end{aligned}$$

So, we can define inductively the HPol inverse sequence of polyhedra

$$\mathcal{M}^*(X) = \{|\mathcal{R}_{2\varepsilon_n}(A_n)|, |p^*_{n,n+1}|, \mathbb{N}\}.$$

We need to prove that this sequence is equivalent to the previous one, i.e., the inverse system $\mathcal{M}^*(X)$ is isomorphic to $A\mathcal{M}(X)$. In order to define a morphism between the systems we are going to use the simplicial map that always exists between a simplicial complex K an its barycentric subdivision $\rho : K' \to K$. The vertex $x = \{x_0, x_1, \ldots, x_s\}$ of K' (and simplex of K) has image the vertex x_s of K. For example, the image of a simplex $\sigma = \langle \{x_0\}, \{x_0, x_1\}, \{x_0, x_1, x_2\} \rangle$ of K' will be $\rho(\sigma) = \langle x_0, x_1, x_2 \rangle$ a simplex of K. The realization of this map $|\rho| : |K'| = |K| \to |K|$ is homotopic to the identity [4]. This map, component by component, induces a morphism of systems $\rho : A\mathcal{M}(X) \to \mathcal{M}^*(X)$. We have to see that for all $n \in \mathbb{N}$, the following diagram is commutative up to homotopy.

$$\begin{array}{ccc} |\mathcal{R}'_{2\varepsilon_n}(A_n)| & \xleftarrow{|p_{n,n+1}|} & |\mathcal{R}'_{2\varepsilon_{n+1}}(A_{n+1})| \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ |\mathcal{R}_{2\varepsilon_n}(A_n)| & \xleftarrow[|p^*_{n,n+1}|]{} & |\mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1})| \end{array}$$

Let $x \in |\mathcal{R}'_{2\varepsilon_{n+1}}(A_{n+1})|$, x belongs to a simplex $\sigma = \langle D_0, D_1, \ldots, D_s \rangle \in \mathcal{R}'_{2\varepsilon_{n+1}}(A_{n+1})$. We need to calculate he images of σ by the simplicial maps $\rho \circ p_{n,n+1}$ and

[4] We could have defined the simplicial map choosing any vertex as image and the realization would have the same homotopic properties (see [48]).

$p^*_{n,n+1} \circ \rho$. We have to describe explicitly some elements. We denote:

$$D_s = \underbrace{\underbrace{\underbrace{a_0^0, a_1^0, \dots, a_{r_0}^0}_{D_0}, a_0^1, a_1^1, \dots, a_{r_1}^1}_{D_1}, \dots,}_{D_{s-1}} a_0^s, a_1^s, \dots, a_{r_s}^s,$$

$$p_{n,n+1}(D_s) = \underbrace{\underbrace{\underbrace{q_{A_n}(a_0^0) \cup \dots \cup q_{A_n}(a_{r_0}^0)}_{p_{n,n+1}(D_0)} \cup q_{A_n}(a_0^1) \cup \dots \cup q_{A_n}(a_{r_1}^1)}_{p_{n,n+1}(D_1)} \cup \dots \cup}_{p_{n,n+1}(D_{s-1})} q_{A_n}(a_0^s) \cup \dots \cup q_{A_n}(a_{r_s}^s),$$

$$\begin{aligned} q_{A_n}(a_{r_0}^0) &= \{b_0^0, b_1^0, \dots, b_{t_0}^0\}, \\ q_{A_n}(a_{r_1}^1) &= \{b_0^1, b_1^1, \dots, b_{t_0}^1\}, \\ &\dots \\ q_{A_n}(a_{r_s}^s) &= \{b_0^s, b_1^s, \dots, b_{t_s}^s\}. \end{aligned}$$

Or, alternatively, for $k = 0, 1 \dots, s$,

$$\begin{aligned} D_k &= \bigcup_{j0}^{k} \bigcup_{i=0}^{r_j} a_i^j = D_{k-1} \cup \bigcup_{i=0}^{r_k} a_i^k, \\ p_{n,n+1}(D_k) &= \bigcup_{j0}^{k} \bigcup_{i=0}^{r_j} q_{A_n}(a_i^j) = p_{n,n+1}(D_{k-1}) \cup \bigcup_{i=0}^{r_k} q_{A_n}(a_i^k), \\ q_{A_n}(a_i^j) &= \bigcup_{l=0}^{t_j} b_l^j. \end{aligned}$$

Then, we have

$$\begin{aligned} \langle D_0, D_1, \dots, D_s \rangle &\xrightarrow{p_{n,n+1}} \langle p_{n,n+1}(D_0), p_{n,n+1}(D_1), \dots, p_{n,n+1}(D_s) \rangle \xrightarrow{\rho} \langle b_{t_0}^0, b_{t_1}^1, \dots, b_{t_s}^s \rangle = \sigma_1, \\ \langle D_0, D_1, \dots, D_s \rangle &\xrightarrow{\rho} \langle a_{r_0}^0, a_{r_1}^1, \dots, a_{r_s}^s \rangle \xrightarrow{p^*_{n,n+1}} \langle b_{i_0}^0, b_{i_1}^1, \dots, b_{i_s}^s \rangle = \sigma_2. \end{aligned}$$

But σ_1 and σ_2 lie in a common simplex, say $\sigma_1 \cup \sigma_2 = \langle b_{t_0}^0, b_{t_1}^1, \dots, b_{t_s}^s, b_{i_0}^0, b_{i_1}^1, \dots, b_{i_s}^s \rangle$.

Indeed, it is a simplex of $\mathcal{R}'_{2\varepsilon_n}(A_n)$ because

$$\begin{aligned} \mathrm{d}(b^i_{t_i}, b^j_{i_j}) &\leqslant \mathrm{d}(b^i_{t_i}, a^i_{r_i}) + \mathrm{d}(a^i_{r_i}, a^j_{r_j}) + \mathrm{d}(a^j_{r_j}, b^j_{i_j}) \\ &< \gamma_n + 2\varepsilon_{n+1} + \gamma_n < 2\varepsilon_n. \end{aligned}$$

So, $\rho : A\mathcal{M}(X) \to \mathcal{M}^*(X)$ is a morphism of systems. Moreover, ρ is equivalent to the identity as morphism of systems (see [47], page 6) because the equivalent condition is trivially satisfied because, for all $n \in \mathbb{N}$,

$$\rho, \mathrm{id} : |\mathcal{R}'_{2\varepsilon_n}(A_n)| \longrightarrow |\mathcal{R}_{2\varepsilon_n}(A_n)|$$

are homotopic maps as we pointed out. Then $A\mathcal{M}(X)$ and $\mathcal{M}^*(X)$ are isomorphic.

It just remains to prove that the inverse systems $\mathcal{V}^*(X)$ and $\mathcal{M}^*(X)$ are isomorphic. Now both systems deal with Vietoris Rips complexes (without any barycentric subdivision). We will just write some inclusions between the systems and see that it works. So, if we consider, for each $n \in \mathbb{N}$, the obvious inclusion $j_n : \mathcal{R}_{2\varepsilon_n}(A_n) \to \mathcal{R}_{2\varepsilon_n}(X)$, which is a simplicial map, it defines (with its realizations on the polyhedra) a morphism of systems $j : \mathcal{M}^*(X) \to \mathcal{V}^*(X)$. To see this, we check that, for every $n \in \mathbb{N}$, the diagram

$$\begin{array}{ccc} |\mathcal{R}_{2\varepsilon_n}(A_n)| & \xleftarrow{p^*_{n,n+1}} & |\mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1})| \\ \downarrow{\scriptstyle |j_n|} & & \downarrow{\scriptstyle |j_{n+1}|} \\ |\mathcal{R}_{2\varepsilon_n}(X)| & \xleftarrow[i_{2\varepsilon_n, 2\varepsilon_{n+1}}]{} & |\mathcal{R}_{2\varepsilon_{n+1}}(X)| \end{array}$$

commutes, up to homotopy. Indeed, every $x \in |\mathcal{R}_{2\varepsilon_n}(A_n)|$ belongs to a simplex $\sigma = \langle a_0, a_1, \ldots, a_s \rangle \in \mathcal{R}_{2\varepsilon_n}(A_n)$. The images of the simplex are

$$\begin{aligned} &\langle a_0, a_1, \ldots, a_s \rangle \xrightarrow{p^*_{n,n+1}} \langle b_0, b_1, \ldots, b_s \rangle \xrightarrow{j_n} \langle b_0, b_1, \ldots, b_s \rangle = \sigma_1, \quad b_i \in q_{A_n}(a_i), \\ &\langle a_0, a_1, \ldots, a_s \rangle \xrightarrow{j_{n+1}} \langle a_0, a_1, \ldots, a_s \rangle \xrightarrow{i^*_{n,n+1}} \langle a_0, a_1, \ldots, a_s \rangle = \sigma_2 \end{aligned}$$

which lies on a common simplex $\sigma_1 \cup \sigma_2 = \langle b_0, b_1, \ldots, b_s, a_0, a_1, \ldots, a_s \rangle$ because for every $i, j \in \{0, 1, \ldots, s\}$ we have

$$\begin{aligned} \mathrm{d}(a_i, b_j) &\leqslant \mathrm{d}(a_i, a_j) + \mathrm{d}(a_j, b_j) < \\ &< 2\varepsilon_{n+1} + \gamma_n < 2\frac{\varepsilon_n - \gamma_n}{2} + \gamma_n = \varepsilon_n \end{aligned}$$

which means $\sigma_1 \cup \sigma_2 \in \mathcal{R}_{2\varepsilon_n}(X)$, so the two maps are homotopic. This morphism of systems is, in fact, an isomorphism. To see this, we need to use Morita's lemma (Theorem 3). Roughly speaking, all we need is a diagonal map making the last diagram factorizing through it. So, we define, for every $n \in \mathbb{N}$, a simplicial map

$$\begin{aligned} g_n : \mathcal{R}_{2\varepsilon_{n+1}}(X) &\longrightarrow \mathcal{R}_{2\varepsilon_n}(A_n) \\ x &\longmapsto a \in q_{A_n}(x), \end{aligned}$$

which is well defined because $a, a' \in q_{A_n}(x)$ implies that

$$\mathrm{d}(a, a') \leqslant \mathrm{d}(a, x) + \mathrm{d}(x, a') < 2\varepsilon_n,$$

and simplicial because if $g_n(\langle x_0, x_1, \ldots, x_s\rangle) = \langle a_0, a_1, \ldots, a_s\rangle$, then, for every $i, j \in \{0, 1, \ldots, s\}$, we have

$$\begin{aligned} \mathrm{d}(a_i, a_j) &\leqslant \mathrm{d}(a_i, x_i) + \mathrm{d}(x_i, x_j) + \mathrm{d}(x_j, a_j) < \\ &< \gamma_n + 2\varepsilon_{n+1} + \gamma_n < 2\varepsilon_n. \end{aligned}$$

The realization of this simplicial map is our diagonal that makes the diagram conmutative up to homotopy:

$$\begin{array}{ccc} |\mathcal{R}_{2\varepsilon_n}(A_n)| & \xleftarrow{p^*_{n,n+1}} & |\mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1})| \\ \downarrow |j_n| & \nwarrow^{g_n} & \downarrow |j_{n+1}| \\ |\mathcal{R}_{2\varepsilon_n}(X)| & \xleftarrow[i_{2\varepsilon_n, 2\varepsilon_{n+1}}]{} & |\mathcal{R}_{2\varepsilon_{n+1}}(X)| \end{array}$$

The up-right subdiagram commutes because if

$$\begin{aligned} &\langle a_0, a_1, \ldots, a_s\rangle \xrightarrow{p^*_{n,n+1}} \langle b_0, b_1, \ldots, b_s\rangle, \quad b_i \in q_{A_n}(a_i), \\ &\langle a_0, a_1, \ldots, a_s\rangle \xrightarrow{j_{n+1}} \langle a_0, a_1, \ldots, a_s\rangle \xrightarrow{g_n} \langle b'_0, b'_1, \ldots, b'_s\rangle, \quad b'_i \in q_{A_n}(a_i), \end{aligned}$$

then $d(b_i, b'_j) < 2\varepsilon_n$, and the two maps are homotopic. Finally, the down-left subdiagram commutes because if we write

$$\langle x_0, x_1, \ldots, x_s\rangle \xrightarrow{g_n} \langle a_0, a_1, \ldots, a_s\rangle \xrightarrow{j_n} \langle a_0, a_1, \ldots, a_s\rangle, \quad a_i \in q_{A_n}(x_i),$$
$$\langle x_0, x_1, \ldots, x_s\rangle \xrightarrow{i_{2\varepsilon_n, 2\varepsilon_{n+1}}} \langle x_0, x_1, \ldots, x_s\rangle,$$

then, $d(x_i, a_j) \leqslant \ldots < \varepsilon_n$ and we are done ✓

So, as conjetured in [3], the inverse sequence $A\mathcal{M}(X)$ represents the shape of X so we can compute all the shape invariants using it. For example, if we apply the singular homology functor to our sequence, we obtain that the inverse limit of the resulting sequence is the Čech homology of X. We will formalize this later. First, let us define more inverse sequences with other kinds of polyhedra.

The Alexandrov-McCord sequence provides a completely constructible process to compute the Čech homology or any other shape invariant of X. But it is done using Vietoris Rips complexes, which are very easy to define, but whose homology is very hard to compute. We want to find different kinds of simplicial complexes and use the main construction to find sequences of these polyhedra which also represent shape properties of our space, with better computational behavior. We have seen that we can use the main construction to define maps between the corresponding Vietoris Rips complexes, giving us $\mathcal{M}^*(X)$. Now, we can adapt this to different kind of complexes.

2.3.2 The Čech approximative sequence

We construct an Hpol-expansion of X with nerves of coverings based on our finite approximations. Let us consider the main construction on the compact metric space X. For every $n \in \mathbb{N}$, consider the set of open balls

$$B_n = \{B_n(a) = B(a, \varepsilon_n) : a \in A_n\}$$

which is a covering of X since A_n is a ε_n approximation. We can consider the nerves of these coverings $\mathcal{N}(B_n)$ and define the maps

$$\begin{array}{rcl} p_{B_n,B_{n+1}} : \mathcal{N}(B_{n+1}) & \longrightarrow & \mathcal{N}(B_n) \\ B_{n+1}(a) & \longmapsto & B_n(b), \;\; b \in q_{A_n}(a). \end{array}$$

This is a simplicial map. Let $\langle \mathrm{B}_{n+1}(a_0), \mathrm{B}_{n+1}(a_1), \dots, \mathrm{B}_{n+1}(a_s)\rangle$ be a simplex of $\mathcal{N}(B_{n+1})$, so $\mathrm{B}_{n+1}(a_0) \cap \mathrm{B}_{n+1}(a_1) \cap \dots \cap \mathrm{B}_{n+1}(a_s) \neq \emptyset$. Let $x \in X$ be a point of this intersection, that means $\mathrm{d}(x, a_i) < \varepsilon_{n+1}$ for all $i = 0, 1, \dots, s$. Let us write the image of this simplex as $\langle \mathrm{B}_n(b_0), \mathrm{B}_n(b_1), \dots, \mathrm{B}_n(b_s)\rangle$, with $b_i \in q_{A_n}(a_i)$ for all $i = 0, 1, \dots, s$. Then, since

$$\begin{array}{rcl} \mathrm{d}(x, b_i) & \leqslant & \mathrm{d}(x, a_i) + \mathrm{d}(a_i, b_i) < \\ & < & \varepsilon_{n+1} + \gamma_n < \dfrac{\varepsilon_n - \gamma_n}{2} + \gamma_n < \dfrac{\varepsilon_n + \gamma_n}{2} < \varepsilon_n, \end{array}$$

we obtain $x \in \mathrm{B}_n(b_0) \cap \mathrm{B}_n(b_1) \cap \dots \cap \mathrm{B}_n(b_s) \neq \emptyset$ therefore $\langle \mathrm{B}_n(b_0), \mathrm{B}_n(b_1), \dots, \mathrm{B}_n(b_s)\rangle$ is a simplex of $\mathcal{N}(B_n)$. The realization of these maps

$$|p_{B_n,B_{n+1}}| : |\mathcal{N}(B_{n+1})| \longrightarrow |\mathcal{N}(B_n)|$$

are up to homotopy well defined maps, since if $b, b' \in q_{A_n}(a)$, with $a \in A_{n+1}$, then $a \in \mathrm{B}_n(b) \cap \mathrm{B}_n(b')$ so the two different images are contiguous, and then homotopic, hence, define an Hmap. So, we obtain the inverse sequence of polyhedra

$$A\check{\mathcal{C}}(X) = \{|\mathcal{N}(B_n)|, |p_{B_n,B_{n+1}}|\}$$

which will be called the *Čech approximative sequence*. As the Alexandrov-McCord sequence, we see that

Proposition 13. *The Čech approximative sequence $A\check{\mathcal{C}}(X)$ is an HPol expansion of X.*

Proof. To prove this, we need to see an isomorphism with another Hpol expansion. We will use the well known Čech expansion. This is the inverse system in HPol,

$$A\check{\mathcal{C}}(X) = \{|\mathcal{N}(U)|, |g_{U,V}|, \Lambda\},$$

where Λ is the set of all open coverings of X, ordered by refinement, and for every pair of coverings $U, V \in \Lambda$, such that V refines U, the Hmaps $|g_{U,V}|$ are the (up to homotopy) realizations of the simplicial maps

$$\begin{aligned} g_{U,V} : \mathcal{N}(V) &\longrightarrow \mathcal{N}(U) \\ V_\alpha &\longmapsto U_\alpha \end{aligned}$$

with $V_\alpha \subset U_\alpha$. To see the isomorphism with our sequence we will find a cofinal sequence of this system more similar to our sequence. First of all, we observe that the set of open coverings of X consisting of $\{\mathrm{B}_n\}_{n\in\mathbb{N}}$ are a cofinal directed subset of Λ. Indeed, if $a \in A_{n+1}$, for every $b \in q_{A_n}(a)$ we have that $\mathrm{B}_{n+1}(a) \subset \mathrm{B}_n(b)$, since for every $c \in \mathrm{B}_{n+1}(a)$,

$$\begin{aligned} \mathrm{d}(b,c) &\leqslant \mathrm{d}(b,a) + \mathrm{d}(a,c) < \\ &< \gamma_n + \varepsilon_{n+1} < \gamma_n \frac{\varepsilon_n - \gamma_n}{2} = \frac{\varepsilon_n + \gamma_n}{2} < \varepsilon_n. \end{aligned}$$

That means B_{n+1} refines B_n for every $n \in \mathbb{N}$. So, we can use this new set of indexes to define the inverse sequence

$$A\check{\mathcal{C}}^*(X) = \{|\mathcal{N}(B_n)|, |g_{B_n,B_{n+1}}|\},$$

which is isomorphic to $A\check{\mathcal{C}}(X)$ and then an Hpol expansion of X. Now, for every $n \in \mathbb{N}$, the maps

$$p_{B_n,B_{n+1}}, g_{B_n,B_{n+1}} : |\mathcal{N}(B_{n+1})| \longrightarrow |\mathcal{N}(B_{n+1})|,$$

are homotopic: If $x \in |\mathcal{N}(B_{n+1})|$ then x is in contained in a unique simplex

$$\sigma = \langle B_{n+1}(a_0), B_{n+1}(a_1), \ldots, B_{n+1}(a_s)\rangle$$

with

$$B_{n+1}(a_0) \cap B_{n+1}(a_1) \cap \ldots \cap B_{n+1}(a_s) \neq \emptyset.$$

Let us write

$$p_{B_n,B_{n+1}}(\sigma) = \langle B_n(b_0), B_n(b_1), \ldots, B_n(b_s)\rangle$$

where, for every $i = 0, \ldots s$, $b_i \subset q_{A_n}(a_i)$ and then $B_{n+1}(a_i) \subset B_n(b_i)$, and

$$g_{B_n,B_{n+1}}(\sigma) = \langle B_n(c_0), B_n(c_1), \ldots, B_n(c_s) \rangle$$

where for every $i = 0, \ldots s$, $B_{n+1}(a_i) \subset B_n(c_i)$. It is clear now that $p_{B_n,B_{n+1}}(\sigma) \cup g_{B_n,B_{n+1}}(\sigma)$ is simplex of $\mathcal{N}(B_{n+1})$ because

$$\emptyset \neq \bigcap_{i=0}^{s} B_{n+1}(a_i) \subset \bigcap_{i=0}^{s} B_n(b_i) \cap \bigcap_{i=0}^{s} B_n(c_i).$$

So the maps $p_{B_n,B_{n+1}}, g_{B_n,B_{n+1}}$ are contiguous and hence homotopic. Then the identity is a morphism between the inverse sequences $A\check{\mathcal{C}}(X)$ and $\check{\mathrm{C}}^*(X)$, so they are isomorphic, and we are done ✓

2.3.3 The witness approximative sequence

The witness complex is a simplicial complex constructed over a finite set of points with nice computational properties. Its simplices are sets of points which are close enough to a point that acts as a witness for them. They do not depend on the 1-skeleton (as the Vietoris Rips complexes) and they do not produce so high dimensional simplexes as Vietoris Rips or Čech complexes. To see the definition and some properties of these complexes, see [17]. Now, we define them in our context. Let us consider the main construction over the compact metric space X. For every $n \in \mathbb{N}$, consider the simplicial complex $\mathcal{W}_n$ whose vertex set are the points of the ε_n-approximation A_n and the simplices are sets of points $\{a_0, a_1, \ldots, a_s\} \subset A_n$ such that every subset $\{a_{i_0}, \ldots, a_{i_r}\}$ satisfies that there exists an $x \in X$, the witness, such that

$$\sum_{j=0}^{r} \mathrm{d}(x, a_{i_j}) < (r+1)\varepsilon_n.$$

It is clear from the definition that this is indeed a simplicial complex. Now, we want to define maps between the witness complexes asociated to different approximations in order to define a sequence of polyhedra based on witness complexes. The idea here is that these maps are defined in a way that they "preserve" the witness for each set of points. Let us define,

$$\begin{aligned} \omega_{n,n+1} : \mathcal{W}_{n+1} &\longrightarrow \mathcal{W}_n \\ a &\longmapsto b \in q_{A_n}(a). \end{aligned}$$

It is a simplicial map: Let us suppose that the simplex $\sigma = \langle a_0, a_1, \ldots, a_s \rangle$ is mapped to $\langle b_0, b_1, \ldots, b_s \rangle$. Consider the subset $\{b_{i_0}, \ldots, b_{i_r}\}$. There exists a witness $x \in X$ for the corresponding subset $\{a_{i_0}, \ldots, a_{i_r}\}$ of the simplex σ and we claim that it is also a witness for its image. So, we estimate the sum

$$\begin{aligned} \sum_{j=0}^{r} \mathrm{d}(x, b_{i_j}) &\leqslant \sum_{j=0}^{r} \left(\mathrm{d}(x, a_{i_j}) + \mathrm{d}(a_{i_j}, b_{i_j}) \right) < (r+1)\varepsilon_{n+1} + \sum_{j=0}^{r} \gamma_n < \\ &< (r+1)\frac{\varepsilon_n - \gamma_n}{2} + (r+1)\gamma_n = (r+1)\frac{\varepsilon_n + \gamma_n}{2} < (r+1)\varepsilon_n \end{aligned}$$

and conclude that the map is simplicial. As in previous cases, the realization of this simplicial map is a well defined map: If $b, b' \in q_{A_n}(a)$ then $\mathrm{d}(b, a) + \mathrm{d}(b', a) < 2\varepsilon_n$ (here a is acting as a witness to prove that $\langle b, b' \rangle$ is a simplex in $\mathcal{W}_n$) so the two possible definitions are contiguous maps so they are in the same homotopic class of maps. We obtain then an inverse sequence of polyhedra called the *witness approximative sequence*:

$$A\mathcal{W}(X) = \{|\mathcal{W}_n|, |\omega_{n,n+1}|\}.$$

As before, we will prove

Proposition 14. *The sequence $A\mathcal{W}(X)$ is an HPol expansion of X.*

Proof. We will see that it is isomorphic to $M^*(X)$. For every $n \in \mathbb{N}$, the identitity map defined on the vertices of the witness complex

$$\begin{aligned} f_n : \mathcal{W}_n &\longrightarrow \mathcal{R}_{2\varepsilon_n}(A_n) \\ a &\longmapsto a, \end{aligned}$$

is a simplicial map. Indeed, if $\sigma = \langle a_0, a_1, \ldots, a_s \rangle$ then $f_n(\sigma) = \sigma$ is a simplex in $\mathcal{R}_{2\varepsilon_n}(A_n)$ since $\operatorname{diam}\{a_0, a_1, \ldots, a_s\} < 2\varepsilon_n$ because for every pair $a_i, a_j \in \sigma$

there exists a point $x \in X$ such that

$$\mathrm{d}(a_i, a_j) \leqslant \mathrm{d}(a_i, x) + \mathrm{d}(x, a_j) < 2\varepsilon_n.$$

The realizations of these maps are a map between the sequences $A\mathcal{W}(X)$ and $M^*(X)$ since the diagram

$$\begin{array}{ccc} |\mathcal{W}_n| & \xleftarrow{|\omega_{n,n+1}|} & |\mathcal{W}_{n+1}| \\ \downarrow{\scriptstyle |f_n|} & & \downarrow{\scriptstyle |f_{n+1}|} \\ |\mathcal{R}_{2\varepsilon_n}(A_n)| & \xleftarrow[|p^*_{n,n+1}|]{} & |\mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1})| \end{array}$$

is commutative up to homotopy. Let x be a point of $|\mathcal{W}_{n+1}|$, then it belongs to a unique simplex $\sigma = \langle a_0, a_1, \dots, a_s \rangle$ of $\mathcal{W}_{n+1}$. Let us write the images of this simplex as

$$f_n \circ \omega_{n,n+1}(\sigma) = \langle b_0, b_1, \dots, b_s \rangle$$

and

$$p^*_{n,n+1} \circ f_{n+1}(\sigma) = \langle c_0, c_1, \dots, c_s \rangle$$

where $b_i, c_i \subset q_{A_n}(a_i)$ for $i = 0, 1, \dots, s$. Then

$$\langle b_0, b_1, \dots, b_s, c_0, c_1, \dots, c_s \rangle$$

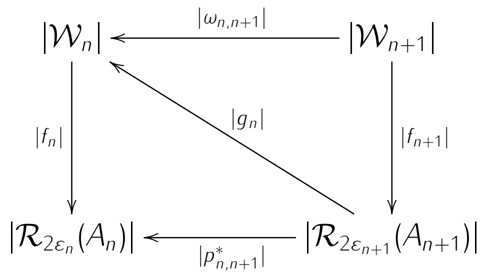
is a simplex of $\mathcal{R}_{2\varepsilon_n}(A_n)$ because it has diameter less than $2\varepsilon_n$. Indeed, for every pair of vertexes b_i, c_j we have that

$$\begin{aligned} \mathrm{d}(b_i, c_j) &\leqslant \mathrm{d}(b_i, a_i) + \mathrm{d}(a_i, a_j) + \mathrm{d}(a_j, b_j) < \\ &< 2\gamma_n + 2\gamma_{n+1} < \varepsilon_n + \gamma_n < 2\varepsilon_n. \end{aligned}$$

So, the two compositions are contiguous, hence its realizations homotopic. Now, in order to apply Morita's lemma, we define the following simplicial maps:

$$\begin{aligned} g_n : \mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1}) &\longrightarrow \mathcal{W}_n \\ a &\longmapsto b \in q_{A_n}(a) \end{aligned}$$

whose realizations make the following diagram commutes for every $n \in \mathbb{N}$:



We have to prove several facts. First of all, the map just defined is simplicial: Let $\sigma = \langle a_0, a_1, \dots, a_s \rangle$ be a simplex of $\mathcal{R}_{2\varepsilon_{n+1}}(A_{n+1})$, and write $g_n(\sigma) = \langle b_0, b_1, \dots, b_s \rangle$. Consider any subset $\{b_{i_0}, \dots, b_{i_r}\}$. In this case, we will not use the witness for the corresponding subset $\{a_0, a_1, \dots, a_s\}$. We will just use $x = a_0$ as witness. So, we evaluate the sum

$$\sum_{j=0}^{r} \mathrm{d}(b_{i_j}, a_0) \leqslant \sum_{j=0}^{r} \left(\mathrm{d}(a_0, a_{i_j}) + \mathrm{d}(a_{i_j}, b_{i_j}) \right) < \sum_{j=0}^{r} (2\varepsilon_{n+1} + \gamma_n) < (r+1)\varepsilon_n$$

and conclude that $g_n(\sigma)$ is a simplex of the witness complex. Again, the realization of this simplicial map is a well defined up to homotopy map, because for two different $b, b' \in q_{A_n}(a)$, $\langle b, b' \rangle$ is a simplex of $\mathcal{W}_n$. It only remains to prove that, in fact, the two triangular diagrams abobe commute up to homotopy. For the upper-right one, consider the simplex $\sigma = \langle a_0, a_1, \dots, a_s \rangle$ of $\mathcal{W}_{n+1}$ and write for its images $\omega_{n,n+1}(\sigma) = \langle b_0, b_1, \dots, b_s \rangle$ and $g_n \circ f_{n+1}(\sigma) = \langle b'_0, b'_1, \dots, b'_s \rangle$. The union of the images, $\langle b_0, b_1, \dots, b_s, b'_0, b'_1, \dots, b'_s \rangle$, is a simplex of $\mathcal{W}_n$. The subset $\{b_{i_0}, \dots, b_{i_{r_1}}, b'_{j_0}, \dots, b'_{j_{r_2}}\}$ has a corresponding one $\{a_{i_0}, \dots, a_{i_{r_1}}, a_{j_0}, \dots, a_{j_{r_2}}\}$ with $x \in X$ as witness, so

$$\sum_{k=0}^{r_1} \mathrm{d}(a_{i_k}, x) + \sum_{k=0}^{r_2} \mathrm{d}(a_{j_k}, x) < (r_1 + r_2 + 2)\varepsilon_n.$$

Then,

$$\begin{aligned}\sum_{k=0}^{r_1} d(b_{i_k}, x) + \sum_{k=0}^{r_1} d(b_{j_k}, x) &\leqslant \sum_{j=0}^{r_1} d(b_{i_k}, a_{i_k}) + d(a_{i_k}, x) + \sum_{j=0}^{r_2} d(b_{j_k}, a_{j_k}) + d(a_{j_k}, x) < \\ &< (r_1 + r_2 + 2)\varepsilon_{n+1} + (r_1 + r_2 + 2)\gamma_n < (r_1 + r_2 + 2)\varepsilon_n.\end{aligned}$$

so $\omega_{n,n+1}$ and $g_n \circ f_{n+1}$ are contiguous maps and their realizations homotopic. To prove the lower-left commutativity we consider a simplex $\sigma = \langle a_0, a_1, \ldots, a_s \rangle$ and observe that the images $f_n \circ g_n(\sigma) = \{b_0, \ldots, b_s\}$ and $p^*_{n,n+1}(\sigma) = \{b'_0, \ldots, b'_s\}$ are in the same simplex (their union). This is so, because for any b_i, b'_j in that union, we have,

$$d(b_i, b'_j) \leqslant d(b_i, a_i) + d(a_i, a_j) + (a_j, b'_j) < 2\gamma_n + 2\varepsilon_{n+1} < 2\varepsilon_n,$$

and that means that the diameter of the union makes it a simplex and then the maps $f_n \circ g_n$ and $p^*_{n,n+1}$ are contiguous ✓

From this proof, it is readily seen that every simplex of the witness complex is indeed a simplex of the Vietoris Rips complex. But the converse is not true, so the witness complex allways will have less simplexes (and simplexes of less or equal dimension) than the Vietoris Rips one. Moreover, with the idea of approximation of compact metric spaces in mind, it makes sense to consider a set of points a simplex only if there is a point close enough to all of them. This make its use better for simplicity and computational purposes.

