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Non-Linear Sets in Real Analysisand Algebraic Genericity

(Conjuntos no Lineales en Análisis Real y Genericidad Algebraica)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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**Departamento de Análisis Matemático
y Matemática Aplicada**

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Memoria para optar al grado de doctor
presentada por

María Elena Martínez Gómez

Bajo la dirección de

Pablo Jiménez Rodríguez,
Gustavo Adolfo Muñoz Fernández,
y
Juan Benigno Seoane Sepúlveda

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Sobre esta Tesis

Fruto del trabajo de estos años han sido los siguientes artículos de investigación (o bien publicados o bien aceptados y pendientes de publicación):

- Ciesielski, Krzysztof C.; Martínez-Gómez, María E.; Seoane-Sepúlveda, Juan B.
  "Big" continuous restrictions of arbitrary functions.
  American Mathematical Monthly 126 (2019), no. 6, 547–552.

- Jiménez-Rodríguez, Pablo; Martínez-Gómez, María E.; Muñoz-Fernández, Gustavo A.; Seoane-Sepúlveda, Juan B.
  Describing multiplicative convex functions.
  Journal of Convex Analysis 27 (2020), no. 3, 935–942.

- Bernal-González, Luis; Fernández-Sánchez, Juan; Martínez-Gómez, María E.; Seoane-Sepúlveda, Juan B.
  Banach spaces and Banach lattices of singular functions.
  Studia Mathematica (accepted for publication, 2020).

- Fernández-Sánchez, Juan; Martínez-Gómez, María E.; Muñoz-Fernández, Gustavo A.; Seoane-Sepúlveda, Juan B.
  Algebraic genericity and special properties within sequence spaces and series.
  The Rocky Mountain Journal of Mathematics (accepted for publication, 2020).

- Jiménez-Rodríguez, Pablo; Martínez-Gómez, María E.; Muñoz-Fernández, Gustavo A.; Seoane-Sepúlveda, Juan B.
  Generalizing Multiplicative Convex Functions.
  Journal of Convex Analysis (accepted for publication, 2020).

- Jiménez-Rodríguez, Pablo; Martínez-Gómez, María E.; Muñoz-Fernández, Gustavo A.; Seoane-Sepúlveda, Juan B.
  Injectiveness and discontinuity of multiplicative convex functions.
  Mathematics 2021, 9, 1035. https://doi.org/10.3390/math9091035.

Contents

Resumen

El título de esta tesis engloba el estudio de dos temas fundamentales en los que se ha trabajado en los últimos años. Los resultados que se han obtenido, que son el fruto del arduo trabajo llevado a cabo durante los últimos tres años, están relacionados con los siguientes temas:

- *Genericidad algebraica y lineabilidad:* Este tema consiste en el estudio de las estructuras algebraicas contenidas en determinados conjuntos de un espacio vectorial o un álgebra. En este sentido, estudiamos problemas de lineabilidad y algebrabilidad para ciertas clases de espacios de sucesiones y series. Así como, la clase de funciones singulares reales en el intervalo unitario. Este tema ha demostrado ser extremadamente fructífero en la última década y esto dió lugar a que la *American Mathematical Society* introdujese las referencias

  15A03 : Espacios vectoriales, independencia lineal, grado, lineabilidad.
  46B87 : Lineabilidad en analisis funcional.

  en su última revisión y actualización de la *Mathematical Subject Classification 2020*.

- *Convexidad:* una función $f : V \to \mathbb{R}$ (donde V es un espacio vectorial sobre $\mathbb{R}$) se dice que es convexa si, para todo $x, y \in V$ y $0 \leq \lambda \leq 1$, tenemos:

  $$f(\lambda x + (1-\lambda)y) \leq f(x) + (1-\lambda)f(y).$$

  En relación con esta definición, en [80], se propuso una definición de funciones multiplicativas convexas centrándose en la media geométrica. Se propone una definición centrándose en la media aritmética y se estudian en profundidad.

Esta memoria está dividida en dos partes. Temas de lineabilidad son la cuestión central y el nexo de unión de los tres capítulos de la primera parte. En el primer capítulo se da una prueba corta y simple de un clásico resultado de Henry Blumberg. En el segundo capítulo estudiamos los problemas de lineabilidad y algebrabilidad para ciertas clases de espacios de sucesiones y series. En el último capítulo se estudia la clase de funciones singulares desde el punto de vista de la lineabilidad. La convexidad es el tema que genera el estudio realizado en los tres capítulos de la segunda parte. En el primer capítulo se estudian las funciones multiplicativas convexas con la condición extra $f(1) = 1$. En el segundo capítulo se generalizan y se inicia el estudio sin la condición de $f(1) = 1$. En el último capítulo de esta parte se estudian la inyectividad de las funciones multiplicativas convexas y el conjunto de estas funciones que son discontinuas.

A continuación damos una descripción de lo que se hace en cada una de las dos partes de esta tesis, por capítulos:

Parte I

Capítulo 1

En este capítulo tratamos un resultado de Henry Blumberg de 1922 que afirma que para toda función $f : \mathbb{R} \to \mathbb{R}$, existe un conjunto denso $D \subset \mathbb{R}$ tal que la restricción $f|D$ es continua. En particular, se da una prueba nueva y corta de este resultado. Este trabajo viene motivado por el estudio realizado en las memorias del TFG y TFM de la candidata en las que se estudian las funciones $\mathcal{SZ}$[1] desde el punto de vista de la lineabilidad y espaciabilidad. Estas funciones son funciones cuya restricción a cualquier conjunto de cardinal $\mathfrak{c}$ es discontinua.

Capítulo 2

En este capítulo estudiamos los problemas de lineabilidad y algebrabilidad para ciertas clases de espacios de sucesiones y series. En particular, extendemos algunos resultados de algebrabilidad en el contexto de cuerpos p-ádicos. También proporcionamos una prueba completa de una pregunta abierta (previamente contestada incorrectamente) en la $\mathfrak{c}$ algebrabilidad de la clase de sucesiones cuyo conjunto de puntos de acumulación es un espacio de Cantor, es decir, es homeomorfo al conjunto de Cantor.

Capítulo 3

[1] Funciones conocidas como “de Sierpiński-Zygmund.”

Estudiamos la clase de funciones singulares reales en el intervalo unitario, es decir, las funciones de variación acotada continua que tienen derivada nula en casi todas partes, desde el punto de vista de lineabilidad. En particular, grandes subespacios vectoriales cerrados, grandes álgebras lineales y grandes retículos de Banach viven, a excepción del cero, dentro de varias subclases de la misma. Estas subclases están relacionadas, entre otras propiedades, al tamaño del conjunto cero, a la monotonía en ninguna parte, o a la existencia de puntos no críticos. También la familia de funciones continuas quasi-constantes se analiza bajo ese punto de vista. Además, se estudia lo que sucede en este contexto, cuando uno pasa de la topología de variación acotada a la topología de convergencia uniforme.

Parte II

Capítulo 4

Las funciones convexas multiplicativas [80] han estado imitando el comportamiento de las funciones convexas, pero centrándose en la media geométrica, en lugar de la media aritmética. En este capítulo introducimos una noción diferente de función multiplicativa convexa que se centra en las operaciones aritméticas. Estudiamos las funciones resultantes con la condición $f(1) = 1$, llegando a dar una caracterización de las mismas.

Capítulo 5

Este capítulo es una continuación natural del trabajo iniciado en el capítulo anterior. Se estudian las funciones multiplicativas convexas, sus propiedades de continuidad y sus generalizaciones (sin la condición $f(1) = 1$). Se presentan algunos resultados de genericidad algebraica que relacionan el tema con la lineabilidad, tema principal de la parte I de la memoria. También se plantean algunas preguntas abiertas.

Capítulo 6

Estudiamos el conjunto de funciones multiplicativas convexas. En concreto, nos centramos en las propiedades de la inyectividad y discontinuidad. Demostraremos que una función multiplicativa convexa no constante es (como mucho) $2-$inyectiva. Construimos funciones multiplicativas convexas que son discontinuas sobre un conjunto infinitos de puntos y estudiamos la dimensión algebraica del tronco de cono que forman.

Abstract

The title of this dissertation encompasses the study of two disparate topics that have been worked on. All the results that have been obtained in this dissertation, as the fruit of three years of tedious work, are related to the following fields within Mathematical Analysis:

- *Algebraic genericity and lineability:* This is the study of the algebraic structure within certain sets in a linear space or an algebra. In this sense, we study lineability and algebrability problems of sequences spaces and series. Just as, for the class of real singular functions on the unit interval. This topic has shown to be extremely fruitful in the last decade and this resulted in the American Mathematical Society introducing references

  15A03 : Vector spaces, linear dependence, rank, lineability.
  46B87 : Lineability in functional analysis.

  in its latest Mathematical Subject Classification 2020.

- *Convexity:* a function $f : V \to \mathbb{R}$ (where V is a vector space over sobre $\mathbb{R}$) is called convex if, whenever $x, y \in V$ and $0 \le \lambda \le 1$, we have:

  $$f(\lambda x + (1 - \lambda)y) \le f(x) + (1 - \lambda)f(y).$$

  In connection with this definition, in [80], a definition of multiplicative convex functions was proposed focusing on the geometric average. A definition focusing on the arithmetic average is also proposed and studied in depth.

This dissertation is divided into two parts. Lineability problems represent the core and the connecting link of the three chapters in the first part. The first chapter provides a short and simple proof of a classic result by Henry Blumberg. In the

second chapter we study the lineability and algebrability problems for certain classes of sequence and series spaces. In the last chapter the class of singular functions is studied from the point of view of lineability. Convexity is the theme that generate the study carried out in the three chapters of the second part. In the first chapter, we study multiplicative convex functions with the additional condition $f(1) = 1$. In the second chapter, the latter are generalized and the study begins without the condition of $f(1) = 1$. The last chapter of this part focuses on the injectivity of multiplicative convex functions and the set of these functions that are discontinuous.

We provide next a brief description of the content of each one of these chapters:

Part I

Chapter 1

In this chapter we discuss an amazing 1922 result of Henry Blumberg stating that *for an arbitrary $f : \mathbb{R} \to \mathbb{R}$, there is a dense $D \subset \mathbb{R}$ such that the restriction $f|D$ is continuous.* In particular, we provide a new short proof of this theorem. This work is motivated by the study carried out in the reports of Master's thesis and final degree project in which $\mathcal{SZ}$ functions are studied from the point of view of lineability and spaciability. These functions are functions whose restriction to any set of cardinal $\mathfrak{c}$ is discontinuous.

Chapter 2

In this chapter we study the lineability and algebrability problems for certain classes of sequence spaces and series. In particular, we extend some algebrability results to the context of p-adic fields. We also provide a complete proof of an (previously inaccurately answered) open question on the $\mathfrak{c}$-algebrability of the class of sequences whose set of accumulation points is a Cantor space, i.e., it is homeomorphic to the Cantor set.

Chapter 3

In this chapter, the class of real singular functions on the unit interval, that is, those continuous bounded variation functions having null derivative almost everywhere, is studied from the point of view of lineability. In particular, large closed vector subspaces, large linear algebras and large Banach lattices are found to live, except for zero, inside several subclasses of it. These subclasses are related, among other properties, to the size of the zero set, to nowhere monotonicity, or to the existence of noncritical points. Also the family of continuous functions being constant on full measure sequences of sets is analyzed under this point of view. Moreover, it is studied what happens in this context when one moves from bounded variation topology to uniform convergence topology.

Part II

Chapter 4

Multiplicative convex functions ([80]) have been mimicking the behaviour of convex functions, but focusing on geometric average, instead of arithmetic average. In this chapter we propose a different definition which focuses on the arithmetic operations taking part and we study the resulting functions. We study the resulting functions with the condition $f(1) = 1$, a characterization is also provided.

Chapter 5

This chapter is a natural continuation of the ongoing work started in the previous chapter. We study multiplicative convex functions, their continuity properties and their generalizations (without the condition $f(1) = 1$). Some results regarding algebraic genericity are also presented, this results related the topic with lineability, principal issue of part I. Also some open questions are posed.

Chapter 6

In this chapter we study the set of multiplicative convex functions. More particularly, we focus on the properties of injectiveness and discontinuity. We will show that a not constant multiplicative convex function is at most 2-injective and construct multiplicative convex functions which are discontinuous over a set with infinite points and we study the algebraic dimension of the truncated cone that they form.

Part I

Lineability

Chapter 1

“Big” Continuous Restrictions of Arbitrary Functions

1.1 Introduction

As soon as a student is introduced to the notion of continuity for one variable real-valued functions, $f:\mathbb{R}\to\mathbb{R}$, it is natural to note that not all such maps are (everywhere) continuous. Perhaps the most natural examples illustrating this are maps having just a single jump discontinuity, such as the famous characteristic function $\chi_{(0,\infty)}:\mathbb{R}\to\{0,1\}$ of $(0,\infty)$. Most undergraduate students are, usually, pleased after learning such examples, without even wondering whether anything “worse” could happen. However, some students may inquire if an arbitrary $f:\mathbb{R}\to\mathbb{R}$ must have “a lot” of points of continuity, as $\chi_{(0,\infty)}$ does. Fortunately, there is yet another simple example of a function f that is, actually, discontinuous at every point: the characteristic function $\chi_{\mathbb{Q}}$ of the set $\mathbb{Q}$ of all rational numbers, known as the Dirichlet function, and named after P. Dirichlet (1805–1859). This example would surely satisfy all but the most curious students. However, such extremely curious (probably graduate) students may notice that the restriction $f|\mathbb{Q}^c$ of $f=\chi_{\mathbb{Q}}$ to the (very big) set $\mathbb{Q}^c:=\mathbb{R}\setminus\mathbb{Q}$ of irrational numbers is still continuous. A natural question arises: *Must something like this be true for every function $f:\mathbb{R}\to\mathbb{R}$?*

In the early 20th century Henry Blumberg (1886–1950, see Figure 1.1), a Russian-American mathematician, proved the following astonishing result [27].

Theorem 1.1. *For every $f:\mathbb{R}\to\mathbb{R}$ there exists a dense subset D of $\mathbb{R}$ such that $f|D$ is continuous.*

Of course, the key property of the set D in Theorem 1.1 is that it is “big,” in the sense that it is dense in $\mathbb{R}$. However, the set D provided in the construction is just countable. Consequently, a natural question is whether the existence of an even bigger set D in the theorem above can always be ensured.

A negative answer to this last question was given only a year later, in the 1923 paper [88] by two Polish mathematicians, Wacław Sierpiński (1882–1969) and Antoni Zygmund[1] (1900–1992); see Figure 1.2. More particularly, they proved the following result (where $\mathfrak{c}$ denotes the cardinality of the continuum, that is, of $\mathbb{R}$). Any function as in the following theorem is nowadays called a *Sierpiński–Zygmund* (or just *SZ-*) *function.*

Theorem 1.2. *There exists a function* $f\colon\mathbb{R}\to\mathbb{R}$ *such that* $f|S$ *is discontinuous for every* $S\subset\mathbb{R}$ *of cardinality* $\mathfrak{c}$.

Thus, by Theorem 1.2, the countable set D constructed in the proof of Theorem 1.1 is the best we can do within the standard axiom system ZFC (the Zermelo–Fraenkel axioms with the axiom of choice) of set theory. Indeed, under the *continuum hypothesis CH*,[2] if f is an SZ-function, then any set D with continuous $f|D$

[1] After the Second World War Zygmund worked in the United States.

[2] Recall that CH, the statement that *there is no cardinal number between* $\mathfrak{c}$ *and* ω (where ω is the cardinality of $\mathbb{N}$), is independent of the usual axioms ZFC of set theory.

Figure 1.1: H. Blumberg in 1914 (courtesy of Dr. George Blumberg and the Blumberg family).

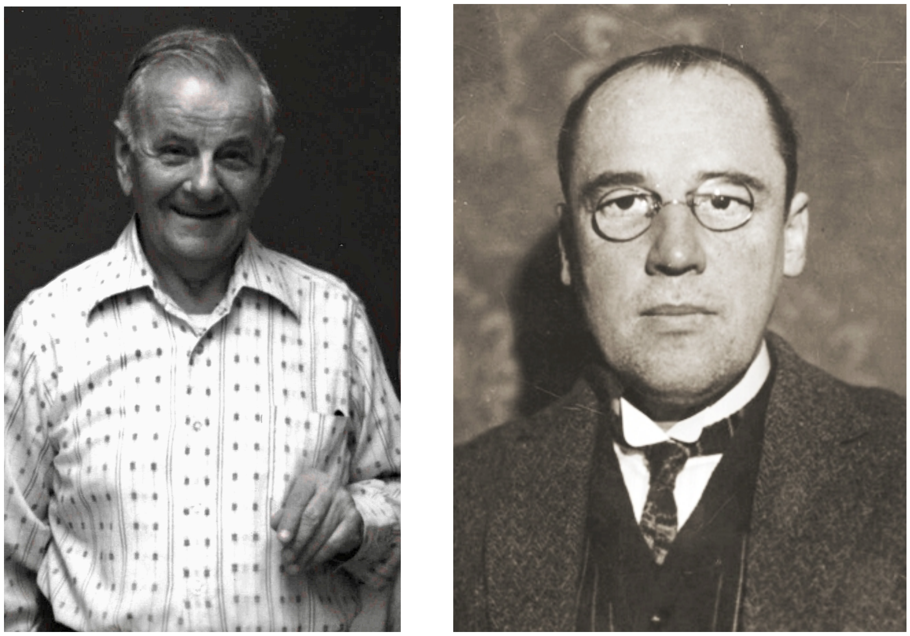

Figure 1.2: A. Zygmund in 1980 Summer Symposium in Real Analysis (courtesy of the *Real Analysis Exchange*) and W. Sierpiński.

must be countable, as it has cardinality less than $\mathfrak{c}$. Still, one might wonder if under the negation of the continuum hypothesis something more can be said about the cardinality of the set D from Blumberg's theorem. However, even $\neg CH$ does not decide anything definitive on the possible size of D. Specifically, this follows from the following two results.

(1) In a model of ZFC obtained by adding at least ω_2 Cohen reals, the continuum hypothesis fails, while *there exists an* $f\colon\mathbb{R}\to\mathbb{R}$ *for which* $f|X$ *is discontinuous for every uncountable* $X\subset\mathbb{R}$. This has been proved by Gruenhage (see the work of Recław [83, Theorem 4]) and Shelah [86, §2]. Of course, in such a model of ZFC the set D from Blumberg's theorem can be at most countable, while ¬CH holds.

(2) Under Martin's axiom MA, *for every function* $f\colon\mathbb{R}\to\mathbb{R}$ *and every infinite cardinal* $\kappa<\mathfrak{c}$ *there exists a* κ*-dense set* $X\subset\mathbb{R}$ *(i.e., such that* $X\cap(a,b)$ *has cardinality* κ *for every* $a<b$*) for which* $f|X$ *is continuous.* This was proved by Baldwin [13]. In particular, under MA+¬CH, which is consistent with ZFC, the set D from Blumberg's theorem can actually be ω_1-dense.

Another possible generalization of Theorem 1.1 studied in the literature is whether there is a model of ZFC in which the set D (not necessarily dense) can be always chosen either of second category or of positive Lebesgue outer measure. Of course, neither of these holds either in the model from (1) or under MA (since, under MA, every set of cardinality less than $\mathfrak{c}$ is both meager and of measure 0). But each of these questions has a positive answer. A model of ZFC in which *for every $f:\mathbb{R}\to\mathbb{R}$ there exists a second category set D with $f|D$ continuous* is constructed in a 1995 paper [86] of Shelah. It is easy to see that this property implies that the set D can also be of second category in every nonempty open set in $\mathbb{R}$ (see, e.g., [34, Theorem 2.10]). In the measure case, Rosłanowski and Shelah proved, in a 2006 paper [84], that it is consistent with ZFC that *for every $f:\mathbb{R}\to\mathbb{R}$ there exists a continuous function $g:\mathbb{R}\to\mathbb{R}$ that agrees with f on a set D of positive Lebesgue outer measure.* Of course, for $f=\chi_{(0,\infty)}$, this last set D cannot be dense. But, even if we require only that $f|D$ be continuous, such a D cannot be expected to be of positive outer measure in every nonempty open set in $\mathbb{R}$. This is prevented by an example of Brown [30]. (Compare also [34, Theorem 2.11].)

There is also a multitude of other generalizations of Blumberg's theorem (e.g., concerning functions between topological spaces X and Y). See, for example, [29, 59, 69, 75]. To see these results from a more general real analysis perspective, see [34, 73].

1.2 The proofs.

The proof of Blumberg's theorem relies on the following lemma from [27]. (See also [73].) For $f:\mathbb{R}\to\mathbb{R}$, a point $x\in\mathbb{R}$ is said to be *f-pleasant* provided for every open $B\ni f(x)$ there is an open $U_x^B\ni x$ such that the set $f^{-1}(B)$ is categorically dense in U_x^B (i.e., $f^{-1}(B)\cap V$ is of second category for every nonempty open $V\subset U_x^B$).

Lemma 1.3. *For every $f:\mathbb{R}\to\mathbb{R}$ the set P_f of all f-pleasant points is residual (i.e., it contains an intersection of countably many dense open sets) in $\mathbb{R}$.*

Proof. Let $\mathcal{B}$ be a countable basis for $\mathbb{R}$. For every $B\in\mathcal{B}$ let

$$E_B := \left\{x\in f^{-1}(B) : f^{-1}(B) \text{ is not categorically dense in any open } U\ni x\right\}$$

and notice that E_B is of first category. Indeed, it is a union of two first category sets: $W\cap E_B$, where $W=\bigcup\{V\in\mathcal{B}: V\cap E_B \text{ is of first category}\}$, and $\mathrm{bd}(W)\cap E_B$ (where $\mathrm{bd}(W)$ is the boundary of W).

Since $E:=\bigcup_{B\in\mathcal{B}} E_B$ is of first category, it is enough to show that $\mathbb{R}\setminus E\subset P_f$. To see this, fix an $x\in\mathbb{R}\setminus E$ and an open $W\ni f(x)$. Choose $B\in\mathcal{B}$ with $f(x)\in B\subset W$. Since $x\notin E_B$, there is an open $U_x^B\ni x$ such that $f^{-1}(B)$ is categorically dense in U_x^B. Then $f^{-1}(W)\supset f^{-1}(B)$ is also categorically dense in U_x^B; that is, $U_x^W := U_x^B$ is as needed. □

New proof of Blumberg's Theorem. Let $\mathcal{B} = \{B_n : n < \omega\}$ be a basis for $\mathbb{R}$. We construct, by induction on $n < \omega$, the sequences $\langle x_n \in B_n \cap P_f \setminus \{x_i : i < n\} : n < \omega\rangle$ and $\langle\langle U_k^n, V_k^n\rangle \in \mathcal{B}^2 : k \le n < \omega\rangle$, aiming for $D := \{x_n : n < \omega\}$ to be our desired set. The continuity of $f|D$ is ensured by the properties of the constructed sets U_k^n and V_k^n: each family $\{V_i^n : n < \omega\}$ will form a basis of $\mathbb{R}$ at $f(x_i)$ and each D-open $U_i^j \cap D \ni x_i$ will be contained in $f^{-1}(V_i^j)$.

To ensure this, we will assume that for every $n < \omega$ and $i \le j \le n$, $k \le \ell \le n$ with $j \le \ell$:

(a_n) $f^{-1}(V_i^j)$ is categorically dense in U_i^j, $x_i \in U_i^j \cap f^{-1}(V_i^j)$, and V_i^j has diameter less than 2^{-j};

(b_n) if $U_i^j \cap U_k^\ell \neq \varnothing$ and $\langle i, j\rangle \neq \langle k, \ell\rangle$, then $j < \ell$ and $U_k^\ell \times V_k^\ell \subset U_i^j \times V_i^j$.

These properties guarantee that $D := \{x_n : n < \omega\}$ is as needed. Indeed, D is dense, since it intersects every $B_n \in \mathcal{B}$. Each family $\{V_i^n : n < \omega\}$ will form a basis of $\mathbb{R}$ at $f(x_i)$, since each open set V_i^n contains x_i and the diameters of V_i^j go to 0 as $j \to \infty$. Thus, to show that $f|D$ is continuous at x_i, it is enough to show that f maps each D-open $U_i^j \cap D \ni x_i$ into V_i^j. To see this, fix an $x_k \in D \cap U_i^j$. We cannot have $k < i$, since then x_k would belong to disjoint U_k^j and U_i^j. By (a_j), we have $f(x_i) \in V_i^j$. Thus, assume that $i < k$. Then $x_k \in U_i^j \cap U_k^k$ and, by (b_k), $f(x_k) \in V_k^k \subset V_i^j$, as needed.

To make the nth step in our construction, choose a nonempty interval $\hat{B}_n \subset B_n$ such that, for every $i \le j < n$, $\hat{B}_n$ is either contained in U_i^j or it is disjoint from U_i^j. Let

$$\mathcal{F}_n := \{U_i^j : i \le j < n \ \&\ \hat{B}_n \subset U_i^j\}.$$

If $\mathcal{F}_n \neq \varnothing$, then $n > 0$ and, by (b_{n-1}), $\mathcal{F}_n$ contains a smallest element, say U_κ^μ. We choose

$$x_n \in \hat{B}_n \cap P_f \cap f^{-1}(V_\kappa^\mu) \setminus \{x_i : i < n\}.$$

This choice can be made since $\hat{B}_n \subset U_\kappa^\mu$ is open and nonempty, $f^{-1}(V_\kappa^\mu)$ is categorically dense in U_κ^μ, and $P_f \setminus \{x_i : i < n\}$ is residual. If $\mathcal{F}_n = \varnothing$, take $x_n \in \hat{B}_n \cap P_f \setminus \{x_i : i < n\}$.

To finish the construction we first choose, for each $k \le n$, a V_k^n as an open interval containing $f(x_k)$ of length less than 2^{-k} small enough such that if $f(x_k) \in V_i^j$ for some $i \le j < n$, then $V_k^n \subset V_i^j$. The existence of sets U_k^n, $k \le n$, satisfying (a_n) follows from $\{x_i : i \le n\} \subset P_f$. Shrinking them if necessary, we can also ensure that they are pairwise disjoint and that if, for some $i \le j < n$, $x_k \in U_i^j$, then $U_k^n \subset U_i^j$. These choices ensure that (a_n) and (b_n) are satisfied. $\square$

Construction of a Sierpiński–Zygmund function. The key fact needed in the construction is the following result of Kuratowski (1896–1980), see, e.g., [71, p. 16]:

(E) For every continuous g from an $S \subset \mathbb{R}$ into $\mathbb{R}$ there exists a G_δ-set $G \supset S$ and a continuous extension $\bar{g}: G \to \mathbb{R}$ of g. In particular, g admits a Borel extension $\hat{g}: \mathbb{R} \to \mathbb{R}$.

Indeed, for every $x \in \mathrm{cl}(S)$ define

$$\mathrm{osc}_g(x) := \inf\{\mathrm{diam}(g[U \cap S]): U \ni x \text{ is open}\}$$

and notice that $G := \{x \in \mathrm{cl}(S): \mathrm{osc}_g(x) = 0\}$ contains S and is a G_δ-set in $\mathbb{R}$ since $G := \bigcap_{n \in \mathbb{N}} W_n$, where each set $W_n := \{x \in \mathrm{cl}(S): \mathrm{osc}_g(x) < 1/n\}$ is open. Now, if $\mathrm{cl}(g)$ is the closure in $\mathbb{R}^2$ of the graph of g, then $\bar{g} = \mathrm{cl}(g) \cap (G \times \mathbb{R})$ is the graph of our desired function $\bar{g}$. A Borel extension $\hat{g}$ of $\bar{g}$ can be defined to be 0 on $\mathbb{R} \setminus G$.

To construct a Sierpiński–Zygmund function $f: \mathbb{R} \to \mathbb{R}$, let $\{x_\xi: \xi < \mathfrak{c}\}$ be an enumeration, with no repetition, of $\mathbb{R}$ and let $\{\hat{g}_\xi: \xi < \mathfrak{c}\}$ be an enumeration of all Borel functions from $\mathbb{R}$ to $\mathbb{R}$. For every $\xi < \mathfrak{c}$ define $f(x_\xi)$ so that

$$f(x_\xi) \in \mathbb{R} \setminus \{\hat{g}_\zeta(x_\xi): \zeta < \xi\}.$$

This defines our SZ-function. Indeed, if $f|S$ is continuous for some $S \subset \mathbb{R}$ then, by (E), there exists a Borel extension $\hat{g}: \mathbb{R} \to \mathbb{R}$ of $f|S$. Let $\zeta < \mathfrak{c}$ be such that $\hat{g}_\zeta = \hat{g}$. Then $S \subset \{x_\xi: \xi \leq \zeta\}$, since $f(x_\xi) \neq \hat{g}_\zeta(x_\xi) = \hat{g}(x_\xi)$ for every $\xi > \zeta$. Thus, S has cardinality $< \mathfrak{c}$, as needed, and we are done. $\square$

Chapter 2

Algebraic genericity and special properties within sequence spaces and series

2.1 Introduction and preliminaries

This chapter contributes to the search for large vector spaces of sequences and series having certain special (or pathological) property. Let us recall, for the sake of completeness, the following definitions of lineability and algebrability, that shall be recurrent throughout this Ph.D. dissertation.

This terminology of lineable and spaceable coined by V.I. Gurariy and it was first introduced in [9, 85] (Figure 2.1). There has been plenty of work in this direction since its appearance about a decade ago. As a matter of fact, this notion was (just recently) introduced by the American Mathematical Society under the MSC2020 15A03 and 46B87 references.

Definition 2.1. *Assume that X is a vector space, that α is a cardinal number and that $A \subset X$. Then A is said to be:*

- lineable *if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$, and*

Figure 2.1: Vladimir Ilyich Gurariy (1935–2005) was born in Kharkov (Ukraine). In 1991 he moved to the USA and worked in Kent State University (Ohio) until his passing.

- α-lineable *if there exists a vector space* M *with* $\dim(M) = \alpha$ *and* $M \setminus \{0\} \subset A$.

If, in addition, X *is a topological vector space, then* A *is said to be* spaceable *whenever there is a closed infinite dimensional vector subspace* M *of* X *satisfying* $M \setminus \{0\} \subset A$.

As introduced in [8], A *is called* dense-lineable *if* $A \cup \{0\}$ *contains a dense vector subspace.*

Trivially, spaceability implies lineability and, if X *is infinite-dimensional, then dense-lineability implies lineability too.*

Finally, when X *is a topological vector space contained in some (linear) algebra then* A *is called:*

- algebrable *if there is an algebra* M *so that* $M \setminus \{0\} \subset A$ *and* M *is infinitely generated, that is, the cardinality of any system of generators of* M *is infinite.*
- α-algebrable *if there is an* α*-generated algebra* M *with* $M \setminus \{0\} \subset A$.
- strongly α-algebrable *if there exists an* α*-generated* free *algebra* M *with* $M \setminus \{0\} \subset A$.

Of course, strong α-algebrability implies α-algebrability, which implies α-lineability. However, in general, the converse implications do not hold, see, e.g., [7,17,25]. Recall that these notions of algebrability and their variants first appeared in [10,11,15].

The interested reader may also consult [7–10,17,20,22,25,33,35,37,45,53,55–58, 67,68,77,85] for a complete account on lineability, spaceability, algebrability and related topics.

