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Three classical problems in Mathematical Analysis

(Tres problemas clásicos en Análisis Matemáticos)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Memoria para optar al grado de doctor
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Hernán Javier Cabana Méndez

Bajo la dirección de los doctores

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y

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- Bernal-González, L.; Cabana-Méndez, H. J.; Muñoz-Fernández, G. A.; Seoane-Sepúlveda, J. B. *Ordering among the topologies induced by various polynomial norms.* **Bull. Belg. Math. Soc. Simon Stevin** 26 (2019), no. 4, 481–492.

- Bernal-González, L.; Cabana-Méndez, H. J.; Muñoz-Fernández, G. A.; Seoane-Sepúlveda, J. B. *Universality of sequences of operators related to Taylor series.* **J. Math. Anal. Appl.** 474 (2019), no. 1, 480–491.

- Cabana-Méndez, H. J.; Muñoz-Fernández, G. A.; Seoane-Sepúlveda, J. B. *Connected polynomials and continuity.* **J. Math. Anal. Appl.** 462 (2018), no. 1, 298–304.

- García, D.; Cabana-Méndez, H. J.; Maestre, M.; Muñoz-Fernández, G. A.; Seoane-Sepúlveda, J. B. *A new approach towards estimating the n-dimensional Bohr radius.* **Preprint (2020)**.

Contents

Resumen

El título de esta tesis alude al estudio de tres problemas clásicos. Los resultados que se han obtenido, que son el fruto del arduo trabajo llevado a cabo durante los últimos cuatro años, están relacionados, total o parcialmente, con algunos de estos tres temas clásicos del Análisis Matemático:

- *Espacios de Banach de polinomios:* Este es un tema muy extenso, como cualquier lector avezado entenderá enseguida. Siendo más concretos, estudiamos propiedades relacionadas con la continuidad de polinomios en espacios de Banach. También estudiamos la relación existente entre las topologías inducidas por diversas normas en un espacio de polinomios.

- *Genericidad Algebraica y lineabilidad:* Este tema consiste en el estudio de las estructuras algebraicas contenidas en determinados conjuntos de un espacio vectorial o un álgebra. En este sentido, aportamos una solución completa a un problema planteado por V. Gurariy poco después del año 2000. De hecho, la solución proporcinada aquí es, en realidad, una generalización del problema original formulado por V. Gurariy. También investigamos problemas vinculados a la noción de genericidad algebraica en el contexto de sucesiones de operadores relacionados con series de Taylor.

- *El problema clásico del radio de Bohr:* Como principal aportación a este problema, damos una acotación (inferior) del radio de Bohr n-dimensional para el polidisco $\mathbb{D}^n$ que mejora estimaciones previas.

Esta memoria está dividida en cinco capítulos. El tema de *polinomios* es la cuestión central de los capítulos 1 y 4. De alguna forma, también aparece en el capítulo 3. Los problemas resueltos en los capítulos 2 y 3 están claramente vinciladas con la *genericidad algebraica y lineabilidad*. También aparecen problemas de

lineabilidad en el capítulo 4. Por último, el capítulo 5 está enteramente dedicado al *problema del radio de Bohr*.

A continuación damos una breve descripción de lo que se hace en cada capítulo.

Capítulo 1.

Este capítulo inicial se titula *"Caracterización de la continuidad de polinomios en un espacio normado"*.

En pocas palabras, lo que hacemos es darle una vuelta más a un resultado elemental del Análisis Real: En particular, una función $f \in \mathbb{R}^{\mathbb{R}}$ es continua si, y solo si f transforma cojuntos compactos y conexos de la recta real en conjuntos compactos y conexos de la rexta real. En realidad este resultado también se pude probar en un contexto más amplio, y de hecho es cierto para funciones entre espacios normados (reales o complejos). Cuando se consideran únicamente polinomios $P : E \to \mathbb{K}$, donde E es un $\mathbb{K}$-espacio normado, entonces se probó en 2012 que P es continuo si, y solo si P transforma conjuntos compactos de E en conjuntos compactos de $\mathbb{K}$. En el caso en que $\mathbb{K} = \mathbb{C}$, en este capítulo demostramos que P es continuo si y solo si P transforma conjuntos conexos de E en conjuntos conexos de $\mathbb{C}$. Aunque también hemos estudiado el caso real de este problema, solo hemos conseguido dar una respuesta parcial. Así, cuando $\mathbb{K} = \mathbb{R}$ este problema sigue, en esencia, abierto.

Capítulo 2.

En el segundo capítulo de la tesis, titulado *Solución a un problema abierto de Vladimir I. Gurariy*, tratamos el conceto de lineabilidad. Esta noción ha atraído el interés de una parte significativa de la comunidad matemática de todo el mundo durante las dos últimas décadas. Este capítulo tiene como particularidad más destacada el contener una solución de una cuestión parcialmente resuelta por V. I. Gurariy (1935–2005) en 2003. El problema aludido está relacionado con la dimensión del espacio vectorial más grande posible que se puede considerar en un conjunto de funciones continuas que cumplen cierta propiedad. Siendo más concretos, si V es un subespacio de $\mathcal{C}(\mathbb{R})$ tal que sus elementos no nulos alcanzan el máximo en un único punto, entonces probamos que $\dim(V) \leq 2$. Es más, aportamos una generalización del resultado anterior en los siguientes términos: Si $m \in \mathbb{N}$ y V_m es un subespacio de $\mathcal{C}(\mathbb{R})$ cuyos elementos no nulos alcanzan su máximo en exactamente m puntos, entonces $\dim(V_m) \leq 2$ para $m > 1$. Tratándose de un problema estrechamente relacionado con el Análisis Real, no es de extrañar que en su solución hayamos precisado de herramientas sacadas de Topología General, Geometría y Análisis

Complejo, entre las que cabe destacar el uso de particiones de variedades y el Teorema de Moore, entre otras.

Capítulo 3.

El tercer capítulo de la tesis se encuadra en el contexto de los polinomios y la teoría de operadores. Su título, *Universalidad de sucesiones de operadores relacionados con series de Taylor* incluye la noción de universalidad. Este concepto, a su vez relacionado con el de genericidad algebraica, lineabilidad y residualidad, ha sido profusamente estudiado a lo largo del último siglo. En este capítulo investigamos la universalidad de las sucesiones de sumas parciales de operadores asociadas a la serie de Taylor de una función holomorfa. Es preciso destacar el hecho de que las series de Taylor se evalúan en un conjunto preestablecido de puntos y que se considera como variable el centro de la serie. También se estudia el comportamiento de la suceción de operadores asociada a las sumas parciales de las series de potencias que no están ligadas a una función entera.

Capítulo 4.

Aquí retomamos el estudio de espacios de Banach de polinomios, y más concretamente nos centramos en las normas polinomiales. De hecho, *Normas polinomiales* es el título de este capítulo. En particular estudiamos las topologías generadas por diversas normas polinomiales, estableciendo qué relaciones existe entre ellas. Hablando con más precisión, consideramos el espacio de todos los polinomios complejos de una variable, $\mathcal{P}$, al que dotamos de las siguientes normas:

$$\|p\|_{D_r} := \sup\{|p(z)| : |z| < r\}, \text{ y } \|p\|_1 := \sum_{i=0}^{n} |a_i|,$$

donde $p(z) = \sum_{i=0}^{n} a_i z^i$.

En este capítulo demostramos que, si $0 < \varepsilon < \varepsilon' < 1 < r < r'$, entonces

$$\|\cdot\|_{D_\varepsilon} \prec \|\cdot\|_{D_{\varepsilon'}} \prec \|\cdot\|_{D_1} \prec \|\cdot\|_1 \prec \|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}},$$

donde $\prec$ representa el orden natural parcial (estricto) de las topologías inducidas por las normas consideradas. El tema de la lineabilidad también aparece en este capítulo.

Capítulo 5.

El último capítulo de la tesis, titulado *Cálculo del radio de Bohr n-dimensional* está dedicado a encontrar una estimación del radio de Bohr de la familia de las funciones holomorfas en el polidisco n-dimensional. La estimación que obtenemos, usando un método original, es la mejor conocida que nosotros sepamos.

Como es habitual, $\mathcal{H}(\mathbb{D})$ representa el espacio de las funciones analíticas en el disco unidad abierto $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ del plano complejo $\mathbb{C}$. En 1914, H. Bohr demostró que cualquier función $f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ tal que $f(\mathbb{D}) \subset \mathbb{D}$, cumple

$$\sum_{k=0}^{\infty} |a_k z^k| \leq 1, \tag{$*$}$$

siempre que $|z| \leq \frac{1}{6}$. El número K_1 definido como el mejor radio para el cual lo anterior ocurre, es decir,

$$K_1 := \sup\{r \in [0,1) : (*) \text{ se cumple para todo } f \in \mathcal{H}(\mathbb{D}) \\ \text{tal que } f(\mathbb{D}) \subset \mathbb{D} \text{ y todo } z \text{ con } |z| \leq r\},$$

recibe el nombre de *radio de Bohr* para $\mathbb{D}$. Según los resultados del propio Bohr, $K_1 \geq \frac{1}{6}$. Con posterioridad, Wiener, Riesz y Schur determinaron, de forma independiente, que el valor exacto de K_1 está dado por $K_1 = \frac{1}{3}$.

En 1997 Boas y Khavinson estudiaron, para cada $n \in \mathbb{N} := \{1, 2, \ldots\}$ el concepto de *radio de Bohr n-dimensional*, representado por K_n, para el polidisco $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. Como es de prever, K_n se define como el número más grande r que satisface la desigualdad $\sum_\alpha |c_\alpha z^\alpha| < 1$ para todo z con $\|z\|_\infty < r$ y todo $f(z) = \sum_\alpha c_\alpha z^\alpha \in \mathcal{H}(\mathbb{D}^n)$ tal que $|f(z)| < 1$ para cada $z \in \mathbb{D}^n$. Aquí α es una n-tupla $(\alpha_1, \ldots, \alpha_n)$ de enteros no negativos, z es una n-tupla $(z_1, \ldots, z_n)$ de números complejos, $\|z\|_\infty = \max\{|z_1|, \ldots, |z_n|\}$, y z^α representa el producto $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Boas y Khavinson demostraron las siguientes acotaciones de K_n:

> Para cada $n \in \mathbb{N}$ con $n \geq 2$, el radio de Bohr n-dimensional K_n cumple
>
> $$\frac{1}{3\sqrt{n}} < K_n < 2\sqrt{\frac{\log n}{n}}.$$

La ámplia literatura existente sobre este tema recoge aproximaciones al valor de K_n en los casos $n = 1$ o $n \geq 2$ en múltiples situaciones, como por ejemplo cuando se consideran dominios más generales que $\mathbb{D}$ o $\mathbb{D}^n$, o cuando se consideran subclases específicas de funciones holomorfas o relacinadoas con las funciones holomorfas, como el caso de las funciones armónicas, estre otras.

También se ha considerado el caso de funciones holomorfas en dominios contenidos en espacios de dimensión infinita o funciones holomorfas que toman valores en un espacio vectorial.

En lo referente al comportamiento asintótico de K_n cuando $n \to \infty$, se ha demostrado recientemente que el crecimiento asintótico exacto de K_n viene dado por

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{(\log n)/n}} = 1.$$

A pesar del hecho de que el límite anterior proporciona una descripción precisa de cómo se comporta K_n asintóticamente, se deconoce por completo cuáles son los valores exactos de K_n para $n \geq 2$. En este capítulo nos centramos en proporcionar una acotación inferior de K_n que mejore $\dfrac{1}{3\sqrt{n}} < K_n$, proporcionando la mejor estimación inferior conocida de K_n.

Abstract

The title of this dissertation alludes to the study of three classical problems of mathematical analysis. All the results that have been obtained as the fruit of four years of hard work are related, wholly or partially to at least one of the following three fields:

- *Banach spaces of polynomials:* This is a vast field as the educated reader knows well. In particular we have studied continuity properties of polynomials on Banach spaces and topological relationships among polynomial spaces.

- *Algebraic genericity and lineability:* This is the study of the algebraic structure within certain sets in a linear space. We give an answer to a question posed by Gurariy in the early 2000's and, as a matted of fact, we prove a generalization to the question formulated by Gurariy. We also link the notion of algebraic genericity to the study of sequences of operators related to Taylor series.

- *The classical Bohr radius problem:* We provide an estimated on the n-dimensional Bohr radius for the polydisk $\mathbb{D}^n$ that improves other previous estimates.

This dissertation is divided into five main chapters. *Polynomials* appear in chapters 1 and 4. This topic is present to in some sense in chapter 3. *Algebraic genericity and lineability* are the main topic of chapters 2 and 3, although chapter 4 contains too lineability questions. Finally, chapter 5 is devoted entirely and solely to the *Bohr radius problem.*

We provide next a brief description of the content of each chapter.

Chapter 1.

This initial chapter is entitled *"Characterization of Continuity of Polynomials on Normed Spaces"*.

In a nutshell, we "revolve" around a classical real analysis result: A function $f \in \mathbb{R}^{\mathbb{R}}$ is continuous if and only if f maps continua (compact, connected sets) to continua. The same holds for mappings between any two (real or complex) normed spaces. However, when we restrict ourselves to polynomials $P : E \to \mathbb{K}$, where E is a $\mathbb{K}$-normed space, then it was proved in 2012 that P is continuous if and only if it transforms compact sets into compact sets. In this chapter we show that (if $\mathbb{K} = \mathbb{C}$) P is continuous if and only if it transforms connected sets into connected sets. Although we also provide some partial results for $\mathbb{K} = \mathbb{R}$, the general case in the real setting remains still an open question.

Chapter 2.

In the second chapter of this dissertation, entitled *Answering an open question of Vladimir I. Gurariy*, we deal with the notion of lineability, notion that has (for the past two decades) attracted the attention of the mathematical community all over the World. This chapter has the peculiarity that solves a question that was partially answered by V. I. Gurariy (1935–2005) in 2003. This question is related to the largest possible dimension of a vector space of continuous functions enjoying certain special properties. More particularly, if V stands for a subspace of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V attains its maximum at one (and only one) point, then we prove that $\dim(V) \leq 2$. Moreover, we generalize the previous result in the following terms: If $m \in \mathbb{N}$ and V_m stands for a subspace of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V_m attains its maximum at m (and only m) points, then $\dim(V_m) \leq 2$ for $m > 1$ as well. Besides being a problem closely related to real analysis, this problem actually needs the use of tools from General Topology, Geometry and Complex Analysis, such as decompositions (or partitions) of manifolds or Moore's Theorem, among others.

Chapter 3.

The third chapter in this thesis falls into the area of polynomials and operator theory. Its title, *Universality of sequences of operators related to Taylor series* considers, this time, the notion of universality. This notion, linked to that of algebraic genericity, lineability and residuality, has been the object of deep study for the last century. On this occasion, the universality of a sequence of operators associated to the partial sums of the Taylor series of a holomorphic function is investigated. The emphasis is put on the fact that the Taylor series are evaluated at a prescribed point and the variable is the center of the expansion. The dynamics of the sequence of operators linked to the partial sums of a power series that is not generated by an entire function is also studied.

Chapter 4.

This time we keep studying Banach space polynomials and Polynomial norms (its title). More particularly we focus on the topologies that can be consider within them and on "sorting them out". More precisely, consider the following two norms in the vector space $\mathcal{P}$ of all complex polynomials:

$$\|p\|_{D_r} := \sup\{|p(z)| : |z| < r\}, \text{ and } \|p\|_1 := \sum_{i=0}^{n} |a_i|,$$

where $p(z) = \sum_{i=0}^{n} a_i z^i$.

In this chapter we show that, if $0 < \varepsilon < \varepsilon' < 1 < r < r'$, then

$$\|\cdot\|_{D_\varepsilon} \prec \|\cdot\|_{D_{\varepsilon'}} \prec \|\cdot\|_{D_1} \prec \|\cdot\|_1 \prec \|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}},$$

where $\prec$ represents the natural (strict) partial order in their corresponding induced topologies. Some lineability results are also studied in this chapter.

Chapter 5.

The final chapter of this dissertation, entitled *Estimating the n-dimensional Bohr radius*, is devoted to the study of a new estimate for the Bohr radius of the family of holomorphic functions in the n-dimensional polydisk. This estimate, obtained via a new approach, is sharper than those that are known up to date.

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. In 1914, H. Bohr proved that any function $f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ satisfies

$$\sum_{k=0}^{\infty} |a_k z^k| \le 1, \tag{$*$}$$

whenever $|z| \le \frac{1}{6}$. The number K_1 defined as the best radius for which this happens, that is,

$$K_1 := \sup\{r \in [0,1) : (*) \text{ holds for all } f \in \mathcal{H}(\mathbb{D}) \\ \text{such that } f(\mathbb{D}) \subset \mathbb{D} \text{ and all } z \text{ with } |z| \le r\},$$

is called the *Bohr radius* for $\mathbb{D}$. Then $K_1 \ge \frac{1}{6}$. Subsequently later, Wiener, Riesz and Schur, independently established the exact value $K_1 = \frac{1}{3}$.

In 1997 Boas and Khavinson introduced, for each $n \in \mathbb{N} := \{1, 2, \dots\}$ the *n-dimensional Bohr radius* K_n for the polysdisk $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. As expected, K_n is defined as the largest number r satisfying $\sum_\alpha |c_\alpha z^\alpha| < 1$ for all z with $\|z\|_\infty < r$ and all $f(z) = \sum_\alpha c_\alpha z^\alpha \in \mathcal{H}(\mathbb{D}^n)$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}^n$. Here α denotes an n-tuple $(\alpha_1, \dots, \alpha_n)$ of nonnegative integers, z stands for an n-tuple $(z_1, \dots, z_n)$ of complex numbers, $\|z\|_\infty = \max\{|z_1|, \dots, |z_n|\}$, and z^α denotes the product $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. They showed the following bilateral estimate:

> For every $n \in \mathbb{N}$ with $n \ge 2$, the n-dimensional Bohr radius K_n satisfies
> $$\frac{1}{3\sqrt{n}} < K_n < 2\sqrt{\frac{\log n}{n}}.$$

Approximations for the value of K_n, in the cases $n = 1$ or $n \ge 2$, have been given in domains more general than $\mathbb{D}$ or $\mathbb{D}^n$ (with appropriate definitions for such domains), for specific subclasses of holomorphic functions, for functions related to holomorphic ones (such as harmonic functions, among others), and even for holomorphic functions on domains contained in infinite dimensional spaces or for vector-valued analytic functions. Concerning the asymptotic behaviour of K_n when $n \to \infty$, it was recently proved that the exact asymptotic behaviour of K_n is established, namely,

$$\lim_{n \to \infty} \frac{K_n}{\sqrt{(\log n)/n}} = 1.$$

Despite the fact that the above limit gives a very precise description of the asymptotic behavior of the sequence (K_n), no exact value of K_n is known for any $n \ge 2$. In this chapter of the dissertation we focus on the (non-asymptotical) lower estimate $\dfrac{1}{3\sqrt{n}} < K_n$, providing the best known to date lower estimate for K_n.

Chapter 1

Characterization of Continuity of Polynomials on Normed Spaces

For convenience let us represent the space of all mappings $f : \mathbb{R} \to \mathbb{R}$ by $\mathbb{R}^{\mathbb{R}}$. The starting point of this chapter in the dissertation is the following (not very famous) characterization of continuity in $\mathbb{R}^{\mathbb{R}}$:

Theorem 1.1. *A function $f \in \mathbb{R}^{\mathbb{R}}$ is continuous if and only if the following two conditions hold:*

1. *For every compact set $C \subset \mathbb{R}$, we have that $f(C)$ is also compact, and*
2. *for every connected set $C \subset \mathbb{R}$ (i.e., for every interval C), we have that $f(C)$ is also connected.*

In every basic course in Analysis of one real variable it is shown that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then it fulfills (1) and (2) above. For a proof of the reverse implication we refer to Velleman [89]. Actually, the following holds as a particular case of a result by Hamlett [61] (see also [68] and [90]):

Theorem 1.2. *A functional $f : E \to \mathbb{R}$, where E is a normed space is continuous if and only if f transforms compact sets and connected sets of E into compact sets and connected sets of $\mathbb{R}$, respectively.*

In [53] the authors studied the algebraic size of the sets of mappings on the real line satisfying one and only one of the conditions (1) and (2). In order to understand the main conclusions of [53] it might be necessary to briefly introduce the concept of lineability (we will not delve too much on this, since it will be the object of full study of the next chapter).

Definition 1.3. *If E is a linear space and λ is a cardinal number, we say that $M \subset E$ is λ-lineable if there exists a λ-dimensional linear subspace V of E such that $V \subset M \cup \{0\}$. If λ is an infinite cardinal, we simply say that M is lineable.*

Recall that $\mathfrak{c}$, as usual, stands for the cardinality of $\mathbb{R}$. Then, in [53] the following was proved:

Theorem 1.4. *Let*

$$\mathcal{A}_1 = \{f \in \mathbb{R}^{\mathbb{R}} : \ f \text{ maps compact sets into compact sets}\} \quad \text{and}$$
$$\mathcal{A}_2 = \{f \in \mathbb{R}^{\mathbb{R}} : \ f \text{ maps connected sets into connected sets}\}.$$

Then both $\mathcal{A}_1 \setminus \mathcal{A}_2$ and $\mathcal{A}_2 \setminus \mathcal{A}_1$ are $2^{\mathfrak{c}}$-lineable, and this is optimal (in terms of dimension). Moreover, both $(\mathcal{A}_1 \setminus \mathcal{A}_2) \cup \{0\}$ and $(\mathcal{A}_2 \setminus \mathcal{A}_1) \cup \{0\}$ contain a $2^{\mathfrak{c}}$-dimensional space of nowhere continuous functions.

The above result is very much related to the question we consider in this chapter. We need to introduce the notion of polynomial on a normed space. Given a normed space E over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, a map $P : E \to \mathbb{K}$ is an *n-homogeneous polynomial* if there is an n-linear mapping $L : E^n \to \mathbb{K}$ for which $P(x) = L(x, \ldots, x)$ for all $x \in E$. In this case it is convenient to write $P = \widehat{L}$. According to a well-known algebraic result, for every n-homogeneous polynomial $P : E \to \mathbb{K}$ there exists a unique symmetric n-linear mapping $L : E^n \to \mathbb{K}$ such that $P = \widehat{L}$. When this happens, L is called the polar of P.

We let $\mathcal{P}_a({}^nE)$, $\mathcal{L}_a({}^nE)$ and $\mathcal{L}_a^s({}^nE)$ denote, respectively, the linear spaces of all scalar-valued, n-homogeneous polynomials on E, the scalar-valued, n-linear mappings on E and the symmetric, scalar-valued, n-linear mappings on E. More generally, a map $P : E \to \mathbb{K}$ is a *polynomial of degree at most n* if

$$P = P_0 + P_1 + \cdots + P_n,$$

where $P_k \in \mathcal{P}_a({}^kE)$ $(1 \leq k \leq n)$, and $P_0 : E \to \mathbb{K}$ is a constant function. The polynomials of degree at most n on E are denoted by $\mathcal{P}_{n,a}(E)$.

Polynomials on a finite dimensional normed space are always continuous; however, the same statement is not valid for infinite dimensional normed spaces. Boundedness is a characteristic property of continuous polynomials on a normed space. The interested reader can find in [49] the details on the proof of the latter fact, as well as a complete and modern account on polynomials on normed spaces. In particular, it is well-known that $P \in \mathcal{P}_{n,a}(E)$ is continuous if and only if P is bounded on the open unit ball of E, denoted by B_E. This fact allows us to endow the space of contintinuous polynomials on E of degree at most n, represented by $\mathcal{P}_n(E)$, with the following norm:

$$\|P\| = \sup\{|P(x)| : x \in \mathsf{B}_E\}.$$

The space of continuous n-homogeneous polynomials on E is denoted by $\mathcal{P}(^nE)$.

The following result refines Theorem 1.2 when restricting our attention to polynomials (see [53]):

Theorem 1.5. *If E is a real normed space, a polynomial $P \in \mathcal{P}_{n,a}(E)$ is continuous if and only if P transforms compact sets into compact sets.*

In [53] it was conjectured that a similar result to Thereom 1.5 for polynomials that transform connected sets into connected sets also holds, i.e., $P \in \mathcal{P}_{n,a}(E)$ is continuous if and only if it transforms connected sets into connected sets. The conjecture was proved (see [53, Proposition 2.2]) for real polynomials in $\mathcal{P}_a(^nE)$ with $n = 1, 2$. In this chapter we show that the conjecture is also true for polynomials in $\mathcal{P}_{2,a}(E)$, being E a real normed space, and for polynomials in $\mathcal{P}_{n,a}(E)$ for every $n \in \mathbb{N}$ whenever E is a complex normed space.

1.1 The complex case

From the classical Maximum Modulus Principle one can obtain the following result.

Theorem 1.6. *Let $\Omega \subset \mathbb{C}$ be an open, connected and bounded set and let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. If there is a $z_0 \in \Omega$ such that $|f(z_0)| \geq \max_{\partial\Omega} |f|$ then f is constant. Moreover* $\max_{\partial\Omega} |f| = \max_{\overline{\Omega}} |f|$.

Corollary 1.7. *Let $\Omega \subset \mathbb{C}$ be open, connected and bounded, and let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. If f is non-constant then, for every $z \in \Omega$, there is $\eta \in \partial\Omega$ such that $|f(\eta)| > |f(z)|$.*

Theorem 1.8. *Let $f \in \mathcal{H}(\mathbb{C})$ be non-constant. Then for all $z_1 \in \mathbb{C}$ and $r, R > 0$ with $|f(z_1)| > r$, there exists a continua (connected compact set) $B \subset \mathbb{C}$ such that $z_1 \in B$, $B \cap \partial D(z_1, R) \neq \varnothing$ and $|f(z)| > \frac{r}{2}$ for every $z \in B$.*

Proof. Let $R > 0$. Since $|f|$ is uniformly continuous over $\overline{D(z_1, R)}$, there exists $\delta > 0$ such that

$$\Big||f(z)| - |f(\eta)|\Big| < r/4, \tag{1.1}$$

for all $z, \eta \in \overline{D(z_1, R)}$ with $|z - \eta| < \delta$. Next, define z_n in $\overline{D(z_1, R)}$ by recursion as follows:

Suppose $z_1, \dots, z_k$ have already been defined in $D(z_1, R)$. Now set

$$D_k = D(z_k, \delta) \cap D(z_1, R),$$
$$M_k = \{z \in \partial D_k : |f(z)| = \max_{\partial D_k} |f|\}.$$

Observe that $\partial D_k \subset \partial D(z_k, \delta) \cup \partial D(z_1, R)$ and $z_k \in D_k$. Two cases are possible now:

1. If $M_k \cap \partial D(z_1, R) \neq \varnothing$, we choose $z_{k+1} \in M_k \cap \partial D(z_1, R)$ and we stop the process.
2. If $M_k \cap \partial D(z_1, R) = \varnothing$, we choose $z_{k+1} \in M_k \cap \partial D(z_k, \delta)$ and the process continues. Notice that in this case $z_{k+1} \in M_k \cap D(z_1, R)$. Also, since f is not constant in D_k (otherwise f would be constant on $\mathbb{C}$ by the Identity Principle), the fact that $z_k \in D_k$ and $|f(z_{k+1})| = \max_{\partial D_k} |f|$ imply that $|f(z_{k+1})| > |f(z_k)|$ by virtue of the Maximum Modulus Principle (Theorem 1.6).

Let us now show that this process, actually, has an end, that is, there exists $N \in \mathbb{N}$ such that $z_N \in \partial D(z_1, R)$. On the contrary, we would have that $z_n \in D(z_1, R)$ for all $n \in \mathbb{N}$. Then (z_n) has a convergent subsequent (z_{i_n}). Let $z_0 \in \overline{D(z_1, R)}$ be the limit of (z_{i_n}). By construction we have $|f(z_n)| < |f(z_{n+1})|$ for all $n \in \mathbb{N}$, which implies that $|f(z_n)| < |f(z_0)|$, for every $n \in \mathbb{N}$. If we choose $s \in \mathbb{N}$ such that $|z_{i_s} - z_0| < \delta$, then $z_0 \in D_{i_s}$ and, also,

$$\max_{\partial D_{i_s}} |f| = |f(z_{i_s+1})| < |f(z_0)| \le \max_{\overline{D_{i_s}}} |f| = \max_{\partial D_{i_s}} |f|,$$

which is a contradiction.

