UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Métodos computacionales en topología y sistemas dinámicos

Computational methods in topology and dynamical systems

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Pedro José Chocano Feito

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Computational Methods in Topology and Dynamical Systems

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A mi Kankun.

Et ignotas animum dimittit in artes.
Ovidio, *Metamorfosis viii, 18*

And then she said the ancient phrase:
"We're like the dreamer who dreams
and then lives inside the dream" I told
her I understood and then she said:
"but who is the dreamer?"
Gordon Cole

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Abstract

Alexandroff spaces are topological spaces satisfying that the arbitrary intersection of open sets is an open set. The theory of Alexandroff spaces is becoming a significant part of topology since it can be used to model and solve mathematical problems of different nature. They were first considered in [1] under the name of *diskrete Räume* (discrete spaces), where P.S. Alexandroff proved that they can also be treated as combinatorial objects. Concretely, he proved that Alexandroff spaces are in bijective correspondence with preordered sets. Therefore, topological notions such as continuity or homotopy can be expressed in terms of combinatorial notions. However, Alexandroff spaces have some poor topological properties. For instance, if X is an Alexandroff space being T_1 or a Fréchet space, then its topology is the discrete topology.

Finite topological spaces are a very important example of Alexandroff spaces. Due to its finitude, at a first glance, they can seem uninteresting from an algebraic point of view. Surprisingly, M.C. McCord proved in [75] that they have the same singular and homotopy groups of finite simplicial complexes. For example, there exists a finite topological space with four points satisfying that its fundamental group is the same as the fundamental group of the circle. The same situation holds for Alexandroff spaces and not necessarily finite simplicial complexes. We can say that Alexandroff spaces are as interesting as simplicial complexes from an algebraic point of view. A study of the homotopy type of finite topological spaces was made in [103], a generalization of this study was made in [67]. Two main monographs on the issue of the algebraic aspects of the topology of finite spaces are [74] and [12]. The first one, due to J.P. May, is the result of some REU programs developed by the author at the University of Chicago. The other one is, essentially, the Ph. D. thesis of J.A. Barmak (under the supervision of E.G. Minian).

Since finite topological spaces can be seen as partially ordered sets and have a finite number of points, it follows that they are good candidates to develop computational methods. Nevertheless, from a more theoretical point of view they are also very interesting. For instance, Quillen's conjecture (see [91, 104] and [12, Chapter 8]) or the Andrews-Curtis conjecture (see [4] and [12, Chapter 11]) can be stated in terms of finite topological spaces. In addition, interesting results regarding to topological spaces that have the fixed point property have been obtained in [14]. J.A. Barmak proved that the fixed point property is not a weak homotopy invariant. For every polyhedron K there exists a topological space which is weak homotopy equivalent to K and has the fixed point property. Finite topological spaces can also be used to solve a classical problem of realization of groups, see [25, 106, 10] or [16, 15]. Hence, we can say that they have strong relations with the theory of finite groups.

Recently, an interest in finding applications to the study of dynamical systems using the theory of finite topological spaces has grown up, see for example [20] or [68]. This is due to the difficulty of describing the exact behavior of the solutions or orbits of a general dynamical system. Topological methods such as the Conley index (see [32, 84, 105] or [51]) have been developed to overcome this difficulty. But the computation of these

methods is far from being reachable in each case. One possible approach to solve this situation is to define a combinatorial dynamical system on a combinatorial object, e.g., a finite topological space or a Lefschetz complex. Then, adapt classical invariants to this framework. Finally, find a discretization of the original dynamical system to obtain a combinatorial dynamical system and derive from here information, see [85] or [21] for more details. Moreover, some tools of differential topology like Morse theory (see [78]) have been adapted in a combinatorial way to the context of finite topological spaces in [80] and [46], and simplicial complexes in [48, 49, 50]. For simplicial complexes this combinatorial approach has been used for computational aspects in a fruitful manner, see for example [38] and [59]. From this it can be thought that a similar situation also holds for dynamical systems.

Objectives

The original idea of this work is to establish a proper theoretical framework to develop computational tools to study dynamical systems. To do this task some ideas of the shape theory are considered. Notice that shape theory fits better than homotopy theory in order to study dynamical objects such as attractors or repellers. This is because of the poor local properties that can have these spaces, e.g., Lorentz attractor. Homotopy theory works very nice for topological spaces with "good" local properties but not necessarily for topological spaces with "bad" local properties. The classical example is the Warsaw circle, it has all its homotopy groups trivial, but it is not contractible. Shape theory appears as a generalization of homotopy theory. It provides a better description or classification. One of the main ideas is to think about a topological space X as a family of "easy" topological spaces that are approximating X. In this case, the "easy" topological spaces are polyhedra and they are closely related to finite topological spaces. Then it is natural to follow a similar approach using finite spaces as it was done in [82] and [31]. The first task is to adapt these ideas to get a description of a compact metric space in terms of finite topological spaces. Once it is accomplished the previous task, it is natural to wonder if shape theory can be reformulated using finite spaces. A positive answer could lead to a sort of computational shape theory.

On the one hand, the groups of automorphisms of finite topological space should be analyzed to study dynamical systems on them. For example, a discrete dynamical system for a finite topological X space is given by a homeomorphism $f : X \to X$. On the other hand, it is not difficult to show that the only continuous dynamical system defined on a finite T_0 topological spaces is the trivial one. This is due to the lack of room, there is only a finite number of points. The usual definition of dynamical system does not work properly in this context and then new definitions must be given. To sum up, the ideas to be developed in this work are the following:

- Develop a theory to reconstruct or approximate a compact metric space using finite topological spaces.
- Study the groups of automorphisms of finite topological spaces.
- Give new definitions of dynamical system to the framework of finite topological spaces.
- Find a discretization of classical dynamical systems and see how this discretization recovers some of the original information.

Results

Chapter 1 serves as an introduction to the theory that will be used in subsequent chapters. The introduction has two well-differentiated parts. The first one deals with basic results and theory of Alexandroff spaces. The second one is a brief introduction to shape theory and results related to it.

In Chapter 2 we give a new inverse sequence to reconstruct the homotopy type of compact metric spaces. Concretely, given a compact metric space X we construct an inverse sequence of finite topological spaces satisfying that its inverse limit contains a homeomorphic copy of X which is a strong deformation retract. We study the relations of our inverse sequence with other similar constructions ([31] and [82]) and propose a computational method to reconstruct the homology groups of a compact metric space. Then, a new description of shape theory is given and some other shape theoretical questions are treated. Finally, we define a topological category SW (SH) that classifies compact metric spaces by their shape and finite topological spaces by their weak homotopy type (homotopy type).

In Chapter 3 problems of realization of groups in topological categories are solved. We prove that every group can be realized as the group of homeomorphisms or self-homotopy equivalences or pointed self-homotopy equivalences of a topological space. In addition, we also prove that every homomorphism of groups can be expressed in terms of the natural homomorphism between the group of homeomorphisms and the group of self-homotopy equivalences of a topological space. Other realization problems involving homotopy groups and homology groups are considered. To do this, we find a family of asymmetric topological spaces satisfying that each member is weak homotopy equivalent to a point. From this we derive some consequences of the relations between the groups mentioned above. We get that some techniques used to study the group of self-homotopy equivalences of CW-complexes cannot be adapted to general topological spaces. The topological spaces used throughout this chapter are Alexandroff spaces. Even though Alexandroff spaces and CW-complexes are closely related, there are huge differences in terms of the groups of self-homotopy equivalences.

In Chapter 4 we propose a method to define dynamical systems for finite topological spaces. From a dynamical point of view single valued maps are not appropriate to model a complex behavior. Then a special class of multivalued maps is introduced, the so-called Vietoris-like multivalued maps. For this special class of multivalued maps, we develop a general Lefschetz fixed point theorem and some coincidence theorems. The usual continuous maps between finite topological spaces can be seen as Vietoris-like multivalued maps, so the classical Lefschetz fixed point theorem can also be obtained as a special case of the general one. In addition, it is not required any hypothesis of continuity in the definition of a Vietoris-like multivalued map. Therefore, the flexibility of this sort of multivalued maps is very high and they can be used to model more complex dynamical systems. We apply the theory of Vietoris-like multivalued maps to approximate discrete dynamical systems defined on polyhedra. To do this we use a sequence of finite topological spaces and Vietoris-like multivalued maps. Finally, we define a category enclosing the theory of Vietoris-like multivalued maps. To get this category we give a new definition of homotopy. The new homotopy generalizes some other definitions of homotopy for multivalued maps and also the classical definition of homotopy for continuous maps between finite topological spaces. Moreover, we give some remarks about a possible concept of topological degree for this setting.

Chapters 2, 3 and 4 can be read independently. The second subsection of Chapter 1 is only referenced in Chapter 2. Therefore it can be omitted in order to read Chapter 3 and Chapter 4. At the beginning of each chapter there is an introduction recalling basic theory, notation and a brief discussion of each topic.

Conclusions and further work

We have obtained a computational method to approximate discrete dynamical systems defined on polyhedra using finite topological spaces. But we use the inverse sequence obtained in [31], which works only for simplicial complexes. It is desirable to adapt this method to discrete dynamical systems defined on general compact metric spaces. The idea is to use the new inverse sequence obtained in Chapter 2. In that way, more general results could be obtained.

A Lefschetz fixed point theorem for Vietoris-like multivalued maps has been obtained. Then it can be interesting to get an axiomatization of the Lefschetz number as in the classical setting, see [7]. A similar axiomatization has been carried out for finite topological spaces and continuous maps in [23].

The theory of Vietoris-like multivalued maps seems a very promising starting point to get the notion of continuous and discrete dynamical system in this framework. Then other classical notions such as the fixed point index or the Conley index may be adapted. Once it is accomplished the previous task, it would be very interesting to see the relations with the original classical notions in the method of approximating dynamical systems given in the fourth chapter. Moreover, it also remains to do a deeper study of the Lefschetz number obtained for a Vietoris-like multivauled map. This is, a study of periodic orbits or other relations with the topology of the space. Another question to be analyzed regarding to the fixed point theory in this framework is the following. Is the fixed point property for Vietoris-like multivalued maps a homotopy invariant? The fixed point property for single-valued maps is a homotopy invariant for finite T_0 topological spaces, see [12, Chapter 10]. But in general, this question does not necessarily have a positive answer. For compact CW-complexes and continuous maps the fixed point property is not a homotopy invariant, see [69]. Similarly, it can also be treated the combinatorial counterpart to some coincidence problems in the classical setting such as the one treated in [54] or [44], i.e., given two continuous maps, do there exist continuous maps f', g' such that f is homotopic to f', g is homotopic to g' and f', g' do not have coincidence points? Furthermore, an adaptation of the theory of Vietoris-like maps and multivalued to general Alexandroff spaces could be done.

This work has a strong theoretical point of view, but the implementation of the methods described throughout this dissertation is something to be done. The construction given in the second chapter can lead to a sort of computational shape theory and the method proposed in the fourth chapter can describe a dynamical system by its approximations. Furthermore, the method obtained to reconstruct homology groups of compact metric spaces could give applications to topological data analysis.

The results of the third chapter are somehow the ones that are closed since a positive answer has been obtained for the problems. Remarkably, it has been proved that an algebraic notion such as the homomorphism of groups can be expressed in terms of groups related to a topological space. Alexandroff spaces are proved to be perfect blocks to realize different groups related to topological spaces. One problem that has not been treated throughout this chapter is the cardinality of the topological spaces. By way of illustration, what is the minimum number of points that are needed to get a finite topological space realizing a finite group G as its group of homeomorphisms? In this direction, there is a recent manuscript, see [15]. Another problem that could be considered is the study of realization problems for other algebraic structures. For instance, the combinatorial version of the problem treated in [35] could be considered.

Resumen

Los espacios de Alexandroff son espacios topológicos que satisfacen la propiedad adicional de que la intersección arbitraria de abiertos es de nuevo un abierto. La teoría de espacios de Alexandroff se está convirtiendo en una parte significativa dentro de la topología dado que dichos espacios pueden ser usados para modelizar y resolver diferentes problemas matemáticos. La primera vez que fueron considerados fue en [1] bajo el nombre de *diskrete Räume* (espacio discreto), en dicho artículo se demostró que estos espacios podían ser tratados como objectos combinatorios. Concretamente, se probó que los espacios de Alexandroff se pueden poner en correspondencia biyectiva con los conjuntos preordenados. Por tanto, algunas nociones topológicas como la continuidad o la homotopía pueden ser descritas en términos combinatorios. Por otro lado, los espacios de Alexandroff tienen algunas propiedades topológicas muy pobres. Por ejemplo, si X es un espacio de Alexandroff satisfaciendo el axioma de separación T_1 o siendo un espacio de Fréchet, entonces la topología definida en X es la discreta.

Los espacios topológicos finitos son un ejemplo importante de espacios de Alexandroff. Debido a su finitud, a primera vista pueden parecer poco interesantes desde un punto de vista algebraico topológico. Sorprendentemente, en [75], se demuestra que éstos tienen los mismos grupos de homología singular y homotopía que complejos simpliciales finitos. Por ejemplo, existe un espacio topológico con solo cuatro puntos cuyo grupo fundamental es el mismo que el de la circunferencia. La misma situación descrita para los grupos de homología y homotopía se mantiene para los espacios de Alexandroff y complejos simpliciales no necesariamente finitos. Así pues, los espacios de Alexandroff pueden ser tan interesantes como los complejos simpliciales desde un punto de vista algebraico. En [103], es llevado a cabo un estudio sobre el tipo de homotopía de los espacios topológicos finitos. Recientemente, en [67], se obtiene una generalización de dichos resultados para espacios de Alexandroff. Hay dos referencias principales que tratan cuestiones algebraicas en espacios topológicos finitos, [74] y [12]. La primera referencia, debida a J.P. May, es el resultado de algunos programas REU desarrollados por el autor en la universidad de Chicago. La segunda referencia es esencialmente la tesis de J.A. Barmak bajo la supervisión de E.G. Minian.

Los espacios topológicos finitos son buenos candidatos para desarrollar métodos computacionales que sirvan en la resolución de algunos problemas de distinta naturaleza. Esto último se debe a que pueden ser tratados como objetos combinatorios y a que tienen un cardinal finito. Sin embargo, desde un punto de vista más teórico siguen siendo de gran interés. Por ejemplo, la conjetura de Quillen (ver [91, 104] y [12, Chapter 8]) o la conjetura de Andrews-Curtis (ver [4] y [12, Chapter 11]) pueden ser reformuladas mediante la teoría de espacios topológicos finitos. Además, interesantes resultados relacionados con espacios que poseen la propiedad de punto fijo pueden obtenerse como es el caso en [14], donde se demuestra que la propiedad de punto fijo no es un invariante para la noción de homotopía débil, esto se debe a que para todo poliedro K existe un espacio de Alexandroff satisfaciendo la propiedad de punto fijo con el mismo tipo de homotopía débil que K. Los

espacios topológicos finitos también se pueden usar para resolver problemas clásicos de realización de grupos, ver [25, 106, 10, 16, 15], lo que implica que hay una fuerte relación entre la teoría de grupos finitos y la de los espacios topológicos finitos.

Recientemente, el interés en encontrar aplicaciones al estudio de sistemas dinámicos mediante la teoría de espacios topológicos finitos ha aumentado, ver por ejemplo [20] o [68]. Este hecho se debe a la dificultad que reside en describir el comportamiento exacto de órbitas o soluciones para un sistema dinámico general. Métodos topológicos como el índice de Conley (ver [32, 84, 105] o [51]) se han desarrollado para superar esta dificultad. Sin embargo, el cómputo de estos métodos en algunos casos se encuentra lejos de poder ser alcanzado. Una posible aproximación para resolver esta situación consiste en definir un sistema dinámico de manera combinatoria sobre objetos también combinatorios como puedan ser los espacios topológicos finitos o los complejos de Lefschetz. A partir de ahí, adaptar algunos invariantes clásicos como el ya mencionado índice de Conley. Finalmente, encontrar una discretización del sistema dinámico original para obtener uno combinatorio y derivar a partir del combinatorio información del original, ver [85] o [21] para más detalles. Además, algunas herramientas de la topología diferencial como la teoría de Morse [78] han sido adaptadas de manera combinatoria a el contexto de espacios topológicos finitos, ver [80] y [46], y a complejos simpliciales, ver [49, 50]. Para complejos simpliciales, este enfoque combinatorio ha sido usado de manera fructífera para tratar algunas cuestiones de carácter computacional, ver por ejemplo [38] o [59]. Por ello, resulta razonable pensar que una situación similar pueda ocurrir en sistemas dinámicos.

Objetivos

La idea original de este trabajo consiste en establecer un marco teórico adecuado para el desarrollo de herramientas computacionales en el estudio de sistemas dinámicos. Para realizar esta tarea, algunas ideas de la teoría de la forma son usadas. Es también importante observar que la teoría de la forma encaja mejor que la teoría de homotopía a la hora de estudiar algunos objetos dinámicos como puedan ser atractores o repulsores. Este último hecho se debe a que en general dichos objetos dinámicos no necesariamente tienen que tener buenas propiedades locales, véase el atractor de Lorenz. La teoría de homotopía funciona bien para espacios con buenas propiedades locales pero no necesariamente para los que no las tengan. Por tanto, la teoría de la forma nace como una generalización de la teoría de homotopía que es capaz de dar una mejor descripción o clasificación. Una de las ideas principales consiste en pensar en un espacio topológico X como si fuera una familia de espacios topológicos "sencillos" que aproximan a X. En este caso, los espacios "sencillos" que se usan son poliedros. A su vez, como los poliedros están relacionados de manera directa con los espacios topológicos finitos es natural intentar seguir un enfoque que use espacios topológicos finitos en vez de poliedros como es hecho en [31] y [82]. Por tanto, la primera tarea a realizar consiste en adaptar estas ideas para obtener una nueva descripción de compactos métricos en términos de espacios topológicos finitos. Una vez logrado este objetivo, es natural plantear si la teoría de la forma puede ser reformulada usando espacios topológicos finitos. En caso de ser posible, se obtendría una especie de teoría de la forma computacional.

Por un lado, el grupo de automorfismos de espacios topológicos finitos debe ser analizado para el estudio de sistemas dinámicos. Por ejemplo, un sistema dinámico discreto para un espacio topológico finito X es dado por un homeomorfismo $f : X \to X$. Por otro lado, no es complicado deducir que el único sistema dinámico continuo que se puede definir sobre un espacio topológico finito es el trivial. Esto último se debe a la falta de espacio porque solamente disponemos de una cantidad finita de puntos. Así pues, las definiciones usuales de sistema dinámico no funcionan demasiado bien en este contexto, nuevas definiciones deben darse para adaptar la noción de sistema dinámico y a su vez otras nociones relacionadas con ello. En resumen, las ideas a ser desarrolladas en este trabajo son las siguientes:

- Desarrollar una teoría de reconstrucción o aproximación para compactos métricos usando espacios topológicos finitos.
- Estudiar el grupo de automorfismos de espacios topológicos finitos.
- Dar definiciones de sistema dinámico para este contexto combinatorio y adaptar algunas nociones clásicas.
- Encontrar una discretización para sistemas dinámicos clásicos y ver cómo esta discretización recupera información del sistema original.

Resultados

El Capítulo 1 de esta tesis sirve como introducción a la teoría que será usada en capítulos posteriores. La introducción contiene dos partes bien diferenciadas. La primera de ellas repasa resultados básicos de la teoría de espacios de Alexandroff. La segunda es una somera introducción a la teoría de la forma y resultados relacionados con ella.

En el Capítulo 2, una nueva construcción para reconstruir el tipo de homotopía de compactos métricos mediante el uso de espacios topológicos finitos es dada. En concreto, dado un compacto métrico X, se construye una sucesión inversa de espacios topológicos finitos satisfaciendo que su límite inverso contiene una copia homeomorfa de X que es un retracto fuerte de deformación de X. Luego, relaciones con otras construcciones similares ([31] y [82]) son estudiadas. Una nueva descripción de la teoría de la forma usando la construcción propuesta es obtenida y otros aspectos relacionados con la teoría de la forma son analizados. Finalmente, se construye una categoría SW (SH) que clasifica compactos métricos por su tipo de forma y espacios topológicos finitos por su tipo de homotopía débil (homotopía).

En el Capítulo 3, algunos problemas de realización de grupos en categorías topológicas son resueltos. En concreto, se demuestra que todo grupo puede realizarse como el grupo de homeomorfismos (auto-equivalencias homotópicas o auto-equivalencias homotópicas punteadas) de un espacio topológico. Además, también se demuestra que todo homomorfismo de grupos puede ser expresado como el homomorfismo natural entre el grupo de homeomorfismos y el grupo de auto-equivalencias homotópicas de un espacio topológico. Otros problemas de realización involucrando grupos de homología y homotopía son resueltos. Para lograr esta meta, una familia asimétrica de espacios topológicos satisfaciendo que cada elemento tiene el tipo de homotopía débil de un punto es encontrada. A partir de aquí, se derivan algunas consecuencias entre las relaciones que hay en los grupos previamente mencionados. Una de ellas prueba que las técnicas usadas para el estudio del grupo de auto-equivalencias homotópicas de CW-complejos no pueden ser adaptadas a espacios topológicos generales. Los espacios topológicos usados en este capítulo son espacios de Alexandroff. Por tanto, a pesar de que los espacios de Alexandroff y los CW-complejos están relacionados, existen enormes diferencias en lo que respecta a el grupo de auto-equivalencias homotópicas.

En el Capítulo 4, se propone un método para adaptar la noción de sistema dinámico discreto en espacios topológicos finitos. El primer enfoque consiste en desarrollar una teoría de punto fijo apropiada. Desde un punto de vista dinámico, las aplicaciones univaluadas no son apropiadas en este contexto para modelizar comportamientos complejos. Por ese motivo, se introduce una clase especial de aplicaciones multivaluadas, las nombradas aplicaciones multivaluadas Vietoris-like. Para esta clase especial de aplicaciones multivaluadas se desarrolla un teorema general de punto fijo de Lefschetz y teoremas de coincidencia. Las aplicaciones continuas usuales entre espacios topológicos finitos son un ejemplo particular de aplicación multivaluada Vietoris-like. Además, ninguna hipótesis de continuidad es requerida en la definición de aplicación multivaluada Vietoris-like, este hecho dota de gran flexibilidad a este tipo de aplicaciones multivaluadas, lo que has hace más idóneas para modelizar sistemas dinámicos más complejos. Una aplicación de la teoría desarrollada para aplicaciones multivaluadas Vietoris-like se obtiene para sistemas dinámicos discretos definidos sobre poliedros. En concreto, se propone un método computacional para aproximar sistemas dinámicos discretos sobre poliedros mediante una sucesión de espacios topológicos finitos y aplicaciones multivaluadas. Por último, se define una categoría que englobe a la teoría de aplicaciones multivaluadas Vietoris-like. Para definir dicha categoría, se introduce una nueva noción de homotopía que a su vez generaliza otras nociones de homotopía para aplicaciones multivaluadas y también a la noción clásica de homotopía para funciones continuas entre espacios finitos. Además, se dan algunas observaciones sobre el concepto de grado topológico para el contexto de espacios finitos.

Los Capítulos 2, 3 y 4 pueden ser leídos de manera independiente. La segunda subsección del Capítulo 1 es solamente referenciada en el Capítulo 2.

Conclusiones y trabajo futuro

Se ha conseguido desarrollar un método para aproximar sistemas dinámicos definidos sobre poliedros mediante el uso de espacios topológicos finitos. Aunque una de las construcciones usadas en dicho método se apoya en la construcción hecha en [31], que funciona solamente para complejos simpliciales. Sería bueno lograr adaptar el método propuesto usando la construcción hecha en el segundo capítulo de este trabajo, de esa manera se podrían aproximar sistemas dinámicos discretos definidos sobre compactos métricos y resultados más generales se podrían obtener.

Se ha desarrollado un teorema de punto fijo de Lefschetz para aplicaciones multivaluadas Vietoris-like, una axiomatización del número de Lefschetz como en el contexto clásico para poliedros y aplicaciones continuas [7] podría llevarse a cabo. Para aplicaciones continuas y espacios topológicos finitos, una axiomatización se ha adaptado en [23].

Las aplicaciones multivaluadas Vietoris-like parecen un prometedor punto de partida para definir la noción de sistema dinámico continuo y discreto en el marco de espacios topológicos finitos. Por tanto, a partir de ahí, otras nociones clásicas como el índice de punto fijo o índice de Conley pueden ser adaptadas. Una vez se hayan adaptado estos invariantes, sería muy interesante conocer las relaciones que tienen con las nociones clásicas en el método de aproximación descrito en el cuarto capítulo de esta tesis. Además, queda todavía por hacer un estudio más profundo de los números de Lefschetz obtenidos para una aplicación multivaluada Vietoris-like, por ejemplo, un estudio de órbitas periódicas u otras relaciones con la topología del espacio. Otra cuestión interesante a ser tratada en lo que concierne a teoría de punto fijo es la siguiente. ¿Es la propiedad de punto fijo para aplicaciones multivaluadas Vietoris-like un invariante homotópico? La propiedad de punto fijo para aplicaciones continuas univaluadas es un invariante homotópico en espacios topológicos finitos T_0, véase [12, Chapter 10]. Pero en general esta cuestión no tiene necesariamente una respuesta afirmativa. Por ejemplo, si consideramos CW-complejos compactos y aplicaciones continuas, la propiedad de punto fijo no es un invariante homotópico como se prueba en [69]. De forma análoga al marco clásico, algunos problemas de coincidencia como el planteado en [54] o [44] pueden ser tratados en su versión combinatoria, es decir, dadas dos aplicaciones continuas f y g ¿Existen aplicaciones continuas f' y g' satisfaciendo que f es homotópica a f, g es homotópica a f y f' no tiene puntos de coincidencia con g'? También queda por adaptar la teoría de aplicaciones multivaluadas Vietoris-like al marco más general de espacios de Alexandroff.

Este trabajo tiene un punto de vista más teórico, pero una implementación de los métodos descritos a lo largo de la tesis es algo que queda por hacer. Un ejemplo de ello es implementar la construcción hecha en el segundo capítulo para intentar obtener una especie de teoría de la forma computacional, o bien el método descrito en el cuarto capítulo para describir un sistema dinámico mediante aproximaciones. Además, el método obtenido para aproximar los grupos de homología de un compacto métrico podría servir para obtener aplicaciones en análisis topológico de datos.

Se puede decir que los resultados obtenidos en el tercer capítulo son los más cerrados en el sentido de que respuestas afirmativas se han obtenido a las preguntas planteadas. Es notable, el hecho de que una noción puramente algebraica como es la de homomorfismo de grupos pueda ser expresada en términos de grupos relacionados con espacios topológicos. Los espacios de Alexandroff han resultado ser bloques perfectos para tratar distintos problemas de realización de grupos. Un problema que no ha sido tratado a lo largo del tercer capítulo es la cardinalidad de los espacios empleados. A modo de ilustración ¿Cuál es el mínimo número de puntos necesarios para realizar a un grupo finito G como grupo de homeomorfismos de un espacio topológico? En esta dirección véase [15]. Otro problema a ser tratado puede ser el de la realización de otras estructuras algebraicas mediante el uso de espacios topológicos. Por ejemplo, la versión combinatoria del problema tratado en [35] podría considerarse.

Table of Contents

Chapter 1

Preliminaries

In this chapter we recall basic definitions and results from the literature. We prove auxiliary propositions that will be used in subsequent chapters and fix notation. The first section is devoted to the theory of Alexandroff spaces and finite topological spaces, a complete exposition of this topic can be found in [12] and [74]. In Section 1.2 we introduce the categorical approach that was used to define the shape theory in [72, 73], we also recall the intrinsic description of shape theory introduced in [98], which was reformulated in [3]. Another categorical approach to define shape theory can be seen in [33].

1.1 Theory of Alexandroff spaces

1.1.1 Basic definitions and properties

Definition 1.1.1. *An Alexandroff space X is a topological space with the property that the arbitrary intersection of open sets is open.*

Remark 1.1.2. *Every finite topological space is an Alexandroff space.*

Remark 1.1.3. *Let X be an Alexandroff space and let $x \in X$, U_x denotes the intersection of every open set containing x. By Definition 1.1.1, U_x is a minimal open set, that is, if V is an open set containing x, then $U_x \subseteq V$. Analogously, F_x denotes the intersection of every closed set containing x. We get that F_x is a minimal closed set.*

Example 1.1.4. We consider $X = \{A, B, C, D\}$ and the topology τ given by $\tau = \{\{X\}, \emptyset, \{A, C, D\}, \{B, C, D\}, \{C\}, \{D\}, \{C, D\}\}$. We have $U_A = \{A, C, D\}$, $U_B = \{B, C, D\}$, $U_C = \{C\}$ and $U_D = \{D\}$. In Figure 1.1.1 we present a schematic drawing of U_A, U_B, U_C and U_D.

Definition 1.1.5. *Let $(X, \leq)$ be a partially ordered set or poset. A lower (upper) set $S \subset X$ is a set satisfying that if $x \in X$ and $y \leq x$ ($y \geq x$), then $y \in S$.*

Let $(X, \leq)$ be a partially ordered set and let $x, y \in X$. If $x < y$ are such that there is no $z \in X$ satisfying $x < z < y$, then we write $x \prec y$. Let $max(X)$ ($min(X)$) denote the maximum (minimum) of X if it exists. Let $maximal(X)$ ($minimal(X)$) denote the set of maximal (minimal) points of X. A map $f : X \to Y$ between two posets is order-preserving if for every $x \leq x'$ in X then $f(x) \leq f(x')$ in Y.

Example 1.1.6. We consider the set of real numbers $\mathbb{R}$ with its usual order. It is easy to deduce that there are no $x, y \in \mathbb{R}$ satisfying that $x \prec y$.

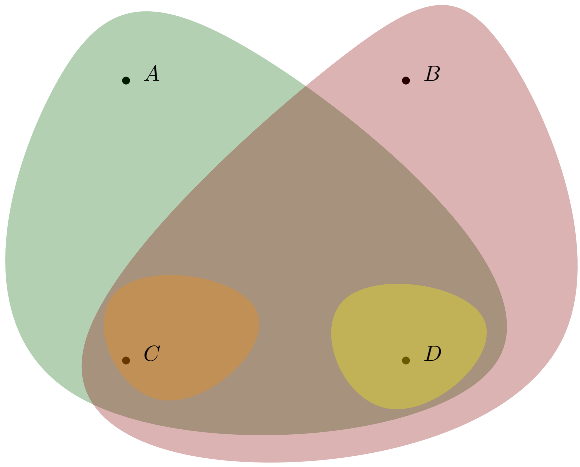

Figure 1.1.1: Schematic drawing of U_A, U_B, U_C and U_D.

Theorem 1.1.7 ([1]). *For a poset $(X, \leq)$, the family of lower (upper) sets of $\leq$ is a T_0 topology on X, that makes X an Alexandroff space. For a T_0 Alexandroff space, the relation $x \leq_\tau y$ if and only if $U_x \subseteq U_y$ ($U_y \subseteq U_x$) is a partial order on X. Moreover, the category of T_0 Alexandroff spaces and the category of posets are isomorphic.*

Consequently, T_0 Alexandroff spaces and partially ordered sets can be seen as the same object from two different perspectives. From now on, every Alexandroff space satisfies the T_0 separation axiom. We will treat Alexandroff spaces and partially ordered sets as the same object without explicit mention. The main reason to consider only Alexandroff spaces satisfying the T_0 separation axiom is the following result.

Proposition 1.1.8. *If X is an Alexandroff space satisfying the T_1 separation axiom, then the topology defined on X is the discrete topology.*

The partial order given in Theorem 1.1.7 is called the natural order, while the partial order given in parenthesis is called the opposite order. If there is no mention to the partial order considered, then we assume that we are working with the natural order.

Remark 1.1.9. *Let X be an Alexandroff space and let $x \in X$. Then $U_x = \{y \in X | y \leq x\}$ and $F_x = \{y \in X | y \geq x\}$.*

Definition 1.1.10. *Let X be an Alexandroff space. We say that X is locally finite if for every $x \in X$, then U_x is a finite set.*

Example 1.1.11. The set of the natural numbers $\mathbb{N}$ with its usual order is an Alexandroff space that is locally finite.

Definition 1.1.12. *Let X be a locally finite Alexandroff space. The Hasse diagram of X, denoted by $H(X)$, is a directed graph. The vertices of $H(X)$ are given by the points of X. There is an edge between $x, y \in X$ if $x \prec y$, where the orientation of the edge goes from the lower element to the upper element. Let $H_u(X)$ denote the undirected graph given by the Hasse diagram of X.*

We omit the orientation of the edges of subsequent Hasse diagrams because we assume an upward orientation. The condition of being locally finite required in Definition 1.1.12 is relevant because not every Alexandroff space admits a Hasse diagram. For instance, the Alexandroff space considered in Example 1.1.6.

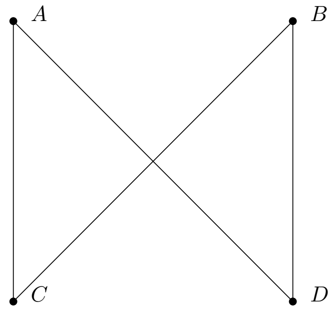

Figure 1.1.2: Hasse diagram of X.

Example 1.1.13. Let X be the finite topological space considered in Example 1.1.4. We get that $A, B > C, D$. In Figure 1.1.2 we present the Hasse diagram of X.

Topological notions can be expressed in a combinatorial way due to Theorem 1.1.7.

Proposition 1.1.14. *Let $f : X \to Y$ be a map between two Alexandroff spaces. Then f is a continuous map if and only if f is order-preserving.*

Proposition 1.1.15. *Let $f, g : X \to Y$ be two continuous maps between Alexandroff spaces. If $f(x) \leq g(x)$ for every $x \in X$, then f and g are homotopic.*

Proposition 1.1.16. *Let $f, g : X \to Y$ be two continuous maps between finite topological spaces. Then f and g are homotopic if and only if there exists a finite sequence of continuous maps from X to Y such that $f(x) = f_0(x) \leq f_1(x) \geq f_2(x) \leq ... \geq f_n(x) = g(x)$ for every $x \in X$.*

Proposition 1.1.17. *If $x_0, x_1, ..., x_n$ are points in an Alexandroff space X such that x_i is comparable to x_{i+1} for every $i = 0, .., n-1$, then there exists a continuous path from x_0 to x_n.*

Proposition 1.1.18. *Let X be an Alexandroff space. Then X is connected if and only if X is path-connected.*

1.1.2 Homotopy types of Alexandroff spaces, a combinatorial approach

We recall the combinatorial techniques developed in [103] and [67] to study the homotopy type of Alexandroff spaces.

Definition 1.1.19. *Let X be an Alexandroff space and let $x \in X$. Then x is a down (up) beat if $U_x \setminus \{x\}$ has a maximum ($F_x \setminus \{x\}$ has a minimum).*

Proposition 1.1.20. *Let X be an Alexandroff space. If $x \in X$ is a beat point, then $X \setminus \{x\}$ is a strong deformation retract of X.*

The notion of beat point was introduced originally for finite topological spaces in [103] but it can be extended to Alexandroff spaces.

Theorem 1.1.21. *Let X be a finite topological space such that it does not have beat points. Then a continuous map $f : X \to X$ is homotopic to the identity map if and only if f is the identity map.*

Definition 1.1.22. *A finite topological space X is a minimal finite topological space if X does not have beat points. The core of a finite topological space X is a strong deformation retract of X that it is also a minimal finite topological space.*

Remark 1.1.23. *Every finite topological space X admits a core. We only need to remove the beat points one by one until there are none left.*

By Theorem 1.1.21 we can obtain that the cores are unique up to homeomorphism. Concretely, two finite topological spaces are homotopy equivalent if and only if they have homeomorphic cores.