The aim of this chapter is to study the lineability/algebrability problem for certain classes of sequence spaces and series. In particular, we extend a result from [3] to the setting of p-adic fields. Also, we solve a problem posed in [16] and continue with the line proposed in [21]. First we provide a number of definitions and notations that, although rather usual, we shall need in this chapter. Next, in Section 3.2 we shall present and prove the main results of this chapter (Theorems 2.5, 2.9, 2.10, 2.11, 2.12, and 2.14).

Let us, then, start off with recalling several notions we shall need from now on. As usual ω denotes the cardinal of $\mathbb{N}$, $\mathfrak{c}$ the cardinal of $\mathbb{R}$ and in general $\operatorname{card}(X)$ denotes the cardinal of X. The concepts and terminology appearing in the following definition can be found in [43].

Definition 2.2. *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we have that*

1. *$\ell_\infty(\mathbb{K})$ stands for the vector space of bounded sequences, c denotes the subspace of convergent sequences, c_0 denotes the subspace of sequences converging to 0, and c_{00} is the subspace of c_0 consisting of sequences that are eventually zero.*

2. *$BS(\mathbb{K})$ is the Banach space of the series $\sum_i a_i$ satisfying that*

$$sup\left\{\left|\sum_{i=1}^{n} a_i\right| : n \in \mathbb{N}\right\} < \infty,$$

   endowed with the norm given by

$$\|(a_i)_i\| = sup\left\{\left|\sum_{i=1}^{n} a_i\right| : n \in \mathbb{N}\right\} < \infty.$$

3. *The subspace of $BS(\mathbb{K})$ consisting of all the convergent series is denoted $CS(\mathbb{K})$, which is a closed subspace of $BS(\mathbb{K})$.*

4. *The subspace of $CS(\mathbb{K})$ formed by the unconditionally convergent series, $UC(\mathbb{K})$, is not closed in $CS(\mathbb{K})$. On the other hand, $UC(\mathbb{K})$ endowed with the norm*

$$\|(a_i)_i\| = sup\left\{\left|\sum_{i\in F} a_i\right| : F \subset \mathbb{N}, F \text{ finite}\right\} < \infty$$

   is a Banach space.

5. *The set of the conditionally convergent series shall be denoted by $CC(\mathbb{K})$.*

Next, and for the sake of completeness, let us also recall certain concepts regarding p-adic numbers and convergence of series within the p-adic field $\mathbb{Q}_p$ (see, e.g., [60, 70]).

Definition 2.3. 1. *Given a prime number p, the p-adic absolute value $|\cdot|_p$ in $\mathbb{Q}$ is defined as follows: for any non-zero $x \in \mathbb{Q}$, there is a unique integer n allowing us to write $x = p^n(\frac{a}{b})$, where none of the integers a and b is divisible by p. Observe that if the numerator and denominator of x in lowest terms do not contain p as a factor, then n is 0. Thus, we define*

$$|x|_p = \begin{cases} p^{-n} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

2. *In $\mathbb{Q}$ we have the non-Archimedean distance $d_p(x, y) = |x - y|_p$. It is known that $(\mathbb{Q}, d_p)$ is not a complete metric space, [60, 70]. The completion of $(\mathbb{Q}, d_p)$ is the $\mathbb{Q}_p$ field with a metric that we also denote d_p.*

3. *A series $\sum_{n=1}^{\infty} a_n$ is said to be convergent in $\mathbb{Q}_p$ if the sequence of its partial sums converges in $\mathbb{Q}_p$, i.e., if*

$$\lim_{m\to\infty} |S_{m+1} - S_m|_p = \lim_{m\to\infty} |a_{m+1}|_p = 0$$

where $S_m = \sum_{n=1}^{m} a_n$.

The series is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|_p$ converges in $\mathbb{R}$.

We shall denote by $CS(\mathbb{Q}_p)$ the space of the convergent series of $\mathbb{Q}_p$ and by $CC(\mathbb{Q}_p)$ the subspace of $CS(\mathbb{Q}_p)$ of the non-absolutely convergent series. The following definition is rather standard (see, [3]).

Definition 2.4. *A family $\{A_\alpha : \alpha \in I\}$ of infinite subsets of $\mathbb{N}$ is called almost disjoint if $A_\alpha \cap A_\beta$ is finite whenever $\alpha, \beta \in I$ and $\alpha \neq \beta$.*

2.2 The main results

In [3, Theorem 2.1] the authors showed that $CS(\mathbb{K})$ contains a vector space E, of dimension $\mathfrak{c}$, such that every $x \in E \setminus \{0\}$ is a conditionally convergent series. Besides this previous result, they also stated that $\mathrm{span}\{E \cup c_{00}\}$ is, actually, an algebra and its elements are either elements of c_{00} or conditionally convergent series.

We are going to show that the proof of [3, Theorem 2.1] can be used to see that the result is also true for $\mathbb{K} = \mathbb{Q}_p$. However, in $\mathbb{Q}_p$ we do not have the concept of conditionally convergent series and, thus, we shall work with series which are convergent but not absolutely convergent.

Theorem 2.5. *$CS(\mathbb{Q}_p)$ contains a vector space E verifying the following properties:*

1. *For every $x \in E \setminus \{0\}$, we have that $x \in CC(\mathbb{Q}_p)$.*
2. $\dim(E) = \mathfrak{c}$.
3. $\mathrm{span}\{E \cup c_{00}\}$ *is an algebra and its elements belong to either c_{00} or to $CC(\mathbb{Q}_p)$.*

Proof. Let us define $\{a_n\}_{n\in\mathbb{N}}$ by $a_1 = p$ and $a_n = p^k$ where k satisfies the following:

$$\sum_{j=1}^{k-1} j^j < n \leq \sum_{j=1}^{k} j^j.$$

Its easy to see that $\sum_{n\in\mathbb{N}} a_n$ converges ($\lim_{n\to\infty} |a_n|_p = 0$). Let us see that it is not absolutely convergent. We have that

$$\sum_{n\in\mathbb{N}} |a_n|_p = \sum_{k\in\mathbb{N}} \left(\frac{k}{p}\right)^k$$

which shows clearly that $\{a_n\}_{n\in\mathbb{N}}$ is not absolutely convergent.

Analogously we can prove that $\{a_n^r\}_{n\in\mathbb{N}}$ is a convergent series but it is not absolutely convergent.

Let us take a family $(A_\alpha)_{\alpha\in I}$ of almost disjoint subsets of $\mathbb{N}$ with $\mathrm{card}(I) = \mathfrak{c}$ (see, e.g., [3]) and define, for every $\alpha \in I$, the sequence given by

$$x_{\alpha,i} = \begin{cases} x_n & \text{if } i = n\text{th element of } A_\alpha, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Next, define the space

$$E = \mathrm{span}\{x_\alpha^r : r \in \mathbb{N} \text{ and } \alpha \in I\}.$$

Now, we assume that

$$z = \sum_{i \in S} \sum_{m=1}^{M} b_{\alpha_i,m} x_{\alpha_i}^{r_m} = 0$$

for some $M \in \mathbb{N}$, $\{r_1, \ldots, r_M\} \subset \mathbb{N}$ and $\alpha_i \in I$ for every $i \in S$ with $\mathrm{card}(S) < \omega$. Then, for every $i \in S$, there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$,

$$\sum_{m=1}^{M} b_{\alpha_i,m} x_{\alpha_i,n}^{r_m} = 0.$$

Therefore, we have that $\sum_{m=1}^{M} b_{\alpha_i,m} x_n^{r_m} = 0$ for every $n > n_0$. But there exists k_0 such that if $k > k_0$ then $\sum_{m=1}^{M} b_{\alpha_i,m} p^{k r_m} = 0$, i.e., the polynomial $\sum_{m=1}^{M} b_{\alpha_i,m} y^{r_m}$ has infinite zeros. Then $b_{\alpha_i,m} = 0$ for every $m \in \{1, \ldots, M\}$. Since the latter is proved for every $i \in S$, we have that $b_{\alpha_i,m} = 0$ for every $m \in \{1, \ldots, M\}$ and $i \in S$. Therefore, $\dim(E) = \mathfrak{c}$.

If $z \neq 0$, it easy to see that $z \in CC(Q_p)$, and since $x_\alpha^{k_1} \cdot x_\beta^{k_2} \in c_{00}$ for every $\alpha \neq \beta$ and $k_1, k_2 \in \mathbb{N}$, we have that $\mathrm{span}\{E \cup c_{00}\}$ is an algebra. □

Some of the results in [3] are refined in [16]. In particular, it is proved that $\{x : x \in \ell_\infty(\mathbb{C})$ and x is a divergent sequence$\}$ is $\mathfrak{c}$-strongly algebrable. Interestingly, this result turns out to be true in a real setting too, as we will see in this section. In order to do so, we need to introduce some notions typical from Ergodic Theory.

Definition 2.6 ([92]). *Suppose $(X_1, \mathcal{B}_1, m_1)$ and $(X_2, \mathcal{B}_2, m_2)$ are probability spaces.*

1. *A transformation $T : X_1 \to X_2$ is measurable if $T^{-1}(\mathcal{B}_2) \subset \mathcal{B}_1$ (i.e., $B_2 \in \mathcal{B}_2 \Rightarrow T^{-1}B_2 \in \mathcal{B}_1$).*

2. *A transformation $T : X_1 \to X_2$ is measure-preserving if T is measurable and $m_1(T^{-1}(B_2)) = m_2(B_2)$ for every $B_2 \in \mathcal{B}_2$.*

Definition 2.7 ([92]). *Let $(X, \mathcal{B}, m)$ be a probability space and T a measure-preserving transformation of $(X, \mathcal{B}, m)$. T is called ergodic if the only members B of $\mathcal{B}$ with $T^{-1}B = B$ satisfy $m(B) = 0$ or $m(B) = 1$.*

Example 2.8. *A well-known example of a continuous transformation that is ergodic is $T_2 : [0,1] \to [0,1]$ defined by*

$$T_2(t) = \begin{cases} 2t & \text{if } x \in [0, 1/2] \\ 2 - 2t & \text{if } x \in]1/2, 1]. \end{cases}$$

Given a transformation $T:[0,1]\to[0,1]$ and a set I with $\text{card}(I)=\mathfrak{c}$ we define $T^I:[0,1]^I\to[0,1]^I$ by $T^I(x)=\{T(x_i)\}_{i\in I}$.

If we consider the Lebesgue measure λ in $[0,1]$, we can define in $[0,1]^I$ the product measure given in [18, Chapter II]. Remember that this measure is defined in the σ-algebra generated by the sets of the form $\prod_{i\in I} J_i$ where $J_i=[0,1]$ excepting in a finite set of indexes for which $J_i=[a_i,b_i[$. The class of this sets is denoted by $\mathcal{C}$. We will denote this measure by Λ. If $T:[0,1]\to[0,1]$ is a λ-measure preserving transformation, it can be proved that T^I is also a Λ-measure preserving transformation. To do that, notice that it is true for the sets in $\mathcal{C}$ that generate the σ-algebra in $[0,1]^I$.

Also, we can adapt the proof of [92, Theorem 1.24] to the case $[0,1]^I$, by taking into consideration that result is true for $A=B_1\cap B_2$ with $B_1,B_2\in\mathcal{C}$.

Observe that [92, Theorem 1.24] holds for weakly-mixing functions. In our version we consider strongly-mixing functions, proving that T^I is strongly-mixing and then, T^I is ergodic.

Theorem 2.9. *$\{x : x\in\ell_\infty(\mathbb{R})$ and x is a divergent sequence$\}$ is $\mathfrak{c}$-strongly algebrable.*

Proof. Given a transformation $T:[0,1]\to[0,1]$, since T^I is ergodic, we can choose $z\in[0,1]^I$ such that, for every $A=\prod_{i\in I} J_i\in\mathcal{C}$ we have that

$$\lim_{n\to\infty}\frac{\text{card}(\{j:(T^I)^{(j)}(z)\in A\})}{n}=\prod(b_i-a_i),$$

where the product is considered in the set of indexes in which $J_i\neq[0,1]$, $f^{(j)}$ denotes the iteration of composition j times, and if $f:[0,1]^I\to\mathbb{R}$ is continuous then

$$\lim_{n\to\infty}\frac{\sum_{j=1}^n f((T^I)^{(j)}(z))}{n}=\int_{[0,1]^I} f d\Lambda,$$

where Λ denotes the product of the Lebesgue measure. Observe that the set with these properties is of Λ−measure 1.

Let us define the sequences x_α with $\alpha\in I$ as $x_{\alpha,n}=T^{(n)}(z_\alpha)$, where $(T^I)^{(n)}(z)=\{x_{\alpha,n}\}_{\alpha\in I}=\{T^{(n)}(z_\alpha)\}_{\alpha\in I}$.

It is clear that $x_\alpha\in\ell_\infty(\mathbb{R})$ for every $\alpha\in I$. Since every x_α is dense in $[0,1]$, we have that x_α is not convergent for every $\alpha\in I$. Thus, we need to see that the set $\{x_\alpha:\alpha\in I\}$ is algebraically independent.

Suppose that

$$\sum_{s=1}^{m} c_s \prod_{\alpha\in I_s} x_\alpha^{\alpha_s} = 0,$$

where $m \in \mathbb{N}$, I_s is a finite subset of I and $c_s \in \mathbb{R} \setminus \{0\}$ for every $s \in \{1,\dots,m\}$. $\alpha_s \in \mathbb{N}$ for every $\alpha \in I_s$ and every $s \in \{1,\dots,m\}$. We now take an element of $A \in \mathcal{C}$ such that if $J_i \neq [0,1]$ then $J_i = [0, b_i[$ and we define

$$f_A(y) = \left(\sum_{s=1}^{m} c_s \prod_{\alpha\in I_s} y_\alpha^{\alpha_s}\right)\chi_A(y)$$

and

$$f(y) = \sum_{s=1}^{m} c_s \prod_{\alpha\in I_s} y_\alpha^{\alpha_s}.$$

So, when A satisfies that $\bigcup_{s=1}^{m} I_s$ is the set of indexes in which $J_i \neq [0,1]$, we have that

$$\begin{aligned} 0 &= \frac{\sum_{j=1}^{k}(\sum_{s=1}^{m} b_s \prod_{\alpha\in I_s} a_{\alpha,j}^{\alpha_s})\chi_A((T^I)^{(j)}(z))}{k} \\ &= \frac{\sum_{j=1}^{k}\sum_{s=1}^{m} b_s \prod_{\alpha\in I_s}(T^{(j)}(x_\alpha))^{\alpha_s}\chi_A((T^I)^{(j)}(z))}{k} \\ &= \frac{\sum_{j=1}^{k} f_A((T^I)^{(j)}(z))}{k}. \end{aligned}$$

Since f_A can be discontinuous, we have to investigate the behaviour of

$$\frac{\sum_{j=1}^{k} f_A((T^I)^{(j)}(z))}{k}.$$

When $0 < b$, we define the functions $\chi_{b,r}$ in such a way that they coincide with $\chi_{[0,b[}$ except for the set $]b - 1/r, b[$, where the function is linear and its graph joins the points $(b-1/r, 1)$ and $(b, 0)$. The function $\chi_{b,r}$ is continuous and converges pointwise to $\chi_{[0,b[}$. We define $\chi_{A,r}(y) = \prod_{i=1}^{t} \chi_{b_i,r}(y_n)$. In addition, we define $A_r = \prod_{i\in I} J_i' \in \mathcal{C}$ with $J_i' = [0,1]$ when $J_i = [0,1]$, and $J_i' = [a_i, b_i - 1/r[$ when $J_i = [a_i, b_i[$. The set $A \backslash A_r$ is a continuity set (its boundary is of zero measure).

Using $\chi_{A,r}$, we introduce the function $f_{A,r}(y) = (\sum_{s=1}^{m} c_s \prod_{\alpha\in I_s} y_\alpha^{\alpha_s})\chi_{A,r}(y)$.

Fix $\varepsilon > 0$, and take r large enough, for $k > k_0$. We have

$$\begin{aligned} &\left|\frac{\sum_{j=1}^{k} f_A((T^I)^{(j)}(z))}{k} - \frac{\sum_{j=1}^{k} f_{A,r}((T^I)^{(j)}(z))}{k}\right| \\ &= \left|\frac{\sum_{j=1}^{k}(f_A - f_{A,r})((T^I)^{(j)}(z))}{k}\right| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\sum_{j=1}^{k} 2M_f \chi_{A\setminus A_r}((T^I)^{(j)}(z))}{k} \\
&< \varepsilon/3.
\end{aligned}
$$

where M_f is the maximum value of f (this maximum exists because f is a continuous function on a compact set). The inequality follows as a consequence of taking r large enough, the fact that $A\setminus A_r$ is a continuity set and Portmanteau's Theorem (See [26, Th. 2.1]).

Also, since $f_{A,r}$ is continuous, we have that

$$\left| \int_{[0,1]^I} f_{A,r} d\Lambda - \frac{\sum_{j=1}^{k} f_{A,r}((T^I)^{(j)}(z))}{k} \right| < \varepsilon/3.$$

Now, applying the Dominated Convergence Theorem, we obtain

$$\left| \int_{[0,1]^I} f_{A,r} d\Lambda - \int_{[0,1]^I} f_A d\Lambda \right| < \varepsilon/3.$$

From the Triangle Inequality and the inequalities above we conclude that

$$\left| \frac{\sum_{j=1}^{k} f_A((T^I)^{(j)}(z))}{k} - \int_{[0,1]^I} f_A d\Lambda \right| < \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{k} f_A((T^I)^{(j)}(z))}{k} = \int_{[0,1]^I} f_A d\Lambda.$$

As $\bigcup_{s=1}^m I_s$ is a finite set, t, and f is a polynomial that depends only on the "t coordinates", we can see the integral as an integral in $[0,1]^t$. However, these integrals are zero if and only if $f = 0$ so that $c_s = 0$ for every $s \in \{1, \ldots, m\}$. $\square$

The set

$$E_{\text{Cantor}} := \left\{ \begin{array}{ll} x \in l^\infty : & \text{LIM}(x) \text{ is the union of a finite set and} \\ & \text{a homeomorphic set to the Cantor set} \end{array} \right\},$$

where $\text{LIM}(x)$ is the set of all limit points of x, is studied in [16]. The authors prove that E_{Cantor} is strongly $\mathfrak{c}$−algebrable and ask (Problem 8.3) whether the set

$$E_C = \{x \in l^\infty : \text{LIM}(x) \text{ is homeomorphic to the Cantor set}\}$$

is $\mathfrak{c}$−algebrable.

We solve the latter question below using Brouwer's characterization of the sets that are homeomorphic to the Cantor set (see [71]). Namely, those sets are metrizable, compact, perfect and totally disconnected topological spaces.

Theorem 2.10. *E_C is strongly $\mathfrak{c}$−algebrable*

Proof. If $x = \{x_n\}$ is a sequence and $s \in \mathbb{R}$, we use the notation $x^s = \{x_n^s\}$. Supponse $\mathcal{H} \subset]1,2[$ is a Hammel basis of $\mathbb{R}$ over the rationals. Let $a = \{a_n\}$ be a sequence that takes infinitely many times each of the elements of the set $C \cap \mathbb{Q}$, where C is the Cantor set. Hence $\mathrm{LIM}(a) = C$. Now consider the family of sequences a^h with $h \in \mathcal{H}$. Since $\mathrm{LIM}(a^h) = C^h = \{c^h : c \in C\}$, we have that $a^h \in E_C$.

We see now that the a^h's are algebraically independent. If they were not we could take a set $\{h_1, \dots, h_s\} \subset \mathcal{H}$, nonzero real numbers c_i and nonnegative integers $n_{j,i}$ such that

$$\sum_{i=1}^{k} c_i a^{h_1 n_{1,i}} \dots a^{h_s n_{s,i}} = 0.$$

We have

$$\sum_{i=1}^{k} c_i a^{h_1 n_{1,i}} \dots a^{h_s n_{s,i}} = \sum_{i=1}^{k} c_i a^{h_1 n_{1,i} + \dots + h_s n_{s,i}} = \sum_{i=1}^{k} c_i a^{t_i}.$$

Consider the function $f(z) = \sum_{i=1}^{k} c_i z^{t_i}$, which is holomorphic on a certain neighborhood of 1. If $\sum_{i=1}^{k} c_i a^{t_i} = 0$, then f vanishes on a set such that 1 is one of its accumulation points. Applying the identity principle to f, it follows that f is identically null on a neighborhood of 1, that is, $c_i = 0\ \forall i$. This proves the algebraic independence.

On the other hand $\sum_{i=1}^{k} c_i a^{h_1 n_{1,i}} \dots a^{h_s n_{s,i}} \in E_C$. In this case $\mathrm{LIM}(x) = f(C)$. We show below that $f(C)$ is homeomorphic to C. Here is where we use Brouwer's characterization of the Cantor spaces.

1. It is obvious that $f(C)$ is metrizable and compact.
2. If $p \in f(C)$, there exists $c \in C$ such that $p = f(c)$. Then we can choose a convergent subsequence $\{a_{\sigma(n)}\}$ converging to c. The sequence $\{f(a_{\sigma(n)})\}$ converges to p. If p were isolated, then we would have that $f(a_{\sigma(n)}) = p$ for n large enough. Again, the Identity Principle tells us that c_i is zero for every i. We conclude that $f(C)$ is perfect.
3. The mapping $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is absolutely continuous, and hence, it maps sets of measure zero into sets of measure zero. Since C is of measure zero, so is $f(C)$. This shows that the only open convex sets are singletons.

□

The previous result holds as well if we replace $\mathbb{R}$ by $\mathbb{Q}_p$, as we are about to see. First, define

$$E_C(\mathbb{Q}_p) := \{x \in l^\infty(\mathbb{Q}_p) : \mathrm{LIM}(x) \text{ is homeomorphic to the Cantor set}\}.$$

A proof of the following result was already shown in [14, Theorem 5.6]. However, it contained some inaccuracies and was not fully complete. We now proved it here below in full detail.

Theorem 2.11. *$E_C(\mathbb{Q}_p)$ is strongly $\mathfrak{c}$−algebrable*

Proof. Let $\{a_n\}$ be an enumeration of $\mathbb{Q} \cap p\mathbb{Z}_p$, $\mathcal{H} \subset \mathbb{Z}_p$ a Hamel basis of $\mathbb{Q}_p$ over $\mathbb{Q}$ and the functions

$$f_\beta : p\mathbb{Z}_p \to \mathbb{Z}_p, \quad f_\beta(x) = \exp(\beta x),$$

with $\beta \in \mathbb{Z}_p$. Define the sequences b_h given by $b_{h,n} = f_h(a_n)$ for $n \in \mathbb{N}$. Let us prove that the b_h's are algebraically independent. Assume that $\sum_{i=1}^n c_{\beta_i} b_{\beta_i} = 0$, that is $\sum_{i=1}^n c_{\beta_i} f_{\beta_i}(a_n) = 0$. Since $\{a_n\}$ es dense in $p\mathbb{Z}_p$, it follows that $\sum_{i=1}^n c_{\beta_i} f_{\beta_i}(x) = 0$ in $p\mathbb{Z}_p$. Also, observe that $\sum_{i=1}^n c_{\beta_i} f_{\beta_i}(x)$ is in $\mathbb{Q}_p[[x]]$ and it has infinitely many zeros. Hence it must be identically zero by the Strassman Theorem.

Now, using the series expansion of exp and the fact that $\sum_{i=1}^n c_{\beta_i} f_{\beta_i}(x)$ is identically null, we have a system of infinitely many equations such that it turns into a Vandermonde system if we truncate it. Hence, $\sum_{i=1}^n c_{\beta_i} f_{\beta_i}(x) = 0$ if and only if $c_{\beta_i} = 0$ for each i.

We have that $\mathrm{LIM}(\sum_{i=1}^n c_{\beta_i} b_{\beta_i}) = F(p\mathbb{Z}_p)$. Let us see that $F(p\mathbb{Z}_p)$ is homeomorphic to C. In order to prove it, we use once again Brouwer's characterization. It is straightforward that $F(p\mathbb{Z}_p)$ is metrizable, compact and completely disconnected. It only remains to prove that it does not have isolated points. If d is an isolated point in $F(p\mathbb{Z}_p)$, choose $q \in p\mathbb{Z}_p$ such that $d = f(q)$. There is a neighborhood of q where F vanishes. Hence, by Strassman's Theorem F is identically null. $\square$

To finish, and in an attempt to link the results within this manuscript with the recent ones from [21] regarding highly tempering infinite matrices (that is, infinite matrices that not only preserve convergence and limits of sequences but also convert divergent sequences into a convergent sequence) we would like to present two more results (Theorems 2.12 and 2.14) sharing this direction of work.

Theorem 2.12. *There exists a vector space E of $CC(\mathbb{R})$ that is dense in $\mathbb{R}^\mathbb{N}$ (with the product topology). Moreover, $CC(\mathbb{R})$ is $\mathfrak{c}$-dense-lineable.*

Proof. First, let a be a conditionally convergent series with $a_1 \neq 0$ and $\{q_n\}_{n\in\mathbb{N}}$ a numeration of $\mathbb{Q} \cap [0,1]$. For every $\alpha \in [0,1]$, we are going to choose sequences b_α such that $b_{\alpha,i} \overset{i\to\infty}{\longrightarrow} \alpha$ and satisfying the following properties:

- $b_{\alpha,i} \in \mathbb{Q} \cap [0,1]$ for every $i \in \mathbb{N}$.
- If $b_{\alpha,i} = q_{n_{\alpha_i}}$ and $b_{\alpha,i+1} = q_{n_{\alpha_{i+1}}}$, then $n_{\alpha_i} < n_{\alpha_{i+1}}$.
- $b_{q_n,1} = q_n$ for every $n \in \mathbb{N}$.

Let us put

$$B_{q_n} = \{q_m : \exists i \in \mathbb{N},\ \text{such that } b_{q_n,i} = q_m\}$$

and

$$A_{q_n} = \{m : q_m \in B_{q_n}\}.$$

Notice that the first element of A_{q_n} is n. Now, we define x_{q_n} for every $n \in \mathbb{N}$ as $x_{q_n,i} = a_m$ if i is the m-th element of A_{q_n} and $x_{q_n,i} = 0$ otherwise.

We can write $x_{q_1}, x_{q_2}, \dots$ as the rows of an infinite upper triangular matrix

$$\begin{array}{cc} x_{q_1} & \longrightarrow \\ x_{q_2} & \longrightarrow \\ x_{q_3} & \longrightarrow \\ \vdots & \end{array} \qquad \begin{pmatrix} a_1 & * & * & * & \cdots \\ 0 & a_1 & * & * & \cdots \\ 0 & 0 & a_1 & * & \cdots \\ \vdots & \vdots & 0 & \ddots & \cdots \end{pmatrix}.$$

Let $y \in \mathbb{R}^{\mathbb{N}}$, then for every $n \in \mathbb{N}$ there exists $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $y_j = \sum_{i=1}^{n} \lambda_i x_{q_i,j}$ for every $j \in \{1, \dots, n\}$. Therefore,

$$E = \operatorname{span}(\{x_{q_1}, x_{q_2}, \dots, x_{q_n}, \dots\})$$

is dense in $\mathbb{R}^{\mathbb{N}}$.

Next, and for the second statement, if we set $E_1 = \operatorname{span}(\{x_\alpha : \alpha \in [0,1]\})$ with x_α defined as $x_{\alpha,i} = a_m$ if i is the m-th element of A_α (where $\{A_\alpha\}_{\alpha\in[0,1]\setminus\mathbb{Q}}$ is an almost disjoint family of subset of $\mathbb{N}$) and $x_{\alpha,i} = 0$ otherwise, then it is clear that E_1 is dense in $\mathbb{R}^{\mathbb{N}}$ since $E \subset E_1$. Furthermore, the dimension of E_1 as vector space is $\mathfrak{c}$ (See [3]). □

Finally, we are going to study the Cesàro summability of the set $\ell_\infty(\mathbb{R})$. To this end we recall the definition of uniformly distributed series.

Definition 2.13. *The sequence* $\{x_n\}_{n\in\mathbb{N}} \subset [0,1[$ *of real numbers is said to be uniformly distributed if for every* $a, b \in \mathbb{R}$ *with* $0 \leq a < b \leq 1$ *we have*

$$\lim_{n\to\infty} \frac{\operatorname{card}([a,b[\cap\{x_1,x_2,\ldots,x_n\})}{n} = b - a.$$

Theorem 2.14. *There exists* $E \subset \ell_\infty(\mathbb{R})$ *that is dense in* $\mathbb{R}^{\mathbb{N}}$ *with dimension* $\mathfrak{c}$ *and such that for every* $x \in E \setminus \{0\}$, x *is not convergent but* $Cx \in c_0$, *where* C *is the Cesàro matrix*

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Proof. Let $\{q_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1[$ uniformly distributed (See [74, Corollary 4.2, p. 135]). Let us take $x \in \ell_\infty(\mathbb{R})$ not convergent with $x_1 \neq 0$, then we take an almost disjoint family $\{A_\alpha\}_{\alpha\in[0,1]\setminus\mathbb{Q}}$ and we choose $\{A_{q_n}\}_{n\in\mathbb{N}}$ as in the proof of Theorem 2.12.

We are going to prove that A_α has density zero for every $\alpha \in]0,1[$ (recall that the density of a set $X \subset \mathbb{N}$ is

$$d(X) = \lim_{n\to\infty} \frac{\operatorname{card}(X \cap \{1,2,\ldots,n\})}{n}$$

provided this limit exists).

Let us write $B_\alpha = \{q_n : n \in A_\alpha\}$. As $\{q_n\}_{n\in\mathbb{N}}$ is uniformly distributed, then for every $\alpha \in (0,1)$ and ε small enough we have that

$$\lim_{n\to\infty} \frac{\operatorname{card}(]\alpha-\varepsilon,\alpha+\varepsilon[\cap\{q_1,q_2,\ldots,q_n\})}{n} = 2\varepsilon.$$

Then

$$\limsup_{n\to\infty} \frac{\operatorname{card}(]\alpha-\varepsilon,\alpha+\varepsilon[\cap B_\alpha \cap \{q_1,q_2,\ldots,q_n\}))}{n} \leq 2\varepsilon.$$

Since $\operatorname{card}(B_\alpha \setminus]\alpha-\varepsilon,\alpha+\varepsilon[) < \omega$, we have that

$$\begin{aligned} 0 \leq \liminf_{n\to\infty} & \frac{\operatorname{card}(B_\alpha \cap \{q_1,q_2,\ldots,q_n\}))}{n} \\ \leq \limsup_{n\to\infty} & \frac{\operatorname{card}(B_\alpha \cap \{q_1,q_2,\ldots,q_n\}))}{n} \leq 2\varepsilon. \end{aligned}$$

Then,

$$\lim_{n\to\infty} \frac{B_\alpha \cap \{q_1,\ldots,q_n\}}{n} = 0.$$

However, and since $\operatorname{card}(B_\alpha \cap \{q_1, \ldots, q_n\}) = \operatorname{card}(A_\alpha \cap \{1, \ldots, n\})$, we have that $d(A_\alpha)$ exists and it is actually zero.