2.3.4 The Dowker approximative sequence

Let us recall the simplicial complexes defined by Dowker in [22] for a given relation on two sets. Given two sets X and Y and a relation between the two sets R, i.e., a subset of the cartesian product $R \subset X \times Y$, we define two simplicial complexes K^X and L^Y: A finite subset σ of elements of X is a simplex of K^X if there exists an element $y \in Y$ related with every element $x \in \sigma$. On the other hand, a finite subset τ of elements of Y is a simplex of L^Y if there exists an element $x \in X$ related with every element of $y \in \sigma$. Note that there is a kind of duality in these definitions. It is readily seen that K^X and K^Y are simplicial complexes. In that paper it is shown that this two complexes have the same homology. Moreover it is proven the following

Theorem 13 (Dowker)**.** *The realizations of the simplicial complexes $|K^X|$ and $|L^Y|$ have the same homotopy type.*

In [22] this is used to prove that the Čech and Vietoris homology for general topological spaces are isomorphic. Moreover, in shape theory, it is used to show that the standard Čech and Vietoris systems for any topological space are isomorphic. The power of Dowker Theorem lies in the generality of its formulation. We only need two sets and a relation and we obtain two homotopical simplicial complexes. We can reformulate it in the context of Alexandrov spaces as follows. Consider an Alexandrov T_0 space given by the poset $(X, \leqslant)$. Let us consider the relation $R \subset X \times X$ given by $xRy \Leftrightarrow x \leqslant y$. Then, we have the following simplicial complexes:

$$\begin{aligned} \sigma = \langle x_0, \ldots, x_r \rangle \in K^X &\iff \exists y \in X : x_0, \ldots, x_r \leqslant y \\ \tau = \langle y_0, \ldots, y_s \rangle \in L^X &\iff \exists x \in X : x \leqslant y_0, \ldots, y_s \end{aligned}$$

Recall that, for every Alexandrov T_0 space, we can construct the McCord complex, which, with the same notation, we can write as

$$\rho = \langle z_0, \ldots, z_p \rangle \in \mathcal{K}(X) \iff z_0 \leqslant \ldots \leqslant z_p.$$

Remark 9. As simplicial complexes, $\mathcal{K}(X) \subset K^X, L^X$.

Now, we adapt this to our special context. Let X be a compact metric space and suppose the main construction done. Consider, for every $n \in \mathbb{N}$, the finite T_0 spaces $X_n = U_{2\varepsilon_n}(A_n)$ and the relation defined above with the order given by the upper semifinite topology, i.e.,

$$CRD \iff C \subset D.$$

So, the simplicial complexes K^{X_n}, L^{X_n} are

$$\begin{aligned} \sigma = \langle C_0, \ldots, C_r \rangle \in K^{X_n} &\iff \exists D \in X_n : C_0, \ldots, C_r \subset D \\ \tau = \langle D_0, \ldots, D_s \rangle \in L^{X_n} &\iff \exists C \in X_n : C \subset D_0, \ldots, D_s. \end{aligned}$$

With the same notation, the McCord complex $\mathcal{K}(X)$ is defined by

$$\rho = \langle F_0, \ldots, F_p \rangle \in \mathcal{K}(X) \Longleftrightarrow F_0 \subset F_1 \subset \ldots \subset F_p.$$

We would like to define inverse sequences based on this simplicial complexes. To do so, we observe that the continuous maps $p_{n,n+1} : X_{n+1} \to X_n$ can be used to define simplicial maps:

$$\begin{array}{rcl} p^K_{n,n+1} : K^{X_{n+1}} & \longrightarrow & K^{X_n} \\ C & \longmapsto & p_{n,n+1}(C), \end{array} \qquad \begin{array}{rcl} p^L_{n,n+1} : L^{X_{n+1}} & \longrightarrow & L^X_n \\ C & \longmapsto & p_{n,n+1}(C). \end{array}$$

These maps are simplicial since

$$\begin{array}{rcl} C_0, \ldots, C_r \subset D & \Longrightarrow & p_{n,n+1}(C_0), \ldots, p_{n,n+1}(C_r) \subset p_{n,n+1}(D), \\ D \subset C_0, \ldots, C_s & \Longrightarrow & p_{n,n+1}(D) \subset p_{n,n+1}(C_0), \ldots, p_{n,n+1}(C_s). \end{array}$$

This allows us to define the inverse sequences of polyhedra:

$$A\mathcal{D}_u(X) = \{|K^{X_n}|, |p^K_{n,n+1}|\}, \qquad A\mathcal{D}_l(X) = \{|L^{X_n}|, |p^L_{n,n+1}|\},$$

called respectively the *upper and lower Dowker approximative sequences*.

Proposition 15. *The upper and lower approximative Dowker sequences are Hpol expansions of X.*

Proof. We will just prove that K^X is an HPol expansion. The proof for L^X is completely dual (in the sense that the proof is exactly the same but using de dual property that defines the simplices for the complexes in this case). In order to prove this, we will see that K^X is isomorphic, as HPol sequence, to the approximative McCord sequence. First of all, we see that, for every $n \in \mathbb{N}$, the inclusion map i_n of $\mathcal{K}(X_n)$ in K^{X_n} give us a morphism between the corresponding sequences. Indeed, for every $n \in \mathbb{N}$, it is easy to see that the following diagram commutes[5]

$$\begin{array}{ccc}
|\mathcal{K}(X_n)| & \xleftarrow{|p_{n,n+1}|} & |\mathcal{K}(X_{n+1})| \\
\Big\downarrow{\scriptstyle |i_n|} & & \Big\downarrow{\scriptstyle |i_{n+1}|} \\
|K^{X_n}| & \xleftarrow[|p^K_{n,n+1}|]{} & |K^{X_{n+1}}|
\end{array}$$

In order to apply Morita's lemma, we would like to define a (diagonal) map from $K^{X_{n+1}}$ to $\mathcal{K}(X_n)$ but it seems there is no (appropiate) simplicial map between these spaces. Alternatively, we can define a simplicial map from the barycentric subdivision,

$$\begin{aligned} g_n : (K^{X_{n+1}})' &\longrightarrow \mathcal{K}(X_{n+1}) \\ \{C_0, \dots, C_s\} &\longmapsto \bigcup_{i=0}^{s} p_{n,n+1}(C_i), \end{aligned}$$

where there exists $C \in X_n$, such that, $\{C_0, \dots, C_s\} \subset C$, so we have that

$$\bigcup_{i=0}^{s} p_{n,n+1}(C_i) \subset p_{n,n+1}(C),$$

hence the map is well defined. To see that it is simplicial, let us write some notation. Let $C^j = \{C_0^j, C_1^j, \dots, C_j^{s_j}\}$ be a set of points $C_i^j \in X_{n+1}$, we can write a simplex of $(K^{X_{n+1}})'$ as

$$\langle C^0, C^0 \cup C^1, \dots, C^0 \cup C^1 \cup C^r \rangle.$$

The image of this simplex by g_n is

$$\left\{ \bigcup_{i=0}^{s_0} p_{n,n+1}(C_i^0), \bigcup_{j=0}^{1}\bigcup_{i=0}^{s_j} p_{n,n+1}(C_i^j), \dots, \bigcup_{j=0}^{r}\bigcup_{i=0}^{s_j} p_{n,n+1}(C_i^0) \right\},$$

[5]Note that we just need them to commute up to homotopy but actually both compositions are exactly the same map.

which is a simplex of $\mathcal{K}(X_n)$ since, for every $k = 0, \dots, r-1$,

$$\bigcup_{j=0}^{k}\bigcup_{i=0}^{s_j} p_{n,n+1}(C_i^j) \subset \bigcup_{j=0}^{k+1}\bigcup_{i=0}^{s_j} p_{n,n+1}(C_i^j).$$

Now we are going to prove that the realization of this map satisfies the Morita's lemma making the following diagram commutative:

$$\begin{array}{ccc} |\mathcal{K}(X_n)| & \xleftarrow{|p_{n,n+1}|} & |\mathcal{K}(X_{n+1})| \\ {\scriptstyle |i_n|}\downarrow & \nwarrow^{|g_n|} & \downarrow{\scriptstyle |i_{n+1}|} \\ |K^{X_n}| & \xleftarrow[|p^K_{n,n+1}|]{} & |K^{X_{n+1}}| \end{array}$$

In order to prove this we need to use the barycentric subdivision of our simplicial complexes. For every simplicial complex K there is a simplicial map from its barycentric subdivision

$$\begin{aligned} \rho : K' &\longrightarrow K \\ \langle x_0, \dots, x_s \rangle &\longmapsto x_s \end{aligned}$$

whose realization is homotopic to the identity. Moreover, every simplicial map $f : K \to L$ induces a simplicial map between its barycentric subdivisions $f' : K' \to L'$. See [48] for details. Using this, we define the following maps:

$$(i_{n,n+1})' : \mathcal{K}'(X_{n+1}) \longrightarrow (K^{X_{n+1}})'$$

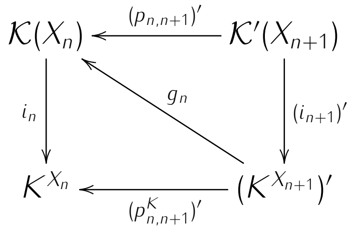
is the simplicial map induced in the barycentric subdivisions by $i_{n,n+1}$.

$$(p_{n,n+1})' : \mathcal{K}'(X_{n+1}) \longrightarrow \mathcal{K}(X_n)$$

is the composition $p_{n,n+1} \circ \rho$. And finally,

$$(p^K_{n,n+1})' : (K^{X_{n+1}})' \longrightarrow K^{X_n}$$

is the composition $p^K_{n,n+1} \circ \rho$. Now, we just need to show that the following diagrams of simplicial maps are contiguous:



The upper right diagram is not only contiguous but commutative. With the notation on simplexes above, if $\sigma = \langle C^0, C^0 \cup C^1, \ldots, C^0 \cup \ldots \cup C^r \rangle \in \mathcal{K}(X_{n+1})$ with $C_i^j \subset C_{i+1}^j$ for every $j = 0, \ldots, r$ and $i = 0, \ldots, s_j - 1$, is a simplex of $\mathcal{K}'(X_{n+1})$, then

$$i_n \circ p'_{n,n+1}(\sigma) = \langle p_{n,n+1}(C^0_{s_0}), p_{n,n+1}(C^1_{s_1}), \ldots, p_{n,n+1}(C^r_{s_r} \rangle = (p^k_{n,n+1})'(\sigma).$$

The lower left diagram is contiguous because if $\tau = \langle C^0, C^0 \cup C^1, \ldots, C^0 \cup \ldots \cup C^r \rangle$ is a simplex of $(K^{X_{n+1}})'$, then

$$i_n \circ g_n(\tau) = \langle p_{n,n+1}(C^0_{s_0}), p_{n,n+1}(C^0_{s_0} \cup C^1_{s_1}), \ldots, p_{n,n+1}(C^0_{s_0} \cup \ldots \cup C^r_{s_r}) \rangle,$$

and

$$(p^K_{n,n+1})'(\tau) = \langle p_{n,n+1}(C^0_{s_0}), p_{n,n+1}(C^1_{s_1}), \ldots, p_{n,n+1}(C^r_{s_r}) \rangle.$$

The union of both images is a simplex of K^{X_n} since every vertex is contained in $p_{n,n+1}(C^0_{s_0} \cup \ldots \cup C^r_{s_r})$. Thus the maps $i_n \circ g_n$ and $(p^K_{n,n+1})'$ are contiguous hence its realizations homotopic, and we are done ✓

All these inverse approximative sequences share the property that they are defined in terms of a sequence of adjusted finite approximations obtained by the main construction. Given a compact metric space (X, d) , we will say that an inverse sequence of polyhedra $\{K_n, p_{n,n+1}\}$ is an *polyhedral approximative sequence* of X, if it is any of the inverse sequences defined on this section. Now, we formalize the idea that these inverse sequences

Corollary 3. *Let (X, d) be a compact metric space and $K = \{K_n, p_{n,n+1}\}$ a polyhedral approximative sequence of X. Consider the induced inverse sequence of groups $F(K) = \{F(K_n), F(p_{n,n+1})\}$, where F is the singular n-th homology* *or homotopy functor. Then, the inverse limit of $F(K)$ is the n-th Čech homology or shape group, respectively.*

2.4 Persistent errors

Given an inverse sequence of polyhedra, we want to measure the "error" of each term respect the inverse limit, at least in terms of homology. The idea is to infer some information about the Čech homology of some compact metric space using finite cuts of the inverse sequence that defines it (in terms of inverse limit), in the same way that Taylor polynomials approximate non linear functions up to some error. Here, polyhedra will play the role of polynomials that approximate any compact metric space.

Let us consider any inverse sequence of polyhedra,

$$K_0 \xleftarrow{p_{0,1}} K_1 \xleftarrow{p_{1,2}} \dots \xleftarrow{p_{n-1,n}} K_n \xleftarrow{p_{n,n+1}} \dots$$

with inverse limit $X = \varprojlim\{K_i, p_{i,i+1}\}$. We write the n-proyection of the limit as $p_n : K \to K_n$. Let us take, in each polyhedron K_i the p-th homology group $H_p(K_i)$ and the induced maps $p^*_{i,i+1}$. We will write the groups as $H_i := H_p(K_i)$ and the maps as $q_{i,i+1} := p^*_{i,i+1}$ and $q_{i,j}$ with $i < j$ by composition. We obtain therefore an inverse sequence of finitely generated abelian groups

$$H_0 \xleftarrow{q_{0,1}} H_1 \xleftarrow{q_{1,2}} \dots \xleftarrow{q_{n-1,n}} H_n \xleftarrow{q_{n,n+1}} \dots$$

whose inverse limit is the p-th Čech homology group of X, written $H = \check{H}_p(X)$.

In these conditions, fix $p \geqslant 0$ (which will be omited from notation from now on) and $n \in \mathbb{N}$, and define, for every $n < m$, the $(n,m) - th$ *group of persistent homology* as

$$H_{n,m} := \text{im}(q_{n,m}) = q_{n,m}(H_m).$$

The inclusion allows us to obtain an inverse sequence of persistent homology groups:

$$H_{0,1} \xleftarrow{i} H_{1,2} \xleftarrow{i} \dots \xleftarrow{i} H_{n,n+1} \xleftarrow{i} \dots$$

As an inverse sequence defined by inclusion maps, the inverse limit is the intersection of all of them,

$$\bigcap_{m=n+1}^{\infty} H_{n,m} = q_n(H).$$

The persistent group $H_{n,m}$ is a normal subgroup of H_n for every $n < m$. So, we can take the quotient of groups $E_{n,m} := \frac{H_n}{H_{n,m}}$, that we will call the (n,m)-th persistent error. The idea of this group is that it measures the validity of H_n seen from the H_m perspective. Moreover, as $H_{n,m+1}$ is also a normal subgroup of $H_{n,m}$,we therefore obtain a natural homomorphism between the quotients

$$\begin{array}{rccc} g_{m,m+1}: & E_{n,m+1} & \longrightarrow & E_{n,m} \\ & h + H_{n,m+1} & \longmapsto & h + H_{n,m}. \end{array}$$

(Or, using a different notation, $[h]_{H_{n,m+1}} \mapsto [h]_{H_{n,m}}$). By composition, we obtain an inverse sequence of errors

$$E_{n,n+1} \xleftarrow{g_{n+1,n+2}} E_{n,n+2} \xleftarrow{g_{n+2,n+3}} \ldots \xleftarrow{g_{n+m-1,n+m}} E_{n+m,n+m+1} \xleftarrow{g_{n+m+1,n+m+2}} \ldots,$$

with an inverse limit, denoted by E_n^i and called the *inductive n-error*. In an ideal of understanding this error in the infinity we define the *n-th real error*, $E_n := \frac{H_n}{q_n(H)}$. In general, these two errors in the limit we have just defined, E_n^i and E_n, are different, but we can do some comparations.

Proposition 16. *There is an injective homomorphism of groups* $\varphi : E_n \to E_n^i$.

Proof. We define the map with

$$h + q_n(H) \longmapsto (h + H_{n,m}, h + H_{n,m+1}, \ldots).$$

It is well defined, because if $h - h' \in q_n(H) = \bigcap H_{n,m}$ then $h - h'$ represents the null class in every group $E_{n,m}$. It is injective because if

$$(h_1 + H_{n,m}, h_1 + H_{n,m+1}, \ldots) = (h_2 + H_{n,m}, h_2 + H_{n,m+1}, \ldots)$$

then $h_1 - h_2 \in H_{n,m}$, $h_1 - h_2 \in H_{n,m+1}, \ldots$, so $h_1 - h_2 \in \bigcap H_{n,m}$ and $h_1 + q_n(H) = h_2 + q_n(H)$ ✓

Moreover, for some kind of spaces, this two errors are actually the same group.

Proposition 17. *Consider an inverse sequence of polyhedra*

$$K_0 \xleftarrow{p_{0,1}} K_1 \xleftarrow{p_{1,2}} \dots \xleftarrow{p_{n-1,n}} K_n \xleftarrow{p_{n,n+1}} \dots$$

with inverse limit X. If X is movable (see section 1.3.2) then, for every $n \in \mathbb{N}$ we have $E_n^i = E_n$.

Proof. If X is movable, then, the inverse sequence that defines it as inverse limit is also movable, so it is its induced homology sequence. So, the last has the Mittag-Leffer property, that is: For every $n \in \mathbb{N}$ there exists $m \geqslant n$ such that for every $r \geqslant m$ we have that $q_{n,r}(H_r) = q_{n,m}(H_m)$, i.e., expressed with persistent homology groups, $H_{n,r} = H_{n,m}$. So, from m, all the persistent groups are the same, so $q_n(H) = \bigcap H_{n,m} = H_{n,m}$. So, the inverse sequence of errors, from m is constant and equal to $E_{n,m} = E_n = E_n^i$ ✓

Sometimes we can define a third kind of error. If there exists a homomorphism of groups $f : H_n \to H_n$ such that $f(H_{n,m}) \subset H_{n,m+1}$, we can define the map

$$\begin{array}{rlcl} l_{m,m+1}: & E_{n,m} & \longrightarrow & E_{n,m+1} \\ & h + H_{n,m} & \longmapsto & f(h) + H_{n,m+1}. \end{array}$$

This map is well defined because if $h - h' \in H_{n,m}$, we get $f(h) - f(h') = f(h - h') \in H_{n,m+1}$. By composition, we can form the direct sequence

$$E_{n,n+1} \xrightarrow{l_{n+1,n+2}} E_{n,n+2} \xrightarrow{l_{n+2,n+3}} \dots \xrightarrow{l_{n+m-1,n+m}} E_{n+m,n+m+1} \xrightarrow{l_{n+m+1,n+m+2}} \dots,$$

and then, a direct limit, denoted by E_n^d. But not every inverse sequence satisfies this property.

Remark 10. For inverse sequences of type

$$X \xleftarrow{p} X \xleftarrow{p} \dots \xleftarrow{p} X \xleftarrow{p} \dots,$$

there exists the limit E_n^d. In this particular case, the map $p^* : H_n \to H_n$ induced in the homology groups plays the role of f, because

$$p^*(H_{n,m}) = p^*p^*_{n,m}(H_m) = p^*p^*_{n+1,m+1}(H_{m+1}) = p^*_{n,m+1}(H_{m+1}) = H_{n,m+1}.$$

Example 2. The dyadic solenoid. This space can be defined as the inverse limit of the inverse sequence

$$\mathbb{S}^1 \overset{2}{\leftarrow} \mathbb{S}^1 \overset{2}{\leftarrow} \dots$$

where, considering $\mathbb{S}^1$ as the complex unit circle, the map "2" means the exponential map e^{2iz}. The induced homology sequence of order 1 is

$$\mathbb{Z} \overset{2}{\leftarrow} \mathbb{Z} \overset{2}{\leftarrow} \dots$$

with inverse limit $\{0\}$. That means that the Čech homology of the dyadic solenoid is trivial. If we consider here our errors, we obtain that for every $n < m$, the persistent homology groups are $H_{n,m} = (2^{m-n})\mathbb{Z}$ (integer multiples of 2^{m-n}). If we consider the quotients to obtain the (n, m)-errors, we obtain the finite groups $E_{n,m} = \frac{\mathbb{Z}}{(2^{m-n})\mathbb{Z}} = \mathbb{Z}_{m-n}$. The natural map between these errors sends each element to its class in the image group. So, for example, the first map will be

$$\begin{array}{ccc} \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_2 \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 1 \\ 2 & \longmapsto & 0 \\ 3 & \longmapsto & 1 \end{array}$$

With these maps, we can define the inverse sequence

$$\mathbb{Z}_2 \longleftarrow \mathbb{Z}_4 \longleftarrow \mathbb{Z}_8 \longleftarrow \dots \longleftarrow \mathbb{Z}_{2^n} \longleftarrow \dots$$

which inverse limit $E^i_n = \mathcal{D}_2$ is the dyadic integers group. In this case, we can also obtain the direct sequence construction. Here, the maps will send each element to the class of this element multiplyed by two. Then, the first map will be

$$\begin{array}{ccc} \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_4 \\ 0 & \longmapsto & 0 \\ 1 & \longmapsto & 2 \end{array}$$

So, the direct sequence has

$$\mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_8 \longrightarrow \ldots \longrightarrow \mathbb{Z}_n \longrightarrow \ldots$$

has direct limit the Prüfer 2-group $E_n^d = \mathbb{Z}(2^\infty)$ (the set of roots of the unity of some power of two). It turns out that the Prüfer group (with the discrete topology) is the Pontryagin dual of the compact group of the dyadic integers.

Question 1. For what class of spaces can we obtain this duality of the errors?

Example 3. The computational Warsaw circle (see next section).

2.5 Example: The computational Warsaw circle

In this section we will perform the main construction on the Warsaw circle in order to apply the theory previously developed. The Warsaw circle is the paradigmatic example of shape theory. It is can be defined as the image in $\mathbb{R}^2$ of the map $f(x) = \sin\left(\frac{1}{x}\right)$ between 0 and $\frac{2}{\pi}$, together with its closure (with the topology as subespace of $\mathbb{R}^2$), that is, the segment joining $(0,-1)$ and $(0,1)$, and any simple arc (meaning not intersecting itself or the rest of the space) joining the point $(0,-1)$ with $(\frac{1}{\pi},-1)$. See figure 2.1. For computational purposes, we are going to define and work with the following homeomorphic copy of the Warsaw circle. Consider, in $\mathbb{R}^2$, the following segments[6]:

$$\begin{aligned}
a_n &= \left(\frac{1}{2^{2n-2}}, 1\right) - \left(\frac{1}{2^{2n-1}}, 1\right), \\
b_{n1} &= \left(\frac{1}{2^{2n-2}}, \frac{1}{2}\right) - \left(\frac{1}{2^{2n-2}}, 1\right), \\
b_{n2} &= \left(\frac{1}{2^{2n-1}}, 1\right) - \left(\frac{1}{2^{2n-1}}, \frac{1}{2}\right), \\
c_n &= \left(\frac{1}{2^{2n-1}}, \frac{1}{2}\right) - \left(\frac{1}{2^{2n}}, \frac{1}{2}\right).
\end{aligned}$$

[6]The notation for the segments is $(a,b)-(c,d)$, meaning the segment joining these two points.

Then, the computational Warsaw circle is

$$\mathcal{W} = (1,0) - (0,0) - (1,0) - (1,1) \bigcup_{n\in\mathbb{N}} a_n \bigcup_{n\in\mathbb{N}\setminus\{1\}} b_{n1} \bigcup_{n\in\mathbb{N}} b_{n2} \bigcup_{n\in\mathbb{N}} c_n.$$

Despite of this complex definition, this subespace of $\mathbb{R}^2$ is very easy and intuitive to understand: See figure 2.1. We can think about it as a one piece drawing: Starting from the point $(0,1)$, go one unit south, one unit east and one unit north. And now, approximate to the segment $(0,1)-(0,\frac{1}{2})$, alternating from hight 1 and $\frac{1}{2}$, and reducing the approximation to the half of the previous one. I.e., from the point we were, go half unit west, half unit south, quarter unit west, half unit north, $\frac{1}{8}$ unit west, half unit south, $\frac{1}{16}$ unit west, half unit north,...,half unit south, $\frac{1}{2^{2n}}$ unit west, half unit north, $\frac{1}{2^{2n+1}}$ unit west, half unit south,... and so on.

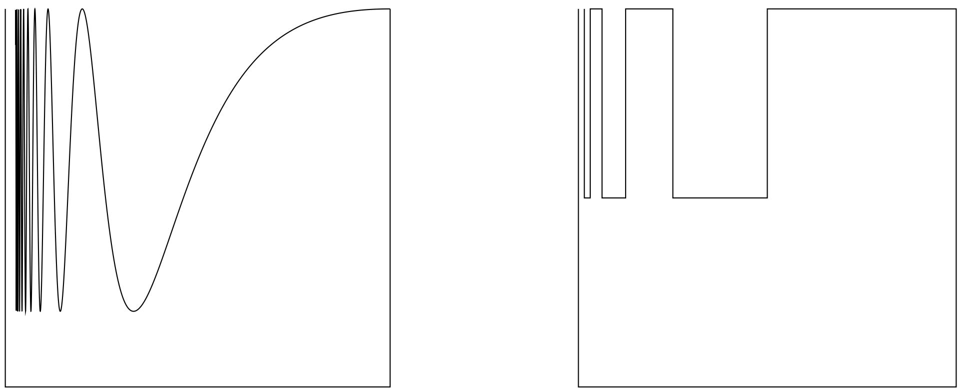


Figure 2.1: The Warsaw circle and the computational Warsaw circle.

We now perform the general construction on $\mathcal{W}$. The diameter of $\mathcal{W}$ is $M=\sqrt{2}$. Then, we can select $\varepsilon_1 = 2\sqrt{2} > M$, and $A_1 = \{(0,0)\}$, so $\gamma_1 = \sqrt{2}$. In the second step, we take $\varepsilon_2 = \frac{\sqrt{2}}{2^3} < \min\left\{\frac{\varepsilon_1-\gamma_1}{2}, \frac{M}{2}\right\} = \frac{\sqrt{2}}{2}$. To get an ε_2 approximation of $\mathcal{W}$, we explain the process better than giving just the coordinates of the points. Consider the intersection of a grid of side $\frac{1}{2^{3-1}}$, $G_2 = \left\{(\frac{l}{2^{3-1}}, \frac{m}{2^{3-1}}) \in \mathbb{R}^2 : l,m \in \mathbb{Z}\right\}$ with $\mathcal{W}$. See figure 2.2. Every point of $\mathcal{W}$, not in the upper left square of the grid, and the one just below it, are at distance less or equal to $\frac{1}{2^3} < \varepsilon_2$. Concerning the two mentioned squares, we see that every point of $\mathcal{W}$ inside them are at distance less than ε_2, except the two centers of the squares, which are exactly at this distance. So, we add these two points

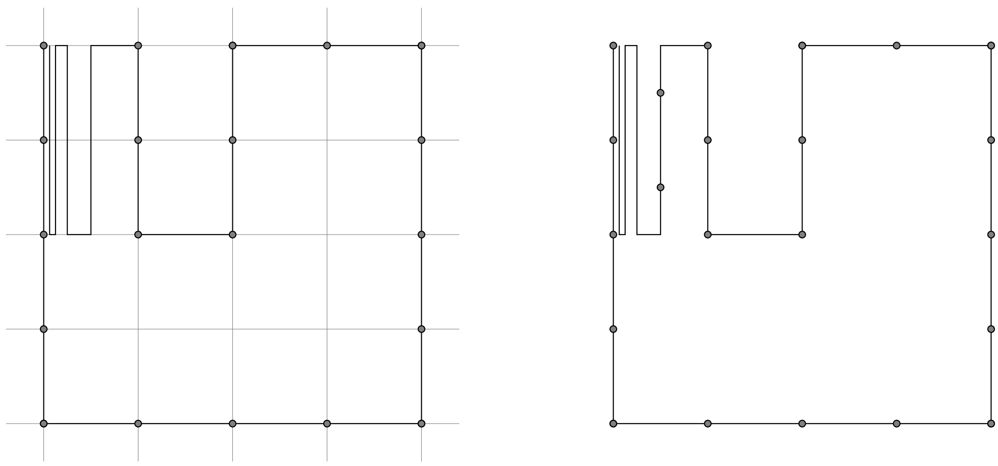


Figure 2.2: The intersection of the grid G_2 with $\mathcal{W}$ and the ε_2 approximation of $\mathcal{W}$.

and then have an ε_2 approximation of $\mathcal{W}$,

$$A_2 = (G_2 \cap \mathcal{W}) \cup \left\{ \left(\frac{1}{2^3}, 1 - \frac{1}{2^3} \right), \left(\frac{1}{2^3}, 1 - \frac{3}{2^3} \right) \right\}.$$

From the picture, we can easily see that $\gamma_2 = \frac{1}{2^3}$, and that we can select $\delta_2 = \frac{\sqrt{2}}{2^3}$. Then, we pick[7] $\varepsilon_3 = \frac{\sqrt{2}}{2^6} < \min\left\{\frac{\varepsilon_2 - \gamma_2}{2}, \frac{\delta_2}{2}\right\} = \frac{\sqrt{2}-1}{2^4}$. To obtain an ε_3 approximation of $\mathcal{W}$, we proceed as before. Consider the grid of side $\frac{1}{2^{6-1}}$, $G_3 = \left\{ (\frac{l}{2^{6-1}}, \frac{m}{2^{6-1}}) \in \mathbb{R}^2 : l, m \in \mathbb{Z} \right\}$ and its intersection with $\mathcal{W}$. Then add the centers of the upper left square of the grid, and the $15 = 2^4 - 1$ below it (16 points of $\mathcal{W}$ in total), to obtain an ε_3 approximation of $\mathcal{W}$ (see figure 2.3),

$$A_3 = (G_3 \cap \mathcal{W}) \cup \left\{ \left(\frac{1}{2^6}, 1 - \frac{1}{2^6} \right), \left(\frac{1}{2^6}, 1 - \frac{3}{2^6} \right), \dots, \left(\frac{1}{2^6}, 1 - \frac{31}{2^6} \right) \right\}.$$

Now, it is again clear from the picture, that $\gamma_3 = \frac{1}{2^6}$ and $\delta_3 = \frac{\sqrt{2}}{2^6}$. We can continue this process to the infinity in the same way. In general, let $\varepsilon_n = \frac{\sqrt{2}}{2^{3n-3}}$. Consider the grid of side $\frac{1}{2^{3n-4}}$, $G_n = \left\{ \left(\frac{l}{2^{3n-4}}, \frac{m}{2^{3n-4}} \right) \in \mathbb{R}^2 : l, m \in \mathbb{Z} \right\}$. Then, its

[7] We want some regularity on the epsilon approximations. All of them will be of the form $\frac{\sqrt{2}}{2^k}$. In this case, there is no k lower than 6 the inequality. This will be proven for the general case, later.