Corollary 1.1.24. *Let $f : X \to X$ be a continuous map and let X be a minimal finite topological space. Then f is a homeomorphism if and only f is a homotopy equivalence.*

Remark 1.1.25. *Let X be a finite topological space. It is easy to identify beat points looking at the Hasse diagram of X. A beat point is a vertex of $H(X)$ for which there is only one edge that enters or exits.*

Example 1.1.26. Consider the topological space given by the Hasse diagram of Figure 1.1.3. We have that E is an up and down beat point and F is an up beat point. We remove E. Now, F is still an up beat point so we can remove it without changing the homotopy type of X. The remaining space does not have beat points, it is a minimal finite topological space, which implies that it is a core of X.

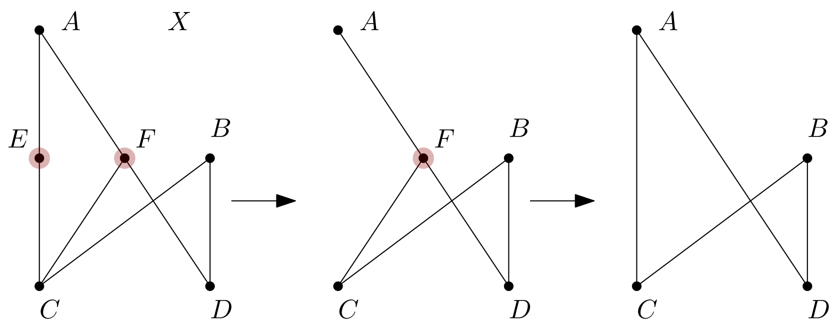

Figure 1.1.3: Removing beat points of X.

The previous results cannot be extended easily to Alexandroff spaces, they require new definitions that generalize the previous ones.

Definition 1.1.27. *Let X be an Alexandroff space. Then $r : X \to X$ is a comparative retraction if r is a retraction in the usual sense and for all $x \in X$, $r(x) \geq x$ or $r(x) \leq x$. The class of all comparative retractions is denoted by $\mathcal{C}$. The topological space X is called a $\mathcal{C}$-core if there is no other retraction $r : X \to X$ in $\mathcal{C}$ other than the identity $id : X \to X$.*

Remark 1.1.28. *If X is an Alexandroff space with a beat point x, then X is not a $\mathcal{C}$-core. Suppose that x is an up beat point, which implies that $F_x \setminus \{x\}$ has a minimum x'. We consider $r : X \to X$ given by $r(x) = x'$ and $r(y) = y$ for every $y \in X \setminus \{x\}$. It is clear that r is a comparative retraction and it is not the identity map. A similar argument can be used for a down beat point.*

Definition 1.1.29. *A $\mathcal{C}$-core X is locally a core if for every $x \in X$ there exists a finite set $A_x \subseteq X$ containing x such that for every $y \in A_x$, then $|A_x \cap maximal(\{z \in X | z < y\})| \geq 2$ if y is not minimal in X and $|A_x \cap minimal(\{z \in X | z > y\})| \geq 2$ if y is not maximal in X.*

Remark 1.1.30. *If X is a minimal finite topological space, then X is locally a core.*

Given two topological spaces X and Y, the space of continuous maps from X to Y equipped with the compact-open topology is denoted by $C(X,Y)$.

Theorem 1.1.31 ([67]). *If X is a locally core, then there is no map in $C(X,X)$ homotopic to the identity other than the identity.*

Corollary 1.1.32. *Let X be locally a core and let $f: X \to X$ be a continuous map. Then f is a homeomorphism if and only if f is a homotopy equivalence.*

Given a topological space X, $Aut(X)$ denotes the group of homeomorphisms of X and $\mathcal{E}(X)$ denotes the group of homotopy classes of self-homotopy equivalences of X.

Corollary 1.1.33. *If X is locally a core, then $Aut(X)$ and $\mathcal{E}(X)$ are isomorphic.*

Proof. We define $\varphi : Aut(X) \to \mathcal{E}(X)$ given by $\varphi(f) = [f]$, where $[f]$ denotes the homotopy class of f. We have that φ is clearly well-defined and a homomorphism of groups. If $\varphi(f)$ is the homotopy class of the identity map, then we get that f is the identity map by Theorem 1.1.31. Therefore, φ is a monomorphism of groups. If f is a self-homotopy equivalence of X, then there exists a continuous map $g : X \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity map id_X. By Theorem 1.1.31, we obtain that $g \circ f = id_X$ and $f \circ g = id_X$. Thus, it can be deduced that f is indeed a homeomorphism. From here, we obtain that φ is an isomorphism of groups. □

Remark 1.1.34. *If X is locally a core, then each homotopy class of the group $\mathcal{E}(X)$ contains exactly one element. We will refer and treat an element $[f] \in \mathcal{E}(X)$ just as $f \in Aut(X)$ and vice versa. We identify both groups so $Aut(X) = \mathcal{E}(X)$.*

1.1.3 Weak homotopy types of Alexandroff spaces

A key concept in the theory of Alexandroff spaces is the notion of weak homotopy equivalence.

Definition 1.1.35. *Let $f : X \to Y$ be a continuous map between two topological spaces. Then f is a weak homotopy equivalence if it induces isomorphism on homotopy groups. Two topological spaces are weak homotopy equivalent if there exists a sequence of spaces $X = X_0, X_1, ..., X_n = Y$ such that there exists a weak homotopy equivalence $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$ for every $0 \leq i \leq n-1$.*

Remark 1.1.36. *Weak homotopy equivalences satisfy the 2-out-3 property, that is, let $f : X \to Y$ and $g : Y \to Z$ be two maps, if 2 of the 3 maps $f, g, g \circ f$ are weak homotopy equivalences, then so is the third.*

The weak homotopy type of finite topological spaces can be studied from a combinatorial point of view as it was done in the previous subsection. In [18], a generalization of beat point is introduced. J.A. Barmak and E.G. Minian define also simple homotopy for finite spaces.

Definition 1.1.37. *Let X be a finite topological space and $x \in X$. Then x is a down (up) weak beat point if $U_x \setminus \{x\}$ ($F_x \setminus \{x\}$) is contractible.*

Proposition 1.1.38. *Let X be a finite topological space and let $x \in X$. If x is a down (up) weak beat point, then the inclusion $i : X \setminus \{x\} \to X$ is a weak homotopy equivalence.*

Definition 1.1.39. *Let X be a finite topological space and let $Y \subset X$. It is said that X collapses to Y by an elementary collapse $X \searrow Y$ if Y is obtained from X by removing a weak beat point. Given two finite topological spaces X and Y, X collapses to Y if there is a sequence $X = X_1, X_2, ..., X_n = Y$ of finite topological spaces such that for each $1 \leq i < n$, X_i collapse to X_{i+1} by an elementary collapse $X_i \searrow X_{i+1}$. A finite topological space is collapsible if it collapses to a point. Two finite topological spaces X and Y are simple homotopy equivalent if there is a sequence $X = X_1, ..., X_n = Y$ of finite topological spaces such that for each $1 \leq i < n$, $X_i \searrow X_{i+1}$ or $X_{i+1} \searrow X_i$.*

Lemma 1.1.40. *Homotopy equivalent finite topological spaces are simple homotopy equivalent.*

A different approach was made in [75]. One way to compute the homology groups or homotopy groups of Alexandroff spaces is to study simplicial complexes.

Definition 1.1.41. *Let X be an Alexandroff space. The order complex or McCord complex $\mathcal{K}(X)$ is a simplicial complex. The simplices are given by the totally ordered subsets of X. $|\mathcal{K}(X)|$ denotes the geometric realization of $\mathcal{K}(X)$.*

Remark 1.1.42. *Let X be a finite topological space. Then X can be equipped with two partial orders, the natural and the opposite. It is simple to check that the McCord complex of X does not depend on the partial order chosen. If X^o denotes X with the opposite partial order and X denotes X with the natural partial order, then $|\mathcal{K}(X^o)| = |\mathcal{K}(X)|$.*

Let X be an Alexandroff space. If $u \in |\mathcal{K}(X)|$, then u is contained in a minimal open simplex $(x_0, ..., x_r)$, where $x_i \in X$ and $x_0 < ... < x_r$. Hence, $f_X : |\mathcal{K}(X)| \to X$ given by $f_X(u) = x_0$ is well-defined. Furthermore, if $g : X \to Y$ is a continuous map between two Alexandroff spaces, then g induces a natural simplicial map $\mathcal{K}(g) : \mathcal{K}(X) \to \mathcal{K}(Y)$.

Theorem 1.1.43. *Let X be an Alexandroff space. Then $f_X : |\mathcal{K}(X)| \to X$ is a weak homotopy equivalence. Moreover, if $g : X \to Y$ is a continuous map between two Alexandroff spaces, then $f_Y \circ \mathcal{K}(g) = g \circ f_X$.*

The proof of Theorem 1.1.43 relies in the following theorem proved in [40]. Before stating this theorem, we provide a definition.

Definition 1.1.44. *An open cover $\mathcal{U}$ of a space B will be called basis-like if whenever $U, V \in \mathcal{U}$ and $x \in U \cap V$, there exists $W \in \mathcal{U}$ such that $x \in W \subset U \cap V$.*

Remark 1.1.45. *Every Alexandroff space X admits a basis-like open cover given by $\mathcal{U} = \{U_x | x \in X\}$.*

Theorem 1.1.46. *Suppose p is a map of a space E into a space B for which there exists a basis-like open cover $\mathcal{U}$ of B satisfying the following condition: for each $U \in \mathcal{U}$, the restriction $p_{|p^{-1}(U)} : p^{-1}(U) \to U$ is a weak homotopy equivalence. Then p itself is a weak homotopy equivalence.*

There is also a correspondence that assign to each simplicial complex an Alexandroff space that is weak homotopy equivalent to it.

Definition 1.1.47. *Let L be a simplicial complex. Then $\mathcal{X}(L)$ denotes the poset given by the subset relation, where the points are the simplices of L.*

Remark 1.1.48. *If L is a simplicial complex, then $\mathcal{K}(\mathcal{X}(L))$ is the barycentric subdivision of L. Hence, $|L| = |\mathcal{K}(\mathcal{X}(L))|$*

Let $g : L \to K$ be a simplicial map. We consider $\mathcal{X}(g) : \mathcal{X}(L) \to \mathcal{X}(K)$ given by $\mathcal{X}(g)(x) = g(\sigma) = y$, where x is the point in $\mathcal{X}(L)$ given by the simplex σ and y is the point in $\mathcal{X}(K)$ given by the simplex $g(\sigma)$.

Theorem 1.1.49. *Let L be a simplicial complex. Then $f_{\mathcal{X}(L)} : |L| \to \mathcal{X}(L)$ is a weak homotopy equivalence. Furthermore, if $g : L \to K$ is a simplicial map between two simplicial complexes, then $\mathcal{X}(g) : \mathcal{X}(L) \to \mathcal{X}(K)$ is a continuous map satisfying that $f_{\mathcal{X}(K)} \circ g$ is homotopic to $\mathcal{X}(g) \circ f_{\mathcal{X}(L)}$.*

Remark 1.1.50. *By construction, $\mathcal{K}$ and $\mathcal{X}$ are covariant functors. They are related by the relation given in Remark 1.1.48*

Example 1.1.51. Let X be the finite topological space considered in Example 1.1.4. Hence, $\mathcal{K}(X)$ is a triangulation of S^1. We define L given by the 2-dimensional simplex. In Figure 1.1.4 we have the Hasse diagram of $\mathcal{X}(L)$.

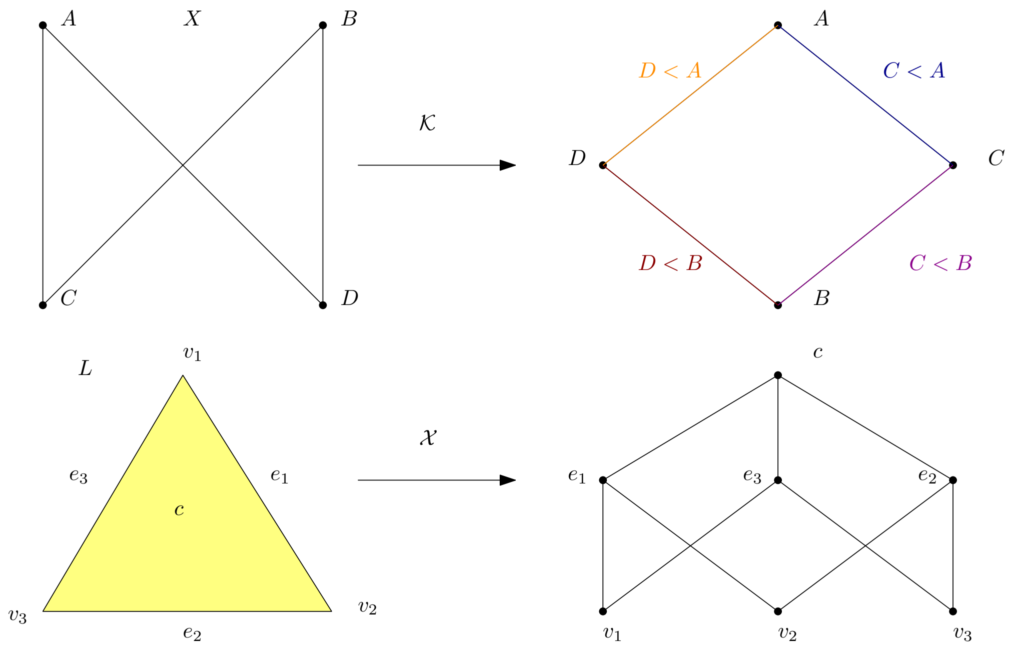

Figure 1.1.4: Representations of L, $\mathcal{X}(L)$, X and $\mathcal{K}(X)$.

Remark 1.1.52. *If X is a finite topological space collapsible, then $\mathcal{K}(X)$ is also collapsible.*

The definition of collapse as well as simple homotopy type for simplicial complexes are the classical ones, where we have the elementary collapses for simplicial complexes introduced in [110].

In [12, Corollary 5.2.8], it can be found a characterization of being a contractible finite topological space in terms of simplicial complexes. The notion of strong collapse is given in [12, Chapter 5] or [19]

Theorem 1.1.53. *A finite T_0 topological space X is contractible if and only if $\mathcal{K}(X)$ is strong collapsible.*

Definition 1.1.54. *The Euler characteristic of a finite poset $(X, \leq)$ is defined as the alternate sum of the number of i-chains, where an i-chain in X is a chain $v_0 < ... < v_i$ with $i+1$ elements in X. The Euler characteristic of X is denoted by $\chi(X)$. Then*

$$\chi(X) = \sum_{i=0} (-1)^i |\{v_0 < ... < v_i | v_j \in X \text{ for every } j\}|.$$

Remark 1.1.55. *Let X be a finite topological space. The Euler characteristic defined in Definition 1.1.54 coincides with the classical Euler characteristic of $\mathcal{K}(X)$.*

Finally, we give a definition that will be used in subsequent chapters.

Definition 1.1.56. *Let K and L be two simplicial complexes and let $f, g : |K| \to |L|$ be two continuous maps. Then f is simplicially close to g if and only if for every x, there exists a simplex σ_x in L containing in its closure $f(x)$ and $g(x)$.*

Proposition 1.1.57. *Let K and L be two simplicial complexes and let $f, g : |K| \to |L|$ be two continuous maps. If f is simplicially close to g, then f is homotopic to g.*

1.1.4 Basic constructions for Alexandroff spaces

We recall some natural constructions with Alexandroff spaces to obtain new Alexandroff spaces.

Definition 1.1.58. *The **product** of two Alexandroff spaces X and Y is given by the Cartesian product $X \times Y$, where it is considered the product order, i.e., $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$.*

Proposition 1.1.59. *Let X_c and Y_c be cores of finite topological spaces X and Y respectively. Then $X_c \times Y_c$ is a core of $X \times Y$.*

Proposition 1.1.60. *Let X and Y be Alexandroff spaces. Then $|\mathcal{K}(X \times Y)|$ is homeomorphic to $|\mathcal{K}(X)| \times |\mathcal{K}(Y)|$.*

Definition 1.1.61. *Let $f : X \to Y$ be a map between finite topological spaces. The non-Hausdorff **mapping cylinder** $B(f)$ is defined as the following finite topological space. The underlying set is the disjoint union $X \sqcup Y$. $B(f)$ keeps the given orderings within X and Y and for $x \in X$, $y \in Y$, $x \leq y$ in $B(f)$ if $f(x) \leq y$ in Y.*

Lemma 1.1.62. *Let $f : X \to Y$ be a continuous map between finite topological spaces. Then Y is a strong deformation retract of $B(f)$.*

Definition 1.1.63. *The non-Hausdorff **join** of two Alexandroff spaces is the disjoint union $X \sqcup Y$ keeping the given ordering within X and Y and setting $x \leq y$ for every $x \in X$ and $y \in Y$. The join of X and Y is denoted by $X \circledast Y$.*

Proposition 1.1.64. *Let X and Y be Alexandroff spaces. Then $\mathcal{K}(X \circledast Y) = \mathcal{K}(X) * \mathcal{K}(Y)$, where $*$ denotes the usual join of simplicial complexes. If X and Y are finite topological spaces and one of them is collapsible, then $\mathcal{K}(X \circledast Y)$ is collapsible.*

Proposition 1.1.65. *If X and Y are Alexandroff spaces, then $Aut(X * Y) = Aut(X) \times Aut(Y)$.*

There is also a definition of barycentric subdivision for finite topological spaces, see for instance [58] and [109].

Definition 1.1.66. *Let X be a finite topological space. The **finite barycentric subdivision** of X is given by $\mathcal{X}(\mathcal{K}(X))$.*

The finite barycentric subdivision of a finite topological space X is denoted by X'. The n-th finite barycentric subdivision of X is denoted by X^n.

Lemma 1.1.67. *Let X be a finite topological space. Then X and X' are simple homotopy equivalent.*

Remark 1.1.68. *Let X be a finite topological space. Then X' can be seen as the poset whose points are the chains in X, i.e., if $x' \in X'$, then x' is given by a chain $x_1 < ... < x_n$ in X. Hence, the partial order defined on X' is given by the subset relation.*

There is a natural map between a finite topological space X and its finite barycentric subdivision X'. We define $h : X' \to X$ given by $h(x') = x_n$, where x' is a chain $x_1 < ... < x_n$ in X. We have that $h_{n+1,n} : X^{n+1} \to X^n$ denotes the corresponding natural map between the finite barycentric subdivisions.

Proposition 1.1.69. *Let X be a finite topological space. Then $h : X' \to X$ is a weak homotopy equivalence.*

Proposition 1.1.70. *Let X be a finite topological space. Then $|\mathcal{K}(h)| : |\mathcal{K}(X')| \to |\mathcal{K}(X)|$ is simplicially close to the identity map, which means that h induces the identity map in homology.*

Example 1.1.71. Let X be the finite topological considered in Example 1.1.4. In Figure 1.1.5 we present a schematic draw of $h : X' \to X$.

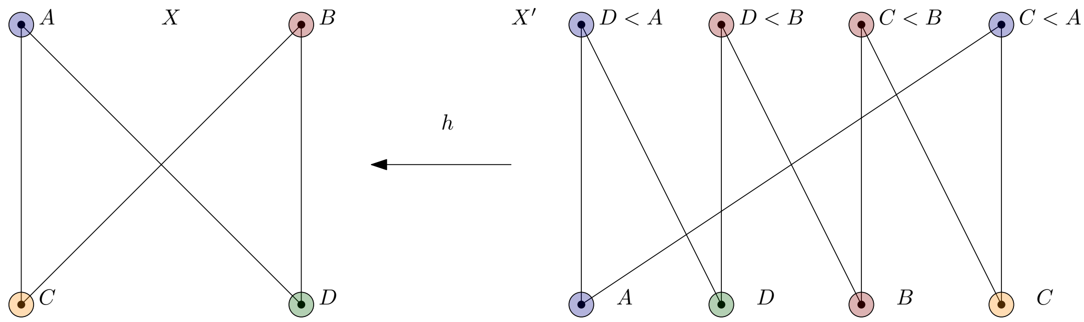

Figure 1.1.5: Hasse diagrams of X and X'.

For a complete exposition and more constructions involving Alexandroff spaces see [12] and [74].

1.1.5 Properties preserved by homeomorphisms

We introduce auxiliary definitions and enunciate some properties that are preserved by homeomorphisms.

Definition 1.1.72. *Given a finite topological space X, the height $ht(X)$ is one less than the maximum number of elements in a chain of X. The height of a point x in a locally finite Alexandroff space is given by $ht(U_x)$. For a general Alexandroff space X the height of a point $x \in X$ is defined as ∞ if U_x contains a chain without a minimum and $ht(U_x)$ otherwise.*

Definition 1.1.73. *Given a finite topological space X and $x \in X$. Let $P_x = (E_x, S_x)$ denote the cardinal numbers $E_x = |\{y \in X | y \prec x\}|$ and $S_x = |\{y \in X | y \succ x\}|$.*

Remark 1.1.74. *Let X be an Alexandroff space and let $x \in X$. If $P_x = (1, b)$ or $P_x = (a, 1)$, then x is a beat point.*

Example 1.1.75. Let us consider the set of the real numbers $\mathbb{R}$ with its usual order. For every $x \in \mathbb{R}$ we get $ht(x) = \infty$ because $... < x - n < ... < x - 2 < x - 1 < ... < x$ is a chain that does not have a minimum. Since there is no $y \in \mathbb{R}$ satisfying that $x \prec y$ or $y \prec x$, it follows that $P_x = (0, 0)$. Now, let us consider $X = \mathbb{N} \cup \{*\}$, where we are considering $\mathbb{N}$ just as a set with any partial order or additional structure and we declare $n < *$ for every $n \in \mathbb{N}$. Then $U_* = X$ but $ht(*) = 1$. Furthermore, $P_* = (|\mathbb{N}|, 0)$. In Figure 1.1.6 we present the Hasse diagram of X.

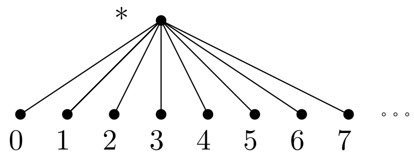

Figure 1.1.6: Hasse diagram of X.

Proposition 1.1.76. *Let X be an Alexandroff space and let $f : X \to X$ be a homeomorphism.*

1. *If $x, y \in X$ are such that $x \prec y$, then $f(x) \prec f(y)$.*
2. *If $x \in X$, then $ht(x) = ht(f(x))$.*
3. *If $x \in X$ is a maximal (minimal) point, then $f(x)$ is a maximal (minimal) point.*
4. *If $x \in X$, then $P_x = P_{f(x)}$.*
5. *If $x \in X$ is a down (up) weak beat point, then $f(x)$ is a down (up) weak beat point.*

Proof. We prove (1). We argue by contradiction. Suppose there exists $z \in X$ such that $f(x) < z < f(y)$. Therefore, $x < f^{-1}(z) < y$, which entails a contradiction.

We have that (3),(4) and (5) follow easily.

We prove (6). Suppose that x is a down weak beat point. It is simple to get that $f(U_x) = U_{f(x)}$, which implies that $f(U_x \setminus \{x\}) = U_{f(x)} \setminus \{f(x)\}$. From here, it can be deduced the desired result. □

1.2 Shape Theory

1.2.1 Geometrical motivation

The goal of shape theory is to study global properties of topological spaces. The idea is to generalize the homotopy theory. Homotopy theory works very nice for topological spaces having good local properties, e.g., CW-complexes. An example of the previous assertion is the following result: a continuous map $f : X \to Y$ between two CW-complexes that induces isomorphisms on all homotopy groups is indeed a homotopy equivalence, see [111, 112]. However, a similar result for general topological spaces does not hold, see Section 1.1. Nevertheless, the classical example is the Warsaw circle W. The Warsaw circle is a subset of $\mathbb{R}^2$ given by

$$W = \{(x, \sin(\frac{1}{x}))|x \in (0, 2\pi]\} \cup \{(0, y)|y \in [-1, 1]\} \cup C,$$

where C is an arc of $\mathbb{R}^2$ joining $(2\pi, 0)$ and $(0, -1)$ such that $C \setminus \{(2\pi, 0), (0, -1)\}$ and $\{(x, \sin(x))|x \in (0, 2\pi]\} \cup \{(0, y)|y \in [-1, 1]\}$ are disjoint sets. We have that W is a compact metric space that it is connected and arc-connected but not locally connected. Moreover, the homotopy groups of W are trivial since every continuous map $f : S^n \to W$, where S^n denotes the n-dimensional sphere and $n \in \mathbb{N}$, is null-homotopic. Therefore, the continuous map $f : * \to W$ given by $f(*) = (0, 0)$ induces isomorphisms on all homotopy groups but W is not contractible, which implies that f is not a homotopy equivalence. From a global point of view, the Warsaw circle seems very similar to a circle. In an informal way, we could say that the Warsaw circle seems a bad draw of a circle. But there is no continuous map from one to another being non-trivial. If we look at small neighborhoods of W as a subspace of $\mathbb{R}^2$, then we can observe that they have the same homotopy type of a circle. In Figure 1.2.1 we have a representation of the Warsaw circle and a small neighborhood of W.

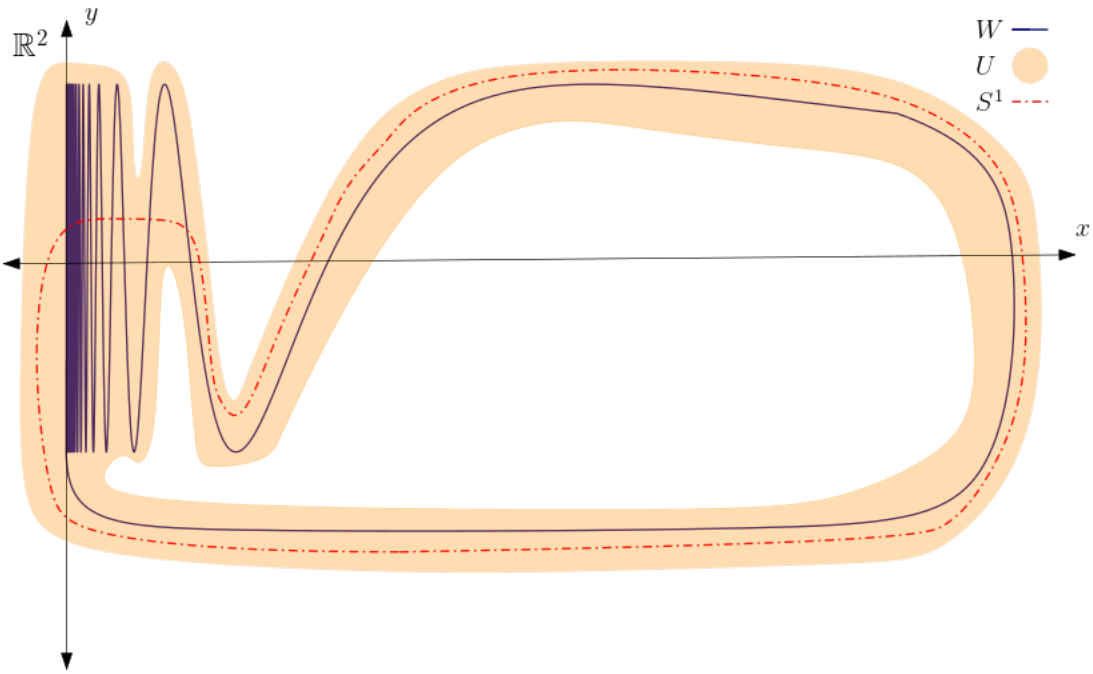

Figure 1.2.1: The Warsaw circle, a small neighborhood U of W and a strong deformation retract $S^1 \subset U$ of U.

The idea to overcome the lack of morphisms is to consider morphisms from a system of neighborhoods that approximates the compact metric space. In that way, we will be able to get more morphisms and even to say that the Warsaw circle and the circle have the same shape although they are not homotopy equivalent. To develop this idea to general compact metric spaces is necessary to embed them in a good ambient space. The original idea used in [27] is to use the Hilbert cube. Once the compact metric space is embedded there, it is considered a decreasing sequence of neighborhoods. Then, the morphisms between two compact metric spaces are given in terms of morphisms between the sequences.

Shape theory provides a classification for compact metric spaces weaker than the homotopy theory. It coincides with homotopy theory when it is applied to CW-complexes or ANR's. One possible application of shape theory is to the study of dynamical systems. This is due to the lack of good local properties for some invariant sets, e.g., the Lorenz attractor. We recall shape theory in subsequent subsections. Instead of following the classical approach of [27], we follow the categorical approach of [73].

1.2.2 Inv-Categories and Pro-Categories

Given a partially ordered set $(\Lambda, \leq)$, a subset $\Lambda' \subset \Lambda$ is cofinal in Λ if for each $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda'$ such that $\lambda < \lambda'$. A partially ordered set $(\Lambda, \leq)$ is a directed set if for any $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\lambda_1 \leq \lambda$ and $\lambda_2 \leq \lambda$.

Definition 1.2.1. *Let $\mathcal{C}$ be an arbitrary category. An inverse system in the category $\mathcal{C}$ consists of a directed set Λ, that is called the index set, of an object X_λ from $\mathcal{C}$ for each $\lambda \in \Lambda$ and of a morphism $p_{\lambda,\lambda'} : X_{\lambda'} \to X_\lambda$ from $\mathcal{C}$ for each pair $\lambda \leq \lambda'$. Moreover, one requires that $p_{\lambda,\lambda}$ is the identity map and that $\lambda \leq \lambda'$ and $\lambda' \leq \lambda''$ implies $p_{\lambda,\lambda'} \circ p_{\lambda',\lambda''} = p_{\lambda,\lambda''}$. An inverse system is denoted by $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$, where the X_λ's are called the terms and $p_{\lambda,\lambda'}$ are called the bonding maps of $\mathbb{X}$.*

Remark 1.2.2. *An inverse system indexed by the natural numbers with its usual order is called an inverse sequence and it is denoted by $\mathbb{X} = (X_n, p_{n,n+1})$. In an inverse sequence $(X_n, p_{n,n+1})$, it suffices to know the morphisms $p_{n,n+1} : X_{n+1} \to X_n$ for every $n \in \mathbb{N}$ because the remaining bonding maps are obtained by composition.*

$$X_1 \xleftarrow[p_{1,2}]{} X_2 \xleftarrow[p_{2,3}]{} X_3 \xleftarrow[p_{3,4}]{} X_4 \xleftarrow[p_{4,5}]{} \cdots \xleftarrow[p_{n-1,n}]{} X_n \xleftarrow[p_{n,n+1}]{} \cdots$$

Remark 1.2.3. *Every object X of $\mathcal{C}$ can be seen as an inverse system (X), where every term is X and every bonding map is the identity map. This inverse system is called rudimentary system.*

A morphism of inverse systems $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda) \to \mathbb{Y} = (Y_\mu, q_{\mu,\mu'}, M)$ consists of a function $\phi : M \to \Lambda$ and of morphisms $f_\mu : X_{\phi(\mu)} \to Y_\mu$ in $\mathcal{C}$ for each $\mu \in M$ such that whenever $\mu \leq \mu'$, then there exists $\lambda \in \Lambda$ satisfying that $\lambda \geq \phi(\mu), \phi(\mu')$, for which $f_\mu \circ p_{\phi(\mu),\lambda} = q_{\mu,\mu'} \circ f_{\mu'} \circ p_{\phi(\mu'),\lambda}$, that is, the following diagram commutes.

$$\begin{tikzcd}
X_{\phi(\mu)} \arrow[d, "f_\mu"] & X_\lambda \arrow[l, "p_{\phi(\mu),\lambda}"'] \arrow[r, "p_{\phi(\mu'),\lambda}"] & X_{\phi(\mu')} \arrow[d, "f_{\mu'}"] \\
Y_\mu & & Y_{\mu'} \arrow[ll, "q_{\mu,\mu'}"]
\end{tikzcd}$$

A morphism of inverse systems is denoted by $(f_\mu, \phi) : \mathbb{X} \to \mathbb{Y}$. Let $\mathbb{Z} = (Z_\nu, r_{\nu,\nu'}, N)$ be an inverse system and let $(g_\nu, \psi) : \mathbb{Y} \to \mathbb{Z}$ be a morphism of systems. Then the composition $(g_\nu, \psi) \circ (f_\mu, \phi) = (h_\nu, \chi) : \mathbb{X} \to \mathbb{Z}$ is given as follows: $\chi = \phi \circ \psi : N \to \Lambda$ and $h_\nu = g_\nu \circ f_{\psi(\nu)} : X_{\chi(\nu)} \to Z_\nu$. It is routine to check that the composition is well-defined and associative. The identity morphism $(id_\lambda, id_\Lambda) : \mathbb{X} \to \mathbb{X}$ is given by the identity map $id_\Lambda : \Lambda \to \Lambda$ and the identity morphisms $id_\lambda : X_\lambda \to X_\lambda$. It is easy to check that $(f_\mu, \phi) \circ (id_\lambda, id_\Lambda) = (f_\mu, \phi)$ and $(id_\mu, id_M) \circ (f_\mu, \phi) = (f_\mu, \phi)$. Thus, we have a category inv-$\mathcal{C}$, whose objects are all inverse systems in $\mathcal{C}$ and whose morphisms are the morphisms of inverse systems described above.

Let $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$ and $\mathbb{Y} = (Y_\lambda, q_{\lambda,\lambda'}, \Lambda)$ be two inverse systems over the same directed set Λ. A morphism of systems (f_λ, φ) is a level morphisms of systems provided φ is the identity map and for $\lambda \leq \lambda'$ the following diagram commutes.

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"] & X_{\lambda'} \arrow[l, "p_{\lambda,\lambda'}"'] \arrow[d, "f_{\lambda'}"] \\
Y_\lambda & Y_{\lambda'} \arrow[l, "q_{\lambda,\lambda'}"]
\end{tikzcd}$$

Given two morphisms $(f_\mu, \phi), (f'_\mu, \phi') : \mathbb{X} \to \mathbb{Y}$, we write that $(f_\mu, \phi) \sim (f'_\mu, \phi')$ if and only if each $\mu \in M$ admits $\lambda \in \Lambda$ satisfying that $\lambda \geq \phi(\mu), \phi'(\mu)$ and that the following diagram commutes.

$$\begin{tikzcd}
X_{\phi(\mu)} \arrow[dr, "f_\mu"'] & X_\lambda \arrow[l, "p_{\phi(\mu),\lambda}"'] \arrow[r, "p_{\phi'(\mu),\lambda}"] & X_{\phi'(\mu)} \arrow[dl, "f_\mu'"] \\
& Y_\mu &
\end{tikzcd}$$

Again, it is routine to check that $\sim$ is an equivalence relation. We have the category pro-$\mathcal{C}$ for the category $\mathcal{C}$. The objects of pro-$\mathcal{C}$ are all inverse systems in $\mathcal{C}$. A morphism $f : \mathbb{X} \to \mathbb{Y}$ is an equivalence class of morphisms of systems with respect to the equivalence relation $\sim$.

Let Λ be a directed set. If $\Lambda' \subset \Lambda$ is a directed set and $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$ is an inverse system, then $\mathbb{X}' = (X_\lambda, p_{\lambda,\lambda'}, \Lambda')$ is a subsystem of $\mathbb{X}$. There is a natural morphism (i_λ, i) given by $i(\lambda) = \lambda$ and $i_\lambda = id_\lambda : X_\lambda \to X_\lambda$ for every $\lambda \in \Lambda'$. The morphism $i : \mathbb{X}' \to \mathbb{X}$ represented by (i_λ, i) is called the restriction morphism.

Theorem 1.2.4. *If Λ' is cofinal in Λ, then the restriction morphism $i : \mathbb{X}' \to \mathbb{X}$ is an isomorphism in pro-$\mathcal{C}$.*

Finally, we recall Morita's lemma, see [83] or [73, Chapter 2, Theorem 5].