Finally, for every $\alpha \in [0,1]$ define x_α as $x_{\alpha,i} = a_m$ if i is the m-th element of A_α and $x_{\alpha,i} = 0$ otherwise. Also, define $E = \operatorname{span}(\{x_\alpha : \alpha \in [0,1]\})$. Then for every $z \in E$, we have that z is not convergent and, since it is bounded and zero except for a set of density zero, Cx converges to zero. Proceeding as in Theorem 2.12 we see that $\operatorname{span}(\{x_{q_1}, x_{q_2}, \ldots, x_{q_n}, \ldots\})$ is dense in $\mathbb{R}^{\mathbb{N}}$. $\square$

Chapter 3

Banach spaces and Banach lattices of singular functions

3.1 Introduction, notation and preliminaries

As we mentioned in the previous chapter, the search for large algebraic structures inside non-linear families of mathematical objects has become a trend in functional analysis since the beginning of this millennium. This chapter intends to shed light on this line of research, as we did in Chapter 2, in the specific realm of real continuous functions defined on the unit interval $[0,1]$, with focus on the so-called singular functions.

We start with some notation and several known preliminary results. As usual, λ will stand for the Lebesgue measure in the Borel σ-algebra $\mathcal{B}$ in $[0,1]$. In general, and unless otherwise specified, the measures we are going to consider are defined in $\mathcal{B}$, any set we use will be in $\mathcal{B}$ and we shall be working with functions in $\mathbb{R}^{[0,1]}$. The indicator function of a subset $A \subset [0,1]$ will be denoted by $\mathbf{1}_A$. For background about measures, the reader is referred to, for instance, the book [62].

The symbol $\mathcal{C}$ will stand for the set of all continuous $[0,1] \longrightarrow \mathbb{R}$. It is a Banach space under the norm $\|\cdot\|_\infty$ of uniform convergence. Moreover, we set $\mathcal{C}_0 := \{f \in \mathcal{C} : f(0) = 0\}$, which is a closed (so Banach) vector subspace of $\mathcal{C}$. Finally, we shall consider the vector space $\mathcal{CBV} :=$ {continuous functions of bounded variation $[0,1] \longrightarrow \mathbb{R}$} as well as its subspaces $\mathcal{CBV}_0 := \{f \in \mathcal{CBV} : f(0) = 0\}$ and $\mathcal{CBV}_{0,1} := \{f \in \mathcal{CBV} : f(0) = 0 = f(1)\}$. Then $\mathcal{CBV}$ becomes a Banach space under the total variation norm $\|f\|_F := |f(0)| + \mathrm{Var}_{[0,1]}(f)$, and both $\mathcal{CBV}_0, \mathcal{CBV}_{0,1}$ are closed in $\mathcal{CBV}$ (so they are also Banach spaces). It is straightforward that convergence in $\|\cdot\|_F$ is strictly stronger than convergence in $\|\cdot\|_\infty$.

It is well known that every $f \in \mathcal{CBV}_0$ can be expressed as the difference of two continuous monotone functions, and that every function of bounded variation and continuous from the right has an associated signed measure μ_f. For a family $\{f_i\}_{i\in J} \in \mathcal{CBV}_0$ we will use $\{\mu_i\}_{i\in J}$ to represent the corresponding family of signed measures when there is no room for confusion. Also, if μ is a signed measure then $|\mu|$ will denote the measure of the total variation of μ, that is, for every $B \in \mathcal{B}$ we have

$$|\mu|(B) = \sup\left\{\sum_{i=1}^{p} |\mu(B_j)| : \right.$$
$$\left. B = \bigcup_{j=1}^{p} B_j;\ B_1, \dots, B_p \in \mathcal{B} \text{ mutually disjoint; } p \in \mathbb{N}\right\}.$$

Of course, $|\mu_f| = \mu_f$ if f is nondecreasing.

Recall that a *Riesz space,* also called a vector lattice, is a partially ordered (with, say, the order $\preceq$) vector space X where the order structure is a lattice, that is, the order $\preceq$ satisfies the following properties for every pair of vectors $x, y \in X$: there is a supremum $x \vee y \in X$; for any $z \in X$ and any scalar $\alpha \geq 0$, the fact $x \preceq y$ implies $x + z \preceq y + z$ and $\alpha x \preceq \alpha y$. Then the existence of infimum $x \wedge y \in X$ is automatically satisfied; namely, $x \wedge y = -((-x) \vee (-y))$. A *Banach lattice* is a Riesz space X endowed with a norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ is a Banach space and $|x| \preceq |y|$ implies $\|x\| \leq \|y\|$, where $|z| := z \vee (-z)$. See, for instance, the book [76] for fundamentals of Banach lattices.

In $\mathcal{CBV}_0$, with the norm of the bounded variation, we can define an structure of Banach lattice by using the following order:

$$f \succeq g \text{ \textit{if and only if} } f - g \text{ \textit{is a nondecreasing function.}}$$

Note that this is equivalent to say that $f \succeq 0$ if and only if f is nondecreasing. We have that

$$f \vee g = (f-g)^+ + g \quad \text{and} \quad f \wedge g = -((-f) \vee (-g)), \tag{3.1}$$

where

$$f^+(x) = \sup\left\{\sum_{i=1}^{n} \max\{0, f(x_i) - f(x_{i-1})\} : \right.$$
$$\left. \text{with } 0 = x_0 < x_1 < \cdots < x_n = x,\ n \in \mathbb{N}\right\}.$$

In addition, we have the following property: If $f, g \geq 0$ then $\|f+g\|_F = \|f\|_F + \|g\|_F$.

One of the main tools we shall use is that the space $(\mathcal{CBV}_0, \|\cdot\|_F, \leq)$ and the space of finite signed measures without atoms (endowed with the norm $\|\mu\|_M := |\mu|([0,1])$ and the natural order $\leq$ such that $\mu \leq \nu$ if and only if $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{B}$) are *lattice isometric:* see [4].

The definitions of lineability theory 2.1, introduced in chapter 2, will be important in the development of this chapter.

Note that if X is contained in some commutative algebra, then a set $B \subset X$ is a generating set of some free algebra contained in A if and only if for any $N \in \mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1, \ldots, f_N \in B$, we have $P(f_1, \ldots, f_N) \in A \setminus \{0\}$.

In 2016 Oikhberg [81] introduced the notion of latticeability. Namely, a subset A of a Banach lattice X is said to be *latticeable* whenever there is an infinite dimensional sublattice M such that $M \subset A \setminus \{0\}$. Results on latticeability can be seen in [28, 81]. In the next definition, we sharpen this concept by considering cardinalities.

Definition 3.1. *Let X be a Riesz space, A a subset of X, and α a cardinal number. Then:*

1. *A is said to be* α-latticeable *if $A \cup \{0\}$ contains a Riesz space of dimension α.*
2. *If X is a normed Riesz space and $A \cup \{0\}$ contains a Banach lattice of dimension α, the set A is said to be* α-B-latticeable.

This chapter is organized as follows. Section 3.2 is devoted to study lineability properties of the class of singular functions in $\mathcal{CBV}$, that is, functions of bounded variation having derivative equal to zero almost everywhere. In fact, several subclasses are analyzed, related to the size of the zero set, to the nonexistence of intervals of monotonicity, and to the existence of many noncritical points. In Section 3.3, the lineability, in its diverse degrees, of the family of the so-called quasi-constant functions and of a subfamily of it is investigated. Finally, in Section 3.4, the spaceability of some of the previous families when bounded variation convergence is moved to uniform convergence topology will be studied; also the algebrability of the family of sequences of strongly singular functions converging in the latter sense but not in the former one is established.

3.2 Singular functions

This section is devoted to analyze the diverse degrees of lineability of the class of singular functions and several subclasses of it. Under this point of view, singular functions were first studied by Balcerzak *et al.* [12].

The history of singular functions can be traced back to 1884 when Cantor and Schefer published two works introducing the currently known as Cantor function. (See [44].) Since then, these functions have been considered within different frameworks of study. Among others, we find them within conjugations between representation systems (see, e.g., [42, 82, 87, 90]) in the form of Fourier series, as Riesz product [93], or functions that appear dealing with self similar fractal sets in Harmonic Analysis [40, 41].

We first recall the concept of singular function. The abbreviation "λ-a.e." stands for "almost everywhere with respect to the Lebesgue measure λ".

Definition 3.2. *A function $f \in \mathcal{CBV}$ is called* singular *if $f'(x)$ exists and is equal to zero λ-a.e. on $[0,1]$. If, in addition, f is non-constant on any non-degenerate subinterval of $[0,1]$, the function f is said to be* strongly singular. *The set of strongly singular functions will be denoted by $\mathcal{S}$.*

3.2.1 Lineability of $\mathcal{S}$

It is well known that, if $f \in \mathcal{CBV}$ is nondecreasing and satisfies $f([0,1]) = [0,1]$, then the following are equivalent: (a) f is a singular function (not necessarily from $\mathcal{S}$); (b) there exists $B \in \mathcal{B}$ such that $\lambda(B) = 0$ and $\lambda(f(B)) = 1$ or, equivalently, $\mu_f(B) = 1$. The last property can be re-phrased by saying that λ and μ_f are mutually singular probabilities. In general, two signed measures μ_1 and μ_2 defined on $\mathcal{B}$ are said to be *mutually singular* if there exists $B \in \mathcal{B}$ satisfying $|\mu_1|(B) = 0$ and $|\mu_2|([0,1] \setminus B) = 0$. For a pair of functions $f, g \in \mathcal{CBV}$, we shall say that are mutually singular whenever μ_f and μ_g are.

The following theorem provides a constructive way to generate closed vector spaces of singular functions.

Theorem 3.3. *Let $\{f_i\}_{i \in I} \subset \mathcal{CBV}_0$ be an infinite family of functions with associated measures μ_i $(i \in I)$ satisfying the following properties:*

(i) *The functions f_i are singular and mutually singular.*

(ii) *Their measures of total variation are probabilities, that is,*

$$|\mu_i|([0,1]) = 1 \ \textit{for all} \ i \in I.$$

(iii) *For every non-degenerate interval* $\mathbb{I} \subset [0,1]$*, we have*

$$\inf_{i\in I} |\mu_i|(\mathbb{I}) > 0.$$

Let us define

$$\mathcal{A} := \overline{\operatorname{span}}\,\{f_i : i \in I\},$$

where the closure is taken in the variation norm $\|\cdot\|_F$*. Assume that* $f \in \mathcal{A}$*. Then the following holds:*

(a) $f \in \mathcal{S} \cup \{0\}$.

(b) *There are a countable subset* $L \subset I$ *and a set* $\{c_k : k \in L\} \subset \mathbb{R}$ *such that* $\sum_{k\in L} |c_k| < +\infty$ *and* $f = \sum_{k\in L} c_k f_k$.

Proof. Since $f \in \mathcal{A}$, there is a sequence $\{F_n\}_{n\geq 1} \subset \operatorname{span}\{f_i : i \in I\}$ such that $\|F_n - f\|_F \underset{n\to\infty}{\longrightarrow} 0$. And for each $n \in \mathbb{N}$ there exists a finite set $I_n \subset I$ (the case $I_n = \varnothing$ is not discarded) as well as nonzero reals $c_{n,i}$ $(i \in I_n)$ satisfying $F_n = \sum_{i\in I_n} c_{n,i} f_i$. Equivalently, $\nu_n = \sum_{i\in I_n} c_{n,i}\mu_i$, where ν_n is the associated signed measure of F_n. Note that the expression $F_n = \sum_{i\in I_n} c_{n,i} f_i$ is unique because, due the mutual singularity of the f_i's, these functions are linearly independent. Moreover, $\|\nu_n - \mu_f\|_M \to 0$ as $n \to \infty$ and, in particular, $|\nu_n|(B) \to |\mu_f|(B)$ $(n \to \infty)$ for all $B \in \mathcal{B}$.

Assume first that f is constant on some non-degenerate interval $\mathbb{I} \subset [0,1]$. Let $\theta_{\mathbb{I}} := \inf_{i\in I} |\mu_i|(\mathbb{I})$. By the assumption (iii), $\theta_{\mathbb{I}} > 0$. Now, $|\mu_f|(\mathbb{I}) = 0$. Since the measures μ_i are mutually singular, we have that

$$|\nu_n|(\mathbb{I}) = \sum_{i\in I} |c_{n,i}|\,|\mu_i|(\mathbb{I}) \geq \theta_{\mathbb{I}} \cdot \sum_{i\in I_n} |c_{n,i}|.$$

But $|\nu_n|(\mathbb{I}) \to |\mu_f|(\mathbb{I}) = 0$ as $n \to \infty$, from which we derive that

$$\sum_{i\in I_n} |c_{n,i}| \to 0$$

as $n \to \infty$. On the other hand, we have

$$0 \leq |\nu_n|([0,1]) = \sum_{i\in I_n} |c_{n,i}|\,|\mu_i|([0,1]) = \sum_{i\in I_n} |c_{n,i}|.$$

Letting $n \to \infty$ we get $|\mu_f|([0,1]) = \lim_{n\to\infty} |\nu_n|([0,1]) = 0$, so that $f = 0$.

Consequently, in order to prove (a), it is enough to show that any given $f \in \mathcal{A}$ is singular, that is, $f' = 0$ λ-a.e. on $[0,1]$. For this, we need to prove that μ_f and the Lebesgue measure λ are mutually singular. With this aim, take for each $i \in I$ a set $B_i \in \mathcal{B}$ satisfying $|\mu_i|(B_i) = 1$ and $\lambda(B_i) = 0$. This is possible thanks to assumptions (i) and (ii). Then the set

$$B := \bigcup_{n\geq 1} \bigcup_{i\in I_n} B_i$$

is a λ-null set since it is a countable union of λ-null sets. Therefore

$$\begin{aligned} |\mu_f|([0,1] \setminus B) &= \lim_{n\to\infty} |\nu_n|([0,1] \setminus B) \\ &= \lim_{n\to\infty} \sum_{i\in I_n} |c_{n,i}|\,|\mu_i|([0,1] \setminus B) \\ &= \lim_{n\to\infty} \sum_{i\in I_n} |c_{n,i}| \cdot 0 = 0. \end{aligned}$$

Then $|\mu_f|([0,1] \setminus B) = 0 = \lambda(B)$, which proves the desired mutual singularity of μ_f and λ.

In order to prove (b), let us define $L := \bigcup_{n\geq 1} I_n$, which is plainly a countable subset of I. For each $n \in \mathbb{N}$, we can write $F_n = \sum_{k\in L} c_{n,k} f_k$, or equivalently, $\nu_n = \sum_{k\in L} c_{n,k}\mu_k$, where we have defined $c_{n,k} := 0$ if $k \in L \setminus I_n$. In the norm of the total variation, the measures μ_k can be seen as independent atoms due to the fact that they are mutually singular. With this in mind, if we could prove that

$$f = \sum_{k\in L} c_k f_k \tag{A}$$

for an appropriate family $\{c_k\}_{k\in L} \subset \mathbb{R}$, then we would obtain

$$|\mu_f|([0,1]) = \sum_{k\in L} |c_k|.$$

Since f has bounded variation, $|\mu_f|([0,1])$ is finite, and so $\sum_{k\in L} |c_k|$ is finite too. It remains to show that (A) is true.

With this aim, observe that (A) is equivalent to

$$\mu_f = \sum_{k\in L} c_k \mu_k \tag{B}$$

for some sequence $(c_k) \subset \mathbb{R}$. For each pair $j,l \in L$ with $j \neq l$, there is $B_{j,l} \in \mathcal{B}$ such that $|\mu_j|(B_{j,l}) = 1$ and $|\mu_l|(B_{j,l}) = 0$. Then the Borel sets $C_j := \bigcap_{j\in L\setminus\{l\}} B_{j,l}$ $(j \in L)$ satisfy $|\mu_j|(C_j) = 1$ and $|\mu_l|(C_j) = 0$ for all $l \in L \setminus \{j\}$. Notice that we can assume that the C_j's are pairwise disjoint: indeed, if this were not the case, then we first would identify $L = \{1, 2, ..., N\}$ or $L = \{1, 2, 3, \dots\} = \mathbb{N}$, and then we would set $\widetilde{C_1} := C_1$, $\widetilde{C_2} := C_2 \smallsetminus C_1$, $\widetilde{C_3} := C_3 \smallsetminus (C_1 \cup C_2)$, and so on; it is easy to check that the $\widetilde{C_j}$'s are mutually disjoint and still satisfy $|\mu_j|(\widetilde{C_j}) = 1$ and $|\mu_l|(\widetilde{C_j}) = 0$ $(j \neq l)$.

Then if $D \in \mathcal{B}$ is a Borel set with $D \subset [0,1] \smallsetminus \bigcup_{j \in L} C_j$, then $\mu_k(D) = 0$ for all $k \in L$, and so $\nu_n(D) = \sum_{k \in L} c_{n,k}\mu_k(D) = 0$ for all $n \in \mathbb{N}$. Hence $\mu_f(D) = \lim_{n\to\infty} \nu_n(D) = 0$. Then if we expand μ_f as a generalized sum $\mu_f = \sum_{i \in I} c_i \mu_i$, we could take $c_i = 0$ for every $i \in I \smallsetminus L$, since each μ_k $(k \in L)$ is concentrated in C_k. Now, for fixed $l \in L$ we have $|\mu_l|(C_l) = 1 \neq 0$; then there is a Borel set $D_l \subset C_l$ with $\mu_l(D_l) \neq 0$. But $|\mu_k|(D_l) = 0$ for all $k \in L \smallsetminus \{l\}$, which entails $\nu_n(D_l) = c_{n,l}\mu_l(D_l)$. On the other hand, $\nu_n(D_l) \longrightarrow \mu_f(D_l)$ as $n \to \infty$. Then $c_{n,l} \longrightarrow \dfrac{\mu_f(D_l)}{\mu_l(D_l)}$ as $n \to \infty$ (observe that, by the uniqueness of the limit, the quotient $\dfrac{\mu_f(A)}{\mu_l(A)}$ is independent of the set $A \subset C_l$, provided that $\mu_l(A) \neq 0$). Therefore, by selecting

$$c_l := \frac{\mu_f(D_l)}{\mu_l(D_l)} \quad (l \in L),$$

it is easy to verify that $\mu_f = \sum_{k \in L} c_k \mu_k$, that is (B): just take $E \in \mathcal{B}$, divide it into countably many mutually disjoint sets $E = (E \cap \bigcup_{j \in L} C_j) \cup \bigcup_{l \in L}(E \cap C_j)$, compute each ν_n $(n \in \mathbb{N})$ as well as $\sum_{k \in L} c_k \mu_k$ at every part P of this union and, finally, let $n \to \infty$, take into account that $\mu_n(P) \to \mu_f(P)$, and add up the resulting equalities over all P. The proof is finished. □

Remark 3.4. Under the assumptions of Theorem 3.3, the functions f_i are mutually singular, so for different $i, j \in I$ we have

$$\|f_i - f_j\|_F = \|\mu_i - \mu_j\|_M = \|\mu_i\|_M + \|\mu_j\|_M = 2.$$

Consequently, if the family $\{f_i\}_{i \in I}$ is *uncountable,* the Banach space $\mathcal{A}$ is *not separable.*

Let us now see some applications of this result. In order to do that, we shall use the functions S_i for $i \in (0,1)$ (terminology borrowed from [39]). These functions are known differently in the literature. Maybe, the best known are De Rham and Lebesgue functions. Each S_i is characterized by being the only function $g \in [0,1]^{[0,1]}$ enjoying

$$g(x) = \begin{cases} i\, g(2x) & \text{if } x \in [0,1/2] \\ i + (1-i) g(2x-1) & \text{if } x \in [1/2,1]. \end{cases} \tag{3.2}$$

Specifically, each function S_i is defined as

$$S_i(x) = \sum_{n=0} i^{m_n - n}(1-i)^n, \tag{3.3}$$

provided that $\sum_{n=0}^{\infty} 2^{-m_n}$ is the dyadic expansion of $x \in [0,1]$. Note that $S_i(0) = 0 \le S_i(x) \le S_i(1) = 1$ for all $x \in [0,1]$. It is well known that the functions S_i are continuous, singular and nondecreasing, that their associated measures μ_i (which are probabilities) are mutually singular and that, for the intervals $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$ ($k \in \{0,1,\dots,2^{n-1}\}$, $n \in \mathbb{N}$), we have $\mu_i\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) = i^c(1-i)^u$, where c and u depend only on k (see [39]). In addition, when $i < i'$, we have that $S_i(x) < S_{i'}(x)$ for $x \in (0,1)$.

It was proved in [12] that the set $\mathcal{S}$ *is spaceable.* Now, we improve this result by establishing its latticeability.

Theorem 3.5. *The set $\mathcal{S}$ is $\mathfrak{c}$-B-latticeable in $\mathcal{CBV}$.*

Proof. As for mere spaceability, we apply Theorem 3.3 with $I := (0,1/2)$ and $f_i := S_i$. As seen above, hypotheses (i) and (ii) in the mentioned theorem are satisfied. As for (iii), we have to show that, given a non-degenerate interval $\mathbb{I} \subset [0,1]$, there is $\theta_{\mathbb{I}} > 0$ satisfying $\mu_i(\mathbb{I}) \ge \theta_{\mathbb{I}}$ for all $i \in (0,1/2)$. To this end, note that there exist $n \in \mathbb{N}$ and $k \in \{0,\dots,2^n - 1\}$ such that $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \subseteq \mathbb{I}$. Hence $\mu_i(\mathbb{I}) \ge \mu_i\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) = i^c(1-i)^u$. Fix any $\alpha \in (0,1/2)$. As the function $t \mapsto t^c(1-t)^u$ is continuous and positive on $[\alpha, 1/2]$, it reaches a minimum bigger than zero. It is enough to select $\theta_{\mathbb{I}}$ as such a minimum.

Let us show that the Banach space

$$\mathcal{A} = \overline{\operatorname{span}}\,\{S_i : i \in (\alpha, 1/2)\}$$

that we have just obtained by using Theorem 3.3 yields in fact ($\mathfrak{c}$-B) latticeability. Observe first that $\dim(A) = \mathfrak{c}$ because this dimension is not less than $\operatorname{card}(\alpha,1/2) = \mathfrak{c}$ but not greater that $\operatorname{card}(\mathcal{C}) = \mathfrak{c}$. Since $\mathcal{CBV}_0$ is a Banach lattice by itself, it is sufficient to prove that $f \vee g,\ f \wedge g \in \mathcal{A}$ for all $f, g \in \mathcal{A}$. To this end, fix two functions $f, g \in \mathcal{A}$. According to Theorem 3.3, there are a countable set $L \subset (\alpha, 1/2)$ as well as real numbers c_k, d_k ($k \in L$) such that

$$f = \sum_{k \in L} c_k S_k \quad \text{and} \quad g = \sum_{k \in L} d_k S_k.$$

Let $J := \{k \in L : c_k > d_k\}$. Using the lattice isometry with the signed measures, we obtain that $(f-g)^+ = \sum_{k \in J}(c_k - d_k)S_k$ is in $\mathcal{A}$. Therefore, keeping in mind (3.1) and the fact that $\mathcal{A}$ is a vector space, we get that $f \vee g$ and $f \wedge g$ belong to $\mathcal{A}$, as required. $\square$

Remark 3.6. Since $(\alpha, 1/2)$ is uncountable, we get from Remark 3.4 that the existing Banach sublattice in $\mathcal{CBV}$ is *nonseparable.*

3.2.2 Singular functions and zeros

In [38] (see also [49]) the authors study the subset of smooth functions in $\mathbb{R}$ having an uncountable set of zeros. Let us show that the set of strongly singular functions is also a large set in the sense of lineability. The subset of $\mathbb{R}^{[0,1]}$ consisting of all functions with uncountably many zeros that are not constant in any subinterval of $[0,1]$ will be denoted by $\mathcal{U}$.

We shall make use of the following two special transforms. For a function $f : [0,1] \to \mathbb{R}$, we define the new function f^* as follows:

$$f^*(x) = \begin{cases} f(2x) & x \in [0,1/2] \\ 1 - f(2x-1) & x \in (1/2,1]. \end{cases}$$

Given a function $T \in \mathcal{CBV}_{01}$, we define $\mathcal{F}_T : \mathcal{CBV}_{01} \longrightarrow \mathcal{CBV}_{01}$ as follows:

$$\mathcal{F}_T(g)(x) = \begin{cases} g(3x)/3 & x \in [0,1/3] \\ T(3x-1) & x \in [1/3,2/3] \\ g(3x-2)/3 & x \in [2/3,1]. \end{cases} \tag{3.4}$$

It is easy to see that $\mathcal{F}_T$ is a contraction under the distance of the total variation that has a unique fixed point. This fixed point will be represented by $\mathbf{T}$.

If the total variation of T is t and that of $\mathbf{T}$ is $\mathbf{t}$,taking into account that

$$\mathbf{T}(x) = \begin{cases} \mathbf{T}(3x)/3 & x \in [0,1/3] \\ T(3x-1) & x \in [1/3,2/3] \\ \mathbf{T}(3x-2)/3 & x \in [2/3,1]. \end{cases} \tag{3.5}$$

we have that $\mathbf{t} = t + \frac{\mathbf{t}}{3} + \frac{\mathbf{t}}{3}$. That is, $\mathbf{t} = 3t$.

Theorem 3.7. *The set $\mathcal{S} \cap \mathcal{U}$ is spaceable in $\mathcal{CBV}$. In particular, $\mathcal{CBV} \cap \mathcal{U}$ is spaceable in $\mathcal{CBV}$.*

Proof. If we use for each $i \in (0,1/2)$ the above described transform $\mathcal{F}_{S_i^*}$ then we obtain the function $\mathbf{S}_i^*$ as its unique fixed point. This function is continuous and of bounded variation. Moreover, it has an uncountable amount of zeros: indeed, $\mathbf{S}_i^*$ is null on the Cantor set C. Hence every function in $\mathcal{A} := \overline{\text{span}}\{\mathbf{S}_i^* : 0 < i < 1/2\}$ vanishes on C and, by adapting the reasoning of Theorem 3.3 to these functions, we get that every nonzero member of $\mathcal{A}$ is strongly singular. This entails the desired spaceability. □

Remark 3.8. Note that, again, the existing closed subspace in $\mathcal{S} \cap \mathcal{U}$ is *nonseparable.*

3.2.3 Singular functions and monotonicity

By $\mathcal{SNM}$ we shall denote the set of all strongly singular functions of bounded variation that are not monotone in any subinterval of $[0,1]$. The next assertion shows that spaceability still holds when nowhere monotonicity is added to strong singularity.

Theorem 3.9. *The set $\mathcal{SNM}$ is spaceable in $\mathcal{CBV}$.*

Proof. Consider the family of functions $f_i := (S_i - S_{1-i})/2$ $(0 < i < 1/2)$. Let us denote by μ_i (μ'_i, resp.) the associated measure to f_i (to S_i, resp.). Then $\mu_i = (\mu'_i - \mu'_{1-i})/2$ and, by mutual singularity, we have $|\mu_i| = (\mu'_i + \mu'_{1-i})/2$. In particular, the $|\mu_i|$'s are probabilities. If $\mathbb{I}$ is a non-degenerate subinterval of $[0,1]$ and $0 < \alpha < \frac{1}{2}$, then

$$\inf_{i \in (0,1/2)} |\mu_i|(\mathbb{I}) = \frac{1}{2} \cdot \inf_{i \in (\alpha,1/2)} (\mu'_i(\mathbb{I}) + \mu'_{1-i}(\mathbb{I})) > 0$$

by the same argument given in the proof of Theorem 3.5. Then Theorem 3.3 yields that $\mathcal{A} := \overline{\text{span}}\,\{f_i : i \in (\alpha, 1/2)\}$ is a closed vector space contained, except for zero, in $\mathcal{S}$. Moreover, the mutual singularity of the μ_i's gives that $\mathcal{A}$ is infinite dimensional (in fact, it is non-separable).

Now, fix a function $f \in \mathcal{A} \setminus \{0\}$. It remains to show that f is not monotone at any subinterval of $[0,1]$. According to Theorem 3.3, there are a countable set $L \subset (\alpha, 1/2)$ and reals c_i $(i \in L)$ such that $f = \sum_{i \in L} c_i f_i$ in the $\|\cdot\|_F$-topology. Moreover, if μ_f stands for the measure associated to f, we have $\mu_f = \sum_{i \in L} c_i \mu_i$. Since $f \neq 0$, there is $t \in L$ such that $c_t \neq 0$.

Now, since the probabilities μ'_i $(0 < i < 1)$ are mutually singular and L is countable, there are (see, for instance, the construction made in the proof of Theorem 3.3) mutually disjoint Borel sets C_j $(j \in L \cup (1 - L) =: T)$ such that $\mu'_j(C_j) = 1$ for all $j \in T$ and $\mu'_j(C_l) = 0$ for all $j, l \in T$ with $j \neq l$ (in particular, $\mu'_j(C_{1-j}) = 0$ for all $j \in T$). If f were monotone on some non-degenerate interval $\mathbb{I} \subset [0,1]$ (without loss of generality we may assume that f is increasing) then, on the one hand, we would have $0 \le \mu_f(C_t \cap \mathbb{I}) = c_t\, \mu'_t(C_t \cap \mathbb{I})/2$ and, on the other hand, $0 \le \mu_f(C_{1-t} \cap \mathbb{I}) = -c_t\, \mu'_{1-t}(C_{1-t} \cap \mathbb{I})/2$. Finally, note that

$$\mu'_t(C_t \cap \mathbb{I}) = \mu'_t(\mathbb{I}) > 0 < \mu'_{1-t}(\mathbb{I}) = \mu'_{1-t}(C_{1-t} \cap \mathbb{I}).$$

Since $c_t \neq 0$, we have a contradiction. Hence, f cannot be monotone on $\mathbb{I}$, as required. $\square$

3.2.4 Singular functions and derivability

The strongly singular functions that appear in most researches has the property that when it admits finite derivative then it is zero and it does not take any other value. In [50] (see also [51, 52]) some examples of functions without this property were created. That is, the latter functions are strongly singular functions such that there exist uncountable many points in which the derivative exists as a real number and it is not zero; in other words, such points are noncritical. Let us see that in the lineability sense this family of functions is not small.

Definition 3.10. *By $\mathcal{SD}$ we denote the set of strongly singular functions f such that there exists an uncountable set $A_f \subset [0,1]$ in which the derivative exists and it is nonzero.*

Lemma 3.11. *Assume that $\alpha, \beta > 0$ and that $f, g : [0,1] \to \mathbb{R}$ are two continuous, strictly increasing and strongly singular functions such that $f(0) = 0 = g(0)$, $f(1) = 1 = g(1)$ and f, g are mutually singular. Suppose also that $\varphi, \psi : [0,1] \to \mathbb{R}$ are strictly increasing, absolutely continuous functions satisfying $\varphi(0) = 0 = \psi(0)$, $\varphi(1) = 1 = \psi(1)$. Then the functions $F := \varphi \circ f$, $G := \psi \circ g$ are continuous, strictly increasing, strongly singular, mutually singular and satisfy $F(0) = 0 = G(0)$, $F(1) = 1 = G(1)$.*

Proof. It is clear that, except for mutual singularity, it is enough to prove that all properties in the conclusion hold for F. That F is continuous, strictly increasing and satisfies $F(0) = 0$ and $F(1) = 1$ is immediate from the assumptions. Let us show that is strongly singular. Since F is strictly increasing, it is nonconstant on any non-degenerate subinterval of $[0,1]$. Then it suffices to prove that F is singular. To this end, note that the singularity of f is equivalent to the existence of a set $E \in \mathcal{B}$ such that $\lambda(E) = 0$ and $\lambda(f(E)) = 1$. Now, as φ is absolutely continuous, then it maps λ-null sets into λ-null sets. Observe that $f([0,1]) = [0,1] = \varphi([0,1])$ and that all three functions f, φ, F are injective. From this and the fact that $\lambda([0,1] \setminus f(E)) = 0$, it follows that

$$\lambda([0,1] \setminus F(E)) = \lambda(\varphi([0,1]) \setminus \varphi(f(E))) = \lambda([0,1] \setminus f(E)) = 0,$$

whence $\lambda(F(E)) = 1$ and we are done.