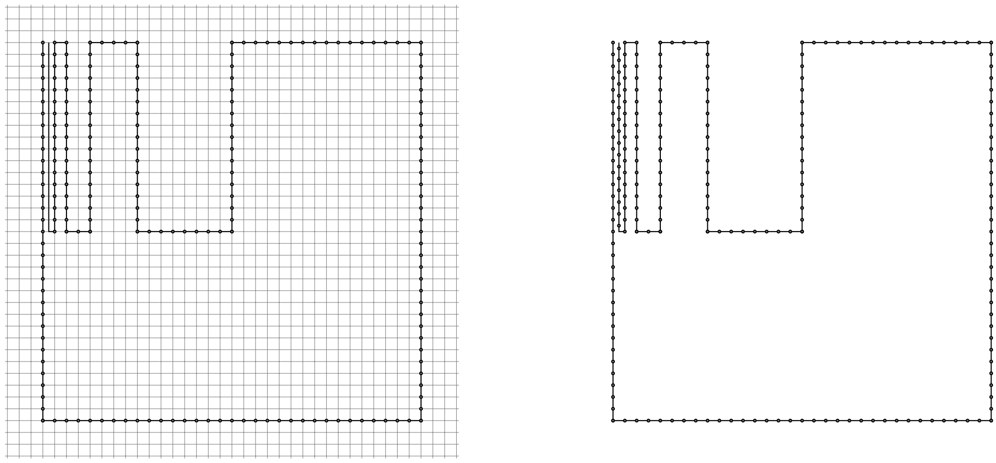


Figure 2.3: The intersection of the grid G_3 with $\mathcal{W}$ and the ε_3 approximation of $\mathcal{W}$.

intersection with $\mathcal{W}$ and the following 2^{3n-5} points, form an ε_n approximation,

$$A_n = (G_n \cap \mathcal{W}) \cup \left\{ \left(\frac{1}{2^{3n-3}}, 1 - \frac{2k-1}{2^{3n-3}} \right) : k = 1, \dots, 2^{3n-5} \right\}.$$

It is clear that, again, $\gamma_n = \frac{1}{2^{3n-3}}$ and $\delta_n = \frac{\sqrt{2}}{2^{3n-3}}$. So, writing[8] $m = 3n-3$, we need

$$\varepsilon_{n+1} < \min\left\{ \frac{\varepsilon_n - \gamma_n}{2}, \frac{\delta_n}{2} \right\} = \frac{\sqrt{2}-1}{2^{m+1}}.$$

We want ε_{n+1} to be of the form $\frac{\sqrt{2}}{2^k}$, so we are looking for $k \in \mathbb{N}$, such that, $\frac{\sqrt{2}}{2^k} < \frac{\sqrt{2}-1}{2^{m+1}}$, i.e., $2^{k-(m+1)} > 2 + \sqrt{2}$. We can estimate $2^{k-(m+1)} > 2 + \sqrt{2} > 2$ so $k > m+2$. But, actually, $k = m+2$ does not satisfy the first inequality, so we can take any $k \geqslant m+3$, and hence, we choose $\varepsilon_{n+1} = \frac{\sqrt{2}}{2^{m+3}} = \frac{\sqrt{2}}{2^{3n}} = \frac{\sqrt{2}}{2^{3(n+1)-3}}$. It is clear, that we can consider an ε_{n+1} approximation as before, intersecting the grid of side $\frac{1}{2^{3(n+1)-4}} = \frac{1}{2^{3n-1}}$, $G_{n+1} = \left\{ \left(\frac{l}{2^{3n-1}}, \frac{m}{2^{3n-1}} \right) \in \mathbb{R}^2 : l, m \in \mathbb{Z} \right\}$ with $\mathcal{W}$ and add $2^{3(n+1)-5} = 2^{3n-2}$ points:

$$A_{n+1} = (G_{n+1} \cap \mathcal{W}) \cup \left\{ \left(\frac{1}{2^{3n}}, 1 - \frac{2k-1}{2^{3n}} \right) : k = 1, \dots, 2^{3n-2} \right\}.$$

[8]The term $3n-3$ relates the exponent of the denominator with the subindex of each ε. We use the m notatin for a moment to understand how the denominator is increased in each step without perturbations of another notations.

Then, $\gamma_{n+1} = \frac{1}{2^{3(n+1)-3}} = \frac{1}{2^{3n}}$, $\delta_{n+1} = \frac{\sqrt{2}}{2^{3(n+1)-3}} = \frac{\sqrt{2}}{2^{3n}}$ and the process is proved to work by induction.

Now, we focus on the Alexandrov-McCord sequence related to this finite approximative sequence. The finite space A_1 is just a point, so its associated simplicial complex is just a vertex. In the second step, we have a more interesting case. In figure 2.4, we have depicted the polyhedron $\mathcal{R}_{2\varepsilon_2}(A_2)$ in two different

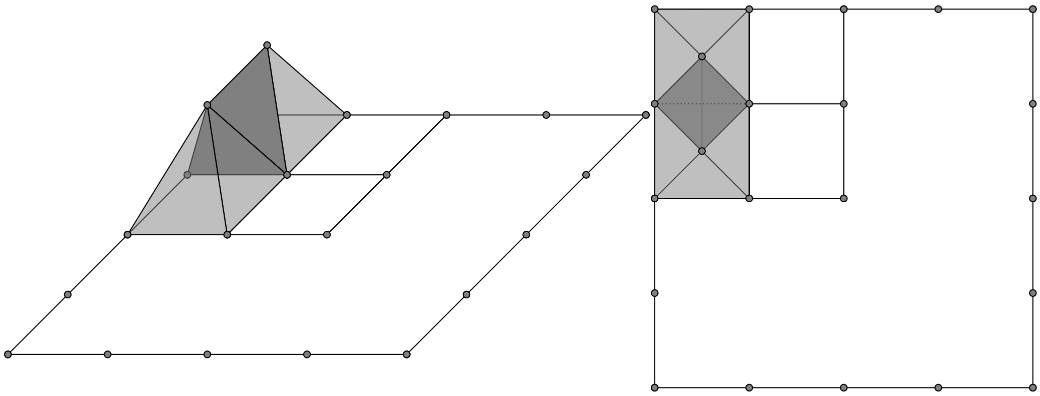


Figure 2.4: The realization of the simplicial complex $\mathcal{R}_{2\varepsilon_2}(A_2)$ in two perspectives: Lateral and Aerial.

perspectives. The barycentric subdivision of this polyhedron is exactly the realization of the simplical complex $\mathcal{K}(U_{2\varepsilon_2}(A_2)) = \mathcal{R}'_{2\varepsilon_2}(A_2)$. Actually, the vertices that are not depicted but belong to the subdivision are the points of the space $U_{2\varepsilon_2} \backslash A_2$. The 1-simplices of this polyhedron are clear from the picture. But there is more structure. First of all there are two empty squares. At their left, there are two piramids whose cusps represent the points added to the intersection of the grid and $\mathcal{W}$. Between the two piramids, there is a tetrahedron sharing one face with each one of them. The four points of the tetrahedron are the two points added and the two points in common of the two squares (the base of each piramid), which, in the approximation, have diameter less than $2\varepsilon_2$, so this tetrahedron is "filled". We have to point out that the piramids are empty, that is, their four faces are simplices that are in the polyhedron, but there is no "solid" base. For the third step, we also depicted the polyhedron $\mathcal{R}_{2\varepsilon_3}(A_3)$ (figure 2.5), whose barycentric subdivision is $\mathcal{K}(U_{2\varepsilon_3}(A_3)) = \mathcal{R}'_{2\varepsilon_3}(A_3)$. The structure of this polyhedron is the same as the previous one. The diference is that it has more 1-simplices, more empty squares (2^4) more piramids (2^4) and more tetrahedrons

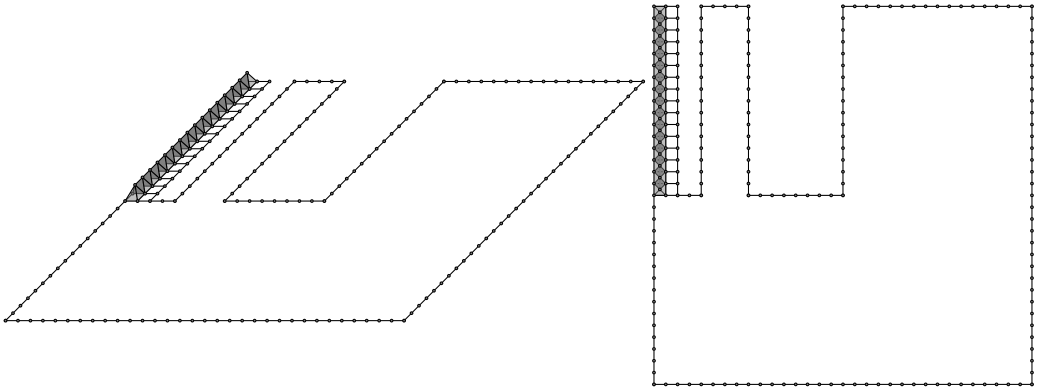


Figure 2.5: The realization of the simplicial complex $\mathcal{R}_{2\varepsilon_3}(A_3)$ in two perspectives: Lateral and Aerial.

$(2^4 - 1)$. In general, for any ε_n approximation we will have the same structure, with 2^{3n-5} squares and piramids and $2^{3n-5} - 1$ tetrahedrons. Concerning the maps, we can use pictures to see where they send the points of the approximations, and the sets of those points, but we will focus our attention on the induced maps in homology which actually will tell us the behavior of the maps.

We now study the previous sequence at the homological level. We will compute the first homology group with coefficients in $\mathbb{Z}$ (with notation $H_1(K) := H_1(K;\mathbb{Z})$) of each polyhedron of the sequence and how the induced homology maps work. For the first approximation, everything is trivial. For the second one, we know that $\mathcal{R}_{2\varepsilon_2}(A_2)$ (in the figure) has the same homotopy type as $\mathcal{K}(U_{2\varepsilon_2}(A_2))$. It has three 1-cycles: The "big" one and the two little squares. There is no more 1-homology on this complex. This is clear from the aerial perspective in figure 2.4. So, the homology group of this polyhedron is just three copies of $\mathbb{Z}$, which we denote $H_1(\mathcal{K}(U_{2\varepsilon_2}(A_2))) \simeq \mathbb{Z}^3$. In the third step, as we can see in picture 2.5, there is again one "big" 1-cycle, and 2^4 small squares. I.e., a total of $2^4 + 1$ copies of $\mathbb{Z}$, so $H_1(\mathcal{K}(U_{2\varepsilon_3}(A_3))) \simeq \mathbb{Z}^{2^4+1}$. We are interested in the map induced in homology by the map

$$p_{2,3} : \mathcal{K}(U_{2\varepsilon_3}(A_3)) \longrightarrow \mathcal{K}(U_{2\varepsilon_2}(A_2)).$$

We need to study, for each 1-cycle, where the vertices are sent by the map[9] An

[9]We can visualize the performance of the map by overlying the pictures of the two consecutive approximations, since the map acts in terms of proximity.

easy reasoning shows that every vertex (included the non drawn ones) in the small 1-cycles of $\mathcal{K}(U_{2\varepsilon_3}(A_3))$ are sent to null homologous cycles in $\mathcal{K}(U_{2\varepsilon_2}(A_2))$ (let us say, they "fall" into the shaded part which is the contratible part). However, the big 1-cicle of $\mathcal{K}(U_{2\varepsilon_3}(A_3))$ is mapped into the "big" one of $\mathcal{K}(U_{2\varepsilon_2}(A_2))$ (actually, it is mapped into something bigger which retracts into this cycle). So, it is clear, that the induced map in homology,

$$(p_{2,3})_* : H_1(\mathcal{K}(U_{2\varepsilon_3}(A_3))) \longrightarrow H_1(\mathcal{K}(U_{2\varepsilon_2}(A_2))),$$

sends the 2^4 generators corresponding to the little squares to zero, and the generator of the "big" 1-cycle to the generator of the "big" one of the target. So, we get that $\mathrm{Im}((p_{2,3})_*) = \mathbb{Z}$. It is readily seen that, if we consider the next step, it will happen the same. In general, the realization of $\mathcal{K}(U_{2\varepsilon_n}(A_n))$ has 2^{3n-5} 1-cycles corresponding to little squares and one "big" 1-cycle. So, $H_1(\mathcal{K}(U_{2\varepsilon_n}(A_n))) \simeq \mathbb{Z}^{2^{3n-5}+1}$. The map induced by

$$p_{n,n+1} : \mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1})) \longrightarrow \mathcal{K}(U_{2\varepsilon_n}(A_n))$$

in homology,

$$(p_{n,n+1})_* : H_1(\mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1}))) \longrightarrow H_1(\mathcal{K}(U_{2\varepsilon_n}(A_n))),$$

sends the 2^{3n-2} 1-cycles corresponding to little squares of $\mathcal{K}(U_{2\varepsilon_{n+1}}(A_{n+1}))$ to zero and the 1-cycle corresponding to the "big" one to the "big" one in the image $\mathcal{K}(U_{2\varepsilon_n}(A_n))$. So, again, the image of the map is $\mathrm{im}((p_{n,n+1})_*) = \mathbb{Z}$. So, we see that in each step, the "big" 1-cycle is the only non-trivial homology that comes from the image of the previous polyhedron. We could say that the "big" cycle is the only one that survives (or persists -see the relation with persistent homology later) in the whole sequence. In terms of the inverse limit, it is clear that the inverse limit of the inverse sequence induced on homology[10]

$$H_1(\mathcal{K}_1) \xleftarrow{(p_{0,1})_*} H_1(\mathcal{K}_2) \xleftarrow{(p_{1,2})_*} \ldots \xleftarrow{(p_{n-1,n})_*} H_1(\mathcal{K}_n) \xleftarrow{(p_{n,n+1})} H_1(\mathcal{K}_{n+1}) \xleftarrow{(p_{n+1,n+2})_*} \ldots .$$

[10] Notation: $\mathcal{K}_n := \mathcal{K}(U_{2\varepsilon_n}(A_n))$

is

$$\varprojlim \{\mathcal{K}_n, (p_{n,n+1})_*\} \simeq \mathbb{Z}.$$

As shown in Theorem 12, this sequence can be used to obtain the Čech homology of the space $\mathcal{W}$, that is, $\check{H}_1(\mathcal{W}) \simeq \mathbb{Z}$.

2.5.1 Persistent errors in the computational Warsaw circle

We know that the Warsaw circle is a movable space (see section 1.3.2). Consequently its first homology induced sequence has the Mittag-Leffler property: For every $n \in \mathbb{N}$ there exists $m \geqslant n$ such that for every $r \geqslant m$ we have that $(p_{n,r})_*(H_1(\mathcal{K}_r)) = (p_{n,m})_*(H_1(\mathcal{K}_m))$. This means, roughly speaking, that avery step n, there exists a further step m, such that all the homology at step n coming from homology at step further than m is equal to the homology coming from m. Or, in other words, all the homology in the step n comes from homology in the step m. We will say that m is the M-L index of n. In our example, it is clear that, for every $n \in \mathbb{N}$, the M-L index is $n+1$, because

$$(p_{n,n+1})_*(H_1(\mathcal{K}_{n+1})) = (p_{n,m})_*(H_1(\mathcal{K}_m)) \simeq \mathbb{Z}$$

for every $m > n+1$. So, in our sequence, all the homology at any step can be founded just going one step further. And, moreover, we now that this is all the homology of the space. This is related with the persistent homology as we will see later.

Concerning the persistent errors in homology, let us fix some $p \geqslant 1$. We adopt the notation from section 2.4. Then, in this case, for every $n < m$, the (n,m)-th persistent homology group is $H_{n,m} \simeq \mathbb{Z}$. So, the (n,m)-th persistent error is

$$E_{n,m} = \frac{H_n}{H_{n,m}} \simeq \frac{\mathbb{Z}^{2^{3n-5}+1}}{\mathbb{Z}} \simeq \mathbb{Z}^{2^{3n-5}}.$$

This means, that the error of the polyhedron $\mathcal{K}_n$ estimating $\mathcal{W}$, seen from $\mathcal{K}_m$, consists of 2^{3n-5} little squares. Or, in other words, $\mathcal{K}_m$ only certifies as proper 1-cycle of $\mathcal{K}_m$ the so called big one (and we know that this is correct from the point of view of the inverse limit). As shown in that section, there is a natural map between the quotients (errors)

$$\begin{array}{rccc} g_{m,m+1}: & E_{n,m+1} & \longrightarrow & E_{n,m} \\ & h+H_{n,m+1} & \longmapsto & h+H_{n,m}. \end{array}$$

In this example, the 1-cycles in $E_{n,m+1}$, i.e., the 1-cycles of H_n not killed by $H_{n,m+1}$ are sent to the 1-cycles of H_n not killed by $H_{n,n}$. These 1-cycles are the same: The corresponding to the little squares. This is so because there is only one 1-cycle in $H_{n,m+1}$ and $H_{n,m}$: The "big" one. That means this map is the identity

$$g_{m,m+1} = \mathrm{id} : \mathbb{Z}^{3n-5} \longrightarrow \mathbb{Z}^{3n-5}$$

and hence the inverse limit of the inverse sequence

$$E_{n,n+1} \xleftarrow{g_{n+1,n+2}} E_{n,n+2} \xleftarrow{g_{n+2,n+3}} \dots \xleftarrow{g_{n+m-1,n+m}} E_{n+m,n+m+1} \xleftarrow{g_{n+m+1,n+m+2}} \dots,$$

is $E_n^i \simeq \mathbb{Z}^{2^{3n-5}}$. As we pointed out before, since $\mathcal{W}$ is movable, this error must coincide with the n-th real error,

$$E_n = \frac{H_n}{(p_n)_*(\check{H}_1(\mathcal{W}))} \simeq \mathbb{Z}^{2^{3n-5}}.$$

2.6 Example: The computational Hawaiian Earring

For computational reasons, in this case, we are going to use not an homeomorphic copy of the Hawaiian Earring but an homotopic (and, hence, with the same shape) one. In this case, we consider the space (see figure 2.6)

$$\mathcal{HE} = \bigcup_{n\in\mathbb{N}\cup\{0\}} \left(0, \frac{1}{2^n}\right) - \left(\frac{1}{2^n}, \frac{1}{2^n}\right) - \left(\frac{1}{2^n}, 0\right) \cup (0,0) - (1,0) \cup (0,0) - (0,1)$$

or

$$\mathcal{HE} = \bigcup_{n\in\mathbb{N}} \square \left(\frac{1}{2^n}, \frac{1}{2^n}\right),$$

where $\square(a,b)$ stands for the square in $\mathbb{R}^2$

$$(0,0) - (2a,0) - (2a,2b) - (0,2b) - (0,0).$$

The idea to obtain the approximations for this space is the same as in the

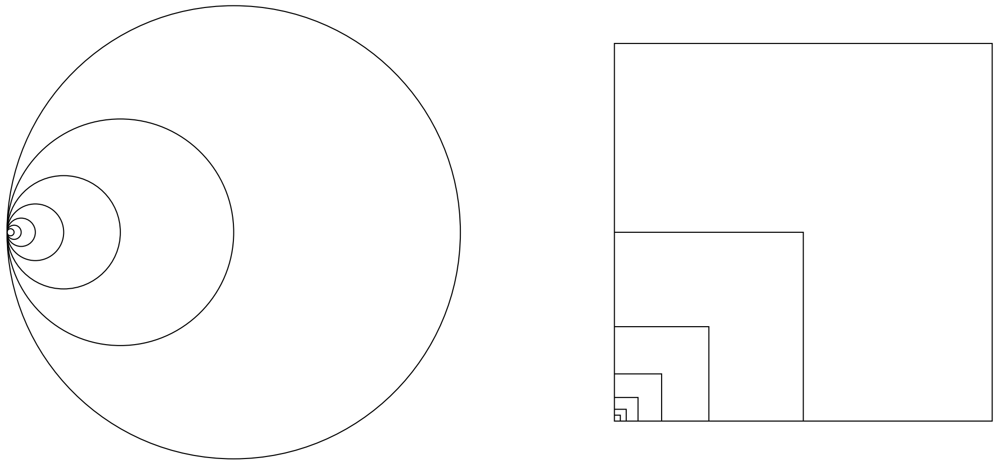


Figure 2.6: The Hawaiian Earring and the computational Hawaiian Earring.

Warsaw circle. Take as points of each approximation the intersection of a grid of corresponding side with the space, and add points where necessary. Moreover, we see that the concrete numbers of the approximation are exactly the same. As in the Warsaw circle, the diameter of $\mathcal{HE}$ is $M = \sqrt{2}$ so we can take $\varepsilon_1 = 2\sqrt{2} > M$ and $A_1 = \{(0,0)\}$. Obviously $\gamma_1 = \sqrt{2}$. So, we select $\varepsilon_2 = \frac{\sqrt{2}}{2^3} < \frac{\sqrt{2}}{2}$. We intersect a grid G_2 of side $\frac{1}{2^2}$ with $\mathcal{HE}$ (see figure 2.7), and

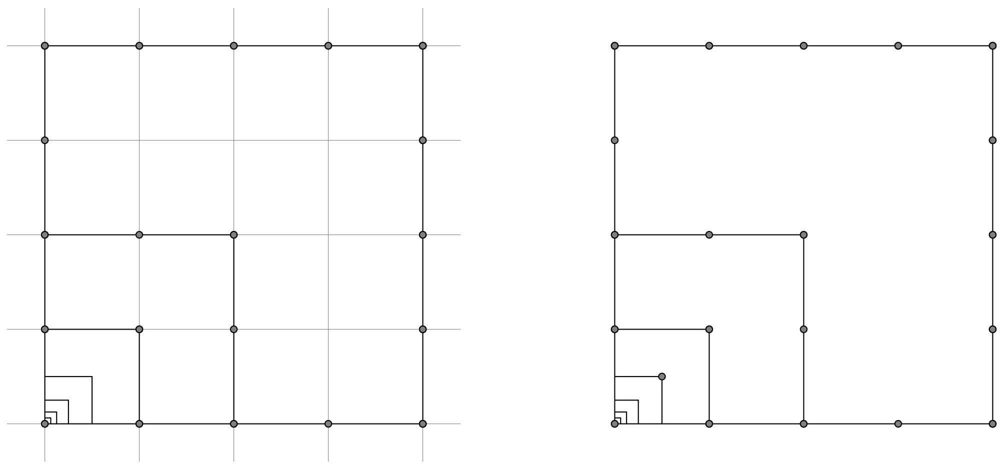


Figure 2.7: The intersection of the grid G_2 with $\mathcal{HE}$ and the ε_2 approximation of $\mathcal{HE}$.

adding the point $\left(\frac{1}{2^3}, \frac{1}{2^3}\right)$, we obtain A_2, an ε_2 approximation of $\mathcal{HE}$. From the picture, it is very easy to see that $\gamma_2 = \frac{1}{2^3}$, and $\delta_2 = \frac{\sqrt{2}}{2^3}$. So, exactly as in the case of the Warsaw circle, we just need to pick $\varepsilon_3 = \frac{\sqrt{2}}{2^6}$. Intersecting a grid G_3 of side $\frac{1}{2^5}$ with $\mathcal{HE}$ and adding the point $\left(\frac{1}{2^6}, \frac{1}{2^6}\right)$, we obtain (see figure 2.8) the ε_3 approximation A_3 of $\mathcal{HE}$. Again, by induction, we see that this process can

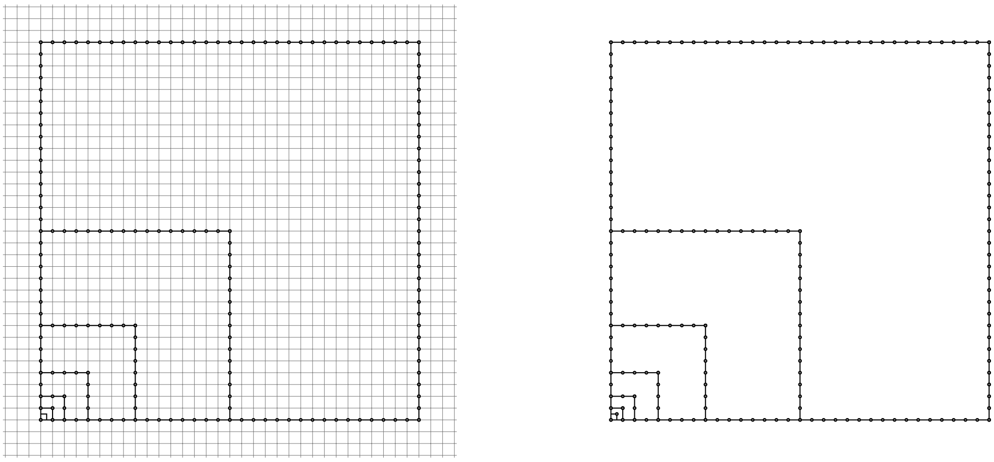


Figure 2.8: The intersection of the grid G_3 with $\mathcal{HE}$ and the ε_3 approximation of $\mathcal{HE}$.

be done indefinitely, with exactly the same numbers as in the Warsaw circle example. So, for every $n > 1$, the finite approximations are defined by:

$$\begin{aligned}
\varepsilon_n &= \frac{\sqrt{2}}{2^{3n-3}}, \\
A_n &= (G_n \cap \mathcal{HE}) \cup \left\{ \left(\frac{1}{2^{3n-3}}, \frac{1}{2^{3n-3}} \right) \right\}, \\
\gamma_n &= \frac{1}{2^{3n-3}}, \quad \delta_n = \frac{\sqrt{2}}{2^{3n-3}}.
\end{aligned}$$

We therefore obtain the sequence of finite spaces from these approximations. Concerning the McCord sequence associated to that sequence, we have the following. In the first step we have only a point as finite space, so the associated polyhedron is just a vertex. For the second and third step, we have depicted in figure 2.9 the realization of the simplicial complexes $\mathcal{R}_{2\varepsilon_2}(A_2)$ and $\mathcal{R}_{2\varepsilon_3}(A_3)$. Then, the corresponding associated simplicial complexes are exactly the barycentric subdivisons of these complexes, that is, $\mathcal{K}(U_{2\varepsilon_2}(A_2)) = \mathcal{R}'_{2\varepsilon_2}(A_2)$ and $\mathcal{K}(U_{2\varepsilon_3}(A_3)) = \mathcal{R}'_{2\varepsilon_3}(A_3)$, respectively. We see that they consist of –from

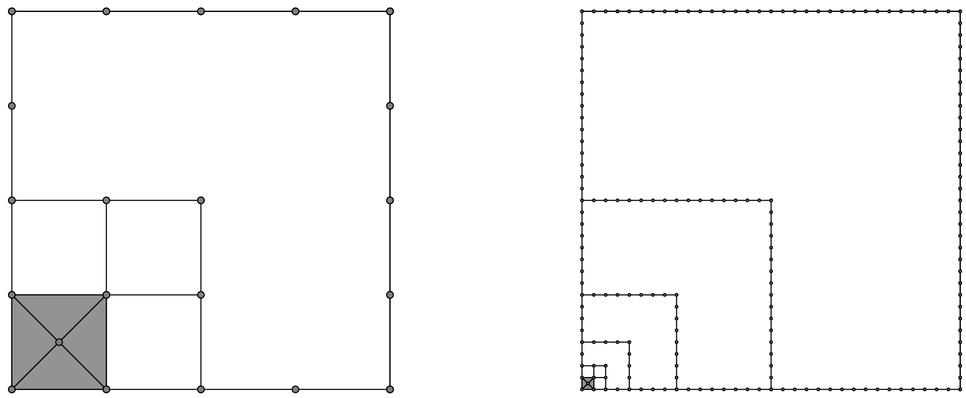


Figure 2.9: The realization of the simplicial complexes $\mathcal{R}_{2\varepsilon_2}(A_2)$ and $\mathcal{R}_{2\varepsilon_3}(A_3)$.

north east to south west- a series of, let us say,"inverted" L's (one in $\mathcal{K}\left(U_{2\varepsilon_2}(A_2)\right)$ and four in $\mathcal{K}\left(U_{2\varepsilon_3}(A_3)\right)$), one more "inverted" L formed by three squares and one more square, filled. It is easy to see that, in each step, we just insert three more "inverted" L's and reduce the size of the three squares forming the "inverted" L and of the filled one. So, we can infer that, in general, the associated McCord polyhedron $\mathcal{K}\left(U_{2\varepsilon_n}(A_n)\right)$ has $3n-5$ "inverted" L's, three squares forming an "inverted" L and one filled square. As before, the maps can be known just overlapping the polyhedra to see where each vertex is sent. But this will be better understood studying the homological situation.

