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TESIS DOCTORAL

Invariants of singularities, generating sequences and toroidal structures

Invariantes de singularidades, sucesiones generatrices y estructuras toroidales

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Tesis Doctoral

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Memoria para optar al grado de Doctor

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Dirigida por

Pedro Daniel González Pérez

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UNIVERSIDAD
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Castigar a los opresores es clemencia,
perdonarlos es barbarie.
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Que no calle el cantor porque el silencio
cobarde apaña la maldad que oprime.
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Resumen

Invariantes de singularidades, sucesiones generatrices y estructuras toroidales

Sinopsis. En las últimas décadas, los ideales multiplicadores y sus números de salto asociados se han convertido en una herramienta importante en el campo de la geometría birracional de variedades algebraicas complejas y en teoría de singularidades.

En esta Tesis Doctoral se describen los ideales multiplicadores y números de salto asociados a un germen irreducible de hipersuperficie quasi-ordinaria y también a una singularidad de curva plana. El enfoque está basado en un teorema de Howald que describe los ideales multiplicadores de una singularidad de hipersuperficie Newton no degenerada en términos de su poliedro de Newton.

Generalizamos el resultado de Howald usando la existencia de una resolución sumergida toroidal para estas singularidades. La estructura de estas resoluciones viene determinada por el tipo topológico sumergido de estas singularidades. El método pasa por describir minuciosamente las sucesiones generatrices asociadas a las valoraciones divisoriales correspondientes a los divisores primos excepcionales en la resolución toroidal sumergida.

El resultado principal en ambos casos es que los ideales multiplicadores están generados por monomios generalizados en las curvas de contacto maximal, también llamadas semiraíces, y sus generalizaciones en el caso de hipersuperficies quasi-ordinarias. Como aplicación de este estudio, obtenemos algoritmos para calcular bases de los ideales multiplicadores y el conjunto de números de salto.

Introducción

En esta Tesis Doctoral estudiamos las singularidades de ciertas clases de variedades algebraicas o analíticas complejas. Un método habitual para estudiar una singularidad es mediante la *resolución de singularidades.* Es un resultado conocido que la resolución de singularidades existe cuando la variedad está definida sobre un cuerpo algebraicamente cerrado de característica 0 ([**Hir64**]). Sin embargo, en general éste sigue siendo un problema abierto cuando el cuerpo base es de característica positiva. Incluso cuando una resolución existe, surge la pregunta de cómo se relacionan los divisores excepcionales de resoluciones diferentes.

Uno de los temas de la teoría de singularidades es clasificación de variedades algebraicas complejas usando *invariantes algebraicos o analíticos* asociados al anillo local de la variedad en el punto singular o la topología del correspondiente enlace, o bien el *tipo topológico* del germen de variedad sumergido en un espacio afín, y entender las relaciones entre estos invariantes.

En este texto estudiamos las singularidades que aparecen en cierto tipo de gérmenes de hipersuperficie de variedades algebraicas o analíticas sumergidas en un espacio afín liso, a saber, gérmenes de curvas planas y de hipersuperficies quasi-ordinarias irreducibles. A tal fin, investigaremos el proceso de resolución embebida mediante modificaciones toroidales. Este enfoque permite dar una descripción concreta de la combinatoria de la resolución en términos de invariantes topológicos de las singularidades aquí consideradas. En particular, esto lleva a la descripci´n detallada de las valoraciones divisoriales que aparecen en estas resoluciones, así como de sus sucesiones generatrices, y proporciona herramientas para describir los ideales multiplicadores y números de salto en términos de la clase topológica sumergida de estos gérmenes.

Singularidades de curvas planas

Sea (S, O) un germen de superficie lisa y $(C, O) \subset (S, O)$ un germen de curva reducida en el punto O. Escogiendo un sistema de coordenadas para la superficie en O el germen de curva viene definido por la ecuación $f(x, y) = 0$, donde $f(x, y)$ es un germen holomorfo $f \in \mathbb{C}\{x, y\}$ en $(0, 0)$. El punto $(0, 0)$ es *singular* si ambas derivadas parciales de f se anulan en $(0, 0)$, en otro caso el punto es liso.

El teorema de Newton-Puiseux asegura que para un germen de curva plana irreducible, también llamado *rama*, se tiene una parametrización de la forma $y = \zeta(x^{1/n})$. Los *exponentes característicos* de una curva son los órdenes en x de la diferencia de parametrizaciones de Newton-Puiseux. En general, $C = \sum C_i$, donde C_i son los factores irreducibles. El *orden de coincidencia* de dos parametrizaciones distintas es el máximo de los órdenes en x de la diferencia de cualesquiera series de Newton-Puiseux series de las mismas. Esta información combinatoria puede organizarse mediante el *árbol de Eggers-Wall*.

Si C es una rama, pueden considerarse las *multiplicidades de intersección* en el origen con curvas que no la contienen como componente. Este subconjunto de los naturales conforma el *semigrupo* de la rama, que admite un conjunto finito de *generadores*. Los generadores del semigrupo pueden expresarse en términos de los exponentes característicos y viceversa. Una *semiraíz* de la rama C es una rama D con el menor nmero de exponentes caractersticos tal que el orden de contacto de C y D es un exponente caracterstico de C. Los generadores del semigrupo de C corresponden a las multiplicidades de intersección de C con las diferentes semiraíces.

Cualquier resolución sumergida de C, también llamada log-resolución, es una composición de explosiones de puntos. Denotemos por E_i las componentes del divisor excepcional reducido $\Psi^{-1}(O) = \sum_{i \in I} E_i$. La transformada total F de la curva es de la forma $F = \sum_{i \in I_\Psi} \mathrm{ord}_{E_i}(C) E_i + \sum_{j=1}^{r} \tilde{C}_j$, donde ord_{E_i} denota la valoración divisorial asociada al divisor E_i y $\tilde{C}_j$ denota la tranformada estricta de la rama C_j de C. El *grafo dual* $\Gamma(\Psi, C)$ asociado a (C, Ψ) tiene vértices que se corresponden a las componentes irreducibles de F, y segmentos uniendo estos vértices en caso de que las correspondientes componentes de F se corten. En el caso de singularidades de curvas planas el grafo dual $\Gamma(\Psi, C)$ es un árbol. Denotemos por V el conjunto de vértices del grafo dual. La *valencia* $v(i)$ de un vértice i es el número de segmentos incidentes a él. Una rama L no contenida entre las componentes irreducibles de C se dice que es una *curva de contacto maximal* de C si Ψ es también una resolución sumergida de $C + L$ y la transformada estricta de L corta a un divisor de valencia 1. Escoger una curva de contacto maximal para cada divisor excepcional E_i de valencia $v(i) = 1$ define un conjunto completo de curvas de contacto maximal $L_0, \ldots, L_k$ para (Ψ, C). Las curvas de contacto maximal generalizan la noción de semiraíces de las componentes de la curva.

Un subconjunto de generadores del ideal maximal del anillo local de la superficia S en O se dice que es una *sucesión generatriz* para una valoración divisorial si los monomios en tales funciones generan los ideales de valoración. Las sucesiones generatrices fueron descritas en mayor generalidad por Spivakovsky ([**Spi90**]). La noción de sucesión generatriz generaliza a un conjunto finito de valoraciones divisoriales y fue caracterizada geométricamente en [DGN08], en términos de las curvas de contacto maximal.

Singularidades de hipersuperficie quasi-ordinaria

Las singularidades de hipersuperficie quasi-ordinaria aparecen clásicamente en el enfoque de Jung para analizar las singularidades de hipersuperficie utilizando la resolución sumergida del discriminante de una proyección finita a un espacio afín. En general, las singularidades de hipersuperficie quasi-ordinaria no son aisladas.

Nosotros nos concentraremos en el estudio de un germen irreducible de hipersuperficie quasi-ordinaria $(H, O) \to (\mathbb{C}^{d+1}, O)$, que viene definido por un polinomio mónico irreducible $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ cuyo discriminante con respecto a Y define un divisor con cruces normales. El teorema de Jung-Abhyankar asegura la existencia de una serie con exponentes fraccionarios $\zeta \in \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$ que es raíz del polinomio f, con $n = \deg_y(f)$ (véase [Jun08, Abh55, Gon00]). Las diferencia de estas raíces tienen exponentes dominantes, llamados *característicos*, que codifican el tipo topológico sumergido del germen gracias a resultados de Gau y Lipman. A una hipersuperficie quasi-ordinaria irreducible se le puede asociar un semigrupo, generalizando la construcción del semigrupo de una curva plana, cuyos generadores vienen determinados por los exponentes característicos (véase [Gon03a]). Este sistema de generadores puede ser asímismo obtenido usando las *semiraíces*, que a grandes rasgos son gérmenes definiendo hipersupeficies quasi-ordinarias parametrizadas por truncaciones adecuadas de ζ. Las semiraíces quasi-ordinarias generalizan la noción de raíz aproximada y las curvas de contacto maximal (véase [AM73, Pop03]).

Se dispone de un proceso de resolución embebida específico para gérmenes de hipersuperficie quasi-ordinaria (véase [Gon03b]). En primer lugar, uno obtiene una normalización sumergida del germen como composición de modificaciones tóricas determinadas por el poliedro de Newton de f y de sus transformadas estrictas (donde las semiraíces aparecen definiendo coordenadas auxiliares adecuadas). Esto produce de modo canónico una resolución sumergida parcial en un *encaje toroidal sin autointersección* dado por un *complejo cónico poliedral con estructura íntegra* Θ. Una resolución sumergida se obtiene entonces componiendo la normalización con cualquier resolución tórica del encaje toroidal, definida por una subdivisión regular Θ_{reg} de Θ. El complejo Θ_{reg} determina el grafo dual en el caso particular de las curvas planas y puede ser entendido como una generalización del mismo al caso de dimensión arbitraria..

Ideales multiplicadores y números de salto

Los ideales multiplicadores son un tema de estudio de la geometría birracional, están relacionados con numerosos invariantes tanto en teoría de singularidades como en álgebra conmutativa. Recordemos su definición y algunas de sus propiedades fundamentales (ver [Laz04, Chapters 11 to 14]).

Sea X una variedad lisa compleja y $\mathfrak{a}$ un haz de ideales en $\mathcal{O}_X$. Recuérdese que un morfismo propio y birracional $\Psi : Y \to X$ es una *log-resolución* del ideal $\mathfrak{a}$ si Y es liso y $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ para cierto divisor efectivo F en Y con cruzamientos normales simples.

Sea K_Ψ el divisor canónico relativo de Ψ, esto es, el divisor definido por el determinante jacobiano de Ψ. El *ideal multiplicador* de $\mathfrak{a}$ para un parámetro fijo $\xi \in \mathbb{Q}_{>0}$ está definido por

$$\mathcal{J}(\mathfrak{a}^\xi) := \Psi_* \mathcal{O}_Y(K_\Psi - \lfloor \xi F \rfloor).$$

Es un hecho conocido que los ideales multiplicadores no dependen de la log-resolución escogida. Las nociones explicadas pueden ser también estudiadas localmente cuando $\mathfrak{a}$ es un ideal del anillo de gérmenes de funciones holomorfas en un punto de la variedad X. Escribamos $F = \sum m_i E_i$

en términos de las componentes irreducibles. Podemos también escribir $K_\Psi = \sum(\lambda_i - 1)E_i$, donde el coeficiente λ_i se denomina la *log-discrepancia* del divisor E_i. Entonces, los ideales multiplicadores se pueden describir valorativamente como:

$$\mathcal{J}(\mathfrak{a}^\xi) = \{h \in \mathcal{O}_X \mid \mathrm{ord}_{E_i}(\Psi^* h) \geq \lfloor \xi m_i \rfloor - (\lambda_i - 1)\ \forall i\}.$$

Existe un conjunto discreto de números racionales ξ_i en los cuales estos ideales cambian. Estos coeficientes se denominan *números de salto* y definen una filtración de ideales

$$\mathcal{O}_X \supsetneq \mathcal{J}(\mathfrak{a}^{\xi_1}) \supsetneq \ldots \supsetneq \mathcal{J}(\mathfrak{a}^{\xi_i}) \supsetneq \ldots,$$

con $0 < \xi_1 < \ldots < \xi_i < \ldots$ y tal que $\mathcal{J}(\mathfrak{a}^\xi) = \mathcal{J}(\mathfrak{a}^{\xi_i})$ para $\xi \in [\xi_i, \xi_{i+1})$, véase [**ELSV04**]. Los números de salto aparecen implícitamente en el trabajo de Libgober ([**Lib83**]), y de Loeser-Vaquié ([**LV90, Vaq92, Vaq94**]) y juegan un papel en el artículo de Artal [**Art94**]. Es importante remarcar que los números de salto verifican ciertas propiedades de periodicidad.

Si D es un divisor efectivo, una log-resolución de D es simplemente una log-resolucin del ideal $\mathcal{O}_X(-D)$. Sea D un divisor efectivo de X. En tal caso denotamos por $\mathcal{J}(\xi D) := \mathcal{J}(\mathcal{O}_X(-D)^\xi)$ los ideales multiplicadores asociados. Se tiene que ξ es un número de salto de D si y sólo si $\xi + 1$ es también un número de salto (véase Lemma 3.26). Como consecuencia, los números de salto de un divisor D están determinados por aquellos en el intervalo $(0, 1]$.

Los números de salto codifican información geométrica, algebraica y topológica relevante y aparecen en multitud de contextos diferentes (véase [**Bud12, ELSV04**]). El número de salto más pequeño, llamado *umbral log canónico* aparece en muchas sitauciones diferentes. Si $0 < \xi \leq 1$ es un número de salto asociado al ideal (f), con $f \in \mathbb{C}[x_1, \ldots, x_n]$, entonces $-\xi$ es una raíz del polinomio de *Bernstein-Sato* de f. Se puede expresar, en términos de los números de salto, una cota uniforme de Artin-Rees del ideal (f), y también se tienen cotas relacionadas para los números de Milnor y Tjurina cuando f tiene singularidades aisladas (véase [**ELSV04**]). El espectro de Hodge, asociado a la cohomología de la fibra de Milnor de f en un punto singular x_0, está relacionado con las multiplicidades de los números de salto de f en x_0 ([**Bud11**]).

Los ideales multiplicadores y los números de salto no son sencillos de calcular. Una excepción a este principio es el caso de los ideales monomiales y las singularidades de hipersuperficie *no degenerada con respecto a su poliedro de Newton*. En ambos casos, Howald prueba que los ideales mutiplicadores son ideales monomiales y los caracteriza en términos de poliedros de Newton ([**How01**]), usando una log-resolución *tórica*.

Motivación y objetivos

La motivación principal de este trabajo es extender los resultados de Howald sobre ideales multiplicadores a otras singularidades usando resoluciones sumergidas toroidales. Esta cuestión tiene varias interpretaciones diferentes que son susceptibles de generalización. En primer lugar, el teorema de Howald proporciona una descripción de los ideales multiplicadores en términos del *poliedro de Newton* del ideal. En los casos estudiados por Howald, la log-resolución tórica factoriza a través de una *resolución parcial sumergida*, dada por la modificación tórica asociada al abanico dual al poliedro de Newton. De ésto se deduce que los números de salto en el caso de una hipersuperficie no degenerada con respecto a su poliedro de Newton solamente dependen de los *divisores tóricos excepcionales* que aparecen en la resolución parcial sumergida cuando $0 < \xi < 1$. . Este razonamiento muestra que nuestro problema está interrelacionado con la cuestión de qué divisores excepcionales son necesarios para describir los números de salto. Las *valoraciones tóricas divisoriales* están determinadas por sus valores en los monomios en las coordenadas afines dadas, esto es, estas coordenadas definen trivialmente sucesiones generatrices para estas valoraciones. El resultado de Howald proporciona una *base monomial* para los ideales multiplicadores de ideales monomiales en las coordenadas afines originales.

En términos más precisos, podemos reformular la siguiente pregunta: Sea $\mathfrak{a}$ un haz de ideales en X y supongamos que existe una resolución parcial sumergida que factoriza la log-resolución de $\mathfrak{a}$. ¿Cómo se relacionan los ideales multiplicadores de $\mathfrak{a}$ con los divisores excepcionales en la resolución parcial sumergida? ¿Podemos describir bases para los ideales multiplicadores en términos de sucesiones generatrices de las valoraciones divisoriales asociadas a tales divisores?

El objetivo es responder a las preguntas previas en el caso de singularidades de curvas planas y de singularidades de hipersuperficies quasi-ordinarias irreducibles, usando la existencia de los métodos de resolución toroidal disponibles.

El enfoque toroidal a la resolución sumergida de curvas planas ha sido estudiado en [**Oka96**, **DO95**, **GGP16**, **GGP19**], mientras que la resolución toroidal de hipersuperficies quasi-ordinarias fue descrita en [**Gon03b**]. El objetivo aquí es entender la estructura toroidal de la resolución sumergida de modo que arroje luz sobre el estudio de los divisores excepcionales que aparecen en ella y permita compararlos con los divisores de la resolución parcial sumergida.

Las sucesiones generatrices de valoraciones divisoriales de curvas planas fueron descritos en [**DGN08**] en términos de las curvas de contacto maximal. Investigaremos su estructura en términos de la resolución toroidal. La primera aparicion del uso herramientas tóricas en el estudio de sucesiones generatrices se encuentra en [**GT00**]. Además, abordaremos la descripción de las sucesiones generatrices en el caso quasi-ordinario en términos de las semiraíces del germen de hipersuperficie. Otro elemento fundamental en la construcción de los ideales multiplicadores son las *log-discrepancias* de las valoraciones divisoriales. En el caso de curvas planas las log-discrepancias fueron descritas en [**FJ04**] and [**GGP18**], mientras que para gérmenes quasi-ordinarios irreducibles fueron estudiadas en [**GG14**].

Contenidos y resultados

Pasamos ahora a una descripción detallada del contenido de esta memoria. En los tres primeros capítulos fijamos notaciones y recordamos algunos resultados conocidos sobre la geometría tórica, las singularidades de curvas planas y teoría básica sobre los ideales multiplicadores.

El Capítulo 1 es un recordatorio de geometría tórica. En primer lugar, trataremos con las definiciones y propiedades fundamentales de las variedades tóricas. Introducimos la variedad tórica normal Z_Σ asociada a un abanico Σ, y recordamos la correspondencia *cono-órbita*, los divisores tóricos invariantes y sus *valoraciones monomiales* asociadas. Después describimos las modificaciones tóricas asociadas a subdivisiones de abanicos. Además, daremos una descripción de los poliedros de Newton y sus funciones soporte, así como su relación con las valoraciones monomiales. Incluímos también algunos lemas concernientes a las funciones soporte de poliedros de Newton que serán clave en la descripción de los ideales multiplicadores.

En el Capítulo 2 discutimos la teoría clásica de invariantes topológicos de una rama plana, empezando con la parametrización de Newton-Puiseux: los generadores del semigrupo, los exponentes característicos y los pares de Newton.

El tipo topológico sumergido de una singularidad de curva plana $(C, 0) \subset (S, O)$ viene codificado por los órdenes de coincidencia de pares de parametrizaciones de Newton-Puiseux de ramas de C relativas a un germen liso R transverso a C (véase [**Wal04**]). Esta información combinatoria se organiza en el *árbol de Eggers-Wall*, $\Theta_R(C)$, de C con respecto a R. Éste es un árbol con raíz cuyos extremos están etiquetados por R y las ramas de C, y está dotado de una función exponente $\mathbf{e}_R$ y una función índice $\mathbf{i}_R$. El árbol tiene un número finito de puntos marcados que corresponden a los órdenes de coincidencia considerados previamente. La función índice provee los grados de las extensiones de Galois asociadas a ciertas truncaciones de las parametrizaciones de Newton-Puiseux de C. Las funciones del árbol comentadas anteriormente determinan la función de contacto $\mathbf{c}_R$, que permite dar fórmulas simples para la multiplicidad de intersección de ramas de C. Las semivaloraciones son una herramienta fundamental en la comprensión de singularidades de curvas planas. Entre ellas están las *valoraciones divisoriales*, ord_E asociada a un divisor excepcional primo E sobre un germen de superficie (S, O). Por otro lado, se tienen las *valoraciones orden de anulación a lo largo de una rama*, asociadas a gérmenes irreducibles de curva plana. Estos dos tipos de semivaloraciones aparecen en la descripción de los ideales multiplicadores. Describiremos las sucesiones generatrices para un conjunto finito de tales valoraciones. Terminamos el capítulo con algunos resultados sobre técnicas de expansion, que jugarán un papel fundamental en la descripción de las sucesiones generatrices.

El Capítulo 3 introduce los *ideales multiplicadores* y los *números de salto* asociados a una variedad compleja sumergida en un espacio afín liso. Comenzaremos dando las definiciones concernientes al *divisor canónico relativo* asociado a una transformación birracional, que es uno de los ingredientes principales en la definición de los ideales multiplicadores. Elegiremos el representante de la clase del divisor canónico relativo dado por la anulación del determinante jacobiano de una log-resolución dada. La estructura local del divisor canónico relativo se codifica mediante un vector de *log-discrepancias* en los hiperplanos coordenados asociados a un sistema adecuado de coordenadas. En este capítulo incluímos también una versión del resultado de Howald para ideales monomiales (Teorema 3.31). Su demostración inspira nuestros teoremas sobre la reducción de condiciones de los ideales multiplicadores, tanto para gérmenes de curvas planas como para gérmenes de hipersuperficies quasi-ordinarias irreducibles.

En el Capítulo 4, formalizaremos la construcción de una *resolución sumergida toroidal* de una curva plana. Precedentes de esta resolución aparecen en [**Oka96, DO95**], aunque seguiremos fundamentalmente el enfoque de [**GGP16**]. Esta parte está también en conexión con el enfoque desarrollado en el reciente estudio [**GGP19**]. Además, estudiaremos las sucesiones generatrices de algunas valoraciones usando la resolución toroidal sumergida.

Empezamos el capítulo recordando algunas propiedades específicas de las variedades tóricas de dimensión dos, incluyendo las resoluciones minimales y los grafos duales. Una resolución sumergida toroidal de una curva plana se describe como composición de modificaciones toroidales asociadas al polígono de Newton de la curva en ciertos sistemas de coordenadas. Cada modificación toroidal está asociada a la mínima subdivisión regular del abanico dual al polígono de Newton de la curva o de sus transformadas estrictas en puntos singulares. Cada modificación reduce la complejidad de la curva, por tanto la composición de un número finito de ellas produce una resolución toroidal sumergida Ψ. Codificamos este proceso mediante una *descomposición adecuada* del árbol de Eggers-Wall de la curva. Una *completación* $\Theta_R(\bar{C})$ del árbol $\Theta_R(C)$ consiste en completar los finales de los niveles de la función índice del árbol. Esto es tanto como considerar ramas auxiliares L_j, $j \in J$, que tienen *contacto maximal* con las componentes de C. La transformada estricta de cada rama auxiliar L_j es lisa e interseca a la fibra excepcional $\Psi^{-1}(O)$ en un punto liso correspondiente a un divisor cuyo vértice en el grafo dual tiene valencia 1. Esto permite dar un homeomorfismo entre la completación de árbol, $\Theta_R(\bar{C})$, y el grafo dual de la curva completada, $\Gamma(\Psi, \bar{C})$, de tal modo que los vértices asociados a divisores E_P, con P un punto marcado del árbol $\Theta_R(C)$, son precisamente los vértices de ruptura (de valencia ≥ 3) en el grafo dual de la resolución sumergida minimal de $\bar{C}$.

La estructura toroidal de la resolución permite asociar a cada punto racional P del árbol de Eggers-Wall $\Theta_R(C)$ un divisor exceptional E_P y su correspondiente valoración divisorial $\nu_P = \text{ord}_{E_P}$. El orden de anulación a lo largo de este divisor puede ser descrito mediante fórmulas combinatorias en las funciones $\mathbf{e}_R$, $\mathbf{i}_R$ y $\mathbf{c}_R$ odel árbol (véase el Teorema 4.70). Esta construcción puede ser generalizada a cualquier punto del árbol, contextualizando la construcción en el marco de la teoría de semivaloraciones sobre una superficie lisa (véase [**GGP18, FJ04**]).

En particular, toda hoja del árbol está asociada a una componente C_j de C, y la valoración correspondiente es el orden de anulación a lo largo de C_j.

La noción de *sucesión generatriz* de una valoración ν fue estudiada por Spivakovsky in [**Spi90**] y generalizada por Delgado, Galindo y Nuñez a una tupla finita de valoraciones [**DGN08**]. Éstos últimos probaron que si elegimos un conjunto finito de valoraciones divisoriales y consideramos su resolución minimal Ψ (en el sentido de la mínima composición de explosiones de puntos tal que todos estos divisores aparecen) entonces un conjunto de curvas de contacto maximal para Ψ son una sucesión generatriz para $\underline{\nu}$. Probaremos que un conjunto de gérmenes $\{z_j\}$ forman una sucesión generatriz para $\underline{\nu}$ si y sólo si toda función h puede ser expandida en términos de aquellos de tal modo que la valoración de h sea la valoración mínima de los términos en su expansión (véase Lemma 4.87). En tal caso, decimos que $\underline{\nu}$ es *monomial* con respecto a los $\{z_j\}$. Este resultado nos permite proveer de bases monomiales a los ideales multiplicadores mediante la elección de expansiones adecuadas para cualquier germen.

Además, daremos una versión de [**DGN08**, Theorem 5]. Probamos que toda valoración ν_P asociada a un punto del árbol, $P \in \Theta_R(\bar{C})$, es monomial con respecto a un conjunto de gérmenes en la completación de contacto maximal $\bar{C}$ de la curva C (Teorema 4.125). Más aún, todo conjunto finito de valoraciones es simultaneamente monomial con respecto a las ramas de la complección maximal, esto es, en un conjunto de curvas de contacto maximal (ver Teorema 4.170):

THEOREM 0.1. *Sea $\underline{\nu} = (\nu_1, \ldots, \nu_s)$ una tupla de valoraciones asociadas a puntos del árbol $\Theta_R(C)$. Entonces $\underline{\nu}$ es monomial con respecto a cualquier conjunto de gérmenes que definan las curvas de contacto maximal de C.*

La metodología utilizada para la prueba de este teorema es diferente de la utilizada en [**DGN08**].

En el caso de una rama, nuestra prueba descansa en el control de los polígonos de Newton de las semiraíces a través del proceso toroidal de resolución y en las propiedades del semigrupo de la rama. En general, generalizamos el caso de una rama utilizando la estructura combinatoria de árbol. Nuestro método incluye la construcción de una *expansión adecuada* para cualquier germen en términos de las curvas de contacto maximal de C. El concepto de adecuada quiere decir aquí que la valoración del germen es igual al mínimo de las valoraciones de los términos en su expansión.

El Capítulo 5 está dedicado a las pruebas de los resultados principales concernientes a los ideales multiplicadores y números de salto de curvas planas. En primer lugar, probamos que las condiciones correspondientes a componentes irreducibles E_i del divisor excepcional que definen vértices de valencia $v(i) \leq 2$ del grafo dual son redundantes en la definición de los ideales multiplicadores (Teorema 5.8):

THEOREM 0.2. *Si $0 < \xi < 1$, entonces*

$$\mathcal{J}(\xi C) = \{h \in \mathcal{O}_{S,O} \mid \mathrm{ord}_{E_i}(h) \geq \lfloor \xi \mathrm{ord}_{E_i}(C) \rfloor - (\lambda_i - 1), \ \textit{con } v(i) \geq 3\}.$$

Para probar este teorema se han de reformular las condiciones valorativas de los ideales multiplicadores en términos de las funciones soporte de la transformada total de C en los distintos pasos del proceso toroidal de resolución asociado a una descomposición de contacto maximal del árbol de Eggers-Wall. Además, hay que aplicar la descripción toroidal de las log-discrepancias para extender el argumento de Howald.

Este teorema es independiente de los resultados de Tucker, Smith y Thompson concernientes a la *contributción de divisores excepcionales a un número de salto*, y proporciona una visión diferente del problema (véase [**Tuc10a, ST07**]).

Posteriormente, daremos una descripción de los generadores de los ideales multiplicadores $\mathcal{J}(\xi C)$ (Teorema 5.16).

THEOREM 0.3. *Sean* $z_0, \ldots, z_k \in \mathcal{O}_{S,O}$ *funciones irreducibles definiendo las curvas de contacto maximal asociadas a* C*. Si* $0 < \xi < 1$*, el ideal multiplicador* $\mathcal{J}(\xi C)$ *está generado por monomios en* $z_0, \ldots, z_k$*.*

La prueba de este teorema se basa en nuestros resultados de monomialización (Corolario 4.160), en términos de las sucesiones generatrices de valoraciones (véase [**DGN08**]). Como consecuencia del resultado anterior, podemos asociar un número de salto a cada *expresión monomial* de la forma $z_0^{i_0} z_1^{i_1} \ldots z_k^{i_k}$ (Corolario 5.22):

COROLLARY 0.4. *Cada monomio de la forma* $\mathcal{M} = z_0^{i_0} z_1^{i_1} \ldots z_k^{i_k}$ *determina un número de salto*

$$\xi_{\mathcal{M}} = \min_{v(i) \geq 3} \left\{ \frac{\nu_{E_i}(\mathcal{M}) + \lambda_{E_i}}{\nu_{E_i}(C)} \right\}.$$

y todo número de salto aparece de este modo.

Nótese que $\xi_{\mathcal{M}}$ es el menor número racional tal que $\mathcal{M} \notin \mathcal{J}(\xi_{\mathcal{M}} C)$.

En particular, el *umbral log-canónico* es el número de salto asociado a $\mathcal{M} = 1$. Como consecuencia probamos que el umbral log-canónico se alcanza en un divisor creado en la primera modificación toroidal de la resolución embebida. Véase también [**ACLM08**].

Pasamos entonces a esbozar cómo nuestros resultados pueden generalizarse para describir los ideales multiplicadores asociados a un ideal de $\mathbb{C}[[x, y]]$. La prueba pasa por dar una descripción de la log-resolución toroida del ideal (compárese con la utilización de transformaciones de Newton en [**CV14**, **CV15**]).

Damos una nueva prueba de las fórmulas descritas por Naie en [**Nai09**] para los números de salto de una rama plana. Nuestra prueba produce una biyección entre el conjunto de números de salto provenientes de las fórmulas de Naie y el conjunto de monomios generalizados en las semiraíces de la rama que producen números de salto menores que uno. Además, probamos la fórmula del cardinal del conjunto de números de salto menores que uno contados con *multiplicidad* de una curva plana (véase la Sección 5.5).

Los resultados principales de esta Tesis Doctoral están plasmados en el Capítulo 6, donde generalizamos los resultados descritos anteriormente para singularidades de curvas planas al caso de un germen irreducible de *hipersuperficie quasi-ordinaria* $(H, O) \subset (\mathbb{C}^{d+1}, O)$.

Con el fin de estudiar los ideales multiplicadores asociados a estas hipersuperficies usamos el proceso de resolución toroidal sumergida introducido por González Pérez in [**Gon03b**]. Como resultado de este proceso se obtiene una *resolución parcial sumergida* en una variedad normal que únicamente tiene singularidades tóricas. Este espacio está dotado de una estructura de *encaje toroidal sin autointersección*, y su complejo cónico con estructura íntegra asociado Θ viene determinado por los exponentes característicos. La resolución sumergida del germen se obtiene como composición de la resolución parcial sumergida con una resolución toroidal del espacio normal ambiente, que viene dada por una subdivisión regular Θ_{reg} de Θ.

La asignación $u \to E_u$ define una biyección entre los rayos del complejo Θ_{reg} y las componentes de la transformada total de la *completación* de H, que es la hipersuperficie reducida cuyas componentes son los hiperplanos coordenados y las semiraíces de H. Entre estas componentes de la transformada total, distinguimos las asociadas a los rayos de Θ:

- Los *rayos finales* de Θ (aquellos que aparecen en un único cono de dimensión máxima del complejo) corresponden a los hiperplanos coordenados $x_k = 0$ para algún $k = 1, \ldots, d$ o a las semiraíces de H.

- Los restantes rayos de Θ corresponden a divisores excepcionales primos en la resolución parcial sumergida y los llamamos *rayos relevantes* y *divisores excepcionales relevantes*, respectivamente (por analogía con [**ST07**]).

Además, mostramos que el complejo cónico poliedral Θ puede ser dotado de una función log-discrepancia. Damos una fórmula combinatoria para las log-discrepancias de cualquier divisor excepcional toroidal E_u, donde u define un rayo de Θ_{reg}, en términos de los exponentes característicos (véase el Lema 6.116).

Tras este trabajo previo de preparación probamos que las condiciones valorativas de los ideales multiplicadores que corresponden a rayos de Θ_{reg}, que no son rayos relevantes de Θ, son asímismo irrelevantes en la definición de los ideales multiplicadores (Teorema 6.122):

THEOREM 0.5. *Si* $0 < \xi < 1$ *se tiene que*

$$\mathcal{J}(\xi H) = \left\{ h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \operatorname{ord}_{E_u}(h) \geq \lfloor \xi \operatorname{ord}_{E_u}(H) \rfloor - (\lambda_u - 1) \right\},$$

donde E_u *recorre los rayos relevantes de* Θ*, i.e., los divisores excepcionales en la resolución parcial sumergida de* H*.*

Generalizamos algunos de los resultados de monomialización de valoraciones divisoriales al caso quasi-ordinario. Utilizando las técnicas de expansión en términos de las semiraíces, el proceso toroidal de resolución parcial sumergida y las propiedades del semigrupo de una hipersuperficie quasi-ordinaria, demostramos que las valoraciones divisoriales de los divisores excepcionales E_u definidos por rayos relevantes de Θ son monomiales en las semiraíces de H (Lema 6.83). Este lemma, combinado con el teorema anterior, permite probar el siguiente resultado sobre los generadores de los ideales multiplicadores de H (Teorema 6.129):

THEOREM 0.6. *Para todo* $0 < \xi < 1$ *el ideal multiplicador* $\mathcal{J}(\xi H)$ *está generado por monomios en las coordenadas* $x_1, \ldots, x_d$ *y las semiraíces* $z_0, \ldots, z_g$*.*

Como consecuencia de los teoremas anteriores damos un método algorítmico para obtener los números de salto de H. Terminamos el capítulo con dos ejemplos del cálculo de los números de salto de las singularidades de hipersuperficies quasi-ordinarias irreducibles.

Comparación con otros enfoques

En los últimos años se han hecho muchos esfuerzos con objeto de mejorar la comprensión de los ideales multiplicadores y los números de salto, especialmente sobre superficies. En varias de las referencias los autores trabajan de modo más general sobre superficies con *singularidades racionales*, a cuyo contexto generalizan los ideales multiplicadores.

Tucker ([**Tuc10a**]) generaliza la noción de *número de salto contribuído por un divisor primo*, introducida por Smith y Thompson ([**ST07**]), como medio para medir qué divisor provoca el salto. En el caso de una rama, Tucker prueba que los únicos divisores que *contribuyen* son los divisores de ruptura de la resolución minimal y proporciona un algoritmo para calcularlos. Tucker sugiere que la noción de contribución está ligada de forma más natural a *divisores reducidos*, e introduce una noción de *contribución crítica*. En el caso de ideales sobre una superficie con singularidades racionales, Tucker da una descripción de la geometría de los divisores que contribuyen críticamente ([**Tuc10b**]), en términos del grafo dual de la log-resolución.

Alberich-Carramiñana, Álvarez Montaner y Dachs-Cadefau ([**AAD16**]) dan un algoritmo para calcular los números de salto y un sistema de generadores de los ideales multiplicadores para un ideal definido sobre una superficie con singularidades racionales. Alberich-Carramiñana, Álvarez Montaner y Blanco ([**AAB17**]) describen un algoritmo para calcular un conjunto de generadores monomiales para los ideales multiplicadores de cualquier ideal plano, y de modo más general para cualquier ideal íntegramente cerrado en el plano. Estos generadores admiten una presentación como monomios generalizados en un conjunto de elementos de contacto maximal asociados a la resolución minimal del ideal. En este caso, se tiene la correspondencia clásica entre ideales íntegramente cerrados y los divisores anti-nef, hecho que los autores usan para calcular la clausura anti-nef de un divisor dado, mediante el *proceso de descarga* estudiado por Casas-Alvero [**CA00**] y posteriormente desarrollado por [**AAD16**]. El Teorema 0.3 fue probado con estos métodos por Alberich-Carramiñana, Álvarez Montaner y Blanco ([**AAB17**]). Los métodos empleados en esta tesis son diferentes y tienen la ventaja de permitir una generalización al caso de hipersuperficies quasi-ordinarias.

En el caso de ramas planas, una demostración del Teorema 0.3 fue obtenida por Guzmán Durán en su reciente Tesis Doctoral ([**GD18**]), mediante el uso de transformadas de Newton. Otra demostración diferente puede encontrarse en el reciente art´culo de Zhang ([**Zha19**]).

En el caso de ramas planas existe además el precedente de las fórmulas para los números de salto en términos de los generadores minimales del semigrupo asociado debida a Naie y Tucker (véase [**Nai09**], la Tesis Doctoral de Tucker [**Tuc10a**] y también la Memoria de Järvilehto [**Jär11**]). Como consecuencia de nuestros resultados, obtenemos una prueba diferente de aquellas fórmulas para los números de salto en el caso de ramas planas (ver Sección 5.4). Una prueba diferente a ambas ha sido descrita de forma independiente por Guzmán en su Tesis Doctoral [**GD18**].

Järvilehto y Hyry dieron una fórmula explícita de los números de salto de un ideal de colongitud finita sobre un anillo local regular de dimensión dos con cuerpo residual algebraicamente cerrado ([**HJ18**]).

El estudio de las sucesiones generatrices está relacionado con las propiedades de los semigrupos de tuplas de valoraciones de curva ([**Del87**]), las series de Poincaré asociadas a un conjunto finito de valoraciones divisoriales y al estudio de las multiplicidades de los números de salto (véanse [**GM10, CDGH10, AADG17**]).

La noción de contribución en un número de salto por un divisor excepcional ha sido estudiada por Baumers, Veys, Smith y Tucker para ideales multiplicadores en dimensiones más altas ([**BVST18**]).

Una generalización del teorema de Howald para ideales multiplicadores en ciertas variedades tóricas singulares fue estudiados por Blickle ([**Bli04**]).

Conclusiones

Los resultados de la presente Tesis proporcionan una respuesta parcial a la pregunta de si los ideales multiplicadores pueden ser calculados usando una resolución parcial en vez de una log-resolución. El proceso toroidal utilizado para encontrar una resolución sumergida para curvas planas y para hipersuperficies quasi-ordinarias irreducibles ha probado ser útil para demostrar que los ideales multiplicadores asociados dependen únicamente de los divisores excepcionales en la resolución parcial sumergida de la variedad en cuestión. Además, las modificaciones toroidales se presentan como una herramienta conveniente para describir las sucesiones generatrices de un conjunto finito de valoraciones divisoriales. Como elemento adicional, nuestro método permite calcular los números de salto asociados a singularidades de curvas planas y de hipersuperficies quasi-ordinarias irreducibles.

Algunas preguntas abiertas

(1) Hyry y Järvilehto proporcionan fórmulas para los números de salto de ideales planos de colongitud finita [**HJ18**]. Un objetivo sería producir fórmulas semejantes para los números de salto de singularidades de curvas planas en términos de la combinatoria

que determina el tipo topológico embebido, generalizando lo que se conoce en el caso de curvas planas.

(2) Patrick Popescu-Pampu propuso generalizar nuestros métodos toroidales al caso de ideales multiplicadores asociados a una variedad sobre una superficie tórica $\mathbb{Q}$-Gorenstein, haciendo uso de la generalización del resultado de Howald en [**Bli04**].

(3) Esperamos que nuestros resultados puedan ser extendidos a gérmenes (no necesariamente reducidos ni irreducibles) de hipersuperficies quasi-ordinarias en un futuro cercano. Nos gustaría probar que los ideales multiplicadores de cualquier germen de hipersuperficie quasi-ordinaria están determinados por las condiciones en los divisores excepcionales de la resolución parcial sumergida. La resolución toroidal sumergida descrita para hipersuperficies quasi-ordinarias irreducibles generaliza a cualquier germen de hipersuperficie quasi-ordinaria (véase [**Gon03b**]). Para probar este resultado sería fundamental describir una función de log-discrepancias sobre el complejo cónico poliedral de la resolución parcial sumergida. Además, es preciso describir las sucesiones generatrices de las valoraciones divisoriales asociadas en términos de las semiraíces adecuadas, generalizando el enfoque de curvas planas.

(4) Entender la relación entre los divisores que contribuyen críticamente los números de salto ([**Tuc10a**]) y la estructura de la log-resolución parcial o sumergida (véanse los Ejemplos de la Sección 6.9). Esto está también relacionado con la pregunta de si un divisor contribuye número de salto si y sólo si no se contrae en el modelo log-canónico ([**BVST18**]).

(5) ¿Existe una relación más profunda entre los números de salto y la geometría tropical, además de las observaciones de la Sección 5.7?

(6) Este trabajo está redactado para variedades algebraicas o analíticas complejas, pero los resultados obtenidos son ciertos sobre cualquier cuerpo algebraicamente cerrado de característica cero. Una posible aplicación del enfoque toroidal sería estudiar hasta qué punto se pueden aplicar los métodos aquí descritos en característica positiva.

Summary

Invariants of singularities, generating sequences and toroidal structures

Abstract. Over the last few decades, multiplier ideals and their associated jumping numbers have become an important topic in the field of birational geometry of complex algebraic varieties and in singularity theory.

In this PhD Thesis we describe the multiplier ideals and the jumping numbers associated with an irreducible germ of quasi-ordinary hypersurface and also with a plane curve singularity. The approach is motivated by a theorem of Howald describing multiplier ideals a Newton non-degenerate hypersurface singularity in terms of a Newton polyhedron.

We prove a version of Howald's result by using that one has toroidal embedded resolutions for these singularities. The structure of these resolutions is determined by the embedded topological type. The method passes by a precise description of the generating sequences associated with the divisorial valuations associated with the exceptional prime divisors appearing in their toroidal embedded resolutions.

The main result in both cases is that multiplier ideals are generated by generalized monomials in the maximal contact curves, also called semi-roots, and their generalizations in the quasi-ordinary hypersurface case. As an application of this study, we obtain algorithms to compute basis of the multiplier ideals and the set of jumping numbers.

Introduction

In this PhD Thesis we study the singularities of certain classes of algebraic or complex analytic varieties. One way to study a given singular variety is by *resolving its singularities*. It is known that resolution of singularities can be found whenever the variety is defined over a field of characteristic 0 ([**Hir64**]). Nonetheless, the general case is still an open problem. Even when such resolutions exist one may ask how the exceptional divisors of different resolutions relate.

Another important topic in singularity theory is the classification of the singularities of complex algebraic varieties by using *algebraic or analytic* invariants associated with the local ring of the variety at a singular point or the topology of the associated link, or the *embedded topology* the germ of the variety embedded in a smooth affine space, and the understanding of the relations between these invariants.

In this manuscript we study singularities appearing on certain hypersurface germs of analytic or algebraic varieties embedded in a smooth affine space, namely, plane curve germs and irreducible quasi-ordinary hypersurfaces. In order to do so, we will investigate an embedded resolution process by toroidal modifications. This approach provides a concrete description of the combinatorics of the resolution in terms of topological invariants of the singularities we consider. In particular, this leads to a better understanding of the divisorial valuations appearing in these resolutions and their generating sequences, and provides us with the tools to describe the associated multiplier ideals and jumping numbers in terms of the embedded topological type of these germs.

Plane curve singularities

The case of singularities of plane curve germs is broadly studied, and there are many related invariants.

Let (S, O) be a germ of smooth surface and $(C, O) \subset (S, O)$ be a reduced plane germ at the point O. By choosing coordinates at O on the smooth surface, the germ of plane curve can be defined as the zero locus of an equation $f(x, y) = 0$, where $f(x, y)$ can be thought as a holomorphic germ $f \in \mathbb{C}\{x, y\}$ at $(0, 0)$. The point $(0, 0)$ is *singular* if both partial derivatives of f vanish at $(0, 0)$, otherwise it is smooth.

By the Newton-Puiseux theorem an irreducible plane curve germ, also called a *branch*, admits a parametrization of the form $y = \zeta(x^{1/n})$, The *characteristic exponents* of the curve are the orders in x of the differences of Newton-Puiseux parametrizations. In general, $C = \sum C_i$, where C_i are irreducible factors. The *order of coincidence* of two distinct branches is the maximum of the orders in x of the differences of any two Newton-Puiseux parametrizations of them. This combinatorial information can be organized by the *Eggers-Wall tree.*

If C is a branch, one can consider the set of *intersection multiplicities* at the origin with curves not containing it as a component. This subset of the naturals is the *semigroup* of the branch, which admits a finite set of *generators*. The generators of the semigroup can be expressed in terms of the characteristic exponents and conversely. A *semi-root* of a branch C is a branch D which has order of contact a characteristic exponent of C and has no further characteristic exponents. The generators of the semigroup of C are the intersection multiplicities of the branch with the different semi-roots.

Any *embedded resolution* of C, also called log-resolution, is a composition of blowings up of points. We denote by E_i the components of the reduced exceptional divisor $\Psi^{-1}(O) = \sum_{i \in I} E_i$. Let F be the total transform of the curve, it is of the form $F = \sum_{i \in I_\Psi} \mathrm{ord}_{E_i}(C) E_i + \sum_{j=1}^{r} \tilde{C}_j$, where ord_{E_i} denotes the divisorial valuations associated with E_i and $\tilde{C}_j$ denotes the strict transform of the branch C_j of C. The *dual graph* $\Gamma(\Psi, C)$ associated with (C, Ψ) has vertices which label the irreducible components of F, and an edge joins two vertices whenever the corresponding components of F intersect. The dual graph $\Gamma(\Psi, C)$ is a tree. Let us denote by V the set of vertices of the dual graph. The *valence* $v(i)$ of a vertex i is the number of edges incident to it. A branch L not contained among the irreducible components of C is a maximal contact curve of C if Ψ is also an embedded resolution of $C + L$ and the strict transform of L intersects a divisor E_i, with $v(i) = 1$. Choosing one *maximal contact curve* at every exceptional divisor E_i with $v(i) = 1$ defines a complete set of maximal contact curves $L_0, \ldots, L_k$ for (Ψ, C). Maximal contact curves generalize the notion of semi-roots of the components of the curve.

Each exceptional divisor defines a *divisorial valuation*, defined by the order of vanishing along the divisor in any model. A subset of generators of the maximal ideal of the local ring of the surface S at O is said to be a *generating sequence* for the valuation if the monomials in those functions generate the valuation ideals. These generating sequences have been described more generally by Spivakovsky ([**Spi90**]). The notion of generating sequence generalizes to a finite set of divisorial valuations in the minimal resolution and it was characterized geometrically by [**DGN08**], in terms of some *maximal contact curves.*

Quasi-ordinary hypersurface singularities

Quasi-ordinary hypersurface singularities appear classically in Jung's approach to to analyze hypersurface singularities by using embedded resolution of the discriminant of a finite projection to an affine space. In general, quasi-ordinary hypersurface singularities are non-isolated, but they share many properties with plane curve singularities.

We will focus on the study of an irreducible germ of quasi-ordinary hypersurface $(H, O) \to (\mathbb{C}^{d+1}, O)$, which can be defined by an irreducible monic polynomial $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ such that its discriminant with respect to y defines a normal crossing divisor. The Jung-Abhyankar theorem ensures the existence of a fractional power series $\zeta \in \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$ root of the polynomial f, with $n = \deg_y(f)$ (see [**Jun08, Abh55, Gon00**]). The differences of these roots have dominating exponents, called *characteristic*, which encode the embedded topology of the germ thanks to Gau and Lipman. To an irreducible quasi-ordinary hypersurface one can associate a semigroup, generalizing the construction of the semigroup of a plane curve, whose generators are defined in terms of the characteristic exponents (see [**Gon03a**]). This system of generators can be obtained by using the *semi-roots*, which are roughly speaking germs defining simpler quasi-ordinary hypersurfaces parametrized by suitable truncations of ζ. Quasi-ordinary semi-roots generalize the notion of approximate roots (see [**AM73, Pop03**]) and the maximal contact curves.

One has an embedded resolution process which is specific for a quasi-ordinary hypersurface germ (see [**Gon03b**]). First, one obtains an embedded normalization of the germ as a composition of toric modifications determined by the Newton polyhedra of f and its strict transforms (where we use the semi-roots to define suitable auxiliary coordinates). This leads in a canonical manner to a partial embedded resolution in a *toroidal embedding without self-intersection* given by a *conic polyhedral complex with integral structure* Θ. An embedded resolution is then obtained by composing the normalization with any toric resolution of the toroidal embedding defined by a regular subdivision Θ_{reg} of Θ. The complex Θ_{reg} determines the dual graph in the particular case of plane curves, and it can be seen as an enriched replacement of it.

Multiplier ideals and jumping numbers

Multiplier ideals have been applied to obtain results in commutative algebra and they are a central notion in the study of birational geometry. Let us recall their definition and some of their basic properties (see [**Laz04**, Chapters 11 to 14]).

Let X be a smooth complex variety and let $\mathfrak{a}$ be an ideal sheaf on $\mathcal{O}_X$. Recall that a proper birational map $\Psi : Y \to X$ is a *log-resolution* of the ideal $\mathfrak{a}$ if Y is non-singular and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F)$ for some effective divisor F on Y with simple normal crossings.

Let K_Ψ be the relative canonical divisor of Ψ, that is, the divisor defined by the jacobian determinant of Ψ. The *multiplier ideal* of $\mathfrak{a}$ for a fixed parameter $\xi \in \mathbb{Q}_{>0}$ is defined by

$$\mathcal{J}(\mathfrak{a}^\xi) := \Psi_* \mathcal{O}_Y(K_\Psi - \lfloor \xi F \rfloor).$$

It is a well-known theorem that the multiplier ideals do not depend on the choice of log-resolution. We write $F = \sum m_i E_i$ in terms of prime components. We can also write $K_\Psi = \sum(\lambda_i - 1)E_i$, where the coefficient λ_i is called the *log-discrepancy* of the divisor E_i. Then, the multiplier ideal has the following valuative description:

$$\mathcal{J}(\mathfrak{a}^\xi) = \{h \in \mathcal{O}_X \mid \operatorname{ord}_{E_i}(\Psi^* h) \geq \lfloor \xi m_i \rfloor - (\lambda_i - 1)\ \forall i\}.$$

There is a discrete set of rational numbers ξ_i at which these ideals change. This coefficients are called the *jumping numbers* and they define a filtration of ideals

$$\mathcal{O}_X \supsetneq \mathcal{J}(\mathfrak{a}^{\xi_1}) \supsetneq \dots \supsetneq \mathcal{J}(\mathfrak{a}^{\xi_i}) \supsetneq \dots,$$

with $0 < \xi_1 < \ldots < \xi_i < \ldots$ and such that $\mathcal{J}(\mathfrak{a}^{\xi}) = \mathcal{J}(\mathfrak{a}^{\xi_i})$ for $\xi \in [\xi_i, \xi_{i+1})$, see [**ELSV04**]. These jumping coefficients appeared implicitly in the work of Libgober ([**Lib83**]), and Loeser-Vaquié ([**LV90, Vaq92, Vaq94**]) and also take a role in Artal's paper [**Art94**]. It is worth noticing that jumping numbers satify certain periodicity properties.

The previous notions can also be studied locally when $\mathfrak{a}$ is an ideal of the ring of germs of holomorphic functions at a point of the smooth variety X.

If D is an effective divisor a log-resolution of $\mathcal{O}_X(-D)$ is basically an embedded resolution of D. Let D be an effective integral divisor on X. In this case we denote by $\mathcal{J}(\xi D) \coloneqq \mathcal{J}(\mathcal{O}_X(-D)^{\xi})$ the associated multiplier ideals. We have that ξ is a jumping number of D if and only if $\xi + 1$ is likewise a jumping number (see Lemma 3.26). As a consequence, the jumping numbers of such a divisor D are fully determined by those in the interval $(0, 1]$.

Jumping numbers encode interesting geometric, algebraic and topological information and arise naturally in many different contexts (see [**Bud12, ELSV04**]). The smallest jumping number, called the *log canonical threshold*, appears in many different situations. If $0 < \xi \leq 1$ is a jumping number associated to the ideal (f), where $f \in \mathbb{C}[x_1, \ldots, x_n]$ then $-\xi$ is a root of the *Bernstein-Sato* polynomial of f. One can express, in terms of the jumping numbers, a uniform Artin-Rees number of the ideal (f), and related bounds for the Milnor and Tjurina numbers when f has isolated singularities (see [**ELSV04**]). The Hodge spectrum, defined in terms of the monodromy and the Hodge filtration on the cohomology of the Milnor fiber of f at a singular point x_0, is characterized in terms of the *multiplicities* of the jumping numbers of f at x_0 (see [**Bud11**]).

Multiplier ideals and jumping numbers are not easy to compute. An exception to this principle is the case of monomial ideals and of *Newton non-degenerate* hypersurface singularities. In these two cases, Howald proved that the multiplier ideals are monomial ideals and characterized them in terms of Newton polyhedra ([**How01**]), by using a *toric* log-resolution.

Motivation and goals

The main motivation of this work is to extend Howald's results on multiplier ideals to other singular varieties using toroidal embedded resolutions. This question has various interpretations which can be generalized. Firstly, Howalds theorem gives a convex description of multiplier ideals in terms of the *Newton polyhedron* of the defining ideal. In the cases studied by Howald, the toric log-resolution factors through a *partial embedded resolution*, given by the toric modification associated to the fan dual to the Newton polyhedron. One deduces that the jumping numbers in these cases only depend on *toric exceptional divisors* which appear in the partial embedded resolution. This shows that our problem is also connected with the question of which exceptional divisors are actually related with the jumping numbers. The associated *toric divisorial valuations* are determined by their values on the monomials in the given affine coordinates, that is, these coordinates define trivially generating sequences for these valuations. Howald's result gives a *monomial basis* for the multiplier ideals of a monomial ideal in terms of the coordinates of the original affine space.

In more concrete terms we can formulate the following question: Assume $\mathfrak{a}$ is an ideal sheaf on X and there exists a partial toroidal embedded resolution factoring the log-resolution of $\mathfrak{a}$, how do the multiplier ideals of $\mathfrak{a}$ relate to the divisors in the partial embedded resolution? Can we describe basis for the multiplier ideals in terms of the generating sequences of the divisorial valuations corresponding to these exceptional divisors?

The goal is to answer to the previous questions in the case of plane curve singularities and irreducible quasi-ordinary hypersurface singularities, using the available toroidal resolution methods.

The toroidal approach to embedded resolution of plane curves is studied in [**Oka96**, **DO95**, **GGP16**, **GGP19**], while the toroidal resolution of quasi-ordinary hypersurfaces was described in [**Gon03b**]. The goal is to understand the toroidal structure of the embedded resolution in order to study the exceptional divisors appearing on it and comparing them with those on the partial embedded resolution.

Generating sequences for divisorial valuations of plane curves were described in [**DGN08**] in terms of the maximal contact curves. We will investigate their structure in terms of the toroidal resolution. The first instance of toric approach to generating sequences seems to be in [**GT00**]. Further, we expect to describe generating sequences for the quasi-ordinary case in terms of the semi-roots of the hypersurface germ. Another fundamental ingredient in the construction of multiplier ideals are the *log-discrepancies* of the divisorial valuations. In the case of plane curves the log-discrepancies are described in [**FJ04**] and [**GGP18**], whereas for irreducible quasi-ordinary germs they were described in [**GG14**]. This suggest the need of a systematic study of the log-discrepancies of exceptional divisors in the toroidal embedded resolutions in order to provide a *toric description* for them.

Contents and results

We now turn to a detailed overview of the content of this work. In the first three chapters we fix notation and recall a series of known results concerning toric geometry, singularities of plane curves and some basic material on multiplier ideals.

Chapter 1 is a review of toric geometry. First, we deal with the basic definition and general properties of toric varieties. We introduce the normal toric variety Z_Σ associated to a fan Σ, and we recall the *orbit-cone* correspondence, the torus invariant divisors and their associated *monomial valuations*. Then, we describe the toric modifications associated with subdivisions of fans. Furthermore, we describe Newton polyhedra and their support functions, and their relation with monomial valuations. We include some basic lemmas on support functions of Newton polyhedra which will be useful in the description of multiplier ideals.

In Chapter 2 we discuss some basic theory of plane curve singularities. We introduce the classical invariants of the topology of a plane branch starting from the Newton-Puiseux parametrization: the generators of the semigroup, the characteristic exponents and the Newton pairs.

The embedded topological type of a plane curve singularity $(C, 0) \subset (S, O)$ is encoded by the orders of coincidence of the pairs of Newton-Puiseux parametrizations of the branches of C relative to a smooth germ R transversal to C (see [**Wal04**]). This combinatorial information is organized in the *Eggers-Wall tree*, $\Theta_R(C)$, of C with respect to R. It is a rooted tree whose ends are labeled by R and the branches of C, which is endowed with an exponent function $\mathbf{e}_R$ and an index function $\mathbf{i}_R$. There is a finite set of marked points whose exponents run through the orders of coincidence considered above. The index function provides the degrees of Galois extensions associated with certain truncations of the Newton-Puiseux parametrizations of C. These two functions determine the contact complexity function $\mathbf{c}$, which provides simple formulas for the intersection multiplicities of pairs of branches of C.

Semivaluations are functions on a ring that provide a measure of size of multiplicity of elements in the ring and provide an important tool to understand the properties of singularities of plane curves. Among them we have *divisorial valuations*, ord_E associated to a prime exceptional divisor E over a germ of smooth surface (S, O). On the other hand, we have *order of vanishing valuations along a branch*, associated to irreducible plane curve germs. These two types of semivaluations are involved in the description of multiplier ideals and we will be interested in generating sequences for a finite set of such valuations. We finish the chapter with some results in expansion techniques, which will play an important role in the description of generating sequences for valuations.

Chapter 3 introduces the *multiplier ideals* and *jumping numbers* associated with a complex variety embedded in a smooth complex variety. We start by giving the definitions concerning the *relative canonical divisor* associated to a birational transformation, which is one of the main ingredients in the description of multiplier ideals. We have to choose the representative of the relative canonical divisor defined by the vanishing of the jacobian determinant of a given log-resolution. The local shape of the relative canonical divisor is encoded by a vector of *log-discrepancies* in the coordinate hyperplanes for a suitable choice of coordinates. We include a slight generalization of Howald's result for monomial ideals (see Theorem 3.31). Its proof gives inspiration for our theorems concerning the reductions of conditions in the multiplier ideals of both plane curve germs and irreducible quasi-ordinary singularities.

In Chapter 4, we formalize the construction of a *toroidal embedded resolution* for a plane curve. Previous works on these resolutions appear in [**Oka96, DO95**], We follow mainly the approach in [**GGP16**]. This part is also connected with the viewpoint developed in the recent survey [**GGP19**]. Further, we study generating sequences of some valuations using the toroidal embedded resolution.

We start this chapter by recalling some specific properties of toric varieties in dimension two, including minimal resolutions and dual graphs. A toroidal embedded resolution of a plane curve is a composition of toroidal modifications associated to the Newton polygon of the curve in certain systems of coordinates. More precisely, each toroidal modification is associated to the minimal regular subdivision of the fan dual to the Newton polygon of the curve and its strict transforms at singular points. Each modification reduces the complexity of the curve, thus the composition of a finite number of them leads to the toroidal embedded resolution Ψ. We encode this toroidal process by a suitable *decomposition* of the Eggers-Wall tree of the curve. A *tree completion* $\Theta_R(\bar{C})$ of the tree $\Theta_R(C)$ consists of completing the ends of the levels of the index function on the tree. This amounts to considering auxiliary branches L_j, $j \in J$, which have *maximal contact* with the branches of C. The strict transform of each auxiliary branch L_j is smooth and intersects the exceptional fiber $\Psi^{-1}(O)$ at a smooth point of a prime component E. The divisor E corresponds to a vertex of valence 1 of the dual graph. This allows to give a homeomorphism between the tree completion $\Theta_R(\bar{C})$ and the dual graph of the complete curve $\Gamma(\Psi, \bar{C})$, in such a way that the vertices associated to those divisors E_P, for P a marked point of the tree $\Theta_R(C)$, are precisely the rupture vertices (of valency ≥ 3) in the dual graph of the minimal embedded resolution of $\bar{C}$.

The toroidal structure of the resolution allows to associate to any rational point P in the Eggers-Wall tree $\Theta_R(C)$ an exceptional divisor E_P and its corresponding divisorial valuation $\nu_P = \mathrm{ord}_{E_P}$. The order of vanishing along such a divisor can be given by combinatorial formulas in the functions $\mathbf{e}_R$, $\mathbf{i}_R$ and $\mathbf{c}_R$ of the tree (see Theorem 4.70). This construction can be generalized to any point of the tree, contextualizing this construction in the theory of semivaluations on a smooth surface (see [**GGP18, FJ04**]). In particular, any leaf of the tree is associated with a component C_j of C, and the corresponding valuation is the order of vanishing along C_j.

The concept of *generating sequence* for a valuation ν was studied by Spivakovsky in [**Spi90**] and further generalized by Delgado, Galindo and Nuñez for several valuations [**DGN08**]. They prove that if we choose a finite set of divisorial valuations and consider its minimal embedded resolution Ψ (the minimal composition ob blow-ups generating all the necessary exceptional divisors), then a set of maximal contact curves for Ψ form a generating sequence for $\underline{\nu}$. We show that a set of germs $\{z_j\}$ is a generating sequence for $\underline{\nu}$ if and only if any function h can be expanded in terms of them in such a way that the valuation of h is the minimum achieved over its expansion (see Lemma 4.87). In this case, we say that $\underline{\nu}$ is *monomial* with respect to the $\{z_j\}$. This result will allow us to give monomial basis for the multiplier ideals by choosing suitable expansions of any germ.

Further, we give a slight generalization of [**DGN08**, Theorem 5]. We prove that any valuation ν_P associated to a point of the tree, $P \in \Theta_R(\bar{C})$, is monomial with respect to the set of germs in the maximal contact completion $\bar{C}$ of the curve C (Theorem 4.125). Even more, any finite set of such valuations is simultaneously monomial with respect to the branches of the tree completion, that is, in a set of curves of maximal contact (see Theorem 4.170):

THEOREM 0.7. *Let $\underline{\nu} = (\nu_1, \dots, \nu_s)$ be a tuple of valuations associated to points in a tree $\Theta_R(C)$. Then $\underline{\nu}$ is monomial with respect to the functions defining the maximal contact curves of C.*

The methods used to proof this result are quite different from those in [**DGN08**].

In the case of a branch, our proof relies on the control of the Newton polygons of the semi-roots through the toroidal embedded resolution process and the properties of the semigroup of the branch. The general case generalizes the case of one branch by using the combinatorics of the tree. Our method includes the construction of a *suitable expansion* for a given germ in terms of the system of maximal contact curves of C. Suitable here means that the valuation of a germ is the minimum of the valuations of the terms in its expansion.

Chapter 5 is devoted to the proofs of the main results about multiplier ideals and jumping numbers of plane curve germs. We prove first that the conditions in the irreducible components E_i of the exceptional divisor which define vertices of valency $v(i) \leq 2$ of the dual graph are redundant in the definition of multiplier ideals (see Theorem 5.8):

THEOREM 0.8. *If $0 < \xi < 1$, then*

$$\mathcal{J}(\xi C) = \{h \in \mathcal{O}_{S,O} \mid \mathrm{ord}_{E_i}(h) \geq \lfloor \xi \mathrm{ord}_{E_i}(C) \rfloor - (\lambda_i - 1), \ \textit{with } v(i) \geq 3\}.$$

In order to prove this theorem one has to reformulate the valuative conditions in the multiplier ideal in terms of the support functions of the total transform of C at the various steps of the toroidal resolution associated with a maximal contact decomposition of the Eggers-Wall tree, and apply an extension of Howald's argument.

This theorem is independent of the results of Tucker, Smith and Thompson on the notion of *contribution of exceptional divisors to a given jumping number*, and sheds a new light on the subjet (see [**Tuc10a, ST07**]).

Then, we give a description of the generators of the multiplier ideals $\mathcal{J}(\xi C)$ (see Theorem 5.16).

THEOREM 0.9. *Let $z_0, \dots, z_k \in \mathcal{O}_{S,O}$ be irreducible functions defining the maximal contact curves associated with C. If $0 < \xi < 1$, the multiplier ideal $\mathcal{J}(\xi C)$ is generated by monomials in $z_0, \dots, z_k$.*

The proof of this theorem is based on our monomialization results (Corollary 4.160), in terms of generating sequences of valuations (see [**DGN08**]). As a consequence of the previous result we can associate a jumping number to any *monomial function* of the form $z_0^{i_0} z_1^{i_1} \dots z_k^{i_k}$ (see Corollary 5.22):

COROLLARY 0.10. *Each monomial of the form $\mathcal{M} = z_0^{i_0} z_1^{i_1} \dots z_k^{i_k}$ determines a jumping number*

$$\xi_{\mathcal{M}} = \min_{v(i) \geq 3} \left\{ \frac{\nu_{E_i}(\mathcal{M}) + \lambda_{E_i}}{\nu_{E_i}(C)} \right\}.$$

and every jumping number is of this form.

Notice that $\xi_{\mathcal{M}}$ is the minimal rational number such that $\mathcal{M} \notin \mathcal{J}(\xi_{\mathcal{M}} C)$.

In particular, the *log-canonical threshold*, is the jumping number associated to $\mathcal{M} = 1$. As a consequence we get that the log-canonical threshold is attained at a divisor created in the first toroidal modification of the toroidal embedded resolution. This is a reformulation of a result in [**ACLM08**].

We sketch how our results can be extended to describe the multiplier ideals associated with an ideal of $\mathbb{C}[[x,y]]$. This passes through a description of a toroidal log-resolution of the ideal (compare with the method of Newton maps in [**CV14, CV15**]).

We give a new proof of the formulas described by Naie in [**Nai09**] for the jumping numbers of a plane branch. Our proof provides a bijection between the set of jumping numbers in Naie's formulas and the set of generalized monomials in the semi-roots of the branch which give jumping numbers smaller than one. In addition, we prove using elementary methods the formula for the number of jumping numbers smaller than one, counted with *multiplicity*, of a plane branch (see Section 5.5).

The main results of this PhD Thesis are given in Chapter 6, where we generalize the previous results about plane curve singularities to the case of an irreducible germ of *quasi-ordinary hypersurface* $(H, O) \subset (\mathbb{C}^{d+1}, O)$.

In order to study the associated multiplier ideals we use the toroidal resolution process introduced by González Pérez in [**Gon03b**]. The output of this process provides a *partial embedded resolution* inside a normal space with only toric singularities. This normal space is endowed of the structure of *toroidal embedding without self-intersection*, and its associated conic polyhedral complex with integral structure Θ is determined by the characteristic exponents. The embedded resolution of the germ is obtained by composing the partial embedded resolution with a toroidal resolution of the normal ambient space, which is obtained by choosing a regular subdivision Θ_{reg} of Θ.

The map $u \to E_u$ defines a bijection between the rays of the complex Θ_{reg} and the components of the total transform of *the completion* of H, which is the reduced hypersurface whose components are the coordinate hyperplanes and the *semi-roots* of H. Among those components of the total transform, we distinguish the ones associated to rays in Θ:

- The *end rays* of Θ (those which appear in a unique maximal dimensional cone of the complex) correspond either to coordinate hyperplanes $x_k = 0$ for $k = 1, \dots, d$ or to semi-roots of H.

- The other rays of Θ correspond to prime exceptional divisors in the partial embedded resolution and we call them *relevant rays* and *relevant exceptional divisors* respectively (by analogy with [**ST07**]).

In addition, we show that the conic polyhedral complex Θ is endowed with a log-discrepancy function, which provides a combinatorial formula for the log-discrepancies of any toroidal exceptional divisor E_u, for u defining a ray of Θ_{reg}, in terms of the characteristic exponents (see Lemma 6.116).

After this preparation we prove that the valuative conditions of the multiplier ideals which correspond to rays of Θ_{reg}, which are not relevant rays of Θ, are also irrelevant in the definition of the multiplier ideals (see Theorem 6.122):

THEOREM 0.11. *If* $0 < \xi < 1$ *one has*

$$\mathcal{J}(\xi H) = \left\{ h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \operatorname{ord}_{E_u}(h) \geq \lfloor \xi \operatorname{ord}_{E_u}(H) \rfloor - (\lambda_u - 1) \right\},$$

where E_u *runs over the relevant rays of* Θ*, i.e., the exceptional divisors in the partial embedded resolution of* H*.*

We generalize some of the results on monomialization of divisorial valuations to the quasi-ordinary case. Using expansion techniques in terms of semi-roots, the toroidal process of partial embedded resolution and the properties of the semigroup of the quasi-ordinary hypersurface, we show that the divisorial valuations of the exceptional divisors E_u defined by relevant rays of Θ are monomial in the semi-roots of H (Lemma 6.83). This lemma combined with the the previous theorem allows us to prove the following result on the generators of the multiplier ideals of H (see Theorem 6.129):

THEOREM 0.12. *For any* $0 < \xi < 1$ *the multiplier ideal* $\mathcal{J}(\xi H)$ *is generated by monomials in the coordinates* $x_1, \ldots, x_d$ *and the the semi-roots* $z_0, \ldots, z_g$.

As a consequence of the previous theorems, we obtain an algorithmic method to obtain the jumping numbers of H. We finish the chapter giving two examples of the computations of jumping numbers of irreducible quasi-ordinary hypersurface singularities.

Comparison with other approaches

In recent years, many efforts have been made in order to better understand multiplier ideals and jumping numbers, especially over surfaces. In many of the references the authors work more generally with surfaces with *rational singularities*, where the definition of multiplier ideals can be extended.

Tucker ([**Tuc10a**]) develops the notion of jumping number contributed by a prime divisor introduced by Smith and Thompson ([**ST07**]), as a way of expressing which divisor provokes the jump. In the case of a plane branch, he shows that the only *contributing divisors* are the rupture divisors of its minimal resolution and he gives an algorithm to compute them. Tucker suggests that the notion of contribution can be better understood by considering *reduced divisors*, and he introduced the notion of *critical contribution*. In the case of ideals over a surface with rational singularities Tucker gives a precise description of the geometry of critically contributing divisors ([**Tuc10b**]), which in the case of plane curves turns out to be related to rupture divisors of the dual graph.

Alberich-Carramiñana, Álvarez Montaner and Dachs-Cadefau ([**AAD16**]) gave an algorithm to compute the jumping numbers and a system of generators of multiplier ideals for any ideal over a surface with rational singularities. Alberich-Carramiñana, Álvarez Montaner and Blanco ([**AAB17**]) provide an algorithm to compute a set of monomial generators for the multiplier ideals of a plane ideal, and more generally of integrally closed ideals. Those generators admit a presentation as generalized monomials in a set of maximal contact elements associated to the minimal resolution of the plane ideal. In this case we have the classical correspondence between integrally closed ideals and anti-nef divisors, and the authors used that multiplier ideals are integrally closed. One has a method to compute the antinef closure of a given divisor, called *unloading procedure* studied by Casas-Alvero [**CA00**] and further developed in [**AAD16**]. Theorem 0.9 was proved previously using these methods by Alberich-Carramiñana, Álvarez Montaner and Blanco ([**AAB17**]). Our methods are quite different from theirs and allow a generalization to the case of quasi-ordinary hypersurfaces.

In the plane branch case, a proof of Theorem 0.9 was also obtained by Guzmán Durán in his recent PhD Thesis ([**GD18**]), by using the method of Newton maps. A different proof can be found in Zhang's recent paper ([**Zha19**]).

In the case of a plane branch there are formulas for the jumping numbers in terms of the minimal generators of its semigroup due to Naie and Tucker (see [**Nai09**], Tucker's Thesis [**Tuc10a**] and also Järvilehto's Memoir [**Jär11**]). As a consequence of our results, we obtain another proof of these formulas for the set of jumping numbers in the case of a plane branch (see Section 5.4). Another proof of these formulas has been given independently by Guzmán in his PhD Thesis [GD18].

Järvilehto and Hyry gave an explicit formula for the jumping numbers of an ideal of finite colength in a two-dimensional regular local ring with an algebraically closed residue field ([HJ18]).

The study of generating sequences is linked with the properties of the semigroup of tuples of curve valuations ([Del87]), the Poincaré series associated to a finite set of divisorial valuations and the study of multiplicities of jumping numbers (see [GM10, CDGH10, AADG17]).

The notion of contribution to a jumping numbers by an exceptional divisors has been studied by Baumers, Veys, Smith and Tucker for multiplier ideals in higher dimensions ([BVST18]).

A generalization of Howald's theorem for multiplier ideals on certain singular toric varieties was given by Blickle ([Bli04]).

Conclusions

The results of this Thesis give a partial answer to the question of whether multiplier ideals may be computed in a less restrictive environment than the log-resolution. The toroidal process used to find embedded resolutions for plane curve germs and irreducible quasi-ordinary hypersurfaces has been useful in order to prove that their multiplier ideals depend only on exceptional divisors appearing in a partial embedded resolution of the variety. Furthermore, toroidal modifications have proved to be a convenient tool to describe generating sequences of a finite set of divisorial valuations. An additional outcome is that our methods allow to algorithmically compute the jumping numbers associated to a plane curve or ideal and to an irreducible quasi-ordinary hypersurface.

Some open questions

(1) Hyry and Järvilehto provide closed formulas for jumping numbers of plane ideals of finite colength [HJ18]. One goal is to give formulas for the jumping numbers of a plane curve singularity in terms of the combinatorics determining the embedded topological type, generalizing those known in the plane branch case.

(2) Patrick Popescu-Pampu proposed to generalize our toroidal methods to the case of multiplier ideals on a $\mathbb{Q}$-Gorenstein toric surface, using the generalization of Howald's result given in [Bli04].

(3) We expect that our results can be extended to germs of (non-necessarily irreducible nor reduced) quasi-ordinary hypersurfaces in the near future. We would like to prove that the multiplier ideals of any quasi-ordinary hypersurface germ are determined by the conditions on exceptional divisors of the partial embedded resolution. The embedded toroidal resolution described for irreducible quasi-ordinary hypersurfaces generalizes to the reduced case (see [Gon03b]). In order to answer this question, it would be important to describe a local function of log-discrepancies in the conic polyhedral complex of the partial embedded resolution, which provides log-discrepancy of toroidal exceptional divisors. Then, we need to describe more precisely the generating sequences of their associated divisorial valuations in terms of a suitable notion of semi-root which generalizes the one in the plane curve case.

(4) Provide a better understanding of the relation between the critical contributions of divisors to jumping numbers ([Tuc10a]) and the structure of a partial log-resolution or embedded resolution (see Examples in Section 6.9). This is also linked with the

question of whether an exceptional divisor contributes to jumping number if and only if it is not contracted in the log-canonical model (see [**BVST18**]).

(5) Is there a deeper relation between jumping numbers and tropical geometry besides the remarks given in Section 5.7?

(6) This work is written for complex algebraic or analytical varieties, but the results obtained are true on any algebraically closed field of characteristic zero. Another posible outcome of the toroidal approach is to study to which extent one can apply the present methods in positive characteristic.

CHAPTER 1

Some background on toric geometry methods

In this chapter we introduce notation and basic definitions of toric geometry, and refer to [**CLS11**, **Ful93**, **Oda88**, **Ewa96**] for proofs and more complete descriptions of toric geometry.

Section 1.1 deals with the basic definition and general properties of toric varieties. We introduce the normal toric variety Z_Σ associated to a fan Σ. Each cone of the fan $\sigma \in \Sigma$ generates a torus orbit $\mathbf{O}_\sigma$, which is the orbit of the torus on the affine chart Z_σ defined by the cone. This orbit is a Zariski closed embedded torus of dimension the codimension of the cone for $\dim \sigma < \mathrm{rk} N$. Thus, each ray τ of the cone generates a torus embedded divisor $\mathbf{O}_\tau$ on the chart. Its closure in the toric variety $\overline{\mathbf{O}}_\tau$ is a torus invariant divisor and the group of torus invariant divisors of the toric variety Z_Σ is generated by those. Each such divisor generates a divisorial valuation ν_τ, which can be computed monomially.