Finally, since f, g are mutually singular, there is a set $A \in \mathcal{B}$ such that $\mu_f(A) = 0$ and $\mu_g([0,1] \setminus A) = 0$. Since f, g are increasing with image $[0,1]$, those conditions are equivalent to $\lambda(f(A)) = 0$ and $\lambda(g(A)) = 1$. Now, by using that φ, ψ are absolutely continuous, we can conclude as in the previous paragraph that $\lambda(F(A)) = \lambda(\varphi(f(A))) = 0$ and $\lambda(G(A)) = \lambda(\psi(g(A))) = 1$. Hence F and G are mutually singular. $\square$

Theorem 3.12. *The set $\mathcal{SD}$ is spaceable in $\mathcal{CBV}$.*

Proof. Recall that, when constructing the classical Cantor set C, in the nth-step, we remove 2^{n-1} open intervals of length $1/3^n$. Let us denote these intervals by $J_{n,r} = (a_{n,r}, b_{n,r})$ $(r = 1, \dots, 2^{n-1})$. Let us represent by C_1 the collection of points of the Cantor set that are not extreme of the intervals $J_{n,r}$.

In [50], the authors construct an example of a function in $\mathcal{SD}$ enjoying the following properties:

1. It is derivable at every point of C_1 and the derivative equals 1.

2. It coincides with the identity at every point of C_1.

The construction used a modification of the identity in the sets $J_{n,r}$ by using strictly increasing singular functions. The fundamental condition that they used for the mentioned modification is that $|S(x) - x| \le 1/3^n$ for all $x \in J_{n,r}$. In particular, it is created a function by using the function $S_{1/(2-3^{-(n+1)})}$ in the intervals $J_{n,r}$. If we use the functions $S_{1/(2-3^{-(ni+1)})}$ with $1 \le i < 2$, we obtain a continuous strictly increasing function $g_i \in \mathcal{SD}$ satisfying conditions (1) and (2) (in particular, $g_i(0) = 0$ and $g_i(1) = 1$). Since $S_{1/(2-3^{-(ni+1)})}$ and $S_{1/(2-3^{-(ni'+1)})}$ are mutually singular when $i \neq i'$, we have that the functions g_i are mutually singular. In addition, by taking intervals of the form $\left[\frac{k}{2^t}, \frac{k+1}{2^t}\right]$ and using an argument similar to the one at the beginning of the proof of Theorem 3.5, we deduce that they satisfy the condition $\inf_{i\in[1,2)} \mu_{g_i}(\mathbb{I}) > 0$ for each non-degenerate interval $\mathbb{I} \subset [0,1]$.

Now, we define $f_i(x) := \dfrac{e^{ig_i(x)} - 1}{e^i - 1}$ for every $i \in [1,2)$. Note that

$$f_i'(x) = g_i'(x) \cdot \frac{ie^{ig_i(x)}}{e^i - 1} = \frac{ie^{ix}}{e^i - 1} \tag{3.6}$$

for every $x \in C_1$. Since we have

$$\begin{aligned} f_i\left(\frac{k+1}{2^t}\right) - f_i\left(\frac{k}{2^t}\right) &= \frac{f_i\left(\frac{k+1}{2^t}\right) - f_i\left(\frac{k}{2^t}\right)}{g_i\left(\frac{k+1}{2^t}\right) - g_i\left(\frac{k}{2^t}\right)} \left(g_i\left(\frac{k+1}{2^t}\right) - g_i\left(\frac{k}{2^t}\right)\right) \\ &= \frac{ie^{i\alpha_{k,i}}}{e^i - 1} \left(g_i\left(\frac{k+1}{2^t}\right) - g_i\left(\frac{k}{2^t}\right)\right) \end{aligned}$$

with $\alpha_{k,i} \in \left(g\left(\frac{k}{2^t}\right), g\left(\frac{k+1}{2^t}\right)\right)$ (for the existence of such $\alpha_{k,i}$'s, just apply the mean value theorem to the function $t \mapsto e^{it}$, together with the strict monotonicity of g_i), we obtain that the f_i's also satisfy

$$\inf_{i\in[1,2)} \mu_{f_i}(\mathbb{I}) > \theta_{\mathbb{I}}.$$

By using Lemma 3.11 (with $\varphi(x) = \frac{e^{ix}-1}{e^i-1}$ for each function $f = g_i$) we have that the functions f_i's are mutually singular and fulfill the conditions of Theorem 3.3. In particular, any nonzero member of the $\|\cdot\|_F$-closure $\mathcal{A}$ of $\operatorname{span}\{f_i : i \in [1,2)\}$ is strongly singular.

In order to obtain functions belonging to $\mathcal{SD}$, we consider a subfamily of $\mathcal{A}$. Namely, we fix any strictly increasing sequence $\{i_k\}_{k\geq 1} \subset [1,2)$ (for instance $i_k = \frac{2k+1}{k+1}$) and define $\widetilde{\mathcal{A}} := \overline{\operatorname{span}}\{f_{i_k} : k \in \mathbb{N}\}$, where the closure is in the norm $\|\cdot\|_F$. Plainly, the members of $\widetilde{\mathcal{A}}$ enjoy the property of being strongly singular. Now, fix a function $f \in \widetilde{\mathcal{A}} \setminus \{0\}$. Our goal is to prove that f' is nonzero on each point of an uncountable set. By using Theorem 3.3, we have for such a function f that

$$f = \sum_{k=1}^{\infty} c_k f_{i_k} \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k| \quad \text{is finite.}$$

for a certain sequence $\{c_k\}_{k\geq 1} \subset \mathbb{R}$. The absolute convergence allows us to express the equality $f(x) = \sum_{k\geq 1} c_k f_{i_k}(x)$ as the difference of two integrals in $\mathbb{N}$; namely, with the positive measures γ_1, γ_2 on the family of all subsets of $\mathbb{N}$ determined by $\gamma_1(\{k\}) = c_k \mathbf{1}_{\mathbb{R}^+}(c_k)$, $\gamma_2(\{k\}) = -c_k \mathbf{1}_{\mathbb{R}^-}(c_{i_k})$ $(k = 1, 2, \dots)$. That is, we have

$$f(x) = \int_{\mathbb{N}} f_{i_k}(x)\, d\gamma_1(k) - \int_{\mathbb{N}} f_{i_k}(x)\, d\gamma_2(k) \quad \text{for every} \quad x \in [0,1].$$

Note that both γ_1, γ_2 are finite. Let us analyze the behaviour of f with respect to the derivative in the set C_1. For every $x_0 \in C_1$ we define the function $h : (k,x) \in \mathbb{N} \times [0,1] \mapsto \mathbb{R}$ by

$$h(k,x) = \begin{cases} \frac{f_{i_k}(x) - f_{i_k}(x_0)}{x - x_0} & x \neq x_0 \\ f'_{i_k}(x_0) & x = x_0. \end{cases}$$

We also define the function $H : x \in [0,1] \setminus \{x_0\} \longmapsto \frac{f(x)-f(x_0)}{x-x_0} \in \mathbb{R}$. Under this notation, we obtain

$$H(x) = \sum_{k=1}^{\infty} c_k h(k,x) = \int_{\mathbb{N}} h(k,x)\, d\gamma_1(k) - \int_{\mathbb{N}} h(k,x)\, d\gamma_2(k).$$

Since

$$\frac{f_{i_k}(x) - f_{i_k}(x_0)}{x - x_0} = \frac{1}{e^i - 1} \frac{e^{ig_i(x)} - e^{ig_i(x_0)}}{x - x_0} \leq \frac{g_i(x) - g_i(x_0)}{x - x_0} 2e^2. \tag{3.7}$$

If we consider that $x \leq g_1(x) \leq g_i(x) \leq g_2(x)$ $\forall (x,i) \in [0,1] \times [1,2)$,

$$\frac{g_i(x)-g_i(x_0)}{x-x_0} \le \begin{cases} 1 & x < x_0 \\ \dfrac{g_2(x)-x_0}{x-x_0} & x > x_0. \end{cases}$$

If we take $K = \max\{\dfrac{g_2(x)-x_0}{x-x_0} : x > x_0\}$, by using (3.7) we can conclude that H is bounded since

$$\frac{f_{i_k}(x)-f_{i_k}(x_0)}{x-x_0} \le K2e^2.$$

By using that

$$\sup\{|f'_{i_k}(x_0)| : k \in \mathbb{N}\} \le \sup\left\{\frac{i(e^{ix_0}-1)}{e^i-1} : i \in [1,2)\right\} \le \frac{2(e^2-1)}{e-1},$$

we obtain that the function h is bounded, so h is dominated by a γ_j-integrable function ($j \in \{1,2\}$) that is independent of the parameter x (because a measurable bounded function is integrable with respect to any finite measure). Hence, by a well-known result about integrals depending on a parameter, we can exchange the integrals with the limit whenever the limits inside the integrals exist. Taking into account (3.6), we obtain as a consequence that there exists

$$f'(x_0) = \lim_{x\to x_0} H(x) = \sum_{k=1}^{\infty} c_k f'_{i_k}(x_0) = \sum_{k=1}^{\infty} \frac{c_k i_k e^{i_k x_0}}{e^{i_k}-1}.$$

Finally, consider the function $\Phi(z) = \sum_{k=1}^{\infty} \frac{c_k i_k e^{i_k z}}{e^{i_k}-1}$ ($z \in \mathbb{C}$), so that $f'(x_0) = \Phi(x_0)$ for all $x_0 \in C_1$. Define $d_k := \frac{c_k i_k}{e^{i_k}-1}$ ($k \ge 1$). Observe that $|d_k| \le |c_k|$ for all $k \in \mathbb{N}$. Thanks to the Weierstrass convergence theorem (see, e.g., [1]), Φ is well defined and entire because, given a compact set $K \subset \mathbb{C}$, there is $R > 0$ such that $K \subset \{|w| < R\}$, and an application of the Weierstrass M-test yields the uniform convergence of the series $\sum_{k=1}^{\infty} d_k e^{i_k z}$ on K: indeed, $|d_k e^{i_k z}| \le |c_k| e^{2R}$ and the majoring series $\sum_{k=1}^{\infty} |c_k| e^{2R}$ converges.

If we proved that Φ is not identically zero, then the Identity Principle for analytic functions would yield that the equality $f'(x_0) = 0$ can only be satisfied at isolated points x_0. Therefore, the set $F := \{x_0 \in C_1 : f'(x_0) = 0\}$ would be finite and, as a consequence, f' is not zero at every point of the uncountable set $C_1 \setminus F$, so $f \in \mathcal{SD}$. Consequently, it remains only to prove that Φ is not the zero function. By way of contradiction, assume that $\Phi(z) = \sum_{k=1}^{\infty} d_k e^{i_k z} = 0$ for all $z \in \mathbb{C}$. Since $f \ne 0$, there is some $c_k \ne 0$, so $d_k \ne 0$. Let $m := \min\{k \in \mathbb{N} : d_k \ne 0\}$. Then

$$d_m + \sum_{k=m+1}^{\infty} d_k e^{(i_k - i_m)x} = 0 \quad \text{for all} \quad x \in (-\infty, 0].$$

Since $\{i_k\}_{k\geq 1}$ is increasing, we have $i_k - i_m > 0$ for all $k > m$, and so $\lim_{x\to-\infty} d_k e^{(i_k-i_m)x} = 0$ for such k's. Now, the Weierstrass M-test (recall that $\sum_{k=1}^{\infty} |d_k| < +\infty$) guarantees the uniform convergence of the series $\sum_{k=m+1}^{\infty} d_k e^{(i_k-i_m)x}$ on $(-\infty, 0]$, which allows us to exchange the summation and the limit when $x \to -\infty$. This yields $d_m + 0 = 0$, which is absurd. □

If for every function f_i of Theorem 3.12 we create the functions f_i^* and $\mathbf{f}_i^*$, then we can get the following result just by adapting the reasoning given in the proof of the mentioned theorem. Recall that $\mathcal{U}$ denotes the family of all functions $[0,1] \to \mathbb{R}$ having uncountable zeros and being nonconstant on any non-degenerate interval of $[0,1]$.

Theorem 3.13. *The set of functions of $\mathcal{SD} \cap \mathcal{U}$ with uncountable zeros is spaceable in $\mathcal{CBV}$.*

To study the algebrability of this set, we shall use the following adapted result which is proved in [49, Theorem 2.6].

Theorem 3.14. *Let $\mathcal{K} \subseteq \mathbb{R}^{[0,1]}$ and $f \in \mathbb{R}^{[0,1]}$ be respectively a family of functions and a function satisfying the following properties:*

(a) *$f \in \mathcal{K}$ is so that $f([0,1])$ has, at least, one accumulation point.*

(b) *The vector space generated by the functions of the form*

$$f(x)^n e^{\alpha f(x)} \quad (n = 1, 2, \ldots;\ \alpha > 0)$$

is contained in $\mathcal{K} \cup \{0\}$.

Then $\mathcal{K}$ is strongly $\mathfrak{c}$-algebrable.

Theorem 3.15. *The sets $\mathcal{SD}$ and $\mathcal{SD} \cap \mathcal{U}$ are strongly $\mathfrak{c}$-algebrable.*

Proof. Trivially, it is enough to prove the strong $\mathfrak{c}$-algebrability for $\mathcal{SD} \cap \mathcal{U}$. In [50] it is constructed a function $J \in \mathbb{R}^{[0,1]}$ that is continuous, strictly increasing, strongly singular, and satisfies $J'(x) = 1$ for an uncountable set of $[0,1]$ (in fact, for a subset C_1 of the Cantor set C). Consider the function $f = \mathbf{J}^*$ (the fixed point of $\mathcal{F}_{J^*}$, see (3.4)). Then $f \in \mathcal{K} := \mathcal{SD} \cap \mathcal{U}$ and the condition (a) in Theorem 3.14 is immediately fulfilled for this function.

It remains to verify (b) for f. It is easy to check that the entire functions $x^n e^{\alpha x}$ $(n \in \mathbb{N},\ \alpha > 0)$ are linearly independent, so any nontrivial finite linear combination of them is a nonzero entire function that is not identically zero and vanishes at the origin. Therefore, in order to apply Theorem 3.14, it is sufficient to prove that for every nonzero entire function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(0) = 0$, the function $g := \varphi \circ f$ belongs to $\mathcal{SD} \cap \mathcal{U}$. Note that φ' is not identically zero either. Since $f \in \mathcal{C}$ and is not constant, the image $f([0,1])$ is a non-degenerate interval $[a,b]$. Let us choose a set $B \in \mathcal{B}$ with $\lambda(B) = 1$ and $f'(x) = 0$ for all $x \in B$. Then the chain rule yields $g'(x) = \varphi'(f(x)) \cdot f'(x) = 0$ on B. Moreover, by the Identity Principle for analytic functions, the respective sets F_1, F_2 of 0-points of φ and φ' in $[a,b]$ are finite. There are uncountable sets $U_1, U_2 \subset [0,1]$ such that $f(x) = 0$ for every $x \in U_1$ and there exists $f'(x) \neq 0$ for every $x \in U_2$. Hence $g(x) = \varphi(f(x)) = \varphi(0) = 0$ for all $x \in U_1$. If g were constant on some non-degenerate interval $\mathbb{I} \subset [0,1]$ then φ would be constant on the non-degenerate interval $f(\mathbb{I})$ (it is not degenerate because $f \in \mathcal{S}$); hence the Identity Principle would imply $\varphi = 0$ on a set that is strictly greater than F_1, a contradiction. Up to the moment, we have got $g \in \mathcal{S} \cap \mathcal{U}$.

Finally, again by the chain rule, there exists $g'(x) = \varphi'(f(x)) \cdot f'(x) \neq 0$ for all $x \in U_2 \setminus f^{-1}(F_2)$. But the function J was constructed in [50] in such a way that $J(x) = x$ for all $x \in C$, in particular for all $x \in C_1$. This carries out that the uncountable set U_2 given above can be chosen to satisfy that $f(U_2) = \mathbf{J}^*(U_2)$ is uncountable. Since F_2 is finite, the set $U_2 \setminus f^{-1}(F_2)$ is uncountable. This finishes the proof. $\square$

3.3 Quasi-constant functions

This section is devoted to study lineability properties of families consisting of continuous functions that are virtually constant in the Lebesgue measure sense.

Definition 3.16. *A function $f \in \mathcal{CBV}$ is said to be* quasi-constant *if it is not constant and there exists a countably many mutually disjoint sets, I_n, such that $\sum_n \lambda(I_n) = 1$ and f is constant on each I_n. The set of such functions f will be denoted by $\mathcal{Q}$.*

Example 3.17. For every $i \in (0,1)$, let us consider the transform $F_i : [0,1]^{[0,1]} \longrightarrow [0,1]^{[0,1]}$ given by

$$F_i(h)(x) = \begin{cases} ih(3x) & 0 \leq x \leq 1/3 \\ i & 1/3 < x < 2/3 \\ i + (1-i)h(3x-2) & 2/3 \leq x \leq 1. \end{cases} \tag{3.8}$$

for each $h \in [0,1]^{[0,1]}$. Then there exists a unique function $\overline{S}_i : [0,1] \to [0,1]$ being a fixed point for F_i, that is, F_i is the unique function h such that $\overline{S}_i(x) = F_i(\overline{S}_i)(x)$

for all $x \in [0,1]$. Then $\overline{S}_i$ is quasi-constant. The symbol γ_i will represent the associated measure of $\overline{S}_i$. Note that for $i = 1/2$ we obtain the so-called *Cantor function* (also known as *Devil's staircase*) $c := \overline{S}_{1/2}$. We shall denote the ternary Cantor set by C.

The following auxiliary result about the family of measures $\{\gamma_i : i \in (0,1)\}$ will be used to reveal lineability properties of the class of quasi-constant functions.

Lemma 3.18. *All measures γ_i $(0 < i < 1)$ are concentrated in C. Furthermore, these measures are mutually singular.*

Proof. We shall make use of the equality $\overline{S}_i = S_i \circ c$. To get it, we are going to see that $S_i \circ c(x) = F_i(S_i \circ c)(x)$. Suppose that $x \in [0,1/3]$. Then

$$S_i \circ c(x) = S_i\left(c(3x)/2\right) = iS_i\left(c(3x)\right).$$

The first equality is consequence of $c = \overline{S}_{1/2}$, while the second one comes from the equation (3.2). In a similar way we get the corresponding equalities when $x \in (1/3,2/3)$ and $x \in [2/3,1]$.

It is well known that c maps $\bigcup_{n,r}(a_{n,r},b_{n,r}]$ to the numbers of $(0,1)$ having finite representation in 2 base, that form a countable set. Therefore the set

$$\overline{S}_i\left(\bigcup_{n,r}(a_{n,r},b_{n,r}]\right) = S_i \circ c\left(\bigcup_{n,r}(a_{n,r},b_{n,r}]\right)$$

is countable. Since $S_i \circ c$ is continuous, we have that $\gamma_i\left(\bigcup_{n,r}(a_{n,r},b_{n,r}]\right) = 0$. That is, γ_i is concentrated in $C' = [0,1] \setminus \bigcup_{n,r}(a_{n,r},b_{n,r}] = C \setminus \{b_{r,n}\}_{r,n}$. The mapping c is a bijection between C' and $[0,1]$. Specifically, if $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in C'$ with $x_n \in \{0,2\}$, we have $c(x) = \sum_{n=1}^{\infty} \frac{x_n/2}{2^n}$. On the other hand, it is well known that S_i maps the null λ-measure set

$$B_i := \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} : \{x_n\}_{n\geq 1} \subset \{0,1\} \text{ and } \lim_{n\to\infty} \frac{|\{k \in \{1,2,\dots,n\} : x_k = 0\}|}{n} = i \right\}$$

onto a set with λ-measure equal to 1 (see [39]), where we have denoted by $|A|$ the cardinality of a set A.

Consequently, we have that $S_i \circ c$ maps the subset of C' given by

$$C_i' := \left\{ x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} : \{x_n\}_{n\geq 1} \subset \{0,2\} \text{ and } \lim_{n\to\infty} \frac{|\{k \in \{1,2,\dots,n\} : x_k = 0\}|}{n} = i \right\}$$

in a set of λ-measure equal to 1. In other words, the measure γ_i is concentrated in C_i'. Finally, as the sets C_i''s are pairwise disjoint, we have that the γ_i's are mutually singular. $\square$

We are now ready to state the following theorem, where lineability properties of the family of quasi-constant functions are gathered. As for the properties involving topological concepts, recall that we are considering $\mathcal{CBV}$ as endowed with the norm $\|\cdot\|_F$ of total variation.

Theorem 3.19. *The sets $\mathcal{Q}$ and $\mathcal{Q}\cap\mathcal{U}$ enjoy the following properties:*

(a) *The set $\mathcal{Q}$ is B-latticeable. In particular, it is spaceable.*

(b) *The set $\mathcal{Q}\cap\mathcal{U}$ is spaceable.*

(c) *The spaces $\mathcal{Q}$ and $\mathcal{Q}\cap\mathcal{U}$ are not separable.*

(d) *The sets $\mathcal{Q}$ and $\mathcal{Q}\cap\mathcal{U}$ are strongly $\mathfrak{c}$-algebrable.*

Proof. (a) Recall that μ_i denotes the measure associated to S_i. By taking into account that c is a bijection between C' and $[0,1]$ and that $\overline{S}_i = S_i\circ c$, we obtain that

$$\gamma_i(B) = \gamma_i(B\cap C') = \mu_i\left(c(B\cap C')\right) = \mu_i\left(c(B)\right)$$

for every Borel set B. Therefore, we can translate properties satisfied by the measures μ_i to the measures γ_i. In particular, the measures γ_i $(0<i<1)$ are singular and mutually singular (as seen in Lemma 3.18), and they are probabilities on $\mathcal{B}$. In addition, if one fix any $\alpha\in(0,1/2)$, then for every non-degenerate interval $\mathbb{I}\subset[0,1]$ we have $\sup_{i\in(\alpha,1/2)}\gamma_i(\mathbb{I}) = \sup_{i\in(\alpha,1/2)}\mu_i(\mathbb{I})>0$. Then we can proceed as in the proof of Theorem 3.5 to conclude that the set

$$\mathcal{A} := \overline{\operatorname{span}}\left\{\overline{S}_i : i\in(\alpha,1/2)\right\} \tag{3.9}$$

is a $\mathfrak{c}$-dimensional Banach lattice contained in $\mathcal{Q}\cup\{0\}$.

(b) Let us consider the family of functions $\mathcal{A}_u := \{\mathbf{A} : A = a^* \text{ with } a\in\mathcal{A}\}$ with $\mathcal{A}$ given in (3.9). Since every $\mathbf{A}\in\mathcal{A}_u$ is zero in the Cantor set, we have that $\mathcal{A}_u\subset\mathcal{U}$. As $\mathcal{A}\subset\mathcal{Q}$, given $a\in\mathcal{A}$, we have that there is a countable amount of subintervals $J^\alpha_{n,r,k}$ of $J_{n,r}$ in which the function $\mathbf{A}$ is constant and $\sum_k \lambda\left(J^\alpha_{n,r,k}\right) = \lambda\left(J_{n,r}\right)$. As this happens at every interval $J_{n,r}$, the total length of the subintervals $J^\alpha_{n,r,k}$ is $\sum_{k,n,r}\lambda(J^\alpha_{n,r,k}) = \sum_{n,r}\lambda(J_{n,r}) = 1$. That is, $\mathcal{A}_u\subset\mathcal{Q}$. We define the map $e:\mathcal{A}\to\mathcal{A}_u$ as

$e(a) = \mathbf{A}$. Since $\|e(a_1) - e(a_2)\|_F = \|e(a_1 - a_2)\|_F = 6\|a_1^* - a_2^*\|_F$, we have that $\mathcal{A}_u$ is closed.

(c) If a subset E of a normed space contains an uncountable subset S satisfying $\inf\{\|u-v\| : u, v \in S,\, u \neq v\} > \gamma > 0$ then E cannot be separable. Moreover, if X is a metric space, $D \subset E \subset X$ and D is not separable, then E is also nonseparable. In our setting, and with the notation of (b), we are going to consider the functions $a_i = \hat{S}_i$ and $\mathbf{A}_i = e(a_i)$. Since the measures μ_{a_i} are mutually singular, we have that $\|a_i^* - a_{i'}^*\|_F = 4$ when $i \neq i'$, we derive that $\|\mathbf{A}_i - \mathbf{A}_{i'}^*\|_F = 12$. Therefore, as $\mathbf{A}_i \in \mathcal{Q} \cap \mathcal{U}$ for every $i \in (0,1)$, we can conclude that the set $\mathcal{Q} \cap \mathcal{U}$ is not separable. Then its superset $\mathcal{Q}$ is also nonseparable.

(d) Plainly, it is enough to show that $\mathcal{Q} \cap \mathcal{U}$ is strongly $\mathfrak{c}$-algebrable. But this is nothing but a new application of Theorem 3.14. Indeed, its condition (a) is trivial because the image of $[0,1]$ under a nonconstant continuous function is a non-degenerate interval. Finally, its condition (b) is also fulfilled, because if f is constant on a set $E \subset [0,1]$ then any function $g = f^n \cdot e^{\alpha f}$ $(n \in \mathbb{N}, \alpha > 0)$ is also constant on E and satisfies $\{\text{zeros of } f\} = \{\text{zeros of } g\}$. $\square$

3.4 Spaceability and uniform convergence

In view of the previous sections, one can wonder what happen if we change the topology of the bounded variation to the uniform convergence topology. It turns to be that most results concerning spaceability are still true.

Theorem 3.20. *The sets $\mathcal{S}$, $\mathcal{Q}$, $\mathcal{SD}$, $\mathcal{S} \cap \mathcal{U}$, $\mathcal{SNM}$ and $\mathcal{Q} \cap \mathcal{U}$ are spaceable in the space $\mathcal{C}$ when endowed with the topology of uniform convergence.*

Proof. We are going to use a remarkable result by Gurariy and Lusky (see [61, pp. 80–81]) asserting that if $\{0 < \lambda_1 < \lambda_2 < \cdots\}$ is a sequence with $\sum_{k=1}^{\infty} 1/\lambda_k < +\infty$ and $\inf_{k \in \mathbb{N}}(\lambda_{k+1} - \lambda_k) > 0$ (for instance, $\lambda_k = k^2$) then $\overline{\text{span}}^{\|\cdot\|_\infty}\{t^{\lambda_k} : k \geq 1\} \subset \mathcal{C}^\omega$, where $\mathcal{C}^\omega$ represents the set of all analytic functions in $(0,1)$. Consequently, the set $\mathcal{C}^\omega \cap \mathcal{C}_0$ is spaceable in $\mathcal{C}$. To sum up, we can choose a $\|\cdot\|_\infty$-closed infinite dimensional vector space $\mathcal{T}$ that is contained in $\mathcal{C}^\omega \cap \mathcal{C}_0$.

Fix $g \in \mathcal{C}$ with $g([0,1]) \subset [0,1]$. Then for each $f \in \mathcal{C}$, the composite function $f \circ g$ makes sense and belongs to $\mathcal{C}$. Now, assume that $g([0,1]) = [0,1]$ and define the set

$$\mathcal{T}_g := \{f \circ g : f \in \mathcal{T}\},$$

which is clearly a vector subspace of $\mathcal{C}$.

Moreover, $\mathcal{T}_g$ is closed. Indeed, assume that $\{h_n\}_{n\geq 1}$ is a sequence in $\mathcal{T}_g$ such that $h_n \to h$ uniformly on $[0,1]$ for some $h \in \mathcal{C}$. Then there is a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{T}$ satisfying $h_n = f_n \circ g$ for all $n \in \mathbb{N}$, and the sequence $\{h_n\}_{n\geq 1}$ is $\|\cdot\|_\infty$-Cauchy. Since $g([0,1]) = [0,1]$, we get $\|h_n - h_m\|_\infty = \|f_n - f_m\|_\infty$ $(m, n \in \mathbb{N})$, whence $\{f_n\}_{n\geq 1}$ is $\|\cdot\|_\infty$-Cauchy too. It follows from the completeness of $\mathcal{C}$ that there is $f \in \mathcal{C}$ such that $f_n \to f$ as $n \to \infty$. Since $\mathcal{T}$ is closed, we obtain $f \in \mathcal{T}$. Define $\widetilde{h} := f \circ g \in \mathcal{C}$. Then $\|h_n - \widetilde{h}\|_\infty = \|f_n - g\|_\infty \to 0$ as $n \to \infty$, and so $h_n \to \widetilde{h}$ uniformly on $[0,1]$. Therefore, the uniqueness of the limit yields $h = \widetilde{h} = f \circ g \in \mathcal{T}_g$, which shows that the last set is closed. Moreover, the fact $g([0,1]) = [0,1]$ implies that if $D \subset \mathcal{T}$ is a linearly independent set then $\{f \circ g : f \in D\}$ $(\subset \mathcal{T}_g)$ is too, from which it is derived that $\mathcal{T}_g$ has infinite dimension.

Assume now that g is strongly singular. Then the chain rule tells us in this case that, for every $h \in \mathcal{T}_g$, we have $h'(x) = 0$ in a set of measure 1. Furthermore, the Identity Principle implies that every nonzero member $h = f \circ g$ of $\mathcal{T}_g$ is nonconstant on any non-degenerate subinterval $\mathbb{I}$ of $[0,1]$: indeed, if h were constant on $\mathbb{I}$, then f would be constant on the non-degenerate interval $g(\mathbb{I})$, so f is constant on $[0,1]$, hence $f = 0$ because $f(0) = 0$, whence $h = 0$, a contradiction. We conclude that if g is strongly singular and $g([0,1]) = [0,1]$ then the set $\mathcal{T}_g$ is a closed infinite dimensional vector subspace of $\mathcal{C}$ all of whose nonzero elements are strongly singular.

Now, we are going to analyze every case considered in the statement. Plainly, it is enough to prove spaceability for the classes $\mathcal{SD}$, $\mathcal{S} \cap \mathcal{U}$, $\mathcal{SNM}$ and $\mathcal{Q} \cap \mathcal{U}$.

Case $\mathcal{SD}$. Consider the strongly singular function $g := J$, the function considered in the proof of Theorem 3.15 and studied in [50]. Since $J(x) = x$ for every $x \in C$ (the Cantor set), we get $J(0) = 0$ and $J(1) = 1$. But J is strictly increasing, hence $g([0,1]) = J([0,1]) = [0,1]$. Then $\mathcal{T}_g$ is a closed infinite dimensional subspace of $\mathcal{C}$ such that $\mathcal{T}_g \setminus \{0\} \subset \mathcal{S}$. Let $h = f \circ g \in \mathcal{T}_g \setminus \{0\}$. It remains to show that h admits a finite nonzero derivative at uncountably many points. Recall that there is an uncountable set $A \subset (0,1)$ satisfying $J'(x) = 1$ for all $x \in A$. Then the chain rule gives $h'(x) = f'(g(x)) \cdot J'(x) = f'(g(x))$ for all $x \in A$. But f is analytic on $(0,1)$ and not identically zero, so it is not identically constant (because $f \in \mathcal{C}_0$). Therefore the Identity Principle implies that the set $Z := \{x \in (0,1) : f'(x) = 0\}$ is discrete, hence countable. Since g is injective, the set $B := g(A) \setminus Z$ is uncountable, and $h'(x) \in \mathbb{R} \setminus \{0\}$ for all $x \in B$.