We study homology where it makes sense, so here we just focus in homology of dimension 1. So, first of all, we give names to the generators: In the simplicial complex associated to the ε_i approximation, $\mathcal{K}(U_{2\varepsilon_i}(A_i))$, let us call ρ_i^j to the homology generator representing the j-th "inverted" L, counting from north east to south west, and λ_i, μ_i, ν_i to the three squares, counted clockwise. Now, in the first non trivial case, the homology is $H_1\left(\mathcal{K}(U_{2\varepsilon_2}(A_2))\right) \simeq \mathbb{Z}^4$ with generators $\rho_2^1, \lambda_2, \mu_2, \nu_2$. The next one is $H_1\left(\mathcal{K}(U_{2\varepsilon_3}(A_3))\right) \simeq \mathbb{Z}^7$ with generators ρ_3^i, with $i = 1, \ldots, 4$, λ_3, μ_3, ν_3. The induced map

$$(p_{2,3})_* : H_1(\mathcal{K}(U_{2\varepsilon_3}(A_3))) \longrightarrow H_1(\mathcal{K}(U_{2\varepsilon_2}(A_2))),$$

sends

$$\begin{aligned} \rho_3^1 &\longmapsto \rho_2^1 \\ \rho_3^2 &\longmapsto \lambda_2 + \mu_2 + \nu_2 \\ \rho_3^3, \rho_3^4, \lambda_3, \mu_3, \nu_3 &\longmapsto 0. \end{aligned}$$

Interpreting this, we see that, in the third approximation, three "new" cycles are created, one is sent to the sum of the three squares, and the other two are sent to zero. This is repeated along all the sequence. For any $n \in \mathbb{N}$, we have that the first homology group is $H_1\left(\mathcal{K}(U_{2\varepsilon_n}(A_n))\right) \simeq \mathbb{Z}^{3n-2}$, with generators ρ_n^i, with $i = 1, \ldots, 3n-5$, λ_n, μ_n, ν_n. The map

$$(p_{n,n+1})_* : H_1(\mathcal{K}(U_{2\varepsilon_{n+1}}(A_n))) \longrightarrow H_1(\mathcal{K}(U_{2\varepsilon_n}(A_n))),$$

acts sending

$$\begin{aligned} \rho_{n+1}^1 &\longmapsto \rho_n^1 \\ &\vdots \\ \rho_{n+1}^{3n-5} &\longmapsto \rho_n^{3n-5} \\ \rho_{n+1}^{3n-4} &\longmapsto \lambda_n + \mu_n + \nu_n \\ \rho_{n+1}^{3n-3}, \rho_{n+1}^{3n-2}, \lambda_{n+1}, \mu_{n+1}, \nu_{n+1} &\longmapsto 0. \end{aligned}$$

It is easy to understand the beaviour of the sequence. In each step, 3 "new" cycles are created, and they have an unique preimage in every further step. So the inverse limit of the sequence is

$$\varprojlim \{\mathcal{K}_n, (p_{n,n+1})_*\} \simeq \mathbb{Z}^\infty$$

and, by Theorem 12, the Čech homology of the space $\mathcal{HE}$ is, $\check{H}_1(\mathcal{HE}) \simeq \mathbb{Z}^\infty$, as we already knew.

2.6.1 Persistent errors in the computational Hawaiian Earring

The Hawaiian earring is also a movable space, but, in contrast with the Warsaw circle, it is not stable. Moreover the latter has the shape of a finite polyhedron (namely $\mathbb{S}^1$), while the former has not. As a movable space, the inverse sequence $\{\mathcal{K}_n, (p_{n,n+1})_*\}$ has the M-L property. An easy induction tells us that, for every $m > n$, $\operatorname{im}\left(p_{n,m}\right)_* \simeq \mathbb{Z}^{3n-4}$, so, the M-L index for every $n \in \mathbb{N}$ is, again, $n+1$. Let us compute the persistent errors. Using the notation from that section, the (n, m)-th persistent homology group, for $1 < n < m$, is $H_{n,m} \simeq \mathbb{Z}^{3n-4}$, so we can compute the (n, m)-th persistent error

$$E_{n,m} = \frac{H_n}{H_{n,m}} \simeq \frac{\mathbb{Z}^{3n-2}}{\mathbb{Z}^{3n-4}} \simeq \mathbb{Z}^2.$$

This two copies of $\mathbb{Z}$ represent the fact that, for every $n \in \mathbb{N}$, all the generators of the group $H_1(\mathcal{K}_n)$, but two, have one (and only one) preimage in every $H_1(\mathcal{K}_n)$. Considering

$$\left\{\rho_n^1, \dots, \rho_n^{3n-5}, \lambda_n + \mu_n + \nu_n, \mu_n, \nu_n\right\}$$

as generators, the last two are the ones without preimage. The map induced in $E_{n,m}$ is clearly the identity map,

$$g_{m,m+1} = \mathrm{id} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$

and the inverse limit of the inverse sequence $\{E_{n,m}, g_{n+m,n+m+1}\}$ is $E_n^i \simeq \mathbb{Z}^2$. Since $\mathcal{HE}$ is movable, this error is equal to the n-th real error,

$$E_n = \frac{H_n}{(p_n)_*(\check{H}_1(\mathcal{HE}))} \simeq \mathbb{Z}^2.$$

Chapter 3

Homotopical and homeomorphic reconstruction

It is clear from the previous chapter that our construction is good to represent the shape invariants of the space. For instance, the Čech homology: We just compute the singular (or simplicial) homology groups of all the terms of the sequence and the inverse limit will be the Čech homology, which is also the singular homology in the case of the space being an ANR. It turns out that according to the McCord correspondence we can just compute the homology of the finite spaces, because they are weakly homotopic to the polyhedra associated to them. Here, we restrict our attention to the sequence of finite spaces. Surprisingly, the inverse limit of this sequence, contains all the topological and homotopical information.

3.1 Finite approximative sequences: Main result

In this section, we prove the main result of this chapter.

Theorem 14. *Let X be a compact metric space. Suppose we perform the main construction of Theorem 8, on X, obtaining a FAS $\{U_{2\varepsilon_n}(A_n), p_{n,n+1}\}$ with inverse limit $\mathcal{X} = \varprojlim\{U_{2\varepsilon_n}(A_n), p_{n,n+1}\}$. Then, there exists a subspace $\mathcal{X}^* \subset \mathcal{X}$ such that $\mathcal{X}^*$ is homeomorphic to X and it is a strong deformation retract of $\mathcal{X}$.*

In order to do this, we will need some technical lemmas about the above construction. Suppose we do the main construction on X and we obtain the inverse sequence of finite spaces and the sequences of numbers with usual notation $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, we write $\overline{\varepsilon_n} = \frac{\varepsilon_n+\gamma_n}{2}$ and $\underline{\varepsilon_n} = \frac{\varepsilon_n-\gamma_n}{2}$. They clearly satisfy $\overline{\varepsilon_n}, \ \underline{\varepsilon_n} < \varepsilon_n$ and $\overline{\varepsilon_n} + \underline{\varepsilon_n} = \varepsilon_n$.

Lemma 6. *For every $n < m$, we have*

$$\sum_{l=n}^{m} \gamma_l < \overline{\varepsilon_n}.$$

Proof. For every $n \in \mathbb{N}$, we have that $\varepsilon_{n+1} < \frac{\varepsilon_n-\gamma_n}{2}$, so $\gamma_n + \varepsilon_{n+1} < \gamma_n + 2\varepsilon_{n+1} < \varepsilon_n$.
Now, let $n < m$ be natural numbers, if we write $m = n + k, \ \ k > 0$, we can apply the previous observation inductively to obtain

$$\begin{aligned} \sum_{l=n}^{m} \gamma_l = \sum_{i=0}^{k} \gamma_{n+i} \ &< \ \left(\sum_{i=0}^{k-1} \gamma_{n+i}\right) + \varepsilon_{n+k} < \left(\sum_{i=0}^{k-2} \gamma_{n+i}\right) + \varepsilon_{n+k-1} < \ldots < \\ &< \ \gamma_n + \varepsilon_{n+1} < \gamma_n + \frac{\varepsilon_n - \gamma_n}{2} = \frac{\varepsilon_n + \gamma_n}{2} \quad \checkmark \end{aligned}$$

Remark 11. The previous lemma gives us a bound in terms of the lower term, so it is readily seen that the infinite sum converges, and

$$\sum_{l=n}^{\infty} \gamma_l < \varepsilon_n.$$

Lemma 7. *For every $n > 1$, $\varepsilon_n < \frac{M}{2^{n-1}}$, where $M = diam(X)$.*

Proof. We proceed by induction over n. The first case is clear, $\varepsilon_2 < \min\left\{\frac{\varepsilon_1 - M}{2}, \frac{M}{2}\right\} < \frac{M}{2}$. Now, let us suppose that $\varepsilon_n < \frac{M}{2^{n-1}}$. Then $\varepsilon_{n+1} < \frac{\varepsilon_n-\gamma_n}{2} < \frac{\varepsilon_n}{2} < \frac{M}{2^n}$ $\checkmark$

Proposition 18. *Let $n < m$ be a pair of natural numbers. Let $a_n \in A_n, \ \ a_m \in A_m$ be two points of X such that $a_n \in p_{n,m}(\{a_m\})$. Then $d(a_n, a_m) < \overline{\varepsilon_n}$.*

Proof. Let us write $m = n + k, \ \ k > 0$. The relation between the points means that there exists a chain of points between them. That is, there exist $a_{n+1} \in A_{n+1}, \ldots, a_{n+k-1} \in A_{n+k-1}$ such that $a_n \in p_{n,n+1}(\{a_{n+1}\}), \ldots, a_{n+k-1} \in$

$p_{n+k-1,n+k}(\{a_{n+k}\})$. Using the previous proposition, we can now estimate

$$d(a_n, a_m) \leqslant \sum_{i=0}^{k} d(a_{n+l}, a_{n+i+1}) \leqslant \sum_{i=0}^{k} \gamma_{n+i} < \overline{\varepsilon_n} \quad \checkmark$$

Before starting the proof of the theorem, let us reinterpret this inverse limit as sequences of points in 2^X, with the Haussdorff distance, that is, as sequences of points in 2^X_H. We consider the inverse limit of the finite spaces. We will write the points of this limit as sequences $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ ($\{C_n\}$ for short), where, for every $n \in \mathbb{N}$, $C_n \in U_{2\varepsilon_n}(A_n)$, and, for every pair $n < m$, $p_{n,m}(C_m) = C_n$. We have to think about this sequences as sets of points of each ε-approximation, related by a notion of proximity. It turns out that these sequences converge to points of X. To have a notion of measure and see this, we will use the Hausdorff distance of the hyperspace 2^X of non-empty closed subsets of X. It can by defined in the following way: For $C, D \in 2^X$ closed sets of X, we will say that the Hausdorff distance of C and D is $d_H(C, D) < \varepsilon$ if $C \subset B(D, \varepsilon)$ and $D \subset B(C, \varepsilon)$[1]. We are going to prove the following

Proposition 19. *Every point of the inverse limit $\{C_n\} \in \mathcal{X}$ is a Cauchy sequence in 2^X_H that converges to a singleton $\{x\}$, with $x \in X$.*

Proof. First of all, we see that, in terms of the Hausdorff metric, the diference between two elements of the sequence can be bounded in terms of the lower index. Let $\{C_n\} \in \mathcal{X}$ be a point of the inverse limit. Then, the Hausdorff distance between terms of the sequence C_n and C_m, with $n < m$, is $d_H(C_n, C_m) < \overline{\varepsilon_n}$. For the first condition, given $c_n \in C_n$, there exists $c_m \in C_m$ such that $c_n \in p_{n,m}(\{c_m\})$, and then $d(c_n, c_m) < \overline{\varepsilon_n}$ by the previous lemma. Analogously, for $c_m \in C_m$ we can take $c_n \in p_{n,m}(C_m)$ and the distance satisfies the second condition.

Now, the sequence of closed sets $\{C_n\} \in \mathcal{X}$ is a Cauchy sequence in 2^X_H. For any $\varepsilon > 0$, it suffices to consider $n_0 \in \mathbb{N}$ such that $\varepsilon_{n_0} < \varepsilon$ and then, for every $n, m > n_0$, we have $d_H(C_n, C_m) < \varepsilon_{n_0} < \varepsilon$.

[1]Here, $B(C, \varepsilon)$ is the generalized ball of radius ε, i.e., the set of points $x \in X$ for which there exists a point c of C at distance $d(x, c) < \varepsilon$ or, equivalently, is the union of balls of radius ε and center any point of C, that is, $B(C, \varepsilon) = \bigcup_{c\in C} B(c, \varepsilon)$.

It remains to prove that every sequence $\{C_n\} \in \mathcal{X}$ converges to a singleton $\{x\}$ of X in the Hausdorff metric. The sequence is Cauchy in the compact metric (and hence complete) space 2^X_H, so there exists a unique limit $C \in 2^X$. The diameter of this point of the hyperspace is

$$\mathrm{diam}(C) = \mathrm{diam}(\lim_n C_n) = \lim_n \mathrm{diam}(C_n) < \lim_n 2\varepsilon_n = 0$$

because of the continuity of the diameter function regarding to the Hausdorff metric (see [55]). So $C = \{x\}$, with $x \in X$ ✓

Remark 12. The meaning of $\{C_n\} \in \mathcal{X}$ converging to a set with only one point $\{x\} \subset X$ is that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$, such that, for every $n > n_0$, $\mathrm{d_H}(\{x\}, C_n) < \varepsilon$, i.e., $C_n \subset \mathrm{B}(\{x\}, \varepsilon)$ and $x \in \mathrm{B}(C_n, \varepsilon)$. But, the first condition, meaning $x \in \bigcap_{c\in C_n} \mathrm{B}(c, \varepsilon)$, implies the second one, $x \in \bigcup_{c\in C_n} \mathrm{B}(c, \varepsilon)$. Henceforth, we will say that $\{C_n\}$ converges to x (written $\{C_n\} \underset{H}{\rightarrow} x$) for the convergence of $\{C_n\}$ to $\{x\}$ with the Hausdorff metric and we will write $\mathrm{d_H}(x, C_n)$ for $\mathrm{d_H}(\{x\}, C_n)$, for simplicity.

We have the following trivial facts relating the Hausdorff distance on the hyperspace of a metric space and the original distance on the space, for distances between points and closed sets.

Proposition 20. *Let X be a metric space, for every pair of points $x, y \in X$ and pair of closed subsets $D \subset C \subset X$, we have:*

i) $d_H(x, y) = d(x, y)$.

ii) $d_H(x, C) = sup\{d(x, c) : c \in C\} \geqslant inf\{d(x, c) : c \in C\} = d(x, C)$.

iii) $d_H(x, D) \leqslant d_H(x, C)$ *but* $d(x, D) \geqslant d(x, C)$.

The last property can be interpreted in some sense as a better behaviour of the Hausdorff distance with respect to the upper semifinite topology.

Remark 13. We can even bound the distances to the limit. If $\{C_n\} \in \mathcal{X}$ is a point of the inverse limit converging to a point $x \in X$ in the Hausdorff metric, then, for every $n \in \mathbb{N}$, $\mathrm{d_H}(x, C_n) < \varepsilon_n$. This is so because, if we consider an $m > n$ such that $\mathrm{d_H}(x, C_m) < \underline{\varepsilon_n}$, then we can write

$$\mathrm{d_H}(x, C_n) \leqslant \mathrm{d_H}(x, C_m) + \mathrm{d_H}(C_m, C_n) < \underline{\varepsilon_n} + \overline{\varepsilon_n} = \varepsilon_n.$$

This measure allows us to understand this sequences from another point of view: If $\{C_n\} \in \mathcal{X}$ is such a sequence, then we know there exists an $x \in X$ such that $\{C_n\}$ converges to $\{x\}$ in the Hausdorff metric. But, from the previous remark, we see that, for every $n \in \mathbb{N}$, $x \in \bigcap_{c \in C_n} \mathrm{B}(c, \varepsilon_n)$. So, we can see x as the infinite intersection over all natural numbers:

$$x = \bigcap_{n \in \mathbb{N}} \left(\bigcap_{c \in C_n} \mathrm{B}(c, \varepsilon_n) \right).$$

Proof of Theorem 14. Now, we can define a map $\varphi : \mathcal{X} \to X$ from the inverse limit $\mathcal{X}$ to the original space X. We do this assigning to every sequence $\{C_n\} \in \mathcal{X}$ the unique point x in the limit $\{x\} = \lim\{C_n\}$ with the Hausdorff metric. The map $\varphi : \mathcal{X} \to X$, sending $\{C_n\}$ to x is continuous. We will see that it is continuous at every point. Let $\{C_n\} \in \mathcal{X}$ such that $x = \lim\{C_n\}$ in the Hausdorff metric. Then, consider a neighborhood U of x inside X. Now we want to find a neighborhood of $\{C_n\}$ in $\mathcal{X}$ with image contained in U. There exists an $\varepsilon > 0$ such that $x \in \mathrm{B}(x, \varepsilon) \subset U$. Let us consider $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, $\varepsilon_n < \frac{\varepsilon}{2}$. We claim that the basic open neighborhood

$$V = \left(2^{C_1} \times 2^{C_2} \times \ldots \times 2^{C_{n_0}} \times U_{2\varepsilon_{n_0+1}}(A_{n_0+1}) \times \ldots \right) \cap \mathcal{X}$$

is the desired neighborhood of $\{C_n\}$ in $\mathcal{X}$. So, let $\{D_n\} \in V$ with $\{D_n\} \underset{H}{\to} y$. Then we have

$$\mathrm{d_H}(x, y) \leqslant \mathrm{d_H}(x, D_{n_0}) + \mathrm{d_H}(D_{n_0}, y) \leqslant \mathrm{d_H}(x, C_{n_0}) + \mathrm{d_H}(D_{n_0}, y) < 2\varepsilon_{n_0},$$

so $y = \varphi(\{D_n\}) \in \mathrm{B}(x, \varepsilon) \subset U$.

Moreover, the map $\varphi : \mathcal{X} \to X$ is surjective. For every $x \in X$, we shall construct an element of the inverse limit explicitly. To do so, let $x \in X$ and consider, for every $n \in \mathbb{N}$, the sets $X^n = \mathrm{B}(x, \varepsilon_n) \cap A_n$. These sets are finite and non-empty, because, for every $n \in \mathbb{N}$, A_n is a finite ε_n-approximation. Now we define, for every $n \in \mathbb{N}$,

$$X_n^* = \bigcap_{m > n} p_{n,m}(X^m),$$

which are non-empty sets, as an intersection of a nested collection of finite (hence closed) sets in a compact space. To show that it is indeed a nested sequence, we need to prove that, for every $x \in X$ and $n < m$, $p_{n,m+1}(X^{m+1}) \subset p_{n,m}(X^m)$. We first show that, for every $m \in \mathbb{N}$, $p_{m,m+1}(X^{m+1}) \subset X^m$. Let $d \in p_{m,m+1}(X^{m+1})$, then, there is an element $c \in X^{m+1}$ such that $d \in p_{m,m+1}(\{c\})$, so $\mathrm{d}(x,c) < \varepsilon_{m+1}$ and $\mathrm{d}(c,d) < \overline{\varepsilon_m}$ and we get

$$\mathrm{d}(x,d) \leqslant \mathrm{d}(x,c) + \mathrm{d}(c,d) < \underline{\varepsilon_m} + \overline{\varepsilon_m} = \varepsilon_m,$$

meaning that $d \in X^m$. Now, it follows directly that

$$p_{n,m+1}(X^{m+1}) = p_{n,m}(p_{m,m+1}(X^{m+1})) \subset p_{n,m}(X^m).$$

The sequence $X^* = \{X^*_n\}$ is an element of the inverse limit $\mathcal{X}$. This is so because, for every $n \in \mathbb{N}$, $\mathrm{diam} X^*_n < 2\varepsilon_n$ (by construction, $X^*_n \subset X^n$) and, for every pair $n < m$, we have $p_{n,m}(X^*_m) = X^*_n$. We just need to prove it for two consecutive terms, i.e., we want to prove that, for every $n \in \mathbb{N}$, $p_{n,n+1}(X^*_{n+1}) = X^*_n$, and the result follows inductively. The last assertion relies on the following fact[2]: For every $n \in \mathbb{N}$ there exist an integer $*(n) > n$ such that, for every $m \geqslant *(n)$, $X^*_n = p_{n,m}(X^m)$. The proof goes by construction. For every $z \in X^n \backslash X^*_n$ there exists $n_z \in \mathbb{N}$ such that, for every $m \geqslant n_z$, $z \notin p_{n,m}(X^m)$. Considering (because X^n is finite set)

$$*(n) = \max\left\{n_z : z \in X^n \backslash X^*_n\right\} = \min\left\{m \in \mathbb{N} : p_{n,m}(X^m) = X^*_n\right\},$$

we have the desired result. The function $* : \mathbb{N} \to \mathbb{N}$ is a non decreasing function. Considering any $m \geqslant *(n+1)$ is elementary to see that

$$p_{n,n+1}(X^*_{n+1}) = p_{n,n+1}(p_{n+1,m}(X^m)) = p_{n,m}(X^m) = X^*_n,$$

as wanted. We claim that $\varphi(X^*) = x$. For every $\varepsilon > 0$, consider n_0 such that $\varepsilon_{n_0} < \varepsilon$. Then, for every $n > n_0$, we have that $\mathrm{d_H}(x, X^*_n) < \varepsilon_n < \varepsilon$, because, for every $x^* \in X^*_n$, $\mathrm{d}(x,x^*) < \varepsilon_n$, and then, $X^*_n \subset \mathrm{B}(x,\varepsilon_n)$ and $x \in \mathrm{B}(X^*_n, \varepsilon_n)$.

The proof of the surjectivity gives us an important element of the inverse limit related with each $x \in X$. By construction, this element of the inverse limit is

[2]This is a kind of Mittag-Leffer property for these elements of the inverse limit.

maximal in the following sense: For every $\{C_n\} \in \mathcal{X}$, such that $x = \varphi(\{C_n\})$, we have that $C_n \subset X_n^*$, for every $n \in \mathbb{N}$. Indeed, for every $m \in \mathbb{N}$, $d_H(x, C_m) < \varepsilon_m$ so $C_m \subset B(x, \varepsilon_m) \subset X^m$. Now, given $n \in \mathbb{N}$, for every $m > *(n)$, $C_n = p_{n,m}(C_m) \subset p_{n,m}(X^m) = X_n^*$. Actually, we can alternatively define X_n^* just with this property as

$$X_n^* = \bigcup_{\{C_n\} \in \varphi^{-1}(x)} C_n,$$

because of the maximal property and that $\varphi(\{X_n^*\}) = x$.

The previous construction allows us to define a map on the other direction, $\phi : X \to \mathcal{X}$ with $\phi(x) = \{X_n^*\}$. To prove that this map is continuous in every point, let us consider a neighborhood V of $\{X_n^*\}$ in $\mathcal{X}$. We know that there exists a neighborhood of the form

$$W = \left(2^{X_1^*} \times 2^{X_2^*} \times \ldots \times 2^{X_r^*} \times U_{2\varepsilon_{r+1}}(A_{r+1}) \times \ldots\right) \cap \mathcal{X}$$

such that $W \subset V$. We need to find points close enought to x, that is, an open neighborhood $U \subset X$ such that $\phi(U) \subset W$. We do this by the following construction. First of all, consider $s = *(r)$. We use the following notation, not to be confused with the usual topological notation:

$$\begin{aligned}
\overline{X^s} &:= \overline{B}(x, \varepsilon_s) \cap A_s \ \ (\text{where } \overline{B}(x, \varepsilon_s) = \{y \in X : d(x, y) \leqslant \varepsilon_s\}), \\
\partial X^s &:= \left(\overline{B}(x, \varepsilon_s) \setminus B(x, \varepsilon_s)\right) \cap A_s, \\
X^s_\delta &:= B(x, \varepsilon_s + \delta), \ \ \text{for } \delta \in (-\varepsilon_s, \infty).
\end{aligned}$$

Let us consider the distance from x to the closest point of A_s that is not in $\overline{X^s}$,

$$\varepsilon_s^+(x) = \min\left\{d(x, a) : a \in A_s \setminus \overline{X^s}\right\} = d(x, A_s \setminus \overline{X^s}) > \varepsilon_s.$$

If there is not such a point, the proof is easier, just consider $\varepsilon_s^+(x) = 2\varepsilon_s$. In general, we claim the following (see figure 3.1):

i) For every $\delta < \varepsilon_s^+(x) - \varepsilon_s$, $\overline{X^s} = X^s_\delta$: If $c \in X^s_\delta$ then $d(x, c) < \varepsilon_s + \delta < \varepsilon_s^+(x)$, so $c \in \overline{X^s}$.

ii) For every $\delta < \underline{\varepsilon_s}$ we have that, for every $y \in B(x, \delta)$, $p_{s,s+1}(Y^{s+1}) \subset$

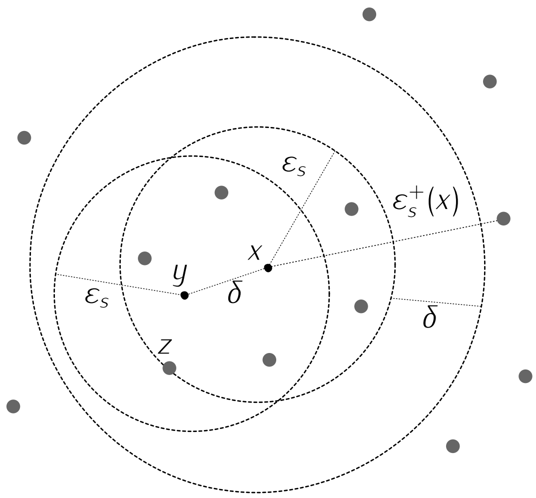


Figure 3.1: For points y, close enough to x, we do not add exterior points of X^s, when we consider its ε_s-neighborhoods. Possibly, some points of the boundary $z \in \partial X^s$ are included, but they are not the image of any point in the next step.

$Y^s \setminus \partial X^s$: Consider $z \in \partial X^s$ and $b \in Y^{s+1}$. Then,

$$\varepsilon_s = \mathrm{d}(x, z) \leqslant \mathrm{d}(x, y) + \mathrm{d}(y, b) + \mathrm{d}(b, z) < 2\underline{\varepsilon_s} + \mathrm{d}(b, z),$$

so $\mathrm{d}(b, z) > \gamma_s$, and then $z \notin p_{s,s+1}(Y_{s+1})$. That means $p_{s,s+1}(Y^{s+1}) \cap \partial X^s = \emptyset$, hence $p_{s,s+1}(Y^{s+1}) \subset Y^s \setminus \partial X^s$.

The desired neighborhood of x is $U = \mathrm{B}(x, \delta)$ with $\delta < \min\left\{\varepsilon_s^+(x) - \varepsilon_s, \underline{\varepsilon_s}\right\}$.

For $y \in \mathrm{B}(x, \delta)$, we have that $Y^s \subset X^s_\delta = \overline{X^s}$ and that

$$Y_r^* \subset p_{r,s+1}(Y^{s+1}) = p_{r,s}(p_{s,s+1}(Y^{s+1})) \subset p_{r,s}(Y^s \setminus \partial X^s) \subset p_{r,s}(X^s) = X_r^*.$$

For $n < r$, $Y_n^* = p_{n,r}(Y_r^*) \subset p_{n,r}(X_r^*) = X_n^*$. Then $\{Y_n^*\} \subset W$ and hence the map $\phi : X \to \mathcal{X}$ is continuous.

This map is clearly injective. Suppose we have two diferent points x, y of X. Then, they are at distance, let us say, $\varepsilon = \mathrm{d}(x, y)$. Consider $s \in \mathbb{N}$ such that $\varepsilon_s < \frac{\varepsilon}{2}$. Then, for every $n > s$, we have that $\mathrm{B}(x, \varepsilon_n) \cap \mathrm{B}(y, \varepsilon_n) = \emptyset$, that is $X_n \cap Y_n = \emptyset$. So, necesarilly, we have that $X_n^* \cap Y_n^* = \emptyset$, which implies that $\{X_n^*\} \neq \{Y_n^*\}$, being the map injective.

If we consider the restriction to the image $\mathcal{X}^* = \phi(X)$, then $\phi : X \to \mathcal{X}^*$ is bijective. But, it is easy to see that $\mathcal{X}^*$ is Hausdorff. If we consider two different points $\{X_n^*\}, \{Y_n^*\} \in \mathcal{X}^*$, then there exist $x \neq y$ such that $\{X_n^*\} = \phi(x)$ and $\{Y^*\} = \phi(y)$. Repeating the last proof, we obtain an $s \in \mathbb{N}$ such that, for every $n > s$ we have $X_n^* \cap Y_n^* = \emptyset$. So, we claim that the neighborhoods

$$\left(2^{X_1^*} \times \ldots \times 2^{X_s^*} \times 2^{X_{s+1}^*} \times U_{2\varepsilon_{s+2}}(A_{s+2}) \times \ldots\right) \cap \mathcal{X}^*$$

and

$$\left(2^{Y_1^*} \times \ldots \times 2^{Y_s^*} \times 2^{Y_{s+1}^*} \times U_{2\varepsilon_{s+2}}(A_{s+2}) \times \ldots\right) \cap \mathcal{X}^*$$

of $\{X_n^*\}$ and $\{Y_n^*\}$ respectively in $\mathcal{X}^*$, are disjoint. Hence $\mathcal{X}^*$ is Hausdorff. Then, as a bijective and continuous map between a compact Hausdorff space and a Hausdorff space, we get that the map $\phi : X \to \mathcal{X}^*$ is a homeomorphism.

So, we have that $\mathcal{X}^*$ is a homeomorphic copy of X inside $\mathcal{X}$. Now, we will see that $\mathcal{X}^*$ is a strong deformation retract of $\mathcal{X}$. To do so, we consider the compositions of the maps defined above. It is very easy to see that $\varphi \cdot \phi : X \to X$ is the identity map. It is not that easy to see that the map $\phi \cdot \varphi : \mathcal{X} \to \mathcal{X}$ is homotopic to the identity $1_{\mathcal{X}}$. We will write the homotopy explicitly: It is the map $H : \mathcal{X} \times [0,1] \to \mathcal{X}$ given by

$$H(\{C_n\}, t) = \begin{cases} \{C_n\} & \text{if} \quad t \in [0,1), \\ \phi \cdot \varphi(\{C_n\}) & \text{if} \quad t = 1. \end{cases}$$

We only need to show the continuity at the points $(\{C_n\}, 1) \in \mathcal{X} \times [0,1]$. Let us write $\phi \cdot \varphi(\{C_n\}) = \phi(x) = \{X_n^*\}$. Consider any neighborhhod V of $\{X_n^*\}$ in $\mathcal{X}$. We can obtain a neighborhood of $\{X_n^*\}$ of the form

$$W = \left(2^{X_1^*} \times \ldots \times 2^{X_r^*} \times U_{2\varepsilon_{r+1}}(A_{r+1}) \times \ldots\right) \cap \mathcal{X}$$

such that $W \subset V$. As we have done in a previous proof, we consider $s = *(r)$, $\varepsilon_s^+(x) = \mathrm{d}(x, A_s \setminus \overline{X^s})$, and $\delta < \min\left\{\varepsilon_s^+(x) - \varepsilon_s, \underline{\varepsilon_s}\right\}$. Select $t > s$ such that $\varepsilon_t < \frac{\delta}{2}$. We claim that the neighborhood

$$U = \left(2^{C_1} \times 2^{C_2} \times \ldots \times 2^{C_t} \times U_{2\varepsilon_{t+1}}(A_{t+1}) \times \ldots\right) \cap \mathcal{X}$$

of $(\{C_n\}, 1)$ in $\mathcal{X} \times [0,1]$ satisfies $H(U \times [0,1]) \subset W$. Let $(\{D_n\}, t) \in U \times [0,1]$

where $D_n \subset C_n$ for $n = 1, \ldots, t$. Then, if $t < 1$, $H(\{D_n\}, t) = \{D_n\} \subset W$, because $r < s < t$ and $D_n \subset C_n \subset X_n^*$ for $n = 1, \ldots, s$. On the other hand, if $t = 1$, then $H(\{D_n\}, 1) = \phi \cdot \varphi(\{D_n\}) = \phi(y) = \{Y_n^*\}$. This implies that $\{Y_n^*\} \in W$. To see why, first observe that, for every $d \in D_t \subset C_t \subset X_t^*$, $\mathrm{d}(x, y) \leqslant \mathrm{d}(x, d) + \mathrm{d}(d, y) < 2\varepsilon_t < \delta$. Then, again as before, $Y^t \subset X^t \cup \partial X^t$ and $Y_r^* \subset X_r^*$, so $\{Y_n^*\} \in W$, and the homotopy is then continuous. The space $\mathcal{X}^*$ is a strong deformation retract of $\mathcal{X}$ and the proof of the theorem is finished ✓

This theorem lead us easily to the following

Corollary 4. *For every compact metric space, there exists a sequence of finite spaces whose limit has the same homotopy type.*

Let us see an example of the main construction and the theorem. Also, observe that the map $\varphi : \mathcal{X} \to X$, defined in the previous proof, is not injective:

Example 4. Let $X = [0, 1]$ be the unit interval with the usual metric d. We will do our construction in a way that it will be easy to find more than one point of the inverse limit converging to the points of X. Concretely, we are going to use, as finite approximations of the unit interval, subdivisions of it in powers of $\frac{1}{3}$. The conditions of the construction will force us to take, for each subdivision, an small enough subdivision of the next step. But we will be able to compute how small it has to be. The diameter of X is $M = 1$. Let $\varepsilon_1 = 2 > M$ and $A_1 = \{0\}$. Obviously $U_{2\varepsilon_1}(A_1) = \{0\}$. Then, it is easy to see that $\gamma_1 = 1$.