Therefore $r < |f(z_1)| < \ldots < |f(z_N)|$ and, by (1.1), we have that $|f(z)| > 3r/4$ for all $z \in \cup_{n=1}^{N-1} D_n$. If we define $B = \cup_{n=1}^{N-1} \overline{D_n}$, then B is compact and connected since the $\overline{D_n}$'s are continua with $z_n, z_{n+1} \in \overline{D_n}$ $(1 \le n \le N-1)$, $|f(z)| \ge 3r/4 > r/2$ for all $z \in B$, $z_N \in B \cap \partial D(z_1, R)$ and, of course, $z_1 \in B$. This concludes the proof. □

Theorem 1.9. *Let $P \in \mathbb{C}[z]$. Recall that $\mathbb{C}[z]$ is a quite standard notation for the space of complex polynomials of one variable (in Chapter 4 we will use the alternative notation $\mathcal{P}$). If there are $r > 0$ and $\eta_1, \eta_2 \in \mathbb{C}$ with*

$$\min\{|P(\eta_1)|, |P(\eta_2)|\} > r,$$

then there is a continua B such that $\eta_1, \eta_2 \in B$ and, for all $z \in B$, $|P(z)| > r/2$.

Proof. If P is constant, there is nothing to prove. Let us suppose that P is non-constant. We have that $K = \{z \in \mathbb{C} : |P(z)| \le r/2\}$ is bounded, thus there is $d > 0$ such that $K \subset \overline{D(0, d)}$. If we choose $R > \max\{d + |\eta_1|, d + |\eta_2|\}$, by Theorem 1.8 there are two connected subsets B_1 and B_2 such that (for $i = 1, 2$) $\eta_i \in B_i$, $|P(z)| > r/2$ for all $z \in B_i$ and there is a $\eta_i' \in B_i$ with $R = |\eta_i' - \eta_i|$, and $|\eta_i'| \ge R - |\eta_i| > d$. Therefore we can construct a continuous curve $\gamma : [0, 1] \to \mathbb{C} \setminus \overline{D(0, d)}$ such that $\gamma(0) = \eta_1'$ and $\gamma(1) = \eta_2'$. We just need to put $B = B_1 \cup B_2 \cup \gamma([0, 1])$ to conclude the proof. □

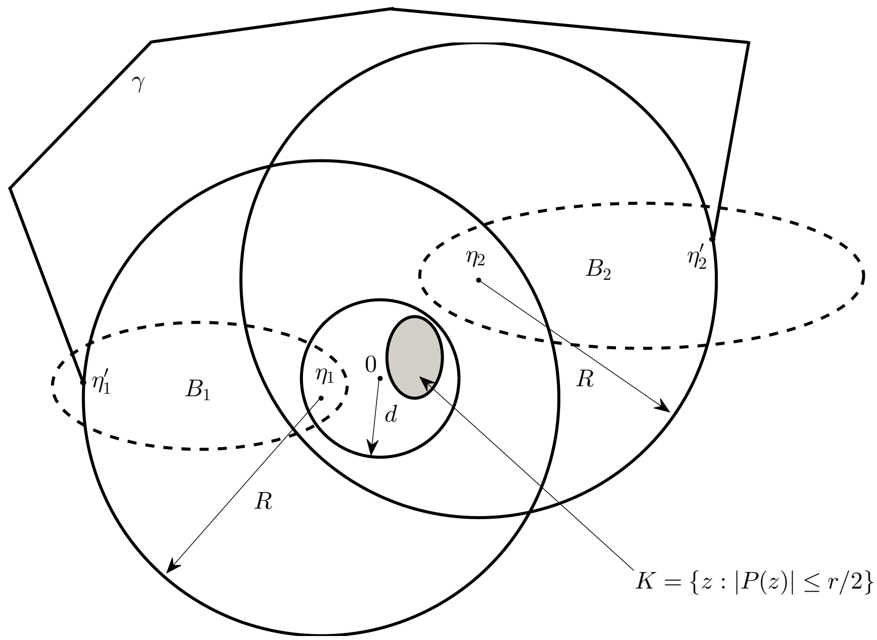

Figure 1.1: Setting of the proof of Theorem 1.9

Theorem 1.10. *Let $r > 0$ and let F be a $\mathbb{C}$-normed space. Also, let $P = \sum_{j=0}^{m} P_j$ with $P_j \in \mathcal{P}({}^jF)$ and $P_0 \in \mathbb{C}$ be such that there are $x, y \in F$ with* $\min\{|P(x)|, |P(y)|\}| > r$. *Then there exists a continua A such that $x, y \in A$ and, for all $a \in A$, $|P(a)| > r/2$.*

Proof. We define $\varphi : \mathbb{C} \to F$ as $\varphi(z) := zx + (1 - z)y$ and consider the complex polynomial in one variable given by $Q(z) = P(\varphi(z))$. Since $|Q(0)| = |P(y)|$, $|Q(1)| = |P(x)| > r$, by Theorem 1.9 there is a continua $B \subset \mathbb{C}$ such that $0, 1 \in B$ and, for all $z \in B$, $|Q(z)| > r/2$. Then it follows that $A = \varphi(B)$ is a connected and compact subset of F such that $x, y \in A$ and for all $a \in A$, $|P(a)| > r/2$. $\square$

Theorem 1.11. *Let E be a $\mathbb{C}$-normed space and let $P = \sum_{j=0}^{m} P_j$ with $P_j \in \mathcal{P}_a({}^jE)$ and $P_0 \in \mathbb{C}$. If P is not continuous then there is a connected subset C of E such that $P(C)$ is not connected.*

Proof. We can assume, without loss of generality, that $P_0 = 0$ by replacing P by $P - P_0$ if necessary. Since P is not continuous, there exists a sequence (x_n) such that $\lim_n x_n = 0$ and, for every $n \in \mathbb{N}$, $|P(x_n)| > r > 0$. Observe that the x_n's cannot be contained in a finite dimensional space. Therefore we can assume that they are pairwise linearly independent. This guaranties that the spaces $F_n = \operatorname{span}\{x_n, x_{n+1}\}$ with $n \in \mathbb{N}$ are not trivial. By Theorem 1.10, for every $n \in \mathbb{N}$ there exists a connected subset A_n of F_n such that $x_n, x_{n+1} \in A_n$ and for every $x \in A_n$ we have $|P(x)| > r/2$. Thus, it follows that both $A = \cup_{n \in \mathbb{N}} A_n$ and $C = A \cup \{0\}$ are connected, but $|P(x)| > r/2$ for all $x \in A$. Since $P(0) = 0$ we have that $P(C)$ is not connected. $\square$

Finally, as a corollary, we have the desired result.

Corollary 1.12. *Let E be a $\mathbb{C}$-normed space and $P \in \mathcal{P}_a(E)$. P is continuous if and only if P transforms connected subsets of E into connected subsets of $\mathbb{C}$.*

1.2 The real case

In [53] it was proved that if E is a real normed space and $P \in \mathcal{P}({}^nE)$ (for $n = 1, 2$) then P transforms connected sets of E into intervals if and only if P is continuous. The proof provided in [53], however, cannot be adapted to homogeneous polynomials of higher degrees, not to mention non-homogeneous polynomials. We show below that, at least for polynomials of degree at most 2 (homogeneous or not), connectedness is equivalent to continuity.

Theorem 1.13. *If E is a real normed space and $P \in \mathcal{P}_{2,a}(E)$, then P is continuous if and only if it transforms cennected sets of E into connected sets of $\mathbb{R}$.*

Proof. Connectedness is always a consequence of continuity, so let us assume that P transforms connected sets of E into connected sets of $\mathbb{R}$. Suppose that $P = P_2 + P_1 + P_0$, where $P_n \in \mathcal{P}_a({}^nE)$, with $n = 1, 2$, and P_0 is a real number. Notice that $P - P_0$ transforms connected sets into intervals as well, so we can assume, replacing P by $P - P_0$ if necessary, that $P_0 = 0$. If P were not continuous, we would have the following cases:

Case 1: P_1 is continuous but P_2 is not or vice versa.

Assume first that P_1 is continuous and P_2 is non-continuous. Since P_2 is not continuous, we can construct a sequence (x_k) in B_E such that $\lim_{k\to\infty} x_k = 0$ and either $\lim_{k\to\infty} P_2(x_k) = -\infty$ or $\lim_{k\to\infty} P_2(x_k) = \infty$. Replacing P by $-P$ if needed, we can assume that $\lim_{k\to\infty} P_2(x_k) = \infty$. We can also choose the x_k's so that the sequence $(P_2(x_k))$ is strictly increasing and $P_2(x_1) > 0$. Furthermore, if $L \in \mathcal{L}_a^s({}^2E)$ is the polar of P_2, we can assume that $L(x_k, x_{k+1}) \geq 0$ replacing x_{k+1} by $-x_{k+1}$ if necessary. Notice that this possible alternation of signs does not alter the fact $(P_2(x_k))$ is an strictly increasing divergent sequence of positive numbers because $P_2(x_k) = P_2(-x_k)$ for all $k \in \mathbb{N}$. Since

$$\begin{aligned}
P_2(\lambda x_k + (1-\lambda)x_{k+1}) &= \lambda^2 P_2(x_k) + 2\lambda(1-\lambda)L(x_k, x_{k+1}) + (1-\lambda)^2 P_2(x_{k+1}) \\
&\geq \lambda^2 P_2(x_k) + (1-\lambda)^2 P_2(x_{k+1}) \\
&\geq \left[\lambda^2 + (1-\lambda)^2\right] P_2(x_k)
\end{aligned}$$

$$\geq \frac{1}{2}P_2(x_k), \tag{1.2}$$

for every $\lambda \in [0,1]$, we have that $P_2(x) \geq P_2(x_k)/2$ for every $x \in [x_k, x_{k+1}]$, where $[x_k, x_{k+1}]$ is the segment with endpoints x_k and x_{k+1}. Now choose $N \in \mathbb{N}$ such that

$$\frac{1}{2}P_2(x_k) > \|P_1\| + 1,$$

for all $k \geq N$ (recall that $\lim_{k\to\infty} P_2(x_k) = \infty$). Define $C = \bigcup_{k=N}^{\infty}[x_k, x_{k+1}]$ and $C^* = C \cup \{0\}$, which is a connected set. Therefore, if $x \in C$ and $k \in \mathbb{N}$ is such that $x \in [x_k, x_{k+1}]$, we have that

$$P(x) = P_2(x) + P_1(x) \geq \frac{1}{2}P_2(x_k) - \|P_1\| > 1,$$

from which $P(C) \subset (1,\infty)$. Since $P(0) = 0$, it follows that $P(C^*)$ is not connected.

The previous proof can be adapted to the case in which P_2 is continuous but P_1 is not. Indeed, as in the previous case, there exists a sequence (x_k) in B_E converging to 0 such that $(P_1(x_k))$ is strictly increasing and divergent. By linearity, it is straightforward that

$$P_1(\lambda x_k + (1-\lambda)x_{k+1}) \geq P_1(x_k),$$

for every $k \in \mathbb{N}$. As above, let $N \in \mathbb{N}$ such that

$$P_1(x_k) > \|P_2\| + 1,$$

for every $k \geq N$. The proof now proceeds as previously.

Case 2, Both P_1 and P_2 are not continuous: Observe that $Z = \ker(P_1)$ is dense in E because P_1 is unbounded. The polynomial P_2 can be continuous on Z or not.

In the case where P_2 is not continuous on Z, we can construct (as in case 1) a sequence (x_k) in $\mathsf{B}_Z \subset \mathsf{B}_E$ such that $\lim_{k\to\infty} x_k = 0$ and $(P_2(x_k))$ is an strictly increasing divergent sequence of positive real numbers with $P_2(x) \geq P_2(x_k)/2$ for every $x \in [x_k, x_{k+1}]$ and every $k \in \mathbb{N}$. If we define $C = \bigcup_{k=1}^{\infty}[x_k, x_{k+1}]$ and $C^* = C \cup \{0\}$, which is a connected set in Z, we have that $P(C) \subset [P_2(x_1)/2, \infty)$ whereas $P(0) = 0$. Since $P_2(x_1) > 0$, we have shown that $P(C^*)$ is not connected.

Now, if P_2 is continuous on Z, let $L = \sup\{|P_2(z)| : z \in \mathsf{B}_Z\} < \infty$. Since P_2 is not bounded on B_E, there exists $x_0 \in \mathsf{B}_E$ (then $\|x_0\| < 1$) such that $|P_2(x_0)| > L$. We can assume that $P_2(x_0)$ has the same sign as $P_1(x_0)$, changing the sign of x_0 if needed. Therefore $|P(x_0)| = |P_2(x_0)| + |P_1(x_0)| \geq |P_2(x_0)| > L$. Since $x_0 \in \mathsf{B}_E$ and Z is dense, there exists a sequence (z_k) in B_Z with $\lim_{k\to\infty} z_k = x_0$. Define $C = \cup_{k=1}^{\infty}[z_k, z_{k+1}]$, which is a connected subset of B_Z and $C^* = C \cup \{x_0\}$, which is connected. Then, if $z \in C$, $|P(z)| = |P_2(z)| \leq L$, whereas $|P(x_0)| > L$. Therefore $P(C^*)$ cannot be connected. □

To finish, it is in order to make a couple of comments on the real case and the proof of Theorem 1.13.

Remark 1.14. *Suppose E is a real normed space and assume we manage to prove that any polynomial $P \in \mathcal{P}_n(E)$ that transforms connected sets into intervals is continuous. Then Theorem 1.11 would follow immediately by considering the underlying real space within a complex space. However, this seems, as the reader might have guessed, quite a hard issue to tackle.*

Remark 1.15. *The proof given of Theorem 1.13 has been adapted from the proof of* [53, Proposition 2.2]*. Observe that we have corrected a flaw appearing in the proof of* [53, Proposition 2.2]*. In particular, the factor $\frac{1}{2}$ appearing in* (1.2) *is missing in* [53, Proposition 2.2]*. This, in fact, is a minor mistake that can easily be mended without altering the validity of that proof.*

Chapter 2

Answering an open question of Vladimir I. Gurariy

2.1 Introduction

Let X be any topological vector space and M any subset of X. We say that M is *spaceable* if $M \cup \{0\}$ contains a *closed* infinite dimensional subspace. The set M shall be called *lineable* if $M \cup \{0\}$ contains an infinite dimensional linear (not necessarily closed) space. At times, we shall be more specific, referring to the set M as κ-lineable if it contains a vector space of dimension κ (finite or infinite cardinality).

These notions of *lineability* and *spaceability* were originally coined by V. I. Gurariy and they first appeared in [8, 60, 88]. During the last decade, many authors have invested a lot of effort in studying *special* cases of lineable sets and *pathological* real-valued functions (see, e.g., [7, 22, 33, 38, 50, 54, 55]).

Let us recall that most results obtained in this theory are "positive" results, in the sense that the sets that authors have considered were (usually) lineable. Thus, nowadays, some authors aim for finding (nontrivial) nonlineable subsets. This theory has experienced a fast development in the last decade. However, there are many problems still unsolved. Here, we shall solve one of them, posed by V. I. Gurariy during a Non-linear Analysis Seminar at Kent State University (Kent, Ohio, USA) in the Academic Year 2003/2004 (and that has resisted the efforts of many authors until now). Let us give some preliminaries on the problem we are dealing with.

Figure 2.1: Vladimir Ilyich Gurariy (Kharkov, Ukraine, 1935). In 1991 he moved to the USA and worked in Kent State University (Ohio) until his passing in 2005. Photograph courtesy of Larisa Lev Altshuler.

Let $A \subseteq \mathbb{R}$ and denote by $\widehat{\mathcal{C}}(A)$ the subset of $\mathcal{C}(A)$ of functions attaining their maximum at a unique point and, unless said otherwise, endowed with the usual sup norm. In [60] it was shown that $\widehat{\mathcal{C}}[0,1]$ is not 2-lineable. This could certainly be called *surprising* if we keep in mind that the set $\widehat{\mathcal{C}}[0,1]$ is a very large subset –in a topological sense– of $\mathcal{C}[0,1]$. To be more specific, we have the following proposition, that was communicated by V. I. Gurariy during a Non-linear Analysis Seminar at Kent State University (Kent, Ohio, USA) in the Fall of 2004 and proved in detail in [29].

Proposition 2.1. *The set $\widehat{\mathcal{C}}[0,1]$ is a G_δ-dense subset of $\mathcal{C}[0,1]$.*

Thus, roughly speaking, the functions from $\widehat{\mathcal{C}}[0,1]$ are *everywhere* within $\mathcal{C}[0,1]$, but there is not even a 2-dimensional vector space of such functions. Also in [60], the authors proved (constructively) that $\widehat{\mathcal{C}}(\mathbb{R})$ is 2-lineable. Namely, they took the two linearly independent functions $f(x)$, $g(x)$ defined on $\mathbb{R}$ as

$$f(x) := \mu(x)\cos(4\arctan(|x|)) \quad \text{and} \quad g(x) := \mu(x)\sin(4\arctan(|x|)),$$

where μ is the real valued continuous function defined on $\mathbb{R}$ by

$$\mu(x) = \begin{cases} e^x & \text{if } t \leq 0, \\ 1 & \text{if } t \geq 0, \end{cases}$$

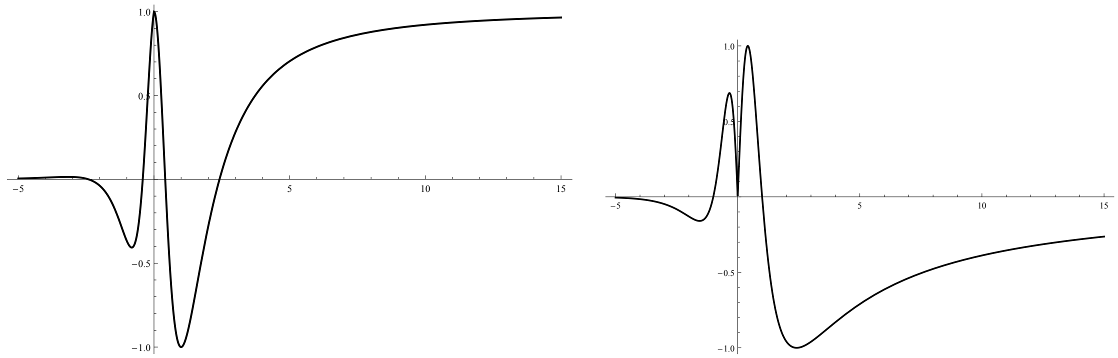

Figure 2.2: Plots of $f(t)$ and $g(t)$, respectively.

and, then, considered the 2-dimensional vector space given by $V = \text{span}\{f(x), g(x)\}$ (see Figure 2.2). It can be seen, quite easily, that $V \notin \widehat{\mathcal{C}}(\mathbb{R}) \cup \{0\}$.

During the last mentioned Seminar, Gurariy posed the following problem (see also the paper [60] by Gurariy and Quarta, where this question is addressed as well).

Question 2.2. *Is there an n-dimensional vector space, with $n > 2$, every nonzero element of which belongs to $\widehat{\mathcal{C}}(\mathbb{R})$ (or even an infinite dimensional vector space, for that matter)?*

Summarizing, the main results obtained in [60] are the following:

(A) There is a 2-dimensional linear subspace of $C[a, b)$ contained in $\widehat{C}[a, b) \cup \{0\}$.

(B) There is a 2-dimensional linear subspace of $C(\mathbb{R})$ contained in $\widehat{C}(\mathbb{R}) \cup \{0\}$.

(C) There is no 2-dimensional linear subspace of $C[a, b]$ contained in $\widehat{C}[a, b] \cup \{0\}$.

Also, later, in further attempts to give an answer to the Question 2.2, in [29], the authors proved the following result, that generalizes (C) above.

Theorem 2.3. *If K is a compact subset of $\mathbb{R}^n$ and if V is a subspace of $C(K)$ inside $\widehat{C}(K) \cup \{0\}$ then $\dim(V) \leq n$.*

One of the interesting points from the proof of Theorem 2.3 is that, whereas Gurariy and Quarta used classical real analysis tools in [60], Theorem 2.3 requires a topological technique (namely, the Borsuk–Ulam Theorem). More recently, Cariello, in his Ph.D. dissertation, was able to reduce the above problem to a topological conjecture [32, §4].

Unfortunately, no more advances have been obtained regarding the original Question 2.2. Here, we shall close this problem in the negative by showing that $\widehat{\mathcal{C}}(\mathbb{R})$ is not 3-lineable. In other words, if V stands for a subspace of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V attains its maximum at only one point, then $\dim(V) \leq 2$.

Again, the technique we shall employ is far from being a classical real analysis tool. We shall use a combination of tools from General Topology, Geometry and Complex Analysis. In particular we shall use decompositions (or partitions) of manifolds, topic that dates back to the work of R.L. Moore in the 1920s (see [77]), and that was renewed by results of R.H. Bing in the 1950s. This area of research has proven to be of extreme importance to the recent characterization of higher-dimensional manifolds in terms of elementary topological properties. In particular we shall make use of Moore's Theorem (Theorem 2.27).

In this chapter we shall have a section which contains several topological results that may not have a great inherent interest, but that play a crucial role in the proof of the main results. The statement and proof of the main results are given later on. There, a chain of results of topological nature leads us to the application of Moore's Theorem, which finally proves the nonexistence of a 3-dimensional subspace in $\widehat{\mathcal{C}}(\mathbb{R}) \cup \{0\}$.

We also show an even stronger result: actually, we solve a generalization of Gurariy's problem. If we let V_m stand for a subspace of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V_m attains its maximum at m (and only m) points, then we show that V_m can be constructed having dimension 2, for every $m \in \mathbb{N}$, but it cannot be constructed having dimension 3. In other words, the subset of $\mathcal{C}(\mathbb{R})$ of functions attaining their maximum at m (and only m) points is 2-lineable but not 3-lineable for every $m \in \mathbb{N}$. It is relevant to mention that the approach employed to tackle the problem for $m = 1$ does not work for $m \geq 2$.

2.2 Some preliminaries on Algebraic Geometry

Given an arbitrary topological space X, a quotient space is, intuitively speaking, the result of "gluing together" certain points of X. Let $\mathcal{F}$ be a decomposition of X into non-empty subsets. For every element F of $\mathcal{F}$, all the points of F are identified topologically as a unique point that we will denote simply as F in $\mathcal{F}$, giving to $\mathcal{F}$ the following topology $\tau := \{\mathcal{U} \subset \mathcal{F} : \bigcup_{F \in \mathcal{U}} U$ is open in $X\}$. The topological space $(\mathcal{F}, \tau)$ is called the *quotient space* constructed with the topological space X and its decomposition into non-empty subsets $\mathcal{F}$.

Another way to see what the quotient space is would be the following: Let X be a topological space and let $\mathcal{F}$ be a decomposition of X into non-empty susbets. Consider the decomposition map $p : X \to \mathcal{F}$ defined as $p(x) := F$, where F is the unique element of $\mathcal{F}$ such that x is in F. The set $\mathcal{F}$, in principle, is just a family of subsets without any topological substrate. There is no possible way to say whether p is continuous or not unless a topology in $\mathcal{F}$ is defined. If we consider the finest topology τ_p in $\mathcal{F}$ that turns p into a continuous mapping, then τ_p and τ coincide. We conclude that the quotient space is in fact the finest topology in $\mathcal{F}$ that turns the decomposition mapping p into a continuous function.

When X is a "geometrical space" like an interval or a finite-dimensional manifold, if we consider an arbitrary decomposition $\mathcal{F}$, the quotient space may be seem a very strange topological space but, as a matter of fact, the quotient space is often homeomorphic to another "geometrical space". For example, if X is the interval $[0,1]$ and we consider the decomposition $\mathcal{F} := \{\{x\} : x \in (0,1)\} \bigcup \{\{0,1\}\}$, then the quotient space is homeomorphic to S^1. The latter is easily seen since, if we join the points 0 and 1 (the extremes of the interval $[0,1]$) then the interval $[0,1]$ becames, topologically, S^1. On the other hand, if f is a surjective, continuous, and closed function between two topological spaces X and Y, if we consider the following decomposition in X: $\mathcal{F} := \{f^{-1}(y) : y \in Y\}$, then the quotient space is homeomorphic to Y. Further, if f is a surjective and continuous function between a compact topological space X and a Hausdorff space Y, then, under these conditions, f is automatically a closed map, so the quotient space given by X and the decomposition $\mathcal{F} := \{f^{-1}(y) : y \in Y\}$ is homeomorphic to Y. As we see, when we have a topological space X and a decomposition $\mathcal{F}$, in certain cases, the quotient space may be a well-known topological space. As we will see, when X is a specific finite-dimensional manifold and $\mathcal{F}$ satisfies certain conditions, the quotient space will be a finite-dimensional manifold, moreover, it will be homeomorphic to X.

Here X will always be a 2-manifold like either the plane $\mathbb{C}$ or the sphere S^2. If a decomposition into non-empty subsets $\mathcal{F}$ is considered in X, we would like to find out what conditions on $\mathcal{F}$ should be assumed so that the quotient space is homeomorphic to X. First of all, if the quotient space is homeomorphic to X then it has to be a Hausdorff space. Therefore, in particular each point has to be a closed subset of the quotient space. However, as it has been seen above, the decomposition map $p : X \to \mathcal{F}$ is continuous. We conclude that for each $F \in \mathcal{F}$, $F = p^{-1}(F)$ is a closed subset of X. Furthermore, in the case where $X = S^2$, since X is compact we deduce that each $F \in \mathcal{F}$ is a closed subset, and hence it is compact.

On the other hand, if $\mathcal{F}$ is a decomposition of X into compact subsets of X, if the quotient space, that is the result of identifying each $F \in \mathcal{F}$ with a point, is homeomorphic to X, then, intuitively, we can assume that each $F \in \mathcal{F}$ has to be equivalent to a point in some topological sense. For example, we may assume that each $F \in \mathcal{F}$ may be a contractible subset or almost a connected set that does not separate X. Let us observe that with these conditions there is little difference between the cases $X = \mathbb{C}$ and $X = S^2$, because if we consider S^2 as $\overline{\mathbb{C}} := \mathbb{C} \bigcup \{\infty\}$ and $\mathcal{F}$ is a decomposition into compact and connected subsets of S^2, none of which separates S^2, then for one fixed $F \in \mathcal{F}$, $\overline{\mathbb{C}} \smallsetminus F \simeq S^2 \smallsetminus F$ is connected. Therefore, by elementary complex variable analysis, $\Omega := \overline{\mathbb{C}} \smallsetminus F$ is a simply connected domain contained in $\mathbb{C}$. However, it is well-known that if $\Omega \subsetneq \mathbb{C}$, then there exists a bijective and biholomorphic function $h : \Omega \to D(0,1)$ (in particular each simply connected domain of $\mathbb{C}$ is homeomorphic to $\mathbb{C}$). Thus $S^2 \smallsetminus F \simeq \mathbb{C}$ and naturally $\mathcal{F} \smallsetminus \{F\}$ is a decomposition into compact subsets of $S^2 \smallsetminus F$. Finally we shall prove that (with their respective identification topologies) $\mathcal{F} \smallsetminus \{F\} \simeq \mathbb{C}$ if and only if $\mathcal{F} \simeq S^2$. Indeed, if $\mathcal{F} \simeq S^2$ then naturally $\mathcal{F} \smallsetminus \{F\} \simeq S^2 \smallsetminus \{p\} \simeq \mathbb{C}$. Reciprocally if $\mathcal{F} \smallsetminus \{F\} \simeq \mathbb{C}$ then we can see, almost intuitively, that $\mathcal{F} = \{F\} \bigcup \mathcal{F} \smallsetminus \{F\} \simeq \{\infty\} \bigcup \mathbb{C} = \overline{\mathbb{C}} \simeq S^2$.