Case $\mathcal{S} \cap \mathcal{U}$. Consider the function $g := \mathbf{J}^* \in \mathcal{S} \cap \mathcal{U}$ this time, where $\mathbf{J}^*$ is the function given in the proof of Theorem 3.15, that is, the fixed point of $\mathcal{F}_{J^*}$, and J is, again, the function studied in [50]. Since J is strongly singular and $J([0,1]) = [0,1]$, we derive that g shares the same properties, that is, it is strongly singular and $g([0,1]) = [0,1]$. Then $\mathcal{T}_g$ is a closed infinite dimensional subspace of $\mathcal{C}$ such that $\mathcal{T}_g \setminus \{0\} \subset \mathcal{S}$. Let $h = f \circ g \in \mathcal{T}_g \setminus \{0\}$. Since g has uncountable many zeros, the same holds for h, because $f(0) = 0$. To sum up, we obtain $\mathcal{T}_g \setminus \{0\} \subset \mathcal{S} \cap \mathcal{U}$, as desired.

Case $\mathcal{SNM}$. Fix any $i \in (0, 1/2)$ and define the function

$$g := \frac{S_{1-i} - S_i}{\beta},$$

where $\beta := \max_{[0,1]}(S_{1-i} - S_i)$. Since being strongly singular is stable under scalings and translations, we can derive as in the proof of Theorem 3.9 that g is strongly singular. Since it also satisfies $g([0,1]) = [0,1]$, we derive that $\mathcal{T}_g$ is a closed infinite dimensional vector subspace of $\mathcal{C}$ satisfying $\mathcal{T}_g \setminus \{0\} \subset \mathcal{S}$. Recall that g is nowhere monotone. Let $h := f \circ g \in \mathcal{T}_g \setminus \{0\}$. In particular, $f \neq 0$. We need to show that h is nowhere monotone. Assume, by way of contradiction, that there is a non-degenerate interval $\mathbb{I} \subset [0,1]$ such that h is monotone on $\mathbb{I}$. Since g is strongly singular, it is not constant on $\mathbb{I}$, so $\mathbb{I}_1 := g(\mathbb{I})$ is a non-degenerate subinterval of $[0,1]$. Now, f is analytic and, in addition, nonconstant (because $f \in \mathcal{C}_0$, and $f \neq 0$). Then the Identity Principle implies the existence of a non-degenerate interval $\mathbb{I}_2 \subset \mathbb{I}_1$ on which f' is nonzero and of constant sign, so f is injective and strictly monotone on $\mathbb{I}_2$. Finally if f^{-1} denotes the inverse of the restricted mapping $f : \mathbb{I}_2 \to \mathbb{I}_3$, where $\mathbb{I}_3 := f(\mathbb{I}_2)$, then f^{-1} is strictly monotone on $\mathbb{I}_3$ and $g = f^{-1} \circ h$ on $\mathbb{I} \cap g^{-1}(\mathbb{I}_2)$. Since g is continuous, the set $\mathbb{I} \cap g^{-1}(\mathbb{I}_2)$ contains some non-degenerate interval $\mathbb{I}_4$. As a composition of two monotone functions, g would be monotone on $\mathbb{I}_4$, which is absurd.

Case $\mathcal{Q} \cap \mathcal{U}$. Take a function $g \in \mathcal{Q} \cap \mathcal{U}$ with $g([0,1]) = [0,1]$ (for instance, one of the functions T_i considered in the proof of Theorem 3.19) and construct the corresponding set $\mathcal{T}_g$, which is a closed infinite dimensional vector subspace of $\mathcal{C}$. By using that that the members of $\mathcal{T} \setminus \{0\}$ are analytic, and nonconstant on any non-degenerate subinterval of $[0,1]$, we can obtain as in the previous cases that every nonzero member of $\mathcal{T}_g \setminus \{0\}$ belongs to both $\mathcal{Q}$ and $\mathcal{U}$. $\square$

To finish this chapter, we shall state and prove the following algebrability theorem about the convergence of sequences of $\mathcal{S}^{\mathbb{N}}$. For recent results about lineability in function sequence spaces, see [6, 31, 32]. In order to study the existence of algebras, we endow the space of function sequences $(\mathbb{R}^{[0,1]})^{\mathbb{N}}$ with the coordenatewise multiplication.

Prior to the theorem, we need the following auxiliary assertion.

Lemma 3.21. *Let X, Y, Z be three metric spaces, such that Y is compact. Assume that $f, f_n : X \to Y$ and $g, g_n : Y \to Z$ $(n \in \mathbb{N})$ are mappings satisfying that $f_n \longrightarrow f$ uniformly on X, $g_n \longrightarrow g$ uniformly on Y, and the family $\{g_n : n \in \mathbb{N}\}$ is uniformly equicontinuous. Then $g_n \circ f_n \longrightarrow g \circ f$ uniformly on X.*

Proof. Since there is no chance of confusion, we denote by d the distance in all three spaces X, Y, Z. Fix $\varepsilon > 0$. Then there is $n_1 \in \mathbb{N}$ such that $d(g_n(y), g(y)) < \varepsilon/2$ for all $n \geq n_1$ and all $y \in Y$. The assumptions imply that g is uniformly continuous on Y. Therefore there is $\delta_1 > 0$ with the property that $d(g(y), g(y')) < \varepsilon/2$ whenever $d(y, y') < \delta_1$. Moreover, there is $\delta_2 > 0$ such that, if $y, y' \in Y$ and $d(y, y') < \delta_2$, then $d(g_n(y), g_n(y')) < \varepsilon/2$ for all $n \in \mathbb{N}_0$. Now, if $\delta := \min\{\delta_1, \delta_2\}$, then the uniform convergence $f_n \to f$ implies the existence of $n_2 \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \delta$ for all $x \in X$ and all $n \geq n_1$. Hence $d(g(f_n(x)), g((f(x))) < \varepsilon/2$ whenever $x \in X$ and $n \geq n_0$, where $n_0 := \max\{n_1, n_2\}$. Finally, the triangle inequality yields that

$$\begin{aligned} d((g_n \circ f_n)(x), (g \circ f)(x)) &\leq d(g_n(f_n(x)), g_n(f(x))) \\ &\quad + d(g_n(f(x)), g(f(x))) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

provided that $n \geq n_0$ and $x \in X$, as desired. □

Theorem 3.22. *The set of sequences of $\mathcal{S}^{\mathbb{N}}$ converging under the uniform convergence topology but not converging under the topology generated by $\|\cdot\|_F$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $\mathcal{H} \subset [1, 2]$ be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$, and $\mathcal{G}$ be a division of $\mathcal{H}$ in $\mathfrak{c}$ sets of cardinal ω, the cardinality of $\mathbb{N}$. If $\gamma \in \mathcal{G}$, we can state a bijection between γ and $\mathbb{N}$. Let us represent by $h'_{\gamma,n}$ the element of γ to which the positive integer n is assigned. Now, we can modify $h'_{\gamma,n}$ by multiplying it by an appropriate rational, say $q_{\gamma,n}$, in such a way that the sequence $\left\{q_{\gamma,n} h'_{\gamma,n}\right\}_{n \geq 1}$ converges to a point $h_\gamma \in \mathcal{H}$ with $h_\gamma \neq h_{\gamma'}$ when $\gamma \neq \gamma'$. Then we write $h_{\gamma,n} := q_{\gamma,n} h'_{\gamma,n}$ for each $n \in \mathbb{N}$. Note that

$$\{h_{\gamma,n} : \gamma \in \mathcal{G} \text{ and } n \in \mathbb{N}\}$$

is still a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For future purposes, without loss of generality we can assume that all $q_{\gamma,n}$'s are in $[1, 2]$, so that all $h_{\gamma,n}$'s are in $[1, 4]$. In addition, we can also assume that, given $\gamma \in \mathcal{G}$, the $q_{\gamma,n}$'s may be selected so that one has $h_{\gamma,n} \geq h_\gamma$ for all but finitely many $n \in \mathbb{N}$.

Next, let us select the sequence $\{i_n\}_{n \geq 1} := \{\frac{1}{4} + \frac{1}{3n}\}_{n \geq 1} \subset (0, 1/2)$. Note that it consists of pairwise different terms, converging to $\frac{1}{4} \in (0, 1/2)$. From the chain rule it is derived that if $f : [0, 1] \to [0, +\infty)$ is in $\mathcal{S}$ and $g : [0, +\infty) \to \mathbb{R}$ is derivable and one-to-one (in particular, if $g(x) = (1+x)^\alpha$ with $\alpha \geq 1$) then $g \circ f \in \mathcal{S}$ too. With this in mind, we define

$$d_\gamma = \{d_{\gamma,n}\}_{n\geq 1} \in \mathcal{S}^{\mathbb{N}} \quad \text{by} \quad d_{\gamma,n} := (1+S_{i_n})^{h_{\gamma,n}},$$

where S_i $(0 < i < 1/2)$ is defined by (3.3). Let $\mathcal{A}$ be the linear algebra generated by the set $\{d_\gamma : \gamma \in \mathcal{G}\}$. The theorem will be proved along the following four steps:

(a) $\mathcal{A}$ is freely generated by the d_γ's, that is, if $\gamma_1, \dots, \gamma_N \in \mathcal{G}$ are pairwise different and P is a polynomial of N variables such that $P(d_{\gamma_1}, \dots, d_{\gamma_N}) = 0$, then $P = 0$.

(b) Each nonzero member of $\mathcal{A}$ is a sequence of strongly singular functions.

(c) Each nonzero member of $\mathcal{A}$ converges uniformly on $[0,1]$ to some function.

(d) Each nonzero member of $\mathcal{A}$ does not converge in the norm $\|\cdot\|_F$.

Proof of (a). For such a polynomial P, there are a nonempty finite set $L \subset (\mathbb{N} \cup \{0\})^N \setminus \{(0,\dots,0)\}$ and reals $c_{\mathbf{m}}$ $(\mathbf{m} = (m_1,\dots,m_N) \in L)$ such that $P(x_1,\dots,x_N) = \sum_{\mathbf{m}\in L} c_{\mathbf{m}} x_1^{m_1} \cdots x_N^{m_N}$. Denote $z = \{z_n\}_{n\geq 1} := P(d_{\gamma_1}, \dots, d_{\gamma_N})$. Then for every $n \in \mathbb{N}$ we have

$$\begin{aligned} z_n &= \sum_{\mathbf{m}\in L} c_{\mathbf{m}}\, d_{\gamma_1,n}^{m_1} \cdots d_{\gamma_N,n}^{m_N} \\ &= \sum_{\mathbf{m}\in L} c_{\mathbf{m}}\, (1+S_{i_n})^{m_1 h_{\gamma_1,n}} \cdots (1+S_{i_n})^{m_N h_{\gamma_N,n}} \\ &= \sum_{\mathbf{m}\in L} c_{\mathbf{m}}\, (1+S_{i_n})^{\rho(\mathbf{m},n)}, \end{aligned}$$

where $\rho(\mathbf{m},n) := \sum_{j=1}^N m_j h_{\gamma_j,n}$. Observe that, since we are using a Hamel basis, we have that for every $n \in \mathbb{N}$, the $\rho(\mathbf{m},n)$'s $(\mathbf{m} \in L)$ are pairwise different. Of course, they are ≥ 1. If $z = 0$ then $z_n = 0$ for all $n \in \mathbb{N}$. In particular, for $n = 1$ we get $\sum_{\mathbf{m}\in L} c_{\mathbf{m}} (1+S_{i_1})^{\rho(\mathbf{m},1)} = 0$. But, by a well-known result due to Pólya (see for instance [63, Corollary 3.2]), a "generalized polynomial"

$$\alpha_0 + \alpha_1 x^{\lambda_1} + \cdots + \alpha_p x^{\lambda_p}$$

(with $\alpha_j \in \mathbb{R}$ and $0 < \lambda_1 < \cdots < \lambda_p$) has only finitely many A-points in $[0,+\infty)$ except that $\alpha_0 = A$ and $\alpha_1 = \cdots = \alpha_p = 0$. Since the function $1 + S_{i_1}$ assumes all values of $[1,2]$, it follows that $c_{\mathbf{m}} = 0$ for all $\mathbf{m} \in F$, and so $P = 0$.

Proof of (b). Let $z = \{z_n\}_{n\geq 1} \in \mathcal{A} \setminus \{0\}$, so that, as in the previous paragraph, there are $\varnothing \neq L \subset (\mathbb{N} \cup \{0\})^N \setminus \{(0,\dots,0)\}$ and $\{c_{\mathbf{m}} : \mathbf{m} \in L\} \subset \mathbb{R} \setminus \{0\}$ with $z_n = \sum_{\mathbf{m}\in L} c_{\mathbf{m}} (1+S_{i_n})^{\rho(\mathbf{m},n)}$, where $\rho(\mathbf{m},n) = \sum_{j=1}^N m_j h_{\gamma_j,n}$ $(n \in \mathbb{N})$. Again by Pólya's result, we have, on the one hand, that for each $n \in \mathbb{N}$ the function z_n cannot be constant on any non-degenerate subinterval of $[0,1]$. On the other hand, there is $B \in \mathcal{B}$ such that $\lambda(B) = 1$ and $S'_{i_n}(x) = 0$ for all $x \in B$. Furthermore, we have that $z_n = \varphi_n \circ S_{i_n}$, where $\varphi_n : [0,+\infty) \to \mathbb{R}$ is the differentiable function given by

$$\varphi_n(x) = \sum_{\mathbf{m} \in L} c_{\mathbf{m}} (1+x)^{\rho(\mathbf{m},n)}. \tag{3.10}$$

Then the chain rule yields $z'_n(x) = \varphi'_n(S_{i_n}(x)) \cdot S'_{i_n}(x) = 0$ for all $x \in B$, which proves that each z_n is strongly singular, as required.

Proof of (c). Let $z = \{z_n\}_{n\geq 1} = \left\{ \sum_{\mathbf{m}\in L} c_{\mathbf{m}} (1+S_{i_n})^{\rho(\mathbf{m},n)} \right\}_{n\geq 1} \in \mathcal{A} \setminus \{0\}$ be as above. Observe that $\rho(\mathbf{m},n) \longrightarrow \sum_{j=1}^N m_j h_{\gamma_j} =: \rho(\mathbf{m})$ as $n \to \infty$, for every $\mathbf{m} \in L$. Define the function

$$\Phi := \sum_{\mathbf{m} \in L} c_{\mathbf{m}} (1+S_{0.25})^{\rho(\mathbf{m})},$$

which is clearly continuous on $[0,1]$. According to (3.3), we have

$$S_{i_n}(x) = \sum_{j=0} i_n^{m_j(x)-j}(1-i_n)^j \quad (n \in \mathbb{N})$$

and

$$S_{0.25}(x) = \sum_{j=0} (0.25)^{m_j(x)-j}(0.75)^j,$$

where $\sum_{j=0}^\infty 2^{-m_j(x)}$ is the dyadic expansion of $x \in [0,1]$. Recall that $i_n = 0.25 + \frac{1}{3n}$, hence $i_n^{m_j(x)-j}(1-i_n)^j \to (0.25)^{m_j(x)-j}(0.75)^j$ $(n\to\infty)$ for every $(j,x) \in (\mathbb{N}\cup\{0\}) \times [0,1]$. In addition, $|i_n^{m_j(x)-j}(1-i_n)^j| \leq (0.75)^j$ $(x \in [0,1],\ n \in \mathbb{N},\ j \geq 0)$, and the series $\sum_{j=0}^\infty (0.75)^j$ converges and does not depend on (x,n). Hence $S_{i_n} \to S_{0.25}$ $(n\to\infty)$ uniformly on $[0,1]$.

Now, observe that $\rho(\mathbf{m},n), \rho(\mathbf{m}) \in [1,\alpha]$ for all $n \in \mathbb{N}$ and all $\mathbf{m} \in L$, where we have set $\alpha := 4\sum_{j=1}^N m_j$. Note that, for each $\mathbf{m} \in L$, one has $\rho(\mathbf{m},n) - \rho(\mathbf{m}) \geq 0$ for all but finitely many $n \in \mathbb{N}$. Define the spaces $X := [0,1]$, $Y := [0,1]$, $Z := \mathbb{R}$, endowed with the usual distance, as well as the functions

$$f := S_{0.25},\ g(x) := \sum_{\mathbf{m}\in L} c_{\mathbf{m}}(1+x)^{\rho(\mathbf{m})},\ f_n := S_{i_n} \ \text{ and } \ g_n := \varphi_n \ (n \in \mathbb{N}),$$

where φ_n is given by (3.10). Fix $\mathbf{m} \in L$. Then for all but finitely many $n \in \mathbb{N}$ we have

$$\sup_{x\in[0,1]} |(1+x)^{\rho(\mathbf{m},n)} - (1+x)^{\rho(\mathbf{m})}| \leq 2^{\rho(\mathbf{m})} \cdot |2^{\rho(\mathbf{m},n)-\rho(\mathbf{m})} - 1| \longrightarrow 0 \quad (n\to\infty),$$

which proves the uniform convergence of $(1+x)^{\rho(\mathbf{m},n)}$ to $(1+x)^{\rho(\mathbf{m})}$ on $[0,1]$. Since finite summations keep uniform convergence, we get that $g_n \to g$ uniformly on Y.

Also, under the given notation, we have already obtained that $f_n \to f$ uniformly on X. Let us show that the family $\{g_n\}_{n\geq 1}$ is uniformly equicontinuous on Y. For this, it is enough to prove that it is uniformly Lipschitz, and this property, in turn, would become apparent (just use the mean value theorem) as soon as we will be able to show that $\sup_{n\in\mathbb{N}} \sup_{[0,1]} |g_n'(x)| < +\infty$. This follows from the simple computation

$$|g_n'(x)| \leq \sum_{\mathbf{m}\in L} |c_{\mathbf{m}}| \rho(\mathbf{m}, n)(1+x)^{\rho(\mathbf{m},n)-1} \leq \alpha \cdot 2^{\alpha-1} \cdot \sum_{\mathbf{m}\in L} |c_{\mathbf{m}}| < +\infty,$$

which is valid for all $(n, x) \in \mathbb{N} \times [0,1]$.

Putting all together, an application of Lemma 3.21 yields $z_n = g_n \circ f_n \underset{n\to\infty}{\longrightarrow} g \circ f = \Phi$ uniformly on $[0,1]$, as desired.

Proof of (d). Finally, we have to show that $z = \{z_n\}_{n\geq 1}$ cannot converge with the total variation. Indeed, since this convergence is stronger than uniform convergence, the unique possible $\|\cdot\|_F$-limit would be the function Φ defined above. Observe that, reasoning as in (b) and taking into account that the exponents $\rho(\mathbf{m})$ $(\mathbf{m} \in F)$ are pairwise different (because the h_{γ_j}'s are pairwise different and belong to the basis $\mathcal{H}$), we have that Φ is strongly singular, hence nonconstant, and so $\mathrm{Var}_{[0,1]}(\Phi) > 0$. Now, since $i_n \neq 0.25$ $(n \in \mathbb{N})$, we get that the associated measures of S_{i_n} and $S_{0.25}$ are mutually singular. Hence, for each $n \in \mathbb{N}$, the functions z_n and Φ are mutually singular too. Observe that $z_n(0) = \sum_{\mathbf{m}\in F} c_{\mathbf{m}} = \Phi(0)$ for all $n \in \mathbb{N}$, and recall that $\mathrm{Var}_{[0,1]}(f+g) = \mathrm{Var}_{[0,1]}(f) + \mathrm{Var}_{[0,1]}(g)$ if μ_f and μ_g are mutually singular. Consequently,

$$\begin{aligned}\|z_n - \Phi\|_F &= |z_n(0) - \Phi(0)| + \mathrm{Var}_{[0,1]}(z_n - \Phi) \\ &= \mathrm{Var}_{[0,1]}(z_n) + \mathrm{Var}_{[0,1]}(\Phi) > \mathrm{Var}_{[0,1]}(\Phi) > 0\end{aligned}$$

for all $n \in \mathbb{N}$. Then $\|z_n - \Phi\|_F \not\to 0$, and the theorem is proved. $\square$

Part II

Convexity

Chapter 4

Describing multiplicative convex functions

4.1 Introducction

The theory of convex functions keeps playing a central role in operator theory, in real analysis and in some realms of applied mathematics, such as management science or optimization theory (we refer the interested reader to [5, 19, 79] for applications of the property of convexity). Since the beginning of their study by Jensen, they have been thoroughly described and many of their properties have been unveiled.

We recall that a function $f : V \to \mathbb{R}$ (where V is a vector space over $\mathbb{R}$) is called **convex** if, whenever x, $y \in V$ and $0 \le \lambda \le 1$, we have

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda)f(y).$$

The study of convex functions has extended to the consideration of other inequalities. In this direction, Niculescu proposed the following definition (see [80]):

Let $I \subseteq (0, \infty)$ be an interval. A function $f : I \to (0, \infty)$ is called **multiplicative convex** if, for every $x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f(x^{1-\lambda} y^{\lambda}) \le f(x)^{1-\lambda} f(y)^{\lambda}. \tag{4.1}$$

Functions satisfying inequality (4.1) are also known as *GG-convex functions*, since they are supposed to substitute the arithmetic mean in the inequality that defines the convex functions by the geometric mean. Yet, the definition of the multiplicative convex functions could be regarded as a way of upgrading the operations that take part in the definition of convex functions. In this direction, we see that the addition operation turns into multiplication operation, and the multiplication operation turns into power operation.

If we adopt this point of view, the upgrade is yet not complete. λ and $\mu = 1 - \lambda$ are related by $\lambda + \mu = 1$. If we keep in mind that addition turns into multiplication, then the exponents λ and μ should be related by $\lambda \cdot \mu = 1$. Following that idea, we propose the following variation for the definition of multiplicative convex functions:

Definition 4.1. *Let $f : (0, \infty) \to [0, \infty)$ be such that $f(1) = 1$. We will say that f is* **multiplicative convex** *if, for every $\mu > 0$ and $x, y \geq 0$ we have*

$$f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu}. \tag{4.2}$$

In particular, by setting $\mu = 1$, we obtain $f(xy) \leq f(x)f(y)$ for every $x, y \geq 0$.

The aim of this chapter is to study the properties that the multiplicative convex functions, in the setting of definition 4.1, verify. In fact, we will be able to completely describe what those functions look like.

4.2 Describing multiplicative convex functions

We will start by studying some of the features that the inequality introduced in Definition 4.1 enjoys. The first result tells us about the increasing/decreasing behavior of such functions.

Theorem 4.2. *Let f be a multiplicative convex function. Then, f is either monotone or it decreases until $x = 1$ and then increases (for simplicity, we will call the functions of the third kind increasing-decreasing functions).*

Proof. Assume there is $0 < \theta < 1$ so that $f(\theta) \leq 1$. Let us show that in that case $f(\psi) \leq 1$ for every $0 < \psi < 1$. Having shown that, we will have also proved that if there is $0 < \theta < 1$ so that $f(\theta) > 1$, then $f(\psi) > 1$ for every $0 < \psi < 1$ (otherwise it contradicts the previous statement). Indeed, let $0 < \psi \leq \theta^2$. Then, we can find $1 \leq \mu$ so that $\psi = \theta^{\mu + 1/\mu}$ and, hence,

$$f(\psi) = f(\theta^\mu \theta^{1/\mu}) \leq f(\theta)^\mu f(\theta)^{1/\mu} \leq 1.$$

Next, let $\theta^2 < \psi < \theta$. We can find $1 \leq \mu < 2$ with $\psi = \theta^\mu$ and, therefore,

$$f(\psi) = f(\theta^\mu 1^{1/\mu}) \leq f(\theta)^\mu f(1)^{1/\mu} \leq 1.$$

Finally, if $\theta < \psi < 1$, then there is $\mu \geq 1$ so that $\psi = \theta^{1/\mu}$ and hence

$$f(\psi) = f(1^\mu \theta^{1/\mu}) \leq f(\theta)^{1/\mu} \leq 1.$$

Assume then that $x < y$. We can write

$$f(x) = f\left(xy\frac{1}{y}\right) \leq f(y) f\left(\frac{x}{y}\right) \leq f(y)$$

and hence f is increasing.

Let us define $g(x) = f(1/x)$. Then, g is also a multiplicative convex function. Furthermore, if there is $x > 1$ so that $f(x) \leq 1$, then $g(1/x) \leq 1$ with $\frac{1}{x} < 1$ and hence g is increasing (and therefore f is decreasing).

If, on the contrary, there is $x > 1$ with $f(x) > 1$, then we would have that $f(x) > 1$ for every $x > 1$ (again, using the function g, which has to be higher than 1 over $(0,1)$). Hence, if $1 < x < y$, then we can find $0 < \mu < 1$ so that $x = y^\mu$ and therefore

$$f(x) = f(y^\mu 1^{1/\mu}) \leq f(y)^\mu \leq f(y),$$

so that f is increasing on $(1,\infty)$. Via another argument involving the function $g(x)$, we would conclude that f is decreasing on $(0,1)$.

□

The following Lemma will be crucial in the theorems to come.

Lemma 4.3. *Let f be a multiplicative convex function and $q \in \mathbb{Q}^+$. Then, $f(x^q) = f(x)^q$*

Proof. Assume, first, that $q \in \mathbb{N}$. Since $f(xy) \leq f(x)f(y)$, we obtain $f(x^q) \leq f(x)^q$. On the other hand,

$$f(x) = f((x^q)^{1/q}) \leq f(x^q)^{1/q} f(1)^q = f(x^q)^{1/q},$$

so that $f(x)^q \leq f(x^q) \leq f(x)^q$.

If now q is of the form $\frac{1}{n}$, with $n \in \mathbb{N}$, we obtain, similarly,

$$f(x^{1/n}) \leq f(x)^{1/n} f(1)^n = f(x)^{1/n},$$

being f multiplicative convex. Conversely, $f(x) = f((x^{1/n})^n) \leq f(x^{1/n})^n$, from which $f(x^{1/n}) \geq f(x)^{1/n}$.

For the general case, $f(x^q) = f(x^{n/m}) = f(x^{1/m})^n = f(x)^{n/m}$. □

Theorem 4.4. *Let f be a multiplicative convex function. Then, f is continuous.*

Proof. We claim first that f is continuous at $x = 1$. Indeed, otherwise we can find a sequence $\{x_n\}_{n=1}^{\infty}$ and a number $M > 0$ so that either $x_n \uparrow 1$ or $x_n \downarrow 1$, and $|f(x_n) - 1| > M$ for every $n \in \mathbb{N}$.

Assume that f is increasing. If we are in the case $x_n \uparrow 1$, then $M < 1 - f(x_n) < 1$, so that $f(x_n) < 1 - M < 1$.

Let $n \in \mathbb{N}$. Then, we can find $m_n \in \mathbb{N}$ so that $x_1 \le x_{m_n}^n$. Then,

$$f(x_1) \le f(x_{m_n}^n) = f(x_{m_n})^n \to 0 \quad \text{as } n \to \infty$$

and hence $f(x_1) = 0$. On the other hand,

$$1 = f(x_1 \frac{1}{x_1}) \le f(x_1) f(\frac{1}{x_1}) = 0,$$

and we reach a contradiction.

If we are in the case of $x_n \downarrow 1$, then $f(x_n) - 1 > M$ for every n, so that $f(x_n) > M + 1 > 1$. Since f is increasing, we can also deduce that $f(y) > M + 1$ for every $y > 1$.

On the other hand, if $y > 1$, we can find $\mu \ge 1$ so that $f(y)^{1/\mu} < M + 1$. Let $x < 1$ so that $x^\mu y^{1/\mu} > 1$. Then,

$$M + 1 < f(x^\mu y^{1/\mu}) \le f(x)^\mu f(y)^{1/\mu} < M + 1$$

(let us recall that $f(z) < 1$ for every $z < 1$ if f is an increasing multiplicative convex function), reaching a contradiction.

Assume next that f is decreasing. Then, we just need to apply the previous study to the function $g(x) = f(\frac{1}{x})$.

Assume finally that f is increasing-decreasing. Since, in that case, g is also an increasing-decreasing multiplicative convex function, we may assume, w.l.o.g., that we can find a sequence $x_n \uparrow 1$ and $M > 0$ so that $f(x_n) > M + 1$, for every $n \in \mathbb{N}$. Again, for $n \in \mathbb{N}$, we can find $m_n \in \mathbb{N}$ so that $x_{m_n}^n \ge x_1$. Then,

$$f(x_1) \ge f(x_{m_n}^n) = f(x_{m_n})^n > (M + 1)^n \to \infty \quad \text{as } n \to \infty,$$

and the claim is proved also for this case.

We will prove continuity in the general case. Let $x \in (0,\infty)$ and $\varepsilon > 0$. Since f is continuous at 1, we can find $0 < \delta < 1$ so that, if $|1-\zeta| < \delta$, then $|1 - f(\zeta)| < \frac{\varepsilon}{f(x)}$.

Let us assume first that f is increasing (which will also cover the case of f being decreasing, via the use of the function $g(x) = f(\frac{1}{x})$) and let $y > 0$ so that $|x-y| < \frac{\delta x}{2} < \frac{x}{2}$. If $y < x$, then

$$\frac{x}{y} - 1 = \frac{x-y}{y} < \frac{\frac{\delta x}{2}}{\frac{x}{2}} = \delta,$$

so that

$$0 < f\left(\frac{x}{y}\right) - 1 < \frac{\varepsilon}{f(x)}.$$

On the other hand,

$$\begin{aligned} 0 < f(x) - f(y) &= f\left(y\frac{x}{y}\right) - f(y) \\ &\le f(y)\left[f\left(\frac{x}{y}\right) - 1\right] \\ &\le f(x)\left[f\left(\frac{x}{y}\right) - 1\right] < \varepsilon. \end{aligned}$$

For the case of f decreasing-increasing, we may reproduce the argument for f increasing and $x < y$ to show that f is continuous from the right. We can conclude the situation for $y < x$ using the function $g(x) = f(\frac{1}{x})$. □

With Theorem 4.4, we can extend Lemma 4.3 to positive exponents:

Lemma 4.5. *Let f be a multiplicative convex function. Then, $f(x^t) = f(x)^t$ for every $t > 0$.*

Proof. If $t > 0$, we can find a sequence $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}^+$ so that $q_n \to t$ as $n \to \infty$. Using the fact that f is continuous,

$$f(x^t) = f\left(x^{\lim_{n\to\infty} q_n}\right) = \lim_{n\to\infty} f(x^{q_n}) = \lim_{n\to\infty} f(x)^{q_n} = f(x)^t.$$

□

We are now ready to describe the way the multiplicative convex functions are, as in Definition 4.1.

Theorem 4.6. *Let $f : (0,\infty) \to [0,\infty)$. Then, f is a multiplicative convex function if and only if it can be written of the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{4.3}$$

where a, b, a' and b' satisfy the following conditions:

1. $0 < a < 1$ *and* $a' > 1$.
2. *If* $b < 1$*, then* $\log_b(b') \leq \log_a(a') < 0$ *(which, in particular, implies* $b' > 1$*).*
3. *If* $b > 1$*, then* $\log_b(b') \geq \log_a(a')$.

Proof. Assume first f is multiplicative convex. Let us fix $0 < x_0 < 1$ and $x_1 > 1$.