For the next step, we need $\varepsilon_2 < \min\left\{\frac{\varepsilon_1-\gamma_1}{2}, \frac{M}{2}\right\} = \frac{1}{2}$. Let us pick $\varepsilon_2 = \frac{1}{3}$ and $A_2 = \{\frac{k}{3}, k = 0, \ldots, 3\}$. Then $U_{2\varepsilon_2}(A_2) = A_2 \cup \left\{\{\frac{k}{3}, \frac{k+1}{3}\}, k = 0, \ldots, 2\}\right\}$, because, for $k < k'$ we have $\mathrm{d}(\frac{k}{3}, \frac{k'}{3}) < \frac{2}{3}$ if and only if $k' - k < 2$, that is, $k' - k$ is 0 or 1. Now $\gamma_2 = \frac{1}{6}$ (the worst thing it could happen is to be exactly in the middle of two points of the approximation which gives the quoted distance). Also, we can claim that $\delta_2 < \frac{1}{3}$. If a set C of points of X has diameter less than $\frac{1}{3}$, it is contained in an interval $[c_1, c_2]$ of length less than this quantity. Now, if $c_1 = c_2 = C$ then $A_2(C)$ is one point or two consecutive points of A_2. If $c_1 < c_2$ it is clear from the construction that $\mathrm{d}(c_i, A_2) < \gamma_2$ for $i = 1, 2$. Then $\mathrm{diam}(A_2(C)) \leqslant \mathrm{d}(A_2, c_1) + \mathrm{d}(c_1, c_2) + \mathrm{d}(c_2, A_2) < \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = \frac{2}{3}$.

We have to pick an $\varepsilon_3 < \min\left\{\frac{\varepsilon_2-\gamma_2}{2}, \frac{\delta_2}{2}\right\} = \frac{1}{3\cdot 2^2}$. Let us select $\varepsilon_3 = \frac{1}{3^3}$ and $A_3 = \left\{\frac{k}{3^3},\ k = 0, \ldots, 3^3\right\}$. Then $U_{2\varepsilon_3}(A_3) = A_3 \cup \left\{\{\frac{k}{3^3}, \frac{k+1}{3^3}\},\ k = 0, \ldots, 3^3 - 1\right\}$

because, for $k < k'$ we have $d(\frac{k}{3^3}, \frac{k'}{3^3}) < \frac{2}{3^3}$ if and only if $k - k' < 2$, that is k and k' are the same or consecutive integers. Now, $\gamma_3 = \frac{1}{2\cdot 3^3}$ (the middle of interval argument holds again), and $\delta_3 < \frac{1}{3^3}$ because if $\operatorname{diam}([c_1, c_2]) < \frac{1}{3^3}$ then

$$\operatorname{diam}(A_3(c_1) \cup A_3(c_2)) \leqslant d(A_3(c_1), c_1) + d(c_1, c_2) + d(c_2, A_3(c_2)) < \frac{1}{2\cdot 3^3} + \frac{1}{3^3} + \frac{1}{2\cdot 3^3} = 2\varepsilon_3.$$

Now, we will select $\varepsilon_4 < \min\left\{\frac{\varepsilon_3 - \gamma_3}{2}, \frac{\delta_3}{2}\right\} = \min\left\{\frac{\frac{1}{3^3} - \frac{1}{2\cdot 3^3}}{2}, \frac{\frac{1}{3^3}}{2} = \frac{1}{3^3\cdot 2^2}\right\}$. So, we can take $\varepsilon_4 = \frac{1}{3^5}$.

Following this process, we can take, for an arbitrary $n \in \mathbb{N}$, $\varepsilon_n = \frac{1}{3^{2n-3}}$ and $A_n = \left\{\frac{k}{3^{2n-3}},\ \ k = 0, \dots, 3^{2^n - 3}\right\}$ is an ε_n–approximation of $[0, 1]$. Observe that

$$U_{2\varepsilon_n}(A_n) = A_n \cup \left\{\left\{\frac{k}{3^{2n-3}}, \frac{k+1}{3^{2n-3}}\right\},\ \ k = 0, \dots, 3^{2n-3} - 1\right\}.$$

We can calculate, as in the previous steps, the numbers to continue the process.The maximum distance of a point of the unit interval to one of the approximation is reached in the middle of any interval formed by two consecutive points of the approximation, so $\gamma_n = \frac{1}{2\cdot 3^{2n-3}}$. Again, $\delta_n < \frac{1}{3^{2n-1}}$ because ,if $\operatorname{diam}([c_1, c_2]) < \frac{1}{3^{2n-3}}$, then

$$\begin{aligned} \operatorname{diam}(A_n(c_1) \cup A_n(c_2)) &\leqslant d(A_n(c_1), c_1) + d(c_1, c_2) + d(c_2, A_n(c_2)) \\ &< \frac{1}{2\cdot 3^{2n-3}} + \frac{1}{3^{2n-3}} + \frac{1}{2\cdot 3^{2n-3}} = 2\varepsilon_n. \end{aligned}$$

Next step will consist of taking

$$\varepsilon_{n+1} < \min\left\{\frac{\frac{1}{3^{2n-3}} - \frac{1}{2\cdot 3^{2n-3}}}{2}, \frac{\frac{1}{3^{2n-3}}}{2}\right\}$$

So we are allowed to choose $\varepsilon_{n+1} = \frac{1}{3^{2n-3+2}} = \frac{1}{3^{2(n+1)-3}}$ and

$$A_{n+1} = \left\{\frac{k}{3^{2(n+1)-3}},\ \ k = 0, \dots, 3^{2(n+1)-3}\right\}$$

is an ε_{n+1}–approximation of $[0, 1]$ and then this construction can be done in this way for every $n \in \mathbb{N}$.

Considering this construction done, we observe that there are points of the interval with only one point in the preimage by φ, i.e., there is only one point of the inverse limit $\mathcal{X}$ converging to it in the Hausdorff metric. To clarify this, let us see what happens at $x = 0 \in [0,1]$. It is obvious that the point $(0,0,\ldots) \in \mathcal{X}$ converges to 0 in the Hausdorff metric. If we want a different element of the inverse limit converging to 0, it is natural to think that we could use the fact that $\lim_{n\to\infty} \frac{1}{3^{2n-3}} = 0$ to obtain it, but it turns out that

$$\left(0, \frac{1}{3}, \frac{1}{3^3}, \ldots, \frac{1}{3^{2n-3}}, \frac{1}{3^{2(n+1)-3}}, \ldots\right) \notin \mathcal{X}$$

because, for every $n \in \mathbb{N}$, $p_{n,n+1}\left(\frac{1}{3^{2(n+1)-3}}\right) = 0$. If we try to construct the "maximal" element X^* of the inverse limit with image $x = 0$ we obtain that, for every $n \in \mathbb{N}$,

$$X_n = \mathrm{B}(0, \varepsilon_n) \cap A_n = \{0\}$$

and $p_{n,m}(0) = 0$ for every $n < m$. So $X_n^* = \bigcap_{n<m} p_{n,m}(X_m) = 0$ for every $n \in \mathbb{N}$. Then, $X^* = (0,0,\ldots)$. Then, every element of $\mathcal{X}$ converging to 0 should be "contained" in this one, in the way we explained before. But there is no possibility appart from X^*.

Actually, every point in A_n for some $n \in \mathbb{N}$ has this property. For any of them, let us say $x = \frac{k}{3^{2n-1}}$, we have

$$X_n = \mathrm{B}\left(\frac{k}{3^{2n-1}}, \frac{1}{3^{2n-1}}\right) \cap A_n = \left\{\frac{k}{3^{2n-1}}\right\}$$

and $p_{n,m}(x) = x$ for $n < m$ and $p_{n,n-1}(x) = 0$. So, the only element of $\mathcal{X}$ converging to x is $X^* = \left(0, \ldots, 0, \frac{k}{3^{2n-1}}, \frac{k}{3^{2n-1}}, \ldots\right)$. The subset of the interval consisting of these points, $\bigcup_{n\in\mathbb{N}} A_n$, is dense in the unit interval.

If we choose a point not in this subset, for example $x = \frac{1}{2}$, we obtain a different result. First of all, we know that $\frac{1}{2}$ is not going to be a point of the approximation, for any $n \in \mathbb{N}$, because if that was true, then $\frac{1}{2} = \frac{k}{3^{2n-3}}$ and that implies $3^{2n-3} = 2k$ which is impossible. Now we claim that, for every $n \in \mathbb{N}$, $\frac{1}{2}$ is at the same distance of two consecutive points of the approximation and, because of that, both minimize its distance to the approximation. This is true

because
$$\frac{k+1}{3^{2n-3}} - \frac{1}{2} = \frac{1}{2} - \frac{k}{3^{2n-3}} \Longleftrightarrow k = \frac{3^{2n-3}-1}{2}.$$
Now, let us construct the "maximal" element for this point. We get:

$$\begin{aligned} X_1 &= 0, \\ X_2 &= \mathrm{B}\left(\frac{1}{2}, \frac{1}{3}\right) \cap A_2 = \left\{\frac{1}{3}, \frac{2}{3}\right\}, \\ &\dots \\ X_n &= \mathrm{B}\left(\frac{1}{2}, \frac{1}{3^{2n-3}}\right) \cap A_n = \left\{\frac{\frac{3^{2n-3}-1}{2}}{3^{2n-3}}, \frac{\frac{3^{2n-3}+1}{2}}{3^{2n-3}}\right\}, \\ &\dots \end{aligned}$$

It is easy to see that $p_{n,m}(X_m) = X_n$ for every $n < m$ so $X_n^* = X_n$ for every $n \in \mathbb{N}$. So, for $x = \frac{1}{2}$,

$$X^* = \left(0, \left\{\frac{1}{3}, \frac{2}{3}\right\}, \left\{\frac{14}{3^3}, \frac{15}{3^3}\right\}, \dots, \left\{\frac{\frac{3^{2n-3}-1}{2}}{3^{2n-3}}, \frac{\frac{3^{2n-3}+1}{2}}{3^{2n-3}}\right\}, \dots\right)$$

which obviously converges to $\frac{1}{2}$ with the Hausdorff metric. But now, we can see that each term has two elements and the maps $p_{n,m}$ are sending the first to the first and the second to the second, so we can consider the sequences

$$\begin{aligned} C_1 &= \left(0, \left\{\frac{1}{3}\right\}, \left\{\frac{14}{3^3}\right\}, \dots, \left\{\frac{\frac{3^{2n-3}-1}{2}}{3^{2n-3}}\right\}, \dots\right) \\ C_2 &= \left(0, \left\{\frac{2}{3}\right\}, \left\{\frac{15}{3^3}\right\}, \dots, \left\{\frac{\frac{3^{2n-3}+1}{2}}{3^{2n-3}}\right\}, \dots\right) \end{aligned}$$

and claim that $C_1,\ C_2 \in \mathcal{X}$ and both converge to $\frac{1}{2}$. So this point has exactly three points of the inverse limit in its inverse image by φ, being that map not injective.

We can actually say more about the injectivity of φ.

Proposition 21. *Let X be a compact metric space and suppose we do the main construction. If $x \in X$ satisfies that there exists an $n_0 \in \mathbb{N}$, with $x \in A_n$ for every $n \geqslant n_0$, then the cardinality of $\varphi^{-1}(x)$ is one.*

Proof. Let us suppose we obtain the sequences $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$ performing the main construction. We are going to prove that, if $x \in X$ belongs to A_n for every $n \geqslant n_0$, then

$$X^* = \left(p_{1,n_0}(x), \ldots, p_{n_0-1,n_0}(x), x, x, \ldots\right).$$

So, there are no more points $C \in \mathcal{X}$ satisfying $\varphi(C) = x$, appart from X^* (because of the maximality of X^*). For $n \geqslant n_0$, we have that $x \in X_n = \mathrm{B}(x, \varepsilon_n) \cap A_n$, and then, $x \in X_n^*$ for every $n \geqslant n_0$, because $p_{n,m}(x) = x$ for every $n_0 \leqslant n < m$. So X^* has the form

$$X^* = \left(p_{1,n_0}(X^*_{n_0}), \ldots, p_{n_0-1,n_0}(X^*_{n_0}), X^*_{n_0}, X^*_{n_0+1}, \ldots\right).$$

Now we prove that $X_n^* = \{x\}$ for every $n \geqslant n_0$. Let $y_0 \in X_{n_0}$. Then, $y \in X^*_{n_0}$ if and only if there exists, for every $i \in \mathbb{N}$, $y_i \in A_{n_0+i}$ such that $y \in p_{n_0+i,n_0+i+1}(y_{i+1})$ and $y_i \in X_{n_0+1}$ for every $i \in \mathbb{N}$. We are going to see that, if there is a chain of points satisfiying the first condition, they cannot satisfy the second. So, let us suppose there exists a chain $y_i \in A_{n_0+i}$, for every $i \in \mathbb{N}$ such that one belongs to the image of the following. For the sake of simplicity, let us write $d_i := \mathrm{d}(x, y_i)$ for $i \in \mathbb{N}$, (and $d_0 = \mathrm{d}(x, y_0)$). For every $i \in \mathbb{N}$, y_{i+1} is closer (or at the same distance) to y_i than to x, so we have

$$d_{i+1} \geqslant \mathrm{d}(y_i, y_{i+1}) < \gamma_{n_0+i}.$$

Moreover, it is obvious that $d_i \leqslant d_{i+1} + \mathrm{d}(y_{i+1}, y_i)$, i.e., $d_i - d_{i+1} \leqslant \mathrm{d}(y_{i+1}, y_i)$. Combining this with the previous observation, we get $d_i - d_{i+1} < \gamma_{n_0+1}$. On the other hand, we have that, for every $i \in \mathbb{N}$, $d_i \leqslant d_{i+1} + \mathrm{d}(y_{i+1}, y_i) \leqslant 2d_{i+1}$, so $d_{i+m} \geqslant \frac{d_i}{2^m}$. We supposed $y_0 \in X_{n_0}$, so $\varepsilon_{n_0} - d_0 > 0$. We claim that, for every $i \in \mathbb{N}$,

$$\varepsilon_{n_0+i} - d_i < \frac{2\varepsilon_{n_0} - (i+2)d_0}{2^{i+1}}.$$

We prove it by induction. The first case is

$$\begin{aligned}
\varepsilon_{n_0+1} - d_1 &< \frac{\varepsilon_{n_0} - \gamma_{n_0}}{2} - d_1 < \frac{\varepsilon_{n_0} - d_0}{2} - \frac{d_1}{2} \\
&\leqslant \frac{\varepsilon_{n_0} - d_0}{2} - \frac{d_0}{2^2} = \frac{2\varepsilon_{n_0} - 3d_0}{2^2}.
\end{aligned}$$

Now, suppose the hypothesis of induction is satisfied, and we check

$$\begin{aligned}\varepsilon_{n_0+i+1} - d_{i+1} &< \frac{\varepsilon_{n_0+i} - \gamma_{n_0+i}}{2} - d_{i+1} < \frac{\varepsilon_{n_0+i} - d_i}{2} - \frac{d_{i+1}}{2} \\ &< \frac{2\varepsilon_{n_0} - (i+2)d_0}{2^{i+2}} - \frac{d_0}{2^{i+2}} = \frac{2\varepsilon_{n_0} - (i+3)d_0}{2^{i+2}}.\end{aligned}$$

It is obvious that there exists an $i \in \mathbb{N}$ such that $(i+3)d_0 > 2\varepsilon_{n_0}$. For this i, we have that $\varepsilon_{n_0} - d_i < 0$, so $y_i \notin X_{n_0+i}$, and then, $y_0 \notin X^*_{n_0}$. We conclude $X^*_{n_0} = \{x\}$ and the same argument can be applied to show that $X^*_n = \{x\}$, for every $n \geqslant n_0$ ✓

3.2 Some special features

In view of Proposition 21, it is natural to look for FAS making the map φ the more "injective" posible, i.e., injective in the largest possible set of points. The first observation we can do is

Remark 14. For every FAS of X, the set $\bigcup_{n\in\mathbb{N}} A_n$ is always dense in X. For each open set $U \subset X$ there exists $x \in U$ and $\varepsilon > 0$ such that $\mathrm{B}(x,\varepsilon) \subset U$. Let us select $n_0 \in \mathbb{N}$ such that $\mathrm{B}(x, \varepsilon_{n_0}) \subset \mathrm{B}(x,\varepsilon)$. Then, for any $a \in A_{n_0}$ with $\mathrm{d}(x,a) < \varepsilon_{n_0}$, we have that $\mathrm{B}(x,\varepsilon) \cap A_{n_0} \neq \emptyset$. This also shows that every compact metric space has a countable dense subset.

We want to apply Proposition 21 to obtain inyectivity in a dense subset of X. We can obtain the following

Construction 1. For every compact metric space X, there exists a FAS with $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$: If $M = \mathrm{diam}(X)$, let us consider $\varepsilon_1 > M$, and $A_1 = \{x\}$ with $x \in X$. Then, consider ε_2 with the usual rule. Now, for A_2, we take the union $A_2 = A'_2 \cap A_1$ where A'_2 is a ε_2 approximation of X, then so is A_2. We can proceed in this way for every $n \in \mathbb{N}$. If we have that A_n is a ε_n approximation of X, consider γ_n and δ_n and take ε_{n+1} as allways. Then consider A_{n+1} as the union $A'_{n+1} \cap A_n$ where A'_{n+1} is a ε_{n+1} approximation of X and, hence, A_{n+1} too. In this way, we obtain the desired FAS of X.

The best situations we can have are the following.

3.2.1 Countable spaces

Proposition 22. *Let X be a countable metric space. Then there exists a FAS of X such that the inverse limit $\mathcal{X}$ is homeomorphic to X.*

Proof. We can write $X = \{x_1, x_2, \ldots, x_n, \ldots\}$. We just need to find a FAS for X satisfying $x_n \in A_n$ and $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$. The first condition gives us $\bigcup_{n\in\mathbb{N}} A_n = X$ and the second one will make φ injective on the set

$$\varphi^{-1}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \varphi^{-1}(X) = \mathcal{X},$$

and then, $\varphi : \mathcal{X} \to X$ will be a homeomorphism. There are many ways of doing so. We can just do the general construction forcing each A_n to contain x_n and A_m, for every $m < n$. For example, if we consider, for every $n \in \mathbb{N}$, the numbers

$$r(n) = \min\left\{i \in \mathbb{N} : \{x_1, \ldots, x_i\} \text{ is a } \varepsilon_n \text{ approximation of } X\right\},$$

it is clear that $r(n+1) \geqslant r(n)$ and then we can write the approximations as

$$\begin{aligned} A_1 &= \{x_1\} \\ A_2 &= \{x_1, \ldots, x_{r(2)}\} \\ &\ldots \\ A_n &= \{x_1, \ldots, x_{r(n)}\} \\ &\ldots \end{aligned}$$

and we are done ✓

Now we face the case of proper dense subsets of X. First of all, we observe the following

Remark 15. For every dense subset $Y \subset X$ of a compact metric space, and every $\varepsilon > 0$, there exists an ε-approximation $A \subset Y$: Let us consider the covering $\{B(x, \frac{\varepsilon}{2}) : x \in X\}$ and a finite subcovering $\{B(x_1, \frac{\varepsilon}{2}), \ldots, B(x_k, \frac{\varepsilon}{2})\}$. Now we take $y_1, \ldots, y_k \in Y$ such that $d(x_i, y_i) < \frac{\varepsilon}{2}$ for every $i = 1, \ldots, k$, so $\{y_1, \ldots, y_k\}$ is an ε-approximation of X.

We can state the main result in this direction

Proposition 23. *For every countable dense subset of a compact metric space, $Y \subset X$, there exists a FAS of X such that there is a dense subset of $\mathcal{X}$ which is homeomorphic to Y.*

Proof. If we write $Y = \{y_1, y_2, \ldots, y_n, \ldots\}$, it is easy to obtain a FAS of X such that $A_n \subset A_{n+1}$, for every $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} A_n = Y$. For example, using the previous remark, we can take as approximations

$$\begin{aligned}
A_1 &= \{y_1\} \\
A_2 &= \{y_2\} \cup A_1 \cup A_2' \text{ with } A_2' \subset Y \text{ an } \varepsilon_2\text{-approximation of } X, \\
A_3 &= \{y_3\} \cup A_2 \cup A_3' \text{ with } A_3' \subset Y \text{ an } \varepsilon_3\text{-approximation of } X, \\
&\ldots \\
A_n &= \{y_n\} \cup A_{n-1} \cup A_n' \text{ with } A_n' \subset Y \text{ an } \varepsilon_n\text{-approximation of } X, \\
&\ldots
\end{aligned}$$

If we restrict the map $\varphi : \mathcal{X} \to X \supset Y = \bigcup_{n\in\mathbb{N}} A_n$ to the set $\varphi^{-1}(Y)$ we obtain that

$$\varphi\,|_{\varphi^{-1}(Y)}\colon \varphi^{-1}(Y) \longrightarrow Y$$

is injective and hence a homeomorphism. So φ^{-1} is the desired set. We have the inclusions $\varphi^{-1}(Y) \subset \mathcal{X}^* \subset \mathcal{X}$, by construction (recall Proposition 21). Now, to see that $\varphi^{-1}(Y)$ is dense in $\mathcal{X}^*$. Let V be any open set of $\mathcal{X}^*$ and $C \in V$ any point of it, where $C = (C_1, C_2, \ldots, C_n, \ldots)$. Choose an open neighborhood from the basis

$$C \in W = (2^{C_1} \times \ldots \times 2^{C_m} \times U_{2\varepsilon_{m+1}}(A_{m+1}) \times \ldots) \cap \mathcal{X} \subset V$$

and select any $c \in C_m$. Then, $c^* = (\ldots, c, c, \ldots)$, because $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$. So $c^* \in W \cap \varphi^{-1}(Y) \subset V \cap \varphi^{-1}(Y)$, which implies $\overline{\varphi^{-1}(Y)} = \mathcal{X}^*$ ✓

Remark 16. The inclusion $\varphi^{-1}(Y)$ of last proposition is proper: Recall example 4 where, $Y = \bigcup_{n\in\mathbb{N}} A_n$ where $A_n = \left\{\frac{k}{3^{2n-3}} : k = 0, 1, \ldots, 3^{2n-3}\right\}$ and, while $\frac{1}{2}^*$ is obviously an element of $\mathcal{X}^*$, it does not belong to $\varphi^{-1}(Y)$, since $\frac{1}{2}$ does not belong to any approximation A_n.

3.2.2 Ultrametric spaces

An ultrametric space X is a metric space with an extra property of the distance. Instead of satisfying just the triangle inequality, they satisfy the strong triangle inequality, that is:

$$\forall x, y, z \in X, \quad d(x, y) \leqslant \max \{d(x, z), d(y, z)\} .$$

This inequality gives us some properties that make the ultrametric spaces very special ones. For example, it is satisfied[3]

· Every triangle is isosceles, with the non equal side smaller than the other two.

· For every $x, y \in X$ and $\varepsilon \geqslant \delta > 0$, $B(x, \varepsilon) \cap B(y, \delta) \neq \emptyset$ implies that $B(y, \delta) \subset B(x, \varepsilon)$.

We want to show that, for the case of ultrametric spaces, there exist FASs such that they recover the topological type of the space. The key idea here, is that for those spaces there are very special approximations:

Lemma 8. *Let X be a compact ultrametric space. For every $\varepsilon > 0$, there exists an ε-approximation of X, $\{x_1, \ldots, x_k\}$, such that $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) = \emptyset$ for every $i \neq j$.*

Proof. The covering by open balls $\{B(x, \varepsilon) : x \in X\}$ of X has a finite subcover $\{B(x_1, \varepsilon), \ldots, B(x_k, \})$. So, $\{x_1, \ldots, x_k\}$ is an ε-approximation of X. Now for any $i \neq j$ such that $B(x_i, \varepsilon) \cap B(x_j, \varepsilon) \neq \emptyset$ it turns out that $B(x_i, \varepsilon) = B(x_j, \varepsilon)$ ✓

We can state the main theorem about ultrametric spaces.

Theorem 15. *Let X be a compact ultrametric space, then, there exists a FAS $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$, such that $\mathcal{X} = X$.*

Proof. Let us consider any FAS of X satisfying the property stated in the previous lemma. Then, for every $x \in X$ and every $n \in \mathbb{N}$, we have that $\mathrm{card}(q_{A_n}(x)) = 1$: Let us suppose that $a_1, a_2 \in q_{A_n}(x)$. Then, $d(x, a_1), d(x, a_2) < \gamma_n < \varepsilon_n$ but,

[3] See chapter 2 of [59] for more properties and detailed proofs about ultrametric spaces.

in that case, we will have that $x \in \mathrm{B}(a_1, \varepsilon_n) \cap \mathrm{B}(a_2, \varepsilon_n)$ which is not possible. Then, $q_{A_n} : X \to A_n$ is actually a single valued continuous map. Moreover, if we restric to A_{n+1}, we obtain that

$$q_{A_n} \mid_{A_{n+1}} = p_{n,n+1} \mid_{A_{n+1}} : A_{n+1} \longrightarrow A_n$$

is a continuous map. So, it makes sense to write the following diagram,

$$\begin{array}{ccc} X & & \\ {\scriptstyle q_{A_{n+1}}}\downarrow & \searrow^{q_{A_n}} & \\ A_{n+1} & \xrightarrow[p_{n+1,n}]{} & A_n, \end{array}$$

which, moreover, is commutative (compare with Proposition 11). If it would not be, then there would exist $a_1, a_2 \in A_n$ with $q_{A_n}(x) = a_1$ and $p_{n,n+1} q_{A_{n+1}}(x) = a_2$. Clearly, $\mathrm{d}(x, a_1) < \varepsilon_n$, but also

$$\begin{aligned} \mathrm{d}(x, a_2) &\leqslant \mathrm{d}(x, q_{A_{n+1}}(x)) + \mathrm{d}(q_{A_{n+1}}(x), p_{n,n+1} q_{A_{n+1}}(x)) < \\ &< \gamma_{n+1} + \gamma_n < \varepsilon_{n+1} + \gamma_n < \frac{\varepsilon_n - \gamma_n}{2} + \gamma_n < \varepsilon_n. \end{aligned}$$

and this is imposible, since then $x \in \mathrm{B}(a_1, \varepsilon_n) \cap \mathrm{B}(a_2, \varepsilon_n)$. Adding that q_{A_n} is always a surjective map distinguishing points of X (see corollary 3 on page 61 of [47]), we have that X is the inverse limit $X = \lim_{\leftarrow}(A_n, p_{n,n+1})$. Now, it remains to see that every element of the inverse limit $C = (C_1, C_2, \ldots, C_n, \ldots) \in \mathcal{X}$ satisfies that $\mathrm{card}(C_n) = 1$ for every $n \in \mathbb{N}$. If not, for any pair $a_1, a_2 \in C_n$ we would have that $\mathrm{d}(x, a_1), \mathrm{d}(x, a_2) < \varepsilon$, with $x = \lim_H \{C_n\}$, which, again, is not possible. So, we have that

$$\mathcal{X} = \lim_{\leftarrow} \left(U_{2\varepsilon_n}(A_n), p_{n,n+1} \right) = \lim_{\leftarrow} (A_n, p_{n,n+1}) = X \quad \checkmark$$

3.3 Previous work on finite approximations

Our construction is a sequence of finite spaces, which, in the limit, are able to reflect the shape and homotopy properties of the original space. Its main features are:

- It is internal: It is constructed from the space itself, without need of external ambient spaces to approximate them. We use the hyperspace, which is constructed just in terms of the compac metric space.

- It is constructive: Given a space explicitly, we can actually select points for each approximation. This is important, since it allows us to perform explictly the construction over the space, and possibly determine some topological structure, up to some error.

There exist previous results on the approximation of topological spaces by finite spaces. This is an old theme, but nowadays it is becoming more important because of its role in the emerging field of computational topology. In this section we will review some of these results and compare them with ours.

3.3.1 Approximation of compact polyhedra

There is a paper of E. Clader [19] where the following theorem is proved:

Theorem 16 (E. Clader). *Every compact polyhedron is homotopy equivalent to the inverse limit of a sequence of finite spaces.*

The proof consists of taking as finite spaces the vertices of the barycentric subdivisions of the simplicial complexes defining the compact polyhedron. Given a simplicial complex, the McCord correspondence assigns a finite T_0 space. Clader assigned the opposite topology to these finite spaces. That is, consider for every $n \in \mathbb{N}$, the n-th barycentric subdivision $K^{(n)}$ and the finite space $F_n = \mathcal{X}(K^n)^{op}$, that is, the n-th barycentric subdivision of the finite space $\mathcal{X}(K)$ with the opposite topology of that assigned by McCord. Then, there is a natural map p_n from $|K|$ to each F_n, because every point of $|K|$ belongs to a unique simplex of $K^{(n)}$. For every $n > 1$, there is a map $q_n : F_n \to F_{n-1}$ closing the diagram with p_n and p_{n-1}. Then, it is shown that the polyhedron is a retract of the inverse limit of these finite spaces and maps. Note that every compact polyhedron is a compact metric space (for details of the metric, see, for example the appendix on polyhedra of [47]). So, this theorem is a particular case of corollary 7.

3.3.2 Finite approximations and Hausdorff reflections

In a series of papers, R. Kopperman *et al.* ([37], [38]) proved the following theorem about finite approximations.

Theorem 17 (R. Kopperman, R. Wilson)**.** *Every compact Hausdorff space is the Hausdorff reflection of the inverse limit of an inverse system of finite spaces.*

The finite spaces involved in this proof are constructed in a very theoretical way. It is considered the set of all possible open coverings of the space and then the spaces are defined with a boolean algebra on the open sets of the coverings. This theorem has the advantage that it is very general: It works for any compact Hausdorff space, with no need for a metric. But it has the desadvantage that it is not constructible and that we loose a lot of intuition with the Hausdorff reflection.