Theorem 1.2.5. *Let $\mathcal{C}$ be a category and let $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda'}, \Lambda)$ and $\mathbb{Y} = (Y_\lambda, q_{\lambda,\lambda'}, \Lambda)$ be inverse systems over the same index set Λ. Let $f : \mathbb{X} \to \mathbb{Y}$ be a morphism of pro-$\mathcal{C}$ given by a level morphism of systems $(f_\lambda) : \mathbb{X} \to \mathbb{Y}$. Then the morphism f is an isomorphism of pro-$\mathcal{C}$ if and only if every $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ and a morphism $g_\lambda : Y_\lambda \to X_\lambda$ of $\mathcal{C}$ such that the following diagram commutes.*

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"] & X_{\lambda'} \arrow[l, "p_{\lambda,\lambda'}"'] \arrow[d, "f_{\lambda'}"] \\
Y_\lambda & Y_{\lambda'} \arrow[l, "q_{\lambda,\lambda'}"] \arrow[ul, "g_\lambda"]
\end{tikzcd}$$

1.2.3 Expansions

Definition 1.2.6. *Let $\mathcal{T}$ be a category and let $\mathcal{P}$ be a subcategory of $\mathcal{T}$. For an object X of $\mathcal{T}$, a $\mathcal{T}$-expansion of X (with respect to $\mathcal{P}$) is a morphism in pro-$\mathcal{T}$ of X to an inverse system $\mathbb{X} = (X_\lambda, p_{\lambda,\lambda',\Lambda})$ in $\mathcal{T}$, $p : X \to \mathbb{X}$, satisfying that for any inverse system $\mathbb{Y} = (Y_\mu, q_{\mu,\mu',M})$ in the subcategory $\mathcal{P}$ and any morphism $h : X \to \mathbb{Y}$ in pro-$\mathcal{T}$, there exists a unique morphism $f : \mathbb{X} \to \mathbb{Y}$ in pro-$\mathcal{T}$ such that $h = f \circ p$, i.e., the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{X} \arrow[dr, "f"] & X \arrow[l, "p"] \arrow[d, "h"] \\
& \mathbb{Y}
\end{tikzcd}$$

It is said that p is a $\mathcal{P}$-expansion of X provided $\mathbb{X}$ and f are in pro-$\mathcal{P}$.

Proposition 1.2.7. *Let $p : X \to \mathbb{X}$ and $p' : X \to \mathbb{X}'$ be two $\mathcal{P}$-expansions of the same object X. Then there is a unique morphism $i : \mathbb{X} \to \mathbb{X}'$ such that $i \circ p = p'$.*

Proposition 1.2.8. *If $i : \mathbb{X} \to \mathbb{X}'$ is an isomorphism in pro-$\mathcal{T}$ and $p : X \to \mathbb{X}$ is an expansion, then $i \circ p : X \to \mathbb{X}'$ is also an expansion.*

Definition 1.2.9. *Let $\mathcal{T}$ be a category and let $\mathcal{P}$ be a subcategory. The subcategory $\mathcal{P}$ is dense in the category $\mathcal{T}$ provided every object X in $\mathcal{T}$ admits a $\mathcal{P}$-expansion $p : X \to \mathbb{X}$.*

Let Top denote the topological category, i.e., the objects are the topological spaces and the morphisms are the continuous maps. Let $HTop$ denote the homotopy category of topological spaces, that is, the objects are the topological spaces and the morphisms are the homotopy classes of continuous maps. Finally, $HPol$ denotes the homotopy category of polyhedra, which is a subcategory of $HTop$.

Theorem 1.2.10. *The homotopy category of polyhedra $HPol$ is a dense subcategory of the homotopy category $HTop$.*

We describe a classical $HPol$-expansion for a compact metric space X, the Čech expansion. Since for a compact metric space X every open cover is a normal cover, we prefer to omit the definition of normal cover for simplicity.

Definition 1.2.11. *Given a topological space X and open covers $\mathcal{U}$, $\mathcal{V}$ of X. Then $\mathcal{V}$ is finer than $\mathcal{U}$ if and only if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{V}$ is finer than $\mathcal{U}$.*

Proposition 1.2.12. *Any two open covers $\mathcal{U}'$,$\mathcal{U}''$ of a compact metric space X admit an open cover $\mathcal{U}$ which refines both.*

Hence, the set of open covers is a directed set under the relation of being a finer cover.

Definition 1.2.13. *Given an open cover $\mathcal{U}$ of a compact metric space X. The nerve of $\mathcal{U}$, denoted by $N(\mathcal{U})$, is a simplicial complex. Its vertices are the elements U of $\mathcal{U}$ and $U_0, ..., U_n \in \mathcal{U}$ span a simplex of $N(\mathcal{U})$ whenever $U_0 \cap ... \cap U_n \neq \emptyset$.*

Given a compact metric space X and two open covers $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U} \leq \mathcal{V}$. We get a natural simplicial map $p_{\mathcal{U},\mathcal{V}} : N(\mathcal{V}) \to N(\mathcal{U})$, a vertex $V \in \mathcal{V}$ is mapped to a vertex $U \in \mathcal{U}$ if $V \subseteq U$. This simplicial map is called projection and determines a continuous map $|N(\mathcal{V})| \to |N(\mathcal{U})|$ for which it is used the same notation $p_{\mathcal{U},\mathcal{V}}$. If $\mathcal{U} \leq \mathcal{V} \leq \mathcal{W}$ are open covers for X, then $p_{\mathcal{U},\mathcal{V}} \circ p_{\mathcal{V},\mathcal{W}} : N(\mathcal{W}) \to N(\mathcal{U})$ is also a projection. Projections are not unique, but it can be shown the following result.

Proposition 1.2.14. *Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a compact metric space X. Any two projections $p, p' : |N(\mathcal{V})| \to |N(\mathcal{U})|$ are contiguous and thus also homotopic.*

Definition 1.2.15. *Let $\mathcal{U}$ be an open cover of a compact metric space X. A canonical map $p : X \to |N(\mathcal{U})|$ is a map such that for each $U \in \mathcal{U}$, $p^{-1}(St(U, N(\mathcal{U}))) \subseteq U$, where $St(U, N(\mathcal{U}))$ is the star of the vertex U in $N(\mathcal{U})$.*

Proposition 1.2.16. *If X is a compact metric space and $\mathcal{U}$ is an open cover of X, then there exists a canonical map $X \to |N(\mathcal{U})|$. Moreover, if $p, p' : X \to |N(\mathcal{U})|$ are canonical maps, then p and p' are contiguous and thus also homotopic.*

Given a compact metric space X, $C(X) = (X_{\mathcal{U}}, [p_{\mathcal{U},\mathcal{U}'}], \Lambda)$ is called the Čech system of X , where Λ is the set of all open covers of X ordered by the relation of being a finer cover, $X_{\mathcal{U}} = |N(\mathcal{U})|$ is the nerve of $\mathcal{U}$ and $[p_{\mathcal{U},\mathcal{U}'}]$ is the unique homotopy class to which belong the projections $p_{\mathcal{U},\mathcal{U}'} : |N(\mathcal{U}')| \to |N(\mathcal{U})|$. For every $U \in \Lambda$, $[p_{\mathcal{U}}] : X \to X_{\mathcal{U}}$ is the unique homotopy class of the canonical mapping $p_{\mathcal{U}} : X \to X_{\mathcal{U}}$. It can be proved that $p_{\mathcal{U},\mathcal{U}'} \circ p_{\mathcal{U}'} = p_{\mathcal{U}}$. Then, $p = (p_{\mathcal{U}}) : X \to C(X)$ is a morphism of pro-$HTop$.

Theorem 1.2.17. *Let X be a compact metric space. The morphism $p : X \to C(X)$ of pro-$HTop$ is a $HPol$-expansion.*

1.2.4 The shape category and topological shape

Let $\mathcal{T}$ be an arbitrary category and let $\mathcal{P}$ be a dense subcategory. Let $p : X \to \mathbb{X}$ and $p' : X \to \mathbb{X}'$ be $\mathcal{P}$-expansions of X and let $q : Y \to \mathbb{Y}$ and $q' : Y \to \mathbb{Y}'$ be $\mathcal{P}$-expansions of Y. The morphisms $f : \mathbb{X} \to \mathbb{Y}$ and $f' : \mathbb{X}' \to \mathbb{Y}'$ in pro-$\mathcal{P}$ are equivalent $f \sim f'$ if and only if the following diagram in pro-$\mathcal{P}$ commutes.

$$\begin{tikzcd} \mathbb{X} \arrow[r, "i"] \arrow[d, "f"] & \mathbb{X}' \arrow[d, "f'"] \\ \mathbb{Y} \arrow[r, "j"] & \mathbb{Y}' \end{tikzcd}$$

Where i and j are the morphisms of Proposition 1.2.7. It can be proved that $\sim$ is an equivalence relation.

The shape category for $(\mathcal{T}, \mathcal{P})$ is denoted by $Sh_{(\mathcal{T},\mathcal{P})}$ or Sh, its objects are all the objects of $\mathcal{T}$. The morphisms $X \to Y$ of Sh are equivalence classes with respect to $\sim$ of morphisms $f : \mathbb{X} \to \mathbb{Y}$ in pro-$\mathcal{P}$. Hence, a shape morphism $F : X \to Y$ is given by a diagram as follows.

$$\begin{tikzcd} \mathbb{X} \arrow[d, "f"] & X \arrow[l, "p"'] \\ \mathbb{Y} & Y \arrow[l, "q"'] \end{tikzcd}$$

The composition of shape morphisms $F : X \to Y$ and $G : Y \to Z$ is defined by composing representatives $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$. The identity shape morphism $id : X \to X$ is defined by $(id_\lambda, id_\Lambda) : \mathbb{X} \to \mathbb{X}$.

It can be proved that for every morphism $f : X \to Y$ in $\mathcal{T}$ and for $\mathcal{P}$-expansions $p : X \to \mathbb{X}$ and $q : Y \to \mathbb{Y}$, there is a unique morphism $f' : \mathbb{X} \to \mathbb{Y}$ in pro-$\mathcal{P}$ such that the following diagram commutes in pro-$\mathcal{T}$.

$$\begin{tikzcd} \mathbb{X} \arrow[d, "f'"] & X \arrow[l, "p"'] \arrow[d, "f"] \\ \mathbb{Y} & Y \arrow[l, "q"'] \end{tikzcd}$$

If $p' : X \to \mathbb{X}'$ and $q' : Y \to \mathbb{Y}'$ are other $\mathcal{P}$-expansions and $\overline{f} : \mathbb{X}' \to \mathbb{Y}'$ is another morphism in pro-$\mathcal{P}$ obtained as before. Then $f' \sim \overline{f}'$, which implies that for every $f \in \mathcal{T}(X, Y)$ it can be associated a shape morphism f' that will be denoted by $S(f)$. If we declare $S(X) = X$, then a covariant functor $S : \mathcal{T} \to Sh$ is obtained. This functor is called the shape functor.

The topological shape is defined as $Sh_{(HTop,HPol)}$. Hence, a shape morphism $F : X \to Y$ between topological spaces X and Y is given by a morphism $f : \mathbb{X} \to \mathbb{Y}$ in pro-$HPol$, where $p : X \to \mathbb{X}$ and $q : Y \to \mathbb{Y}$ are $HPol$-expansions of X and Y respectively.

The shape functor $S : HTop \to Sh$ fixes objects and induces a bijection of $HTop(X, P) \to Sh(X, P)$ for any X in $HTop$ and P in $HPol$. Concretely,

Proposition 1.2.18. *If X and Y are topological spaces that have the same homotopy type, then they have the same shape ($sh(X) = sh(Y)$).*

1.2.5 Inverse limits and shape theory

Instead of giving the general definition of inverse limit we prefer to give the following one for simplicity.

Definition 1.2.19. *Let $\mathbb{X} = (X_n, p_{n,n+1})$ be an inverse sequence in the topological category. The inverse limit of $\mathbb{X}$ is a subspace of $\Pi_{n\in\mathbb{N}} X_n$ which consists of all points x satisfying $\pi_n(x) = p_{n,m}(\pi_m(x))$ for every $m \geq n \in \mathbb{N}$, where $\pi_i : \Pi_{n\in\mathbb{N}} X_n \to X_i$ is the natural projection and $\Pi_{n\in\mathbb{N}} X_n$ is considered with the Tychonoff topology.*

Given an inverse sequence $\mathbb{X} = (X_n, p_{n,n+1})$ in the topological category. Let $\lim \mathbb{X}$ denote the inverse limit of $\mathbb{X}$. Moreover, for every $n \in \mathbb{N}$ there is a natural continuous map $p_n : \lim \mathbb{X} \to X_n$ given by $p_n = \pi_{n_{|\lim \mathbb{X}}}$, which is called projection. It can be deduced that $p = (p_n)$ is a morphism in pro-Top from the rudimentary system (X) to $\mathbb{X}$.

Theorem 1.2.20. *Every compact metric space X is the inverse limit of an inverse sequence of compact polyhedra $\mathbb{X} = (X_n, p_{n,n+1})$.*

Let X be the inverse limit of an inverse sequence $\mathbb{X}$ in Top and let $p : X \to \mathbb{X}$ be the morphism in pro-Top given by the projections. If we apply the homotopy functor H that keeps the objects of Top fixed and sends continuous maps to its homotopy classes, then we get a morphism $Hp : X \to H\mathbb{X}$ in pro-$HTop$, where $H\mathbb{X} = (X_n, [p_{n,n+1}])$. The following result relates more clearly the relations between the notion of inverse limit and shape theory.

Theorem 1.2.21. *Let $\mathbb{X}$ be an inverse sequence of compact polyhedra. If X is the inverse limit of $\mathbb{X}$ and $p : X \to \mathbb{X}$ is the morphism in pro-Top given by the projections, then $Hp : X \to H\mathbb{X}$ is a $HPol$-expansion.*

1.2.6 Hyperspaces and a different description of shape theory

Given a compact metric space (X, d), where d denotes the metric, $2^X = \{C \subseteq X | C$ is non-empty and closed$\}$ is called the hyperspace of X. For every open set $U \subseteq X$ we consider $B(U) = \{C \in 2^X | C \subset U\}$. The family $B = \{B(U) | U \subseteq X$ is open $\}$ is a base for the upper semifinite topology on 2^X. In addition, there is a natural metric that can be defined on 2^X, the Hausdorff metric d_H. If $C, D \in 2^X$, then the Hausdorff metric is defined as follows:

$$d_H(C, D) = \inf\{\epsilon > 0 | C \subseteq \mathcal{B}(D, \epsilon),\ D \subseteq \mathcal{B}(C, \epsilon)\},$$

where $\mathcal{B}(C, \epsilon)$ denotes the generalized ball of radius ϵ, that is, if $C \in 2^X$, then $\mathcal{B}(C, \epsilon) = \{x \in X | d(x, C) < \epsilon\}$. The hyperspace of X with the Hausdorff metric is a compact metric space. We recollect in the following proposition some properties of the Hausdorff metric.

Proposition 1.2.22. *Let (X, d) be a compact metric space and let $(2^X, d_H)$ be the hyperspace of X with the Hausdorff metric.*

- $d_H(x, y) = d(x, y)$ *if* $x, y \in X$.
- $d_H(x, C) = \sup\{d(x, c) | c \in C\} \geq \inf\{d(x, c) | c \in C\} = d(x, C)$ *if* $x \in X$ *and* $C \in 2^X$.
- $d_H(x, D) \leq d_H(x, C)$ *but* $d(x, D) \geq d(x, C)$ *if* $x \in X$ *and* $C, D \in 2^X$, *where* $D \subseteq C$.

Moreover, (X,d) can be identified with $\phi(X) \subset 2^X$, where $\phi : X \to 2^X$ is given by $\phi(x) = \{x\}$. This map is well-defined since X is T_2. It is not difficult to check that ϕ is an isometry onto its image and then $\phi(X)$ is homeomorphic to X. Therefore, (X,d) is embedded in $(2^X, d_H)$. We consider $U_\epsilon = \{C \in 2^X | diam(C) < \epsilon\}$, where $diam(C)$ denotes the diameter of C and ϵ is a positive real value. Furthermore, the closure operator can be described easily. If $C \in 2^X$, then the closure of C is given by $\overline{\{C\}} = \{D \in 2^X | C \subset D\}$. For a complete exposition about this topic, see [88].

We recall some properties of hyperspaces with the upper semifinite topology studied in [3] and [81].

Proposition 1.2.23. *Let (X,d) be a compact metric space. Then the family $\mathcal{U} = \{U_\epsilon\}_{\epsilon>0}$ is a base of open neighborhoods of X inside 2^X.*

Lemma 1.2.24. *Let Z and T be compact metric spaces and let $h : Z \to 2^T$ be a continuous map. Then $h^* : 2^Z \to 2^T$ given by $h^*(C) = \bigcup_{c \in C} f(c)$ is well-defined and continuous.*

Theorem 1.2.25. *Let X and Y be compact metric spaces. If $H : X \times [0,1] \to 2^Y$ and $h : 2^X \to 2^Y$ are continuous maps such that $H(x,0) = h_{|X}(\{x\})$, then there exists a map $\overline{H} : 2^X \times [0,1] \to 2^Y$ satisfying the following properties:*

1. *$\overline{H}(C,0) = h(C)$ for all $C \in 2^X$.*
2. *$\overline{H}_{|X \times [0,1]} = H$.*
3. *$\overline{H}$ is continuous.*
4. *If $H(x,t) \in \mathcal{U}_\epsilon(Y)$ for all $(x,t) \in X \times [0,1]$, then it can be chosen $\gamma > 0$ such that $\overline{H}(U_\gamma(X) \times [0,1]) \subset U_\epsilon(Y)$.*

Now, we recall the description of the shape theory that was given in [3] using the results obtained in [98].

Definition 1.2.26. *Let X and Y be compact metric spaces. A sequence of continuous functions $\overline{f} = \{f_k : X \to 2^Y\}_{n \in \mathbb{N}}$ is said to be an approximative map from X to Y if for every neighborhood U of the canonical copy Y in 2^Y there exists $k_0 \in \mathbb{N}$ such that f_k is homotopic to f_{k+1} in U for all $k \geq k_0$.*

Definition 1.2.27. *Let $\overline{f}$ and $\overline{g}$ be approximative maps. Then $\overline{f}$ is homotopic to $\overline{g}$ if for each open neighborhood U of the canonical copy Y in 2^Y there exists n_0 such that f_n is homotopic to g_n in U for every $n \geq n_0$.*

Theorem 1.2.28. *The set of all homotopy classes of approximative maps from X to Y is in bijective correspondence with the set of shape morphisms from X to Y.*

Remark 1.2.29. *It can be said that this approach substitutes the Hilbert cube by hyperspaces in the original idea mentioned in Subsection 1.2.1.*

Chapter 2

Reconstruction of compact metric spaces and shape theoretical properties

Given a compact metric space X, we associate to it an inverse sequence of finite T_0 topological spaces such that its inverse limit has the same homotopy type of X. We give a method to reconstruct the homology groups of X and study computational aspects. We propose a different description of shape theory. Finally, we construct two categories. The first (second) one classifies compact metric spaces by their shape and compact Alexandroff spaces by their weak homotopy type (homotopy type).

2.1 Introduction

Approximation of topological spaces is an old theme in geometric topology. There are two approaches that can be followed. Given a topological space X. One approach is to find a simpler topological space Y such that X and Y share some topological properties (compactness, homotopy type, etc.) or algebraic properties (homology and homotopy groups, etc.). Polyhedra have been used for this purpose, see for instance [71]. In this direction, it is important to remark the following classical result, which is known as the nerve theorem. The idea is to use good covers to construct a simplicial complex that reconstructs the homotopy type of X, see [26] and [76].

Theorem 2.1.1 (Nerve theorem)**.** *If $\mathcal{U}$ is an open cover of a paracompact space X such that every non-empty intersection of finitely many sets in $\mathcal{U}$ is contractible, then X is homotopy equivalent to the nerve of $\mathcal{U}$.*

Example 2.1.2. Let us consider the topological space $X \subseteq \mathbb{R}^2$ given in Figure 2.1.1 and the open cover $\mathcal{U}$ given by U_1, U_2, U_3, U_4 and all the possible intersections of them. It is clear that $N(\mathcal{U})$ has the same homotopy type of X.

This approach using the nerve theorem has a drawback, it is not always easy to find an open cover satisfying the hypothesis required. Recently, interesting results have been obtained in this direction modifying the notion of good cover. In [56], the hypothesis of being a good cover is relaxed. Using computational tools such as persistent homology, results about the reconstruction of the homology of the original space were obtained. Moreover, for Riemannian manifolds, results of reconstruction using Vietoris-Rips complexes were obtained in [61].

A different approach is to approximate X studying the inverse limit of an inverse sequence. The idea is the following. The bigger $n \in \mathbb{N}$ is, the better the term X_n to approximate X is. Indeed, if an inverse sequence is indexed by a finite totally ordered set N, then the inverse limit is homeomorphic to the term indexed by the maximum of N.

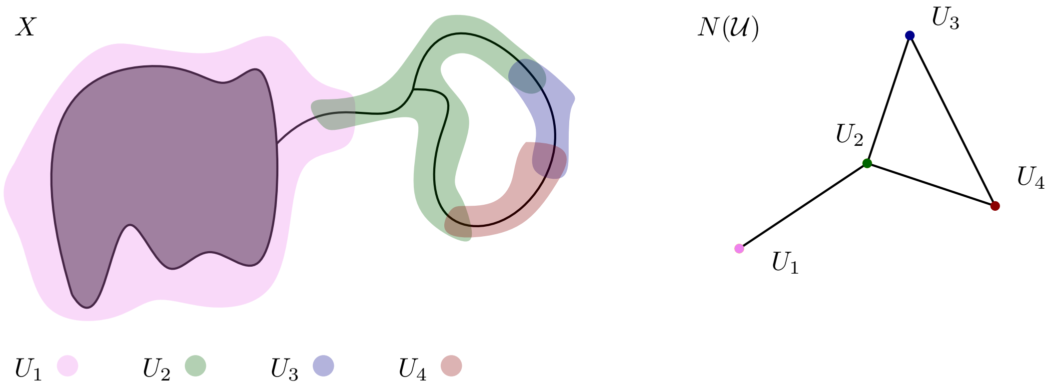

Figure 2.1.1: X, schematic representation of $\mathcal{U}$ and $N(\mathcal{U})$.

Example 2.1.3. Let us consider the Hawaiian earring, that is,

$$\mathbb{H} = \bigcup_{n=1}^{\infty} \{(x, y) \in \mathbb{R}^2 | (x - \frac{1}{n})^2 + y^2 = (\frac{1}{n})^2\}.$$

We consider $X_n = \bigcup_{i=1}^{n} \{(x, y) \in \mathbb{R}^2 | (x - \frac{1}{i})^2 + y^2 = (\frac{1}{i})^2\}$. It is clear that $X_n \subseteq X_{n+1}$ for every $n \in \mathbb{N}$. We also consider $p_{n,n+1} : X_{n+1} \to X_n$ given by $p_{n,n+1}(x, y) = (x, y)$ if $(x, y) \in X_n \subseteq X_{n+1}$ and $p_{n,n+1}(x, y) = (0, 0)$ if $(x, y) \in X_{n+1} \setminus X_n$. We have an inverse sequence $(X_n, p_{n,n+1})$ satisfying that its inverse limit is homeomorphic to $\mathbb{H}$. If N denotes the totally ordered set $\{1, ..., N\}$, then the inverse limit of $(X_n, p_{n,n+1}, N)$ is X_N. Hence, the higher the value of N is, the better the inverse limit of $(X_n, p_{n,n+1}, N)$ approximates $\mathbb{H}$.

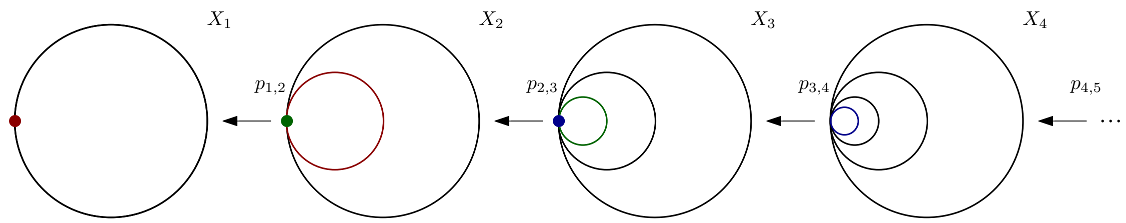

Figure 2.1.2: Inverse sequence $(X_n, p_{n,n+1})$ and schematic representation of $p_{n,n+1}$.

As mentioned earlier, polyhedra have been good candidates to get results of approximation. In [31], it is proved that finite topological spaces are also good candidates. Concretely, E. Clader proved that given a compact polyhedron X, there exists a natural inverse sequence of finite topological spaces such that its inverse limit contains a homeomorphic copy of X which is a strong deformation retract. In [81, 82], it is proved a generalization of this result to compact metric spaces. In [24], a similar result is obtained for topological spaces satisfying that are locally compact, paracompact and Hausdorff spaces, where non-finite Alexandroff spaces are considered. Throughout this chapter, we will restrict our discussion to compact metric spaces and finite topological spaces. Given a compact metric space (X, d), we recall the Main Construction [2] or Finite Approximative Sequence (FAS) for X [82]. A FAS for a compact metric space X is an inverse sequence of finite topological spaces that reconstructs the homotopy type of X. We recall notation and some technical results before giving the construction.

Definition 2.1.4. *Let (X,d) be a compact metric space and let ϵ be a positive real number. A finite subset A of X is an ϵ-approximation of X if for every $x \in X$ there exists $a \in A$ satisfying that $d(x,a) < \epsilon$.*

In the conditions of Definition 2.1.4, $\mathcal{U}_\gamma(A) = \{C \subseteq A | diam(C) < \gamma\}$ is a finite poset for every positive real value γ, the partial order is given by the subset relation, that is, $C \leq D$ if and only if $C \subseteq D$. If (X,d) is a compact metric space, then it is easy to check that for every $\epsilon > 0$ there exists an ϵ-approximation A of X.

Lemma 2.1.5. *Let (X,d) be a compact metric space. If ϵ is a positive real value and A is an ϵ-approximation of X, then there exists $0 < \epsilon' < \epsilon$ such that for every ϵ'-approximation A' of X the map $p : \mathcal{U}_{2\epsilon'}(A') \to \mathcal{U}_{2\epsilon}(A)$ given by $p(C) = \bigcup_{c \in C}\{a \in A | d(c,a) = d(c,A)\}$ is continuous. Moreover, ϵ' can be chosen satisfying $\epsilon' < \frac{\epsilon - \gamma}{2}$, where $\gamma = sup\{d(x,A) | x \in X\}$.*

As an immediate consequence of Lemma 2.1.5, it can be obtained the so-called Main Construction or FAS for a compact metric space.

Proposition 2.1.6 (Main Construction or FAS). *Let (X,d) be a compact metric space. There exists an inverse sequence $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$, where $\{\epsilon_n\}_{n \in \mathbb{N}}$ is a sequence of decreasing positive real values satisfying that $\epsilon_{n+1} < \frac{\epsilon_n - \gamma_n}{2}$, $\{A_n\}_{n \in \mathbb{N}}$ is a sequence of ϵ_n-approximations of X, $\gamma_n = sup\{d(x,A_n) | x \in X\}$ and $p_{n,n+1} : \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1}) \to \mathcal{U}_{2\epsilon_n}(A_n)$ is given by $p_{n,n+1}(C) = \bigcup_{c \in C}\{a \in A | d(c,a) = d(c,A)\}$.*

One important property of this inverse sequence relies on its inverse limit.

Theorem 2.1.7. *Let (X,d) be a compact metric space and let $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ be a FAS for X. Then the inverse limit of $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ contains a homeomorphic copy of X which is a strong deformation retract.*

In addition, if we apply the McCord functor to a FAS $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ for a compact metric space X, then we get an inverse sequence of polyhedra $(\mathcal{K}(\mathcal{U}_{2\epsilon_n}(A_n)), |\mathcal{K}(p_{n,n+1})|)$ or an object in pro-$HPol$.

Theorem 2.1.8. *Let (X,d) be a compact metric space and let $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ be a FAS for X. Then $(\mathcal{K}(\mathcal{U}_{2\epsilon_n}(A_n)), |\mathcal{K}(p_{n,n+1})|)$ is a $HPol$-expansion of X.*

If we consider the opposite order on the finite T_0 topological spaces of a FAS for a compact metric space, then the inverse limit does not need to preserve the good properties that were obtained in Theorem 2.1.7. From a set theoretical point of view, the inverse limit is the same independently of the partial order chosen, but from a topological point of view the topologies of the inverse limits are different.

Example 2.1.9. Let us consider the unit interval $I = [0,1]$. We consider the same FAS for I that was chosen in [81, Example 4]. Namely, $A_1 = \{0\}$, $\epsilon_1 = 2$, $\epsilon_n = \frac{1}{3^{2n-3}}$ and $A_n = \{\frac{k}{3^{2n-3}} | k = 0, ..., 3^{2n-3}\}$ for every $n \in \mathbb{N} \setminus \{1\}$. Then, $\mathcal{U}_{2\epsilon_n}(A_n) = A_n \cup \{\{\frac{k}{3^{2n-3}}, \frac{k+1}{3^{2n-3}}\} | k = 0, ..., 3^{2n-3} - 1\}$. We denote by $\varphi : \mathcal{I}^u \to I$ the map obtained in Theorem 2.1.7, where $\mathcal{I}^u$ denotes the inverse limit of $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$. Since $C = (C_1, C_2, C_3, ..., C_n, ...) \in \mathcal{I}^u$ can be seen as a sequence in the hyperspace of I (2^I) with the Hausdorff distance, φ is defined by sending $(C_1, C_2, C_3, ...)$ to its convergent point $\{x\} \in 2^I$, where $x \in I$. If $x \in \bigcup_{n \in \mathbb{N}} A_n$, then $\varphi^{-1}(x)$ has cardinality one. If $x \in I \setminus \bigcup_{n \in \mathbb{N}} A_n$, then $\varphi(x)^{-1} = \{C, D, X\}$, where $C_i, D_i \subset X_i$ for every i. For a complete exposition of the previous assertion, see [81, Chapter 3]. In Figure 2.1.3 we present a schematic description of the above situation.

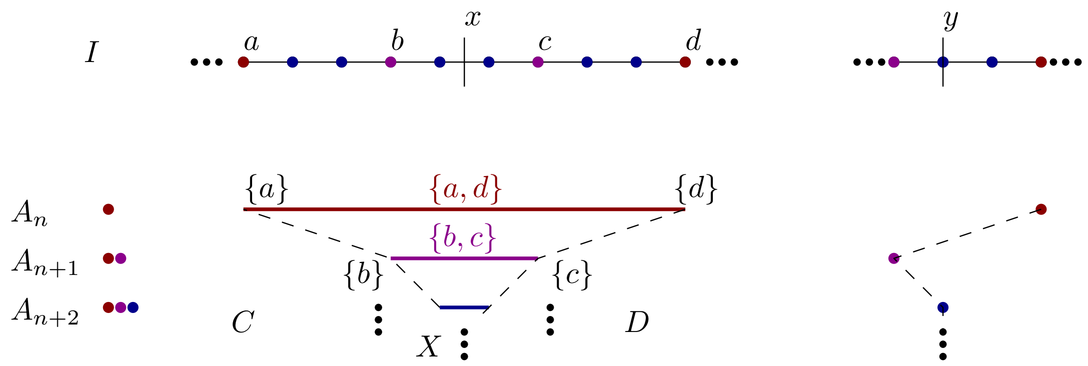

Figure 2.1.3: Schematic description of $\varphi^{-1}(x)$ and $\varphi^{-1}(y)$, where $x \in I \setminus \{\bigcup_{n\in\mathbb{N}} A_n\}$ and $y \in \bigcup_{n\in\mathbb{N}} A_n$.

If we consider the opposite partial order in every term of the inverse sequence and $x \in I \setminus \bigcup_{n\in\mathbb{N}} A_n$, then $X_i <_o C_i$ and $X_i <_o D_i$ for every i, where $\varphi(x)^{-1} = \{C, D, X\}$. We denote by $\mathcal{I}^o$ the inverse limit of the inverse sequence of finite topological spaces with the opposite partial order. Therefore, the identity map $id : \mathcal{I}^o \to \mathcal{I}^u$ is not continuous. We argue by contradiction. Let us consider $x = \frac{1}{2}$, we have that every open neighborhood U of $C \in \mathcal{I}^o$ contains X because $X_i <_o C_i$ for every i. On the other hand, we can consider $V = U_{C_1} \times U_{C_2} \times \cdots \times U_{C_n} \times \mathcal{U}_{2\epsilon_{n+1}(A_{n+1})} \times \mathcal{U}_{2\epsilon_{n+2}(A_{n+2})} \times \cdots$, where U_{C_i} denotes the minimal open neighborhood of $C_i \in \mathcal{U}_{2\epsilon_i(A_i)}$. We get that V is an open neighborhood of $id(C)$ and does not contain X because $X_i > C_i$ for every i, which entails a contradiction.

This example shows the following. If we want to keep a similar result changing the partial order of the terms, then we need to get a different construction. The opposite partial order has been used recently to find applications to the study of dynamical systems, see for example [68]. Furthermore, the inverse sequence of finite topological spaces obtained in [31] also uses the opposite order.

The organization of this chapter is as follows. In Section 2.2 we construct an analog of the FAS for a compact metric space using the opposite partial order. This construction has more advantages from a computational viewpoint. The different stages of the construction are implemented in classical algorithms or algorithms used to calculate persistent homology. A discussion about this issue is done in Section 2.5. In Section 2.3, we study the properties of the inverse limit of the inverse sequence obtained in Section 2.2. Namely, we prove that the inverse limit reconstructs the homotopy type of the compact metric space chosen. Since the inverse sequences obtained are not strictly equal, we give a result of uniqueness in Section 2.4. Then we use this result to get a method of approximation of homology groups. However, this method does not reconstruct the homotopy type, but it is more suitable for computational reasons. In addition, we study the relation of our inverse sequence and the Main Construction. We also study the relation with the inverse sequence given in [31]. In Section 2.6, we give a different description of shape theory. The definition of core for a finite topological space is extended to inverse sequences of finite topological spaces. We give an application to shape theory in Section 2.7. In Section 2.8, we define a category that classifies compact metric spaces by their shape and finite topological spaces by their weak homotopy type. We also get a stronger version that classifies compact metric spaces by their shape and finite topological spaces by their homotopy type. In Section 2.9, Theorem 2.1.8 is proved with a shorter proof than the one given in [81].

2.2 Finite approximative sequences with the opposite order

Given a compact metric space (X,d), we construct an inverse sequence of finite T_0 topological spaces using the opposite order. Let ϵ and γ be positive real values and let A be an ϵ-approximation of X. From now on, if there is no explicit mention of the partial order considered on $\mathcal{U}_\epsilon(A)$, then it is considered the partial order given as follows: $C \leq D$ if and only $D \subseteq C$.

Lemma 2.2.1. *Let (X,d) be a compact metric space. If ϵ is a positive real value and A is an ϵ-approximation of X, then for every $\epsilon' < \frac{\epsilon-\gamma}{2}$ and every ϵ'-approximation A' of X the map $q : \mathcal{U}_{4\epsilon'}(A') \to \mathcal{U}_{4\epsilon}(A)$ given by $q(C) = \bigcup_{x\in C} \mathcal{B}(x,\epsilon) \cap A$ is well-defined and continuous, where $\gamma = sup\{d(x,A)|x \in X\}$.*

Proof. Since X is a compact space, it follows that γ exists. We check that q is well-defined. Let us take $C \in \mathcal{U}_{4\epsilon'}(A')$, which implies that $diam(C) < 4\epsilon'$. If $x, y \in q(C)$, then there exist $c_x, c_y \in C$ satisfying that $x \in \mathcal{B}(c_x,\epsilon)$ and $y \in \mathcal{B}(c_y,\epsilon)$. Therefore, we have

$$d(x,y) \leq d(x,c_x) + d(c_x,c_y) + d(c_y,y) < \epsilon + 4\epsilon' + \epsilon < 2\epsilon + 2(\epsilon - \gamma) < 4\epsilon,$$

which implies that $diam(q(C)) < 4\epsilon$. The continuity of q follows trivially. □

Remark 2.2.2. *In Lemma 2.2.1, we can consider $\epsilon' < \frac{\epsilon}{2}$ for simplicity and the result also holds true.*

From here, we can get the desired inverse sequence.