Let as continue with the question about the assumptions we have to consider on $\mathcal{F}$ so that the quotient space is homeomorphic to X. So far we have consider very reasonable conditions on the decomposition $\mathcal{F}$. Our main concern emerges from the fact although X is a Hausdorff space (since it is a 2-manifold), we cannot dedude that the quotient space is a Hausdorff space as well. Therefore it is necessary to consider another condition on $\mathcal{F}$ to resolve this. Let us suppose that F_1 and F_2 are two different elements of $\mathcal{F}$. Since X is a normal topological space, it is well-known that there exist two open disjoint subsets U_1, U_2 of X such that $F_i \subset U_i$ for each $i = 1, 2$. However if we now define (for each $i = 1, 2$)

$$V_i := \bigcup_{F \in \mathcal{F}, F \subset U_i} F,$$

then V_1 and V_2 are two disjoint, non-empty subsets of X and, of course, $p^{-1}(\mathcal{V}_i) = V_i$, where $\mathcal{V}_i := p(V_i)$. Observe that $\mathcal{V}_i$ $(i = 1, 2)$ are open in the quotient space if and only if V_i $(i = 1, 2)$ are open in X. We conclude that if we assume this condition we will have that the quotient space will be a Hausdorff space. In Daverman's book [40] this condition is called "upper semicontinity". We present next some equivalent statements to upper semicontinuity:

Proposition 2.4. *Let $\mathcal{F}$ be a decomposition of a topological space X into closed subsets. The following statements are equivalent:*

1. *$\mathcal{F}$ is upper semicontinuous.*
2. *For each open subset U in X the set $V := \bigcup_{F \in \mathcal{F}, F \subset U} F$ is an open subset in X.*
3. *The decomposition map $p : X \to \mathcal{F}$ is closed.*

As for the case $X = \mathbb{C}$, in the 1920s and 1930s, R. L. Moore proved the following sufficient condition on $\mathcal{F}$ so that the quotient is $X = \mathbb{C}$:

Theorem 2.5 (Moore). *If $\mathcal{F}$ is an upper semicontinuous decomposition of the plane $\mathbb{C}$ into continua, none of which separates $\mathbb{C}$, then $\mathcal{F}$ is homeomorphic to $\mathbb{C}$.*

It has already been mentioned that the case $X = S^2$ is very similar. As far as we are concerned, the case $X = S^2$ is more important and, as a matter of fact, we will have to deal with it in this section.

Next we provide provide a brief account of some auxiliary results from topology and complex analysis that will be crucial in order to proceed with the proof of our main result.

2.3 Preliminary results from topology and complex analysis

We begin with a couple of propositions on convex analysis and topology. Recall first that the *convex hull* (or convex envelope) of a set X of points in $\mathbb{R}^n$ is defined as the smallest convex set that contains X, and it is denoted by $\mathrm{co}\{X\}$ (see, e.g., [73]).

Proposition 2.6. *Let A be a 2-dimensional Euclidean affine space and let $\varnothing \neq C \subset A$ be a convex, compact set. Then exactly one of the following properties holds:*

1. *C is a segment.*
2. *$Int(C) \neq \varnothing$.*

Proof. Let us suppose that C is not a segment. Since $card(C) \geq 2$, let $x, y \in C$ with $x \neq y$. Now, consider the straight line r defined by x and y. Since C is not a segment, there is $z \in C$ that does not belong to r. Consider now the convex hull of x, y, z, denoted $\mathrm{co}\{x, y, z\}$. We have that $C \supset Int(\mathrm{co}\{x, y, z\}) \neq \varnothing$, from what we have that $Int(C) \neq \varnothing$. □

The following result is the 3-dimensional analogue of Proposition 2.6.

Proposition 2.7. *Let A be a 3-dimensional Euclidean affine space and let $\varnothing \neq C \subset A$ be a convex, compact set. Then exactly one of the following properties holds:*

1. *C lies in a plane.*

2. $Int(C) \neq \varnothing$.

Proof. Suppose that C does not lie in a plane. Let $x, y \in C$, with $x \neq y$, and let r be the straight line defined by x and y. Since C does not lie in a plane, we have that there exists $z \in C$, and not belonging to r, such that the Euclidean affine space E generated by x, y, z is a plane with $E \subset A$. Next, take $w \in C \setminus E$. We now have that the tetrahedral of vertices x, y, z, w lies in C and, thus, $\mathrm{co}\{x, y, z, w\} \subset C$. Hence, as we inferred in the proof of Proposition 2.6, $Int(C) \neq \varnothing$. □

The following result shall also be of crucial importance for our purposes.

Theorem 2.8. *Let A be a n-dimensional Euclidean affine space and let $C \subset A$ be a convex, compact subset such that $Int(C) \neq \varnothing$. We have that ∂C is homeomorphic to S^{n-1} ($\partial C \simeq S^{n-1}$).*

Proof. Let $x_0 \in C$ and $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset C$. Since C is convex, given $u \in S^{n-1}$ there is a unique $x_u \in \partial C$ such that

$$\partial C \cap \{x_0 + tu \,:\, t > 0\} = \{x_u\}.$$

Also, notice that the mapping

$$h : S^{n-1} \to \partial C$$

given by $h(u) \coloneqq x_u$ is a bijection. Moreover, given any sequence $\{u_n\}_{n \in \mathbb{N}} \subset S^{n-1}$ with $u_n \overset{n\to\infty}{\longrightarrow} u$ we have that $x_{u_n} \overset{n\to\infty}{\longrightarrow} x_u$. Therefore, h is the homeomorphism we are looking for. □

Next, let us recall some well known-results from Complex Analysis (that can be found in, for instance, [72]). These shall also be of need in what follows. Recall that a domain –that is, a connected open nonempty set $\Omega \subset \mathbb{C}$– is said to be *simply connected* if it contains the geometrical interior of any closed Jordan curve contained in it.

Proposition 2.9. *Let $\Omega \subset \mathbb{C}$ be a domain, then it is simply connected if and only if $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$ is connected.*

Corollary 2.10. *Let $\varnothing \neq F \subset S^2$ be a closed connected set. We have that, if $S^2 \setminus F$ is connected, then there exists a simply connected domain $\Omega \subset \mathbb{C}$ with $S^2 \setminus F \simeq \Omega$.*

Theorem 2.11. *If $\Omega \subset \mathbb{C}$ is a simply connected domain then $\Omega \simeq \mathbb{C}$.*

2.4 The solution to Gurariy's original question

Here, we shall build a series of constructions in order to prove that $\widehat{\mathcal{C}}(\mathbb{R})$ is not 3-lineable (Theorem 2.33). The basic idea is to proceed by contradiction, assuming that there is a 3-dimensional linear space $E \subset \mathcal{C}(\mathbb{R})$ of functions such that every nonzero function $f \in E$ attains its maximum at one (and only one) point $t_0 \in \mathbb{R}$. As usual, we endow E with the sup norm, defined by

$$\|f\|_\infty := \sup\{|f(t)| : t \in \mathbb{R}\}.$$

Also, if S_E denotes the unit sphere of E, the dual space of E, E', shall be endowed with the dual norm

$$\|x'\| := \sup\{\langle x', f\rangle : f \in S_E\}.$$

If $t \in \mathbb{R}$, we represent the linear evaluation form on E at t as δ_t, that is, $\delta_t(f) = f(t)$ for every $f \in E$.

Let us begin this section with some simple, although necessary, results. We denote by w^* the weak topology on the dual of a topological vector space.

Proposition 2.12. *The mapping $\varphi : \mathbb{R} \to E'$ given by $\varphi(t) = \delta_t$ is continuous.*

Proof. Let $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a convergent sequence of limit t. Thus, for every $f \in E \subset \mathcal{C}(\mathbb{R})$, we have

$$\langle \delta_{t_n}, f\rangle = f(t_n) \overset{n\to\infty}{\longrightarrow} f(t) = \langle \delta_t, f\rangle,$$

from which we have that $\delta_{t_n} \overset{n\to\infty}{\longrightarrow} \delta_t$ in $(E', w^*) = (E', \|\cdot\|) = E'$. □

Corollary 2.13. *The set*

$$J = \{t \in \mathbb{R} : \exists f \in S_E \text{ such that } \max\{f(s) : s \in \mathbb{R}\} = f(t)\}$$

is closed.

Proof. Let $\{t_n\}_{n\in\mathbb{N}} \subset J$ be a convergent sequence $t_n \overset{n\to\infty}{\longrightarrow} t \in \mathbb{R}$. For every $n \in \mathbb{N}$ there is $f_n \in S_E$ such that

$$\max\{f_n(s) : s \in \mathbb{R}\} = f_n(t_n).$$

Since S_E is compact, we can assume (taking a subsequence if needed) that $f_n \overset{n\to\infty}{\longrightarrow} f$ for some $f \in S_E$ which, by Proposition 2.12, gives

$$\langle \delta_{t_n}, f_n\rangle \overset{n\to\infty}{\longrightarrow} \langle \delta_t, f\rangle,$$

implying that

$$\begin{aligned} f(t) &= \langle \delta_t, f\rangle \\ &= \lim_{n\to\infty} \langle \delta_{t_n}, f_n\rangle \\ &= \lim_{n\to\infty} \max\{f_n(s) : s \in \mathbb{R}\} \\ &= \max\{f(s) : s \in \mathbb{R}\}, \end{aligned}$$

and, thus, $t \in J$. This finishes the proof. Notice that the last equality we have used the fact (left as a simple exercise) that under the hypothesis we are dealing with, the identity

$$\lim_{n\to\infty} \max\{f_n(s) : s \in \mathbb{R}\} = \max_{s\in\mathbb{R}} \left\{ \lim_{n\to\infty} f_n(s) \right\}$$

holds (although this is not true in general). □

The set J defined in Corollary 2.13 will play a crucial role in this section and in the proof of the main result. Note that, by definition, $J \neq \varnothing$ and $card(J) \geq 2$ (as a matter of fact, it can be proved that $card(J) = \mathfrak{c}$, but this will not be needed here).

Also, from now on we shall need to refer to the following set:

$$C := \bigcap_{f\in S_E} \{x' \in E' : \langle x', f\rangle \leq \max\{f(s) : s \in \mathbb{R}\}\}. \tag{2.1}$$

Let us now see some properties of the previous set.

Proposition 2.14. *The set C is a continuum. In particular, it is convex, compact and $Int(C) \neq \varnothing$.*

Proof. We have the following:

1. C is convex and closed since, by definition, C is the intersection of closed and convex sets.

2. C is bounded. To see this, take $x' \in C$. Then

$$\langle x', f\rangle \leq \max\{f(s) : s \in \mathbb{R}\} \leq \|f\|_\infty = 1,$$

   for all $f \in S_E$. Therefore $\|x'\| \leq 1$ for all $x' \in C$.

3. $Int(C) \neq \varnothing$. Indeed, assume that $Int(C) = \varnothing$. By Proposition 2.7, C is contained in a plane. Hence there exist $f \in E \setminus \{0\}$ and $c \in \mathbb{R}$ such that $\langle x', f\rangle = c$ for every $x' \in C$. Observe that $\{\delta_t : t \in \mathbb{R}\} \subset C$. Then $f(t) = \langle \delta_t, f\rangle = c$ for all $t \in \mathbb{R}$. This contradicts the fact that $f \in E \setminus \{0\}$ attains its maximum at a single point.

□

Corollary 2.15. $\partial C \simeq S^2$.

Proof. It follows from Proposition 2.14 and Theorem 2.8. □

Lemma 2.16. $\partial C = \bigcup_{f \in S_E} \{x' \in C : \langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\}$.

Proof. The fact that

$$\bigcup_{f \in S_E} \{x' \in C : \langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\} \subset \partial C$$

is clear.

Now take $x' \in \partial C$. Then there is a sequence $\{x'_n\}_{n \in \mathbb{N}}$ in $E' \setminus C$ such that $\lim_{n\to\infty} x'_n = x'$. Hence, for each $n \in \mathbb{N}$ there is $f_n \in S_E$ such that $\langle x'_n, f_n\rangle \geq \max\{f_n(s) : s \in \mathbb{R}\}$. Since S_E is compact, we can assume (taking an appropriate subsequence if necessary) that $\{f_n\}_{n \in \mathbb{N}}$ converges to an element f of S_E. It follows that $\langle x', f\rangle \geq \max\{f(s) : s \in \mathbb{R}\}$ and therefore $\langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\}$ (notice that $x' \in C$). □

Definition 2.17. *Consider the set J defined in Corollary 2.13. For every $t \in J$, we define the set*

$$F_t = \{x' \in C : \ \exists f \in S_E \text{ with } \langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t)\}.$$

Lemma 2.18. *Assume that H is an affine plane such that $Int_H(D) \neq \varnothing$, where $D = C \cap H$ and $Int_H(D)$ stands for the interior of D in H. Then one and only one of the following statements is true:*

(a) $Int_H(D) \subset Int(C)$ and $\partial_H(D) \subset \partial C$, where $\partial_H(D)$ represents the boundary of D in the topological subspace H.

(b) There exists $f \in S_E$ such that

$$H = \{x' \in E' : \langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\}.$$

In particular, by Lemma 2.16, we have $D \subset \partial C$.

Proof. Notice that D is compact, so $D = \partial_H D \cup Int_H(D)$. This shows that (a) and (b) do not hold simultaneously. To finish we have to prove that either (a) or (b) is true. Assume that (a) does not hold. Since $\partial_H D \subset \partial C$ is always true, there exists $y' \in Int_H(D)$ such that $y' \in \partial C$. By Lemma 2.16 we can find $f \in S_E$ such that

$$\langle y', f\rangle = \max\{f(s) : s \in \mathbb{R}\}.$$

If $\pi = \{x' \in E : \langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\}$, we need to prove that $H = \pi$. If $H = \pi$ were not true, the set $\pi \cap H$ would be a line passing through y' as in Figure 2.3. This line divides the plane H into two halves. Let us choose z' in the upper half plane in such a way that $z' \in D$ (see Figure 2.3). For this z' we would have

$$\langle z', f\rangle > \max\{f(s) : s \in \mathbb{R}\},$$

which implies that $z' \notin C$. This contradicts the fact that $D \subset C$. Therefore (b) holds whenever we assume that (a) is not true, which concludes the proof. □

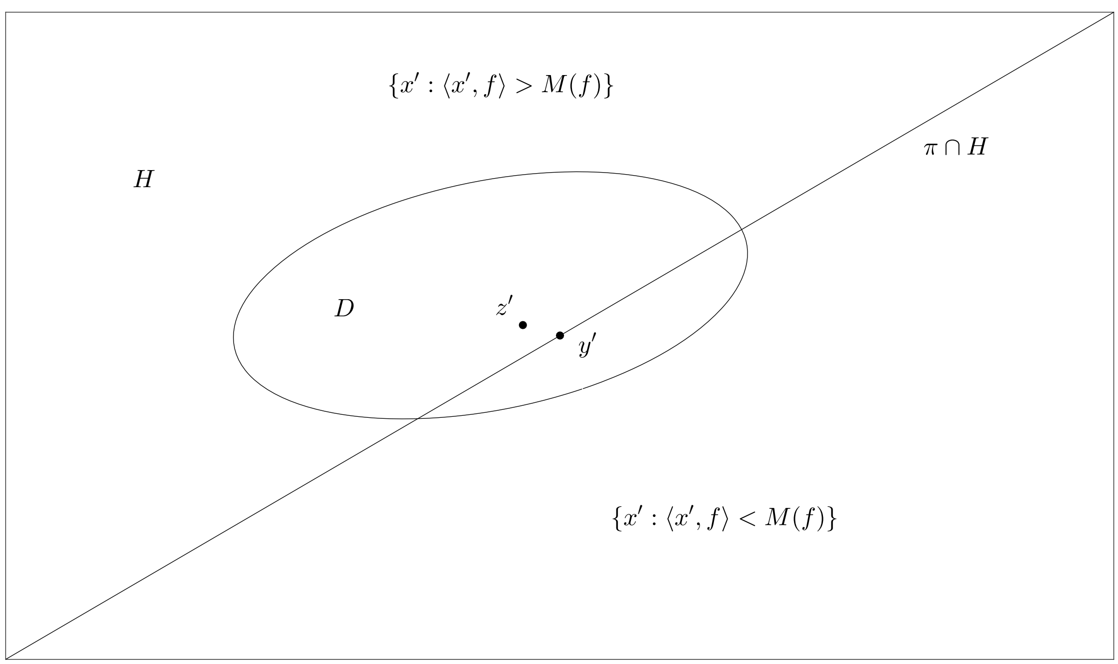

Figure 2.3: Illustration of elements in the proof of Lemma 2.18. Here, $M(f)$ stands for $\max\{f(s) : s \in \mathbb{R}\}$.

The following theorem shall deal with describing some properties of the set F_t given in Definition 2.17.

Theorem 2.19. *The following statements hold true:*

1. $\partial C = \bigcup_{t\in J} F_t$.
2. $\{F_t\}_{t\in J}$ *is a family of pairwise disjoint closed sets.*
3. $\delta_t \in F_t$ *for every* $t \in J$ *and* $[\delta_t, x'] \subset F_t$ *for all* $x' \in F_t$. *In particular,* F_t *is a nonempty connected set. Here* $[\delta_t, x']$ *stands for the segment with endpoints* δ_t *and* x'.
4. $\partial C \setminus F_t$ *is path-connected for all* $t \in J$ *and therefore it is connected.*

Proof. We prove each statement separately:

1. It follows easily from Lemma 2.16 and the definition of J.
2. Let $\{x'_n\}_n$ be a sequence in F_t such that $\lim_{n\to\infty} x'_n = x'$ and choose $f_n \in S_E$ such that $\langle x'_n, f_n\rangle = \max\{f_n(s) : s \in \mathbb{R}\} = f_n(t)$ for every $n \in \mathbb{R}$. Since S_E is compact, we can assume (considering a subsequence if necessary) that $\lim_n f_n = f$. Then

$$\begin{aligned}\langle x', f\rangle &= \lim_{n\to\infty} \langle x'_n, f_n\rangle \\ &= \lim_{n\to\infty} \max\{f_n(s) : s \in \mathbb{R}\} \\ &= \max\{f(s) : s \in \mathbb{R}\}.\end{aligned}$$

   On the other hand $\max\{f_n(s) : s \in \mathbb{R}\} = f_n(t)$ for all $n \in \mathbb{N}$, from which $\max\{f(s) : s \in \mathbb{R}\} = f(t)$. It follows that $\langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t)$ and hence $x' \in F_t$. The previous reasoning shows that F_t is closed for all $t \in J$.

   Suppose now that there exist $t_1, t_2 \in J$ with $t_1 \neq t_2$ such that $F_{t_1} \cap F_{t_2} \neq \varnothing$. Let $x' \in F_{t_1} \cap F_{t_2}$ and $f_1, f_2 \in S_E$ be such that

$$\langle x', f_i\rangle = \max\{f_i(s) : s \in \mathbb{R}\} = f_i(t_i) \quad (i = 1, 2).$$

   Choose $\lambda, \mu > 0$ such that $\lambda f_1 + \mu f_2 \in S_E$. Since f_i attains it maximum only at t_i, $i = 1, 2$, we have:

   (a) $\lambda f_1(t) + \mu f_2(t) < \lambda \max\{f_1(s) : s \in \mathbb{R}\} + \mu \max\{f_2(s) : s \in \mathbb{R}\}$, for all $t \in \mathbb{R} \setminus \{t_1, t_2\}$.

   (b) $\lambda f_1(t_1) + \mu f_2(t_1) = \lambda \max\{f_1(s) : s \in \mathbb{R}\} + \mu f_2(t_1) < \lambda \max\{f_1(s) : s \in \mathbb{R}\} + \mu \max\{f_2(s) : s \in \mathbb{R}\}$.

   (c) $\lambda f_1(t_2) + \mu f_2(t_2) = \lambda f_1(t_2) + \mu \max\{f_2(s) : s \in \mathbb{R}\} < \lambda \max\{f_1(s) : s \in \mathbb{R}\} + \mu \max\{f_2(s) : s \in \mathbb{R}\}$.

From (a), (b) and (c) above it follows that

$$\begin{aligned}\max\{(\lambda f_1 + \mu f_2)(s) : s \in \mathbb{R}\} &< \lambda \max\{f_1(s) : s \in \mathbb{R}\} \\ &\quad + \mu \max\{f_2(s) : s \in \mathbb{R}\} \\ &= \lambda\langle x', f_1\rangle + \mu\langle x', f_2\rangle \\ &= \langle x', \lambda f_1 + \mu f_2\rangle,\end{aligned}$$

from which $x' \notin C$. We arrive at a contradiction.

3. By definition of J it is obvious that $\delta_t \in F_t$ for all $t \in J$. Now, if $x' \in F_t$ there exists $f \in S_E$ such that

$$\langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t).$$

Therefore, since C is convex, $\lambda\delta_t + (1-\lambda)x' \in C$ for every $\lambda \in [0,1]$ and hence

$$\begin{aligned}\langle \lambda\delta_t + (1-\lambda)x', f\rangle &= \lambda f(t) + (1-\lambda) f(t) \\ &= f(t) = \max\{f(s) : s \in \mathbb{R}\}.\end{aligned}$$

Consequently $[\delta_t, x'] \subset F_t$.

4. We will prove that for a fixed $t \in J$ and $x', y' \in \partial C \setminus F_t$ there exists a continuous path $\gamma : [0,1] \to \partial C \setminus F_t$ such that $\gamma(0) = x'$ and $\gamma(1) = y'$. Two cases will be considered:

   - *Case 1.* The points x', y', δ_t are aligned: As in the proof of Theorem 2.8 we can find $p \in Int(C)$ such that p does not lie in the straight line defined by x', y', δ_t. Therefore, if H is the convex hull of $\{x', y', \delta_t, p\}$, then H is a plane. Observe that $D = C \cap H$ is a compact, convex subset of H that cannot be a segment since $x', y', \delta_t, p \in D$ and x', y', δ_t, p generate H. Now, from Proposition 2.6 we have that $Int_H(D) \neq \varnothing$. Also $p \in Int(C)$. Hence, using Lemma 2.18 we arrive at $int_H D \subset Int(C)$ and $\partial_H D \subset \partial C$.

     Let us now show that $\partial_H D \cap F_t$ is connected. Indeed, let $z' \in (\partial_H D) \cap F_t$. By (3) and the convexity of D we have that

     $$[\delta_t, z'] \subset D \cap F_t \subset F_t \subset \partial C.$$

     Therefore $[\delta_t, z'] \subset (\partial_H D) \cap F_t$ since $Int_H(D) \subset Int(C)$ and $\partial_H D \subset \partial C$. Then $[\delta_t, z'] \subset (\partial_H D) \cap F_t$ for all $z' \in (\partial_H D) \cap F_t$ proving that $(\partial_H D) \cap F_t$ is connected.

     On the other hand $\partial_H D \simeq S^1$ by Theorem 2.8. Putting this together with the fact that $(\partial_H D) \cap F_t$ is connected it follows that $(\partial_H D) \setminus F_t$ is path-connected. Hence there exists a continuous path $\gamma : [0,1] \to (\partial_H D) \setminus F_t \subset (\partial C) \setminus F_t$ such that $\gamma(0) = x'$ and $\gamma(1) = y'$.

- *Case 2.* The points x', y', δ_t are not aligned. In this case, consider H to be the affine plane determined by x', y', and δ_t. By Proposition 2.6 we have that, if $D = H \cap C$, $Int_H(D) \neq \varnothing$. Thus, by Lemma 2.18 we have that:

  (I) Either $\partial_H D \subset \partial C$ and $Int_H(D) \subset Int(C)$,

  (II) or there exists $f \in S_E$ such that

  $$H = \{z' \in E' : \langle z', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\}.$$

  Let us suppose that (II) holds. Then, since

  $$x', y', \delta_t \in H = \{z' \in E' : \langle z', f\rangle = \max\{f(s) : s \in \mathbb{R}\}\},$$

  we have that

  $$x', y' \in H = \{z' \in E' : \langle z', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t)\},$$

  having that (since $x', y' \in C$) $x', y' \in F_t$, a contradiction. Therefore we have that, by exclusion, (I) holds.

Next, proceeding as in *Case 1* above we have that there exists a continuous mapping

$$\gamma : [0,1] \longrightarrow (\partial_H D) \smallsetminus F_t \subset (\partial C) \smallsetminus F_t$$

such that $\gamma(0) = x'$ and $\gamma(1) = y'$.

□

From now on we denote ∂C by X. Then X is a compact metric space. By the previous theorem, $\{F_t\}_{t \in J}$ is a partition of X into nonempty continua. The following proposition shall provide us with more properties enjoyed by the set $\{F_t\}_{t \in J}$.

Proposition 2.20. *We have* $X \smallsetminus F_{t_0} \simeq \mathbb{C}$ *for all* $t_0 \in J$.

Proof. We know from Corollary 2.8 that $S^2 \simeq X$. Then $F_{t_0} \subset S^2$ where F_{t_0} has to be understood as the image of F_{t_0} by the homomorphism existing between X and S^2. By Theorem 2.19 we know that F_{t_0} and $S^2 \smallsetminus F_{t_0}$ are connected. Applying now Corollary 2.10 and Theorem 2.11 we have that there is a simply connected domain $\Omega \subset \mathbb{C}$ such that $X \smallsetminus F_{t_0} \simeq S^2 \smallsetminus F_{t_0} \simeq \Omega \simeq \mathbb{C}$. □

Lemma 2.21. *Let* $\{t_n\}_{n \in \mathbb{N}} \subset J$ *be such that* $\delta_{t_n} \overset{n\to\infty}{\longrightarrow} y' \in F_t$. *Given any sequence* $\{x'_n\}_{n \in \mathbb{N}}$ *with* $x'_n \in F_{t_n}$ *for every* $n \in \mathbb{N}$, *we have that, if* $x'_n \overset{n\to\infty}{\longrightarrow} x'$ *and* $t \in J$ *satisfies* $y' \in F_t$, *then* $x' \in F_t$.

Proof. For every $n \in \mathbb{N}$ there is $f_n \in S_E$ such that

$$\langle x', f_n \rangle = \max\{f_n(s) : s \in \mathbb{R}\} = f_n(t_n) = \langle \delta_{t_n}, f \rangle.$$

Taking a subsequence if needed, we can assume that $f_n \overset{n\to\infty}{\longrightarrow} f \in S_E$, from which we have

$$\lim_{n\to\infty} \langle x'_n, f_n \rangle = \lim_{n\to\infty} \max\{f_n(s) : s \in \mathbb{R}\} = \lim_{n\to\infty} \langle \delta_{t_n}, f_n \rangle$$

and, thus,

$$\langle x', f \rangle = \max\{f(s) : s \in \mathbb{R}\} = \langle y', f \rangle.$$

Therefore, we have that

$$\max\{f(s) : s \in \mathbb{R}\} = \langle y', f \rangle = f(t).$$

By Theorem 2.19, and since $y' \in F_t$, we have that $y' \notin F_s$ for every $s \in J \setminus \{t\}$. Hence $\langle x', f \rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t)$, from which $x' \in F_t$, and the proof is finished. □

The following proposition makes use of the notion of *upper semicontinuous decomposition*. Although its proof is self-contained, we refer the reader to the work [40] for more information and a detailed study of this notion. First we define upper semicontinuous decomposition.

Definition 2.22. *Let S be a topological space and let G be a partition of S into closed sets. Then G is upper semicontinuous if and only if for every open set $U \subset S$ the set*

$$U^* = \bigcup_{g \in G,\, g \subset U} g$$

is an open subset of S.

Proposition 2.23. *For every $t_0 \in J$ we have that*

$$\{F_t\}_{t \in J \setminus \{t_0\}}$$

is an upper semicontinuous partition into continua of $X \setminus F_{t_0}$.

Proof. By Theorem 2.19 it can be seen that $\{F_t\}_{t \in J \setminus \{t_0\}}$ is a decomposition of continua of $X \setminus F_{t_0}$. To see that this decomposition is upper semicontinuous we just need to show that for every $U \subset X \setminus F_{t_0}$ open in $X \setminus F_{t_0}$, the set

$$V = \bigcup_{t \in J \setminus \{t_0\},\, F_t \subset U} F_t$$

is open in $X \setminus F_{t_0}$ (see, e.g., [40]). Suppose, by contradiction, that there is an open set U in $X \setminus F_{t_0}$ such that the corresponding V defined above is not open in $X \setminus F_{t_0}$. Then, we would have that there are $x' \in F_t \subset U$ (for some $t \in J \setminus \{t_0\}$) and

$\{x'_n\} \subset (X \smallsetminus F_{t_0}) \smallsetminus V$ with $x'_n \overset{n\to\infty}{\longrightarrow} x'$. This implies that, for every $n \in \mathbb{N}$, $F_{t_n} \smallsetminus U \neq \varnothing$ (where $x'_n \in F_{t_n}$). Thus, for every $n \in \mathbb{N}$, we can select $y'_n \in F_{t_n} \smallsetminus U$.