If $0 < x < 1$, we can find $\mu > 0$ so that $x = x_0^\mu$. More concretely, we can write $\mu = \log_{x_0}(x) > 0$. Therefore,

$$f(x) = f(x_0^\mu) = f(x_0)^\mu = f(x_0)^{\log_{x_0}(x)}.$$

Similarly, we can write $f(x) = f(x_1)^{\log_{x_1}(x)}$ for $x > 1$, so we can identify $a = x_0 \in (0,1)$, $b = f(x_0)$, $a' = x_1 > 1$, $b' = f(x_1)$.

Notice now that we can find $\mu \geq 1$ so that $x_0^\mu x_1^{1/\mu} < 1$. Then,

$$\begin{aligned} f(x_0^\mu x_1^{1/\mu}) &= f(x_0)^{\log_{x_0}(x_0^\mu x_1^{1/\mu})} \\ &= \left[f(x_0)^{\log_{x_0}(x_0)}\right]^\mu \left[f(x_0)^{\log_{x_0}(x_1)}\right]^{1/\mu} \\ &= f(x_0)^\mu \left[f(x_0)^{\log_{x_0}(x_1)}\right]^{1/\mu}. \end{aligned}$$

On the other hand, $f(x_0^\mu x_1^{1/\mu}) \leq f(x_0)^\mu f(x_1)^{1/\mu}$. Therefore, $f(x_0)^{\log_{x_0}(x_1)} \leq f(x_1)$ and hence

$$\log_{x_0}(x_1) \begin{cases} \leq \log_{f(x_0)}(f(x_1)) & \text{if } f(x_0) > 1, \\ \geq \log_{f(x_0)}(f(x_1)) & \text{if } f(x_0) < 1. \end{cases}$$

Reciprocally, let $f(x)$ be defined as Equation (4.2), with a, a', b and b' satisfying the corresponding conditions. Assume first $b < 1$ (so that $\log_b(b') \leq \log_a(a')$).

Let $x, y > 0$ and $\mu > 0$ so that $x^\mu y^{1/\mu} > 1$ (we may assume, w.l.o.g., that $y > 1$). Then,

$$f(x^\mu y^{1/\mu}) = b'^{\log_{a'}(x^\mu y^{1/\mu})} = \left[b'^{\log_{a'}(x)}\right]^\mu \left[b'^{\log_{a'}(y)}\right]^{1/\mu}.$$

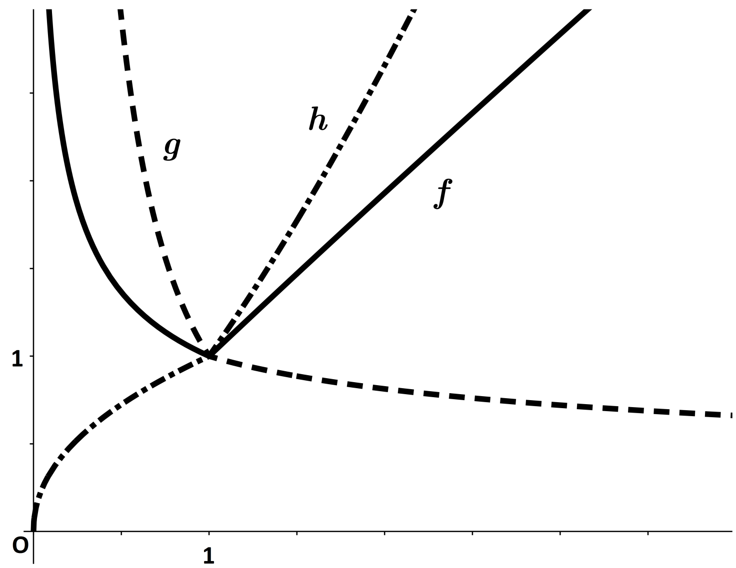

Figure 4.1: The functions are defined as in Equation (4.2) by using the following constants: $a_f = 0.27$, $b_f = 1.8$, $a'_f = 2.35$, $b'_f = 2.25$, $a_g = 0.45$, $b_g = 3.45$, $a'_g = 2.62$, $b'_g = 0.75$, $a_h = 0.18$, $b_h = 0.45$, $a'_h = 1.86$, and $b'_h = 2.4$

If $x > 1$, we obtain trivially $f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu}$.

If $x < 1$, then we need to show that $b'^{\log_{a'}(x)} \leq b^{\log_a(x)}$, which is equivalent to show that

$$\log_{a'}(x)\log_b(b') \geq \log_a(x), \tag{4.4}$$

since $b < 1$.

Now, having that $a' > 1$, we obtain that $\log_{a'}(x) < 0$ and therefore

$$\log_{a'}(x)\log_b(b') = \frac{\log_a(x)}{\log_a(a')}\log_b(b') \geq \log_a(x),$$

as desired.

Assume next that we are in the situation of $x^\mu y^{1/\mu} < 1$ (so that, w.l.o.g., $x < 1$). Then,

$$f(x^\mu y^{1/\mu}) = \left[b^{\log_a(x)}\right]^\mu \left[b^{\log_a(y)}\right]^{1/\mu}$$

and we only need to show that, if $y > 1$, then $b^{\log_a(y)} \leq b'^{\log_{a'}(y)}$, which is equivalent to show that

$$\log_a(y) \geq \log_{a'}(y)\log_b(b').$$

On the other hand, notice that $a, b < 1$, so $\log_a(y)$, $\log_a(a') < 0$ and hence

$$\log_a(y) \geq \log_a(y)\frac{\log_b(b')}{\log_a(a')} = \log_{a'}(y)\log_b(b').$$

If now $b > 1$ (so $\log_b(b') \geq \log_a(a')$) we have that the conditions 1,2,3 imply

$$\frac{\log_a(x)}{\log_a(a')}\log_b(b') = \log_{a'}(x)\log_b(b') \leq \log_a(x) \quad \text{for every } 0 < x < 1,$$
$$\frac{\log_a(y)}{\log_a(a')}\log_b(b') = \log_{a'}(y)\log_b(b') \geq \log_a(y) \quad \text{for every } y > 1.$$

Notice again that $\log_a(a') < 0$ and therefore $\dfrac{\log_b(b')}{\log_a(a')} \leq 1$. □

The previous theorem characterizes the multiplicative convex functions and allows us, through the use of appropriate values of the constants, to graphically show different types of multiplicative convex functions (See Figure 4.1).

In the definition 4.1 we make the distinction $f(1) = 1$. But since $f(1) = f(1 \cdot 1) \leq f(1)^2$, we can deduce that $f(1) \geq 1$. On the other hand, if $C > 1$ and f is multiplicative convex, then Cf verifies (4.2). Then, we wonder if every function that verify (4.2) can be set as Cf with C a constant higher than 1 and f a function as in Theorem 4.6. That is, if a function f satisfies (4.2) would we have that $g(x) = f(x)/f(1)$ verifies (4.2)? We can state, by a negative answer, that the distinction $f(1) = 1$ is essential in Theorem 4.6.

Remark 4.7. *The function $f(x) = x + 1$ is multiplicative convex (condition (4.2)) but $g(x) = \dfrac{f(x)}{2}$ is not. Indeed, we have to see that for every $\mu > 0$ and $x, y \geq 0$ we have*

$$x^\mu y^{1/\mu} + 1 \leq (x+1)^\mu (y+1)^{1/\mu}.$$

If $\mu = 1$, then $(x+1)(y+1) = xy + x + y + 1 \geq xy + 1$. Next, if $\mu > 1$ (it will be analogous for $\mu < 1$) we will have that

$$(x+1)^\mu (y+1)^{1/\mu} \geq (x^\mu + 1)(y+1)^{1/\mu}.$$

Then, if $y \geq 1$ we have that $1 \leq y^{1/\mu} \leq (y+1)^{1/\mu}$ so,

$$(x^\mu + 1)(y+1)^{1/\mu} \geq (x^\mu + 1)y^{1/\mu} = x^\mu y^{1/\mu} + y^{1/\mu} \geq x^\mu y^{1/\mu} + 1$$

And, if $y < 1$, we have that $y^{1/\mu} \leq 1 \leq (y+1)^{1/\mu}$, so

$$(x^\mu + 1)(y+1)^{1/\mu} \geq (x^\mu + 1)1 = x^\mu + 1 \geq x^\mu y^{1/\mu} + 1$$

Therefore $f(x) = x + 1$ verifies condition (4.2).

Next, let us see that $g(x)$ does not verify (4.2). Taking $x = y = \frac{1}{2}$ and $\mu = 10$, we have that, on the one hand,

$$\frac{x^\mu y^{1/\mu} + 1}{2} = \frac{\left(\frac{1}{2}\right)^{10+1/10} + 1}{2} \approx 0.50045,$$

and, on the other hand,

$$\left(\frac{x+1}{2}\right)^\mu \left(\frac{y+1}{2}\right)^{1/\mu} = \left(\frac{3}{4}\right)^{10+1/10} \approx 0.05471,$$

and we are done, since we have just showed that, for this previous choice of x, y, and μ we actually have

$$g(x^\mu y^{1/\mu}) > g(x)^\mu g(y)^{1/\mu}.$$

Chapter 5

Generalizing multiplicative convex functions

5.1 Introduction and new notation

In chapter 4, the following definition was proposed.

Definition 5.1. *Let $f : (0, \infty) \to [0, \infty)$ be a function. We will say that f is* **multiplicative convex** *(or f is an mc–function, for short) if, for every $\mu > 0$ and $x, y > 0$ we have*

$$f(x^{\mu} y^{1/\mu}) \leq f(x)^{\mu} f(y)^{1/\mu}. \tag{5.1}$$

The inequality in (5.1) implies that $f(1) \geq 1$. In chapter 4 the following distinction was made:

Definition 5.2. *A function $f : (0, \infty) \to [0, \infty)$ is said to be* **1-multiplicative convex** *(or f is a mc1–function) if, for every $\mu > 0$ and $x, y > 0$ we have*

$$f(x^{\mu} y^{1/\mu}) \leq f(x)^{\mu} f(y)^{1/\mu}.$$

and the function satisfies also the condition $f(1) = 1$.

We will denote

$$\mathcal{MC} := \{f : (0, \infty) \to [0, \infty) \,|\, f \text{ is a mc–function}\}$$
$$\mathcal{MC}_1 := \{f : (0, \infty) \to [0, \infty) \,|\, f \text{ is a mc1–function}\}.$$

In chapter 4 the property $f(1) = 1$ was of great importance to give the following characterization of $mc1$–functions:

Theorem 5.3. [64, Theorem 2.5] *Let $f : (0,\infty) \to [0,\infty)$. Then, f is a $mc1$–function if and only if it can be written in the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{5.2}$$

where a, b, a' and b' satisfy the following conditions:

1. $0 < a < 1$ *and* $a' > 1$.
2. *If* $b < 1$, *then* $\log_b(b') \leq \log_a(a') < 0$ *(which, in particular, implies* $b' > 1$*).*
3. *If* $b > 1$, *then* $\log_b(b') \geq \log_a(a')$.

In the direction of generalizing $mc1$–functions into mc–functions we can find the following remark.

Remark 5.4 ([64])**.** *If f is a $mc1$–function and $C > 1$ then $g(x) = Cf(x)$ is a mc–function (but not $mc1$).*

The problem of the characterization of the mc-functions which are not $mc1$ remains open, but a natural approach would be to relate both classes. We can then find the following natural question:

Problem 5.5 ([64])**.** *If f is a mc–function, is $g(x) = \frac{f(x)}{f(1)}$ a $mc1$–function?*

That was negatively answered with the following example.

Example 5.6. [64] *The function $f(x) = x + 1$ is a mc–function but $g(x) = \dfrac{f(x)}{2}$ is not $mc1$.*

In section 5.2 we shall focus on the study of certain features of mc–functions in order to see some of the algebraic properties of the set $\mathcal{MC}$. In the following sections, we shall use those properties to describe the algebraic structure of this set and we provide an example of a discontinuous mc-function, which leads to the conclusion that the mc-property does not imply continuity. We will then focus our attention to the question of whether continuous mc-functions may be all obtained via basic manipulations of $mc1$-functions. Some results regarding algebraic genericity are also presented and open questions will be posed in the last section. This work is a natural continuation of [64].

5.2 Some algebraic properties of the set $\mathcal{MC}$

Before stating the main points of this section, let us mention a well-known result that will be crucial in some of the proofs presented here. The following lemma can be found in [46] and it is very well known, however we include it here for self-containment.

Lemma 5.7. *For $1 \le p \le \infty$, let ℓ_p denote the space*

$$\ell_p = \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

endowed with the usual norm, which we denote by $\|\cdot\|_p$. Then, ℓ_p may be continuously embedded into ℓ_q for $1 \le p \le q \le \infty$.

In particular, for every $1 \le p \le q \le \infty$ and $x \in \ell_p$, $\|x\|_q \le \|x\|_p$

A natural, and straightforward, consequence of the previous lemma is the following result.

Corollary 5.8. *If a, b are positive numbers and $\mu \ge 1$, then $a^\mu + b^\mu \le (a+b)^\mu$.*

The first algebraic property of $\mathcal{MC}$ indicates that this set is closed under addition:

Lemma 5.9. *If f and g are mc–functions, then $h(x) = f(x) + g(x)$ in a mc–function as well.*

Proof. Let f, g be mc-functions, $x, y \in (0, \infty)$ and $\mu \ge 1$, then

$$\begin{aligned}
(f+g)(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}}) + g(x^\mu y^{\frac{1}{\mu}}) \le f(x)^\mu f(y)^{\frac{1}{\mu}} + g(x)^\mu g(y)^{\frac{1}{\mu}} \\
&\le f(x)^\mu (g(y) + f(y))^{\frac{1}{\mu}} + g(x)^\mu (f(y) + g(y))^{\frac{1}{\mu}} \\
&= (f(x)^\mu + g(x)^\mu)(g(y) + f(y))^{\frac{1}{\mu}} \\
&\le \big(f(x) + g(x)\big)^\mu (g(y) + f(y))^{\frac{1}{\mu}} \\
&= [(f+g)(x)]^\mu [(f+g)(y)]^{\frac{1}{\mu}}
\end{aligned}$$

Therefore, $f + g$ is a mc–function. $\square$

Corollary 5.10. *If f is a mc–function and $c \ge 1$, then $g(x) = f(x) + c$ is a mc–function as well.*

For the case $g(x) = f(x) + c$ with $c \in (0,1)$ we will study first the case where f is a $mc1$-function.

Lemma 5.11. *If f is a $mc1$–function and $f(x) \geq 1$ for every $x \in (0,\infty)$, then $g(x) = f(x) + c$ is a mc-function for $0 < c < 1$.*

Proof. We know that f is decreasing in $(0,1)$ and increasing in $(1,\infty)$, so we can proof that:

$$f(x)^\mu + c \leq (f(x) + c)^\mu \tag{5.3}$$

If we define $h(x) = (f(x) + c)^\mu - f(x)^\mu - c$, then $h'(x) = \mu f'(x)[f(x) + c)^{\mu-1} - f(x)^{\mu-1}]$. Due to the monotonicity of f we have that $h(x)$ has an absolute minimum in $x = 1$ but $h(1) = (1 + c)^\mu - (1 + c) \geq 0$ for every $\mu \geq 1$, therefore $h(x) \geq 0$ for every $x \in (0,\infty)$ and we have (5.3).

Lets see that g is a mc–function. Since $1 \leq f(y)^{1/\mu}$ we have

$$\begin{aligned} g(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}}) + c \leq f(x)^\mu f(y)^{\frac{1}{\mu}} + c \leq f(x)^\mu f(y)^{\frac{1}{\mu}} + cf(y)^{\frac{1}{\mu}} \\ &= f(y)^{\frac{1}{\mu}}(f(x)^\mu + c) \leq (f(y) + c)^{\frac{1}{\mu}}(f(x)^\mu + c) \\ &\leq (f(x) + c)^\mu (f(y) + c)^{\frac{1}{\mu}} = g(x)^\mu g(y)^{\frac{1}{\mu}}, \end{aligned}$$

where the latter inequality is due to (5.3). □

Notice that, according to Theorem 5.3, the situation described in Lemma 5.11 refers to the decreasing-increasing type of functions. For the other two types, we have the following:

Lemma 5.12. *If f is a monotone $mc1$–function then $g(x) = f(x) + c$ is not a mc–function for $0 < c < 1$.*

Proof. We are going to suppose that f is increasing (using that if f is a decreasing and mc–function, then $f(\frac{1}{x})$ is increasing and mc as well and we can generalize the result). So that we can use the characterization given in Theorem 5.3 with $a \in (0,1)$, $a' > 1$, $b < 1$ and $b' > 1$.

Next, taking

$$x = y = a^{\frac{\log \frac{1-c}{4}}{\log b}} \in (0,1)$$

and $\mu = 1$ we have that

$$\begin{aligned} g(xy) &= b^{\log_a(xy)} + c, \\ g(x)g(y) &= b^{\log_a(xy)} + c(b^{\log_a x} + b^{\log_a y} + c), \qquad \text{but} \\ b^{\log_a x} + b^{\log_a y} + c &= \frac{1-c}{4} + \frac{1-c}{4} + c = \frac{1+c}{2} < 1. \end{aligned}$$

Therefore $g(xy) > g(x)g(y)$ and $g(x)$ is not a mc–function. □

Hence, being closed under addition by a positive constant smaller than 1 depends on the function: $f(x) = x + c$ is not mc for any $c < 1$ but $g(x) = x + 1 + c$ is mc for every $c > 0$.

The following result tells us that $\mathcal{MC}$ has structure as an algebra:

Lemma 5.13. *If f and g are mc–functions, then $h(x) = f(x)g(x)$ is a mc–function.*

Proof. Let f an g be mc-functions and let $h(x) = f(x)g(x)$.

$$\begin{aligned} h(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}})g(x^\mu y^{\frac{1}{\mu}}) \le f(x)^\mu f(y)^{\frac{1}{\mu}} g(x)^\mu g(y)^{\frac{1}{\mu}} \\ &= (f(x)g(x))^\mu (f(y)g(y))^{\frac{1}{\mu}} = h(x)^\mu h(y)^{\frac{1}{\mu}} \end{aligned}$$

□

Corollary 5.14. *Let f be a mc-function and $\lambda \ge 1$. Then, λf is a mc–function as well.*

Lemma 5.15. *If f and g are mc-functions and f is increasing then $f(g(x))$ is a mc–function.*

Proof. $f(g(x^\mu y^{\frac{1}{\mu}})) \le f(g(x)^\mu g(y)^{\frac{1}{\mu}}) \le f(g(x))^\mu f(g(y))^{\frac{1}{\mu}}$ □

Corollary 5.16. *If f is a mc-function then f^n is a mc–function for every $n \in \mathbb{N}$.*

With respect to convergence of mc–functions, we have the following result:

Lemma 5.17. *Let $f(x) = \lim_{n\to\infty} f_n(x)$ with $f_n \in \mathcal{MC}$ (resp. $f_n \in \mathcal{MC}_1$) for every $n \in \mathbb{N}$. Then, $f(x)$ is a mc–function (resp. a mc1–function).*

Proof. Just notice that

$$f(x^\mu y^{1/\mu}) = \lim_{n\to\infty} f_n(x^\mu y^{1/\mu}) \le \lim_{n\to\infty} f_n(x)^\mu f_n(y)^{1/\mu} = f(x)^\mu f(y)^{1/\mu}.$$

If, furthermore, $f_n(x) = 1$ for every n, $f(1) = \lim_{n\to\infty} f_n(1) = 1$. □

The previous battery of examples, specially Lemma 5.9, Lemma 5.13 and Corollary 5.14, describes a very specific structure for the $\mathcal{MC}$ functions. We will need to introduce some considerations before carrying on with the results.

Definition 5.18. [72] *A cone is a set P endowed with two operations, addition and multiplication by positive scalars, which fulfill the usual properties (commutativity, associativity, existence of neutral element, etc.).*

If this set contains a linearly independent subset of infinite cardinality, the cone is said to be infinite. The dimension of the cone is the maximal possible cardinality of such a linearly independent set.

The theory of cones provides a different scoop to work in settings more general than the usual Banach or vector spaces. We refer the interesting reader to [72], where the authors make an introduction to the functional analysis on cones, overviewing the basic notions and definitions and studying how some of the classical results on Banach spaces (such as the Hahn-Banach Theorem) can be extended to this theory.

In particular, cones may be useful when being closed under scalar multiplication is not fulfilled but for positive scalars. In this direction, let us briefly recall the recently introduced notion of *lineability* (see, e.g., [2,7,9,20,25,37,45,85]).

Definition 5.19. *Let X be a vector space, $M \subseteq X$ and let μ any (finite or infinite) cardinal number. We say that M is $\mu-$lineable if $M \cup \{0\}$ contains a vector space of dimension μ. If X is a topological vector space, we shall say that M is $\mu-$spaceable if $M \cup \{0\}$ contains a closed vector space of dimension μ. We say that M is $\mu-$coneable if $M \cup \{0\}$ contains a cone of dimension μ.*

The idea behind Definition 5.19 is the search of algebraic structure on certain sets (whose elements usually verify some pathological property) of the greatest possible dimension. Depending on the set under study, some of the structures introduced in 5.19 will be allowed and some will not. For example (see, [67]):

Theorem 5.20. *There exist two closed cones of dimension $\mathfrak{c}$, C_1, C_2, with the property that, if $f \in C_1 \setminus \{0\}$ and $g \in C_2 \setminus \{0\}$, then:*

1. $f \in D[-1,1]$.
2. $g \in D[-1,1]$.
3. $f * g \notin D[-1,1]$.

Nevertheless, two analogous structures can not be found if we are requiring them to be closed vector spaces (so the spaceability analogue does not exist).

For the properties we are interested in this mansucript we can not search for cones, since being closed under multiplication by positive scalars is not ensured. Because of this, we have to consider another definition.

Definition 5.21. *Let X be a vector space and $M \subseteq X$. We say that M is an infinite dimensional* **truncated cone** *if M fulfills the following properties:*

1. *M is closed under addition.*
2. *M is closed under multiplication by scalar no less than 1.*
3. *M contains a set of infinite cardinality of linearly independent elements.*

The maximal possible cardinality of a set like in the property (3) is the **dimension** *of the truncated cone.*

If, furthermore, we can define a multiplication on X (satisfying the usual properties on itself and with respect to addition) and M is closed under products, then we say that M is an **algebraic truncated cone**. *In that case,*

1. *the* **linear dimension** *of the truncated cone will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on linearly independent elements.*
2. *The* **algebraic dimension** *will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on algebraic independent elements (that is, so that the only polynomial vanishing on them is the null polynomial).*

Trivially we obtain that the algebraic dimension is not greater than the linear dimension.

The relationship between truncated cones and cones, appart from the usual geometric one, can be represented in the following:

Proposition 5.22. *Let A be an infinite dimensional truncated cone of linear dimension μ. Then $A - A = \{a_1 - a_2 : a_i \in A\}$ is an infinite dimensional cone of the same dimension.*

Proof. Let $a_1, a_2, b_1, b_2 \in A$, $a \in A - A$ and $\lambda \geq 0$. Then

$$\lambda a = \lambda(a_1 - a_2) = [(\lambda+1)a_1 + a_2] - [(\lambda+1)a_2 + a_1] \in A - A,$$
$$(a_1 - a_2) + (b_1 - b_2) = (a_1 + b_1) - (a_2 + b_2) \in A - A.$$

Let next $\{f_i : i \in \Gamma\} \subseteq A$ be a linearly independent set of cardinality μ, $i^0, i^{(1)} \in \Gamma$ be different elements and $h : (\Gamma \setminus \{i^{(0)}\}) \times (\Gamma \setminus \{i^{(1)}\}) \to \Gamma$ be a bijection. If we define the set

$$B = \left\{ f_{h(i,i^{(0)})} - f_{h(i^{(1)},i)} : i \in \Gamma \setminus \{i^{(0)}, i^{(1)}\} \right\}$$

then $\mathrm{card}(B) = \mu$ and we claim that $B \subseteq A - A$ is a linearly independent set: Indeed, assume $\lambda_k \in \mathbb{R}$, $i_k \in \Gamma \setminus \{i^{(0)}, i^{(1)}\}$ for $1 \leq k \leq n$. Then, we notice that the elements $h(i_k, i^{(0)})$, $h(i^{(1)}, i_k) \in \Gamma$ are all different and therefore

$$0 = \sum_{k=1}^{n} \lambda_k \left(f_{h(i_k,i^{(0)})} - f_{h(i^{(1)},i_k)} \right) = \sum_{k=1}^{n} \lambda_k f_{h(i_k,i^{(0)})} - \sum_{k=1}^{n} \lambda_k f_{h(i^{(1)},i_k)}$$

leads to the conclusion $\lambda_k = 0$ for every $1 \leq k \leq n$. □

Let us come back to our object of study and focus our attention on continuous mc–functions. We can summarize the main results of this section so far in the following theorem:

Theorem 5.23. *Let $A = C(0,\infty) \cap \mathcal{MC}$. Then,*

1. *A is closed for the compact-open topology.*
2. *A is an algebraic truncated cone.*

The question that we may ask ourselves is if the truncated cone from Theorem 5.23 is of infinite dimension and, in which case, what is the maximal dimension we can consider.

Theorem 5.24. *The set A defined in Theorem 5.23 is a truncated cone of algebraic dimension $\mathfrak{c}$ (the continuum), that is, of the largest possible dimension.*

Proof. Let us define, for $b \in (0,1)$,

$$f_b(x) = \begin{cases} b^{\log_{1/2}(x)} & \text{if } 0 < x \leq 1, \\ \left(\frac{1}{b}\right)^{\log_2(x)} & \text{if } x > 1. \end{cases}$$

Then, by Theorem 5.3, f_b is a $mc1$–function for every $b \in (0,1)$. Let us show, to finish with, that this set is algebraically independent. Indeed, let $\lambda_i \in \mathbb{R}$, $b_j \in (0,1)$ and $k_{i,j} \in \mathbb{N}$ for $1 \le i \le n$ and $1 \le j \le m$. We define

$$g(x) = \sum_{i=1}^{n} \prod_{j=1}^{m} f_{b_j}(x)^{k_{i,j}}.$$

Assume then that $g(x) = 0$. In particular, evaluating in $g(\frac{1}{2^s})$ for $0 \le s \le n-1$ we obtain

$$\begin{aligned} 0 = g(1) &= \sum_{i=1}^{n} \lambda_i \\ 0 = g\left(\frac{1}{2}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}} \\ 0 = g\left(\frac{1}{2^2}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{2k_{i,1}} b_2^{2k_{i,2}} \cdot \ldots \cdot b_m^{2k_{i,m}} \\ &= \sum_{i=1}^{n} \lambda_i \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)^2 \end{aligned} \tag{5.4}$$

and so on, also having that:

$$\begin{aligned} 0 = g\left(\frac{1}{2^{n-1}}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{(n-1)k_{i,1}} b_2^{(n-1)k_{i,2}} \cdot \ldots \cdot b_m^{(n-1)k_{i,m}} \\ &= \sum_{i=1}^{n} \lambda_i \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)^{n-1}. \end{aligned} \tag{5.5}$$

Denote $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $h_i = \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_1^2 & h_2^2 & h_3^2 & \cdots & h_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1^{n-1} & h_2^{n-1} & h_3^{n-1} & \cdots & h_n^{n-1} \end{bmatrix}.$$

Then, the system given by (5.4) and (5.5) may be summarized as $A\Lambda^T = 0$, where A is a Van der Monde type matrix and therefore it is invertible, leading us to conclude $\Lambda = (0, \ldots, 0)$ □

5.3 Discontinuous mc–functions

In this section we focus on the question whether every mc-function is continuous or not. The following example shows that there are mc-functions that are not continuous.

Example 5.25. $f(x) = \begin{cases} 2 & if \quad x < 1 \\ 4 & if \quad x \geq 1 \end{cases}$ *is a mc–function which is not continuous.*

Proof. $f(x^\mu y^{\frac{1}{\mu}}) \leq 4 \leq 2^{(\mu + \frac{1}{\mu})} \leq f(x)^\mu f(y)^{\frac{1}{\mu}}$ □

Following Example 5.25, we generalize it in our next result:

Theorem 5.26. *Let $\alpha > 1$, $\alpha < \beta \leq \alpha^2$ and f be a mc1–function. Define the function*

$$g(x) = \begin{cases} \alpha f(x) & \textit{if } 0 < x \leq 1, \\ \beta f(x) & \textit{if } x > 1. \end{cases}$$

Then g is a discontinuous mc–function.

Proof. It is trivial to prove that g is discontinuous, since

$$\lim_{x \to 1^-} g(x) = \alpha < \beta = \lim_{x \to 1^+} g(x).$$

Let $x, y > 0$ and $\mu > 0$.

Assume first that $x^\mu y^{1/\mu} \leq 1$. Then

$$\begin{aligned} g(x^\mu y^{1/\mu}) &= \alpha f(x^\mu y^{1/\mu}) \leq \alpha f(x)^\mu f(y)^{1/\mu} \\ &\leq \alpha^{\mu + 1/\mu} f(x)^\mu f(y)^{1/\mu} = [\alpha f(x)]^\mu [\alpha f(y)]^{1/\mu} \\ &\leq g(x)^\mu g(y)^{1/\mu}. \end{aligned}$$

If now $x^\mu y^{1/\mu} > 1$ and $x, y > 1$, then $g(x^\mu y^{1/\mu}) \leq g(x)^\mu g(y)^{1/\mu}$, analogously as in the previous case.

If $x \leq 1$ (without loss of generality), then

$$\begin{aligned} g(x^\mu y^{1/\mu}) &= \beta f(x^\mu y^{1/\mu}) \leq \alpha^2 f(x^\mu y^{1/\mu}) \\ &\leq \alpha^{\mu + 1/\mu} f(x)^\mu f(y)^{1/\mu} = [\alpha f(x)]^\mu [\alpha f(y)]^{1/\mu} \\ &\leq g(x)^\mu g(y)^{1/\mu}. \end{aligned}$$

□

The functions defined in Theorem 5.26 are all discontinuous at $x = 1$. Concerning other points of discontinuity, we may prove the following result:

Proposition 5.27. *Let $x_0 > 0$. Then, there exists a mc–function that is discontinuous at x_0*

Proof. Assume first $x_0 < 1$ and let f be an increasing $mc1$–function and g be a function from Theorem 5.26. Then,

$$h_{x_0}(x) = g\Big(\frac{1}{f(x_0)}f(x)\Big)$$

is a mc–function which is discontinuous at x_0.

If now $x_0 > 1$, then we may just define

$$h^{(x_0)}(x) = h_{\frac{1}{x_0}}\Big(\frac{1}{x}\Big).$$

□

Even though we can not ensure that the set of discontinuous mc–functions is a truncated cone (being closed under multiplication is not guaranteed), we can take the idea from Definition 5.19 and search for certain algebraic structures:

Proposition 5.28. *The set*

$$A = \{f : (0,\infty) \to [0,\infty) : \ f \text{ is a discontinuous mc–function}\}$$

contains an algebraic truncated cone of algebraic dimension $\mathfrak{c}$.