Theorem 2.2.3. *Let (X,d) be a compact metric space. There exists an inverse sequence $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$, where $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of decreasing positive real values with $\epsilon_{n+1} < \frac{\epsilon_n-\gamma_n}{2}$, $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of ϵ_n-approximations of X, $\gamma_n = sup\{d(x,A_n)|x \in X\}$ and the map $q_{n,n+1} : \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1}) \to \mathcal{U}_{4\epsilon_n}(A_n)$ is given by $q_{n,n+1}(C) = \bigcup_{x\in C} \mathcal{B}(x,\epsilon_n) \cap A_n$.*

Proof. We start with $\epsilon_1 > diam(X)$. Then, A_1 can be taken as $A_1 = \{a\}$ for some $a \in X$, we define $\gamma_1 = \sup\{d(x,A_1)|x \in X\}$. Applying Lemma 2.2.1, we can obtain $\epsilon_2 < \frac{\epsilon_1-\gamma_1}{2}$, a ϵ_2-approximation A_2 of X and a continuous map $q_{1,2} : \mathcal{U}_{4\epsilon_2} : (A_2) \to \mathcal{U}_{4\epsilon_1}(A_1)$. It suffices to repeat this method inductively to conclude. □

The inverse sequence obtained in Theorem 2.2.3 is called Finite Approximative Sequence with Opposite order (FASO). For simplicity, when there is no confusion, $(\mathcal{U}_n, q_{n,n+1})$ denotes $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ and we will also omit the subscript of the bonding maps.

Example 2.2.4. We construct a FASO for the unit circle S^1. We consider the unit circle in the complex plane with the geodesic distance, $S^1 = \{z \in \mathbb{C}||z| = 1\}$. We get a FASO for S^1 by steps.

Step 1. We consider $\epsilon_1 = 3\pi > diam(S^1) = \pi$ and $A_1 = \{e^{2\pi i}\}$, which is clearly an ϵ_1-approximation of S^1. Then, $\gamma_1 = \pi$ and $\mathcal{U}_1 = A_1$.

Step 2. We consider $\epsilon_2 = \frac{\pi}{2} < \frac{3\pi-\pi}{2}$ and $A_2 = \{a^2_k = e^{\frac{2\pi k i}{4}}|k = 0,1,2,3\}$, which is clearly an ϵ_2-approximation of S^1. Then, $\gamma_2 = \frac{\pi}{4}$ and $\mathcal{U}_2 = \{2^{A_2}\}$, where 2^{A_n} denotes the power set of S minus the empty set.

Step 3. We consider $\epsilon_3 = \frac{\pi}{16} < \frac{\pi}{8}$ and $A_3 = \{a^3_k = e^{\frac{2\pi k i}{32}}|k = 0,1,...,31\}$, which is clearly an ϵ_3-approximation. Then, $\gamma_3 = \frac{\pi}{32}$ and $\mathcal{U}_3 = \{2^{a^3_i,a^3_{i+1},a^3_{i+2},a^3_{i+3}}\}_{i=0,...,31}$, where the subindices are considered modulo 32 and $2^{a^3_i,a^3_{i+1},a^3_{i+2},a^3_{i+3}}$ denotes the power set of

$\{a_i^3, a_{i+1}^3, a_{i+2}^3, a_{i+3}^3\}$ minus the empty set. The last statement is true due to the fact that $d(a_i^3, a_{i+1}^3) = \frac{\pi}{16}$ for every $i, i+1$ modulo 32 and $4\epsilon_3 = \frac{\pi}{4}$.

Step n. We consider $\epsilon_n = \frac{\pi}{2^{3n-5}}$ and $A_n = \{a_k^n = e^{\frac{2\pi k i}{2^{3n-4}}} | k = 0, ..., 2^{3n-4} - 1\}$, which is clearly an ϵ_n-approximation of S^1. Therefore, we obtain that $\gamma_n = \frac{\pi}{2^{3n-4}}$ and $\mathcal{U}_n = \{2^{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n}\}_{i=0,...,2^{3n-4}-1}$, where the subindices are considered modulo 2^{3n-4} and $2^{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n}$ denotes the power set of $\{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n\}$ minus the empty set. The last statement is true due to the fact that $d(a_i^n, a_{i+1}^n) = \frac{\pi}{2^{3n-5}}$ for every $i, i+1$ modulo 2^{3n-4} and $4\epsilon_n = \frac{\pi}{2^{3n-7}}$.

For a schematic representation of the minimal points of $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$, see Figure 2.2.1. Each arc represents a minimal point. Red, green and blue arcs represent the minimal points of $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$, respectively.

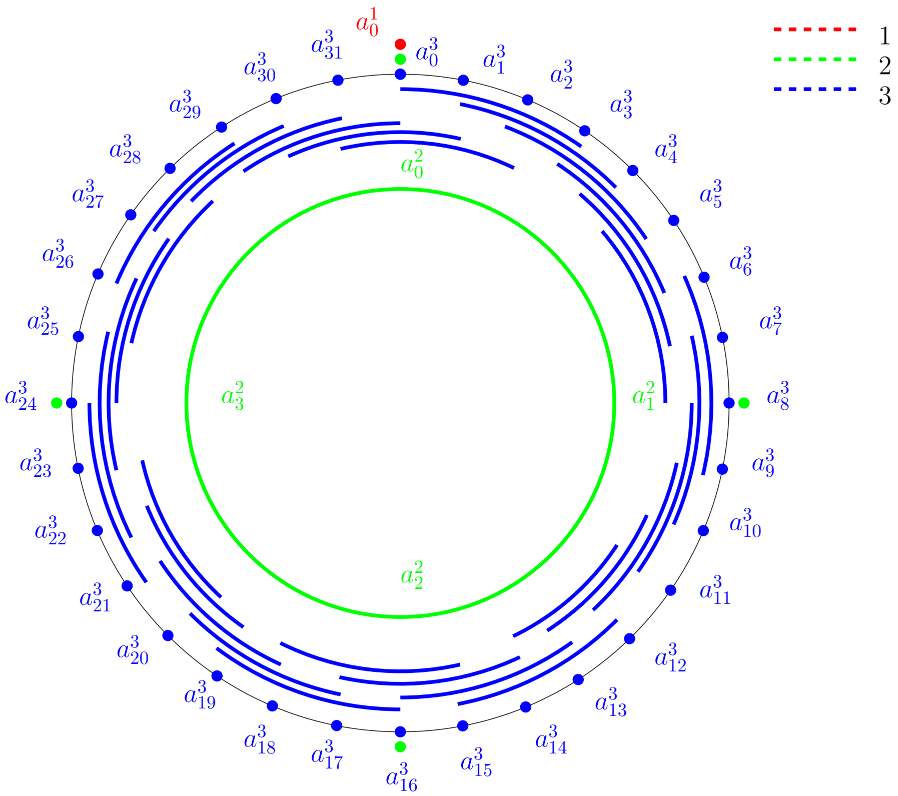

Figure 2.2.1: Diagram of the minimal points of $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ in S^1.

Given a compact metric space (X, d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X, there is natural map $q_n : X \to \mathcal{U}_n$ given by $q_n(x) = \mathcal{B}(x, \epsilon_n) \cap A_n$ for every $n \in \mathbb{N}$, .

Proposition 2.2.5. *Given a compact metric space (X, d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X. The following diagram commutes up to homotopy for every $n \in \mathbb{N}$.*

$$\begin{tikzcd}
& X \arrow[dl, "q_n"'] \arrow[dr, "q_{n+1}"] & \\
\mathcal{U}_{4\epsilon_n}(A_n) & & \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1}) \arrow[ll, "q_{n,n+1}"]
\end{tikzcd}$$

Then $q = (q_n) : X \to (\mathcal{U}_n, q_{n,n+1})$ is a morphism in pro-HTop.

Proof. Firstly, we prove that q_n is continuous and well-defined for every $n \in \mathbb{N}$. If $x \in X$, then the diameter of $q_n(x)$ is less than $2\epsilon_n$, which implies that $q_n(x) \in \mathcal{U}_{4\epsilon_n}(A_n)$. Now, we prove the continuity of q_n. We consider $\gamma = \max\{d(x,b) | b \in q_n(x)\}$ and $0 < \delta < \epsilon_n - \gamma$. We have that γ is well-defined since $q_n(x)$ is a finite set. For every $y \in \mathcal{B}(x,\delta)$ we have that $q_n(x) \subseteq q_n(y)$. We prove the last assertion. If $b \in q_n(x)$, then we get $d(x,b) \leq \gamma$. Therefore,

$$d(y,b) < d(y,x) + d(x,b) < \epsilon_n - \gamma + \gamma = \epsilon_n.$$

From here, the continuity of q_n follows easily since we have that $q_n(\mathcal{B}(x,\delta)) \subseteq U_{q_n(x)}$, where $U_{q_n(x)}$ denotes the minimal open neighborhood of $q_n(x)$.

We consider $h : X \to \mathcal{U}_n$ given by $h(x) = q_n(x) \cup q_{n,n+1}(q_{n+1}(x))$. We prove that $h(x)$ has diameter less than $4\epsilon_n$. Let us take $a \in q_n(x)$ and $b \in q_{n,n+1}(q_{n+1}(x))$. By construction, $d(a,x) < \epsilon_n$ and there exist $C \in \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1})$ and $c \in C$ such that $c \in q_{n+1}(x) = C$ and $b \in q_{n,n+1}(c)$, which implies that $d(x,c) < \epsilon_{n+1} < \frac{\epsilon_n}{2}$ and $d(c,b) < 2\epsilon_n$. We have

$$d(a,b) < d(a,x) + d(x,c) + d(c,b) < \epsilon_n + \frac{\epsilon_n}{2} + 2\epsilon_n < 4\epsilon_n.$$

From this, we get that $h = q_n \cup q_{n,n+1} \circ q_{n+1} : X \to \mathcal{U}_{4\epsilon_n}(A_n)$ is well-defined. The continuity of h follows trivially. In addition, $q_n(x), q_{n,n+1}(q_{n+1}(x)) \subseteq q_n(x) \cup q_{n,n+1}(q_{n+1}(x))$ for every $x \in X$. It is easy to deduce that the diagram commutes up to homotopy. We only prove one of the two homotopies because the other one is similar. We consider $H : X \times [0,1] \to \mathcal{U}_{4\epsilon_n}(A_n)$ given by

$$H(x,t) = \begin{cases} q_n(x) \cup q_{n,n+1}(q_{n+1}(x)) & \text{if } t \in [0,1) \\ q_n(x) & \text{if } t = 1. \end{cases}$$

It suffices to verify the continuity at $t = 1$. If $(x,1) \in X \times [0,1]$, then we consider the minimal open neighborhood of $H(x,1)$, i.e., $U_{H(x,1)}$. By the continuity of q_n, there exists an open neighborhood V of x with $H(V,1) \subset U_{H(x,1)}$. By construction, if $t \neq 1$, then we have $H(V,t) = h(V) \supset H(V,1) = q_n(V)$ so $h(y) \in U_{q_n(x)}$ for every $y \in V$. Thus, $V \times [0,1]$ is an open neighborhood of $(x,1)$ in $X \times [0,1]$ satisfying $H(V,[0,1]) \subset U_{H(x,1)}$, which implies the continuity of H at $(x,1)$. □

Given a compact metric space (X,d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X. We can change the partial order defined on the terms of the inverse sequence $(\mathcal{U}_n, q_{n,n+1})$, that is, for every $n \in \mathbb{N}$ we consider the other possible partial order defined on $\mathcal{U}_n$. We get that the bonding maps of the inverse sequence are also continuous but Proposition 2.2.5 does not hold true. This is due to the following result.

Proposition 2.2.6. *Let (X,d) be a non-empty connected compact metric space and let Y be a finite topological space. If $f : X \to Y$ is continuous and it is also continuous when the other possible order on Y is considered, then f is the constant map.*

Proof. Let us consider $x \in X$ and the minimal open neighborhood containing $f(x)$ for the natural order and opposite order, that is, $U_{f(x)}$ and $F_{f(x)}$ respectively. By the continuity of f, there exist open sets V_x and W_x containing x such that $f(V_x) \subseteq U_{f(x)}$ and $f(W_x) \subseteq F_{f(x)}$. Therefore, $f(V_x \cap W_x) \subseteq U_{f(x)} \cap F_{f(x)} = \{f(x)\}$, which implies that f is a locally constant map. Since X is connected, if follows that f is a constant map. □

2.3 Properties of the inverse limit of a FASO for a compact metric space

Given a compact metric space (X, d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X. We study properties of the inverse limit of $(\mathcal{U}_n, q_{n,n+1})$, denoted by $\mathcal{X}$. Firstly, we prove that $\mathcal{X}$ is non-empty. Despite the fact that this result can be deduced from [102, Theorem 2], we prefer to describe specific elements of $\mathcal{X}$.

For every $x \in X$, we consider

$$X_*^n = \bigcup_{m>n} q_{n,m}(A_m(x)),$$

where $A_m(x) = \{a \in A_m | d(x, a) = d(x, A_m)\}$. The sequence $\{X_*^n\}_{n\in\mathbb{N}}$ is a candidate to be an element of $\mathcal{X}$.

Proposition 2.3.1. *If $x \in X$, then $X_*^n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$.*

Proof. We prove that if $x \in X$, then $X_*^n \subseteq \mathcal{B}(x, 2\epsilon_n) \cap A_n$ for every $n \in \mathbb{N}$. We verify that $q_{n,m}(A_m(x)) \subset \mathcal{B}(x, 2\epsilon_n) \cap A_n$ for all $m > n$. As a consequence, it can be deduced that $diam(X_*^n) < 4\epsilon_n$ and $X_*^n \in \mathcal{U}_{4\epsilon_n}(A_n)$.

If $a_n \in q_{n,m}(A_m(x))$, then there exists a sequence $\{a_t\}_{n\le t\le m}$ with $a_t \in A_t$ and $a_t \in q_{t,t+1}(a_{t+1})$, so $d(a_t, a_{t+1}) < \epsilon_t$. In addition, $a_m \in A_m(x)$, which means $d(a_m, x) < \epsilon_m$. Therefore,

$$\begin{aligned} d(a_n, x) &< d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \cdots + d(a_{m-1}, a_m) + d(a_m, x) < \\ &< \epsilon_n + \epsilon_{n+1} + \cdots + \epsilon_{m-1} + \epsilon_m < \epsilon_n + \frac{\epsilon_n}{2} + \cdots + \frac{\epsilon_n}{2^{m-1-n}} + \frac{\epsilon_n}{2^{m-n}} = \\ &= \epsilon_n(1 + \sum_{i=1}^{m-n} \frac{1}{2^i}) < 2\epsilon_n. \end{aligned}$$

□

The idea of the following lemmas is to show that X_*^n, which is an infinite union of sets, stabilizes for every $n \in \mathbb{N}$.

Lemma 2.3.2. *If $x \in X$, then $A_n(x) \subset q_{n,n+1}(A_{n+1}(x))$ for every $n \in \mathbb{N}$.*

Proof. If $a_n \in A_n(x)$ and $a_{n+1} \in A_{n+1}(x)$, then we have $d(a_n, x) \le \gamma_n$ and $d(a_{n+1}, x) < \epsilon_{n+1}$. We obtain the following relations

$$d(a_n, a_{n+1}) < d(a_n, x) + d(x, a_{n+1}) < \gamma_n + \epsilon_{n+1} < \gamma_n + \frac{\epsilon_n - \gamma_n}{2} = \frac{\epsilon_n}{2} + \frac{\gamma_n}{2} < \epsilon_n.$$

Thus, we get $a_n \in \mathcal{B}(a_{n+1}, \epsilon_n) \cap A_n = q_{n,n+1}(a_{n+1})$. □

Lemma 2.3.3. *If $x \in X$, then $q_{n,m}(A_m(x)) \subset q_{n,m+1}(A_{m+1}(x))$ for every $m > n$.*

Proof. We know that $q_{n,m+1}(A_{m+1}(x)) = q_{n,m}(q_{m,m+1}(A_{m+1}(x)))$. By Lemma 2.3.2, we have that $A_m(x) \subset q_{m,m+1}(A_{m+1}(x))$. On the other hand, $q_{n,m}$ is a continuous map between finite topological spaces so $q_{n,m}$ preserves the subset relation. Therefore, if we apply $q_{n,m}$ to $A_m(x) \subset q_{m,m+1}(A_{m+1}(x))$, then we get $q_{n,m}(A_m(x)) \subseteq q_{n,m}(q_{m,m+1}(A_{m+1}(x))) = q_{n,m+1}(A_{m+1}(x))$. □

Proposition 2.3.4. *If $x \in X$, then for every $n \in \mathbb{N}$ there exists $n_* > n$ such that for all $m \ge n_*$ we have $X_*^n = q_{n,m}(A_m(x))$.*

Proof. By Proposition 2.3.1, we know that $X_*^n \subset \mathcal{B}(x, 2\epsilon_n) \cap A_n$ so X_*^n is a finite set. By Lemma 2.3.3, if $a \in X_*^n$, then there exists $n_a \in \mathbb{N}$ such that $a \in q_{n,m}(A_m(x))$ for every $m \geq n_a$. We consider

$$n_* = max\{n_a | a \in X_*^n \quad and \quad a \in q_{n,m}(A_m(x)) \quad \text{with} \quad m \geq n_a\},$$

where n_* is well-defined since X_*^n is a finite set. From here, it follows the desired result. □

Finally, we prove that $\{X_*^n\}_{n\in\mathbb{N}}$ is an element of the inverse limit $\mathcal{X}$. Then, we also prove a connection between the elements of $\mathcal{X}$ and X.

Proposition 2.3.5. *If $x \in X$, then $\{X_*^n\}_{n\in\mathbb{N}} \in \mathcal{X}$.*

Proof. It suffices to check that $q_{n,n+1}(X_*^{n+1}) = X_*^n$, the general case follows inductively. By Proposition 2.3.4, if $s > n_*, (n+1)_*$, then $q_{n,s}(A_s(x)) = X_*^n$ and $q_{n+1,s}(A_s(x)) = X_*^{n+1}$. Thus,

$$q_{n,n+1}(X_*^{n+1}) = q_{n,n+1}(q_{n+1,s}(A_s(x))) = q_{n,s}(A_s(x)) = X_*^n.$$

□

Proposition 2.3.6. *If $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{X}$, then $\{C_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the hyperspace of X with the Hausdorff distance 2_H^X that converges to $\{x\} \in 2_H^X$ for some $x \in X$. In addition, $d_H(x, C_n) < 2\epsilon_n$ for every $n \in \mathbb{N}$.*

Proof. Firstly, we check that $d_H(C_n, C_m) < 2\epsilon_n - \frac{\gamma_n}{2}$ for every $n, m \in \mathbb{N}$ satisfying $m \geq n$. If $c_n \in C_n$, then there exists a sequence $\{c_t\}_{n\leq t\leq m}$ with $c_t \in A_t$ and $c_t \in q_{t,t+1}(c_{t+1})$. We get $d(c_t, c_{t+1}) < \epsilon_t$. We obtain

$$\begin{aligned} d(c_n, c_m) &< d(c_n, c_{n+1}) + \cdots + d(c_{m-1}, c_m) < \epsilon_n + \cdots + \epsilon_{m-1} < \\ &< \epsilon_n + \frac{\epsilon_n - \gamma_n}{2} + \cdots + \frac{\epsilon_n - \gamma_n}{2^{m-1-n}} = \epsilon_n(1 + \sum_{i=1}^{m-1-n} \frac{1}{2^i}) - \gamma_n \sum_{i=1}^{m-1-n} \frac{1}{2^i} \\ &< 2\epsilon_n - \frac{\gamma_n}{2}, \end{aligned}$$

which implies that $C_n \subset \mathcal{B}(C_m, 2\epsilon_n - \frac{\gamma_n}{2})$. If $c_m \in C_m$, then we can repeat the same argument to show that $C_m \subset \mathcal{B}(C_n, 2\epsilon_n - \frac{\gamma_n}{2})$. Therefore, for every $\epsilon > 0$ there exists $s \in \mathbb{N}$ such that for every $n, m > s$ we have $d_H(C_n, C_m) < \epsilon$. It suffices to consider s satisfying that $2\epsilon_s - \frac{\gamma_s}{2} < \epsilon$. We have proved that $\{C_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a compact metric space 2_H^X so $\{C_n\}_{n\in\mathbb{N}}$ converges to an element $C \in 2_H^X$. It is important to recall that $diam(C_n) \leq 4\epsilon_n$ because $C_n \in \mathcal{U}_{4\epsilon_n}(A_n)$. Due to the fact of the continuity of the diameter function regarding to the Hausdorff metric, we have

$$diam(C) = diam(\lim_{n\to\infty}(C_n)) = \lim_{n\to\infty}(diam(C_n)) < \lim_{n\to\infty} 4\epsilon_n = 0.$$

Thus, $C = \{x\}$ for some $x \in X$.

We have shown that $d_H(C_n, C_m) < 2\epsilon_n - \frac{\gamma_n}{2}$ for every $m, n \in \mathbb{N}$ satisfying $m \geq n$. We know that $\{C_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence that converges to $\{x\}$. Therefore, for $\frac{\gamma_n}{2}$ there exists $n_0 > n$ such that for every $m > n_0$ we get $d_H(x, C_m) < \frac{\gamma_n}{2}$. From this, we obtain the following relations

$$d_H(x, C_n) < d_H(x, C_m) + d_H(C_m, C_n) < \frac{\gamma_n}{2} + 2\epsilon_n - \frac{\gamma_n}{2} = 2\epsilon_n.$$

□

Proposition 2.3.7. *If $x \in X$, then $\{X_*^n\}_{n\in\mathbb{N}} \in \mathcal{X}$ converges to $\{x\}$ in 2_H^X.*

Proof. It is an immediate consequence of Proposition 2.3.1 and Proposition 2.3.6. □

Furthermore, we also get that $\{X_*^n\}_{n\in\mathbb{N}}$ is somehow minimal with respect to the elements of $\mathcal{X}$ that converge to the same point.

Proposition 2.3.8. *If $\{C_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ converges to $\{x\}$ for some $x \in X$, then $X_*^n \subseteq C_n$ for every $n \in \mathbb{N}$.*

Proof. We prove that $A_n(x) \subset C_n$ for all $n \in \mathbb{N}$. We know that $q_{n,n+1}(C_{n+1}) = C_n$. By Proposition 2.3.6, if $c_{n+1} \in C_{n+1}$, then we have that $d_H(x, C_{n+1}) < 2\epsilon_{n+1}$. Therefore, $d(x, c_{n+1}) < 2\epsilon_{n+1}$. In addition, for every $a_n \in A_n(x)$ we get $d(x, a_n) \leq \gamma_n$. We have

$$d(a_n, c_{n+1}) < d(a_n, x) + d(x, c_{n+1}) < \gamma_n + 2\epsilon_{n+1} < \gamma_n + \epsilon_n - \gamma_n = \epsilon_n,$$

so $a_n \in \mathcal{B}(c_{n+1}, \epsilon_n) \cap A_n = q_{n,n+1}(c_{n+1}) \subset C_n$. We can conclude that $A_n(x) \subset C_n$. We take $s > n_*$ where n_* is given by Proposition 2.3.4. We know that $q_{n,s}(A_s(x)) = X_*^n$. On the other hand, we have proved that $A_s(x) \subset C_s$. If we apply $q_{n,s}$ to the previous content, then we get the desired result. This is due to the fact that $q_{n,s}$ is a continuous map between finite topological spaces, which means that it preserves the subset relation. Thus,

$$X_*^n = q_{n,s}(A_s(x)) \subset q_{n,s}(C_s) = C_n.$$

□

Now, we can define a map φ between the inverse limit of $(\mathcal{U}_n, q_{n,m})$ and X. The map $\varphi : \mathcal{X} \to X$ sends each element of the inverse limit to its convergent point given by Proposition 2.3.6, that is, $\varphi(\{C_n\}_{n\in\mathbb{N}}) = x$ where $\lim_{n\to\infty} C_n = \{x\}$.

Proposition 2.3.9. *The map $\varphi : \mathcal{X} \to X$ is surjective and continuous.*

Proof. The surjectivity is given by the construction of $\{X_*^n\}_{n\in\mathbb{N}}$ and Proposition 2.3.7, so it only remains to show the continuity. For each open neighborhood V of $\varphi(\{C_n\}_{n\in\mathbb{N}}) = x$, we can take $\delta > 0$ such that $\mathcal{B}(x, \delta) \subset V$. We consider the open neighborhood of $\{C_n\}_{n\in\mathbb{N}}$ given as follows

$$W = (U_{C_1} \times U_{C_2} \times \cdots \times U_{C_{n_0}} \times \mathcal{U}_{4\epsilon_{n_0+1}(A_{n_0+1})} \times \ldots) \cap \mathcal{X},$$

where $U_{C_t} = \{D \in \mathcal{U}_{4\epsilon_t(A_t)} | C_t \subset D\}$ denotes the minimal open neighborhood of C_t in $\mathcal{U}_{4\epsilon_t}(A_t)$. We consider n_0 satisfying that for every $n \geq n_0$ we obtain that $\epsilon_n < \frac{\delta}{4}$. Suppose that $\{D_n\}_{n\in\mathbb{N}} \in W$ with $\{D_n\} \to \{y\}$. We verify that $y \in \mathcal{B}(x, \delta)$. By construction, $C_n \subset D_n$ for every $n \leq n_0$. By the second part of Proposition 2.3.6 and the previous observation, we get

$$\begin{aligned} d(x, y) = d_H(x, y) < d_H(x, C_{n_0}) + d_H(C_{n_0}, y) < d_H(x, C_{n_0}) + d_H(D_{n_0}, y) < \\ < 2\epsilon_{n_0} + 2\epsilon_{n_0} < 4\epsilon_{n_0} < \delta, \end{aligned}$$

where we are using the properties of the Hausdorff metric given in Proposition 1.2.22. □

We can also define a map ϕ between X and $\mathcal{X}$ given by the construction made at the beginning, i.e., $\phi(x) = \{X_*^n\}_{n\in\mathbb{N}}$.

Proposition 2.3.10. *The map ϕ is injective and continuous.*

Proof. We show the continuity of ϕ. For each open neighborhood V of $\phi(x) = \{X_*^n\}_{n\in\mathbb{N}}$ we can find an open neighborhood of the form

$$W = (U_{X_*^1} \times U_{X_*^2} \times \cdots \times U_{X_*^n} \times \mathcal{U}_{4\epsilon_{n+1}(A_{n+1})} \times \dots) \cap \mathcal{X}$$

such that $W \subset V$. By Proposition 2.3.4, for every X_*^n there exists n_* satisfying that for every $s > n_*$ we get $q_{n,s}(A_s(x)) = X_*^n$. We fix a value $s > n_*$. Now, we consider $\delta < \epsilon_s - \gamma_s - \epsilon_{s+1}$, where we have $\epsilon_s - \gamma_s - \epsilon_{s+1} > 0$ because $\epsilon_{s+1} < \frac{\epsilon_s - \gamma_s}{2}$. The idea is to verify that for every $y \in \mathcal{B}(x, \delta)$ we get $\phi(y) = \{Y_*^n\}_{n\in\mathbb{N}} \in W$. By the continuity of $q_{m,n}$ for all $m < n$, it suffices to check that $X_*^n \subset Y_*^n$ because $X_*^m = q_{m,n}(X_*^n) \subset q_{m,n}(Y_*^n) = Y_*^m$. From here, we would get $\phi(y) \in W$.

We prove that $A_s(x) \subset q_{s,s+1}(A_{s+1}(y))$. If $a_{s+1} \in A_{s+1}(y)$, then we have $d(y, a_{s+1}) < \epsilon_{s+1}$. On the other hand, if $b_s \in A_s(x)$, then we know that $d(x, b_s) \le \gamma_s$. Hence,

$$d(b_s, a_{s+1}) < d(b_s, x) + d(x, y) + d(y, a_{s+1}) < \gamma_s + \delta + \epsilon_{s+1} < \gamma_s + \epsilon_s - \gamma_s - \epsilon_{s+1} + \epsilon_{s+1} = \epsilon_s,$$

which means that $b_s \in \mathcal{B}(a_{s+1}, \epsilon_s) \cap A_s = q_{s,s+1}(a_{s+1})$. We obtain $A_s(x) \subset q_{s,s+1}(A_{s+1}(y))$, as we wanted. If we apply $q_{n,s}$ to the previous content, then we get

$$X_*^n = q_{n,s}(A_s(x)) \subset q_{n,s}(q_{s,s+1}(A_{s+1}(y))) = q_{n,s+1}(A_{s+1}(y)) \subset Y_*^n = \bigcup_{m>n} q_{n,m}(A_m(y)).$$

We prove the injectivity of ϕ. Suppose that $x \neq y$, which implies that $d(x, y) > 0$. We take n_0 such that for all $n > n_0$ we get $\epsilon_n < \frac{d(x,y)}{16}$. By Proposition 2.3.1, we know that $X_*^n \subset \mathcal{B}(x, 2\epsilon_n)$ and $Y_*^n \subset \mathcal{B}(y, 2\epsilon_n)$. If $X_*^n \cap Y_*^n \neq \emptyset$, then we can take $a \in X_*^n \cap Y_*^n$. Hence, $d(a, x), d(a, y) < 2\epsilon_n$ and we get a contradiction since

$$16\epsilon_n < d(x, y) < d(x, a) + d(a, y) < 2\epsilon_n + 2\epsilon_n < 4\epsilon.$$

Thus, $X_*^n \cap Y_*^n = \emptyset$ and we conclude that $\phi(x) \neq \phi(y)$. □

We consider $\phi(X) = \mathcal{X}_* \subseteq \mathcal{X}$. We will prove that $\mathcal{X}_*$ is homeomorphic to X and a strong deformation retract of $\mathcal{X}$. Firstly, we verify that $\mathcal{X}_*$ is a homeomorphic copy of X in $\mathcal{X}$ but before stating this result, we prove a property of $\mathcal{X}_*$.

Proposition 2.3.11. *The topological space $\mathcal{X}_*$ is a Hausdorff space.*

Proof. Let us take $\phi(x) = \{X_*^n\}_{n\in\mathbb{N}} \neq \{Y_*^n\}_{n\in\mathbb{N}} = \phi(y)$, so $x \neq y$ and $d(x, y) > 0$. We consider n_0 such that for every $n \ge n_0$, we get $\epsilon_n < \frac{d(x,y)}{16}$. Furthermore, we know that $X_*^n \cap Y_*^n = \emptyset$ for all $n \ge n_0$ by the proof of Proposition 2.3.10. We consider the following open neighborhoods for $\{X_*^n\}_{n\in\mathbb{N}}$ and $\{Y_*^n\}_{n\in\mathbb{N}}$

$$V_1 = (U_{X_*^1} \times U_{X_*^2} \times \cdots \times U_{X_*^{n_0}} \times \mathcal{U}_{4\epsilon_{n_0+1}}(A_{n_0+1}) \times \dots) \cap \mathcal{X}_*$$
$$V_2 = (U_{Y_*^1} \times U_{Y_*^2} \times \cdots \times U_{Y_*^{n_0}} \times \mathcal{U}_{4\epsilon_{n_0+1}}(A_{n_0+1}) \times \dots) \cap \mathcal{X}_*.$$

We argue by contradiction, suppose that the intersection of V_1 and V_2 is non-empty. Then, there exists $\{Z_*^n\}_{n\in\mathbb{N}} \in V_1 \cap V_2$ such that $X_*^m, Y_*^m \subset Z_*^m$ for every $m \le n_0$. We consider $x_{n_0} \in X_*^{n_0}$ and $y_{n_0} \in Y_*^{n_0}$. It follows that

$$\begin{aligned} 16\epsilon_{n_0} < d(x, y) &< d(x, x_{n_0}) + d(x_{n_0}, y_{n_0}) + d(y_{n_0}, y) < \\ &< 2\epsilon_{n_0} + d(x_{n_0}, y_{n_0}) + 2\epsilon_{n_0} = 4\epsilon_{n_0} + d(x_{n_0}, y_{n_0}). \end{aligned}$$

Therefore, $12\epsilon_{n_0} < d(x_{n_0}, y_{n_0})$ but $\{Z_n\}_{n\in\mathbb{N}} \in \mathcal{X}_*$. We get that $diam(Z_{n_0}) < 4\epsilon_{n_0}$, which leads to a contradiction. □

Theorem 2.3.12. *The compact metric space X is homeomorphic to $\mathcal{X}_*$.*

Proof. We have that $\phi : X \to \mathcal{X}_*$ is a continuous bijective map between a compact Hausdorff space and a Hausdorff space. Thus, ϕ is a homeomorphism. $\square$

Remark 2.3.13. *We can also prove Theorem 2.3.12 without Proposition 2.3.11. We have that $\varphi_{|\mathcal{X}_*} : \mathcal{X}_* \to X$ is a continuous and bijective function. On the other hand, $\phi : X \to \mathcal{X}_*$ is a continuous and bijective function that verifies $\phi \circ \varphi_{|\mathcal{X}_*} = id_{\mathcal{X}_*}$.*

Theorem 2.3.14. *The topological space $\mathcal{X}_*$ is a strong deformation retract of $\mathcal{X}$.*

Proof. It is easy to check that $\varphi \circ \phi : X \to X$ is the identity map. We will check that $\phi \circ \varphi : \mathcal{X} \to \mathcal{X}$ is homotopic to the identity map $id_{\mathcal{X}}$. We consider $H : \mathcal{X} \times I \to \mathcal{X}$ given by

$$H(\{C_n\}_{n\in\mathbb{N}}, t) = \begin{cases} \{C_n\}_{n\in\mathbb{N}} & t \in [0,1) \\ \phi(\varphi(\{C_n\}_{n\in\mathbb{N}})) & t = 1, \end{cases}$$

where I denotes the unit interval. To study the continuity of H, it suffices to prove the continuity at the point $(\{C_n\}_{n\in\mathbb{N}}, 1) \in \mathcal{X} \times I$. For every neighborhood W of $\phi(\varphi(\{C_n\}_{n\in\mathbb{N}})) = \{X_*^n\}_{n\in\mathbb{N}}$ we can obtain a neighborhood V of the form

$$V = (U_{X_*^1} \times U_{X_*^2} \times \cdots \times U_{X_*^r} \times \mathcal{U}_{4\epsilon_{r+1}}(A_{r+1}) \times \dots) \cap \mathcal{X}$$

such that $V \subset W$. By the continuity of $\phi \circ \varphi$ we know that there exists an open neighborhood U of $\{C_n\}_{n\in\mathbb{N}}$ with $\phi(\varphi(U)) \subset V$. We take $\{D_n\}_{n\in\mathbb{N}} \in U$ and we denote $\phi(\varphi(\{D_n\}_{n\in\mathbb{N}})) = \{Y_*^n\}_{n\in\mathbb{N}}$. By continuity, $\{Y_*^n\}_{n\in\mathbb{N}} \in V$, so $X_*^n \subset Y_*^n$ for every $n \leq r$. By Proposition 2.3.8, we also know $Y_*^m \subset D_m$ for every $m \in \mathbb{N}$. Concretely, $X_*^n \subset D_n$ for every $n \leq r$. Therefore, $H(\{D_n\}_{n\in\mathbb{N}}, t) = \{D_n\}_{n\in\mathbb{N}} \in V$ when $t \in [0,1)$ and $H(\{D_n\}_{n\in\mathbb{N}}, 1) = \{Y_*^n\}_{n\in\mathbb{N}} \in V$. Thus, $U \times I$ satisfies that $H(U \times I) \subset V$. $\square$

We update the example introduced in Section 2.2 with the theory developed in this section. Since the elements of $\mathcal{X}_*$ have a constructive description, they can be computed.