Next, taking a subsequence if necessary, we can assume that

$$\delta_{t_n} \overset{n\to\infty}{\longrightarrow} z' \in X$$

and

$$y'_n \overset{n\to\infty}{\longrightarrow} y' \in X.$$

Therefore, there exists $t' \in J$ such that $z' \in F_{t'}$. Now, by Lemma 2.21, $x', y' \in F_{t'}$ implies $t' = t$, which yields $y' \in F_t$. This is a contradiction because the fact $\{y'_n\}_{n\mathbb{N}} \subset (X \smallsetminus F_{t_0}) \smallsetminus U$ implies $y' \in (X \smallsetminus F_{t_0}) \smallsetminus U$, which gives $y' \notin U \supset F_t$. The proof is concluded. □

Proposition 2.24. *For every pair of points $t_1, t_2 \in J$, the set $X \smallsetminus (F_{t_1} \cup F_{t_2})$ is connected.*

Proof. Using Proposition 2.20, we obtain

$$X \smallsetminus F_{t_1} \simeq \mathbb{C} \simeq S^2 \smallsetminus \{(0,0,1)\}.$$

Therefore, there exists a homeomorphism $g_1 : X \smallsetminus F_{t_1} \to S^2 \smallsetminus \{(0,0,1)\}$. We can extend g_1 to X by defining $g_1(x') = (0,0,1)$ for $x' \in F_{t_1}$. Let us show that $g_1 : X \to S^2$ is continuous and surjective. Surjectivity is obvious. To prove continuity, observe that g_1 restricted to $X \smallsetminus F_{t_1}$ is continuous and that $X \smallsetminus F_{t_1}$ is open. Then g_1 is continuous in $X \smallsetminus F_{t_1}$. Now suppose that g_1 is not continuous at a given point $x' \in F_{t_1}$. Hence there exists a sequence (x'_n) in X such that $x'_n \overset{n\to\infty}{\longrightarrow} x'$ but $(g_1(x'_n))$ does not converge to $g_1(x') = (0,0,1)$. Considering a subsequence if necessary we can assume that $(g_1(x'_n))$ converges to a certain $u \in S^2 \smallsetminus \{(0,0,1)\}$ and $g_1(x'_n) \neq (0,0,1)$ for all $n \in \mathbb{N}$. Since $(x'_n) \subset X \smallsetminus F_{t_1}$ and $g_1|_{X\smallsetminus F_{t_1}} : X \smallsetminus F_{t_1} \to S^2 \smallsetminus \{(0,0,1)\}$ is a homeomorphism, there exists $y' \in X \smallsetminus F_{t_1}$ such that $x'_n \overset{n\to\infty}{\longrightarrow} y'$, which contradicts the fact that (x'_n) also converges to x'.

Next, notice that, by Theorem 2.19, the set $X \smallsetminus F_{t_2}$ is connected, and so is $g_1(X \smallsetminus F_{t_2})$. It is also simple to check that $\{g_1(F_{t_2}), g_1(X \smallsetminus F_{t_2})\}$ is a partition of S^2 in which $g_1(F_{t_2})$ is compact (and, thus, closed) and, therefore, $g_1(X \smallsetminus F_{t_2})$ is open. Now, by Corollary 2.10 and Theorem 2.11,

$$g_1(X \smallsetminus F_{t_2}) \simeq \mathbb{C} \simeq S^2 \smallsetminus \{(0,0,1)\}.$$

Thus, there exists an homeomorphism

$$g_2 : g_1(X \smallsetminus F_{t_2}) \to S^2 \smallsetminus \{(0,0,1)\},$$

that can be extended to S^2 by defining $g_2(u) := (0,0,1)$ if $u \in g_1(F_{t_2})$, having that $g_2 : S^2 \to S^2$ is continuous and surjective. Taking now the composite mapping $g = g_2 \circ g_1$, we see (by construction and by Theorem 2.19) that $\delta_{t_i} \in F_{t_i}$, $g_{|X \setminus (F_{t_1} \cup F_{t_2})}$ in an homeomorphism onto its image and $g(F_{t_i}) = \{g(\delta_{t_i})\}$ for $i = 1, 2$. Therefore, and since g is surjective,

$$X \setminus (F_{t_1} \cup F_{t_2}) \simeq S^2 \setminus \{g(\delta_{t_1}), g(\delta_{t_2})\},$$

and, consequently, $X \setminus (F_{t_1} \cup F_{t_2})$ is connected. □

Corollary 2.25. *For every $t_0 \in J$ we have that $X \setminus F_{t_0} \simeq \mathbb{C}$. That homeomorphism transforms $\{F_t\}_{t \in J \setminus \{t_0\}}$ into an upper semicontinuous decomposition of $\mathbb{C}$ into continua that do not separate $\mathbb{C}$.*

Proof. It is the result of applying Propositions 2.20, 2.23, and 2.24. □

In the rest of the chapter we shall be using the concept of *identification topology.* Recall that this topology is defined as follows: Let $\mathcal{X}$ be a topological space, $\mathcal{Y}$ a partition of $\mathcal{X}$ and the surjective mapping $\pi : \mathcal{X} \to \mathcal{Y}$ given by $\pi(x) = y$, where $x \in y$. Then the identification topology in $\mathcal{Y}$, denoted by T_π, is defined as

$$T_\pi = \{U \subset \mathcal{Y} : \pi^{-1}(U) \text{ is open in } \mathcal{X}\}.$$

We can consider the identification topology in the family $\mathcal{F} := \{F_t : t \in J\}$ induced by the surjective mapping $p : X \to \mathcal{F}$ defined by $p(x') = F_t$ whenever $x' \in F_t$. Notice that the F_t's are topological continua for every $t \in J$ by Theorem 2.19.

Similarly, if $t_0 \in J$ then we can also consider the partition of $X \setminus F_{t_0}$ given by $\mathcal{F}_{t_0} := \{F_t : t \in J \setminus \{t_0\}\}$, and the initial topology in $\mathcal{F}_{t_0}$ induced by the surjective mapping $p_{t_0} : X \setminus F_{t_0} \to \mathcal{F}_{t_0}$ defined by $p_{t_0} := p_{|X \setminus F_{t_0}}$.

Proposition 2.26. *Let $t_0 \in J$ and $U \subset \mathcal{F}_{t_0}$. The following statements are equivalent:*

1. *U is T_p-open in $\mathcal{F}_{t_0}$.*
2. *U is T_p-open.*
3. *U is $T_{p_{t_0}}$-open.*

Proof. That (1) and (2) are equivalent follows from the fact that $\mathcal{F}_{t_0}$ is a T_p-open set. On the other hand, U is $T_{p_{t_0}}$-open if and only if $p_{t_0}^{-1}(U)$ is open in $X \setminus F_{t_0}$, or equivalently, if $p^{-1}(U) = p_{t_0}^{-1}(U)$ is open in X. Hence, (2) is equivalent to (3). □

The previous result justifies the identification $(\mathcal{F}_{t_0}, T_{p_{t_0}}) = (\mathcal{F}_{t_0}, T_p)$ for every $t_0 \in J$. We will use from now on the notation $\mathcal{F}_{t_0}$ to denote $(\mathcal{F}_{t_0}, T_{p_{t_0}})$ or $(\mathcal{F}_{t_0}, T_p)$.

Now, let us recall a result by Moore (see, e.g., [40, 77]) which will be of crucial importance in order to achieve our main result (this version of the theorem is stated for the complex plane).

Theorem 2.27. *Assume that G is an upper semicontinuous decomposition of the plane $\mathbb{C}$ into continua, none of which separates $\mathbb{C}$. If G is endowed with the identification topology, then G is homeomorphic to $\mathbb{C}$.*

Theorem 2.28. *For every $t_0 \in J$, $\mathcal{F}_{t_0} \simeq \mathbb{C}$.*

Proof. It is the result of, first, applying Corollary 2.25 and, then, Moore's theorem (Theorem 2.27). □

Lemma 2.29. *The topological space $\mathcal{F}$ satisfies the first axiom of countability.*

Proof. Let $F_t \in \mathcal{F}$ with $t \in J$. Fix $t_0 \in J \setminus \{t\}$. Then $F_t \in \mathcal{F}_{t_0}$, which is an open set and $\mathcal{F}_{t_0} \simeq \mathbb{C}$. Hence F_t possesses a countable basis of open neighborhoods. □

Theorem 2.30. $\mathcal{F} \simeq S^2$.

Proof. Let us fix $t_\infty \in J$. By Lemma 2.29 we know that $\mathcal{F}_{t_\infty} \simeq \mathbb{C} \simeq S^2 \setminus \{(0,0,1)\}$. Consider a homeomorphism $h : \mathcal{F}_{t_\infty} \to S^2 \setminus \{(0,0,1)\}$. If we extend h to $\mathcal{F}$ by defining $h(F_{t_\infty}) := (0,0,1)$, h is continuous at F_t for all $t \in J \setminus \{t_\infty\}$ since $\mathcal{F}_{t_\infty}$ is open and $h_{|\mathcal{F}_{t_\infty}}$ is continuous by definition. It remains to prove that h is also continuous at F_{t_∞}.

Assume h is not continuous at F_{t_∞}. Since $\mathcal{F}$ enjoys the fist axiom of countability, there exists $\{F_{t_n}\} \subset \mathcal{F}$ such that $F_{t_n} \longrightarrow F_{t_\infty}$ but $h(F_{t_n})$ does not converge to $h(F_{t_\infty})$. By compacity of S^2 we can take for granted that $\{F_{t_n}\} \subset \mathcal{F}_{t_\infty}$ and $h(F_{t_n}) \longrightarrow u \in S^2 \setminus \{(0,0,1)\}$ (considering a subsequence if needed). Now, $h : \mathcal{F}_{t_\infty} \to S^2 \setminus \{(0,0,1)\}$ is a homeomorphism, so there exists $t \in J \setminus \{t_\infty\}$ such that $h(F_t) = u$ and $F_{t_n} \longrightarrow F_t$. Taking $t_0 \in J$ with $t_0 \neq t, t_\infty, t_n$ for all $n \in \mathbb{N}$ we see that $\{F_{t_n}\} \subset \mathcal{F}_{t_0}$ and $F_{t_n} \longrightarrow F_{t_\infty}$. Therefore $\mathcal{F}_{t_0}$ would not be a Haussdorff space, contradicting the fact that $\mathcal{F}_{t_0} \simeq \mathbb{C}$.

Therefore $h : \mathcal{F} \to S^2$ is continuous. Also, since h is bijective and $\mathcal{F}$ is compact, we conclude that h is a homeomorphism. □

Remark 2.31. *From now on we will consider the mapping $\gamma : J \to X$ defined by $\gamma(t) = \delta_t$ for every $t \in J$. Observe that γ is continuous from Proposition 2.12.*

We are now ready to prove that there are no 3-dimensional spaces in $\widehat{\mathcal{C}}[a,b)\cup\{0\}$ or $\widehat{\mathcal{C}}(\mathbb{R})\cup\{0\}$. The proof of the latter statement is based on the construction of a continuous bijection $q: J \to S^2$ and the following remark:

Remark 2.32. *Suppose $q : J \to S^2$ is a continuous bijection. Notice that the sets $J_n = J \cap [-n, n]$ are compact since J is closed (see Corollary 2.13). Then $q_{|J_n}$ would be continuous, injective and closed. Then $q_{|J_n} : J_n \to q(J_n)$ is a homeomorphism. Hence $Int(q(J_n)) = \varnothing$ and $q(J_n)$ would be closed. Since $S^2 = \bigcup_{n=1}^{\infty} q(J_n)$, S^2 would be a first category space, which is impossible because S^2 is a Baire space.*

Theorem 2.33. *Let $a, b \in \mathbb{R}$, $a < b$. Let V stand for a subspace of either $\mathcal{C}[a,b)$ or $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V attains its maximum at one (and only one) point. Then $dim(V) \leq 2$.*

Proof. The existence of a 3-dimensional space in $\widehat{\mathcal{C}}[a,b)\cup\{0\}$ or $\widehat{\mathcal{C}}(\mathbb{R})\cup\{0\}$ implies, as we have seen throughout the chapter, the existence of the following three mappings:

1. $\gamma : J \to X$, which is continuous.

2. $p : X \to \mathcal{F}$, which is continuous and surjective.

3. $h : \mathcal{F} \to S^2$ which is a homeomorphism.

Then the composition $q := h \circ p \circ \gamma$ turns out to be a continuous bijection between J and S^2. Indeed:

- q is injective: If t_1, t_2 are two distinct elements of J then $p(\gamma(t_1)) = p(\delta_{t_1}) = F_{t_1} \neq F_{t_2} = p(\delta_{t_2}) = p(\gamma(t_1))$, and since h is injective we have $q(t_1) = h(p(\gamma(t_1))) = h(F_{t_1}) \neq h(F_{t_2}) = h(p(\gamma(t_2))) = q(t_2)$.

- q is surjective: Let $u \in S^2$. Then there exists $t \in J$ such that $h(F_t) = u$ because h is surjective. Hence $q(t) = h(p(\gamma(t))) = h(p(\delta_t)) = h(F_t) = u$.

- q is continuous because it is the composition of continuous functions.

Thus we arrive at a contradiction by Remark 2.32. $\square$

2.5 The generalization of Gurariy's problem

It is natural to wonder whether a similar problem could be considered for values of $m \geq 2$, that is:

> *Let $m \geq 2$. If $m \in \mathbb{N}$ and V_m stands for a subspace of mappings in $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V_m attains its maximum at m (and only m) points, then ... How big can $\dim(V_m)$ get?*

Here we shall show that $\dim(V_m) \leq 2$. Let us first show first that there actually exists a 2-dimensional vector space of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in it attains its maximum at m (and only m) points.

Example 2.34. *First of all, it is clear that for every $\lambda, \mu \in \mathbb{R}$ with $(\lambda, \mu) \neq (0,0)$, there is a unique point $t_0 \in [0, 2\pi)$ such that $\{t_0, t_0 + 2\pi, \ldots, t_0 + (m-1)2\pi\}$ is the set of points of $[0, 2\pi m)$ at which the function $\lambda \sin t + \mu \cos t$ attains its maximum.*

Next, consider the functions

$$f(t) = \cos(4m \arctan t) \quad \text{and} \quad g(t) = \sin(4m \arctan t).$$

It is a simple exercise to check that every nontrivial linear combination of f and g above attains its maximum at exactly m points in $[0, \infty)$.

Secondly, take

$$h_1(t) = \begin{cases} e^t f(-t) & \text{if } t \leq 0, \\ f(t) & \text{if } t \geq 0, \end{cases}$$

and

$$h_2(t) = \begin{cases} e^t g(-t) & \text{if } t \leq 0, \\ g(t) & \text{if } t \geq 0, \end{cases}$$

Having defined h_1 and h_2 (see Figure 2.4) as elements of $\mathcal{C}(\mathbb{R})$, the linear space we are looking for is, precisely, $\mathrm{span}\{h_1, h_2\}$.

In the remaining of this section we shall focus on proving that if V_m is a subspace of $\mathcal{C}(\mathbb{R})$ whose non-zero elements attain their maximum at exactly m points, with $m > 1$, then $\dim(V_m) \leq 2$. As in Section 3, we shall proceed by contradiction, assuming that there is a 3-dimensional linear space $E \subset \mathcal{C}(\mathbb{R})$ of mappings such that every nonzero function $f \in E$ attains its maximum at m (and only m) points. Again, we endow E with the sup norm $\|f\|_\infty$ and, if S_E denotes the unit sphere of E, the dual space E' of E will be endowed with the dual norm $\|x'\| := \sup\{\langle x', f\rangle : f \in S_E\}$.

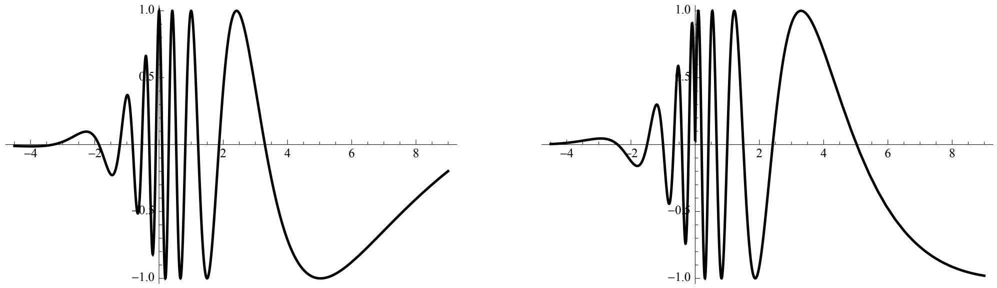

Figure 2.4: Plots of $h_1(t)$ and $h_2(t)$, respectively, for $m = 4$.

As in the previous section, we represent the linear evaluation form on E at the point $t \in \mathbb{R}$ as δ_t, that is, $\delta_t(f) = f(t)$ $(f \in E)$.

Recall from the previous section that the set

$$J := \left\{ t \in \mathbb{R} : \exists f \in S_E \text{ such that } \max\{f(s) : s \in \mathbb{R}\} = f(t) \right\}$$

is closed and that, by definition, $J \neq \varnothing$ and $card(J) \geq 2$.

Proposition 2.35. *For every $t_1 \in J$ there is a unique set of $m-1$ points*

$$\{t_2, \ldots, t_m\} \subset J$$

such that, given any function $f \in S_E$, the following are equivalent:

1. *There is $i_0 \in \{1, \ldots, m\}$ such that $f(t_{i_0}) = \max\{f(t) : t \in \mathbb{R}\}$.*
2. *For every $i \in \{1, \ldots, m\}$, $f(t_i) = \max\{f(t) : t \in \mathbb{R}\}$.*

Proof. We shall first prove the existence part. Since $t_1 \in J$, there is $g \in S_E$ such that $g(t_1) = \max\{g(t) : t \in \mathbb{R}\}$ and, thus, there exist $t_2, \ldots, t_m \in J$ such that $\{t \in \mathbb{R} : g(t) = \max\{g(s) : s \in \mathbb{R}\}\} = \{t_1, \ldots, t_m\}$. Next, given $f \in S_E$, we have (2) implies (1) above is trivial. Let us see that (1) implies (2).

In order to see this, let $\lambda, \mu > 0$ such that $\lambda f + \mu g \in S_E$. Notice that

$$\max\{\lambda f(s) + \mu g(s) :\ s \in \mathbb{R}\} \leq \lambda \max\{f(s) :\ s \in \mathbb{R}\} + \mu \max\{g(s) :\ s \in \mathbb{R}\}$$

and $\lambda f(t_{i_0}) + \mu g(t_{i_0}) = \lambda \max\{f(s) : s \in \mathbb{R}\} + \mu \max\{g(s) : s \in \mathbb{R}\}$.

Therefore, we obtain the following equivalence:

$\lambda f + \mu g$ attains its maximum at t if and only if f and g attain their corresponding maxima at t.

Consequently, we have that

$$\{t \in \mathbb{R} : \lambda f(t) + \mu g(t) = \max\{\lambda f(s) + \mu g(s) : s \in \mathbb{R}\}\} \subset \{t_1, \dots, t_m\},$$

and, for being a set of cardinality m, we have that the following identity holds

$$\left\{t \in \mathbb{R} : \lambda f(t) + \mu g(t) = \max\{\lambda f(s) + \mu g(s) : s \in \mathbb{R}\}\right\} = \{t_1, \dots, t_m\},$$

and, finally, by the previous equivalence, we have that f attains its maximum at $t_1, \dots, t_m$.

Next, let us see about the uniqueness. Suppose we have two distinct sets of $m-1$ points, $t_2, \dots, t_m$ and $s_2, \dots, s_m$, with the property. Since $t_1 \in J$ there exists $g \in S_E$ with $g(t_1) = \max\{g(t) : t \in \mathbb{R}\}$. Thus, notice that

$$\{t_1, t_2, \dots, t_m\} \cup \{t_1, s_2, \dots, s_m\} \subset \left\{t \in \mathbb{R} : g(t) = \max\{g(s) : s \in \mathbb{R}\}\right\},$$

is a set of cardinality m, and, therefore we can conclude that

$$\{t_1, t_2, \dots, t_m\} = \{t_1, s_2, \dots, s_m\},$$

as required. $\square$

The following definition shall be crucial in what follows. As usual, we denote by $\mathcal{P}(J)$ the family of subsets of J.

Definition 2.36. *We define the mapping* $\psi : J \to \mathcal{P}(J)$ *as*

$$\psi(t_1) = \{t_1, \dots, t_m\},$$

where $t_2, \dots, t_m$ *are the points defined in Proposition 2.35 (and related to* t_1 *in the natural way given in the previous proposition). We shall also denote, for short,* $H = \psi(J)$.

Remark 2.37. *For every* $h = \{t_1, \dots, t_m\} \in H$*, we have* $\psi(t_i) = h$ *for every* $i \in \{1, \dots, m\}$.

Proposition 2.38. *For every* $h_1, h_2 \in H$*, if* $h_1 \cap h_2 \neq \varnothing$ *then* $h_1 = h_2$.

Proof. Let $t \in h_1 \cap h_2$. Then, by the previous remark, we have $h_1 = \psi(t_i) = h_2$. $\square$

Corollary 2.39. *The set* H *is a partition of* J *in subsets of cardinality* m.

Recall now the definition of the set C from the previous section:

$$C = \bigcap_{f \in S_E} \left\{ x' \in E' : \langle x', f \rangle \leq \max\{f(s) : s \in \mathbb{R}\} \right\}.$$

We have already proved that C is a convex and compact set with nonempty interior, which, together with Theorem 2.8 shows the following.

Corollary 2.40. *$\partial C \simeq S^2$.*

The following definition is the analogue of the earlier Definition 2.17.

Definition 2.41. *Given $h \in H$, we define the set*

$$F_h = \left\{ x' \in C : \ \exists f \in S_E \ \textit{with} \ \langle x', f \rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t) \ \forall t \in h \right\}.$$

The following theorem shall deal with describing some properties of the set F_h given in Definition 2.41. Its proof is similar to that of Theorem 2.19, so we leave the details to the reader.

Theorem 2.42. *The following statements hold true:*

1. *$\partial C = \bigcup_{h \in H} F_h$.*
2. *$\{F_h\}_{h \in H}$ is a family of pairwise disjoint closed sets.*
3. *For every $h \in H$ and every $t \in h$, one has that $\delta_t \in F_h$ and $[\delta_t, x'] \subset F_h$ for all $x' \in F_h$. In particular, F_h is a nonempty connected set. Here $[\delta_t, x']$ stands for the segment with endpoints δ_t and x'.*
4. *$\partial C \setminus F_h$ is path-connected for all $h \in H$ and therefore it is connected.*

From now on we denote ∂C by X. Then X is a compact metric space. By the previous theorem, $\{F_h\}_{h \in H}$ is a partition of X into nonempty continua. The following proposition, whose proof can be mimicked from Proposition 2.20 in the previous section, shall provide us with more properties enjoyed by the set $\{F_h\}_{h \in H}$ that will be used later.

Proposition 2.43. *For all $h_0 \in H$, we have $X \setminus F_{h_0} \simeq \mathbb{C}$.*

The following lemma is analogous to Lemma 2.21.

Lemma 2.44. *Let* $\{t_n\}_{n\in\mathbb{N}} \subset J$ *be such that* $\delta_{t_n} \overset{n\to\infty}{\longrightarrow} y' \in F_h$. *Given any sequence* $\{x'_n\}_{n\in\mathbb{N}}$ *with* $x'_n \in F_{h_n}$ *where* $t_n \in h_n$ *for every* $n \in \mathbb{N}$, *we have that, if* $x'_n \overset{n\to\infty}{\longrightarrow} x'$ *and* $h \in H$ *satisfies* $y' \in F_h$, *then* $x' \in F_h$.

Proof. For every $n \in \mathbb{N}$ there is $f_n \in S_E$ such that

$$\langle x', f_n\rangle = \max\{f_n(s) : s \in \mathbb{R}\} = f_n(t_n) = \langle \delta_{t_n}, f\rangle.$$

Taking a subsequence if needed, we can assume that $f_n \overset{n\to\infty}{\longrightarrow} f \in S_E$, from which we have

$$\lim_{n\to\infty} \langle x'_n, f_n\rangle = \lim_{n\to\infty} \max\{f_n(s) : s \in \mathbb{R}\} = \lim_{n\to\infty} \langle \delta_{t_n}, f_n\rangle$$

and, thus,

$$\langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = \langle y', f\rangle.$$

Since $y' \in F_h$ and, by Theorem 2.42, $y' \notin F_{h'}$ for all $h' \in H \smallsetminus \{h\}$, we have

$$\max\{f(s) : s \in \mathbb{R}\} = \langle y', f\rangle = f(t),$$

for every $t \in h$. Therefore $\langle x', f\rangle = \max\{f(s) : s \in \mathbb{R}\} = f(t)$, for all $t \in h$, from which $x' \in F_h$, and the proof is finished. □

The following proposition, which is analogous to Proposition 2.23, makes use of the notion of upper semicontinuous decomposition given in Section 3.

Proposition 2.45. *For every* $h_0 \in H$ *we have that*

$$\{F_h\}_{h\in H\smallsetminus\{h_0\}}$$

is an upper semicontinuous partition into continua of $X \smallsetminus F_{h_0}$.

Proof. From Theorem 2.42 it follows that $\{F_h\}_{h\in H\smallsetminus\{h_0\}}$ is a decomposition of continua of $X \smallsetminus F_{h_0}$. To see that this decomposition is upper semicontinuous we just need to show that for every $U \subset X \smallsetminus F_{h_0}$ open in $X \smallsetminus F_{h_0}$, the set

$$V = \bigcup_{h\in H\smallsetminus\{h_0\},\, F_h\subset U} F_h$$

is open in $X \smallsetminus F_{h_0}$ (see e.g. [40]). Suppose, by contradiction, that given U, an open set in $X \smallsetminus F_{h_0}$, the corresponding V defined above is not open in $X \smallsetminus F_{h_0}$. Then, we would have that there are $x' \in F_h \subset U$ (for some $h \in H \smallsetminus \{h_0\}$) and $\{x'_n\} \subset (X \smallsetminus F_{h_0}) \smallsetminus V$ with $x'_n \overset{n\to\infty}{\longrightarrow} x'$. This implies that, for every $n \in \mathbb{N}$, $F_{h_n} \smallsetminus U \neq \varnothing$ (where $x'_n \in F_{h_n}$). Thus, for every $n \in \mathbb{N}$, let us take $y'_n \in F_{h_n} \smallsetminus U$.

Next, taking a subsequence if necessary, we can assume that

$$\delta_{t_n} \overset{n\to\infty}{\longrightarrow} z' \in X, \text{ where } t_n \in h_n$$

and that

$$y'_n \overset{n\to\infty}{\longrightarrow} y' \in X.$$

Therefore, there exists $h' \in H$ such that $z' \in F_{h'}$. Now, by Lemma 2.44, $x', y' \in F_{h'}$ implies $h' = h$, which gives that $y' \in F_h$. This is a contradiction because the fact $\{y'_n\}_{n\in\mathbb{N}} \subset (X \setminus F_{h_0}) \setminus U$ implies $y' \in (X \setminus F_{h_0}) \setminus U$, which gives $y' \notin U \supset F_h$. □

The following proposition is a direct consequence of Proposition 2.45.

Proposition 2.46. *For every $h_1, h_2 \in H$, the set $X \setminus (F_{h_1} \cup F_{h_2})$ is connected.*

Corollary 2.47. *For every $h_0 \in H$ we have that $X \setminus F_{h_0} \simeq \mathbb{C}$. That homeomorphism transforms $\{F_h\}_{h\in H\setminus\{h_0\}}$ into an upper semicontinuous decomposition of the complex plane $\mathbb{C}$ into continua that do not separate $\mathbb{C}$.*

Proof. It is the result of applying Propositions 2.43, 2.45 and 2.46. □

We can consider the topology in the family $\mathcal{F} := \{F_h : h \in H\}$ induced by the surjective mapping $p : X \to \mathcal{F}$ defined by $p(x') = F_h$ whenever $x' \in F_h$. Notice that the F_h's are topological continua for every $h \in H$ by Theorem 2.42. Similarly, if $h_0 \in H$ we can also consider the partition of $X \setminus F_{h_0}$ given by $\mathcal{F}_{h_0} := \{F_h : h \in H \setminus \{h_0\}\}$, and the identification topology in $\mathcal{F}_{h_0}$ induced by the surjective mapping $p_{h_0} : X \setminus F_{h_0} \to \mathcal{F}_{h_0}$ defined by $p_{h_0} := p_{|X\setminus F_{h_0}}$. Similarly to Proposition 2.26, we also have the following assertion.

Proposition 2.48. *Let $h_0 \in H$ and $U \subset \mathcal{F}_{h_0}$. The following statements are equivalent:*

1. *U is T_p-open in $\mathcal{F}_{h_0}$.*
2. *U is T_p-open.*
3. *U is $T_{p_{h_0}}$-open.*

The previous result, as we did in Section 3, justifies the identification $(\mathcal{F}_{h_0}, T_{p_{h_0}}) = (\mathcal{F}_{h_0}, T_p)$ for every $h_0 \in H$. We will use from now on the notation $\mathcal{F}_{h_0}$ to denote $(\mathcal{F}_{h_0}, T_{p_{h_0}})$ or $(\mathcal{F}_{h_0}, T_p)$.