The idea of a reflection of a topological space is to construct another space, as similar as posible to the first one with an extra separation and a universal map. Somehow it is similar to compactification. Concretely, given a topological space X and a separation property T, we will say that $\mu_X : X \to X_T$ is the *T reflection* of X if μ_X is a continuous map, X_T has the property T and every continuous map $f : X \to Z$ with Z having property T, factors through a map $g : X_T \to Z$, i.e., the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle \mu_X}\downarrow & \nearrow_{g} & \\ X_T & & \end{array}$$

commutes. If the map μ_X is surjective we will say that the reflection is surjective, too. For some properties, the existence of reflectors is a well known fact.

Theorem 18. *(see [54], chapter 14)Let X be a topological space. For T being the separation properties T_i, $i = 0, 1, 2, 3, 3\frac{1}{2}$ there exists a surjective reflection.*

It is easy to see that two reflections of the same space are homeomorphic.

In many cases, reflections are obtained as quotient spaces (not in every case, as for example the Tychonoff functor -or reflection- which is not obtained as a quotient) for a relation. Nevertheless, it is not allways obtained as the obvious relation. As a matter of fact, in order to obtain the Hausdorff reflection we need to define the following relations (see the reference [62], a short and beautiful paper about reflections, where this is showed):

- xR_1y iff for every pair of neighborhoods U_x, U_y of x, y resp., we have $U_x \bigcap U_y \neq \emptyset$.
- xR_2y iff there exist $x = z_1, z_2, \ldots, z_n = y$ such that $z_1R_1z_2R_1 \ldots R_1z_n$.
- xR_3y iff for every $f : X \to Z$, with Z Hausdorff, we have $f(x) = f(y)$.

Then, the Hausdorff reflection of X is the quotient space $X_H = X/R_3$.

We want to compare the Hausdorff reflection of a topological space with the space itself in terms of shape type. As a motivation, we can cite [53], where it is shown that the Tychonoff functor indeed induces the identity morphism in shape. So, a topological space and its Tychonoff reflection have the same shape. We will show the same for the Hausdorff reflection.

Lemma 9. *The Hausdorff reflection of the product $X \times I$, where $I = [0, 1]$, is homeomorphic to $X_H \times I$.*

Proof. Consider the continuous map

$$\begin{array}{rcl} f : X \times I & \longrightarrow & X_H \times I \\ (x, t) & \longmapsto & (\mu_X(x), t), \end{array}$$

which is a quotient map. Moreover, the space $X_H \times I$ is Hausdorff, so there exists a continuous surjective map $h : (X \times I)_H \to X_H \times I$ such that the diagram

$$\begin{array}{ccc} X \times I & \xrightarrow{\quad f \quad} & X_H \times I \\ {\scriptstyle \mu_{X\times I}}\downarrow & \nearrow{\scriptstyle h} & \\ (X \times I)_H & & \end{array}$$

commutes. We see that h is actually a homeomorphism. First of all, h is a quotient map, because f and $\mu_{X\times I}$ are ([28], pag 91). Also, it is an injective map. Indeed, let $[a], [b] \in (X \times I)_H$ such that $h([a]) = h([b]) = ([z], t)$. Considering that $\mu_{X\times I}$ is surjective, there exist $(x, t_1), (x, t_2)$ such that $\mu_{X\times I}(x, t_1) = [a]$ and

$\mu_{X\times I}(y, t_2) = [b]$. Because of the commutativity of the previous diagram we have that

$$([x], t_1) = f(x, t_1) = h(\mu_{X\times I}(x, t_1)) = h([a]) = ([z], t)$$
$$([y], t_2) = f(y, t_2) = h(\mu_{X\times I}(y, t_2)) = h([b]) = ([z], t)$$

so $[x] = [y] = [z]$ and $t_1 = t_2 = t$. For this concrete t, we consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{id\times t} & X \times I \\ \downarrow{\scriptstyle \mu_X} & & \downarrow{\scriptstyle \mu_{X\times I}} \\ X_H & \xrightarrow[g]{} & (X \times I)_H \end{array},$$

which exists for being $\mu_{X\times I} \circ (id \times t) : X \to (X \times I)_H$ a continuous map to a Hausdorff space. We consider the images of x, y by the two different maps of the diagram. As $[x] = [y]$, we obtain that $[a] = [b]$, so h is injective. A quotient and injective map is a homeomorphism ✓

Theorem 19. *For every topological space X, the Hausdorff reflection $\mu_X : X \to X_H$ induces the identity morphism in shape.*

Proof. To show this, we are going to use the characterization of identity morphisms in shape, Theorem 4. So, $\mu_X : X \to X_H$ is the identity morphism in shape if and only if the map

$$\begin{array}{rcl} [X_H, P] & \longrightarrow & [X, P] \\ h & \longmapsto & h \cdot f, \end{array}$$

with P being any metric ANR, is bijective.

It is surjective: Given a map $g : X \to P$, with P ANR and then, Hausdorff, there exists a map $h : X_H \to P$ such that $g = h \cdot \mu_X$, that is, what we wanted. It is injective: Let $h_1, h_2 : X_H \to P$, with P ANR, two continuous maps such that $h_1 \cdot \mu_X$ y $h_2 \cdot \mu_X$ are homotopic, i.e., there exists a continuous map, $G : X \times I \to P$ such that $G(x, 0) = h_1 \cdot \mu_X(x)$ and $G(x, 1) = h_2 \cdot \mu_X(x)$. Being P Hausdorff, there exists a continuous map $F : (X \times I)_H \to P$ such that $G = F \cdot \mu_{X\times I}$. Applying the previous lemma, we get $\mu_{X\times I} = \mu_X \times id$, so we have that the following diagram commutes:

$$\begin{array}{ccc} X \times I & \xrightarrow{G} & P\,. \\ \Big\downarrow{\scriptstyle \mu_X \times id} & \nearrow_{F} & \\ X_H \times I & & \end{array}$$

Then, for every $x \in X$, we have

$$\begin{aligned} F([x],0) &= G(x,0) = h_1 \cdot \mu_X(x) = h_1([x]) \\ F([x],1) &= G(x,1) = h_2 \cdot \mu_X(x) = h_2([x]). \end{aligned}$$

So, h_1 and h_2 are homotopic ✓

Corollary 5. *A topological space X has the same shape than its Hausdorff reflection X_H.*

Note that with Theorem 17 and the result just proved here about the shape of the Hausdorff reflection we will get the following generalization.

Corollary 6. *Every compact Hausdorff space has the same shape as the inverse limit of an inverse system of finite spaces.*

In an attempt of understanding better the Hausdorff reflection of an inverse system of spaces, Kopperman and Wilson proved that the original space is not only the Hausdorff reflection but the set of closed points of the inverse limit. We can prove the same in our construction.

Proposition 24. *For every compact metric space X and every FAS of X, $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$, the space $\mathcal{X}^*$ is just the set of closed points of $\mathcal{X}$. Moreover it is its Hausdorff reflection $\mathcal{X}^* = X_H$.*

Proof. First of all, we are going to characterize, for every $x \in X$ the point of the inverse limit $X^* = \phi(x)$. It is the set

$$X^* = \bigcap_{C \in \varphi^{-1}(x)} \overline{\{C\}}.$$

We divide the proof:

($\subset$) We show here that if $\varphi(X^*) = \varphi(C) = x$ (notation: $X^* = (X_1^*, X_2^*, \ldots)$), then $X^* \in \overline{\{C\}}$. Let $X^* \in V$ an open neighborhood in $\mathcal{X}$. Then, there exists an open neighborhood

$$X^* \in U = (2^{X_1^*} \times 2^{X_2^*} \times \ldots \times 2^{X_r^*} \times U_{2\varepsilon_{r+1}}(A_{r+1}) \times \ldots) \cap \mathcal{X}.$$

But, obviously, $C \in U$, so $C \in U \cap \{C\} \neq \emptyset$.

($\supset$) Let $D = (D_1, D_2, \ldots) \in \bigcap_{C \in \varphi^{-1}(x)} \overline{\{C\}}$.Then $\{D_n\}$ converges to x in the Hausdorf metric. So, for every $D \in U$ open neighborhood, we have that $U \cap \{X^*\} \neq \emptyset$. In particular, for every $r \in \mathbb{N}$ we have neighborhoods of the form

$$(2^{D_1} \times 2^{D_2} \times \ldots \times 2^{D_r} \times U_{2\varepsilon_{r+1}}(A_{r+1}) \times \ldots) \cap \mathcal{X},$$

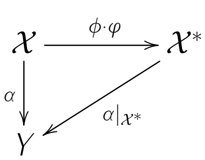
where X^* belongs. So, for every $r \in \mathbb{N}$ we have $X_r^* = D_r$, hence $X^* = D$.

Now to show that $\mathcal{X}^*$ is the set of closed points, first observe that every $X^* \in \mathcal{X}^*$ is $X^* = \bigcap_{C \in \varphi^{-1}(x)} \overline{\{C\}}$, so a closed set. On the other hand, if there is a closed point $C \in \mathcal{X}$, with $\varphi(C) = y$ then $Y^* \in \overline{\{C\}} = \{C\}$ so $C = Y^* \in \mathcal{X}^*$.

To show that $\mathcal{X}^*$ is actually the Hausdorff reflection of $\mathcal{X}$, let us consider a continuous map $\alpha : \mathcal{X} \to Y$ with Ya Hausdorff space. Consider two points $C, C' \in \mathcal{X}$ such that $\varphi(C) = \varphi(C') = x = \varphi(X^*)$, with $X^* \in \mathcal{X}^*$. Then, using the previous characterization of X^*, we have that $X^* \in \overline{\{C\}} \cap \overline{\{C'\}}$. Then, applying the map α, and using that it is continuous and that Y is Hausdorff, we obtain

$$\begin{aligned} \alpha(X^*) &\in \alpha(\overline{\{C\}}) \cap \alpha(\overline{\{C'\}}) \subset \\ &\subset \overline{\{\alpha(C)\}} \cap \overline{\{\alpha(C')\}} = \\ &= \{\alpha(C)\} \cap \{\alpha(C')\}, \end{aligned}$$

so, $\alpha(X^*) = \alpha(C) = \alpha(C')$. Now, we claim that the map $\phi \cdot \varphi : \mathcal{X} \to \mathcal{X}^*$ is actually the Hausdorff reflection of $\mathcal{X}$. This is so, because the map $\alpha \mid_{\mathcal{X}^*} : \mathcal{X}^* \to Y$ makes the diagram



commutative and $\alpha\mid_{\mathcal{X}^*}$ is continuous since $\phi\cdot\varphi$ is a retraction and hence a identification ✓

3.4 Generalization to hyperspaces

Let X be a compact metric space and 2^X_u its hyperspace with the upper semifinite topology. If we perform the main construction on X applying Theorem 8, we obtain sequences $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$, such that, for every $n\in\mathbb{N}$, A_n is an ε_n approximation of X and $\{\varepsilon_n\}$ is a decreasing sequence tending to zero. We also have continuous maps $p_{n,n+1} : U_{2\varepsilon_{n+1}}(A_n) \to U_{2\varepsilon_n}(A_n)$ for every $n\in\mathbb{N}$. These maps can be extended to the hyperspaces $2^{A_n}_u$ (for short, 2^{A_n}) of the approximations with the upper semifinite topology in the obvious way,

$$\begin{aligned} p_{n,n+1} : 2^{A_{n+1}} &\longrightarrow 2^{A_n} \\ C &\longmapsto r_{A_n}(C). \end{aligned}$$

They are continuous, because, as Alexandroff spaces, the order is preserved by the map. Moreover, they make the following diagram, where the maps $i : U_{2\varepsilon_n}(A_n) \hookrightarrow 2^{A_n}$ are just inclusions, commutative:

$$\begin{array}{ccc} 2^{A_{n+1}} & \xrightarrow{p_{n,n+1}} & 2^{A_n} \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ U_{2\varepsilon_{n+1}}(A_{n+1}) & \xrightarrow[p_{n,n+1}]{} & U_{2\varepsilon_n}(A_n) \end{array}$$

We can obtain the hyperspace of X with the upper semifinite topology as an inverse sequence of finite spaces, up to homotopy.

Theorem 20. *Let X be a compact metric space and consider sequences $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$ from the main construction. The inverse limit $\mathcal{A}$ of the inverse sequence $\left\{2^{A_n}, p_{n,n+1}\right\}$ has the homotopy type of 2^X_u. Moreover, there is a subspace $\mathcal{A}^*\subset\mathcal{A}$ homeomorphic to 2^X_u.*

We first prove the following technical and easy lemma, which will be useful in the proof.

Lemma 10. *Let (X, d) be a compact metric space. For every $C, D \subset X$. Then, for every $0 < \delta < \varepsilon$ and $\eta \leqslant \varepsilon - \delta$, we have that, if $D \subset B(C, \eta)$, then $B(D, \delta) \subset B(C, \varepsilon)$. In particular, if $D \subset B(C, \varepsilon - \delta)$, then $B(D, \delta) \subset B(C, \varepsilon)$.*

Proof. Let $d' \in \mathrm{B}(D, \delta)$, there exists $d \in D$ such that $\mathrm{d}(d', c) < \delta$. By hypothesis, there exists $c \in C$ such that $\mathrm{d}(d, c) < \eta$. Then,

$$\mathrm{d}(d', c) \leqslant \mathrm{d}(d', d) + \mathrm{d}(d, c) < \delta + \eta < \varepsilon \quad \checkmark$$

Remark 17. The converse of this lemma is obviously false. For instance, Let X be the interval $I = [0, 1]$, $C = \{0\}$ and $D = \{1\}$. Then $\mathrm{B}\left(D, \frac{1}{2}\right) \subset \mathrm{B}\left(C, \frac{3}{2}\right)$, but $D \in \mathrm{B}(C, 1)$.

Proof of Theorem 20. The proof follows the same steps as Theorem 14, altough we will need to add some extra proofs. The interpretation of the inverse sequence of points with the Hausdorff distance comes from the quoted theorem. Every point of the inverse limit $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ is a Cauchy sequence in 2^X_H converging to a unique point $C \in 2^X$. Also, we have that, in terms of the Hausdorff metric, the diference between two elements of the sequence depends only on the lower one. Let $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ be a point of the inverse limit. Then, the Hausdorff distance between terms of the sequence C_n and C_m, with $n < m$, is $\mathrm{d_H}(C_n, C_m) < \overline{\varepsilon_n}$. This leads (as in the other case) to a bound for the limit: $\mathrm{d_H}(C, C_n) < \varepsilon_n$.

Define the map $\varphi : \mathcal{A} \to 2^X_u$ assigning to every sequence $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ the unique point in the limit $C = \lim\{C_n\}_{n\in\mathbb{N}}$ in the Hausdorff metric. The map $\varphi : \mathcal{A} \to 2^X_u$, sending $\{C_n\}_{n\in\mathbb{N}}$ to C is continuous. As in the previous case, we will show that it is continuous in every point. Let U be an open neighborhood of C in 2^X_u. Assume $C \in B(V) \subset U$, with V open neighborhood containing C. Consider $\varepsilon > 0$ such that $\mathrm{B}(C, \varepsilon) \subset V$. Hence, $B(\mathrm{B}(C, \varepsilon)) \subset B(V)$. Let $n_0 \in \mathbb{N}$ be a natural number satisfying that, for every $n \geqslant n_0$, $\varepsilon_n < \frac{\varepsilon}{2}$. Consider the open neighborhood of $\{C_n\}$,

$$W = \left(2^{C_1} \times 2^{C_2} \times \ldots \times 2^{C_{n_o}} \times 2^{A_{n_0+1}} \times \ldots\right) \cap \mathcal{A}.$$

Let $\{D_n\} \in W$ be an element of this neighborhood an suppose $D = \lim_H\{D_n\}$. Then,

$$D \subset \mathrm{B}(D_{n_0}, \varepsilon_{n_0}) \subset \mathrm{B}(C_{n_0}, \varepsilon_{n_0})\, \mathrm{B}(C, \varepsilon),$$

because $\mathrm{d_H}(D, D_{n_0})$, $D_{n_0} \subset C_{n_0}$ and Lemma 10 applied to the inclusion

$$C_{n_0} \subset \mathrm{B}(C, \varepsilon_{n_0}) \subset \mathrm{B}(C, \varepsilon - \varepsilon_{n_0}),$$

which is true since $\varepsilon_{n_0} < \frac{\varepsilon}{2} < \varepsilon - \varepsilon_{n_0}$. Hence, $D \subset B(V) \subset U$, and φ is continuous.

Now, we define an inverse for φ as in the previous case. For every $C \in 2^X_u$, define for each $n \in \mathbb{N}$, the sets $C^n = \mathrm{B}(C, \varepsilon_n) \cap A_n$, finite and non-empty subsets of X, and

$$C_n^* = \bigcap_{m>n} p_{n,m}(C^m),$$

also non-empty sets (nested sequence of closed sets). To show that it is indeed a nested sequence, is slightly different from the simpler case. We have to show that, for every $C \in 2^X_u$ and $n < m$, $p_{n,m+1}(C^{m+1}) \subset p_{n,m}(C^m)$. As before, it is enough to show that $p_{m,m+1}(C^{m+1}) \subset C^m$. Let us consider an element $d \in p_{m,m+1}(C^{m+1})$. There exists an element $c \in C^{m+1}$ such that $d \in p_{m,m+1}(\{c\})$, and hence, $\mathrm{d}(d, c) < \overline{\varepsilon_m}$. If $c \in C^{m+1}$, there exists $c' \in C$ such that $\mathrm{d}(c, c') < \varepsilon_{m+1}$. Then, we have

$$\mathrm{d}(d, c') \leqslant \mathrm{d}(d, c) + \mathrm{d}(c, c') < \overline{\varepsilon_m} + \varepsilon_{m+1} < \varepsilon_m,$$

and hence $d \in C^m$. To show that actually $\{C_n^*\}$ is an element of the inverse limit $\mathcal{A}$, we just repeat the arguments of the previous case, where we showed that, for every $x \in X$, $\{X_n^*\} \in \mathcal{X}$. It is not used there the fact that the diammeter of the set $\{x\}$ is zero. We also define the map $* : \mathbb{N} \to \mathbb{N}$ as before. What needs to be proved is that $\lim_H\{C_n^*\} = C$. That is, for every $\varepsilon > 0$ we want a natural number n_0 such that, for every $n \geqslant n_0$, $\mathrm{d_H}(C, C_n^*) < \varepsilon$. We claim that, for every $n \in \mathbb{N}$, $\mathrm{d_H}(C, C_n^*) < \varepsilon_n$. By definition, we have that $C_n^* \subset \mathrm{B}(C, \varepsilon_n)$, for every $n \in \mathbb{N}$. On the other hand, we want to show, for every $n \in \mathbb{N}$, that, for every $c \in C$, there is a point $c^* \in C_n^*$ such that $\mathrm{d}(c, c^*) < \varepsilon_n$. Consider

$$c^n = \mathrm{B}(c, \varepsilon_n) \cap A_n \subset \mathrm{B}(C, \varepsilon_n) \cap A_n = C^n,$$

for every $n \in \mathbb{N}$. We know from the proof of Theorem 14, that the set c^n is non-empty and that $p_{n,m}(c^m) \subset c^n$. Moreover, the sets $\bigcap_{m>n} p_{n,m}(c^m)$ are not empty, and, by construction,

$$\bigcap_{m>n} p_{n,m}(c^m) \subset \bigcap_{m>n} p_{n,m}(C^m) = C_n^*,$$

so, any element $c^* \in \bigcap_{m>n} p_{n,m}(c^m)$ satisfies $c^* \in C_n^*$ and $\mathrm{d}(c, c^*) < \varepsilon_n$.

Here, we also have the maximal property: For every $\{C_n\} \in \mathcal{A}$, such that $C = \varphi(\{C_n\})$, we have that $C_n \subset C_n^*$, for every $n \in \mathbb{N}$. It can be proved exactly the same way as in the proof of Theorem 14.

We define the map $\phi : 2_u^X \to \mathcal{A}$ with $\phi(C) = \{C_n^*\}$. This map is shown to be continuous following the same arguments as in the proof of Theorem 14 adapted to a set C instead of a point x. Given an open neighborhood V, containing $\{C_n^*\}$, there is a basic open subset

$$\{C_n^*\} \subset W = \left(2^{C_1^*} \times \ldots \times 2^{C_r^*} \times 2^{A_{r+1}} \times \ldots\right) \cap \mathcal{A} \subset V.$$

Consider $s = *(r)$, and

$$\varepsilon_s^+(C) = \left\{\mathrm{d}(a, C) : a \in A_s \setminus \overline{X^s}\right\}.$$

Then, the open neighborhood $B(U)$ of C, where $U = \mathrm{B}(C, \delta)$, with $\delta < \min\left\{\varepsilon_s^+(C) - \varepsilon_s, \underline{\varepsilon_s}\right\}$, satisfies that, for every $D \in B(U)$, $\phi(D) \subset W \subset V$. Hence ϕ is continuous. Note that we have not required C to be finite anywhere, and it is enough to be closed, so the domain of the map is the whole hyperspace.

Next, we show that $\phi : 2_u^X \to \mathcal{A}$ is injective. First, we are going to prove the following claim: If $C, D \in 2_u^X$ satisfy $C \subsetneq D$, then, $C_n^* \subset D_n^*$ for every $n \in \mathbb{N}$ and there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$, $C_n^* \subsetneq D_n^*$: Indeed, if $C \subset D$, then $C^n \subset D^n$ for every $n \in \mathbb{N}$, and then $C_n^* \subset D_n^*$. If there is no $n_0 \in \mathbb{N}$ such that $C_{n_0}^* \neq D_{n_0}^*$, then $\{C_n^*\} = \{D_n^*\}$ and $C = D$. For every $n > n_0$, $C_n^* \neq D_n^*$, because if not, $C_{n_0}^* = p_{n_0,n}(C_n^*) = p_{n_0,n}(D_n^*) = D_{n_0}^*$, and that is impossible. Now, for the injectivity, if $C \neq D$, we can assume without loosing the generality that there exists $c \in C \setminus D$. Consider $\varepsilon = \mathrm{d}(c, D)$. For $s > 0$ satisfying $\varepsilon_s < \frac{\varepsilon}{2}$, we have $c^n \cap D^n = \emptyset$ for every $n > s$, so $c_n^* \cap D_n^* = \emptyset$ for every $n > s$. Because of the previous claim, we have, $c_n^* \subset C_n^*$ for every $n \in \mathbb{N}$. Hence, $\{C_n^*\}$ and $\{D_n^*\}$ must be different.

Note, in a similar fashion as in Theorem 14, that $\mathcal{A}^* = \phi(\mathcal{A}) = \left\{\{C_n^*\} : C \in 2_u^X\right\}$. In contrast with that theorem, $\mathcal{A}^*$ is not necesarily Hausdorff, because, for example, if $C, D \in 2_u^X$ are subsets such that $C \subset D$, then it is not possible to separate $\{C_n^*\}$ and $\{D_n^*\}$ by open neighborhoods. Nevertheless, the maps $\varphi : \mathcal{A}^* \to 2_u^X$ and $\phi : 2_u^X \to \mathcal{A}^*$ are mutually inverse continuous bijections, so $\mathcal{A}^*$ is homeomorphic to 2_u^X.

It remains to prove that $\mathcal{A}$ retracts to $\mathcal{A}^*$, and hence as 2_u^X. Repeat, step by step, this part of the proof of Theorem 14. With the same notation, we have that

$$H(\{D_n\}, 1) = \phi \circ \varphi(\{D_n\}) = \phi(D) = \{D_n^*\} \in W.$$

Then $D \subset \mathrm{B}(C, \delta)$ and hence, $D^s = C^s$, so $D_r^* \subset C_r^*$ and we are done ✓

So, we obtain:

Corollary 7. *For every compact metric space X, there exists an inverse sequence of finite spaces whose inverse limit has the same homotopy type as 2_u^X.*

Remark 18. Note that the finite spaces are precisely subsets of the hyperspace 2_u^X, and that the hyperspace 2_u^X is an Alexandroff space. This is related with the next chapter, where we deal with this kind of inverse limit approximations for Alexandroff spaces.

Moreover, these inverse sequences are related in the following way. For every $n \in \mathbb{N}$, the inclusion $i_n : U_{2\varepsilon_n}(A_n) \hookrightarrow 2^{A_n}$ induces a map in the limit $f : \mathcal{X} \to \mathcal{A}$ making this diagram commutative.

$$\begin{array}{ccccccccccc}
2^{A_1} & \longleftarrow & 2^{A_2} & \longleftarrow & \dots & \longleftarrow & 2^{A_n} & \longleftarrow & 2^{A_{n+1}} & \longleftarrow \dots & \mathcal{A} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow f \\
U_{2\varepsilon_1}(A_1) & \longleftarrow & U_{2\varepsilon_2}(A_2) & \longleftarrow & \dots & \longleftarrow & U_{2\varepsilon_n}(A_n) & \longleftarrow & U_{2\varepsilon_{n+1}}(A_{n+1}) & \longleftarrow \dots & \mathcal{X}
\end{array}$$

This map is essentialy (up to homotopy type of the spaces) the embedding of the canonical copy of X in the hyperspace 2_u^X, in the sense that this diagram of homeomorphisms is commutative:

$$\begin{array}{ccc} 2^X_u & \xrightarrow{\phi} & \mathcal{A}^* \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f} \\ 2^X_1 & \xrightarrow[\phi]{} & \mathcal{X}^* \end{array}$$

Chapter 4

Universal hyperspaces

In the previous chapters we have used extensively finite spaces that are subsets of some hyperspaces with the upper semifinite topology. It is precisely this topology what makes this finite spaces so useful. Here, we propose the same hyperspace, in a more abstract setting, to be universal (in a sense that we will see later) for Alexandroff spaces. The upper semifinite topology is posed, even having poor topological properties, as a good ambient space to set (algebraic) topological results with the advantage that it is defined via the space itself. Moreover, the easy handling of the upper semifinite topoloy, allows us to express, very easily, some complicated properties.

4.1 Hyperspaces of discrete spaces

We will define hyperspaces with universality properties for some classes of Alexandroff spaces. In [5], the authors describe an embedding for every Tychonov space in its hyperspace with the upper semifinite topology. They relate properties of the space with properties of the hyperspace using that, although the topology of the hyperspace is non-Hausdorff, it is very easy to manipulate. In [6], the same authors describe a special neighborhood system, for the embedding of a compact metric space in its upper semifinite hyperspace to get results in the shape theory for compacta. Here we also use hyperspaces with the upper semifinite topology, but with a slightly different point of view: Given a topological space, we define hyperspaces of sets with the discrete topology.

Then, some subspaces of these hyperspaces will describe the topology of the original space. So, we focus in the subsets of the hyperspace, more than in the topology of the hyperspace itself.

Recall the following notation: For an Alexandroff space X, we write B_x for the minimal neighborhood of the point $x \in X$. For every topological space X, let 2_X be the set of non-empty closed subsets of X. The upper semifinite topology is generated by the base

$$B(U) = \{C \in 2^X : C \subset U\}, \quad U \text{ open in } X.$$

First of all, we quote the following result about hyperspaces of Alexandroff spaces.

Proposition 25. *For every Alexandroff space X, the hyperspace 2^X_u is an Alexandroff space.*

Proof. As an Alexandroff space, every point $x \in X$ has a minimal neighborhood B_x. Now, consider a point $C \in 2^X_u$. This point consist of a set of points $C = \{x_j\}_{j \in J}$, $x_j \in X$. For every $j \in J$, consider the open neighborhood $B(B_{x_j})$ in 2^X_u. Then we claim that

$$B_C = \bigcup_{j \in J} B(B_{x_j})$$

is the minimal neighborhood of C. Consider any basic open neighborhood $B(U)$ of C. Then, $C \subset U$, so $x_j \in U$ for every $j \in J$, and then, $B_{x_j} \subset U$. Then it is clear, for every $j \in J$, that $B(B_{x_j}) \subset U$ and hence $B_C \subset B(U))$ ✓

Example 5. The converse of this proposition is not true, even for T_0 Alexandroff spaces, as shown in the following example: Consider the unit interval $I = [0, 1]$ with the topology having as proper open sets the half intervals $[0, t)$ with $t \in (0, 1)$. Now consider the subspace $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with the subspace topology. It is a T_0 non-Alexandroff space, since the (of course infinite) intersection of open sets

$$\bigcap_{t \in (0,1)} ([0, t) \cap X) = \{0\}$$

is not an open set. But, it turns out that the hyperspace 2^X_u is Alexandroff. Every proper closed set is of the form $X\backslash[0,t)$, with $t \in (0,1)$, i.e., $X_n = \left\{\frac{1}{n}, \frac{1}{n-1}, \ldots, 1\right\}$ with $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, the only open set containing X_n is X, so every point of 2^X_u has $B(X) = 2^X_u$ as minimal neighborhood.

Let X be a set with the discrete topology. We can consider the hyperspace of non-empty (closed)[1] subsets 2^X with the upper semifinite topology. We will write $2^{X_d}_u$ to denote this topological space.

Proposition 26. *For every set X, the space $2^{X_d}_u$ is a T_0 Alexandroff*[2] *space.*

Proof. To show that it is Alexandroff, we need to find, for every point of the space, a minimal neighborhood. Every subset C of X is open with the discrete topology, so the basis element of C is $B_C = 2^C$. It is easy to see that

$$B_C = \bigcap_{C \subset U \text{open in} X} B(U)$$

so, it contains C and it is contained in any open neighborhood of C. Hence $2^{X_d}_u$ is an Alexandroff space with minimal neighborhoods 2^C for every $C \in 2^{X_d}_u$. It is T_0 because, for every pair of different points $C, D \in 2^{X_d}_u$, there are two possibilities: If $C \subsetneq D$, then $C \in B_C \not\ni D$. If $C \not\subset D$ and $D \not\subset C$ then we have both $C \in B_C \not\ni D$ and $C \notin B_D \in D$. The space $2^{X_d}_u$ is not T_1 because for every two points such that $C < D$, that is, $C \subset D$, every neighborhood of D contains $B_D \supset B_C \ni C$ ✓

Remark 19. As a T_0 Alexandroff space, the partial order induced in $2^{X_d}_u$ is, for every $C, D \in 2^{X_d}_u$, $C \leqslant D$ if and only if $C \subset D$.

We now consider the *power of finite sets of* X, that is,

$$2^{X_d}_f = \{C \in 2^{X_d}_u : \text{card}(C) \text{ is finite}\} \subset 2^{X_d}_u$$

which receives the subspace topology, so

$$\{B(U) \cap 2^{X_d}_f : U \text{ open in } X\}$$

[1] In this case, it is not a necessary condition, since every subset is open and closed.