We also introduce the toric modification $\psi_{\Sigma'} : Z_{\Sigma'} \to Z_\Sigma$ induced by a subdivision Σ' of the fan Σ. The toric structure allows to describe the exceptional and discriminant loci combinatorially.

In Section 1.2, we deal with Newton polyhedra defined on a toric variety. We describe them in terms of supporting half-spaces associated to rays (valuations) in their dual fan. This allows to give a valuative description of any Newton polyhedron with respect to the primitive integral generators of its dual fan.

1.1. Introduction to Toric Varieties

1.1.1. Cones, fans and lattices. Let N be a d-dimensional lattice, i.e., a free $\mathbb{Z}$-module of rank d. We denote by

$$N_\mathbb{R} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$$

the real vector space associated with the lattice. An element u of N which is not a non-trivial multiple of another lattice vector will be called **primitive**. Let S be a finite subset of $N_\mathbb{R}$. We will say that $\sigma \in N_\mathbb{R}$ is a **convex polyhedral cone** if there is a finite set $S_\sigma \subset N_\mathbb{R}$ such that

$$\sigma = \mathbb{R}_{\geq 0} \langle S_\sigma \rangle = \{ \sum_{v \in S_\sigma} a_v v \mid a_v \geq 0 \}.$$

We denote by $M := \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice of N and by $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ the real vector space spanned by the lattice M. The pairing

$$\langle v, u \rangle := v(u), \text{ for } v \in N_\mathbb{R}, \text{ and } M_\mathbb{R}. \tag{1.1}$$

is bilinear in both entries.

The dual cone $\sigma^\vee$ (respectively, its orthogonal cone $\sigma^\perp$) of σ is the set

$$\sigma^\vee = \{ u \in M_\mathbb{R} \mid \langle v, u \rangle \geq 0 \text{ (resp } \langle v, u \rangle = 0) \ \forall u \in \sigma \}.$$

We say that S_σ spans σ and we often denote σ by $\mathbb{R}_{\geq 0} S_\sigma$. The **dimension** of the cone σ is the dimension of the vector space $\mathbb{R}\sigma$ spanned by σ. This cone is called **rational** for the lattice N if $S \subset N$. It is called **strictly convex** if σ contains no lines. A cone σ in $N_\mathbb{R}$ is called **regular** for the lattice N if it is spanned by a subset of a basis of N. Recall that vectors $u_1, u_2, \dots, u_d$ form a **basis** of N if the map

$$\mathbb{Z}^d \to N, \quad (a_1, \dots, a_d) \mapsto a_1 u_1 + \dots + a_d u_d,$$

is an isomorphism. Then we will speak about the coordinates $(a_1, \dots, a_d)$ with respect to this basis of the vector $u = \sum a_i u_i$. Notice that u is primitive if and only if its coordinates with respect to a basis are coprime. A basis of N is also a basis of $N_\mathbb{R}$. We often denote it simply by $\mathbb{R}^d_{\geq 0}$ the cone spanned by the elements of the basis $u_1, \dots, u_d$, if it is clear from the context.

DEFINITION 1.2. A **fan** Σ in $N_\mathbb{R}$ is a collection of rational strictly convex cones which is closed under taking faces and under intersection. Its **support** is the union of its cones

$$\mathrm{Supp}(\Sigma) = \underset{\sigma \in \Sigma}{\cup} \sigma.$$

The i skeleton of the fan, $\Sigma^{(i)}$, is the subset of i dimensional cones of Σ. We denote by Σ_{prim} the set of primitive integral vectors $v_j \in N$ generating the rays of the fan $\tau_j \in \Sigma^{(1)}$.

A fan is **regular** if any cone in it is regular.

We often abuse slightly of notation by denoting with the same letter the cone σ and the fan consisting of the set of faces of σ.

1.1.2. Affine toric varieties. Let $\sigma \subset N_\mathbb{R}$ is a strictly convex rational polyhedral cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup (by Gordan's Lemma, see [CLS11, Proposition 1.2.17]). We now associate to S_σ its **semigroup algebra**

$$\mathbb{C}[S_\sigma] = \left\{ \sum_{finite} a_v \chi^v \mid v \in S_\sigma, \, a_v \in \mathbb{C} \right\}.$$

In particular, we have the inclusion $S_\sigma \subset S_{\{0\}} = \{0\}^\vee \cap M = M$. Hence, $\mathbb{C}[S_\sigma] \subset \mathbb{C}[M]$. If $v_1, \dots, v_d$ is a basis of the lattice M then we get that the map $\mathbb{C}[x_1^{\pm 1}, \dots, x_d^{\pm 1}] \to \mathbb{C}[M]$, which sends $t_i \to \chi^{v_i}$ is an isomorphism. It follows that $\mathbb{C}[S_\sigma]$ is an integral domain.

DEFINITION 1.3. Let $\sigma \subset N_\mathbb{R}$ be a strictly convex rational polyhedral cone. Then the affine variety

$$Z_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]),$$

is the **affine toric variety** associated to σ. A (non-necessarily normal) affine toric variety over the field $\mathbb{C}$ is of the form

$$Z^\Lambda = \mathrm{Spec}\mathbb{C}[\Lambda],$$

where Λ is a subsemigroup of finite type of the lattice $M = \mathbb{Z}\Lambda$ which generates it as a group.

The affine variety $Z_\sigma = Z^{\sigma^\vee \cap M}$ is normal and, in fact, every normal affine toric variety arises this way (see [CLS11, Theorem 1.3.5]).

The closed points of Z^Λ correspond to semigroup homomorphisms $\Lambda \to \mathbb{C}$, where $\mathbb{C}$ is considered as a semigroup with respect to multiplication. The action of the torus $T^M := T_N$ on Z^Λ is defined by multiplication of the corresponding homomorphisms of semigroups.

The normalization of Z^Λ is obtained from the inclusion $\Lambda \longrightarrow \mathbb{R}_{\geq 0}\Lambda \cap (\mathbb{Z}\Lambda)$. The action of the torus has a fixed point if and only if the cone $\mathbb{R}_{\geq 0}\Lambda$ is strictly convex. In this case, the 0-dimensional orbit of the torus action, is reduced to this point, which is defined by the ideal $(x^u \mid u \in \Lambda \setminus 0) \subset \mathbb{C}[\Lambda]$.

1.1.3. Normal toric varieties. Let us start by recalling the abstract definition of a normal toric variety.

DEFINITION 1.4. A normal **toric variety** is a normal algebraic variety X such that it contains the d-dimensional torus $(\mathbb{C}^*)^d$ as a dense open Zariski subset, and such that the action of $(\mathbb{C}^*)^d$ on itself extends to an action on X.

We can define also a toric variety associated to a fan. Given a fan Σ and cones $\tau \prec \sigma \in \Sigma$, we have $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$. This inclusion induces an open immersion $Z_\tau \subset Z_\sigma$. For a pair of cones $\sigma, \sigma' \in \Sigma$, we have open immersions $Z_{\sigma \cap \sigma'} \subset Z_\sigma, Z_{\sigma'}$ and, if we denote by $Z_{\sigma\sigma'}, Z_{\sigma'\sigma}$ the respective images, an isomorphism $Z_{\sigma\sigma'} \simeq Z_{\sigma'\sigma}$. The **toric variety** associated to the fan Σ, Z_Σ, is described locally as above with these gluing conditions, and it is separated and normal (see [**CLS11**, Theorem 3.1.5]). The toric variety Z_Σ is smooth if and only if every cone $\sigma \in \Sigma$ is regular ([**CLS11**, Theorem 3.1.19]). One can prove that every normal toric variety according to the Definition 1.4 is associated to certain fan.

1.1.4. The completion of the local rings of an affine toric variety. Let σ be a rational strictly convex cone of dimension $d = \mathrm{rk} N$. The set of formal power series with exponents in S_σ is a ring, which we denote by $\mathbb{C}[[S_\sigma]]$. The ring of convergent power series with exponents in S_σ, which we denote by $\mathbb{C}\{S_\sigma\}$, is the subring of $\mathbb{C}[[S_\sigma]]$ of absolutely convergent power series in a neighborhood of the distinguished point o_σ of the affine toric variety Z_σ.

LEMMA 1.5. [**Gon00**, Lemme 1]. *The local algebra of germs of holomorphic functions at* (Z_σ, o_σ) *is isomorphic to* $\mathbb{C}\{S_\sigma\}$.

Since Z_σ is a normal variety, we deduce that $\mathbb{C}\{S_\sigma\}$ is local and integrally closed.

EXAMPLE 1.6. Let $N \simeq \mathbb{Z}^d$ with basis $u_1, \dots, u_d$ and let $\rho = \mathrm{Cone}(u_1, \dots, u_d) = \mathbb{R}^d_{\geq 0}$. The affine toric variety Z_ρ is the affine space $\mathbb{C}^d$ and the cone ρ corresponds to the distinguished point $o_\rho = O$, the origin. The algebra of holomorphic functions at $(\mathbb{C}^d, O)$ is $\mathbb{C}\{x_1, \dots x_d\}$, where $x_i := \chi^{\check{u}_i}$, and $\check{u}_1, \dots, \check{u}_d$ denotes the dual basis of $u_1, \dots, u_d$.

1.1.5. Torus orbits. Let Σ be a fan defining a toric variety Z_Σ.

Recall that closed points of Z_σ, for $\sigma \in \Sigma$, are in bijective correspondence with semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}^*$ ([**CLS11**, Prop 1.3.1]). For each cone σ we have its distinguished point is defined by the semigroup homomorphism:

$$m \in S_\sigma \longmapsto \begin{cases} 1 & \text{if } m \in S_\sigma \cap \sigma^\perp = M \cap \sigma^\perp, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

The action of the torus T^M on each open subset Z_σ of Z_Σ, glue up to define an action on Z_Σ. The fan Σ is in bijection with the set of orbits of the torus action on Z_Σ. This bijection, called **orbit-cone correspondence**, sends a cone σ in the fan Σ to the orbit

$$\mathbf{O}_\sigma := T_N \cdot o_\sigma \subset Z_\Sigma.$$

See [**CLS11**, Theorem 3.2.6].

If $\dim(\sigma) = \mathrm{rk}(N)$ the orbit, $\mathbf{O}_\sigma$, is reduced to the special point. If $\dim(\sigma) < \mathrm{rk}(N)$ we have an exact sequence of lattices

$$0 \longrightarrow M(\sigma) \longrightarrow M \xrightarrow{j} M_\sigma \longrightarrow 0,$$

where $M(\sigma) = M \cap \sigma^{\perp}$, with dual exact sequence

$$0 \longrightarrow N_\sigma \xrightarrow{j_*} N \longrightarrow N(\sigma) \longrightarrow 0,$$

where N_σ is the lattice spanned by $\sigma \cap N$ of rank equal to the dimension of the cone, $\mathrm{rk}(N_\sigma) = \dim(\sigma)$. The semigroup S_{σ,N_σ} of σ seen as a rational cone with respect to the lattice N_σ, is defined as the intersection of the dual cone of σ, as a subset of $(N_\sigma)_{\mathbb{R}}$, with the dual lattice $N_\sigma^* = M_\sigma$. We have that:

$$S_{\sigma,N_\sigma} \simeq j(\sigma^\vee \cap M).$$

Lemma 1.8. [**CLS11**, Proposition 3.3.11]. *If we choose a splitting, $M \simeq M_\sigma \oplus M(\sigma)$, we obtain a semigroup isomorphism,*

$$\sigma^\vee \cap M \simeq M(\sigma) \oplus (\sigma^\vee_{N_\sigma} \cap M_\sigma),$$

inducing an isomorphism of $\mathbb{C}$-algebras,

$$\mathbb{C}[\sigma^\vee \cap M] \simeq \mathbb{C}[M(\sigma)] \underset{\mathbb{C}}{\otimes} \mathbb{C}[\sigma^\vee_{N_\sigma} \cap M_\sigma],$$

which defines an isomorphism of toric varieties

$$Z_{\sigma,N} \simeq \mathbf{O}_{\sigma,N} \times Z_{\sigma,N_\sigma}. \tag{1.9}$$

This isomorphism depends on the choice of an isomorphism $M \simeq M_\sigma \oplus M(\sigma)$.

If $\tau \prec \sigma$ is a face, then $Z_\tau \subset Z_\sigma$ is an open subset and the orbit $\mathbf{O}_\sigma$ is contained in the closure of $\mathbf{O}_\tau$ in Z_σ, since $\sigma^\perp \subset \tau^\perp$, thus the closure of the orbit of τ in Z_Σ is $\overline{\mathbf{O}}_\tau = \bigcup_{\tau \preceq \sigma} \mathbf{O}_\sigma$. The orbit closure $\overline{\mathbf{O}}_\tau$ is a normal toric variety with respect to the lattice $N(\tau) = N/N_\tau$ ([**CLS11**, Proposition 3.2.7]).

The singular locus of Z_Σ is equal to the union of orbits of non-regular cones,

$$\mathrm{Sing}(Z_\Sigma) = \bigcup_{\sigma \ non-reg} \mathbf{O}_\sigma.$$

See [**CLS11**, Proposition 11.1.2].

1.1.6. T_N- Invariant divisors and divisorial valuations. By the orbit-cone correspondence, the rays (1-dimensional cones) of a fan, Σ in $N_{\mathbb{R}}$, correspond to irreducible divisors of the normal toric variety Z_Σ, which are invariant by the action of the torus.

For a primitive integral vector $u \in N$ spanning a ray of Σ we denote by $E_u = \overline{\mathbf{O}}_{\mathbb{R}_{\geq 0} u}$ the irreducible torus-invariant divisor corresponding to it. The torus of Z_Σ is equal to $T_N = Z_\Sigma \backslash \cup E_u$, where u runs through the set Σ_{prim} of primitive integral vectors of the fan Σ.

Let v be an element in M. Since $\chi^v \in \mathbb{C}.[M]$ maps T_N to $\mathbb{C}^*$, we can regard χ^v as a rational function on Z_Σ, which is non-vanishing on T_N. Hence, the divisor of χ^v is supported on $\bigcup_{u \in \Sigma_{\mathrm{prim}}} E_u$. Since Z_Σ is normal and E_u is irreducible, the order of vanishing along the divisor E_u is a valuation, which we denote by ord_{E_u}. One has that

$$\mathrm{ord}_{E_u}(\chi^v) = \langle v, u \rangle\,, \tag{1.10}$$

(see [**Ful93**, Section 3.3] or [**CLS11**, Proposition 4.1.1]).

The divisor of χ^v is given by

$$\mathrm{div}(\chi^v) = \sum_{u \in \Sigma_{\mathrm{prim}}} \langle v, u \rangle\, E_u. \tag{1.11}$$

The divisors of the form $\sum a_u E_u$ are precisely the **divisors invariant under the torus action** of Z_Σ.

EXAMPLE 1.12. The affine plane $\mathbb{C}^2$, equipped with the affine coordinates (x, y) is an example of toric variety. The torus $T_2 = (\mathbb{C}^*)^2$ is an open dense subset which acts on $\mathbb{C}^2$ by multiplication coordinate-wise, and this action extends the product operation on T^2 as an algebraic group. The coordinate lines $R = V(x)$ and $L = V(y)$ are invariant for this action. More generally any invariant divisor is of the form $nR + mL = \operatorname{div}(x^n y^m)$, for $(n, m) \in \mathbb{Z}^2$. It follows that the group of invariant divisors is a rank 2 free abelian group M with basis R, L.

1.1.7. Toric modifications.

DEFINITION 1.13. A fan Σ' is a subdivision of Σ if both fans have the same support and if any cone of Σ' is contained in a cone of Σ. The subdivision Σ' is regular if any cone of Σ' is regular, and it is a regular subdivision if any regular cone of Σ belongs to Σ'.

Associated to a subdivision Σ' of a fan Σ there is a **toric modification**, i.e., a proper birational map, inducing an isomorphism between their tori and equivariant with respect to the torus action:

$$\psi_{\Sigma'} : Z_{\Sigma'} \longrightarrow Z_\Sigma.$$

EXAMPLE 1.14. Let Σ be a regular subdivision of the cone $\rho = \mathbb{R}^d_{\geq 0}$ with lattice $N = \mathbb{Z}^d$. This subdivision defines a modification

$$\psi_\Sigma : Z_\Sigma \longrightarrow Z_\rho \simeq \mathbb{C}^d.$$

The variety Z_Σ is non-singular, as it is the fan. For any cone of maximal dimension, $\rho \supset \sigma \in \Sigma(d)$, the associated variety Z_σ is isomorphic to $\mathbb{C}^d$, and the restriction of the morphism ψ_Σ,

$$\psi_\sigma : Z_\sigma \longrightarrow Z_\rho,$$

is induced by the semigroup inclusion

$$\mathbb{R}^d_{\geq 0} \cap M \hookrightarrow \sigma^\vee \cap M,$$

where the set $\sigma(1)$ of primitive vectors in the 1 skeleton of σ is a basis of N and its dual basis, say $\{u_i = (a^i_1, \ldots, a^i_d)\}$, is a minimal set of generators of the semigroup $\sigma^\vee \cap M$. These generators give us toric coordinates $y_i := \chi^{u_i}$, for $i = 1, \ldots, d$ to describe ψ_σ:

$$x_1 = y_1^{a^1_1} \ldots y_d^{a^d_1},$$
$$\ldots$$
$$x_d = y_1^{a^1_d} \ldots y_d^{a^d_d}.$$

Since the fan Σ is regular, it is clear that the map π_Σ is an isomorphism over the torus of Z_ρ, $\mathbb{C}^d \setminus \{x_1 \ldots x_d = 0\}$.

A **resolution of singularities** of a variety Z is a proper birational morphism, $\Psi : Z' \to Z$, which is an isomorphism outside the singular locus of Z.

A basic result in toric geometry is that any fan admits a **regular subdivision** (check [**CLS11**, Theorem 11.1.9] and [**Ful93**, Section 2.6 Proposition]). It follows that the resolution of singularities of normal toric varieties is reduced to a combinatorial property, namely, given any fan Σ there is a regular subdivision Σ' of it, and the associated toric modification is a resolution of singularities, (see [**CLS11**, Theorem 11.1.9]).

It is possible to describe the exceptional locus associated to such a toric modification. Taking away a cone $\sigma \in \Sigma$ corresponds geometrically to take away the orbit $\mathbf{O}_\sigma$ from the variety Z_σ, from which one can deduce that its preimage is the union of orbits associated to cones subdividing it

$$\Psi^{-1}(\mathbf{O}_\sigma) = \bigcup_{\substack{\tau \in \Sigma' \\ \tau^\circ \subset \sigma^\circ}} \mathbf{O}_\tau.$$

It follows that the **exceptional fibers**, i.e., the union of subvarieties of dimension ≥ 1 which are mapped to points, are given as the union of maximal cones which are strictly subdivided

$$\bigcup_{\substack{\dim(\sigma)=\mathrm{rk}(N) \\ \sigma \notin \Sigma'}} \Psi^{-1}(\mathbf{O}_\sigma).$$

The **exceptional locus**, i.e., the union of subvarieties mapped to a subvariety of smaller dimension, is the union of closures of orbits of (minimal) cones which subdivide cones in the fan of the target

$$\bigcup_{\sigma \in \Sigma \setminus \Sigma'} \Psi^{-1}(\mathbf{O}_\sigma) = \bigcup_{\substack{\tau \in \Sigma' \setminus \Sigma \\ \tau \ \min}} \overline{\mathbf{O}}_\tau. \tag{1.15}$$

On the target variety, the **discriminant locus**, i.e., the image of the exceptional locus, is equal to the union of closures of (minimal) cones which are subdivided

$$\bigcup_{\substack{\sigma \in \Sigma \setminus \Sigma' \\ \sigma \ \min}} \overline{\mathbf{O}}_\sigma. \tag{1.16}$$

1.2. Newton polyhedra, dual fans and partial resolution

Assume $M \simeq \mathbb{Z}^d$, and denote by $x = (x_1, \dots, x_d)$. Let $f = \sum c_a x^a \in \mathbb{C}\{\underline{x}\}$ be a non zero series. The **Newton polyhedron** of f with respect to the coordinates $\{x_1, \dots, x_d\}$ is

$$\mathcal{N}(f) = \mathrm{Conv}\Bigg(\bigcup_{c_a \neq 0} a + \mathbb{R}^d_{\geq 0}\Bigg).$$

More generally, let $\rho \subset N_\mathbb{R}$ be a strictly convex cone and $f = \sum c_a x^a \in \mathbb{C}\{\rho^\vee \cap M\}$ be a non zero germ of holomorphic function at the special point o_ρ of a normal affine toric variety $Z_\rho = \mathrm{Spec}\{\rho^\vee \cap M\}$.

Definition 1.17. The **support** of the germ f is the set

$$\mathrm{Supp}(f) = \{a \in M \mid c_a \neq 0\}. \tag{1.18}$$

Its **Newton polyhedron** is

$$\mathcal{N}_\rho(f) = \mathrm{Conv}\Bigg(\bigcup_{a \in \mathrm{Supp}(f)} a + \rho^\vee\Bigg) \subset M_\mathbb{R}.$$

We will drop the subscript of the cone in the Newton polyhedron whenever it is clearly determined by the context.

Remark 1.19. The Newton polyhedron of a product is the Minkowski sum of the Newton polyhedra of its factors. In particular, if the Newton polyhedron of the product has only one vertex, the same holds for each of the factors.

Let $u \in \rho$ be a vector, it determines a dual face of the Newton polyhedron defined by the minimal of its support function along the polyhedron

$$\mathcal{F}_u(\mathcal{N}(f)) = \{v \in \mathcal{N}(f) \mid \langle v, u\rangle = \min_{v' \in \mathcal{N}(f)} \langle v', u\rangle\}.$$

All faces of the polyhedron can be recovered this way and it is easy to see that compact faces of $\mathcal{N}(f)$ correspond to $u \in \rho^{\circ}$.

Dually, any face $\mathcal{F}$ of the polyhedron $\mathcal{N}(f)$ has an associated cone

$$\sigma(\mathcal{F}) = \{u \in \rho \mid \langle v, u\rangle = \min_{v' \in \mathcal{N}(f)} \langle v', u\rangle \;\; \forall v \in \mathcal{F}\}.$$

The set of cones $\sigma(\mathcal{F})$, for $\mathcal{F}$ running through the set of faces of the polyhedron $\mathcal{N}(f)$, define a subdivision $\Sigma(f) = \Sigma(\mathcal{N}(f))$ of the cone ρ called the **dual fan**. The relative interiors of the cones in $\Sigma(f)$ are equivalence classes of vectors in ρ by the relation $u \sim u' \Leftrightarrow \mathcal{F}_u = \mathcal{F}_{u'}$.

Definition 1.20. The function

$$\begin{aligned} \Phi_{\mathcal{N}(f)} : \rho^{\vee} &\longrightarrow \mathbb{R}_{\geq 0} \\ u &\longmapsto \min_{v \in \mathcal{N}(f)} \langle u, v\rangle \,, \end{aligned} \tag{1.21}$$

is the **support function** of the polyhedron $\mathcal{N}(f)$.

The support function is linear on each cone of the dual fan $\Sigma(f)$ (see [**Ewa96**, Lemma 5.9]). By definition, if $\sigma \in \Sigma(f)$ is a maximal dimensional cone there is a unique vertex $v_\sigma \in \mathcal{N}(f)$, such that

$$\Phi_{\mathcal{N}(f)}(u) = \langle u, v_\sigma\rangle \,, \text{ for all } u \in \sigma.$$

Indeed, for any vector a in the interior of the cone σ, the face $\mathcal{F}_a(\mathcal{N}(f))$ is $\{v_\sigma\}$.

As a consequence of the previous discussion we get:

Lemma 1.22. *Let $u \in \rho \cap N$ be a primitive integral vector and let us denote by $\nu_u = \mathrm{ord}_{E_u}$ its associated divisorial valuation (Subsection 1.1.6). Then, we have the following expression for the value of the valuation ν_u on the polynomial f:*

$$\nu_u(f) = \Phi_{\mathcal{N}(f)}(u). \tag{1.23}$$

In particular, if u belongs to a maximal dimensional cone $\sigma \in \Sigma(f)$ then we have the relation:

$$\nu_u(f) = \langle u, v_\sigma\rangle = \Phi_{\mathcal{N}(f)}(u). \tag{1.24}$$

The support function completely determines the Newton polyhedron (see [**Ewa96**, Theorem 6.8]). Indeed, for a given $a \in \rho$ the half-space

$$H^{+}_{a,\Phi_{\mathcal{N}(f)}(a)} := \left\{u \in M \mid \langle a, u\rangle \geq \Phi_{\mathcal{N}(f)}(a)\right\}, \tag{1.25}$$

contains the Newton polyhedron. The intersection

$$H^{+}_{a,\Phi_{\mathcal{N}(f)}(a)} \cap \mathcal{N}(f)$$

is the **face** $\mathcal{F}_a(\mathcal{N}(f))$ **of** $\mathcal{N}(f)$ **supported by** a. Then, we get the Newton polygon as intersection of half-spaces

$$\mathcal{N}(f) = \bigcap_{a \in \rho \cap N} H^{+}_{a,\Phi_{\mathcal{N}(f)}(a)} \tag{1.26}$$

(more generally see [**Ewa96**, Theorem 3.8]).

The Newton polyhyedron of f has a finite number of facets (faces of maximal dimension), which are orthogonal to the rays of the fan $\Sigma(C)$. It follows that

$$\mathcal{N}(f) = \bigcap_{a \in \Sigma(f)_{\text{prim}}} H^+_{a,\Phi_{\mathcal{N}(f)}(a)}, \tag{1.27}$$

where $\Sigma(f)_{\text{prim}}$ denotes the set of primitive integral vectors spanning the rays of the fan $\Sigma(f)$.

Corollary 1.28. *Let $f, g \in \mathbb{C}\{\rho \cap N\}$ be two germs at (X_ρ, o_ρ). Denote by $\mathcal{N}(f)$ (resp. $\mathcal{N}(g)$) the Newton polyhedron of f (resp. g) with respect to (ρ, N). The following two conditions are equivalent:*

(1) $\mathcal{N}(g) \subset \mathcal{N}(f)$,
(2) $\nu_a(g) \geq \nu_a(f)$, for all $a \in \Sigma(f)_{\text{prim}}$.

In addition, the following two conditions are equivalent:

(1) $\mathcal{N}(g)$ is contained in the interior of $\mathcal{N}(f)$,
(2) $\nu_a(g) > \nu_a(f)$, for all $a \in \Sigma(f)_{\text{prim}}$.

Proof. A vector $v \in M$ belongs to $\mathcal{N}(f)$ if and only if $\nu_a(x^v) \geq \nu_a(f)$ for all $a \in \Sigma(f)_{\text{prim}}$. This holds by using the description of $\mathcal{N}(f)$ given by Equation (1.27) combined with the reformulation in terms of monomial valuations (see Subsection 1.1.6). Then, the Newton polyhedron $\mathcal{N}(g)$ is contained in the Newton polyhedron of f if and only this condition holds for every vector v in the support of f. This happens if and only if $\nu_a(g) \geq \nu_a(f)$, for all $a \in \Sigma(f)_{\text{prim}}$.

A vector v is contained in the interior of $\mathcal{N}(C)$ if and only if it lies in the interior of every half-space describing the Newton polygon of C, i.e., $\nu_a(x^v) > \nu_a(f)$ for all $a \in \Sigma(f)_{\text{prim}}$. The second equivalence follows then by the same argument as the first one. ☐

Lemma 1.29. *Let $\mathcal{N}$, $\mathcal{N}_I$, $I \in J$ be Newton polyhedra for J a finite set. The following statements are equivalent:*

(1) $\mathcal{N}$ is equal to the convex hull of the union of $\mathcal{N}_I$, $I \in J$.
(2) $\Phi_{\mathcal{N}}(a) = \min\{\Phi_{\mathcal{N}_I}(a) \mid I \in J\}$ for every primitive vector in $\rho \cap N$.

Proof. (2) $\Rightarrow$ (1). A face of a Newton polyhedron is always supported by a vector $a \in \rho \cap N$. The inequality $\Phi_{\mathcal{N}}(a) \leq \Phi_{\mathcal{N}_I}(a)$ implies that $\mathcal{N}_I \subset \mathcal{N}$. If $b \in \mathcal{N}$ is a vertex, then there exists a vector $a \in \rho \cap N$ such that $\{b\}$ is the face of $\mathcal{N}$ supported by a. By the hypothesis (2) there exists $I_0 \in J$ such that $\Phi_{\mathcal{N}}(a) = \Phi_{\mathcal{N}_{I_0}}(a)$. Taking into account the inclusion $\mathcal{N}_{I_0} \subset \mathcal{N}$ we get that the face of $\mathcal{N}_{I_0}$ supported by a must be contained in the face of $\mathcal{N}$ supported by a, which is equal to $\{b\}$. That is, b is a vertex of $\mathcal{N}_{I_0}$.

(2) $\Rightarrow$ (1). Let $\mathcal{N}$ be the convex hull of $\mathcal{N}_I$ for $I \in J$. By definition, $\mathcal{N}$ is the intersection of convex sets containing all the $\mathcal{N}_I$. Assume that there exists a vector $a \in \rho \cap N$ such that $\Phi_{\mathcal{N}}(a) > \Phi_{\mathcal{N}_I}(a)$ for all $I \in J$. Then

$$\mathcal{N} \cap H^+_{a, \min\limits_{I \in J}\{\Phi_{\mathcal{N}_I}(a)\}}$$

is a convex set which contains all the $\mathcal{N}_I$ but is strictly contained in $\mathcal{N}$, contradicting the definition of $\mathcal{N}$. ☐

We say that a fan Σ supported on the cone ρ is **compatible** with a set of holomorphic functions $f_1, \dots, f_s \in \mathbb{C}\{\rho^\vee \cap M\}$ (or that it is compatible with their Newton polyhedra, $\mathcal{N}(f_i)$) if it subdivides the fan of its product, $\Sigma(f_1 \dots f_s)$. Notice that a cone in the fan $\Sigma(f_1 \dots f_s)$ is intersection of cones of the fans $\Sigma(f_i)$ and therefore Σ is also compatible with all the germs f_i. If Σ is compatible with $\mathcal{N}(f_1 \dots f_s)$ and $\sigma \in \Sigma$, then any vector $u \in \sigma^\circ$ defines the same face, $\mathcal{F}_\sigma$, of $\mathcal{N}(f_1 \dots f_s)$.

Definition 1.30. Let $0 \neq f = \sum c_a x^a \in \mathbb{C}\{\rho^\vee \cap M\}$ be a holomorphic germ and $\mathcal{F}$ a compact face of its Newton pelyhedron. The symbolic restriction of f to $\mathcal{F}$ is the sum of those terms in the face

$$f_{|\mathcal{F}} = \sum_{a \in \mathcal{F}} c_a x^a \in \mathbb{C}[\rho^\vee \cap M].$$

The **Newton principal part** $f_{|\mathcal{N}(f)}$ is the sum of those terms of f having exponents lying on the compact faces of its Newton polyhedron.

Notice that the Newton principal part of f does not change if we change the ring $\mathbb{C}[[\rho^\vee \cap M]]$ by extending the lattice M.

Let now Σ be any fan supported on a cone ρ, defining a modification $\psi_\Sigma : Z_\Sigma \to Z_\rho$. Let $V \subset Z_\rho$ be a subvariety such that the intersection of the discriminant locus of ψ_Σ with each irreducible component V_i of V is nowhere dense in V_i. For an irreducible V this condition holds if the intersection of V with the torus is an open dense subset of V.

Definition 1.31. The **strict transform** $V_\Sigma \subset Z_\Sigma$ is the subvariety of $\psi_\Sigma^{-1}(V)$ such that the restriction $V_\Sigma \to V$ is a modification.

The modification ψ_Σ is an isomorphism over the torus $T_N \subset Z_\rho$ and the strict transform is the closure of $\psi_1^{-1}(V \cap T_N)$ in Z_Σ.

Definition 1.32. If the fan Σ is regular, the toric map ψ_Σ is a **toric embedded pseudo-resolution** of V if the restriction $V_\Sigma \to V$ is a modification such that the strict transform V_Σ is non-singular and transversal to the orbit stratification of the exceptional locus of ψ_Σ. The modification is a **toric embedded resolution** of V if the restriction to the strict transform is an isomorphism outside the singular locus of V.

If ψ_Σ is only a pseudo-resolution we can only guarantee that the restriction $V_\Sigma \to V$ is an isomorphism outside the intersection of V with the discriminant locus of ψ_Σ, i.e., the discriminant locus contains singular locus of V but is not necessarily equal to it.

Definition 1.33. If Σ is a (not necessarily regular) subdivision of ρ, the toric morphism $\psi_\Sigma : Z_\Sigma \to Z_\rho$ is a **partial toric embedded resolution** of V if for any regular subdivision Σ' of Σ the map $\psi_{\Sigma'} \circ \psi_\Sigma$ is an embedded resolution of V.

CHAPTER 2

Basic notions about plane curve singularities

2.1. Basic notation and definitions

We begin by fixing some notation.

We denote by $\mathbb{C}\{x_1, \dots, x_d\}$ the local ring of convergent power series in the variables $x_1, \dots, x_d$ and by $\mathbb{C}[[x_1, \dots, x_d]]$ its completion with respect to the maximal ideal. If $k = (k_1, \dots, k_d)$ we often denote $x_1^{k_1} \cdots x_k^{k_d}$ by x^k and we use also the notation $\mathbb{C}\{x\}$ (resp. $\mathbb{C}[[x]]$) for these rings respectively. We denote by $\mathbb{C}\{\{x\}\}$ (resp. $\mathbb{C}((x))$) their fields of fractions.

Let $f = \sum a_k x^k \in \mathbb{C}\{x\} \setminus \{0\}$ be a germ. We denote by $f_m = \sum_{|k|=m} a_k x^k$ the sum of terms of degree m in the expansion of f (where $|k| = k_1 + \cdots + k_d$). We have the expansion

$$f = \sum_{m \geq 0} f_m$$

The **order** of f is the integer

$$o(f) = \min\{m \mid f_m \neq 0\}.$$

A series $f \in \mathbb{C}\{x_1, \dots, x_d, y\}$ is **regular of order** n **in** y if the order of the series $f(0, \dots, 0, y) \in \mathbb{C}\{y\}$ is equal to n.

Theorem 2.1 (Weierstrass Preparation Theorem). *If $f \in \mathbb{C}\{x_1, \dots, x_d, y\}$ is regular of order n in y, then there exists a unique monic polynomial*

$$P \in \mathbb{C}\{x_1, \dots, x_d\}[y] \tag{2.2}$$

and a unit $u \in \mathbb{C}\{x_1, \dots, x_d, y\}$ such that

$$f = u \cdot P \tag{2.3}$$

See [**Wal04**, Theorem 2.2.2] or [**Che78**, Section 6.3] for a proof in dimension two, or more generally [**ZS60**, Theorem 5, Chapter VII, Section 1], [**Nag62**, Theorem 45.3].

The polynomial P of the above theorem is said to be a **distinguished polynomial** in y. As a consequence of Weierstrass Preparation Theorem, the germ defined by $f = 0$ in a sufficiently small neighborhood of the origin in $\mathbb{C}^{d+1}$ coincides with the germ defined by its associated Weierstrass polynomial, (see [**ZS60**, Corollary 2, Chapter VII, Section 1]).

2.2. The Puiseux characteristic of a plane branch

Let S be a smooth complex algebraic or analytic surface and O a closed point on it. The germ (S, O) is isomorphic to $(\mathbb{C}^2, O)$ and the ring of germs of holomorphic functions on S at O, $\mathcal{O}_{(S,O)}$ is isomorphic to the power series ring $\mathbb{C}\{x, y\}$.

We consider local coordinates (x, y) at O, i.e., an ordered pair of representatives of transversal smooth curves at O. A germ of a complex analytic curve $(C, O) \subset (S, O)$ is defined by a representative, which we will denote by $f_C \in \mathbb{C}\{x, y\}$, defined up to multiplication by a unit of the ring $\mathbb{C}\{x, y\}$. A local equation of (C, O) is given by $f_C(x, y) = 0$. Similarly, if f is a holomorphic function, we will denote by $V(f)$ its associated curve germ. If C is irreducible, meaning that f is irreducible, we will say that it is a **branch**. The **multiplicity** at O of C is the order of its representative, f_C, and we denote it by $m_O(C)$. Often we will speak of the germ C instead of (C, O). If the order of $f(0, y)$ is equal to n Weierstrass Preparation Theorem 2.1 ensures that there exists another representative $f'_C \in \mathbb{C}\{x\}[y]$, which is monic and has degree n in y.

If $m = o(f_C) = m_O(C)$ and $f_C = \sum_{k \geq m} f_k$, then the homogeneous polynomial f_m defines the **tangent cone** of C. The irreducible factors of f_m are of degree one, since $\mathbb{C}$ is algebraically closed, and define the **tangent lines** of the germ C at O. The germ C is smooth if and only if $m_O(C) = 1$.

Definition 2.4. We say that a pair (R, L) of transversal smooth branches at a smooth surface (S, O) is a **cross** at O.

Given such a cross we can always consider a pair of local coordinates at $\mathcal{O}_{(S,O)}$ such that $R = V(x)$ and $L = V(y)$. If (x', y') verify also that $R = V(x')$ and $L = V(y')$ then there are units $u, v \in \mathbb{C}\{x, y\}$ such that $x' = xu$ and $y' = yv$. If $h \in \mathcal{O}$ then we can expand it as a power series $h = f'(x', y')$ with $f' \in \mathbb{C}\{x', y'\}$, and also as a series $h = f(x, y)$, with $f \in \mathbb{C}\{x, y\}$. Obviously, we have the relation

$$g(x, y) = f(xu, yv),$$

which implies that the Newton polygon of h with respect to (x, y) coincides with the one with respect to (x', y'). For this reason we speak about the **Newton polygon** of $h \in \mathcal{O}$, with respect to the cross (R, L). We denote it by $\mathcal{N}_{R,L}(h)$, whenever it is necessary to emphasize the role of the pair (R, L). Similarly if C is a plane curve singularity we will denote $\mathcal{N}_{R,L}(f_C)$ also by $\mathcal{N}_{R,L}(C)$.

Definition 2.5. If C, D are two germs of curves at O in a smooth surface S their **intersection multiplicity** is

$$(C, D)_O = \dim_{\mathbb{C}} \left(\mathbb{C}[[x, y]]/(f_C, f_D)\right). \tag{2.6}$$

A smooth germ D is **transversal** to C at O if $(C, D)_O = m_O(C)$ and it is **tangent** to C at O if $(C, D)_O > m_O(C)$.

Let us consider the following ring of fractional power series

$$\mathbb{C}\{x^{1/\mathbb{N}}\} := \bigcup_{m \geq 1} \mathbb{C}\{x^{1/m}\}.$$

We define similarly the ring $\mathbb{C}[[x^{1/\mathbb{N}}]]$.

The Newton-Puiseux theorem is a fundamental result for plane curves. See [**Che78**, Chapter 8] or [**Wal04**, Chapter 2].

Theorem 2.7. *Let $f \in \mathbb{C}\{x\}[y]$ be a monic polynomial in y of degree $n \in \mathbb{N}^*$. Then, the polynomial f has n roots in the ring $\mathbb{C}\{x^{1/\mathbb{N}}\}$. In addition, if f is irreducible then there is a root $\zeta(x^{1/n}) \in \mathbb{C}\{x^{1/n}\}$ and f factors as:*

$$f = \prod_{\eta \in G_n} \left(y - \zeta(\eta \cdot x^{1/n})\right), \tag{2.8}$$

where $G_n = \{\eta \in \mathbb{C}^* \mid \eta^n = 1\}$.

We denote by $\mathrm{Zer}(f)$ the set of roots of f.

If C is a branch, the local ring $\mathcal{O}_C = \mathbb{C}\{x, y\}/(f_C)$ is an integral domain. Let us denote by K_C the field of fractions of $\mathcal{O}_C$. The **integral closure** $\overline{\mathcal{O}}_C$ of $\mathcal{O}_C$ in K_C is a **discrete valuation**

ring isomorphic to the ring of convergent power series $\mathbb{C}\{t\}$. If $\mathrm{ord}_y f_C(0, y) = n$ then the Newton-Puiseux theorem provides a parametrization $\phi : \mathcal{O}_C \to \mathbb{C}\{t\}$ by setting $t = x^{1/n}$:

$$\begin{cases} x &= t^n \\ y &= \zeta(t). \end{cases} \tag{2.9}$$

We identify $\overline{\mathcal{O}}_C$ with $\mathbb{C}\{t\}$ by this parametrization, and we consider on this ring the valuation ord_t, which takes the order in t of a series. The parametrization (2.9) is **primitive**, that is, the greatest common divisor of n and the exponents appearing in $z(t)$ is equal to one.

Let us introduce the notion of *characteristic exponents* of a primitive parametrization (2.9). Start from

$$\zeta(t) = \sum c_j t^j. \tag{2.10}$$

Denote $b_0 = e_0 = n$ and define recursively

$$b_i = \min\{j \mid c_j \neq 0 \text{ and } e_{i-1} \not| j\} \text{ and } e_i = \gcd(e_{i-1}, b_i). \tag{2.11}$$

Since the parametrization (2.9) is primitive there exists an integer $g \in \mathbb{N}$ such that $e_g = 1$. By construction we have that e_i divides e_{i-1} and in addition:

$$n_i = \frac{e_{i-1}}{e_i} > 1 \text{ and } e_{i-1} = n_i e_i, \text{ for } i = 1, \dots, g \text{ where we set } n_0 := 1. \tag{2.12}$$

DEFINITION 2.13. The **characteristic exponents** $\mathcal{E}(C)$ of C with respect to the coordinates (x, y) are the fractional exponents:

$$\alpha_i := \frac{b_i}{b_0}, \text{ for } i = 1, \dots, g, \tag{2.14}$$

which appear in the series $\zeta(x^{1/n})$. Let us write the expansion of $\zeta(x^{1/n})$ in the form:

$$\zeta(x^{\frac{1}{n}}) = x^{\frac{b_1}{b_0}} \left(c_{b_1} + \ldots + x^{\frac{b_2 - b_1}{b_0}} \left(c_{b_2} + \ldots + x^{\frac{b_g - b_{g-1}}{b_0}} \left(c_{b_g} + \cdots \right) \cdots \right) \right) \tag{2.15}$$

By expressing the following quotients as irreducible fractions we obtain:

$$\frac{b_k - b_{k-1}}{b_0} = \frac{m_k}{n_1 \dots n_k} \text{ for } k = 1, \dots, g. \tag{2.16}$$

The set of pairs $\{(n_k, m_k)\}$ is called the set of **Newton pairs** of C with respect to (x, y).

The terminology *Newton pairs* comes from [**EN85**, Appendix to Chapter 1].

It is easy to see that the Newton pairs determine the characteristic exponents and viceversa. Note also that we can obtain the characteristic exponents by taking the orders of the difference of distinct roots of f_C:

$$\{\mathrm{ord}_x(\zeta - \zeta') \mid \zeta \neq \zeta', \zeta, \zeta' \in \mathrm{Zer}(f)\} = \{\alpha_1, \dots, \alpha_g\}.$$

If C and D are two distinct branches we can also compare the roots of f_C with the roots of f_D, with respect to the coordinates (x, y):

DEFINITION 2.17. Consider two distinct branches C, D on (S, O). The **order of contact** of C and D with respect to the coordinates (x, y) is

$$\kappa(C, D) := \max\{\mathrm{ord}_x(\zeta_C - \zeta_D) \mid \zeta_C \in \mathrm{Zer}(f_C),\ \zeta_D \in Zer(f_D)\} \in \mathbb{Q}^*.$$

REMARK 2.18. Note that this function is symmetric, $\kappa(C, D) = \kappa(D, C)$. Also observe that, as a consequence of Theorem 2.7, it can be computed by fixing any root $\zeta_C \in Zer(f_C)$ and varying the root ζ_D in $Zer(f_D)$.

Definition 2.19. We say that the coordinates (x, y) are **generic** for the branch C if the smooth germ $R = V(x)$ is transversal to C.

Example 2.20. Let C be the branch defined by $y^6 - x^{13} = 0$. The branch C has only one characteristic exponent $\frac{13}{6}$ with respect to (x, y). Set $z = y - x^2$. One can check that the characteristic exponents of C with respect to the local coordinates (z, x) are $\frac{1}{2}, \frac{7}{12}$. This can be seen as a consequence of the *inversion formulas* for the characteristic exponents (see [**Abh67, GGP17**]).

Remark 2.21. As is explained in the previous example the characteristic exponents of the branch C depend on the choice of the coordinates (x, y). The same also happens for the and the contact $\kappa(C, D)$ between the branches C and D. The dependency on the coordinates is explained by the following proposition:

Proposition 2.22. *Let (x, y) and (x', y') be two choices of local coordinates at a point O of a smooth surface S such that $V(x) = V(x')$. Let C and D be two smooth branches in (S, O). Then:*

(1) The characteristic exponents of C with respect to (x, y), coincide with the characteristic exponents of C with respect to (x', y').
(2) The order of contact of C and D with respect to (x, y) coincide with the order of contact of C and D with respect to (x', y').

Based on Proposition 2.22 we use the following notation and terminology to emphasize the dependence of the notions of characteristic exponent and order of contact on the choice of the auxiliary smooth germ $R = V(x)$:

Definition 2.23. We denote the set $\mathcal{E}(C)$ introduced in Definition 2.13 by $\mathcal{E}_R(C)$ and we call it the set of characteristic exponents of C with respect to $R = V(x)$. We denote also by $\kappa_R(C, D)$ the number $\kappa(C, D)$ introduced in Definition 2.17 and we call it the order of contact $\kappa_R(C, D)$ of the branches C and D with respect to R. Similarly, we will speak about the Newton pairs of C with respect to R and so on.

In what follows when we speak about the characteristic exponents of a plane branch C, we mean with respect to a fixed smooth germ R.

Definition 2.24. Let C be a branch on a smooth surface (S, O) and let us denote by R a smooth branch. A k-th **semi-root** of C with respect to R is a branch L_k such that

$$(L_k, R)_O = n_0 n_1 \cdots n_k \tag{2.25}$$

and

$$\kappa_R(L_k, C)_O = \alpha_{k+1}. \tag{2.26}$$

where $\alpha_1, \ldots, \alpha_g$ denote the characteristic exponents of C with respect to R and the integers n_i, defined in (2.12), are the first entry of the Newton pairs of C with respect to R. A **complete sequence of semi-roots** of C with respect to R is:

$$L_0, L_1, \ldots, L_g,$$

where L_k is a k-semi-root of C with respect to R, for $0 \leq k \leq g - 1$ and $L_g := C$.

By abuse of notation we call also a k-th semi-root any holomorphic germ defining it.

One can build a k-th semi-root of C by truncating the parametrization $\zeta(x^{1/n})$ as follows. The fractional power series

$$\zeta_k(x^{1/n}) = \sum_{j < b_{k+1}} c_j x^{j/n}, \text{ for } k = 0, \ldots g - 1, \tag{2.27}$$

parametrizes a k-th semi-root of C with respect to $R = V(x)$. Notice that the condition (2.26) holds by definition, while the condition (2.25) holds because the series $\zeta_k(x^{1/n})$ belongs to the ring $\mathbb{C}\{x^{1/n_1\cdots n_k}\}$. In our approach, it will be important to have the freedom to make a *suitable choice* of semi-roots. In particular, we will consider other semi-roots different from those obtained by truncations.

The set $\Gamma(C)$ consisting of intersection multiplicity numbers of C with germs at the origin not having C as a component is a subsemigroup of $(\mathbb{N}, +)$:

DEFINITION 2.28. The **semigroup** of the branch C is the set

$$\Gamma(C) = \{(C, D)_O \mid C \not\subset D\}.$$

REMARK 2.29. Let C be a branch and $\phi : \mathcal{O}_C \to \mathbb{C}\{t\}$ a primitive parametrization of it. If D is a germ of plane curve then one has

$$(C, D)_O = \text{ord}_t(\phi \circ f_D). \qquad (2.30)$$

(see [**Wal04**, Lemma 4.3.1] for a proof).

THEOREM 2.31. *The semigroup of the branch C is generated by*

$$\bar{b}_0, \bar{b}_1, \ldots \bar{b}_g.$$

where $\bar{b}_0 := b_0$ and the elements $\bar{b}_j$ are defined inductively by the formulas

$$\bar{b}_{k+1} - n_k \bar{b}_k = b_{k+1} - b_k. \qquad (2.32)$$

If we choose R to be transversal to C, this set of generators is the minimal one.

Proof of these facts can be found in [**Wal04**, Thm 4.3.5] and [**Zar06**, Thm 3.9.].

Notice that $\bar{b}_0 = (R, C)_O$ where $R = V(x)$. The proof of the previous theorem uses the following proposition, which shows that the intersection numbers of C with the semi-roots are the elements $\bar{b}_1, \ldots, \bar{b}_g$.

PROPOSITION 2.33. *If L_j is a j-th semi-root of the branch C with respect to R, for $0 \leq j \leq g - 1$, then:*

$$(C, L_j)_O = \bar{b}_{j+1}.$$

REMARK 2.34. For $0 \leq k \leq g$ it holds that

$$\gcd(b_0, b_1, \ldots, b_k) = e_k = \gcd(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_k).$$

(where e_k is the number defined by (2.11)). The order of $\bar{b}_k \mod e_{k-1}\mathbb{Z}$ is equal to n_k and, in addition, $n_k \bar{b}_k \in \mathbb{N}\bar{b}_0 + \cdots + \mathbb{N}\bar{b}_{k-1}$ (see [**Wal04**, Lemma 4.3.6] and [**Zar06**, Lemma 2.2.1]).

Let us study the notion of *conductor* of the semigroup $\Gamma(C)$:

DEFINITION 2.35. There exists an element $\mathsf{c} \in \mathbb{N}$ such that for all $i \in \mathbb{Z}$, $i \geq \mathsf{c}$ implies $i \in \Gamma(C)$. This number is called the **conductor** of the semigroup.

The conductor can be equivalently defined as the natural number such that the ideal (t^{c}) is the largest ideal of $\mathbb{C}\{t\}$ contained in $\mathcal{O}_C$. Zariski deduces a formula for the conductor (see Theorem 3.9 in [**Zar06**] and the explanations after it).

LEMMA 2.36. *The conductor $\mathsf{c}(C)$ of the semigroup of the branch, $\Gamma(C)$ is equal to*

$$\mathsf{c}(C) = n_g \bar{b}_g - b_g - e_0 + 1. \qquad (2.37)$$

REMARK 2.38. The proof of this result in [**Zar06**] is stated in terms of generic coordinates but the formula holds in general.

One has the following fundamental property.

THEOREM 2.39. *The characteristic exponents of the branch C are independent of the choice of generic coordinates (x, y) for C. In this case, the elements $\bar{b}_0, \bar{b}_1, \ldots \bar{b}_g$, determined in terms of any choice of generic coordinates with respect to C, form an ordered minimal generating system of the semigroup $\Gamma(C)$.*

There are several notions of equivalence between plane curve singularities. From a topological view-point we can consider the following definition:

DEFINITION 2.40. Two germs of curves at the origin C, D in (S, O), are **topologically equisingular** if there exists a homeomorphism germ $\sigma : (S, O) \to (S, O)$ such that it restricts to a germ homeomorphism $C \to D$.

THEOREM 2.41. [**Zar68**, Thm. 2.1]. *Two branches are topologically equisingular if and only if they have the same generic characteristic exponents.*

In fact, the previous theorem has a general form for reduced plane curves.

THEOREM 2.42. [**CA00**, Theorem 3.8.6]. *The topological equisingularity class of a reduced plane curve is determined by the characteristics exponents of their branches and their intersection multiplicities with each other.*

REMARK 2.43. The notation

$$\beta_i := b_i, \text{ and } \bar{\beta}_i := \bar{b}_i, \text{ for } i = 1, \ldots, g,$$

is commonly used in many references when dealing with the characteristic exponents of a branch C and the generators of the semigroup $\Gamma(C)$, with respect to generic coordinates.

2.3. Resolution of curves by blowing-up

We begin by describing the blowing up of a point (the origin) in a smooth complex surface.

Let us consider the projective space $\mathbb{P}^1(\mathbb{C})$ as the space of lines through a point O on a smooth surface S. Choosing coordinates (x, y) at O, it is customary to cover this projective space by two affine charts, U_x and U_y, corresponding to lines in the open sets $x \neq 0$ and $y \neq 0$, respectively. One takes homogeneous coordinates $(u : v)$ on the projective space $\mathbb{P}^1(\mathbb{C})$, corresponding to the line defined by

$$xv - yu = 0.$$

This equation defines a non-singular algebraic surface $\mathrm{Bl}_O(S) \subset S \times \mathbb{P}^1$. The **blowing up** of O in S is the projection:

$$\pi_O : \mathrm{Bl}_O(S) \longrightarrow S.$$

The fiber of the origin $\pi_O^{-1}(O)$ is the entire projective line $\mathbb{P}^1$, since the equation involving u and v vanishes identically, while the fiber over any other closed point reduces to a closed point because the coordinates x, y determine uniquely the ratio between u, v. A basic property of the blowing up is that it does not depend on the choice of the coordinates (x, y) at O (see [**Wal04**, Lemma 3.2.1]).

The surface $\mathrm{Bl}_O(S)$ is covered by two affine charts. On the open set where $u \neq 0$, we can take coordinates $x_1 = x, y_1 = \frac{v}{u}$, so the map on it is described by

$$\begin{array}{rcl} x & = & x_1 \\ y & = & x_1 y_1. \end{array} \tag{2.44}$$

Similarly on the open chart where $v \neq 0$ the map is described by

$$\begin{array}{rcl} x & = & x_2 y_2 \\ y & = & y_2. \end{array} \tag{2.45}$$

Notice that on the first chart the exceptional curve $\pi_O^{-1}(O)$ is defined by the equation $x_1 = 0$ (while on the second chart it is defined by $y_2 = 0$). This comes from a more general fact, namely, the blown-up space (the center of blow-up) is transformed into a subspace defined locally by one equation, i.e., a divisor, which is called the **exceptional divisor** of the blowing up.

Suppose one has an analytic germ f at O and we expand it with respect to local coordinates (x, y), as a sum of homogeneous polynomials:

$$f(x, y) = \sum_{j \geq m} f_j(x, y) \in \mathbb{C}\{x, y\}$$

In the chart $U = \{u \neq 0\}$ we have:

$$f \circ \pi_O(x_1, y_1) = x_1^m (\sum_{j \geq m} x_1^{j-m} f_j(1, y_1)) \tag{2.46}$$

and there is a similar expansion in the other chart. Looking at the zero set of it, we see that in each chart it contains the exceptional divisor with multiplicity m and a curve on the surface on $\mathrm{Bl}_O(S)$. This last part is called the **strict transform** $\tilde{C}$ of the original curve C.

The intersection multiplicity of the strict transform $\tilde{C}$ of C with the exceptional divisor E is equal to the multiplicity of C. This property holds since the strict transform $\tilde{C}$ meets the exceptional divisor in finitely many points $(u : v) \in \mathbb{P}^1$ where $f_m(u, v) = 0$. The strict transform of f meets the exceptional divisor at the points in the projective space defined by the lines in the tangent cone at the origin of the curve C. In particular, if C and C' are two branches with different tangent lines then their strict transforms are disjoint. If C is a smooth then $\tilde{C}$ is transversal to the exceptional divisor and we have an isomorphism $\tilde{C} \to C$ induced by the blowing up (see [**Wal04**, Lemma 3.4.1]). We can also check these relations by passing to local coordinates. By formula (2.46), a point $\tilde{O} \in E \cap \tilde{C}$ can be seen in the chart U with coordinates, $(0, t_{\tilde{O}})$, where $t_{\tilde{O}}$ is a root of the polynomial $f_m(1, y) \in \mathbb{C}[y]$ of multiplicity $e_{\tilde{O}} = (E, \tilde{C})_{\tilde{O}}$. We obtain the inequality

$$m_{\tilde{O}}(\tilde{C}) \leq (E, \tilde{C})_{\tilde{O}} = e_{\tilde{O}}, \tag{2.47}$$

with equality if E is transversal to $\tilde{C}$ at $\tilde{O}$. It follows that the sum of multiplicities of the strict transform at points in the exceptional divisor is less or equal to the multiplicity of C, i.e.,

$$\sum_{\tilde{O} \in \tilde{C} \cap E} m_{\tilde{O}}(\tilde{C}) \leq \sum_{\tilde{O} \in \in \tilde{C} \cap E} e_{\tilde{O}} = \deg f_m(1, y) = m = m_O(C). \tag{2.48}$$

A collection of curves in a smooth surface S is said to have **normal crossings** if each curve is smooth, no three curves meet in a point, and any intersection of two of them is transverse.

DEFINITION 2.49. A **modification** $\pi : T \to S$ between two smooth surfaces is a proper and birational map. Let C be a plane curve singularity in a smooth surface (S, O). An **embedded resolution** of (C, O) is a proper modification $\pi : T \to S$ where T is smooth, and if $E = \pi^{-1}(O)$ the restriction of π to $T \setminus E \to S \setminus \{O\}$ is an isomorphism, and $\pi^{-1}(C)$ has normal crossings.

We discuss the iteration of blowing ups centered at exceptional points over O which are required in order to provide an embedded resolution of a plane curve singularity C.

We set $S_1 = \mathrm{Bl}_O(S)$ and we denote $E^{(1)}$ the exceptional curve $\pi_O^{-1}(O)$. Choose a point $O_1 \in E^{(1)}$ and denote by $\pi_{O_1} : S_2 \to S_1$ the blowing up and denote by $\pi_2 = \pi_O \circ \pi_{O_1}$ their composition. Inductively, assume we have defined morphism $\pi_{k-1} : S_{k-1} \to S$ between smooth surfaces. Then, we choose a point $O_k \in \pi_{k-1}^{-1}(O)$ and we set $\pi_k = \pi_{k-1} \circ \pi_{O_k}$. By the above discussion the exceptional divisor $E^{(k)}$ of $\pi_k : S_k \to S$ defined by $\pi_k^{-1}(O)$ has normal crossings. Assume that we start with a singular curve C at O. At each step k we denote by $C^{(k)}$ the strict transform of C at S_k. In addition, we will suppose that the point $O_k \in C^{(k)} \cap E^{(k)}$ is a singular point of $C^{(k)}$.

THEOREM 2.50. [**Wal04**, Theorem 3.3.1 and Theorem 3.4.4]. *In the situation described above there is an integer k_0 such that $C^{(k)}$ is smooth for all $k \geq k_0$. In addition, there is an integer k_1 such that the reduced divisor defined by $\pi_k^{-1}(C)$ has normal crossings for all $k \geq k_1$.*

An embedded resolution is also called **good resolution** or **log-resolution**. The curves in $\pi^{-1}(C)$ are the components of the strict transform of C and the exceptional curves. Theorem 2.50 guarantees that there is a minimal integer k_1 such that $\pi_{k_1}^{-1}(C)$ has normal crossings. We say in this case that the resolution π_{k_1} is the **minimal good resolution** of C.

REMARK 2.51. Theorem 2.50 implies that an embedded resolution of C is obtained by a composition of blowings up of points. Zariski proved that if $\pi : T \to S$ is a modification between two smooth surfaces then π is a composition of blow-ups of points (check [**Bea96**] Lemma II.11). In particular, any irreducible component exceptional locus of the modification is a smooth rational curve (that is, isomorphic to $\mathbb{P}^1$).

DEFINITION 2.52. We say that P is an **infinitely near point** over O if there is a modification $\pi : T \to S$, such that P belongs to the exceptional divisor $\pi^{-1}(O)$.

We sketch the proof of Theorem 2.50 (see [**Wal04**] and [**Tei07**]). If we start with a plane curve C of multiplicity m at O and after blowing up it there is a point $\tilde{O} \in E \cap \tilde{C}$ which is also of multiplicity m then then by (2.48) it is the only infinitely near point of C in E. In order to prove resolution of singularities of plane curves, we have to show that this last situation cannot persist indefinitely in a sequence of blowing ups. Let us consider first the case when C is a singular plane branch. Let us take local coordinates (x, y) such that the smooth curve $R = V(x)$ is transversal to C and $L = V(y)$ is a 0-th-semi-root of C with respect to R. The smooth branch L has *maximal contact* with C, with the terminology of [**Tei07**]. In this case the parametrization (2.9) is of the form:

$$\left\{ \begin{array}{lcl} x & = & t^{b_0} \\ y & = & \sum\limits_{j \geq b_1} c_j t^j, \end{array} \right. \tag{2.53}$$

where $b_0 = m_O(C) < b_1$. If $1 \leq s \leq \lfloor \frac{b_1}{b_0} \rfloor$ we get that the strict transform $C^{(j)}$ of C after j blowing ups has a parametrization, with respect to suitable coordinates (x_s, y_s) of the form:

$$\left\{ \begin{array}{lcl} x_s & = & t^{b_0} \\ y_s & = & \sum\limits_{j \geq b_1 - jb_0} c_j t^j, \end{array} \right.$$

This implies that if $s = \lfloor \frac{b_1}{b_0} \rfloor$ then the remainder of the division of b_1 by b_0 is $b_1 - sb_0$ thus:

$$b_0 = m_O(C) > m_{O^{(s)}}(C^{(s)}) = b_1 - sb_0.$$

If follows that the iteration of blowing ups provides an embedded resolution of plane branches. Indeed, one has that the generic characteristic exponents of a plane branch determine the multiplicity sequence $(m_{O^{(s)}}(C^{(s)}))_{s \geq 0}$ (see [**Wal04**, Theorem 3.5.6]). In the general case, it remains to prove that if a plane curve singularity consists of smooth branches we can separate their strict transforms after a sequence of blowing ups. If C and D are two smooth branches in (S, O) it is easy to check using local coordinates that the intersection number of the strict transforms of C and D after blowing up O is one less than the intersection number of C and D at O (see [**Wal04**, Theorem 3.4.4] or apply Lemma 2.55 below). This shows that the intersection numbers of $C^{(j)}$ and $D^{(j)}$ is eventually equal to zero.

REMARK 2.54. Notice that blowing up a point O is an isomorphism outside O. A *reduced* algebraic plane curve has *finitely many* singular points. One gets a resolution of its singularities by performing a sequence of blowing ups over its singular points.

We introduce the notion of total transform of a curve C on a smooth surface S by a modification $\pi : T \to S$. The **total transform** $\pi^* C$ is the divisor of $f_C \circ \pi$, where $f_C = 0$ is a local equation for C at O.

The following lemma gives the relation of the intersection numbers and self-intersection numbers of exceptional components after blowing up.

LEMMA 2.55. [**Wal04**, Lemma 8.1.2]. *Let us denote by $\pi : T \longrightarrow S$ the blowing up of a point O on a smooth surface. If C is a curve through O of multiplicity m, then we have*

$$\pi^*(C) = \tilde{C} + mE, \tag{2.56}$$

where $\tilde{C}$ denotes the strict transform and E denotes the exceptional divisor. If C, D are two curves in S. Then, we have:

$$\begin{array}{rcl} E^2 = (E \cdot E) & = & -1, \\ (E \cdot \pi^* C) & = & 0, \\ (\pi^* C \cdot \pi^* D) & = & (C, D)_O. \end{array}$$

More generally:

LEMMA 2.57. *Let $\pi : \widetilde{S} \to S$ a proper birational morphism between smooth surfaces which is an isomorphism outside O. Denote by $\pi^{-1}(O) = \cup_{j \in J} E_j$ the decomposition of the exceptional divisor $\pi^{-1}(O)$ as a union of irreducible components. For D, D' divisors in S one has the following properties:*

(1) $\pi^*(D) = \widetilde{D} + \sum_{j \in J} \nu_{E_j}(D) E_j$, where ν_{E_j} is the order of vanishing of D over the divisor E_j (see Definition 2.74 below).
(2) $(D, D')_O = \pi^*(D) \cdot \pi^*(D')$, where $\cdot$ denotes the intersection form for divisors on $\tilde{S}$.
(3) $\pi^*(D) \cdot Z = 0$, for any integral divisor Z supported on $\pi^{-1}(O)$.

Check [**Sha94a**, Chapter 4, Section 3.2, Theorem 2] for a proof.

PROPOSITION 2.58. *Let $\pi : \widetilde{S} \to S$ be a proper birational morphism between smooth surfaces which is an isomorphism outside O. Assume that $\pi = \psi \circ \pi'$, where $\psi : \widetilde{S} \to S'$ and $\pi' : S' \to S$. Then, the intersection number of two exceptional divisors on S' equals the intersection number of their total transforms on $\widetilde{S}$.*

One can compute self-intersection numbers by using the following proposition, which is a consequence of the previous lemmas.

PROPOSITION 2.59. *Let E be a compact irreducible and reduced smooth curve on a smooth surface S and let p be a point on E. Denote by $\tilde{E}$ the strict transform of E after blowing up p. Then: $\tilde{E}^2 = E^2 - 1$.*

PROOF. Let us denote by π the blowing up of p in S and by E_p its exceptional divisor. By (2.56) one has

$$\pi^*(E) = \tilde{E} + E_p$$

since p is a smooth point of E.

By Lemma 2.55 we have $\tilde{E} \cdot E_p = 1$ and $E_p^2 = -1$. Now we expand

$$(\pi^* E)^2 = (\tilde{E} + E_p)^2 = \tilde{E}^2 + 2\tilde{E} \cdot E_p + E_p^2 = \tilde{E}^2 + 1 \tag{2.60}$$

By Proposition 2.58 we have $(\pi^* E)^2 = E^2$ (one has to extend Lemma 2.57 to consider also the self-intersection of compact irreducible curves on a smooth surface as in Proposition 2.58). This implies the result. □

We introduce below the notion of dual graph associated to a modification of a smooth surface and also the dual graph associated to an embedded resolution of a germ of curve.

DEFINITION 2.61. Let $D_j, j \in J$ be the irreducible components of a curve D with normal crossings on a surface T. The **dual graph** of $D_j, j \in J$ the combinatorial graph $G(D)$ with vertex set J. The edges of $G(D)$ are in bijection with the singular points of D. If p is a singular point of D then there are unique elements $j, k \in J$ such that $p \in E_j \cap E_k$. Then $\mathcal{E}_p = \{j, k\}$ is the corresponding edge of $G(D)$. The **valency** of a vertex j of $G(D)$ is the number of edges incident to j (where loop edges of the form $\mathcal{E} = \{j, j\}$ count twice). A prime component D_j with j of valency ≥ 3 is called a **rupture divisor** of the dual graph.

DEFINITION 2.62. Let $\pi : (T, E) \to (S, O)$ be a modification of smooth surfaces which is an isomorphism outside the point O.

- The **dual graph** of π is the combinatorial graph $G(\pi^{-1}(O))$. We denote it by $G(\pi)$.
- If C is a germ of curve in S at O and if π is an embedded resolution of C, the **dual graph** of (π, C) is $G(\pi^{-1}(C))$. We denote it by $G(\pi, C)$.
- The **weighted dual graph** of $G(\pi)$ or of $G(\pi, C)$ has the vertices corresponding to the exceptional component E_j decorated with its self-intersection number E_j^2.

REMARK 2.63. The exceptional curve of π is of the form $\pi^{-1}(O) = \cup_{j \in J} E_j$ where the E_j are rational curves (see Remark 2.51). The dual graphs $G(\pi^{-1}(C))$ and $G(\pi^{-1})$ are **finite trees**, that is, they are connected graphs with a finite number of vertices and with no cycles. Usually, the components corresponding to components of the strict transform of C are denoted by an arrow on the graphical representation of $G(\pi, C)$.

One has the following characterization of the minimal embedded resolution of C.

LEMMA 2.64. [**DO95**, Proposition 3.10]. *Let $\pi : (T, E) \to (S, O)$ be an embedded resolution of curve (C, O). Then, π is isomorphic to the minimal embedded resolution of C if and only if for any exceptional component E_j, with self-intersection equal to -1, the valency of the corresponding vertex in the dual graph $G(\pi, C)$ is ≥ 3.*

Any compact irreducible smooth rational curve with self-intersection -1 on a smooth surface can be blown down (collapsed to a point). The fact that any -1 curve has to intersect at least three components of the total transform corresponds to the fact that, blowing it down, creates a non-normal crossing divisor.

DEFINITION 2.65. Let $\pi : (T, E) \to (S, O)$ be a modification and denote by E_i an irreducible component of E. A branch K_i in (S, O) is a **curvetta** at E_i if the strict transform of K_i by π is smooth and intersects the reduced exceptional divisor of $\pi^{-1}(O)$ at a smooth point which belongs to E_i.

Note that K_i being a curvetta at E_i means that π is an embedded resolution of K_i and that the strict transform of K_i intersects the exceptional prime divisor E_i.

DEFINITION 2.66. Let $\pi : (T, E) \to (S, O)$ be a modification. Let us denote by $\{E_i \mid i \in I_1\}$ the set of irreducible components of $\pi^{-1}(O)$ such that their corresponding vertex in $G(\pi)$ has valency 1. For every $i \in I_1$ choose a curvetta K_i at E_i according to Definition 2.65. Then, we say that the branches K_i, for $i \in I_1$, define a set of **maximal contact curves** of π.

REMARK 2.67. If π is a minimal resolution of a branch C, then the maximal contact curves of π are in bijection with the minimal set of generators of the semigroup of the branch C, namely, one maps K_i to the intersection number $(C \cdot K_i)$, for $i \in I_1$.

2.4. Semivaluations

Semivaluations provide an important tool for understanding the properties of singularities of plane curves.

We denote by $\mathcal{O}$ the formal local ring of S at O, by $\mathcal{F}$ its field of fractions and by $\mathcal{M}$ the maximal ideal of $\mathcal{O}$. We consider the interval $[0, \infty] = \mathbb{R}_+ \cup \{\infty\}$ with the usual order.

DEFINITION 2.68. A **semivaluation of** $\mathcal{O}$ is a function $\nu : \mathcal{O} \to [0, \infty]$ such that:

(1) $\nu(fg) = \nu(f) + \nu(g)$ for all $f, g \in \mathcal{O}$;
(2) $\nu(f + g) \geq \min(\nu(f), \nu(g))$ for all $f, g \in \mathcal{O}$;
(3) $\nu(\lambda) := \begin{cases} 0 & \text{if } \lambda \in \mathbb{C}^*, \\ \infty & \text{if } \lambda = 0. \end{cases}$

If in addition $\nu(\mathcal{M}) \subset \mathbb{R}_+^* \cup \{\infty\}$ we say that the semivaluation ν is **centered at** O. The semivaluation ν is a **valuation** if it takes the value ∞ only at 0.

REMARK 2.69. We do not consider the general notion of valuation, which takes values on ordered groups, as in [**ZS60**].

REMARK 2.70. Let $a \in \mathcal{O}$ be a unit, in other words, if (x, y) is a system of coordinates, $\mathcal{O} \simeq \mathbb{C}\{x, y\}$, $a(0, 0) \neq 0$. Let a^{-1} be the inverse of a, that is, $1 = aa^{-1}$. By the first property in Definition 2.68, we have $\nu(aa^{-1}) = \nu(a) + \nu(a^{-1})$, and by the third property, $\nu(1) = 0$. Thus, $\nu(a) = -\nu(a^{-1})$, but since by definition semivaluations are non-negative, this implies that $\nu(a) = \nu(a^{-1}) = 0$ for any unit $a \in \mathcal{O}^*$.

DEFINITION 2.71. Let us fix a branch C defined by $f = 0$. If $h \in \mathcal{O}$, $h \neq 0$ we set

$$\nu_C(h) = a \text{ if } h = f^a g, \text{ with } \gcd(f, g) = 1.$$

This makes sense since $\mathcal{O}$ is a unique factorization domain (UFD for short). We set $\nu_C(0) = \infty$. The function ν_C is called the **vanishing order valuation along the branch** C.

Notice that $\nu_C(h)$ is the coefficient of C in the divisor of h. In particular, ν_C is a valuation, although not centered at O.

DEFINITION 2.72. The **intersection semivaluation defined by a branch** C is defined by:

$$I_C(h) = (C, C_h)_O, \text{ for } h \in \mathcal{O},$$

where C_h is the germ defined by h at O.

Note that I_C is a semi-valuation since, if C is a component of C_h then $I_C(h) = \infty$.

EXAMPLE 2.73. The **multiplicity valuation** $\nu_O : \mathcal{O} \to \mathbb{R}$. If $h \in \mathcal{O}$ and $h \neq 0$ then

$$\nu_O(h) = \max\{n \in \mathbb{N} \mid h \in \mathcal{M}^n\}.$$

In terms of local coordinates (x, y) one expands h as a series in $\mathbb{C}[[x, y]]$. Then, $\nu_O(h)$ is the order of this series. Let us consider the blowing up $\pi_O : (\mathrm{Bl}_O S, E_O) \to (S, O)$. Let us consider a chart of the blow-up

$$\begin{array}{rcl} x & = & x_1 y_1, \\ y & = & x_1. \end{array}$$

Then E_O is defined by $x_1 = 0$ on this chart. For any $h \in \mathcal{O}$ one can evaluate the order of vanishing of h along E_O by

$$\nu_{E_O}(h) := \mathrm{ord}_u(h \circ \pi)$$

By formula (2.46) we see that the value $\nu_{E_O}(h)$ is equal to the multiplicity $m_O(h)$ of h at O. For this reason we also use the notation ν_{E_O} for the valuation ν_O. This is the basic case of the fundamental notion of divisorial valuation explained below.

DEFINITION 2.74. Let $\pi : T \to S$ be a modification of smooth surfaces over O. Let E be an irreducible component of the exceptional divisor $\pi^{-1}(O)$. Then, for any $h \in \mathcal{O}$ we set $\nu_E(h)$ equal to the order of vanishing along E of $h \circ \pi$. The function ν_E is the **divisorial valuation**, ν_E, associated to E. If D is an effective divisor at S, defined by $h_D = 0$ for some $h_D \in \mathcal{O}$, then we denote

$$\nu_E(D) := \nu_E(h_D).$$

REMARK 2.75.

(1) We know that the modification π in the previous definition is a sequence of blowing up points. In particular, there is a modification, $\pi' : T' \to S$, and an infinitely near point, $P \in (\pi')^{-1}(O)$, such that $E = E_P$ is the exceptional divisor of the blowing up of P in T'. It follows that the value of the valuation ν_E at the function h is equal to the multiplicity of $h \circ \pi'$ at the point P.

(2) With the notation of Definition 2.74, if $\pi' : T' \to T$ is a modification of smooth surfaces and we denote by D' the strict transform of the irreducible divisor D on T' then we have $\nu_D = \nu_{D'}$.

In most general terms one can associate a divisorial valuation to any prime (irreducible) divisor on a normal variety.

REMARK 2.76. Let X be a normal algebraic variety, $D \subset X$ an irreducible divisor and $\mathfrak{p}_D$ its associated prime ideal sheaf. Then $((\mathcal{O}_X)_{\mathfrak{p}_D}, \mathfrak{m} = \mathfrak{p}_D(\mathcal{O}_X)_{\mathfrak{p}_D})$ is a discrete valuation ring with valuation ν_D. It follows that $\nu_D : \mathcal{O}_X \to [0, \infty]$, where $\nu_D(h) = max\{n \in \mathbb{N} \mid h \in \mathfrak{m}^n\}$, is a valuation in the sense of Definition 2.68 (see [**Eis95**, section 11.2] for details).