Again, as in Section 3, we have that the following assertion is the result of, first, applying Corollary 2.47 and, then, Moore's Theorem (Theorem 2.27).

Theorem 2.49. *For every $h_0 \in H$, $\mathcal{F}_{h_0} \simeq \mathbb{C}$.*

Of course, as we previously showed, we also have:

Theorem 2.50. $\mathcal{F} \simeq S^2$.

Remark 2.51. *From now on we will consider the mapping $\gamma : J \to X$ defined by $\gamma(t) = \delta_t$ for every $t \in J$. Observe that γ is continuous from Proposition 2.12. Moreover, we observe that $\gamma^{-1}(F_h) = h$ for all $h \in H$.*

Definition 2.52. *Let us consider the following three mappings:*

1. $\gamma : J \to X$, *which is continuous.*
2. $p : X \to \mathcal{F}$, *which is continuous and surjective.*
3. $g : \mathcal{F} \to S^2$, *which is an homeomorphism.*

We define the mapping $q : J \to S^2$ as $q := g \circ p \circ \gamma$.

The following result and its corresponding corollary is also a consequence of previous calculations from Section 3 as well.

Proposition 2.53. *By construction, the mapping q is continuous and, for every $u \in S^2$, there exists a unique $h \in H$ such that $q^{-1}(u) = h$.*

Corollary 2.54. *If $m = 1$ then the mapping $q : J \to S^2$ is a bijection.*

Now, we are ready to, finally, tackle the case $m \geq 2$. That is, if $m \in \mathbb{N}$ and V_m stands for a subspace of $\mathcal{C}(\mathbb{R})$ of mappings such that every nonzero function in V_m attains its maximum at m (and only m) points, then $\dim(V_m) \leq 2$.

Let us recall that, by Corollary 2.39, H is a partition of J into sets of cardinality m with $m \geq 2$.

Definition 2.55. *Given $h \in H$, we define*

$$\sigma(h) := \min\{|t - s| : t, s \in h \text{ and } t \neq s\}.$$

Definition 2.56. *Let $n \in \mathbb{N}$. We denote by J_n the set*

$$J_n = \bigcup_{h \in H_n} h,$$

where $H_n = \left\{h \in H : \sigma(h) \geq \frac{1}{n} \text{ and } h \subset [-n, n]\right\}$.

Remark 2.57. *It is clear that, for every $h \in H$, we have that $\sigma(h) > 0$. Then one can select an $n = n(h) \in \mathbb{N}$ such that $\sigma(h) \geq 1/n$. As h is a finite set, the number n can be chosen so that, in addition, $n \geq \max\{|t| : t \in h\}$. Then $h \in H_n$. Since $J = \bigcup_{h \in H} h$, we obtain $J = \bigcup_{n \in \mathbb{N}} J_n$.*

Theorem 2.58. *For every $n \in \mathbb{N}$ the set J_n is compact.*

Proof. Since J_n is bounded, we only need to show that J_n is closed. With this aim, let $\{t_{1,k}\}_{k \in \mathbb{N}} \subset J_n$ with $t_{1,k} \xrightarrow{k \to \infty} t_1$. Let us recall that, since by Corollary 2.13, the set J is closed, we have $t_1 \in J$ and, therefore, there is a unique $h \in H$ with $t_1 \in h$.

Next, let $h_k = \{t_{1,k}, \ldots, t_{m,k}\} \in H_n$ for every $k \in \mathbb{N}$. Since $J \cap [-n, n]$ is compact, we can assume (taking a subsequence if needed) that for every $i \in \{2, \ldots, m\}$ there is $t_i \in J \cap [-n, n]$ such that $t_{i,k} \longrightarrow t_i$. Thus, by Proposition 2.12, for every $i \in \{1, \ldots, m\}$ we have that $\delta_{t_{i,k}} \longrightarrow \delta_{t_i}$ as $k \to \infty$.

Now we have that $\delta_{t_{1,k}} \longrightarrow \delta_{t_1} \in F_h$ and, for every $i \in \{2, \ldots, m\}$, $\delta_{t_{i,k}} \longrightarrow \delta_{t_i}$ with $\delta_{t_{i,k}} \in F_{h_k}$ for each $k \in \mathbb{N}$. Hence, by Lemma 2.44, we also have that for every $i \in \{2, \ldots, m\}$, $\delta_{t_i} \in F_h$ and, therefore, $t_2, \ldots, t_m \in h$.

Moreover, notice that for every $i, j \in \{1, \ldots, m\}$ with $i \neq j$, we have

$$|t_i - t_j| = \lim_{k \to \infty} |t_{i,k} - t_{j,k}| \geq \frac{1}{n}.$$

We can, thus, conclude that $t_1, \ldots, t_m$ are all pairwise different and, therefore, $h = \{t_1, \ldots, t_m\} \subset [-n, n]$ and $\sigma(h) \geq \frac{1}{n}$. Hence, $h \in H_n$ and $t_1 \in h \subset J_n$, which proves that J_n is closed. □

Proposition 2.59. *Let $q : J \to S^2$ be the mapping from Definition 2.52. Then, for every $n \in \mathbb{N}$, the set $q(J_n)$ has empty interior in S^2.*

Proof. Suppose, by contradiction, that there exists $n \in \mathbb{N}$ such that $q(J_n)$ has nonempty interior. That is, there exists $u \in Int(q(J_n))$. By Proposition 2.53 and by the construction of J_n we can find $h \in H$ with $h \subset J_n$ such that $q^{-1}(u) = h = \{t_1, \ldots, t_m\}$.

Next, for $i \in \{1, \ldots, m\}$, let $a_i < t_i < b_i$ such that $b_i - a_i < \frac{1}{n}$. Notice that

$$A := q\Big(J_n \setminus \bigcup_{i=1}^{m} (a_i, b_i)\Big)$$

is compact, and hecen closed too. Moreover, $u \notin A$. Thus,

$$u \in Int(q(J_n)) \setminus A \subset B := q(J_n) \setminus A,$$

and, therefore,

$$u \in Int(B) \subset B \subset q\Big(\bigcup_{i=1}^{m}[a_i, b_i] \cap J_n\Big).$$

Now, given $i \in \{1, \dots, m\}$, the mapping

$$q_{|[a_i,b_i]\cap J_n} : [a_i, b_i] \cap J_n \longrightarrow q([a_i, b_i] \cap J_n)$$

is continuous and closed. Let us see that it is also a bijection.

Suppose, by way of contradiction, that there exist $t, t' \in [a_i, b_i] \cap J_n$, $t \neq t'$, such that $q(t) = q(t')$. By Proposition 2.53 there exists $h' \in H$ with $h' \subset J_n$ and so that $t, t' \in h'$. Thus,

$$\sigma(h') \leq |t - t'| \leq b_i - a_i < \frac{1}{n},$$

reaching a contradiction. Therefore the previous mapping $\Psi := q_{|[a_i,b_i]\cap J_n}$ is continuous, bijective and closed, thus it is a homeomorphism. Hence, for every $i \in \{1, \dots, m\}$, the set $q([a_i, b_i] \cap J_n)$ is compact and, in addition, it has empty interior in S^2.

Finally, applying the Baire category theorem, we obtain that

$$q\Big(\bigcup_{i=1}^{m}[a_i, b_i] \cap J_n\Big) = \bigcup_{i=1}^{m} q([a_i, b_i] \cap J_n)$$

has empty interior, which is absurd, since

$$u \in Int(B) \subset B \subset q\Big(\bigcup_{i=1}^{m}[a_i, b_i] \cap J_n\Big).$$

This contradiction concludes the proof of the proposition. □

Summarizing, under the assumption that there exists such a 3-dimensional linear space E, we obtain by using Remark 2.57 that the sphere

$$S^2 = q(J) = q\Big(\bigcup_{n\in\mathbb{N}} J_n\Big) = \bigcup_{n\in\mathbb{N}} q(J_n)$$

is the countable union of compact (so closed) sets with empty interior, which is absurd due to the Baire category theorem. This completes the result and the chapter itself.

Chapter 3

Universality of sequences of operators related to Taylor series

3.1 Introduction, preliminaries and background

Universal Taylor series and universal sequences of differential operators have been largely investigated along the last decades; see [12, 13, 15, 35, 36, 58], [7, Chapter 3] and the references contained in them. This chapter deals with specific points inside both topics, which are, in a certain sense, connected. We will use notation that is mostly standard, so that the reader who is already acquainted with it may skip the next three paragraphs.

Throughout this chapter, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}_\infty$ and $B(z_0, r)$ will represent, respectively, the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the field of rationals, the real line, the complex plane, the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, the extended complex plane $\mathbb{C} \cup \{\infty\}$, and the open ball $\{z \in \mathbb{C} : |z - z_0| < r\}$ with center z_0 and radius r. By a domain we mean a nonempty connected open set $G \subset \mathbb{C}$. We say that a domain G is simply connected whenever $\mathbb{C}_\infty \setminus G$ is connected. For any domain G, the vector space $\mathcal{H}(G)$ of holomorphic functions $G \to \mathbb{C}$ is endowed with the topology of uniform convergence on compact subsets of G. It is well-known (see, e.g. [39]) that, under this topology, $\mathcal{H}(G)$ becomes an F-space, that is, a complete metrizable topological vector space. Moreover, $\mathcal{H}(G)$ is separable. If K is a compact subset of $\mathbb{C}$, then $A(K)$ will stand for the space of all continuous functions $K \to \mathbb{C}$ that are holomorphic in the interior K° of K. The set $A(K)$ becomes a separable Banach space under the norm $\|f\|_\infty = \max_{z \in K} |f(z)|$, that generates the topology of uniform convergence on K. By $\overline{A}$ we denote the closure in a topological space X of a subset $A \subset X$.

Some additional terminology, borrowed from the theories of lineability and of linear chaos, will be needed. For background on them, the reader may consult [7, 8, 13, 21, 22, 50, 58, 88]. Assume that X and Y are (Hausdorff) topological vector spaces. Then a subset $A \subset X$ is said to be *dense-lineable* (*spaceable,* resp.) in X whenever there is a dense (a closed infinite dimensional, resp.) vector subspace M of X such that $M \setminus \{0\} \subset A$.

Let us denote by $L(X,Y)$ the space of all continuous linear mappings $X \to Y$, and by $L(X)$ the space $L(X,X)$ of all operators on X. A sequence $(T_n)_n \subset L(X,Y)$ is said to be *hypercyclic* (or *universal*) provided that there is a vector $x_0 \in X$ –called hypercyclic or universal for $(T_n)_n$– such that the orbit $\{T_n x_0 : n \in \mathbb{N}\}$ of x_0 under $(T_n)_n$ is dense in Y. An operator $T \in L(X)$ is said to be *hypercyclic* if the sequence $(T^n)_n$ of its iterates is hypercyclic. The corresponding sets of hypercyclic vectors will be respectively denoted by $HC((T_n)_n)$ and $HC(T)$. A sequence $(T_n)_n \subset L(X,Y)$ is said to be *transitive* (*mixing,* resp.) provided that, given two nonempty open sets $U \subset X, V \subset Y$, there is $n_0 \in \mathbb{N}$ such that $T_{n_0}(U) \cap V \neq \varnothing$ (such that $T_n(U) \cap V \neq \varnothing$ for all $n \geq n_0$, resp.). From Birkhoff Transitivity Theorem (see, e.g., [58]), we have that, provided that X and Y are F-spaces and Y is separable, a sequence $(T_n)_n \subset L(X,Y)$ is transitive if and only if $HC((T_n)_n)$ is residual (in fact, a dense G_δ subset) in X. Moreover, $(T_n)_n$ is mixing if and only if any subsequence $(T_{n_k})_k$ is transitive.

Let $G \subset \mathbb{C}$ be a domain with $G \neq \mathbb{C}$, $\zeta \in G$ and $f \in \mathcal{H}(G)$. Then f is said to be a *universal Taylor series* with center ζ provided that it satisfies the following property: For every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a (strictly increasing) sequence $(\lambda_n) \subset \mathbb{N}$ such that

$$\lim_{n\to\infty} \sup_{z\in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0,$$

where $S(N, f, \zeta)$ represents the Nth partial Taylor sum of f at ζ, that is,

$$S(N,f,\zeta)(z) = \sum_{j=0}^{N} \frac{f^{(j)}(\zeta)}{j!}(z-\zeta)^j \qquad (z \in \mathbb{C},\ N \in \mathbb{N}_0).$$

This concept dates back to Nestoridis [79], who studied a kind of universality which was slightly stronger than the one considered by Luh [69, 70] and Chui and Parnes [37] (where K is supposed not to cut $\overline{G}$). The set of universal Taylor series in G with center ζ is denoted by $U(G,\zeta)$. It is proved in [79] that $U(\mathbb{D},0)$ is a dense G_δ subset of $\mathcal{H}(\mathbb{D})$, and this is generalized in [80] by showing that $U(G,\zeta)$ is a dense G_δ subset of $\mathcal{H}(G)$ for any simply connected domain G and any $\zeta \in G$. Now, for a domain $G \subset \mathbb{C}$, let $U(G)$ denote the family of all functions $f \in \mathcal{H}(G)$ satisfying that, for every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a sequence $(\lambda_n) \subset \mathbb{N}_0$ such that, for every compact set $L \subset G$, one has

$$\lim_{n\to\infty} \sup_{\zeta\in L} \sup_{z\in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0.$$

Obviously, $U(G) \subset U(G,\zeta)$ for all $\zeta \in G$. It is shown in [80] that $U(G)$ is a dense G_δ subset of $\mathcal{H}(G)$ if G is simply connected, in [74] that $U(G) = \varnothing$ if G is not simply connected, and in [78] that $U(G,\zeta) = U(G)$ if G is simply connected and ζ is any point of G.

According to [87], Nestoridis posed the question of whether the universality of Taylor series is preserved if we fix the point of evaluation z (without loss of generality, we may assume $z = 0$) and the center ζ of expansion is variable. To be more specific, the question is whether the set

$$\mathcal{S}(G) := \left\{ f \in \mathcal{H}(G) : \{\widetilde{T_n} f\}_{n\geq 0} \text{ is dense in } \mathcal{H}(G) \right\}$$

is not empty, where

$$(\widetilde{T_n} f)(\zeta) := \sum_{j=0}^{n} \frac{f^{(j)}(\zeta)}{j!} (-\zeta)^j \quad (\zeta \in G,\, n \geq 0). \tag{3.1}$$

We remark the connection: $\mathcal{S}(G) = HC((\widetilde{T_n})_n)$, where we are considering $\widetilde{T_n} \in L(\mathcal{H}(G))$ $(n \geq 0)$. It is proved in [87, Section 4] that $\mathcal{S}(G)$ is always a G_δ subset of $\mathcal{H}(G)$ (the proof is there given for a simply connected domain G, but it can be extended to any domain, just by replacing the dense sequence (p_j) of polynomials by a dense sequence in $\mathcal{H}(G)$, which exists thanks to the separability of $\mathcal{H}(G)$), that $\mathcal{S}(G) = \varnothing$ if $0 \in G$ and that, if G is simply connected, then $\mathcal{S}(G)$ is either empty or dense (so either empty or residual). In [87] the broader class

$$\mathcal{S}_t(G) := \left\{ f \in \mathcal{H}(G) : \overline{\{\widetilde{T_n} f\}_{n\geq 0}} \supset \{\text{constants}\} \right\}$$

is also considered, and it is shown to be a G_δ subset of $\mathcal{H}(G)$. Once again, $\mathcal{S}_t(G) = \varnothing$ if $0 \in G$. Moreover, if G is simply connected and $0 \notin G$, then $\mathcal{S}_t(G)$ is dense (hence residual) in $\mathcal{H}(G)$. Recently, Panagiotis [81] has answered the conjecture by Nestoridis (see [87]) in the affirmative by proving that $\mathcal{S}(G) \neq \varnothing$ in the special case where G is an open disc not containing 0.

In this chapter we prove –with methods that are rather different from those in [81]– that the condition $0 \notin G$ *characterizes* the non-vacuousness of $\mathcal{S}(G)$ if G is simply connected. In fact we shall study the universality of sequences that are more general than $(\widetilde{T_n})$. Finally the dynamics of the sequence of differential operators generated by a *power series* with finite radius of convergence is investigated, and lineability properties of the corresponding sets of universal functions are shown.

3.2 Universality of Taylor-like series

In this section, the hypercyclicity of the sequence of operators $\widetilde{T_n}$ $(n \geq 0)$ given by (3.1) will be studied. In order to tackle the problem, we shall adopt a slightly general point of view, by considering the following more general families of operators.

For each $(a, n, f, z) \in \mathbb{C} \times \mathbb{N}_0 \times \mathcal{H}(G) \times G$, we set

$$(T_{a,n} f)(z) := \sum_{j=0}^{n} \frac{f^{(j)}(z)}{j!} \cdot (az)^j. \tag{3.2}$$

Note that $\widetilde{T_n} = T_{-1,n}$. From the continuity of the derivative operator D $(Df := f')$, it follows that every $T_{a,n}$ is a well defined continuous linear mapping $\mathcal{H}(G) \to \mathcal{H}(G)$, that is, $(T_{a,n})_n \subset L(\mathcal{H}(G))$ for all $a \in \mathbb{C}$. We start with a necessary condition for universality. As usual, ∂A represents the boundary of a set $A \subset \mathbb{C}$.

Proposition 3.1. *Let $a \in \mathbb{C}$. Assume that $G \subset \mathbb{C}$ is a domain, and that the sequence of operators $T_{a,n} : \mathcal{H}(G) \to \mathcal{H}(G)$ $(n \in \mathbb{N})$ defined by* (3.2) *is universal. Then we have:*

(a) $0 \notin G$, *and*

(b) $|a| \geq \sup_{z \in G} \dfrac{\operatorname{dist}(z, \partial G)}{|z|}$.

Proof. (a) By hypothesis, there is $f \in HC((T_{a,n})_n)$. Proceeding by way of contradiction, assume that $0 \in G$. Consider the constant function $g(z) := 1 + f(0)$. Then there would exist a sequence $(n_k) \subset \mathbb{N}$ such that $T_{n_k} f \to g$ $(k \to \infty)$ uniformly on every compact set $K \subset G$. In particular, for $K = \{0\}$, we would obtain

$$f(0) = \frac{f^{(0)}(0)}{0!} = (T_{n_k} f)(0) \longrightarrow g(0) = 1 + f(0) \quad \text{as} \quad k \to \infty,$$

which is clearly absurd.

(b) We proceed, again, by way of contradiction, so that we are simultaneously assuming $|a| < \sup_{z \in G} \frac{\operatorname{dist}(z, \partial G)}{|z|}$ and the existence of an $f \in HC((T_{a,n})_n)$. Then there exists $z_0 \in G$ such that $|a| < \frac{R}{|z_0|}$, where $R := \operatorname{dist}(z_0, \partial G)$. Therefore $B(z_0, R) \subset G$. Consequently, the Taylor expansion $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ holds in $B(z_0, R)$ for our function f. Due to the hypercyclicity of f, some subsequence of $(T_{a,n} f)_n$ should tend in the compact set $K = \{z_0\} \subset G$ to any prescribed constant, in particular, to the constant $1 + f((a+1)z_0)$: this is, indeed, a well defined number because $|(a+1)z_0 - z_0| = |az_0| < R$, and so $(a+1)z_0 \in B(z_0, R) \subset G$. However,

$$(T_{a,n}f)(z_0) = \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}(az_0)^j = \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}((a+1)z_0 - z_0)^j$$
$$\longrightarrow \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}((a+1)z_0 - z_0)^n = f((a+1)z_0)$$

as $n \to \infty$, which is the sought-after contradiction. □

Remark 3.2. 1. In the case $a = -1$, condition (a) above was already obtained in [87], and (b) is always satisfied as soon as $0 \notin G$, because we would have $|-1| \cdot |z| = |z| = |z - 0| \geq \mathrm{dist}\,(z, \partial G)$ for all $z \in G$.

2. From condition (a) in Proposition 3.1 one derives as in the last remark that $|z| = |z-0| \geq \mathrm{dist}\,(z, \partial G)$ for all $z \in G$. Then we have $\sup_{z \in G} \frac{\mathrm{dist}\,(z, \partial G)}{|z|} \leq 1$. Therefore, according to (b), if $|a| < 1$ and G is a domain such that some sequence $(z_n) \subset G$ satisfies $\lim_{n\to\infty} \frac{\mathrm{dist}\,(z_n, \partial G)}{|z_n|} = 1$ (for instance $G = B(c, |c|)$, where $c \in \mathbb{C} \setminus \{0\}$), then $(T_{a,n})$ is not universal on $\mathcal{H}(G)$. Another example in which $(T_{a,n})$ is not universal (even though $0 \notin G$) is obtained when G is a sector $\{re^{i\theta} : r > 0, 0 < \theta < \alpha\}$ $(0 < \alpha < 2\pi)$ and $|a| < \sin \frac{\alpha}{2}$.

In order to provide sufficient conditions for universality, we distinguish two cases, namely, $a \neq -1$ and $a = -1$. The reason is that the approaches of the proofs are rather different. Note that we obtain in fact (see Theorem 3.10 below) a characterization of universality in the case $a = -1$: this follows from Proposition 3.1 and the fact that the condition $G \cap (a+1)G = \varnothing$ given in the next theorem means $0 \notin G$ in that case. As usual, we have set $cS := \{cz : z \in S\}$ for $c \in \mathbb{C}$, $S \subset \mathbb{C}$.

The auxiliary results contained in the next lemma are needed to face the case $a \neq -1$. If $M \subset \mathbb{N}_0$ is an infinite set and $G \subset \mathbb{C}$ is a domain, then we denote by $U(G, M)$ the family of all functions $f \in \mathcal{H}(G)$ satisfying that, for every compact set $K \subset \mathbb{C} \setminus G$ with $\mathbb{C} \setminus K$ connected, and every $g \in A(K)$, there exists a strictly increasing sequence $(\lambda_n) \subset M$ such that, for every compact set $L \subset G$, one has

$$\lim_{n\to\infty} \sup_{\zeta \in L} \sup_{z \in K} |S(\lambda_n, f, \zeta)(z) - g(z)| = 0.$$

Note that $U(G, \mathbb{N}_0) = U(G)$.

Lemma 3.3. *Let $G \subset \mathbb{C}$ be a simply connected domain with $G \neq \mathbb{C}$, and $M \subset \mathbb{N}_0$ be an infinite subset. Then the following holds:*

(a) *$U(G,M)$ is a dense G_δ subset of $\mathcal{H}(G)$.*

(b) *$U(G)$ is dense-lineable in $\mathcal{H}(G)$.*

(c) *$U(G)$ is spaceable in $\mathcal{H}(G)$.*

Proof. Part (a) is a refinement of an assertion of [80] given in Section 1, and it is a consequence of Theorem 3.4 in [74] just by choosing A = the infinite unit matrix there.

Part (b) can be derived from Theorem 6 in [12]. In fact, we only need the conclusion (ii) of such theorem (for $l = 0$), together with the property that –thanks to Mergelyan's Approximation Theorem (see, e.g., [52])– the set of entire functions is dense in $A(K)$, provided that K is a compact subset of $\mathbb{C}$ with connected complement.

Part (c) follows from the just mentioned denseness property together with Theorem 4.2 in [75] (see also [34]). We need only the conclusion (i) (for $l = 0$) of this theorem. □

Remark 3.4. In 2005, Bayart established the dense-lineability ([10]) and the spaceability ([11]) of $U(\mathbb{D})$.

Theorem 3.5. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $T_{a,n} : \mathcal{H}(G) \to \mathcal{H}(G)$ $(n \in \mathbb{N})$ defined by (3.2), where $a \in \mathbb{C} \setminus \{-1\}$. If $G \cap (a+1)G = \varnothing$ then we have:*

(a) *The sequence $(T_{a,n})$ is mixing (hence universal).*

(b) *The set $HC((T_{a,n}))$ is dense-lineable and spaceable in $\mathcal{H}(G)$.*

Proof. (a) To show that $(T_{a,n})$ is mixing, we are going to prove that, for every fixed sequence $M = \{n_1 < n_2 < n_3 < \cdots\} \subset \mathbb{N}_0$, the set $HC((S_k)_{k\ge1})$ is residual in $H(G)$, where we have set $S_k := T_{a,n_k}$. According to Lemma 3.3(a), it is enough to prove that $U(G,M) \subset HC((S_k)_{k\ge1})$ or, equivalently, that for each $f \in U(G,M)$ the orbit $\{S_k f : k \in \mathbb{N}\}$ is dense in $H(G)$. Since G is simply connected, the set of polynomials is dense in $H(G)$. Therefore it is sufficient to exhibit, for every fixed polynomial P, a sequence $(k(l))_l \subset M$ such that $S_{k(l)} f \longrightarrow P$ $(l \to \infty)$ uniformly on compacta in G. Choose an increasing sequence of compact sets $\{L_l\}_{l\ge1}$ such that $G = \bigcup_{l\ge1} L_l$ and every set $\mathbb{C} \setminus L_l$ is connected; this is possible due to the simple connectedness of G (see, e.g., [85, Chapter 13]). Then every compact set $L \subset G$ is contained in some $L_{l(L)}$.

Fix f and P as above. Since $a+1 \neq 0$, the set $(a+1)G$ is a simply connected domain contained in $\mathbb{C} \setminus G$. Moreover, each set $K_l := (a+1)L_l$ is compact, $\mathbb{C} \setminus K_l$ is connected and $K_l \subset \mathbb{C} \setminus G$. In addition, every mapping $z \in K_l \longmapsto P(\frac{z}{a+1}) \in \mathbb{C}$ belongs to $A(K_l)$. Thus, there is $m_l = n_{k(l)} \in M$ such that

$$\sup_{\zeta \in L_l} \sup_{z \in K_l} \left| S(m_l, f, \zeta)(z) - P\Big(\frac{z}{a+1}\Big) \right| < \frac{1}{l}.$$

It is evident that (m_l) can be selected so as to be strictly increasing. Notice that we have, in particular, that $|S(m_l, f, z)((a+1)z) - P(z)| < 1/l$ for all $z \in L_l$. But

$$S(m_l, f, z)((a+1)z) = \sum_{j=0}^{m_l} \frac{f^{(j)}(z)}{j!}((a+1)z - z)^j = (S_{k(l)} f)(z).$$

On the other hand, given a compact set $L \subset G$, there is $l_0 \in \mathbb{N}$ such that $L \subset L_l$ for all $l \geq l_0$. This yields $\sup_{z \in L} |(S_{k(l)} f)(z) - P(z)| < 1/l$ for all $l \geq l_0$ and, consequently, $\lim_{l \to \infty} \sup_{z \in L} |(S_{k(l)} f)(z) - P(z)| = 0$, which proves the desired uniform convergence.

(b) This follows from Lemma 3.3(b,c) together with the fact that

$$U(G) \subset HC((T_{a,n})),$$

proved in the preceding paragraph (with $M = \mathbb{N}_0$). □

For instance, if Π is one of the two open half-planes determined by a straight line passing through the origin and G is any simply connected domain contained in Π, then $G \cap (-G) = \varnothing$, and so the sequence $(T_{-2,n})$ is universal on $H(G)$.

Remark 3.6. Contrary to the case $a = -1$ (Theorem 3.10), we do not know whether or not the condition $G \cap (a+1)G = \varnothing$ in Theorem 3.5 is necessary for the universality of $(T_{a,n})$.

For any meromorphic function R we will consider the set $\mathcal{P}_R$ of its poles in the extended plane, that is, $\mathcal{P}_R = \{z \in \mathbb{C}_\infty : R(z) = \infty\}$. The following three lemmas will be used in the proof of our main result, with which we conclude this section.