Example 2.3.15. We study the elements of the inverse limit of the FASO constructed in Example 2.2.4 for S^1. By construction, we have that $A_n \subset A_{n+1}$ for every $n \in \mathbb{N}$. We study two cases to get a description of $\mathcal{X}_*$. Firstly, we show one useful property.

Assertion. If $a_k^n, a_{k+1}^n \in A_n$, where $k, k+1 \in \{0, 1, ..., 2^{3n-4} - 1\}$, then a_k^n and a_{k+1}^n are in an arc of S^1 formed by two consecutive points $a_l^{n-1}, a_{l+1}^{n-1} \in A_{n-1}$ for some l such that the length of the arc is $\frac{\pi}{2^{3(n-1)-5}}$.

Proof. We can determine l solving the following equation: $\frac{2\pi k}{2^{3n-4}} = \frac{2\pi l}{2^{3n-7}}$. Then, $l = \frac{k}{2^3}$. We have two possibilities:

- $l \in \mathbb{N}$, which implies that $a_k^n \in A_{n-1}$ and the result follows easily.
- l is not a natural number. We define l^* as the integer part of l. We will check that $a_{l^*}^{n-1}, a_{l^*+1}^{n-1} \in A_{n-1}$ are the desired points. We know that $k = 8a + r$ for some a and $r < 8$. Therefore,

$$\frac{2\pi(k+1)}{2^{3n-4}} = \frac{2\pi m}{2^{3n-7}} \Rightarrow m = \frac{k+1}{8} = a + \frac{r+1}{8}.$$

If $\frac{r+1}{8}$ is an integer, we are in the first case. We suppose that it is not an integer number. Then, $\frac{r+1}{8} < 1$ and the integer part of m is exactly l^*.

□

If $x \in X$, then $\{X_*^n\}_{n\in\mathbb{N}}$ can only be of two different forms.

Case 1: $x \in S^1$ satisfying that $x = a_s^n \in A_n$ for some $n \in \mathbb{N}$. Therefore, $a_s^n \in A_m$ for every m satisfying $n \leq m$. We want to describe $\varphi(x) = \{X_*^n\}_{n\in\mathbb{N}}$. On the one hand, it is clear that $A_m(a_s^n) = a_s^n$ for every $m \geq n$. On the other hand, $q_{m,m+1}(A_{m+1}(a_s^n)) = q_{m,m+1}(a_s^n) = \mathcal{B}(a_s^n, \epsilon_m) \cap A_m = a_s^n$ for every $m \geq n$ because $d(a_s^n, A_m \setminus \{a_s^n\}) = \epsilon_m$. From here, we can deduce that $X_*^m = \{a_s^n\}$ for every $m \geq n$. We study X_*^m with $m < n$. By the previous observation, $q_{m,t}(A_t(a_s^n)) = q_{m,n}(a_s^n)$ if $t > n$. Clearly, a_s^n is between two consecutive points $a_k^{n-1}, a_{k+1}^{n-1} \in A_{n-1}$. Therefore, $q_{n-1,n}(a_s^n) = \{a_k^{n-1}, a_{k+1}^{n-1}\}$. By the previous assertion, a_k^{n-1} and a_{k+1}^{n-1} are between two consecutive points $a_k^{n-2}, a_{k+1}^{n-2} \in A_{n-2}$ so $q_{n-2,n-1}(\{a_k^{n-1}, a_{k+1}^{n-1}\}) = \{a_k^{n-2}, a_{k+1}^{n-2}\}$. Thus, we can deduce that $X_*^m = \{a_k^m, a_{k+1}^m\}$ for every $m < n$ and $m \neq 1$ because $X_*^1 = \{e^{2\pi i}\}$. In Figure 2.3.2 we present a schematic draw of the above description.

$$\{X_*^n\}_{n\in\mathbb{N}} = (\{e^{2\pi i}\}, \{a_k^1, a_{k+1}^1\}, \{a_k^2, a_{k+1}^2\}, \ldots, \{a_k^{n-1}, a_{k+1}^{n-1}\}, \{a_s^n\}, \{a_s^n\}, \ldots\})$$

Case 2: $x \in S^1$ such that $x \notin A_n$ for every $n \in \mathbb{N}$. We are interested in the following property: for every n there exists $m > n$ such that $A_m(x) \notin A_{m-1}$. We argue by contradiction. Suppose $A_m(x) \in A_{n+1}$ for every $m > n$, which implies $x \in A_{n+1}$ and then a contradiction. By Proposition 2.3.4, we know that for each n there exists $n_* > n$ such that $\{X_*^n\} = q_{n,m}(A_m(x))$ for every $m > n_*$. On the other hand, there exists $s > n_*$ with $A_s(x) \notin A_{s-1}$ and $\{X_*^n\} = q_{n,s}(A_s(x))$. We get that $A_s(x)$ is at most two consecutive points $a_k^s, a_{k+1}^s \in A_s$. Furthermore, by the previous property, a_k^s, a_{k+1}^s are between two consecutive points $a_k^{s-1}, a_{k+1}^{s-1} \in A_{s-1}$. Hence, $q_{s-1,s}(\{a_k^s, a_{k+1}^s\}) = \{a_k^{s-1}, a_{k+1}^{s-1}\}$. Thus, we can deduce that $X_*^n = \{a_k^n, a_{k+1}^n\}$ where $a_k^n, a_{k+1}^n \in A_n$ and x lies in the arc formed by a_k^n and a_{k+1}^n. In Figure 2.3.1 we present a schematic draw of the above description.

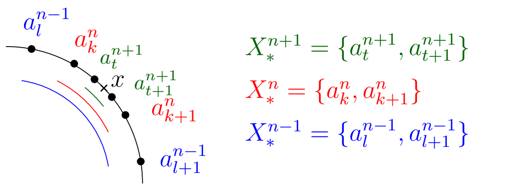

Figure 2.3.1: Schematic situation when $x \notin A_n$ for every $n \in \mathbb{N}$.

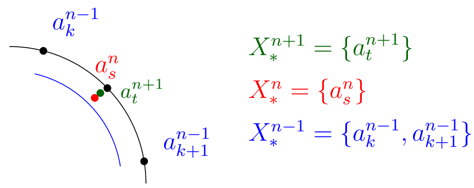

Figure 2.3.2: Schematic situation when x is equal to $a_n \in A_n$ but $x \notin A_{n-1}$.

2.4 Uniqueness of the FASO constructed for a compact metric space and relations with other constructions

Let (X, d) be a compact metric space. Then a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X is not unique since it depends on the values of ϵ_n and the points of A_n that we chose. We also know that the main results obtained in Section 2.3 do not depend on the points chosen. A FASO $(\mathcal{U}_n, q_{n,n+1})$ for X can be seen as an object of pro-*HTop* because $(\mathcal{U}_n, q_{n,n+1})$ is an inverse sequence of finite T_0 topological spaces.

Theorem 2.4.1. *Let (X, d) be a compact metric space. If $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is a FASO for X and $(\mathcal{U}_{4\delta_n}(B_n), q_{n,n+1})$ is a different FASO for X, then $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ and $(\mathcal{U}_{4\delta_n}(B_n), q_{n,n+1})$ are isomorphic in pro-HTop.*

Proof. Firstly, we define a candidate to be an isomorphism. We consider $I:\mathbb{N}\to\mathbb{N}$ given by $I(n)=min\{l\in\mathbb{N}|\delta_l<\frac{\epsilon_n}{16}\}$ and $I_n:\mathcal{U}_{4\delta_{I(n)}}(B_{I(n)})\to\mathcal{U}_{4\epsilon_n}(A_n)$ given by $I_n(C)=\bigcup_{x\in C}\mathcal{B}(x,\epsilon_n)\cap A_n$. We prove that (I_n,I) is a morphism in pro-$HTop$. For simplicity, we omit some subscripts when there is no confusion.

We prove that I_n is well-defined for every $n\in\mathbb{N}$. Suppose $C\in\mathcal{U}_{4\delta_{I(n)}}(B_{I(n)})$. If $x,y\in I_n(C)$, then there exist $a_x,a_y\in C$ such that $x\in I_n(a_x)$ and $y\in I_n(a_x)$. We obtain that $d(x,a_x),d(y,a_y)<\epsilon_n$. We also have that $diam(C)<4\delta_{I(n)}<\frac{\epsilon_n}{4}$, which implies that

$$d(x,y)<d(x,a_x)+d(a_x,y)+d(a_y,y)<\epsilon_n+\frac{\epsilon_n}{4}+\epsilon_n<4\epsilon_n.$$

Thus, I_n is well-defined for every $n\in\mathbb{N}$. The continuity of I_n follows trivially because I_n clearly preserves the order. Now, we check that for every $m\geq n$, where $n,m\in\mathbb{N}$, the following diagram is commutative up to homotopy.

$$\begin{tikzcd}
\mathcal{U}_{4\delta_{I(n)}}(B_{I(n)}) \arrow[d, "I_n"] & \mathcal{U}_{4\delta_{I(m)}}(B_{I(m)}) \arrow[l, "q_{I(n),I(m)}"'] \arrow[d, "I_m"] \\
\mathcal{U}_{4\epsilon_n}(A_n) & \mathcal{U}_{4\epsilon_m}(A_m) \arrow[l, "q_{n,m}"]
\end{tikzcd}$$

Suppose $C\in\mathcal{U}_{4\delta_{I(m)}}(B_{I(m)})$. If $x\in I_n(q(C))$, then there exist $a_x\in B_{I(n)},b_x\in C$ satisfying that $a_x\in q(b_x)$ and $x\in I_n(a_x)$. Therefore, $d(b_x,a_x)<2\delta_{I(n)}<\frac{\epsilon_n}{8}$ and $d(a_x,x)<\epsilon_n$. If $y\in q(I_m(C))$, then there exist $a_y\in A_m$, $b_y\in C$ such that $y\in q(a_y)$ and $a_y\in I_m(b_y)$. We get $d(y,a_y)<2\epsilon_n$ and $d(a_y,b_y)<\epsilon_m<\frac{\epsilon_n}{2}$. We know $d(b_y,b_x)<4\delta_{I(m)}<\frac{\epsilon_m}{4}<\frac{\epsilon_n}{8}$. Hence,

$$\begin{aligned}d(x,y)<&d(x,a_x)+d(a_x,b_x)+d(b_x,b_y)+d(b_y,a_y)+d(a_y,y)\\ <&\epsilon_n+\frac{\epsilon_n}{8}+\frac{\epsilon_n}{8}+\frac{\epsilon_n}{2}+2\epsilon_n<4\epsilon_n.\end{aligned}$$

We can conclude that $diam(I_n(q(C))\cup q(I_m(C)))<4\epsilon_n$ for every $C\in\mathcal{U}_{4\delta_{I(m)}}(B_{I(m)})$. Thus, $I_n\circ q\cup q\circ I_m:\mathcal{U}_{4\delta_{I(m)}}(B_{I(m)})\to\mathcal{U}_{4\epsilon_n}(A_n)$ is well-defined, continuous and satisfies $I_n(q(C))$, $q(I_m(C))\subset I_n(q(C))\cup q(I_m(C))$ for every $C\in\mathcal{U}_{4\delta_{I(m)}}(B_{I(m)})$. From here, we get that the previous diagram is commutative up to homotopy.

We consider $T:\mathbb{N}\to\mathbb{N}$ given by $T(n)=min\{l\in\mathbb{N}|\epsilon_l<\frac{\delta_n}{16}\}$ and $T_n:\mathcal{U}_{4\epsilon_{T(n)}}(A_{T(n)})\to\mathcal{U}_{4\delta_n}(B_n)$ given by $T_n(C)=\bigcup_{x\in C}\mathcal{B}(x,\delta_n)\cap B_n$. It suffices to repeat the same arguments used before to get that (T_n,T) is a well-defined morphism in pro-$HTop$.

We prove that $(T_n,T)\circ(I_n,I)$ is homotopic to $(id_\delta,id):(\mathcal{U}_{4\delta_n(B_n)},q_{n,n+1})\to(\mathcal{U}_{4\delta_n(B_n)},q_{n,n+1})$, where (id_δ,id) denotes the identity morphism.

We consider $m\in\mathbb{N}$ satisfying that $m>I(T(n))$. Hence, we need to show that the following diagram is commutative up to homotopy, where id denotes the identity map.

$$\begin{tikzcd}
\mathcal{U}_{4\epsilon_{T(n)}}(A_{T(n)}) \arrow[drr, "T_n"'] & \mathcal{U}_{4\delta_{I(T(n))}}(B_{I(T(n))}) \arrow[l, "I_{T(n)}"'] & \mathcal{U}_{4\delta_m}(B_m) \arrow[l, "q_{I(T(n)),m}"'] \arrow[r, "q_{n,m}"] & \mathcal{U}_{4\delta_n}(B_n) \arrow[dl, "id"] \\
 & & \mathcal{U}_{4\delta_n}(B_n) &
\end{tikzcd}$$

Suppose that $C\in\mathcal{U}_{4\delta_m}(B_m)$. If $x\in q(C)$, then there exists $c_x\in C$ such that $x\in q(c_x)$ and $d(x,c_x)<2\delta_n$. If $y\in T(I(q(C)))$, then there exist $a_y\in A_{T(n)}$, $b_y\in B_{I(T(n))}$ and $c_y\in C$ such that $y\in T(a_y)$, $a_y\in I(b_y)$, $b_y\in q(c_y)$. Therefore, $d(y,a_y)<\delta_n$, $d(a_y,b_y)<$

$\epsilon_{I(n)} < \frac{\delta_n}{16}$, $d(b_y, c_y) < 2\delta_{I(T(n))} < \frac{\epsilon_{I(n)}}{8} < \frac{\delta_n}{128}$. On the other hand, $diam(C) < 4\delta_m$ and $d(c_x, c_y) < 4\delta_m < 2\delta_{I(T(n))} < \frac{\delta_n}{128}$, which implies

$$\begin{aligned} d(x,y) <& d(x,c_x) + d(c_x,c_y) + d(c_y,b_y) + d(b_y,a_y) + d(a_y,y) \\ <& 2\delta_n + \frac{\delta_n}{128} + \frac{\delta_n}{128} + \frac{\delta_n}{16} + \delta_n < 4\delta_n. \end{aligned}$$

We get $diam(T(I(q(C))) \cup q(C)) < 4\delta_n$ for every $C \in \mathcal{U}_{4\delta_m}(B_m)$. Thus, $h = T \circ I \circ q \cup q : \mathcal{U}_{4\delta_m}(B_m) \to \mathcal{U}_{4\delta_n}(B_n)$ is a well-defined and continuous map. Furthermore, $I(I(q(C))), q(C) \subset h(C)$ for every $C \in \mathcal{U}_{4\delta_m}(B_m)$, which implies that the diagram is commutative up to homotopy.

Repeating the same arguments, it can be deduced that $(I_n, I) \circ (T_n, T)$ is homotopic to the identity morphism $(id_\epsilon, id) : (\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1}) \to (\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$. □

We study the relations between the inverse sequences considered throughout this chapter as objects of pro-$HTop$. Given a compact metric space (X, d), it can be considered the FAS or Main Construction $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ for X. We consider the opposite order for each term of this inverse sequence. Then we can compare this inverse sequence with a FASO $(\mathcal{U}_{4\delta_n}(B_n), q_{n,n+1})$ for X.

Theorem 2.4.2. *Given a compact metric space (X, d). If $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ is a FAS for X, where each term is considered with the opposite order, then every FASO $(\mathcal{U}_{4\delta_n}(B_n), q_{n,n+1})$ for X is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ in pro-$HTop$.*

Proof. It is easy to show that $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is a FASO for X. By Theorem 2.4.1, $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is isomorphic to $(\mathcal{U}_{4\delta_n}(B_n), q_{n,n+1})$. Therefore, it suffices to show that $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$.

We have a natural inclusion $i_n : \mathcal{U}_{2\epsilon}(A_n) \to \mathcal{U}_{4\epsilon}(A_n)$ for every $n \in \mathbb{N}$. The following diagram is commutative up to homotopy due to the fact that $p(i(C)) \subseteq i(q(C))$ for every $C \in \mathcal{U}_{2\epsilon_n}(A_n)$.

$$\begin{tikzcd} \mathcal{U}_{2\epsilon_n}(A_n) \arrow[d, "i"] & \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1}) \arrow[l, "p_{n,n+1}"'] \arrow[d, "i"] \\ \mathcal{U}_{4\epsilon_n}(A_n) & \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1}) \arrow[l, "q_{n,n+1}"] \end{tikzcd}$$

Then, we have a level morphism between the two inverse sequences considered. For every $m > n$ we define $g_n : \mathcal{U}_{4\epsilon_m}(A_m) \to \mathcal{U}_{2\epsilon_n}(A_n)$ given by

$$g_n(C) = \overline{\mathcal{B}}(C, \gamma_n) \cap A_n = \bigcup_{x \in C} \overline{\mathcal{B}}(x, \gamma_n) \cap A_n,$$

where $\overline{\mathcal{B}}(x, \gamma_n)$ denotes the closed ball of radius γ_n and $\gamma_n = \sup\{d(x, A_n) | x \in X\}$. We prove that g_n is well-defined. If $x, y \in g_n(C)$, then there exist $c_x, c_y \in C$ with $x \in g_n(c_x), y \in g_n(c_y)$. We obtain that $d(x, c_x), d(y, c_y) \leq \gamma_n$. We also know that $diam(C) < 4\epsilon_m$, which implies $d(c_x, c_y) < 4\epsilon_m < 2\epsilon_n - 2\gamma_n$. Therefore,

$$d(x,y) < d(x,c_x) + d(c_x,c_y) + d(c_y,y) < \gamma_n + 2\epsilon_n - 2\gamma_n + \gamma_n = 2\epsilon_n.$$

The continuity of g_n for every $n \in \mathbb{N}$ follows trivially. Since $g_n(i(C)) \subset p(C)$ and $q(C) \subset i(g_n(C))$, we get that $g_n \circ i$ is homotopic to p and q is homotopic to $i \circ g_n$. By Theorem 1.2.5, we get the desired result. □

From Theorem 2.4.2 and Theorem 2.1.8 we get the following the result. If the McCord functor $\mathcal{K}$ is applied to a FASO $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ for X, then a $HPol$-expansion of X is obtained. On the other hand, given a compact metric space (X, d) and a FASO $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ for X. We can consider a decreasing sequence of positive real values $\{\tau_n\}_{n\in\mathbb{N}}$ in the hypothesis of Remark 2.2.2, which is clearly less restrictive. For every $n \in \mathbb{N}$ we consider a τ_n-approximation B_n for X. We get an inverse sequence $(\mathcal{U}_{4\tau_n}(B_n), q_{n,n+1})$. Repeating the same arguments used in the proof of Theorem 2.4.1, we can prove that $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is isomorphic to $(\mathcal{U}_{4\tau_n}(B_n), q_{n,n+1})$. This result is important for computational reasons because we can drop a hard hypothesis to check, i.e., it is not necessary to compute $\gamma_n = sup\{d(x, A_n)|x \in X\}$. Removing this hypothesis, we cannot expect to get Theorem 2.3.12 and Theorem 2.3.14. This is due to the fact that $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ and $(\mathcal{U}_{4\tau_n}(B_n), q_{n,n+1})$ are just isomorphic in pro-$HTop$ and not in pro-Top. Since both inverse sequences are isomorphic in pro-$HTop$, if follows that $(\mathcal{U}_{4\tau_n}(B_n), q_{n,n+1})$ serves to approximate algebraic invariants such as the homology groups. If we apply the homology functor to $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ and $(\mathcal{U}_{4\tau_n}(B_n), q_{n,n+1})$, then we get two inverse sequences of groups that are isomorphic in the pro-category of groups. This means that both inverse limits are isomorphic. Thus, the new inverse sequence can be used to approximate or get the Čech homology groups of X. For the case X is a CW-complex the singular homology groups and Čech homology groups are the same.

Thus, we have a method to reconstruct homology groups. In the following proposition we sum up the previous comments.

Proposition 2.4.3. *Let X be a compact metric space and let $m \in \mathbb{N}$. Then there exists an inverse sequence $(\mathcal{U}_{4\tau_n}(A_n), q_{n,n+1}$, where $\tau_{n+1} \leq \frac{\tau_n}{2}$ and A_n is a τ_n-approximation for every $n \in \mathbb{N}$, satisfying that the inverse limit of $(H_m(\mathcal{U}_{4\epsilon_n}(A_n)), q_{n,n+1*})$ is isomorphic to $\check{H}_m(X)$.*

Remark 2.4.4. *Similarly, we can use other functors to reconstruct other groups. For instance, shape groups or Čech cohomology groups.*

Given a compact polyhedron K, we can obtain an inverse sequence of finite topological spaces using the theory developed or we can apply the construction given in [31]. Let $(X^n, h_{n,n+1})$ denote this construction, where X^n denotes the n-th finite barycentric subdivision of $\mathcal{X}(K)$ with the opposite order and $h_{n,n+1}$ is the natural map considered in Subsection 1.1.4.

Theorem 2.4.5. *Let K be a compact polyhedron. If $(\mathcal{U}_{4\epsilon_n}(A_n), q_{n,n+1})$ is a FASO for K, then there is a natural morphism $(i_n, id) : (X^n, h_{n,n+1}) \to (\mathcal{U}_{4\epsilon_n}, q_{n,n+1})$ in pro-$HTop$.*

Proof. We construct a new FASO $(\mathcal{U}_{4\sigma_n}(B_n), q_{n,n+1})$ for K and a morphism (i_n, id) : $(X^n, h_{n,n+1}) \to (\mathcal{U}_{4\sigma_n}(B_n), q_{n,n+1})$. Hence, by Theorem 2.4.1, we get the desired result.

We start taking $\sigma_1 > diam(K)$, B_1 is the set of vertices of K and $X^1 = \mathcal{X}(K)$. We consider $\delta_1 = \sup\{d(x, B_1)|x \in K\}$ and $\sigma_2 < \frac{\sigma_1 - \delta_1}{2}$. Applying barycentric subdivisions of K, we obtain that there exists $t_2 \in \mathbb{N}$ such that the set of vertices of K^{t_2} is a σ_2-approximation B_2 of K, $X^2 = \mathcal{X}(K^{t_2})$ is a subposet of $\mathcal{U}_{4\sigma_2}(B_2)$ and $q_{2,1} \circ i \geq i \circ h_{t_2,t_1}$, where i denotes the inclusion map with abuse of notation. We can deduce the last assertion using that the diameters of the simplices obtained after a barycentric subdivision are smaller than the original ones, see [60] or [100]. In Figure 2.4.1 there is an example of the situation described above.

Arguing inductively, we can obtain a FASO $(\mathcal{U}_{4\sigma_n}(B_n), q_{n,n+1})$ for K such that X^{t_n} is a subposet of $\mathcal{U}_{4\sigma_n}(B_n)$. The set $\{t_n\}_{n\in\mathbb{N}}$ is a cofinal subset of $\mathbb{N}$. Then, $(X^n, h_{n,n+1})$ is

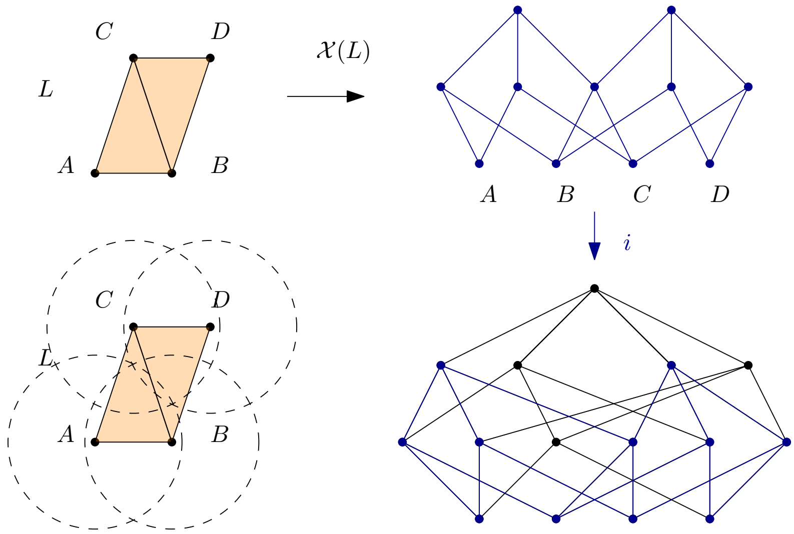

Figure 2.4.1: Representation of the simplicial complex L and Hasse diagrams of $\mathcal{X}(L)$ and $\mathcal{U}_{4\epsilon_n}(\{A, B, C, D\})$.

isomorphic to $(X^{t_n}, h_{n,n+1}, \{t_n\}_{n\in\mathbb{N}})$ in pro-*HTop*. We have that the following diagram commutes up to homotopy.

$$\begin{tikzcd}
\mathcal{U}_{4\sigma_n}(B_n) & \mathcal{U}_{4\sigma_{n+1}}(B_{n+1}) \arrow[l, "q_{n,n+1}"'] \\
X^{t_n} \arrow[u, "i"] & X^{t_{n+1}} \arrow[l, "h_{t_n,t_{n+1}}"] \arrow[u, "i"']
\end{tikzcd}$$

□

Remark 2.4.6. *In general, the continuous inclusion $i : X^{t_n} \to \mathcal{U}_{4\sigma_n}(B_n)$ considered in the proof of Theorem 2.4.5 is not a homotopy equivalence. From Example 2.7.5 we can deduce that X^{t_n} and $\mathcal{U}_{4\sigma_n}(B_n)$ are not homotopy equivalent.*

2.5 Computational aspects of a FASO for a compact metric space and implementation

We recall the notion of Vietoris-Rips complex, [108]. Given a finite set of points $S \subseteq \mathbb{R}^n$ for some $n \in \mathbb{N}$ and a positive real value ϵ. The Vietoris-Rips complex is a simplicial complex given as follows $V_\epsilon(S) = \{\sigma \subseteq S | d(u, v) \leq \epsilon, \ \forall u \neq v \in \sigma\}$. In Figure 2.5.1 we have an example of a Vietoris-Rips complex. For a complete introduction, see for example [42].

Let us assume that X is a compact metric space embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Given an ϵ_n-approximation A_n, the problem of finding $\mathcal{U}_{4\epsilon_n}(A_n)$ is equivalent to construct the Vietoris-Rips complex $V_{4\epsilon_n}(A_n)$. This is due to the fact that $\mathcal{U}_{4\epsilon_n}(A_n)$ is the face poset of $V_{4\epsilon_n}(A_n)$, that is, $\mathcal{X}(V_{4\epsilon_n}(A_n)) = \mathcal{U}_{4\epsilon_n}(A_n)$. Furthermore, we get that $V_{4\epsilon_n}(A_n)$ has the same weak homotopy type of $\mathcal{U}_{4\epsilon_n}(A_n)$ by Theorem 1.1.49. In [114], it is obtained an algorithm to get Vietoris-Rips complexes. Furthermore, it is also compared with other algorithms in terms of computational time. The experiments show that the algorithm introduced in [114] is the fastest. It is also important to observe that this algorithm has

Figure 2.5.1: A set of points $S \subset \mathbb{R}^2$ and Vietoris-Rips complex of S for some positive real value ϵ.

two phases. In the first one, the 1-skeleton of the Vietoris-Rips complex is constructed. Once it is obtained the 1-skeleton, the problem to obtain the entire simplicial complex is a combinatorial one and it is related to the computation of clique complexes. A clique is a set of vertices in a graph that induces a complete subgraph. A clique complex has the maximal cliques of a graph as its maximal simplices.

If we have computed $\mathcal{U}_{4\epsilon_n}(A_n)$ and $\mathcal{U}_{4\epsilon_{n+1}}(A_{n+1})$, then we need to get the map $q_{n,n+1}$. This problem is equivalent to a variation of a classical one in computational geometry, the ϵ-nearest neighborhood problem. Namely, given a set of points $S \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$, a point $q \in \mathbb{R}^n$ and a positive value ϵ, find the set $N = \{x \in S | d(x, y) < \epsilon\}$. This problem has been treated largely in the literature with different approaches, see for instance [52] or [8]. Using one of the previous algorithms it can be obtained $q_{n,n+1}$ over A_{n+1}. For the remaining points in $\mathcal{U}_{4\epsilon_{n+1}}(A_{n+1})$, the description of $q_{n,n+1}$ is purely combinatorial since it is the union of the images of the points obtained before.

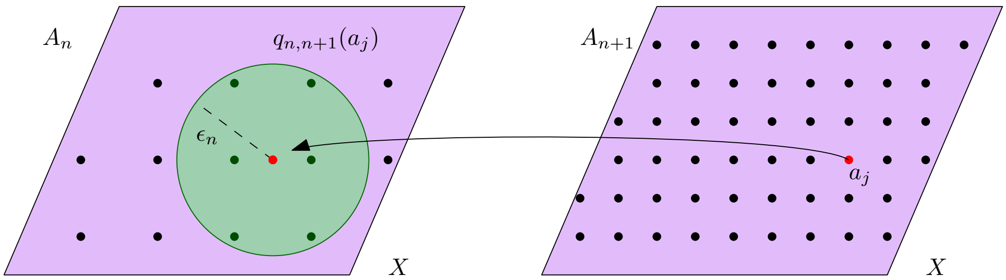

Figure 2.5.2: Schematic description of $q_{n,n+1}(a_j)$, where $a_j \in A_{n+1}$.

Thus, if $X \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$. we have the following steps:

1. Find a sequence of positive values $\{\epsilon_n\}_{n\in\mathbb{N}}$ and an ϵ_n-approximation A_n for every $n \in \mathbb{N}$ satisfying the conditions required in the construction of a FASO for X.

2. Compute $\mathcal{U}_{4\epsilon_n}(A_n)$ for every $n \in \mathbb{N}$.

3. Obtain $q_{n,n+1} : \mathcal{U}_{4\epsilon_{n+1}}(A_{n+1}) \to \mathcal{U}_{4\epsilon_n}(A_n)$ for every $n \in \mathbb{N}$.

One possible approach to get the first step is to consider a sequence of grids. We have seen that the second step is equivalent to the construction of a Vietoris-Rips complex and the third step is equivalent to a classical problem in computational geometry. Then, combining classical algorithms we can obtain a FASO for a compact metric space embedded in $\mathbb{R}^m$.

Remark 2.5.1. *Step 1 can be simplified using Remark 2.2.2 and Proposition 2.4.3. We lose information about X but we get a simpler algorithm.*

The method proposed throughout this chapter can also be used for data analysis. Instead of having a compact metric space embedded in $\mathbb{R}^n$ we could have just a set of points S in $\mathbb{R}^n$. Using the metric inherit from $\mathbb{R}^n$, we can speak about ϵ-approximations for the data set S. In this case, higher values of the inverse sequence tend to recover the data set as a disjoint union of points, i.e., there exists m such that for every $n \geq m$ we get $\mathcal{U}_{4\epsilon_n}(A_n) = \{S\}$.

We present an easy example. We assume that we cannot get in a deterministic way the points of the approximation, that is, we can obtain data from experiments, or we only know that the data follow a distribution.

Example 2.5.2. We consider two squares,

$$D = \{0\} \times [0, 2] \cup \{1\} \times [0, 2] \cup [0, 1] \times \{0\} \cup [0, 1] \times \{1\} \cup [0, 1] \times \{2\}.$$

We can consider $\epsilon_n = \frac{1}{2^{2(n-1)}}$ by Remark 2.2.2 and Proposition 2.4.3. Suppose that the data that can be taken to approximate D follow a continuous uniform distribution. In Figure 2.5.3 we have the data obtained for the first observations, that is, A_1, A_2, A_3 and A_4.

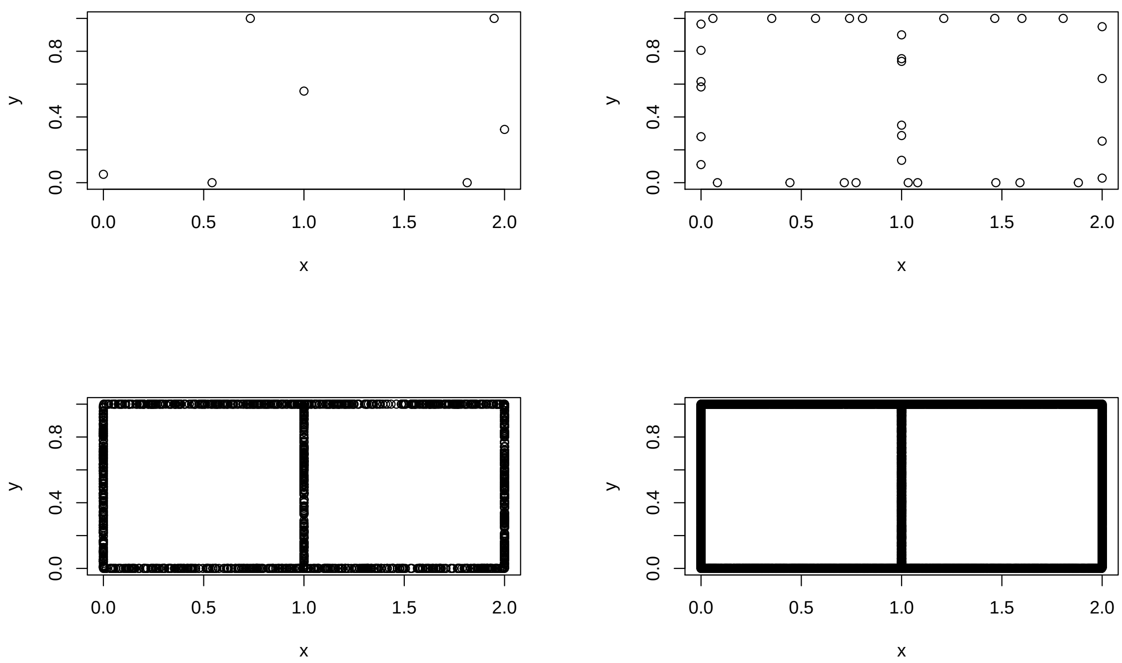

Figure 2.5.3: A_1, A_2, A_3 and A_4

We compute the dimensions of the homology groups of $\mathcal{U}_{4\epsilon_i}(A_i)$ for $i = 1, .., 4$, see Table 2.5.1. It seems that with a few steps we get a good representation of D at least from a homological viewpoint. On the other hand, we know that different choices of the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ or observations lead to the same results due to the results obtained in Section 2.4. In addition, Proposition 2.4.3 guarantees that this method is appropriate.

In Example 2.5.2 we can observe that the dimensions of the homology groups stabilize very soon. This can happen due to the good properties of the topological space D, which is a CW-complex. We provide an example for which this behavior does not happen.

Table 2.5.1: Dimensions of homology groups.

	H_0	H_1	H_2
$\mathcal{U}_{4\epsilon_1}(A_1)$	1	0	0
$\mathcal{U}_{4\epsilon_2}(A_2)$	1	2	0
$\mathcal{U}_{4\epsilon_3}(A_3)$	1	2	0
$\mathcal{U}_{4\epsilon_4}(A_4)$	1	2	0

Example 2.5.3. We consider the Cantor set $C \subset [0,1]$. The exact construction is as follows. From the closed interval $E_1 = [0,1]$, first remove the open interval $(\frac{1}{3}, \frac{2}{3})$ leaving $E_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. From E_2, delete the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. E_3 is the remaining 4 closed intervals. From E_3, remove middle thirds as before obtaining E_4. Hence, $C = \cap_{i=1}^{\infty} E_i$. For a complete introduction and properties about the Cantor set C, see [101].

Again, it can be taken $\epsilon_n = \frac{1}{2^{2(n-1)}}$ by Remark 2.2.2 and Proposition 2.4.3. We will use as approximations the endpoints of the closed intervals that remain after removing open intervals. For example, $A_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. We define A_n given by the endpoints of E_{n+1}. It is not difficult to show that the endpoints of the remaining closed intervals belong to C and are an ϵ_n-approximation for every $n \in \mathbb{N}$. We present in Figure 2.5.4 the first approximations, i.e., A_1 and A_2.