Proof. Similarly as in Theorem 5.24, we define, for $b \in (0,1)$,

$$f_b(x) = \begin{cases} 2b^{\log_{1/2}(x)} & \text{if } 0 < x \le 1, \\ 4\left(\frac{1}{b}\right)^{\log_2(x)} & \text{if } x > 1. \end{cases}$$

Each f_b is a discontinuous mc–function, by means of Theorem 5.26, and they form a linearly independent set (the proof follows similarly as the proof in Theorem 5.24). We can just define the following truncated cone

$$T = \left\{\lambda \prod_{i=1}^{n} f_{b_i}(x) \ : \ \lambda \ge 1,\, n \in \mathbb{N}\, b_1,\dots,b_n \in (0,1)\right\},$$

finishing the proof. □

5.4 Other continuous mc–functions

Multiplicative convexity is closed under the majority of the algebraic operations, similarly as continuity. In this section we will examine whether the set of $mc1$–functions can serve as a system of generators of the set

$$A = C(0,\infty) \cap \mathcal{MC},$$

that is, if we can obtain every element of the set A by means of algebraic operations over elements from $\mathcal{MC}_1$.

Lemma 5.29. *The function $h(x) = f(x) + c$ with $f \in \mathcal{MC}_1$, $f(x) \geq 1$ for every $x \in (0,\infty)$ and $c \in (0,1)$ verifies that $h \in A$ but can not be set as $\sum_{n=1}^{m} \lambda_n f_n(x)$ with $f_n \in \mathcal{MC}_1$ and $\lambda_n \in [1,\infty)$.*

Proof. We have shown in Lemma 5.11 that if f is a $mc1$–function that satisfies $f(x) \geq 1$ for every $x \in (0,\infty)$ and $c \in (0,1)$, then $h(x) = f(x)+c$ is also a mc–function. Since f is continuous we have that $h \in A$.

Notice that $h(1) = 1 + c < 2$. If $m > 1$, $\lambda_1, \ldots, \lambda_m \geq 1$ and $f_1, \ldots, \lambda_m$ are $mc1$–functions, then $\sum_{n=1}^{m} \lambda_n f_n(x) > 2$, so that if $h(x) = \sum_{n=1}^{m} \lambda_n f_n(x)$ then $m = 1$ and the function h will be a product of a $mc1$–function and a constant bigger than 1. Let see that the function $g(x) = \frac{f(x)+c}{1+c}$ is not $mc1$. Taking $\mu = 1$, $x = y = a^{log_b 2} \in (0,1)$.

$$\begin{aligned}
g(x)g(y) &= \left(\frac{f(x)+c}{1+c}\right)^2 = \left(\frac{b^{log_a a^{log_b 2}} + c}{1+c}\right)^2 = \left(\frac{2+c}{1+c}\right)^2 \\
&= \frac{c^2+4c+4}{c^2+2c+1} \\
g(xy) &= \frac{f(xy)+c}{1+c} = \frac{b^{log_a (a^{log_b 2})}+c}{1+c} \\
&= \frac{b^{2log_b 2}+c}{1+c} = \frac{4+c}{1+c} = \frac{c^2+5c+4}{c^2+2c+1}
\end{aligned}$$

Since $g(xy) > g(y)g(x)$, $g \notin \mathcal{MC}_1$.

$\square$

5.5 Open questions

We finish this work by posing some open directions of study in order to continue this ongoing research theory of multiplicative convex functions.

1. The cardinality of the set $\{f : \mathbb{R} \to \mathbb{R}\}$ is $2^{\mathfrak{c}}$. What is the cardinality of $\mathcal{MC}$?

2. In Proposition 5.28 we proved the existence of an algebraic truncated cone of algebraic dimension $\mathfrak{c}$ contained in $\mathcal{MC} \smallsetminus C(0,\infty)$. What is the maximum possible algebraic dimension of such a truncated cone (if it is greater than $\mathfrak{c}$)?

3. Is $\mathcal{MC} \smallsetminus C(0,\infty)$ a truncated cone itself? That is, is $\mathcal{MC} \smallsetminus C(0,\infty)$ closed under addition and multiplication?

4. What can be said about the general behavior of not continuous mc–functions? Can they be of the three types described in [64] for $mc1$–functions (monotone or decreasing-increasing type)?

5. What can be said about the set of points of discontinuity of a multiplicative convex function, in general?

Chapter 6

Injectiveness and discontinuity of multiplicative convex functions

6.1 Introduction

In chapter 5 it was shown that the distinction between the sets $\mathcal{MC}$ and $\mathcal{MC}1$ (namely, $f(1) = 1$) was crucial. In fact, the condition $f(1) = 1$ suffices to completely describe the set $\mathcal{MC}1$:

Theorem 6.1. [64] *Let $f : (0, \infty) \to [0, \infty)$. Then, f is a mc1–function if and only if it can be written of the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{6.1}$$

where a, b, a' and b' satisfy the following conditions:

1. $0 < a < 1$ *and* $a' > 1$.
2. *If* $b < 1$, *then* $\log_b(b') \leq \log_a(a') < 0$ *(which, in particular, implies* $b' > 1$*).*
3. *If* $b > 1$, *then* $\log_b(b') \geq \log_a(a')$.

On the other hand, requiring $f(1) > 1$ (for a mc–function can not have $f(1) < 1$) implied a huge difference with respect to the previous situation, where everything was under control: for one thing, $\mathcal{MC}$ is closed under addition, product, multiplication by a scalar no smaller than 1 and composition (if the first function in the composition is not decreasing), but it was also proved the existence of discontinuous mc–functions:

Theorem 6.2. [65] *Let $\alpha > 1$, $\alpha < \beta \leq \alpha^2$ and f be a mc1−function. Define the function*

$$g(x) = \begin{cases} \alpha f(x) & \text{if } 0 < x \leq 1, \\ \beta f(x) & \text{if } x > 1. \end{cases}$$

Then g is a discontinuous mc−function.

The following theorem can be used to prove the existence of a mc−function which is discontinuous at any given point. We include the constructive proof for the sake of completeness.

Proposition 6.3. [65] *Let $x_0 > 0$. Then, there exists a mc−function that is discontinuous at x_0*

Proof. Assume first $x_0 < 1$ and let f be a $mc1$−function and g be an increasing discontinuous function from Theorem 6.2. Then,

$$h_{x_0}(x) = g\Big(\frac{1}{f(x_0)} f(x)\Big)$$

is a mc−function which is discontinuous at x_0.

If now $x_0 > 1$, then we may just define

$$h^{(x_0)}(x) = h_{\frac{1}{x_0}}\Big(\frac{1}{x}\Big).$$

$\square$

Given the aim of this chapter, we shall introduce some notation to denote the set of points of discontinuity of functions.

Definition 6.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We shall denote*

$$\mathfrak{D}(f) = \{x \in \mathbb{R} : f \textit{ is discontinuous on } x\}.$$

This set has some regular algebraic properties. For example, what follows is a standard Calculus exercise:

Proposition 6.5. *Let f and g be two functions so that $\mathfrak{D}(f) \cap \mathfrak{D}(g) = \varnothing$. Then,*

$$\mathfrak{D}(f+g) = \mathfrak{D}(fg) = \mathfrak{D}(f) \cup \mathfrak{D}(g).$$

In general, we can not ensure that, given a mc–function f, αf is also a mc–function for $\alpha \geq 0$ but for $\alpha \geq 1$. Because of this, we have to consider the following definition from chapter 5.

Definition 6.6. [65] *Let X be a vector space and $M \subseteq X$. We say that M is an infinite dimensional* **truncated cone** *if M fulfills the following properties:*

1. *M is closed under addition.*
2. *M is closed under multiplication by scalar no less than 1.*
3. *M contains a set of infinite cardinality of linearly independent elements.*

The maximal posible cardinality of a set like in the property (3) is the **dimension** *of the truncated cone.*

If, furthermore, we can define a multiplication on X (satisfying the usual properties on itself and with respect to addition) and M is closed under products, then we say that M is an **algebraic truncated cone***. In that case,*

1. *the* **linear dimension** *of the truncated cone will be the maximal posible cardinality so that there exists a subset of such cardinality and consisting on linearly independent elements.*
2. *The* **algebraic dimension** *will be the maximal posible cardinality so that there exists a subset of such cardinality and consisting on algebraic independent elements (that is, so that the only polynomial vanishing on them is the null polynomial).*

Trivially we obtain that the algebraic dimension is not greater than the linear dimension.

In this chapter there are two main aims: the first one is to complete a description of the set of general mc–functions in a similar way as how the set of $mc1$–functions was described in 4 (where, before succeeding in giving a complete characterization of the set, the authors proved that $mc1$–functions were continuous and either monotone or not increasing-not decreasing). The results in Section 6.2 will lead to the conclusions that a general mc–function is of one of the mentioned two behaviors (monotone or not increasing-not decreasing), which in particular implies that the set $\mathcal{MC}$ is of cardinality $\mathfrak{c}$ and that a mc–function is continuous but a set at most countable.

The second aim is to provide examples of mc–functions which are discontinuous over an infininte set of points. The main result in Section 6.3 is held by some propositions and lemmas encompassed in Number Theory.

The following two classical results will be of great importance for those two goals.

Theorem 6.7. [54, Darboux-Froda's Theorem] *The set of points of discontinuity of a monotone function is at most countable.*

Theorem 6.8. [78] *The cardinality of the set of monotone functions is* $\mathfrak{c}$.

Section 6.4 follows the line of action shown in 5 and focuses on the existence of certain algebraic structures whose non-zero elements fulfill some properties.

6.2 Study of the injectiveness of a mc-function

Theorem 6.9. *Let f be a mc–function. Then, there is no $0 < \theta_1 < \theta_2$ so that $f(\theta_1) = f(\theta_2) = c < 1$. In other words, if f is a multiplicative convex function and $A = f^{-1}(0,1)$, then $f_{|A}$ is injective.*

Proof. Assume first that we can find such points and that $1 < \theta_1, \theta_2$. We claim that we must have $\theta_2 < \theta_1^4$. Indeed, if $\theta_2 \geq \theta_1^4$, then

$$\begin{aligned}
&1 = \log_{\theta_2}(\theta_2) \geq 4\log_{\theta_2}(\theta_1) \quad \text{so} \\
&0 \leq 1 - 4\log_{\theta_2}(\theta_1)
\end{aligned}$$

and therefore we can define

$$\mu = \frac{1 + \sqrt{1 - 4\log_{\theta_2}(\theta_1)}}{2} > 0,$$

which is a solution to $\theta_1^{1/\mu}\theta_2^{\mu} = \theta_2$. This allows us to conlcude

$$f(\theta_2) = f(\theta_1^{1/\mu}\theta_2^{\mu}) \leq f(\theta_1)^{1/\mu} f(\theta_2)^{\mu} = f(\theta_2)^{\mu + 1/\mu},$$

finding a contradiction to the condition $f(\theta_2) < 1$ and proving our claim.

Define next

$$g(x) = f\Big(\frac{\theta_1^4}{\theta_2}x\Big).$$

Then,

$$g\Big(\frac{\theta_1\theta_2}{\theta_1^4}\Big) = g\Big(\frac{\theta_2^2}{\theta_1^4}\Big),$$

so by our previous claim,

$$\begin{aligned}
&\frac{\theta_2^2}{\theta_1^4} < \Big(\frac{\theta_1\theta_2}{\theta_1^4}\Big)^4 = \frac{\theta_1^4\theta_2^4}{\theta_1^{16}} \quad \text{from which} \\
&\theta_1^8 < \theta_2^2,
\end{aligned}$$

reaching a contradiction.

If $0 < \theta_1 < 1 < \theta_2$, then $\log_{\theta_2}(\theta_1) < 0$ and therefore such number

$$\mu = \frac{1 + \sqrt{1 - 4\log_{\theta_2}(\theta_1)}}{2}$$

is always well-defined and positive, so we can again reach the contradiction $f(\theta_2) \leq f(\theta_2)^{\mu + 1/\mu}$.

Any other situation may be reduced to one of the previous two via the auxiliary function $h(x) = f(1/x)$. □

Theorem 6.10. *There is no mc-function, f, so that we can find $1 < \theta_1 < \theta_2$ such that* $\max\{f(1), f(\theta_2)\} < f(\theta_1)$.

Proof. If we choose

$$\mu = \log_{\theta_2}(\theta_1) < 1$$

then it must be

$$f(\theta_1) \leq f(\theta_2)^{\mu} f(1)^{1/\mu}.$$

Let us choose $0 < a, b < 1$, $a', b' > 1$, and $\alpha > 1$ so that

$$\begin{aligned} \log_b(b') &= \log_a(a'), \\ \alpha b'^{\log_{a'}(\theta_1)} f(\theta_1) &> f(1), \\ b'^{\log_{a'}\left(\frac{\theta_1}{\theta_2}\right)} f(\theta_1) &> f(\theta_2), \\ \alpha^{2-(2\mu+1/\mu)} &> \frac{f(\theta_2)^{\mu} f(1)^{1/\mu}}{f(\theta_1)}. \end{aligned} \tag{6.2}$$

Then, if we define the function

$$g(x) = \begin{cases} \alpha b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ \alpha^2 b'^{\log a'(x)} & \text{if } x > 1 \end{cases}$$

we can apply Theorems 6.1 and 6.2 to conclude that g is a (discontinuous) mc–function. Hence, $h(x) = f(x)g(x)$ would also be a mc–function. Now, we can deduce the following chain of equivalent inequalities:

$$\begin{aligned}
\alpha^{2-\left(2\log_{\theta_2}(\theta_1)+1/\log_{\theta_2}(\theta_1)\right)} &> \frac{f(\theta_2)^{\log_{\theta_2}(\theta_1)} f(1)^{1/\log_{\theta_2}(\theta_1)}}{f(\theta_1)}, \\
\alpha^2 f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} \left[\alpha f(1)\right]^{1/\log_{\theta_2}(\theta_1)}, \\
\alpha^2 b^{\log_a(\theta_1)} f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} b^{\log_a(\theta_1)} \left[\alpha f(1)\right]^{1/\log_{\theta_2}(\theta_1)}, \\
\alpha^2 b^{\log_a(a')\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} b^{\log_a(a')\log_{a'}(\theta_2)\log_{\theta_2}(\theta_1)} \\
&\quad \cdot \left[\alpha f(1)\right]^{1/\log_{\theta_2}(\theta_1)}, \\
\alpha^2 b^{\log_b(b')\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 b^{\log_b(b')\log_{a'}(\theta_2)} f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} \\
&\quad \cdot \left[\alpha f(1)\right]^{1/\log_{\theta_2}(\theta_1)}, \\
\alpha^2 b'^{\log_{a'}(\theta_1)} f(\theta_1) &> \left[\alpha^2 b'^{\log_{a'}(\theta_2)} f(\theta_2)\right]^{\log_{\theta_2}(\theta_1)} \left[\alpha f(1)\right]^{1/\log_{\theta_2}(\theta_1)}, \\
h(\theta_1) &> h(\theta_2)^{\mu} h(1)^{1/\mu}.
\end{aligned}$$

On the other hand, applying the requisites over a, a', b, b' and α from equation (6.2),

$$\begin{aligned}
h(\theta_1) = \alpha^2 b'^{\log_{a'}(\theta_1)} f(\theta_1) &> \max\{\alpha^2 b'^{\log_{a'}(\theta_2)} f(\theta_2), \alpha f(1)\} \\
&= \max\{f(\theta_2), f(1)\},
\end{aligned}$$

reaching a contradiction.

□

Corollary 6.11. *Let f be an mc–function and $\theta_1 < \theta_2 < \theta_3$. Then, it can not be $\max\{f(\theta_1), f(\theta_3)\} < f(\theta_2)$.*

Proof. Assume otherwise. If $\theta_1 = 1$, then this corollary is just Theorem 6.10. For $1 < \theta_1$, consider $g_1(x) = f(\theta_1 x)$. For $\theta_3 > 1$, consider $g_2(x) = f\left(\frac{\theta_3}{x}\right)$.

For $\theta_3 < 1$, consider first $\tilde{g}_3(x) = f\left(\frac{1}{x}\right)$ and then $g_3(x) = \tilde{g}_3\left(\frac{1}{\theta_3}x\right)$.

In any case, g_i would be a mc–function which contradicts Theorem 6.10 □

Corollary 6.12. *Let f be a mc–convex function which is not locally constant (that is, for every real number x and interval I with x in I we can find $x_1 \neq x_2$ also in I so that $f(x_1) \neq f(x_2)$).*

Then, f is at most 2-injective (meaning that for every real number y $\#f^{-1}(y) \leq 2$).

Proof. Let y be a real number and x_1, x_2, x_3 be so that $f(x_1) = f(x_2) = f(x_3)$. Because f is not locally constant, we can find $x_1 < x^{(1)} < x_2 < x^{(2)} < x_3$ so that $f(x^{(1)}) \neq f(x_2) \neq f(x^{(2)})$ and this is a contradiction with Theorem 6.10. □

The following Corollary generalizes Theorem 4.2 from 4:

Corollary 6.13. *Let f be a mc–function (continuous or not). Then, f is either monotone or not increasing-not decreasing.*

Proof. This Corollary is easily proved if we consider the situation where f is not monotone and we apply Theorem 6.10 repeatedly. □

In conclusion, Theorem 6.10 and its corresponding corollaries, in combination with Theorem 6.8, lead to the following Theorem, which answers several questions posted in chapter 5:

Theorem 6.14. *The set $\mathcal{MC}$ has cardinality $\mathfrak{c}$.*

In particular, this implies that the algebraic structures considered in chapter 5 are of the greatest possible dimension.

6.3 On the set of points of discontinuity of a mc–function

It is obvious that, if $A = \{x_1, x_2, \dots, x_n\}$ is any finite set and, for $1 \leq k \leq n$, f_{x_k} is a mc–function that is discontinuous at x_k, then $g(x) = f_{x_1}(x) + f_{x_2}(x) + \dots + f_{x_n}(x)$ is discontinuous over the set A. The question we will focus on now is whether it is posible to find a mc–function which is discontinuous over an infinite set. We remark that, taking Theorem 6.3 and Corollary 6.13 into account, the set A must be countable.

The main result of this section is as follows:

Theorem 6.15. *Let $\mathcal{X} = \{x_n\}_{n=1}^{\infty} \subseteq (0,1)$ be a decreasing sequence. Then, there exists a mc–function which is discontinuous on $\mathcal{X}$.*

Before giving the proof, we will need some preliminary lemmas and definitions.

Lemma 6.16. *Let $\{f_\alpha\}_{\alpha\in\Gamma} \subseteq \mathcal{MC}$ and assume that, for $x > 0$,*

$$\{f_\alpha(x) : \alpha \in \Gamma\}$$

is a bounded set.

Then,

$$g(x) = \sup_{\alpha\in\Gamma}\{f_\alpha(x)\}$$

is a mc–function.

Proof. We shall use the following fact, if $\{y_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ are bounded sequences, then

$$\lim_{n\to\infty}\sup_{k\geq n} y_k z_k \leq \Big(\lim_{n\to\infty}\sup_{k\geq n} y_k\Big)\Big(\lim_{n\to\infty}\sup_{k\geq n} z_k\Big).$$

Let now $y, z, \mu > 0$. Then, we can find a sequence $\{\alpha_n\}_{n=1}^\infty \subseteq \Gamma$ so that $g(y^\mu z^{1/\mu}) = \lim_{n\to\infty} f_{\alpha_n}(y^\mu z^{1/\mu})$. Now,

$$\begin{aligned}
g(y^\mu z^{1/\mu}) &= \lim_{n\to\infty} f_{\alpha_n}(y^\mu z^{1/\mu}) \\
&\leq \lim_{n\to\infty}\sup_{k\geq n} f_{\alpha_k}(y^\mu z^{1/\mu}) \\
&\leq \lim_{n\to\infty}\sup_{k\geq n}\Big[f_{\alpha_k}(y)^\mu f_{\alpha_k}(z)^{1/\mu}\Big] \\
&\leq \Big[\lim_{n\to\infty}\sup_{k\geq n} f_{\alpha_k}(y)^\mu\Big]\Big[\lim_{n\to\infty}\sup_{k\geq n} f_{\alpha_k}(z)^{1/\mu}\Big] \\
&\leq g(y)^\mu g(z)^{1/\mu}.
\end{aligned}$$

□

Definition 6.17. *We will define the following elements, for $n \geq 1$:*

$$\begin{aligned}
a_n &= \Big(\frac{x_n}{x_{n+1}}\Big)^{x_{n+1}}, \\
\alpha_n &= \prod_{k\geq n} a_k, \\
\beta_n &= \prod_{k\geq n} a_k^2.
\end{aligned}$$

Lemma 6.18. *For every $n \geq 1$, α_n and β_n are well-defined.*

Proof. We will just show that β_1 is well-defined.

Indeed, the product that defines β_1 converges if and only if $\sum_{k=1}^{\infty} \log(a_k^2)$ converges. Now,

$$\begin{aligned}
\sum_{k=1}^{\infty} \log(a_k^2) &= 2 \sum_{k=1}^{\infty} \log\left[\left(\frac{x_k}{x_{k+1}}\right)^{x_{k+1}}\right] = 2 \sum_{k=1}^{\infty} x_{k+1} \log\left(\frac{x_k}{x_{k+1}}\right) \\
&= 2 \sum_{k=1}^{\infty} x_{k+1} \log\left(1 + \frac{x_k - x_{k+1}}{x_{k+1}}\right) \le 2 \sum_{k=1}^{\infty} x_{k+1} \frac{x_k - x_{k+1}}{x_{k+1}} \\
&= 2 \sum_{k=1}^{\infty} (x_k - x_{k+1}) \le 4x_1.
\end{aligned}$$

□

Lemma 6.19 (Steiner's problem, [89])**.** *The maximum of the function $f(x) = x^{1/x}$ is attained at $x = e$. In fact, the function f is increasing on $(0, e)$ and decreasing on (e, ∞)*

Lemma 6.20. *Let $n \ge 3$ and $1 \le m \le n - 2$. Then,*

$$\left(\frac{x_n}{x_m}\right)^{x_m} \prod_{k=m}^{n-1} a_k < \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k$$

Proof. Just notice that

$$\begin{aligned}
\left(\frac{x_n}{x_m}\right)^{x_m} \prod_{k=m}^{n-1} a_k &= \left(\frac{x_n}{x_m}\right)^{x_m} a_m \prod_{k=m+1}^{n-1} a_k \\
&= \left(\frac{x_n}{x_m}\right)^{x_m} \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k \\
&= \left(\frac{x_n}{x_m}\right)^{x_m} \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \left(\frac{x_{m+1}}{x_n}\right)^{x_{m+1}} \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k \\
&= \left(\frac{x_n}{x_m}\right)^{x_m - x_{m+1}} \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k \\
&< \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}} \prod_{k=m+1}^{n-1} a_k,
\end{aligned}$$

based on the fact that $\{x_k\}_{k=1}^{\infty}$ is a decreasing sequence. □

Lemma 6.21. *For every $n \ge 1$ and $m \ge n + 2$,*

$$\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}} \left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}} \cdot \ldots \cdot \left(\frac{x_{m-1}}{x_m}\right)^{x_m} > \left(\frac{x_n}{x_m}\right)^{x_m}$$

Proof. We will proceed via induction on m. If $m = n+2$, then

$$\begin{aligned}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}} &= \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}-x_{n+2}}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+2}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}} \\ &= \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}-x_{n+2}}\left(\frac{x_n}{x_{n+2}}\right)^{x_{n+2}} \\ &> \left(\frac{x_n}{x_{n+2}}\right)^{x_{n+2}}.\end{aligned}$$

Assuming the result is true for m, then

$$\begin{aligned}&\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\left(\frac{x_{n+1}}{x_{n+2}}\right)^{x_{n+2}}\cdot\ldots\cdot\left(\frac{x_{m-1}}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad> \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad= \left(\frac{x_n}{x_m}\right)^{x_m-x_{m+1}}\left(\frac{x_n}{x_m}\right)^{x_{m+1}}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \\ &\quad> \left(\frac{x_n}{x_{m+1}}\right)^{x_{m+1}},\end{aligned}$$

as desired. ∎

Corollary 6.22. *For every $n \geq 1$ and $m \geq n+2$,*

$$\alpha_m\left(\frac{x_n}{x_m}\right)^{x_m} < \alpha_{n+1}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}} \quad \textit{and}$$
$$\beta_m\left(\frac{x_n}{x_m}\right)^{x_m} < \beta_{n+1}\left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}$$

Theorem 6.23. *Let $y_n \to y_0$ and $\{f_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^{\mathbb{R}}$. Then,*

$$\sup_{k\in\mathbb{N}} \lim_{n\to\infty}\left\{\sup_{m\geq n} f_k(y_m)\right\} \leq \lim_{n\to\infty}\sup_{m\geq n}\left\{\sup_{k\in\mathbb{N}} f_k(y_m)\right\}.$$

Proof. Let $l, n \in \mathbb{N}$. We notice that

$$f_l(y_n) \leq \sup_{k\in\mathbb{N}} f_k(y_n).$$

Therefore,

$$\sup_{m\geq n} f_l(y_m) \leq \sup_{m\geq n}\left\{\sup_{k\in\mathbb{N}} f_k(y_m)\right\}$$

and hence

$$\lim_{n\to\infty}\left\{\sup_{m\geq n} f_l(y_m)\right\} \leq \lim_{n\to\infty}\sup_{m\geq n}\left\{\sup_{k\in\mathbb{N}} f_k(y_m)\right\}$$

for every $l \in \mathbb{N}$. Finally,

$$\sup_{k\in\mathbb{N}} \lim_{n\to\infty}\left\{\sup_{m\geq n} f_k(y_m)\right\} \leq \lim_{n\to\infty}\sup_{m\geq n}\left\{\sup_{k\in\mathbb{N}} f_k(y_m)\right\}.$$

∎

Corollary 6.24. *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions so that, for every $x \in \mathbb{R}$, $\{f_k(x) : k \geq 1\}$ is a bounded set and $\lim_{x\to x_0} f_k(x)$ exists for every $k \geq 1$. Define the function $g(x) = \sup_{k\geq 1} f_k(x)$. Then, if $\lim_{x\to x_0} g(x)$ also exists,*

$$\sup_{k\geq 1} \lim_{x\to x_0} f_k(x) \leq \lim_{x\to x_0} g(x) = \lim_{x\to x_0} \sup_{k\geq 1} f_k(x). \tag{6.3}$$

Remark 6.25. *The inequality in (6.3) cannot be turned into an equality: as a counterexample, we choose the functions*

$$f_k(x) = 1 - \left(\frac{1}{1+x}\right)^k,\ x > -1,\ k \geq 0.$$

Then, $\lim_{x\to 0} f_k(x) = 0$ for every $k \geq 1$, so that $\sup_{k\geq 1} \lim_{x\to 0} f_k(x)$. On the other hand, $\sup_{k\geq 1} f_k(x) = 1$ for every $k \geq 1$, $x > -1$, so that $\lim_{x\to 0} \sup_{k\geq 1} f_k(x) = 1$.

proof of Theorem 6.15. Define, given $m \in \mathbb{N}$, the following function:

$$f_m(t) = \begin{cases} \alpha_m \left(\frac{t}{x_m}\right)^{x_m} & \text{if } 0 < t \leq x_m, \\ \beta_m \left(\frac{t}{x_m}\right)^{x_m} & \text{if } t > x_m. \end{cases}$$

The function f_m is multiplicative convex and it is discontinuous at $t = x_m$, since it has been defined following the contruction from Proposition 6.3.

We will show that the function $g(t) = \sup\{f_n(t) : n \in \mathbb{N}\}$ is discontinuous on the set $\mathcal{X}$. First of all, for $t > 0$ the set $\{f_n(t) : n \in \mathbb{N}\}$ is bounded, since

$$\begin{aligned} &\alpha_n, \beta_n \leq \beta_1 \leq e^{4x_1}, \\ &t^{x_n} \leq \begin{cases} 1 & \text{if } 0 < t \leq 1, \\ t & \text{if } t > 1, \end{cases} \quad \text{for every } n \geq 1 \text{ and} \\ &x_n^{-x_n} \leq e^{1/e} \text{ because of Lemma 6.19.} \end{aligned}$$

Let $n, m \in \mathbb{N}$. Then,

$$f_m\left(x_n\right) = \begin{cases} \alpha_m \left(\frac{x_n}{x_m}\right)^{x_m} & \text{if } 1 \leq m \leq n, \\ \beta_m \left(\frac{x_n}{x_m}\right)^{x_m} & \text{if } m \geq n+1. \end{cases}$$

Notice that, if $m < n$

$$\begin{aligned}
\alpha_m\left(\frac{x_n}{x_m}\right)^{x_m} &= \left(\frac{x_n}{x_m}\right)^{x_m}\prod_{k\geq m} a_k \\
&= \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}}\left(\frac{x_{m+1}}{x_{m+2}}\right)^{x_{m+2}}\cdot\ldots \\
&\quad \ldots\cdot\left(\frac{x_{n-1}}{x_n}\right)^{x_n}\prod_{k\geq n}\left(\frac{x_k}{x_{k+1}}\right)^{x_{k+1}} \\
&= \left(\frac{x_n}{x_m}\right)^{x_m}\left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}}\left(\frac{x_{m+1}}{x_{m+2}}\right)^{x_{m+2}}\cdot\ldots\cdot\left(\frac{x_{n-1}}{x_n}\right)^{x_n}\alpha_n \\
&< \left(\frac{x_n}{x_{n-1}}\right)^{x_{n-1}}\left(\frac{x_{n-1}}{x_n}\right)^{x_n}\alpha_n \\
&< \alpha_n,
\end{aligned}$$

via Lemma 6.20.

If $m \geq n+2$, and using Corollary 6.22, we can prove that

$$\begin{aligned}
\beta_n &= \left(\frac{x_n}{x_{n+1}}\right)^{2x_{n+1}}\beta_{n+1} \\
&> \left(\frac{x_n}{x_{n+1}}\right)^{x_{n+1}}\beta_{n+1} \\
&> \left(\frac{x_n}{x_m}\right)^{x_m}\beta_m.
\end{aligned}$$

In conclusion, we can say that $g(x_n) < \beta_n$.

On the other hand,

$$\lim_{x\to x_n^+} f_m(x) = \begin{cases} \alpha_m\left(\frac{x_n}{x_m}\right)^{x_m} & \text{if } 1\leq m\leq n-1, \\ \beta_m\left(\frac{x_n}{x_m}\right)^{x_m} & \text{if } m\geq n. \end{cases}$$

With a similar argument as before, $\sup\left\{\lim_{x\to x_n^+} f_m(x) : m\geq 1\right\} = \beta_n$.