Lemma 3.7. *Let $G \subset \mathbb{C}$ be a simply connected domain such that $0 \notin G$. Then the family $\mathcal{R}_0$ of rational functions R with $\mathcal{P}_R \subset \{0\}$ is a dense subset of $H(G)$.*

Proof. As a consequence of the Runge Approximation Theorem, if A is a subset of $\mathbb{C}_\infty$ containing exactly one point in each connected component of $\mathbb{C}_\infty \setminus G$, then the family of rational functions R with $\mathcal{P}_R \subset A$ is a dense subset of $H(G)$ (see, e.g., [85, Chapter 13]). In our case, the set $\mathbb{C}_\infty \setminus G$ is connected and $0 \in \mathbb{C}_\infty \setminus G$, so it is enough to choose $A = \{0\}$. □

Lemma 3.8. *Assume that X and Y are separable F-spaces. Let $(T_n) \subset L(X,Y)$ be a mixing sequence. Then $HC((T_n))$ is dense-lineable.*

Proof. In [19] it is proved that, if X and Y are metrizable separable topological vector spaces and (T_n) is a sequence in $L(X,Y)$ such that $HC((T_{n_k}))$ is dense for every sequence $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$, then $HC((T_n))$ contains, except for 0, a dense vector subspace of X. The conclusion of this lemma follows from the fact that being mixing implies transitivity of each subsequence (T_{n_k}), and this in turn is equivalent to the denseness of each set $HC((T_{n_k}))$ (in fact, all that is needed is X to be, in addition, a Baire space). □

Lemma 3.9. *Let $G \subset \mathbb{C}$ be a simply connected domain with $0 \notin G$, and M be an infinite subset of $\mathbb{N}_0$. Then the set*

$$\mathcal{S}_{t,M}(G) := \left\{ f \in H(G) : \overline{\{\widetilde{T_n} f\}_{n \in M}} \supset \{constants\} \right\}$$

is dense in $H(G)$.

Proof. In [87, Theorem 4.7], the statement of the lemma is proved for the case $M = \mathbb{N}_0$ by showing that $U(G) \subset \mathcal{S}_t(G) = \mathcal{S}_{t,\mathbb{N}_0}(G)$. With the same approach it can be seen that $U(G,M) \subset \mathcal{S}_{t,M}(G)$. But, by Lemma 3.3, the set $U(G,M)$ is dense in $\mathcal{H}(G)$. Thus, $\mathcal{S}_{t,M}(G)$ is dense too. □

Theorem 3.10. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $\widetilde{T_n} : \mathcal{H}(G) \to \mathcal{H}(G)$ $(n \in \mathbb{N})$ defined in* (3.1). *Then the following properties are equivalent:*

(a) $0 \notin G$.

(b) *The sequence $(\widetilde{T_n})$ is universal, that is, $\mathcal{S}(G) \neq \varnothing$.*

(c) *The sequence $(\widetilde{T_n})$ is mixing.*

(d) *The set $\mathcal{S}(G)$ is residual in $\mathcal{H}(G)$.*

(e) *The set $\mathcal{S}(G)$ is dense-lineable in $\mathcal{H}(G)$.*

Proof. Recall that $\mathcal{S}(G) = HC((\widetilde{T_n})_{n \geq 0})$, where

$$\widetilde{T_n} f(z) = \sum_{j=0}^{n} \frac{f^{(j)}(z)}{j!} (-z)^j.$$

The implication (b) $\Rightarrow$ (a) has been already proved in [87] (alternatively, see Proposition 3.1), while (c) $\Rightarrow$ (b) is trivial because any mixing sequence of operators on a separable F-space is universal. On the other hand, the implications (d) $\Rightarrow$ (b) and (e) $\Rightarrow$ (b) are also evident because if a set is dense then it is, trivially, nonempty. That (c) $\Rightarrow$ (d) is a consequence of the fact that mixing implies transitive. And (c) $\Rightarrow$ (e) follows from Lemma 3.8 as applied to our sequence $(\widetilde{T_n})$ and $X = \mathcal{H}(G) = Y$.

Consequently, all we need to prove is that (a) implies (c). So, we assume $0 \notin G$. Our goal is to show that $(\widetilde{T_n})_{n\in\mathbb{N}_0}$ is mixing. This is equivalent to show that $(\widetilde{T_n})_{n\in M}$ is transitive for every infinite subset $M \subset \mathbb{N}_0$. With this aim, fix such a subset M as well as two nonempty open sets U, W of $\mathcal{H}(G)$. We should find $n_0 \in M$ such that $\widetilde{T_{n_0}}(U) \cap W \neq \varnothing$. Recall that the family of all sets of the form

$$V(f, K, \epsilon) = \{g \in \mathcal{H}(G) : |g(z) - f(z)| < \epsilon \text{ for all } z \in K\}$$

($f \in \mathcal{H}(G)$, $\epsilon > 0$, K a compact subset of G) is an open basis for the topology of $\mathcal{H}(G)$. Now, recall that since G is simply connected, the set $\mathcal{P}$ of all polynomials and the set $\mathcal{R}_0$ (Lemma 3.7) are dense in $\mathcal{H}(G)$. Moreover, we have $V(f, K, \epsilon) \subset V(f, L, \alpha)$ if $K \supset L$ and $\epsilon < \alpha$. Then there are $\epsilon > 0$, $P \in \mathcal{P}$, $R \in \mathcal{R}_0$ and a compact subset $K \subset G$ such that $U \supset V(P, K, \epsilon)$ and $W \supset V(R, K, \epsilon)$.

Thus, we should search for an $m \in M$ enjoying the property that there is a function $f \in \mathcal{H}(G)$ such that $f \in V(P, K, \epsilon)$ and $\widetilde{T_m} f \in V(R, K, \epsilon)$ or, equivalently, such that

$$|f(z) - P(z)| < \epsilon \text{ and } |(\widetilde{T_m} f)(z) - R(z)| < \epsilon \text{ for all } z \in K. \tag{3.3}$$

Let $p := \text{degree}(P)$. On the one hand, if $n \geq p$ and $z \in \mathbb{C}$, we obtain from the Taylor expansion that

$$\begin{aligned}(\widetilde{T_n} P)(z) &= \sum_{j=0}^{n} \frac{P^{(j)}(z)}{j!}(-z)^j = \sum_{j=0}^{p} \frac{P^{(j)}(z)}{j!}(-z)^j \\ &= \sum_{j=0}^{p} \frac{P^{(j)}(z)}{j!}(0 - z)^j = P(0).\end{aligned} \tag{3.4}$$

On the other hand, there are $b_0, b_1, \dots, b_q \in \mathbb{C}$ such that

$$R(z) = b_0 + \frac{b_1}{z} + \cdots + \frac{b_q}{z^q} =: b_0 + R_0(z).$$

According to Lemma 3.9, we can find a function $\varphi \in \mathcal{H}(G)$ and an infinite subset $M_0 \subset M$ such that

$$|\varphi(z)| < \frac{\epsilon}{2} \text{ and } |(\widetilde{T_n}\varphi)(z) - (-P(0) + b_0)| < \epsilon \quad (z \in K,\, n \in M_0). \tag{3.5}$$

Now, since $K \subset G$ is compact and $0 \notin G$, we can find $C_K \in (0,1)$ such that

$$|z| > C_K \quad \text{for all} \quad z \in K. \tag{3.6}$$

Since M_0 is infinite, we can choose $m \in M_0$ (hence $m \in M$) satisfying

$$m > p \quad \text{and} \quad m > \frac{2q \cdot \max_{1\le k\le q} |b_k|}{\epsilon \cdot C_K^q}. \tag{3.7}$$

For each $k \in \{1, \ldots, q\}$, let us define the numbers d_k and a_k by

$$d_k := \sum_{j=0}^{m} \frac{k(k+1)\cdots(k+j-1)}{j!} \quad \text{and} \quad a_k := \frac{b_k}{d_k}, \tag{3.8}$$

with the convention $\frac{k(k+1)\cdots(k+j-1)}{j!} := 1$ if $j = 0$. Observe that $d_k \ge m+1$ for all $k \in \{1, \ldots, q\}$. We also define the function

$$f := P + \varphi + S, \text{ where } S(z) := \frac{a_1}{z} + \cdots + \frac{a_q}{z^q}. \tag{3.9}$$

Obviously, $f \in \mathcal{H}(G)$. Let $\psi_k(z) := z^{-k}$ for $k \in \mathbb{N}$. An easy computation gives $\widetilde{T_m}\psi_k = d_k\psi_k$. Hence, by linearity, $\widetilde{T_m}S = \sum_{k=1}^q a_k d_k \psi_k = \sum_{k=1}^q b_k\psi_k = R_0$. On the one hand, we have by (3.5), (3.6), (3.7), (3.8), (3.9) and the triangle inequality that, for all $z \in K$,

$$\begin{aligned}|f(z) - P(z)| &\le |\varphi(z)| + |S(z)| \le \frac{\epsilon}{2} + \sum_{k=1}^{q} \left|\frac{a_k}{z^k}\right| \\ &= \frac{\epsilon}{2} + \sum_{k=1}^{q} \frac{|b_k|}{|d_k z^k|} \le \frac{\epsilon}{2} + \sum_{k=1}^{q} \frac{|b_k|}{m\, C_K^k} \\ &< \frac{\epsilon}{2} + \frac{q \cdot \max_{1\le k\le q}|b_k|}{m\, C_K^q} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

On the other hand, from (3.4), (3.5), (3.7), (3.9), the triangle inequality and the linearity of $\widetilde{T_m}$ we get for all $z \in K$ that

$$\begin{aligned}|(\widetilde{T_m}f)(z) - R(z)| &= \left|(\widetilde{T_m}P)(z) + (\widetilde{T_m}\varphi)(z) + (\widetilde{T_m}S)(z) - b_0 - \sum_{k=1}^{q} \frac{b_k}{z^k}\right| \\ &\le |P(0) + (\widetilde{T_m}\varphi)(z) - b_0| + \left|(\widetilde{T_m}S)(z) - R_0(z)\right| < \epsilon + 0 = \epsilon.\end{aligned}$$

Consequently, (3.3) holds for the chosen function f, and we are done. □

Question 3.11. *Let $G \subset \mathbb{C}$ be a simply connected domain with $0 \notin G$. Is $\mathcal{S}(G)$ spaceable?*

3.3 Differential polynomials associated to power series

Let $G \subset \mathbb{C}$ be a domain. We can associate to each polynomial $P(z) = \sum_{k=0}^N a_k z^k$ with complex coefficients a_k a differential operator $P(D) = \sum_{k=0}^N a_k D^k \in L(\mathcal{H}(G))$, where $D^k f = f^{(k)}$ for $k \in \mathbb{N}_0$. Then $P(D)f = \sum_{k=0}^N a_k f^{(k)}$. Therefore, any (formal) power series $\sum_{n=0}^\infty c_n z^n$ (or, that is the same, any sequence $\mathbf{c} = (c_n) \in \mathbb{C}^{\mathbb{N}_0}$) defines, in a natural way, a sequence $\{T_{\mathbf{c},n}\}_{n\geq 0}$ of operators on $\mathcal{H}(G)$ given by $T_{\mathbf{c},n} = \sum_{j=0}^n c_j D^j$, that is,

$$(T_{\mathbf{c},n} f)(z) = \sum_{j=0}^n c_j f^{(j)}(z) \quad (f \in \mathcal{H}(G)). \tag{3.10}$$

Then it is natural to ask for the universality of such a sequence.

However, before going on, it is worth mentioning that there are some restrictions on the desired universality. For instance, if the series $\sum_{n=0}^\infty c_n z^n$ is "very convergent", we should not get our hopes up too much. To be more explicit, assume that $\Phi(z) = \sum_{n=0}^\infty c_n z^n$ is an entire function of subexponential type, that is, given $\epsilon > 0$, there is a constant $K = K(\epsilon) \in (0, +\infty)$ such that $|\Phi(z)| \leq K\, e^{\epsilon|z|}$ for all $z \in \mathbb{C}$. Then the infinite order differential operator $\Phi(D) = \sum_{n=0}^\infty c_n D^n$ is well defined on $\mathcal{H}(G)$; see, e.g., [17] (in fact, it makes sense on $\mathcal{H}(\mathbb{C})$ if Φ is just of exponential type, that is, if there are constants $A, B \in (0, +\infty)$ satisfying $|\Phi(z)| \leq A\, e^{B|z|}$ for all $z \in \mathbb{C}$). The corresponding sequence $\{T_{\mathbf{c},n}\}_{n\geq 0}$ of operators satisfies

$$T_{\mathbf{c},n} f \longrightarrow \Phi(D) f = \sum_{k=0}^\infty c_k f^{(k)} \quad (n \to \infty)$$

uniformly on compacta in G, so we have a kind of "anti-hypercyclicity" in this case.

With this in mind, we have got a partial positive result (Theorem 3.13) by assuming that $\mathbf{c}$ is *not* the sequence of Taylor coefficients of an entire function (i.e., $\limsup_{n\to\infty} |c_n|^{1/n} > 0$) as well as some "angular" behavior of these coefficients. The remaining cases in which the series $\sum_{n=0}^\infty c_n z^n$ does not define an entire function of subexponential type stay –as far as we know– as an open problem. For the proof, we need the following lemma, which is in the line of the eigenvalue criteria given in [20, 23, 56]. However, the lemma cannot be deduced from those criteria. Moreover, its content might be of some interest by itself. By $\mathrm{span}(A)$ we represent the linear span of a subset A of a vector space.

Lemma 3.12. *Assume that X is a separable F-space and that $(T_n)_{n\geq 0} \subset L(X)$. Suppose that there are subsets $D, E \subset X$ satisfying the following conditions:*

(a) *D and* $\operatorname{span}(E)$ *are dense in* X.

(b) *For each* $d \in D$, *the sequence* $\{T_n d\}_{n \geq 0}$ *converges in* X.

(c) *Each* $e \in E$ *is an eigenvector of every* T_n $(n \geq 0)$, *with eigenvalue* $\lambda(T_n, e)$, *say.*

(d) $\lim_{n \to \infty} \lambda(T_n, e) = \infty$ *for all* $e \in E$.

Then $(T_n)_n$ *is mixing and the set* $HC((T_n)_n)$ *is dense-lineable in* X.

Proof. The second conclusion follows from Lemma 3.8. As for the first conclusion, we want to prove that every subsequence (T_{n_k}) of (T_n) is transitive. Let us denote $R_k := T_{n_k}$ for $k \in \mathbb{N}$.

In order to show that (R_k) is transitive, fix two nonempty open sets $U, V \subset X$. Our goal is to exhibit an $m \in \mathbb{N}$ such that $R_m(U) \cap V \neq \varnothing$. By the denseness of D assumed in (a), there is $d \in D \cap U$. It follows from (b) the existence of a vector $f \in X$ such that $R_k d \to f$ as $k \to \infty$. Now, by the denseness of $\operatorname{span}(E)$ this time, there is $e \in \operatorname{span}(E) \cap (V - f)$, because the translate $V - f$ of V is also open and nonempty. Since $e \in \operatorname{span}(E)$, we can find finitely many scalars μ_j and vectors $e_j \in E$ $(j = 1, \dots, q)$ such that $e = \sum_{j=1}^{q} \mu_j e_j$. Thanks to (c) and (d), we have $R_k e_j = \lambda(T_{n_k}, e_j) e_j$ and $\lim_{k \to \infty} \lambda(T_{n_k}, e_j) = \infty$ for all $j \in \{1, \dots, q\}$. In particular, there is $k_1 \in \mathbb{N}$ such that $\lambda(T_{n_k}, e_j) \neq 0$ for all $k \geq k_1$ and all $j \in \{1, \dots, q\}$. Next, for any $k \geq k_1$, we define

$$x_k := d + \sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} e_j.$$

Since $\frac{\mu_j}{\lambda(T_{n_k}, e_j)} \to 0$ $(k \to \infty)$ for $j \in \{1, \dots, q\}$, it follows from the continuity of the multiplication by scalars in a topological vector space that $x_k \to d + 0 = d$ as $k \to \infty$. As $d \in U$ and U is open, there exists $k_2 \geq k_1$ such that $x_k \in U$ for all $k \geq k_2$. Finally, we get

$$\begin{aligned} R_k x_k &= R_k d + R_k \Big(\sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} e_j \Big) = R_k d + \sum_{j=1}^{q} \frac{\mu_j}{\lambda(T_{n_k}, e_j)} R_k e_j \\ &= R_k d + \sum_{j=1}^{q} \mu_j e_j = R_k d + e \longrightarrow f + e \quad \text{as} \quad k \to \infty. \end{aligned}$$

Since $f + e \in f + (V - f) = V$ and V is open, one can find $k_3 \geq k_2$ such that $R_k x_k \in V$ for all $k \geq k_3$. Consequently, we obtain $R_m(U) \cap V \neq \varnothing$ as soon as we choose $m := k_3$. This had to be shown. $\square$

We are now ready to state our final theorem.

Theorem 3.13. *Let $G \subset \mathbb{C}$ be a simply connected domain, and consider the sequence of operators $T_{\mathbf{c},n} : \mathcal{H}(G) \to \mathcal{H}(G)$ $(n \in \mathbb{N}_0)$ defined in* (3.10)*, where $\mathbf{c} = (c_n)_{n\geq 0}$ satisfies the following conditions:*

(i) $\limsup_{n\to\infty} |c_n|^{1/n} > 0$.

(ii) *There exist $\alpha \in \mathbb{R}$ and a sequence $(\theta_n)_{n\geq 0} \in \mathbb{R}^{\mathbb{N}_0}$ with*

$$\min\Big\{\limsup_{n\to\infty} |\theta_n|, \limsup_{n\to\infty} \Big|\theta_n - \frac{\pi}{2}\Big|, \limsup_{n\to\infty} |\theta_n - \pi|, \limsup_{n\to\infty} \Big|\theta_n - \frac{3\pi}{2}\Big|\Big\} < \frac{\pi}{2}$$

such that $\arg c_n = n\,\alpha + \theta_n$ whenever $c_n \neq 0$.

Then $(T_{\mathbf{c},n})$ is mixing and, in particular, universal. Moreover, the set $HC((T_{\mathbf{c},n}))$ is dense-lineable in $\mathcal{H}(G)$.

Proof. The second part of the conclusion follows from the first one and Lemma 3.8. Hence, our goal is to prove that $(T_{\mathbf{c},n})$ is mixing. We will use Lemma 3.12 with $X := \mathcal{H}(G)$, $T_n := T_{\mathbf{c},n}$ $(n \geq 0)$, $D := \mathcal{P} = \{\text{polynomials}\}$ and $E := \{e_\lambda : \lambda \in \{t\,e^{-i\alpha} : t > R\}\}$, where $e_a(z) := e^{az}$ $(a \in \mathbb{C})$ and R is the radius of convergence of the power series $\sum_{n=0}^\infty c_n z^n$, that is, $R = (\limsup_{n\to\infty} |c_n|^{1/n})^{-1}$. Observe that $0 \leq R < +\infty$ by (i), which yields $E \neq \varnothing$.

On the one hand, the denseness of D in X follows from the simple connectedness of G. On the other hand, it is known (see, e.g., [58, Lemma 2.34]) that if $\Lambda \subset \mathbb{C}$ is a set with an accumulation point, then $\operatorname{span}(\{e_\lambda : \lambda \in \Lambda\})$ is dense in $\mathcal{H}(\mathbb{C})$, and hence in $\mathcal{H}(G)$ due to Runge's approximation theorem and the simple connectedness of G. Consequently, $\operatorname{span}(E)$ is dense in X and condition (a) of Lemma 3.12 is fulfilled. Now, if $P \in \mathcal{P}$ and $N = \operatorname{degree}(P)$ then $P^{(n)} = 0$ for all $n > N$, and so $T_n P = \sum_{j=0}^N c_j P^{(j)} := Q$ for all $n \geq N$. Hence $T_n P \to Q$ as $n \to \infty$, which tells us that condition (b) in Lemma 3.12 is also satisfied. As for condition (c), notice that $e_\lambda^{(n)} = \lambda^n e_\lambda$ for all $\lambda \in \mathbb{C}$ and all $n \in \mathbb{N}_0$, which entails $T_n e_\lambda = \lambda(T_n, e_\lambda) e_\lambda$, where $\lambda(T_n, e_\lambda) = \sum_{j=0}^n c_j \lambda^j$, that is, each $e_\lambda \in E$ is in fact an eigenvector for all T_n. Let us verify, finally, condition (d) in Lemma 3.12.

For this, take any $n \in \mathbb{N}_0$ and any $\lambda = t\,e^{-i\alpha}$ with $t > R$. From (ii), at least one of the following inequalities is true: $\limsup_{n\to\infty} |\theta_n| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \frac{\pi}{2}| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \pi| < \frac{\pi}{2}$, $\limsup_{n\to\infty} |\theta_n - \frac{3\pi}{2}| < \frac{\pi}{2}$. Suppose that the first inequality holds. Then there is $N \in \mathbb{N}$ such that $\sup_{n>N} |\theta_n| < \frac{\pi}{2}$. Let $\gamma := \inf_{n>N} \cos\theta_n$. Note that $\gamma > 0$. Let $n > N$. Also by (ii) and the triangle inequality, we can estimate:

$$\begin{aligned}
|\lambda(T_n, e_\lambda)| &= \Big|\sum_{j=0}^{n} c_j\, \lambda^j\Big| = \Big|\sum_{j=0}^{n} |c_j|\, e^{i(j\alpha+\theta_j)}(t\, e^{-i\alpha})^j\Big| = \Big|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}\Big| \\
&\geq \mathrm{Re}\Big(\sum_{j=N+1}^{n} |c_j|\, t^j\, e^{i\,\theta_j}\Big) - \sum_{j=0}^{N} |c_j|\, t^j \\
&= \sum_{j=0}^{n} |c_j|\, t^j\, \cos\theta_j - \sum_{j=0}^{N} |c_j|\, t^j \\
&\geq \gamma \cdot \sum_{j=0}^{n} |c_j|\, t^j - \sum_{j=0}^{N} |c_j|\, t^j \longrightarrow +\infty \quad \text{as} \quad n \to \infty
\end{aligned}$$

because the series with positive terms $\sum_{n=0}^{\infty} |c_n| t^n$ diverges: indeed, $t > R$, the radius of convergence. If $\limsup_{n\to\infty} |\theta_n - \frac{\pi}{2}| < \frac{\pi}{2}$ holds, the reasoning is similar by considering $\gamma := \inf_{n>N} \sin\theta_n$ and taking imaginary parts instead of real parts. The remaining third and four cases $\limsup_{n\to\infty} |\theta_n - \pi| < \frac{\pi}{2}$ and $\limsup_{n\to\infty} |\theta_n - \frac{3\pi}{2}| < \frac{\pi}{2}$ are analogous, just by considering the inequalities $|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}| \geq \mathrm{Re}\,(\sum_{j=N+1}^{n} -|c_j|\, t^j\, e^{i\,\theta_j}) - \sum_{j=0}^{N} |c_j|\, t^j$ $|\sum_{j=0}^{n} |c_j|\, t^j\, e^{i\,\theta_j}| \geq \mathrm{Im}\,(\sum_{j=N+1}^{n} -|c_j|\, t^j\, e^{i\,\theta_j}) - \sum_{j=0}^{N} |c_j|\, t^j$ and letting $\gamma := \inf_{n>N} |\cos\theta_n|$, $\gamma := \inf_{n>N} |\sin\theta_n|$, respectively. Thus, (d) is satisfied and the proof is concluded. □

Corollary 3.14. *Let $G \subset \mathbb{C}$ be a simply connected domain, and assume that $\mathbf{c} = (c_n)_{n\geq 0}$ is a sequence satisfying $c_n \geq 0$ for all $n \geq 0$ and $\limsup_{n\to\infty} c_n^{1/n} > 0$. Then $(T_{\mathbf{c},n})$ is mixing on $\mathcal{H}(G)$.*

Remark 3.15. 1. For instance, the sequence of operators on $\mathcal{H}(G)$ given by $\big\{\sum_{k=0}^{n} (k+i)(1+i)^k D^k\big\}_{n\in\mathbb{N}_0}$ is universal, for any simply connected domain $G \subset \mathbb{C}$.

2. In [56] the hypercyclicity of a nonscalar operator $\Phi(D)$ on $\mathcal{H}(\mathbb{C})$ is established, which in particular yields Birkhoff's Theorem [24] and MacLane's Theorem [71] on hypercyclicity of the translation operator and the derivative operator, respectively. Note that this is equivalent to the universality of the sequence $(\Phi^n(D))$. Concerning universality of sequences of differential operators not being the iterates of a single one, the reader can find a number of results in [18, 20, 23, 83], but none of them covers Theorem 3.13. Moreover, the set $HC(\Phi(D))$ is spaceable, as proved by Petersson, Shkarin and Menet [76, 82, 86] (see also [58, Section 10.1]). This fact together with the results of this section motivates the next and final question.

Question 3.16. *Let $G \subset \mathbb{C}$ be a simply connected domain. Under what conditions is $HC((T_{\mathbf{c},n}))$ spaceable in $\mathcal{H}(G)$?*

Chapter 4

Polynomial norms

4.1 Introduction and preliminaries

Let us denote by $\mathcal{P}$ and $\mathcal{P}_n$, respectively, the vector spaces of all complex polynomials and all complex polynomials of degree at most $n \in \mathbb{N}$. Since $\mathcal{P}_n$ is finite dimensional, all norms defined on $\mathcal{P}_n$ are equivalent. In other words, if $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms defined on $\mathcal{P}_n$, there there exist constants $k(n), K(n) > 0$ such that

$$k(n)\|p\|_a \leq \|p\|_b \leq K(n)\|p\|_a \tag{4.1}$$

for all $p \in \mathcal{P}_n$. Inequalities of this type have been studied in the past for several polynomial norms. For instance, we can endow $\mathcal{P}$ with the following norms:

1. $\|p\|_{D_r} := \sup\{|p(z)| : |z| < r\}$, and
2. $\|p\|_1 := \sum_{i=0}^{n} |a_i|$,

where p is given by $p(z) = \sum_{i=0}^{n} a_i z^i$, $a_0, \ldots, a_n \in \mathbb{C}$, $r > 0$, and $D_r = r\mathbb{D}$ with $\mathbb{D}$ being the open unit disk. The optimal constants $k(n,r), K(n,r') > 0$ in (4.1), where $r, r' > 0$, $\|\cdot\|_a = \|\cdot\|_{D_r}$ and $\|\cdot\|_b = \|\cdot\|_{D_{r'}}$ are known (see for instance [28] and [84] for a complete account on polynomials and polynomial inequalities). A natural question would be whether or not $\|\cdot\|_{D_r}$ and $\|\cdot\|_{D_{r'}}$ are equivalent too in $\mathcal{P}$. The answer is no. However we can establish a relationship between the topologies induced by $\|\cdot\|_{D_r}$ and $\|\cdot\|_{D_{r'}}$ in $\mathcal{P}$.

Given two norms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathcal{P}$, we can define a relation representing the natural partial order $(\leq)$ in their respective induced topologies $T_{\|\cdot\|}$ and $T_{\|\cdot\|'}$ as follows.

Definition 4.1. *We say that* $\|\cdot\| \leq \|\cdot\|'$ *if the following three equivalent statements hold:*

(a) *There exists a constant* $K > 0$ *such that, for all* $p \in \mathcal{P}$*, we have* $\|p\| \leq K\|p\|'$.

(b) *The identity operator* $I : (\mathcal{P}, \|\cdot\|') \to (\mathcal{P}, \|\cdot\|)$ *is continuous.*

(c) $T_{\|\cdot\|'}$ *is finer than* $T_{\|\cdot\|}$*, that is,* $T_{\|\cdot\|} \subset T_{\|\cdot\|'}$.

Remark 4.2. The relation $\leq$ is *not* really a partial order *on* $\mathcal{P}$. If we consider any two equivalent norms $\|\cdot\|, \|\cdot\|'$ on $\mathcal{P}$ then $\|\cdot\| \leq \|\cdot\|'$ and $\|\cdot\|' \leq \|\cdot\|$. Then we should always see $\leq$ as the natural partial order *on their induced topologies,* that is,

$$\|\cdot\| \leq \|\cdot\|' \text{ if and only if } T_{\|\cdot\|} \subset T_{\|\cdot\|'}.$$

We also consider the corresponding strict order relation, that is:

Definition 4.3. *Given two norms* $\|\cdot\|$ *and* $\|\cdot\|'$ *in* $\mathcal{P}$*, we say that* $\|\cdot\| < \|\cdot\|'$ *if* $\|\cdot\| \leq \|\cdot\|'$ *but* $\|\cdot\|' \not\leq \|\cdot\|$.

The content of the next proposition is well-known.

Proposition 4.4. *Let* $\|\cdot\|$ *and* $\|\cdot\|'$ *be two norms on a vector space* Z*. The following are equivalent:*

1. $\|\cdot\| < \|\cdot\|'$.
2. *The identity operator* $I : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ *is continuous but it is not a topological isomorphism.*

A very simple way of proving that $\|\cdot\| < \|\cdot\|'$, is by means of compact operators.

Lemma 4.5. *Let* $(E, \|\cdot\|')$*,* $(F, \|\cdot\|)$ *be normed spaces and* $Z \subset E, F$ *be an infinite dimensional vector space. Suppose that* $T : (E, \|\cdot\|') \to (F, \|\cdot\|)$ *be a linear operator with* $T|_Z = I$*, the identity operator. If* T *is a compact operator then* $\|\cdot\| < \|\cdot\|'$ *on* Z.