EXAMPLE 2.77. Let us fix a vector $w = (a, b) \in \mathbb{R}^2_{>0}$. Let us fix local coordinates (x, y) at O, which provide an isomorphism of $\mathcal{O}$ with the power series ring $\mathbb{C}[[x, y]]$. In term of these choices, we can define a valuation $\nu_w : \mathcal{O} \to [0, \infty]$. It is enough to define its value on a power series $h = \sum_{i,j} c_{i,j} x^i y^j \in \mathbb{C}[[x, y]]$. We set

$$\nu_w(h) = \min\{\langle (a, b), (i, j) \rangle \mid c_{i,j} \neq 0\}. \tag{2.78}$$

In this formula, we consider w as a linear form on the real vector space $\mathbb{R}^2$ spanned by the exponents of elements in $\mathbb{C}[[x, y]]$, and the pairing $\langle (a, b), (i, j) \rangle := ai + bj$ denotes the value of w on (i, j). We say that the valuation ν_w is **monomial** in terms of x and y. We will see below that if a, b are to coprime integers then ν_w is actually a divisorial valuation (see Remark 4.19).

The following result allows us to determine the values of a divisorial valuation ν_E at a branch C_i of C in terms of the intersection multiplicities with a suitable curvetta at E (see Definition 2.65).

PROPOSITION 2.79. *Let ν_E be a divisorial valuation. Assume that C is a germ of curve at (S, O) and let $\pi : T \to S$ be an embedded resolution of C where the divisor E appears. Let K be a branch at (S, O) such that:*

(1) π is an embedded resolution of $C \cup K$,
(2) The strict transform of K intersects the exceptional prime divisor E.

Then, for any irreducible component C_i of C:

$$\nu_E(C_i) = (C_i, K)_O. \tag{2.80}$$

PROOF. We consider K as an effective divisor and notice that it is enough to prove it when C is a branch. The pullback of K and C by π is of the form:

$$\pi^*(C) = \widetilde{C} + \sum \nu_{E_i}(C)E_i, \quad \pi^*(K) = \widetilde{K} + \sum \nu_{E_i}(K)E_i,$$

where the E_i are the prime exceptional components in $\pi^{-1}(O)$, and $\widetilde{C}$ and $\widetilde{K}$ denote the strict transforms of C and K. By (2.) in Lemma 2.57 we can compute the intersection number $(K, C)_O$ by considering the pullbacks of these divisors on T:

$$(K, C)_O = \pi^*(K) \cdot \pi^*(C) = (\widetilde{K} + \sum_i \nu_{E_i}(K)E_i) \cdot (\widetilde{C} + \sum_i \nu_{E_i}(C)E_i).$$

By (3.) in Lemma 2.57 this simplifies as

$$(K, C)_O = (\widetilde{K} \cdot \widetilde{C}) + (\widetilde{K} \cdot (\sum_i \nu_{E_i}(C)E_i))$$

Now the hypothesis that π is an embedded resolution of $C \cup K$ implies that the strict transforms $\widetilde{K}$ and $\widetilde{C}$ do not intersect at T, that is, $(\widetilde{K} \cdot \widetilde{C}) = 0$. Thus

$$(K, C)_O = (\widetilde{K} \cdot (\sum_i \nu_{E_i}(C)E_i)) = \sum_i \nu_{E_i}(C)(\widetilde{K} \cdot E_i).$$

Our hypothesis on π imply that the strict transform $\widetilde{K}$ intersects only the component E of $\pi^{-1}(O)$ and this intersection is transversal. This means that

$$(\widetilde{K} \cdot E_i) = \begin{cases} 0 & \text{if } E_i \neq E, \\ 1 & \text{if } E_i = E. \end{cases}$$

Then, it follows that $(K, C)_O = \nu_E(C)$. □

2.5. Eggers-Wall Trees

Eggers-Wall trees represent a way of encoding the topological equisingularity class of a reduced plane curve singularity.

DEFINITION 2.81. Let C be a branch on S, different from the smooth germ $R = V(x)$. The **Eggers-Wall tree** $\Theta_R(C)$ of the branch C relative to R is a compact oriented segment endowed with:

- an increasing homeomorphism $\mathbf{e}_{R,C} : \Theta_R(C) \to [0, \infty]$, called the **exponent function**;
- **marked points**, which are by definition the points whose values by the exponent function are the characteristic exponents of C relative to R, as well as the smallest exponent end of $\Theta_R(C)$, labeled by R, and the greatest point, labeled by C.

- an **index function** $\mathbf{i}_{R,C} : \Theta_R(C) \to \mathbb{N}$, which associates to each point $P \in \Theta_R(C)$ the index of $(\mathbb{Z}, +)$ in the subgroup of $(\mathbb{Q}, +)$ generated by 1 and the characteristic exponents of C which are strictly smaller than $\mathbf{e}_{R,C}(P)$.

REMARK 2.82. If the characteristic exponents of C with respect to R are $\alpha_1, \dots, \alpha_g$ we denote also $\alpha_0 = 0$ and $\alpha_{g+1} = \infty$. Then, the **marked points** on $\Theta_R(C)$ are the points $A_j = \mathbf{e}_{R,C}^{-1}(\alpha_j)$ for $0 \leq j \leq g+1$. The index function $\mathbf{i}_{R,C}$ takes the constant value 1 on $[A_0 = R, A_1]$ and the value $n_1 \cdots n_j$, on each segment $(A_j, A_{j+1}]$ for any $1 \leq j \leq g$ (where the integers n_i are those defined in (2.12)). Using this description we see that $\mathbf{i}_{R,C}(P)$ can be seen as the smallest common denominator of the exponents of a Newton-Puiseux root of f_C which are strictly less than $\mathbf{e}_{R,C}(P)$.

EXAMPLE 2.83. Figure 1 shows the Eggers-Wall tree of a plane branch C with g characteristic exponents.

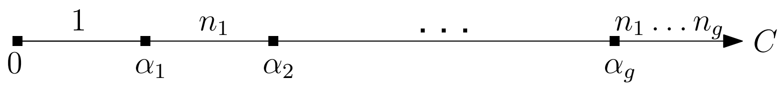

FIGURE 1. The Eggers-Wall tree of a plane branch. On the line are represented the values of the index function, together with the values of the exponent function at the marked points.

Now assume that C is a reduced germ of plane curve with branches $C_1, \dots, C_r$, different from R. Let us define the set $\Theta_R(C)$ by following gluing procedure on the disjoint union $\bigsqcup \Theta_R(C_i)$. For each pair of different branches C_i and C_j we identify the segments

$$[\mathbf{e}_{R,C_i}^{-1}(0), \mathbf{e}_{R,C_i}^{-1}(\kappa_R(C_i, C_j))] \subset \Theta_R(C_i) \text{ and } [\mathbf{e}_{R,C_j}^{-1}(0), \mathbf{e}_{R,C_j}^{-1}(\kappa_R(C_i, C_j))] \subset \Theta_R(C_j),$$

by the unique exponent-preserving homeomorphism between them. This construction is well defined if $r \geq 3$ since by Remark 2.18, for all $1 \leq i < j < l \leq r$, one has

$$\kappa_R(C_i, C_l) \geq \min\{\kappa_R(C_i, C_j), \kappa_R(C_j, C_l)\}.$$

The various exponent functions $\mathbf{e}_{C_i,R}$ associated with the components C_i of C glue up and define an increasing surjection:

$$\mathbf{e}_R : \Theta_R(C) \to [0, \infty],$$

called the **exponent function**. Similarly, the various index functions $\mathbf{i}_{C_i,R}$ associated with the components C_i of C glue up and define map

$$\mathbf{i}_R : \Theta_R(C) \to \mathbb{N}$$

called the **index function**.

DEFINITION 2.84. Let C be a germ of reduced plane curve, which does not contain a fixed smooth branch R as a component. The **Eggers-Wall tree** $\Theta_R(C)$ of the reduced plane curve C with respect to R is the tree defined by the above construction, endowed with the exponent function and the index function. The other ends of this tree are labeled by the branches of C.

Let us introduce some vocabulary and notation to speak about the tree $\Theta_R(C)$.

NOTATION 2.85. The topology of the tree $\Theta_R(C)$ determines certain special points. The **valency** of a point P in the tree is the numbers of *germs of segments* of the form $[P, Q)$ (as in Definition 2.61). A **ramification point** is a point of valency > 2. The **ends** are the points of valency 1. One is labeled by R, the **root**, while the others are labeled by the branches of C and are called the **leaves** of C. The **interior points** are those which are not ends.

Other points of the tree are determined by the index function. The set of **marked points** of the tree, $\Upsilon \subset \Theta_R(C)$, is the set of points of discontinuity of the index function together with the ramification points and the ends of the tree. That is, the marked points are the images of the marked points of the individual trees and the images of $\mathbf{e}^{-1}_{R,C_i}(\kappa_R(C_i, C_j)) \subset \Theta_R(C_i)$ for $1 \leq i < j \leq r$, by the projection onto $\Theta_R(C)$ defined by the identifications defined above. For each $1 \leq i \leq r$ we have an embedding of the Eggers-Wall tree $\Theta_R(C_i)$ in $\Theta_R(C)$. We denote by Υ° to the set of interior marked points, i.e., the union of the points of discontinuity of the index function and the ramification points of the tree.

We consider the **partial order** on the tree $\Theta_R(C)$ defined by $P \leq_R Q$ if $[R, P] \subset [R, Q]$, and we will often denote it by $P \leq Q$, whenever R is clear from the context.

If $O, P, Q \in \Theta_R(C)$ the center of the **tripod** $\langle O, P, Q\rangle$ defined by these points, is the intersection of the segments joining the points pairwise, i.e., $\langle O, P, Q\rangle = [P, Q] \cap [O, Q] \cap [P, O]$. Notice that this definition makes sense even if some of the points O, P, Q coincide.

If $d \geq 1$ is a value of the index function, then **level** d of the index function,

$$\Theta^d_R(C) := \mathbf{i}^{-1}_R(d) \subset \Theta_R(C),$$

may be non-connected. Notice that the level $\Theta^1_R(C)$ is a connected closed subtree of $\Theta_R(C)$ containing the root.

A point $P \in \Theta_R(C)$ is **rational** (resp. **irrational**) if $\mathbf{e}_R(P) \in \mathbb{Q}_{>0}$ (resp. if $\mathbf{e}_R(P) \in \mathbb{R}_{>0} \setminus \mathbb{Q}$). A point which is either rational or irrational is called interior point of the tree.

The following properties of the tripod are elementary.

LEMMA 2.86. *Let $P, P', Q, Q' \in \Theta_R(C)$.*

$$\text{If } P \leq_R P', \text{ then we get } \langle R, Q, P\rangle \leq_R \langle R, Q, P'\rangle, \tag{2.87}$$

$$\text{If } P \leq_R P' \text{ and } P \notin [R, Q], \text{ then we get } \langle R, Q, P\rangle = \langle R, Q, P'\rangle. \tag{2.88}$$

$$\text{If } \langle R, Q, P\rangle \leq_R Q', \text{ then we get } \langle R, Q, P\rangle \leq_R \langle R, Q, Q'\rangle. \tag{2.89}$$

We exemplify the previous properties in Figure 2. The left drawing in Figure 2 exhibits the second property. Notice that, if $P \in [R, Q]$, $\langle R, P, Q\rangle = P$, and if $P \leq_R P'$, then $\langle R, P', Q\rangle = \langle P, P', Q\rangle$. The third property is a consequence of this reasoning, and it is shown in the right drawing of Figure 2.

NOTATION 2.90. Let P be a rational point of $\Theta_R(C)$. Let us take any branch C_i of C such that $P \in [R, C_i]$ (where we identify C_i with its corresponding point of the tree). We denote by $\mathbf{i}^+_R(P)$ the index of $(\mathbb{Z}, +)$ in the subgroup of $(\mathbb{Q}, +)$ generated by 1, the characteristic exponents of a branch C_i of C which are less than $\mathbf{e}_R(P)$ and $\mathbf{e}_R(P)$ itself. We call the integer $\mathbf{i}^+_R(P)$ the **extended index** of P.

If C_j is another branch of C such that $P \in [R, C_j]$, then it follows that $P \leq \langle R, C_i, C_j\rangle$. This implies that the extended index of P does not depend on the choice of a branch of C_i of C such that $P \in [R, C_i]$.

The notion of attaching map allows us to study the Eggers-Wall tree when we add another branch:

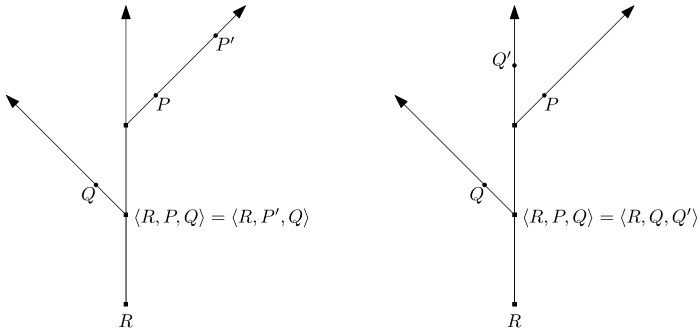

FIGURE 2. The Eggers-Wall tree of a plane branch. The image on the left represents points P, P', Q as in 2.88. The image on the right represents also the point Q' as in 2.89.

DEFINITION 2.91. Let C be a reduced plane curve singularity on a smooth surface (S, O) and let R be a smooth reference branch, different from the components $C_1, \dots, C_r$ of C. If D is other branch we define the **attaching map**:

$$\pi^D_{R,C} : \Theta_R(D) \to \Theta_R(C), \quad \pi^D_{R,C}(P) = \max_{\leq_R}\{\langle R, P, C_i\rangle \mid 1 \leq i \leq r\}, \tag{2.92}$$

where the points $\langle R, P, C_i\rangle$ are viewed as points of the tree $\Theta_R(C + D)$. In particular, the point $\pi^D_{R,C}(D)$ is the **attaching point** of the branch D to the Eggers-Wall tree $\Theta_R(C)$.

If C_i is a branch of C, then the attaching point $\pi^D_{R,C_i}(D)$ is the unique point of ramification of the tree $\Theta_R(C_i \cup D)$, which has exponent $\kappa_R(C_i, D)$. Furthermore, the points $\langle R, P, C_i\rangle$ belong to $\Theta_R(D)$, thus all of them are comparable with respect to $\leq_R$ and so the maximum exists. By definition, $[R, \pi^D_{R,C}(D)] \subset [R, C_i]$ for some C_i, hence if $P \in [R, \pi_{R,C}(D)]$, its image under the attaching map is itself, that is, $P = \pi^D_{R,C}(P)$.

EXAMPLE 2.93. Consider $f = (y^2 - x^3)^2 - 4x^5y - x^7$ and the associated curve $C = \{f = 0\}$. Applying Newton-Puiseux one checks that its roots are

$$\zeta = (\eta_4 x)^{\frac{3}{2}} + (\eta_4 x)^{\frac{7}{4}}$$

where, η_4 varies among the 4-th roots of 1, so the characteristic exponents are $\varepsilon(C) = \{\frac{3}{2}, \frac{7}{4}\}$. Notice that $L_0 = \{y = 0\}$ is a 0-th semi-root, and $L_1 = V(y^2 - x^3)$ is a 1-th semi-root for C. Figure 3 shows the Eggers-Wall tree $\Theta_R(C \cup L_0 \cup L_1)$.

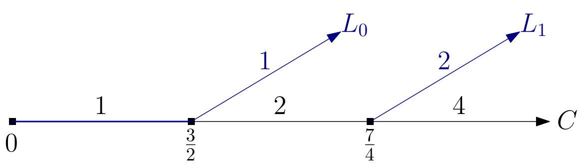

FIGURE 3. $\Theta_R(C \cup L_0 \cup L_1)$

The center of the tripod $P = \langle R, L_0, C\rangle$ is the unique point verifying $\mathbf{e}_R(P) = \dfrac{3}{2}$.

EXAMPLE 2.94. Consider the reduced curve $C = \cup_{i=1}^{5} C_i$, where the branch C_i is defined by the Newton Puiseux series ζ_i:

$$\zeta_1 = x^{\frac{3}{2}} + x^{\frac{11}{4}} + x^{\frac{31}{8}},$$
$$\zeta_2 = x^{\frac{3}{2}} + x^{\frac{11}{4}} + 2x^{\frac{31}{8}} + x^{\frac{49}{12}},$$
$$\zeta_3 = x^{\frac{3}{2}} + x^{\frac{11}{4}} + x^{\frac{15}{4}},$$
$$\zeta_4 = x^{\frac{3}{2}} + x^{\frac{5}{2}},$$
$$\zeta_5 = x^{\frac{3}{2}} + x^{\frac{5}{2}} + x^{\frac{17}{6}}.$$

Figure 4 shows its Eggers-Wall tree.

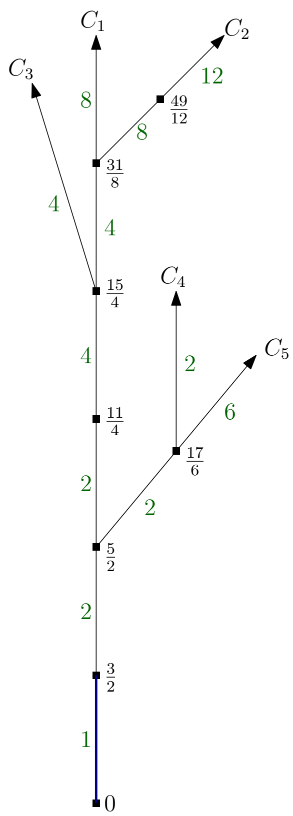

FIGURE 4. Eggers-Wall tree of C in Example 2.94

Note that $\frac{5}{2}, \frac{15}{4}$ are not characteristic exponents, but just orders of contact. On the other hand $\frac{17}{6}$ is a characteristic exponent of C_5 but not of C_4, but $\frac{31}{8}$ is a common characteristic exponent of C_1 and C_2.

EXAMPLE 2.95. Let us consider the example 2.20 where we consider the branch $C = V(f)$, where $f = y^6 - x^{13}$. Denote by $R = V(x)$ and $R' = V(z)$ where $z = y - x^2$. In Figure 5, we compare the trees $\Theta_R(C \cup R')$ and $\Theta_{R'}(C \cup R)$. The exponent of the ramification point $\langle R, R', C \rangle$ is a characteristic exponent of C only when seen on the tree $\Theta_{R'}(C \cup R)$. This characteristic exponent reflects the fact that the smooth germ R is tangent to C.

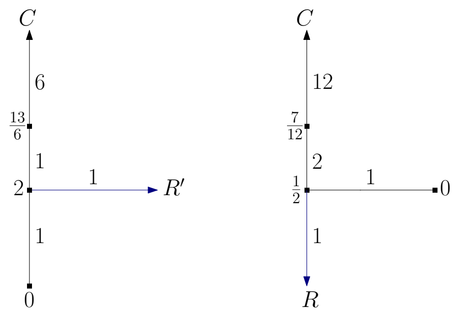

FIGURE 5. $\Theta_R(C \cup R')$ and $\Theta_{R'}(C \cup R)$

2.5.1. The contact function. The notion of contact of two branches with respect to R is related with the notion of intersection multiplicity (see [**Wal04**, Theorem 4.1.6]):

THEOREM 2.96. *Let R, C, D be three distinct branches on a smooth surface (S, O), with R smooth. Let us denote by $\alpha_1, \dots, \alpha_g$ the characteristic exponents of C with respect to R with the associated integers e_j defined in (2.12). If $\alpha_k \leq \kappa_R(C, D) \leq \alpha_{k+1}$ then:*

$$(C, D)_O = (D, R)_O \left((e_0 - e_1)\alpha_1 + \ldots + (e_{k-1} - e_k)\alpha_k + e_k \kappa_R(C, D)\right). \tag{2.97}$$

Notice that by definition $(C, R)_O = e_0 = b_0$. Then, dividing by $(C, R)_O \cdot (D, R)_O$ on both sides of (2.97) provides the equality:

$$\frac{(C, D)_O}{(D, R)_O\,(C, R)_O} = \frac{(e_0 - e_1)}{e_0}\alpha_1 + \ldots + \frac{(e_{k-1} - e_k)}{e_0}\alpha_k + \frac{e_k}{e_0}\kappa_R(C, D).$$

Taking into account that

$$\frac{e_j}{e_0} = \frac{1}{n_1 \cdots n_j},$$

by definition (2.12), this formula can be rewritten in the form:

$$\frac{(C,D)_O}{(D,R)_O\,(C,R)_O} = \alpha_1 + \frac{1}{n_1}(\alpha_2 - \alpha_1) + \cdots + \frac{1}{n_1 \ldots n_{k-1}}(\alpha_k - \alpha_{k-1}) + \frac{1}{n_1 \ldots n_k}(\kappa_R(C, D) - \alpha_k). \tag{2.98}$$

The advantage of this reformulation is that we can use the right-hand side of formula (2.98) to define a function $\mathbf{c}_R$ on the Eggers Wall tree $\Theta_R(C)$ of the branch C. Recall that we denote by A_j, the marked points of the tree $\Theta_R(C)$ where $\mathbf{e}_R(A_j) = \alpha_j$ for $j = 1, \dots, g$, A_0 is the point labeled by R and A_{g+1} denotes the point labeled by C (see Remark 2.82).

DEFINITION 2.99. Let C be a branch and R a smooth branch on a smooth surface (S, O). Let P be a point on $\Theta_R(C)$. The **contact function** $\mathbf{c}_{R,C} : \Theta_R(C) \to [0, \infty]$ is defined for a point $P \in \Theta_R(C)$ such that $A_k \leq P \leq A_{k+1}$ by:

$$\mathbf{c}_{R,C}(P) := \frac{\mathbf{e}_R(A_1) - \mathbf{e}_R(A_0)}{\mathbf{i}_R(A_1)} + \cdots + \frac{\mathbf{e}_R(A_k) - \mathbf{e}_R(A_{k-1})}{\mathbf{i}_R(A_k)} \frac{\mathbf{e}_R(P) - \mathbf{e}_R(A_k)}{\mathbf{i}_R(P)} \tag{2.100}$$

If C is a reduced plane curve singularity with branches $C_1, \dots, C_r$, the contact functions $\mathbf{c}_{R,C_j}$ are compatible with the gluing procedure in the definition of $\Theta_R(C)$ (see Definition 2.84), that is, the **contact function**:

$$\mathbf{c}_R : \Theta_R(C) \to [0, \infty],$$

given by $\mathbf{c}_R(P) := \mathbf{c}_{R,C_j}(P)$ if $P \in \Theta_R(C_j)$, is well defined.

Notice that the contact function $\mathbf{c}_R$ is a continuous strictly increasing surjection. By Theorem 2.96 the contact function allows us to recover the intersection multiplicities $(C_i, C_j)_O$ of the components of C:

Corollary 2.101. *Let C be a reduced plane curve singularity on a smooth surface (S, O). Let R be a smooth branch different from the components of C. If C_i and C_j are two branches of C, then:*

$$(C_i, C_j)_0 = \mathbf{i}_R(C_i)\, \mathbf{i}_R(C_j)\, \mathbf{c}_R(\langle R, C_i, C_j\rangle),$$

where $\langle R, C_i, C_j\rangle$ denotes the center of the tripod determined by these three branches on $\Theta_R(C)$.

Proof. We consider formula (2.98) applied to $C = C_i$ and $D = C_j$. Notice that by definition $\mathbf{e}_R(\langle R, C_i, C_j\rangle) = \kappa_R(C_i, C_j)$. Then, we compare (2.98) with (2.100) and we get that

$$\mathbf{c}_R(\langle R, C_i, C_j\rangle) = \frac{(C_i, C_j)_O}{(R, C_i)_O (R, C_j)_O}.$$

Finally, notice that for any branch C_l of C one has $\mathbf{i}_R(C_l) = (R, C_l)_O$. □

Example 2.102. Coming back to Example 2.93, Figure 3 shows the values of the contact function at the marked points. For instance, using Corollary 2.101, one has that the intersection multiplicity at the origin of L_1 and C:

$$(C, L_1)_0 = \mathbf{i}_R(L_1)\mathbf{i}_R(C)\mathbf{c}_R(\langle R, L_1, C\rangle) = 2 \cdot 4 \cdot \frac{13}{8} = 13.$$

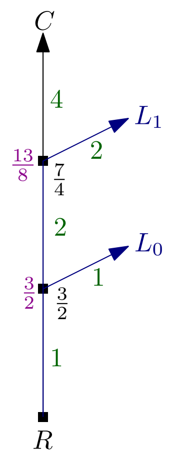

Figure 6. The Eggers-Wall tree of a curve with the values of the contact function at the marked points

2.5.2. The Eggers Wall tree and the Newton polygons. Following [**Tei95**], we say that a Newton polygon is **elementary** if it has only one compact edge with vertices at the coordinate axes.

It is uniquely determined by its length m and its height n as indicated in Figure 7, thus it is denoted by $\left\{\frac{m}{n}\right\}$.

The **inclination** of the Newton polygon $\mathcal{N}(f)$ is the rational number $\frac{m}{n}$. Notice that $\frac{m}{n}$ may not be an irreducible fraction. Proposition 2.106 below shows that it makes sense to consider the extremal cases $\left\{\frac{1}{\infty}\right\} := \mathcal{N}(x)$ and $\left\{\frac{\infty}{1}\right\} := \mathcal{N}(y)$, which have only one vertex.

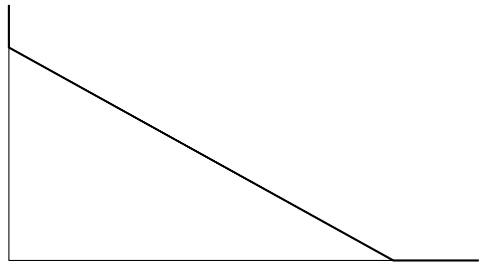

FIGURE 7. The Newton polygon $\{\frac{m}{n}\}$.

DEFINITION 2.103. If $\mathcal{N}$ and $\mathcal{N}'$ are polyhedra, their **Minkowski sum** is

$$\mathcal{N} + \mathcal{N}' = \{a + b \mid a \in \mathcal{N},\ b \in \mathcal{N}'\}.$$

PROPOSITION 2.104. *If $f = f_1^{a_1} \dots f_r^{a_r}$ with $f_i \in \mathbb{C}\{x, y\}$ irreducible and if the Newton polygons of each f_i is $\left\{\frac{m_i}{n_i}\right\}$ then:*

$$\mathcal{N}(f) = \sum_{i=1}^{r} a_i \left\{\frac{m_i}{n_i}\right\}, \tag{2.105}$$

where the sum is a Minkowski sum.

Check [**Tei95**, page 873] for further details.

The Proposition 2.108, which relates the Newton polygon and the Eggers-Wall tree, is a generalization of the following well-known property:

PROPOSITION 2.106. *The Newton polygon of an irreducible series is elementary of the form $\{\frac{m}{n}\}$, where $n = \mathrm{ord}_y(f(0, y))$ and $m = \mathrm{ord}_x(f(x, 0))$.*

PROPOSITION 2.107. *Let (R, L) be a cross at a smooth surface (S, O). Let $C = \sum_{i=1}^{r} C_i$ be a reduced plane curve singularity, such that all its branches C_i are different from R. Then, the Newton polygon of C with respect to (R, L) is the Minkowski sum:*

$$\mathcal{N}_{R,L}(C) = \sum_{i=1}^{r} \mathbf{i}_R(C_i) \left\{\frac{\mathbf{e}_R(\langle R, L, C_i\rangle)}{1}\right\}. \tag{2.108}$$

PROOF. By Proposition 2.104 it is enough to prove the result when C is a branch. If $C = L$ then (2.108) holds trivially by the definitions. Assume that the branch C is different from L. Choose a pair of representatives (x, y) of the cross (R, L). They define a system of coordinates for S at O such that $V(x) = R$ and $V(y) = L$. Since L is given by $y = 0$, by Newton-Puiseux Theorem 2.7 the parametrization of C in these coordinates is given by

$$\begin{cases} x &= t^n \\ y &= \sum\limits_{j \geq m} b_j t^j, \end{cases} \quad \text{with } b_m \neq 0. \tag{2.109}$$

with $(R, C)_O = n$ and $L = (L, C)_O = m$. If $e = \gcd(n, m)$ then the Newton polygon of C is

$$\mathcal{N}_{R,L}(C) = \left\{\frac{m}{n}\right\} = e \cdot \left\{\frac{\kappa_R(C, L)}{1}\right\},$$

where $\kappa_R(C, L) = \frac{m}{n}$ is the order of contact of C and L with respect to R. Then, the equality (2.108) holds in this case, since by definition:

$$\kappa_R(C, L) = \mathbf{e}_R(\langle R, L, C\rangle) \text{ and } e = \mathbf{i}_R(C).$$

□

REMARK 2.110. With notation as in Proposition 2.107, let $C = \sum_{i=1}^{r} a_i C_i$ be a not necessarily reduced plane curve. Then, the Newton polygon of C with respect to (R, L) is the Minkowski sum

$$\mathcal{N}_{R,L}(C) = \sum_{i=1}^{r} a_i \mathbf{i}_R(C_i) \left\{ \frac{\mathbf{e}_R(\langle R, L, C_i \rangle)}{1} \right\} \tag{2.111}$$

with the multiplicities a_i of the components of C.

The notion of attaching point given in Definition 2.91 is very useful to describe the Newton polygons from the Eggers Wall tree:

PROPOSITION 2.112. *Let C be a reduced plane curve singularity on a smooth surface (S, O) and let R be a smooth reference branch, different from the components of C. Let L be a smooth branch transversal to R.*

(1) If the exponent of the attaching point $\pi_{R,C}(L)$ belongs to $\mathbb{N}^*$, then $\pi_{R,C}(L)$ is an interior point of the index 1 subtree $\Theta_R^1(C)$.
(2) Otherwise, when the exponent of $\pi_{R,C}(L)$ belongs to $(\mathbb{Q}_+^* \cup \{\infty\}) \setminus \mathbb{N}$, then $\pi_{R,C}(L)$ is an end of the index 1 subtree $\Theta_R^1(C)$.

PROOF. Since L is transversal to R, the segment $[R, L]$ is contained in the index 1 subtree $\Theta_R^1(C \cup L)$. Since the index function of $\Theta_R(C \cup L)$ extends the index function of $\Theta_R(C)$, the index of the attaching point $\pi_{R,C}(L)$ must be equal to one.

By definition of attaching point there is a branch C_i of C such that $\pi_{R,C}(L) = \langle R, L, C_i \rangle$. By definition the exponent of this point is the order of contact $\kappa_R(C_i, L)$.

- If $\kappa_R(C_i, L)$ is an integer, we are in the first case.

- Otherwise, $\kappa_R(C_i, L)$ is a rational non-integer, i.e., it is the first characteristic exponent of the branch C_i or $\kappa_R(C_i, L) = \infty$, that is, $L = C_i$. In both cases, $\pi_{R,C}(L)$ is an end of the index 1 subtree $\Theta_R^1(C)$. □

The following remark explains the relation with the semi-roots.

REMARK 2.113. If L is a smooth branch, then L is a 0-th semi-root of any branch C_j of C such that the exponent of $\langle R, C_j, L \rangle$ does not belong to $\mathbb{N}$. If $\mathbf{e}_R(\langle R, C_j, L \rangle) = \infty$ then $L = C_j$.

The previous property motivates the following definition:

DEFINITION 2.114. Let C be a reduced plane curve singularity with branches C_j for $1 \leq j \leq r$ and let R be a smooth branch which is not a component of C. A smooth branch L has **maximal contact with C relative to R** if L is transversal to R and the the point $\pi_{R,C}(L)$ is an end of the level $\Theta_R^1(C)$.

REMARK 2.115. Let C be a reduced plane curve and L a smooth curve with maximal contact with C with respect to R. Then for any component C_i of C such that $\kappa_R(C_i, L) \in \mathbb{Q} \setminus \mathbb{N}$ and for any other smooth curve L' transversal to R, the orders of contact satisfy $\kappa_R(C_i, L') \leq \kappa_R(C_i, L)$ since by Proposition 2.112 the attaching point of L is an end of the index level 1 subtree $\Theta_R^1(C_i)$.

As a consequence of Proposition 2.107, when we consider coordinates (x, y) where $R = V(x)$ and $L = V(y)$ has maximal contact with C relative to R, then the Newton polygon $\mathcal{N}_{R,L}(C)$ is determined by the marked points of a segment in the Eggers-Wall tree of C:

COROLLARY 2.116. *Let C be a reduced plane curve singularity, such that all its branches are different from a fixed smooth branch R. If L has maximal contact with C relative to R then* *the inclinations of the compact faces of the Newton polygon* $\mathcal{N}_{R,L}(C)$ *are the exponents of the marked points of the segment* $[R, \pi_{R,C}(L)]$ *in* $\Theta_R(C)$.

PROOF. This is a consequence of Proposition 2.107 and the fact that L has maximal contact implies that the sets of marked points of $\Theta_R(C \cup L)$ and $\Theta_R(C)$ are equal. □

2.6. Expansions

Abhyankar and Moh applied and developed the expansion of functions using *approximate roots* in the study of algebraic curves (see [**Abh77**, **Abh89**, **AM75**, **AM73**]). Following [**Pop03**] and [**Pop04**, Section 7] we introduce some expansion results:

LEMMA 2.117. *Let* A *be an integral domain and let* n_k *be integers with* $n_0 = 1$ *and* $n_k > 1$ *for* $1 \leq k \leq g$. *Let* $z_k \in A[y]$ *be monic polynomials of degree* $n_0 \dots n_k$ *for* $1 \leq k \leq g$. *Then, any polynomial* $h \in A[y]$ *has a unique finite expansion:*

$$h = \sum c_{i_0 \dots i_g} \cdot z_0^{i_0} \dots z_g^{i_g}, \tag{2.118}$$

with $c_{i_0 \dots i_g} \in A$, *where the* g*-tuples* $(i_0, \dots, i_g)$ *verify*

$$0 \leq i_k < n_{k+1} \text{ for } k \in \{0, \dots, g-1\} \tag{2.119}$$

and $i_g \leq \lfloor \frac{\deg_y(h)}{\deg_y(z_g)} \rfloor$.

PROOF. Let $h \in A[y]$ be a polynomial. First, one performs Euclidean division of h by z_g and of the successive quotients by z_g, until one obtains a quotient with degree in y less than $e_0 = n_0 \dots n_g$. Thus, one gets the z_g-adic expansion of h, which has the form

$$h = \sum c_{i_g}(x, y) \cdot z_g^{i_g},$$

with $i_g \leq \lfloor \frac{\deg_y(h)}{\deg_y(z_g)} \rfloor$ and $c_{i_g} \in A[y]$ of degree $< n_0 \dots n_g$. Then one iterates this process by making at each step the z_{k-1}-adic expansions of the coefficients in the previous expansion. The coefficients obtained at the last step are obtained as remainders of division by a monic polynomial of degree one in $A[y]$, thus these coefficients belong to A.

The unicity of the expansion (2.118) follows from the unicity of Euclidean division, since the degrees in y of terms $c_{i_0 \dots i_g} z_0^{i_0} \dots z_g^{i_g}$ are pairwise distinct, which is a consequence of the following result. □

LEMMA 2.120. *Let us consider a sequence of integers* $n_i > 1$, *for* $1 \leq i \leq g$. *Let* $I = (i_0, \dots, i_g)$, $I' = (i'_0, \dots, i'_g)$ *be two* g*-tuples verifying*

$$0 \leq i_k, i'_k < n_{k+1} \text{ for } k \in \{0, \dots, g-1\}.$$

Then, if

$$i_0 + i_1 n_1 + i_2 n_1 n_2 + \dots + i_g n_1 \dots n_g = i'_0 + i'_1 n_1 + i'_2 n_1 n_2 + \dots + i'_g n_1 \dots n_g$$

we must have $I = I'$.

PROOF. Let us reason by contradiction, assume that $I \neq I'$. Then, there exists $l \in \{0, \dots, g\}$ maximal such that $i_l \neq'_l$, i.e.: $i_k = i'_k$ for all $k > l$ and $i_l \neq i'_l$. Suppose, for example, that $i_l > i'_l$. Then, we obtain:

$$(i_l - i'_l) n_1 \dots n_l = \sum_{k=0}^{l-1} (i'_k - i_k) n_1 \dots n_k \leq \sum_{k=0}^{l-1} (n_{k+1} - 1) n_1 \dots n_k = n_1 \dots n_l - 1,$$

and so $(i_l - i'_l) < 1$, which is a contradiction since both $i_l, i'_l \in \mathbb{N}$. □

We end this section by giving a consequence of the Weierstrass Division Theorem (see [**ZS60**, Chapter VII, Theorem 5] and [**Ebe07**, Theorem 2.2]).

LEMMA 2.121. *Let $f \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ be a Weierstrass polynomial of degree n in y and order order $r > 0$. Then, any $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ has a unique expansion*

$$h = \sum_{k \geq 0} P_k f^k, \tag{2.122}$$

with $P_k \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ of degree $< n$.

PROOF. The assertion is clear if $h = 0$ so assume that $h \neq 0$. Since the ring $\mathbb{C}\{x_1, \ldots, x_d, y\}$ is a unique factorization domain, there exists an integer $s \geq 0$ such that

$$h = h_{s-1} \cdot f^s, \quad \text{with } h_s \notin (f). \tag{2.123}$$

By the Weierstrass division theorem there are are unique elements $h_s \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ and $P_s \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ of degree $< n$ such that

$$h_{s-1} = h_s \cdot f + P_s.$$

Substituting in (2.123) on obtains

$$h = P_s \cdot f^s + h_s \cdot f^{s+1}.$$

Then one applies Weierstrass division theorem to the quotient h_s,

$$h_s = h_{s+1} \cdot f + P_{s+1},$$

where $h_{s+1} \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ and $P_{s+1} \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ of degree $< n$. By successive applications one gets

$$h = P_s \cdot f^s + P_{s+1} \cdot f^{s+1} + \ldots + P_q \cdot f^q + h_{q+1} \cdot f^{q+1}.$$

By assumption on the order r of f, if we denote by $\mathfrak{m} = (x_1, \ldots, x_d, y)$ to the maximal ideal of $\mathbb{C}\{x_1, \ldots, x_d, y\}$, then $h_{q+1} \cdot f^{q+1} \in \mathfrak{m}^{r(s+1)}$ while $P_j \cdot f^j \in \mathfrak{m}^{rs}$. Thus, the infinite sum $\sum_{k \geq 0} P_k f^k$ converges and it is the required expansion. □

We call the expansion (2.122) the f**-adic expansion** of h.

CHAPTER 3

Some basic definitions on multiplier ideals

Given a smooth variety X over $\mathbb{C}$ and an ideal sheaf $\mathfrak{a}$ on X, one can attach to $\mathfrak{a}$ a family of *multiplier ideals*, $\mathcal{J}(\mathfrak{a}^{\xi})$, parametrized by positive rational numbers $\xi \in \mathbb{Q}_{\geq 0}$. One has a discrete strictly increasing sequence of positive rational numbers $(\xi_s)_{s\geq 1}$, called *jumping numbers* of C, such that $\mathcal{O}_{C_X} \supsetneq \mathcal{J}(\xi_1 C)$ and $\mathcal{J}(\xi C) = \mathcal{J}(\xi_i C) \supsetneq \mathcal{J}(\xi_{i+1} C)$ for every $\xi \in [\xi_i, \xi_{i+1})$ (see [**Laz04**]).

These objects have received special attention in the past three decades. Nadel [**Nad90**] defined multiplier ideals in terms of local convergence of certain integrals and proved some Kodaira-type vanishing theorem for them. In commutative algebra they were introduced by Lipman [**Lip93**] under the name of adjoint ideals in connection with the so-called Brianon-Skoda theorem and in relation to questions of integral closure of ideals. In algebraic geometry, multiplier ideals already appear in Esnault and Viehweg work on cohomology vanishing theorems ([**EV92**]). Multiplier ideals have been successfully used in many areas of mathematics. For instance, they have led to some uniformity results in local algebra (see [**ELS03, ELS01**]) and they have been an important tool in birational geometry (see [**EM06**]). They show some important connections with many invariants like the mixed Hodge spectrum, the roots of the Bernstein-Sato polynomial or the poles of Igusa zeta function (see [**ELSV04, Bud11**]). We refer the reader to [**Laz04**, Chapters 11 to 14] for an extended description of their applications.

Jumping numbers encode interesting geometric, algebraic and topological information, and arise naturally in many different contexts (see [**Laz04, Bud12, ELSV04**]). These numerical invariants already appear in [**Lib82, LV90**]. The smallest jumping number, called the *log canonical threshold*, appears in many different situations. If $0 < \xi \leq 1$ is a jumping number associated to the ideal (f), where $f \in \mathbb{C}[x_1, \ldots, x_n]$ then $-\xi$ is a root of the Bernstein-Sato polynomial of f (see [**ELSV04**]). One can express in terms of the jumping numbers a uniform Artin-Rees number of the ideal (f), and related bounds for the Milnor and Tjurina numbers when f has isolated singularities (see [**ELSV04**]). The Hodge spectrum, defined in terms of the monodromy and the Hodge filtration on the cohomology of the Milnor fiber of f at a singular point x_0, is characterized in terms of the *multiplicities* of the jumping numbers of f at x_0 (see [**Bud11**]).

This chapter aims to give the necessary definitions and properties related to multiplier ideals and jumping numbers. In Section 3.1, we introduce the relative canonical divisor associated to a modification of smooth varieties and the log-discrepancy of an exceptional divisor. In Section 3.2, we define the multiplier ideals and jumping numbers associated to an ideal sheaf and give a general overview of their properties. Section 3.3 is devoted to Howald's theorem and the description of multiplier ideals of monomial ideals.

3.1. The relative canonical divisor and log-discrepancies

Let X be a smooth complex quasi-projective variety (i.e., reduced and of finite type) of dimension n. The sheaf of regular n-forms $\bigwedge^n \Omega$ on X is locally free of rank one hence it determines a Cartier divisor, called a **canonical divisor** K_X, which is unique up to linear equivalence of divisors (see [**Sha94b**, Chapter VI, Section 1.4]). The associated linear equivalence class is called the **canonical class** of X. Let ω be a nonzero meromorphic n-form, so that locally

$$\omega = f dx_1 \wedge \ldots \wedge dx_n,$$

where $x_1, \ldots, x_n$ are local coordinates on X and $0 \neq f$ is a meromorphic function. Then, one can check that the divisor of ω, which is

$$\operatorname{div}(\omega) = \text{zeros} - \text{poles of } \omega$$

is a canonical divisor of X. If ω' is another nonzero meromorphic n-form then $\operatorname{div}(\omega) - \operatorname{div}(\omega')$ is the divisor of a meromorphic function. In order to give a meromorphic n-form it suffices to give a rational n-form on an open affine chart with coordinates $(x_1, \ldots, x_n)$, say $\omega = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$ and to check what it looks like on other affine charts. For instance, if $(y_1, \ldots, y_n)$ define coordinates on another chart and if $y_i = y_i(x_1, \ldots, x_n)$, for $i = 1, \ldots, n$ on an open subset, then we use that

$$dy_1 \wedge \ldots \wedge dy_n = \det \left(\frac{\partial y_i}{\partial x_j} \right) dx_1 \wedge \ldots \wedge dx_n.$$

to express ω in terms of $(y_1, \ldots, y_n)$ (see [**Sha94a**, Chapter 2, Section 6.3]).

A canonical divisor K_X of X can be defined also when the variety X is normal, but in this case K_X is a Weil divisor, which is not necessarily a Cartier divisor.

Definition 3.1. Let $\Psi : Y \longrightarrow X$ be a proper birational morphism between smooth complex varieties. The **relative canonical divisor** of the modification Ψ is the divisor K_Ψ defined by the vanishing of the jacobian determinant of Ψ:

$$K_\Psi = \operatorname{div}(\det(\operatorname{Jac}(\Psi))).$$

One writes the coefficients of this divisor in the form:

$$K_\Psi = \sum_E (\lambda_E - 1)E,$$

where E runs over the components of the critical locus of Ψ, seen as a reduced divisor, and the positive integer λ_E is called the **log-discrepancy** of the exceptional divisor E.

Remark 3.2. One has that K_Ψ is the unique effective divisor supported on the critical locus of Ψ, which belongs to the divisor class $K_Y - \Psi^* K_X$, modulo linear equivalence, where K_X and K_Y are canonical divisors of X and Y, respectively. In order to have this property one can choose an arbitrary representative K_Y of the canonical class on Y and then set $K_X := \Psi_*(K_Y)$ as a representative of the canonical class on X.

Remark 3.3. In this work we will often consider the case when $X = \mathbb{C}^n$ is the affine n-dimensional space. By definition, K_Ψ is the divisor associated to the pullback of the form $\omega = dx_1 \wedge \ldots \wedge dx_n$ by Ψ, for any choice of coordinates $(x_1, \ldots, x_n)$ on X. This is because if we take a chart of Y with coordinates $(y_1, \ldots, y_n)$ then an easy computation shows that $\Psi^*\omega$ is defined by

$$\det(\operatorname{Jac}(\Psi)) dy_1 \wedge \ldots \wedge dy_n.$$

Assume in addition that the components of K_Ψ which pass through the origin of this chart are among the coordinate hyperplanes $V(y_i)$, for $i = 1, \ldots, n$. Then, the restriction of $\Psi^*\omega$ to this chart is of the form

$$y_1^{\lambda_1 - 1} \cdots y_n^{\lambda_n - 1} \cdot \epsilon \cdot dy_1 \wedge \ldots \wedge dy_n, \text{ with } \epsilon \text{ a unit.}$$

We say that $\underline{\lambda} := (\lambda_1, \ldots, \lambda_n)$ is the **vector of log-discrepancies** of K_Ψ associated with $(y_1, \ldots, y_n)$. By definition, $\lambda_k = 1$ if and only if $V(y_k)$ is not contained in the support of K_Ψ.

For convenience we often say that $\underline{1} = (1, \dots, 1)$ is the vector of log-discrepancies associated with the coordinates $(x_1, \dots, x_n)$ of X and the form $dx_1 \wedge \dots \wedge dx_n$.

In order to compute log-discrepancies in a composition of toroidal modifications we will use the following result, which computes the pullback of a d-form with zeros in a normal crossing divisor.

PROPOSITION 3.4. *Let $(X, 0)$ be a germ of smooth affine n-dimensional complex space with a system of coordinates $x_1, \dots, x_n$, equipped with a holomorphic n-form*

$$\omega = x_1^{\lambda_1 - 1} \dots x_n^{\lambda_n - 1} dx_1 \wedge \dots \wedge dx_n. \tag{3.5}$$

Let us set $\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$. Let $a_1, \dots, a_n \in \mathbb{Z}^n_{\geq 0}$ be a basis of the lattice $\mathbb{Z}^n$ defining the regular cone $\sigma = \operatorname{cone}(a_1, \dots, a_n)$. Consider the birational monomial map $\psi : \mathbb{C}^n(\sigma) \to X$ defined by

$$x_1 = y_1^{a_{1,1}} y_2^{a_{2,1}} \dots y_n^{a_{n,1}},$$
$$\vdots$$
$$x_n = y_1^{a_{1,n}} y_2^{a_{2,n}} \dots y_n^{a_{n,n}}.$$

Then:

$$\psi^* \omega = y_1^{\langle \underline{\lambda}, a_1 \rangle - 1} \dots y_n^{\langle \underline{\lambda}, a_n \rangle - 1} dy_1 \wedge \dots \wedge dy_n.$$

PROOF. Notice that, if we write $\omega = h\, dx_1 \wedge \dots \wedge dx_n$, then $\psi^*\omega = \psi^*(h)\ \psi^*(dx_1 \wedge \dots \wedge dx_n)$. Let us denote by $\underline{1} = (1, \dots, 1)$. One can easily show by direct computation that

$$y_1 \dots y_n\ \psi^*(dx_1 \wedge \dots \wedge dx_n) = y_1^{\langle \underline{1}, a_1 \rangle} \dots y_n^{\langle \underline{1}, a_n \rangle} dy_1 \wedge \dots \wedge dy_n.$$

Now, for the holomorphic function $h = x_1^{\lambda_1 - 1} \dots x_n^{\lambda_n - 1}$,

$$\psi^* h = y_1^{\langle \underline{\lambda} - \underline{1}, a_1 \rangle} \dots y_n^{\langle \underline{\lambda} - \underline{1}, a_n \rangle}.$$

Thus, we have

$$\begin{aligned} y_1 \dots y_n\ \psi^*\omega &= \psi^*(h)\ y_1 \dots y_n\ \psi^*(dx_1 \wedge \dots \wedge dx_n) \\ &= y_1^{\langle \underline{\lambda} - \underline{1}, a_1 \rangle} \dots y_n^{\langle \underline{\lambda} - \underline{1}, a_n \rangle} y_1^{\langle \underline{1}, a_1 \rangle} \dots y_n^{\langle \underline{1}, a_n \rangle} dy_1 \wedge \dots \wedge dy_n, \end{aligned}$$

or, equivalently,

$$\psi^* \omega = y_1^{\langle \underline{\lambda}, a_1 \rangle - 1} \dots y_n^{\langle \underline{\lambda}, a_d \rangle - 1} dy_1 \wedge \dots \wedge dy_n.$$

□

REMARK 3.6. Let $X = \mathbb{C}^n$ and consider ω as in (3.5). Let Ψ be the toric modification defined by a regular fan Σ subdividing the cone $\mathbb{R}^n_{\geq 0}$. Denote by Σ^{prim} the set of primitive vectors spanning the rays of Σ. Recall from Subsection 1.1.6 that the toric divisors associated to rays in the fan generate the group of torus invariant divisors on the variety (see Subsection 1.1.6). By Proposition 3.4 we have that the divisor defined by $\Psi^*(\omega)$ is equal to:

$$\sum_{u \in \Sigma^{\text{prim}}} \left(\nu_{D_u}(x_1^{\lambda_1} \dots x_n^{\lambda_n}) - 1 \right) D_u = \sum_{u \in \Sigma^{\text{prim}}} \left(\langle u, \underline{\lambda} \rangle - 1 \right) D_u,$$

where D_u is the torus invariant divisor associated to the ray $\mathbb{R}_{\geq 0} u$.

REMARK 3.7. With the hypotheses and notation of Proposition 3.4, assume in addition that the divisor of ω in (3.5) is the restriction to an affine chart of the relative canonical divisor of some morphism $\Phi : X \to X_0$, and $\underline{\lambda}$ is the vector of log-discrepancies associated to $(x_1, \ldots, x_n)$. Then, Proposition 3.4 implies that the vector of log-discrepancies of the relative canonical divisor of $\psi \circ \Phi$, associated with $(y_1, \ldots, y_n)$ is

$$(\langle \underline{\lambda}, a_1 \rangle, \ldots, \langle \underline{\lambda}, a_n \rangle). \tag{3.8}$$

3.2. The definitions of multiplier ideals and jumping numbers

In this section we give the definition of multiplier ideals and jumping numbers on a smooth variety X defined over $\mathbb{C}$ and discuss some of their basic properties. In the next section, we discuss the monomial case (Howald's Theorem) and we give a slight generalization of it.

We refer to [**Laz04**] as a general reference for multiplier ideals and their applications. See also [**Dem01**, **Lip93**, **Nad90**, **Siu05**].

We will focus on the algebraic definition of multiplier ideals, but we start introducing the analytic definition:

DEFINITION 3.9 (Analytic Multiplier Ideal). Consider a smooth algebraic variety X defined over $\mathbb{C}$ and an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$. Let $f_1, \ldots, f_r$ be a local set of generators of the ideal $\mathfrak{a}$ The **analytic multiplier ideal** associated to $\mathfrak{a}$ and the coefficient $\xi \in \mathbb{Q}_+^*$ is

$$\mathcal{J}_{an}(X, \mathfrak{a}^\xi) =_{\text{locally}} \left\{ h \in \mathcal{O}_X \,\middle|\, \frac{|h|^2}{(\sum |f_i|^2)^\xi} \text{ is locally integrable} \right\}.$$

In order to give the algebraic definition of multiplier ideals we need some notation:

DEFINITION 3.10. A (Weil) divisor $D = \sum_i D_i$ on a smooth variety X of dimension n is a simple normal crossing divisor if any component D_i of D is smooth and for every point $p \in X$ there exist local coordinates $x_1, \ldots, x_n$ at p such that the equation of D is given locally by the vanishing of $x_1 \ldots x_r$ for some $r < n$, i.e., D is locally a union of coordinate hyperplanes intersecting transversally.

DEFINITION 3.11 (log-resolution of an ideal sheaf). Consider a smooth algebraic variety X defined over $\mathbb{C}$ and an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$. A log-resolution of $\mathfrak{a}$ is a modification (proper and birational map) $\Psi : Y \to X$ with Y smooth such that:

$$\Psi^* \mathfrak{a} = \mathcal{O}_Y(-F)$$

with F an effective divisor such that $F + E$ is a **simple normal crossing divisor**, where E denotes the exceptional locus of Φ.

Recall that the exceptional locus E of the modification Φ above is the divisor of the jacobian determinant of Φ.

Because we are working over a field of characteristic zero, $\mathbb{C}$, log-resolution for any ideal sheaf always exist according to a fundamental result of Hironaka (see [**Hir64**]).

For a rational number $a \in \mathbb{Q}$, we denote by $\lfloor a \rfloor$ the greatest integer $\leq a$. Let $D = \sum_j a_j D_j$ be a divisor with rational coefficients supported on the prime divisors D_j. We denote by $\lfloor D \rfloor := \sum_j \lfloor a_j \rfloor D_j$, which is a divisor with integral coefficients.

DEFINITION 3.12 (Algebraic Multiplier Ideal). Let $\Psi : Y \to X$ be a log-resolution of an ideal sheaf $\mathfrak{a}$ of $\mathcal{O}_X$. The **multiplier ideal** sheaf $\mathcal{J}(\mathfrak{a}^\xi)$ of $\mathfrak{a}$ with coefficient $\xi \in \mathbb{Q}_{>0}$ is defined by:

$$\mathcal{J}(\mathfrak{a}^\xi) := \Psi_* \mathcal{O}_Y(K_\Psi - \lfloor \xi F \rfloor).$$

The definition of multiplier ideals $\mathcal{J}(\mathfrak{a}^\xi)$ relies on the choice of a log-resolution of $\mathfrak{a}$, but one can show that they are independent of it:

THEOREM 3.13. [**Laz04**, Theorem 9.2.8]. *The multiplier ideals $\mathcal{J}(\mathfrak{a}^\xi)$ are independent of the log-resolution used to construct them.*

The proof uses that any two log-resolutions can be dominated by a third, which reduces the problem to showing that if Ψ is a log-resolution, then the multiplier ideal does not change by passing to a log-resolution dominating Ψ.

One can reformulate the definition of multiplier ideals in terms of valuations. If E_i is a prime divisor on Y we denote by ν_{E_i} the vanishing order valuation along E_i. Notice that when E_i is contained in the support of $F + K_\Psi$, it may happen that E_i is the strict transform of a divisor on X or otherwise E_i must be contained in the exceptional divisor of Ψ.

REMARK 3.14. Let us write

$$\begin{array}{lcl} F & = & \sum r_i E_i, \\ K_\Psi & = & \sum (\lambda_{E_i} - 1) E_i, \end{array} \tag{3.15}$$

where the E_i are the prime divisors in the support of $E + F$ on Y. Observe that $\lambda_{E_i} - 1 > 0$ only when E_i is contained in the exceptional divisor E of Ψ. Notice that $h \in \mathcal{J}(\mathfrak{a}^\xi)$ if and only if $\nu_{E_i}(h) \geq \lfloor \xi r_i \rfloor - (\lambda_{E_i} - 1)$ for every prime divisor E_i in the support of $E + F$. This condition does not allow h to have any poles on X, that is, $\mathcal{J}(\mathfrak{a}^\xi) \subset \mathcal{O}_X$ is an ideal sheaf. It follows from this that:

$$\mathcal{J}(\mathfrak{a}^\xi) = \{h \in \mathcal{O}_X \mid \nu_{E_i}(h) \geq \lfloor \xi r_i \rfloor - (\lambda_{E_i} - 1) \text{ for all } i\}. \tag{3.16}$$

We often use the following arithmetic property:

REMARK 3.17. Let $a \in \mathbb{Z}$ be an integer and $b \in \mathbb{Q}$ a rational number. Then the condition

$$a \geq \lfloor b \rfloor \text{ is equivalent to } a > b - 1. \tag{3.18}$$

Using the arithmetic property (3.18) and the equality (3.16) we obtain the following reformulation of the definition of multiplier ideals:

PROPOSITION 3.19. *With the above notation we have:*

$$\mathcal{J}(\mathfrak{a}^\xi) = \{h \in \mathcal{O}_X \mid \nu_{E_i}(h) + \lambda_{E_i} > \xi \ \nu_{E_i}(\mathfrak{a}) \text{ for all } i\}. \tag{3.20}$$

THEOREM 3.21. [**Laz04**, Theorem 9.3.42]. *In the above settings, the analytic multiplier ideal is the analytic sheaf determined by the algebraic multiplier ideal.*

The properties of the *jumping numbers* associated with multiplier ideals are intensively studied in [**ELSV04**]. Let us define them:

LEMMA 3.22. *Let X be an smooth algebraic variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal sheaf. Then, there exists a strictly increasing discrete sequence (ξ_i) of positive rational numbers such that for any $\xi \in \mathbb{Q}_+^*$ with*

$$\xi_i \leq \xi < \xi_{i+1},$$

one has

$$\mathcal{J}(\mathfrak{a}^{\xi_i}) = \mathcal{J}(\mathfrak{a}^\xi) \supsetneq \mathcal{J}(\mathfrak{a}^{\xi_{i+1}}).$$

The numbers ξ_i are called the ***jumping numbers*** *associated with $\mathfrak{a}$.*

PROOF. Fix a log-resolution Ψ of the ideal $\mathfrak{a}$ such that

$$\begin{array}{rcl} F & = & \sum r_i E_i, \\ K_\Psi & = & \sum(\lambda_{E_i} - 1)E_i. \end{array} \tag{3.23}$$

By the description of multiplier ideals in (3.16), it follows that the set of jumping numbers is contained in the set of candidates

$$\{\xi \in \mathbb{Q}_{>0} \mid \xi r_i = \lfloor \xi r_i \rfloor, \text{ for some } i\},$$

i.e., the set of rational numbers such that some condition changes. Since i varies in a finite set, this set of candidates is discrete and so is the set of jumping numbers. Furthermore, if we have two consecutive candidates and a rational number $\xi \in \mathbb{Q}_{>0}$ in between, $\xi_i \leq \xi < \xi_{i+1}$, all the conditions in (3.16) remain the same. A jumping number ξ_i is a candidate such that there exist some function $h \in \mathcal{O}_X$ such that

$$\nu_{E_i}(h) + \lambda_{E_i} \leq \xi_i r_i \text{ and } \nu_{E_i}(h) + \lambda_{E_i} > (\xi_i - \varepsilon) r_i \text{ for any } \varepsilon > 0.$$

□

DEFINITION 3.24. The smallest jumping number ξ_1 is called the **log-canonical threshold**.

REMARK 3.25. Let D be an effective integral divisor on X. Since X is smooth D is a Cartier divisor, so it determines a line bundle (see [**Sha94b**, Chapter VI, Section 1.4])

$$\mathcal{O}_X(-D) := \{h \in \mathcal{O}_X \mid \operatorname{div}(h) - D \geq 0\}.$$

Notice that a log-resolution of D is a slightly weaker version of an embedded resolution. Usually an embedded resolution is required to meet the additional condition of its restriction to the strict transform of D being an isomorphism outside the singular locus of D. We denote by

$$\mathcal{J}(\xi D) := \mathcal{J}(\mathcal{O}_X(-D)^\xi),$$

the multiplier ideals of the ideal sheaf $\mathcal{O}_X(-D)$. One can extend this definition to the case of effective $\mathbb{Q}$-divisors as in [**Laz04**, Chapter 9], though we do not need this more general notion here.

LEMMA 3.26.

(1) Let D be an effective divisor on a smooth variety X. Then, 1 is a jumping number of the multiplier ideals of D.
(2) Let D be an effective integral divisor on a variety X. Then $\xi > 0$ is a jumping number of D if and only if $\xi + 1$ is so.

PROOF. It is enough to prove the result locally, when X is affine and $D = V(f)$ for a function $f \in \mathcal{O}_X$. As explained in Remark 3.25 $\mathcal{O}_X(-D)$ is locally principal, $\mathcal{O}_X(-D) = (f)$. Clearly, $\Psi^* D = \mathcal{O}_Y(-\Psi^* D)$ where $F = \Psi^* D$ is an integral divisor. Thus,

$$\begin{aligned} \mathcal{J}(D) &= \Psi_* \mathcal{O}_Y(K_\Psi - F) \\ &= \Psi_* \left(\mathcal{O}_Y(K_\Psi) \otimes \Psi^* \mathcal{O}_X(-D)\right) \\ &= \mathcal{O}_X \otimes \mathcal{O}_X(-D) \\ &= (f). \end{aligned}$$

On the other hand, let $x \in D_i$ be a general point on any of the components of $D = \sum a_i D_i$. The log-resolution Ψ is an isomorphism outside the singular locus of D, thus

$$\nu_{D_i}(\mathcal{J}(\xi D)) < a_i \text{ for } 0 < \xi < 1.$$

Therefore, $\mathcal{J}(\xi D) \subsetneq (f)$.

Using a similar argument for any $\xi \in \mathbb{Q}_{>0}$ we get that

$$\mathcal{J}(\xi D) = \mathcal{J}((\xi - \lfloor \xi \rfloor)D) \otimes \mathcal{O}_X(-\lfloor \xi \rfloor D) = \mathcal{J}((\xi - \lfloor \xi \rfloor)D) \otimes (f)^{\lfloor \xi \rfloor}$$

and that $\mathcal{J}(\xi D) \subsetneq (f)^{\lfloor \xi \rfloor + 1}$ since

$$\nu_{D_i}(\mathcal{J}(\xi D)) < (\lfloor \xi \rfloor + 1)a_i.$$

□

The above lemma implies that all the the jumping numbers associated with D are determined by the finitely many lying in the unit interval $(0, 1]$.

Definition 3.27. If D is an effective divisor on X we denote by $\Xi(D)$ the set of jumping numbers of the multiplier ideals of D which are ≤ 1. The **jumping length** of D is the cardinality of the set $\Xi(D)$.

Some properties of the jumping length are explained in [**ELSV04**].

More generally, the jumping numbers of an ideal sheaf satisfy some periodicity properties (see [**ELSV04**, Proposition 1.12] and [**Laz04**, Example 9.3.24 and Section 11.1.A] for further details).

Lemma 3.28. *Let X be a smooth complex variety of dimension d and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-trivial ideal sheaf. If $\xi > d-1$ then ξ is a jumping number of $\mathfrak{a}$ if and only if $(\xi+1)$ is a jumping number of $\mathfrak{a}$.*

Proof. We can assume that X is an affine variety. Suppose ξ is a jumping number of $\mathfrak{a}$. Then, we can find a regular function $h \in \mathcal{O}_X$ such that $h \in \mathcal{J}(\mathfrak{a}^{\xi-\varepsilon})$ for any $\varepsilon > 0$ but such that $h \notin \mathcal{J}(\mathfrak{a}^{\xi})$. Let $g \in \mathfrak{a}$ be a general element. By definition, $gh \in \mathcal{J}(\mathfrak{a}^{\xi+1-\varepsilon})$ for any $\varepsilon > 0$ but $gh \notin \mathcal{J}(\mathfrak{a}^{\xi})$. It follows that $\xi + 1$ is a jumping number of $\mathfrak{a}$.

For the converse, fix an integer $n > d - 1$. A theorem of Skoda ([**Laz04**, Theorem 11.1]) asserts that for any rational $\xi \in \mathbb{Q}_{>0}$ we have

$$\mathcal{J}(\mathfrak{a}^{n+\xi+1}) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{n+\xi}).$$

This shows that if $n + \xi$ is not a jumping number then $n + \xi + 1$ is not a jumping number. □

Remark 3.29. If X is a smooth variety and f is a germ of complex analytic function at $x \in X$ then the above definitions of multiplier ideals and jumping numbers of (f) generalize to this local setting (see [**ELSV04**, Remark 1.26]). We will apply this without further comment in the following chapters.

3.3. Multiplier ideals in the monomial case

In this section we give a slight generalization of a theorem of Howald describing the multiplier ideals associated with monomial ideals (check [**How01**]).

First, let us introduce the Newton polygon of an ideal.

Definition 3.30. Let $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n}$ be an ideal with generators $f_1, \ldots, f_r$. The Newton polyhedron of $\mathfrak{a}$ is the convex hull of the union of Newton polygons of its generators f_i.

Assume now that $\mathfrak{a}$ is a monomial ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Let Σ be a regular subdivision of the dual fan associated to the Newton polyhedron of $\mathfrak{a}$. The associated toric modification

$$\Psi : Y \to X = \mathbb{C}^n$$

is a log-resolution of $\mathfrak{a}$, hence there is an effective divisor F such that $\Psi^*\mathfrak{a} = \mathcal{O}_Y(-F)$ and $F + K_\Psi$ has simple normal crossings. Indeed, $F + K_\Psi$ is torus invariant divisor of Y, seen as a toric variety defined by the fan Σ. That is, its support is contained in the set of torus invariant divisors E_u, $u \in I$, which is in bijective correspondence with the set Σ^{prim} of primitive vectors spanning the rays of Σ (see Subsection 1.1.6). Thus, we can write

$$F = \sum_{u \in \Sigma^{\text{prim}}} r_u E_u, \text{ with } r \in \mathbb{N}.$$

PROPOSITION 3.31. *Let us consider the holomorphic form:*

$$\omega = x_1^{\lambda_1} \dots x_n^{\lambda_n} dx_1 \wedge \dots \wedge dx_n,$$

where $\underline{\lambda} := (\lambda_1, \dots, \lambda_n) \in (\mathbb{N}^*)^n$ *and* $\xi \in \mathbb{Q}_{>0}$. *With the above notation the set*

$$\mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma) = \{h \in \mathcal{O}_X \mid \nu_{E_u}(h) + \langle u, \underline{\lambda} \rangle - 1 \geq \lfloor \xi r_u \rfloor, \quad u \in \Sigma^{\text{prim}}\}.$$

is an ideal of $\mathcal{O}_X$ *which is independent of the choice of regular subdivision* Σ *and we have that:*

$$\mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma) = \left\langle \underline{x}^v \in \mathbb{C}\{x_1, \dots x_n\} \mid v + \underline{\lambda} \in \text{Int}\,(\xi \mathcal{N}(\mathfrak{a})) \right\rangle. \tag{3.32}$$

PROOF. Let us denote by $\mathcal{N}$ the Newton polyhedron of the monomial ideal $\mathfrak{a}$, by Σ_0 its dual fan, and by Φ the support function of $\mathcal{N}$. We can describe the Newton polyhedron as an intersection of hyperplanes defined by the rays in the dual fan Σ_0, as in Formula 1.26,

$$\mathcal{N} = \bigcap_{u \in \Sigma_0^{\text{prim}}} H^+_{u,\Phi(u)}, \tag{3.33}$$

where $\Sigma_{0,\text{prim}}$ denotes the set of primitive vectors spanning the rays of Σ_0, and we wrote

$$H^+_{u,\Phi}(u) = \{v \in \mathbb{R}^n \mid \langle u, v \rangle \geq \Phi(u)\}.$$

The proof of Corollary 1.28 generalizes to our n-dimensional situation. Since Σ is a subdivision of Σ_0, we obtain the equivalences:

$$v \in \text{Int}(\mathcal{N}) \Leftrightarrow \langle u, v \rangle > \Phi(u), \ \forall u \in \Sigma_{0,\text{prim}} \Leftrightarrow \langle u, v \rangle > \Phi(u), \ \forall u \in \Sigma_{\text{prim}}. \tag{3.34}$$

Using the defining properties of valuations it is easy to see that $\mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma)$ is an ideal of $\mathcal{O}_X$. As we explained above the divisor F is invariant with respect to the torus action on Y. The same property applies to the the divisor $K_{\Psi,\omega} := \sum_{u \in \Sigma_{\text{prim}}} (\langle u, \underline{\lambda} \rangle - 1) E_u$ of the form $\Psi^*(\omega)$ by Remark 3.6. These observations imply that $\mathcal{J}(\mathfrak{a}^c, \omega, \Sigma)$ is generated by monomials:

$$\mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma) = \langle \underline{x}^v \in \mathbb{C}[\underline{x}] \mid \nu_{E_u}(\underline{x}^v) + \langle u, \underline{\lambda} \rangle - 1 \geq \lfloor \xi \nu_{E_u}(\mathfrak{a}) \rfloor, \ \forall u \in \Sigma_{\text{prim}} \rangle\,.$$

Taking into account that $\nu_{E_u}(\underline{x}^v) = \langle u, v \rangle$ and $\nu_{E_u}(\mathfrak{a}) = \Phi(u)$ we get:

$$\begin{aligned} \mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma) &\overset{(3.18)}{=} \left\langle \underline{x}^v \mid \langle u, v + \underline{\lambda} \rangle > \xi \Phi(u), \ \forall u \in \Sigma_{\text{prim}} \right\rangle \\ &\overset{(3.34)}{=} \left\langle \underline{x}^v \mid v + \underline{\lambda} \in \text{Int}(\xi \mathcal{N}) \right\rangle, \end{aligned}$$

since $\xi \Phi$ is the support function of $\xi \mathcal{N}$. □

REMARK 3.35. Since the ideal $\mathcal{J}(\mathfrak{a}^\xi, \omega, \Sigma)$ is independent of Σ we denote it simply by $\mathcal{J}(\mathfrak{a}^\xi, \omega)$. Notice that, because of Corollary 1.28, in order to check if a monomial $\underline{x}^v$ belongs to $\mathcal{J}(\mathfrak{a}^\xi, \omega)$ it is enough to check the conditions

$$\nu_{E_u}(\underline{x}^v) + \langle u, \underline{\lambda} \rangle - 1 \geq \lfloor \xi \nu_u(\mathfrak{a}) \rfloor,$$

hold for u running through $\Sigma_{0,\text{prim}}$.

REMARK 3.36. In the particular case $\underline{\lambda} = (1, \dots, 1)$ in Theorem 3.31 it follows that $\mathcal{J}(\mathfrak{a}^\xi, \omega)$ is the multiplier ideal $\mathcal{J}(\mathfrak{a}^\xi)$ and we recover the statement of Howald's theorem in [**How01**].

Remark 3.37. Blickle gave a further generalization of Howald's theorem in [**Bli04**].

Remark 3.38. In [**How03**], Howald generalizes his result to non-degenerate polynomials. A polynomial f is said to be non-degenerate if for any face $\mathcal{F}$ of its Newton polygon $\mathcal{N}(f)$, the differential of the terms lying of the face does not vanish over the torus $(\mathbb{C}^*)^d$. Howald shows that this condition means that the log-resolution of the monomial ideal generated by the terms in the expansion of f is also a log-resolution for f. Furthermore, they share the same multiplier ideals for rationals $\xi < 1$, which allows to characterize the jumping numbers of the ideal (f).

CHAPTER 4

Toroidal resolutions and generating sequences of divisorial valuations

4.1. Toric surfaces and toric modifications of $\mathbb{C}^2$

In this section we describe some particular features of normal toric surfaces.

Recall from Example 1.12 that the affine plane $\mathbb{C}^2$, equipped with the affine coordinates (x, y) is an example of toric variety. The torus $T_2 = (\mathbb{C}^*)^2$ is an open dense subset which acts on $\mathbb{C}^2$ by multiplication coordinate-wise, and this action extends the product operation on T^2 as an algebraic group. The coordinate lines $R = V(x)$ and $L = V(y)$ are invariant for this action.

Definition 4.1. If $M = \mathbb{Z}^2$ then $M_{\mathbb{R}} = \mathbb{R}^2$, and we simply denote by $\check{\mathbb{R}}^2$ the dual vector space $N_{\mathbb{R}}$ and by $\check{\mathbb{Z}}^2 \subset \check{\mathbb{R}}^2$ the dual lattice $N_{\mathbb{R}}$. We denote by $\mathbb{R}^2_{\geq 0}$ the cone spanned by the canonical basis of $\mathbb{Z}^2$. Its dual cone $\check{\mathbb{R}}^2_{\geq 0}$ is spanned by the dual basis, which is the canonical basis of $\check{\mathbb{Z}}^2$. The **slope** of a nonzero vector $u = (a_1, a_2) \in \check{\mathbb{Z}}^2$, or of the **ray** $\mathbb{R}_{\geq 0} u$, is $a_2/a_1 \in \mathbb{Q} \cup \{\pm\infty\}$.

More generally, a two-dimensional strictly convex rational cone $\sigma = \mathbb{R}_{\geq 0}(a, b) \subset N_{\mathbb{R}}$ is spanned by two primitive vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in N. The cone σ is regular if and only if

$$\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \pm 1. \tag{4.2}$$

If the cone σ is regular the toric variety $S(\sigma)$ is isomorphic to the affine plane. We denote it by $\mathbb{C}^2_\sigma$ to recall that it is equipped with the affine coordinates (u_σ, v_σ), defining the T_2-invariant divisors. We denote by D_a the torus invariant divisor defined by $u_\sigma = 0$ on $\mathbb{C}^2_\sigma$ (we define similarly D_b).

If in addition, $\sigma \subset \check{\mathbb{R}}^2_{\geq 0}$ then by (4.2) we have a **monomial map** $\psi(\sigma) : \mathbb{C}^2_\sigma \to \mathbb{C}^2$ given by:

$$\begin{aligned} x &= u_\sigma^{a_1} v_\sigma^{b_1}, \\ y &= u_\sigma^{a_2} v_\sigma^{b_2}, \end{aligned} \tag{4.3}$$

which is an isomorphism over the torus T_N.

Proposition 4.4. *If Σ is a fan subdividing $\check{\mathbb{R}}^2_{\geq 0}$, the monomial maps associated to its two-dimensional cones glue up to define a proper birational morphism*

$$\psi(\Sigma) : S(\Sigma) \longrightarrow \mathbb{C}^2 \tag{4.5}$$

which is equivariant with respect to the action of the torus.

Definition 4.6. The map (4.5) is the **toric modification** defined by the subdivision Σ.

Remark 4.7. If the fan Σ is not regular, there is also an associated toric modification, $\psi(\Sigma) : S(\Sigma) \longrightarrow \mathbb{C}^2$ though the surface $S(\Sigma)$ is not smooth, but just normal.

Example 4.8. Blowing up. If Σ is the subdivision of $\mathbb{R}^2_{\geq 0}$ along the ray $\mathbb{R}_{\geq 0}(1, 1)$ then the toric modification $\psi(\Sigma)$ has two charts (2.44) and (2.45), thus it is the blowing up of $0 \in \mathbb{C}^2$.

It is a standard fact in toric geometry that the group of torus invariant divisors is free and generated by divisors associated with 1-dimensional rays in the fan (see Subsection 1.1.6).

In particular, if $\Sigma \subset N_{\mathbb{R}}$ is a regular fan subdividing $\check{\mathbb{R}}^2_{\geq 0}$, for each $a \in \Sigma^{(1)}$ we have a torus invariant divisors D_a, which appears, on those charts $\mathbb{C}^2_\sigma$ for each two dimensional cone $\sigma \in \Sigma$ such that $a \in \sigma$. By definition (4.3) the divisor D_a is exceptional for $\psi(\Sigma)$, that is, $\psi(\Sigma)(D_a) = 0$, if and only if a belongs to the interior of the cone $\check{\mathbb{R}}^2_{\geq 0}$, and in this case D_a is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. The invariant divisors D_a and D_b appearing on the chart $\mathbb{C}^2_\sigma$, for $\sigma = \mathbb{R}_{\geq 0}(a, b)$ intersect transversally at the origin of the corresponding chart $\mathbb{C}^2_\sigma$. By (4.3) the divisor D_{u_1} associated to the ray $\mathbb{R}_{\geq 0}u_1$ in the boundary of the cone $\check{\mathbb{R}}^2_{\geq 0}$ is the strict transform of the divisor $R = V(x)$ (and similarly for D_{u_2} with respect to the coordinate line $L = V(y)$).