[2] If X is discrete, then it is an Alexandroff space, and we already know that the hyperspace is Alexandroff. We prove it explicitly for this case, in order to find the minimal basis and we also show that the hyperspace is also T_0.

is a basis for its topology. As a subspace, $2_f^{X_d}$ is a T_0 Alexandroff space, with minimal neighborhoods 2^C, for each $C \in 2_f^{X_d}$. Define, for every $r \in \mathbb{N}$, the set

$$2_r^{X_d} = \left\{ C \in 2_f^{X_d} : \operatorname{card} C \leqslant r \right\}$$

of points of the hyperspace $2_u^{X_d}$ with a bounded (by r) number of elements.

Remark 20. For every $r \in \mathbb{N}$, the space $2_r^{X_d}$ is an open subset of $2_f^{X_d}$. This is so because, for every $C \in 2_r^{X_d}$, its minimal neighborhood 2^C is contained in the space $2_r^{X_d}$. Moreover, for every pair $r \leqslant s$, we have the inclusion $2_r^{X_d} \subset 2_s^{X_d}$.

Example 6. In general, the space $2_f^{X_d}$ is not T_1. For example, for the set of natural numbers $\mathbb{N}$, we have that $2_f^{\mathbb{N}}$ is not T_1 because, for example, the minimal neighborhood of $\{1, 2, 3\}$ contains $\{1, 2\}$.

Of course, the only outstanding information of the set X that is kept in the hyperspaces $2_u^{X_d}$ and $2_f^{X_d}$ is the cardinality. That is, we have the following

Proposition 27. *Let X, Y be sets with cardinalities ω_X and ω_Y respectively. Then, the following are equivalent:*

(i) $\omega_X = \omega_Y$.

(ii) $2_u^{X_d}$ *is homeomorphic to* $2_u^{Y_d}$.

(iii) $2_f^{X_d}$ *is homeomorphic to* $2_f^{Y_d}$.

Despite this fact, we will use the set notation instead of dealing just with cardinalities for the sake of simplicity.

We study some topological properties of these hyperspaces. For instance, the space $2_f^{X_d}$ is strongly deformable to any of its points.

Proposition 28. *For every set X, the space $2_f^{X_d}$ is contractible. Moreover it can be strongly retracted to any of its points.*

Proof. Let A be any point of $2_f^{X_d}$, and $\bullet$ a single point; we will prove that $2_f^{X_d}$ can be retracted to that point. Let us consider the following maps

$$\begin{array}{rccc} p: & 2_f^{X_d} & \longrightarrow & \bullet \\ & C & \longmapsto & \bullet \end{array} \qquad \begin{array}{rccc} p: & \bullet & \longrightarrow & 2_f^{X_d} \\ & \bullet & \longmapsto & A. \end{array}$$

The composition of maps $p \cdot i : \bullet \to \bullet$ is the identity. And the composition

$i \cdot p : 2_f^{X_d} \to 2_f^{X_d}$ is not the identity, in general, but it turns out that it is homotopic to it. Indeed, the map $H : 2_f^{X_d} \times [0,1] \to 2_f^{X_d}$ defined by

$$H(x,t) = \begin{cases} A & \text{if } t \in [0, \frac{1}{2}), \\ C \cup A & \text{if } t = \frac{1}{2}, \\ A & \text{if } t \in (\frac{1}{2}, 1] \end{cases}$$

is an homotopy between the maps $H(*,0) = i \cdot p(*)$ and $H(*,1) = \text{id}$. We shall prove that this map is continuous everywhere. Let $(C,t) \in 2_f^{X_d} \times [0,1]$.

- If $t \in [0, \frac{1}{2})$ then $H(C,t) = A$. Let V be any neighborhood of A, we have that $A \in B_A \subset V$. The neighborhood of (C,t) given by $U = 2_f^{X_d} \times [0, \frac{1}{2})$ satisfies $H(U) = A \in V$.

- If $t \in (\frac{1}{2}, 1]$ then $H(C,t) = C$. Let V be a neighborhood of C, we know that $C \in B_C \subset V$. The neighborhood $U = 2_f^{X_d} \times (\frac{1}{2}, 1]$ of (C,t) satisfies $H(U) = C \in V$.

- Finally, $H(C, \frac{1}{2}) = C \cap A$. For any neighborhood V of $C \cup A$ we can claim that $C \cup A \in B_{C \cup A} \subset V$ so the image of the neighborhood of $(C, \frac{1}{2})$ given by $U = 2_u^{X_d} \times [0,1]$ satisfies $H(U) = C \cup A \in V$ ✓

This is quite non-evident since this space is highly non-homogeneous.

Definition 8. A topological space X is said to be homogeneous if, for every two points $x, y \in X$, there is a homeomorphism $f : X \to X$ such that $f(x) = y$. In other words, the group of self homeomorphisms of X is transitive in X.

We characterize the homeomorphisms of our space in order to measure its unhomogeneity.

Proposition 29. *Let X be any set. Then a function $f : 2_f^{X_d} \to 2_f^{X_d}$ is a homeomorphism if and only if there exists a bijection $\gamma : X \to X$ such that $f = 2^\gamma$. That is, the homeomorphism group of $2_f^{X_d}$ is isomorphic to the group of permutations of card(X) elements.*

Proof. This is a direct consecuence of proposition 2.8 of [5], because every set with the discrete topology is a Tychonov space. In this particular situation, the proof is simpler, as shown.

Consider we have a bijection γ of X and define $f : 2_f^{X_d} \to 2_f^{X_d}$ as $f = 2^\gamma$. Then, for every $C \in 2_f^{X_d}$, we have $f(C) = \cup_{c\in C}\gamma(c)$. The map f is continuous and open, since, for every $C \in 2_f^{X_d}$, we have

$$f(2^C) = \bigcup_{D\subset C}\left(\bigcup_{c\in D}\gamma(c)\right) = 2^{\bigcup_{c\in C}\gamma(c)} = 2^{f(C)}.$$

It is clearly injective. If $C \neq D$, let us suppose that there exists $d \in D\backslash C$. Then $\gamma(d) \in f(D)\backslash f(C)$. Finally, f is surjective: For every $C \in 2_f^{X_d}$, $C = \bigcup_{c\in\gamma^{-1}(C)}\gamma(c) = f(\gamma^{-1}(C))$. We conclude that f is a homeomorphism. On the other hand, let $f : 2_f^{X_d} \to 2_f^{X_d}$ be a homeomorphism. Consider a point of $2_f^{X_d}$ consisting of only one point of X, that is $\{x\} \in 2_f^{X_d}$. Let us write $f(\{x\}) = C \in 2_f^{X_d}$. Then, f^{-1} is a continuous map sending C to $\{x\}$, so $f^{-1}(2^C) \subset 2^{\{x\}} = \{x\}$. But f^{-1} must be injective so card $\left(2^C\right) = 1$, hence $C = \{y\}$ with $y \in X$. That means there exists a function $\gamma : X \to X$ such that $f(\{x\}) = \gamma(x)$ for every $x \in X$. This function must be a bijection, since f is. Now, let us consider $C \in 2_f^{X_d}$. Since f is continuous, we have $\gamma(c) = f(\{c\}) \subset f(C)$ for every $c \in C$, that is, $\bigcup_{c\in C}\gamma(c)$. On the other hand, since f^{-1} is continuous, for every $d \in f(C)$ we have $\gamma^{-1}(d) = f^{-1}(\{d\}) \subset C$, and hence $d \in \bigcap_{c\in C}\gamma(c)$. We conclude $f(C) = \bigcap_{c\in C}\gamma(c)$, i.e., $f = 2^\gamma$ ✓

As an immediate corollary we obtain

Corollary 8. *Let X be any set and consider $C, D \in 2_f^{X_d}$. Then there exists a homeomorphism $f : 2_f^{X_d} \to 2_f^{X_d}$ with $f(C) = D$ if and only if card$(C) =$ card(D).*

Proof. If $f : 2_f^{X_d} \to 2_f^{X_d}$ is a homeomorphism then, by the previous proposition, there exists a bijection $\gamma : X \to X$ such that $f = 2^\gamma$. It is clear that the elevation of a bijection must preserve the cardinal of the elements.

The opposite implication is straightforward because it is always possible to extend bijections to sets with the same cardinal. If card$(C) =$ card(D), then we can define two bijections $\alpha : C \to D$ and $\beta : X\backslash C \to X\backslash D$. Now we can define a bijection $\gamma : X \to X$ on the whole set by

$$\gamma(x) = \begin{cases} \alpha(x) & \text{if } x \in C, \\ \beta(x) & \text{if } x \in X\backslash C. \end{cases}$$

Since it is a bijection, then $f = 2^\gamma$ is a homeomorphism sending C to D, as required ✓

Remark 21. From the previous proof we can deduce that there exist exactly card(C)!·card($X\setminus C$)! different homemorphisms (the combination of possible choices for the bijections α and β) sending C to D.

Remark 22. The last proposition and its corollary remain true if we replace $2_f^{X_d}$ with $2_u^{X_d}$. Nothing in the proofs actually changes.

Local finiteness is a property of topological spaces closely related to Alexandroff spaces.

Definition 9. A topological space X is called *locally finite* if, for every $x \in X$, there exists a finite neighborhood $x \in U \subset X$. We will say that an Alexandroff space A is *strongly locally finite*[3] if, for every $a \in A$, the set of points related to a, that is,

$$\{b \in A : b \leqslant a \text{ or } a \leqslant b\},$$

is finite (equivalently, for every $a \in A$, B_a and $\overline{\{a\}}^A$ are finite sets).

Remark 23. Finite topological spaces are always locally finite. For Alexandroff spaces, strong local finiteness implies local finiteness.

Remark 24. For every infinite set X, the hyperspace $2_f^{X_d}$ is not finite, is locally finite but it is not strongly locally finite: For every $C \in 2_f^{X_d}$, the minimal neighborhood 2^C is a finite open set containing C. But C is contained in an infinite number of elements of $2_f^{X_d}$.

It turns out that locally finite spaces are nothing but an special class of Alexandroff spaces.

Proposition 30. *Every topological space is a locally finite space if and only if it is an Alexandroff space with finite minimal neighborhoods.*

Proof. Let X be a locally finite space. Let us consider, for $x \in X$, a finite open neighborhood $x \in U \subset X$. We claim that

$$B_U = \bigcap_{x \in B \subset U \text{ open}} B$$

[3] This notion of local finiteness comes from the paper [39].

is the minimal open neighborhood for x. Note that it is not empty, because $B_U \subset U \ni x$, and open, because it is a finite intersection (remember U is finite) of open sets. It is the minimal neighborhood of x because, if V is another open neighborhood of x, then $V \cap U \subset U$ is an open neighborhood of x, and hence $B_U \subset V$. Finally, the construction does not depend on the choice of the finite open neighborhood of x. If we use a different one, say U', then $U \cap U' \subset U, U'$, so $B_U \subset U'$ and $B_{U'} \subset U$ which implies, respectively, that $B_{U'} \subset B_U$ and $B_U \subset U'$, so $B_U = B_{U'} = B_x$ is well defined. The converse is obviously true ✓

Compactness and paracompactness are easily characterized in these hyperspaces:

Proposition 31. *Let X be any set. The following statements are equivalent:*

(i) X is finite.

(ii) $2_f^{X_d}$ is compact.

(iii) $2_f^{X_d}$ is paracompact.

(iv) $2_f^{X_d}$ is strongly locally finite.

Proof. The implications (i)⇒(ii)⇒(iii) are obvious.

(iii)⇒(iv) If $2_f^{X_d}$ is paracompact, then the minimal open covering $\left\{2^C : C \in 2_f^{X_d}\right\}$ must be locally finite, so, for every $C \in 2_f^{X_d}$, $C \subset D$ for a finite number of points $D \in 2_f^{X_d}$ (or, in other words, for every $C \in 2_f^{X_d}$, the closure $\overline{\{C\}}^{2_f^{X_d}}$ is finite). Since 2^C is always finite too, $2_f^{X_d}$ space is strongly locally finite.

(iv)⇒(i) If X was infinite then, for every $C \in 2_f^{X_d}$, we would have that $X \backslash C$ would be infinite, and then $C \subset C \cup D$ for every $D \in X \backslash C$, making $\overline{\{C\}}^{2_f^{X_d}}$ infinite, which is impossible ✓

We can look for the smallest compact space containing $2_f^{X_d}$. For non-Hausdorff spaces, there is an analogous to the concept of compactification called the Alexandroff extension[4], which is defined in the same way.

[4] In the definition of compactification is usually assumed that the space is, at least, Hausdorff, in order to ensure that the compactification has some desired properties.

Definition 10. Let X be any topological space and ∞ any object not in X. Consider the set $X^* = X \cup \{\infty\}$ with open sets the open sets of X and the subsets $\infty \in U$ such that $X \backslash U$ is closed and compact. The inclusion map $c : X \to X^*$ is then called the *Alexandroff extension* of X.

Proposition 32 (Properties of the Alexandroff extension)**.** *Let X be a topological space and $c : X \to X^*$ its Alexandroff extension. Then:*

(i) The space X^* is compact.

(ii) The map $c : X \to X^*$ is continuous and open.

(iii) If X is not compact, $c(X)$ is dense in X^*.

Definition 11. Let X be a topological space and $*$ any point not in X. The non-Hausdorff cone is the space $X \cup \{\}$ with proper open sets the open sets of X.

Remark 25. In general, for every topological space X, the topology of the Alexandroff extension is finer that the one in the non-Hausdorff cone.

In order to find the Alexandroff extension of our space, we need the following lemma.

Lemma 11. *If X is an infinite set, there are no closed and compact subsets of $2_f^{X_d}$.*

Proof. Consider a non-empty subset $B \subset 2_f^{X_d}$ and suppose it is closed and compact. Consider a point $a \in B$, then $\overline{\{a\}} \subset \overline{B} = B$, being a closed subset of a compact space, is compact. But this is not possible: Consider the open covering $\bigcup_{\{a\} \subset C} 2^C$ of $\overline{\{a\}}$, and suppose there is a finite subcovering, say $\left\{2^{C_1}, \dots, 2^{C_s}\right\}$. Then, for every $D \in C \backslash X$, we have $\{a\} \cup D \in \overline{\{a\}}$ but $\{a\} \cup D \notin \left\{2^{C_1}, \dots, 2^{C_s}\right\}$, so there are no possible finite subcoverings ✓

Remark 26. It turns out that given any set X, the Alexandroff extension and the non-Hausdorff cone of the hyperspace $2_f^X \subset 2_u^X$ are exactly the same topological space.

Proposition 33. *The subspace $2_f^{X_d} \cup \{X\} \subset 2_u^{X_d}$ is the Alexandroff extension of $2_f^{X_d}$.*

Proof. We will show that they have exactly the same open sets. As a subspace, $2_f^{X_d} \cup \{X\}$ has as open sets the intersections with open sets of $2_u^{X_d}$. Let U be an open set of $2_u^{X_d}$, and consider its intersection with $2_f^{X_d} \cup \{X\}$.

- If $X \notin U$, then the intersection is $U \cap 2_f^{X_d}$.
- If $X \in U$, then U has to be 2^X, so the intersection is the whole set $2_f^{X_d} \cup \{X\}$.

On the other hand, as the Alexandroff extension, the open sets are the open sets of $2_f^{X_d}$ and the sets V, containing X, such that $2_f^{X_d} \backslash V$ is closed and compact. But, by the previous lemma, the only possibility is the empty set, so there is only one open set more, $2_f^{X_d} \cup \{X\}$ ✓

Now we give a description of the Alexandroff extension in terms of an inverse limit of subspaces of $2_f^{X_d}$.

Theorem 21. *Let X be any set. Consider the hyperspace of finite subsets $2_f^{X_d} \subset 2_u^{X_d}$ with the upper semifinite topology. The Alexandroff extension of $2_f^{X_d}$ is homeomorphic to an inverse limit of an inverse system of finite spaces.*

Remark 27. The proof of this theorem for the case $2_f^{\mathbb{N}}$ (or, equivalently, when the cardinal of X is countable) is simpler and more intuitive. Even the statement of the theorem we want to prove is then easier, because we just need a sequence (instead of a system) of finite spaces. We include it here and we recommend the reader to check this proof in order to understand the general case.

Theorem 22. *The Alexandroff extension of $2_f^{\mathbb{N}}$ is homeomorphic to an inverse limit of an inverse sequence of finite spaces.*

Proof. We should think about this space as an countable cone over the point $\{1\}$. This allows us to understand what follows. The natural numbers are totally ordered. In this case, there is a sequence of ordered-by-inclusion open sets (of the basis)

$$B_{\{1\}} \subset B_{\{1,2\}} \subset B_{\{1,2,3\}} \subset \dots$$

such that

$$\bigcup_{s=2,\dots,\infty} B_{\{1,\dots,s\}} = 2_f^{\mathbb{N}}.$$

Despite of this, they are not a basis: For example, the point $\{1,4\} \in B_{\{1,2,4\}}$ but it is imposible to find an s such that $\{1,4\} \in B_{\{1,\dots,s\}} \subset B_{\{1,2,4\}}$. We want to construct an inverse sequence in terms of this ordered chain. We can define a natural map (a kind of "collapse"-this is not formal!) from every element of the chain to a lower one. For every $n \in n \in \mathbb{N}$, define a map $p_{n,n+1} : 2^{\{1,\dots,n,n+1\}} \to 2^{\{1,\dots,n\}}$ as

$$p_{n,n+1} = \begin{cases} C & \text{if } n+1 \notin C, \\ \{1,\dots,n\} & \text{if } n+1 \in C. \end{cases}$$

This map is continuous: Suppose we have a pair of points of $2^{\{1,\dots,n,n+1\}}$, namely $C \subset D$. Then, there are three different cases:

- If $n+1 \in C \subset D$, then $p_{n,n+1}(C) = p_{n,n+1}(D) = \{1,\dots,n\}$.
- If $n+1 \notin C, D$, then $p_{n,n+1}(C) = C \subset D = p_{n,n+1}(D)$.
- If $n+1 \notin C$ but $n+1 \in D$, then $p_{n,n+1}(C) = C \subset \{1,\dots,n\} = p_{n,n+1}(D)$.

So, $p_{n,n+1}$ is continuous for every $n \in n \in \mathbb{N}$.

Now it makes sense to ask what is the inverse limit of the inverse sequence

$$\left\{2^{\{1,\dots,n\}}, p_{n,n+1}\right\}$$

Table 4.1 shows a visualization of the first elements of the sequence: From the table, we see that each element $\{a_1,\dots,a_s\}$ (suppose ordered) of $2^{\mathbb{N}}_f$ is represented in the inverse limit as an element that begins at the a_s-th term of the sequence. But there is an element of the inverse sequence that is not any of the previosly described, namely

$$(\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\dots).$$

So, we claim that the inverse limit $\mathcal{N}$ of the inverse sequence $\left\{2^{\{1,\dots,n\}}, p_{n,n+1}\right\}$ is homeomorphic to the subspace $2^{\mathbb{N}}_f \cup \{\mathbb{N}\} \subset 2^{X_d}_u$.

For every $C \in 2^{\mathbb{N}}_f$, let us write the maximum of its elements as $m(C) = \max\{c_i \in C\}$. Define the following map, $h : 2^{\mathbb{N}}_f \cup \{\mathbb{N}\} \to \mathcal{N}$ as

$$\begin{aligned} h(\mathbb{N}) &= (\{1\},\{1,2\},\{1,2,3\},\dots), \\ h(C) &= (\{1\},\{1,2\},\dots,\{1,2,\dots,m(C)-1\},C,C,\dots), \text{ for every } C \in 2^{\mathbb{N}}_f. \end{aligned}$$

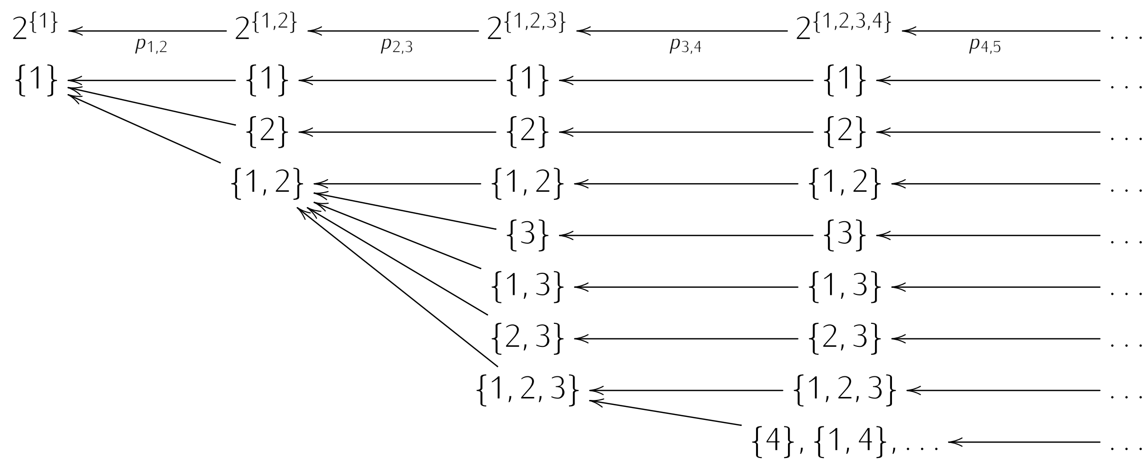


Table 4.1: Visualization of the inverse limit of $\left\{2^{\{1,\ldots,n\}}, p_{n,n+1}\right\}$.

We will show that this map is a homeomorphism between the two spaces.

1. h is well defined: It is obvious that $h(\mathbb{N}) \in \mathcal{N}$. And, for every $C \in 2^{\mathbb{N}}_f$, $h(C) \in \mathcal{N}$, since

   - for $n \leqslant m(C) - 2$, $p_{n,n+1}(\{1, \ldots, n, n+1\}) = \{1, \ldots, n\}$,
   - $p_{m(C)-1,m(C)}(C) = \{1, 2, \ldots, m(C) - 1\}$ and
   - for $n \geqslant m(C)$, $p_{n,n+1}(C) = C$.

2. h is injective, as easily checked from the definition.

3. h is surjective: Consider $(C_1, C_2, \ldots,) \in \mathcal{N}$. Two possibilities:

   - If, for every $n \in \mathbb{N}$, $C_n \neq C_{n+1}$, then $C_n = \{1, 2, \ldots, n\}$ for every $n \in \mathbb{N}$ and $h(\mathbb{N}) = (C_1, C_2, \ldots)$.
   - If there exists $n_0 \in \mathbb{N}$, such that, $C_{n_0} = C_{n_0+1}$, let us suppose that it is the minimum satisfying this condition and then: For every $n < n_0$ we have that $C_n \neq C_{n+1}$, so $C_n = \{1, \ldots, n\}$. For every $n > n_0$, $n \notin C_n$, so $C_{n+1} = C_n$. In this case, $h(C_{n_0}) = (C_1, C_2, \ldots)$.

4. h is continuous: Let us divide the proof in two cases.

- Every open neighborhood U of $h(\mathbb{N})$ must be $U = \left(2^{\{1\}} \times 2^{\{1,2\}} \times \dots\right) \cap \mathcal{N} = \mathcal{N}$ and $h\left(2_f^{\mathbb{N}} \cup \{\mathbb{N}\}\right) \subset U$.
- For every $C \in 2_f^{\mathbb{N}}$, consider an open neighborhood W of $h(C)$ in $\mathcal{N}$. Then

$$h(C) \in V = \left(2^{\{1\}} \times 2^{\{1,2\}} \times \dots \times 2^{\{1,\dots,m(C)\}} \times 2^C \times 2^{\{1,\dots,m(C)+2\}} \times \dots\right) \cap \mathcal{N} \subset W.$$

The open neighborhood 2^C of C satisfies that $h(2^C) \subset V$, since, for every $D \in 2^C$, $h(D) \in V$, because $m(D) \leqslant m(C)$.

5. h is an open map: It sends every basic open set to an open set. Namely, $h(2_f^{\mathbb{N}} \cup \{\mathbb{N}\}) = \mathcal{N}$ and, for every $C \in 2_f^{\mathbb{N}}$,

$$h(2^C) = \left(2^{\{1\}} \times 2^{\{1,2\}} \times \dots \times 2^{\{1,\dots,m(C)\}} \times 2^C \times 2^{\{1,\dots,m(C)+2\}} \times \dots\right) \cap \mathcal{N}.$$

$\subset$ For every $D \in 2^C$, $h(D) \in \mathcal{N}$ by definition, and

$$h(D) \in \left(2^{\{1\}} \times 2^{\{1,2\}} \times \dots \times 2^{\{1,\dots,m(C)\}} \times 2^C \times 2^{\{1,\dots,m(C)+2\}} \times \dots\right),$$

because, for $n = 2, \dots, m(D) - 1$, $\{1, \dots, n\} \in 2^{\{1,\dots,n\}}$, for $n = m(D), \dots, m(C)$, $D \in 2^{\{1,\dots,n\}}$, $D \in 2^C$ (case $n = m(C) + 1$) and, for $n \geqslant m(C) + 2$, $D \in 2^{\{1,\dots,n\}}$.

$\supset$ Any element of this set is of the form

$$A = \left(p_{1,m(C)+1}(D), p_{2,m(C)+1}(D), \dots, p_{m(C),m(C)+1}(D), D, p^{-1}_{m(C)+1,m(C)+2}(D), \dots\right),$$

for some $D \in 2^C$. But, it is easy to see that

$$A = (\{1\}, \{1,2\}, \dots, \{1, \dots, m(D) - 1\}, D, D, \dots) = h(D).$$

The map h, as a continuous open bijection, is a homeomorphism. ✓

Now we proceed with the general case.

Proof of Theorem 21. We define first the inverse system. Consider the directed set $2_f^{X_d}$ in which $C \leqslant C'$ if $C \subset C'$. As objects, we consider the finite spaces 2^C, for every $C \in 2_f^{X_d}$. For every pair $C \leqslant C'$, define the map $p_{C,C'} : 2^{C'} \to 2^C$ as

$$p_{C,C'}(D) = \begin{cases} D & \text{if } D \subset C \\ C & \text{if } D \not\subset C \end{cases}$$

for every $D \subset C'$. This map is continuous, because, for every pair $D \subset D'$ such that $D, D' \subset C'$, we have:

If $D \subset D' \subset C$, then $h(D) = D \subset D' = h(D')$,

if $D \subset C$ but $D' \not\subset C$, then $h(D) = D \subset C = h(D')$ and

if $D, D' \not\subset C$, then $h(D) = C = h(D')$.

Let us write $\mathcal{X}$ for the inverse limit of the inverse system $\left\{2^C, p_{C,C'}, 2_f^{X_d}\right\}$. We define a map

$$h : 2_f^{X_d} \cup \{X\} \longrightarrow \mathcal{X}$$

as follows. For $X \in 2^X \cup \{X\}$, $(h(X))_C = C$, for every $C \in 2_f^{X_d}$. For every $D \in 2^X \cup \{X\}$,

$$(h(D))_C = \begin{cases} D & \text{if } D \subset C \\ C & \text{if } D \not\subset C \end{cases}$$

We will show that h is a homeomorphism and, in order to do that, we need to show several things.

1. h is a well defined map.

   This is almost trivial in the case of the image of X, because it is $(h(X))_C = C$, for every $C \in 2_f^{X_d}$, and $p_{C,C'}(C') = C$ for every $C \leqslant C'$. So $h(X) \in \mathcal{X}$.

   For every $D \in 2_f^{X_d}$, we have that, for every pair $C \leqslant C'$,

$$p_{C,C'}\left((h(D))_{C'}\right) = \begin{cases} p_{C,C'}(D) & \text{if } D \subset C' \\ p_{C,C'}(C') & \text{if } D \not\subset C' \end{cases} = \begin{cases} D & \text{if } D \subset C \\ C & \text{if } D \not\subset C \\ C & \text{if } D \not\subset C' \end{cases} = (h(D))_C\,,$$

   hence $h(D) \in \mathcal{X}$.

2. h is continuous.

The only possible open neighborhood of $h(X)$ is $\mathcal{X}$.

For every $D \in 2_f^{X_d}$, let us consider an open neighborhood $h(D) \subset W$ in $\mathcal{X}$. There exists an open set $V \subset \prod_{C \in 2_F^X} 2^C$ such that $h(D) \in V \cap \mathcal{X} \subset W$ with

$$(V)_C = \begin{cases} 2^D & \text{if } C \in \{C_1, \ldots, C_n\} \\ 2^C & \text{if not,} \end{cases}$$

for some $n \in \mathbb{N}$ and with $D \subset C_1, \ldots, C_n$. We claim that $h(2^D) \subset V \cap \mathcal{X}$: For every $A \in 2^D$, we have that, if $C \in \{C_1, \ldots, C_n\}$, then $A \subset D \subset C$, so $(h(A))_C = A \in 2^D$. If not, $(h(A))_C$ could be A or C, but both are in 2^C.

3. h is surjective. Let A be an element of $\mathcal{C}$.

   If, for every $C \in 2_f^{X_d}$, $(A)_C = C$, then $h(X) = A$.

   If there exists $C \in 2_f^{X_d}$ such that $(A)_C = D \subsetneq C$, then we compute the rest of the projections as follows.

   - For every $C' \leqslant C$,

   $$p_{C',C}(D) = \begin{cases} D & \text{if } D \subset C' \\ C' & \text{if } D \not\subset C'. \end{cases}$$

   - For every $C \leqslant C'$, $p_{C,C'}(D) = D$.
   - If C and C' are not related ($C \nleqslant C'$ and $C' \nleqslant C$), then we know that there exists $C'' \geqslant C, C'$ (because 2_F^X with the subset relation is a directed set), and then, $p_{C,C''}(D) = D$ so

   $$p_{C',C''}(D) = \begin{cases} D & \text{if } D \subset C' \\ C' & \text{if } D \not\subset C', \end{cases}$$

   and hence

   $$(A)_C = \begin{cases} D & \text{if } D \subset C \\ C & \text{if } D \not\subset C, \end{cases}$$

   so $A = h(D)$.

4. h is injective. Let $C \neq D$ two points of $2_f^{X_d} \cup \{X\}$. If one of the two points is X, it is clear that the images are different. If both are points of $2_f^{X_d}$,

then consider $E \geqslant C, D$ and then we have $C = (h(C))_E \neq (h(D))_E = D$.

5. h is an open map. Let $D \in 2_f^{X_d}$ a point and 2^D its minimal open neighborhood. We claim that $h(2^D) = V \cap \mathcal{X}$, with

$$(V)_C = \begin{cases} 2^D & \text{if } C = D \\ C & \text{if not.} \end{cases}$$

   - $\subset$ For every $E \subset D$, we have $(h(E))_D = E \in 2^D$.
   - $\supset$ Let A be a point of the intersection $V\mathcal{X}$. Then $(A)_D = B \subset D$. Because of the surjectivity we have $A = h(B)$, with $B \in 2^D$.