Proof. Since T is compact, T is continuous and, thus, the operator $I = T|_Z : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is also continuous. Moreover, by Riesz's Theorem (see, e.g., [63]) there exist $\varepsilon_0 > 0$ and a sequence $\{p_n\}_{n\in\mathbb{N}} \subset B_{(Z,\|\cdot\|')}$ with $\|p_n - p_m\|' \geq \varepsilon_0 > 0$. Since T is a compact operator, we can assume, passing to a subsequence if necessary, that $\{T(p_n) = p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Z, \|\cdot\|)$. Therefore the operator $I : (Z, \|\cdot\|) \to (Z, \|\cdot\|')$ does not transform Cauchy sequences into Cauchy sequences and it cannot be uniformly continuous (nor continuous, by the linearity of I). Hence its inverse $I : (Z, \|\cdot\|') \to (Z, \|\cdot\|)$ is not a topological isomorphism. By the previous proposition, we conclude that $\|\cdot\| < \|\cdot\|'$ on Z. $\square$

On the one hand, it follows from the triangle inequality that $\|p\|_{D_1} \leq \|p\|_1$ for all $p \in \mathcal{P}$ (and, thus, $\|\cdot\|_{D_1} \preceq \|\cdot\|_1$). On the other hand, we shall prove that, for all $r > 1$, there exists a constant $K(r) > 0$ such that $\|p\|_1 \leq K(r)\|p\|_{D_r}$ for $p \in \mathcal{P}$ (and, thus, $\|\cdot\|_{D_1} \preceq \|\cdot\|_1 \preceq \|\cdot\|_{D_r}$ for all $r > 1$).

It might seem intuitive the fact that, if $r \to 1^+$, then

$$\|\cdot\|_{D_1} \preceq \|\cdot\|_1 \preceq \|\cdot\|_{D_r} \longrightarrow \|\cdot\|_{D_1}$$

and that, as a consequence, the norms $\|\cdot\|_{D_1}$ and $\|\cdot\|_1$ are really equivalent. However, and as we will also prove, this is *not* true. We shall prove that, although none of these previous norms are actually equivalent in any sense, what we do have is that

$$\|\cdot\|_{D_\varepsilon} \prec \|\cdot\|_{D_{\varepsilon'}} \prec \|\cdot\|_{D_1} \prec \|\cdot\|_1 \prec \|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}}$$

for every $0 < \varepsilon < \varepsilon' < 1 < r < r'$. Moreover, it is provided a rather general criterion about topological largeness of sets arising naturally when comparing two norms. The notation will be rather usual and the tools we employ are classical ones from the fields of Topology and Complex Variables.

4.2 The results

First of all, we shall need some additional notation.

Definition 4.6. *For every $r > 0$, we denote $\mathcal{H}_b(D_r) := \{f \in \mathcal{H}(D_r) : \|f\|_{D_r} < +\infty\}$, where $\mathcal{H}(D_r)$ stands for the space of all holomorphic functions on D_r. Here*

$$\|f\|_{D_r} = \sup\{|f(z)| : |z| < r\}.$$

Remark 4.7. We consider $\mathcal{H}_b(D_r)$ as the Banach space $(\mathcal{H}_b(D_r), \|\cdot\|_{D_r})$ and, naturally, if $0 < r < r'$ then, by the Identity Principle, we may consider $\mathcal{H}_b(D_{r'})$ as a subset of $\mathcal{H}_b(D_r)$.

Definition 4.8. *If $r' > r > 0$, we define the linear operator $I_{r,r'} : \mathcal{H}_b(D_{r'}) \to \mathcal{H}_b(D_r)$ as $I_{r,r'}(f) = f$.*

Now, we can obtain the first of one main results. By B_X we will denote the closed unit ball of a normed space X.

Theorem 4.9. *Assume that $0 < r < r'$. Then the following holds:*

1. *The ball $B_{\mathcal{H}_b(D_{r'})}$ is compact in $\mathcal{H}_b(D_r)$.*

2. $I_{r,r'}$ *is a compact operator.*

3. $\|\cdot\|_{D_r} \prec \|\cdot\|_{D_{r'}}$ *on* $\mathcal{P}$.

Proof. Obviously (1) implies (2) and, by Lemma 4.5 (with $E = \mathcal{H}_b(D_{r'})$, $F = \mathcal{H}_b(D_r)$, $Z = \mathcal{P}$, $\|\cdot\| = \|\cdot\|_{D_r}$, and $\|\cdot\|' = \|\cdot\|_{D_{r'}}$), (2) implies (3). So we only have to prove (1).

With this aim, let $\{f_n\}_{n\in\mathbb{N}} \subset B_{\mathcal{H}_b(D_{r'})}$. By Montel's Theorem (see, e.g., [57]) there exist a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ and an $f \in \mathcal{H}(D_{r'})$ such that $f_{n_k} \longrightarrow f$ uniformly in compact subsets of $D_{r'}$. We conclude that $f \in B_{\mathcal{H}_b(D_{r'})}$ and $f_{n_k} \longrightarrow f$ in $\mathcal{H}(D_r)$. So $B_{\mathcal{H}_b(D_{r'})}$ is compact in $\mathcal{H}_b(D_r)$. □

On the other hand, if we now consider $r > 1$, we have that, for all $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}_b(D_r)$, its radius of convergence is not less than $r > 1$, so $\sum_{n=0}^{\infty} |a_n| < +\infty$. This allows us to consider the linear operator given in the next definition, where ℓ_1 denotes the set of all absolutely summable sequences of complex numbers, which becomes a Banach space when endowed with the norm $\|(a_n)_{n\geq 0}\| = \sum_{n=0}^{\infty} |a_n|$.

Definition 4.10. *For all* $r > 1$ *we define the operator* $I_r : \mathcal{H}_b(D_r) \to \ell_1$ *as* $I_r(f) := (a_n)_{n\geq 0}$, *where* f *is as above.*

In order to prove that I_r is continuous, it will be useful to recall some basic concepts and results related to the compact-open topology.

Definition 4.11. *Let* $r > 0$, $f \in \mathcal{H}(D_r)$, $K \subset D_r$ *be a compact subset and* $\varepsilon > 0$. *We define* $B_f(K,\varepsilon) := \{g \in \mathcal{H}(D_r) : \sup\{|g(z) - f(z)| : z \in K\} \leq \varepsilon\}$.

Theorem 4.12. *Let* T_c *be the compact-open topology in* $\mathcal{H}(D_r)$. *Then we have:*

1. *The family* $\{B_f(K,\varepsilon) : f \in \mathcal{H}(D_r), \epsilon > 0, K \text{ compact} \subset D_r\}$ *is a neighborhood base for* $(\mathcal{H}(D_r), T_c)$.

2. $f_n \longrightarrow f$ *in* $(\mathcal{H}(D_r), T_c)$ *if and only if* $f_n \longrightarrow f$ *uniformly on compact subsets of* D_r.

3. $(\mathcal{H}(D_r), T_c)$ *is a completely metrizable space, hence a Baire space.*

Definition 4.13. *Let* $r > 1$. *For every* $N \in \mathbb{N}$, *we denote*
$F_N := \left\{f \in \mathcal{H}(D_r) : \sum_{n=0}^{\infty} |a_n| \leq N \text{ with } f(z) = \sum_{n=0}^{\infty} a_n z^n\right\}$.

Remark 4.14. Notice that $\mathcal{H}(D_r) = \bigcup_{N\in\mathbb{N}} F_N$.

The easy proof of the following result is left to the reader. If α is an scalar and S is a subset of a vector space, then αS stands for $\{\alpha x : x \in S\}$.

Lemma 4.15. *Assume that* $r > 1$ *and* $R > 0$. *Let* $\{(a_{i,n})_{i\geq 0}\}_{n\in\mathbb{N}} \subset RB_{\ell_1}$ *be a sequence such that*

$$\lim_{n\to\infty} a_{i,n} = a_i$$

for all $i \in \mathbb{N}$. *Then* $\sum_{i=0}^{\infty} |a_i| \leq R$.

In the following theorem, we collect a number of properties of the sets F_N given in Definition 4.13.

Theorem 4.16. *Let* $r > 1$. *We have:*

(a) *The set* F_N *is a closed subset of* $(\mathcal{H}(D_r), T_c)$ *for all* $N \in \mathbb{N}$.

(b) *There exists an* $N \in \mathbb{N}$ *such that:*

  (1) F_N *has non-empty interior in* $(\mathcal{H}(D_r), T_c)$.

  (2) $0 \in int_{(\mathcal{H}(D_r),T_c)} F_N$.

  (3) *There exists* $\varepsilon > 0$ *such that* $\varepsilon B_{\mathcal{H}_b(D_r)} \subset F_N$.

Proof. (a) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} f_n = f$ in $(\mathcal{H}(D_r), T_c)$, where $f_n(z) = \sum_{i=0}^{\infty} a_{i,n} z^i$ and $f(z) = \sum_{i=0}^{\infty} a_i z^i$. Since $f_n \longrightarrow f$ uniformly in compact subsets of D_r, by Weiestrass' Theorem (see [57]), $\lim_{i\to\infty} f_n^{(i)}(0) = f^{(i)}(0)$ for all $i \in \mathbb{N} \cup \{0\}$, and hence $\lim_{i\to\infty} a_{i,n} = a_i$. By Lemma 4.15, $\sum_{i=0}^{\infty} |a_i| \leq N$ and so $f \in F_N$.

(b) Part (1) follows from (a), Theorem 4.12(3) and Remark 4.14.

(2) By (1) and Theorem 4.12(1), there exist $f \in \mathcal{H}(D_r)$, a compact set $K \subset D_r$ and $\varepsilon > 0$ such that $f + B_0(K, \varepsilon) = B_f(K, \varepsilon) \subset F_N$. Since F_N is a symmetric set, we get

$$-2f + B_f(K, \varepsilon) = -f + B_0(K, \varepsilon) = B_{-f}(K, \varepsilon) = -B_f(K, \varepsilon) \subset F_N$$

and, since F_N is convex, $B_0(K, \varepsilon) = -f + B_f(K, \varepsilon) \subset F_N$.

(3) Obviously $\varepsilon B_{\mathcal{H}_b(D_r)} \subset B_0(K, \varepsilon) \subset F_N$. □

Now, one of our main results can be easily derived:

Theorem 4.17. *Assume that* $r > 1$. *Then the following holds:*

(a) *The linear operator* $I_r : \mathcal{H}_b(D_r) \to \ell_1$ *is compact.*

(b) $\|\cdot\|_1 \prec \|\cdot\|_{D_r}$ *on* $\mathcal{P}$.

Proof. By Theorem 4.16(b)(3), I_r is a bounded operator. Fix any $d \in (1, r)$. We can see I_r as the composition $I_r = I_d I_{d,r}$. Since I_d is continuous and $I_{d,r}$ is compact, I_r is compact. This proves (a). Finally, (b) follows from (a) and Lemma 4.5. □

Given a normed space E, we shall denote by $\overline{E}$ its completion.

Remark 4.18. Let us recall that:

1. $\overline{(\mathcal{P}, \|\cdot\|_1)} = \ell_1$.
2. $\overline{(\mathcal{P}, \|\cdot\|_{D_1})} = \mathcal{H}(D_1) \cap \mathcal{C}(\overline{D_1}) =: \mathcal{A}(D_1)$, the disk algebra.

The content of the following auxiliary assertion is well known.

Lemma 4.19. *Let* E *and* F *be normed spaces and let* $T : E \to F$ *be a linear and continuous operator. Then the following holds:*

1. *There exists a unique linear continuous operator* $\overline{T} : \overline{E} \to \overline{F}$ *such that* $\overline{T}|_E = T$.
2. *If* T *is a topological isomorphism then* $\overline{T}$ *is also a topological isomorphism.*

We denote by $I : \mathcal{P} \to \mathcal{P}$ the identity mapping $I(P) = P$, where the space $\mathcal{P}$ on the left should be thought as identified with c_{00}, the space of eventually zero complex sequences.

Corollary 4.20. *The linear operator* $\overline{I} : (a_n)_{n\in\mathbb{N}} \in \ell_1 \mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}(D_1)$ *is continuous and injective.*

Proof. By using the Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\overline{D_1}$. Since each term $a_n z^n$ is continuous on $\overline{D_1}$, so is its sum f. Moreover, the Weierstrass convergence theorem guarantees that f is holomorphic in D_1, so that $f \in \mathcal{A}(D_1)$ and the mapping $(a_n)_{n\in\mathbb{N}} \in \ell_1 \mapsto f \in \mathcal{A}(D_1)$ is well defined and, obviously, linear. That this mapping equals $\overline{I}$ is clear because its restriction to $\mathcal{P}$ equals I (via the identification $\mathcal{P} = c_{00}$), and $\mathcal{P}$ is dense both in ℓ_1 and $\mathcal{A}(D_1)$. The continuity of $\overline{I}$ is derived from Lemma 4.19(1), while its injectivity follows from the uniqueness of the Taylor coefficients around 0. □

However, is $\overline{I}$ also topological isomorphism? In order to answer this question, let us focus on the following four conjectures.

Conjecture 4.21.
(CI) *$\|\cdot\|_1$ and $\|\cdot\|_{D_1}$ are equivalent norms in $\mathcal{P}$.*

(CII) *The linear operator $\overline{I} : \ell_1 \to \mathcal{A}(D_1)$ is a topological isomorphism.*

(CIII) *For every $f \in \mathcal{A}(D_1)$ there exists $(a_i)_{i\in\mathbb{N}} \in \ell_1$ such that $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in \overline{D_1}$.*

(CIV) *The set*

$$A := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| < +\infty \ \text{ where } \ f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\} \tag{4.2}$$

is of second category in $\mathcal{A}(D_1)$.

Proposition 4.22. *The previous four conjectures* (CI), (CII), (CIII) *and* (CIV) *are equivalent.*

Proof. To start with, the facts (CII) $\Longrightarrow$ (CIII) and (CIII) $\Longrightarrow$ (CIV) are straightforward.

• (CI) is equivalent to (CII): $\|\cdot\|_1$ and $\|\cdot\|_{D_1}$ are equivalent norms in $\mathcal{P}$ if and only if $I : (\mathcal{P}, \|\cdot\|_1) \to (\mathcal{P}, \|\cdot\|_{D_1})$ is a topological isomorphism and, by Remark 4.18 and Lemma 4.19, the last property is equivalent to the fact that $\overline{I} : \ell_1 \to \mathcal{A}(D_1)$ is a topological isomorphism.

• (CIV) $\Longrightarrow$ (CII): The set $\overline{I}(\ell_1) = A$ is a second category set. Now, since $\overline{I}$ is linear, continuous and injective, the Banach–Schauder Theorem (Open Mapping Theorem) implies that $\overline{I}$ is a topological isomorphism. □

Proposition 4.23. (CIII) *is false, and so are* (CI), (CII) *and* (CIV) *by Proposition 4.22.*

Proof. The following result can be found in [62, §6]: There exists $f \in \mathcal{A}(D_1)$ such that $\sum_{i=0}^{\infty} |a_i| = +\infty$, where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$. This disproves (CIII). □

Since (CI) is false but we have $\|\cdot\|_{D_1} \le \|\cdot\|_1$, we obtain another promised result:

Theorem 4.24. *$\|\cdot\|_{D_1} \prec \|\cdot\|_1$ on $\mathcal{P}$.*

Corollary 4.25. $\|I_r\| \longrightarrow +\infty$ *as* $r \to 1^+$.

Proof. By way contradiction, suppose that there exist $K > 0$ and a sequence $\{r_n\}_{n\in\mathbb{N}}$ with $r_n \to 1^+$ such that $\|I_{r_n}\| \le K$ for all $n \in \mathbb{N}$. We have that, for every $n \in \mathbb{N}$ and for all $p \in \mathcal{P}$,

$$\|p\|_1 = \|I_{r_n}(p)\|_1 \le K\|p\|_{D_{r_n}} \longrightarrow K\|p\|_{D_1} \quad \text{as} \quad n \to \infty.$$

Thus, $\|p\|_1 \le K\|p\|_{D_1}$ and so $\|\cdot\|_1 \le \|\cdot\|_{D_1}$, which is absurd. □

Let A be the set defined in (4.2). Since (CIV) is false, A is a first category set. We will show that A enjoys, actually, a nice topological structure; namely, A is an $\mathcal{F}_\sigma$ set. Note that, in addition, A is dense since it contains the class $\mathcal{P}$.

Let $f \in \mathcal{A}(D_1)$ with $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$. We know that its radius of convergence is at least 1. Then, for all $\varepsilon \in (0,1)$, we obtain $\sum_{i=0}^{\infty} |a_i|\varepsilon^i < +\infty$. Thus, we can define the following operator.

Definition 4.26. *For every* $\varepsilon \in (0,1)$ *we define the linear operator*

$$i_\varepsilon : \mathcal{A}(D_1) \longrightarrow \ell_1$$

as

$$i_\varepsilon(f) = (a_i\varepsilon^i)_{i\ge 0},$$

where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ *for every* $z \in D_1$.

Proposition 4.27. *For every* $\varepsilon \in (0,1)$, *we have that* i_ε *is a compact operator.*

Proof. We are going to define a linear operator

$$T_\varepsilon : \mathcal{A}(D_1) \to \mathcal{H}_b(D_{1/\varepsilon}).$$

For this, making the substitution $z = \varepsilon\omega$ we set $T_\varepsilon(f)(\omega) := f(\varepsilon\omega)$. Now, since

$$\|T_\varepsilon(f)\|_{D_{1/\varepsilon}} = \|f\|_{D_1}$$

for every $f \in \mathcal{A}(D_1)$, we have that T_ε is continuous. Moreover, we can see i_ε as the composition $i_\varepsilon = I_{1/\varepsilon}T_\varepsilon$, where $I_{1/\varepsilon}$ is a compact operator. To sum up, i_ε is the composition of a compact operator and of a continuous operator, from which we conclude that it is compact. □

Corollary 4.28. *For every* $\varepsilon \in (0,1)$ *and every* $M > 0$*, we have that*

$$C_{M,\varepsilon} := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i|\varepsilon^i \leq M \text{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\}$$

is a closed subset of $\mathcal{A}(D_1)$.

Proof. It suffices with noticing that $C_{M,\varepsilon} = i_\varepsilon^{-1}(MB_{\ell_1})$. □

Corollary 4.29. *The set*

$$C_M := \left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| \leq M \text{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\} \tag{4.3}$$

is closed for every $M > 0$.

Proof. Let us show that $C_M = \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. It is clear that $C_M \subset \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. Let us see that $C_M \supset \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$. If $f \in \bigcap_{\varepsilon\in(0,1)} C_{M,\varepsilon}$ (where $f(z) = \sum_{i=0}^{\infty} a_i z^i$ for all $z \in D_1$) then (taking a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n \to 1^-$) we have that, for every $n \in \mathbb{N}$, $\sum_{i=0}^{\infty} |a_i|\varepsilon_n^i \leq 1$. Finally, by Lemma 4.15, $\sum_{i=0}^{\infty} |a_i| \leq 1$ and $f \in C_M$, which concludes the proof. □

Theorem 4.30. *The set* A *defined in* (4.2) *is an* $\mathcal{F}_\sigma$ *set of first category in* $\mathcal{A}(D_1)$. *Hence the set*

$$\left\{ f \in \mathcal{A}(D_1) : \sum_{i=0}^{\infty} |a_i| = +\infty \text{ where } f(z) = \sum_{i=0}^{\infty} a_i z^i \ \ \forall z \in D_1 \right\}$$

is a dense G_δ *set, so residual in* $\mathcal{A}(D_1)$.

Proof. It suffices with noticing that A is of first category due to Propostion 4.23 and to the fact that we can write $A = \bigcup_{N\in\mathbb{N}} C_N$, where the C_N's are given in (4.3). □

To finish this chapter, and inspired by the last theorem and its proof, we can furnish a rather general criterion of topological largeness inside normed spaces, see Theorem 4.33 below. Actually, the criterion also contains assertions about algebraic largeness.

Let us denote by $[-\infty, +\infty]$ the extended real line, endowed with the order topology. Recall that, if X is a topological space, a mapping $\Phi : X \to [-\infty, +\infty]$ is called *lower semicontinuous* (see, e.g., [30] for concepts and properties) whenever, given any $\alpha \in \mathbb{N}$, the set $\{x \in X : \Phi(x) > \alpha\}$ is open. If X is a metric space, this is equivalent to $\Phi(x_0) \leq \liminf_{x\to x_0} \Phi(x)$ for all $x_0 \in X$.

In the next definition, we are considering on $[0, +\infty]$ the natural extension of the usual order in $[0, +\infty)$.

Definition 4.31. *Let X be a vector space and $\Phi : X \to [0, +\infty]$ be a lower semi-continuous mapping. We say that Φ is an* extended norm on X *provided that the following properties are satisfied:*

(i) $\Phi(x) = 0$ *if and only if* $x = 0$.

(ii) *If* $\{\Phi(x), \Phi(y)\} \subset [0, +\infty)$ *then* $\Phi(x + y) \le \Phi(x) + \Phi(y)$.

(iii) *If* $\Phi(x) < +\infty$ *and* α *is a scalar then* $\Phi(\alpha x) = |\alpha|\Phi(x)$.

It is easy to see that, under the notation of the last definition, the set

$$X_\Phi := \{x \in X : \Phi(x) < +\infty\}$$

is a vector subspace of X and that the restriction of Φ to X_Φ is a norm on X_Φ.

The following concepts, which are taken from the theory of lineability (see [7] for background) are also needed.

Definition 4.32. *Assume that X is a vector space and that $A \subset X$. We say that A is* lineable *if it contains, except for zero, an infinite dimensional vector space. If X is, in addition, a topological vector space, then A is said to be* dense-lineable (spaceable, resp.) *in X provided that it contains, except for zero, a dense (a closed infinite dimensional, resp.) vector subspace.*

Theorem 4.33. *Assume that $(X, \|\cdot\|)$ is a Banach space and that Φ is an extended norm on X. Let us denote*

$$A_\infty := X \setminus X_\Phi = \{x \in X : \Phi(x) = +\infty\}.$$

Then the following holds:

(a) *If* $\Phi \not\lesssim \|\cdot\|$ *on* X_Φ, *then the set* A_∞ *is residual in* X.

(b) *If* X_Φ *is dense in* X, $\|\cdot\| \prec \Phi$ *on* X_Φ *and* (X_Φ, Φ) *is a Banach space, then* A_∞ *is spaceable in* X. *If, in addition,* $(X, \|\cdot\|)$ *is separable, then* A_∞ *is dense-lineable too.*

Proof. (a) We have to prove that X_Φ is of first category in X. For this, note that $X_\phi = \bigcup_{n=1}^\infty F_n$, where we have set $F_\alpha := \{x \in X : \Phi(x) \le \alpha\}$ for every $\alpha \in (0, +\infty)$. Therefore it suffices to show that each set F_α is closed and has empty interior in X. That F_α is closed is derived from the openness of $X \setminus F_\alpha = \{x \in X : \Phi(x) > \alpha\}$, which in turn comes from the assumption of lower semicontinuity for Φ.

In order to prove that F_α has empty interior in X, assume, by way of contradiction, that there are $x_0 \in X$ and $R > 0$ such that $\{x \in X : \|x - x_0\| \le R\} \subset F_\alpha$ or, that is the same, $\Phi(x) \le \alpha$ for every $x \in X$ satisfying $\|x - x_0\| < R$. Since $\Phi \not\precsim \|\cdot\|$ on X_Φ, we can find a sequence $\{x_n\}_{n\ge 1} \subset X_\Phi$ such that $\Phi(x_n) > n\|x_n\|$ for all $n \in \mathbb{N}$. Note that $x_n \ne 0$, and so $\|x_n\| > 0$ $(n = 1, 2, \dots)$. Select an $N \in \mathbb{N}$ with $N > 2\alpha/R$ and define $x := x_0 + \frac{R}{\|x_N\|} x_N$. Note, on the one hand, that $\|x - x_0\| = R$, which implies $\Phi(x) \le \alpha$. But, on the other hand, since Φ is a norm on X_Φ, thanks to the triangle inequality we get

$$\Phi(x) \ge \Phi\Big(\frac{R}{\|x_N\|} x_N\Big) - \Phi(x_0) > RN - \alpha > 2\alpha - \alpha = \alpha,$$

which is absurd. This proves the residuality of A_∞.

(b) Here we shall make use of the following facts. The first of them is a special case of Theorem 3.3 in [67], while the second one can be found in [21, Theorem 2.5]:

(1) *Let Y be a Banach space and X be a Fréchet space. If $T : Y \to X$ is a continuous linear mapping and $T(Y)$ is not closed in X, then the complement $X \setminus T(Y)$ is spaceable in X.*

(2) *Let X be a metrizable separable topological vector space and Y be a vector subspace of X. If $X \setminus Y$ is lineable, then $X \setminus Y$ is dense-lineable in X.*

Let us apply (1) with $Y := (X_\Phi, \Phi)$ and $T := I : x \in X_\Phi \mapsto x \in X$, the inclusion mapping, which is linear, but also continuous because $\|\cdot\| \le \Phi$ on X_Φ. Observe that, under this notation, $A_\infty = X \setminus T(Y)$. Assume, via contradiction, that $T(Y) = X_\Phi$ is closed in X. Since $\|\cdot\| \prec \Phi$, we have in particular that $\Phi \not\precsim \|\cdot\|$ on X_Φ. Hence, by part (a), A_∞ is residual in X, so nonempty. But $X_\Phi = X$ because X_Φ is dense and closed, which entails $A_\infty = \varnothing$, that is absurd. Consequently, $T(Y)$ is not closed in X and (1) tells us that A_∞ is spaceable.

Finally, if we assume that $(X, \|\cdot\|)$ is separable, the dense-lineability of A_∞ follows from the above result (2) (with $Y := X_\Phi$) and the fact that spaceability implies lineability. □

Remark 4.34. Theorem 4.30 follows from Theorem 4.33(a) just by taking $X := \mathcal{A}(D_1)$, $\|f\| := \sup_{z\in D_1} |f(z)|$ and $\Phi(f) := \sum_{n=0}^\infty |a_n|$, where $f \in \mathcal{A}(D_1)$ and $f(z) = \sum_{n=0}^\infty a_n z^n$ for all $z \in D_1$. Since $\|\cdot\| \prec \Phi$ on the space $A := \big\{f \in \mathcal{A}(D_1) : \sum_{n=0}^\infty |a_n| < +\infty\big\}$, we obtain in particular that $\Phi \not\precsim \|\cdot\|$. Then the unique property to be checked is the lower semicontinuity of Φ. For this, observe that each mapping

$$S_n : f \in \mathcal{A}(D_1) \longmapsto \sum_{k=0}^n |a_k| = \sum_{k=0}^n \Big|\frac{f^{(k)}(0)}{k!}\Big| \in \mathbb{R} \quad (n \in \mathbb{N})$$

is continuous, due to the Weierstrass convergence theorem for derivatives and the fact that convergence in $\mathcal{A}(D_1)$ implies uniform convergence on compacta (hence convergence at 0). In particular, each S_n is lower semicontinuous. But, evidently, $\Phi = \sup\{S_n : n \in \mathbb{N}\}$, and the supremum of a family of lower semicontinuous functions is known to be lower semicontinuous (see [30]). The spaceability (already proved in [67]) and the dense lineability of $\left\{f \in \mathcal{A}(D_1) : \sum_{n=0}^{\infty} |a_n| = +\infty\right\}$ follow from Theorem 4.33(b) since the set A is dense in the (separable) space $\mathcal{A}(D_1)$ and (A, Φ) is a Banach space.

Chapter 5

Estimating the n-dimensional Bohr radius

5.1 Introduction

As in the previous chapters, denote the space of all analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ by $\mathcal{H}(\mathbb{D})$. In 1914, H. Bohr [27] proved that any function $f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{D})$ such that $f(\mathbb{D}) \subset \mathbb{D}$ satisfies

$$\sum_{k=0}^{\infty} |a_k z^k| \leq 1 \tag{5.1}$$

whenever $|z| \leq \frac{1}{6}$. The number K_1 defined as the best radius for which this happens, that is,

$$\begin{aligned} K_1 := \sup\{r \in [0,1) : (5.1) \text{ holds for all } f \in \mathcal{H}(\mathbb{D}) \\ \text{such that } f(\mathbb{D}) \subset \mathbb{D} \text{ and all } z \text{ with } |z| \leq r\}, \end{aligned}$$

is called the *Bohr radius* for $\mathbb{D}$. Then $K_1 \geq \frac{1}{6}$. Subsequently later, Wiener, Riesz and Schur, independently established the exact value $K_1 = \frac{1}{3}$. A detailed account of the development of the topic can be read in the survey article [2] and the references therein.