Remark 4.9. The dual graph $G(\psi(\Sigma), R \cup L)$ can be embedded in Σ as the union of the segments $[u, v]$, such that $u, v \in \Sigma^{(1)}$ span a cone of Σ. Similarly, the dual graph of the morphism $G(\psi(\Sigma))$ can be embedded in Σ as the union of the segments $[u, v]$, such that $u, v \in \Sigma^{(1)} \cap \mathbb{R}^2_{>0}$. To see this, notice that the divisors D_u and D_v intersect if and only if the cone σ spanned by the vector u and v belong to Σ. The point of intersection is the origin of the chart $\mathbb{C}^2_\sigma$. The divisor D_u is exceptional if and only if $u \in \mathbb{R}^2_{>0}$.

Example 4.10. Let us consider the cone $\check{\mathbb{R}}^2_{\geq 0}$ with ray generators u_1, u_2 corresponding to the cross (R, L). We refine it by introducing the rays spaned by $a_1 = (1, 1)$, $a_2 = (1, 2)$ and $a_3 = (2, 3)$. Every cone in the refinement is regular, for example $\mathbb{R}_{\geq 0}(a_2, a_3)$ has determinant $3 - 4 = -1$. The induced morphism $\psi(\Sigma)$ can be factored as the blow up of the origin, π_1 which creates the exceptional divisor D_{a_1}, the blow up of the point of intersection of $R = D_{u_1}$ and D_{a_1}, π_2 which creates the exceptional divisor D_{a_2}, and the blow up of the point of intersection of D_{a_1} and D_{a_2}, π_3 which creates a new exceptional divisor D_{a_3}. Figure 1 shows the fan Σ, the dual graph and the enrichment of the graph with the initial coordinates. The black segment on the fan Σ is isomorphic to the enriched dual graph.

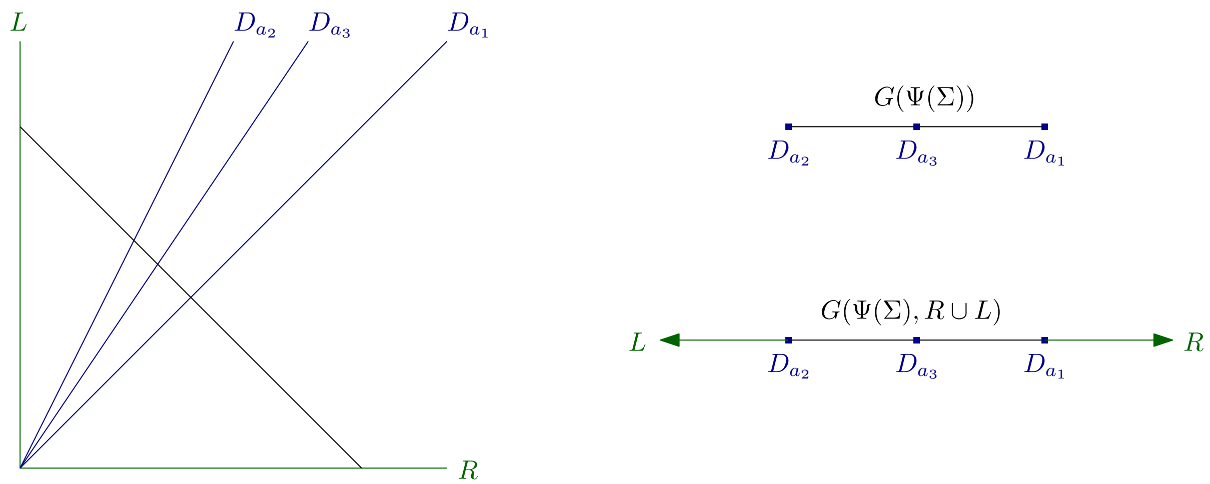

Figure 1. A fan Σ refining $\check{\mathbb{R}}_{\geq 0}$, its dual graph and its enrichment.

Now we want to describe the self-intersections of the exceptional components (see [CLS11, Theorem 10.4.4]). In order to do this, we use Lemma 2.57. First let us compute the total transform of L by $\psi(\Sigma)$ in term of the various local charts. Let $\sigma = \mathbb{R}_{\geq 0}(a, b)$ be a regular cone spanned by the primitive vectors $a = (a_1, a_2), b = (b_1, b_2)$ and consider the local chart of the modification (4.3)

$$\begin{aligned} x &= u^{a_1} v^{b_1}, \\ y &= u^{a_2} v^{b_2}. \end{aligned} \tag{4.11}$$

Note that the second component of the vector is gives the order of vanishing of the curve L, so the total transform is

$$\psi(\Sigma)^* L = \tilde{L} + \sum_{a \in \Sigma^{(1)}} a_2 D_a \tag{4.12}$$

Assume $a \in \Sigma^{(1)}$ is not an element of the basis u_1, u_2 of N, so that its associated divisor, D_a is exceptional. Notice that there exist two regular cones σ_1, σ_2 such that $\mathbb{R}_{\geq 0} a = \sigma_1 \cap \sigma_2$. Say $\sigma_1 = \text{Cone}(a, b)$ and $\sigma_2 = \text{Cone}(a, c)$. By the third assertion of Lemma 2.57 we get

$$0 = \psi(\Sigma)^* L \cdot D_a = (a_2 D_a + b_2 D_b + c_2 D_c) \cdot D_a. \tag{4.13}$$

Recall that, since the cones σ_i are regular, $Z_{\sigma_i} = \mathbb{C}^2$, so that the intersection $D_a \cdot D_b = D_a \cdot D_c = 1$. Thus

$$a_2 D_a \cdot D_a = -(b_2 D_b + c_2 D_c) \cdot D_a = -b_2 - c_2.$$

If we do the same reasoning for R instead of L, we obtain

$$a_1 D_a \cdot D_a = -(b_1 D_b + c_1 D_c) \cdot D_a = -b_1 - c_1.$$

This shows in particular, that there exists a positive integer k such that $b + c = ka$ and then the self-intersection of the divisor D_a is $D_a \cdot D_a = -k$.

Example 4.14. Suppose we want to compute the self-intersection of the divisors in Figure 1. By the above reasoning, if we want to compute it for D_{a_3}, we only have to write the ray generator $a_3 = (2, 3)$ in terms of the minimal generators of the adjacent rays, $a_1 = (1, 1)$ and $a_2 = (1, 2)$. It follows that

$$D_{a_3} \cdot D_{a_3} = -1, \; D_{a_1} \cdot D_{a_1} = -3, \; D_{a_2} \cdot D_{a_2} = -2. \tag{4.15}$$

Remark 4.16. (See [**Oda88**, Proposition 1.19], [**Ful93**, Section 2.6] and [**CLS11**, Section 10.2]). A fundamental property, specific of dimension 2, is that any fan $\Sigma \subset N_{\mathbb{R}}$ has a minimal regular subdivision Σ_{reg}, in the sense that any other regular refinement of the fan is also a refinement of Σ_{reg}. If the cone $\sigma \in \Sigma$ is non-regular its minimal regular subdivision is obtained by taking the rays spanned by the integral points on the polygon $\mathcal{P}_\sigma$, which is the union of the compact edges of the convex hull of the set $\sigma \cap N \setminus \{0\}$.

By convexity if a and b are two consecutive integral vectors on the polygon $\mathcal{P}_\sigma$ then the triangle with vertices $0, a, b$ contains no points of N other than its vertices. This implies that there is no lattice point in the polygon with vertices $0, a, b, a + b$, aside from the vertices, hence the determinant (4.2) is ± 1 and a, b form a basis of N. For the minimality, if b, a, c are three consecutive integral vectors on this polygon, there exists an integer $k \geq 1$ such that $b + c = ka$ (as shown previously in the computation of self-intersections). The case $k = 1$ cannot happen by convexity (the vector a cannot belong to the polygon $\mathcal{P}_\sigma$ in this case). Thus the divisors associated to interior points in the regularization of a cone have self-intersection ≤ -2. The minimality property is characterized by the fact that any exceptional divisor has self-intersection ≤ -2 (see [**Ful93**, Section 2.6]).

One can also construct the minimal regular subdivision of a non-regular cone σ by using continued fractions. With an appropriate choice of basis e_1, e_2 of N, one can assume that

$\sigma = \mathrm{Cone}(e_2, de_1 - ke_2)$ with $d > k > 0$ and $\gcd(d,k) = 1$. One can consider the **Hirzebruch-Jung continued fraction expansion** of $\frac{d}{k}$,

$$\frac{d}{k} = b_1 - \frac{1}{b_2 - \frac{1}{\cdots - b_r}},$$

where the terms $b_i \in \mathbb{N}$ are uniquely determined by the constraint $b_2, \dots, b_r > 0$. The Hirzebruch-Jung continued fraction expansion is sometimes denoted by

$$\frac{d}{k} = [b_1, \dots, b_r]^-.$$

Then, one can define recursively a sequence of approximations by defining a sequence

$$P_0 = 1, \quad Q_0 = 0,$$
$$P_1 = b_1, \quad Q_1 = 1,$$

and for all $2 \leq i \leq r$

$$P_i = bP_{i-1} - P_{i-2}, \quad Q_i = b_i Q_{i-1} - Q_{i-2}.$$

Then $\frac{P_i}{Q_i} = [b_1, \dots, b_i]^-$ and that $P_{i-1}Q_i - Q_{i-1}P_i = 1$ for all $1 \leq i \leq r$. With these integers, one constructs a set of vectors:

$$u_0 = e_2,\ u_i = P_{i-1}e_1 - Q_{i-1}e_2,$$

and cones $\sigma_i = \mathrm{Cone}(u_{i-1}, u_i)$ for $1 \leq i \leq r+1$. Then, the fan consisting of the set of cones σ_i and their faces is the minimal regular refinement of σ. It turns out that the set of integral points of $\mathcal{P}_\sigma$ coincides with the set of vectors $\{u_0, u_1, \dots u_{r+1}\}$. Check [**CLS11**, Section 10.2] for further details or [**Pop11**] for a visual interpretation of usual continued fractions (with + instead of - signs).

Example 4.17. Let us subdivide $\mathbb{R}_{\geq 0}$ by a ray generated by the vector $u = (2,3)$. The resulting cones, σ_1, σ_2 are not regular. Figure 2 shows the convex hulls $\mathrm{Conv}(\sigma_i \cap N \setminus \{0\})$ with the primitive elements marked by thicker points. If we subdivide the cones with the ray generated by the irreducible elements, we get the regular fan of example 4.10.

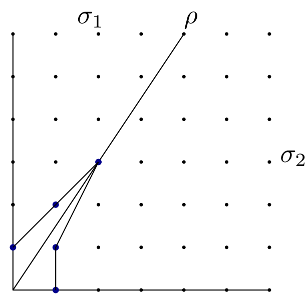

Figure 2. A fan Σ refining $\mathbb{R}_{\geq 0}$, its dual graph and its enrichment

In particular, if Σ is a non-regular fan subdividing $\mathbb{R}^2_{\geq 0}$ then taking the minimal regular subdivision Σ_{reg} provides the toric modification:

$$\psi_{\Sigma_{\mathrm{reg}}} : S(\Sigma_{\mathrm{reg}}) \to \mathbb{C}^2. \tag{4.18}$$

This modification can be factored as a composition of blow ups of infinitely near points which are invariant by the action of the torus (see Example 4.10). For an explanation on how to factor the modification into blow ups we recommend the reading of [**Pop11**, Section 5] and [**GGP19**, Section 4.3].

REMARK 4.19. Let $a = (a_1, a_2)$ be a primitive vector in $\mathbb{R}^2_{>0}$ and denote by Σ the subdivision of $\mathbb{R}^2_{\geq 0}$ defined by adding the ray $\mathbb{R}_{\geq 0}a$. The divisorial valuation associated with the irreducible component D_a of the exceptional divisor of the modification (4.18) coincides with the monomial valuation ν_a of Example 2.77. Indeed, choose a regular refinement Σ of the subdivision. Then one considers a regular cone $\sigma \in \Sigma$ containing $\mathbb{R}_{\geq 0}(a, b)$ and looks at the monomial map (4.3). The order of vanishing at D_a of $x \circ \psi(\sigma)$ (resp. $y \circ \psi(\sigma)$) is equal to a_1 (resp. a_2). More generally, if $h = \sum c_{ij} x^i y^j \in \mathbb{C}[x, y]$ then the order of vanishing along D_a of $h \circ \psi(\sigma)$ is equal to

$$\nu_a(h) = \mathrm{ord}_{u_\sigma} \left(\sum c_{ij} u_\sigma^{a_1 i + a_2 j} v_\sigma^{b_1 i + b_2 j} \right) = \min_{c_{ij} \neq 0} \{a_1 i + a_2 j\}.$$

4.2. Toroidal modifications and the minimal embedded resolution

In this section we explain how the choice of a pair of smooth transversal branches R and L at a point O on a smooth surface S and a reduced plane curve singularity C determines a modification of S over O. If (x, y) are local coordinates such that $R = V(x)$ and $L = V(y)$ then this modification can be expressed by monomial maps as in the toric case studied in Chapter 1. It is defined in terms of the dual fan associated with the Newton polygon of $f_C(x, y)$. This fan is the subdivision of the cone $\mathbb{R}^2_{\geq 0}$, obtained by adding the rays orthogonal to the compact faces of the Newton polygon of $f_C(x, y)$. The slopes of these rays are the exponents of marked points in the Eggers-Wall tree of $\Theta_R(C \cup L)$, which belong to the segment $[R, L]$ (see Proposition 2.107). The modification we consider is defined by the minimal regularization of the previous fan. We describe some *maximal contact* choices for L and explain the resulting construction of a embedded resolution as a composition of toroidal modifications. This resolution is the minimal embedded resolution of C, when R is transversal to C. The third part of the section shows that to a toric modification we can associate a transformation of the Eggers-Wall tree $\Theta_R(C)$. In this fashion, the minimal embedded resolution corresponds to a suitable segment decomposition of the tree. Many results of this section are based on the preprint book [**GGP16**, Chapter 7].

Let (R, L) define a cross at a point O of a smooth surface S. The Newton polygon of a plane curve singularity C lies in the real vector space spanned by the rank two lattice $M_{R,L}$ with basis R, L (see Definition 2.4 and Example 1.12).

For simplicity, we often use this basis to identify the lattice $M_{R,L}$ with $\mathbb{Z}^2$, the cone $\mathbb{R}_{\geq 0}(R, L)$ spanned by R and L simply by $\mathbb{R}^2_{\geq 0}$ and use the notation explained in the Remark 4.1. The modification (4.21) introduced below in terms of a subdivision of $\check{\mathbb{R}}^2_{\geq 0}$, is defined as (4.5), in terms of the local coordinates (x, y) defining the cross (R, L):

DEFINITION 4.20. Let Σ be a subdivision of the cone $\check{\mathbb{R}}^2_{\geq 0}$. The **toroidal modification of S defined by Σ, with respect to the cross** (R, L) is

$$\psi_{R,L}(\Sigma) : S(\Sigma)_{R,L} \to S. \tag{4.21}$$

We will often simplify this notation writing $\psi(\Sigma)$ or $S(\Sigma)$ instead of $\psi_{R,L}(\Sigma)$ or $S(\Sigma)_{R,L}$ when the cross is (R, L) is clear from the context.

DEFINITION 4.22. Let (R, L) define a cross at a point O of a smooth surface S. Let $P \in \Theta_R(L)$ be a point of the tree. For 1 and 2 we assume that P has rational exponent $e_R(P) = \frac{m}{n}$, where $\gcd(m, n) = 1$.

(1) The **divisorial valuation** associated to P is the monomial divisorial valuation ν_P determined by $\nu_P(x) = n$ and $\nu_P(y) = m$ (see Example 2.77 and Equation 1.10).

(2) Let Σ be a regular subdivision of $\check{\mathbb{R}}^2_{\geq 0}$ having a ray of slope $e_R(P)$. We denote by $E_P := D_{(n,m)}$ the corresponding exceptional prime divisor in the model $S(\Sigma)_{R,L}$ of S (see Remark 4.19).
(3) If $P \in [R, L]$ is an irrational point, that is its exponent β is an irrational number, we denote by ν_P the monomial valuation $\nu_P(x) = 1$ and $\nu_P(y) = \beta$.

Remark 4.23. If Σ and Σ' are two regular subdivisions of the cone $\check{\mathbb{R}}^2_{\geq 0}$ on which appears the ray of slope $\frac{m}{n}$, we will slightly abuse notation by using the same symbol for the corresponding divisor seen on the models $S(\Sigma)_{R,L}$ and $S(\Sigma')_{R,L}$. Since the ends of the Eggers-Wall trees are labeled by branches, it makes sense to label these exceptional divisors by their corresponding rational points of the Eggers Wall segment $[R, L]$, as in Definition 4.22. We will extend this relation to arbitrary Eggers-Wall trees in Notation 4.65.

Definition 4.24. Let C be a plane curve singularity at (S, O) and (R, L) be a cross at O. The **dual fan** $\Sigma_{R,L}(C)$ of the Newton polygon $\mathcal{N}_{R,L}(C)$ is the refinement of $\check{\mathbb{R}}^2_{\geq 0}$ by subdividing it along the rays which are orthogonal to the edges of $\mathcal{N}_{R,L}(C)$. If the cross (R, L) is clear from context we will simply denote this fan by $\Sigma(C)$ as we do in the case of the Newton polygon $\mathcal{N}(C)$.

Remark 4.25. Since the Newton polygon of a product $h = h_1 \dots h_r$ is the Minkowski sum of the Newton polygons of its factors (see Proposition 2.104), it follows that the dual fan $\Sigma(C_h)$ is the refinement of $\check{\mathbb{R}}^2_{\geq 0}$ by subdividing it along the rays orthogonal to the edges of each $\mathcal{N}(C_{h_i})$.

Definition 4.26. Let C be a reduced plane curve singularity in a smooth surface (S, O) and (R, L) be a cross at O. Let Σ be the minimal regular subdivision of the fan $\Sigma_{R,L}(C)$. The **toroidal modification of S defined by C, with respect to the cross** (R, L) is the modification (4.21), where Σ is taken to be the minimal regular subdivision of the fan $\Sigma_{R,L}(C)$.

Remark 4.27. By Proposition 2.107 the slopes of the rays orthogonal to the compact edges of the Newton fan $\Sigma_{R,L}(C)$ are the exponents of the marked points in the interior of segment $[R, L]$ of the Eggers-Wall tree $\Theta_R(C \cup L)$. Those marked points are of the form $\langle R, L, C_i \rangle$, for C_i a branch of C.

Proposition 4.28. *With the hypotheses and notation introduced above, we denote by $\psi : T \to S$ the toroidal modification of S with respect to the reduced plane curve singularity C and the cross (R, L). Then, the strict transform C_i' of a branch C_i of $C - R - L$ intersects the exceptional curve at a single point O_i in the smooth locus of $\psi^{-1}(R \cup L)$. More precisely, the point O_i belongs to the component E_{P_i} of $\psi^{-1}(O)$, where $P_i = \langle R, L, C_i \rangle$.*

In addition, if C_j is another branch of $C - R - L$ then one has $O_i = O_j$ if and only if

$$P_i = P_j <_R \langle R, C_i, C_j \rangle . \tag{4.29}$$

and if L has maximal contact with C relative to R then

$$(E_{P_i}, C')_{O_i} < (R, C)_O. \tag{4.30}$$

Proof. Let $\sigma := \mathbb{R}_{\geq 0}(a, b)$ be a cone in Σ, where $a = (a_1, a_2), b = (b_1, b_2)$ are primitive integral vectors. Since σ is regular we can assume that

$$a_1 b_2 - a_2 b_1 = 1. \tag{4.31}$$

Choose a system of coordinates (x, y) representing the cross (R, L). In the chart defined by σ the map ψ is given by

$$x = x_1^{a_1} x_2^{b_1},$$
$$y = x_1^{a_2} x_2^{b_2}.$$

Note that the toric divisor D_a is a component of the exceptional curve if and only if $a_1a_2 \neq 0$. If $a=(1,0)$ (resp. $a=(0,1)$) then one has that $D_a = R'$ (resp.$D_a = L'$).

Let C_i be an irreducible component of C not equal to R or L. By Proposition 2.107, we have that the Newton polygon of C_i is of the form:

$$\mathcal{N}_{R,L}(C_i) = \mathbf{i}_R(C_i)\left\{\frac{\mathbf{e}_R(\langle R,L,C_i\rangle)}{1}\right\}.$$

Actually, if $\mathbf{e}_R(\langle R,L,C_i\rangle) = \frac{m}{n}$ with $\gcd(m,n)=1$ and if we denote $\mathbf{i}_R(C_i) = e_i$ for simplicity, then the polynomial f_{C_i} is of the form:

$$f_{C_i} = (y^n - \theta_i x^m)^{e_i} + \sum_{nr+ms>nme_i} \theta_{r,s}x^r y^s, \text{ with } \theta \neq 0.$$

Since the fan Σ refines the Newton fan $\Sigma(C)$, both linear forms a and b must reach the minimal value on the polygon of C_i at the same vertex, say at $(0, me_i)$. Then we can factor its total transform

$$\psi^*(f_{C_i}) = x_1^{a_1me_i}x_2^{b_1me_i}[(x_1^{a_2n-a_1m}x_2^{b_2n-b_1m} - \theta_i)^{e_i} + g].$$

The strict transform of C_i is defined on this chart by the vanishing of

$$[((x_1^{a_2n-a_1m}x_2^{b_2n-b_1m} - \theta_i)^{e_i} + g], \text{ with } g \text{ divisible by } x_1.$$

Assume that $D_a = V(x_1)$ is an exceptional curve for ψ (that is, $a \neq (1,0),(0,1)$). Then, the linear form a reaches its minimum on $\mathcal{N}(C_i)$ on a compact face which can be either a vertex or an edge:

(1) If a is orthogonal to the compact edge of $\mathcal{N}(C_i)$, then $a=(n,m)$ and by (4.31) one has $b_2n - b_1m = 1$ and $a_2n - a_1m = 0$. Then, we get:

$$\psi^*(f_{C_i}) = x_1^{nme_i}x_2^{b_2me_i}[(x_2 - \theta_i)^{e_i} + g].$$

This means that strict transform of C_i intersects only the component D_a at the point O_i of this chart, defined by $x_1 = 0$ and $x_2 = \theta_i$. It follows also that $({C_i}', D_a)_{O_i} = e_i \leq e_i n$.

(2) Otherwise, $a_2n - a_1m > 0$. In this case, the strict transform of C_i does not intersect the component D_a.

Since D_a or D_b must be a component of the exceptional curve, it follows that the origin of the chart $D_a \cap D_b$ is not in the strict transform C_i'. Hence the intersection point of C_i' with $\psi^{-1}(O)$ is a smooth point of $\psi^{-1}(R \cup L)$.

Assume now that $a \in \check{\mathbb{R}}^2_{>0}$ is a primitive vector defining a ray in the dual fan $\Sigma(C)$, that is, a is orthogonal to a compact edge of the Newton polygon $\mathcal{N}(C)$.

The previous discussion shows that the branches C_i' and C_j' go through the same point, $O_i = O_j$ of D_a if and only if $\mathbf{e}_R(\langle R,L,C_i\rangle) = \mathbf{e}_R(\langle R,L,C_j\rangle) = \frac{a_2}{a_1}$ and $\theta_i = \theta_j$. These equalities hold if and only if the first nonzero term of the Newton Puiseux series of C_i, with respect to x, coincides with those of C_j. Translating these equivalences in terms of order of coincidence we get that the strict transforms C_i' and C_j' go through the same point of the exceptional fiber if and only if

$$P_i = \langle R,L,C_i\rangle = P_j = \langle R,L,C_j\rangle <_R \langle R,C_i,C_j\rangle\,.$$

We deduce the following bounds for the intersection multiplicity:

$$(C', D_a)_{O_i} = \sum_{P_j = P_i < \langle R,C_i,C_j\rangle}^{\mathbf{e}_R(P_j)=\frac{a_2}{a_1}} e_j$$

and

$$(C', D_a)_{O_i} \leq a_1 \left(\sum_{P_j = P_i}^{\mathbf{e}_R(P_j) = \frac{a_2}{a_1}} e_j \right) \leq (C, R)_O, \tag{4.32}$$

where the middle term in (4.32) is the intersection multiplicity of R with the branches of C whose strict transform pass through D_a. The left inequality in in Formula (4.32) is an equality if and only if $a_1 = 1$ and there is only one term in the sum.

Assume now that L is is a smooth branch which has maximal contact with C relative to R. Then, the points $P_i = \langle R, L, C_i \rangle$ are marked points of $\Theta_R(C)$ and two cases may occur:

- The exponent $\mathbf{e}_R(P_i)$ is an integer, that is, $a_1 = 1$. This implies that P_i is a ramification of subtree of index 1 of $\Theta_R(C)$. Hence, there are at least two branches C_i and C_j such that $P_i = P_j = \langle R, C_i, C_j \rangle$ In terms of Formula 4.32 we get more than one term in the sum thus $(C', D_a)_{O_i} < (C, R)_O$.
- The exponent $\mathbf{e}_R(P_i) = \frac{a_2}{a_1}$ is a characteristic exponent of C_i of C. Then, $a_1 > 1$ and by Formula 4.32 we get $(C', D_a)_{O_i} < (C, R)_O$.

□

Remark 4.33. With the hypotheses of the previous proposition, the strict transform R' of R (respectively, L' of L) intersects transversally the reduced exceptional divisor at a point belonging to the irreducible exceptional component corresponding to the primitive integral vector in $\Sigma \cap \check{\mathbb{R}}^2_{>0}$, with the lowest (resp. the greatest) slope, as explained in Remark 4.9.

4.2.1. Toroidal resolution. By Definition 2.114 if we start with a reduced plane curve singularity C and a smooth branch R then we can choose a smooth branch L which has maximal contact with C relative to R. After applying the toroidal modification of S with respect to C and (R, L):

$$\psi_1 : S_1 \to S,$$

we get that the strict transform C' of C may be singular at the point O' of intersection of C' with the reduced exceptional divisor $E_1 = \psi^{-1}(O)$ and at no other point. By Proposition 4.28 the germ $R_1 = (E_1, O')$ is a smooth branch and by (4.30) the intersection multiplicity $(R_1, C')_{O'}$ is smaller than $(C, R)_O$. This shows that the iteration of this procedure leads to a situation where this intersection multiplicity becomes one, that is, the strict transform becomes smooth and transversal to the exceptional divisor. This leads to the following Corollary:

Corollary 4.34. *Let C be a reduced germ of plane curve singularity in a smooth surface (S, O) and let R be a smooth branch. The modification*

$$\Psi : T \to S \tag{4.35}$$

obtained as the composition of the toroidal modifications in the following algorithm is an embedded resolution of C.

(1) Choose a cross (R, L) at the origin such that L has maximal contact with C relative to R.

(2) Denote by $\psi_1 : S_1 \to S$ the toroidal modification of S with respect to C and (R, L). If $\psi_1^*(C)$ has normal crossings, stop. Otherwise, for each point of $\tilde{O} \in E^{(1)} \cap C'$, where the total transform $\psi_1^*(C)$ does not have normal crossings denote by $\tilde{R}$ the germ of the reduced exceptional divisor $E^{(1)}$ at $\tilde{O}$. Then, choose a smooth branch $\tilde{L}$ at $\tilde{O}$ such that $\tilde{L}$ has maximal contact with the strict transform of C relative to $\tilde{R}$ and go to step 1.

Definition 4.36. We call the modification (4.35) obtained by the above algorithm a **maximal contact toroidal resolution of C with respect to R.**

Remark 4.37. If we remove the maximal contact hypotheses in Corollary 4.34, we still obtain a toroidal resolution of C with respect to R although we may need additional modifications. This more general situation is considered in the recent paper [**GGP19**] where different ways of describing the combinatorics of a plane curve singularity are compared.

Remark 4.38. In order to apply the construction given in Corollary 4.34 one has to start with a smooth branch L which has maximal contact with C relative to R, that is, the attaching point $\pi^L_{R,C}(L)$ of L to the tree $\Theta_R(C)$ is an end of $\Theta^1_R(C)$, the level 1 of the index function $\mathbf{i}_R$ on the Eggers-Wall tree $\Theta_R(C)$ (see Proposition 2.112 and Definition 2.114). If follows from the definitions that if L and L' are two smooth branches with the same attaching point $\pi^L_{R,C}(L) = \pi^{L'}_{R,C}(L')$ then the Eggers-Wall trees $\Theta_R(C \cup L)$ and $\Theta_R(C \cup L')$ are the same up to replacing the label of L by the one of L'.

When we start with a reference branch R which is transversal to C we have:

Proposition 4.39. [**Gon03b**, Section 3.2]. *If we start with a reference branch R which is transversal to C then any maximal contact toroidal resolution of C with respect to R is the minimal embedded resolution of C.*

Proof. We sketch the proof explaining the application of Lemma 2.64. Assume we are in the first step of the resolution and (R, L) is a cross at (S, O), such that L has maximal contact with C relative to R. We denote by Σ the minimal regularization of the fan $\Sigma(C)$.

By Remark 4.16, only the divisors associated to rays in the interior of $\Sigma(C)$ can have self-intersection -1. If such a ray has non-integral slope then the associated divisor intersects at least three other components of the total transform of C.

Recall that the exceptional divisors which intersect R and L are associated to rays of Σ with minimal and maximal slope (check Remark 4.33). We discuss the cases when some of these rays belong also to $\Sigma(C)$.

- Since R is transversal to C and L has maximal contact with C, there is no ray of slope < 1 in $\Sigma(C)$ and the minimal slope > 0 of a ray of $\Sigma(C)$ cannot be an integer.

- If L is not a component of C, the ray of maximal slope on $\Sigma(C)$ has non-integral slope, since L has maximal contact with C.

- If L is a component of C, we have to consider also the case when the ray of maximal slope in Σ belongs to $\Sigma(C)$. In this case, the associated divisor intersects at least three other components of the total transform of C.

This shows that an exceptional divisor at the first step of the resolution, either has self-intersection ≤ -2 or it intersects at least three other components of the total transform.

In the other steps of the toroidal resolution we have that the reference branch is an exceptional divisor, i.e., a component of the total transform and the argument is similar to the one used in the third case above. □

4.2.2. Transformations of the Eggers-Wall tree through the toric resolution. It makes sense to consider now the Eggers-Wall tree of the strict transform of C with respect to the reduced exceptional divisor $\psi_1^{-1}(O)$ since the strict transform of C only passes through smooth points of this divisor. Indeed, by Proposition 4.28 those branches C_i whose end belongs to the same connected component of $\Theta_R(C \cup L) \setminus [R, L]$ pass through the same point of the exceptional divisor $\psi_1^{-1}(O)$.

Definition 4.40. Let θ be a closed subtree of $\Theta_R(C)$ and let R_θ be its smallest element with respect to $\leq_R$. We define the **renormalized exponent function** and **index function** on θ:

$$\left\{ \begin{array}{ll} \mathbf{i}_{R_\theta} : \theta \to \mathbb{N}, & \mathbf{i}_{R_\theta}(P) = \frac{\mathbf{i}_R(P)}{\mathbf{i}^+_R(R_\theta)}, \\ \mathbf{e}_{R_\theta} : \theta \to [0, \infty], & \mathbf{e}_{R_\theta}(P) = \mathbf{i}^+_R(R_\theta) \cdot (\mathbf{e}_R(P) - \mathbf{e}_R(R_\theta)), \end{array} \right. \tag{4.41}$$

where $\mathbf{i}^+_R(R_\theta)$ is the extended index of Notation 2.90.

Proposition 4.42. *Let C be a reduced plane curve and R a smooth reference branch at (S, O). Let L be a smooth branch transversal to R and consider the toroidal modification of S with respect to the C and the cross (R, L), $\psi : T \to S$. Then, the renormalized functions on the Eggers-Wall tree of the strict transform of C at each point of intersection with the exceptional locus with respect to the corresponding exceptional component are obtained by the formulas in Definition 4.40.*

Proof. Choose a system of coordinates (x, y) representing the cross (R, L) and let us consider an irreducible component D of C. By Proposition 4.28, the strict transform of D only intersects the exceptional component P, for $P = \langle R, D, L\rangle$, corresponding to the ray of slope $\mathbf{e}_R(P) = \frac{m}{n}$ at a single point, say O_D (check Definition 4.22). Indeed, let σ be the cone in Σ generated by (n, m), (a_1, a_2) the primitive integral vectors such that

$$na_2 - ma_1 = 1. \tag{4.43}$$

In the chart defined by σ the map ψ is given by

$$\begin{aligned} x &= u^n v^{a_1}, \\ y &= u^m v^{a_2}. \end{aligned} \tag{4.44}$$

Then, if we denote by $e_0 = ne_1$,

$$\psi^*(f_D) = x_1^{nme_0} x_2^{a_2 me_0} \left[(x_2 - \theta_i)^{e_1} + g\right], \text{ with } g \text{ divisible by } x_1.$$

If we choose variables $(x_1, y_1 = x_2 - \theta)$, we see that the multiplicity of the strict transform of f_D is now e_1.

By Newton-Puiseux Theorem 2.7, since f_D is irreducible, there is a root $\zeta(x^{1/e_0}) \in \mathbb{C}\{x^{1/e_0}\}$ and f_D factors as:

$$f_D = \prod_{\eta \in G_{e_0}} \left(y - \zeta(\eta \cdot x^{1/e_0})\right), \tag{4.45}$$

where we denote by $e_0 = \mathbf{i}_R(D)$, $G_{e_0} = \{\eta \in \mathbb{C}^* \mid \eta^{e_0} = 1\}$ and

$$\zeta(t) = c_b t^b + \sum_{\alpha > b} c_\alpha t^\alpha,$$

with $c_b \neq 0$. Notice that $b/e_0 = m/n$ as irreducible fraction and $\gcd(e_0, b) = e_1$.

By applying (4.44) to the product (4.45) we obtain

$$\psi^* f = \prod_{\eta \in G_{e_0}} \left[u^m v^{a_2} - \left(c_b \eta^b u^m v^{a_1 b/e_0} + \sum_{\alpha > b} c_\alpha \eta^\alpha u^{n\alpha/e_0} v^{a_1 \alpha/e_0}\right)\right],$$

which can be rewritten as

$$\psi^* f = u^{m_1 e_0} v^{a_1 b} \prod_{\eta \in G_{e_0}} \left[v^{1/n} - \left(c_b \eta^b + \sum_{\alpha > b} c_\alpha \eta^\alpha u^{(\alpha - b)/e_1} v^{a_1(\alpha - b)/e_0}\right)\right]. \tag{4.46}$$

Now, we consider the change of variables

$$u = \frac{x_2}{(y_2 - c_b)^{a_1}} \tag{4.47}$$

$$v = (y_2 - c_b)^n. \tag{4.48}$$

Notice that we can expand

$$v = \sum_{k=0}^{n} \binom{n}{k} y_2^k c_b^{n-k} = y_2^n + \ldots + \theta.$$

Thus, $v - \theta = y_2 \varepsilon$, where ε is a unit, and this is indeed a well defined change of variables around $x_2 = y_2 = 0$, since $(y_2 - c_b)^{a_1}$ is also a unit. Let us fix a determination $v^{1/n} = y_2 - c_b$. With this choice, the total transform of a factor in (4.45) under (4.44) becomes

$$\psi^* \eta = x_2^m \left[y_2 - c_b - \left(c_b \eta^b + \sum_{\alpha > b} c_\alpha \eta^\alpha x_2^{(\alpha - b)/e_1} \right) \right].$$

For such a factor to be a non unit, it must happen that $c_b = c_b \eta^b$, i.e., $\eta^b = 1$.

We claim that

$$\{\eta \in \mathbb{C} \mid \eta^{e_0} = \eta^b = 1\} = G_{e_1}. \tag{4.49}$$

Indeed, since $e_0 = ne_1$, $b = me_1$, if it happens that $\eta^{e_1} = 1$, then $\eta^{e_0} = (\eta^{e_1})^n = 1$ and $\eta^b = (\eta^{e_1})^m = 1$. Conversely, since $e_1 = \gcd(e_0, b)$, by Bezout's identity there exist $k_1, k_2 \in \mathbb{Z}$ such that $e_1 = k_1 e_0 + k_2 b$. Thus, if η verifies $\eta^{e_0} = \eta^b = 1$ then $\eta^{e_1} = (\eta^{e_0})^{k_1} (\eta^b)^{k_2} = 1$.

It follows that exactly e_1 of the factors if (4.45) become non units, and the strict transform of f_D is given by the product

$$\prod_{\eta \in G_{e_1}} \left(y_2 - \sum_{\alpha > b} c_\alpha \eta^\alpha x_2^{(\alpha - b)/e_1} \right).$$

Furthermore, the previous factors are a set of conjugates, since $\eta^\alpha = \eta^{\alpha - b}$ by (4.49).

Note that, for a given characteristic exponent $\alpha_i > b$ of f_D, its strict transform has a characteristic exponent $\mathbf{i}_R^+(P)(\alpha_i - b/e_0)$ (see Definition 4.40). In fact, the set of characteristic exponents of the strict transform of D is equal to the set of characteristic exponents of D which are greater than P. Thus, the Eggers-Wall tree of the strict transform of D at P, $\Theta_P(D)$, is obtained from $\Theta_R(D)$ by eliminating the segment $[R, P]$ and endowing $\Theta_P(D)$ with the renormalized index and exponent functions.

Since the coefficients of the Puiseux expansions of the irreducible components of C do not change, it follows that the previous reasoning extends to reduced curves, preserving characteristic exponents and contact exponents. □

In Definition 2.114 we introduced the notion of a smooth branch with maximal contact with C relative to R. We will see below how the following definition is in a way a generalization of it.

Definition 4.50. Let C be a plane curve singularity, R a smooth branch and K a branch at (S, O). Let $Q \in \Theta_R(C)$ be a point with exponent $\mathbf{e}_R(Q) \in \mathbb{Q}_{\geq 0} \cup \{\infty\}$. Assume that the attaching point of K to $\Theta_R(C)$ is Q, that is, $\pi^K_{R,C}(K) = Q$. The branch K has **maximal contact** at Q if $\mathbf{i}_R(Q) = \mathbf{i}_R(K)$.

Remark 4.51. (1) By definition the only maximal contact branch at R is R itself.

(2) Notice that if $Q = C_i$ is a branch of C then a branch having maximal contact at Q, according to the above definition, must be equal to C_i.

(3) If $Q \in \Theta_R(C)$ is a point of discontinuity of the index function, then there is a branch C_i of C such that Q is also a point of discontinuity of the index function in restriction to the segment $[R, C_i]$. That is, it corresponds to a characteristic exponent of C_i, with respect to R.

If K is a branch having maximal contact at Q then $Q = \pi^K_{R,C}(K)$ is the attaching point of K to $\Theta_R(C)$ and by definition $\mathbf{i}_R(Q) = \mathbf{i}_R(K)$. This implies that K is a semi-root of the branch C_i of C.

In addition, if there is a branch C_j of C which has maximal contact at Q, then it is the only branch with this property. In order to check this suppose that K is a branch different from C_j such that $\pi^K_{R,C}(K) = Q$. Then $\langle R, C_j, K\rangle = Q$, thus the exponent $\mathbf{e}_R(Q)$ must appear in the Newton-Puiseux expansion of K, since it does not in the one of C_j, and so $\mathbf{i}_R(C_j) = \mathbf{i}_R(Q) < \mathbf{i}^+_R(Q) = \mathbf{i}_R(K)$. Thus K cannot have maximal contact at Q.

(4) Lejeune-Jalabert considered also a notion of singular curves with maximal contact in [**Lej73**].

The following theorem shows that the renormalized index and exponent functions characterize the Eggers-Wall forest (the finite disjoint union of trees) of the strict transform of C by the toroidal modification of S with respect to C and (R, L). We refer to [**Gon03b**, Section 3.2] and Proposition 4.28 for the proof of assertion (1). Assertion (2) follows from (1) and the Definition 4.40 of the renormalization.

THEOREM 4.52. *Denote by $\psi_1 : S_1 \to S$ the toroidal modification of S with respect to C and (R, L). The Eggers-Wall forest associated to the strict transform of $C - R - L$ by ψ_1, with respect to the reduced exceptional divisor $\psi_1^{-1}(O)$, is a disjoint union of Eggers-Wall trees. Each one of them is the closure θ of a connected component of $\Theta_R(C \cup L) \setminus [R, L]$. In addition:*

(1) Each component θ has a unique smallest element R_θ with respect to $\preceq_R$. The Eggers-Wall tree θ is rooted at R_θ, has index function $\mathbf{i}_{R_\theta}$ and exponent function $\mathbf{e}_{R_\theta}$ (see (4.41)). The ends of θ are labeled by the strict transforms of the branches of C which pass through the point O_θ of intersection with the exceptional divisor of ψ_1.

(2) A branch K has maximal contact at an end Q of the level $\mathbf{i}_R(P) = \mathbf{i}^+_R(R_\theta)$ of the subtree θ if and only if its strict transform K' by ψ is a smooth branch having maximal contact at the point Q, seen as a point of the Eggers-Wall tree θ (cf. Definitions 4.50 and 2.114).

Thanks to Theorem 4.52 we can describe a finite set of auxiliary branches that we need to consider at the various steps of a toroidal resolution of C, with respect to R, in terms of a maximal contact decomposition of the Eggers-Wall tree $\Theta_R(C)$ and of its maximal contact completion (see Definitions 4.53 and 4.54 below).

The steps of the construction of a toroidal resolution of C with respect to R in Corollary 4.34, start by choosing a smooth branch L with maximal contact with C relative to R. This defines an end $Q_0 = \pi^L_{R,C}(L)$ of the level one of the index function on $\Theta_R(C)$. Then, by Proposition 4.28 we consider the germs of the strict transform C' of C at the points of intersection of C' with exceptional divisors R_P, for P an interior marked point of the segment $[R, L]$. By Theorem 4.52 we have a bijection $\theta \to O_\theta$ between the set of clausures of the connected components of $\Theta_R(C) \setminus [R, L]$ which contain the point R_P and the set of points of intersection of C' with R_P in such a way that the Eggers-Wall tree of (C', O_θ) with respect to R_P is isomorphic to the tree θ equipped with the renormalized exponent and index function at R_P. Applying again the procedure of Corollary 4.34, we have to choose a smooth branch L'_θ with maximal contact with (C', O_θ), relative to R_P. By Theorem 4.52 it is equivalent to choosing a branch L_θ with maximal contact at an end Q_θ of the level $\mathbf{i}_R(P) = \mathbf{i}^+_R(R_P)$ of θ and then taking its strict transform. After iterating this procedure, we obtain a finite set of segments $[R_j, Q_j]$, $j \in J$ of the tree $\Theta_R(C)$ starting with $[R := R_0, Q_0]$, and then $[R_\theta, Q_\theta]$ and so on. At the terminal steps of this process we get that the Eggers-Wall trees of the strict transform are segments without marked points on them. The output of this procedure is a *maximal contact decomposition* of the Eggers-Wall tree $\Theta_R(C)$ according to Definition 4.53 below.

DEFINITION 4.53. A **maximal contact decomposition** of the Eggers-Wall tree $\Theta_R(C)$ is a finite family of segments $[R_j, Q_j]$, $j \in J$ contained in $\Theta_R(C)$ such that

(1) $\cup_{j \in J}[R_j, Q_j] = \Theta_R(C)$,
(2) the segment $(R_j, Q_j]$ is contained in the level $\mathbf{i}_R(P) = \mathbf{i}_R(Q_j)$ and Q_j is an end of this level.
(3) The open segments (R_j, Q_j), $j \in J$, are disjoint.
(4) Each segment $[R_j, Q_j]$ is considered with the marked points of $\Theta_R(C)$ contained in it, together with the restriction of the renormalized exponent function $\mathbf{e}_{R_j}$.

Similarly, the segments $[R_j, L_{Q_j}]$, $j \in J$ obtained by this procedure, starting with $[R_0 = R, L_0 = L]$, and then $[R_\theta, L_\theta]$ and so on, define a maximal contact decomposition of the maximal contact completion of C with respect to R, according to Definition 4.54 below.

DEFINITION 4.54. The **maximal contact completion** of a reduced plane curve singularity C with respect to R is a curve of the form $\bar{C} = \sum_Q L_Q + R$, where L_Q is a branch which has maximal contact with C at Q relative to R, and where Q runs over the ends of the levels of the index function on $\Theta_R(C)$ (see Definition 4.50). We say that the curve C is **tree-complete** with respect to R if it coincides with its maximal contact completion.

Notice that the branches of C are components of the maximal contact completion $\bar{C}$ by definition and that the trees $\Theta_R(\bar{C})$ and $\Theta_R(C)$ have the same marked points in their interiors.

REMARK 4.55.

(1) If $[R_j, Q_j]$, $j \in J$, is a maximal contact decomposition of $\Theta_R(C)$ then it follows from the definitions that $[R_j, L_{Q_j}]$, $j \in J$ is a maximal contact decomposition of $\Theta_R(\bar{C})$. The marked points of both decompositions are the same as the marked points of $\Theta_R(C)$.
(2) If $\bar{C}$ and $\bar{C}'$ are two maximal contact completions of C with respect to R then the Eggers-Wall trees $\Theta_R(\bar{C})$ and $\Theta_R(\bar{C}')$, equipped with exponents and index functions, are the same up to replacing the labels of the branches L_Q by one of the L'_Q, where Q runs over the ends of the levels of the index function on $\Theta_R(C)$. This is a consequence of Remark 4.38 and Theorem 4.52 and induction.
(3) A maximal contact decomposition $[R_j, L_{Q_j}]$, $j \in J$ of $\Theta_R(\bar{C})$, has a unique **initial segment** $[R_0 = R, L_{Q_0}]$ which contains R. In addition, if $j \in J$, $j \neq 0$ then there is a unique element $p(j) \in J$ such that R_j belongs to $(R_{p(j)}, L_{Q_{p(j)}})$. We say that $[R_{p(j)}, L_{Q_{p(j)}}]$ is the **predecessor** of $[R_j, L_{Q_j}]$ in the decomposition.

As a corollary of the above discussion we obtain that:

PROPOSITION 4.56.

(1) Any maximal contact toroidal resolution process of C with respect to R determines a maximal contact decomposition of the tree $\Theta_R(\bar{C})$.
(2) Conversely, any maximal contact decomposition of the tree $\Theta_R(\bar{C})$ determines a maximal contact toroidal resolution of C with respect to R.
(3) Any maximal contact toroidal resolution for C is also a maximal contact toroidal resolution for $\bar{C}$, with respect to R.

EXAMPLE 4.57. Let $C = L_g$ be a plane branch and R a smooth reference branch. Its Eggers-Wall tree was already shown in Figure 1 of Chapter 2. It has one maximal contact decomposition with segments

$$[R = R_0, R_1],\ [R_1, R_2], \dots, [R_g, L_g], \tag{4.58}$$

where R_i is the point of exponent the i-th characteristic exponent of C, α_i.

The Eggers-Wall tree of its maximal contact completion, $\Theta_R(\bar{L_g})$, is shown in Figure 8. It has one maximal contact decomposition with segments

$$[R = R_0, L_0],\ [R_1, L_1], \dots, [R_g, L_g]. \tag{4.59}$$

The maximal contact decompositions of $\Theta_R(L_g)$ and $\Theta_R(\bar{L_g})$ appear in Figure 3.

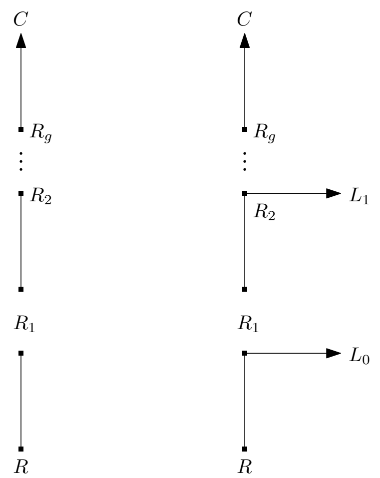

FIGURE 3. Maximal contact decomposition of the tree of $\Theta_R(L_g)$ (on the right) and of $\Theta_R(\bar{L_g})$ (on the left).

EXAMPLE 4.60. Consider the curve in Example 2.94 of Chapter 2. We choose maximal contact curves for the index levels $1, 2, 4, 8$ of the curve C_2 to obtain a complete tree, $\Theta_R(C + L_0 + L_1 + L_2 + L_3)$. The complete tree $\Theta_R(\bar{C})$ is shown in Figure 4.

We choose a maximal contact decomposition of the complete tree $\Theta_R(\bar{C})$, $\{[R_i, L_i]\}_{i=1}^{8}$, where $R_0 = R$, R_1 is the marked point of exponent $\frac{3}{2}$, R_2 the one with exponent $\frac{11}{4}$, $R_3 = R_8$ the one with exponent $\frac{5}{2}$, R_5 the one with exponent $\frac{31}{8}$, R_6 the one with exponent $\frac{15}{4}$ and R_7 the one with exponent $\frac{17}{6}$, while $C_4 = L_4$, $C_2 = L_5$, $C_3 = L_6$, $C_5 = L_7$, $C_1 = L_8$. The maximal contact decomposition is shown in Figure 5.

4.2.3. Relation between the dual graph and the Eggers-Wall tree. In this section we show how we can build the dual graph of a toroidal resolution of C from the its Eggers-Wall tree, with respect to R.

Denote by $\bar{C}$ a maximal contact completion of C with respect to R. We start by taking a maximal contact decomposition $[R_j, L_{Q_j}]$, $j \in J$ of $\Theta_R(\bar{C})$ defining a toroidal resolution

$$\Psi : T \to S$$

of $(C, O) \subset (S, O)$ with respect to R. By Definition 4.53 each segment of the decomposition is equipped with the renormalized exponent function:

$$\mathbf{e}_{R_j} : [R_j, L_{Q_j}] \to [0, \infty],$$

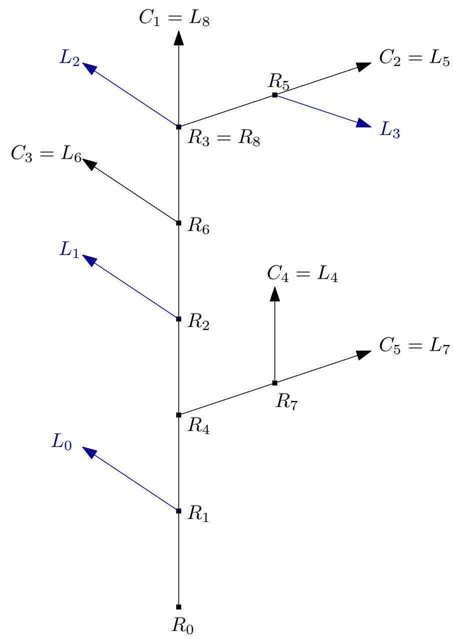

FIGURE 4. The maximal contact completion of the curve in Example 2.94.

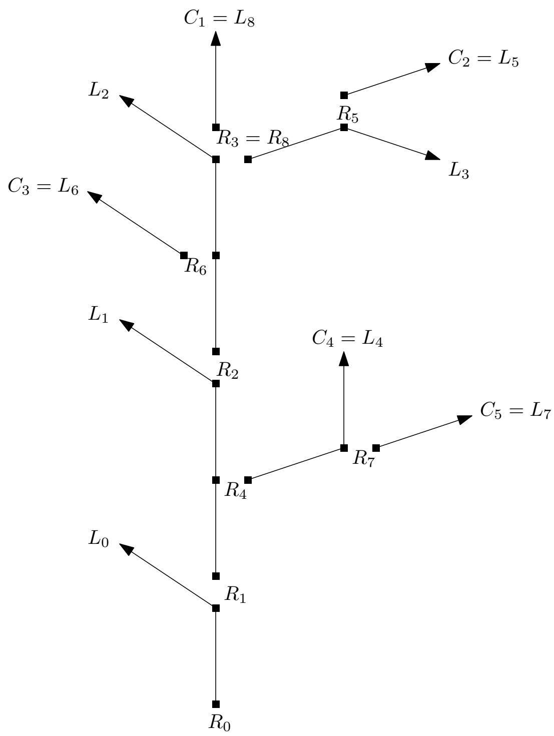

FIGURE 5. The maximal contact decomposition of the curve in Example 2.94.

and the marked points of $\Theta_R(\bar{C})$ contained in it remain as marked points of this segment. At some step of the toroidal resolution process, we have a cross defined by R_j and the strict transform of L_{Q_j}. We consider the fan Σ_j, subdividing $\check{\mathbb{R}}^2_{\geq 0}$, whose rays have slopes the renormalized exponents $\mathbf{e}_{R_j}(Q)$, for Q running over the marked points of $[R_j, L_{Q_j}]$. Finally we consider the minimal regular subdivision Σ_j^{reg} of Σ_j.

DEFINITION 4.61. The **enriched set of marked points** of the segment $\mathcal{E}_j \subset [R_j, L_{Q_j}]$ is the image under $\mathbf{e}_{R_j}^{-1}$ of the set of slopes of rays of the fan Σ_j^{reg} defined above. Each point $P \in \mathcal{E}_j$ corresponds to an exceptional divisor of ψ which we denote also by P abusing slightly of notation as in Notation 4.65. The total set of marked points of the tree $\Theta_R(\bar{C})$ is the union $\mathcal{E} = \cup_{j \in J} \mathcal{E}_j$.

As a consequence of the previous discussion and of Remark 4.9 we recover the resolution graph $G(\Psi, \bar{C})$ by gluing the segments of the decomposition equipped with their enriched sets of marked points:

PROPOSITION 4.62. *The combinatorial tree defined by the tree $\Theta_R(\bar{C})$, seen as a topological space equipped with the set of marked points $\mathcal{E} = \cup_{j \in J} \mathcal{E}_j$, is isomorphic to the dual graph $G(\Psi, \bar{C})$ of the toroidal resolution. The isomorphism sends a marked point P in $\mathcal{E}$ to the vertex of $G(\Psi, \bar{C})$ corresponding to the exceptional divisor E_P, and preserves the labelings of the ends by the branches of $\bar{C}$.*

As a consequence of the Proposition 4.62 we have:

REMARK 4.63. If $\Psi : T \to S$ is a maximal contact toroidal resolution of C with respect to R, one can take a set of maximal contact curves of Ψ, according to Definition 2.66, as a subset of the set of branches of the maximal contact completion $\bar{C}$ of C with respect to R. Namely, for any prime exceptional divisor E_i of Ψ defining a vertex of valency one in the dual graph $G(\Psi)$, we choose a branch of $\bar{C}$ whose strict transform intersects E_i. Then, the chosen branches form a maximal contact set of branches for Ψ.

EXAMPLE 4.64. Let C be the curve in Example 2.94 of Chapter 2, and $\{[R_i, L_i]\}_{i=1}^{8}$ the maximal contact decomposition considered in Example 4.60.

We consider the minimal regular subdivisions of each fan Σ_i whose rays have slopes the renormalized exponents for marked points in $[R_i, L_i]$. For instance, the segment $[R_0, L_0]$ has a marked points of exponent $\frac{3}{2}$, the minimal regular subdivision of the associated fan has two more rays of slopes 1 and 2 (as in Figure 2). Figure 6 shows the enriched sets of marked points for each segment in the decomposition. The composition of the toroidal modifications yields a toroidal resolution Ψ of the complete curve $\bar{C}$. gluing the segments of the decomposition with the enriched sets of marked points we recover the dual graph embedded on the tree as shown in Figure 7.

4.3. Divisorial Valuations and Eggers-Wall trees

Let C be a plane curve singularity and R a smooth branch on the smooth surface (S, O). We will assume in this section that R is not a branch of C. Let us fix a completion $\bar{C}$ of C with respect to R and a maximal contact decomposition $\mathcal{D}$ of $\Theta_R(\bar{C})$.

Another output of a toroidal resolution is that there are some exceptional divisors corresponding to the marked points in the interior of the tree $\Theta_R(C)$. If P is a rational point of $\Theta_R(C)$ then it belongs to the interior of some segment $[R_j, L_j]$ in the maximal contact decomposition $\mathcal{D}$. If P belongs to the initial segment of the decomposition then ν_P is defined in Notation 4.22. Otherwise, at some step of the toroidal resolution process we have $P \in (R_j, L_j)$, for $j \neq 0$. Since by construction R_j is a marked point of the tree $\Theta_R(C)$, we can assume that its exceptional prime divisor E_{R_j} and its associated divisorial valuation ν_{R_j} are defined by induction. At a certain step of the maximal contact toroidal resolution process, we have a cross

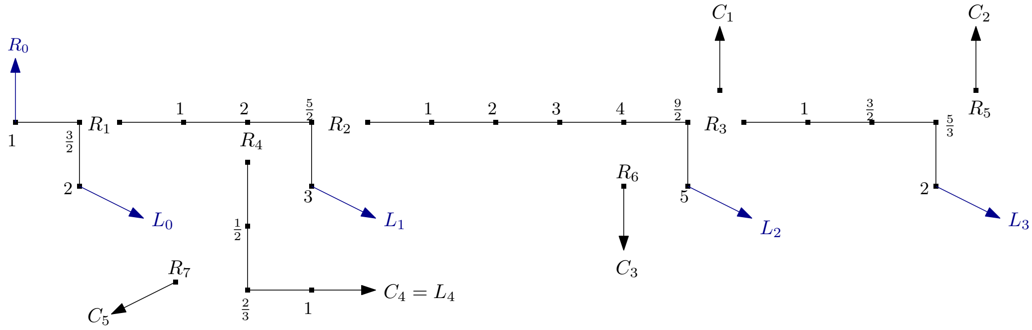

FIGURE 6. The enriched set of marked points for a maximal contact decomposition of the tree $\Theta_R(\bar{C})$ for the curve in Example 2.94. In black the renormalized exponents of the enriched set of marked points.

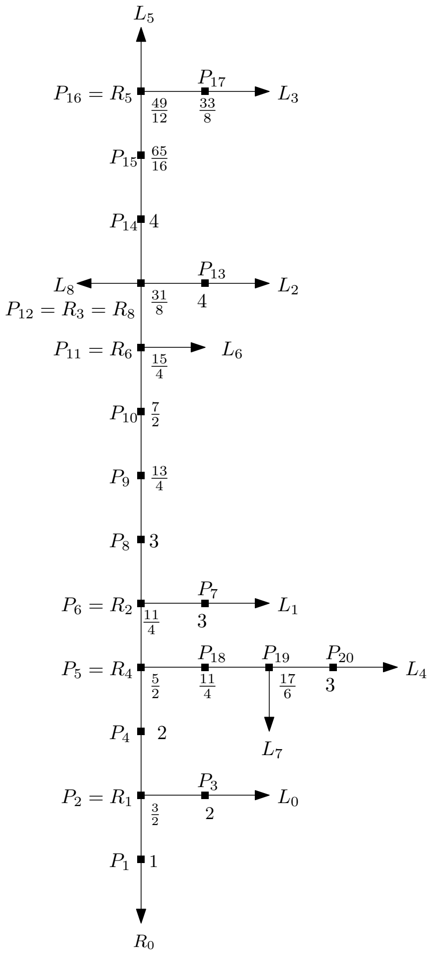

FIGURE 7. The dual graph $G(\Psi, \bar{C})$ embedded on the complete tree $\Theta_R(\bar{C})$ for the curve in Example 2.94.

(R_j, L_j) at an infinitely near point O_j of O defined by R_j and the strict transform of L_j. Let us denote by Σ_j a regularization of the subdivision of $\check{\mathbb{R}}^2_{\geq 0}$, whose rays have slopes the renormalized exponents $\mathbf{e}_{R_j}(Q)$, for Q running over the marked points of $[R_j, L_{Q_j}]$. The fan Σ_j defines a toroidal map in the resolution process. Then, we say that the exceptional prime divisor of the toroidal resolution corresponding to the ray of the fan Σ_j of slope $\mathbf{e}_{R_j}(P)$ is associated to P. We denote it by E_P. If P is not a marked point, we proceed in the same way by replacing Σ_j by the regularization of the subdivision of Σ_j by adding the ray of slope $\mathbf{e}_R(P)$. The following notation extends Notation 4.22.

NOTATION 4.65. Let P be a rational point of the Eggers-Wall tree $\Theta_R(C)$. We denote by E_P the exceptional prime divisor corresponding to P in a maximal contact toroidal resolution process of C with respect to R. We denote by ν_P the associated divisorial valuation. As a consequence of Theorem 4.70 below the exceptional divisor corresponding to P and its associated divisorial valuation ν_P do not depend on the choice of maximal contact decomposition. If P is an end of the tree $\Theta_R(C)$, that is, P is the point with label D equal to R or a branch of C, we denote also by ν_P the vanishing order valuation ν_D along the branch D (see Definition 2.71).

Let D be an effective divisor on S defined by some element f_D in the maximal ideal of $\mathcal{O}$. In what follows we will be interested in the Newton polygon of the total transform of D at some intermediate step of the toroidal resolution, with respect to the cross (R_j, L_j). Let us introduce a notation for this:

DEFINITION 4.66. If D is a plane curve at (S, O) we denote by

$$\mathcal{N}_{R_j,L_j}(D),$$

or also by $\mathcal{N}_{R_j,L_j}(f_D)$, the Newton polygon of the total transform of $\Psi_j^*(D)$ with respect to the cross (R_j, L_j), where Ψ_j is the composition of the toroidal modifications required in order to build the cross (R_j, L_j).

Notice that the valuations ν_{R_k} and ν_{L_k} are associated to the cross (R_k, L_k).

The following Proposition is a consequence of Lemma 1.22 and the above definitions:

PROPOSITION 4.67. *Let $P \in \Theta_R(\bar{C})$ be a rational point. Then, there is a segment $[R_j, L_j]$ of the maximal contact decomposition $\mathcal{D}$ of $\Theta_R(\bar{C})$ such that $P \in [R_j, L_j)$. Denote $\mathbf{e}_{R_k}(P) = \frac{m}{n}$ with $\gcd(n, m) = 1$. With the above notation we have the following expression:*

$$\nu_P(D) = \Phi_{\mathcal{N}_{R_j,L_j}(D)}(n, m).$$

REMARK 4.68. Keep the notation of Proposition 4.67 above. Let $P = L_j$ for some branch L_j in $\bar{C}$. We can describe the vanishing order valuation ν_{L_j} via the support function by taking $u = (0, 1)$ the primitive integral vector representing the ray of L_j in the local toric structure defined by the cross (R_j, L_j):

$$\nu_{L_j}(D) = \Phi_{\mathcal{N}_{R_j,L_j}(D)}(0, 1).$$

We will also use the intersection numbers with suitable *curvettas* in order to describe the values of ν_P. The following notion generalizes Definition 2.65 as we will explain later on.

DEFINITION 4.69. Let C be a plane curve singularity, R a smooth branch and K a branch at (S, O). Let $Q \in \Theta_R(C)$ be a point with exponent $\mathbf{e}_R(Q) \in \mathbb{Q}_{>0} \cup \{\infty\}$. A branch K is a **curvetta** at Q if the attaching point of K to $\Theta_R(C)$ is Q, that is $\pi^K_{R,C}(K) = Q$, and, in addition $\mathbf{i}^+_R(Q) = \mathbf{i}_R(K)$ (see Definition 2.90).

THEOREM 4.70. *Let C be a plane curve singularity at a smooth surface (S, O) and R a smooth branch through it. Let $P \in \Theta_R(C)$ be a point of rational exponent. Then, the divisorial* *valuation ν_P (see Notation 4.65) has the property that for every branch D of a maximal contact completion $\bar{C}$ with respect to R we have:*

$$\nu_P(D) = \mathbf{i}_R(D)\, \mathbf{i}_R^+(P)\, \mathbf{c}_R(\langle R, P, D\rangle). \tag{4.71}$$

Proof. Let us consider a curvetta K at P according to Definition 4.69. We can assume that K is not a component of C. Then, P is marked point of the tree $\Theta_R(C+K)$.

Denote by $\bar{C}$ a maximal contact completion of C with respect to R. Let $[R_j, L_{Q_j}]$, $j \in J$ be a maximal contact decomposition of $\Theta_R(\bar{C})$. Then, there is a unique segment in the decomposition which contains the point P in its interior. One obtains a maximal contact decomposition associated to the maximal contact completion of $C+K$ by marking the point P on this segment and by adding then the additional segment $[P, K]$, without marked points on it (see Definition 4.53).

By Theorem 4.52 and the definitions, the strict transform of K by the toroidal resolution of $C+K$ corresponding to this decomposition, is smooth and transversal to the exceptional divisor corresponding to P. In this situation we can apply Proposition 2.79. It follows that for any branch D of $\bar{C}$ different from K one has:

$$\nu_P(D) = (D, K)_O.$$

Then, the assertion follows by using the description of intersection numbers of branches of $\Theta_R(C+K)$ given by Corollary 2.101:

$$(D, K)_O = \mathbf{i}_R(D)\mathbf{i}_R(K)\mathbf{c}_R(\langle R, K, D\rangle) = \mathbf{i}_R(D)\mathbf{i}_R^+(P)\mathbf{c}_R(\langle R, P, D\rangle).$$

□

Remark 4.72. Theorem 4.70 also implies that the values of the divisorial valuation ν_P on the ends of $\Theta_R(\bar{C})$ are independent on the choice of the maximal contact decomposition of $\Theta_R(\bar{C})$. Actually, the valuation ν_P is determined by its values on the ends of the tree $\Theta_R(\bar{C})$ (see Theorem 4.125).

Remark 4.73. We will also define a valuation ν_P associated to a point of irrational exponent of $\Theta_R(C)$ in Section 4.7.

Remark 4.74. Let K be a curvetta at a rational point of the Eggers-Wall tree according to Definition 4.69 and let $\pi : T \to S$ be a maximal contact toroidal resolution of $C+K$. Then, K is a curvetta at the exceptional divisor E_P according to Definition 2.65.

4.4. Generating sequences of a tuple of valuations

We will only be considering valuations with value group contained in $\mathbb{R}$.

Let ν be a valuation of $\mathcal{O}_{S,O}$. The **value semigroup** of ν is

$$S\nu = \{\nu(f) \mid f \in \mathcal{O} \setminus \{0\}\} \subset \mathbb{R}_{\geq 0}.$$

For each $c \in \mathbb{R}_{\geq 0}$ the following set

$$I_c^\nu = \{f \in \mathcal{O} \setminus \{0\} \mid \nu(f) \geq c\},$$

is an ideal of $\mathcal{O}$, called a **valuation ideal** of ν.

The following concept was introduced in [**Spi90**, Definition 1.1].

DEFINITION 4.75. Let $\{z_j\}_{j\in J} \subset \mathcal{O}$ be a subset of the maximal ideal of $\mathcal{O}$. We say that $\{z_j\}_{j\in J}$ is a **generating sequence** for ν if for any $c \in \mathbb{R}_{\geq 0}$, the ideal I_c is generated by:

$$\left\{ \prod_{\substack{j\in J_0\subset J \\ J_0 \text{ finite}}} z_j^{b_j} \mid b_j \in \mathbb{Z}_{\geq 0},\ \sum b_j \nu(z_j) \geq c \right\}. \tag{4.76}$$

The generating sequence $\{z_j\}_{j\in J}$ is said to be **minimal** whenever any proper subset of it fails to be a generating sequence.

REMARK 4.77. Notice that if ϵ_j are units in $\mathcal{O}$ then $\{\epsilon_j \cdot z_j\}_{j\in J}$ is a generating sequence iff $\{z_j\}_{j\in J}$ is a generating sequence. For this reason we say, in more geometrical terms, that $\{L_j\}_{j\in J}$ is a generating sequence when $L_j = V(z_j)$.

LEMMA 4.78. *Let ν be a valuation of $\mathcal{O}$. Let $z_0, \ldots, z_r \in \mathcal{O}$ be a finite generating sequence for ν. Then, any nonzero function $h \in \mathcal{O}$ admits a finite expansion of the form*

$$h = \sum_{a_I \neq 0} a_I \cdot z_0^{i_0} \ldots z_r^{i_r}, \ \textit{with } a_I \in \mathcal{O}, \tag{4.79}$$

in such a way that

$$\nu(h) = \min_{I=(i_0,\ldots,i_r)} \nu(a_I \cdot z_0^{i_0} \ldots z_r^{i_r}).$$

In addition, if $I^* = (i_0^*, \ldots, i_r^*)$ is an index such that $a_{I^*} \neq 0$ and

$$\nu\left(a_{I^*} z_0^{i_0^*} \ldots z_r^{i_r^*}\right) = \min_{a_I\neq 0} \left\{ \nu\left(z_0^{i_0} \ldots z_r^{i_r}\right)\right\},$$

then,

$$\nu(h) = \nu\left(z_0^{i_0^*} \ldots z_r^{i_r^*}\right). \tag{4.80}$$

PROOF. Let us assume that $z_0, \ldots, z_r$ is a generating sequence for ν and let $h \in \mathcal{O}$ be a nonzero function. If $\nu(h) = c$, by definition of generating sequence, h can be expanded as a finite sum

$$h = \sum_{I=(i_0,\ldots,i_r)} a_I \cdot z_0^{i_0} \ldots z_r^{i_r},$$

in such a way that $\nu(z_0^{i_0} \ldots z_r^{i_r}) \geq c$ for all I with $a_I \neq 0$. Let us denote by $I^* = (j_0^*, i_0^*, \ldots, i_r^*)$ an index such that $a_{I^*} \neq 0$ and

$$\nu\left(a_{I^*} z_0^{i_0^*} \ldots z_r^{i_r^*}\right) = \min_{a_I\neq 0} \left\{ \nu\left(z_0^{i_0} \ldots z_r^{i_r}\right)\right\}. \tag{4.81}$$

By property (2) in Definition 2.68, we get:

$$\begin{aligned} \nu(h) &\geq \min_{a_I\neq 0}\{\nu(a_I \cdot z_0^{i_0} \ldots z_r^{i_r})\} \\ &= \nu(a_{I^*} z_0^{i_0^*} \ldots z_r^{i_r^*}) = \nu(a_{I^*}) + \nu(z_0^{i_0^*} \ldots z_r^{i_r^*}) \\ &\geq \nu(a_{I^*}) + c. \end{aligned}$$

Since $\nu(h) = c$ by hypothesis, this implies that $\nu(a_{I^*}) \leq 0$, and since $a_{I^*} \in \mathcal{O}$ it follows that $\nu(a_{I^*}) \geq 0$, hence $\nu(a_{I^*}) = 0$, that is, a_I^* is a unit in $\mathcal{O}$. It follows from this that $\nu(h) = \nu(z_0^{i_0^*} \ldots z_r^{i_r^*})\}$. □

The concept of generating sequence can be generalized to a finite set of valuations. Let $\underline{\nu} = (\nu_1, \ldots, \nu_s)$ be a tuple of valuations of $\mathcal{O}$. One defines similarly the semigroup of values of $\underline{\nu}$ as:

$$S^{\underline{\nu}} = \{\underline{\nu}(f) = (\nu_1(f), \ldots, \nu_s(f)) \mid f \in \mathcal{O} \setminus \{0\}\}.$$

One has $S\underline{\nu} \subset \mathbb{R}^s_{\geq 0}$. If $\underline{c} = (c_1, \ldots, c_s) \in \mathbb{R}^s_{\geq 0}$ then one has also that the sets

$$I^{\underline{\nu}}_{\underline{c}} = \{f \in \mathcal{O} \setminus \{0\} \mid \underline{\nu}(f) \geq \underline{c}\}$$

are ideals of $\mathcal{O}$, called **valuation ideals** of $\underline{\nu}$.

Remark 4.82. Let $\underline{a} \geq \underline{b}$ for $\underline{a}$ and $\underline{b} \in \mathbb{R}^s$. We define the the partial order relation $\underline{a} \geq \underline{b}$ by $a_i \geq b_i$ for every index $i \in \{1, \ldots, s\}$. We write $\underline{a} > \underline{b}$ if $\underline{a} \geq \underline{b}$ and there is some index $j \in \{1, \ldots, s\}$ such that $a_j > b_j$.

Remark 4.83. In [DGN08], the authors define the valuation ideals $I^{\underline{\nu}}_{\underline{c}}$ of $\underline{\nu}$ restricting to tuples $\underline{c}$ in the value semigroup. Instead, we allow tuples $\underline{c} \in \mathbb{R}^s_{\geq 0}$.

The following concept was introduced in [Spi90, Definition 1.1] for one valuation.

Definition 4.84. A set of elements $\{z_j\}_{j \in J} \subset \mathcal{O}$ in the maximal ideal is a **generating sequence** of $\underline{\nu}$ for every $\underline{c} \in \mathbb{R}^c_{\geq 0}$ the ideal $I^{\underline{\nu}}_{\underline{c}}$ is generated by:

$$\{\prod_{j \in J_0} z_j^{b_j} \mid J_0 \subset J,\ \# J_0 < \infty,\ b_j \in \mathbb{Z}_{\geq 0},\ \sum_{j \in J_0} b_j \underline{\nu}(z_j) \geq \underline{c}\}.$$

A generating sequence is **minimal** if every proper subset of it fails to be a generating sequence.

Remark 4.85. If $\{z_j\}_{j \in J}$ is a generating sequence, then it forms a set of generators of the maximal ideal of $\mathcal{O}$.

Remark 4.86. We define the *graded algebra associated to* $\underline{\nu}$ by

$$\mathrm{gr}_{\underline{\nu}} \mathcal{O} := \bigoplus_{\underline{m} \in \mathbb{Z}^s_{\geq 0}} I^{\underline{\nu}}_{\underline{c}} / I^{\underline{\nu}}_{\underline{m}+\underline{1}},$$

where $\underline{1} = (1, \ldots, 1)$. Generating sequences of a family $\underline{\nu}$ of valuation and its graded algebra are closely related. Indeed, assuming the valuations of ν_i are divisorial [DGN08, Theorem 4] asserts that a system of generators $\{z_j\}_{j \in J}$ of the maximal ideal is a generating sequence for $\underline{\nu}$ if and only if their classes $\{\overline{z}_j\}_{j \in J}$ generate the graded algebra $\mathrm{gr}_{\underline{\nu}} \mathcal{O}$.

The proof of the following Lemma is analogous to the one of Lemma 4.78.

Lemma 4.87. *Let $\underline{\nu} = (\nu_1, \ldots, \nu_s)$ be a tuple of valuations of $\mathcal{O}$, and (x, y) local coordinates at O. Assume that $z_0, \ldots, z_r \in \mathcal{O}$ is a finite generating sequence for $\underline{\nu}$. Then, any nonzero function $h \in \mathcal{O}$ admits a finite expansion of the form*

$$h = \sum a_I \cdot z_0^{i_0} \ldots z_r^{i_r}, \text{ with } a_I \in \mathcal{O},$$

in such a way that for every $1 \leq j \leq s$ there exists an index $I^{(j)} = (i_0^{(j)}, \ldots, i_r^{(j)})$ with $a_{I^{(j)}} \neq 0$ such that

$$\nu_j(h) = \nu_j(z_0^{i_0^{(j)}} \ldots z_r^{i_r^{(j)}}) = \min\{\nu_j(a_I \cdot z_0^{i_0} \ldots z_r^{i_r})\}.$$

Proof. Let us assume that $z_0, \dots, z_r$ is a generating sequence for $\underline{\nu}$ and let $h \in \mathcal{O}$ be a nonzero function. If $\underline{\nu}(h) = \underline{c}$, by definition of generating sequence, h can be expanded as a finite sum

$$h = \sum_{I=(i_0,\dots,i_r)} a_I \cdot z_0^{i_0} \dots z_r^{i_r},$$

in such a way that $\underline{\nu}(z_0^{i_0} \dots z_r^{i_r}) \geq \underline{c}$ for all I with $a_I \neq 0$. In particular, this means that $\nu_j(z_0^{i_0} \dots z_r^{i_r}) \geq c_j$ for all I with $a_I \neq 0$ and for every $j = 1, \dots, s$. Lemma 4.78 implies that for every $1 \leq j \leq s$ there exists an index $I^{(j)} = (i_0^{(j)}, \dots, i_r^{(j)})$ with $a_{I^{(j)}} \neq 0$ such that

$$\nu_j(h) = \nu_j\left(z_0^{i_0^{(j)}} \dots z_r^{i_r^{(j)}}\right) = \min\left\{\nu_j\left(a_I \cdot z_0^{i_0} \dots z_r^{i_r}\right)\right\}.$$

□

Let $\underline{\nu} = (\nu_1, \dots, \nu_s)$ be a tuple of divisorial valuations of $\mathcal{O}$. We say that a modification $\Psi : T \to S$ is a **minimal embedded resolution** of $\underline{\nu}$ if Ψ is a composition of the minimal number of blowing ups of points such that the exceptional divisors E_i with $\nu_i = \nu_{E_i}$ appear on T for $i = 1, \dots, s$.