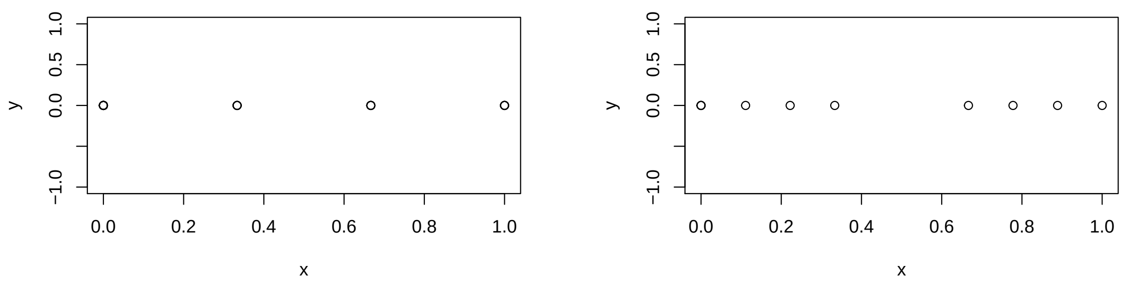

Figure 2.5.4: A_1 and A_2.

In Table 2.5.2 we present the dimensions of the homology groups for the first values of the sequence $\{\mathcal{U}_{4\epsilon_n}(A_n)\}_{n\in\mathbb{N}}$. It can be observed that if n increases, then the dimension of the 0-dimensional homology group for $\mathcal{U}_{4\epsilon_n}(A_n)$ increases. The previous behavior is expected since the Cantor set is a totally disconnected topological space. Therefore, in each step we are locating more components.

Table 2.5.2: Dimensions of homology groups.

	$\mathcal{U}_1$	$\mathcal{U}_2$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_5$	$\mathcal{U}_6$	$\mathcal{U}_7$	$\mathcal{U}_8$
H_0	1	1	1	3	7	31	63	127
H_1	0	0	0	0	0	0	0	0

The computations for Example 2.5.2 and Example 2.5.3 have been carried out by the software R.

2.6 New description of the shape theory

We use the theory developed earlier to establish a new description of shape theory based on [98] and [3]. Given a compact metric space X, we choose a FASO $(\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ for X. By Theorem 2.4.1, it makes sense to consider this choice. Then, for every compact metric space we can fix a FASO. Concretely, we have a map T that sends each compact metric space to an object in pro-$HTop$. We will show that the set of morphisms in pro-$HTop$ between the FASOs for two compact metric spaces X and Y is in bijective correspondence with the set of shape morphisms between X and Y.

For technical reasons, every term of a FASO considered in this section has the natural order, that is, $A \leq B$ if and only if $A \subseteq B$. The reason to consider the natural order relies on the results that were recalled in Subsection 1.2.6. These results were obtained for the upper semifinite topology in hyperspaces. This topology coincides with the natural order for finite topological spaces. On the other hand, the main result also holds true if we consider the opposite order as we will see in Remark 2.6.8. Before giving the proof of the desired result, we prove technical propositions and lemmas.

Lemma 2.6.1. *Let (X,d) be a compact metric space and let A be an ϵ-approximation of X. We consider the natural order in $\mathcal{U}_{4\epsilon}(A)$.*

1. *The map $p : X \to \mathcal{U}_{4\epsilon}(A)$ given by $p(x) = \{a \in A | d(x,a) = d(x,A)\}$ is well-defined and continuous.*

2. *If $f, g : \mathcal{U}_{2\epsilon}(Y) \to \mathcal{U}_{4\epsilon}(A)$ are continuous maps and $f \cup g$ given by $(f \cup g)(x) = f(x) \cup g(x)$ is well-defined, then $f \cup g$ is also continuous. Moreover, f is homotopic to $f \cup g$ and g is homotopic to $f \cup g$.*

Proof. (1) The proof follows easily from [81, Lemma 2].

(2) The continuity is easy to check. Let x be a point of X. The minimal open neighborhood that contains $f(x) \cup g(x)$ also contains the minimal open neighborhoods of $f(x)$ and $g(x)$. Since f and g are continuous maps, there exist open sets W and V containing x such that $f(W) \subseteq U_{f(x)}$ and $g(V) \subseteq U_{g(x)}$. From here, the continuity of $f \cup g$ at x follows trivially. For the second part, we consider $H : \mathcal{U}_{2\epsilon}(Y) \times [0,1] \to \mathcal{U}_{4\epsilon}(A)$ given by $H(x,t) = f(x)$ if $t \in [0,1)$ and $H(x,t) = f(x) \cup g(x)$ if $t = 1$. We only need to check the continuity of H at points of the form $(x,1)$, where $x \in X$. Let us consider $(x,1) \in \mathcal{U}_{2\epsilon}(Y) \times [0,1]$. Since $f \cup g$ is continuous, there exists an open set V containing x such that $f(V) \cup g(V) \subseteq U_{f(x)\cup g(x)}$. Concretely, $f(V) \subseteq U_{f(x)\cup g(x)}$. Then, $V \times I$ is an open set of $\mathcal{U}_{2\epsilon}(Y) \times I$ containing x and satisfying that $H(V \times I) \subseteq U_{f(x)\cup g(x)}$, which implies the desired result. The proof to show that g is homotopic to $f \cup g$ is the same. □

Remark 2.6.2. *Repeating similar arguments than the ones used in the second part of Lemma 2.6.1, it can be obtained the following: if $f, g : X \to \mathcal{U}_{2\epsilon}(Y)$ are continuous maps such that $f \cup g$ is well-defined, then $f \cup g$ is continuous, f is homotopic to $f \cup g$ and g is homotopic to $f \cup g$.*

In the following proposition, we get a constructive method to get a morphism in pro-$HTop$ induced by the homotopy class of an approximative map.

Proposition 2.6.3. *Let (X,d) and (Y,l) be compact metric spaces. If $[\overline{f}] : X \to Y$ is the homotopy class of an approximative map, then there exists a natural morphism $T([\overline{f}]) : T(X) \to T(Y)$ in pro-$HTop$.*

Proof. We fix notation, $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ and $T(Y) = (\mathcal{U}_{4\epsilon_n}(B_n), q_{n,n+1})$. Firstly, we prove that for every $n \in \mathbb{N}$ the map $r_n : \mathcal{U}_{2\epsilon_n}(Y) \to \mathcal{U}_{4\epsilon_n}(B_n)$ given by

$$r_n(C) = \bigcup_{x \in C} \{b \in B_n | l(x,b) = l(x, B_n)\}$$

is well-defined and continuous. If $x, y \in r_n(C)$ for some $C \in \mathcal{U}_{2\epsilon_n}(Y)$, then there exist $c_x, c_y \in C$ such that $x \in r_n(c_x)$ and $y \in r_n(c_y)$. We have $l(x, c_x), l(y, c_y) < \epsilon_n$ and $l(c_x, c_y) < 2\epsilon_n$. Hence, we get

$$l(x,y) < l(x,c_x) + l(c_x,c_y) + l(c_y,y) < \epsilon_n + 2\epsilon_n + \epsilon_n,$$

which implies that r_n is well-defined. By Lemma 1.2.24, we get that r_n is continuous since $r_n = p^*_{n|\mathcal{U}_{2\epsilon_n}(Y)}$, where p_n is the continuous map considered in Lemma 2.6.1 and p^*_n denotes the extension of p_n to the hyperspace of Y defined in Lemma 1.2.24.

Now, we prove the commutativity up to homotopy of the following diagram, where i denotes the inclusion map.

$$\begin{tikzcd}
\mathcal{U}_{2\epsilon_n}(Y) \arrow[r, "r_n"] & \mathcal{U}_{4\epsilon_n}(B_n) \\
\mathcal{U}_{2\epsilon_{n+1}}(Y) \arrow[r, "r_{n+1}"] \arrow[u, "i"] & \mathcal{U}_{4\epsilon_{n+1}}(B_{n+1}) \arrow[u, "q_{n,n+1}"']
\end{tikzcd}$$

We consider $C \in \mathcal{U}_{2\epsilon_{n+1}}(Y)$. If $x \in q_{n,n+1}(r_{n+1}(C))$, then there exist $a_x \in B_{n+1}$, $b_x \in C$ such that $x \in q_{n,n+1}(a_x)$ and $a_x \in r_{n+1}(b_x)$. Therefore, we get,

$$l(x, b_x) < l(x, a_x) + l(a_x, b_x) < \epsilon_n + \frac{\epsilon_n}{2}.$$

If $y \in r_n(i(C))$, then there exists $b_y \in C$ such that $y \in r_n(b_y)$. Since $l(b_x, b_y) < 2\epsilon_{n+1} < \epsilon_n$, we have

$$l(x,y) < l(x,b_x) + l(b_x,b_y) + l(b_y,y) < \epsilon_n + \epsilon_n + \frac{\epsilon_n}{2} + \epsilon_n.$$

Thus, we have shown that $h = r_n \circ i \cup q_{n,n+1} \circ r_{n+1} : \mathcal{U}_{2\epsilon_{n+1}}(Y) \to \mathcal{U}_{4\epsilon_n(B_n)}$ is well-defined. The continuity of h follows by Lemma 2.6.1. Again, by Lemma 2.6.1, we get the commutativity up to homotopy of the previous diagram.

Let us consider $f = {f_k : X \to 2^Y}_{k \in \mathbb{N}} \in [\overline{f}]$. By Proposition 1.2.23, for every open neighborhood U of Y in 2^Y there exists n such that $U_{2\epsilon_n}(Y) \subseteq U$. By Definition 1.2.26, there exists $s(n)$ such that f_m is homotopic to f_{m+1} in $U_{2\epsilon_n}(Y) \subseteq U$ for every $m \geq s(n)$. Let H denote the homotopy between $f_{s(n)}$ and $f_{s(n)+1}$. By Theorem 1.2.25, there exists $\gamma_n > 0$ such that $\overline{H}(\mathcal{U}_{\gamma_n}(X) \times I) \subseteq \mathcal{U}_{2\epsilon_n}(Y)$. We also denote by f the map $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = \min\{l \in \mathbb{N} | 4\delta_l < \gamma_n\}$, it is clear that f is well-defined and satisfies that $f(n) \leq f(m)$ for every $n \leq m$ in $\mathbb{N}$. For every natural number n we consider $f_n : \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \to \mathcal{U}_{4\epsilon_n}(B_n)$ given by $f_n = r_n \circ f^*_{s(n)} \circ i$, where $f^*_{s(n)}$ denotes the extension of $f_{s(n)}$ to the hyperspace of X given in Lemma 1.2.24 and $i : \mathcal{U}_{4\delta_{f(n)}}(B_n) \to \mathcal{U}_{\gamma_n}(X)$ denotes the inclusion. By construction, it is immediate to get that f_n is well-defined and continuous for every $n \in \mathbb{N}$. To check that $(f_n, f) : T(x) \to T(Y)$ is a morphism in pro-$HTop$, we need to verify the commutativity up to homotopy of the following diagram.

$$\begin{tikzcd}
\mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \arrow[r, "i"] & \mathcal{U}_{\gamma_n}(X) \arrow[r, "f^*_{s(n)}"] & \mathcal{U}_{2\epsilon_n}(Y) \arrow[r, "r_n"] & \mathcal{U}_{4\epsilon_n}(B_n) \\
\mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)}) \arrow[r, "i"] \arrow[u, "q"] & \mathcal{U}_{\gamma_{n+1}}(X) \arrow[r, "f^*_{s(n+1)}"] \arrow[u, "i"] & \mathcal{U}_{2\epsilon_{n+1}}(Y) \arrow[r, "r_{n+1}"] \arrow[u, "i"] & \mathcal{U}_{4\epsilon_{n+1}}(B_{n+1}) \arrow[u, "q"]
\end{tikzcd}$$

We check the commutativity up to homotopy of the first square. We consider $C \in \mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)})$. If $x \in i(q(C)) = q(C)$, then there exists $a_x \in C$ such that $x \in q(a_x)$, which implies $d(x, a_x) < 2\delta_{f(n)} < \frac{\gamma_n}{2}$. If $y \in i(i(C)) = C$, then $d(y, a_x) < 4\delta_{f(n+1)} < \frac{\gamma_n}{2}$. We get $d(x, y) < \gamma_n$. Therefore, $q(C) \subseteq q(C) \cup C$ for every $C \in \mathcal{U}_{4\delta_{f(n+1)}}(A_{f(n+1)})$, where $q \cup id : \mathcal{U}_{4\delta_{f(n+1)}} \to \mathcal{U}_{\gamma_n}(X)$ is well-defined and continuous. By Lemma 2.6.1, the commutativity up to homotopy of the first square can be deduced. The second square is commutative up to homotopy by construction. The commutativity up to homotopy of the third square was proved at beginning.

If $g \in [\overline{f}]$, then we can repeat the same construction to get $(g_n, g) : T(X) \to T(Y)$. We prove that (f_n, f) is equivalent to (g_n, g) as morphisms in pro-$HTop$. To do this, given a natural number n we need to verify the commutativity up to homotopy of the following diagram for some $m \geq f(n), g(n)$.

$$\begin{tikzcd}
 & \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \arrow[r, "i"] & \mathcal{U}_{\gamma_n}(X) \arrow[r, "f^*_{s(n)}"] \arrow[dd, "i"] & \mathcal{U}_{2\epsilon_n}(Y) \arrow[dd, "id"] \arrow[dr, "r_n"] & \\
\mathcal{U}_{4\delta_m}(A_m) \arrow[ur, "q"] \arrow[dr, "q"'] & & & & \mathcal{U}_{4\epsilon_n}(B_n) \\
 & \mathcal{U}_{4\delta_{g(n)}}(A_{g(n)}) \arrow[r, "i"] & \mathcal{U}_{\tau_n}(X) \arrow[r, "g^*_{h(n)}"] & \mathcal{U}_{2\epsilon_n}(Y) \arrow[ur, "r_n"'] &
\end{tikzcd}$$

We define $m = \max\{f(n), g(n)\}$. Without loss of generality, we can suppose that $\gamma_n \leq \tau_n$ and $m = f(n)$. We study the commutativity up to homotopy of the first square. We consider $C \in \mathcal{U}_{4\delta_m}(A_m)$. We assume that $f(n) \neq g(n)$ since the other case follows trivially. If $y \in i(q(C)) = q(C)$, then there exists $a_y \in C$ such that $y \in q(a_y)$. We get $d(a_y, y) < 2\delta_{g(n)} < \frac{\tau_n}{2}$ and $d(x, a_y) < 4\delta_{f(n)} < 2\delta_{g(n)} < \frac{\tau_n}{2}$ for every $x \in C$. Thus, we have

$$d(x, y) < d(a_x, a_y) + d(a_y, y) < \tau_n$$

and $i \circ q \cup i \circ i : \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \to \mathcal{U}_{\tau_n}(X)$ is well-defined. By Lemma 2.6.1, we get the desired result. The commutativity up to homotopy of the second square follows due to the fact that f and g are homotopic approximative maps by Theorem 1.2.25 and the choice of $s(n)$ and $h(n)$. For every $m \geq s(n), h(n)$ we have that $f_{s(n)}$ is homotopic to f_m in $\mathcal{U}_{2\epsilon_n}(Y)$ and $g_{h(n)}$ is homotopic to g_m in $\mathcal{U}_{2\epsilon_n}(Y)$. The third square commutes trivially. $\square$

We obtain a method to get a homotopy class of an approximative map from a morphism in pro-$HTop$ between two FASOs.

Proposition 2.6.4. *Let (X, d) and (Y, l) be compact metric spaces. If $(f_n, f) : T(X) \to T(Y)$ is a morphism in pro-$HTop$, then there exists a natural homotopy class of an approximative map $E(f_n) : X \to Y$ induced by (f_n, f).*

Proof. We fix notation, $T(X) = (\mathcal{U}_{4\delta_n}(A_n), q_{n,n+1})$ and $T(Y) = (\mathcal{U}_{4\epsilon_n(B_n)}, q_{n,n+1})$. Firstly, we consider $p_n : X \to \mathcal{U}_{4\delta_n}(A_n)$ given by $p_n(x) = \{a \in A | d(x, a) = d(x, A)\}$ for every $n \in \mathbb{N}$, that is, the continuous map considered in Lemma 2.6.1. We prove that the following diagram commutes up to homotopy.

$$\begin{tikzcd}
 & X \arrow[dl, "p_n"'] \arrow[dr, "p_{n+1}"] & \\
\mathcal{U}_{4\delta_n}(A_n) & & \mathcal{U}_{4\delta_{n+1}(A_{n+1})} \arrow[ll, "q_{n,n+1}"]
\end{tikzcd}$$

If $a_x \in p(q(x))$, then there exists $b_x \in A_{n+1}$ with $a_x \in p(b_x)$ and $b_x \in p(x)$. Hence, $d(x, b_x) < \delta_{n+1} < \frac{\delta_n}{2}$ and $d(a_x, b_x) < \delta_n$. If $c_x \in p(x)$, then we get $d(x, c_x) < \delta_n$. Thus,

$$d(c_x, a_x) < d(c_x, x) + d(x, b_x) + d(b_x, a_x) < 3\delta_n,$$

which implies that $q_{n,n+1} \circ p_{n+1} \cup p_n : X \to \mathcal{U}_{4\delta_n}(A_n)$ is well-defined. Applying Lemma 2.6.1, we get that the diagram commutes up to homotopy.

We construct the candidate to be an approximative map. We consider $F = \{F_k : X \to 2^Y\}_{k\in\mathbb{N}}$ given by $F_k = f_k \circ p_{f(k)} : X \to \mathcal{U}_{4\epsilon_k}(B_k)$. For every open neighborhood U of the canonical copy of Y in 2^Y, there exists $4\epsilon_m$ such that $\mathcal{U}_{4\epsilon_m}(B_m) \subset \mathcal{U}_{4\epsilon_m}(Y) \subseteq U$ by Proposition 1.2.23. To prove that F is an approximative map we only need to check that F_m is homotopic to F_{m+1} in U, which is equivalent to show the commutativity up to homotopy of the following diagram.

$$\begin{tikzcd}
X \arrow[r, "p_{f(m)}"] & \mathcal{U}_{4\delta_{f(m)}}(A_{f(m)}) \arrow[r, "f_m"] & \mathcal{U}_{4\epsilon_m}(B_m) \\
X \arrow[u, "id"] \arrow[r, "p_{f(m+1)}"] & \mathcal{U}_{4\delta_{f(m+1)}}(A_{f(m+1)}) \arrow[u, "q"] \arrow[r, "f_{m+1}"] & \mathcal{U}_{4\epsilon_{m+1}}(B_{m+1}) \arrow[u, "q"]
\end{tikzcd}$$

The commutativity up to homotopy of the first square follows from the commutativity up to homotopy of the diagram at the beginning. The second square commutes up to homotopy since (f_n, f) is a morphism in pro-$HTop$. Thus, we deduce that F is an approximative map. We denote by $E(f_n)$ the homotopy class generated by the approximative map F. We verify that E is well-defined, that is, if (g_n, g) is equivalent to (f_n, f) as morphisms in pro-$HTop$, then the induced approximative map $G = \{G_k = g_k \circ p_{f(k)} : X \to 2^Y\}_{k\in\mathbb{N}}$ is homotopic to $F = \{F_k = f_k \circ p_{f(k)} : X \to 2^Y\}_{k\in\mathbb{N}}$.

For every open neighborhood U of Y in 2^Y there exists $4\epsilon_n$ such that $\mathcal{U}_{4\epsilon_n}(Y) \subseteq U$ by Proposition 1.2.23. By hypothesis, for every n there exists $m \geq f(n), g(n)$ such that $f_n \circ q_{f(n),m}$ is homotopic to $g_n \circ q_{g(n),m}$. We have the following diagram.

$$\begin{tikzcd}
X \arrow[d, "p_{f(n)}"] & X \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "p_m"] & X \arrow[d, "p_{g(n)}"] \\
\mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \arrow[dr, "f_n"'] & \mathcal{U}_{4\delta_m}(A_m) \arrow[l, "q_{f(n),m}"'] \arrow[r, "q_{g(n),m}"] & \mathcal{U}_{4\delta_{g(n)}}(A_{g(n)}) \arrow[dl, "g_n"] \\
& \mathcal{U}_{4\epsilon_n}(B_n) &
\end{tikzcd}$$

It is clear that every square commutes up to homotopy. From here, we obtain the desired result. $\square$

Lemma 2.6.5. *Let (X, d) and (Y, l) be compact metric spaces. If $[\overline{f}] : X \to Y$ is the homotopy class of an approximative map, then $E(T([\overline{f}])) = [\overline{f}]$.*

Proof. Let $\overline{f} = \{\overline{f}_k : X \to 2^Y\}_{k\in\mathbb{N}}$ denote the approximative map that generates $[\overline{f}]$. We consider a representative (f_n, f) of $T([\overline{f}])$ induced by $\overline{f}$ and a representative $\overline{f}' = \{\overline{f}'_k : X \to 2^Y\}_{k\in\mathbb{N}}$ of $E(T(f_n))$ induced by (f_n, f). Then, $\overline{f}'_k = f_k \circ p_{f(k)}$, where $f_k = r_k \circ \overline{f}^*_{s(k)} \circ i$, see the proof of Proposition 2.6.3. For every open neigbhorhood U of Y in 2^Y there exists

$\mathcal{U}_{4\epsilon_n}(Y) \subseteq U$ by Proposition 1.2.23. We check that the following diagram is commutative up to homotopy.

$$\begin{tikzcd}
X \arrow[r, "p_{f(n)}"] \arrow[d, "\overline{f}_{s(n)}"] & \mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \arrow[r, "i"] & \mathcal{U}_{\gamma_n}(X) \arrow[dl, "\overline{f}^*_{s(n)}"] \\
\mathcal{U}_{2\epsilon_n}(Y) \arrow[d, "i"] & \mathcal{U}_{2\epsilon_n}(Y) \arrow[l, "id"] \arrow[dl, "r_n"] & \\
\mathcal{U}_{4\epsilon_n}(Y) & &
\end{tikzcd}$$

Let $x \in X$, we know that $diam(\overline{f}_{s(n)}(x)) < 2\epsilon_n$. We have $d(x,a) < \delta_{f(n)} < \gamma_n$ for every $a \in p(x)$. Thus, $2\epsilon_n > diam(\overline{f}^*_{s(n)}(i(p(x)) \cup \{x\})) = diam(\overline{f}^*_{s(n)}(i(p(x))) \cup \overline{f}_{s(n)}(x))$ so $h = \overline{f}_{s(n)} \cup \overline{f}^*_{s(n)} \circ i \circ p_{f(n)} : X \to \mathcal{U}_{\epsilon_n}(Y)$ is continuous and well-defined. In addition, $\overline{f}_{s(n)}(x), \overline{f}^*_{s(n)}(i(p_{f(n)}(x))) \subseteq h(x)$ for every $x \in X$. By Remark 2.6.2, we get the commutativity up to homotopy of the first square. It is clear that $diam(i(C) \cup r_n(C)) < 4\epsilon_n$ for every $C \in \mathcal{U}_{2\epsilon_n}(Y)$. Therefore, we can repeat the previous argument to show that r_n is homotopic to i.

By construction and hypothesis, for every $m \geq s(n), n$ we have that $\overline{f}_m$ is homotopic to $\overline{f}_{s(n)}$ in U and $\overline{f}'_n$ is homotopic to $\overline{f}'_m$ in U, which implies the desired result. $\square$

Lemma 2.6.6. *Let (X,d) and (Y,l) be compact metric spaces. If $(f_n, f) : T(X) \to T(Y)$ is a morphism in pro-HTop, then $T(E(f_n)) = (f_n, f)$.*

Proof. We consider a representative (f'_n, f') of $E(T(f_n))$, where (f'_n, f') is induced by an approximative map F induced by (f_n, f). Given $n \in \mathbb{N}$, we need to verify that for every $l \geq f'(n), f(n)$ the following diagram commutes up to homotopy.

$$\begin{tikzcd}
\mathcal{U}_{4\delta_{f(n)}}(A_{f(n)}) \arrow[dr, "f_n"'] & \mathcal{U}_{4\delta_l}(A_l) \arrow[l, "q_{f(n),l}"'] \arrow[r, "q_{f'(n),l}"] & \mathcal{U}_{4\delta_{f'(n)}}(A_{f'(n)}) \arrow[dl, "f'_n"] \\
& \mathcal{U}_{4\epsilon_n(B_n)} &
\end{tikzcd}$$

Without loss of generality, we can assume that $f'(n) > f(n)$. Then, we need to check the commutativity up to homotopy of the following diagram, where $h = (f_{s(n)} \circ p_{f(s(n))})^*$.

$$\begin{tikzcd}
\mathcal{U}_{4\delta_{f'(n)}}(A_{f'(n)}) \arrow[r, "i"] \arrow[d, "q"] & \mathcal{U}_{\gamma_n}(X) \arrow[r, "h"] & \mathcal{U}_{2\epsilon_n}(Y) \arrow[r, "r_n"] & \mathcal{U}_{4\epsilon_n}(B_n) \\
\mathcal{U}_{4\delta_{f(n)}}(B_{f(n)}) \arrow[urrr, "f_n"] & & &
\end{tikzcd}$$

By construction, $h(C) = \bigcup_{c \in C} f_{s(n)}(p_{f(s(n))}(c))$ for every $C \in \mathcal{U}_{4\delta_{f'(n)}}(A_{f'(n)})$. We have $f'(n) \geq f(s(n))$ and $s(n) \geq n$. Then, it is easy to show that $q_{f'(n),f(s(n))}$ is homotopic to $p_{f(s(n))} \circ i$ because $p_{f(s(n))}(i(c)) \subseteq q_{f'(n),f(s(n))}(c)$ for every $c \in C$. In addition, r_n restricted to the image of $h \circ i$ is homotopic to $q_{s(n),n}$ because $r_n(a) \subseteq q_{s(n),n}(a)$. Since (f_n, f) is a morphism in pro-$HTop$, it follows the commutativity up to homotopy of the diagram. $\square$

Theorem 2.6.7. *Let (X, d) and (Y, l) be compact metric spaces. The set of shape morphisms between X and Y is in bijective correspondence with the set of morphisms in pro-HTop between $T(X)$ and $T(Y)$.*

Proof. We consider the constructions made in Proposition 2.6.3 and Proposition 2.6.4. Thus, the result is an immediate consequence of Lemma 2.6.5 and Lemma 2.6.6. □

Remark 2.6.8. *The result of Theorem 2.6.7 also holds true if the FASOs considered have the opposite order. This is due to the following fact. Let $(X_n, q_{n,n+1})$ and $(Y, p_{n,n+1})$ be inverse sequences of finite topological spaces. If $(f_n, f) : (X_n, q_{n,n+1}) \to (Y, p_{n,n+1})$ is a morphism in pro-HTop, then (f_n, f) is also a morphism in pro-HTop when it is considered the other possible partial order for every term of $(X_n, q_{n,n+1})$ and $(Y_n, p_{n,n+1})$.*

We have obtained a computational description of shape theory based on finite topological spaces. From an algebraic point of view, it is not surprising to get this description since the homology and homotopy groups of finite posets are the same groups of simplicial complexes. Nevertheless, every finite topological space has trivial shape, see Proposition 2.8.1.

2.7 Core of an inverse sequence of finite spaces and an application to shape theory

For finite topological spaces there exists the notion of core, which was recalled in Subsection 1.1.2. We generalize this notion to inverse sequences of finite topological spaces.

Definition 2.7.1. *Let $(X_n, t_{n,n+1})$ be an inverse sequence of finite topological spaces. The core of $(X_n, t_{n,n+1})$ is given by $(C_n, r_n \circ t_{n,n+1} \circ i_{n+1})$, where C_n is the core of X_n, $i_n : C_n \to X_n$ denotes the inclusion and $r_n : X_n \to C_n$ is a retraction satisfying $r_n \circ i_n = id_{C_n}$ and $i_n \circ r_n$ is homotopic to id_{X_n}. We say that $C(X_n, t_{n,n+1})$ is the core of $(X_n, t_{n,n+1})$.*

Given a finite topological space X. If X^c denotes the core of X, then X and X^c are isomorphic in *HTop*. We prove the analogue result for an inverse sequence of finite topological spaces.

Theorem 2.7.2. *Let $(X_n, q_{n,n+1})$ be an inverse sequence of finite topological spaces. Then $(X_n, q_{n,n+1})$ is isomorphic to $C(X_n, q_{n,n+1})$ in pro-HTop.*

Proof. We fix notation, $C(X_n, q_{n,n+1}) = (C_n, h_{n,n+1})$, where $h_{n,n+1} = r_n \circ t_{n,n+1} \circ i_{n+1}$. There is a natural morphism in pro-*HTop* between $(C_n, h_{n,n+1})$ and $(\mathcal{U}_n, q_{n,n+1})$ induced by the inclusions, that is, $i : \mathbb{N} \to \mathbb{N}$ is the identity map and $i_n : C_n \to X_n$ is the inclusion. It is trivial to check that (i_n, i) is a well-defined morphism since the following diagram is commutative up to homotopy.

$$\begin{tikzcd} C_n \arrow[d, "i_n"] & C_{n+1} \arrow[l, "h_{n,n+1}"'] \arrow[d, "i_{n+1}"] \\ X_n & X_{n+1} \arrow[l, "q_{n,n+1}"] \end{tikzcd}$$

Now, we construct a sequence $\{g_n : X_{n+1} \to C_n\}_{n\in\mathbb{N}}$ of continuous maps making the following diagram commutative up to homotopy.

$$\begin{tikzcd} C_n \arrow[d, "i_n"'] & C_{n+1} \arrow[l, "h_{n,n+1}"'] \arrow[d, "i_{n+1}"] \\ X_n & X_{n+1} \arrow[l, "q_{n,n+1}"] \arrow[ul, "g_n"'] \end{tikzcd}$$

For every $n \in \mathbb{N}$ we consider $g_n : X_{n+1} \to C_n$ given by $g_n = r_n \circ q_{n,n+1}$. By construction, we have $g_n \circ i_{n+1} = r_n \circ q_{n,n+1} \circ i_{n+1}$ and $h_{n,n+1} = r_n \circ q_{n,n+1} \circ i_{n+1}$, which implies the commutativity of the first triangle. We also have that $i_n \circ g_n = i_n \circ r_n \circ q_{n,n+1}$. Therefore, $i_n \circ g_n$ is homotopic to $q_{n,n+1}$ and we get the commutativity up to homotopy of the second triangle. By Theorem 1.2.5, we get the desired result. $\square$

Remark 2.7.3. *Given an inverse sequence of finite topological spaces $(X_n, q_{n,n+1})$. Suppose L_n is a strong deformation retract of X_n for every $n \in \mathbb{N}$. Following the same arguments used before, we can obtain an inverse sequence where the terms are given by L_n. Repeating the proof of Theorem 2.7.2, it can be deduced that this new inverse sequence is isomorphic to $(X_n, q_{n,n+1})$ in pro-HTop.*

Example 2.7.4. We consider $X = \{A, B\}$, where we declare that $A < B$. Let X^n denote the n-th finite barycentric subdivision of X, see Definition 1.1.66. We have a natural inverse sequence given by $(X^n, h_{n,n+1})$, where $h_{n,n+1} : X^{n+1} \to X^n$ is the natural map considered in Subsection 1.1.4. It is easily seen that the core of X^n is homotopy equivalent to one point for every $n \in \mathbb{N}$. This implies that $(X^n, h_{n,n+1})$ is isomorphic to the rudimentary system given by one point. In Figure 2.7.1 we have an schematic representation.

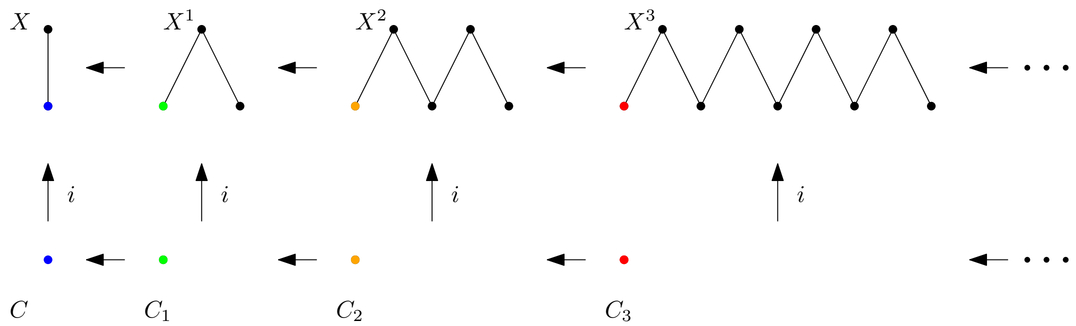

Figure 2.7.1: Schematic illustration of the inverse sequence $(X^n, h_{n,n+1})$ and its core.

We can study the core of a FASO for a compact metric space X. We update the example introduced in Section 2.2.

Example 2.7.5. We get the core of the FASO obtained in Example 2.2.4.

Step 1. Since $\mathcal{U}_1$ has one element, it follows that $C_1 = \mathcal{U}_1$.

Step 2. We get that $\mathcal{U}_2$ is contractible because it has a minimum, which is $\{a_0^2, a_1^2, a_2^2, a_3^2\}$. Then, $C_2 = \{a_0^2, a_1^2, a_2^2, a_3^2\}$.

Step 3. We identify beat points of $\mathcal{U}_3$. Then, we remove them one by one. The resulting topological space is C_3. If x is of the form $\{a_i^3, a_{i+2}^3, a_{i+3}^3\}$ or $\{a_i^3, a_{i+1}^3, a_{i+3}^3\}$, where $i \in \{0, ..., 31\}$ and the subindices are modulo 32, then x is a down beat point.

This is due to the fact that $U_x \setminus \{x\} = \{a_i^3, a_{i+1}^3, a_{i+2}^3, a_{i+3}^3\}$. After removing these beat points, we get that $x_i = \{a_i^3, a_{i+2}^3\}$ and $y_i = \{a_i^3, a_{i+3}^3\}$, where $i \in \{0, ..., 31\}$ and the subindices are modulo 32, are down beat points because $U_{x_i} \setminus \{x_i\} = \{a_i^3, a_{i+1}^3, a_{i+2}^3\}$ and $U_{y_i} \setminus \{y_i\} = \{a_i^3, a_{i+1}^3, a_{i+2}^3, a_{i+3}^3\}$. There are no more beat points. Thus, C_3 consists of points of the form $\{a_i^3, a_{i+1}^3, a_{i+2}^3, a_{i+3}^3\}$, $\{a_i^3, a_{i+1}^3, a_{i+2}^3\}$, $\{a_i^3, a_{i+1}^3\}$ and $\{a_i^3\}$, where $i \in \{0, ..., 31\}$ and the subindices are modulo 32. In Figure 2.7.2 we present the Hasse diagram of the minimal closed set of $\mathcal{U}_3$ containing $\{a_0^3, a_1^3, a_2^3, a_3^3\} \in C_3$, where we have omitted the superscripts for simplicity.

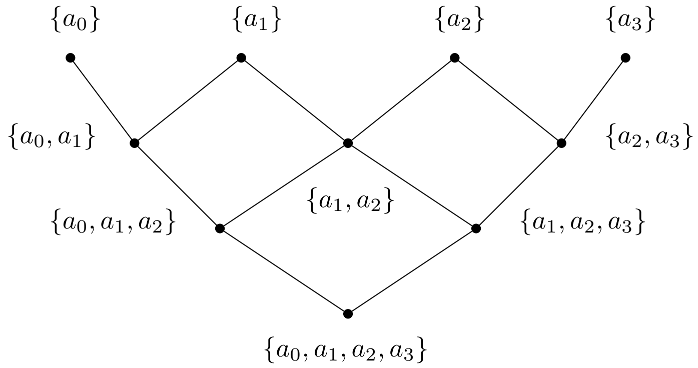

Figure 2.7.2: Hasse diagram of the minimal closed set of $\mathcal{U}_3$ containing $\{a_0^3, a_1^3, a_2^3, a_3^3\} \in C_3$.