In 1997 Boas and Khavinson [26] introduced for each $n \in \mathbb{N} := \{1, 2, \dots\}$ the *n-dimensional Bohr radius* K_n for the polysdisk $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. As expected, K_n is defined as the largest number r satisfying $\sum_{\alpha} |c_\alpha z^\alpha| < 1$ for all z with

$\|z\|_\infty < r$ and all $f(z) = \sum_\alpha c_\alpha z^\alpha \in \mathcal{H}(\mathbb{D}^n)$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}^n$. Here α denotes an n-tuple $(\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, z stands for an n-tuple $(z_1, \ldots, z_n)$ of complex numbers, $\|z\|_\infty = \max\{|z_1|, \ldots, |z_n|\}$, and z^α denotes the product $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. In [26, Theorem 2] the following bilateral estimate is proved.

Theorem 5.1. *For every $n \in \mathbb{N}$ with $n \geq 2$, the n-dimensional Bohr radius K_n satisfies*

$$\frac{1}{3\sqrt{n}} < K_n < 2\sqrt{\frac{\log n}{n}}. \tag{5.2}$$

Approximations for the value of K_n, in the cases $n = 1$ or $n \geq 2$, have been given in domains more general than $\mathbb{D}$ or $\mathbb{D}^n$ (with appropriate definitions for such domains), for specific subclasses of holomorphic functions, for functions related to holomorphic ones (such as harmonic functions, among others), and even for holomorphic functions on domains contained in infinite dimensional spaces or for vector-valued analytic functions (see, e.g., [1,3–6,9,16,25,26,31,42,45,47,48,51,59,64–66] and the references contained in them). For information about the state of the art on Bohr radii we refer to [46]. Concerning the asymptotic behaviour of K_n when $n \to \infty$, it was proved in [41] that $K_n \geq c\sqrt{\log n/(n \log\log n)}$ for some constant $c > 0$, while in [43] it was shown that $K_n = b_n\sqrt{(\log n)/n}$ with $1/\sqrt{2} + o(1) \leq b_n \leq 2$. Finally, in [14], the exact asymptotic behaviour of K_n is established, namely,

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{(\log n)/n}} = 1.$$

Despite the fact that the above limit gives a very precise description of the asymptotic behavior of the sequence (K_n), no exact value of K_n is known for any $n \geq 2$. In this chapter we focus on the (non-asymptotical) lower estimate $\dfrac{1}{3\sqrt{n}} < K_n$ of (5.2). In Remark 1 of the paper [26] itself a small improvement is stated, namely, K_n is not less than the solution $r > 0$ of the equation

$$r + \sum_{k=2}^{\infty} \binom{n+k-1}{k}^{1/2} r^k = \frac{1}{2}.$$

However, no further non-asymptotical enhancement has been given since then.

The aim of this chapter is to provide such an improved lower estimate. Moreover, it will be shown that the new estimate is better than the previously mentioned estimate found in [41] and, finally, that it is not less than an absolute constant times the n-dimensional Bohr radius. We shall begin by establishing a number of preliminary assertions. The main result of this chapter shall be Theorem 5.14.

5.2 Some preliminary results and notation

Let $k, n \in \mathbb{N}$. Define $N_k(n)$ as the combinatorial number

$$N_k(n) = \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!\,k!}.$$

Since $\lim_{k\to\infty} \frac{N_k(n)}{N_{k+1}(n)} = 1$ for each fixed $n \in \mathbb{N}$, we derive from Cauchy–Hadamard's formula that the radius of convergence of the series

$$\sum_{k=2}^{\infty} \sqrt{N_k(n)}\, x^k$$

equals 1. Therefore, the function

$$f_n(x) := x + \sum_{k=2}^{\infty} \sqrt{N_k(n)}\, x^k \tag{5.3}$$

is well-defined and analytic in $(-1,1)$, and hence in $[0,1)$ as well.

Lemma 5.2. *For every $n \in \mathbb{N}$, let $f_n : [0,1) \to \mathbb{R}$ be the restriction to $[0,1)$ of the function defined by equation (5.3). We have*

(a) *$f_n'(x) > 0$ for all $x \in (0,1)$.*

(b) *f_n is strictly increasing.*

(c) *f_n is injective.*

(d) $\lim_{x\to 1^-} f_n(x) = +\infty$.

(e) *For every $M \in (0,+\infty)$ there exists a unique $S \in (0,1)$ such that $f_n(S) = M$.*

Proof. (a) follows straightforwardly because all coefficients of the series defining f_n are positive. Also, (a) implies (b), and (b) implies (c). Finally, (d) implies the existence part of (e) (the uniqueness part follows from (c)) by the intermediate mean value and the fact $f_n(0) = 0$. As for (d), since $N_k(n) \geq 1$ for all n we obtain that $f(x) \geq \sum_{n=1}^{\infty} x^n = x/(1-x)$ and the conclusion follows. □

The following definition makes sense by taking $M = \frac{1}{2}$ in Lemma 5.2(e).

Definition 5.3. *For every $n \in \mathbb{N}$, we denote by S_n the unique positive solution –a fortiori, in $(0,1)$– of the equation:*

$$S_n + \sum_{k=2}^{\infty} \sqrt{N_k(n)} S_n^k = \frac{1}{2}. \tag{5.4}$$

In other words, S_n is the estimate given in [26, Remark 1] (see Section 1). In [26] it is not explicitly proved that S_n is a better lower estimate than $\frac{1}{3\sqrt{n}}$. We do it in the following proposition for the sake of completeness.

Proposition 5.4. *For every $n \in \mathbb{N}$, $n \geq 2$, we have $S_n > \dfrac{1}{3\sqrt{n}}$.*

Proof. Since $N_k(n) < n^k$ for all $k \in \mathbb{N}$, $k \geq 2$, we have

$$\begin{aligned} f_n\left(\frac{1}{3\sqrt{n}}\right) &= \sum_{k=1}^{\infty} \sqrt{N_k(n)} \left(\frac{1}{3\sqrt{n}}\right)^k < \sum_{k=1}^{\infty} \sqrt{n^k} \left(\frac{1}{3\sqrt{n}}\right)^k \\ &= \sum_{k=1}^{\infty} (1/3)^k = \frac{1}{2} = f_n(S_n). \end{aligned}$$

Thus, $\dfrac{1}{3\sqrt{n}} < S_n$ due to Lemma 5.2(b). □

Prior to going on, note that for all $n, k \in \mathbb{N}$ we have the linear isomorphism $\mathcal{P}(^k\mathbb{C}^n) \approx \mathbb{C}^{N_k(n)}$. We recall that $\mathcal{P}(^k\mathbb{C}^n)$ is the space of all k-homogeneous polynomials of n complex variables. Also $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. Here, as in previous chapters, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol $\Lambda(n,k)$ will stand for the set of all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ such that $|\alpha| = k$, while $\Delta(n,k)$ will represent the set $\{\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n : |\nu_1| + \cdots + |\nu_n| \leq k\}$, where, as usual, $\mathbb{Z}$ is the set of all integers. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\nu = (\nu_1, \ldots, \nu_n)$, we set $\nu\, x := \nu_1 x_1 + \cdots + \nu_n x_n$. Note that $\Lambda(n,k)$ contains $N_k(n)$ elements. We also denote $\mathbb{T} = \partial\mathbb{D} = \{z : |z| = 1\}$, the unit circle. The remaining of this section is devoted to provide a number of properties of homogeneous polynomials that will be used to prove our results.

Definition 5.5. Let $p \in \mathcal{P}(^k\mathbb{C}^n)$ with $p(z) = \sum_{\alpha \in \Lambda(n,k)} a_\alpha z^\alpha$. We define

1. $\|p\|_2 = \left(\int_{[0,1]^n} |p(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n})|^2 \, dt_1 \ldots dt_n\right)^{\frac{1}{2}}$,

   which is equal to $\left(\sum_{\alpha \in \Lambda(n,k)} |a_\alpha|^2\right)^{\frac{1}{2}}$.

2. $\|p\|_\infty = \sup\{|p(z)| : z \in \mathbb{T}^n\}$.

3. $\|p\|_1 = \sum_{\alpha\in\Lambda(n,k)} |a_\alpha|$.

It is very well-known that $\|\cdot\|_\infty$, $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on $\mathcal{P}({}^k\mathbb{C}^n)$.

The following lemma, whose proof is elementary and so left to the reader, will be used several times along this section.

Lemma 5.6. *For all* $p \in \mathcal{P}({}^k\mathbb{C}^n)$ *with* $p(z) = \sum_{\alpha\in\Lambda(n,k)} a_\alpha z^\alpha$ *the following holds*

1. $\|p\|_2 \le \|p\|_\infty$*, and* $\|p\|_2 = \|p\|_\infty$ *if and only if* $|p|$ *is constant on the circle product* $\mathbb{T}^n$*.*

2. $\|p\|_\infty \le \|p\|_1$*.*

3. $\|p\|_1 \le \sqrt{N_k(n)}\|p\|_2$*, and* $\|p\|_1 = \sqrt{N_k(n)}\|p\|_2$ *if and only if for all* $\alpha \in \Lambda(n,k)$*,* $|a_\alpha| = \frac{\|p\|_2}{\sqrt{N_k(n)}}$*.*

Definition 5.7. A mapping $T : \mathbb{R}^n \to \mathbb{C}$ is a *trigonometric polynomial of degree* k *on* n *variables* if there exists a finite family $\{c_\nu\}_{\nu\in\Delta(n,k)} \subset \mathbb{C}$ such that $T(x) = \sum_{\nu\in\Delta(n,k)} c_\nu e^{i\nu x}$ for all $x \in \mathbb{R}^n$.

Note that norms like $\|\cdot\|_\infty$, $\|\cdot\|_1$, $\|\cdot\|_2$ above can be similarly defined on the vector space of trigonometric polynomials given in the last definition. It is straightforward to check, and well-known, that (with the previous notation) $T = 0$ if and only if $c_\nu = 0$ for all $\nu \in \Delta(n,k)$.

Next, let us adopt some notation. For each $p \in \mathcal{P}({}^k\mathbb{C}^n)$, we set

$$T_p(x) := |p(e^{ix_1}, \ldots, e^{ix_n})|^2 \quad (x = (x_1, \ldots, x_n) \in \mathbb{R}^n),$$

that is, $T_p(x) = p(e^{ix_1}, \ldots, e^{ix_n})\overline{p(e^{ix_1}, \ldots, e^{ix_n})}$. It is also straightforward to see that T_p is a trigonometric polynomial of degree $2k$ on n variables. And if $p(z) = \sum_{\alpha\in\Lambda(n,k)} a_\alpha z^\alpha$ and $j \in \{1, \ldots, n\}$, then we denote

$$p_j(z) = \sum_{\alpha\in\Lambda(n,k)} b_\alpha z^\alpha,$$

where $b_\alpha = \alpha_j a_\alpha$ for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n,k)$.

For each $j \in \mathbb{N}$, we denote by e_j the n-tuple $(0, 0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 at the jth place, so that the n-tuple ke_j consists of zeros except for k at the jth place.

Lemma 5.8. *Assume that $n \geq 2$. Let $p \in \mathcal{P}(^k\mathbb{C}^n)$ with*

$$p(z) = \sum_{\alpha \in \Lambda(n,k)} a_\alpha z^\alpha$$

satisfying that there exist $j, j' \in \{1, \dots, n\}$ with $j \neq j'$ such that $a_{ke_j} \neq 0 \neq a_{ke_{j'}}$. Then $\|p\|_2 < \|p\|_\infty$.

Proof. We have that

$$\begin{aligned}
\frac{\partial T_p}{\partial x_j}(x) &= i p_j(e^{ix_1}, \dots, e^{ix_n})\overline{p(e^{ix_1}, \cdots, e^{ix_n})} - i p(e^{ix_1}, \dots, e^{ix_n})\overline{p_j(e^{ix_1}, \dots, e^{ix_n})} \\
&= i b_{ke_j}\overline{a_{ke_{j'}}} e^{ik(x_j - x_{j'})} + \cdots - i a_{ke_{j'}}\overline{b_{ke_j}} e^{ik(x_{j'} - x_j)} - \cdots \\
&= ik a_{ke_j}\overline{a_{ke_{j'}}} e^{ik(x_j - x_{j'})} + \cdots - ik a_{ke_{j'}}\overline{a_{ke_j}} e^{ki(x_{j'} - x_j)} - \cdots .
\end{aligned}$$

Since $a_{ke_j} a_{ke_{j'}} \neq 0$ it follows that $\frac{\partial T_p}{\partial x_j} \neq 0$. Thus T_p is non-constant. However T_p is nothing but $|p|^2$ on $\mathbb{T}^n$. From Lemma 5.6(1) we conclude that $\|p\|_2 < \|p\|_\infty$. □

Proposition 5.9. *Let $n \in \mathbb{N}$ with $n \geq 2$. Then for all $p \in \mathcal{P}(^k\mathbb{C}^n)$ with $\|p\|_\infty = 1$ we have*

$$\|p\|_1 < \sqrt{N_k(n)}.$$

Proof. By Lemma 5.6 we obtain

$$\|p\|_1 \leq \sqrt{N_k(n)}\|p\|_2 \leq \sqrt{N_k(n)}.$$

Let us suppose, by way of contradiction, that there exists $p \in \mathcal{P}(^k\mathbb{C}^n)$, $p(z) = \sum_{\alpha \in \Lambda(n,k)} a_\alpha z^\alpha$, with $\|p\|_\infty = 1$ such that $\|p\|_1 = \sqrt{N_k(n)}$. By the previous inequalities we deduce that

$$\|p\|_1 = \sqrt{N_k(n)}\|p\|_2 = \sqrt{N_k(n)}.$$

Then $\|p\|_2 = 1 = \|p\|_\infty$ and, by Lemma 5.6(3), for all $\alpha \in \Lambda(n,k)$ there exists $t_\alpha \in \mathbb{R}$ such that $a_\alpha = \frac{1}{\sqrt{N_k(n)}} e^{it_\alpha}$. Finally, by Lemma 5.8 we conclude that $1 = \|p\|_2 < \|p\|_\infty = 1$, a contradiction. □

Definition 5.10. We recall that for each pair $n, k \in \mathbb{N}$, *the Sidon constant* $S(k,n)$ is defined as

$$\begin{aligned}
S(k,n) &= \sup\Big\{ \sum_{|\alpha|=k} |c_\alpha| : P(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha \in \mathcal{P}(^k\mathbb{C}^n) \text{ and } |P(z)| \leq 1,\ \forall z \in \mathbb{D}^n \Big\} \\
&= \inf\{M > 0 : \|P\|_1 \leq M\|P\|_\infty \text{ for all } P \in \mathcal{P}(^k\mathbb{C}^n)\}.
\end{aligned}$$

Clearly $S(k,n) \geq 1$ for every k,n. Moreover, $S(1,n) = 1$ for all n. Indeed, if $P(z) = \sum_{j=1}^n c_j z_j$, consider $t_j \in [0,2\pi]$ such that $|c_j| = e^{it_j} c_j$ for $j = 1, \ldots n$. We have that

$$\sum_{j=1}^n |c_j| = \sum_{j=1}^n c_j e^{it_j} = P(e^{it_1}, \ldots, e^{it_n}) \leq \|P\|_\infty.$$

Hence $S(1,n) \leq 1$. Clearly the Sidon constant coincides with the norm of the identity operator

$$I : (\mathcal{P}(^k\mathbb{C}^n), \|\cdot\|_\infty) \longrightarrow (\mathcal{P}(^k\mathbb{C}^n), \|\cdot\|_1). \tag{5.5}$$

As $\mathcal{P}(^k\mathbb{C}^n)$ is finite dimensional, $S(k,n)$ is actually a maximum, i.e., for each $k \in \mathbb{N}$, there exists a polynomial $P_k \in \mathcal{P}(^k\mathbb{C}^n)$ satisfying $\|P_k\|_\infty = 1$ and $\|P_k\|_1 = S(k,n)$.

Corollary 5.11. *For each pair $n,k \in \mathbb{N}$ with $n \geq 2$, we have*

$$0 < S(k,n) < \sqrt{N_k(n)}.$$

5.3 The new estimate for the n-dimensional Bohr radius

The following lemma is needed in order to prove our estimate. It is an application of a classical result by Wiener (see e.g. [46, Lemma 8.4, p.183], or [26, Theorem 3]). Actually it is, implicitly, contained in the proof of [43, Theorem 2] as well. We give the proof here for the sake of completeness.

Lemma 5.12. *Let $f(z) = \sum_\alpha c_\alpha z^\alpha$ be an analytic function of modulus less than 1 in the n-dimensional polydisk $\mathbb{D}^n$. Then the following holds*

$$\sum_{|\alpha|=k} |c_\alpha| \leq S(k,n)(1 - |c_0|^2),$$

for all $k \geq 1$.

Proof. For every $j \in \mathbb{N}$ we consider the polynomial $p_j(z) := \sum_{|\alpha|=j} c_\alpha z^\alpha$ and for a fixed k take $u = (u_1, \ldots, u_n) \in \mathbb{T}^n$ such that $|p_k(u)| = \|p_k\|_\infty$. Now, for $\omega \in \mathbb{D}$ we define

$$g(\omega) := f(u_1\omega, \ldots, u_n\omega) = \sum_{j=0}^\infty b_j\, \omega^j.$$

Then g is an analytic function in $\mathbb{D}$ of modulus less than 1 satisfying $b_j = p_j(u)$ for all j. Now, Wiener's result ([46, Lemma 8.4, p.183]) yields

$$\|p_k\|_\infty = |b_k| \leq 1 - |b_0|^2 = 1 - |c_0|^2,$$

for all k. Finally, by definition,

$$\sum_{|\alpha|=k} |c_\alpha| \le S(k,n)\|p_k\|_\infty \le S(k,n)(1-|c_0|^2).$$

□

Analogously to Lemma 5.2, and taking into account that $S(k,n) < \sqrt{N_k(n)}$ for $n \ge 2$ (Corollary 5.11), we obtain that the function

$$\widetilde{f_n}(x) := \sum_{k=1}^{\infty} S(k,n)\, x^k \tag{5.6}$$

is well-defined and analytic in $(-1,1)$, it is strictly increasing on $[0,1)$ and satisfies $\widetilde{f_n}(0) = 0$ and $\widetilde{f_n}(x) > x$ for all $x \in (0,1)$. Then, for every $M \in (0,1)$ there exists a unique $H \in (0,1)$ such that $\widetilde{f_n}(H) = M$. Taking, in particular, $M = \frac{1}{2}$, the following definition makes sense.

Definition 5.13. *For every $n \in \mathbb{N}$, we denote by H_n the unique positive solution –a fortiori, in $(0,1)$– of the equation*

$$H_n + \sum_{k=2}^{\infty} S(k,n) H_n^k = \frac{1}{2}. \tag{5.7}$$

The next statement shows that H_n is a lower estimate for the Bohr radius that is better than S_n (we shall later also show that H_n is sharper than that from [41]).

Theorem 5.14. *Let $n \in \mathbb{N}$ with $n \ge 2$, and consider the n-dimensional Bohr radius K_n as well as the numbers S_n, H_n that had been respectively defined by (5.4) and (5.7). Then we have*

$$S_n < H_n \le K_n \le 3H_n.$$

Proof. Since $S(k,n) < \sqrt{N_k(n)}$, for all $k \in \mathbb{N}$, we get $\widetilde{f_n}(x) < f_n(x)$ for all $x \in (0,1)$, where f_n, $\widetilde{f_n}$ are respectively defined by (5.3) and (5.6). Observe that

$$f_n(S_n) = \frac{1}{2} = \widetilde{f_n}(H_n) < f_n(H_n).$$

Then the inequality $S_n < H_n$ follows from the strict monotonicity of f_n.

Next, we prove $H_n \le K_n$. For this, assume that $f(z) := \sum_\alpha c_\alpha z^\alpha$ is an analytic function of modulus less than 1 in the polydisk $\mathbb{D}^n$. Take a point $z = (z_1, \ldots, z_n) \in$

$\mathbb{D}^n$ with $|z_j| \le H_n$ for all $j \in \{1, \dots, n\}$. From Lemma 5.12, we obtain

$$\begin{aligned}\sum_{\alpha} |c_\alpha z^\alpha| &\le |c_0| + \sum_{k=1}^{\infty} \sum_{|\alpha|=k} |c_\alpha| H_n^k \\ &\le |c_0| + \sum_{k=1}^{\infty} (1 - |c_0|^2) S(k,n) H_n^k \\ &= |c_0| + (1 - |c_0|^2) \cdot \left(\sum_{k=1}^{\infty} S(k,n) H_n^k \right) \\ &= |c_0| + (1 - |c_0|^2) \cdot \frac{1}{2} \le 1.\end{aligned}$$

From the definition of K_n we infer that $H_n \le K_n$.

Now, let us show that $K_n \le 3H_n$. In order to do this, fix $n \in \mathbb{N}$ with $n \ge 2$. It is well-known that $K_n < K_1 = \frac{1}{3}$. Let us fix $\delta \in (3, \frac{1}{K_n})$. Then $\frac{1}{\delta} < 1 - \frac{2}{\delta}$, so that we can choose $r \in (\frac{1}{\delta}, 1 - \frac{2}{\delta})$. Also, for each k, we take a polynomial $P_k \in \mathcal{P}({}^k\mathbb{C}^n)$ satisfying $\|P_k\|_\infty = 1$ and $\|P_k\|_1 = S(k,n)$. The series

$$\sum_{k=1}^{\infty} r^k P_k(z)$$

converges absolutely and uniformly on $\mathbb{D}^n$. Hence it defines a holomorphic function F in $\mathbb{D}^n$. This function satisfies

$$|F(z)| \le \sum_{k=1}^{\infty} r^k = \frac{r}{1-r} \quad \text{for all } z \in \mathbb{D}^n.$$

Let us denote by $\sum_\alpha c_\alpha z^\alpha$ the monomial expansion of F on the n-dimensional polydisk.

Let $R := \delta H_n$. Observe that $0 < R \le \delta K_n < 1$. By using $\delta r > 1$ and $\delta > \frac{2}{1-r}$, we obtain for any $z \in R\,\mathbb{T}^n$ the following:

$$\begin{aligned}\sum_{\alpha} |c_\alpha z^\alpha| &= \sum_{k=1}^{\infty} \sum_{|\alpha|=k} |c_\alpha z^\alpha| = \sum_{k=1}^{\infty} \sum_{|\alpha|=k} |c_\alpha| (\delta H_n)^k \\ &= \sum_{k=1}^{\infty} (\delta H_n)^k r^k \|P_k\|_1 = \sum_{k=1}^{\infty} r^k \delta^k H_n^k S(k,n) \\ &= \delta r \cdot \sum_{k=1}^{\infty} (r\delta)^{k-1} H_n^k S(k,n) > \delta r \cdot \sum_{k=1}^{\infty} S(k,n) H_n^k \\ &= \delta r \cdot \frac{1}{2} > \frac{r}{1-r} \ge \|F\|_\infty.\end{aligned}$$

According to the definition of K_n, this implies $\delta H_n = R \ge K_n$. Letting $\delta \to 3^+$, we get $K_n \le 3H_n$. This completes the proof. $\square$

Remark 5.15. *Notice that trivially, $S(k,1) = 1$ for all $k \in \mathbb{N}$. Therefore H_1 is nothing but the unique solution, in $(0,1)$, of the equation $\sum_{k=1}^{\infty} x^k = 1/2$. Moreover, $\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$ and, consequently, $H_1 = \frac{1}{3} = K_1$, obtaining that*

$$H_n \leq K_n \leq 3H_n$$

also holds for $n = 1$.

Finally, we show that for every $n \geq 2$, our estimate H_n is sharper than that from [41], which was the best known estimate for low dimensions up to date.

Consider now in $\mathbb{C}^n$ any norm $\|\cdot\|$ (like $\|\cdot\|_1$, $\|\cdot\|_2$ or $\|\cdot\|_\infty$ defined above) and for each $k \in \mathbb{N}$ now $\mathcal{P}({}^k\mathbb{C}^n)$ stands for the finite dimensional Banach space of all k-homogeneous polynomials $P(z) = \sum_{|\alpha|=k} c_\alpha z^\alpha$ $(z \in \mathbb{C}^n)$ endowed with the norm $\|P\| := \sup_{\|z\|\leq 1} |P(z)|$. We denote by $\chi_{\text{mon}}(\mathcal{P}({}^k\mathbb{C}^n))$ the unconditional basic constant for all monomials z^α $(|\alpha| = k)$, that is,

$$\chi_{\text{mon}}(\mathcal{P}({}^k\mathbb{C}^n)) = \sup\left\{\Big\|\sum_{|\alpha|=k} |c_\alpha| z^\alpha\Big\| : \Big\|\sum_{|\alpha|=k} c_\alpha z^\alpha\Big\| \leq 1\right\}.$$

The following inequality is given in [44, (0.5)] (see also [45, (2.5)]):

$$\frac{1}{3 \cdot \sup_k (\chi_{\text{mon}}(\mathcal{P}({}^k\mathbb{C}^n)))^{1/k}} \leq K(B_{\mathbb{C}^n}),$$

where $K(B_{\mathbb{C}^n})$ is the Bohr radius of $B_{\mathbb{C}^n}$. In the case $\|\cdot\| = \|\cdot\|_\infty$ we have that $B_{\mathbb{C}^n}$ is the polydisk $\mathbb{D}^n$, so $K(B_{\mathbb{C}^n}) = K_n$. But, in addition, it is clear that

$$\chi_{\text{mon}}(\mathcal{P}({}^k\mathbb{C}^n), \|\cdot\|_\infty) = S(k,n).$$

Consequently, one has $\frac{1}{3C_n} \leq K_n$, where

$$C_n := \sup\{(S(k,n))^{1/k} : k \in \mathbb{N}\}.$$

In Theorem 1.1 of Defant–Frerick's paper [41], it is proved the existence of an absolute constant $c > 0$ such that

$$\frac{1}{c}\sqrt{\frac{\ell(n)}{n}} \leq K_n \quad \text{for all} \quad n \geq 2, \tag{5.8}$$

where $\ell(n) := \max\{l \in \mathbb{N} : l^l \leq n\}$. From this and the fact that $\ell(n) \geq \widetilde{c}\sqrt{\log n/\log\log n}$ for another absolute constant $\widetilde{c} > 0$, it is derived that $K_n \geq C\sqrt{\log n/(n \log\log n)}$ for some absolute constant $C > 0$. Now, we show that our estimate is better than the one in [41].

Proposition 5.16. *If $C > 0$ is the above constant, then*

$$C\sqrt{\log n/(n\log\log n)} < H_n \le K_n \quad \textit{for all} \quad n \ge 2.$$

Proof. The right hand inequality $H_n \le K_n$ has been proved in Theorem 5.14. Concerning the left hand inequality, note that in the proof of Theorem 1.1 of [41] the authors showed in fact, by using (3.1) of their article, that $\frac{1}{c}\sqrt{\frac{\ell(n)}{n}} \le \frac{1}{3C_n}$, where $c > 0$ is the constant in (5.8) above.

Fix $n \ge 2$. Firstly, we are going to show that there exists $k \in \mathbb{N}$ such that $(S(k,n))^{1/k} < C_n$. Indeed, if this were not the case, then $S(k,n) = C_n^k$ for all $k \in \mathbb{N}$. In particular, since $\|\cdot\|_\infty = \|\cdot\|_1$ for homogeneous polynomials of degree 1, we would obtain $1 = S(1,n) = C_n$, so $S(k,n) = 1$ for all $k \in \mathbb{N}$, which is false.

The function $\widetilde{f}_n(x) = \sum_{k=1}^\infty S(k,n)\, x^k$ satisfies that $\widetilde{f}_n(H_n) = \frac{1}{2}$, and

$$\widetilde{f}_n\left(\frac{1}{3C_n}\right) = \sum_{k=1}^\infty S(k,n)\cdot\frac{1}{3^k C_n^k} = \sum_{k=1}^\infty \frac{S(k,n)}{C_n^k}\cdot\frac{1}{3^k} < \sum_{k=1}^\infty \frac{1}{3^k} = \frac{1}{2} = \widetilde{f}_n(H_n).$$

Since $\widetilde{f}_n$ is strictly increasing, we get $\dfrac{1}{3C_n} < H_n$. Consequently, $H_n > \dfrac{1}{c}\sqrt{\dfrac{\ell(n)}{n}}$ which, together with the earlier comments, leads to the desired inequality. □

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