Spivakovsky described the elements of the generating sequence of a divisorial valuation [**Spi90**]. Delgado, Galindo and Núñez generalize Spivakovsky's results to a tuple of divisorial valuations. They proved that:

Theorem 4.88. [**DGN08**, Theorem 5]. *Let $\underline{\nu} = (\nu_1, \dots, \nu_s)$ be a tuple of divisorial valuations of the local ring $\mathcal{O}$ and denote by $\Psi : T \to S$ a minimal embedded resolution of $\underline{\nu}$. Let $z_j \in \mathcal{O}$, for $j = 0, \dots, r$ be irreducible elements such that the branches defined by them define a set of maximal contact curves of Ψ (see Definition 2.66). Then, $(z_0, \dots, z_r)$ form a minimal generating sequence for $\underline{\nu}$.*

Corollary 4.89. *Let $\Psi : T \to S$ be the minimal embedded resolution of a plane curve singularity C. Consider the tuple of divisorial valuations $\underline{\nu} := (\nu_1, \dots, \nu_s)$ defined by those exceptional prime divisors which correspond to vertices of valence ≥ 3 on the tree $G(\pi, C)$. Let R be a smooth branch transversal to C. Choose a maximal contact completion $\bar{C}$ of C with respect to R. Denote by $z_j \in \mathcal{O}$, for $j = 0, \dots, r$ irreducible elements such that their product defines $\bar{C}$. Then, $(z_0, \dots, z_r)$ is a generating sequence of $\underline{\nu}$.*

Proof. Since R is transversal to C then the minimal embedded resolution of C can be built as a maximal contact toroidal resolution of C with respect to R.

Proposition 4.62 implies that we have a bijection between the set of ramification points of the tree $\Theta_R(\bar{C})$ and the set of exceptional prime divisors which define vertices of valency ≥ 3 of the tree $G(\Psi, C)$. Thus, if $P_1, \dots, P_s$ are the ramification points of the tree $\Theta_R(\bar{C})$ we can assume that $\nu_i = \nu_{P_i}$ for $i = 1, \dots, s$. Notice that the points $P_1, \dots, P_s$ are the marked points of the tree $\Theta_R(C)$ of valency > 1.

If E_i is a component of $\Psi^{-1}(O)$ which intersects the strict transform of C, it must be of the form $E_{P_{j(i)}}$ for some $1 \leq j(i) \leq s$. By definition this implies that π is a minimal embedded resolution of $\underline{\nu}$.

By Remark 2.66 we can take a set of maximal contact curves for Ψ among the irreducible components of $\bar{C}$. Then, Theorem 4.88 implies that the sequence $(z_0, \dots, z_r)$ contains a minimal generating sequence for $\underline{\nu}$, that is, $(z_0, \dots, z_r)$ is a generating sequence for $\underline{\nu}$. □

The following Lemma can be seen as a partial converse of Lemma 4.78, where for practical reasons we replace the local ring $\mathbb{C}\{x, y\}$ by the polynomial ring $\mathbb{C}\{x\}[y]$. This Lemma will be important in order to describe generating functions in terms of expansions in Section 4.5 (see Seccion 2.6 of Chapter 2).

LEMMA 4.90. *Let ν be a valuation of $\mathcal{O}_{S,O}$ and (x, y) local coordinates at O. Let us consider elements $x, y = z_0, \dots, z_r \in \mathbb{C}\{x\}[y] \subset \mathcal{O}$. Assume that for any $0 \neq h \in \mathbb{C}\{x\}[y]$ we have an expansion*

$$h = \sum_{I=(i_0,i_1,\dots,i_r)} a_I(x) \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

such that

$$\nu(h) = \min_I \{\nu(\, a_I(x)\, z_0^{i_0} \dots z_r^{i_r}\,)\}. \tag{4.91}$$

Then, $(x, z_0, \dots, z_r)$ form a generating sequence for ν.

PROOF. We identify the ring $\mathcal{O}$ with $\mathbb{C}\{x, y\}$ by using the local coordinates (x, y). Let us consider $0 \neq H \in \mathbb{C}\{x, y\}$ with $H(0, 0) = 0$. Using that $\mathbb{C}\{x, y\}$ is a UFD we factor $H = x^d H'$, with $H'(0, y) \neq 0$ and $d \geq 0$. By the Weierstrass Preparation Theorem 2.1 applied to H', there exists a unit $\varepsilon \in \mathbb{C}\{x, y\}$ and a Weierstrass polynomial $h \in \mathbb{C}\{x\}[y]$ such that $H = \varepsilon \cdot h \cdot x^d$. It follows that:

$$\nu(H) = \nu(\varepsilon) + \nu(x^a h) = \nu(x^d) + \nu(h),$$

since the valuation of a unit is equal to zero. Then, using the hypothesis we get a finite expansion

$$H = \sum_{I=(i_0,i_1,\dots,i_r)} \varepsilon \cdot a_I(x) \cdot x^d \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

with $\nu(a_I(x) \cdot x^d \cdot z_0^{i_0} \dots z_r^{i_r}) \geq \nu(H)$. This implies that $x, y = z_0, \dots, z_r \in \mathcal{O}$ is a generating sequence for ν. □

COROLLARY 4.92. *Let $\underline{\nu}$ be a tuple of valuations of $\mathcal{O}_{S,O}$ and (x, y) local coordinates at O. Let us consider elements $x, y = z_0, \dots, z_r \in \mathbb{C}\{x\}[y] \subset \mathcal{O}$. Assume that for any $0 \neq h \in \mathbb{C}\{x\}[y]$ we have an expansion*

$$h = \sum_{I=(i_0,i_1,\dots,i_r)} a_I(x) \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

such that for every $1 \leq j \leq s$ there exists an index $I^{(j)} = (i_0^{(j)}, \dots, i_r^{(j)})$ with $a_{I^{(j)}} \neq 0$ such that

$$\nu_j(h) = \nu_j(z_0^{i_0^{(j)}} \dots z_r^{i_r^{(j)}}) = \min\{\nu_j(a_I \cdot z_0^{i_0} \dots z_r^{i_r})\}.$$

Then, $(x, z_0, \dots, z_r)$ form a generating sequence for $\underline{\nu}$.

PROOF. As before, we start by identifying $\mathcal{O}$ with $\mathbb{C}\{x, y\}$ by using the local coordinates (x, y). Using the arguments in the proof of Lemma 4.90 and the hypothesis we obtain a finite expansion

$$H = \sum_{I=(i_0,i_1,\dots,i_r)} \varepsilon \cdot a_I(x) \cdot x^d \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

such that for every $1 \leq j \leq s$, $\nu_j(H) \geq \nu_j(a_I(x) \cdot x^d \cdot z_0^{i_0} \dots z_r^{i_r})$, i.e.,

$$\underline{\nu}(a_I(x) \cdot x^d \cdot z_0^{i_0} \dots z_r^{i_r}) \geq \underline{\nu}(H).$$

This implies that $x, y = z_0, \dots, z_r \in \mathcal{O}$ is a generating sequence for $\underline{\nu}$. □

Throughout the following sections the smooth reference branch $R = V(x)$ is fixed.

4.5. Valuations and Newton polyhedra via toroidal resolutions

The aim of this section is to provide a different proof of the results of Delgado, Galindo and Nuñez.

We introduce first the notation that we will use in this section.

NOTATION 4.93. Let (S, O) be a germ of smooth surface and R a smooth reference branch. Let L_g be a branch with g characteristic exponents with respect to R. Denote by $L_0, \dots, L_g$ a complete system of semi-roots of L_g with respect to R (see Definition 2.24), and by $R_1 < \dots < R_g$ the points of discontinuity of the index function $\mathbf{i}_R$ on the segment $[R, L_g]$.

REMARK 4.94. Let L_g be as in Notation 4.93. The Eggers-Wall tree of its maximal contact completion, $\Theta_R(\bar{L}_g)$, is shown in Figure 8. Its maximal contact decomposition (see Example 4.57) has segments

$$[R = R_0, L_0],\ [R_1, L_1], \dots, [R_g, L_g]. \tag{4.95}$$

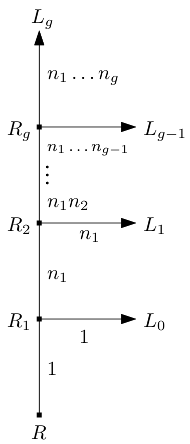

FIGURE 8. Maximal contact completion of the tree of a plane branch.

For each $1 \le i \le g$, one has that:

$$R_i \in [R_{i-1}, L_{i-1}] \text{ and } \mathbf{i}_R(R_i) = \mathbf{i}_R(L_{i-1}),$$

and the renormalized exponent is written in terms of Newton pairs (Definition 2.13)

$$\mathbf{e}_{R_{i-1}}(R_i) = \frac{m_i}{n_i}. \tag{4.96}$$

LEMMA 4.97. *With the Notation 4.93, let us consider the sequence of generators $\bar{b}_0, \dots, \bar{b}_g$ of the semigroup of the branch L_g, with respect to R. Then:*

$$\begin{array}{rcl} \nu_{R_g}(R) & = & \bar{b}_0; \\ \nu_{R_g}(L_{j-1}) & = & \bar{b}_j, \text{ for } 1 \le j \le g; \\ \nu_{R_g}(L_g) & = & \mathbf{i}_{R_{g-1}}(L_g)\bar{b}_g. \end{array}$$

More generally, if D is a branch such that $\pi_{R,L_g}(D) = R_g$, then

$$\nu_{R_g}(D) = \mathbf{i}_{R_{g-1}}(D)\bar{b}_g.$$

PROOF. By definition we have that L_g is a curvetta at R_g. By Proposition 4.56, we have that an embedded resolution of $\bar{C}$ such that L_g verifies the conditions in Proposition 2.79. Thus,

$$\nu_{R_g}(R) = (R, L_g)_O = \bar{b}_0$$

and

$$\nu_{R_g}(L_{j-1}) = (L_{j-1}, L_g)_O \overset{(2.33)}{=} \bar{b}_j, \text{ for } 1 \leq j \leq g.$$

In particular, by Corollary 2.101 we get:

$$\bar{b}_g = (L_{g-1}, L_g)_O \overset{(2.101)}{=} \mathbf{i}_R(L_g)\mathbf{i}_R(L_{g-1})\mathbf{c}_R(R_g) \tag{4.98}$$

If D is a branch such that $\pi_{R,L_g}(D) = R_g$, then by the same argument as before: $\nu_{R_g}(D) = (D, L_g)_O$ and (2.101) we get

$$(D, L_g)_O \overset{(2.101)}{=} \mathbf{i}_R(L_g)\mathbf{i}_R(D)\mathbf{c}_R(R_g) \overset{(4.98)}{=} \frac{\mathbf{i}_R(D)}{\mathbf{i}_R(L_{g-1})}\bar{b}_g = \mathbf{i}_{R_{g-1}}(D)\bar{b}_g,$$

where we decompose $\mathbf{i}_R(D) = \mathbf{i}_R^+(R_{g-1})\mathbf{i}_{R_{g-1}}(D)$, and we use that $\mathbf{i}_R^+(R_{g-1}) = \mathbf{i}_R(L_{g-1})$.

If K is a curvetta at R_g different from L_g then by Proposition 2.79:

$$\nu_{R_g}(L_g) = (L_g, K)_O = \nu_{R_g}(K).$$

Since by definition $\mathbf{i}_{R_{g-1}}(K) = \mathbf{i}_{R_{g-1}}(L_g)$, the formula for $\nu_{R_g}(L_g)$ follows. □

We will consider now the expansions of elements of the ring $\mathbb{C}\{x\}[y]$ in terms of a complete sequence of semi-roots of a branch (see Section 2.6).

DEFINITION 4.99. Let us consider elements $z_0, \dots, z_g \in \mathbb{C}\{x\}[y]$ representing the branches $L_0, \dots, L_g$. If $0 \neq h \in \mathbb{C}\{x\}[y]$ an expansion

$$h = \sum c_{i_0 \dots i_g}(x) \cdot z_0^{i_0} \dots z_g^{i_g}, \tag{4.100}$$

of the form (2.118) is called an **expansion with respect to the semi-roots** $L_0, \dots, L_g$. We often denote by

$$\mathcal{M}_I(h) := c_{i_0 \dots i_g}(x) \cdot z_0^{i_0} \dots z_g^{i_g},$$

or simply by $\mathcal{M}_I$, if h is clear from the context, the term of the expansion (2.118) corresponding to the index $I = (i_0, \dots, i_g)$.

LEMMA 4.101. *Let L_g be a plane branch with g characteristic exponents with respect to a smooth reference branch R. Then, $R, L_0, \dots, L_g$ is a generating sequence for the vanishing order valuations ν_{L_g} and ν_R.*

PROOF. In both cases we use Lemma 4.90 to apply the expansions of $0 \neq h \in \mathbb{C}\{x\}[y]$ in terms of the semi-roots. By Definition 2.71, the value of ν_{L_g} on a generalized monomial is $\nu_{L_g}(c_{i_0 \dots i_g}(x) z_0^{i_0} \dots z_g^{i_g}) = i_g$. If $0 \neq h \in \mathbb{C}\{x\}[y]$ then we get $\nu_{L_g}(h)$ from the values of ν_{L_g} on the terms of the expansion (4.100), namely: $\nu_{L_g}(h) = \min\{i_g \mid c_{i_0 \dots i_g} \neq 0\}$.

Similarly, we get for ν_R that: $\nu_R(c_{i_0 \dots i_g}(x) z_0^{i_0} \dots z_g^{i_g}) = \nu_R(c_{i_0 \dots i_g})$. If $0 \neq h \in \mathbb{C}\{x\}[y]$ then we get $\nu_R(h)$ from the values of ν_R on the terms of the expansion (4.100), namely: $\nu_R(h) = \min\{\nu_R(c_{i_0 \dots i_g}(x)) \mid c_{i_0 \dots i_g} \neq 0\}$. □

The following classical notion will be useful for studying the properties of the expansions:

DEFINITION 4.102. Recall that the **lexicographical order** on $\mathbb{Z}^2$ is defined by

$$(a, b) \leq_{\text{lex}} (c, d) \Leftrightarrow \begin{cases} \text{if } a \leq c \\ \text{or } a = c, \; b \leq d. \end{cases} \tag{4.103}$$

The **leading term** of a series $0 \neq h = \sum c_{a,b} x^a y^b \in \mathbb{C}\{x, y\}$ is

$$\text{LT}(h) = x^{a_0} y^{b_0} \text{ with } (a_0, b_0) = \min_{\leq_{\text{lex}}}\{(a, b) \mid (a, b) \in \text{Supp}(h)\}.$$

We only use the notion of leading term with respect to the lexicographical order.

NOTATION 4.104. If D is a plane curve at (S, O) defined by $f_D \in \mathcal{O}$ we denote, similarly as in Definition 4.66, by

$$\mathrm{LT}_{R_k, L_k}(D)$$

(or also by $\mathrm{LT}_{R_k,L_k}(f_D)$) the leading term of $\Psi_k^*(f_D)$, with respect to local coordinates (x_k, y_k) defining a cross (R_k, L_k) appearing in the toroidal resolution process.

EXAMPLE 4.105. Let $z_0^{i_0} \dots z_g^{i_g}$ be a monomial in the semi-roots as in Definition 4.99. Then the leading term is

$$\mathrm{LT}(z_0^{i_0} \dots z_g^{i_g}) = y^{i_0 + i_1 n_1 + \dots + i_g n_1 \dots n_g}.$$

where the $n_1, \dots, n_g$ are the integers defined in (2.12) for the branch L_g with respect to R.

REMARK 4.106. Let (x, y) be local coordinates at (S, O) representing the cross (R, L). If C is a plane curve singularity at (S, O), then $\mathrm{LT}(f_C(x, y))$ is the monomial defined by the smallest vertex of the Newton polygon $\mathcal{N}_{R,L}(C)$ with respect to the lexicographical order. For this reason, we denote $\mathrm{LT}(f_C(x, y))$ also by $\mathrm{LT}_{R,L}(C)$ or $\mathrm{LT}_{R,L}(f_C)$.

LEMMA 4.107. *With the notation 4.93, if $0 \leq k \leq g$ we denote by $\bar{b}_0, \dots, \bar{b}_k$ the sequence of generators of the semigroup of L_k with respect to R. Then, we have:*

$$\left\{ \begin{array}{lcllcl} \mathcal{N}_{R_k,L_k}(R) & = & (\bar{b}_0, 0) & + & \mathbb{R}^2_{\geq 0}, \\ \mathcal{N}_{R_k,L_k}(L_0) & = & (\bar{b}_1, 0) & + & \mathbb{R}^2_{\geq 0}, \\ \dots & \dots & \dots & & \\ \mathcal{N}_{R_k,L_k}(L_{k-1}) & = & (\bar{b}_k, 0) & + & \mathbb{R}^2_{\geq 0}, \\ \mathcal{N}_{R_k,L_k}(L_k) & = & (\mathbf{i}_{R_{k-1}}(L_k)\bar{b}_k, 1) & + & \mathbb{R}^2_{\geq 0}. \end{array} \right. \tag{4.108}$$

In addition, if $k < g$, let us denote

$$\boldsymbol{e}_{R_k}(R_{k+1}) = \frac{m_{k+1}}{n_{k+1}} \text{ as irreducible fraction.}$$

Then, for $k < j \leq g$, we have:

$$\mathcal{N}_{R_k,L_k}(L_j) = (\nu_{R_k}(L_j), 0) + \mathbf{i}_{R_k}(L_j) \left\{ \frac{m_{k+1}/n_{k+1}}{1} \right\}, \tag{4.109}$$

and

$$\mathrm{LT}_{R_k,L_k}(L_j) = x_k^{\nu_{R_k}(L_j)} y_k^{\mathbf{i}_{R_k}(L_j)}. \tag{4.110}$$

In addition, if $\mathcal{M}_I := c_{i_0 \dots i_g}(x) \cdot z_0^{i_0} \dots z_g^{i_g}$ then we obtain that

$$\mathrm{LT}_{R_k,L_k}(\mathcal{M}_I) = x_k^{\nu_{R_k}(\mathcal{M}_I)} y_k^{i k + i_{k+1} \mathbf{i}_{R_k}(L_{k+1}) + \dots + i_g \mathbf{i}_{R_k}(L_g)}. \tag{4.111}$$

PROOF. By Lemma 4.97 we have that

$$\nu_{R_k}(R) = \bar{b}_0, \quad \nu_{R_k}(L_0) = \bar{b}_1, \quad \nu_{R_k}(L_1) = \bar{b}_2, \quad \dots, \quad \nu_{R_k}(L_{k-1}) = \bar{b}_k, \tag{4.112}$$

and if $k \leq j \leq g$ then

$$\nu_{R_k}(L_j) = \mathbf{i}_{R_{k-1}}(L_j)\bar{b}_k.$$

Taking into account the monomial form of the maps defining the toroidal resolution we get that the Newton polygons (4.108) have only one vertex of the required form.

If $k < j \leq g$, it follows from Theorem 4.52 and Proposition 2.107 that the Newton polygon $\mathcal{N}_{R_k,L_k}(L_j)$ has the form (4.109). Formula (4.110) follows from (4.109) and the Definitions 4.66. Finally (4.111) is consequence of (4.110) and of (4.108). ☐

LEMMA 4.113. *Let $I = (i_0, \dots, i_g)$ and $I' = (i'_0, \dots, i'_g)$ be two indices corresponding to nonzero terms $\mathcal{M}_I(h)$ and $\mathcal{M}_{I'}(h)$ in the expansion of $h \in \mathbb{C}\{x\}[y]$ with respect to $L_0, \dots, L_g$ (see Definition 4.99). If*

$$\mathrm{LT}_{R_k, L_k}(\mathcal{M}_I(h)) = \mathrm{LT}_{R_k, L_k}(\mathcal{M}_{I'}(h))$$

then $\mathcal{M}_I(h) = \mathcal{M}_{I'}(h')$.

PROOF. By (4.111) the hypothesis implies that

$$i_k + i_{k+1}\mathbf{i}_{R_k}(L_{k+1}) + \cdots + i_g\mathbf{i}_{R_k}(L_g) = i'_k + i'_{k+1}\mathbf{i}_{R_k}(L_{k+1}) + \cdots + i'_g\mathbf{i}_{R_k}(L_g)$$

Notice that $\mathbf{i}_{R_k}(L_j) = n_{k+1} \cdots n_j$ for $k < j \leq g$ (where n_i, $i = 1, \dots, g$, denote the integers (2.12) associated to the branch L_g, with respect to R). Since $n_i > 1$, by Lemma 2.120 we conclude that

$$i_j = i'_j, \text{ for } j \geq k. \tag{4.114}$$

On the other hand, by (4.111) the hypothesis also imply the equality of valuations $\nu_{R_k}(\mathcal{M}_I) = \nu_{R_k}(\mathcal{M}_{I'})$. Taking into account (4.114) we get:

$$\nu_{R_k}(c_{i_0 \dots i_g}(x) \cdot z_0^{i_0} \dots z_{k-1}^{i_{k-1}}) = \nu_{R_k}(c_{i'_0 \dots i'_g}(x) \cdot z_0^{i'_0} \dots z_{k-1}^{i'_{k-1}}).$$

Let us denote $j_0 = \mathrm{ord}_x(c_{i_0 \dots i_g})$ and $j'_0 = \mathrm{ord}_x(c_{i_0 \dots i_g})$. Thanks to (4.112) the previous equality can be rewritten as:

$$j_0\bar{b}_0 + \sum_{0 \leq j < k} i_j\bar{b}_{j+1} = j'_0\bar{b}_0 + \sum_{0 \leq j < k} i'_j\bar{b}_{j+1},$$

where $\bar{b}_0, \dots, \bar{b}_k$ is the sequence of generators of the semigroup of L_k with respect to R. It follows that

$$(i_{k-1} - i'_{k-1})\bar{b}_k = (j'_0 - j_0)\bar{b}_0 + \sum_{0 \leq l < k-1} (i'_l - i_l)\bar{b}_{l+1}, \tag{4.115}$$

By the definition of the expansion $|i_{k-1} - i'_{k-1}| < n_k$. If $i_{k-1} - i'_{k-1} \neq 0$ then (4.115) would contradict Remark 2.34. We can apply inductively the same argument and we obtain that $i_j = i'_j$ for $j = 0, \dots, k-1$ (and also $j_0 = j'_0$). This shows that $I = I'$ hence the corresponding terms $\mathcal{M}_I(h)$ and $\mathcal{M}_{I'}(h')$ are also equal. □

LEMMA 4.116 (*Toric* description of valuations). *Let C, D be plane branches and let $\langle R, C, D\rangle$ be the ramification point on the tree $\Theta_R(C \cup D)$. Denote by R_k be the root of the index level of $\langle R, C, D\rangle$ on the tree $\Theta_R(C \cup D)$. Let P be a rational point of the tree $\Theta_R(C \cup D)$ such that $R_k <_R P \leq_R \langle R, C, D\rangle$. Let us write its renormalized exponent $\mathbf{e}_{R_k}(P) = \frac{m}{n}$ with $\gcd(n, m) = 1$. Then:*

$$\nu_P(D) = n \cdot \nu_{R_k}(D) + m \cdot \mathbf{i}_{R_k}(D).$$

PROOF. This hypothesis implies that

$$\mathbf{i}_R(P) = \mathbf{i}^+_R(R_k) \tag{4.117}$$

By definition (see (4.41)) we have:

$$\mathbf{i}^+_R(P) = \mathbf{i}^+_R(R_k)\mathbf{i}^+_{R_k}(P), \tag{4.118}$$

$$\mathbf{i}^+_{R_k}(P) = n \text{ and } m = \mathbf{i}^+_{R_k}(P)\,\mathbf{e}_{R_k}(P), \tag{4.119}$$

and

$$\mathbf{i}_R(D) = \mathbf{i}^+_R(R_k)\,\mathbf{i}_{R_k}(D). \tag{4.120}$$

By Formula (4.71) and the hypothesis $P = \langle R, C, D\rangle$ we have that

$$\nu_P(D) = \mathbf{i}_R(D)\,\mathbf{i}_R^+(P)\,\mathbf{c}_R(P).$$

Since $R_k \leq P$, by definition of the contact function (see (2.100)), we can write :

$$\mathbf{c}_R(P) = \mathbf{c}_R(R_k) + \frac{\mathbf{e}_R(P) - \mathbf{e}_R(R_k)}{\mathbf{i}_R(P)}. \tag{4.121}$$

Since $R_k \leq D$ it follows that $\langle R, R_k, D\rangle = R_k$ and

$$\begin{aligned} n \cdot \nu_{R_k}(D) &\overset{(4.119)}{=} \mathbf{i}_R^+(P)\nu_{R_k}(D) \\ &\overset{(4.71)}{=} \mathbf{i}_R^+(P)\mathbf{i}_R^+(R_k)\mathbf{i}_R(D)\mathbf{c}_R(\langle R, R_k, D\rangle) \\ &\overset{(4.120)}{=} \mathbf{i}_R^+(P)\mathbf{i}_R(D)\mathbf{c}_R(R_k). \end{aligned}$$

Similarly, we get that

$$\begin{aligned} m \cdot \mathbf{i}_{R_k}(D) &\overset{(4.119)}{=} \mathbf{i}_{R_k}(D)\mathbf{i}_{R_k}^+(P)\,\mathbf{e}_{R_k}(P) \\ &\overset{(4.41)}{=} \mathbf{i}_{R_k}(D)\mathbf{i}_{R_k}^+(P)\,\mathbf{i}_R^+(R_k)\,(\mathbf{e}_R(P) - \mathbf{e}_R(R_k)) \\ &\overset{(4.118)}{=} \mathbf{i}_{R_k}(D)\mathbf{i}_R^+(P)\,(\mathbf{e}_R(P) - \mathbf{e}_R(R_k)) \\ &\overset{(4.120)}{=} \mathbf{i}_R(D)\mathbf{i}_R^+(P)\tfrac{\mathbf{e}_R(P)-\mathbf{e}_R(R_k)}{\mathbf{i}_R^+(R_k)} \\ &\overset{(4.117)}{=} \mathbf{i}_R(D)\mathbf{i}_R^+(P)\tfrac{\mathbf{e}_R(P)-\mathbf{e}_R(R_k)}{\mathbf{i}_R(P)}. \end{aligned} \tag{4.122}$$

Adding these two equalities, taking into account (4.121), provides the required relation. □

DEFINITION 4.123. Let ν be a valuation of $\mathcal{O}_{S,O}$ and (x, y) local coordinates at O. Consider a set of generators, $x, z_0, \dots, z_r \subset \mathbb{C}\{x\}[y]$, of the maximal ideal, i.e., $(x, y) = (x, z_0, \dots, z_g)$. Then **the valuation ν is monomialized at $h \in \mathbb{C}\{x\}[y]$, with respect to $z_0, \dots, z_r$** if there exists an expansion

$$h = \sum_{I=(i_0,i_1,\dots,i_r)} a_I(x) \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

such that

$$\nu(h) = \min_I \left\{ \nu\left(a_I(x)\, z_0^{i_0} \dots z_r^{i_r}\right)\right\}. \tag{4.124}$$

We say that ν is a **monomial valuation with respect to** $z_0, \dots, z_r \subset \mathbb{C}\{x\}[y]$ if ν is monomialized at every function $h \in \mathbb{C}\{x\}[y]$, with respect to $z_0, \dots, z_r$. If $L_i = V(z_i)$ for $i = 0, \dots, r$, we say also that ν is a monomial valuation with respect to $L_0, \dots, L_r$.

4.6. Another proof of the monomialization of divisorial valuations

In Section 4.3 we showed that to any rational point P in the tree of a plane curve germ $\Theta_R(C)$ we can associate a divisorial valuation ν_P. If P is an irrational point, we will associate in Section 4.7 a valuation which is monomial at a suitable level (segment) of a maximal contact decomposition of the tree. If P is an end, we associate to it the order of vanishing semivaluation for the corresponding branch.

The aim of this section is to show that the semivaluation ν_P is a *monomial valuation* with respect to a suitable sequence of semi-roots. For P an interior point (either rational or irrational), this can be seen as an aspect of the theory studied by Spivakovsky in [**Spi90**, Definition 1.1 and Theorem 8.6]. We have the following result.

THEOREM 4.125 (Monomialization of divisorial valuations). *With Notation 4.93, let P be a rational point in the segment (R, L_g). Then, the divisorial valuation ν_P is a monomial valuation with respect to $L_0, L_1, \ldots, L_g$.*

PROOF. We keep the notation of the previous discussion. In particular there is an integer $0 \leq k \leq g$ such that $P \in (R_k, L_k)$.

- If $k = g$ then it follows from (4.108) that if $\mathcal{M}_I$ is a generalized monomial in the expansion of a function $h \in \mathbb{C}\{x\}[y]$, then the Newton polygon $\mathcal{N}_{R_g, L_g}(\mathcal{M}_I)$ has only one vertex corresponding to its leading term $\mathrm{LT}_{R_g,L_g}(\mathcal{M}_I)$. Then, by Lemma 4.113 applied to $D = R, L_0, \ldots, L_{g-1}$ and $C = L_g$ we get that

$$\nu_P(h) = \min_{c_I \neq 0}\{\nu_P(\mathcal{M}_I)\}.$$

- If $k < g$ since $P \in [R, L_g]$ our assumptions imply that $P \in (R_k, R_{k+1}]$. It follows that

$$\mathbf{e}_{R_k}(P) = \frac{m}{n} \leq \mathbf{e}_{R_k}(R_{k+1}) = \frac{m_{k+1}}{n_{k+1}}. \tag{4.126}$$

This implies that the valuation with respect to P of a generalized monomial

$$\mathcal{M}_I = c_{i_0 \ldots i_g}(x)\, z_0^{i_0} \ldots z_g^{i_g}$$

is attained at the leading term:

$$\nu_P(\mathcal{M}_I) = \nu_P\left(\mathrm{LT}_{R_k,L_k}(\mathcal{M}_I)\right) = \big\langle (n, m), (\nu_{R_k}(\mathcal{M}_I), \deg_{\mathrm{y_k}}(\mathcal{M}_I))\, \big\rangle. \tag{4.127}$$

Let $\mathcal{M}_I$ and $\mathcal{M}_{I'}$ be two different generalized monomials in the expansion of a function $0 \neq h \in \mathbb{C}\{x\}[y]$ such that $\nu_P(\mathcal{M}_I) = \nu_P(\mathcal{M}_{I'})$. By Lemma 4.113, we have $\mathrm{LT}_k(\mathcal{M}_I) \neq \mathrm{LT}_k(\mathcal{M}_{I'})$. By (4.127) this implies

$$\nu_P(\mathcal{M}_I + \mathcal{M}_{I'}) = \nu_P(\mathcal{M}_I).$$

Since we have only finitely many terms in the expansion, we apply inductively this argument to get the assertion:

$$\nu_P(h) = \min_{c_I \neq 0}\big\{\nu_P(\mathcal{M}_I)\big\}.$$

□

REMARK 4.128. The divisor R_{k+1} appears in the minimal resolution of L_{k+1}, but notice that in order to monomialize the associated divisorial valuation we just need the set of semi-roots $L_0, \ldots, L_k$. In the sense of Definition 4.75 the set $\{x, z_0, \ldots, z_k\}$ is a minimal generating sequence of $\nu_{R_{k+1}}$. Notice that enlarging a generating sequence by adding more elements produces another generating sequence.

LEMMA 4.129. *Let L_g denote a plane branch with g characteristic exponents with respect to the smooth reference branch R. With Notation 4.93, let us fix an integer $0 \leq k \leq g$. Assume that $0 \neq h \in \mathbb{C}\{x\}[y]$ is a polynomial such that*

$$\langle R_k, L_k, C_{h_i} \rangle \leq_R L_g, \tag{4.130}$$

for every irreducible factor h_i of h. Then:

(1) The Newton polygon $\mathcal{N}_{R_k,L_k}(h)$ is equal to the convex hull of the Newton polygons

$$\mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h)),$$

with $\mathcal{M}_I(h)$ denoting the terms of the expansion of h with respect to $L_0, \ldots, L_g$ (see Definitions 4.66 and 4.99).

(2) If $P \in [R_k, L_k]$ is a rational point or if $P = L_k$, then:

$$\nu_P(h) = \min_I\{\nu_P(\mathcal{M}_I(h))\}. \tag{4.131}$$

PROOF. Let P be a rational point in the segment $[R_k, L_k]$. Let us express its renormalized exponent as a primitive fraction: $\mathbf{e}_{R_k}(P) = \frac{a_P}{b_P}$ with $\gcd(a_P, b_P) = 1$. If $0 \neq h \in \mathbb{C}\{x\}[y]$ then $\nu_P(h)$ is equal to the value of the support function $\Phi_{\mathcal{N}_{R_k,L_k}(h)}$ of the Newton polygon $\mathcal{N}_{R_k,L_k}(h)$ on the primitive vector (b_P, a_P), that is,

$$\nu_P(h) = \Phi_{\mathcal{N}_{R_k,L_k}(h)}(b_P, a_P). \tag{4.132}$$

If $P = L_k$ the same assertions holds taking $(b_P, a_P) = (0, 1)$.

Let us denote by

$$\{(b_{P_i}, a_{P_i}) \mid i = 0, \dots, r\}$$

the set of of primitive vectors which are orthogonal vectors to the edges of the Newton polygon $\mathcal{N}_{R_k,L_k}(h)$. We label the corresponding rational points in such a way that

$$P_0 = R_k < P_1 < \cdots < P_r = L_k.$$

If $1 \leq i < r$ then there is an irreducible factor h_i of h such that

$$P_i = \langle R_k, L_k, C_{h_i} \rangle \,,$$

and the hypothesis implies that

$$P_i \in [R_k, L_g]. \tag{4.133}$$

We prove first the inclusion:

$$\mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h)) \subset \mathcal{N}_{R_k,L_k}(h). \tag{4.134}$$

We start by showing that (4.131) holds for $P = L_k$ (recall that we denote $\nu_{L_k} := \nu_{L_k}$). We distinguish the cases $k = g$ and $k < g$:

- If $k < g$ the hypothesis (4.130) implies that $\nu_{L_k}(h) = 0$, hence $\nu_{L_k}(\mathcal{M}_I(h)) = 0$, for all I, since $\nu_{L_k}(h) \geq \min_I\{\nu_{L_k}(\mathcal{M}_I(h))\} \geq 0$.
- If $k = g$ then (4.131) holds for $P = L_g$ by Lemma 4.101.

In addition, the equality (4.131) holds for $P \in \{R_k, P_1, \dots, P_{r-1}\}$ by Theorem 4.125, since $P \leq L_g$ by the hypothesis (4.133).

If $P \in \{R_k, P_1, \dots, P_r, L_k\}$ we have shown that (4.131) holds, that is, one has

$$\nu_P(h) \leq \nu_P(\mathcal{M}_I(h)), \tag{4.135}$$

for every term $\mathcal{M}_I(h)$ in the expansion of h with respect to $L_0, \dots, L_g$. Thanks to (4.132), the inequality (4.135) translates into an inequality of support functions

$$\Phi_{\mathcal{N}_{R_k,L_k}(h)}(b_P, a_P) \leq \Phi_{\mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h))}(b_P, a_P).$$

The inclusion (4.134) follows from this by applying Corollary 1.28.

We prove now the statement (1). Taking into account the inclusion (4.134) it is enough to prove that for every vertex (c, d) of $\mathcal{N}_{R_k,L_k}(h)$ there exists an index I_0 such that $(c, d) \in \mathcal{N}_{R_k,L_k}(\mathcal{M}_{I_0}(h))$. Assume by contradiction that $(c, d) \notin \mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h))$, for every term $\mathcal{M}_I(h)$. Taking into account that $h = \sum_I \mathcal{M}_I(h)$, we would obtain that $(c, d) \notin \mathcal{N}_{R_k,L_k}(h)$, a contradiction.

Finally, we prove that (4.131) holds for every rational point $P \in [R_k, L_k]$. The assertion (1), reformulated in terms of support functions, provides the equality:

$$\Phi_{\mathcal{N}_{R_k,L_k}(h)}(b_P, a_P) = \min_I\{\Phi_{\mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h))}(b_P, a_P)\}$$

(see Lemma 1.29). This ends the proof of (4.131), since

$$\nu_P(h) = \Phi_{\mathcal{N}_{R_k,L_k}(h)}(b_P, a_P) \tag{4.136}$$

and

$$\nu_P(\mathcal{M}_I(h)) = \Phi_{\mathcal{N}_{R_k,L_k}(\mathcal{M}_I(h))}(b_P, a_P) \tag{4.137}$$

by (4.132). □

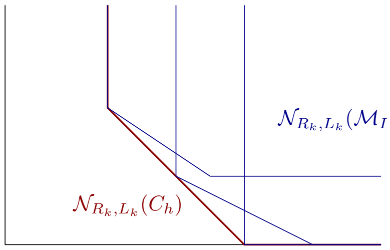

FIGURE 9. This figure is an illustration of the proof of Lemma 4.129. It shows a possible shape of the Newton polygon of $\mathcal{N}_{R_k,L_k}(h)$, in comparison with some Newton polygons of terms $\mathcal{N}_{R_k,L_k}(\mathcal{M}_I)$ when $k < g$. Notice that the inclination of the compact edge of the Newton polygon of h is less or equal than the inclination of the compact edge of the Newton polygon of the $\mathcal{M}_I$.

REMARK 4.138. If $P \in [R_k, L_k]$ Lemma 4.129 proves that the value $\nu_P(h)$ is monomialized with respect to $L_0, \dots, L_g$, for those functions h satisfying the hypotheses. This is weaker than saying that ν_P is a monomial valuation with respect to $L_0, \dots, L_g$ (see Definition 4.123).

EXAMPLE 4.139. Let us consider local coordinates (x, y) at O and branches

$$R = V(x), \quad L_0 = V(y), \quad L_1 = V(y^2 - x^7), \quad L_1' = V(y^2 - x^3).$$

One has

$$\langle R, L_0, L_1' \rangle < L_1$$

hence the hypothesis of Lemma 4.129 are satisfied for L_1 and $k = 0$, see Figure 10.

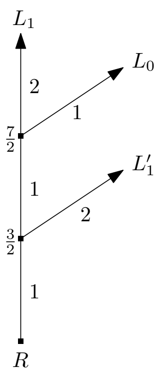

FIGURE 10. The Eggers-Wall tree $\Theta_R(L_0 + L_1 + L_1')$.

The expansion of $f_{L_1'} = y^2 - x^3$ with respect to f_{L_0}, f_{L_1} (Definition 4.99) has terms

$$f_{L_1} = y^2 - x^7, x^7, -x^3.$$

The Newton polygon of each of these terms is contained in the Newton polygon of $f_{L_1'}$, and the convex hull of the the Newton polygons of the terms is equal to the Newton polygon of $f_{L_1'}$, as stated by Lemma 4.129. However, the expansion of f_{L_1} with respect $f_{L_0}, f_{L_1'}$ has terms

$$f_{L_1'} = y^2 - x^3, x^3, -x^7,$$

but the Newton polygons the term $f_{L_1'}$ and of x^3 are not contained in the Newton polygon of f_{L_1} (see Figure 11).

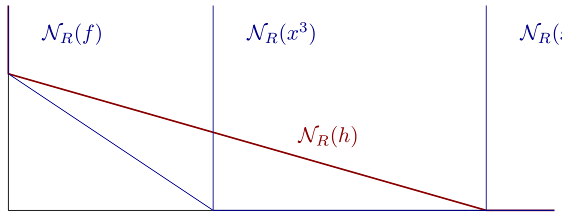

FIGURE 11. The Newton polygon of $h = f_{L_1}$, in comparison with the Newton polygons of $f = f_{L_1'}$, of x^3 and of x^7.

COROLLARY 4.140. *With the notation of Lemma 4.129, let $P \in \Theta_R(\sum_{k=0}^{g} L_k)$ be a rational point such that for every irreducible factor h_i of h*

$$\langle R, P, C_{h_i} \rangle \leq L_g.$$

Then, the valuation ν_P is monomialized at h with respect to $L_0, \dots, L_g$.

PROOF. Let $[R_k, L_k]$ be the segment of the maximal contact decomposition of $\Theta_R(\sum_{k=0}^{g} L_k)$ containing P in its interior (see Figure 12). By (2.87) we get:

$$\langle R, P, C_{h_i} \rangle \overset{(P \preceq L_k)}{\preceq_R} \langle R, L_k, C_{h_i} \rangle \overset{(R \preceq R_k)}{\preceq_R} \langle R_k, L_k, C_{h_i} \rangle \,. \tag{4.141}$$

The hypothesis is equivalent to saying that the attaching point of C_{h_i} to the tree $\Theta_R(\sum_{k=0}^{g} L_k)$ does not belong to the segment $(R_{k+1}, L_k]$ (see Figure 12). This condition implies that the centers of the tripods in (4.141) are all contained in the segment $[R, L_g]$, thus $\langle R_k, L_k, C_{h_i} \rangle \leq L_g$. Then, the assertion follows by Lemma 4.129. □

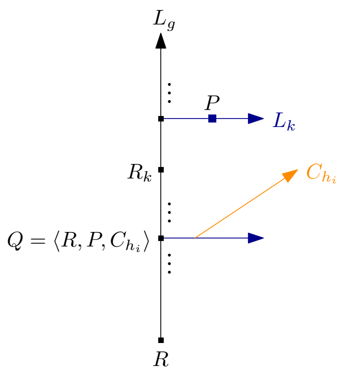

FIGURE 12. A point $P \in \Theta_R(\bar{L}_g)$ such that $Q = \langle R, P, C_{h_i} \rangle \leq L_g$.

Let us introduce a graphical characterization of generating sequence:

DEFINITION 4.142. Let (C, O) be a plane curve germ at a smooth surface (S, O) and R a smooth branch at O. Denote by $\bar{C}$ a maximal contact completion of C with respect to R. Let P be a leaf of the Eggers-Wall tree $\Theta_R(C)$. We say that a sequence of branches $L_0, \dots, L_g$ of $\bar{C}$ is a **generating sequence at** P if $P = L_g$ and if $R_1 < \cdots < R_g$ are the points of discontinuity of the index function $\mathbf{i}_R$ on $[R, L_g]$, then for each $1 \leq i \leq g$, one has that L_{i-1} is a branch of $\bar{C}$ such that

$$R_i \in [R, L_{i-1}] \text{ and } \mathbf{i}_R(R_i) = \mathbf{i}_R(L_{i-1}).$$

LEMMA 4.143. *Let C be a plane curve singularity at (S, O) and R a smooth reference branch. We denote by L_g a branch of C with g characteristic exponents with respect to R. If for every irreducible factor h_i of $h \in \mathbb{C}\{x\}[y]$, the attaching point of C_{h_i} to the tree $\Theta_R(C)$ is $\leq_R L_g$, then, for any rational point Q of $\Theta_R(C)$ the valuation ν_Q is monomialized at h with respect to any generating sequence at L_g.*

PROOF. Let us denote by $\bar{C}$ a maximal contact completion of C with respect to R and by $L_0, \dots, L_g$ a generating sequence at L_g according to Definition 4.142. Recall that the attaching point of C_{h_i} to $\Theta_R(C)$ is

$$\pi_{R,C}^{C_{h_i}}(C_{h_i}) = \max_{<_R} \{\langle R, C_j, C_{h_i}\rangle \mid C_j \text{ branch of } C\}. \tag{4.144}$$

• Let us consider first the case $Q \in \Theta_R(\sum_{k=0}^{g} L_k)$. Then, there exists a branch C_j of C such that $Q \leq C_j$ since $Q \in \Theta_R(C)$, and then

$$\langle R, Q, C_{h_i}\rangle \overset{(2.87)}{\leq} \langle R, C_j, C_{h_i}\rangle \overset{(4.144)}{\leq} \pi_{R,C}^{C_{h_i}}(C_{h_i}) \leq L_g, \tag{4.145}$$

where the last inequality is given by the hypothesis. Then, we get the result by applying Corollary 4.140.

• Secondly, we consider the case $Q \notin \Theta_R(\sum_{k=0}^{g} L_k)$.

Let C_j be a branch of C such that $Q \leq C_j$. Denote by P the attaching point of C_j to the tree $\Theta_R(\sum_{k=0}^{g} L_k)$ (see Figure 13).

Since $Q \notin \Theta_R(\sum_{k=0}^{g} L_k)$ and since $P \leq Q$ we have that

$$\langle R, P, L_i\rangle \overset{(2.88)}{=} \langle R, Q, L_i\rangle, \text{ for } i = 0, \dots, g. \tag{4.146}$$

Similarly, for any branch C_{h_i} of C_h, the hypothesis $\pi_{R,C}^{C_{h_i}}(C_{h_i}) \leq L_g$ implies that $Q \notin [R, C_{h_i}]$ hence by (2.88) one has:

$$\langle R, Q, C_{h_i}\rangle \overset{(2.88)}{=} \langle R, P, C_{h_i}\rangle. \tag{4.147}$$

We notice that:

$$\nu_Q(R) = \mathbf{i}_R^+(Q) = \mathbf{i}_R^+(P)\mathbf{i}_P^+(Q) = \mathbf{i}_P^+(Q)\nu_P(R). \tag{4.148}$$

If D is a branch such that $\langle R, P, D\rangle = \langle R, Q, D\rangle$ then by Theorem 4.70 we obtain:

$$\begin{cases} \nu_Q(D) & = & \mathbf{i}_R^+(Q)\mathbf{i}_R(D)\mathbf{c}_R(\langle R, Q, D\rangle) \\ & = & \mathbf{i}_R^+(Q)\mathbf{i}_R(D)\mathbf{c}_R(\langle R, P, D\rangle) \\ & \overset{(4.148)}{=} & \mathbf{i}_P^+(Q)\mathbf{i}_R^+(P)\mathbf{i}_R(D)\mathbf{c}_R(\langle R, P, D\rangle) \\ & \overset{(4.70)}{=} & \mathbf{i}_P^+(Q)\nu_P(D). \end{cases} \tag{4.149}$$

By (4.147) we can apply (4.149) to $D = C_{h_i}$, and using the additivity property of valuations with respect to products of functions (see Definition 2.68), we get:

$$\nu_Q(C_h) = \mathbf{i}^+_P(Q)\nu_P(C_h). \tag{4.150}$$

By (4.146) we apply (4.149) to L_i for $i = 0, \dots, g$ and we have:

$$\nu_Q(L_i) = \mathbf{i}^+_P(Q)\nu_P(L_i), \text{ for } i = 0, \dots, g.$$

We get from this and from (4.148), using again the additivity of valuations with respect to products of functions, that

$$\nu_Q(\mathcal{M}_I(h)) = \mathbf{i}^+_P(Q)\nu_P\left(\mathcal{M}_I(h)\right). \tag{4.151}$$

for any term $\mathcal{M}_I(h)$ appearing in the expansion of C_h with respect to $L_0, \dots, L_g$.

By applying the result in the first case to P we get

$$\nu_P(h) = \min_I \left\{\nu_P\left(\mathcal{M}_I(h)\right)\right\}. \tag{4.152}$$

Putting all these relations together we get the desired equality:

$$\begin{aligned}
\nu_Q(h) &\overset{(4.150)}{=} \mathbf{i}^+_P(Q)\nu_P(C_h) \\
&\overset{(4.152)}{=} \mathbf{i}^+_P(Q)\min_I \left\{\nu_P\left(\mathcal{M}_I(h)\right)\right\} = \min_I \left\{\mathbf{i}^+_P(Q)\nu_P\left(\mathcal{M}_I(h)\right)\right\} \\
&\overset{(4.151)}{=} \min_I \left\{\nu_Q\left(\mathcal{M}_I(h)\right)\right\}.
\end{aligned}$$

□

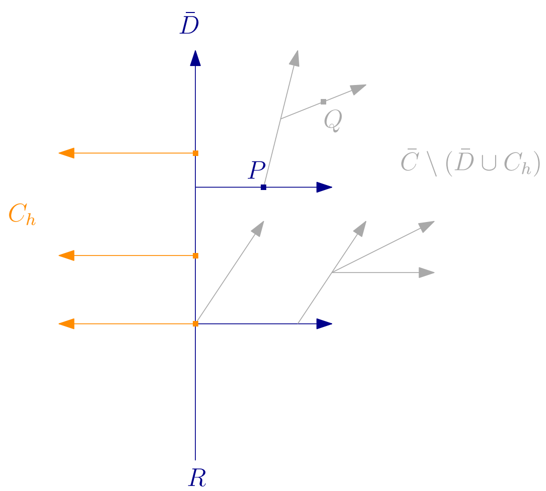

FIGURE 13. The tree $\Theta_R(\bar{C} \cup C_h)$. A point $Q \in \Theta_R(\bar{C})$ and its projection P into the tree $\Theta_R(\bar{D})$.

DEFINITION 4.153. Let C be a reduced plane curve germ with irreducible components $C_1, \dots, C_r$ at the smooth surface (S, O). Take local coordinates (x, y) at O with $R = V(x)$. If $h \in \mathbb{C}\{x\}[y]$ we say that a factorization

$$h = H_1 \cdots H_r$$

is **adequate for** C **with respect to** R if for $1 \leq k \leq r$ and for every irreducible factor $h_{k,i}$ of H_k the attaching point of $C_{h_{k,i}}$ to the tree $\Theta_R(C)$ is $\leq_R C_k$.

Adequate factorizations are not unique as we can check from the following example.

EXAMPLE 4.154. Let $C = C_1 + C_2$ be a reduced plane curve germ as $(\mathbb{C}^2, 0)$ and let (R, L) be a cross providing a system of coordinates $\{x, y\}$, Let $f_1 = (y^2 - x^3)^2 + yx^6$ and $f_2 = y^3 - x^5$ representatives of C_1, C_2 respectively. We choose complete sequences of semi-roots $\{L_0, L_1, C_1 = L_2^{(1)}\}$ with $L_1 = V(y^2 - x^3)$, and $\{L_0, C_2 = L_1^{(2)}\}$.

Let $h = h_1 h_2 h_3$ be a germ such that $h_1 = y^4 - x^3$, $h_2 = y^3 - 2x^5$ and $h_3 = f_1 - z_1 x^4$. We depict the tree $\Theta_R(\bar{C} + C_h)$ in Figure 14. Notice that h_3 must be expanded in terms of C_1 and h_2 in terms of C_2 in order to verify definition 4.153. Then if we write $H_1 = h_1 h_3$ $H_2 = h_2$ we obtain an adequate factorization of h with respect to C and if we write $H_1' = h_3$ and $H_2' = h_1 h_2$ we get a different adequate factorization.

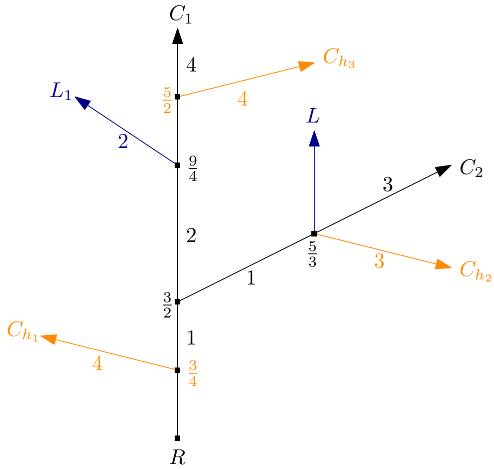

FIGURE 14. The tree $\Theta_R(\bar{C} \cup C_h)$ in Example 4.154.

DEFINITION 4.155. Let C be a reduced plane curve with irreducible components $C_1, \ldots, C_r$. Let us choose for every branch C_k of C a generating sequence $L_0^{(k)}, \ldots, L_{g(k)}^{(k)}$ at C_k in the maximal completion $\bar{C}$ of C (see Definition 4.142). Let $h \in \mathbb{C}\{x\}[y]$ and let

$$h = H_1 \cdots H_r$$

be an adequate factorization for C with respect to R (see Definition 4.153). Denote by

$$H_k = \sum_{I^{(k)}} a_{I^{(k)}} \mathcal{M}_{I^{(k)}}^{(k)}, \tag{4.156}$$

the expansion of H_k with respect to $L_0^{(k)}, \ldots, L_{g(i)}^{(k)}$ for $1 \leq k \leq r$. Then, we say that the expansion:

$$h = \prod_{k=1}^{r} \left(\sum_{I^{(k)}} a_{I^{(k)}} \mathcal{M}_{I^{(k)}}^{(k)} \right) = \sum_{I^{(1)}, \ldots, I^{(r)}} a_{(I^{(1)}, \ldots, I^{(r)})} \mathcal{M}_{I^{(1)}}^{(1)} \cdots \mathcal{M}_{I^{(r)}}^{(r)}, \tag{4.157}$$

is an **adequate expansion** of h with respect to the generating sequences $L_0^{(k)}, \ldots, L_{g(k)}^{(k)}$, $k = 1, \ldots, r$. For short, we will say that this is an **adequate expansion of h with respect to $\bar{C}$** and that a monomial appearing in expression (4.157) **is an adequate monomial with respect to $\bar{C}$**.

REMARK 4.158. Notice that by definition there are only finitely many terms in the expansion (4.157). The branches in the generating sequence $L_0^{(k)}, \dots, L_{g(k)}^{(k)}$ at C_k are chosen inside the maximal contact completion $\bar{C}$ of C with respect to R, there may be coincidences with other branches of other generating sequences at C_j for $k \neq j$. The branches $L_0^{(k)}, \dots, L_{g(k)}^{(k)}$, for $k = 1, \dots, r$, are precisely the ends of $\Theta_R(\bar{C})$.

THEOREM 4.159. *Let C be a plane curve germ with irreducible components $C_1, \dots, C_r$ at the smooth surface (S, O) and let (R, L) be a cross at O defined by local coordinates (x, y). Let $h \in \mathbb{C}\{x\}[y]$ and consider an adequate expansion (4.157). Then, for any rational point $P \in \Theta_R(C)$ we have that the valuation ν_P is monomialized at h, that is,*

$$\nu_P(h) = \min_{I^{(1)}, \dots, I^{(r)}} \nu_P(\mathcal{M}^{(1)}_{I^{(1)}} \cdots \mathcal{M}^{(r)}_{I^{(r)}}).$$

PROOF. If $h = H_1 \cdots H_r$ is an adequate factorization of h then by Lemma 4.143, the valuation ν_P is monomialized at H_k with respect to the expansion $L_0^{(k)}, \dots, L_{g(k)}^{(k)}$. That is, we have that

$$\nu_P(H_k) = \min_{I^{(k)}} \nu_P(\mathcal{M}_{I^{(k)}}(H_k)).$$

Since $\nu_P(h) = \sum_{i=1}^k \nu_P(H_k)$ and taking into account that $\nu_P(\mathcal{M}_{I^{(k)}}(H_k)) \geq 0$, we get

$$\nu_P(h) = \sum_{i=1}^k \min \left\{ \nu_P \left(\mathcal{M}_{I^{(k)}}(H_k) \right) \right\} = \min \left\{ \sum_{i=1}^k \nu_P \left(\mathcal{M}_{I^{(k)}}(H_k) \right) \right\} = \min \left\{ \nu_P \left(\prod_{i=1}^k \mathcal{M}_{I^{(k)}}(H_k) \right) \right\}.$$

□

As a consequence of the above theorem we obtain the following result, which is a slight generalization of Theorem 4.88 ([**DGN08**, Theorem 5]). It considers a tuple of valuations consisting on divisorial valuations (associated to rational points) and order of vanishing semivaluations along a branch (associated to end points of the tree).

COROLLARY 4.160. *Let $P_1, \dots, P_s$ be rational points or ends of $\Theta_R(\bar{C})$. Then, the defining functions of the branches of the maximal contact completion $\bar{C}$ of C are a generating sequence for $(\nu_{P_1}, \dots, \nu_{P_s})$ (see Definition 4.84).*

4.7. Eggers-Wall tree embedding in the semivaluation space

We defined divisorial valuations associated to rational points of the Eggers-Wall tree, and valuations associated to the leaves. In this section we define valuations associated to irrational points, which are not divisorial, although they can be understood as limits of divisorial valuations.

Let C be a reduced plane curve singularity, R a smooth reference branch and $\bar{C} = \sum L_Q$ be a maximal contact completion (see Definition 4.54). Let $[R_j, L_{Q_j}]$ be a maximal contact decomposition of $\Theta_R(\bar{C})$ (see Definition 4.53).

Let $P \in \Theta_R(\bar{C})$ be an irrational point (see Notation 2.85). There is a unique segment in the decomposition which contains P in its interior, $P \in [R_k, L_k]$. We define the valuation ν_P in the smooth surface given by (R_k, L_k) as we did in Definition 4.22:

DEFINITION 4.161. Let $\mathbf{e}_{R_k}(P) = \beta$ be an irrational number, and (x_k, y_k) the local system of coordinates associated to the cross (R_k, L_k). We define the **valuation ν_P associated to the irrational point P** as the monomial valuation such that $\nu_P(x_k) = 1$ and $\nu_P(y_k) = \beta$, that is, ν_P is the monomial valuation associated to the vector $(1, \beta)$ with respect to (x_k, y_k).

We extend now ν_P to a valuation on (S, O). Let D be a plane curve at (S, O), then

$$\nu_P(D) = \Phi_{\mathcal{N}_{R_k,L_k}(D)}((1,\beta)). \tag{4.162}$$

We can extend the result in Theorem 4.70 by setting the extended index function at P to be its index at P,

$$\mathbf{i}_R^+(P) \coloneqq \mathbf{i}_R(P). \tag{4.163}$$

Notice that, if R_k is the root of the index level of P, $\mathbf{i}_R(P) = \mathbf{i}_R^+(R_k)$.

THEOREM 4.164. *Let C be a plane curve singularity at a smooth surface (S, O) and R a smooth branch through it. Let $P \in \Theta_R(C)$ be an irrational point. Then, the valuation ν_P has the property that for every branch D of a maximal contact completion $\bar{C}$ with respect to R we have:*

$$\nu_P(D) = \mathbf{i}_R(D)\,\mathbf{i}_R^+(P)\,\mathbf{c}_R(\langle R, P, D\rangle). \tag{4.165}$$

PROOF. Let $\mathcal{D}$ be a maximal contact decomposition of $\Theta_R(\bar{C})$ and let $[R_k, L_k] \in \mathcal{D}$ the unique segment such that $P \in (R_k, L_k)$. Let D be a branch in $\bar{C}$. By Lemma 2.57, we can describe the Newton polygon of D with respect to (R_k, L_k) as the Minkowski sum of the Newton polygons of its strict transform and its exceptional part,

$$\mathcal{N}_{R_k,L_k}(D) = \mathcal{N}_{R_k}(x_k^{\nu_{R_k}(D)}) + \mathcal{N}_{R_k,L_k}(\tilde{D}). \tag{4.166}$$

Let us denote by $P_{k,D} = \langle R, L_k, D\rangle$. By Theorem 4.52, the strict transform of D passes through the point O_k of intersection of R_k, L_k if and only if $R_k <_R \langle R, L_k, D\rangle$. Let us assume that $P_{k,D} > R_k$, then the Newton polygon of the strict transform $\tilde{D}$ is

$$\mathcal{N}_{R_k,L_k}(\tilde{D}) = \mathbf{i}_{R_k}(D) \left\{ \frac{\mathbf{e}_{R_k}(P_{k,D})}{1} \right\}, \tag{4.167}$$

as we showed in Proposition 2.107. This Newton polygon has two vertices corresponding to

$$y_k^{\mathbf{i}_{R_k}}(D) \text{ and } x_k^{\mathbf{i}_{R_k}(D)\mathbf{e}_{R_k}(P_{k,D})}. \tag{4.168}$$

By Definition 4.161,

$$\nu_P(D) = \Phi_{\mathcal{N}_{R_k,L_k}(D)}((1,\beta)) = \begin{cases} \nu_{R_k}(D) + \beta \mathbf{i}_{R_k}(D) & \text{if } P_{k,D} >_R P, \\ \nu_{R_k}(D) + \mathbf{i}_{R_k}(D)\mathbf{e}_{R_k}(P_{k,D}) & \text{if } P_{k,D} <_R P. \end{cases}$$

Since $P_{k,D} > R_k$, we can expand the contact function as we did in Lemma 4.116,

$$\mathbf{c}_R(P_{k,D}) = \mathbf{c}_R(R_k) + \frac{\mathbf{e}_R(P_{k,D}) - \mathbf{e}_R(R_k)}{\mathbf{i}_R(P_{k,D})}. \tag{4.169}$$

Thus, we write

$$\begin{aligned} \mathbf{i}_R(D)\,\mathbf{i}_R^+(P)\,\mathbf{c}_R(\langle R, P, D\rangle) &\overset{(4.71)}{=} \nu_{R_k}(D) + \mathbf{i}_R(D)\,\mathbf{i}_R^+(P)\,\frac{\mathbf{e}_R(P_{k,D}) - \mathbf{e}_R(R_k)}{\mathbf{i}_R(P_{k,D})} \\ &\overset{(4.122)}{=} \nu_{R_k}(D) + \mathbf{i}_{R_k}(D)\,\mathbf{e}_{R_k}(P_{k,D}). \end{aligned}$$

If $P < P_{k,D}$, then $\langle R, P, D\rangle = P$ and $\mathbf{e}_{R_k}(P) = \beta$, thus

$$\nu_P(D) = \mathbf{i}_R(D)\mathbf{i}_R^+(P)\mathbf{c}_R(P).$$

If $P_{k,D} < P$, then $\langle R, P, D\rangle = P_{k,D}$, thus

$$\nu_P(D) = \mathbf{i}_R(D)\mathbf{i}_R^+(P)\mathbf{c}_R(P_{k,D}).$$

□

Now we can extend Theorem 4.159 to any point of the tree.

THEOREM 4.170. *Let C be a plane curve germ with irreducible components $C_1, \dots, C_r$ at the smooth surface (S, O) and let (R, L) be a cross at O defined by local coordinates (x, y). If $h \in \mathbb{C}\{x\}[y]$, then for any point $P \in \Theta_R(C)$ the valuation ν_P is monomialized at h by any adequate expansion (4.157) of h, that is,*

$$\nu_P(h) = \min_{I^{(1)},\dots,I^{(r)}} \nu_P\left(\mathcal{M}^{(1)}_{I^{(1)}} \cdots \mathcal{M}^{(r)}_{I^{(r)}}\right).$$

PROOF. By using the Definition of the irrational valuation in terms of the support function we can extend the monomialization in Theorem 4.125 to irrational points in (R, L_g), (see (4.126) and (4.127)).

Part (2) in Lemma 4.129 generalizes to irrational points of the tree by applying Definition 4.161 in (4.136) and in (4.137). Thus Corollary 4.140 also extends to valuations associated to irrational points of the tree.

Lemma 4.143 requires us to apply Theorem 4.164, by using the description in terms of the contact function in (4.149). □

REMARK 4.171. In [**GGP18**], the authors construct the valuations associated to points in the tree in a different way. Let $R = V(x)$ be a smooth reference branch, $\xi \in \mathbb{C}\{x^{1/\mathbb{N}}\}$ and $\alpha \in (0, \infty]$. First consider the set of Newton-Puiseux series which coincide with ξ up to exponent α,

$$\mathcal{NP}_R(\xi, \alpha) := \{\eta \in \mathbb{C}\{x^{1/\mathbb{N}}\} \mid \mathrm{ord}_x(\xi - \eta) \geq \alpha\}.$$

Let C be a branch, $R = V(x)$ a smooth reference branch and $\xi \in \mathbb{C}\{x^{1/\mathbb{N}}\}$ a root of D. Let $P \in \Theta_R(C)$ with $\mathbf{e}_R(P) \in (0, \infty]$. For each germ $f \in \mathcal{O}_{S,O}$ define:

$$\nu^{\xi,\alpha}(f) := \inf\{\mathrm{ord}_x(f(x, \eta)) \;\mid\; \eta \in \mathcal{NP}_R(\xi, \alpha)\}$$

Set also $\nu^{\xi,0} := \nu_R$. One can show $\nu^{\xi,\infty} = I_{C_\xi}$ is the intersection semivaluation defined by the branch C_ξ.

It turns out that, for P an interior point of the tree, $\nu^{\xi,\alpha}$ is a valuation and one has: $\nu_P = \mathbf{i}^+_R(P)\nu^{\xi,\alpha}$ (see [**GGP18**, Lemma 8.7. and Proposition 8.9.]).

Let us denote by $\mathcal{V}_R$ the set of semivaluations ν on (S, O) such that $\nu(R) = 1$. In [**GGP18**, Theorem 8.19] the authors proved that the map $V_R : \Theta_R(C) \to \mathcal{V}_R$, defined by $V^R_P = \nu^{\xi,\alpha}$, can be seen as an increasing embedding of rooted trees, and that the semivaluation space can be thought of as the projective limit of Eggers-Wall trees under this embeddings.

CHAPTER 5

On generators of multiplier ideals of a plane curve

In this chapter we study multiplier ideals of curves on a smooth surface.

In Section 5.2 we give the main results of the work for plane curves. First, we prove that the set of divisorial conditions associated to marked points of the tree is sufficient to describe multiplier ideals (Theorem 5.8). A converse was proved in [**ST07**, Theorem 3.1], where the authors show that all of those divisors contribute to a jumping number. Theorem 5.16 shows that we can find a monomial basis for each multiplier ideals formed by generalized monomials in the branches of maximal contact of the plane curve. We observe that the monomiality of multiplier ideals can be proved independently of Theorem 5.8. As a consequence of both results we obtain that for every jumping number there exists a (not necessarily unique) generalized monomial satisfying that the jumping number is the minimal rational number such that the monomial does not belong to the multiplier ideals.

Further in the Chapter we give some relations with previous works on this topic. On [**ACLM08**] the authors prove that the log-canonical threshold is attained at a divisor which appears in the first toric modification of C for a suitable cross at the origin. In the first section of this chapter we give a new proof of this fact relating this fact with the index level one of the Eggers-Wall tree (see Theorem 5.40). Section 5.3 generalizes the results in Section 5.2 for plane ideals using results proved in [**CV14, CV15**]. In Section 5.4, we relate our results to Naie's formulas for jumping numbers of plane branches ([**Nai09**]), showing that there exists a bijection between the set of jumping numbers given by Naie and monomials in the semi-roots associated to jumping numbers smaller than one. Section 5.6 explains the relation between our monomialization of multiplier ideals and the computation of a basis described in [**AAB17**]. Section 5.5 gives a formula for the cardinality of jumping numbers smaller than one counted with multiplicity (see [**AADG17**] for a deep study of multiplicity of jumping numbers).

In Section 5.7 we introduce the notation of tropical semirings to show that the jumping numbers can be computed as the integral values of a tropical polynomial associated to Formula 5.20.

In Section 5.8 the reader will find several examples computing the jumping numbers of plane curves and an example of the jumping numbers of plane ideals.

5.1. Eggers-Wall description of log-discrepancies

As shown in Section 4.3, each rational point $P \in \Theta_R(C)$ has an associated exceptional divisor E_P (see Notation 4.65). We denote the log-discrepancy of the exceptional divisor E_P by λ_P (see Definition 3.1). In this section we give a combinatorial formula for the log-discrepancy in terms of the exponent and the extended index.

A different proof of this Proposition was given in [**GGP18**, Proposition 8.17], based on valuative arguments in [**FJ04**] (check also [**Jon15**, Chapter 7]).

Proposition 5.1. *Let C be a plane curve singularity and R a smooth branch. Then, for any rational point P on $\Theta_R(C)$ we have the following formula for the log-discrepancy of the* *exceptional divisor* E_P*:*

$$\lambda_P = \mathbf{i}^+_R(P)\,(1 + \mathbf{e}_R(P))\,.$$

PROOF. Let $\mathcal{D}$ be a maximal contact decomposition of the tree $\Theta_R(C)$. There is a unique segment $[R_k, P_k]$ of the decomposition $\mathcal{D}$ such that $P \in (R_k, L_k)$. One can write $\mathbf{e}_{R_k}(P) = \frac{m}{n}$ with $n = \mathbf{i}^+_{R_k}(P)$ (see Definition 4.40). The exceptional divisor E_P is the toric divisor associated with the ray of slope (n, m), in terms of the cross (R_k, L_k). The vector of log-discrepancies associated with the cross (R_k, L_k) is of the form $(\lambda_{R_k}, 1)$, since L_k is not in the exceptional locus and $\lambda_{L_k} = 1$. By Proposition 3.4 we obtain:

$$\lambda_P = \langle (\lambda_{R_k}, 1), (n, m) \rangle = n\lambda_{R_k} + m. \tag{5.2}$$

(see also the explanations in Remark 3.7). If $R_k = R$ then $\lambda_{R_k} = 1$ by definition since R is not an exceptional curve. In this case (5.2) can be reformulated as:

$$\lambda_P = \langle (1,1), (n, m) \rangle = n + m = \mathbf{i}^+_R(P)\,(\mathbf{e}_R(P) + 1)\,.$$

This implies that the statement is true when P is an interior point of the level one of the index function, since we can chose a maximal contact decomposition $\mathcal{D}$ of $\Theta_L(C)$ such that $P \in (R, L)$ with $[R, L] \in \mathcal{D}$. Let us prove the result by induction on levels of the index function. By induction hypothesis applied to R_k we have that:

$$\lambda_{R_k} = \mathbf{i}^+_R(R_k)\,(1 + \mathbf{e}_R(R_k))\,. \tag{5.3}$$

Combining (5.2) and (5.3) we get:

$$\begin{aligned}\lambda_P &= \Big(\lambda_{R_k} + \mathbf{e}_{R_k}(P)\Big)\mathbf{i}^+_{R_k}(P) \\ &= \left(1 + \mathbf{e}_R(R_k) + \frac{\mathbf{e}_{R_k}(P)}{\mathbf{i}^+_R(R_k)}\right)\mathbf{i}^+_R(R_k)\mathbf{i}^+_{R_k}(P) \\ &= \Big(1 + \mathbf{e}_R(P)\Big)\mathbf{i}^+_R(P),\end{aligned} \tag{5.4}$$

since $\mathbf{i}^+_R(P) = \mathbf{i}^+_R(R_k)\mathbf{i}^+_{R_k}(P)$ and $\mathbf{e}_R(P) = \mathbf{e}_R(R_k) + \frac{\mathbf{e}_{R_k}(P)}{\mathbf{i}^+_R(R_k)}$ by Proposition 4.42. $\square$

REMARK 5.5. In Section 4.2.3 we explained how to describe the dual graph of a toroidal resolution $\Psi : Y \to S$ of C with respect to R in terms of the enriched set $\mathcal{E}$ of marked points of the Eggers-Wall tree $\Theta_R(C)$ (see Definition 4.61). Then, by Proposition 5.1 we get the following formula for the relative canonical divisor of Ψ:

$$K_\Psi = \sum_{P \in \mathcal{E}} (\lambda_P - 1)P, \text{ where } \lambda_P = \mathbf{i}^+_R(P)\,(\mathbf{e}_R(P) + 1)\,.$$

EXAMPLE 5.6. Let C be the curve in Example 2.94 of Chapter 2, and $\{[R_i, L_i]\}_{i=1}^8$ the maximal contact decomposition considered in Example 4.60. Let $\{P_i\}$ be the set of points in the dual graph of the embedded resolution Ψ of $\bar{C}$, as in Figure 7.

Let $P_{16} \in [R_3, L_3]$ be the point with exponent $\mathbf{e}_R(P_{16}) = \frac{49}{12}$. According to Proposition 5.1

$$\lambda_{P_{16}} = 24\left(\frac{49}{12} + 1\right) = 122.$$

On the other hand,

$$\lambda_{R_3} = 8\left(\frac{31}{8} + 1\right) = 39,$$

while $\lambda_{L_3} = 1$ since its strict transform is not contained in the exceptional locus of Ψ. Thus, using Remark 3.7 we can also compute its log-discrepancy by

$$\lambda_{P_{16}} = \langle (39, 1), (3, 5) \rangle = 122,$$

where $\mathbf{e}_{R_3}(P_{16}) = \frac{5}{3}$ and $(39, 1)$ is the vector of log-discrepancies on (R_3, L_3).

By Remark 5.5 we have that the relative canonical divisor of Ψ is of the form:

$$\begin{aligned} K_\Psi =& 1E_{P_1} + 4E_{P_2} + 2E_{P_3} + 2E_{P_4} + 6E_{P_5} + 14E_{P_6} + 7E_{P_7} + \\ & 15E_{P_8} + 16E_{P_9} + 17E_{P_{10}} + 18E_{P_{11}} + 38E_{P_{12}} + 19E_{P_{13}} + \\ & 39E_{P_{14}} + 80E_{P_{15}} + 121E_{P_{16}} + 40E_{P_{17}} + 14E_{P_{18}} + 22E_{P_{19}} + 3E_{P_{20}}. \end{aligned}$$

5.2. Multiplier Ideals of Plane Curves

In this section we prove a version of Howald's theorem for plane curve germs. We show that the set of conditions in the definition of multiplier ideals is equivalent to the set of conditions associated to marked points in the Eggers-Wall tree of the curve. Furthermore, using the results about generating sequences of Chapter 4 Section 4.4 (see Corollary 4.89), we show that multiplier ideals are monomial ideals in terms of monomials in the maximal contact branches. As a consequence, we obtain that the jumping numbers can be computed from the conditions associated to marked points in the adequate monomials.

We begin by fixing some notation which will be used for the rest of the section.

Let C be a plane curve germ and let R be a smooth reference branch **transversal** to C on a smooth surface (S, O). Let $\Theta_R(C)$ be its Eggers-Wall tree. The set of **marked points of the tree**, $\Upsilon \subset \Theta_R(C)$, is the union of the points of discontinuity of the index function (characteristic exponents), the ramification points (contact exponents) and the ends of the tree (branches of C). We write Υ° for the set of interior marked points, i.e., the union of the points of discontinuity of the index function and the ramification points of the tree, as in Notation 2.85. For each interior point $P \in \Upsilon$, we denote by E_P its corresponding exceptional divisor (see Chapter 4, Notation 4.65). Recall that those divisors are rupture divisors (see Definition 2.61) of the dual graph of the minimal resolution of the curve, $G(\Psi, C)$ (see Proposition 4.62).

We fix a **maximal contact completion** $\bar{C}$ of C (recall from Definition 4.54 that we set $R \subset \bar{C}$). This is equivalent to adding to C curvettas meeting the exceptional divisors of valency 1 in the dual graph $G(\Psi, C)$. For each end L of $\Theta_R(\bar{C})$, we choose a germ $z_L \in \mathcal{O}_{S,O}$ such that $V(z_L) = L$. Recall that if we fix coordinates x, y we can choose z_L to be a Weierstrass polynomial. Let $J = \{L \mid L \text{ is an end of } \Theta_R(\bar{C})\}$. We will say that

$$\mathcal{M} = \prod_{L \in J} z_L^{a_L}, \text{ with } a_L \in \mathbb{Z}_{\geq 0}, \tag{5.7}$$

is a **generalized monomial** for $\bar{C}$. Finally, we fix a maximal contact decomposition $\mathcal{D}$ of $\Theta_R(\bar{C})$.

Theorem 5.8 (On the conditions defining the multiplier ideals of a plane curve). *Assume that C is a reduced plane curve and let $0 < \xi < 1$ be a rational number. In order to define the multiplier ideal of C for $\xi < 1$ it is sufficient to consider the conditions in the set of interior marked points of its Eggers-Wall tree, i.e.,*

$$\mathcal{J}(\xi C) = \{h \in \mathcal{O}_{S,O} \mid \nu_P(h) \geq \lfloor \xi \nu_P(C) \rfloor - (\lambda_P - 1) \textit{ for all } P \in \Upsilon^\circ\}. \tag{5.9}$$

Proof. Let $\mathcal{D}$ be a maximal contact decomposition of the tree $\Theta_R(\bar{C})$ (see Definition 4.53). Recall that a maximal contact decomposition determines a log-resolution of the curve C, say Ψ.