   Hence h is open since $V \cap \mathcal{X}$ is open ✓

4.2 Embeddings into hyperspaces

Let us recall this result in [5] about embeddings of a space into its hyperspace with the upper semifinite topology:

Proposition 34. *Let X be a Tychonov space. The map $\phi : X \to 2_u^X$ given by $\phi(x) = \{x\}$ is a topological embedding. Moreover $\phi(X)$ (called the canonical copy of X) is dense in 2_u^X.*

Note that the hyperspaces used in this proposition have the upper semifinite topology given by the topology of X (in contrast to our case, in which they have the discrete topology). This is used to stablish the quoted embedding. We would like to embed topological spaces in the hyperspaces $2_u^{X_d}$ and $2_f^{X_d}$. It is obvious that the same map is not useful here. In fact, we have the following anti-embeddability result:

Proposition 35. *Let X be a topological space. Then, the map $\phi : X \to 2_f^{X_d}$, defined by $\phi(x) = \{x\}$, is continuous if and only if X has the discrete topology.*

Proof. If it is continuous then, for every $x \in X$, there exists a neighborhood of x, say U, such that $\phi(U) \subset 2^{\{x\}} = \{x\}$. If y is another point of X lying in U, then its image would be x, but this is not possible. Then, $U = \{x\}$ is open in X so it is discrete ✓

We will need to find a new class of spaces and a different map. Every subspace of $2_u^{X_d}$ is a T_0 Alexandroff space, so the following proposition is natural.

Proposition 36. *For every T_0 Alexandroff space X, there exists a topological embedding $\rho : X \hookrightarrow 2_u^{X_d}$. If the space X is also locally finite, then the embedding is into $2_f^{X_d}$. Moreover, the embedding is as an open subset if and only if X has the discrete topology.*

Proof. We define the map

$$\begin{aligned} \rho : X &\longrightarrow 2_u^{X_d} \\ x &\longmapsto B_x. \end{aligned}$$

It is obviously well defined. If X is locally finite, we know by Proposition 30 that the minimal neighborhoods are finite so then, actually, the image is in $2_f^{X_d}$. It is a continuous map, because

$$x \leqslant y \Longleftrightarrow B_x \subset B_y \Longleftrightarrow \rho(x) \leqslant \rho(y). \tag{4.1}$$

It is injective, because, for two different points $x \neq y$ in X, since X is T_0, there exists an open neighborhood of one not containing the other, let us say $x \in U \not\ni y$. Then, $x \in B_x \not\ni y$ so $B_x \neq B_y$ and hence $\rho(x) \neq \rho(y)$. It remains to show that the map restricted to its image $\rho : X \to \rho(X)$ is a homeomorphism. But this is trivial because of relation (4.1) ✓

As we know, the hyperspaces $2_u^{X_d}$ and $2_f^{X_d}$ are determined by the cardinal of the space X. So, we can generalize this embeddings a little bit.

Proposition 37. *Let X be any set. The hyperspace $2_u^{X_d}$ ($2_f^{X_d}$) is universal for every T_0 Alexandroff (locally finite) space Y with card$(Y) \leqslant$ card(X).*

Proof. Consider a bijection of Y with a subset of X, $\alpha : Y \to Z \subset X$. We define the map

$$\begin{aligned} \rho : Y &\longrightarrow 2_u^{X_d} \\ y &\longmapsto \left\{\alpha(y_i) : y_i \in B_y\right\}. \end{aligned}$$

This map is shown to be well defined, continuous, injective and a homeomorphism if considered onto its image in the same way as in the previous proposition, because relation (4.1) holds again ✓

Example 7. We really need the T_0 condition. For example, the finite space $X = \{1, 2, 3\}$, with open sets $\tau = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, is not T_0 and the map $\rho : X \to 2_f^{\mathbb{N}}$ sending x to B_x is not injective.

Remark 28. What we are really doing is to consider every POSET as a family of subsets and inclusions, which is quite natural.

Using the Alexandroff-McCord functors, we can actually embed every T_0 Alexandroff space up to weak homotopy equivalence.

Proposition 38. *For every (locally finite) T_0 Alexandroff space X, there is an embedding $\varphi : Y \to 2_u^{X_d}(2_f^{X_d})$ where Y is a topological space weakly homotopically equivalent to X and $\varphi(Y)$ is open in $2_u^{X_d}(2_f^{X_d})$.*

Proof. We define $Y = \mathcal{X}(\mathcal{K}(X))$ and the map $\varphi : Y \to 2_u^{X_d}$ is just the identity map. It is obvious that, every element of Y belongs to $2_u^{X_d}$. The continuity, bijectivity onto the image and the continuity of the inverse are also trivial. We just need to show that $\varphi(Y)$ is open in $2_u^{X_d}$. Let C be a point of $\varphi(Y)$. Then $C \in 2^C \subset \varphi(Y)$, because, for every $D \in 2^C$ we have $D \in \mathcal{X}(\mathcal{K}(X))$ ✓

As a direct corollary, we obtain an embedding of the Alexandroff space associated to every simplicial complex.

Corollary 9. *Let K be a (finite) simplicial complex. Then, there exists an embedding of $\mathcal{X}(K)$ as an open subset of $2_u^V(2_f^V)$, where V is the discrete set of vertices. The embedded copy contains the set of singletons of vertices 2_1^V.*

4.3 The simplicial neighborhoods category

Let K be a simplicial complex with vertex set V. As we saw in corollary 9, we can identify the simplicial complex with an open subspace $U = \mathcal{X} \subset 2_f^V$ of a hyperspace, such that it contains a canonical copy of the vertex set, that is, $2_1^V = \{\{v_1\}, \ldots, \{v_n\}\} \subset U$. We will say that U is a simplicial neighborhood of the vertex set 2_1^V.

We will review some properties and examples of simplicial complexes from this point of view.

Definition 12. Let $U \subset 2^V_f$ be a simplicial neighborhood. We will say that U is locally finite if, for every $C \in U \cap 2^V_1$ the closure $\overline{\{C\}}^U$ is finite.

Given any simplicial neighborhood $U \subset 2^V_F$, we will say that $C \in U$ is a q-simplex if $C \in 2^V_{q+1} \backslash 2^V_q$. We also say that C has dimension q. For any two elements $C, D \in U$ satisfying $C \leqslant D$, we say that C is a face of D and a proper face if $C \neq D$. Moreover, if C has dimension q, we say that C is a q-face of D.

Example 8. Some examples from the list of examples of simplicial complexes from the book [63]

1. The empty set $\emptyset$ is a simplicial neighborhod.
2. For every set V, 2^V_F is a simplicial neighborhood.
3. Let C be a point of a simplicial neighborhood. Its set of faces, $\overline{C} = \{D \in U : D \leqslant C\}$, is a simplicial neighborhood, because $\overline{C} = 2^C$.
4. For $C \in U$, the set of proper faces of C, $\dot{C} = \{D \in U : D \lneqq C\}$ is a simplicial neighborhood. This is so, because $2^C \backslash \{C\} = \bigcup_{D \lneqq C} 2^D$ is open.
5. For every simplicial neighborhood $U \subset 2^V_f$, its q-dimensional skeleton $U_q = U \cap 2^V_{q+1}$, being an intersection of open sets, is a simplicial neighborhood.
6. Let X be any set. Consider a family $\mathcal{W} = \{W_\alpha\}$ of subsets $W_\alpha \subset X$. The nerve $\mathcal{N}(\mathcal{W})$ of $\mathcal{W}$, is the simplicial neighborhood of $2^{\mathcal{W}}_f$ defined by

$$\{W_{\alpha_0}, \ldots, W_{\alpha_q}\} \in \mathcal{N}(\mathcal{W}) \Longleftrightarrow W_{\alpha_0} \cap \ldots \cap W_{\alpha_q} \neq \emptyset.$$

   It is an open set since every point $\{W_{\alpha_0}, \ldots, W_{\alpha_q}\} \in \mathcal{N}(\mathcal{W})$ has an open neighborhood $2^{\{W_{\alpha_0}, \ldots, W_{\alpha_q}\}} \subset \mathcal{N}(\mathcal{W})$.

We can define a notion of dimension exactly in the same way it is defined for simplicial complexes. Let $U \subset 2^V_f$. We will say that U has dimension $\emptyset$ if $U = \emptyset$, dimension n if $U \subset 2^V_{n+1}$ and dimension ∞ if $U \not\subset 2^V_n$ for every $n \in \mathbb{N}$. We will say that U is finite as simplicial neighborhood if it is finite as a set.

Given any simplicial neighborhood $U \subset 2^V_f$, a simplicial subneighborhood $W \subset U$ is just an open space contained in U. We will say that a simplicial subneighborhood $W \subset U$ is full if, for every $C \in U$ satisfying $2^C \cap 2^V_1 \subset W$, the closures in both spaces are the same, $\overline{\{C\}}^U = \overline{\{C\}}^W$.

Example 9. More examples from [63].

1. For every $q \in \mathbb{N}$, the q-skeleton U_q is a simplicial subneighborhood of $U \subset 2^V_f$. For $p \leqslant q$, U_p is a simplicial subneighborhood of U_q.

2. For every C in a simplicial neighborhood $U \subset 2^V_F$, we have that $\dot{C} \subset \overline{C} \subset U$ are simplicial subneighborhoods.

3. Consider a family $\{U_j\}_{j\in J}$ of simplicial subneighborhoods of a simplicial neighborhood U. Then the union $\bigcup_{j\in J} U_j$ and the intersection $\bigcap_{j\in J} U_j$ are simplicial subneighborhoods of U.

4. For $A \subset X$, and $\mathcal{W} = \{W_\alpha\}$ with $W_\alpha \subset X$, the nerve of A, defined as $\mathcal{N}_A(\mathcal{W}) = \mathcal{N}(\mathcal{W}) \cap 2^A$, is a simplicial subneighborhood of $\mathcal{N}(\mathcal{W})$.

Given a simplical map between simplicial complexes, $\varphi : K_1 \longrightarrow K_2$, we can define a map between the corresponding simplicial neighborhoods $\psi : U_1 \longrightarrow U_2$ as the map $\psi = \mathcal{X}(\varphi)$. That is, the map between the simplicial neighborhoods is defined as

$$\begin{aligned} \psi : U_1 &\longrightarrow U_2 \\ \{v_0, \dots, v_n\} &\longmapsto \{\varphi(v_0), \dots, \varphi(v_n)\} \end{aligned}$$

This map is obviously continuous (as seen in [48]). Moreover, it is an open map, because, for every $C \in U_1$, $\psi(2^C) = 2^{\psi(C)}$. So, it is evident that the application $\mathcal{X}$ is a covariant functor between the category of simplicial complexes and simplicial maps and the category of simplicial neighborhoods of hyperspaces (with the upper semifinite topology) and continuous and open maps between them. From now on, we will call a continuous and open map between simplicial neighborhoods a hypersimplicial map.

We will define now an inverse of this functor. For every simplicial neighborhood $U \subset 2^V_f$, we define a simplicial complex $K = \mathcal{Y}(U)$ as follows: The vertices of K are the points of $U \cap 2^V_1$ and the simplices are the points of $U \backslash 2^V_1$. In this way, every face of a simplex $\tau \subset \sigma$ is a simplex, since, as points in U, if $\sigma \in U$, then $2^\sigma \subset U$, because U is open. For every hypersimplicial map between simplical neighborhoods $\psi : U_1 \to U_2$, we define a simplicial map between the simplicial complexes $\mathcal{Y}(\psi) : \mathcal{Y}(U_1) \to \mathcal{Y}(U_2)$ as $\mathcal{Y}(\psi)(v_1) = v_2$, where $\psi(\{v_1\}) = \{v_2\}$. It is a simplicial map: For every $C \in U$, we have that $\{C\}$ is open in U if and only if C consists of only one element $C = \{v\}$ (because its minimal neighborhood 2^C has to be only C). So, it is clear that $\mathcal{Y}(\psi)$ sends vertices to vertices. Moreover, ψ is continuous, so $C \subset D$ implies $\psi(C) \subset \psi(D)$, and that ensures that the induced map $\mathcal{Y}(\psi)$ sends simplices to simplices. These functors are mutually inverse, so we have the following.

Corollary 10. *The category of simplicial neighborhoods and hypersimplicial maps is equivalent to the category of abstract simplicial complexes and simplicial maps.*

4.3.1 Universality of $2^{X_d}_u$ for shape properties

In this section we will use subsets of a given compact metric space, lying in the hyperspace, to determine the shape properties of the space that are encoded in some way in the hyperspace. Recall from [4] that, for every compact metric space (X, d) , we can define, for every $\varepsilon > 0$, the subsets of X consisting of

$$U_\varepsilon = \{C \subset X \text{ closed} : \mathrm{diam}(C) < \varepsilon\} \subset 2^X_u.$$

In that paper it is shown that the family $\{U_\varepsilon\}$ is a base of open neighborhoods of the canonical copy of X for the topology of 2^X_u. We can define these sets still in the hyperspaces $2^{X_d}_u$ (although the closed condition is unnecessary) and they are also open, because for every $\varepsilon > 0$ and every $C \in U_\varepsilon$, we have $2^C \subset U_\varepsilon$. We can use these sets to determine the Čech homology of our original space X.

Proposition 39. *Let (X, d) be a compact metric space. There exists an inverse system of subspaces of $2^{X_d}_f$ such that their McCord associated inverse system is an HPol expansion of X.*

Proof. We consider, for every $\varepsilon > 0$, the intersection

$$U_\varepsilon^f = U_\varepsilon \cap 2_f^{X_d},$$

which is open, T_0 and Alexandroff. For every pair $\varepsilon' < \varepsilon$ we have the inclusion map

$$i_{\varepsilon,\varepsilon'} : U_{\varepsilon'}^f \longrightarrow U_\varepsilon^f.$$

So, we can consider the inverse system of the McCord associated polyhedra and maps,

$$\left\{|\mathcal{K}(U_\varepsilon^f)|, |\mathcal{K}(i_{\varepsilon,\varepsilon'})|, \mathbb{R}\right\}.$$

As it is shown in corollary 7 of [3] (for a finite set of vertices but the same proof extends to an infinite set of vertices), the simplicial complex $\mathcal{K}(U_\varepsilon^f)$ is isomorphic to the barycentric subdivision of the Vietoris-Rips complex $\mathcal{R}_\varepsilon(X)$, so their realizations are homeomorphic. So, this inverse system is isomorphic to the Vietoris system (see [47] for a description), which is an HPol-expansion of X ✓.

Remark 29. Note that the same construction can be done for every different metric (generating the same topology or not) in the set X. So, in that sense, $2_f^{X_d}$ is universal for every shape property of every possible metric given over X.

Remark 30. Note that we have encoded all the shape information of a compact metric space (actually all the shape information of every possible metric on the set, that makes it compact) in terms of the category of simplicial neighborhoods and hypersimplicial maps.

Chapter 5

Problems, speculations and some scattered results

In this chapter, we present directions for future work, with some observations and small results that have not reached yet the theorem status, but we think that they could be a good starting point.

5.1 Inverse persistence

It is clear (see sections 1.3 and 1.5 in chapter 1) that a persistence module is nothing but an inverse sequence of vector spaces and homomorphisms reversed, in the sense that the sequence grows in the opposite direction. Moreover, if we "cut" the inverse sequence at some step, we obtain a persistence module of finite type and, hence, the corresponding barcode. In this way, we can obtain persistence modules from $HPol$-expansions of spaces and this makes a connection between shape theory and persistent homology theories

Let us consider a compact metric space X and a polyhedral approximative sequence[1] $A\mathcal{K}(X) = \{K_n, p_{nn+1}\}$. Although all these inverse sequences of polyhedra are the realizations of inverse sequences of simplicial complexes and simplicial maps between them, they are not filtrations of simplicial complexes, even obviating the finiteness condition, since the maps involved are not the in-

[1] Any of the polyhedral approximative sequences of section 2.3. Hence, the letter $\mathcal{K}$ stands here for any of the following: $\mathcal{M}, \check{\mathcal{C}}, \mathcal{W}, \mathcal{D}_u$ or $\mathcal{D}_l$.

clusion. But, if we consider, for any $p \in \mathbb{N}$, and a field F, the induced homology inverse sequences $H_p(A\mathcal{K}(X); F)$, denoted $H_p(\mathcal{K}(X))$ for short, they are persistence modules (with maps not induced by the inclusion) of simplicial complexes, but by means of proximity, as indicated in the quoted section.

We propose an alternative process for the construction of persistence modules coming from a point cloud, using our polyhedral approximative sequences. The theoretical foundation of doing so relies on the fact that, if we would have the ideal situation of having ε-approximations for every $\varepsilon > 0$, these sequences will give us all the information concerning the Čech homology. Now for the details. Let $\mathbb{X}$ be a point cloud. We apply our main construction to obtain a FAS $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$ of $\mathbb{X}$. Since $\mathbb{X}$ is finite, $\{\varepsilon_n, A_n, \gamma_n, \delta_n\}_{n\in\mathbb{N}}$ has only a finite number of different approximations: There is an integer s such that, for every $n \geqslant s$,

$$2\varepsilon_n < \max\{\mathrm{d}(x, y) : x, y \in X\},$$

and hence $A_n = A_{n+1} = X$, $U_{2\varepsilon_n}(A_n) = U_{2\varepsilon_{n+1}}(A_{n+1})$ and $p_{nn+1} = id$. So, we have only a finite number s of "changes" in the sequence, that we can be written as

$$U_{\varepsilon_1}(A_1) \xleftarrow{p_{12}} U_{\varepsilon_2}(A_2) \leftarrow \ldots \leftarrow U_{\varepsilon_{s-1}}(A_{s-1}) \xleftarrow{p_{s-1s}} U_{\varepsilon_s}(A_s).$$

Now, consider any of the induced polyhedral approximative sequences of section 2.3,

$$K_1 \xleftarrow{p_{12}} K_2 \leftarrow \ldots \leftarrow K_{s-1s} \xleftarrow{p_{s-1s}} K_s.$$

Its induced p-th singular homology sequence (for a field F)

$$H_p(K_1) \xleftarrow{p_{12}} H_p(K_2) \leftarrow \ldots \leftarrow H_p(K_{s-1s}) \xleftarrow{p_{s-1s}} H_p(K_s)$$

is indeed a persistence module of finite type, so it has an associated barcode $\mathcal{B}_{\mathbb{X}}$. There are several differences between this procedure, that we will informally call *inverse persistence*, and the usual one. We list some of them here:

1. The simplicial complexes used in regular persistence are constructed using all the set of points of the point cloud for every level. In contrast, the simplicial complexes constructed in the inverse persistence are based on subsets of the point cloud. Moreover, we need to add more points to the finite spaces, in order to make the maps between them continuous.

2. The maps used in the finite sequence of polyhedra constructed from the point cloud are always inclusions in regular persistence, but they are not in inverse persistence. Although they are not inclusions, they are consistent is some sense because they are defined in terms of proximity and, as we have seen, they are constructed in a way that, carried until the infinity, captures the Čech homology (or more generally, any shape property) of the space.

For the analysis of the inverse persistence we propose the following steps.

1. Formalize the algorithm outlined here and compare the computational cost with the standard algorithms for persistence on point clouds.

2. Compare the obtained inverse persistence modules and compare them with the regular persistence modules in terms of the concept of *interleaving*, introduced in [18] by Chazal et al.

3. Compare the obtained barcodes from inverse persistence with the ones obtained by regular persistence using the *bottleneck* distance on barcodes (see [20] for definition and main results concerning this distance).

It is expected that the inverse persistence modules have the same behaviour as regular ones in terms of stability (see [20, 18]), because of the shape theoretical framework where they are constructed. We hope this shape approach to persistence to be suitable for real world applications because of its constructibility and its good properties concerning stability.

5.2 The stability problem

It is a well known problem to determine for which spaces a finite set of points determines the homotopy type (or the homology) of the space. It could be roughly posed as follows.

Question 2. Let X be a topological space. What conditions do we have to impose to X in order to claim that there exists a finite set $F \subset X$ and a real number $\varepsilon > 0$ such that $|\mathcal{R}_\varepsilon(F)| \simeq X$ or, at least, for every $p \in \mathbb{N}$ and some commutative ring R with unity, $H_p(|\mathcal{R}_\varepsilon(F)|; R) \cong H_p(X; R)$?

This problem was first proposed by Hausmann for Riemannian manifolds in [35], where he proves the following theorem.

Theorem 23. *Let M be a Riemannian manifold with $r(M) > 0$ (this is an non-negative real number associated to M and related with its curvature by means of some conditions on the geodesics). For every $0 < \varepsilon \leqslant r(M)$, the map $T : |\mathcal{R}_\varepsilon(M)| \to M$ is a homotopy equivalence.*

In [40], Latschev answers in the affirmative the problem posed by Hausmann with the following stronger theorem. It makes use of the *Gromov-Hausdorff distance* between metric spaces X, Y, which is[2]

$$\mathrm{d}_{GH}(X, Y) = \inf_{Z} \inf_{X', Y'} \mathrm{d}_H(X', Y'),$$

for Z every metric space containing X', Y' isometric copies of X, Y.

Theorem 24. *Let X be a closed Riemannian manifold. There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ there is a $\delta > 0$ such that any metric space Y with $d_{GH}(X, Y) < \delta$ (where d_{GH} denotes the Gromov-Hausdorff distance for metric spaces) then $|\mathcal{R}_\varepsilon(Y)|$ is homotopically equivalent to X.*

In a completely different direction, there are some results of Niyogi et al. [56, 57] giving probabilistic bounds for capturing or reconstructing the homology of a submanifold of some $\mathbb{R}^n$ with some probability distribution involved, and considering noise. The geometric properties as a manifold are also extensively used there. In particular, they introduce the *condition number*, a real number asociated to M encoding some local and global curvature considerations.

We are interested in this problem in our context in order to determine what spaces admit inverse persistence modules of finite type. In order to address this problem, we set the following definitions

Definition 13. A polyhedral approximative sequence $A\mathcal{K}(X) = \{|\mathcal{K}_n|, |p_{nn+1}|\}$ for X is said to be *weakly homotopically (shape) stable* provided there is an integer k such that, for every $n > k$, $|\mathcal{K}_n|$ is homotopically (shape) equivalent to X. If, moreover, for every $n > k$, the maps $|p_{nn+1}|$ are homotopic (shape) equivalences,

[2]For some results concerning this notion, see [16], for example.

then $\mathcal{K}(X)$ is called *homotopically (shape) stable*. Finally, a compact metric space X will be called *(weakly) homotopically (shape) approximable* if it has (weakly) homotopically (shape) stable polyhedral approximative sequences.

The first unknown fact is to determine if the word weakly really changes the definitions above, that is:

Question 3. If we perform the main construction to X and we obtain a polyhedral approximative sequence $\{|K_n|, |p_{nn+1}|\}$ with $|K_n| \cong |K_{n+1}|$ $(Sh(|K_n|) = Sh(|K_{n+1}|))$ for some n. Does it follow that the map $|p_{nn+1}| : |K_{n+1}| \to |K_n|$ is a homotopy (shape) equivalence?

Remark 31. Evindently, a (weakly) homotopically stable sequence is also a (weakly) shape stable one. Hence, a (weakly) homotopically approximable space is (weakly) shape approximable. Moreover, for polyhedra, the words "homotopy" and "shape" in these definitions are interchangeable.

Remark 32. The induced inverse sequences of homology groups and homomorphisms of homotopically stable polyhedral approximative sequences are inverse persistent modules of finite type.

Remark 33. For a finite metric space, every polyhedral approximative sequence is homotopically stable.

From Latschev's previous result, we can easily derive the following proposition. As we will make use of finite approximations of our space, the following observation about Gromov-Hausdorff distances between a metric space and a subset will be used.

Remark 34. If $A \subset X$ is a ε approximation of the metric space X, then

$$\mathrm{d}_{GH}(A, X) < \mathrm{d}_H(A, X) \leqslant \varepsilon.$$

Proposition 40. *Every closed Riemannian manifold M admits an Alexandroff-McCord polyhedral approximative sequence weakly homotopically stable.*

Proof. The result is obtained by a slight modification of the main construction for FAS's. Consider the real number ε_0 of Theorem 24 for M. Perform the main construction on M until we get an integer k such that $\varepsilon_k < \varepsilon_0$. Then, by the same theorem, there exists a $\delta(\varepsilon_k)$ such that, for every metric space Y satisfying $\mathrm{d}_{GH}(Y, M) < \delta(\varepsilon_k)$, we have $\mathcal{R}_{\varepsilon_k}(Y)$. Now, consider a η_k approximation A_k, with $\eta_k = \min\{\varepsilon_k, \delta(\varepsilon_k)\}$. Then, since $\mathrm{d}_{GH}(A_k, M) < \delta(\varepsilon_k)$, we have

$$|\mathcal{K}(U_{2\varepsilon_k}(A_k))| = |\mathcal{R}'_{2\varepsilon_k}(A_k)| \simeq |\mathcal{R}_{2\varepsilon_k}(A_k)| \simeq M.$$

Now compute γ_k, δ_k and ε_{k+1} as usual and repeat the previous step. We can repeat this indefinitely to obtain the desired inverse sequence ✓

We want to know whether other weaker properties can be asked to a topological space to be homotopically approximable. We believe that shape properties such as movability, stability or to have the same shape as a finite polyhedron can be crucial here. The example of $\mathbb{S}^1$ from the introduction of [3] is homotopically stable. The examples of sections 2.5 and 2.6, for the computational Warsaw circle and the computational Hawaiian earring are examples of non weakly homotopically stable polyhedral approximative sequences. The (computational) Hawaiian earring cannot be a shape stable space, because its Čech homology is not finitely generated. So, movable spaces are not necessarily shape approximables. But, there are shape stable polyhedral approximative sequences, as shown in the following example. So, the Warsaw circle is shape approximable.

Example 10. The computational Warsaw circle $\mathcal{W}$ of section 2.5 is weakly shape approximable. Indeed, given any FAS of $\mathcal{W}$, we can modify it from any step to obtain a shape stable polyhedral approximative sequence from this point. Let us consider that we have performed the main construction to $\mathcal{W}$ until step $s \geqslant 1$, obtaining a "truncated" FAS $\{U_{2\varepsilon_n}, A_n, \gamma_n, \delta_n\}_{n=1}^{s}$. For the next step, consider first an integer k large enough to satisfy

$$k > \max\left\{1 - \frac{\log(\varepsilon_s - \gamma_s)}{2}, 1 - \frac{\log \delta_s}{2}\right\}$$

which ensures

$$\frac{1}{2^k} < \min\left\{\frac{\varepsilon_s - \gamma_s}{2}, \frac{\delta_s}{2}\right\}. \tag{5.1}$$

Now, consider any $\varepsilon_{s+1} \in \left(\frac{\sqrt{2}}{2^{k+1}}, \frac{1}{2^k}\right)$, for example $\varepsilon_{s+1} = \frac{\sqrt{2}+2}{2^{n+1}}$, which is a valid value because of 5.1. Consider a grid in $\mathbb{R}^2$ of side $\frac{1}{2^k}$, $G(k) = \left\{\left(\frac{n}{2^k}, \frac{m}{2^k}\right) : n, m \in \mathbb{Z}\right\}$.

We take its intersection with the computational Warsaw circle $A_{s+1} = G(k) \cap \mathcal{W}$ as ε_{s+1}-approximation (see figure 5.1) of $\mathcal{W}$. The corresponding Alexandroff-McCord polyhedron $|\mathcal{K}(U_{2\varepsilon_{s+1}}(A_{s+1}))|$ is the barycentric subdivision of the polyhedron depicted in the right side of figure 5.1. As it is readily seen, the homotopy

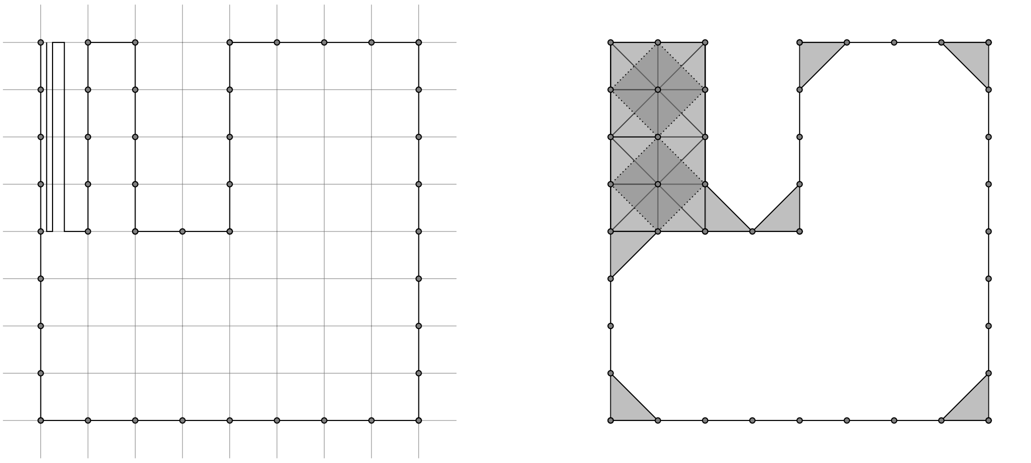


Figure 5.1: On the left, the intersection of the grid $G(k)$ with $\mathcal{W}$ is A_{s+1}. On the right, the polyhedron $|\mathcal{R}_{2\varepsilon_{s+1}}(A_{s+1})|$, where each  represents a tetrahedron.

type of this polyhedron is the same as $\mathbb{S}^1$. So we have

$$Sh(|\mathcal{K}(U_{2\varepsilon_{s+1}}(A_{s+1}))|) = Sh(\mathbb{S}^1) = Sh(\mathcal{W}).$$

Independently of the values of γ_{s+1} and δ_{s+1}, we can apply the same construction to obtain an ε_{s+2}-approximation A_{s+2} of $\mathcal{W}$ such that $|\mathcal{K}(U_{2\varepsilon_{s+1}}(A_{s+1}))|$ has the same shape as $\mathcal{W}$. We can do this process indefinitely to obtain a weakly shape stable polyhedral approximative sequence of $\mathcal{W}$.

Question 4. It is evident that a necessary condition to be a weakly homotopically (shape) approximable space is to have the homotopy (shape) type of a compact polyhedron. Is it a sufficient condition?

In section 2.5 and in Example 5.1, we have computed two different FASs for the computational Warsaw circle with their corresponding Alexandroff-McCord approximative sequences. They are essentially different at the homology level. While the first homology group of polyhedra become more different to the Čech homology of the space in the first sequence, it has the same homology in the second step and further in the second one. But, even in the first case, we observed (see comments about homology in section 2.5) that the image of the homology group at any step by the induced map in homology by the bonding maps gives us the Čech homology of the space. We would like to know if this property is satisfied for any class of compact metric space. We formulate the following tentative result:

Conjecture 1. Let X be a compact metric space with the shape of a compact polyhedron. Then, for every FAS of X, there exist two integers $n < m$ such that the map induced in homology by the bounding map

$$(p_{n,m})_* : H_1(\mathcal{K}(U_{2\varepsilon_m})(A_m)) \longrightarrow H_1(\mathcal{K}(U_{2\varepsilon_n})(A_n))$$

satisfies

$$\mathbf{im}((p_{n,m})_*) \simeq \check{H}(X).$$

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