Hence, by Corollary 6.24

$$\begin{aligned}
\lim_{x\to x_n^+} g(x) &= \lim_{x\to x_n^+}\sup\{f_m(x) : m\geq 1\} \\
&\geq \sup\left\{\lim_{x\to x_n^+} f_m(x) : m\geq 1\right\} \\
&= \beta_n > g(x_n).
\end{aligned}$$

As a consequence, g is not continuous at x_n. $\square$

The following result complements Theorem 6.15 and shows that $\mathfrak{D}(g) = \mathcal{X}$:

Proposition 6.26. *The function g considered in the proof of Theorem 6.15 is continuous on $(0,\infty) \setminus \mathcal{X}$.*

Proof. Let f_m be the functions defined in the proof of Theorem 6.15, $m_0 \geq 1$ and $x_{m_0} < x < x_{m_0-1}$. Then,

$$f_m(x) = \begin{cases} \alpha_m \left(\frac{x}{x_m}\right)^{x_m} & \text{if } m \leq m_0 - 1, \\ \beta_m \left(\frac{x}{x_m}\right)^{x_m} & \text{if } m \geq m_0. \end{cases}$$

If $m \leq m_0 - 1$, then $x < x_m$ so that

$$\begin{aligned} & x_m^{x_{m+1}-x_m} \leq x^{x_{m+1}-x_m}, \quad \text{from which} \\ & \left(\frac{x_m}{x_{m+1}}\right)^{x_{m+1}} \left(\frac{x}{x_m}\right)^{x_m} \leq \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} \quad \text{and therefore} \\ & \alpha_m \left(\frac{x}{x_m}\right)^{x_m} \leq \alpha_{m+1} \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}}, \end{aligned}$$

leading us to conclude that

$$\sup\{f_m(x) : 1 \leq m \leq m_0 - 1\} = \alpha_{m_0-1} \left(\frac{x}{x_{m_0-1}}\right)^{x_{m_0-1}}.$$

If now $m \geq m_0$, then $x_m < x$, which implies that $x^{x_{m+1}-x_m} < x_m^{x_{m+1}-x_m}$. On the other hand, from $x_{m+1} < x_m$ we can deduce that

$$x_m^{x_{m+1}-x_m} < \frac{x_m^{2x_{m+1}-x_m}}{x_{m+1}^{x_{m+1}}},$$

which allows us to deduce that

$$x^{x_{m+1}-x_m} < \frac{x_m^{2x_{m+1}-x_m}}{x_{m+1}^{x_{m+1}}}.$$

This last inequality is equivalent to

$$\left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} < \left(\frac{x_m}{x_{m+1}}\right)^{2x_{m+1}} \left(\frac{x}{x_m}\right)^{x_m},$$

that is,

$$\beta_{m+1} \left(\frac{x}{x_{m+1}}\right)^{x_{m+1}} < \beta_m \left(\frac{x}{x_m}\right)^{x_m}.$$

As a consequence,

$$\sup\{f_m(x) : m \geq m_0\} = \beta_{m_0} \left(\frac{x}{x_{m_0}}\right)^{x_{m_0}}$$

and therefore

$$g(x) = \sup\{f_m(x) : m \geq 1\} = \max\left\{\alpha_{m_0-1}\left(\frac{x}{x_{m_0-1}}\right)^{x_{m_0-1}}, \beta_{m_0}\left(\frac{x}{x_{m_0}}\right)^{x_{m_0}}\right\}.$$

Hence, on $(x_m.x_{m-1})$, g can be expressed as the maximum of two continuous functions and, in conclussion, it is continuous over the interval.

If next $0 < x < x_0 = \lim_{n\to\infty} x_n$ (which in particular implies that $x_0 \neq 0$), then $f_m(x) = \alpha_m\left(\frac{x}{x_m}\right)^{x_m}$ and, same way as before, we would have

$$\alpha_m\left(\frac{x}{x_m}\right)^{x_m} \leq \alpha_{m+1}\left(\frac{x}{x_{m+1}}\right)^{x_{m+1}}.$$

Therefore,

$$g(x) = \lim_{m\to\infty} \alpha_m\left(\frac{x}{x_m}\right)^{x_m} = \left(\frac{x}{x_0}\right)^{x_0},$$

which again is a continuous function.

For the case $x = x_0$ we notice that $g(x_0) = 1$. Assume $\{y_n\}_{n=1}^{\infty}$ is such that $y_n \to x_0$ as $n \to \infty$. If $y_n < x_0$ then $g(y_n) = \left(\frac{y_n}{x_0}\right)^{x_0}$. On the other hand, if $y_n > x_0$ then we can find $\{m_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ so that $x_{m_n} \leq y_n < x_{m_n-1}$. As before,

$$g(y_n) = \max\left\{\alpha_{m_n-1}\left(\frac{y_n}{x_{m_n-1}}\right)^{x_{m_n-1}}, \beta_{m_n}\left(\frac{y_n}{x_{m_n}}\right)^{x_{m_n}}\right\}.$$

Since

$$\lim_{n\to\infty}\left(\frac{y_n}{x_0}\right)^{x_{m_0}} = \lim_{n\to\infty} \alpha_{m_n-1}\left(\frac{y_n}{x_{m_n-1}}\right)^{x_{m_n-1}} = \lim_{n\to\infty} \beta_{m_n}\left(\frac{y_n}{x_{m_n}}\right)^{x_{m_n}} = 1$$

we are able to conclude that $\lim_{n\to\infty} g(y_n) = g(x_0)$.

Finally, for the case $x > x_1$ we would obtain that $f_m(x) = \beta_m\left(\frac{x}{x_m}\right)^{x_m}$ and hence (again, similarly as before)

$$g(x) = \beta_1\left(\frac{x}{x_1}\right)^{x_1},$$

so g is continuous on x. □

Remark 6.27. *If $\mathcal{X} \subseteq (0,1)$ is an increasing sequence, then there exists a mc–function so that $\mathfrak{D}(f) = \mathcal{X}$.*

Indeed, we only need to change the definition of the elements a_n in Definition 6.17 as follows:

$$a_n = \left(\frac{x_{n+1}}{x_n}\right)^{x_n}.$$

Corollary 6.28. *If $\mathcal{X}$ is a monotone sequence, then we can find a mc–function which is discontinuous on $\mathcal{X}$.*

Proof. Let $\mathcal{X}_1 = \mathcal{X} \cap (0,1)$ and $\mathcal{X}_2 = \mathcal{X} \cap [1,\infty)$. We can find two functions f_1 and f_2 so that f_1 is discontinuous only on $\mathcal{X}_1$ and f_2 is discontinuous only on the set $\left\{\frac{1}{x} : x \in \mathcal{X}_2\right\}$.

The required function is then $f(x) = f_1(x) + f_2(\frac{1}{x})$. □

6.4 Algebraic structure on $\mathcal{MC} \setminus C(0,\infty)$

Theorem 6.29. *There exists an algebraic truncated cone of algebraic dimension $\mathfrak{c}$ every non-trivial algebraic combination of which is a mc–function which is discontinuous over an infinite set.*

Proof. Let us consider a $\mathbb{Q}$–linearly independent set of cardinality $\mathfrak{c}$, $\{a_\zeta : \zeta < \mathfrak{c}\} \subseteq (4,6)$. Then, for every $\zeta < \mathfrak{c}$ we consider a decreasing sequence, $\mathcal{X}_\alpha = \{x_{k,\zeta}\}_{k=1}^{\infty}$ converging to $\frac{1}{a_\zeta}$ and so that $x_{1,\zeta} = \frac{1}{4}$.

Using Theorem 6.15 for every $\zeta < \mathfrak{c}$ there exists a mc–function $\tilde{f}_\zeta$ which is discontinuous on $\mathcal{X}_\zeta$.

In particular, taking a look to the proof of Theorem 6.15, the function $\tilde{f}_\zeta$ is defined as $\tilde{f}_\zeta(x) = \sup\{f_{k,\zeta}(x) : k \in \mathbb{N}\}$, where

$$f_{k,\zeta}(x) = \begin{cases} \alpha_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} & \text{if } 0 < x \le x_{k,\zeta}, \\ \beta_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} & \text{if } x > x_{k,\zeta} \end{cases},$$

$$\alpha_{k,\zeta} = \prod_{l \ge n}\left(\frac{x_{l,\zeta}}{x_{l+1,\zeta}}\right),$$

$$\beta_{k,\zeta} = \prod_{l \ge n}\left(\frac{x_{l,\zeta}}{x_{l+1,\zeta}}\right)^2.$$

In particular, if $x > x_{1,\zeta}$ then

$$\tilde{f}_\zeta(x) = \sup\left\{\beta_{k,\zeta}\left(\frac{x}{x_{k,\zeta}}\right)^{x_{k,\zeta}} : k \in \mathbb{N}\right\}.$$

Because of Lemma 6.19 and since $\left\{\frac{1}{x_{k,\zeta}}\right\}_{k=1}^{\infty}$ is an increasing sequence with $\frac{1}{x_{1,\zeta}} = 4 > e$, we obtain that if $x > x_{1,\zeta} = \frac{1}{4}$ then $\tilde{f}_\zeta(x) = \beta_{1,\zeta}(4x)^{1/4}$.

Define, for $\zeta < \mathfrak{c}$, the auxiliary function

$$g_\zeta(x) = \frac{4}{4^{1/4}\beta_{1,\zeta}} x^{a_\zeta - 1/4}$$

From the proof of Lemma 6.18, $\beta_{1,\zeta} < e^{4x_{1,\zeta}} = e$ and therefore

$$1 < \frac{4}{4^{1/4}\beta_{1,\zeta}},$$

so g_ζ is a (continuous) mc–function.

Define then the function

$$f_\zeta(x) = \tilde{f}_\zeta(x) g_\zeta(x).$$

Then f_ζ is a mc–function (for being the product of two mc–functions) which is discontinuous on the set $\mathcal{X}_\zeta$.

Furthermore, if $\frac{1}{4} < x$,

$$f_\zeta(x) = \tilde{f}_\zeta(x) g_\zeta(x) = \beta_{1,\zeta}(4x)^{1/4} \frac{4}{4^{1/4}\beta_{1,\zeta}} x^{a_\zeta - 1/4} = 4x^{a_\zeta}.$$

Let us show that $B = \{f_\zeta : \zeta < \mathfrak{c}\}$ is an algebraically independent set.

Indeed, let $\zeta_1, \zeta_2, \ldots, \zeta_n < \mathfrak{c}$, $\lambda_1, \ldots, \lambda_m$ be non-zero numbers and $N = \{n_{i,j}\}_{i,j=1}^{n,m}$ be a matrix consisting of natural numbers as entries and without two equal columns.

Assume that

$$f = \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}} = 0.$$

Then, we notice that for $\frac{1}{4} < x$ it must be

$$\begin{aligned}
0 = f(x) &= \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}}(x) \\
&= \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} \left(4x^{a_{\zeta_i}}\right)^{n_{i,j}} \\
&= \sum_{j=1}^{m} \lambda_j \left(4^{\sum_{i=1}^{n} n_{i,j}}\right) x^{\sum_{i=1}^{n} a_{\zeta_i} n_{i,j}}.
\end{aligned}$$

Since the elements $a_{\zeta_1}, a_{\zeta_2}, \ldots, a_{\zeta_n}$ are $\mathbb{Q}$–linearly independent and the columns of $N = \{n_{i,j}\}_{i,j=1}^{n,m}$ are different one from each other, we can conclude that the exponents

$$\sum_{i=1}^{n} a_{\zeta_i} n_{i,j}$$

are all different from each other.

Therefore, $f_{|(1/4,\infty)}$ is an identically null extended polynomial (with positive exponents). Hence, all its coefficients must be zero and, in conclusion, if $1 \le j \le m$

$$\lambda_j 4^{\sum_{i=1}^{n} n_{i,j}} = 0, \text{ which implies that } \lambda_j = 0$$

and in conclusion B is algebraically independent.

Let us choose to finish with an element f in the trunkated cone generated by B. Then, we can find $\zeta_1, \zeta_2, \ldots, \zeta_n < \mathfrak{c}$, $\lambda_1, \ldots, \lambda_m \ge 1$ and a matrix consisting of natural numbers as entries and without two equal columns $N = \{n_{i,j}\}_{i,j=1}^{n,m}$ so that

$$f = \sum_{j=1}^{m} \lambda_j \prod_{i=1}^{n} f_{\zeta_i}^{n_{i,j}}.$$

Without loss of generality we may assume that $a_{\zeta_i} < a_{\zeta_{i+1}}$ for every $1 \le i \le n-1$. Then, f_{ζ_i} is continuous on $(0, a_{\zeta_2})$ for every $i \ge 2$.

For any function $h : \mathbb{R} \to \mathbb{R}$ we denote $h^{a^-} = \lim_{x \to a^-} h(x)$ and $h^{a^+} = \lim_{x \to a^+} h(x)$. We can then write

$$f = f_{\zeta_1}^{n_{1,1}} g_1 + \ldots + f_{\zeta_1}^{n_{j,1}} g_j + \ldots + f_{\zeta_1}^{n_{m,1}} g_m$$

where g_j is a mc–function continuous on $(0, a_{\zeta_2})$.

For every $\beta \in \mathcal{X}_{\zeta_i} \cap (0, a_{\zeta_2})$, we have that

$$f^{\beta^+} = (f_{\zeta_1}^{\beta^+})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{j,1}} g_j(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{m,1}} g_1(\beta)$$

and

$$f^{\beta^-} = (f_{\zeta_1}^{\beta^-})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{j,1}} g_j(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{m,1}} g_m(\beta).$$

Since the function f_{ζ_1} is not continuous on β we may assume without loss of generality that $f_{\zeta_1}^{\beta^-} < f_{\zeta_1}^{\beta^+}$ (the procedure would be analogous for the case $f_{\zeta_1}^{\beta^-} > f_{\zeta_1}^{\beta^+}$) and observe that if $f_{\zeta_1}^{\beta^-} < f_{\zeta_1}^{\beta^+}$ then $(f_{\zeta_1}^{\beta^-})^q < (f_{\zeta_1}^{\beta^+})^q$ for every $q \in \mathbb{N} \setminus \{0\}$. We can now write

$$\begin{aligned}
f^{\beta^+} - f^{\beta^-} =& (f_{\zeta_1}^{\beta^+})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^+})^{n_{m,1}} g_m(\beta) \\
& - \left[(f_{\zeta_1}^{\beta^-})^{n_{1,1}} g_1(\beta) + \ldots + (f_{\zeta_1}^{\beta^-})^{n_{m,1}} g_m(\beta) \right] \\
=& g_1(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{1,1}} - (f_{\zeta_1}^{\beta^-})^{n_{1,1}} \right] + \ldots + g_j(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{j,1}} - (f_{\zeta_1}^{\beta^-})^{n_{j,1}} \right] \\
& + \ldots + g_m(\beta) \left[(f_{\zeta_1}^{\beta^+})^{n_{m,1}} - (f_{\zeta_1}^{\beta^-})^{n_{m,1}} \right] \\
>& 0,
\end{aligned}$$

so that f is discontinuous on $\mathcal{X}_{\zeta_1} \cap (0, a_{\zeta_2})$ and the proof is done. $\square$

Bibliography

[1] L. V. Ahlfors, *Complex Analysis*, 3rd ed., McGraw-Hill, London, 1979.

[2] A. Aizpuru, C. Pérez-Eslava, F. J. García-Pacheco, and J. B. Seoane-Sepúlveda, *Lineability and coneability of discontinuous functions on* $\mathbb{R}$, Publ. Math. Debrecen **72** (2008), no. 1-2, 129–139.

[3] A. Aizpuru, C. Pérez-Eslava, and J. B. Seoane-Sepúlveda, *Linear structure of sets of divergent sequences and series*, Linear Algebra Appl. **418** (2006), no. 2-3, 595–598, DOI 10.1016/j.laa.2006.02.041.

[4] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*: *A hitchhiker's guide*, 3rd edition, Springer, Berlin, 2006.

[5] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Generalized convexity and inequalities*, J. Math. Anal. Appl. **335** (2007), no. 2, 1294–1308, DOI 10.1016/j.jmaa.2007.02.016.

[6] G. Araújo, L. Bernal-González, G. A. Muñoz-Fernández, J. A. Prado-Bassas, and J. B. Seoane-Sepúlveda, *Lineability in sequence and function spaces*, Studia Math. **237** (2017), 119–136.

[7] R. M. Aron, L. Bernal González, D. M. Pellegrino, and J. B. Seoane Sepúlveda, *Lineability: the search for linearity in mathematics*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016.

[8] R. M. Aron, F. J. García-Pacheco, D. Pérez-García, and J. B. Seoane-Sepúlveda, *On dense-lineability of sets of functions on* $\mathbb{R}$, Topology **48** (2009), no. 2-4, 149–156, DOI 10.1016/j.top.2009.11.013.

[9] R. M. Aron, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc. **133** (2005), no. 3, 795–803, DOI 10.1090/S0002-9939-04-07533-1.

[10] R. M. Aron, D. Pérez-García, and J. B. Seoane-Sepúlveda, *Algebrability of the set of non-convergent Fourier series*, Studia Math. **175** (2006), no. 1, 83–90, DOI 10.4064/sm175-1-5.

[11] R. M. Aron and J. B. Seoane-Sepúlveda, *Algebrability of the set of everywhere surjective functions on* $\mathbb{C}$, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 1, 25–31.

[12] M. Balcerzak, A. Bartoszewicz, and M. Filipczak, *Nonseparable spaceability and strong algebrability of sets of continuous singular functions*, J. Math. Anal. Appl. **407** (2013), no. 2, 263–269.

[13] Baldwin, S. (1990/91). Martin's axiom implies a stronger version of Blumberg's theorem. *Real Anal. Exchange*. 16(1): 67–73. www.jstor.org/stable/44153681

[14] A. Bartoszewicz, M. Bienias, M. Filipczak, and S. Głąb, *Strong $\mathfrak{c}$-algebrability of strong Sierpiński-Zygmund, smooth nowhere analytic and other sets of functions*, J. Math. Anal. Appl. **412** (2014), no. 2, 620–630, DOI 10.1016/j.jmaa.2013.10.075.

[15] A. Bartoszewicz and S. Głąb, *Strong algebrability of sets of sequences and functions*, Proc. Amer. Math. Soc. **141** (2013), no. 3, 827–835, DOI 10.1090/S0002-9939-2012-11377-2.

[16] A. Bartoszewicz, S. Głąb, and T. Poreda, *On algebrability of nonabsolutely convergent series*, Linear Algebra Appl. **435** (2011), no. 5, 1025–1028, DOI 10.1016/j.laa.2011.02.008.

[17] F. Bastin, J. A. Conejero, C. Esser, and J. B. Seoane-Sepúlveda, *Algebrability and nowhere Gevrey differentiability*, Israel J. Math. **205** (2015), no. 1, 127–143, DOI 10.1007/s11856-014-1104-1.

[18] H. Bauer, *Probability theory*, De Gruyter Studies in Mathematics, vol. 23, Walter de Gruyter & Co., Berlin, 1996. Translated from the fourth (1991) German edition by Robert B. Burckel and revised by the author.

[19] E. F. Bechenbach, *Convex functions*, Bull. Amer. Math. Soc. **54** (1948), 439–460, DOI 10.1090/S0002-9904-1948-08994-7.

[20] L. Bernal-González, H. J. Cabana-Méndez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *On the dimension of subspaces of continuous functions attaining their maximum finitely many times*, Trans. Amer. Math. Soc. **373** (2020), no. 5, 3063–3083, DOI 10.1090/tran/8054.

[21] L. Bernal-González, J. A. Conejero, M. Murillo-Arcila, and J. B. Seoane-Sepúlveda, *Highly tempering infinite matrices*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **112** (2018), no. 2, 341–345, DOI 10.1007/s13398-017-0385-8.

[22] L. Bernal-González, J. A. Conejero, M. Murillo-Arcila, and J. B. Seoane-Sepúlveda, $[\mathcal{S}]$*-linear and convex structures in function families*, Linear Algebra Appl. **579** (2019), 463–483, DOI 10.1016/j.laa.2019.07.003.

[23] L. Bernal-González, P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *On inequalities for convex functions*, J. Convex Anal. **26** (2019), no. 2.

[24] L. Bernal-González, J. Fernández-Sánchez, M. E. Martínez-Gómez, and J. B. Seoane-Sepúlveda, *Banach spaces and Banach lattices of singular functions*, Studia Mathematica, accepted for publication.

[25] L. Bernal-González, D. Pellegrino, and J. B. Seoane-Sepúlveda, *Linear subsets of nonlinear sets in topological vector spaces*, Bull. Amer. Math. Soc. (N.S.) **51** (2014), no. 1, 71–130, DOI 10.1090/S0273-0979-2013-01421-6.

[26] P. Billingsley, *Convergence of probability measures*, Second, Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1999. A Wiley-Interscience Publication.

[27] Blumberg, H. (1922). New properties of all real functions. *Trans. Amer. Math. Soc.* 24(2): 113–128. doi.org/10.2307/1989037

[28] G. Botelho and J. L. P. Luiz, *A note on latticeability in vector-valued sequence spaces*, available at arXiv:3115436 [math.FA], Preprint (2020).

[29] Bradford, J. C., Goffman, C. (1960). Metric spaces in which Blumberg's theorem holds. *Proc. Amer. Math. Soc.* 11(4): 667–670. doi.org/10.2307/2034731

[30] Brown, J. B. (1977). A measure theoretic variant of Blumberg's theorem. *Proc. Amer. Math. Soc.* 66(2): 266–268. doi.org/10.2307/2040943

[31] M. C. Calderón-Moreno, P. J. Gerlach-Mena, and J. A. Prado-Bassas, *Algebraic structure of continuous, unbounded and integrable functions*, J. Math. Anal. Appl. **470** (2019), 348–359.

[32] ———, *Lineability and modes of convergence*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114** (2020), paper no. 18.

[33] D. Cariello and J. B. Seoane-Sepúlveda, *Basic sequences and spaceability in ℓ_p spaces*, J. Funct. Anal. **266** (2014), no. 6, 3797–3814, DOI 10.1016/j.jfa.2013.12.011.

[34] Ciesielski, K. C. (1997). Set-theoretic real analysis. *J. Appl. Anal.* 3(2): 143–190. doi.org/10.1515/JAA.1997.143

[35] K. C. Ciesielski, M. E. Martínez-Gómez, and J. B. Seoane-Sepúlveda, *"Big" continuous restrictions of arbitrary functionss*, American Mathematical Monthly **126** (2019), no. 6, 547–552.

[36] K. C. Ciesielski and J. B. Seoane-Sepúlveda, *A century of Sierpiński-Zygmund functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 4, 3863–3901, DOI 10.1007/s13398-019-00726-0.

[37] K. C. Ciesielski and J. B. Seoane-Sepúlveda, *Differentiability versus continuity: restriction and extension theorems and monstrous examples*, Bull. Amer. Math. Soc. (N.S.) **56** (2019), no. 2, 211–260, DOI 10.1090/bull/1635.

[38] J. A. Conejero, G. A. Muñoz-Fernández, M. Murillo Arcila, and J. B. Seoane-Sepúlveda, *Smooth functions with uncountably many zeros*, Bull. Belg. Math. Soc. Simon Stevin **22** (2015), no. 1, 71–75.

[39] E. de Amo, M. Díaz Carrillo, and J. Fernández-Sánchez, *Singular functions with applications to fractal dimensions and generalized Takagi functions*, Acta Appl. Math. **119** (2012), 129–148.

[40] E. de Amo, M. Díaz Carrillo, and J. Fernández Sánchez, *Harmonic analysis on the Sierpiński gasket and singular functions*, Acta Math. Hungar. **143** (2014), no. 1, 58–74, DOI 10.1007/s10474-013-0373-1.

[41] E. de Amo, M. Díaz Carrillo, and J. Fernández-Sánchez, *A family of singular functions and its relation to harmonic fractal analysis and fuzzy logic*, Open Math. **14** (2016), no. 1, DOI 10.1515/math-2016-0094.

[42] ———, *Pisot numbers and strong negations*, Chaos Solitons Fractals **104** (2017), 61–67, DOI 10.1016/j.chaos.2017.08.002.

[43] J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.

[44] O. Dovgoshey, O. Martio, V. Ryazanov, and M. Vuorinen, *The Cantor function*, Expo. Math. **24** (2006), no. 1, 1–37, DOI 10.1016/j.exmath.2005.05.002.

[45] P. H. Enflo, V. I. Gurariy, and J. B. Seoane-Sepúlveda, *Some results and open questions on spaceability in function spaces*, Trans. Amer. Math. Soc. **366** (2014), no. 2, 611–625, DOI 10.1090/S0002-9947-2013-05747-9.

[46] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books in Mathematics, Springer New York, 2010.

[47] M. Fenoy-Muñoz, J. L. Gámez-Merino, G. A. Muñoz-Fernández, and E. Sáez-Maestro, *A hierarchy in the family of real surjective functions*, Open Math **15** (2017), no. 1, 486-501, DOI 10.1515/MATH-2017-0042.

[48] J. Fernández-Sánchez, M. E. Martínez-Gómez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *Algebraic genericity and special properties within sequence spaces and series*, The Rocky Mountain Journal of Mathematics **96** (2020), accepted for publication.

[49] J. Fernández-Sánchez, G. A. Muñoz-Fernández, D. L. Rodríguez-Vidanes, and J. B. Seoane-Sepúlveda, *Obtaining algebrability in subsets of real functions*, Publ. Math. Debrecen **96** (2020), no. 1-2, 231–244.

[50] J. Fernández Sánchez, P. Viader, J. Paradís, and M. Díaz Carrillo, *A singular function with a non-zero finite derivative*, Nonlinear Anal. **75** (2012), no. 13, 5010–5014.

[51] ______, *A singular function with a non-zero finite derivative on a dense set*, Nonlinear Anal. **95** (2014), 703–713.

[52] ______, *A singular function with a non-zero finite derivative on a dense set with Hausdorff dimension one*, J. Math. Anal. Appl. **434** (2016), no. 1, 713–728.

[53] J. Ferrer, D. García, M. Maestre, and J. B. Seoane-Sepúlveda, *On the zero-set of 2-homogeneous polynomials in Banach spaces*, Linear Multilinear Algebra **67** (2019), no. 10, 1958–1970, DOI 10.1080/03081087.2018.1476448.

[54] A. Froda, *Sur la distribution des propriétés de voisinage des fonctions de variables réelles*, Doctorat d'État, 1929 (fr).

[55] J. L. Gámez-Merino, G. A. Muñoz-Fernández, V. M. Sánchez, and J. B. Seoane-Sepúlveda, *Sierpiński-Zygmund functions and other problems on lineability*, Proc. Amer. Math. Soc. **138** (2010), no. 11, 3863–3876.

[56] J. L. Gámez-Merino, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *Lineability and additivity in* $\mathbb{R}^{\mathbb{R}}$, J. Math. Anal. Appl. **369** (2010), no. 1, 265–272.

[57] J. L. Gámez-Merino, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *A characterization of continuity revisited*, Amer. Math. Monthly **118** (2011), no. 2, 167–170.

[58] J. L. Gámez-Merino and J. B. Seoane-Sepúlveda, *An undecidable case of lineability in* $\mathbb{R}^{\mathbb{R}}$, J. Math. Anal. Appl. **401** (2013), no. 2, 959–962, DOI 10.1016/j.jmaa.2012.10.067.

[59] Goffman, C. (1954). On a theorem of Henry Blumberg. *Michigan Math. J.* 2(1): 21–22.

[60] F. Q. Gouvêa, *p-adic numbers*, 2nd ed., Universitext, Springer-Verlag, Berlin, 1997. An introduction.

[61] V. I. Gurariy and W. Lusky, *Geometry of Müntz spaces and related questions*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2005.

[62] E. Hewitt and K. R. Stromberg, *Real and abstract analysis*, Springer, Berlin, 1965.

[63] G. J. O. Jameson, *Counting zeros of generalized polynomials: Descartes' rule of signs and Laguerre extensions*, Math. Gazette **518** (2006), 223–234.

[64] P. Jiménez-Rodríguez, M. E. Martínez-Gómez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *Describing multiplicative convex functions*, J. Convex Anal. **27** (2020), no. 3.

[65] ______, *Generalizing multiplicative convex functions*, Journal of Convex Analysis (2020).

[66] ______, *Injectiveness and discontinuity of multiplicative convex functions*, Mathematics **9** (2021), no. 1035.

[67] P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, E. Sáez-Maestro, and J. B. Seoane-Sepúlveda, *Algebraic genericity and the differentiability of the convolution*, Journal of Approximation Theory **241** (2019), 86-106, DOI 10.1016/j.jat.2019.01.002.

[68] P. Jiménez-Rodríguez, G. A. Muñoz-Fernández, and J. B. Seoane-Sepúlveda, *On Weierstrass' Monsters and lineability*, Bull. Belg. Math. Soc. Simon Stevin **20** (2013), no. 4, 577–586.

[69] Jordan, F. (2001/2002). Generalizing the Blumberg theorem. *Real Anal. Exchange.* 27(2): 423–439.

[70] S. Katok, *p-adic analysis in comparison with real*, Amer. Math. Soc., Providence, RI, 2003.

[71] A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[72] K. Keimel and W. Roth, *Ordered cones and approximation*, Lecture Notes in Mathematics, vol. 1517, Springer-Verlag, Berlin, 1992. MR1176514

[73] Kharazishvili, A. B. (2006). *Strange Functions in Real Analysis.* Pure and Applied Mathematics 156. Boca Raton, FL: Chapman & Hall/CRC.

[74] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.

[75] López Pellicer, M. (1984). A note on the Blumberg property. *Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid.* 78(1–2): 147–150.

[76] P. Meyer-Nieberg, *Banach lattices*, Universitext, Springer-Verlag, Berlin, 1991.

[77] G. A. Muñoz-Fernández, N. Palmberg, D. Puglisi, and J. B. Seoane-Sepúlveda, *Lineability in subsets of measure and function spaces*, Linear Algebra Appl. **428** (2008), no. 11-12, 2805–2812.

[78] Y. Moschovakis, *Notes on set theory*, 2nd ed., Undergraduate Texts in Mathematics, Springer, New York, 2006. MR2192215

[79] C. T. Ng, *On midconvex functions with midconcave bounds*, Proc. Amer. Math. Soc. **102** (1988), 538–540.

[80] C. P. Niculescu, *Convexity according to the geometric mean*, Math. Inequal. Appl. **2** (2000), 155–167.

[81] T. Oikhberg, *A note on latticeability and algebrability*, J. Math. Anal. Appl. **434** (2016), no. 1, 523–537.

[82] J. Paradís, P. Viader, and L. Bibiloni, *Riesz-Nágy singular functions revisited*, J. Math. Anal. Appl. **329** (2007), no. 1, 592–602, DOI 10.1016/j.jmaa.2006.06.082.

[83] Recław, I. (1993). Restrictions to continuous functions and Boolean algebras. *Proc. Amer. Math. Soc.* 118(3): 791–796. doi.org/10.2307/2160122

[84] Rosłanowski, A., Shelah, S. (2006). Measured creatures. *Israel J. Math.* 151(1): 61–110. doi.org/10.1007/BF02777356

[85] J. B. Seoane, *Chaos and lineability of pathological phenomena in analysis*, ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–Kent State University.

[86] Shelah, S. (1995). Possibly every real function is continuous on a non-meagre set. *Publ. Inst. Math. (Beograd) (N.S.)* 57(71): 47–60.

[87] Y.-G. Shi and Y. Tang, *On conjugacies between asymmetric Bernoulli shifts*, J. Math. Anal. Appl. **434** (2016), no. 1, 209–221, DOI 10.1016/j.jmaa.2015.09.019.

[88] Sierpiński, W., Zygmund, A. (1923). Sur une fonction qui est discontinue sur tout ensemble de puissance du continu. *Fund. Math.* 4: 316–318. doi.org/10.4064/fm-4-1-316-318.

[89] J. Steiner, *Über das größte Product der Theile oder Summanden jeder Zahl*, Journal für die reine und angewandte Mathematik **40** (1850), 208–208.

[90] P. Viader, J. Paradís, and L. Bibiloni, *A new light on Minkowski's* $?(x)$ *function*, J. Number Theory **73** (1998), no. 2, 212–227, DOI 10.1006/jnth.1998.2294.

[91] J. von Neuman, *Ein System algebraisch unabhängiger zahlen*, Math. An. **99** (1928), 134–141.

[92] P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982.

[93] A. Zygmund, *Trigonometric series. Vol. I, II*, 3rd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2002.