Recall that we use the notation $\mathcal{E} = \cup \mathcal{E}_\tau$ for the total enriched set of marked points (check Chapter 4 Definition 4.61). Now we use the valuative description of the multiplier ideal in Formula 3.20 and the maximal contact decomposition of the complete tree we obtain

$$\begin{aligned}\mathcal{J}(\xi C) &= \{h \in \mathcal{O}_{S,O} |\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \mathcal{E}\} \\ &= \bigcap_{\tau \in \mathcal{D}} \{h \in \mathcal{O}_{S,O} |\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \mathcal{E}_\tau\}.\end{aligned}$$

Let us fix a segment $\tau = [R_\tau, L_\tau] \in \mathcal{D}$ and a rational point $P \in \tau$. Consider the divisorial valuation ν_P, which corresponds to a primitive integral vector u_P of slope $\mathbf{e}_{R_k}(P)$ in a regular subdivision Σ of the cone $\mathbb{R}^2_{\geq 0}$. By Remark 3.7, if we denote by $\underline{\lambda}_\tau = (\lambda_{R_\tau}, \lambda_{L_\tau})$ the vector of log-discrepancies associated to (R_τ, L_τ), we compute $\lambda_P = \langle \underline{\lambda}_\tau, u_P \rangle$ and by (1.23), $\nu_P(h) = \Phi_{\mathcal{N}_{R_\tau, L_\tau}(h)}(u_P)$. Since the vector of log-discrepancies corresponds to a monomial $x_\tau^{\lambda_\tau} y_\tau^{\lambda_\tau}$, the Newton polygon $\mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau$ is a translate of $\mathcal{N}_{R_\tau, L_\tau}(h)$ with the same faces. Thus, we have

$$\Phi_{\mathcal{N}_{R_\tau, L_\tau}(h)}(u_P) + \langle \underline{\lambda}_\tau, u_P \rangle = \Phi_{\mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau}(u_P).$$

Furthermore,

$$\Phi_{\xi \mathcal{N}_{R_\tau, L_\tau}(C)} = \xi \Phi_{\mathcal{N}_{R_\tau, L_\tau}(C)}.$$

If we fix a segment $\tau = [R_\tau, L_\tau] \in \mathcal{D}$, the set of conditions on $P \in \tau$ correspond to the minimal regular subdivision of the fan dual to $\mathcal{N}_{R_\tau, L_\tau}(f)$ (Definition 4.66). Thus, using Corollary 1.28 we have

$$\begin{aligned}&\{h \in \mathcal{O}_{S,O} \mid \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \mathcal{E}_\tau\} = \\ &\{h \in \mathcal{O}_{S,O} \mid \Phi_{\mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau}(u_P) > \Phi_{\xi \mathcal{N}_{R_\tau, L_\tau}(C)}(u_P) \text{ for all } P \in \mathcal{E}_\tau\} = \\ &\{h \in \mathcal{O}_{S,O} \mid \mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau \subset \mathrm{Int}(\xi \mathcal{N}_\tau(C))\}.\end{aligned} \tag{5.10}$$

On the other hand, the conditions in the dual fan, which correspond to marked points in the segment τ, characterize the Newton polygon of C. Thus, applying Formula (1.26) and Corollary 1.28 we obtain

$$\begin{aligned}&\{h \in \mathcal{O}_{S,O} \mid \mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau \subset \mathrm{Int}(\xi \mathcal{N}_\tau(C))\} = \\ &\{h \in \mathcal{O}_{S,O} \mid\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \tau\}.\end{aligned} \tag{5.11}$$

Let P_L be the point of the Eggers-Wall tree $\Theta_R(\bar{C})$ corresponding to a branch L such that it is not in C. Since the strict transform $\tilde{L}$ is not contained in the exceptional locus of the resolution Ψ, we have that its log-discrepancy is equal to one, $\lambda_{P_L} = 1$. Thus, the condition on the order of vanishing over L is fulfilled for any holomorphic germ $h \in \mathcal{O}_{S,O}$,

$$\nu_L(h) + \lambda_{P_L} > \nu_L(h) \geq 0.$$

Since by hypothesis $\xi < 1$ and C is reduced, the previous reasoning is valid for all the leaves in $\Theta_R(\bar{C})$. It follows that the conditions in the leaves of $\Theta_R(\bar{C})$ which are not leaves of $\Theta_R(C)$ are redundant,

$$\begin{aligned}&\{h \in \mathcal{O}_{S,O} \mid\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \tau\} = \\ &\{h \in \mathcal{O}_{S,O} \mid\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \tau \cap \Upsilon^\circ\}.\end{aligned}$$

The previous reasoning shows that the conditions associated to interior marked points of the tree are sufficient to describe the multiplier ideals, i.e.,

$$\mathcal{J}(\xi C) = \{h \in \mathcal{O}_{S,O} |\ \nu_P(h) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \Upsilon^\circ\}. \tag{5.12}$$

Notice that (5.12) is equivalent to (5.9) in the statement by property (3.18). □

We obtain from equation (5.10) a version of Howald's result in terms of Newton Polygons:

COROLLARY 5.13. *Let C be a reduced curve and $\xi < 1$ as before. Then*

$$\mathcal{J}(\xi C) = \{h \in \mathcal{O}_{S,O} | \; \mathcal{N}_{R_\tau, L_\tau}(h) + \underline{\lambda}_\tau \subset \operatorname{Int}(\xi \mathcal{N}_{R_\tau, L_\tau}(C)) \;\; \textit{for all } \tau \in \mathcal{D}\}. \tag{5.14}$$

REMARK 5.15. If $C = \sum C_i$, the strict transform of C with respect to the minimal embedded resolution Ψ is the sum of the strict transforms of its components $\tilde{C} = \sum \tilde{C}_i$. Let P_i be the point of the Eggers-Wall tree $\Theta_R(C)$ corresponding to the branch C_i. Since the strict transform $\tilde{C}_i$ of C is not contained in the exceptional locus of the resolution Ψ, its log-discrepancy is equal to one, $\lambda_{P_i} = 1$. Thus, the condition on the order of vanishing at C_i is fulfilled for any holomorphic germ $h \in \mathcal{O}_{S,O}$,

$$\nu_{C_i}(h) + \lambda_{P_i} > \nu_{C_i}(h) \geq 0.$$

THEOREM 5.16 (On the monomiality of multiplier ideals of a plane curve). *Let C be a reduced plane curve. The multipliers ideals of C for $\xi < 1$ are monomial ideals in the generalized monomials of expresion 5.7, i.e.,*

$$\mathcal{J}(\xi C) = \left\langle \mathcal{M} \;\middle|\; \nu_P(\mathcal{M}) + \lambda_P > \xi \nu_P(C) \;\textit{for all } P \in \Upsilon^\circ \right\rangle. \tag{5.17}$$

PROOF. Let $h \in \mathcal{O}$ be a function such that

$$\nu_P(h) > -\lambda_P + \xi \nu_P(f) \text{ for all } P \in \Upsilon^\circ.$$

Taking $\underline{\alpha} = (\lfloor \xi \nu_{P_1}(C) \rfloor - (\lambda_{P_1} - 1), \ldots, \lfloor \xi \nu_{P_s}(C) \rfloor - (\lambda_{P_s} - 1))$ we get that $h \in \mathcal{J}(\xi C)$ if and only if h belongs to the valuation ideal $I^{\underline{\nu}}_{\underline{\alpha}}$ for $\underline{\nu} = (\nu_{P_1}, \ldots, \nu_{P_s})$. By Corollary 4.89 we have that the branches of $\bar{C}$ form a generating sequence for $\underline{\nu}$. By Lemma 4.87 we have a finite expansion of h in terms of the defining functions of the branches of $\bar{C}$ in such a way that every nonzero term of the expansion must belong to the valuation ideal $I^{\underline{\nu}}_{\underline{\alpha}}$. This shows that the multiplier ideal $\mathcal{J}(\xi C)$ is generated by the generalized monomials in the branches of $\bar{C}$ which belong to $I^{\underline{\nu}}_{\underline{a}}$. ☐

REMARK 5.18. Let C be as in Theorem 4.125. Its multiplier ideals are monomial since, by Lemma 3.26

$$\mathcal{J}((1+\xi)C) = (f)\mathcal{J}(\xi C).$$

REMARK 5.19. Let $C = \sum a_i C_i$, $a_i \in \mathbb{N}^*$, be a plane curve germ. The proofs of Theorems 5.8 and 5.16 remain valid for any ξ by considering the vanishing order valuations associated with the ends of $\Theta_R(C)$. Indeed, the set of maximal contact curves of C is a generating sequence both for the tuple of divisorial valuations associated to interior marked points of the tree and for the tuple of valuations associated to all marked points of the tree, which includes vanishing order valuations associated to the ends of $\Theta_R(C)$ (see Lemma 4.101, Corollary 4.160 and Lemma 4.90). Thus, we have

$$\mathcal{J}(\xi C) = \left\langle \mathcal{M} \;\middle|\; \nu_P(\mathcal{M}) + \lambda_P > \xi \nu_P(C) \text{ for all } P \in \Upsilon \right\rangle, \tag{5.20}$$

or, rewriting the conditions in terms of ξ,

$$\mathcal{J}(\xi C) = \left\langle \mathcal{M} \;\middle|\; \frac{\nu_P(\mathcal{M}) + \lambda_P}{\nu_P(C)} > \xi \text{ for all } P \in \Upsilon \right\rangle. \tag{5.21}$$

COROLLARY 5.22 (Generalized monomials determine the jumping numbers). *Each jumping number of the plane curve C corresponds to at least one adequate monomial*

$$\xi_{\mathcal{M}} = \min_{P \in \Upsilon} \left\{ \frac{\nu_P(\mathcal{M}) + \lambda_P}{\nu_P(C)} \right\}. \tag{5.23}$$

Proof. Theorem 5.16 shows that multiplier ideals are monomially generated by generalized monomials in $\bar{C}$ (Definition 4.155). Obviously, a monomial does not belong to a multiplier ideal whenever any of the conditions

$$\nu_P(\mathcal{M}) + \lambda_P > c\nu_P(C), \tag{5.24}$$

is not satisfied. This implies that the monomial no longer belongs to the multiplier ideal $\mathcal{J}(C^c)$ whenever

$$c = \min_{P\in\Upsilon}\left\{\frac{\nu_P(\mathcal{M}) + \lambda_P}{\nu_P(C)}\right\}. \tag{5.25}$$

□

It is well-known that the log-canonical threshold is

$$\mathrm{lct}(C) = \min_{P\in\Upsilon}\left\{\frac{\lambda_P}{\nu_P(C)}\right\}.$$

The following corollary states that the above minimum is attained at the index level one subtree, $\Theta^1_R(C)$.

Corollary 5.26 (The log-canonical threshold is attained at the index level 1 subtree). *The log-canonical threshold of the curve C is attained at a marked point on the index level 1 subtree*

$$\mathrm{lct}(C) = \min_{P\in\Upsilon\cap\Theta^1_R(C)}\left\{\frac{\lambda_P}{\nu_P(C)}\right\}. \tag{5.27}$$

Proof. The log-canonical threshold is the smallest jumping number and so it is associated to a unit considered as a monomial, i.e., $\mathrm{lct}(C) = \xi_{\mathcal{M}}$ with $\mathcal{M} = 1$.

We will proof that, for any marked point $P \in \Theta_R(\bar{C}) \setminus \Theta^1_R(C)$, the root of its index level, say $R_l \in \Theta_R(C)$, describes a smaller candidate, i.e.,

$$\frac{\lambda_{R_l}}{\nu_{R_l}(C)} < \frac{\lambda_P}{\nu_P(C)}. \tag{5.28}$$

- Let us first assume that C is a branch: Recall from (5.4) that we can write the log-discrepancy of P in terms of R_l,

$$\lambda_P = \mathbf{i}^+_{R_l}(P)(\mathbf{e}_{R_l}(P) + \lambda_{R_l}). \tag{5.29}$$

We can also interpret the valuation of C at P in terms of the valuation of C at R_l,

$$\begin{aligned}\nu_P(C) &\overset{(4.70)}{=} \mathbf{i}^+_R(P)\mathbf{i}_R(C)\mathbf{c}_R(\langle R,C,P\rangle)\\ &\overset{(4.120)}{=} \mathbf{i}^+_R(R_l)\mathbf{i}^+_{R_l}(P)\mathbf{i}_R(C)\mathbf{c}_R(\langle R,C,P\rangle)\\ &\overset{(2.100)}{=} \mathbf{i}^+_R(R_l)\mathbf{i}^+_{R_l}(P)\mathbf{i}_R(C)\left(\mathbf{c}_R(R_l) + \tfrac{\mathbf{e}_R(\langle R,C,P\rangle)-\mathbf{e}_R(R_l)}{\mathbf{i}_R(P)}\right)\\ &\overset{(4.70)}{=} \mathbf{i}^+_{R_l}(P)\left(\nu_{R_l}(C) + \mathbf{i}_R(C)\left(\mathbf{e}_R(\langle R,C,P\rangle) - \mathbf{e}_R(R_l)\right)\right).\end{aligned} \tag{5.30}$$

Now, inequality (5.28) is equivalent to

$$\lambda_{R_l}\nu_P(C) < \lambda_P\nu_{R_l}(C). \tag{5.31}$$

Using formulas (5.29) and (5.30) we translate inequality (5.31) into

$$\lambda_{R_l}\left[\mathbf{i}^+_{R_l}(P)\left(\nu_{R_l}(C) + \mathbf{i}_R(C)\left(\mathbf{e}_R(\langle R,C,P\rangle) - \mathbf{e}_R(R_l)\right)\right)\right] < \left[\mathbf{i}^+_{R_l}(P)\left(\mathbf{e}_{R_l}(P) + \lambda_{R_l}\right)\right]\nu_P(C).$$

Simplifying common terms on both sides we obtain

$$\lambda_{R_l}\mathbf{i}_R(C)\left(\mathbf{e}_R(\langle R,P,C\rangle) - \mathbf{e}_R(R_l)\right) < \mathbf{e}_{R_l}(P)\nu_{R_l}(C). \tag{5.32}$$

Using (5.4) for R_l we write

$$\lambda_{R_l} = \mathbf{i}_R^+(R_l)(\mathbf{e}_R(R_l) + 1),$$

and by (4.70)

$$\nu_{R_l}(C) = \mathbf{i}_R^+(R_l)\mathbf{i}_R(C)\mathbf{c}_R(R_l).$$

With those equalities, we rewrite (5.32)

$$(5.33) \qquad (\mathbf{e}_R(R_l) + 1)\, \mathbf{i}_R^+(R_l)\mathbf{i}_R(C)\, (\mathbf{e}_R(\langle R, P, C\rangle) - \mathbf{e}_R(R_l)) < \mathbf{e}_{R_l}(P)\mathbf{i}_R^+(R_l)\mathbf{i}_R(C)\mathbf{c}_R(R_l),$$

which simplifies into

$$(5.34) \qquad (\mathbf{e}_R(R_l) + 1)\, (\mathbf{e}_R(\langle R, P, C\rangle) - \mathbf{e}_R(R_l)) < \mathbf{e}_{R_l}(P)\mathbf{c}_R(R_l).$$

Notice that $\langle R, P, C\rangle < P$, so that $\mathbf{e}_R(\langle R, P, C\rangle) \leq \mathbf{e}_R(P)$. Thus, it is enough to prove the inequality obtained by substituting $\langle R, P, C\rangle$ by P, i.e., inequality

$$(5.35) \qquad (\mathbf{e}_R(R_l) + 1)\, (\mathbf{e}_R(P) - \mathbf{e}_R(R_l)) < \mathbf{e}_{R_l}(P)\mathbf{c}_R(R_l),$$

implies inequality (5.34).

Recall from Definition 4.40 that

$$\mathbf{i}_R^+(R_l)(\mathbf{e}_R(P) - \mathbf{e}_R(R_l)) = \mathbf{e}_{R_l}(P).$$

Thus, inequality (5.35) is equivalent to

$$(5.36) \qquad \mathbf{e}_R(R_l) + 1 < \mathbf{i}_R^+(R_l)\mathbf{c}_R(R_l).$$

By hypothesis $R_l \succeq R_1$ is the root of some level of the index function. Let us rearrange the terms in the contact function (Definition 2.99)

$$(5.37) \qquad \mathbf{c}_R(R_l) = \frac{\mathbf{e}_R(R_l)}{\mathbf{i}_R(R_l)} + \sum_{j=1}^{l-1} \frac{\mathbf{e}_R(R_j)\left(\mathbf{i}_{R_{j-1}}(R_j) - 1\right)}{\mathbf{i}_R^+(R_j)}.$$

Clearly, the exponent function is smaller than the first term in (5.37),

$$(5.38) \qquad \mathbf{e}_R(R_l) < \mathbf{i}_R^+(R_l)\frac{\mathbf{e}_R(R_l)}{\mathbf{i}_R(R_l)}.$$

On the other hand, since we chose R transversal to C,

$$(5.39) \qquad 1 < \mathbf{i}_R^+(R_l)\mathbf{e}_R(R_1) < \mathbf{i}_R^+(R_l)\frac{\mathbf{e}_R(R_1)(\mathbf{i}_R^+(R_1) - 1)}{\mathbf{i}_R(R_1)}.$$

Inequalities (5.38) and (5.39) prove (5.36) which was equivalent to (5.28). Thus, we have proved (5.28) for the branch case.

Since the proof is quite technical, let us translate it for the first characteristic exponent, R_1, and a point in the second index level, $P \in \Theta_R^{n_1}(C)$. Then, inequality (5.36) gives

$$\mathbf{e}_R(R_1) + 1 < \mathbf{i}_R^+(R_1)\mathbf{c}_R(R_1),$$

which in terms of the Newton pairs is

$$\frac{m_1}{n_1} + 1 < n_1\frac{m_1}{n_1}.$$

This is equivalent to the inequality

$$n_1 + m_1 < n_1 m_1.$$

This inequality is true as long as $n_1, m_1 > 1$ and $\gcd(n_1, m_1) = 1$. The fact that $m_1 > 1$ corresponds to the fact that R is transversal to C, while $n_1 > 1$ and $\gcd(n_1, m_1) = 1$ hold by the definition of Newton pairs (see Definition 2.13).

- Now let us turn to the general case: Assume $C = \sum a_i C_i$ is a general plane curve. Let $P \in \Theta_R(C) \setminus \Theta_1(C)$ and let R_l be the root of its index level. We want to show that

$$\frac{\lambda_{R_l}}{\nu_{R_l}(C)} < \frac{\lambda_P}{\nu_P(C)},$$

which is equivalent to

$$\lambda_{R_l}\left(\sum a_i \nu_P(C_i)\right) < \lambda_P\left(\sum a_i \nu_{R_l}(C_i)\right),$$

by the additivity property of semivaluations (Definition 2.68).

If we prove the inequality term by term for every i we are done, but the same reasoning as in the branch case applies here. In particular, if the projections of R_l and P to $\Theta_R(C_i)$ coincide, $\langle R, P, C_i\rangle = \langle R, R_l, C_i\rangle$, the inequality is trivial. Indeed, by Formula 5.30 we have that

$$\nu_P(C_i) = \mathbf{i}^+_{R_l}(P)\nu_{R_l}(C_i).$$

Thus, inequality

$$\lambda_{R_l}\nu_P(C_i) < \lambda_P \nu_{R_l}(C_i)$$

is equivalent to

$$0 < \mathbf{i}^+_{R_l}(P)\mathbf{e}_{R_l}(P),$$

which is trivially satisfied for $R_l < P$. □

In [**ACLM08**] the authors relate the log-canonical threshold with the Newton polygon in the following manner.

THEOREM 5.40. *Given $f \in \mathbb{C}\{x, y\}$, there exists a system of coordinates $\{x, y\}$ at the origin such that*

$$\mathrm{lct}(f) = \frac{1}{t} \text{ for some } (t, t) \in \partial\mathcal{N}(f).$$

We give an alternative proof using Corollary 5.26.

PROOF. We know by Corollary 5.26 that the log-canonical threshold is attained at an exceptional divisor P corresponding to a marked point in $P \in \Theta^1_R(C)$. Choose a system of coordinates such that P belongs to the segment defined by $R = V(x), L = V(y)$,

$$\tau = [R, L] \ni P.$$

Thus, we have the set of inequalities

$$\mathrm{lct}(f) = \frac{\lambda_P}{\nu_P(f)} \leq \frac{\lambda_Q}{\nu_Q(f)}, \ \forall Q \in \tau.$$

Define t consequently:

$$t = \frac{\nu_{E_P}(f)}{\lambda_P}.$$

Using the inequalities above and writing $\mathbf{e}_R(Q) = m_Q/n_Q$, we obtain

$$t\lambda_Q = \langle (n_Q, m_Q), (t, t)\rangle \ \geq \nu_Q(f).$$

Since this happens for any $Q \in \tau = [R, L]$, it follows that (t, t) is in $\mathcal{N}_{R,L}(f)$, while the equality for P implies that (t, t) is in the boundary of the Newton polygon of f. □

5.3. The Plane Ideal Case

The results for multiplier ideals of plane curves in Section Section 5.2 generalize to multiplier ideals of plane ideals. In this section, we give a definition of Eggers-Wall tree for a plane ideal which encodes the resolution of a generic section (generic linear combination of its generators). We explain how a decomposition of the tree describes a process of log-resolution of the plane ideal. A similar object, the *Newton tree*, was considered by Cassou-Noguès and Veys in [**CV14, CV15**]. In their work, the authors consider Newton maps. Each Newton map corresponds to the local toric modification associated to a regular cone subdividing $\mathbb{R}^2_{\geq 0}$ composed with a suitable change of coordinates.

With the tools provided in the previous chapters we can provide a toroidal resolution process for plane ideals.

Let $I \subset \mathcal{O}_S$ be an ideal on a germ of smooth surface. Recall that we are in a Noetherian ring, so ideals are finitely generated. Our claim is that if $I = (h_1, \ldots, h_s)$ is a plane ideal, a log-resolution of the ideal is given by a log-resolution of a generic $\mathbb{C}$-linear combination, g, of its generators (see [**CV14**, Proposition 3.6] and [**CV15**, Theorem 5.1]).

Let us start with some examples:

EXAMPLE 5.41. Let $I = (y^2 - x^3, y^3 - x^2)$ be a plane ideal, and let $g = y^2 - x^2 - x^3 + y^3$ be a generic element. The Newton polygon of g has just one compact edge, and the dual fan corresponds simply to add the ray generated by the vector $(1, 1)$ to the positive quadrant of the lattice. The situation is symmetric, so we just check what happens in the chart given by $\sigma = \mathrm{Cone}((1, 0), (1, 1))$

$$\psi(\sigma)^*(I) = x_1^2(1 - x_1 y_1^3, x_1 - y_1^2) = (x_1^2)$$

which is already a normal crossing divisor, since over $E_1 = \{x_1 = 0\}$ is a unit. Actually, the strict transform of g over E_1 is the union of two branches defined by $y_1 + 1, y_1 - 1 = 0$ which cut transversely E_1 at different points, so both the ideal I and g are log-resolved. For this toric modification we associate a trunk $\tau = [0, 1]$, i.e., the tree of g deprived from its leaves.

The following example allows to check what happens in the case one uses an exceptional divisor which attains the minimum at a single point of the Newton polygon:

EXAMPLE 5.42. Let $I = (xy - x^3, xy - y^3)$ and $g = 2xy - x^3 - y^3$. The Newton polygon of g has in this case 2 compact faces. The case is chosen to be symmetric, let us take $\sigma = \mathrm{Cone}((1, 1), (1, 2))$, then

$$\psi(\sigma)^* g = x_1^2 y_1^3 (2 - x_1 - x_1 y_1^3)$$

which is clearly a unit over the divisor $E_2 = \{x_1 = 0\}$, and the intersection with $E_1 = \{y_1 = 0\}$ happens at $x_1 = 1$. Checking I, we see that the non-exceptional part is a unit over both E_1, E_2. We assign the trunk $\tau = [0, 2]$ with a marked point of exponent $1/2$.

The next example shows an ideal whose generic section is a branch with two characteristic exponents:

EXAMPLE 5.43. Let $I = ((y^2 - x^3)^2, x^5 y)$ be a plane ideal and let $g = (y^2 - x^3)^2 + x^5 y$. The Newton polygon of g consists of a single compact face parallel to the one in the usual cusp, and the dual fan to three blow-ups as usual. For $\sigma = \mathrm{Cone}((1, 1), (2, 3))$ we have

$$\psi(\sigma)^* I = x_1^{12} u_1^4 ((1 - u_1^2)^2, x_1 u_1)$$

obviously $1 - u_1$ does not cut $E_2 = \{u_1 = 0\}$, so the ideal is a unit over it. But choosing $y_1 = 1 - u_1$ we check that, over $E_1 = \{x_1 = 0\}$, the non-exceptional part of the ideal becomes

(y_1^2, x_1), which shows that the ideal is not resolved. Turning back to g

$$\pi^* g = x_1^{12}(y_1^2 + x_1)$$

up to a unit factor. The curve g is not resolved, since it is tangent to the exceptional divisor E_1. Once again, we associate the trunk $\tau = [0, 3/2]$.

Over the cross formed by $E_1, L_1 = \{y_1 = 0\}$ the Newton polygon of g has a single compact face, after the toric modification corresponding to the minimal regularization of the dual fan, one observes the both I and g are resolved. We associate to this toric modification the trunk $\tau = [0, 1/2]$. This is actually the renormalization of the second maximal trunk associated to g, from which by renormalization formula we can reconstruct the Eggers-Wall tree of g and associate it to I, once deprived from its leaf.

In order to apply what we have seen in the examples, we recall the definition of Newton polygon of an ideal.

Definition 5.44. Let $I \subset \mathcal{O}_S$ be an ideal on a germ of smooth surface and let $R = V(x)$, $L = V(y)$ be a cross. Let $I = (h_1, \ldots, h_s)$ be a finite set of generators of the ideal. We define

$$\mathcal{N}_{R,L}(I) = \operatorname{Conv}\left(\bigcup_i \operatorname{Supp}(h_i)\right).$$

The above definition is independent of the choice of generators. Furthermore, it coincides with the one in the examples, in order to avoid cancellations between terms, we just have the union of supports for each generator.

Remark 5.45. The Newton polygon of an ideal coincides with the Newton polygon of a generic element.

The following result can be deduced from [**CV14**, Proposition 3.6], [**CV15**, Theorem 5.1].

Proposition 5.46. *The ideal I is log-resolved by composing the toric modifications associated to the minimal regularization of the dual fan to the corresponding Newton polygon of the ideal at each point lying either on the origin or on the intersection of the weak transform of the ideal with any exceptional divisor created in the process.*

Proof. With the above notation, let us consider the Newton polygon of I and the corresponding minimal regularization of its dual fan. We will differentiate between those rays dual to some compact face of the Newton polygon (i.e., belonging to the dual fan) or the ones strictly in the regularization.

Say u represents a ray which is in the regularization but it does not belong to the dual fan, i.e., its support function attains its minimum at a single vertex of the Newton polygon, so at least one of the generators becomes a unit over the associated exceptional divisor. Thus, over the corresponding exceptional divisor, the toric morphism is a log-resolution. This is the case of Example 5.42.

Now, let us consider a primitive integral vector u of the dual fan. If there exists a generator g_1 for which u attains the minimum only at a vertex of g_1 which appears also as a vertex of $\mathcal{N}(I)$, then the weak transform, $\tilde{I}$, of I is a unit over E_u. If this does not happen, it may exist an intersection $V(\tilde{I}) \cap E_u$ which has to be either empty of a finite set of points.

To this toric modification we associate, as in the examples, a trunk $\tau = [R, L]$ with marked points corresponding to each compact face of the Newton polygon, $\mathcal{N}(I)$.

For each u dual to a compact face and each point $p \in V(\tilde{I}) \cap E_u$ we have to choose a curvetta. We choose a smooth branch of maximal contact with the strict transform of the generic curve

C_g. By what we proved for germs of functions, we know that the degree in the corresponding curvetta of each of the irreducible components of each of the generators is strictly smaller than the degree in the previous curvetta of the corresponding component and generator, or one has to perform changes of coordinates as in the reasoning of the previous section. Following this process we eventually arrive at local degree zero after a finite number of steps. □

The minimal embedded resolution is obtained by choosing R transversal to the ideal I.

Let I be and ideal generated by

$$I = f(h_1, \ldots, h_s).$$

We can glue together the tree corresponding to f and the different trunks appearing in each step of the resolution of I, unnormalizing the functions on them (see Definition 4.40), to obtain a tree. We will call this tree the *Eggers-Wall tree of the ideal* I and denote it by $\Theta_R(I)$. Notice that the ends of this tree are not necessarily leaves. We also denote by $\overline{\Theta}_R(I) = \Theta_R(\bar{C}_g)$ for g a generic element of I. We show the Eggers-Wall tree of an ideal in Example 5.104.

We are ready to generalize Theorem 5.16 and Corollary 5.22 for plane ideals.

Proposition 5.47 (On the monomiality of multiplier ideals of a plane ideal). *Let I be an ideal, $\Theta_R(I)$ its Eggers-Wall tree and $\overline{\Theta}_R(I)$ a maximal contact completion. The multiplier ideals of I are monomial ideals in the generalized monomials of expression 5.7 in the branches of the completion $\overline{\Theta}_R(I)$, i.e.,*

$$\mathcal{J}(I^{\xi}) = \left\langle \mathcal{M} \;\middle|\; \nu_P(\mathcal{M}) + \lambda_P > \xi \nu_P(I) \text{ for all } P \in \Upsilon^{\circ} \right\rangle. \tag{5.48}$$

Remark 5.49. Tucker ([Tuc10b, Tuc10a]) develops the notion of jumping number contributed by a divisor introduced by Smith and Thompson ([ST07]). Tucker says that a reduced divisor G **critically contributes** to the jumping number ξ if

$$\mathcal{J}(\xi C) \subsetneq \Psi_* \mathcal{O}_Y(K_\Psi - \lfloor \xi F \rfloor + G)$$

and no proper subdivisor of G contributes ξ. In his work, he gives a geometric characterization of those divisors, proving that such a divisor must be a connected chain whose ends are, in our language, marked points of the tree.

5.4. Relation with Naie's Formulas

In this section we inspect the set of jumping numbers smaller than one for a plane branch. The main result of this section, Theorem 5.57, gives a different approach to Naie's formulas in [Nai09] based on the results of Chapters 4 and 5.

Let us introduce the Naie set of a plane branch. First, for p, q such that $\gcd(p, q) = 1$ Naie defines the set:

$$\Delta(p, q) = \{ap + bq \mid a, b \in \mathbb{N}^*,\ ap + bq < pq\},$$

and for $m \in \mathbb{N}^*$

$$\Delta^m(p, q) = \bigcup_{k=0}^{m-1} \left(kpq + \Delta(p, q)\right).$$

Let C be a plane curve germ and R a smooth branch transversal to it. Recall that the semigroup of C is

$$\Gamma(C) = \mathbb{N}\bar{b}_0 + \mathbb{N}\bar{b}_1 + \ldots + \mathbb{N}\bar{b}_g.$$

Let $(n_1, m_1), \ldots, (n_g, m_g)$ be the Newton pairs of C. As usual, we write $e_k = n_{k+1} \ldots n_g$ for $0 \le k \le g$.

Definition 5.50. The **Naie set of the branch** C is defined by

$$\Delta(C) = \bigsqcup_{k=1}^{g} \Delta^{e_k}\left(n_k, \frac{\bar{b}_k}{e_k}\right).$$

Naie's result states that there is a surjection, η, from the Naie set to the set of jumping numbers smaller than one,

$$\eta : \Delta(C) \twoheadrightarrow \{\text{jumping numbers smaller than one}\}, \tag{5.51}$$

such that for $(a,b,c) \in \Delta^{e_k}\left(n_k, \frac{\bar{b}_k}{e_k}\right)$,

$$\eta(a,b,c) = \frac{a n_k + b\frac{\bar{b}_k}{e_k} + c n_k \frac{\bar{b}_k}{e_k}}{\operatorname{lcm}(e_{k-1}, \bar{b}_k)} \tag{5.52}$$

Let us denote by $C = L_g$, by $L_0, \ldots, L_g$ a complete system of semi-roots of L_g with respect to R (see Definition 2.24), and by $R_1 < \cdots < R_g$ the points of discontinuity of the index function $\mathbf{i}_R$ on the segment $[R, L_g]$. The Eggers-Wall tree $\Theta_R(\bar{C})$ is shown in Figure 8. It has one maximal contact decomposition with segments

$$[R = R_0, L_0],\ [R_1, L_1], \ldots, [R_g, L_g] \tag{5.53}$$

and for each $1 \leq i \leq g$, one has:

$$R_i \in [R_{i-1}, L_{i-1}] \text{ and } \mathbf{i}_R(R_i) = \mathbf{i}_R(L_{i-1}).$$

Let $L_i = V(z_i)$ $i = 0, \ldots, g$ be a sequence of representatives. We define the set

$$\mathfrak{M}(C) = \{\mathcal{M} = x^j z_0^{i_0} \ldots z_{g-1}^{i_{g-1}} \mid \xi_{\mathcal{M}} < 1\}, \tag{5.54}$$

of generalized monomials of $\bar{C}$ which give jumping numbers smaller than one (with limited exponents as in Lemma 2.117). Recall that we have a surjection

$$\xi : \mathfrak{M}(C) \twoheadrightarrow \{\text{jumping numbers smaller than one}\}, \tag{5.55}$$

such that for a generalized monomial $\mathcal{M}$,

$$\xi(\mathcal{M}) = \xi_{\mathcal{M}} = \min_{P \in \Upsilon} \left\{ \frac{\nu_P(\mathcal{M}) + \lambda_P}{\nu_P(C)} \right\}, \tag{5.56}$$

as in Corollary 5.22.

Theorem 5.57. *There is a bijection $\Lambda : \Delta(C) \to \mathfrak{M}(C)$ such that the following diagram is commutative*

$$\begin{array}{ccc} \Delta(C) & \xrightarrow{\ \Lambda\ } & \mathfrak{M}(C) \\ & \searrow^{\eta} & \downarrow \xi \\ & & \{\textit{jumping numbers of } C \text{ ¡ } 1\}. \end{array}$$

In order to prove the theorem we need some previous results: first, we describe explicitly the elements in $\Delta(C)$ of Definition 5.50, then we describe the conductor of the semigroup of each semi-root, finally we compare the jumping number condition in a given monomial in a particular marked point of the tree with the immediate previous and next ones.

Recall that we denote by $R_1 < \ldots < R_g$ the points of discontinuity of the index function on the tree $\Theta_R(C)$.

LEMMA 5.58. *The semigroup of L_k is equal to the semigroup of values of ν_{R_k},*

$$\Gamma(L_k) = \left\langle \frac{\bar{b}_0}{e_k}, \dots, \frac{\bar{b}_k}{e_k} \right\rangle,$$

where

$$\nu_{R_k}(R) = (R, L_k)_O = \frac{\bar{b}_0}{e_k}$$
$$\nu_{R_k}(L_{i-1}) = (L_{i-1}, L_k)_O = \frac{\bar{b}_i}{e_k} \text{ for } i = 1, \dots, k.$$

Furthermore,

$$\nu_{R_k}(L_{k+j}) = n_k n_{k+1} \dots n_{k+j} \frac{\bar{b}_k}{e_k} \text{ for } j = 0, \dots, g-k.$$

In particular, $\operatorname{lcm}(e_k, \bar{b}_k) = \nu_{R_k}(C)$.

PROOF. The minimal embedded resolution of L_k is the composition of the k toroidal modifications associated to the segments $[R_i, L_i]$ for $i = 0, \dots, k-1$ and by definition we have that L_k is a curvetta at R_k. By Proposition 4.56, we have an embedded resolution of $\bar{L}_k$ such that L_k verifies the conditions in Proposition 2.79. Thus,

$$\nu_{R_k}(R) = (R, L_k)_O$$

and

$$\nu_{R_k}(L_{i-1}) = (L_{i-1}, L_k)_O, \text{ for } 1 \leq i \leq k.$$

By Corollary 2.101 we get:

$$(L_{i-1}, L_k)_O \overset{(2.101)}{=} \mathbf{i}_R(L_{i-1})\mathbf{i}_R(L_k)\mathbf{c}_R(R_i) \overset{(2.33)}{=} \frac{\bar{b}_i}{e_k}, \tag{5.59}$$

where we decompose $\mathbf{i}_R(L_g) = \mathbf{i}_R^+(R_k)\mathbf{i}_{R_k}(L_g)$ and we use that $\mathbf{i}_R^+(R_k) = \mathbf{i}_R(L_k)$, $\mathbf{i}_{R_k}(L_g) = e_k$.

For L_{k+j} we have that $\langle R, L_k, L_{k+j} \rangle = R_k$ and $\mathbf{i}_R(L_{k+j}) = \mathbf{i}_R^+(R_k)\mathbf{i}_{R_k}(L_{k+j})$. By Proposition 4.70,

$$\nu_{R_k}(L_{k+j}) \overset{(4.70)}{=} \mathbf{i}_R(L_k)\mathbf{i}_R(L_{k+j})\mathbf{c}_R(R_k) \overset{(5.59)}{=} \frac{\mathbf{i}_R(L_{k+j})}{\mathbf{i}_R(L_k)}\bar{b}_k = \mathbf{i}_{R_k}(L_{k+j})\bar{b}_k.$$

By Definition (2.11) and Remark 2.34, $\gcd(e_{k-1}, \bar{b}_k) = e_k$, so

$$\operatorname{lcm}(e_{k-1}, \bar{b}_k) = \frac{e_{k-1}\bar{b}_k}{e_k} = n_k \bar{b}_k \overset{(5.59)}{=} \nu_{R_k}(L_g).$$

□

Thus we write

$$\Delta\left(n_k, \nu_{R_k}(L_{k-1})\right) = \{(a,b) \in (\mathbb{N}^*)^2 \mid a n_k + b \nu_{R_k}(L_{k-1}) < \nu_{R_k}(L_k)\}, \tag{5.60}$$

in expression (5.50).

Recall from Lemma 2.36 that for a plane branch C the conductor is

$$\mathsf{c}(C) = n_g \bar{b}_g - b_g - e_0 + 1.$$

We rewrite the previous expression of the conductor of the semigroup $\Gamma(L_k)$:

LEMMA 5.61. *The conductor of the semigroup L_k, for $k = 1, \dots, g$, is* $\mathsf{c}(L_k) = \nu_{R_k}(L_k) - \lambda_{R_k} + 1$.

PROOF. Clearly, it is enough to prove it for $k = g$, $z_g = f$. First, notice that

$$\nu_{R_g}(C) = \nu_{R_g}(z_{g-1})n_g, \tag{5.62}$$

since by definition $\mathbf{i}_{R_{g-1}}(L_g) = n_g$ (check also Lemma 4.97). Thus, the proof boils down to check that

$$\lambda_{R_g} = b_g + e_0.$$

By Definition 2.13 this is equivalent to Proposition 5.1,

$$\lambda_{R_g} = (\alpha_g + 1)e_0.$$

□

Let $0 \leq k \leq g$ be an integer and $\mathcal{M} = x^j y^{i_0} z_1^{i_1} \ldots z_{g-1}^{i_{g-1}}$ a generalized monomial in the semi-roots of C. The **jumping number condition** associated to the monomial $\mathcal{M}$ at the marked point R_k of $\Theta_R(C)$ is

$$\mathtt{C}_k(\mathcal{M}) = \frac{\nu_{R_k}(\mathcal{M}) + \lambda_{R_k}}{\nu_{R_k}(C)}. \tag{5.63}$$

We consider the truncation of $\mathcal{M}$ at the k-th level

$$\mathcal{M}_{|<\mathrm{k}} = x^j y^{i_0} z_1^{i_1} \ldots z_{k-1}^{i_{k-1}}.$$

LEMMA 5.64. *Let* $\sim \in \{<, >, =\}$. *The relation* $\mathtt{C}_k(\mathcal{M}) \sim \mathtt{C}_{k+1}(\mathcal{M})$ *is equivalent to*

$$\nu_{R_k}(\mathcal{M}_{|<\mathrm{k}}) + \lambda_{R_k} \sim \nu_{R_k}(z_k)$$

And there are implications

$$\begin{aligned} \mathtt{C}_k(\mathcal{M}) \leq \mathtt{C}_{k+1}(\mathcal{M}) &\Rightarrow \mathtt{C}_{k+1}(\mathcal{M}) \leq \mathtt{C}_{k+2}(\mathcal{M}), \\ \mathtt{C}_k(\mathcal{M}) \geq \mathtt{C}_{k+1}(\mathcal{M}) &\Rightarrow \mathtt{C}_{k-1}(\mathcal{M}) \geq \mathtt{C}_k(\mathcal{M}). \end{aligned}$$

PROOF. Assume that we have a relation

$$\mathtt{C}_k(\mathcal{M}) = \frac{\nu_{R_k}(\mathcal{M}) + \lambda_{R_k}}{\nu_{R_k}(f)} \sim \frac{\nu_{R_{k+1}}(\mathcal{M}) + \lambda_{R_{k+1}}}{\nu_{R_{k+1}}(f)} = \mathtt{C}_{k+1}(\mathcal{M}) \tag{5.65}$$

We use Theorem 4.70 and Remark 4.116 for valuations and Lemma 5.1 for log-discrepancies in order to express each term with respect to R_k and the pair (n_{k+1}, m_{k+1}).

$$\nu_{R_{k+1}}(C) \overset{(4.116)}{=} n_{k+1}\nu_{R_k}(C) + m_{k+1}e_k. \tag{5.66}$$

By using Lemma 5.58 we obtain

$$\nu_{R_k}(C) = \nu_{R_k}(L_k)e_k. \tag{5.67}$$

Let us write s_k for the degree in y_k of the k-th leading term of $\mathcal{M}$,

$$s_k = \deg_{\mathrm{y_k}}(\mathrm{LT}_{R_k, L_k}(\mathcal{M})) \overset{(4.111)}{=} i_k + n_{k+1}i_{k+1} + \ldots + n_{k+1}\ldots n_{g-1}i_{g-1}. \tag{5.68}$$

Then, by 4.110 we have that

$$\nu_{R_{k+1}}(\mathcal{M}) = n_{k+1}\nu_{R_k}(\mathcal{M}) + m_{k+1}s_k. \tag{5.69}$$

Furthermore, we can apply the additivity property of semivaluations and assertion 3. in Lemma 5.58 to obtain

$$\begin{aligned} \nu_{R_k}(\mathcal{M}) &\overset{(2.68)}{=} \nu_{R_k}(\mathcal{M}_{|<k}) + \nu_{R_k}(\mathcal{M}_{|\geq k}) \\ &\overset{(5.58)}{=} \nu_{R_k}(\mathcal{M}_{|<k}) + \nu_{R_k}(L_k)s_k. \end{aligned} \tag{5.70}$$

Finally, using (5.4) we write the log-discrepancy

$$\lambda_{R_{k+1}} = n_{k+1}\lambda_{R_k} + m_{k+1}. \tag{5.71}$$

By using (5.66), (5.69) and (5.71) we obtain from (5.65) the equivalent relation

$$(\nu_{R_k}(\mathcal{M}) + \lambda_{R_k})\,(n_{k+1}\nu_{R_k}(C) + m_{k+1}e_k) \sim \nu_{R_k}(C)\Big[n_{k+1}\,(\nu_{R_k}(\mathcal{M}) + \lambda_{R_k}) + m_{k+1}\,(s_k+1)\Big].$$

Canceling out coinciding terms we have that (5.65) is equivalent to

$$(\nu_{R_k}(\mathcal{M}) + \lambda_{R_k})\,m_{k+1}e_k \sim \nu_{R_k}(C)m_{k+1}(s_k+1). \tag{5.72}$$

By (5.67) we can expand $\nu_{R_k}(C) = \nu_{R_k}(L_k)e_k$. Simplifying we obtain that (5.65) is equivalent to the relation

$$\nu_{R_k}(\mathcal{M}) + \lambda_{R_k} \sim \nu_{R_k}(L_k)(s_k+1). \tag{5.73}$$

By using (5.70) in (5.73), we can simplify the term $\nu_{R_k}(\mathcal{M}_{|\geq k})$ on both sides. Thus, we obtain that (5.65) is equivalent to the desired relation

$$\mathtt{C}_k(\mathcal{M}) \sim \mathtt{C}_{k+1}(\mathcal{M}) \;\equiv\; \nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k} \sim \nu_{R_k}(L_k), \tag{5.74}$$

which proves the first assertion.

Now we prove the second statement of the Lemma: Assume, for example, that the generalized monomial $\mathcal{M}$ verifies $\mathtt{C}_k(\mathcal{M}) \leq \mathtt{C}_{k+1}(\mathcal{M})$, and we want to prove

$$\mathtt{C}_{k+1}(\mathcal{M}) \leq \mathtt{C}_{k+2}(\mathcal{M}). \tag{5.75}$$

Using 5.74 below, relation (5.75) is equivalent to

$$\nu_{R_{k+1}}(\mathcal{M}_{|<k+1}) + \lambda_{R_{k+1}} \leq \nu_{R_{k+1}}(L_{k+1}).$$

This can be rewritten using (5.69) for $\mathcal{M}_{|<k+1} = \mathcal{M}_{|<k} z_k^{i_k}$ and (5.71)

$$\Big[n_{k+1}\left(\nu_{R_k}(\mathcal{M}_{|<k}) + \nu_{R_k}(L_k)i_k\right) + m_{k+1}i_k\Big] + (n_{k+1}\lambda_{R_k} + m_{k+1}) \leq n_{k+1}\left(n_{k+1}\nu_{R_k}(L_k) + m_{k+1}\right).$$

By reordering we obtain that (5.75) is equivalent to the relation

$$n_{k+1}\left(\nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k}\right) \leq n_{k+1}\,(n_{k+1} - i_k)\,\nu_{R_k}(L_k) + (n_{k+1} - i_k - 1)\,m_{k+1}. \tag{5.76}$$

By hypothesis, the left hand side verifies

$$n_{k+1}\left(\nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k}\right) \leq n_{k+1}\nu_{R_k}(L_k). \tag{5.77}$$

Furthermore, since $i_k < n_{k+1}$, we have that

$$n_{k+1}\nu_{R_k}(L_k) \leq n_{k+1}\,(n_{k+1} - i_k)\,\nu_{R_k}(L_k) + (n_{k+1} - i_k - 1)\,m_{k+1}. \tag{5.78}$$

Combining (5.77) and (5.78), we prove (5.76), which is equivalent to (5.75).

Analogously, assume that the generalized monomial $\mathcal{M}$ verifies $\mathtt{C}_k(\mathcal{M}) \geq \mathtt{C}_{k+1}(\mathcal{M})$ and let us prove that it verifies

$$\mathtt{C}_{k-1}(\mathcal{M}) \geq \mathtt{C}_k(\mathcal{M}). \tag{5.79}$$

By hypothesis, $\mathcal{M}$ verifies $\mathtt{C}_k(\mathcal{M}) \geq \mathtt{C}_{k+1}(\mathcal{M})$, which by (5.74) is equivalent to

$$\nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k} \geq \nu_{R_k}(L_k).$$

We use (5.69) for $\mathcal{M}_{|<k} = \mathcal{M}_{|<k-1} z_{k-1}^{i_{k-1}}$ and (5.71) to rewrite the above relation, so $\mathcal{M}$ satisfies

$$\Big[n_k\left(\nu_{R_{k-1}}(\mathcal{M}_{|<k-1}) + \nu_{R_{k-1}}(L_{k-1})i_{k-1}\right) + m_k i_{k-1}\Big] + \left(n_k\lambda_{R_{k-1}} + m_k\right) \geq n_k\left(n_k\nu_{R_{k-1}}(L_{k-1}) + m_k\right).$$

By reordering the above relation, $\mathcal{M}$ satisfies

$$n_k\left(\nu_{R_{k-1}}(\mathcal{M}_{|<k-1}) + \lambda_{R_{k-1}}\right) \geq n_k\,(n_k - i_{k-1})\,\nu_{R_{k-1}}(L_{k-1}) + (n_k - i_{k-1} - 1)\,m_k. \tag{5.80}$$

Since $i_{k-1} < n_k$, the right hand side of (5.80) satisfies

$$n_k\,(n_k - i_{k-1})\,\nu_{R_{k-1}}(L_{k-1}) + (n_k - i_{k-1} - 1)\,m_k \geq n_k \nu_{k-1}(L_{k-1}). \tag{5.81}$$

Combining (5.80) and (5.81), we obtain

$$\nu_{R_{k-1}}(\mathcal{M}_{|<k-1}) + \lambda_{R_{k-1}} \geq \nu_{k-1}(L_{k-1}),$$

which by (5.74) is equivalent to (5.79). □

We are ready to prove the announced bijection.

Proof of Theorem 5.57. Let us define $\Lambda : \Delta(C) \longrightarrow \mathfrak{M}(C)$. Assume first that we have an element in the Naie set of C of the form

$$(a, b, 0) \in \Delta\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right).$$

We want to assign to such an element a generalized monomial in the semi-roots,

$$\Lambda(a, b, 0) = \mathcal{M} = x^j z_0^{i_0} \ldots z_k^{i_k},$$

such that

$$\eta(a, b, 0) = \xi_{\mathcal{M}},$$

or equivalently,

$$\nu_{R_{k+1}}(\mathcal{M}) + \lambda_{R_{k+1}} = a n_{k+1} + b \nu_{R_{k+1}}(L_k). \tag{5.82}$$

Using identities (5.69) and (5.71) in (5.82) gives

$$(n_{k+1}\nu_{R_k}(\mathcal{M}) + m_{k+1} i_k) + (n_{k+1}\lambda_{R_k} + m_{k+1}) = a n_{k+1} + b\left(n_{k+1}\nu_{R_k}(L_k) + m_{k+1}\right), \tag{5.83}$$

where (n_{k+1}, m_{k+1}) is the $(k+1)$-th characteristic pair of C, so $\gcd(n_{k+1}, m_{k+1}) = 1$. Taking modulo n_{k+1} forces the congruence relation

$$m_{k+1}(i_k + 1 - b) \equiv 0.$$

Since the exponent i_k satisfies $0 \leq i_k < n_{k+1}$, and by definition $0 < b < n_{k+1}$, the congruence implies

$$0 \leq i_k = b - 1 < n_{k+1} - 1. \tag{5.84}$$

Substituting this in equation (5.83) and simplifying by the common factor n_{k+1} yields

$$\nu_{R_k}(\mathcal{M}) = (i_k + 1)\nu_{R_k}(L_k) - \lambda_{R_k} + a,$$

or equivalently,

$$\nu_{R_k}(\mathcal{M}_{|<k}) = \nu_{R_k}(L_k) - \lambda_{R_k} + a. \tag{5.85}$$

Since $a > 0$, the valuation of such a generalized monomial is greater than the conductor of the semigroup of L_k by Lemma 5.61,

$$\nu_{R_k}(\mathcal{M}) \geq \nu_{R_k}(z_k) - \lambda_{R_k} + 1 \geq \mathsf{c}(L_k),$$

so $\nu_{R_k}(\mathcal{M}) \in \Gamma(L_k)$. Using the unique expansion of elements in the semigroup in terms of the generators and Lemma 5.58, we obtain exponents

$$0 \leq j, i_0, \ldots, i_{k-1} \text{ with } 0 \leq i_l < n_{l+1}. \tag{5.86}$$

We set

$$\Lambda(a, b, 0) = \mathcal{M}_{|<k+1} = x^j z_0^{i_0} \ldots z_{k-1}^{i_{k-1}} z_k^{i_k}. \tag{5.87}$$

By equation (5.85) we have

$$\nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k} > \nu_{R_k}(L_k),$$

and by (5.82) and the definition of $\Delta\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right)$,

$$\nu_{R_{k+1}}(\mathcal{M}_{|<k+1}) + \lambda_{R_{k+1}} < \nu_{R_{k+1}}(L_{k+1}).$$

Thus, by Lemma 5.64

$$\eta(a, b, 0) = \xi_{\mathcal{M}_{|<k+1}} = \mathtt{C}_k(\mathcal{M}_{|<k+1}) < 1. \tag{5.88}$$

In general, let us consider an element $(a, b, c) \in \Delta^{e_{k+1}}\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right)$, where $(a, b) \in \Delta\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right)$ and $0 \leq c < e_{k+1}$.

Since $c < e_{k+1} = n_{k+2} \ldots n_g$, c is expanded uniquely (Lemma 2.120) as

$$c = i_{k+1} + n_{k+2} i_{k+2} + \ldots + n_{k+2} \ldots n_{g-1} i_{g-1} \text{ with } 0 \leq i_l < n_{l+1}.$$

Now we set

$$\Lambda(a, b, c) = \mathcal{M} = \Lambda(a, b, 0) \cdot z_{k+1}^{i_{k+1}} \ldots z_{g-1}^{i_{g-1}}, \tag{5.89}$$

which by (5.88) verifies $\eta(a, b, c) = \xi_{\mathcal{M}} = \mathtt{C}_k(\mathcal{M})$ and, since $\nu_{R_{k+1}}(f) = e_{k+1}\nu_{R_{k+1}}(L_k)$, $\xi_{\mathcal{M}} < 1$.

Now we define Λ^{-1}. Let $\mathcal{M} = x^j y^{i_0} z_1^{i_1} \ldots z_{g-1}^{i_{g-1}}$ be a generalized monomial such that it generates jumping number smaller than one at the $k+1$-th relevant exceptional divisor, i.e.,

$$\xi_{\mathcal{M}} = \mathtt{C}_{k+1}(\mathcal{M}) = \frac{\nu_{R_{k+1}}(\mathcal{M}) + \lambda_{R_{k+1}}}{\nu_{R_{k+1}}(C)}.$$

In particular, by Lemma 5.64, we have

1. $\mathtt{C}_{k+1}(\mathcal{M}) < \mathtt{C}_k(\mathcal{M})$ which due to Lemma 5.64 is equivalent to

$$\nu_{R_k}(z_k) < \nu_{R_k}(\mathcal{M}_{|<\mathrm{k}}) + \lambda_{R_k}. \tag{5.90}$$

2. For the same reason $\mathtt{C}_{k+1}(\mathcal{M}) < \mathtt{C}_{k+2}(\mathcal{M})$ is equivalent to

$$\nu_{R_{k+1}}(\mathcal{M}_{|<\mathrm{k}+1}) + \lambda_{R_{k+1}} < \nu_{R_{k+1}}(z_{k+1}). \tag{5.91}$$

Take into account that, due to the minimality of the conductor of the semigroup of L_k and L_{k+1} (5.61) equality is not possible.

We set

$$\begin{aligned} a &= \nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k} - \nu_{R_k}(L_k), \\ b &= i_k + 1, \\ c &= (i_{k+1} + n_{k+2} i_{k+2} + \ldots + n_{k+2} \ldots n_{g-1} i_{g-1}). \end{aligned} \tag{5.92}$$

Clearly $j > 0$, while $i > 0$ because of inequality (5.90). Furthermore,

$$\begin{aligned} a n_{k+1} + b \nu_{R_{k+1}}(L_k) &\overset{(5.92)}{=} (i_k + 1)\nu_{R_{k+1}}(L_k) + n_{k+1}\left(\nu_{R_k}(\mathcal{M}_{|<k}) + \lambda_{R_k} - \nu_{R_k}(L_k)\right) \\ &\overset{(5.66)}{=} \left(\nu_{R_{k+1}}(\mathcal{M}_{|<k}) + i_k \nu_{R_{k+1}}(L_k)\right) + \left(n_{k+1}\lambda_{R_k} + m_{k+1}\right) \\ &\overset{(5.71)}{=} \nu_{R_{k+1}}(\mathcal{M}_{|<k+1}) + \lambda_{R_{k+1}} \\ &\overset{(5.91)}{<} \nu_{R_{k+1}}(z_{k+1}). \end{aligned}$$

This shows that $(a, b, 0) \in \Delta\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right)$.

Since $i_{k+1} < n_{k+2}$ and $i_{k+2} < n_{k+3}$, $i_{k+1} + n_{k+2} i_{k+2} < n_{k+2} n_{k+3}$. Applying recurrently this reasoning, we conclude that $c < e_{k+1}$. Thus, $(a, b, c) \in \Delta^{e_{k+1}}\left(n_{k+1}, \nu_{R_{k+1}}(L_k)\right)$ and we define $\Lambda^{-1}(\mathcal{M}) = (a, b, c)$. By construction, $\xi_{\mathcal{M}} = \eta(a, b, c) < 1$.

Our definitions of Λ, Λ^{-1} are mutually inverse since they are defined by clearing the unknowns in the equations (5.92). □

REMARK 5.93. Notice that for any element in the semigroup there is a unique expansion in the $\bar{b}_k$, with exponents $j, i_0, \dots, i_k$ as in (5.86). Thus, if two generalized monomials result in the same jumping number, they attain the minimum at different levels.

5.5. Multiplicity of jumping numbers

As noted before, the set of jumping numbers of a plane branch in Naie's description may not be all different, but the same jumping number can correspond to two different relevant exceptional divisors. This is what we call multiplicity phenomenon of a jumping number. In [**AADG17**], the authors describe the multiplicities of the jumping numbers of any plane ideal. In this section, we aim to compute the cardinality of the set of jumping numbers smaller than one for a plane branch C.

Let $\mathfrak{a} \subseteq \mathcal{O}_{S,O}$ be a primary ideal and let us denote by $\{\xi_i\}$ its collection of jumping numbers. Since $\mathfrak{a}$ is primary its multipier ideals are primary as well, so they have finite codimension, as $\mathbb{C}$-vector spaces, in $\mathcal{O}_{S,O}$.

DEFINITION 5.94. The multiplicity of a fixed jumping number ξ_i is

$$m(\xi_i) = \dim_{\mathbb{C}} \left(\mathcal{J}(\mathfrak{a}^{\xi_i}) / \mathcal{J}(\mathfrak{a}^{\xi_{i+1}}) \right).$$

Let C be a plane branch with g characteristic exponents with respect to a smooth branch R. Denote by $L_0, \dots, C = L_g$ a complete system of semi-roots of C with respect to R (see Definition 2.24), and by $R_1 < \cdots < R_g$ the points of discontinuity of the index function $\mathbf{i}_R$ on the segment $[R, L_g]$. The Eggers-Wall tree $\Theta_R(\sum_{k=0}^{g} L_k)$ is shown in Figure 8.

Notice that, since C is a plane curve, Lemma 3.26 implies that in order to know the set of jumping numbers, it is enough to know the jumping numbers smaller than one.

The following formula relates the cardinality of the set of jumping numbers smaller than one with the Milnor number and the conductor of the semigroup. Although the technique is different, the result is already known by the relation with the Hodge spectrum (see [**Bud03, Bud12**]).

PROPOSITION 5.95. *The set of jumping numbers of C smaller than one counted with multiplicity has cardinality*

$$\sum_{\xi<1} \operatorname{mult}(\xi) = \frac{\mathsf{c}(C)}{2},$$

where $\mathsf{c}(C)$ is the conductor of the semigroup of C.

PROOF. Let us now work with a plane branch C, let $\Theta_R(C)$ be its Eggers-Wall tree and Υ its set of marked points. By Theorem 5.57 we have that multiplicity can be rewritten for a jumping number ξ_i (check Formula 5.54 for the notation):

$$\#\{\mathcal{M} \in \mathfrak{M}(C) \mid \xi_i = \xi_{\mathcal{M}}\}.$$

For each such $\mathcal{M}$, there exists an interior marked point, $R_k \in \Upsilon$, such that

$$\xi_i = \xi_{\mathcal{M}} = \frac{\nu_{R_k}(\mathcal{M}) + \lambda_{R_k}}{\nu_{R_k}(C)}.$$

We know by Lemma 5.64 that $c\nu_{R_k}(C) - \lambda_{R_k}$ is an element of the semigroup of the $k+1$ semi-root, $\Gamma(L_{k+1})$, so that there is a unique way to write it in terms of the generators of this semigroup, hence this is the only multiplicity phenomenon that may occur.

By Lemma 5.64, we know that if $\nu_{R_k}(\mathcal{M}) + \lambda_{R_k} < \nu_{R_k}(C)$, then

$$\nu_{R_g}(\mathcal{M}) + \lambda_{R_g} < \nu_{R_g}(C).$$

Thus, any jumping number smaller than one has an associated monomial which generates an element in the semigroup of C verifying

$$\nu_{R_g}(C) - \lambda_{R_g} > \nu_{R_g}(\mathcal{M}) \in \Gamma(C).$$

Conversely, any element in the semigroup of the branch, $\Gamma(C)$, which verifies that $\nu_{R_g}(\mathcal{M}) < \nu_{R_g}(C) - \lambda_{R_g}$, generates a jumping number smaller than one, since by Corollary 5.22

$$\xi_{\mathcal{M}} = \min_{1\leq i\leq g}\left\{\frac{\nu_{R_k}(\mathcal{M}) + \lambda_{R_k}}{\nu_{R_k}(C)}\right\} \leq \frac{\nu_{R_g}(\mathcal{M}) + \lambda_{R_g}}{\nu_{R_g}(C)} < 1.$$

Thus, the number of jumping numbers smaller than one counted with multiplicity is

$$\#\{0 \leq a \in \Gamma(C) \mid a < \nu_{R_g}(C) - \lambda_{R_g}\}.$$

According to [**Wal04**, Corollary 4.3.7] the *double point number* of the semigroup of C, $\delta(C)$, i.e., the number of positive integers not in $\Gamma(C)$ is half the conductor,

$$2\delta(C) = \#\{a \notin \Gamma(C)\} = \mathsf{c}(C),$$

and according to Milnor, the conductor coincides with the *Milnor number* (see [**Wal04**, Proposition 6.3.2])

$$\mu(C) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(\partial_x f, \partial_y f) = \mathsf{c}(C).$$

This is, in fact, Milnor formula for an irreducible curve, $\mu(C) = \delta(C) - r(C) + 1$, where $r(C)$ is the number of irreducible factors of C (see [**Wal04**, Theorem 6.5.1]).

Thus, the number of jumping numbers smaller than one counted with multiplicity is equal to half the Milnor Number of the singularity

$$\sum_{\xi<1} \mathrm{mult}(\xi) = \#\{0 \leq a \in \Gamma(C) | a < \nu_{R_g}(C) - \lambda_{R_g}\} = \delta(C) = \frac{\mathsf{c}(C)}{2} = \frac{\mu(C)}{2}.$$

□

5.6. Monomial generators of complete planar ideals

Let (S, O) be a germ of smooth surface, $\mathcal{O}_{S,O}$ the ring of germs of holomorphic functions in a neighborhood of O and let $\mathfrak{m}$ be the maximal ideal at O. Let $\Psi : X \to S$ be a proper birational morphism that is a composition of blow-ups along a set of points $\{\pi_i\}_{i=1,\dots,n}$. For any effective divisor D over X, we define $H_D := \pi_*\mathcal{O}_X(-D)$. If D has exceptional support, H_D is an $\mathfrak{m}$-primary complete ideal of $\mathcal{O}_{S,O}$.

An effective divisor D with exceptional support is called antinef if its intersection with each prime exceptional divisor is not positive, $D \cdot E_i \leq 0$ for $i = 1, \dots, n$. Zariski showed in [**Zar38**] that there exists an isomorphism of semigroups between the set of complete $\mathfrak{m}$-primary ideals and the set of antinef divisors with exceptional support.

In [**AAB17**], the authors use this correspondence together with some properties of the theory of adjacent ideals to conclude that the ideals H_D are generated by monomials in the semi-roots (maximal contact elements), and compute it through an algorithm ([**AAB17**, Algorithm 3.5.]).

If $D = \sum d_i E_i$ is an effective divisor with exceptional support, we have

$$H_D = \{h \in \mathcal{O}_{S,O} \mid \nu_{E_i}(h) \geq d_i \ \forall i = 1, \dots, n\}.$$

Corollary 4.89 and Lemma 4.87 imply that this ideal is generated by monomials in the semi-roots. Indeed, let G be the dual graph of Ψ and consider a complete curve $\bar{C}$ and a smooth reference branch R, so that $\Theta_R(\bar{C})$ corresponds to G. By Corollary 4.89, the components of $\bar{C}$ for a generating sequence for the set $\{\nu_{E_i}\}$. Thus, by Lemma 4.87, for any germ h we have a finite expansion with respect to $\bar{C}$,

$$h = \sum_{I=(i_0,i_1,\dots,i_r)} a_I(x)\cdot z_0^{i_0}\dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x\},$$

in such a way that for every $1 \le j \le s$ there exists an index $I^{(j)} = (i_0^{(j)},\dots,i_r^{(j)})$ with $a_{I^{(j)}} \neq 0$ such that

$$\nu_j(h) = \nu_j\left(z_0^{i_0^{(j)}}\dots z_r^{i_r^{(j)}}\right) = \min\left\{\nu_j\left(a_I\cdot z_0^{i_0}\dots z_r^{i_r}\right)\right\}.$$

Thus $h \in H_D$ if and only if $a_I(x)\cdot z_0^{i_0}\dots z_r^{i_r} \in H_D$ for every I such that $c_I \neq 0$.

EXAMPLE 5.96. Let us consider the complete curve $\bar{C}$ in Example 2.93. Recall that $f_C = (y^2-x^3)^2 - 4x^5y - x^7$, $L_0 = \{y=0\}$ and $L_1 = V(z_1)$ with $z_1 = y^2 - x^3$ form a maximal contact completion of C. The dual graph $G(\Psi,\bar{C})$ of its minimal embedded resolution is shown in Figure 1.

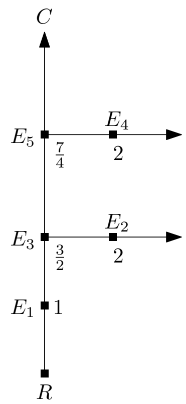

FIGURE 1. The dual graph $G(\Psi,\bar{C})$ of the complete curve $\bar{C}$ in Example 2.93.

Consider the exceptional part in its minimal resolution. Let us denote by $\{E_i\}_{i=1,\dots,5}$ the exceptional divisors in the minimal embedded resolution of C, then

$$D_C := (\Psi^*C)_{\text{exc}} = 4E_1 + 6E_2 + 12E_3 + 13E_4 + 26E_5 = (4,6,12,13,26).$$

By Corollary 4.89, x, y, z_1, f_C form a generating sequence for $\underline{\nu} = (\nu_{E_i})$. In fact, x, y, z_1 form a generating sequence for $\underline{\nu}$ (see Theorem 4.159).

By Lemma 4.87, for any germ h we have a finite expansion with monomials

$$\mathcal{M} = x^a y^b z^c,$$

with $b < 2$. We check whether such a monomial belongs to H_D by the inequality

$$\underline{\nu}(\mathcal{M}) \geq \underline{\nu}(C).$$

The exceptional parts of the curves in the completion is

$$D_R = (1,1,2,2,4),$$
$$D_L = (1,2,3,3,6),$$
$$D_{L_1} = (2,3,6,7,13).$$

We limit the set of monomials to check by choosing an upper limit to the exponents a, c by setting

$$a_{\max} = \max\lceil \tfrac{\nu_i(C)}{\nu_i(x)} \rceil,$$
$$c_{\max} = \max\lceil \tfrac{\nu_i(C)}{\nu_i(z)} \rceil.$$

For each pair (b, c) with $b < 2$, $c \leq c_{\max}$ we choose the minimal a such that $\mathcal{M}$ verifies all the conditions. For example, let $\mathcal{M} = x^a yz$, we check that the minimal a for which $\mathcal{M}$ belongs to the ideal H_{D_C} is $a = 2$, $\mathcal{M} = x^2yz$. Thus we find a set of generators

$$H_D = (x^7, x^5y, x^4z, x^2yz, z^2).$$

In [**AAB17**, Example 3.7.] the authors give a set of generators which contains the above monomials and also the monomials xy^2z, x^4y^2. Notice that we do not consider exponents > 1 in y by the conditions on the expansion (see 2.6).

5.7. Tropical interpretation

It is possible to interpret the conditions for jumping numbers as tropical polynomials (see [**MS15**] for a general introduction to tropical geometry). I thank Patrick Popescu-Pampu for the insight, pointing out the connection between jumping numbers and tropical polynomials.

For a branch C, the set of jumping numbers smaller than one,

$$\Xi(C) = \{\xi < 1 \text{ jumping number of } C\},$$

is contained in the image of the lattice points in a convex polyhedron by a tropical polynomial. Precisely, denote by $N = \mathbb{Z}^{g+1}$ the lattice of exponents in $g + 1$ variables and inside this lattice consider the partially bounded polyhedron

$$\Delta = \{(i_{-1}, i_0, \ldots, i_g) \mid 0 \leq i_{-1}; 0 \leq i_j \leq n_{j+1}\ \forall 0 \leq j < g - 1; 0 \leq i_g\}. \tag{5.97}$$

In fact, to compute $\Xi(C)$ it is enough to choose the bounded polyhedron with $i_g = 0$. Consider now the function

$$H : \mathbb{Z}^{g+1} \to \mathbb{Q},$$

defined by

$$\begin{aligned} H(i_{-1}, \ldots, i_g) &:= \min_{1\leq k\leq g} \left\{ \frac{\nu_{E_k}(\mathcal{M}) + \lambda_k}{\nu_{E_k}(C)} \right\} \\ &= \min_{1\leq k\leq g} \left\{ \frac{\lambda_{R_k}}{\nu_{R_k}(C)} + \sum_{-1\leq j\leq g} \frac{\nu_{E_k}(z_j)}{\nu_{E_k}(C)} i_j \right\} \\ &= \operatorname{trop}\left(\sum_{1\leq k\leq g} \frac{\lambda_{R_k}}{\nu_{R_k}(C)} \prod_{-1\leq j\leq g} i_j^{\frac{\nu_{R_k}(z_j)}{\nu_{R_k}(C)}} \right), \end{aligned}$$

where trop represents tropicalization, i.e., interpret the operations in the field $\mathbb{R}$ as the ones in the tropical semiring $(\mathbb{R}, +, \min)$.

Denote by

$$Q := \sum_{1\leq k\leq g} \left(\frac{\lambda_{R_k}}{\nu_{R_k}(f)} \prod_{-1\leq j\leq g} X_j^{\frac{\nu_{R_k}(z_j)}{\nu_{R_k}(f)}} \right),$$

the 'polynomial' which, by tropicalization, gives H. As a consequence of Corollary 5.22, the image of Δ under H allows to recover the jumping numbers smaller than one of the branch C by discarding the images ≥ 1.

More generally, let C be a plane curve germ and R a smooth reference branch transversal to C. Let $\Theta_R(C)$ be the Eggers-Wall tree of C and Υ its set of marked points. Let $\bar{C} = \{L_\alpha\}_{\alpha \in A}$ be a maximal contact completion of C. Consider the tropical polynomial

$$H\left(\{i_\alpha\}_{\alpha \in A}\right) := \operatorname{trop}\left(\sum_{P \in \Upsilon}\left(\frac{\lambda_P}{\nu_P(C)} \prod_{P \in \Upsilon} i_\alpha^{\frac{\nu_P(L_\alpha)}{\nu_P(C)}}\right)\right).$$

As a consequence of Corollary 5.22 we have:

PROPOSITION 5.98. *The set of jumping numbers of C is the image of $\mathbb{Z}_{\geq 0}^A$ under H.*

5.8. Examples

In this section we describe various examples.

Recall that in order to compute the jumping numbers of a plane curve, according to Lemma 3.26, we just need to compute jumping numbers $\xi \in (0,1)$, since the rest of jumping numbers are fixed by their periodicity.

The following remark is based on the construction of expansions in Chapter 4 Sections 4.4 and 4.6.

REMARK 5.99. Let C be a plane curve germ with components $C_1, \ldots, C_r$ and R be a smooth reference branch not in C. Let h be a Weierstrass polynomial and consider the expansion of h with respect to R, C (Definition 4.155),

$$h = \sum \alpha \mathcal{M}_\alpha,$$

where $\mathcal{M}_\alpha$ is a product of semi-roots. The bounds on the exponents of the semi-roots are inherited from the conditions on the expansion with respect to each component of C.

Consider a semi-root z associated to a curve D in $\bar{C}$ such that it is a semi-root for $C_{l_1}, \ldots, C_{l_k}$ irreducible components of C. Let R_D be the root of the index level of D and P_{l_t} the marked point of $\Theta_R(C_{l_t})$ in the segment $[R_D, D]$, $P_{l_t} = \langle R, D, C_{l_t}\rangle$. Then z appears in the expansion of H with exponent i_z

$$0 \leq i_z \leq \sum_{t=1}^{l_k}\left(\mathbf{i}_{R_D}^{+}(P_{l_t}) - 1\right). \tag{5.100}$$

In this first example we compute the jumping numbers and multiplier ideals for the curve in Example 2.93.

EXAMPLE 5.101. Consider $f = (y^2 - x^3)^2 - 4x^5 y - x^7$ and the associated branch $C = V(f)$. The characteristic exponents are

$$\varepsilon(C) = \left\{\tfrac{3}{2}, \tfrac{7}{4}\right\}.$$

Thus $\{(2,3),(2,1)\}$ are the Newton pairs. For Naie's algorithm one needs the characteristic, $(4,6,13)$, minimal set of generators of the semigroup of the branch, $\Gamma(C)$. The numbers $lcm(4,6) = 12$ and $lcm(gcd(4,6),13) = 26$ appear as the denominators of possible jumping numbers.

The smooth curve $L = V(y)$ is a 0-th semi-root and cusp $L_1 = V(z)$, with $z := y^2 - x^3$, is a 1-st semi-root. A complete sequence of semi-roots for $R = V(x), C$ is given by $\{y, z, f\}$ but for jumping numbers $\xi < 1$ it is enough to consider monomials

$$\mathcal{M} = x^j y^{i_0} z^{i_1},\ i_0, i_1 \in \{0,1\}.$$

We denote by R_1, R_2 the relevant exceptional divisors associated, respectively, to the first and second characteristic exponents of the branch. Table 1 below gives the values of x, each semi-root and log-discrepancy with respect to the aforementioned divisorial valuations.

	x	y	z	f	λ
R_1	2	3	6	12	5
R_2	4	6	13	26	11

Table 1. Orders of vanishing of a complete sequence of semi-roots for f in Example 5.101 with respect to the relevant exceptional divisors.

Then, according to Corollary 5.22, the set of jumping numbers is given by

$$\left\{ \min\left\{ \frac{2i_{-1} + 3i_0 + 6i_1 + 5}{12}, \frac{4i_{-1} + 6i_0 + 13i_1 + 11}{26} \right\} \right\}.$$

Table 2 below shows the set of jumping numbers of C in $(0,1)$. Notice that jumping numbers associated to R_i have denominator $\nu_{R_i}(f)$.

$\mathcal{M}$	1	x	y	x^2	xy	y^2	z	x^3
$\xi_{\mathcal{M}}$	$\frac{5}{12}$	$\frac{15}{26}$	$\frac{17}{26}$	$\frac{19}{26}$	$\frac{21}{26}$	$\frac{23}{26}$	$\frac{11}{12}$	$\frac{25}{26}$

Table 2. List of jumping numbers in Example 5.101.

From Theorem 5.16, we deduce that, given a jumping number associated to a monomial $\mathcal{M}$, the monomials generating a greater jumping number form a generating set for the multiplier ideal. For example

$$\mathcal{J}(\tfrac{5}{12}C) = (x, y, x^2, xy, y^2, z, x^3) = (x, y).$$

After looking for minimal sets of generators, one gets the filtered sequence:

$$\begin{array}{ccccccccc} \mathbb{C}\{x,y\} & \supsetneq & (x,y) & \supsetneq & (x^2,y) & \supsetneq & (x^2,xy,z) & \supsetneq & (x^3,xy,z) \\ (x^3,x^2y,z) & \supsetneq & (x^4,x^2y,z) & \supsetneq & (x^4,x^2y,xz,yz) & \supsetneq & (x^4,x^3y,xz,yz) & \supsetneq & (f). \end{array}$$

Next we give a series of related examples.