Step n. We continue in this fashion, that is, we identify beat points and then we remove them. If x is of the form $\{a_i^n, a_{i+2}^n, a_{i+3}^n\}$ or $\{a_i^n, a_{i+1}^n, a_{i+3}^n\}$, where $i \in \{0, ..., 2^{3n-4} - 1\}$ and the subindices are modulo 2^{3n-4}, then it is a down beat point. This is due to the fact that $U_x \setminus \{x\} = \{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n\}$. After removing these beat points, we get that $x_i = \{a_i^n, a_{i+2}^n\}$ and $y_i = \{a_i^n, a_{i+3}^n\}$, where $i \in \{0, ..., 2^{3n-4} - 1\}$ and the subindices are modulo 2^{3n-4}, are down beat points since $U_{x_i} \setminus \{x_i\} = \{a_i^n, a_{i+1}^n, a_{i+2}^n\}$ and $U_{y_i} \setminus \{y_i\} = \{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n\}$. There are no more beat points. Thus, C_n consists of points of the form $\{a_i^n, a_{i+1}^n, a_{i+2}^n, a_{i+3}^n\}$, $\{a_i^n, a_{i+1}^n, a_{i+2}^n\}$, $\{a_i^n, a_{i+1}^n\}$ and $\{a_i^n\}$, where $i \in \{0, ..., 2^{3n-4}-1\}$ and the subindices are modulo 2^{3n-4}.

Given a compact metric space (X, d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ or FAS $(\mathcal{U}_n, p_{n,n+1})$ for X. In general, the inverse limit of the core of $(\mathcal{U}_n, q_{n,n+1})$ or $(\mathcal{U}_n, p_{n,n+1})$ does not preserve the good properties of the inverse limit of $(\mathcal{U}_n, q_{n,n+1})$ or $(\mathcal{U}_n, p_{n,n+1})$. However, the core of an inverse sequence of finite topological spaces can be used to show that two compact metric spaces have the same shape. We give an example of this.

Example 2.7.6. We consider the computational model of the topologist's sine curve S, that is, we consider

$$\begin{aligned} \overline{b}_n &= (\frac{1}{2^{2n-1}}, \frac{1}{2}) - (\frac{1}{2^{2n-2}}, \frac{1}{2}) \quad n \geq 1, \\ \underline{b}_n &= (\frac{1}{2^{2n}}, \frac{1}{2}) - (\frac{1}{2^{2n-1}}, 0) \quad n \geq 1, \\ a_n &= (\frac{1}{2^n}, \frac{1}{2}) - (\frac{1}{2^n}, 0) \quad n \geq 0, \\ a_\infty &= (0, \frac{1}{2}) - (0, 0), \end{aligned}$$

where $n \in \mathbb{N}$ and $(a, b) - (c, d)$ denotes the segment joining the point (a, b) with (c, d). Therefore,

$$S = a_\infty \cup (\cup_{n\geq 1}\overline{b}_n) \cup (\cup_{n\geq 1}\underline{b}_n) \cup (\cup_{n\geq 0}a_n).$$

In Figure 2.7.3 we present the computational topologist's sine curve. The metric is the one induced as a subspace of $\mathbb{R}^2$. We construct a FAS for S and study at the same time its core.

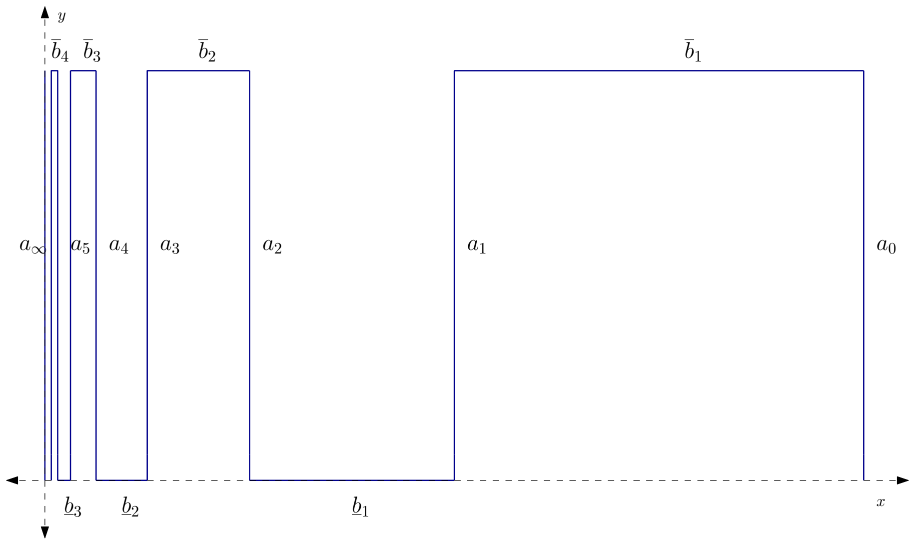

Figure 2.7.3: The computational topologist's sine curve.

Step 1. The diameter of S is $\frac{\sqrt{2}}{2}$, so we can consider $\epsilon_1 = \sqrt{5}$, $A_1 = \{(0, \frac{1}{4})\}$ and $\mathcal{U}_{2\epsilon_1}(A_1) = A_1$. It is easy to check that $\gamma_1 = \frac{\sqrt{5}}{4}$.

Step 2. We consider $\epsilon_2 = \frac{\sqrt{2}}{2^3} < \frac{\epsilon_1 - \gamma_1}{2}$, the grid $G_2 = \{(\frac{l}{2^{3-1}}, \frac{k}{2^{3-1}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_2 with S. There are two points in a_3 that are at distance ϵ_2 to $G_2 \cap S$, which are $b_1 = (\frac{1}{2^3}, \frac{1}{2^3})$ and $b_2 = (\frac{1}{2^3}, \frac{1}{2^3} + \frac{1}{2^2})$. If we add $(0, \frac{1}{2^3})$ and $(0, \frac{1}{2^3} + \frac{1}{2^2})$ to $G_2 \cap T$, then we get an ϵ_2-approximation, i.e.,

$$A_2 = G_2 \cap S \cup \{(0, \frac{1}{2^3}), (0, \frac{1}{2^3} + \frac{1}{2^2})\}$$

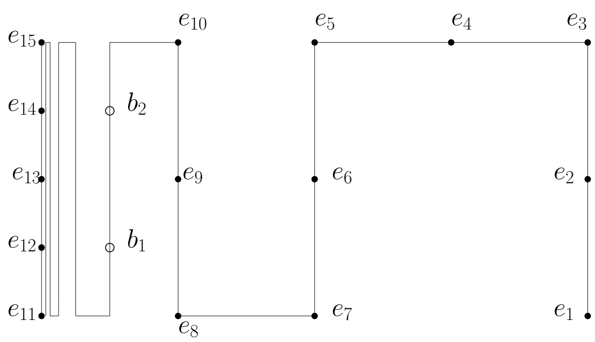

Figure 2.7.4: ϵ_2-approximation A_2 for S.

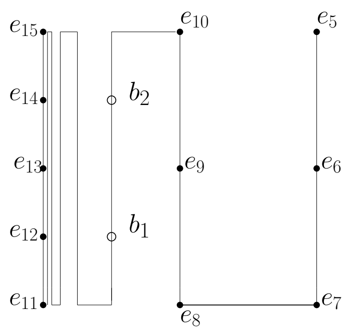

Figure 2.7.5: The set of points A'_2 in S.

We consider $B_2 = \{x \in A_2 | x \in \bar{b}_1 \setminus a_1 \text{ or } x \in a_0\}$, that is, the points of A_2 that lie to the right of a_1. It is easy to observe that $\mathcal{U}_{2\epsilon_2}(A'_2)$ is a strong deformation retract of $\mathcal{U}_{2\epsilon_2}(A_2)$, where $A'_2 = A_2 \setminus B_2$. The last assertion is an immediate consequence of the construction we made of A_2 and the value that we have chosen for ϵ_2. Suppose that $C \in \mathcal{U}_{2\epsilon_2}(A_2)$ contains points of the approximation that lie in B_2, i.e., C is of the form $C = \{e_k, e_{k+1}\}$

or $C = \{e_k\}$. Then $\{e_1\}$ is an up beat point because $F_{\{e_1\}} \setminus \{e_1\} = \{e_1, e_2\}$, so we can remove it without changing the homotopy type of $\mathcal{U}_{2\epsilon_2}(A_2)$. Now, $\{e_1, e_2\}$ is a down beat point since $U_{\{e_1,e_2\}} \setminus \{e_1, e_2\} = \{e_2\}$. Therefore, we can remove it without changing the homotopy type. We can proceed inductively to get the desired result. The point $\{e_5\}$ satisfies that $d(e_5, e_6), d(e_5, e_{10}) = \frac{1}{4} < 2\epsilon_2 = \frac{\sqrt{2}}{2^2}$. We get $\{e_5, e_6\}$, $\{e_5, e_{10}\} \in \mathcal{U}_{2\epsilon_2}(A_2)$, which means that $\{e_5\}$ is not an up beat point. On the other hand, $\{e_5, e_6\}$ and $\{e_5, e_{10}\}$ are clearly not down beat points. A similar argument can be made with the rest of the points in A_2 that lie in a_1. In addition, the map $p_{1,2}$ trivially sends every $C \in \mathcal{U}_{2\epsilon_n}(A'_2)$ to A_1. In Figure 2.7.4 we present A_2. In Figure 2.7.5 we present A'_2.

Step 3. It is easy to get that $\gamma_2 = \frac{1}{2^3}$. Then, we can choose $\epsilon_3 = \frac{\sqrt{2}}{2^6} < \frac{\epsilon_2 - \gamma_2}{2}$. We consider the grid $G_3 = \{(\frac{l}{2^{6-1}}, \frac{k}{2^{6-1}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_3 with S. There are 16 points that are at distance ϵ_3 to A_3, these points lie in a_6. Concretely,

$$\{(\frac{1}{2^6}, \frac{2k+1}{2^6}) | k = 0, 1, 2..., 15\}.$$

We add points of a_∞ to get an ϵ_3-approximation, i.e.,

$$A_2 = (G_3 \cap S) \cup \{(0, \frac{2k+1}{2^6}) | k = 0, 1, 2..., 15\}.$$

We consider $B_3 = \{x \in A_2 | x \in a_l$ with $l = 0, 1, 2, 3$ or $x \in \overline{b}_i$ with $i = 1, 2$ or $x \in \underline{b}_1$ or $x \in \underline{b}_2 \setminus a_4$ $\}$, i.e., the points of A_3 that lie to the right of a_4. Again, it is easy to observe that $\mathcal{U}_{2\epsilon_3}(A'_3)$ is a strong deformation retract of $\mathcal{U}_{2\epsilon_3}(A_3)$ where $A'_3 = A_3 \setminus B_3$. We enumerate from right to left the points of A_3, see Figure 2.7.6. Then $\{e_1\}$ is only covered by $\{e_1, e_2\}$, so we can remove it. Now, $\{e_1, e_2\}$ only covers $\{e_2\}$, so it is a down beat point and we can remove it. If we continue in this fashion, we get the desired result. If x is a point of A_3 that lie in a_4, then it satisfies that there exists another point in a_5 and a_4 that are at distance less than $2\epsilon_3$. If $C \in \mathcal{U}_{2\epsilon_3}(A'_3)$, then C contains points lying in a_4 or a_5. In Figure 2.7.7 we have A'_3.

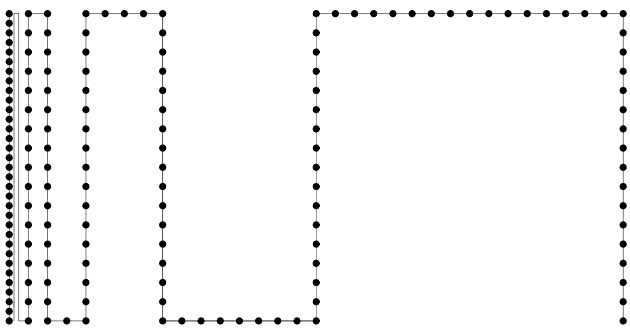

Figure 2.7.6: ϵ_3-approximation A_3 for S.

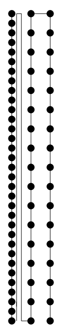

Figure 2.7.7: The set of points A'_3 in S.

Furthermore, the image of the map $p_{2,3} : \mathcal{U}_{2\epsilon_3}(A'_3) \to \mathcal{U}_{2\epsilon_2}(A_2)$ is $\mathcal{U}_{2\epsilon_2}(\overline{A}_2) \subset \mathcal{U}_{2\epsilon_2}(A'_2)$, where $\overline{A}_2 = \{x \in A_2 | x \in a_\infty\}$. On the other hand, $\mathcal{U}_{2\epsilon_2}(\overline{A}_2)$ is contractible to $\{(0, \frac{1}{4})\}$. We have that $\mathcal{K}(\mathcal{U}_{2\epsilon_2}(\overline{A}_2))$ is strong collapsible. Therefore, we obtain the desired result after applying Theorem 1.1.53. In Figure 2.7.8 we present the McCord complex of $\mathcal{U}_{2\epsilon_1}(\overline{A}_1)$, $\mathcal{U}_{2\epsilon_1}(\overline{A}_1)$ and $\mathcal{U}_{2\epsilon_1}(\overline{A}_1)$.

Step n. We consider $\epsilon_n = \frac{\sqrt{2}}{2^{3n-3}}$, the grid $G_n = \{(\frac{l}{2^{3n-4}}, \frac{k}{2^{3n-4}}) \in \mathbb{R}^2 | l, k \in \mathbb{Z}\}$ and the intersection of G_n with S. There are 2^{3n-5} points that lie in a_{3n-3} such that the distance to $G_n \cap S$ is exactly ϵ_n. If we add the following points to $G_n \cap S$, then we get an

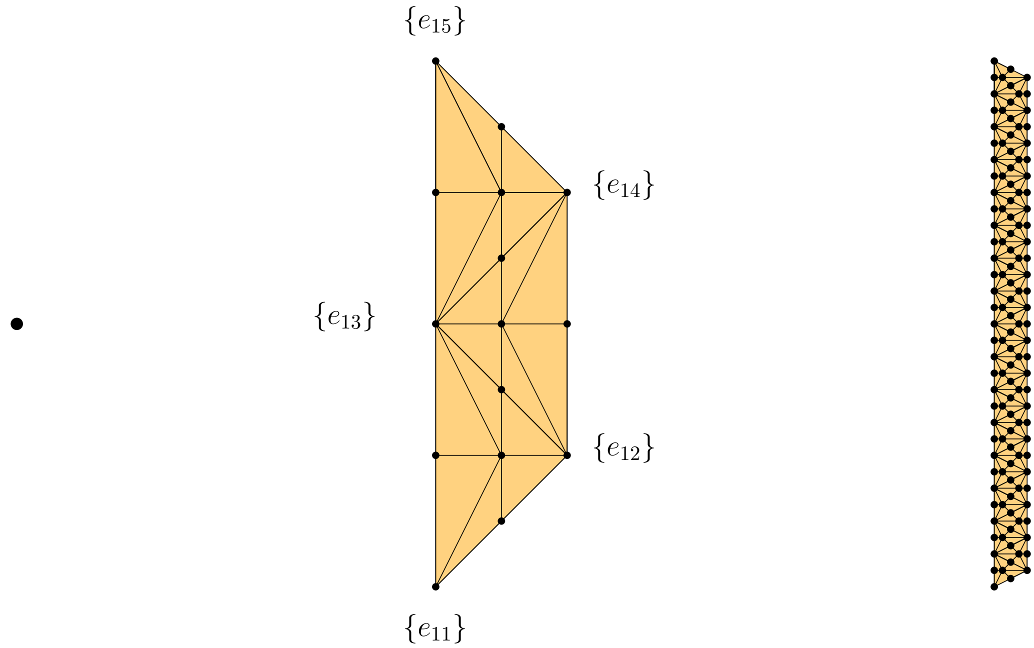

Figure 2.7.8: $\mathcal{K}(\mathcal{U}_{2\epsilon_1}(\overline{A}_1)), \mathcal{K}(\mathcal{U}_{2\epsilon_2}(\overline{A}_2))$ and $\mathcal{K}(\mathcal{U}_{2\epsilon_3}(\overline{A}_3))$.

ϵ_n-approximation

$$A_n = (G_n \cap S) \cup \{(0, \frac{2k+1}{2^{3n-3}}) | k = 0, 1..., 2^{3n-4} - 1\}.$$

Moreover, $\gamma_n = \frac{1}{2^{3n-3}}$. It is simple to show that $\mathcal{U}_{2\epsilon_n}(A_n)$ is homotopy equivalent to $\mathcal{U}_{2\epsilon_n}(A'_n)$, where $A'_n = A_n \setminus B_n$ and B_n consists of points in A_n that lie to the right of a_{3n-5}.

In addition, $p_{n-1,n}$ sends $\mathcal{U}_{2\epsilon_n}(A'_n)$ to $\mathcal{U}_{2\epsilon_{n-1}}(\overline{A}_{n-1}) \subset \mathcal{U}_{2\epsilon_{n-1}}(A_{n-1})$, where $\overline{A}_{n-1} = \{x \in A_{n-1} | x \in a_\infty\}$. The McCord complex $\mathcal{K}(\mathcal{U}_{2\epsilon_{n-1}}(\overline{A}_{n-1}))$ is strong collapsible, so it can be deduced that $\mathcal{U}_{2\epsilon_{n-1}}(\overline{A}_{n-1})$ is contractible to $\{(0, \frac{1}{4})\}$.

Inverse limit. We have constructed a FAS $(\mathcal{U}_{2\epsilon_n}(A'_n), \overline{p}_{n,n+1})$ that is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ due to Remark 2.7.3. We denote by $\mathcal{S}$ the inverse limit of $(\mathcal{U}_{2\epsilon_n}(A'_n), \overline{p}_{n,n+1})$. If $\{D_n\} \in \mathcal{S}$, then we have that $D_n \in \mathcal{U}_{2\epsilon_n}(\overline{A}_n)$ for every $n \in \mathbb{N}$ because the image of $\overline{p}_{n,n+1}$ is precisely $\mathcal{U}_{2\epsilon_n}(\overline{A}_n)$; otherwise we would not have $\overline{p}_{n,n+1}(D_{n+1}) = D_n$. On the other hand, $(\mathcal{U}_{2\epsilon_n}(\overline{A}_n), p^*_{n,n+1})$ is a FAS for $a_\infty = [0, \frac{1}{2}]$, where $p^*_{n,n+1}$ is the restriction of $\overline{p}_{n,n+1}$ to $\mathcal{U}_{2\epsilon_{n+1}}(\overline{A}_{n+1})$. Therefore, it is easy to check that $\mathcal{S}$ and the inverse limit of $(\mathcal{U}_{2\epsilon_n}(\overline{A}_n), p^*_{n,n+1})$ (FAS for interval $[0, \frac{1}{2}]$) are the same. Nevertheless, the topologist's sine curve and the interval $[0, \frac{1}{2}]$ does not have the same homotopy type, which implies that the inverse limits considered before does not have the same homotopy type.

Shape of the topologist's sine curve. By Theorem 2.4.2 we have that every FASO for S is isomorphic to the previous FAS for S, where we are taking the opposite order. By previous arguments, the FAS for S is isomorphic to a FAS for the interval $[0, \frac{1}{2}]$. Again, every FASO for $[0, \frac{1}{2}]$ is isomorphic to the previous FAS for $[0, \frac{1}{2}]$. Thus, we get that the shape of S and $[0, \frac{1}{2}]$ is the same by Theorem 2.6.7.

Remark 2.7.7. *Example 2.7.6 illustrates a way to use this theory to the study of the shape of a compact metric space.*

2.8 Categories SW and SH

Shape theory serves to classify compact Hausdorff spaces but does not work properly for topological spaces with poor separation axioms such as finite topological spaces.

Proposition 2.8.1. *If X is a connected finite topological space, then X has trivial shape.*

Proof. The result is an immediate consequence of the following: every continuous map from a connected finite topological space to a compact metric space is the constant map. We prove the last assertion. Suppose M is a compact metric space. If f is continuous at $x \in X$, then for every open neighborhood V of $f(x)$ there exists an open neighborhood U of x satisfying $f(U) \subseteq V$. We consider a decreasing sequence of positive values $\{\epsilon_n\}$ satisfying that $\lim_{n\to\infty} \epsilon_n = 0$. Since U_x is a minimal open neighborhood for X, it follows that $f(U_x) \subseteq \mathcal{B}(x, \epsilon_n)$ for every $n \in \mathbb{N}$. We have $\bigcap_{n\in\mathbb{N}} \mathcal{B}(x, \epsilon_n) = \{f(x)\}$ and then $f(U_x) = f(x)$. Therefore, f is a locally constant map defined on a connected space, which implies that it is constant. From this, we get that the only $HPol$-expansion of X is the trivial one. $\square$

The idea of this section is to construct categories that serve to classify compact Hausdorff spaces by their shape and finite T_0 topological spaces by their weak homotopy type or homotopy type. Firstly, we construct an auxiliary category M. The objects of M are the finite T_0 topological spaces. If X and Y are objects of M, then we define $Mor(X, Y) = \{f : |\mathcal{K}(X)| \to |\mathcal{K}(Y)| | f \text{ is a continuous map }\}$. It is easy to check that M is a category. Let HM denote the homotopy category of M, that is, the objects of HM are the objects of M and the morphisms of HM are the homotopy classes of the morphisms of M. Again, it is trivial to check that HM is a category.

A finite cover of a compact topological space can be seen as a finite T_0 topological spaces since there is a natural partial order given by $U \leq V$ if and only if $U \subseteq V$. From now on, we will treat open covers as finite topological spaces without explicit mention.

Given a compact space X, we will associate to X an object of pro-HM. We consider $Cov(X) = \{\mathcal{U} \text{ is a finite open cover of } X | \mathcal{U} \text{ is basis-like }\}$. Let $\mathcal{U}, \mathcal{V} \in Cov(X)$. Then we say that $\mathcal{U} \geq_C \mathcal{V}$ if and only if $\mathcal{U}$ refines $\mathcal{V}$ and there exists a continuous map $p_{\mathcal{V},\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ such that for every $U \in \mathcal{U}$, $p_{\mathcal{V},\mathcal{U}}(U)$ contains U. When there is no confusion, we will omit the subscript of $p_{\mathcal{V},\mathcal{U}}$.

Example 2.8.2. Let us consider $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. Let $\mathcal{U}$ denote the open cover given by $U_1 = [0, 1] \times [0, \frac{3}{4})$ and $U_2 = [0, 1] \times (\frac{1}{4}, 1]$. Let $\mathcal{V}$ denote the open cover given by U_1, U_2 and $U_3 = U_1 \cap U_2$. It is clear that $\mathcal{V} \not\geq_C \mathcal{U}$ because there is no continuous map $p : \mathcal{V} \to \mathcal{U}$ satisfying that $U \subseteq p(U)$. In Figure 2.8.1 we present a schematic representation of the situation described above.

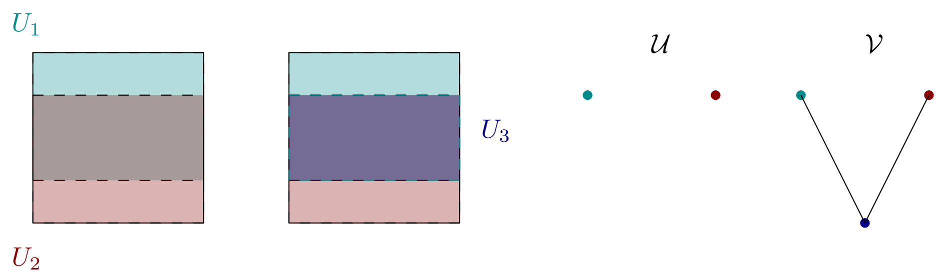

Figure 2.8.1: $X, \mathcal{U}$, $\mathcal{V}$ and Hasse diagrams of $\mathcal{U}$ and $\mathcal{V}$.

Remark 2.8.3. *Let X be a finite T_0 topological space and let $\mathbb{B}(X) = \{U_x | x \in X\}$. Then $\mathbb{B}(X) \in Cov(X)$. In addition, it is easy to get that $\mathbb{B}(X)$ is homeomorphic to X. We have that $\varphi : \mathbb{B}(X) \to X$ given by $\varphi(U) = max(U)$ is a homeomorphism.*

Proposition 2.8.4. *Let X be a finite T_0 topological space. Then $\mathbb{B}(X) \geq_C \mathcal{U}$ for every $\mathcal{U} \in Cov(X)$. Moreover, if there exists $\mathcal{V} \in Cov(X)$ with $\mathcal{V} \geq_C \mathbb{B}(X)$, then $\mathcal{V}$ is homotopy equivalent to $\mathbb{B}(X)$.*

Proof. Let $\mathcal{U} \in Cov(X)$. It is trivial to check that $\mathbb{B}(X)$ refines $\mathcal{U}$, in fact, $\mathbb{B}(X)$ is a basis for the topology of X. We define $p : \mathbb{B}(X) \to \mathcal{U}$ as follows: $p(U_x)$ is the minimum open set in $\mathcal{U}$ containing U_x. We prove that p is well-defined, i.e., $p(U_x)$ exists. Since $\mathbb{B}(X)$ refines $\mathcal{U}$, it follows that there exists at least one open set in $\mathcal{U}$ that contains U_x. If $U_x \in \mathcal{U}$, then $p(U_x) = U_x$ and p is well-defined. If $U_x \notin \mathcal{U}$, then we argue by contradiction. Suppose that there is no minimum open set containing U_x, that is, there are two open sets $U_1, U_2 \in \mathcal{U}$ that are not comparable satisfying $U_x \subseteq U_i$ for $i = 1, 2$ and the property that there is no $W \in \mathcal{U}$ such that $U_x \subseteq W \subseteq U_i$ for $i = 1, 2$. We get that $U_1 \cap U_2$ is non-empty because it contains U_x. By the property of being basis-like, we have $U_1 \cap U_2 = \bigcup_{t \in T} U_t$ for some T, where $U_t \in \mathcal{U}$ for every $t \in T$. On the other hand, $\mathbb{B}(X)$ is a basis for the topology in X, so $U_t = \bigcup_{y \in I_t} U_y$ for every $t \in T$ and some $I_t \subseteq X$. We get that $U_x \subseteq \bigcup_{t \in T} \bigcup_{y \in I_t} U_y$, which means that $x \in U_y$ for some $y \in I_t$ and $t \in T$. Hence, $U_x \subseteq U_y \subseteq U_t \in \mathcal{U}$ but $U_t \subseteq U_i$ for $i = 1, 2$, which leads to contradiction.

We have shown that p is well-defined, it remains to show the continuity of it. It suffices to check that it is order-preserving. Again, we argue by contradiction. Suppose that $p(U_x) \nsubseteq p(U_y)$. By hypothesis, $U_x \subseteq U_y \subseteq p(U_y)$ so $p(U_x) \cap p(U_y) \neq \emptyset$. We repeat the same argument used before to get the contradiction. We have that $p(U_x) \cap p(U_y) = \bigcup_{t \in T} U_t$ for some T, where $U_t \in \mathcal{U}$ for every $t \in T$. On the other hand, $U_t = \bigcup_{z \in I_t} U_z$ for some $I_t \subseteq X$, where $U_z \in \mathbb{B}(X)$ for every $z \in I_t$ and every $t \in T$. We get $U_x \subseteq p(U_x) \cap p(U_y) = \bigcup_{t \in T} \bigcup_{z \in I_t} U_z$, therefore, $U_x \subseteq U_z \subset U_t \subseteq p(U_x)$ for some $t \in T$, which leads to a contradiction with the minimality of $p(U_x)$.

To prove the last part of the proposition we use the characterization of homotopic maps between finite topological spaces. We show that $p_{\mathbb{B},\mathcal{U}} \circ p_{\mathcal{U},\mathbb{B}} \geq id_{\mathcal{U}}$. By construction, for every $U \in \mathcal{U}$ we get that $id(U) = U \subseteq p_{\mathbb{B},\mathcal{U}}(p_{\mathcal{U},\mathbb{B}}(U))$ because $U \subseteq p_{\mathcal{U},\mathbb{B}}(U) \subseteq p_{\mathbb{B},\mathcal{U}}(p_{\mathcal{U},\mathbb{B}}(U))$. We apply the same argument to show $p_{\mathcal{U},\mathbb{B}} \circ p_{\mathbb{B},\mathcal{U}} \geq id_{\mathbb{B}}$. Thus, $\mathbb{B}(X)$ and $\mathcal{U}$ have the same homotopy type. $\square$

Let X be a finite topological space. We consider $\overline{Cov}(X) = \mathbb{B}(X)$ because $\mathbb{B}(X)$ is cofinal. If X is not a finite topological space, then we consider $\overline{Cov}(X) = \{\mathcal{U}$ is a finite open cover of $X |$ for every $U, V \in \mathcal{U}$, $U \cap V \in \mathcal{U}\}$. We have trivially that $\overline{Cov}(X) \subseteq Cov(X)$. In fact, $\overline{Cov}(X)$ is cofinal in $Cov(X)$ with the usual relation of refinement.

Lemma 2.8.5. *Let X be a compact space. Then $(Cov(X), \geq_C)$ is a preordered set. If X is a finite topological space, then $(Cov(X), \geq_C)$ is a directed preordered set. If X is a compact Hausdorff space $(\overline{Cov}(X), \geq_C)$ is a directed preordered set.*

Proof. Suppose X is a compact space. For every $\mathcal{U} \in Cov(X)$ we have the reflexive property, $\mathcal{U} \geq_C \mathcal{U}$, it suffices to take $p : \mathcal{U} \to \mathcal{U}$ as the identity map. The transitive property also holds trivially, we only need to compose the continuous maps given by the relations $\mathcal{U} \geq_C \mathcal{V}$ and $\mathcal{V} \geq_C \mathcal{W}$, where $\mathcal{U}, \mathcal{V}, \mathcal{W} \in Cov(X)$.

The results for the finite can be deduced from Proposition 2.8.4. Thus, let us check the properties for a compact Hausdorff space X. We prove that $(\overline{Cov}(X), \geq_C)$ is also a directed set. For any $\mathcal{U}$, $\mathcal{V} \in \overline{Cov}(X)$ there exists a finite open cover $\mathcal{W}$ that refines $\mathcal{U}$ and

$\mathcal{V}$ in the usual sense [73, Appendix 1, 3.1]. Let $\overline{\mathcal{W}}$ denote the open cover $\mathcal{W}$ with all the possible intersections of open sets from it. Then $\overline{\mathcal{W}}$ clearly refines $\mathcal{W}$ and $\overline{\mathcal{W}} \in \overline{Cov}(X)$. We define $p : \overline{\mathcal{W}} \to \mathcal{U}$ given by

$$p(W) = \bigcap_{V \in U_W} V, \text{ where } U_W = \{V \in \mathcal{U} | W \subseteq V\}.$$

The map p is well-defined because $\mathcal{U} \in \overline{Cov}(X)$, which means that the intersection of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$. Therefore, $p(W) \in \mathcal{U}$ for every $W \in \overline{\mathcal{W}}$. We prove the continuity of p. By hypothesis $W' \subseteq W$ implies $U_W \subseteq U_{W'}$ and therefore $p(W') \subseteq p(W)$. Thus, $\overline{\mathcal{W}} \geq_C \mathcal{U}$ and we only need to repeat the same argument to get $\overline{\mathcal{W}} \geq_C \mathcal{V}$. $\square$

With the usual notion of refinement, we can have the following situation for a finite topological space X. There is a finite cover $\mathcal{U} \in Cov(X)$ satisfying that $\mathcal{U}$ is cofinal in $Cov(X)$ and $\mathcal{U} \neq \mathbb{B}(X)$. On the other hand, $\mathbb{B}(X)$ is also cofinal in $Cov(X)$. We have that $\mathcal{U}$ and $\mathbb{B}(X)$ do not have necessarily the same homotopy type or weak homotopy type as finite topological spaces. Hence, if we consider a cofinal subset in $Cov(X)$, then we make a choice that it is not unique.

Thus, for the finite case, it does not make sense to consider a cofinal subset in $Cov(X)$ with the usual notion of refinement. This is the main reason why we define $\leq_C$, Proposition 2.8.4. In contrast, for compact Hausdorff spaces the relation $\leq_C$ fits well with $\overline{Cov}(X)$, which is a cofinal subset in $Cov(X)$ with the usual notion of refinement. In this case, it makes sense to consider a cofinal subset because we would not have made a choice of an element that determines $Cov(X)$. This is due to the cardinality of $Cov(X)$. For the finite case, $Cov(X)$ is a finite set, i.e., we do not have enough covers, while for the non-finite case the cardinality of $Cov(X)$ is infinite. Therefore, for the non-finite case we consider $Cov(X) = \overline{Cov}(X)$.

Remark 2.8.6. *Let X be a compact Hausdorff space. If $\mathcal{U}$ refines $\mathcal{V}$, then $\mathcal{U} \geq_C \mathcal{V}$, where we consider $p : \mathcal{U} \to \mathcal{V}$ given by $p(U) = \bigcap_{U \subseteq V \in \mathcal{V}} V$. By construction, p is well-defined and continuous. Furthermore, if $q : \mathcal{U} \to \mathcal{V}$ is another map given by the relation $\geq_C$, then it is clear that $p \leq q$, which implies that p is homotopic to q.*

In the following example, we have an open cover $\mathcal{U}$ for a finite topological space X satisfying that $\mathcal{U}$ is not homotopy equivalent to $\mathbb{B}(X)$, $\mathcal{U}$ refines $\mathbb{B}(X)$ and $\mathcal{U} \not\geq_C \mathbb{B}(X)$.

Example 2.8.7. We consider $X = \{a, b, c, d\}$ with the following topology $\tau = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$. It is trivial to check that (X, τ) is a finite T_0 topological space. On the one hand, $\{\{a\}, \{b\}, \{a, b, d\}, \{a, b, d\}\} = \mathbb{B}(X) \in Cov(X)$. We get that $\mathbb{B}(X)$ as a finite topological space is homeomorphic to X, see Remark 2.8.3. On the other hand, we consider $\mathcal{U} = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}\} \in Cov(X)$. Since $\mathcal{U}$ contains a minimum given by $\{a, b\}$, it follows that $\mathcal{U}$ has the same homotopy type of a point. We have that $|\mathcal{K}(X)| = S^1$, which implies that $\mathbb{B}(X)$ is not contractible. In fact, X is a finite model of the circle. It is not difficult to get that $\mathbb{B}(X)$ refines $\mathcal{U}$ and $\mathcal{U}$ refines $\mathbb{B}(X)$ in the usual sense. Moreover, $\mathbb{B}(X) >_C \mathcal{U}$ considering $p : \mathbb{B}(X) \to \mathcal{U}$ given by $p(\{a, b, c\}) = \{a, b, c\}$, $p(\{a, b, d\}) = \{a, b, d\}$ and $p(a) = p(b) = \{a, b\}$. There is no continuous map $q : \mathcal{U} \to \mathbb{B}(X)$ satisfying that $U \subseteq q(U)$ for every $U \in \mathbb{B}(X)$. If it exists, then $q(\{a, b, c\}) = \{a, b, c\}$ and $q(\{a, b, d\}) = \{a, b, d\}$. We only have two options for $q(\{a, b\})$. If $q(\{a, b\}) = q(\{a, b, c\})$, then q is not continuous since $\{a, b, d\} > \{a, b\}$ but $q(\{a, b, d\}) = \{a, b, d\} \not\geq \{a, b, c\} = q(\{a, b\})$. Similarly, we get a contradiction with the continuity of q if $q(\{a, b\}) = \{a, b, d\}$. In Figure 2.8.2, we have the Hasse diagrams of X, $\mathbb{B}(X)$ and $\mathcal{U}$.