Example 5.102. Consider the function $f = (y^2 - x^3)(y^3 - x^5)$. It is a product of two different cusps. A complete sequence of semi-roots is just the cross (R, L) and the factors of f, let us call them L_1, L_2 associated, respectively, to $z_1 = y^2 - x^3$ and $z_2 = y^3 - x^5$. One can apply directly Howald's theorem since the associated curve is resolved by just one toric modification. By applying Corollary 1.28, the set of jumping numbers are attained are the relevant exceptional divisors: R_1 associated to the characteristic exponent of L_1, and R_2 associated to L_2. Table 3 below lists the set of values in the monomials and log-discrepancies with respect to the ν_{R_i}.

Both relevant exceptional divisors are monomial in the system of coordinates (R, L), so one checks that the set of jumping numbers < 1 are given by

$$\left\{ \xi_{x^a y^b} = \min\left\{ \frac{2a + 3b + 5}{6 + 9}, \frac{3a + 5b + 8}{9 + 15} \right\} \right\}.$$

	x	y	z	f_2	λ
R_1	2	3	6	9	5
R_2	3	5	9	15	8

TABLE 3. List of relevant valuations for Example 5.102.

This gives a list of 13 jumping numbers < 1, listed on Table 4. Both the log-canonical threshold, $\operatorname{lct}(f) = \frac{1}{3}$, and the jumping number associated to xy, $\xi_{xy} = \frac{2}{3}$, are attained simultaneously at both exceptional divisors.

$\mathcal{M}$	1	x	y	x^2	xy	x^3	y^2	x^2y	x^4	xy^2	x^3y	y^3	x^5
$\xi_{\mathcal{M}}$	$\frac{5}{15} = \frac{8}{24}$	$\frac{11}{24}$	$\frac{8}{15}$	$\frac{14}{24}$	$\frac{10}{15} = \frac{16}{24}$	$\frac{17}{24}$	$\frac{11}{15}$	$\frac{19}{24}$	$\frac{20}{24}$	$\frac{13}{15}$	$\frac{22}{24}$	$\frac{14}{15}$	$\frac{23}{24}$

TABLE 4. List of jumping numbers in Example 5.102.

EXAMPLE 5.103. Now we substitute the first cusp L_1 in Example 5.102, by a branch C_1 (and a corresponding germ, say f_1) with two characteristic exponents, $\frac{3}{2}, \frac{9}{4}$. We consider the curve $C = C_1 + L_2$. $L_1 = V(z_1)$ is a curvetta for C_1. There is an additional relevant exceptional divisor R_3 associated to the second characteristic exponent of C_1. A complete sequence of semi-roots for R, C is $\{y, z_1, f_1, z_2\}$. Figure 2 shows the completion of the tree, $\Theta_R(\bar{C})$.

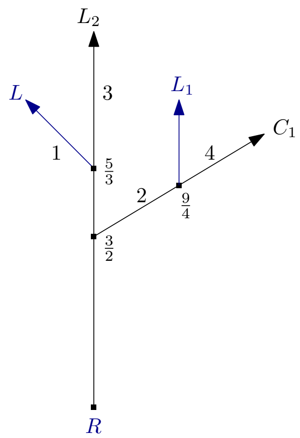

FIGURE 2. The complete tree $\Theta_R(\bar{C})$ for the curve C in Example 5.103.

Table 5 lists the values of the monomials and the log-discrepancy with respect to R_i.

In order to simultaneously monomialize the relevant exceptional divisors of the example we need R, L, L_1, so let us denote $\mathcal{M} = x^a y^b z_1^c$. The set of jumping numbers < 1 for C are given by

$$\left\{ \xi_{\mathcal{M}} = \min \left\{ \frac{2a + 3b + 6c + 5}{21}, \frac{3a + 5b + 9c + 8}{33}, \frac{4a + 6b + 15c + 13}{48} \right\} \right\},$$

which gives a list of 30 jumping numbers listed in Table 6.

	x	y	z	f_1	f_2	f_1f_2	λ
R_1	2	3	6	12	9	21	5
R_2	3	5	9	18	15	33	8
R_3	4	6	15	30	18	48	13

TABLE 5. List of values for Example 5.103.

$\mathcal{M}$	1	x	y	x^2	xy	x^3, z_1	y^2	x^2y	x^4	xz_1
$\xi_{\mathcal{M}}$	$\frac{5}{21}$	$\frac{7}{21}=\frac{11}{33}$	$\frac{8}{21}$	$\frac{14}{33}$	$\frac{10}{21}$	$\frac{17}{33}$	$\frac{25}{48}$	$\frac{27}{48}$	$\frac{28}{48}$	$\frac{20}{33}$

$\mathcal{M}$	x^3y	yz_1	x^5	x^2z_1	x^4y	xyz_1	x^6	x^3z_1	y^2z_1	x^5y
$\xi_{\mathcal{M}}$	$\frac{31}{48}$	$\frac{14}{21}=\frac{22}{33}=\frac{32}{48}$	$\frac{33}{48}$	$\frac{23}{33}$	$\frac{35}{48}$	$\frac{25}{33}$	$\frac{37}{48}$	$\frac{26}{33}$	$\frac{17}{21}$	$\frac{39}{48}$

$\mathcal{M}$	x^2yz_1	x^7	x^4z_1	x^6y	xy^2z_1	x^8	x^3yz_1	y^3z_1	x^5z_1	x^7y
$\xi_{\mathcal{M}}$	$\frac{28}{33}$	$\frac{41}{48}$	$\frac{29}{33}$	$\frac{43}{48}$	$\frac{19}{21}$	$\frac{45}{48}$	$\frac{31}{33}$	$\frac{20}{21}$	$\frac{32}{33}$	$\frac{47}{48}$

TABLE 6. List of jumping numbers for Example 5.103.

EXAMPLE 5.104 (Importance of the Strict Transforsm). Let un consider the germs in Example 5.102, $z_1 = y^2 - x^3$, $z_2 = y^3 - x^5$ and let $g = z_1^2 z_2$. The germ g is a generic section for the ideal

$$I = (y^7 - 2y^5x^3 - y^4x^5 + y^3x^6 + 2y^2x^8, x^9) \ni f.$$

We denote by $L_i = V(z_i = 0)$, $C = 2L_1 + L_2$. A complete sequence of semi-roots for C is given by $\{y, z_1, z_2\}$. Figure 3 shows the Eggers-Wall tree of the completion of C, $\Theta_R(\bar{C})$, and the embedded subtree corresponding to the ideal, $\Theta_R(I)$.

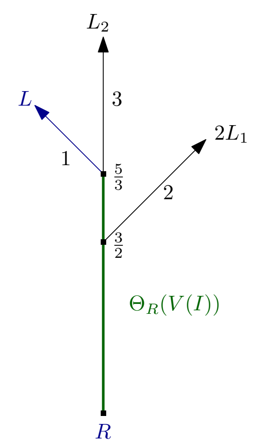

FIGURE 3. $\Theta_R(\bar{C})$ with embedded subtree $\Theta_R(I)$.

A generalized monomial will have the form (see Equation 5.100)

$$\mathcal{M} = x^j y^{i_0} z_1^{i_1} z_2^{i_2} \text{ with } i_0 \leq (2-1)+(3-2).$$

The tree has two marked points corresponding to relevant exceptional divisors R_1, R_2, for which Table 3 gives the values of the monomials and log-discrepancy.

Table 7 below lists the jumping numbers associated to the curve C. Notice that the branch semivaluations associated to L_2 plays a fundamental role.

$\mathcal{M}$	1	x	y	x^2	xy	$y^2, x^3, \ldots$	z_1	xz_1	yz_1	x^2z_1
$\xi^C_{\mathcal{M}}$	$\frac{5}{21}$	$\frac{7}{21}=\frac{11}{33}$	$\frac{8}{21}$	$\frac{14}{33}$	$\frac{10}{21}$	$\frac{1}{2}$	$\frac{17}{33}$	$\frac{20}{33}$	$\frac{14}{21}=\frac{22}{33}$	$\frac{23}{33}$

$\mathcal{M}$	xyz_1	x^3z_1, z_1^2	y^2z_1	x^2yz_1	x^4z_1, xz_1^2	xy^2z_1	x^3yz_1, yz_1^2	y^3z_1	$x^5z_1, x^2z_1^2$
$\xi^C_{\mathcal{M}}$	$\frac{25}{33}$	$\frac{26}{33}$	$\frac{17}{21}$	$\frac{28}{33}$	$\frac{29}{33}$	$\frac{19}{21}$	$\frac{31}{33}$	$\frac{20}{21}$	$\frac{32}{33}$

TABLE 7. The list of jumping numbers for the curve C in Example 5.104.

Table 8 below lists the set of jumping numbers smaller than one for the ideal I. Notice that this list does not determine the whole set of jumping numbers of the ideal.

$\mathcal{M}$	1	x	y	x^2	xy	x^3, z_1	y^2	x^2y
$\xi^I_{\mathcal{M}}$	$\frac{5}{21}$	$\frac{7}{21}=\frac{11}{33}$	$\frac{8}{21}$	$\frac{14}{33}$	$\frac{10}{21}$	$\frac{17}{33}$	$\frac{11}{21}$	$\frac{12}{21}$

$\mathcal{M}$	x^4, xz_1	xy^2	x^3y, y^3, yz_1	x^5, x^2z_1	x^2y^2	$x^4y, \ldots$	xy^3	x^6
$\xi^I_{\mathcal{M}}$	$\frac{20}{33}$	$\frac{13}{21}$	$\frac{14}{21}=\frac{22}{33}$	$\frac{23}{33}$	$\frac{15}{21}$	$\frac{25}{33}$	$\frac{16}{21}$	$\frac{26}{33}$

$\mathcal{M}$	$x^3y^2, \ldots$	$x^5y, \ldots$	x^2y^3	$x^7, \ldots$	$x^4y^2, \ldots$	$x^6y, \ldots$	$x^3y^3, \ldots$	x^8
$\xi^I_{\mathcal{M}}$	$\frac{17}{21}$	$\frac{28}{33}$	$\frac{18}{21}$	$\frac{29}{33}$	$\frac{19}{21}$	$\frac{31}{33}$	$\frac{20}{21}$	$\frac{32}{33}$

TABLE 8. The list of jumping numbers smaller than one for the ideal $I = (y^7 - 2y^5x^3 - y^4x^5 + y^3x^6 + 2y^2x^8, x^9)$ in Example 5.104.

We see that $\operatorname{lct}(f) = \operatorname{lct}(I) = \frac{5}{21}$. In the case of the ideal, there is no strict transform, and so the powers of x generate 8 different jumping numbers, attained at the relevant exceptional divisor R_2, associated to the characteristic exponent of L_2. In particular,

$$\xi^I_{x^3} = \frac{17}{33} > \xi^C_{x^3} = \frac{1}{2}.$$

Furthermore, notice that

$$\xi^C_{x^3} < \xi^C_{z_1} < \xi^I_{x^3} = \xi^I_{z_1}.$$

CHAPTER 6

The quasi-ordinary branch case

In this chapter we study multiplier ideals of an analytically irreducible quasi-ordinary hypersurface singularity $(H, 0) \subset (\mathbb{C}^{d+1}, 0)$. The hypersurface H is defined by the vanishing of a monic polynomial $f \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ whose discriminant is of the form a monomial times a unit. This class of singularities has fractional power series parametrizations $y = \zeta(x_1^{1/n}, \ldots, x_d^{1/n})$ generalizing the plane branch case. These power series have a finite set of *characteristic monomials* which encode the embedded topological type of $(H, 0) \subset (\mathbb{C}^{d+1}, 0)$ (see [**Lip83, Gau88**]). One has the notion of a semigroup of a quasi-ordinary hypersurface, which generalizes the classical semigroup of a plane branch ($d = 1$) (see [**Pop04, Gon03a, KM90**]). A complete sequence of *semi-roots* of f together with the coordinates $x_1, \ldots, x_d$ provide a set of generators for this semigroup. One can built an embedded resolution of H as a composition of toric modifications with respect to suitable coordinates (see [**Gon03b**]). We call it a *toroidal resolution* since one has a natural structure of toroidal embedding. As an output we obtain a set of *toroidal divisors* which are in bijection with the integral points of the *conic polyhedral complex* $\Theta(H)$ associated with the toroidal embedding. We build a piecewise linear continuous function on $\Theta(H)$ which provides the log-discrepancy of the toroidal divisors (Section 6.7). A finite set of *relevant* toroidal divisors is associated with rays of $\Theta(H)$ (Definition 6.65). We prove that the associated divisorial valuations are monomial in the set of semi-roots of H together with the coordinates $x_1, \ldots, x_d$ (see Section 6.5). As a consequence of this study we generalize the results on multiplier ideals of Chapter 5.2 to irreducible quasi-ordinary germs (see Theorems 6.122, 6.129 and Corollary 6.133). In particular, we obtain a combinatorial characterization of the jumping numbers in terms of the relevant toroidal divisors. This characterization allows us to compute the jumping number in some examples at the end of the chapter.

6.1. Quasi-ordinary hypersurface singularities

Quasi-ordinary hypersurface singularities arise classically in Jung's approach to analyzing a hypersurface singularity by using embedded resolution of the discriminant of a finite projection to the affine space.

Let (H, O) be a germ of analytically irreducible complex variety of dimension d, and denote by R its associated analytic algebra. If we consider a finite map germ $H \to H'$, a sufficiently small representative, $(H, O) \longrightarrow (H', O')$ has finite fibers, its image is an open neighborhood of O' and the maximal cardinality of its fibers is equal to the **degree** of the map. The **discriminant locus**, which is the set of points having fibers of cardinality less than the degree, is an analytic subvariety of H', which we can think of as an analytic space or as a germ at O'. Outside the discriminant locus, the map is an **unramified covering**.

Definition 6.1. A germ of complex analytic variety (H, O) is a **quasi-ordinary singularity** if there exists a finite morphism (called the quasi-ordinary projection)

$$\pi : (H, O) \longrightarrow (\mathbb{C}^d, O) \tag{6.2}$$

and local coordinates $(x_1, \dots, x_d)$ at O such that the morphism is an unramified covering over the torus $\{x_1 \dots x_d \neq 0\}$ in a neighborhood of the origin, i.e., the discriminant locus is germ-wise contained in a normal crossing divisor.

The class of quasi-ordinary singularities contains all curve singularities. The Jung-Abhyankar Theorem 6.4 guarantees that R can be viewed as a subring of $\mathbb{C}\{x_1^{1/m}, \dots, x_d^{1/m}\}$ for some integer m (see [**Jun08**] for the original topological proof in the surface case, [**Abh55**, Theorem 3] for an algebraic proof, see also [**Gon00**, **PR12**]). One can understand this result as a generalization of Newton-Puiseux Theorem 2.7 to the case of quasi-ordinary polynomials.

If $H \subset (\mathbb{C}^{d+1}, 0)$ is a quasi-ordinary hypersurface then there exist coordinates $(x_1, \dots, x_d, y)$ such that:

- H is defined by a Weierstrass polynomial $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ of degree n in y,
- the map π in (6.2) is the restriction to H of the projection $(x_1, \dots, x_d, y) \mapsto (x_1, \dots, x_d)$,
- the discriminant $\Delta_y f$ of the polynomial f is of the form a monomial times a unit in the ring $\mathbb{C}\{x_1, \dots, x_d\}$.

Definition 6.3. Let $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ be a non-zero polynomial. We say that f is a **quasi-ordinary polynomial** if the discriminant $\Delta_y f$ of the polynomial f with respect to y is of the form: $\Delta_y f = x^\alpha \cdot \epsilon$, where $\epsilon \in \mathbb{C}\{x_1, \dots, x_d\}$ is a unit.

Theorem 6.4. [**Gon00**, Thèorème 1 and Remarque 1]. *For any quasi-ordinary polynomial $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ there exists a $k \in \mathbb{N}$ such that f has $n = \mathrm{ord}_y(f)$ roots in $\mathbb{C}\{x_1^{1/k}, \dots, x_d^{1/k}\}$. If f is irreducible, we can take $k = n$.*

From now on we fix an irreducible quasi-ordinary polynomial $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ and one of its roots $\zeta \in \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$. The analytic algebra $R = \mathbb{C}\{x_1, \dots, x_d\}[y]/(f)$ is a domain. The inclusion

$$\mathbb{C}\{x_1, \dots, x_d\} \subset \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$$

defines a normal Galois extension, $K \subset K_n$, of the corresponding fields of fractions. The minimal polynomial of the root ζ over K is f, so that $R \simeq \mathbb{C}\{x_1, \dots, x_d\}[\zeta]$, and the field of fractions of R is $K[\zeta]$, since ζ is finite over K. The conjugates ζ^i of ζ by the action of the Galois group $\mathrm{Gal}(K \subset K_n)$ define all the roots, since the extension $K[\zeta] \subset K_n$ is Galois.

The irreducible quasi-ordinary polynomial f has all its roots in the ring $\mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$. The difference $\zeta^s - \zeta^t$ of two different roots of f divides the discriminant $\Delta_y(f) \in \mathbb{C}\{x_1, \dots, x_d\}$, since (see [**GKZ94**, Product formula 1.23])

$$\Delta_y(f) = \prod_{s \neq t} (\zeta^s - \zeta^t).$$

And since the Newton polyhedron of the discriminant has only one vertex, the same applies to the difference of different roots (see Remark 1.19). Therefore we have

$$\zeta^s - \zeta^t = x^{\alpha_{st}} h_{st},$$

where $h_{st} \in \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$ is a unit. The monomials $x^{\alpha_{st}}$ are called **characteristic monomials** and the corresponding exponents, $\alpha_{st} \in \mathbb{Q}^d$, **characteristic exponents**. The roots of a quasi-ordinary polynomial are called **quasi-ordinary branches**.

If $d = 1$ the characteristic exponents coincide with the classical Newton-Puiseux characteristic exponents (see Definition 2.13).

The notion of characteristic monomials can be found in Zariski's work ([**Zar67**]). Many geometrical and topological properties of quasi-ordinary hypersurface singularities can be expressed in terms of the characteristic monomials (see [**Lip65**, **Lip83**, **Lip88**, **Lue83**, **Gau88**, **KM90**, **Pop01**, **Pop04**, **Gon03a**, **Gon03b**, **BGG12**, **GG14**, **ACLM13**, **ACLM05**]).

PROPOSITION 6.5. [**Lip88**, Lemma 5.6]. *The characteristic exponents associated with an irreducible quasi-ordinary polynomial can be reordered so that*

$$\alpha_1 \leq \cdots \leq \alpha_g \tag{6.6}$$

where $\leq$ means coordinate-wise.

We associate to the characteristic exponents a sequence of lattices and integers (be aware of the similarity with Definition 2.17):

DEFINITION 6.7. The **characteristic lattices** and **integers** of a quasi-ordinary branch are

$$\begin{cases} M_0 := = \mathbb{Z}^d, \ M_i = M_{i-1} + \mathbb{Z}\alpha_i, \\ n_0 := 1, \qquad n_i = \#M_i/M_{i-1} \qquad i = 1, \ldots, g. \end{cases}$$

We also define the **characteristic multiplicities**, denoted by $e_{i-1} = n_i \ldots n_g$ for $i = 1, \ldots, g$ and set $n_0 = 1$, and we denote by $N_g \subset \ldots \subset N_1 \subset N_0 = N$ the sequence of dual lattices.

Notice that the sequence of integers corresponds to the first component of the characteristic pairs in the plane branch case (see Definition 2.13).

LEMMA 6.8. [**Lip88**]. *Let f be an irreducible quasi-ordinary polynomial and ζ a root of f, then:*

1. *The characteristic integers satisfy $n_i > 1$ for $i = 1, \ldots, g$ and $\deg_y(f) = e_0$.*
2. *The field of fractions of R is $K[\zeta] = K[x^{\alpha_1}, \ldots, x^{\alpha_g}]$.*

In fact, the characteristic integers and the e_i correspond to degrees of Galois extensions

$$e_i := [K[\zeta] : K[x^{\alpha_1}, \ldots, x^{\alpha_i}]]\,,$$
$$n_i := [K[x^{\alpha_1}, \ldots, x^{\alpha_{i-1}}] : K[x^{\alpha_1}, \ldots, x^{\alpha_i}]]\,.$$

Those fractional power series which are quasi-ordinary branches where characterized by Lipman (see [**Lip83**, Proposition 1.5], [**Gau88**, Proposition 1.3]).

LEMMA 6.9. *Let $\zeta = \sum c_\alpha x^\alpha \in \mathbb{C}\{x_1^{1/n}, \ldots, x_d^{1/n}\}$ with $c_0 = 0$. Then ζ is a quasi-ordinary branch if and only if there exist elements $\alpha_1, \ldots, \alpha_g \in \frac{1}{n}\mathbb{N}^d$ such that*

1. *$\alpha_1 < \ldots < \alpha_g$ and $c_{\alpha_i} \neq 0$ for $i = 1, \ldots, g$.*
2. *If $c_\alpha \neq 0$ then α belongs to the sublattice of $M_{\mathbb{Q}}$ given by $M + \sum_{\alpha_i < \alpha} \mathbb{Z}\alpha_i$.*
3. *α_j is not in the sublattice $M + \sum_{\alpha_i < \alpha_j} \mathbb{Z}\alpha_i$ of $M_{\mathbb{Q}}$ for $j = 1, \ldots, g$.*

If such elements exist, they are uniquely determined by ζ and they are the characteristic exponents of ζ.

We say that a quasi-ordinary branch ζ has well ordered variables if the g-tuples corresponding to the i-th coordinates of the characteristic exponents are ordered lexicographically, i.e.,

$$(\alpha_{1,i}, \ldots, \alpha_{g,i}) \underset{\text{lex}}{\geq} (\alpha_{1,j}, \ldots, \alpha_{g,j}) \ \text{ for } 1 \leq i < j \leq d.$$

Given a quasi-ordinary branch, we can relabel the variables $x_1, \ldots, x_d$ in order to satisfy this condition.

Definition 6.10. A quasi-ordinary branch ζ is **normalized** if it has well ordered variables and, if it happens that the first characteristic exponent has only one non-zero coordinate, then $\alpha_{1,1} > 1$.

This condition in the case of plane curve germs means that the kernel of the projection $(x, y) \mapsto x$ is not contained in the tangent cone of the germ, i.e., the germ $R = V(x)$ is transversal to the curve (Definition 2.5). Lipman proved that any irreducible quasi-ordinary hypersurface can be parametrized by a normalized quasi-ordinary branch. He also proved that the characteristic monomials of a normalized quasi-ordinary branch determine the topological type of the corresponding germ ([**Lip83**]). Gau proved the converse, therefore monomials define a complete invariant of the embedded topological type of the germ (see [**Gau88**]).

Example 6.11. The fractional power series

$$\zeta = x_1^{\frac{3}{2}} x_2^{\frac{1}{2}} + x_1^{\frac{7}{4}} x_2^{\frac{1}{2}}.$$

ζ is a quasi-ordinary branch with characteristic exponents $\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right)$ and $\alpha_2 = \left(\frac{7}{4}, \frac{1}{2}\right)$. It is a root of the quasi-ordinary polynomial

$$f = (y^2 - x_1^3 x_2)^4 + 2x_1^{13} x_2^4 + x_1^{14} x_2^4 - 12x_1^{10} x_2^3 y^2 - 2x_1^7 x_2^2 y^4.$$

The characteristic integers are $n_1 = 2$, $n_2 = 4$.

Example 6.12. The quasi-ordinary polynomial

$$f = (y^2 - x_1^3 x_2)^6 + x_1^{21} x_2^8.$$

defines a quasi-ordinary branch with characteristic exponents $\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right)$, $\alpha_2 = \left(\frac{7}{4}, \frac{2}{3}\right)$, and characteristic integers are $n_1 = 2$ and $n_2 = 6$.

The normalization of an irreducible toric quasi-ordinary hypersurface germ is a toric singularity:

Proposition 6.13. [**Gon03b**, Proposition 14]. *The normalization of an analytically irreducible quasi-ordinary hypersurface is isomorphic to germ of the toric variety $Z_{\mathbb{R}^d_{\geq 0}, N_g}$ at its 0-dimensional orbit.*

One can associate to an irreducible germ H of quasi-ordinary hypersurface a semigroup $\Gamma \subset M_g$. Let the notation be as in Definition 6.7. We introduce first Γ combinatorially by giving its generators by analogy with the plane branch case (see Theorem 2.31):

$$\begin{cases} \gamma_1 = \alpha_1, \\ \gamma_{j+1} = n_j \gamma_j + (\alpha_{j+1} - \alpha_j) \ \text{ for all } j \in \{1, \ldots g\}. \end{cases} \tag{6.14}$$

The semigroup of the branch ζ is:

$$\Gamma = \mathbb{Z}^d_{\geq 0} + \gamma_1 \mathbb{Z}_{\geq 0} + \ldots + \gamma_g \mathbb{Z}_{\geq 0} \subset M_g$$

If $d = 1$ the semigroup Γ is the clasical semigroup of the plane branch (Definition 2.28).
We denote by $\varepsilon^0_1, \ldots, \varepsilon^0_d$ the canonical basis of $\mathbb{Z}^d = M_0$.

Lemma 6.15. [**Gon03a**, Lemma 3.4]. *The semigroup Γ has a unique set of minimal generators, in the sense that no generator belongs to the semigroup spanned by the others. If ζ is normalized this set is the union of the canonical basis $\{\varepsilon^1_0, \ldots, \varepsilon^d_0\}$ of M_0 with $\{\gamma_1, \ldots, \gamma_g\}$.*

Example 6.16. Let us consider the quasi-ordinary hypersurface of Example 6.11. Then, we have:

$$\gamma_1 = \alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right),$$
$$\gamma_2 = 2\left(\frac{3}{2}, \frac{1}{2}\right) + \left(\left(\frac{7}{4}, \frac{3}{2}\right) - \left(\frac{3}{2}, \frac{1}{2}\right)\right) = \left(\frac{13}{4}, 1\right).$$

The following lemma generalizes the properties of the semigroup of a plane branch.

Lemma 6.17. [**Gon03a**, Lemma 3.3]. *With notation as above, we have the following properties:*

(1) The sublattice of M_g generated by $\mathbb{Z}^d_{\geq 0}+\gamma_1\mathbb{Z}_{\geq 0}+\ldots+\gamma_j\mathbb{Z}_{\geq 0}$ is equal to M_j for $0 \leq j \leq g$.

(2) The order of the image of γ_j in M_j/M_{j-1} is n_j for $1 \leq j \leq g$.

(3) $\gamma_{j+1} > n_j\gamma_j$ for $1 \leq j < g$.

(4) If $u \in \rho^\vee \cap M_j$ then $u_j + n_j\gamma_j \in \mathbb{Z}^d_{\geq 0} + \gamma_1\mathbb{Z}_{\geq 0} + \ldots + \gamma_j\mathbb{Z}_{\geq 0}$.

(5) $n_j\gamma_j \in \mathbb{Z}^d_{\geq 0} + \gamma_1\mathbb{Z}_{\geq 0} + \ldots + \gamma_{j-1}\mathbb{Z}_{\geq 0}$ for $1 \leq j \leq g$ and there is a unique expansion
$$n_j\gamma_j = k + l_0\gamma_1 + \ldots + l_j\gamma_{j-1},$$
such that $0 \leq l_i < n_{i+1}$ for $1 \leq i < j$ and $k \in M_0$.

As a consequence of the previous proposition there is a canonical way of writing the elements of the lattice M_g.

Lemma 6.18. [**Pop02**, Lemma 6]. *Every element of the lattice M_g can be written in a unique way as a sum*
$$l_0 + l_1\gamma_1 + \ldots + l_g\gamma_g,$$
where $l_0 \in M_0$ and $0 \leq l_i < n_{i+1}$ for all $i \in \{1, \ldots, g\}$.

Remark 6.19. The semigroup Γ was introduced in [**KM90**] in the case $d = 2$. The semigroup Γ of a quasi-ordinary polynomial f can be seen as the semigroup generated by vertices of Newton polyhedra $\mathcal{N}(h(\zeta))$ for $h \in \mathbb{C}\{x_1, \ldots, x_d\}[y] \setminus (f)$ (see [**Gon03a**, Theorem 3.6]). An alternative approach to the definition of the semigroup of f was given Popescu-Pampu in [**Pop01**] (see also [**Pop04**]). First, he introduced the set C_f of functions $h \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ such that the Newton polyhedron of $h(\zeta)$ has only one vertex γ_h and then defined the semigroup Γ as $\{\gamma_h \mid h \in C_f\}$.

Now we define the concept of semi-root of an irreducible quasi-ordinary germ.

Definition 6.20. A j-th **semi-root** of f is an irreducible quasi-ordinary polynomial $z_j \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ of degree $n_1 \ldots n_j$ such that $z_j(\zeta) = x^{\gamma_{j+1}}\varepsilon$ for a unit $\varepsilon \in \mathbb{C}\{\rho^\vee \cap M_g\}$. A **complete sequence of semi-roots of** f is a sequence of j-th semi-roots z_j, for $j = 0, \ldots, g$ where $z_g := f$. We denote by $V(z_j) = L_j$ for $j = 0, \ldots, g$ the hypersurfaces defined by this sequence.

Remark 6.21. The characteristic lattices provide a canonical way of writing the terms of its roots,

$$\zeta = p_0 + p_1 + \ldots + p_g, \tag{6.22}$$

with $p_0 \in \mathbb{C}\{x_1^{1/n}, \ldots, x_d^{1/n}\}$ and x^α appearing in p_j if $\alpha_j \leq_\rho \alpha$, $\alpha_{j+1} \not\leq \alpha$. It follows from Lemma 6.9 that the truncation $p_0 + p_1 + \ldots + p_i$ is a quasi-ordinary branch with i characteristic exponents. This implies that it is a root of a quasi-ordinary polynomial $q_j \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$.

PROPOSITION 6.23. [**Gon03a**]. *The minimal polynomial q_j of $p_0 + \cdots + p_j$ over the field of fractions of $\mathbb{C}\{x_1, \ldots, x_d\}$ is a quasi-ordinary polynomial in the ring $\mathbb{C}\{x_1, \ldots, x_d\}[y]$ and is a j-th semi-root of f.*

EXAMPLE 6.24. Let us consider the quasi-ordinary branch

$$\zeta = 2x_1 + x_1^{\frac{3}{2}} x_2^{\frac{1}{2}} + x_1^{\frac{7}{4}} x_2^{\frac{1}{2}}.$$

Then, we have $q_0 = y - 2x_1$, $q_1 = (y - 2x_1)^2$ and $q_2 = f$, with

$$f = ((y - 2x_1)^2 - x_1^3 x_2)^4 + 2x_1^{13} x_2^4 + x_1^{14} x_2^4 - 12x_1^{10} x_2^3 (y - 2x_1)^2 - 2x_1^7 x_2^2 (y - 2x_1)^4.$$

See Example 6.11.

Let us denote by $z_0, \ldots, z_g = f$ a complete sequence of semi-roots of f. By Lemma 2.117 any $h \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ admits a finite expansion in terms of a complete sequence $z_0, \ldots, z_g$ of f. More generally we have:

LEMMA 6.25. *Any holomorphic germ $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ has a unique expansion*

$$h = \sum_{m \geq 0} \left(\sum_{finite} a_{I,m}(x) z_0^{i_{0,m}} \ldots z_{g-1}^{i_{g-1,m}} \right) z_g^m, \tag{6.26}$$

with $a_{I,m} \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ and $0 \leq i_{j,m} < n_{j+1}$ for $1 \leq j < g - 1$ and $m \in \mathbb{Z}_{\geq 0}$. In addition, the degrees as polynomials in y of the terms $z_0^{i_{0,m}} \ldots z_{g-1}^{i_{g-1,m}} z_g^m$ are pairwise distinct.

PROOF. Let $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ be a holomorphic germ. By applying Lemma 2.121 to h with respect to z_g we obtain a unique expansion

$$h = \sum_{m \geq 0} P_m z_g^m, \tag{6.27}$$

with $P_k \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ of degree $< n_0 \ldots n_g$ for all $m \in \mathbb{Z}_{\geq 0}$.

Now we apply Lemma 2.117 to every polynomial $P_m \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$. By replacing P_m by its expansion in (6.27) we obtain an expansion of h of the required form. □

DEFINITION 6.28. We say that a series $h \in \mathbb{C}\{t_1, \ldots t_r\}$ has a **dominant monomial** if

$$h = t^\alpha \varepsilon_h,$$

where $\alpha \in \frac{1}{s}\mathbb{Z}^r_{\geq 0}$ for some $s \in \mathbb{Z}_{\geq 0}$ and ε_h is a unit.

We have the following result for the properties of the expansions in terms of the semi-roots.

REMARK 6.29. Notice that if $a_I \in \mathbb{C}\{x_1, \ldots, x_d\}$ has a dominant monomial x^α, then $a_I(z_0(\zeta))^{i_0} \ldots (z_{g-1}(\zeta))^{i_{g-1}}$ also has a dominant monomial $x^{\alpha + i_0\gamma_1 + \ldots + i_{g-1}\gamma_g}$, by definition of semi-root.

The following result is a consequence of (5) in Lemma 6.17.

LEMMA 6.30. *Let $h \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ be a polynomial of degree $< n$ and let*

$$h = \sum a_I z_0^{i_0} \ldots z_g^{i_{g-1}}, \tag{6.31}$$

with $I = (i_0, \ldots, i_g)$, $a_I \in \mathbb{C}\{x_1, \ldots, x_d\}$ and $0 \leq i_j < n_{j+1}$ for $1 \leq j < g$ be its expansion as in Lemma 2.117. If $I \neq I'$ and a_I, $a_{I'}$ have dominant monomials, then

$$a_I(z_0(\zeta))^{i_0} \ldots (z_g(\zeta))^{i_{g-1}} \text{ and } a_{I'}(z_0(\zeta))^{i'_0} \ldots (z_g(\zeta))^{i'_{g-1}}$$

have different dominant monomials.

Remark 6.32. One can check that y is a 0-th semi-root of f if and only if $p_0 = 0$ in the expansion (6.22) of the quasi-ordinary branch ζ. If y is not a 0-th semi-root, setting $y' = z_0 = y + a(x) \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ defines a local coordinate system $(x_1, \ldots, x_d, y')$ at the origin of $\mathbb{C}^d$ and y' is by definition a 0-th semi-root of f.

We assume from now on that y is a 0-th semiroot of the quasi-ordinary polynomial f.

6.2. Toroidal embeddings

Let $\mathbf{X}$ be a normal variety of dimension $d+1$ and let $E = \{E_i\}_{i \in I}$ be a finite set of normal hypersurfaces with complement $\mathbf{U}$ on $\mathbf{X}$. A **toroidal embedding without self-intersection** is defined by requiring the triple $(\mathbf{X}, \mathbf{U}, \mathbf{x})$, at any point $\mathbf{x} \in \mathbf{X}$, to be formally isomorphic to (Z_σ, T, z) for a point z in some normal toric variety Z_σ. This means that there is a formal isomorphism between the completions of the local rings at respective points which sends the ideal of $\mathbf{X} \setminus \mathbf{U}$ to the ideal of $Z_\sigma \setminus T$. The complement of $\mathbf{U}$ in $\mathbf{X}$ is called the **boundary** and we denote it by $\partial \mathbf{X}$.

The variety X is naturally stratified with strata

$$\bigcap_{i \in K} E_i \setminus \bigcup_{i \notin K} E_i,$$

and open stratum $\mathbf{U}$. The **star** of a stratum $\mathbf{S} \subset \mathbf{X}$, $\mathrm{star}(\mathbf{S})$, is the union of the strata containing $\mathbf{S}$ in their closure. We associate to the stratum $\mathbf{S}$ the set $M^{\mathbf{S}}$ of Cartier divisors supported on $\mathbf{S} \setminus \mathbf{U}$ and we denote by $N^{\mathbf{S}}$ its dual group. The semigroup of effective divisors defines, in the real vector space $M^{\mathbf{S}}_{\mathbb{R}}$, a rational convex polyhedral cone and we denote its dual cone by $\rho_{\mathbf{S}} \subset N^{\mathbf{S}}$. If $\mathbf{S}' \in \mathrm{star}(\mathbf{S})$ is a stratum, we have a group homomorphism defined by restriction of Cartier divisors $M^{\mathbf{S}} \to M^{\mathbf{S}'}$ which is surjective. By duality we obtain an inclusion of the dual lattices $N^{\mathbf{S}'} \to N^{\mathbf{S}}$ and, under the associated real vector space map, the cone $\rho_{\mathbf{S}'}$ is mapped onto a face of $\rho_{\mathbf{S}}$. In this way we can associate to a toroidal embedding without self-intersection a **conic polyhedral complex** Θ. A conic polyhedral complex is a finite family of cones, each σ_i contained in a finite dimensional vector space V_i, such that it contains their faces. An integral structure is given by a set of lattices N_i on V_i such that for a face σ_j of σ_i, $\sigma_i \cap N_j = \sigma_j \cap N_j$. Notice that, unlike fans in toric varieties, we do not have an embedding of the various cones in a fixed vector space (see [**KKMSD73**, Chapter II, Definitions 5 and 6]).

Notation 6.33. We denote by $\Theta^{(d)}$ the set of d-dimensional cones of the conic polyhedral complex. In particular, $\Theta^{(1)}$ denotes the set of rays and we will use the notation Θ_{prim} for the set of primitive integral vectors generating the rays of the conic polyhedral complex.

This conic polyhedral complex is combinatorially isomorphic to the cone over the dual complex of intersection of the divisors E_i, in such a way that the strata are in one-to-one correspondence with the cones of the conic polyhedral complex. This generalizes the way of recovering the associated fan from a normal toric variety.

Example 6.34. Let Σ be a fan with respect to the lattice N. The conic polyhedral complex associated to the toroidal embedding (Z_Σ, T_N) is isomorphic to the fan Σ, with the integral structure defined by the lattice N. Recall from Subsection 1.1.6 that the set of T_N-invariant divisors of Z_Σ is equal to $\{D_u\}_{u \in \Sigma^{\mathrm{prim}}}$. Each D_u is a normal toric variety and these divisors define the boundary of the toroidal embedding in this case.

We can define, in an analogous manner to the case of a fan, a regular subdivision of a conic polyhedral complex. Such a finite rational polyhedral subdivision Θ' of Θ induces a **toroidal**

modification, i.e., a normal variety $\mathbf{X}'$ with a toroidal embedding $\mathbf{U}' \subset \mathbf{X}'$ and a modification $\mathbf{X}' \to \mathbf{X}$ provided with a commutative diagram

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & \mathbf{X}' \\ \downarrow & \swarrow & \\ \mathbf{X} & & \end{array}$$

such that $\mathbf{U}' \to \mathbf{U}$ is an isomorphism (check [**KKMSD73**, Theorems 6* and 8*, Chapter II]). That is, a toroidal modification between toroidal embeddings is a complex analytic morphism $\psi : \mathbf{X}' \to \mathbf{X}$, which is locally analytically isomorphic in a boundary-preserving way to a toric morphism, i.e., for any $p \in \mathbf{X}'$ there exist charts of $\mathbf{X}'$ and $\mathbf{X}$ centered at p and $\psi(p)$ respectively, relative to which ψ becomes a toric modification. In particular, if Θ' is a regular subdivision of Θ, then the map $\mathbf{X}' \to \mathbf{X}$ is a resolution of singularities of $\mathbf{X}$ ([**KKMSD73**, Theorem 12*, Chapter II]). The notion of partial toric embedded resolution generalizes to the toroidal case.

6.3. Toric quasi-ordinary singularities

In this section we introduce the notion of *toric quasi-ordinary singularity* considered in [**Gon03b, Gon00**]. The basic idea is to replace the germ $(\mathbb{C}^d, 0)$ in the target of the quasi-ordinary projection (6.2) by a germ of affine toric variety (Z_ρ, o) at the origin (see the definitions in Section 1.1).

First, we recall the notion of relative hypersurface singularity. A finite map germ $(H, O) \longrightarrow (H', O')$ corresponds algebraically to a local homomorphism $R' \to R$ between their respective analytic algebras which gives R the structure of a finite R'-module. In particular, if R is generated by one element ζ over R', there is a surjection $R'[y] \to R$, $y \mapsto \zeta$, which corresponds geometrically to an embedding

$$(H, O) \longrightarrow (H' \times \mathbb{C}, (O', 0)).$$

We say in this case that (H, O) is a relative hypersurface with respect to the base (H', O').

Let N be a rank d lattice and let ρ be a rational strictly convex polyhedral cone in $N_\mathbb{R}$ of dimension d. We denote by M the dual lattice of N and by $(Z_{\rho,N}, o_\rho)$ the germ of normal affine toric variety at its origin (0-dimensional orbit) (see the notation of Section 1.1).

The germ (H, O) is a **toric quasi-ordinary singularity** if there exists a finite morphism

$$(H, O) \to (Z_\rho, o_\rho), \tag{6.35}$$

(called the quasi-ordinary projection) unramified over the torus in a neighbourhood the origin of the normal affine toric variety (Z_ρ, o_ρ). If the germ (H, O) is a toric quasi-ordinary singularity then the quasi-ordinary projection (6.35) corresponds to a finite extension of the analytic algebras $\mathbb{C}\{\rho^\vee \cap M\} \hookrightarrow R$. The germ (H, O) is a toric quasi-ordinary hypersurface if, in addition, there is an element $\zeta \in R$ such that $R = \mathbb{C}\{\rho^\vee \cap M\}[\zeta]$. Then, as we explained above there is an embedding $H \subset Z_\rho \times \mathbb{C}$, which maps O to the origin (0-dimensional orbit) of $Z_\rho \times \mathbb{C}$ seen as a toric variety with respect to the lattice $N' = N \times \mathbb{Z}$.

Any toric quasi-ordinary hypersurface is defined by a quasi-ordinary polynomial in this generalized setting: a monic polynomial $f \in \mathbb{C}\{\rho^\vee \cap M\}[y]$, of degree n such that $f(O, y) = y^n$, whose discriminant $\Delta_y f \in \mathbb{C}$ is of the form a monomial times a unit in the ring $\mathbb{C}\{\rho^\vee \cap M\}$ (check Lemma 6.38 below).

The Jung-Abhyankar theorem extends to this setting (see [**Gon00, PR12**]), namely, if f is an irreducible toric quasi-ordinary polynomial, then the roots of f belong to the ring $\mathbb{C}\{\rho^\vee \cap \frac{1}{n} M\}$. The roots of f, called (toric) quasi-ordinary branches, are characterized combinatorially in a way similar to Lemma 6.9 (see [**Gon03b**, Lemma 12]). Similarly, we can associate to f a finite set of characteristic exponents $\alpha_1, \ldots, \alpha_g \in \rho^\vee \cap \frac{1}{n}M$ and their characteristic lattices and integers. We have an order relation, analogous to (6.6), given by

$$\alpha_1 \leq_\rho \cdots \leq_\rho \alpha_g, \tag{6.36}$$

where $\alpha \leq_\rho \alpha'$ means that $\alpha' \in \alpha + \rho^\vee$, for $\alpha, \alpha' \in M_\mathbb{R}$. Observe that these characteristic exponents are vectors in the lattice $\frac{1}{n}M$, which is given without any distinguished basis. One can define the semigroup associated with a toric quasi-ordinary singularity by

$$\rho^\vee \cap M + \mathbb{Z}_{\geq 0}\gamma_1 + \cdots + \mathbb{Z}_{\geq 0}\gamma_g,$$

where the γ_j are defined by (6.14). In particular, the notion of semi-root generalizes to this setting by considering an irreducible toric quasi-ordinary polynomial $z_j \in \mathbb{C}\{\rho^\vee \cap M\}[y]$.

Remark 6.37. If $f \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$ is an irreducible polynomial then we can consider it as a toric quasi-ordinary polynomial by taking $\rho := \mathbb{R}^d_{\geq 0}$ and $N = \mathbb{Z}^d$. From this point of view, it is convenient to consider the characteristic exponents of a quasi-ordinary branch not only as d-tuples of rational numbers but rather as rational vectors inside a reference cone $\rho^\vee$, with respect to a given reference lattice N. It is useful to consider the product $\mathbb{C}^d \times \mathbb{C}$ as the toric variety $Z_{\rho,N} \times \mathbb{C}$ defined by the cone $\varrho = \rho \times \mathbb{R}_{\geq 0}$ with respect to the lattice $N' = N \times \mathbb{Z}$, dual to the lattice $M' := M \times \mathbb{Z}$, so that we have $\varrho^\vee \cap M' \cong (\rho^\vee \cap M) \times \mathbb{Z}_{\geq 0} = \mathbb{Z}^{d+1}_{\geq 0}$. We will denote by $x^u y^s$ for the monomial associated to $(u, s) \in M \times \mathbb{Z}$.

One has a form of the Weierstrass Preparation Theorem in this situation.

Lemma 6.38. *Let $h \in \mathbb{C}\{\varrho^\vee \cap M'\}$ satisfying $h(o_\rho, y) = y^n$. There exists a unit $\epsilon \in \mathbb{C}\{\varrho^\vee \cap M'\}$ and a polynomial $P \in \mathbb{C}\{\varrho^\vee \cap M\}[y]$ such that $h = \epsilon \cdot P$.*

Proof. Let $u_1, \ldots, u_s$ be a finite system of generators of the semigroup $\varrho^\vee \cap M$. We have a surjective $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \varphi : \mathbb{C}\{X_1, \ldots X_s, Y\} &\longrightarrow \mathbb{C}\{\varrho^\vee \cap M'_1\} \\ X_i &\longmapsto x^{u_i}, \\ Y &\longmapsto y, \end{aligned}$$

which corresponds to a closed immersion $Z_{\varrho, N'_1} \hookrightarrow \mathbb{C}^{d+1}$. Since φ is surjective, there exists a series $\tilde{h} \in \mathbb{C}\{X_1, \ldots X_s, Y\}$ such that $\varphi(\tilde{h}) = h$. This implies that with $\tilde{h}(0, \ldots, 0, Y) = Y^n$. By the Weierstrass Preparation Theorem 2.1 applied to $\tilde{h}$ there exists a unique polynomial $\tilde{P} \in \mathbb{C}\{X_1, \ldots X_s\}[Y]$ of degree n and a unit $\tilde{\epsilon} \in \mathbb{C}\{X_1, \ldots X_s, Y\}$ such that $\tilde{h} = \tilde{\epsilon} \cdot \tilde{P}$. We obtain that $\epsilon := \varphi(\tilde{\epsilon})$ is a unit in the ring $\mathbb{C}\{\varrho^\vee \cap M'\}$, $P := \varphi(\tilde{P})$ is a polynomial of degree n in $\mathbb{C}\{\varrho^\vee \cap M\}[y_1]$ and $h = \epsilon \cdot P$. □

6.4. Toroidal embedded resolution of an irreducible quasi-ordinary hypersurface

In this section we summarize the toroidal embedded resolution of an irreducible germ of quasi-ordinary hypersurface following [Gon03b] and also [GG14].

Let (H, O) be an irreducible germ of quasi-ordinary hypersurface defined by a quasi-ordinary polynomial $f \in \mathbb{C}\{x_1, \ldots, x_d\}[y]$. First, we consider the toric modification ψ_1 defined by the dual fan associated to the Newton polyhedron of f, when y is chosen as a 0-semi-root of f (see Remark 6.32). This fan is a subdivision of the positive orthant $\varrho := \rho \times \mathbb{R}_{\geq 0} = \mathbb{R}^{d+1}_{\geq 0}$. This subdivision is determined by the first characteristic exponent. The first characteristic exponent determines a d-dimensional cone ρ_1 in the interior of ϱ. The strict transform of H by ψ_1 intersects the orbit $\mathbf{O}_{\rho_1}$ and one can show that it is a *toric quasi-ordinary hypersurface* with one less characteristic exponent. The composition of a finite number of such toroidal modifications gives an embedded normalization in a toroidal embedding without self-intersection, with an associated conic polyhedral complex Θ, whose combinatorial structure depends only on the characteristic exponents. The embedded resolution is then obtained by composing the embedded normalization with the toroidal modification associated to a regular subdivision of Θ.

We use the notation of Section 6.1.

NOTATION 6.39. Let $f \in \mathbb{C}\{x_1, \dots, x_d\}[y]$ be an irreducible quasi-ordinary polynomial of degree n defining an irreducible quasi-ordinary hypersurface germ $(H, O) \subset (\mathbb{C}^d, O)$, parametrized by a quasi-ordinary branch $\zeta \in \mathbb{C}\{x_1^{1/n}, \dots, x_d^{1/n}\}$. For each $0 \leq j \leq g$, we choose a j-th semi-root z_j and denote by $L_j = V(z_j)$ the associated irreducible quasi-ordinary hypersurface germ, with $z_g = f$, $L_g = H$ (Definition 6.20). In addition, we will assume that $y = z_0$ (see Remark 6.21). We denote by $\{\alpha_1, \dots, \alpha_g\}$ its characteristic exponents, by $n_1, \dots, n_g$ its characteristic integers and by $e_j = n_{j+1} \cdots n_g$ for $0 \leq j < g$ (Formula 6.6 and Definition 6.7).

We denote by N_0 the lattice $\mathbb{Z}^d$, and by $\epsilon_1^0, \dots, \epsilon_d^0$ its canonical basis, which spans the cone $\rho = \mathbb{R}^d_{\geq 0} \subset (N_0)_{\mathbb{R}}$. We denote by $\varepsilon_1^0, \dots, \varepsilon_d^0$ the dual basis of $M_0 = N_0^* := \check{\mathbb{Z}}^d$, which spans the dual cone $\rho^\vee \subset (M_0)_{\mathbb{R}}$. We denote by $N_j' = N_j \times \mathbb{Z}$ and by $M_j' = M_j \times \mathbb{Z}$ its dual lattice for $0 \leq j \leq g$. We set $\varrho := \rho \times \mathbb{R}_{\geq 0}$, and denote by ρ the face $\rho \times \{0\}$ of ϱ.

Since y is a 0-th semi-root of f, the Newton polyhedron $\mathcal{N}(f) := \mathcal{N}_\varrho(f) \subset (M_0')_{\mathbb{R}}$ has only one compact edge $\mathcal{E}_1$ with vertices $(0, e_0)$ and $(e_0\alpha_1, 0)$ (the proof is similar to the one for curves in Proposition 2.107, see also [**Gon00**]). The dual fan $\Sigma_1 := \Sigma(\mathcal{N}(f))$ is a subdivision of the cone ϱ, with only two $(d+1)$-dimensional cones σ_1^+, σ_1^- corresponding to vertices $(e_0\alpha_1, 0)$ and $(0, e_0)$ respectively. This two cones intersect along the common d-dimensional face ρ_1 (see Figure 1). The support function of the Newton polyhedron $\mathcal{N}(f)$ is defined for a vector $(v, r) \in \varrho$ by

$$\Phi_{\mathcal{N}(f)}(v, r) = \begin{cases} e_1 \langle v, n_1\alpha_1 \rangle & \text{if } (v, r) \in \sigma_1^+, \\ e_1 n_1 r & \text{if } (v, r) \in \sigma_1^-. \end{cases}$$

The polynomial f is of the form

$$f = (y^{n_1} - c_1 x^{n_1\alpha_1})^{e_1} + \dots, \tag{6.40}$$

where $c_1 \in \mathbb{C}^*$ and the terms which are not written lie above the unique compact edge $\mathcal{E}_1$ of $\mathcal{N}(f)$. See Figure 1.

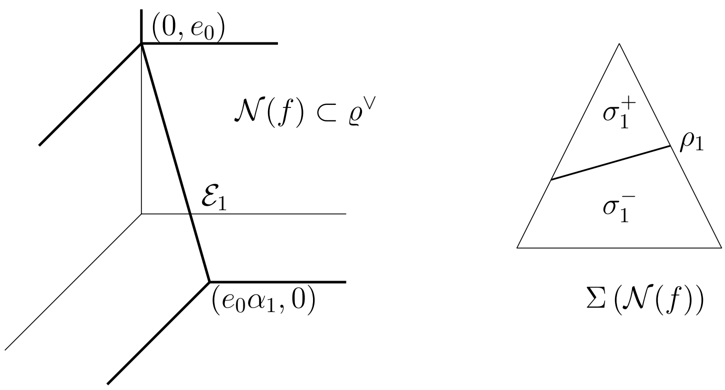

FIGURE 1. The Newton polyhedron $\mathcal{N}(f) \subset (M_0')_{\mathbb{R}}$ of a quasi-ordinary polynomial $f \in \mathbb{C}\{x_1, x_2\}[y]$, and the projectivization of its dual fan Σ_1, which subdivides ϱ. The cone ρ_1 is dual to the compact edge $\mathcal{E}_1$.

We have the toric modification

$$\psi_1 : Z_1 \longrightarrow Z_0,$$

defined by the subdivision Σ_1 of ϱ, with respect to the lattice N_0'.

The orbit $\mathbf{O}_{\rho_1}$ associated to ρ_1, is a one-dimensional torus embedded as a closed subset in the chart $Z_{\rho_1, N_0'} \subset Z_1$. The monomial

$$w_1 := y^{n_1} x^{-n_1 \alpha_1} \in \mathbb{C}[\rho_1^{\perp} \cap M_0'],$$

is a holomorphic function on the chart $Z_{\rho_1, N_0'} \subset Z_1$ and the coordinate ring of $\mathbf{O}_{\rho_1}$ is $\mathbb{C}[w_1^{\pm 1}] = \mathbb{C}[\rho_1^{\perp} \cap M_0']$. By definition of the toric modification, in the chart Z_{ρ_1}, we can factor the transform of f as:

$$(f \circ \psi_1)_{|Z_{\rho_1}} = x^{e_1 n_1 \alpha_1} [(w_1 - c_1)^{e_1} + \ldots], \tag{6.41}$$

where the terms which are not written vanish over the orbit $\mathbf{O}_{\rho_1}$. This implies the following Lemma about the strict transform of H (Definition 1.31):

Lemma 6.42. [Gon03b, Lemma 18]. *The intersection of the strict transform of H by ψ_1 with the exceptional fiber $\psi_1^{-1}(o_\varrho)$ is reduced to a point $o_1 \in \mathbf{O}_{\rho_1}$ counted with multiplicity e_1 (see Notation 6.7). In particular, the intersection of the strict transform with the orbit $\mathbf{O}_\rho$ is transversal if and only if $e_1 = 1$. The strict transform $H^{(1)}$ of the hypersurface H by ψ_1 is a germ at the point o_1.*

Recall that we fixed a 1-semi-root z_1 of f and denote $L_1 = V(z_1)$. By Lemma 6.42, applied to L_1 the strict transform $L_1^{(1)}$ intersects the orbit $\mathbf{O}_{\rho_1}$ at the same point o_1 with multiplicity one.

Let us discuss the local structure of the germ (Z_1, o_1). We have the relation (1.9), which expresses product decomposition of a toric chart in terms of its corresponding orbit. The following Lema provides an explicit description of this descomposition in one particular case:

Lemma 6.43. [Gon03b, Lemma 17]. *The lattice homomorphism*

$$\begin{aligned} \phi : M_0' &\longrightarrow M_1, \\ (v, a) &\longmapsto v + a\alpha_1, \end{aligned}$$

is surjective and has kernel

$$\mathrm{Ker}(\phi) = \rho_1^{\perp} \cap M_0' = (-n_1 \alpha_1, n_1)\mathbb{Z}.$$

If we choose a splitting $M_0' \simeq M_1 \oplus \mathrm{Ker}(\phi)$, we have a semigroup isomorphism

$$\rho_1^{\vee} \cap M_0' \xrightarrow{\sim} (\rho^{\vee} \cap M_1) \times \mathrm{Ker}(\phi), \tag{6.44}$$

which induces an isomorphism

$$Z_{\rho_1, N_0'} \simeq Z_{\rho, N_1} \times \mathbf{O}_{\rho_1, N_0'}. \tag{6.45}$$

The previous lemma states that the pair $(\rho_1, (N_0')_{\rho_1})$ is isomorphic to (ρ, N_1) and, as a consequence, $Z_{\rho_1, (N_0')_{\rho_1}}$ and Z_{ρ, N_1} are isomorphic toric varieties (see Lemma 1.8).

Notice that the vector $(-n_1 \alpha_1, n_1)$ defines the monomial w_1 defined above.

Lemma 6.46. *The ring of germs of holomorphic functions of Z_1 at the point o_1 is isomorphic to $\mathbb{C}\{(\rho^{\vee} \cap M_1) \times \mathbb{Z}_{\geq 0}\}$. The strict transform $L^{(1)}$ is analytically isomorphic to the germ of normal toric variety (Z_{ρ, N_1}, o_1).*

Proof. The point o_1 is defined by the maximal ideal

$$(w_1 - c_1) + (x^u \mid u \in \rho^\vee \cap M_1 \setminus \{0\}),$$

of the local ring of (Z_1, o_1). By (6.45) the chart $Z_{\rho_1, N_0'}$ is isomorphic to $Z_{\rho, N_1} \times \mathbb{C}^*$, where $\mathbb{C}^* \simeq \mathbf{O}_{\rho_1, N_0'}$. Under this isomorphism, the point o_1 corresponds to the point $(o_{\rho_1, N_0'}, c_1)$ of $Z_{\rho, N_1} \times \mathbb{C}^*$, where o_{ρ_1, N_1} is the distinguished point of Z_{ρ, N_1} and $\mathbb{C}^*$ is considered with coordinate w_1. The local ring of the orbit $\mathbf{O}_{\rho_1, N_0'}$ at the point o_1 is isomorphic to $\mathbb{C}\{u_1\}$, where $u_1 = w_1 - c_1$. It follows that

$$\mathcal{O}_{Z_{\rho_1, N_0'}, o_1} \simeq \mathbb{C}\{\rho^\vee \cap M_1\}\{u_1\} \simeq \mathbb{C}\{\varrho^\vee \cap M_1'\}, \tag{6.47}$$

and we have the isomorphism of local germs $(Z_{\rho_1, N_0'}, o_1) \simeq (Z_{\varrho, N_1'}, o_{\varrho, N_1'})$.

By Lemma 6.42 applied to L_1 (see (6.41)) one has

$$y_1 := x^{-n_1 \alpha_1} (z_1 \circ \psi_1)_{|Z_{\rho_1}} = u_1 + \ldots, \tag{6.48}$$

where the terms which are not written in (6.48) vanish on the orbit $\mathbf{O}_{\rho_1, N_0'}$. If we apply Lemma 6.38 to y_1 we obtain that the strict transform of L_1 is given by $u_1 + a_0$, where $a_0 \in \mathbb{C}\{\rho^\vee \cap M_1\}$. The local ring of $(L_1^{(1)}, o_1)$ is

$$\mathbb{C}\{\rho^\vee \cap M_1\}\{u_1\}/(u_1 + a_0) \simeq \mathbb{C}\{\rho^\vee \cap M_1\},$$

the isomorphism being given by sending u_1 to $-a_0$. It follows that the germ of $L_1^{(1)}$ at o_1 is isomorphic to (Z_{ρ, N_1}, o_1) as stated. □

More generally we have:

Proposition 6.49. [Gon03b, Proposition 19]. *The projection*

$$\pi_1 : Z_{\rho, N_1} \times \mathbb{C}^* \longrightarrow Z_{\rho, N_1},$$

restricted to the strict transform of the quasi-ordinary hypersurface H at the point o_1 defines an unramified finite covering of the torus $T_{N_1} \subset Z_{\rho, N_1}$.

The above proposition means that the strict transform $H^{(1)}$ of the quasi-ordinary hypersurface at the point o_1 is a germ of **toric quasi-ordinary hypersurface** relative to the base Z_{ρ, N_1} (see Section 6.3).

Recall that we denote by $z_0, \ldots, z_g = f$ a complete sequence of semi-roots of f and by $V(z_j) = L_j$ the hypersurfaces defined by this sequence for $j = 0, \ldots, g$. The above reasoning also applies to the strict transforms $L_j^{(1)}$ of the semi-roots and moreover we have:

Proposition 6.50. [Gon03b, Proposition 19]. *For any $2 \leq j \leq g$ the following holds:*

1. *The strict transform of the j-th semi-root at the point o_1, $(L_j^{(1)}, o_1)$, is parametrized by a toric quasi-ordinary branch, $\zeta_j^{(1)}$, which has characteristic exponents*

$$\alpha_2 - \alpha_1, \ldots, \alpha_j - \alpha_1,$$

and characteristic integers

$$n_2, \ldots, n_j,$$

with respect to the cone $\rho^\vee$ and with reference lattice M_1.

2. *The strict transform $(L_j^{(1)}, o_1)$ of the j-th semi-root L_j is a $(j-1)$-th semi-root of the strict transform $(H^{(1)}, o_1)$ of the hypersurface H.*

If the number of characteristic exponents is $g > 1$, it follows that the germ $(L_j^{(1)}, o_1)$ is defined by the vanishing of

$$z_j^{(1)} := (x^{-n_1 \dots n_j \alpha_1} z_j \circ \psi_1)_{|Z_{\rho_1}}, \text{ for } j = 2, \dots, g. \tag{6.51}$$

By Lemma 6.38 applied to $z_j^{(1)}$, there exists a unit $\epsilon_j^{(1)} \in \mathbb{C}\{\varrho^\vee \cap M_1'\}$ such that $\epsilon_j^{(1)} z_j^{(1)} = f_j^{(1)}$ where $f_j^{(1)} \in \mathbb{C}\{\rho^\vee \cap M_1\}[y_1]$ is a toric quasi-ordinary polynomial parametrized by the toric quasi-ordinary branch $\zeta_j^{(1)}$ for $2 \leq j \leq g$.

Thus, we can assume that the *strict transform* $f^{(1)}$ of f is an element of $\mathbb{C}\{\rho^\vee \cap M_1\}[y_1]$ (Notice that $\mathbb{C}\{\rho^\vee \cap M_1\}[y_1] \subset \mathbb{C}\{\varrho^\vee \cap M_1'\}$ by Lemma 6.46).

By Proposition 6.50, the term y_1 is a 0-semi-root of $f^{(1)}$ so that its Newton polyhedron,

$$\mathcal{N}_1(f^{(1)}) := \mathcal{N}_\varrho(f^{(1)}) \subset (M_1')_{\mathbb{R}} = (M_1)_{\mathbb{R}} \times \mathbb{R},$$

has a unique compact edge $\mathcal{E}_2$ with vertices $(e_1(\alpha_2 - \alpha_1), 0), (0, e_1)$, getting

$$f^{(1)} = (y_1^{n_2} - c_2 x^{n_2 \alpha_2^{(1)}})^{e_2} + \dots, \tag{6.52}$$

where $c_2 \in \mathbb{C}^*$ and the terms which are not written lie above the edge $\mathcal{E}_2$. The Newton polyhedron $\mathcal{N}(f^{(1)})$ defines a subdivision Σ_2, the dual fan, of the cone ϱ with only two $(d+1)$-dimensional cones, σ_2^+, σ_2^+ corresponding, respectively to the vertices $(e_1(\alpha_2 - \alpha_1), 0), (0, e_1)$ of the compact edge $\mathcal{E}_2$. These cones intersect along the d-dimensional cone ρ_2 which is the cone dual to the unique compact edge of the Newton polyhedron of the strict transform. With respect to these new coordinates we consider the induced toric modification

$$\psi_2 : Z_2 \longrightarrow Z_1,$$

where $\psi_2 := \psi_{\Sigma_2}$, $Z_2 := Z_{\Sigma_2, N_1'}$. We set $w_2 := y_1^{n_2} x^{-n_2 \alpha_2^{(1)}}$. In the chart $Z_{\rho_2, N_1'} \subset Z_2$ one can factorize

$$(f^{(1)} \circ \psi_2)_{|Z_{\rho_2}} = x^{e_2(\alpha_2 - \alpha_1)} \left[(w_2 - c_2)^{e_3} + \dots \right],$$

where the terms which are not written vanish on the orbit $\mathbf{O}_{\rho_2}$. The strict transform $H^{(2)}$ defined by the vanishing of the non-exceptional part

$$x^{-e_2 \alpha_2^{(1)}} (f^{(1)} \circ \psi_2)_{|Z_{\rho_2}},$$

is a germ at the point $o_2 \in \mathbf{O}_{\rho_2}$ defined by a toric quasi-ordinary polynomial $f^{(2)} \in \mathbb{C}\{\rho^\vee \cap M_2\}[y_2]$, where $V(y_2)$ defines the strict transform of L_2.

Proposition 6.50 generalizes to this setting. We iterate this procedure, obtaining a sequence of local toric modifications

$$\psi_{j-1} : Z_j \to Z_{j-1}, \quad \text{for } 1 \leq j \leq g.$$

Each toric modification ψ_{j-1} is defined in terms of a fan Σ_{j-1} subdividing the cone ϱ with respect to the lattice N_{j-1}' into $(d+1)$-dimensional cones σ_j^+, σ_j^-, which intersect along the d-dimensional cone ρ_j for $1 \leq j \leq g$. We denote by Σ_{g+1} the fan of faces of the cone ϱ with respect to the lattice N_g'.

Remark 6.53. If we define the lattice homomorphism

$$\begin{aligned} \phi_j : M_{j-1}' &\longrightarrow M_j, \\ (v, a) &\longmapsto v + a(\alpha_j - \alpha_{j-1}), \end{aligned} \tag{6.54}$$

by applying Lemma 6.43 inductively, we get that the restriction of the dual homomorphism, $\phi_j^* : N_j \to N'_{j-1}$ defines an isomorphism of semigroups

$$(\phi_j^*)_{|\rho \cap N_j} : \rho \cap N_j \to \rho_j \cap N'_{j-1}, \tag{6.55}$$

for $1 \leq j \leq g$. The isomorphism $(\phi_j^*)_{|\rho \cap N_j}$ induces an identification of the pair $(\rho_j, (N'_{j-1})_{\rho_j})$ with the pair (ρ, N_j). Notice that $\rho_j \in \Sigma_j$. We write simply ρ for the face $\rho \times \{0\}$ of ϱ, which can be seen as a cone of the fan Σ_{j+1}, in such a way that $(\rho, N_j) = (\rho, (N'_j)_\rho)$. As a consequence, the germ (Z_j, o_j) is analytically isomorphic to the germ of normal toric variety $(Z_{\varrho, N'_j}, o_\varrho)$ (see Lemma 6.46). The ring of regular functions of the germ $(Z_{\varrho, N'_j}, o_\varrho)$ is $\mathbb{C}\{\varrho^\vee \cap M'_k\}$, where $M'_j = M_j \times \mathbb{Z}_{\geq 0}$ and any monomial is of the form $x^m y_j^l$, with $m \in M_j$, $a \in \mathbb{Z}_{\geq 0}$ and where y_j is the strict transform of z_j on this ring.

Let us denote by $\Psi_j = \psi_1 \circ \ldots \circ \psi_j$ the composition of the toric modifications up to level j.

Definition 6.56. Let $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ be a holomorphic germ. We denote by

$$\mathcal{N}_j(h),$$

the Newton polyhedron of the total transform $\Psi_j^*(h)$ of h at the point o_j, seen as an element of the ring $\mathbb{C}\{\varrho^\vee \cap M'_j\}$.

With these notation we obtain the following information about the total transforms of the semi-roots.

Lemma 6.57. [**GG14**, Lemma 5.8]. *For a level $1 \leq j \leq g$, we have that any semi-root z_i with $i < j$ satisfies*

$$\mathcal{N}_j(z_i) = \mathcal{N}_\varrho(x^{\gamma_{i+1}}).$$

Furthermore,

$$\mathcal{N}_j(z_j) = \mathcal{N}_\varrho(x^{n_j \gamma_j} y_j),$$

and for any semi-root z_k with $j < k$ we have

$$\mathcal{N}_j(z_k) = \mathcal{N}_\varrho\left(x^{n_j \cdots n_k \gamma_j} z_k^{(j)}\right),$$
$$\mathcal{N}_\varrho\left(z_k^{(j)}\right) = \mathcal{N}_j\left(\left(y^{n_{j+1}} - c_{j+1} x^{n_{j+1}(\alpha_{j+1} - \alpha_j)}\right)^{n_{j+2} \cdots n_k}\right)$$

Definition 6.58. The **completion** $\bar{H}$ of the hypersurface H is the union of the coordinate divisors $R_i = V(x_i)$ and a complete sequence of semi-roots $L_j = V(z_j)$

$$\bar{H} = \overset{d}{\underset{i=1}{\cup}} R_i \bigcup \overset{g}{\underset{j=0}{\cup}} L_j.$$

In [**Gon03b**] it is shown that the pair (Z_g, U), consisting of the normal variety Z_g and the complement U of the strict transform of $\bar{H}^{(g)}$ by the modification Ψ_g, defines a toroidal embedding without self-intersection. The associated conic polyhedral complex $\Theta(H)$ can be described combinatorially as follows:

Definition 6.59. Let Σ_{g+1} be the fan of faces of the cone ϱ with respect to the lattice N'_g. We set

$$\Theta(H) := (\bigsqcup_{k=1}^{g+1} \Sigma_k)/_\sim,$$

where $\sim$ is defined by identifying the cones $\rho_k \in \Sigma_k$ with the cone $\rho = \rho \times \{0\}$ of the fan Σ_{k+1}, for $k = 1, \ldots, g$, thanks to the isomorphism of pairs $(\rho_k, N'_{k-1}) \simeq (\rho, N_k)$ (see (6.55)). This identification is extended to pairs of corresponding faces of these cones. The set $\Theta(H)$ is **the conic polyhedral complex** associated to the quasi-ordinary hypersurface germ (H, O).

REMARK 6.60. The support of the complex, $|\Theta|$, is equal to the disjoint union $\bigsqcup_{k=0}^{g} \varrho \cap (N'_k)_{\mathbb{R}}$ modulo the equivalence relation which identifies $(v, 0) \in \varrho \cap (N'_k)_{\mathbb{R}}$ with $\phi_k^*(v) \in \rho_k \cap N'_{k-1} \subset \varrho \cap (N'_{k-1})_{\mathbb{R}}$ for all $v \in \rho \cap N_k$. The **integral vectors** of the complex $\Theta(H)$ are those in

$$\bigsqcup_{k=1}^{g+1} (\varrho \cap N'_{k-1}) / \sim$$

where $\sim$ is induced by the above identifications of vectors.

When we write that a vector $(v, a) \in \varrho \cap (N'_{k-1})_{\mathbb{R}}$ is in the support of Θ, we abuse the notation by taking the representative of its class in $|\Theta|$ as an element of the real vector space $(N'_{k-1})_{\mathbb{R}}$ containing it in the disjoint union $\bigsqcup_{j=0}^{g} \varrho \cap (N'_k)_{\mathbb{R}}$. In order to be more precise we should denote by $(v, a)^{(k-1)}$ this representative of the class of (v, a) in $|\Theta|$.

THEOREM 6.61. [**Gon03b**, Theorem 1]. *The proper morphism obtained by composition of the toric modifications, Ψ_g, gives an embedded normalization of the quasi-ordinary hypersurface $(H, O) \subset (\mathbb{C}^{d+1}, O)$. An embedded resolution of (H, O) is obtained by the composition of Ψ_g with the toroidal modification $\Psi_{\Theta'}$ associated to any regular subdivision Θ' of the conic polyhedral complex $\Theta(H)$.*

The regular subdivision of the conic polyhedral complex is a set of regular subdivisions of the fans Σ_j compatible with the identification $\sim$ of Definition 6.59.

The construction of the conic polyhedral complex is illustrated in Figure 2.

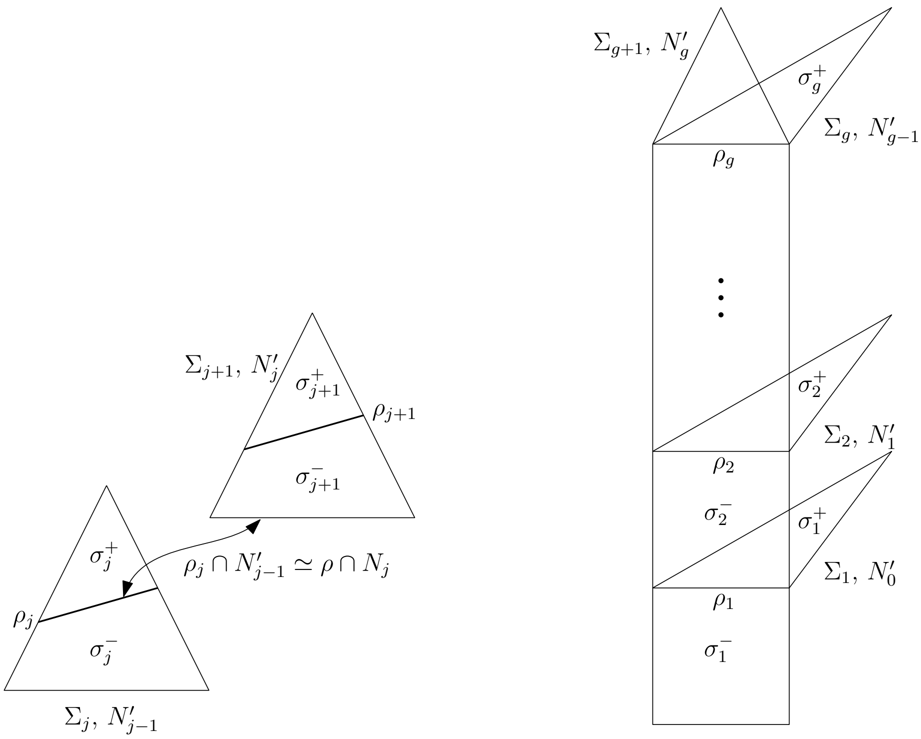

FIGURE 2. The construction of the conic polyhedral complex.

Remark 6.62. In the case of plane curves, the projectivization of the support of the conic polyhedral complex is homeomorphic to to the completion of the Eggers-Wall tree, $\Theta_R(\bar{C})$ (see Definition 4.54).

Definition 6.63. Let v be a an integral vector in the support of the conic polyhedral complex $\Theta(H)$. This means that $v \in |\Sigma_k| \cap N'_{k-1} = \varrho \cap N'_{k-1}$ for some k. We denote by ν_v the associated *torus invariant valuation* of the toric variety $Z_{\varrho, N'_{k-1}}$. If Σ'_k is any regular subdivision of Σ_k on which the ray spanned by v appears then we denote by $D_v^{(k)}$ the associated divisor. We often denote $D_v^{(k)}$ simply by D_v if k is clear from the context. We say that v is of **depth** k if $v \in |\Sigma_k|$, where k is the smallest integer in $\{1, \ldots, g+1\}$ satisfying this condition. If v does not span an end ray of $\Theta(H)$ we say that the divisor $D_v^{(k)}$ is a **toroidal exceptional divisor** of the toroidal resolution process.

Notation 6.64. Recall that the value of the associated monomial valuation ν_v on a function $h \in \mathbb{C}\{\varrho^\vee \cap M'_{k-1}\}$ is also the order of vanishing of $h \circ \psi_{\Sigma'}$ along the divisor D_v (see Subsection 1.1.6). If $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$, then the order of vanishing of h along the divisor D_v is equal to $\nu_v(h \circ \psi_1 \circ \ldots \circ \psi_{k-1} \circ \psi_{\Sigma'})$. This defines a valuation of $\mathbb{C}\{x_1, \ldots, x_d, y\}$ which we denote also by ν_v by abuse of notation.

Definition 6.65. A one-dimensional face $\rho \in \Theta^{(1)}$ is an **end** ray if there exists a unique cone $\sigma \in \Theta^{(d+1)}$ with $\rho \subset \sigma$ (see Notation 6.33). Otherwise, we say that ρ is a **relevant** ray of Θ. Let u be an integral vector in the support of Θ. We say that the ray $\mathbb{R}_{\geq 0} u$ is **exceptional** if it is not an end ray.

Notice that a relevant ray is by definition an exceptional ray such that $\rho \in \Theta^{(1)}$.

Remark 6.66. The above definition provides an analogue of the set Υ° of marked points of the Eggers-Wall tree (Notation 2.85). The end rays of the complex correspond to the ends on the completion of the Eggers-Wall tree of the curve, which are associated to irreducible components of the complete curve. Exceptional rays of the complex correspond to rational points of the Eggers-Wall tree, which have an associated exceptional component (see Notation 4.65). In particular, relevant rays of Θ correspond to ramification points in the completion of the Eggers-Wall tree, i.e., the set of interior marked points Υ°.

Remark 6.67.

(1) The irreducible components of the boundary divisor of the toroidal embedding defined by the conic polyhedral complex Θ, which are the strict transforms of the components in the completion $\bar{H}$ (see Definition 6.61), are in bijection with the end rays of Θ.
(2) If $\mathbb{R}_{\geq 0} u$ is an exceptional ray, we can choose a subdivision of Θ containing it, which provides a model in which the toroidal exceptional divisor D_u appears.
(3) In particular, if $\mathbb{R}_{\geq 0} u$ is an actual ray of Θ, the corresponding toroidal divisor already appears in the partial embedded resolution Ψ_g, and we say that it is a relevant exceptional divisor.

Example 6.68. Figure 3 represents the projectivization of the conic polyhedral complex Θ of the hypersurface germ H defined by the quasi-ordinary polynomial in Example 6.11. On it, we represent the five end rays of Θ by disks, which are associated to the strict transforms of the completion of the germ. Θ has 3 relevant rays, marked by squares, associated to the exceptional divisors $D_1^{(1)}, D_2^{(1)}$ (coming from the first characteristic exponent) and $D_1^{(2)}$ (coming from the second characteristic exponent). We marked with crosses two non-relevant exceptional rays, $\mathbb{R}_{\geq 0} u_1, \mathbb{R}_{\geq 0} u_2$. The vector u_1 belongs to the intersection of the supports of Σ_1, Σ_2, while $u_2 \in |\Sigma_2|$.

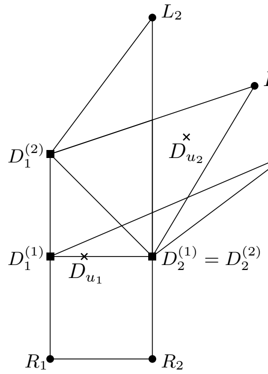

FIGURE 3. End, exceptional and relevant divisors on the conic polyhedral complex of the quasi-ordinary hypersurface in Example 6.11.

In the previous cases (Remark 6.67), we have a divisor associated to a ray of $\Theta(H)$ (or of a subdivision of it). We are going to describe the corresponding valuation (divisorial or vanishing order) in terms of the expansions with respect to the semi-roots. Recall that if $g \in \mathbb{C}\{x_1, \dots, x_d, y\}$ is irreducible and $L = V(g)$ we denote by ord_L the vanishing order valuation along L. This in an immediate extension of Definition 2.71.

6.5. Monomialization of toroidal valuations

In this section, we show that the divisorial valuations associated to the characteristic exponents are monomial in a complete sequence of semi-roots of the hypersurface H. This method generalizes to primitive integral vectors in any of the σ_k^- (see Figure 2).

First, let us generalize Definition 4.123 to $\mathcal{O}_{\mathbb{C}^{d+1},O}$.

DEFINITION 6.69. Let ν be a valuation of $\mathbb{C}\{x_1, \dots, x_d, y\}$. Let us consider a finite set of polynomials $z_0, \dots, z_r \subset \mathbb{C}\{x_1, \dots, x_d\}[y]$ such that $\{x_1, \dots, x_d, z_0, \dots, z_r\}$ generate the maximal ideal. We say that ν is a **monomial valuation with respect to** $z_0, \dots, z_r \subset \mathbb{C}\{x_1, \dots, x_d\}[y]$ if for every $h \in \mathbb{C}\{x_1, \dots, x_d, y\}$ there exists an expansion

$$h = \sum_{I=(i_0,i_1,\dots,i_r)} a_I(x_1, \dots, x_d) \cdot z_0^{i_0} \dots z_r^{i_r}, \text{ with } a_I \in \mathbb{C}\{x_1, \dots, x_d\},$$

such that

$$\nu(h) = \min_I \left\{ \nu \left(a_I(x_1, \dots, x_d)\, z_0^{i_0} \dots z_r^{i_r} \right) \right\}. \tag{6.70}$$

If $L_i = V(z_i)$ for $i = 0, \dots, r$, we say also that ν is a monomial valuation with respect to $L_0, \dots, L_r$.

Let us fix a complete sequence of semi-roots $z_0, \dots, z_g = f$ of the quasi-ordinary polynomial f. Let $h \in \mathbb{C}\{x_1, \dots, x_d, y\}$ be a holomorphic germ. By Lemma 6.25, there exists an expansion of the form

$$h = \sum_{m \geq 0} \left(\sum_{I_m}^{\text{finite}} a_{I_m} z_0^{i_{0,m}} \dots z_{g-1}^{i_{g-1,m}} \right) z_g^m. \tag{6.71}$$

where $a_{I_m} \in \mathbb{C}\{x_1, \dots, x_d\}$ and the exponents $I_m = (i_{0,m}, \dots, i_{g-1,m})$ verify

$$0 \leq i_{j,m} < n_{j+1} \text{ for } j \in \{0, \dots, g-1\}, \quad m \geq 0.$$

Let us write

$$\mathcal{M}_I = a_I z_0^{i_0} \dots z_{g-1}^{i_{g-1}} z_g^{i_g}, \tag{6.72}$$

for a generalized monomial in the expansion (6.71).

DEFINITION 6.73. The **leading term** of a polynomial $h = \sum_{j=0}^{n} c_j y^j \in A[y]$ of degree n is $c_n y^n$.

EXAMPLE 6.74. The leading term with respect to y of a generalized monomial $\mathcal{M}_I$ of the form (6.72) is $a_I y^{n_I}$ where $n_I = i_0 + i_1 n_1 + \dots + i_g n_1 \dots n_g$.

Now we describe the leading terms of the total transforms with respect to $\psi_1 \circ \cdots \circ \psi_k$ of a generalized monomial in the semi-roots at the point o_k of Z_k. Recall that the germ (Z_k, o_k) is analytically isomorphic to the germ of normal toric variety $(Z_{\varrho, N'_k}, o_\varrho)$, so the total transform of a generalized monomial is seen as a germ in $\mathbb{C}\{\varrho^\vee \cap M'_k\}$. A monomial of $\mathbb{C}\{\varrho^\vee \cap M'_k\}$ is of the form $x^m y_j^a$, with $m \in M_j$, $a \in \mathbb{Z}_{\geq 0}$ and y_j denotes the strict transform of z_j at o_j.

Notice that by definition $x_j = x^{\varepsilon_j^0}$ and since $\varepsilon_j^0 \in M_0 \subset M_j$, its total transform is itself. From Lemma 6.57 we know that the total transforms of the semi-roots is of the form:

$$\begin{aligned}
(x^{\varepsilon_j^0} \circ \Psi_k)_{|Z_{\rho_k}} &= x^{\varepsilon_j^0}, \\
(z_i \circ \Psi_k)_{|Z_{\rho_k}} &= x^{\gamma_{i+1}} \times \text{ unit} \qquad \forall i < k, \\
(z_i \circ \Psi_k)_{|Z_{\rho_k}} &= x^{n_k \dots n_i \gamma_k} z_i^{(k)} \times \text{ unit} \quad \forall i \geq k,
\end{aligned} \tag{6.75}$$

where the strict transforms are of the form

$$\begin{aligned}
z_k^{(k)} &= y_k, \\
z_i^{(k)} &= \left(y_k^{n_{k+1}} - c_k x^{n_{k+1}(\alpha_{k+1} - \alpha_k)}\right)^{n_{k+2} \dots n_i} + \dots \;\; \forall i > k,
\end{aligned} \tag{6.76}$$

where the terms which are not written vanish on the orbit $\mathbf{O}_{\rho_j}$ (they lie above the compact edge of the Newton polyhedron). Notice the similarity with the plane curve case in Lemma 4.107.

DEFINITION 6.77. Let $h \in \mathbb{C}\{x_1, \dots, x_d, y\}$ such that its total transform by Ψ_k (Definition 6.56) is a polynomial $h \in \mathbb{C}\{\rho^\vee \cap M_k\}[y_k]$. We call the expression

$$\mathrm{LT}_k(h) = \mathrm{LT}\left((\Psi_k^* h)_{|\varrho, N'_k} \right),$$

the ***k*-th leading term of *h***.

Recall that the total transform of a generalized monomial $\mathcal{M}_I = a_I z_0^{i_0} \dots z_g^{i_g}$ is a polynomial in y_k, since so is each of its terms. Thus, the k-th leading term of a generalized monomial $\mathcal{M}_I$ is $\mathrm{LT}_k(\mathcal{M}_I) = ((\Psi_k^* \mathcal{M}_I))$. According to (6.75) and (6.76), the k-th leading term of $\mathcal{M}_I = a_I z_0^{i_0} \dots z_g^{i_g}$ is equal to

$$\begin{aligned}
\mathrm{LT}_k(a_I z_0^{i_0} \dots z_g^{i_g}) &= \mathrm{LT}_k\left((a_I z_0^{i_0} \dots z_g^{i_g}) \circ \Psi_k \right) \\
&= a_I \cdot x^{i_0 \gamma_1 + \dots + i_{k-1} \gamma_k + n_k s_k \gamma_k} \cdot y_k^{s_k},
\end{aligned} \tag{6.78}$$

where

$$s_k = i_k + n_{k+1} i_{k+1} + \dots + n_{k+1} \cdots n_g i_g. \tag{6.79}$$

Each monomial in the k-th leading term is of the form

$$x^{m_{j(I)}+i_0\gamma_1+\ldots+i_{k-1}\gamma_k+n_k s_k\gamma_k} y_k^{s_k}, \tag{6.80}$$

where $m_{j(I)} \in M_0$ comes from a_I and

$$i_0\gamma_1 + \ldots + i_{k-1}\gamma_k + n_k s_k \gamma_k \tag{6.81}$$

comes from the exceptional part of the total transform of the monomial $z_0^{i_0} \ldots z_g^{i_g}$.

The following result states that different generalized monomials have different k-th leading terms (see Lemma 4.113 for the plane curve analogue).

LEMMA 6.82. *Let $I = (i_0, \ldots, i_g), I' = (i'_0, \ldots, i'_g)$ be two indices corresponding to nonzero terms $\mathcal{M}_I(h), \mathcal{M}_{I'}(h)$ in the expansion of $0 \neq h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ with respect to $z_0, \ldots, z_g$ (Lemma 6.25). If $\mathrm{LT}_k(\mathcal{M}_I) = \mathrm{LT}_k(\mathcal{M}'_I)$, then we have $I = I'$.*

PROOF. By (6.78) the hypothesis implies that $s_k = s'_k$. Since $n_i > 1$, by Lemma 2.120 we conclude that $i_j = i'_j$, for $j \geq k$. If $m_j(I)$ is a vertex of the Newton polyhedra of a_I (viewed in $\mathbb{C}\{x_1, \ldots, x_d, y\}$), then $m_j(I)$ is also a vertex of the polyhedron $\mathcal{N}_k(a_I)$ and vice versa. The hypothesis implies that there exists a vertex of $m'_{j(I')}$ of $a_{I'}$ such that:

$$m_{j(I)} + i_0\gamma_1 + \ldots + i_{k-1}\gamma_k + n_k s_k \gamma_k = m'_{j(I')} + i'_0\gamma_1 + \ldots + i'_{k-1}\gamma_k + n_k s'_k \gamma_k,$$

as in expression (6.81). Thus, Lemma 6.18 concludes $I = I'$. □

The following result generalizes Theorem 4.125.

LEMMA 6.83. *Let $v \in N'_k \cap \sigma_k^-$ be a primitive integral vector and ν its associated divisorial valuation. Then, ν is a monomial valuation with respect to $L_0, \ldots, L_g$ (Definition 4.123).*

PROOF. Since $v \in \sigma_k^-$ the valuation with respect to ν of a generalized monomial

$$\mathcal{M}_I = a_I z_0^{i_0} \ldots z_g^{i_g}$$

is attained at a monomial of the k-th leading term:

$$\begin{aligned} \nu(\mathcal{M}_I) &= \nu(\mathrm{LT_k}\,(\mathcal{M}_I)) \\ &= \left\langle v, \left(m_{j(I)} + i_0\gamma_1 + \ldots + i_{k-1}\gamma_k + n_k s_k \gamma_k, s_k\right)\right\rangle, \end{aligned} \tag{6.84}$$

for some monomial $m_{j(I)}$ of a_I, where $s_k = i_k + n_{k+1} i_{k+1} \ldots + n_{k+1} \cdots n_g i_g \gamma_g$ as in (6.79). Indeed, by (6.75) and (6.76) the Newton polyhedron of the total transform of each z_i with $i > k$, seen as an element of the ring $\mathbb{C}\{\varrho^\vee \cap M'_k\}$, has a single compact edge, homothetic to the one of the total transform of f. As a consequence, their dual fans coincide and are equal to Σ_k, and the cone σ_k^- is, by definition, dual to the vertex corresponding to the leading term of z_i for $i > k$. Since the Newton polyhedron of a product is the Minkowski sum of the Newton polyhedra of its factors, the valuation of a generalized monomial is attained at the leading term (Definition 6.77). By Lemma 1.22, it follows that the valuation is attained at some vertex of the Newton polyhedron of the leading term, given by an expression (6.80).

Let $\mathcal{M}_I, \mathcal{M}'_I$ be two different generalized monomials in the expansion of a germ $0 \neq h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$ such that $\nu(\mathcal{M}_I) = \nu(\mathcal{M}'_I)$. By Lemma 6.82, we have that $\mathrm{LT_k}(\mathcal{M}_I) \neq \mathrm{LT_k}(\mathcal{M}_{I'})$. By formula (6.84) this implies that

$$\nu(\mathcal{M}_I + \mathcal{M}_{I'}) = \nu(\mathcal{M}_I).$$

Since ν is exceptional we have $\nu(z_g) > 0$. Therefore, there exists a k_0 such that $\nu(h) = \nu(z_g^{k_0})$ and using (6.71) we write

$$h = \sum_{0 \leq m \leq k_0} \left(\sum_{I_m}^{\text{finite}} a_{I_m} z_0^{i_{0,m}} \dots z_{g-1}^{i_{g-1,m}} \right) z_g^m + h',$$

where $\nu(h') > \nu(h)$. Thus, $\nu(h) = \nu(h - h')$, and since $h - h' = \sum_I \mathcal{M}_I$ is a finite sum we get by induction that

$$\nu(h) = \min_{a_I \neq 0} \{\nu(\mathcal{M}_I)\}.$$

□

Remark 6.85. The total transforms by $\psi_1 \circ \dots \psi_k$ of the semi-roots $x_1, \dots, x_d, z_0, \dots, z_{k-1}$ on (Z_{ϱ, N'_k}, o_k) have a dominant monomial in $\mathbb{C}\{\varrho^\vee \cap M'_k\}$. We can use this fact, arguing as in Lemma 6.82, to monomialize the valuation ν_v, associated with any primitive integral vector $v \in N'_k \cap \varrho \setminus \{0\}$, in terms of the expansions with respect to the semi-roots $z_0, \dots, z_k$. This applies in particular to the valuations associated with the divisors $D_j^{(k+1)}$ for $j = 1, \dots, d$. Compare with Remark 4.128 in the plane curve case.

6.6. Computations with the relevant exceptional divisors

6.6.1. Description of the integral vectors. In this section we follow [GG14, Section 9] and construct a basis for the vector spaces $(M_j)_\mathbb{Q}$ associated to the characteristic lattices, which allows to give a precise description of the primitive integral vectors in the edges of the cones ρ_j.

We expand the first characteristic exponent in terms of the basis $\{\varepsilon_j^0\}_{j=1}^d$ (see Notation 6.39):

$$\alpha_1 = \sum_{j=1}^d \frac{q_j^1}{p_j^1} \varepsilon_j^0$$

where $\dfrac{q_j^1}{p_j^1}$ are irreducible fractions, and then we set:

$$\epsilon_j^1 = p_j^1 \epsilon_j^0, \quad \text{for } 1 \leq j \leq d.$$

The vectors ϵ_j^1 for $1 \leq j \leq d$ define a basis of a sublattice $\tilde{N}_1 \subset N_1$. The dual basis $\{\varepsilon_j^1\}$ of $\{\epsilon_j^1\}$, which is given by

$$\varepsilon_j^0 = p_j^1 \varepsilon_j^1 \ \text{ for } 1 \leq j \leq d,$$

is a basis of the dual lattice $\tilde{M}_1 \supset M_1$. Then, we have an expansion

$$\alpha_1 = \sum_{j=1}^d q_j^1 \varepsilon_j^1.$$

We suppose that $\tilde{M}_{k-1}$ together with its basis $\{\varepsilon_j^{k-1}\}$ have been defined by induction for $1 \leq k < j$. Then we expand

$$\alpha_k - \alpha_{k-1} = \sum_{j=1}^d \frac{q_j^k}{p_j^k} \varepsilon_j^{k-1},$$

where the fractions $\dfrac{q_j^k}{p_j^k}$ are irreducible. With these notation we define a basis $\{\epsilon_j^k\}$ of a new lattice $\tilde{N}_k \subset N_k$,

$$\epsilon_j^k = p_j^k \epsilon_j^{k-1} = p_j^k \cdots p_j^1 \epsilon_j^0, \quad \text{for } 1 \leq j \leq d, \tag{6.86}$$

and we get a dual basis $\{\varepsilon_j^k\}$ for the dual lattice $\tilde{M}_k \supset M_k$, such that:

$$\varepsilon_j^{k-1} = p_j^k \varepsilon_j^k = p_j^k \cdots p_j^1 \varepsilon_j^0, \quad \text{for } 1 \leq j \leq d. \tag{6.87}$$

Then, we have an integral expansion

$$\alpha_k - \alpha_{k-1} = \sum_{j=1}^{d} q_j^k \varepsilon_j^k. \tag{6.88}$$

Notice that

$$q_j^k = \langle \epsilon_j^k, \alpha_k - \alpha_{k-1} \rangle, \tag{6.89}$$

and we have inclusions $M_k \subset \tilde{M}_k$ and $\tilde{N}_k \subset N_k$.

Remark 6.90. The lattice homomorphism $\phi_k : M'_{k-1} \to M_k$ in (6.54) identifies M_k with the quotient lattice $M'_{k-1}/\mathrm{Ker}(\phi_k)$. Its dual lattice homomorphism,

$$\phi_k^* : N_k \longrightarrow N'_{k-1}$$

is injective and by (6.87) and (6.88) it verifies that $\phi_k^*(\epsilon_j^k) = (p_j^k \epsilon_j^{k-1}, q_j^k)$. The following lemma states that these are the primitive integral vectors of the cone ρ_k for the lattice N'_{k-1} (and also for the lattice $\tilde{N}_{k-1} \times \mathbb{Z}$).

Lemma 6.91. [**GG14**, Lemma 9.2]. *The primitive integral vectors at the edges of the cone ρ_k with respect to the lattice $N'_{k-1} = N_{k-1} \times \mathbb{Z}$ coincide with those with respect to the lattice $\tilde{N}_{k-1} \times \mathbb{Z}$ and are equal to:*

$$u_j^k = (p_j^k \epsilon_j^{k-1}, q_j^k) \ \text{ for } 1 \leq j \leq d. \tag{6.92}$$

Example 6.93. Let

$$\zeta = x_1^{\frac{3}{2}} x_2^{\frac{1}{2}} + x_1^{\frac{7}{4}} x_2^{\frac{1}{2}}$$

be as in Example 6.11, with characteristic exponents

$$\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right), \ \alpha_2 = \left(\frac{7}{4}, \frac{1}{2}\right).$$

By definition $p_1^1 = p_2^1 = 2$ and $q_1^1 = 3$, $q_2^1 = 1$. Now we have

$$\alpha_2 - \alpha_1 = \left(\frac{1}{4}, 0\right) = \frac{1}{2} \varepsilon_1^1.$$

Thus, $p_1^2 = 2$ and $q_1^2 = 1$, while $p_2^2 = 1$ and $q_2^2 = 0$, since its second component is trivial. There are three exceptional divisors E_1^1, E_2^1 corresponding to the first characteristic exponent and E_1^2 corresponding to the first component of the second characteristic exponent in the normalization of $H = V(f)$. Those divisors are associated to primitive integral vectors

$$u_1^1 = (2\epsilon_1^0, 3), u_2^2 = (2\epsilon_1^0, 1) \in N_0,$$
$$u_1^2 = (2\epsilon_1^1, 1) \in N_1.$$

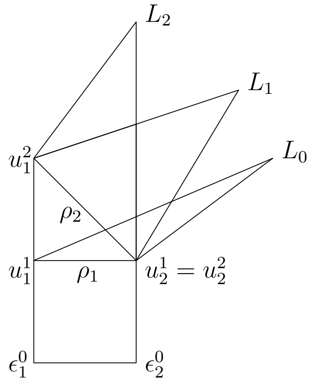

FIGURE 4. Projectivization of the conic polyhedral complex of the surface in Example 6.93 with the primitive integral vectors of the cones ρ_i.

EXAMPLE 6.94. The vector u_1^2 of Example 6.93 is of depth 2, while the vectors u_1^1 and $u_2^1 = u_1^1$ are of depth 1.

NOTATION 6.95. We denote by ν_j^k the divisorial valuation of $\mathbb{C}\{x_1, \ldots, x_d\}[y]$ associated with the primitive vector $u_j^k \in N'_{k-1}$ (see (6.92)), and by $D_j^{(k)} := D_{u_j^k}$ the corresponding divisor, which appears in Z_k (see Remark 6.67).

REMARK 6.96. Notice that we may have $u_j^k = u_j^{k+1}$ in the complex Θ and then $D_j^{(k)} = D_j^{(k+1)}$ (see Example 6.93).

6.6.2. Description of the divisorial valuations.

We have the following result, similar to the one on curves (see Theorem 4.70).

LEMMA 6.97. *We have the following formulas for $k = 0, \ldots, g-1$ and $j = 1, \ldots, d$:*

$$\nu_j^{k+1}(z_i) = \begin{cases} \nu_j^{k+1}(x_j) \cdot \gamma_{i+1,j} & \text{if } i \leq k, \\ n_{k+1} \cdots n_i \cdot \nu_j^{k+1}(x_j) \cdot \gamma_{k+1,j} & \text{if } i > k. \end{cases} \tag{6.98}$$

Furthermore, if $\beta = \sum_{r=1}^{d} \beta_r \varepsilon_r^0 \in M_0$ then

$$\nu_j^{k+1}(x^\beta) = p_j^{k+1} p_j^k \cdots p_j^1 \cdot \beta_j.$$

PROOF. Let $\beta \in M_0$ and let x^β be its associated monomial. Recall that $M_0 \subset M_{k+1}$ and that $u_j^{k+1} \in N'_{k+1}$. Thus, we have

$$\begin{aligned} \nu_j^{k+1}(x^\beta) &= \left\langle (p_j^{k+1} \epsilon_j^k, q_j^{k+1}), (\beta, 0) \right\rangle \\ &= \left\langle p_j^{k+1} \epsilon_j^k, \beta \right\rangle \\ &\overset{(6.86)}{=} p_j^{k+1} p_j^k \cdots p_j^1 \left\langle \epsilon_j^0, \beta \right\rangle \\ &= p_j^{k+1} p_j^k \cdots p_j^1 \cdot \beta_j. \end{aligned} \tag{6.99}$$

Thus, for $1 \leq l \leq d$, we get

$$\nu_j^{k+1}(x_l) = \left\langle \left(p_j^{k+1} \epsilon_j^k, q_j^{k+1} \right), \varepsilon_l^0 \right\rangle = \left\langle \epsilon_j^{k+1}, \varepsilon_l^0 \right\rangle = p_j^{k+1} \cdots p_j^1 \cdot \delta_{j,l}, \tag{6.100}$$

where $\delta_{j,l}$ denotes the Kronecker's delta.

Recall that $D_j^{(k+1)}$ is the divisor associated to the ray generated by the primitive integral vector $u_j^{k+1} = (p_j^{k+1}\epsilon_j^k, q_j^{k+1}) \in N'_{k+1}$ (Lemma 6.91). This ray belongs to the cone $\rho_{j+1} \in \Sigma_{j+1}$, which is the cone dual to the unique compact edge of $\mathcal{N}_j(z_i)$, for $i \geq k+1$ (see (6.76)) As a consequence, the valuation ν_j^{k+1} of z_i is achieved at any point of the this compact edge, in particular at the the leading term of z_i seen as a germ in $\mathbb{C}\{\varrho \cap M'_k\}$ (check Remark 6.53 and the proof of Lemma 6.83). We get from this and Lemma 6.57 that, if $i < k$,

$$\nu_j^{k+1}(z_i) = \left\langle \left(p_j^{k+1}\epsilon_j^k, q_j^{k+1}\right), (\gamma_{i+1}, 0) \right\rangle, \tag{6.101}$$

while for $i \geq k$,

$$\nu_j^{k+1}(z_i) = \left\langle \left(p_j^{k+1}\epsilon_j^k, q_j^{k+1}\right), \left(n_k \cdots n_i \gamma_k, \deg_{y_k}\left(z_i^{(k)}\right)\right) \right\rangle. \tag{6.102}$$

In order to prove the statement (6.98), we consider three different cases.

(1) Let us start with the case $i = k$:

$$\begin{aligned} \nu_j^{k+1}(z_k) &\overset{(6.102)}{=} \left\langle (p_j^{k+1}\epsilon_j^k, q_j^{k+1}), (n_k\gamma_k, 1) \right\rangle \\ &\overset{(6.88)}{=} \left\langle \epsilon_j^{k+1}, n_k\gamma_k + (\alpha_{k+1} - \alpha_k) \right\rangle \\ &\overset{(6.14)}{=} \left\langle \epsilon_j^{k+1}, \gamma_{k+1} \right\rangle \\ &\overset{(6.99)}{=} \nu_j^{k+1}(x_j) \left\langle \epsilon_j^0, \gamma_{k+1} \right\rangle. \end{aligned} \tag{6.103}$$

(2) Now we prove the result for $i > k$:

$$\begin{aligned} \nu_j^{k+1}(z_i) &\overset{(6.102)}{=} \left\langle u_j^{k+1}, (n_k \cdots n_i \gamma_k, n_{k+1} \cdots n_i) \right\rangle \\ &= n_{k+1} \cdots n_i \left\langle u_j^{k+1}, (n_k\gamma_k, 1) \right\rangle \\ &\overset{(6.103)}{=} n_{k+1} \cdots n_i \cdot \nu_j^{k+1}(z_k), \end{aligned}$$

Thus the result follows by the case (1).

(3) For the case $i < k$ we have

$$\begin{aligned} \nu_j^{k+1}(z_i) &\overset{(6.101)}{=} \left\langle \left(p_j^{k+1}\epsilon_j^k, q_j^{k+1}\right), (\gamma_{i+1}, 0) \right\rangle \\ &\overset{(6.88)}{=} \left\langle \epsilon_j^{k+1}, \gamma_{i+1} \right\rangle \\ &\overset{(6.86)}{=} p_j^{k+1} \cdots p_j^{i+2} \left\langle \epsilon_j^{i+1}, \gamma_{i+1} \right\rangle \\ &\overset{(6.103)}{=} p_j^{k+1} \cdots p_j^{i+2} \cdot \nu_j^{i+1}(z_i). \end{aligned}$$

Since by (6.99) we have $\nu_j^{k+1}(x_j)/\nu_j^{i+1}(x_j) = p_j^{k+1} \cdots p_j^{i+2}$, the result follows by the case (1).

□

NOTATION 6.104. Let us denote by

$$\nu_\bullet^{k+1}(h) := (\nu_1^{k+1}(h), \ldots, \nu_d^{k+1}(h))$$

By the previous lemma we can write:

$$\nu_{\bullet}^{k+1}(z_i) = (\nu_1^{k+1}(z_i), \dots, \nu_d^{k+1}(z_i)) \\ = n_0 \cdots n_i \cdot \frac{1}{n_0 \cdots n_{\min\{k,i\}}} \cdot \left(\nu_{\bullet}^{k+1}(x_1 \cdots x_d)\right) * \left(\gamma_{\min\{k+1, i+1\}}\right), \tag{6.105}$$

where the product $*$ is the componentwise product (see Example 6.106).

Example 6.106. Let η be the branch in Example 6.11, with characteristic exponents $\alpha_1 = (\frac{3}{2}, \frac{1}{2})$, $\alpha_2 = (\frac{7}{4}, \frac{1}{2})$. We computed in Example 6.16 the generators of the semigroup, $\gamma_1 = \alpha_1$, $\gamma_2 = (\frac{13}{4}, 1)$. Its conic polyhedral complex appears in Figure 4. Recall from Example 6.93 that $p_1^1 = p_2^1 = p_1^2 = 2$ and $p_2^2 = 1$. Let us denote by z_1 a 1-st semi-root of the quasi-ordinary branch and by ν_2^1 the divisorial valuation associated to the first component of the second characteristic exponent. Using Lemma 6.97 we compute

$$\nu_1^2(z_1) = 2 \cdot \frac{1}{2} \cdot 4 \cdot \tfrac{13}{4} = 13.$$

On the other hand, by (6.105) we have

$$\nu_{\bullet}^1(z_1) = 2 \cdot \frac{1}{1} \cdot (2, 2) * (\tfrac{3}{2}, \tfrac{1}{2}) = (6, 2).$$

Since $u_2^1 = u_2^2$, we have that $\nu_2^2(z_1) = \nu_2^1(z_1) = 2$ (see Remark 6.96 and Figure 4).

6.7. Log-discrepancies of toroidal exceptional divisors

In this section we show that there exists a piecewise continuous function $\lambda : |\Theta| \to \mathbb{R}_{\geq 0}$, which is linear when restricted to each $|\Sigma_j|$, and such that the log discrepancy of a toroidal divisor D_u is equal to $\lambda(u)$. We will use the notation of Section 6.6.1.

Notation 6.107. Let us denote by $\lambda_k = \alpha_k + \lambda_0 \in \rho \cap M_k$ and by $\lambda'_k = (\lambda_k, 1) \in \varrho \cap M'_k$. We call λ'_k the **log-discrepancy vector** of depth k of Θ.

Recall that a vector in the support of Θ is of the form $(v, a) = (v, a)^{(k)} \in \varrho \cap (N'_k)_{\mathbb{R}} = (\rho \cap (N_k)_{\mathbb{R}}) \times \mathbb{R}_{\geq 0}$ for some $0 \leq k \leq g$ (see Remark 6.60).

Definition 6.108. We set $\lambda_0 = \sum_{j=1}^d \varepsilon_j^0$ and $\alpha_0 = 0$. Notice that λ'_k is a linear form on $(N'_k)_{\mathbb{R}}$. We define the **log-discrepancy function of** Θ as, $\lambda : |\Theta| \to \mathbb{R}$,

$$\lambda(v, a) := \left\langle (v, a), \lambda'_k \right\rangle, \text{ if } (v, a) \in \varrho^{\vee} \cap (N'_k)_{\mathbb{R}}. \tag{6.109}$$

Lemma 6.110. *The function λ defined above is a non-negative continuous function, which is linear on the support of Σ_k, for $k = 1, \dots, g+1$ and takes integral values on integral vectors.*

Proof. Let us show that first that λ is well defined. By Definition 6.59 it is enough to show that the restrictions of λ to $\rho \cap N_k$ and $\rho_k \cap N'_{k-1}$ coincide when applied to vectors which are identified in the complex, for $k = 1, \dots, g$. Recall that a vector $(v, 0) \in (\rho \cap (N_k)_{\mathbb{R}}) \times \mathbb{R}$ is identified with $\phi_k^*(v) \in \rho_k \cap (N'_{k-1})_{\mathbb{R}}$ (see Definition 6.59). One has to prove that for any vector $(v, 0) \in (\rho \cap (N_k)_{\mathbb{R}}) \times \mathbb{R}$ the following equality holds

$$\langle (v, 0), (\alpha_k + \lambda_0, 1) \rangle = \langle \phi_k^*(v), (\alpha_{k-1} + \lambda_0, 1) \rangle \tag{6.111}$$

By linearity it is enough to prove this when $v = \epsilon_j^k$, for $j = 1, \dots, d$ since $\{\epsilon_j^k\}_{j=1}^d$ is a basis of $(N_k)_\mathbb{R}$. The identity (6.111) holds for $v = \epsilon_j^k$, for $j = 1, \dots, d$ since:

$$\begin{aligned}\left\langle \phi_k^*(\epsilon_j^k), (\alpha_{k-1} + \lambda_0, 1) \right\rangle &\overset{(6.90)}{=} \left\langle (p_j^k \epsilon_j^{k-1}, q_j^k), (\alpha_{k-1} + \lambda_0, 1) \right\rangle \\ &\overset{(6.86)}{=} \left\langle p_j^k \epsilon_j^{k-1}, \alpha_{k-1} + \lambda_0 \right\rangle + q_j^k \\ &\overset{(6.89)}{=} \left\langle \epsilon_j^k, \alpha_{k-1} + (\alpha_k - \alpha_{k-1}) + \lambda_0 \right\rangle \\ &= \left\langle \epsilon_j^k, \alpha_k + \lambda_0 \right\rangle = \left\langle (\epsilon_j^k, 0), (\alpha_k + \lambda_0, 1) \right\rangle .\end{aligned}$$

It follows that λ is well-defined, and linear on the support of each fan Σ_k. In particular, it is continuous. Since $(\alpha_k + \lambda_0, 1) \in \varrho^\vee \cap M'_k$, it follows that $\lambda(v, a) = \langle (v, a), (\alpha_k + \lambda_0, 1) \rangle \geq 0$ for every vector $(v, a) \in \varrho \cap (N'_k)_\mathbb{R}$. In addition, if $(v, a) \in \varrho \cap N'_k$ then $\lambda(v, a)$ is an integer since $M'_k = \mathrm{Hom}_\mathbb{Z}(N'_k, \mathbb{Z})$ is the lattice dual to N'_k. □

The following result can be seen as a partial multidimensional generalization of a result of Favre and Jonsson, [**FJ04**, Proposition D.1], see also [**GGP18**, Theorem 8.18 and Proposition 8.26] and [**Jon15**, Section7].

Proposition 6.112. *Let us consider a primitive integral vector $(v, a) = (v, a)^{(k)} \in \varrho \cap N'_k$ in the support of Θ. We denote by $D_{(v,a)}^{(k)}$ the associated toroidal divisor. Then, the log-discrepancy of $D_{(v,a)}^{(k)}$ is equal to $\lambda(v, a) = \langle (v, a), \lambda'_k \rangle$.*

Proof. We consider first the case $k = 0$. Let $(v, a) \in \varrho \cap N'_0$ be an integral vector, and let Σ'_1 be a regular subdivision of Σ_1 containing the ray $\mathbb{R}_{\geq 0}(v, a)$. The vector of log-discrepancies of $\mathbb{C}^{d+1}$ is $\lambda'_0 = (1, \dots, 1)$ (see Remark 3.3). By Remark 3.6 we have that the log-discrepancy of $D_{(v,a)}^{(0)}$ is equal to

$$\langle (v, a), \lambda'_0 \rangle = \lambda(v, a).$$

We assume by induction that the statement is true for vectors $(v, a) \in \varrho \cap N'_{k-1}$, that is, the log-discrepancy of the toroidal divisor $D_{(v,a)}^{(k-1)}$ is $\lambda(v, a)$.

Assume that $(v, a) \in \varrho \cap N'_k$. It is easy to see that there exists a d-dimensional cone $\sigma \subset \rho$, which is regular for the lattice N_k, such that $(v, a) \in \sigma' := \sigma \times \mathbb{R}_{\geq 0}$. If $v_1, \dots, v_d$ is the base of N_k which spans the cone σ then $v'_1 := (v_1, 0), \dots, v'_d = (v_d, 0), v'_{d+1} := (0, 1)$ is the base of N'_k which spans the cone σ' (see Figure 5).

Let us denote by ℓ_j the log-discrepancy of $D_{(v_j,0)}^{(k)}$, for $j = 1, \dots, d$. Notice that by construction of the toroidal embedded resolution, the divisor $D_{(0,1)} \subset Z_{\sigma', N'_k}$ is equal to the strict transform of the k-th semi-root, hence its log-discrepancy is equal to one by definition.

By definition of Θ, we identify the vector $\phi_k^*(v_j) \in \varrho \cap (N'_{k-1})_\mathbb{R}$ with $(v_j, 0) \in \varrho \cap (N'_k)_\mathbb{R}$ for $j = 1, \dots, d$ (see Remark 6.60). The induction hypothesis implies that

$$\ell_j = \langle \phi_k^*(v_j), \lambda'_{k-1} \rangle = \langle (v_j, 0), \lambda'_k \rangle ,$$

where the last equality comes from identity (6.111). Let us denote by $u'_1, \dots, u'_d, u'_{d+1} \subset M'_k$ the dual basis of $v'_1, \dots, v'_d, v'_{d+1}$. The above reasoning shows that

$$\ell_1 u'_1 + \dots \ell_d u'_d + u'_{d+1}$$

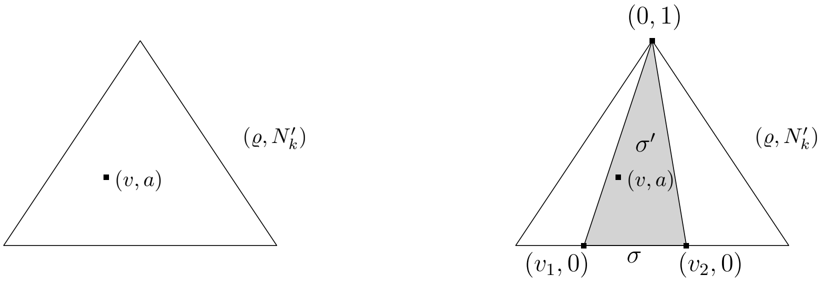

FIGURE 5. Given $(v,a) \in \varrho \cap N'_k$, one can find a regular cone $\sigma \subset (N_k)_{\mathbb{R}}$ such that $(v,a) \in \sigma' = \sigma \times \mathbb{R}_{\geq 0}$.

is the vector of log-discrepancies on Z_{σ', N'_k}. By Remark 3.7 this implies that the log-discrepancy of $D^{(k)}_{(v,a)}$ is equal to

$$\lambda_{D^{(k)}_{(v,a)}} = \langle (v,a), \ell_1 u'_1 + \ldots \ell_d u'_d + u'_{d+1} \rangle. \tag{6.113}$$

By definition $\ell_i = \langle (v_i, 0), \lambda'_k \rangle$ for $i = 1, \ldots, d$. Hence,

$$\lambda'_k = \ell_1 u'_1 + \ldots \ell_d u'_d + u'_{d+1}, \tag{6.114}$$

is the expression of λ'_k in the basis $u'_1, \ldots, u'_{d+1}$ of M'_k. It follows that

$$\begin{aligned} \lambda(v,a) &= \left\langle (v,a), \lambda'_k \right\rangle \\ &\overset{(6.114)}{=} \left\langle (v,a), \ell_1 u'_1 + \ldots \ell_d u'_d + u'_{d+1} \right\rangle \\ &\overset{(6.113)}{=} \lambda_{D^{(k)}_{(v,a)}}. \end{aligned}$$

This completes the proof by induction. □

REMARK 6.115. In [**GG14**, Proposition 7.2] the authors compute the order of the jacobian of compositions of toroidal modifications in the partial embedded resolution, which leads to a different approach to the computation of log-discrepancies.

6.7.1. Computation of log-discrepancies of divisors associated to interior primitive integral vectors of Θ. Let u_j^{k+1} be a ray generator of ρ_{k+1} (Lemma 6.91). Recall that we denote by $D_j^{(k+1)}$ its associated relevant toroidal divisor, by ν_j^{k+1} its divisorial valuation and by $\lambda_j^{k+1} = \lambda_{D_j^{(k+1)}}$ its log-discrepancy.

LEMMA 6.116. *The log-discrepancy λ_j^{k+1} of the divisor $D_j^{(k+1)}$ can be written in terms of the invariants of the branch as*

$$\lambda_j^{k+1} = \nu_j^{k+1}(x_j) \cdot ((\alpha_{k+1})_j + 1).$$

PROOF. By the reasoning in the previous section, we can compute its log-discrepancy by

$$\begin{aligned} \lambda_j^{k+1} &\overset{(6.112)}{=} \langle u_j^{k+1}, \lambda'_k \rangle = \langle u_j^{k+1}, \lambda_{k+1} \rangle \\ &\overset{(6.91)}{=} \langle \epsilon_j^{k+1}, \alpha_{k+1} + \lambda_0 \rangle \\ &\overset{(6.86)}{=} p_j^{k+1} \cdots p_j^1 \cdot \left\langle \epsilon_j^0, \alpha_{k+1} + \lambda_0 \right\rangle. \end{aligned}$$

Since by Lemma 6.97 $\nu_j^{k+1}(x_j) = \nu_j^{k+1}(x^{\varepsilon_j^0}) = p_j^{k+1} \cdots p_j^1$, we can write

$$\lambda_j^{k+1} = \nu_j^{k+1}(x_j) \cdot ((\alpha_{k+1})_j + 1)\,, \tag{6.117}$$

where $(\alpha_{k+1})_j$ is the j-th component of the $(k+1)$-th characteristic exponent seen as a rational vector in M_0 (check Proposition 5.1 for the curve case). □

NOTATION 6.118. Let us denote by

$$\lambda_\bullet^{k+1} := (\lambda_1^{k+1}, \ldots, \lambda_d^{k+1}),$$

and by

$$\nu_\bullet^{k+1}(x_1 \cdots x_d) = \left(\nu_j^{k+1}(x_1), \ldots, \nu_d^{k+1}(x_d)\right).$$

We can pack the divisors corresponding to the ray generators of ρ_k to compute vectorially their log-discrepancies

$$\lambda_\bullet^{k+1} = \left(\nu_\bullet^{k+1}(x^{\lambda_0})\right) * (\alpha_{k+1} + \lambda_0)\,, \tag{6.119}$$

where $\lambda_0 = \sum_{j=1}^d \varepsilon_j^0$ and the product $*$ is the componentwise product (see Example 6.120 below).

EXAMPLE 6.120. Let η be the branch in Example 6.11, with characteristic exponents $\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right)$ and $\alpha_2 = \left(\frac{7}{4}, \frac{1}{2}\right)$. Recall from Example 6.93 that $p_1^1 = p_2^1 = p_1^2 = 2$ and $p_2^2 = 1$. Let us denote by E_1^2 the divisor associated to the first component of the second characteristic exponent. By Lemma 6.116, we compute

$$\lambda_1^2 = 2 \cdot 2 \cdot \left(\frac{7}{4} + 1\right) = 11.$$

On the other hand, using (6.119), we have

$$\lambda_\bullet^1 = (2,2) * \left(\frac{3}{2} + 1, \frac{1}{2} + 1\right) = (5,3).$$

6.8. Multiplier Ideals of an irreducible quasi-ordinary germ

In this section we study the multiplier ideals associated with a quasi-ordinary hypersurface germ. We show that the set of conditions in the divisorial definition of multiplier ideals is equivalent to a set of conditions associated to primitive integral vectors in the conic polyhedral complex Θ. Furthermore, using the results about monomialization of valuations in Section 6.5 we show that multiplier ideals are monomial ideals with respect to a complete sequence of semi-roots for the hypersurface H and the variables $x_1, \ldots, x_d$. As a consequence, we obtain a combinatorial characterization of the jumping numbers.

We keep the notation introduced in the previous sections.

NOTATION 6.121. We will denote by Θ° the set of *relevant rays of* Θ (Definition 6.65) and by $\Theta^\circ_{\mathrm{prim}}$ the corresponding set of *primitive integral generators* of relevant rays of the complex Θ (Notation 6.33). For any $v \in \Theta_{\mathrm{prim}}$ primitive integral vector on the support of Θ we denote by D_v its associated *toroidal divisor* (Definition 6.63), by ν_v its *divisorial valuation* (Notation 6.64 and Definition 6.63) and by λ_v the *log-discrepancy* of D_v (Section 6.7).

The following result generalizes Theorem 5.8 to irreducible quasi-ordinary hypersurfaces.

THEOREM 6.122. *Let $0 < \xi < 1$ be a rational number. Denote by (H, O) an irreducible germ of quasi-ordinary hypersurface. Then, we have the following description of the multiplier ideal $\mathcal{J}(\xi H)$.*

$$\mathcal{J}(\xi H) = \left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \nu_v(h) \geq \lfloor \xi \nu_v(H) \rfloor - (\lambda_v - 1),\ \textit{for all } v \in \Theta^{\circ}_{\text{prim}}\right\}. \tag{6.123}$$

PROOF. Recall from Theorem 6.61 that the composition of the toric modifications, Ψ_g, gives an embedded normalization of the quasi-ordinary hypersurface $(H, O) \subset (\mathbb{C}^{d+1}, O)$ in a toroidal embedding $\mathbf{X}_\Theta$. Furthermore, an embedded resolution Ψ of (H, O) is obtained by the composition of Ψ_g with the modification $\Psi_{\Theta'}$ associated to a regular subdivision Θ' of the conic polyhedral complex $\Theta(H)$.

Now we use the valuative description of the multiplier ideal in Formula 3.20 and the local description of the conic polyhedral complex Θ' by regular fans Σ_j^{reg} subdividing the fans Σ_j, gives

$$\begin{aligned}\mathcal{J}(\xi H) &= \left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} |\ \nu_v(h) + \lambda_v > \xi \nu_v(H) \text{ for all } v \in \Theta'^{\text{prim}}\right\}\\ &= \bigcap_{1 \leq j \leq g+1} \left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} |\ \nu_v(h) + \lambda_v > \xi \nu_v(H) \text{ for all } v \in \Sigma_j^{\text{reg,prim}}\right\}.\end{aligned}$$

Let us fix j, and for the regular fan Σ_j^{reg} we consider a primitive integral vector v. We consider the valuation ν_v associated to it. By Lemma 6.112, there exists a vector $\lambda'_j \in M'_j$ such that $\lambda_v = \langle v, \lambda'_j \rangle$. On the other hand, by Lemma 1.22 we have that $\nu_v(h) = \Phi_{\mathcal{N}_j(h)}(v)$, where $\mathcal{N}_j(h)$ is the Newton polyhedron of the germ h at level j (Definition 6.56). We obtain that

$$\Phi_{\mathcal{N}_j(h)}(v) + \langle v, \lambda'_j \rangle = \Phi_{\mathcal{N}_j(h) + \lambda'_j}(v),$$

and also

$$\Phi_{\xi \mathcal{N}_j(H)} = \xi \Phi_{\mathcal{N}_j(h)}.$$

Thus, the set of conditions on $v \in \Sigma_j^{\text{reg,prim}}$ correspond to a regular subdivision of the fan Σ_j dual to the polyhedron $\mathcal{N}_j(H)$. Using Corollary 1.28 we have

$$\begin{aligned}&\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \nu_v(h) + \lambda_v > \xi \nu_v(H) \text{ for all } v \in \Sigma_j^{\text{reg,prim}}\right\} =\\ &\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \Phi_{\mathcal{N}_j(h)+\lambda'_j}(v) > \Phi_{\xi \mathcal{N}_j(H)}(v) \text{ for all } v \in \Sigma_j^{\text{reg,prim}}\right\} =\\ &\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \mathcal{N}_j(h) + \lambda'_j \subset \text{Int}(\xi \mathcal{N}_j(H))\right\}.\end{aligned} \tag{6.124}$$

On the other hand, the conditions in the dual fan, which correspond to primitive integral vectors defining rays of Σ_j, characterize the Newton polyhedron of H. Thus, applying Formula (1.26) and Corollary 1.28 we obtain

$$\begin{aligned}&\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \mathcal{N}_j(h) + \lambda'_j \subset \text{Int}(\xi \mathcal{N}_j(H))\right\} =\\ &\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid\ \nu_v(h) + \lambda_v > \xi \nu_v(H) \text{ for all } v \in \Sigma_j^{\text{reg,prim}}\right\}.\end{aligned} \tag{6.125}$$

Now, let $\mathbb{R}_{\geq 0} v$ be an end ray of Θ (Definition 6.65) and L be the corresponding branch of the completion of H (Remark 6.67). Since the strict transform $\tilde{L}$ is not contained in the exceptional locus of the resolution Ψ, we have that its log-discrepancy is equal to one, $\lambda_L = 1$. Thus, the condition on the order of vanishing over L is fulfilled for any holomorphic germ $h \in \mathbb{C}\{x_1, \ldots, x_d, y\}$, that is,

$$\nu_L(h) + \lambda_L > \nu_L(h) \geq 0.$$

It follows that the conditions in the end rays of Θ are redundant and we get the equality:

$$\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \ \nu_v(h) + \lambda_v > \xi\nu_v(H) \text{ for all } v \in \Sigma_j^{\text{reg,prim}}\right\} =$$
$$\left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} \mid \ \nu_v(h) + \lambda_v > \xi\nu_v(H) \text{ for all } v \in \Sigma_j^{\text{prim}} \cap \Theta^\circ_{\text{prim}}\right\}.$$

The previous reasoning shows that the conditions associated to primitive integral vectors generating the relevant rays of Θ are sufficient to describe the multiplier ideals, i.e.,

$$\mathcal{J}(\xi H) = \left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} | \ \nu_v(h) + \lambda_v > \xi\nu_v(H), \text{ for all } v \in \Theta^\circ_{\text{prim}}\right\}. \tag{6.126}$$

We have shown (6.126), which is a reformulation of the equality (6.123) in the statement of the theorem by (3.18). □

We obtain from equation (6.124) a version of Howald's result (Proposition 3.31) in terms of Newton polyhedra for quasi-ordinary branches (see Corollary 5.13 for the plane curve analogue). We use Notation 6.107.

COROLLARY 6.127.

$$\mathcal{J}(\xi H) = \left\{h \in \mathcal{O}_{\mathbb{C}^{d+1},O} | \ \mathcal{N}_j(h) + \lambda'_j \subset \text{Int}\left(\xi \mathcal{N}_j(H)\right) \ \textit{for all } 1 \leq j \leq g\right\}. \tag{6.128}$$

The following result states that we can find a monomial basis for the multiplier ideals of a quasi-ordinary branch in terms of the generalized monomials in the semi-roots (compare with Theorem 5.16 in the plane curve case).

THEOREM 6.129. *The multipliers ideals $\mathcal{J}(\xi H)$ of an irreducible germ of quasi-ordinary hypersurface H have a finite basis consisting monomials in $x_1, \dots, x_d, z_0, \dots, z_g$.*

PROOF. Let $h \in \mathcal{O}_{\mathbb{C}^{d+1},O}$ be a germ such that $h \in \mathcal{J}(\xi H)$. By Theorem 6.122 one has

$$\nu_v(h) > -\lambda_v + \xi\nu_v(H) \text{ for all } v \in \Theta^\circ_{\text{prim}}.$$

We consider an expansion in terms of the semi-roots as in (6.26),

$$h = \sum a_I z_0^{i_0} \dots z_g^{i_g}. \tag{6.130}$$

By Lemma 6.83 every non-zero term $\mathcal{M} = a_I z_0^{i_0} \dots z_g^{i_g}$ verifies that $\nu_v(\mathcal{M}) \geq \nu_v(h)$. Thus,

$$\nu_v(\mathcal{M}) \geq \nu_v(h) > -\lambda_v + \xi\, \nu_v(H), \text{ for all } v \in \Theta^\circ_{\text{prim}}.$$

By Theorem 6.122 that means $\mathcal{M} \in \mathcal{J}(\xi H)$. This implies that

$$\mathcal{J}(\xi H) = \left\langle \mathcal{M} \,\middle|\, \nu_v(\mathcal{M}) + \lambda_v > \xi\nu_v \text{ for all } v \in \Theta^\circ_{\text{prim}} \right\rangle. \tag{6.131}$$

□

REMARK 6.132. Let $H = V(f)$ be as in Theorem 6.129. Its multiplier ideals are monomial since, by Lemma 3.26, we have $\mathcal{J}\left((1+\xi)H\right) = (f)\mathcal{J}(\xi H)$.

As a consequence of the previous theorem, we obtain that the jumping numbers are associated to generalized monomials in the semi-roots determined by the divisorial valuations corresponding to relevant rays of the complex Θ (see Definition 6.65, and compare this with 5.22).

Corollary 6.133. *Each jumping number of the plane curve C corresponds to at least one monomial $\mathcal{M}$ in $x_1, \dots, x_d, z_0, \dots, z_g$. Namely, we have that every jumping number is of the form:*

$$\xi_{\mathcal{M}} = \min_{v \in \Theta^{\circ}_{\text{prim}}} \left\{ \frac{\nu_v(\mathcal{M}) + \lambda_v}{\nu_v(H)} \right\}. \tag{6.134}$$

Proof. Theorem 6.129 shows that multiplier ideals are monomially generated by generalized monomials in the semi-roots of f (Lemma 6.25). Obviously, a monomial stops to belong to a multiplier ideal whenever one of the conditions $\nu_v(\mathcal{M}) + \lambda_v > \xi\nu_v(H)$, is not satisfied. This implies that the monomial $\mathcal{M} \notin \mathcal{J}(\xi C)$ whenever

$$\xi \geq \min_{v \in \Theta^{\circ}_{\text{prim}}} \left\{ \frac{\nu_v(\mathcal{M}) + \lambda_v}{\nu_v(H)} \right\}. \tag{6.135}$$

The smallest ξ verifying this condition is $\xi_{\mathcal{M}}$. Since the multiplier ideal are generated by these kind of monomials every jumping number must be of this form. □

Using the notation in (6.95) and Remark 6.67, Formula 6.134 can be rewritten as

$$\xi_{\mathcal{M}} = \min_{\substack{1 \leq k \leq g \\ 1 \leq j \leq d}} \left\{ \frac{\nu^k_j(\mathcal{M}) + \lambda^k_j}{\nu^k_j(H)} \right\}.$$

As an application we include a proof of the formula for the log-canonical threshold of a quasi-ordinary hypersurface (see Corollary 6.140 and [**BGG12**, Theorem 3.1]). This proof generalizes Corollary 5.26 by using similar methods to those in Section 5.4. We make use of Corollary 6.133, the toroidal descriptions of log-discrepancies (Section 6.7) and the description of relevant valuations (Section 6.6) to describe the log-canonical threshold of a quasi-ordinary branch. We introduce first some notation and a lemma.

Notation 6.136. Let $\{\alpha_1, \dots, \alpha_g\}$ be the characteristic exponents of the hypersurface H and let us fix a generalized monomial $\mathcal{M} = x^m z_0^{i_0} \dots z_g^{i_g}$ We denote by l_i the number of non-zero components of the i-th characteristic exponent as a vectr $\alpha_i \in \mathbb{Q}_{\geq 0}$. For $j \in \{1, \dots, d\}$, $k \in \{1, \dots, g\}$ we denote by

$$\mathtt{C}^k_j(\mathcal{M}) = \frac{\nu^k_j(\mathcal{M}) + \lambda^k_j}{\nu^k_j(H)}$$

the *jumping number condition* of the monomial $\mathcal{M}$ on the divisor associated to $u^k_j \in \Theta^{\circ}_{\text{prim}}$. Now let us fix $l \in \{1, \dots, g\}$, we set

$$\mathcal{M}_{|<l} = x^m z_0^{i_0} \dots z_l^{i_l}.$$

The proof of the following lemma is similar to the curve case in Lemma 5.64.

Lemma 6.137. *Let us fix a generalized monomial*

$$\mathcal{M} = x^m z_0^{l_0} \dots z_g^{l_g},$$

for $m \in \mathbb{Z}^d_{\geq 0}$, let $j \in \{1, \dots, d\}$ be a component and let $\sim \in \{<, >, =\}$ be a relation. Then,

$$\mathtt{C}^k_j(\mathcal{M}) \sim \mathtt{C}^{k+1}_j(\mathcal{M}) \;\Leftrightarrow\; \nu^k_j(\mathcal{M}_{|<k}) + \lambda^k_j \sim \nu^k_j(z_k),$$

for any $1 \leq k < g$.

Furthermore, we have the implications

$$\mathtt{C}_j^k(\mathcal{M}) \leq \mathtt{C}_j^{k+1}(\mathcal{M}) \Rightarrow \mathtt{C}_j^{k+1}(\mathcal{M}) \leq \mathtt{C}_j^{k+2}(\mathcal{M}),$$
$$\mathtt{C}_j^k(\mathcal{M}) \geq \mathtt{C}_j^{k+1}(\mathcal{M}) \Rightarrow \mathtt{C}_j^{k-1}(\mathcal{M}) \geq \mathtt{C}_j^k(\mathcal{M}).$$

REMARK 6.138. With notation as in Lemma 6.137,

$$\begin{aligned} & \mathtt{C}_j^k(\mathcal{M}) \geq \mathtt{C}_j^{k+1}(\mathcal{M}) && \text{for all } j \in \{1, \ldots, d\} \\ & \equiv \nu_j^k(\mathcal{M}_{|<k}) + \lambda_j^k \geq \nu_j^k(z_k) && \text{for all } j \in \{1, \ldots, d\} \\ & \equiv \mathcal{N}_k(\mathcal{M}_{|<k}) + \lambda'_k \subseteq \mathcal{N}_k(z_k)\ . \end{aligned} \tag{6.139}$$

The following corollary describes the log-canonical threshold of a quasi-ordinary branch (note the similarity with 5.26).

COROLLARY 6.140. [**BGG12**, Theorem 3.1]. *If H is defined by a normalized branch, its log-canonical threshold of H satisfies:*

$$\operatorname{lct}(f) = \begin{cases} \min\{1, \mathtt{C}_1^1(1)\} & \textit{if } \alpha_{1,1} \neq \frac{1}{n_1}, \textit{ or } g = 1, \\ \min\{\mathtt{C}_1^2(1), \mathtt{C}_{l_1+1}^2(1)\} & \textit{if } \alpha_{1,1} = \frac{1}{n_1}, g\textit{=1 and } l_1 < l_2, \\ \mathtt{C}_1^2(1) & \textit{if } \alpha_{1,1} = \frac{1}{n_1}, \ g\textit{=1 and } l_1 = l_2. \end{cases}$$

PROOF. Comparing conditions $\mathtt{C}_j^1(1)$ for $1 \leq j \leq d$ we obtain

$$\frac{\lambda_1^1}{e_0 q_1^1} < \frac{\lambda_j^1}{e_0 q_j^1} \Leftrightarrow 1 + \frac{1}{\alpha_{1,j}} > 1 + \frac{1}{\alpha_{1,1}},$$

which is satisfied due to the normalization condition on the branch, $\alpha_{1,1} \geq \alpha_{1,j}$.

By Lemma 6.137, $\mathtt{C}_1^1(1) < \mathtt{C}_1^2(1)$ if and only if $\lambda_1^1 < \nu_1^1(z_1)$, which is equivalent to

$$1 + \frac{1}{\alpha_{1,1}} < n_1.$$

We deduce that this is only possible when $\alpha_{1,1} = \frac{1}{n_1}$ and if so

$$\begin{cases} \alpha_{1,j} = \frac{1}{n_1} & \text{for } j \leq l_1, \\ \alpha_{1,j} = 0 & \text{for } j > l_1. \end{cases}$$

This implies in particular that the characteristic exponents of the strict transform of f, $\alpha_{2,j} - \alpha_{1,j}$, verify the normalization condition (Definition 6.10),

$$(\alpha_{2,1} - \alpha_{1,1}) \geq \ldots \geq (\alpha_{2,d} - \alpha_{1,d})\,. \tag{6.141}$$

By Lemma 6.137, comparing $\mathtt{C}_j^2(1) < \mathtt{C}_j^3(1)$ is equivalent to

$$\alpha_{2,j} + 1 < n_1 \cdot n_2 \cdot \left(\alpha_{1,j} + \tfrac{\alpha_{2,1} - \alpha_{1,1}}{n_1}\right) = n_2 \cdot \left(\alpha_{2,j} + \frac{n_1 - 1}{n_1}\right),$$

which is verified since $\alpha_{1,j} < \alpha_{2,j}$ and $n_2 > 1$.

It remains to show is that $C_1^2 \leq C_j^2$ for $j \leq l_1$. By Lemma 6.137, this assertion is equivalent to

$$\frac{\alpha_{2,1} + 1}{n_2 \gamma_{2,1}} < \frac{\alpha_{2,j}}{n_2 \gamma_{2,j}},$$

which is also equivalent to

$$(\alpha_{2,j} + 1) \cdot (n_1 \alpha_{2,j} + (\alpha_{2,j} - \alpha_{1,j})) < (\alpha_{2,j} + 1) \cdot (n_1 \alpha_{2,1} + (\alpha_{2,1} - \alpha_{1,1}))\,.$$

Simplifying, this reduces to

$$n_1\alpha_{2,j} + (\alpha_{2,j} - \alpha_{1,j}) < n_1\alpha_{2,1} + (\alpha_{2,1} - \alpha_{1,1}),$$

which holds true by (6.141) and by the fact that H is defined by a normalized branch. □

6.9. Examples

The methods described in this chapter give an algorithm for computing the jumping numbers of a quasi-ordinary branch. Corollary 6.133 asserts that the jumping numbers are determined by the conditions on the generalized monomials (6.72)

(6.142) $$\mathcal{M} = x^m z_0^{i_0} \dots z_{g-1}^{i_{g-1}} z_g^{i_g},$$

over the relevant exceptional divisors D_v for $v \in \Theta^\circ_{\text{prim}}$ (Remark 6.67), i.e.,

$$\xi_{\mathcal{M}} = \min_{\substack{1\le k\le g\\ 1\le l\le d}} \left\{ \frac{\nu_j^k(\mathcal{M}) + \lambda_j^k}{\nu_j^k(H)} \right\}.$$

One can compute the orders of vanishing of the semi-roots in terms of the invariants of the branch by applying Lemma 6.97 and the log-discrepancies using Lemma 6.116. In this section we compute the jumping numbers of Examples 6.11 and 6.12.

Example 6.143. Let L_2 be the irreducible quasi-ordinary surface of Example 6.11, which has characteristic exponents

$$\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right), \ \alpha_2 = \left(\frac{7}{4}, \frac{1}{2}\right),$$

and with characteristic integers $n_1 = 2$, $n_2 = 4$. The generators of the semigroup were computed in Example 6.16,

$$\gamma_1 = \left(\frac{3}{2}, \frac{1}{2}\right), \ \gamma_2 = \left(\frac{13}{4}, 1\right).$$

The denominators of the rational generators are $p_1^1 = p_2^1 = p_1^2 = 2$, $p_2^2 = 1$ as shown in Example 6.93. The set $\{y = z_0, z_1, f = z_2\}$ is a complete sequence of semi-roots. Let us denote by $L_i = V(z_i)$. The conic polyhedral complex of the surface is represented in Figure 6.

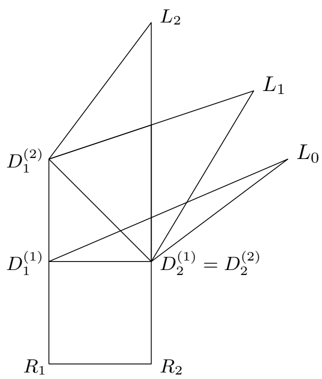

Figure 6. Projectivization of the conic polyhedral complex of the surface in Example 6.143.

	x_1	x_2	y	z_1	f	λ
$D_1^{(1)}$	2	0	3	6	24	5
$D_2^{(1)}$	0	2	1	2	8	3
$D_1^{(2)}$	4	0	6	13	52	11

TABLE 1. Set of valuations of semi-roots for Example 6.143.

In Figure 7 we show a regular subdivision of the fan Σ_1 dual to the Newton polyhedron of f. Theorem 6.122 ensures that the multiplier ideals of L_2 on this level only depend on the divisors $D_1^{(1)}$ associated to the primitive integral vector $(2, 0, 3)$ and $D_2^{(1)}$ associated to $(0, 2, 1)$.

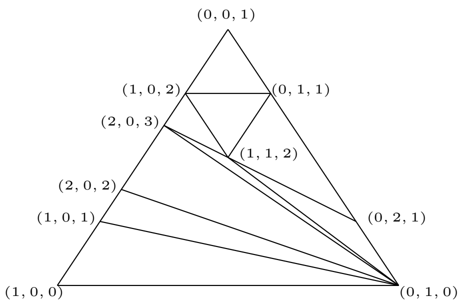

FIGURE 7. A regular refinement of Σ_1, the fan of the first toroidal modification in the normalization of the hypersurface in Example 6.143.

In Table 1, we give the set of valuations of the semi-roots with respect to the relevant exceptional divisors, which one can compute as in Example 6.106, and the log-discrepancy of of each divisor, as in Example 6.120. Notice that there is only three relevant exceptional divisors, since the second coordinate of α_1 and of α_2 coincide, i.e., $D_2^{(2)} = D_2^{(1)}$.

The set of jumping numbers smaller than one of L_2 is given in Table 2, together with the monomial which determines them.

EXAMPLE 6.144. Let L_2 be the quasi-ordinary surface of Example 6.12, which has characteristic exponents

$$\alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right), \alpha_2 = \left(\frac{7}{4}, \frac{2}{3}\right),$$

and whose characteristic integers are $n_1 = 2$ and $n_2 = 6$.

We compute the generators of the semigroup (6.14),

$$\gamma_1 = \alpha_1 = \left(\frac{3}{2}, \frac{1}{2}\right), \gamma_2 = \left(\frac{13}{4}, \frac{7}{6}\right).$$

The set $\{x_1, x_2, y = z_0, z_1, f = z_2\}$ is a complete sequence of semi-roots for the surface L_2. In Figure 8 we depict the projectivization of the conic polyhedral complex of L_2. The denominators appearing in the rational generators of L_2 are $p_1^1 = p_1^2 = p_2^1 = 2$ and $p_2^2 = 3$.

Using Lemma 6.97, we can compute the value of the semi-roots with respect to the relevant exceptional divisors and Equation 6.117 allows us to compute their log-discrepancies. The list of values of the semi-roots with respect to the four relevant exceptional divisors (rays of the conic

$\mathcal{M}$	1	x_1	y	x_1^2	x_1^3	$x_1 y$	$x_1^3 x_2$	z_1	$x_1^2 y$	$x_1^3 y$
$\xi_{\mathcal{M}}$	$\frac{5}{24}$	$\frac{15}{52}$	$\frac{8}{24} = \frac{17}{52}$	$\frac{19}{52}$	$\frac{3}{8}$	$\frac{21}{52}$	$\frac{23}{52}$	$\frac{11}{24}$	$\frac{25}{52}$	$\frac{4}{8}$

$\mathcal{M}$	$x_1^4 x_2$	$x_1 z_1$	$x_1^3 x_2 y$	$y z_1$	$x_1^5 x_2$	$x_1^2 z_1$	$x_1^3 z_1$	$x_1^4 x_2 y$	$x_1 y z_1$	$x_1^6 x_2^2$
$\xi_{\mathcal{M}}$	$\frac{27}{52}$	$\frac{28}{52}$	$\frac{29}{52}$	$\frac{30}{52}$	$\frac{31}{52}$	$\frac{32}{52}$	$\frac{5}{8}$	$\frac{33}{52}$	$\frac{34}{52}$	$\frac{35}{52}$

$\mathcal{M}$	$x_1^3 x_2 z_1$	z_1^2	$x_1^5 x_2 y$	$x_1^2 y z_1$	$x_1^3 y z_1$	$x_1^4 x_2 z_1$	$x_1 z_1^2$	$x_1^3 x_2 y z_1$	$y z_1^2$	$x_1^5 x_2 z_1$
$\xi_{\mathcal{M}}$	$\frac{36}{52}$	$\frac{17}{24}$	$\frac{37}{52}$	$\frac{38}{52}$	$\frac{6}{8} = \frac{39}{52}$	$\frac{40}{52}$	$\frac{41}{52}$	$\frac{42}{52}$	$\frac{43}{52}$	$\frac{44}{52}$

$\mathcal{M}$	$x_1^2 z_1^2$	$x_1^3 z_1^2$	$x_1^4 x_2 y z_1$	$x_1 y z_1^2$	$x_1^6 x_2^2 z_1$	$x_1^3 x_2 z_1^2$	z_1^3	$x_1^5 x_2 y z_1$	$x_1^2 y z_1^2$
$\xi_{\mathcal{M}}$	$\frac{45}{52}$	$\frac{7}{8}$	$\frac{46}{52}$	$\frac{47}{52}$	$\frac{48}{52}$	$\frac{49}{52}$	$\frac{23}{24}$	$\frac{50}{52}$	$\frac{51}{52}$

TABLE 2. Set of jumping numbers smaller than one in Example 6.143.

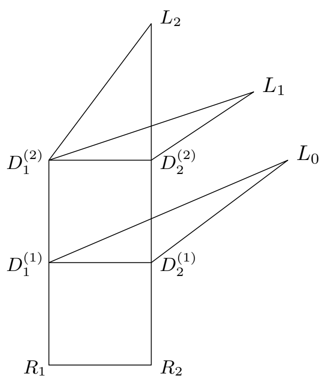

FIGURE 8. Projectivization of the conic polyhedral complex of the surface L_2 in Example 6.144.

	x_1	x_2	y	z_1	f	λ
$D_1^{(1)}$	2	0	3	6	36	5
$D_2^{(1)}$	0	2	1	2	12	3
$D_1^{(2)}$	4	0	6	13	78	11
$D_2^{(2)}$	0	6	3	7	42	10

TABLE 3. Set of valuations of semi-roots and log-discrepancies for Example 6.144.

polyhedral complex) and the log-discrepancies are presented in Table 3. The list of jumping numbers smaller than one appears in Table 4.

$\mathcal{M}$	1	x_1	y	x_1^2	$x_1^2x_2$	x_1y	$x_1^3x_2$	z_1
$\xi_{\mathcal{M}}$	$\frac{5}{36}$	$\frac{15}{78}$	$\frac{17}{78}$	$\frac{10}{42}$	$\frac{19}{78}$	$\frac{21}{78}$	$\frac{23}{78}$	$\frac{11}{36}$

$\mathcal{M}$	x_1^2y	$x_1^2x_2y$	$x_1^4x_2$	x_1z_1	$x_1^3x_2y$	$x_1^5x_2$	yz_1	$x_1^5x_2^2$
$\xi_{\mathcal{M}}$	$\frac{13}{42}$	$\frac{25}{78}$	$\frac{27}{78}$	$\frac{28}{78}$	$\frac{29}{78}$	$\frac{16}{42}$	$\frac{30}{78}$	$\frac{31}{78}$

$\mathcal{M}$	$x_1^2z_1$	$x_1^2x_2z_1$	$x_1^4x_2y$	x_1yz_1	$x_1^6x_2^2$	$x_1^5x_2y$	$x_1^3x_2y$	z_1^2
$\xi_{\mathcal{M}}$	$\frac{17}{42}$	$\frac{32}{78}$	$\frac{33}{78}$	$\frac{34}{78}$	$\frac{35}{78}$	$\frac{19}{42}$	$\frac{36}{78}$	$\frac{17}{36}$

$\mathcal{M}$	$x_1^5x_2^2y$	$x_1^2yz_1$	$x_1^2x_2yz_1$	$x_1^7x_2^2$	$x_1^4x_2z_1$	$x_1^8x_2^2$	$x_1z_1^2$	$x_1^3x_2yz_1$
$\xi_{\mathcal{M}}$	$\frac{37}{78}$	$\frac{20}{42}$	$\frac{38}{78}$	$\frac{39}{78}$	$\frac{40}{78}$	$\frac{22}{42}$	$\frac{41}{78}$	$\frac{42}{78}$

$\mathcal{M}$	$x_1^5x_2z_1$	yz_1^2	$x_1^5x_2^2z_1$	$x_1^2z_1^2$	$x_1^2x_2z_1^2$	$x_1^4x_2yz_1$	$x_1^8x_2^2y$	$x_1yz_1^2$
$\xi_{\mathcal{M}}$	$\frac{23}{42}$	$\frac{43}{78}$	$\frac{44}{78}$	$\frac{24}{42}$	$\frac{45}{78}$	$\frac{46}{78}$	$\frac{25}{42}$	$\frac{47}{78}$

$\mathcal{M}$	$x_1^6x_2^2z_1$	$x_1^5x_2yz_1$	$x_1^3x_2z_1^2$	z_1^3	$x_1^5x_2^2yz_1$	$x_1^2yz_1^2$	$x_1^2x_2yz_1^2$	$x_1^7x_2^2z_1, x_1^1x_2^3$
$\xi_{\mathcal{M}}$	$\frac{48}{78}$	$\frac{26}{42}$	$\frac{49}{78}$	$\frac{23}{36}$	$\frac{50}{78}$	$\frac{27}{42}$	$\frac{51}{78}$	$\frac{52}{78}=\frac{28}{42}$

$\mathcal{M}$	$x_1^4x_2z_1^2$	$x_1^8x_2^2z_1$	$x_1z_1^3$	$x_1^3x_2yz_1^2$	$x_1^5x_2z_1^2$	yz_1^3	$x_1^5x_2^2z_1^2$	$x_1^2z_1^3$
$\xi_{\mathcal{M}}$	$\frac{53}{78}$	$\frac{29}{42}$	$\frac{54}{78}$	$\frac{55}{78}$	$\frac{30}{42}$	$\frac{56}{78}$	$\frac{57}{78}$	$\frac{31}{42}$

$\mathcal{M}$	$x_1^2x_2z_1^3$	$x_1^4x_2yz_1^2$	$x_1^8x_2^2yz_1$	$x_1yz_1^3$	$x_1^6x_2^2z_1^2$	$x_1^5x_2yz_1^2$	$x_1^3x_2z_1^3$	z_1^4
$\xi_{\mathcal{M}}$	$\frac{58}{78}$	$\frac{59}{78}$	$\frac{32}{42}$	$\frac{60}{78}$	$\frac{61}{78}$	$\frac{33}{42}$	$\frac{62}{78}$	$\frac{29}{36}$

$\mathcal{M}$	$x_1^5x_2^2yz_1^2$	$x_1^2yz_1^3$	$x_1^2x_2yz_1^3$	$x_1^7x_2^2z_1^2, x_1^{11}x_2^3z_1$	$x_1^4x_2z_1^3$	$x_1^8x_2^2z_1^2$	$x_1z_1^4$	$x_1^3x_2yz_1^3$
$\xi_{\mathcal{M}}$	$\frac{63}{78}$	$\frac{34}{42}$	$\frac{64}{78}$	$\frac{65}{78}=\frac{35}{42}$	$\frac{66}{78}$	$\frac{36}{42}$	$\frac{67}{78}$	$\frac{68}{78}$

$\mathcal{M}$	$x_1^5x_2z_1^3$	yz_1^4	$x_1^5x_2^2z_1^3$	$x_1^2z_1^4$	$x_1^2x_2z_1^4$	$x_1^4x_2yz_1^3$	$x_1^8x_2^2yz_1^2$	$x_1yz_1^4$
$\xi_{\mathcal{M}}$	$\frac{37}{42}$	$\frac{69}{78}$	$\frac{70}{78}$	$\frac{38}{42}$	$\frac{71}{78}$	$\frac{72}{78}$	$\frac{39}{42}$	$\frac{73}{78}$

$\mathcal{M}$	$x_1^6x_2^2z_1^3$	$x_1^5x_2yz_1^3$	$x_1^3x_2z_1^4$	$x_1^5x_2^2yz_1^3$	$x_1^2yz_1^4$	$x_1^2x_2yz_1^4$
$\xi_{\mathcal{M}}$	$\frac{74}{78}$	$\frac{40}{42}$	$\frac{75}{78}$	$\frac{76}{78}$	$\frac{41}{42}$	$\frac{77}{78}$

TABLE 4. List of jumping numbers smaller than one for the quasi-ordinary surface in Example 6.144.

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