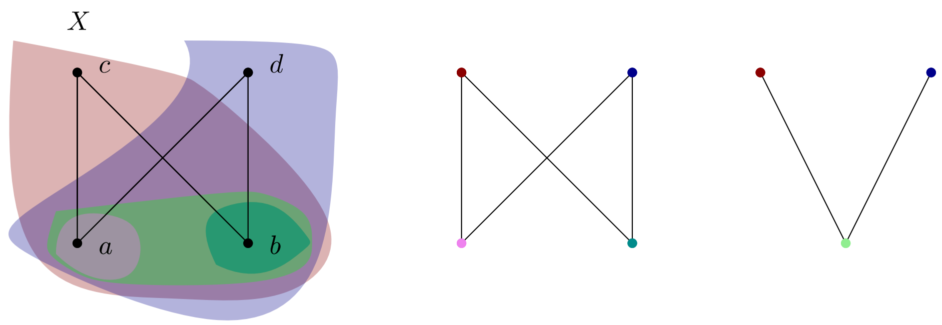

Figure 2.8.2: Hasse diagrams of X, $\mathbb{B}(X)$ and $\mathcal{U}$ and schematic representation of the open sets of $\mathbb{B}(X)$ and $\mathcal{U}$.

For every compact space X we associate an element in pro-$H\mathcal{M}$ given by

$$\mathcal{M}(X) = (\mathcal{U}, \mathcal{K}(p_{\mathcal{U},\mathcal{V}}), Cov(X)),$$

where $\mathcal{U} \in Cov(X)$ is considered as a finite T_0 topological space, $\mathcal{K}(p_{\mathcal{U},\mathcal{V}}) : \mathcal{K}(\mathcal{U}) \to \mathcal{K}(\mathcal{V})$ denotes the induced continuous function given by the McCord functor and $p_{\mathcal{U},\mathcal{V}}$ is a continuous map from $\mathcal{V}$ to $\mathcal{U}$ given by the relation $\leq_C$. It remains to verify that $\mathcal{M}(X)$ is indeed an element of pro-$H\mathcal{M}$.

Lemma 2.8.8. *If X is a compact space, then $\mathcal{M}(X)$ is an object of pro-$H\mathcal{M}$.*

Proof. In Lemma 2.8.5, it is shown that $Cov(X)$ is a directed set. It only remains to show that for every $\mathcal{U}, \mathcal{V}, \mathcal{W} \in Cov(X)$ satisfying that $\mathcal{U} \leq_C \mathcal{V}$ and $\mathcal{V} \leq_C \mathcal{W}$ we get $\mathcal{K}(p_{\mathcal{U},\mathcal{V}}) \circ \mathcal{K}(p_{\mathcal{V},\mathcal{W}}) \simeq \mathcal{K}(p_{\mathcal{U},\mathcal{W}})$.

For the non-finite case, by the definition of $\leq_C$ and Remark 2.8.6, we get that $p_{\mathcal{U},\mathcal{W}}(W) \subseteq p_{\mathcal{U},\mathcal{V}}(p_{\mathcal{V},\mathcal{W}}(W))$ for every $W \in \mathcal{W}$. Therefore, using the characterization of homotopic maps between finite T_0 topological spaces, we have $p_{\mathcal{U},\mathcal{W}} \simeq p_{\mathcal{U},\mathcal{V}} \circ p_{\mathcal{V},\mathcal{W}}$ and the desired result when we use the McCord functor.

For the finite case, we consider $Cov(X) = \mathbb{B}(X)$ since $\mathbb{B}(X)$ is cofinal, see Proposition 2.8.4. Then, $\mathcal{M}(X)$ is a rudimentary system. □

Applying the McCord functor, $\mathcal{M}(X)$ can also be seen as an element of *pro-$HPol$*. Let $\mathcal{K}(X)$ denote the inverse system given by

$$\mathcal{K}(X) = (\mathcal{K}(\mathcal{U}), \mathcal{K}(p_{\mathcal{U},\mathcal{V}}), Cov(X)).$$

Proposition 2.8.9. *If X is a compact Hausdorff space, then $\mathcal{K}(X)$ is a $HPol$-expansion.*

Proof. We prove the result showing that $\mathcal{K}(X)$ is isomorphic to the Čech system $(|N(\mathcal{U})|, N(p_{\mathcal{U},\mathcal{V}}), \overline{Cov}(X))$, which is a $HPol$-expansion of X. In Subsection 1.2.3, it can be seen the definition and properties of the Čech system. Nevertheless, a detailed description about Čech systems for compact Hausdorff spaces can be found in [73, Appendix 1.3].

Firstly, we study the map $N(p_{\mathcal{U},\mathcal{V}})$. We have that $p_{\mathcal{U},\mathcal{V}} : \mathcal{V} \to \mathcal{U}$ satisfies $V \subseteq p(V)$ for every $V \in \mathcal{V}$. Then, $p_{\mathcal{U},\mathcal{V}}$ induces a map between the nerves, i.e., $N(p) : |N(\mathcal{V})| \to |N(\mathcal{U})|$. By construction, $\mathcal{K}(\mathcal{U}) \subseteq N(\mathcal{U})$. Moreover, $|\mathcal{K}(\mathcal{U})|$ is deformation retract of $|N(\mathcal{U})|$ by [76, Lemma 3]. Furthermore, the set of vertices of $\mathcal{K}(\mathcal{U})$ and $N(\mathcal{U})$ is the same. We have that p can be taken satisfying that $N(p)$ and $\mathcal{K}(p)$ are the same map over the set of vertices. In addition, in [73, Appendix 1.3], it is proved that all choices of $N(p)$ are homotopic. Thus, we have $\mathcal{K}(p) = N(p)_{|\mathcal{K}(\mathcal{U})}$.

We also have a level morphism between the two inverse systems given by the natural inclusion i. We will use Theorem 1.2.5 to conclude the proof.

$$\begin{tikzcd} \mathcal{K}(\mathcal{U}) \arrow[d, "i"] & \mathcal{K}(\mathcal{V}) \arrow[l, "\mathcal{K}(p_{\mathcal{U},\mathcal{V}})"'] \arrow[d, "i"'] \\ N(\mathcal{U}) & N(\mathcal{V}) \arrow[l, "N(p_{\mathcal{U},\mathcal{V}})"] \arrow[ul, "g_{\mathcal{U}}"] \end{tikzcd}$$

We consider $g_{\mathcal{U}} = r \circ N(p)$, where $r : |N(\mathcal{U})| \to |\mathcal{K}(\mathcal{U})|$ denotes the retraction considered before. The commutative up to homotopy of the above diagram follows trivially. Applying Theorem 1.2.5, we get the desired result. $\square$

We define the category SW, its objects are compact spaces and if $X, Y \in Obj(\mathcal{SW})$, then $Mor(X, Y) = \{f : \mathcal{M}(X) \to \mathcal{M}(Y) | \ f \text{ is a morphism in pro-}H\mathcal{M}\}$.

Theorem 2.8.10. *Let X, Y be compact Hausdorff spaces. Then X and Y have the same shape if and only if X and Y are isomorphic in $\mathcal{SW}$. Furthermore, let X, Y be finite T_0 topological spaces. Then X is weak homotopy equivalent to Y if and only if X is isomorphic to Y in SW.*

Proof. Suppose X, Y are finite T_0 topological spaces that are isomorphic in $\mathcal{SW}$. We prove that X and Y are weak homotopy equivalent, i.e., there exists a CW-complex Z and weak homotopy equivalences $Z \to X$, $Z \to Y$ [Proposition 4.13, Corollary 4.19, [60]]. By hypothesis, $|\mathcal{K}(\mathbb{B}(X))|$ and $|\mathcal{K}(\mathbb{B}(Y))|$ are homotopy equivalent. In addition, by Theorem 1.1.43 there exist weak homotopy equivalences $|\mathcal{K}(\mathbb{B}(X))| \to X$, $|\mathcal{K}(\mathbb{B}(Y))| \to Y$. From here, we deduce the desired result. Now, we prove the opposite implication, so let us assume that X is weak homotopy equivalent to Y. Hence, $|\mathcal{K}(\mathbb{B}(X))| = |\mathcal{K}(X)|$ and $|\mathcal{K}(\mathbb{B}(Y))| = |\mathcal{K}(Y)|$ are also weak homotopy equivalent. By [60], there exists a CW-complex Z and weak homotopy equivalences $|\mathcal{K}(X)| \leftarrow Z \to |\mathcal{K}(Y)|$. By a well-known theorem of Whitehead, a weak homotopy equivalence between connected CW-complexes is a homotopy equivalence, therefore, we have that the previous weak homotopy equivalences are indeed homotopy equivalences. Then, $|\mathcal{K}(\mathbb{B}(X))|$ is homotopy equivalent to $|\mathcal{K}(\mathbb{B}(Y))|$, so X is isomorphic to Y in $\mathcal{SW}$.

Suppose X and Y are compact Hausdorff spaces that are isomorphic in $\mathcal{SW}$. Then $\mathcal{K}(X)$ and $\mathcal{K}(Y)$ are $HPol$-expansions of X and Y by Proposition 2.8.9. It is clear that $\mathcal{M}(X) \simeq \mathcal{M}(Y)$ implies $\mathcal{K}(X) \simeq \mathcal{K}(Y)$. Thus, X and Y have the same shape. Suppose X and Y are compact Hausdorff spaces that have the same shape. Then, there are two $HPol$-expansions $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $(Y_\lambda, p_{\lambda\lambda'}, \Lambda)$ satisfying that $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is isomorphic to $(Y_\lambda, p_{\lambda\lambda'}, \Lambda)$. For each polyhedra X_λ of $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ we take a triangulation and we apply the McCord functor $\mathcal{X}$ to get a finite T_0 topological space. It is clear that $(\mathcal{X}(X_\lambda), p_{\lambda\lambda'}, \Lambda))$ is an element of pro-$H\mathcal{M}$. In addition, $(|\mathcal{K}(\mathcal{X}(X_\lambda))|, p_{\lambda\lambda'}, \Lambda)$ is isomorphic to $(X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in pro-$HPol$ because $\mathcal{K}(\mathcal{X}(X_\lambda))$ is just the barycentric subdivision of the triangulation given to X_λ, i.e., $|\mathcal{K}(\mathcal{X}(X_\lambda))| = X_\lambda$. In fact, $(\mathcal{X}(X_\lambda), p_{\lambda\lambda'}, \Lambda)$ is isomorphic to $\mathcal{M}(X)$ because $(|\mathcal{K}(\mathcal{U})|, \mathcal{K}(p), \overline{Cov}(X))$ is a $HPol$-expansion of X by Proposition 2.8.9. The same situation holds for Y. Thus, $\mathcal{M}(X)$ is isomorphic to $\mathcal{M}(Y)$ in pro-$H\mathcal{M}$ and we deduce that X is isomorphic to Y in SW. $\square$

Similarly, a category that classifies compact metric spaces by their shape and finite topological spaces by their homotopy type can be obtained. By Section 2.6, we know that there is a description of the shape theory in terms of finite topological spaces.

Given a compact metric space X, we associate an inverse sequence of finite topological spaces $(\mathcal{U}_n, q_{n,n+1})$. In the following section, we prove that $(\mathcal{U}_n, q_{n,n+1})$ is isomorphic to $(\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(X))$. The objects of SH are compact topological spaces. Given a compact metric space X, we consider $(\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(X))$. Let X and Y be compact space. We define $Mor(X,Y) = \{f : (\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(X)) \to (\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(Y))| \ f$ is a morphism in pro-$HTop$ $\}$. It is easy to check that SH is a category. By construction, it is immediate to get the following result.

Theorem 2.8.11. *Let X, Y be compact metric spaces. Then X and Y have the same shape if and only if X and Y are isomorphic in SH. Furthermore, let X, Y be finite T_0 topological spaces. Then X is homotopy equivalent to Y if and only if X is isomorphic to Y in SH.*

2.9 FASOs for a compact metric space and HPol-expansions

Given a compact metric space (X, d) and a FASO $(\mathcal{U}_n, q_{n,n+1})$ for X. If we apply the McCord functor, then we obtain an inverse sequence of polyhedra, i.e., an object of pro-$HPol$. We denote that inverse sequence by

$$\mathcal{AM}_4(X) = (|\mathcal{K}(\mathcal{U}_{4\epsilon_n}(A_n))|, |\mathcal{K}(q_{n,n+1})|).$$

Given a finite partially ordered set X, the McCord complex of X with the opposite order is homeomorphic to the McCord complex of X with the natural order, see Remark 1.1.42. Since $(\mathcal{U}_{4\epsilon_n}(A_n)_n, q_{n,n+1})$ is isomorphic in pro-$HTop$ to the Main Construction $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ with the opposite order, it follows that $\mathcal{AM}_4(X)$ is a $HPol$-expansion of X. This is due the fact that

$$\mathcal{AM}_2(X) = (|\mathcal{K}(\mathcal{U}_{2\epsilon_n}(A_n))|, |\mathcal{K}(p_{n,n+1})|)$$

is a HPol-expansion of X by [81, Theorem 12]. In this section, we present an alternative and shorter proof of this result.

We consider $\mathcal{B}_n = \{\mathcal{B}(x, \epsilon_n) | x \in A_n\}$. By construction, $\mathcal{B}_n$ is an open cover of X. By abuse of notation, $\mathcal{B}_n$ also denotes the open cover that contains all possible intersections of elements of $\mathcal{B}_n$. Hence, for every $U, V \in \mathcal{B}_n$ we get $U \cap V \in \mathcal{B}_n$. We can also treat $\mathcal{B}_n$ as a partially ordered set if we consider the following partial order on $\mathcal{B}_n$: $C \leq D$ if and only if $D \subseteq C$, where $C, D \in \mathcal{B}_n$. From now on, $\mathcal{B}_n$ can be treated as an open cover or as an Alexandroff space without explicit mention. In Figure 2.9.1 there is a schematic illustration of the situation described above.

We consider the power set of A_n, denoted by 2^{A_n}, with the subset relation, that is, $C \leq D$ if and only if $C \subset D$. We define $\varphi : \mathcal{B}_n \to 2^{A_n}$ given by $\varphi(C) = \{x\}_{x \in I}$, where $C = \bigcap_{x \in I} \mathcal{B}(x, \epsilon_n)$ and $I \subseteq A_n$ satisfies that if $C = \bigcap_{x \in J} \mathcal{B}(x, \epsilon_n)$, then $J \subseteq I$.

Lemma 2.9.1. *The map φ is well-defined, continuous and injective. Furthermore, if $C \in \mathcal{B}_n$, then $\varphi(C) \in \mathcal{U}_{2\epsilon_n}(A_n)$.*

Proof. It is clear by construction that φ is well-defined. We show the continuity of φ. Suppose $D, C \in \mathcal{B}_n$ with $C \subseteq D$. Then, $C = \bigcap_{x \in I} \mathcal{B}(x, \epsilon_n) \subseteq \bigcap_{x \in J} \mathcal{B}(x, \epsilon_n) = D$ and $J \subseteq I$. Therefore, $D \leq C$ implies $\varphi(D) \leq \varphi(C)$. Now, we prove the injectivity of φ. If $C = \bigcap_{x \in \varphi(C)} \mathcal{B}(x, \epsilon_n) \neq \bigcap_{x \in \varphi(D)} \mathcal{B}(x, \epsilon_n) = D$, then $\varphi(C) \neq \varphi(D)$.

For the second part of the statement, we consider $C \in \mathcal{B}_n$ and $z \in C = \bigcap_{x \in I} \mathcal{B}(x, \epsilon_n)$. We have that $d(z, x) < \epsilon_n$ for every $x \in I$. If $x, y \in \varphi(C)$, then we get $d(x, y) < d(x, z) + d(z, y) < 2\epsilon_n$. $\square$

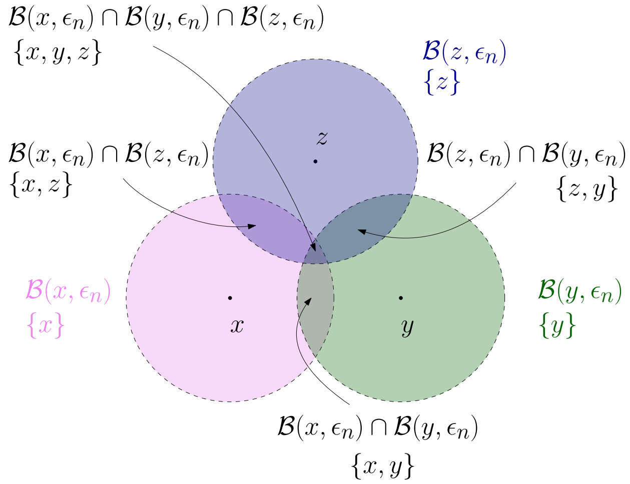

Figure 2.9.1: Schematic illustration of $\mathbb{B}_n$.

We construct a new inverse sequence, the terms are given by $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$. We define $h_{n,n+1}:\mathcal{B}_{n+1}\to\mathcal{B}_n$ given by $h_{n,n+1}(C)=\bigcap_{D\in S(C)}D$, where $S(C)=\{D\in\mathcal{B}_n|C\subset D\}$.

Proposition 2.9.2. *For every $n\in\mathbb{N}$ the map $h_{n,n+1}:\mathcal{B}_{n+1}\to\mathcal{B}_n$ is well-defined, continuous and satisfies that $C\subseteq h_{n,n+1}(C)$.*

Proof. Firstly, we prove that for every $C\in\mathcal{B}_{n+1}$ there exists at least one point $x\in A_n$ such that $C\subset\mathcal{B}(x,\epsilon_n)$. We take $x\in\varphi(C)$ and $y\in p_{n,n+1}(x)$, where $p_{n,n+1}$ is the bonding map of the Main Construction. Therefore, $d(x,z)<2\epsilon_{n+1}$ for every $z\in C$ and $d(x,y)\leq\gamma_n$. Thus,

$$d(z,y)<d(z,x)+d(x,y)<2\epsilon_{n+1}+\gamma_n<\frac{2(\epsilon_n-\gamma_n)}{2}+\gamma_n=\epsilon_n.$$

Concretely, we get that $C\subset\mathcal{B}(y,\epsilon_n)$, which implies that $h_{n,n+1}$ is well-defined. To verify the continuity we only need to check that $h_{n,n+1}$ is order-preserving. If $C\subseteq D$, where $C,D\in\mathcal{B}_{n+1}$, then $S(D)\subseteq S(C)$. From here, we deduce that $h_{n,n+1}$ is order-preserving. The last property is an immediate consequence of the construction of $h_{n,n+1}$. □

By Proposition 2.9.2, we get that $(\mathcal{B}_n,h_{n,n+1})$ is a well-defined inverse sequence of finite T_0 topological spaces.

Given a compact metric space X, we consider $\overline{Cov}(X)=\{\mathcal{U}|\ \mathcal{U}$ is a finite open cover and for every $U,V\in\mathcal{U}$, $U\cap V\in\mathcal{U}$ $\}$. It is easy to show that $\overline{Cov}(X)$ is cofinal in $Cov(X)=\{\mathcal{U}|\ \mathcal{U}$ is a finite open cover $\}$ with the usual notion of refinement, see Subsection 1.2.3. If $\mathcal{U}\in Cov(X)\setminus\overline{Cov}(X)$, then $\overline{\mathcal{U}}$ given by $\mathcal{U}$ and $\{U\cap V|U,V\in\mathcal{U}\}$ refines $\mathcal{U}$. If $\mathcal{U},\mathcal{V}\in\overline{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, then we consider $p_{\mathcal{U},\mathcal{V}}:\mathcal{V}\to\mathcal{U}$ given by $p_{\mathcal{U},\mathcal{V}}(V)=\bigcap_{V\subseteq U\in\mathcal{U}}U$. Repeating the same arguments used in the proof of Proposition 2.9.2, we get that $p_{\mathcal{U},\mathcal{V}}$ is well-defined and continuous. We get that $(\mathcal{U},p_{\mathcal{U},\mathcal{V}},\overline{Cov}(X))$ is an inverse system of finite topological spaces.

Proposition 2.9.3. *Let X be a compact metric space and let $(\mathcal{U}_{2\epsilon_n}(A_n),p_{n,n+1})$ be a FAS for X. Then $(\mathcal{U}_{2\epsilon_n}(A_n),p_{n,n+1})$ is isomorphic to $(\mathcal{U},p_{\mathcal{U},\mathcal{V}},\overline{Cov}(X))$ in pro-HTop.*

Proof. We prove that $(\mathcal{B}_n,h_{n,n+1})$ is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n),p_{n,n+1})$ in pro-*HTop*. Later, we show that $(\mathcal{B}_n,h_{n,n+1})$ is isomorphic to $(\mathcal{U},p_{\mathcal{U},\mathcal{V}},\overline{Cov}(X))$ in pro-*HTop*.

Firstly, we verify that φ is a level morphism, i.e., the following diagram commutes up to homotopy.

$$\begin{tikzcd}
\mathcal{B}_n \arrow[d, "\varphi"] & \mathcal{B}_{n+1} \arrow[l, "h_{n,n+1}"'] \arrow[d, "\varphi"] \\
\mathcal{U}_{2\epsilon_n}(A_n) & \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1}) \arrow[l, "p_{n,n+1}"']
\end{tikzcd}$$

By the proof of Proposition 2.9.2, for every $C \in \mathcal{B}_{n+1}$ we get $C \subset \bigcap_{x \in p_{n,n+1}(\varphi(C))} \mathcal{B}(x, \epsilon_n)$. We have that $h_{n,n+1}(C) \subset \bigcap_{x \in p_{n,n+1}(\varphi(C))} \mathcal{B}(x, \epsilon_n)$. From here, we deduce that $\varphi(h_{n,n+1}(C)) \geq p_{n,n+1}(\varphi(C))$ and the diagram is commutative up to homotopy.

We want to apply Theorem 1.2.5, so we only need to find $g_n : \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1}) \to \mathcal{B}_n$ making the diagrams commutative up to homotopy. We define $g_n(C) = \bigcap_{x \in p_{n,n+1}(C)} \mathcal{B}(x, \epsilon_n)$. We prove that $C \subset \bigcap_{x \in p_{n,n+1}(C)} \mathcal{B}(x, \epsilon_n)$ to verify that the intersection is non-empty and g_n is well-defined. Let us take $c \in C$ and $x \in p_{n,n+1}(c)$, $d(x, c) \leq \gamma_n$. For every $c' \in C$ we get

$$d(x, c') < d(x, c) + d(c, c') < \gamma_n + 2\epsilon_{n+1} < \epsilon_n.$$

Therefore, g_n is well-defined. If $C \leq D$, where $C, D \in \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1})$, then $g_n(D) \subset g_n(C)$ by the definition of g_n, which implies that $g_n(C) \leq g_n(D)$ and g_n is continuous.

For every $C \in \mathcal{U}_{2\epsilon_{n+1}}(A_{n+1})$ we have clearly that $\varphi(g_n(C)) \geq p_{n,n+1}(C)$, so $\varphi \circ g_n$ is homotopic to $p_{n,n+1}$. For every $C \in \mathcal{B}_{n+1}$ we have trivially $h_{n,n+1}(C) \subset g_n(\varphi(C))$, so $h_{n,n+1}(C) \geq g_n(\varphi(C))$ and $h_{n,n+1}$ is homotopic to $g_n \circ \varphi$. Thus, the diagrams are commutative up to homotopy and $(\mathcal{B}_n, h_{n,n+1})$ is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ in pro-*HTop*.

By Lebesgue's number lemma [86], for every $\mathcal{U} \in \overline{Cov}(X)$ there exists ϵ_n such that every subset of diameter less than ϵ_n is contained in some element of $\mathcal{U}$. Therefore, $\mathcal{B}_n$ refines $\mathcal{U}$ and we get that $\{\mathcal{B}_n\}_{n \in \mathbb{N}}$ is cofinal in $\overline{Cov}(X)$. In addition, $(\mathcal{B}_n, h_{n,n+1})$ isomorphic to $(\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(X))$ in *pro-HTop*. □

Now, it is easy to get an alternative proof of Theorem 2.1.8.

Theorem 2.9.4. *Let X be a compact metric space and let $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1})$ be a FAS for X. Then $\mathcal{AM}_2(X)$ is a HPol-expansion of X.*

Proof. We have that $(\mathcal{U}, p_{\mathcal{U},\mathcal{V}}, \overline{Cov}(X))$ is isomorphic to $(\mathcal{U}_{2\epsilon_n}(A_n), p_{n,n+1}, \mathbb{N})$ in pro-*HTop* by Proposition 2.9.3. If we apply the McCord functor, then we get that both inverse systems are isomorphic in *pro-HPol*. However, the first inverse system is a *HPol*-expansion of X by Proposition 2.8.9. □

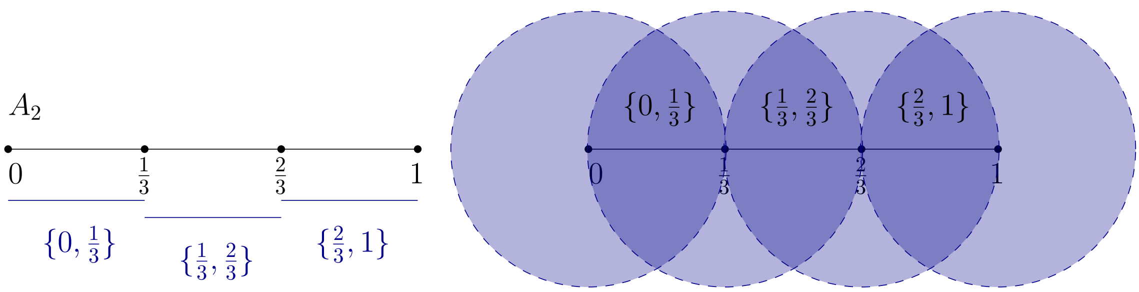

Figure 2.9.2: Schematic representation of $\mathcal{U}_{2\epsilon_2}(A_2)$ and $\mathcal{B}_2$.

Example 2.9.5. Let us consider the same FAS for the unit interval considered in Example 2.1.9. For this particular case, we get that $\mathcal{B}_n$ is exactly $\mathcal{U}_{2\epsilon_n}(A_n)$ for every $n \in \mathbb{N}$. This is because the distance between two consecutive points of the approximation A_n is precisely ϵ_n. In Figure 2.9.2, this situation for the case $n = 2$ can be observed, where we have that $A_2 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $\epsilon_2 = \frac{1}{3}$. The blue lines represent the maximal points of $\mathcal{U}_{2\epsilon_2}(A_2)$.

Chapter 3

Realization problems of groups in topology

In this chapter we study problems of realization of groups and homomorphisms in topological spaces. Let X be a topological space. Then several groups can be associated to it. Given a group G, it is natural to wonder for the existence of a topological space X having G as a group related to it.

3.1 Introduction

We state the main problems that we treat in subsequent sections and introduce notation. Let $\mathcal{C}$ denote an arbitrary category. We have the following classical problem.

Problem 3.1.1. *Given a group G, is there an object X in $\mathcal{C}$ such that the group of automorphisms of X is isomorphic to G?*

We recall the importance of this problem when $\mathcal{C}$ is the category of simple graphs *SimGph*. See [57] for a complete introduction to graph theory. In [66], Problem 3.1.1 was posed for *SimGph*. A positive answer for finite groups was given in [53]. It was proved that a variant of the Cayley graph of a finite group G satisfies that its group of automorphisms is isomorphic to G. The Cayley graph of a group G, which was introduced in [28], provides a way to obtain a representation of a group by means of a colored and directed graph. Given a group G and a generating set S for G, the vertices of the Cayley graph $\Gamma(G, S)$ are the elements of G. For each generator $s \in S$ is assigned a color c_s. For any $g \in G$ and $s \in S$, the vertices corresponding to the elements g and gs are joined by a directed edge from g to gs of color c_s. Then, the edges of the Cayley graph were replaced by some asymmetric graphs, that is, graphs with trivial group of automorphisms. In that way, an undirected and uncolored graph satisfying that its group of automorphisms is isomorphic to G is obtained. In Figure 3.1.1, we present an example of the Cayley graph for the dihedral group D_4 and one possible modification to get a simple graph having D_4 as group of automorphisms. Years later, in two independent papers, [96] and [39], the same result for arbitrary groups is obtained. In addition, J. de Groot also solved Problem 3.1.1 for the topological category *Top*. Namely, given a group G, J. de Groot found a complete, connected, locally connected metric space M of any positive dimension such that its group of homeomorphisms is isomorphic to G. As a corollary, he also obtained that every group is the automorphism group of some commutative ring. The idea to get the previous result goes as follows. Firstly, given a group G, find an easy structure (graph) with the desired property of realization. Then, find asymmetric structures, i.e., a family of pairwise non-homeomorphic topological spaces with trivial groups of homeomorphisms. Finally, replace the edges of the graph by the asymmetric structures to obtain a topological space satisfying the property required.

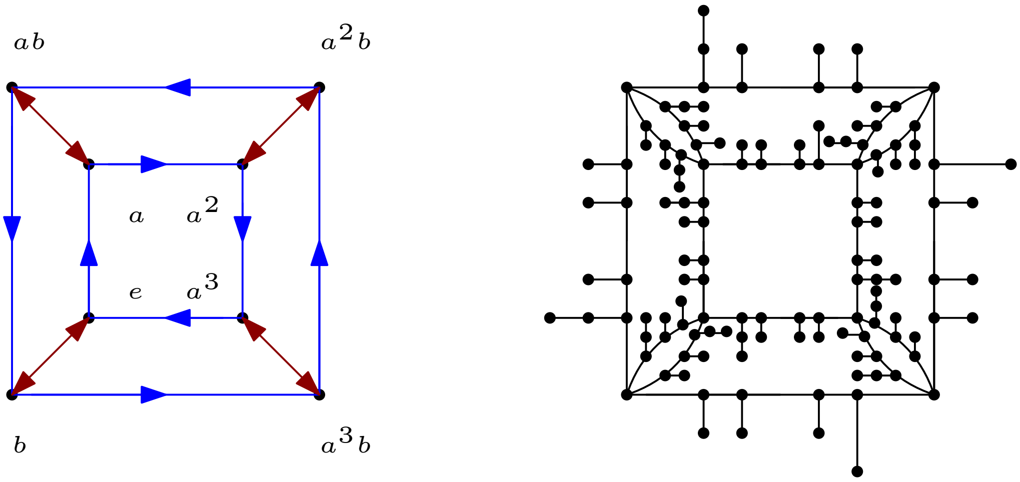

Figure 3.1.1: Cayley graph of D_4 (left) and modified graph having D_4 as a group of automorphisms in *SimGph* (right).

Thus, the realization problem for the category of graphs has played an important role to solve other realization problems. However, the answer to Problem 3.1.1 is not always positive, e.g., the category of groups *Grp*. If we consider the group of integer numbers $\mathbb{Z}$, then there is no group G such that its group of automorphisms is isomorphic to $\mathbb{Z}$. For more cases, see [47] or [70].

In Section 3.2, Section 3.3 and Section 3.4, we solve Problem 3.1.1 for topological categories. Specifically, the topological category *Top*, the homotopical category *HTop* and the pointed cases for the previous categories Top_* and $HTop_*$. Since the solution of Problem 3.1.1 in each category is constructive, we can study some properties of the topological spaces obtained such as the weak homotopy type, the number of points, etc. Some of these properties are studied in Section 3.5. Furthermore, for any group G, we found a family of pairwise non-homeomorphic topological spaces satisfying that their groups of homeomorphisms are precisely G. This result provides more examples of another important problem in mathematics that is formulated in the following question.

Problem 3.1.2. *Given two topological spaces X and Y with isomorphic groups of homeomorphisms, does it follow that X and Y are homeomorphic?*

In general, the answer for this question is negative. For example, the method that we obtain in Section 3.5 answers negatively to this question. There are other examples and well-known cases that also give a negative answer, see for instance [92, 99, 37] or just consider the closed interval $[0, 1]$ and the open interval $(0, 1)$. In the opposite direction, there are some results giving a positive answer for certain classes of topological spaces, see [113], [77] or [94].

In $HPol$ ($HPol_*$), the full subcategory of *HTop* ($HPol_*$) whose objects are all (pointed) topological spaces having the homotopy type of a (pointed) polyhedron, the problem of realizability has appeared in many papers for over fifty years, see [64], [5], [45], [95]. This problem has been placed as the first problem to solve in [6], a list of open problems about groups of self-homotopy equivalences. In this direction, a complete answer for finite groups and the pointed case was obtained in [34]. Furthermore, in [36], the free case has been completely solved using tools of highly algebraic character. In *Top*, there is also a positive answer for finite groups using finite T_0 topological spaces. In [25], given a finite group G with a generating set S, a topological space with $|G|(|G| + 1)$ points having G as a group of homeomorphisms is obtained. In [106], the finite topological space found realizing G has $|G|(2|S|+1)$ points. In [16], a topological space with $|G|(|S|+2)$ points such that its group of homeomorphisms is isomorphic to G is obtained. Recently, in [15], it is improved the previous result and a finite topological space with $4|G|$ points realizing G as a group of homeomorphisms is obtained. On the other hand, in [9, 10], it is shown that for every group G there exists a finite poset with $3|G|$ points having G as a group of homeomorphisms. Despite the fact that this construction is the sharpest one obtained up to now, it relies heavily on finding a generating set satisfying a list of non-trivial conditions. It is proved that for every group there exists a generating set satisfying these conditions but in practice it is not always trivial to get the desired generating set. The result for posets described above is obtained as a corollary of a similar statement for directed graphs, which proves again the importance of graphs in this sort of problems.

In subsequent sections, given a topological space X, we denote by $Aut(X)$ and $\mathcal{E}(X)$, the group of homeomorphisms (Top) and the group of homotopy classes of self-homotopy equivalences ($HTop$) respectively. We also denote by $Aut_*(X)$ and $\mathcal{E}_*(X)$ the pointed cases (Top_* and $HTop_*$). With abuse of notation, let $\mathcal{E}_*(X)$ denote the group of elements of $\mathcal{E}(X)$ inducing the identity on the homology groups of X. It is not difficult to check that $\mathcal{E}_*(X)$ is a normal subgroup of $\mathcal{E}(X)$. Then, the quotient group $\mathcal{E}(X)/\mathcal{E}_*(X)$ can be considered. In [45, Problem 19], A. Viruel raised the question whether an arbitrary group can be realized as $\mathcal{E}(X)/\mathcal{E}_*(X)$. In Section 3.4, we will obtain a positive answer for the combinatorial version of this question, that is, given a finite group G we find a topological space X such that $\mathcal{E}(X)/\mathcal{E}_*(X)$ is isomorphic to G.

Another realization problems of groups in topology that arise in a natural manner are the ones involving homology or homotopy groups.

Problem 3.1.3. *Given a group G (abelian) and a positive integer n ($n > 1$), is there a topological space X such that its n-th homotopy group is isomorphic to G?*

A similar problem can be considered for homology groups. For homotopy groups, a positive answer was obtained considering the so-called Eilenberg-MacLane spaces. Given a group G and a positive integer number n, the Eilenberg-MacLane space is denoted by $K(G,n)$ and satisfies that $\pi_n(K(G,n)) \simeq G$ and $\pi_m(K(G,n)) = 0$ for every $m \neq n$. The analogue case for homology groups is solved using the so-called Moore spaces. Given a group G and a positive integer number n, the Moore space is denoted by $M(G,n)$ and satisfies that $H_n(M(G,n)) \simeq G$ and $H_m(M(G,n)) = 0$ for every $m \neq n$. Both spaces, Eilenberg-Maclane and Moore spaces, are CW-complexes. A complete description of the construction and properties of these spaces can be found for instance in [60].

Combining some of the previous realization problems, we can state new questions.

Problem 3.1.4. *Given G and H two groups, is there a topological space X such that $\mathcal{E}(X) \simeq H$ and $Aut(X) \simeq G$?*

We give a positive answer to this question in Section 3.6, where some properties of the space constructed are studied. Consequently, we have that for a topological space X its group of homeomorphisms has no relations with its group of self-homotopy equivalences and vice versa. In addition, given a topological space X, there is a natural morphism $\tau : Aut(X) \to \mathcal{E}(X)$ sending each homeomorphism f to its homotopy class $[f]$. Therefore, the following problem can be stated.

Problem 3.1.5. *Let $f : G \to H$ be a homeomorphism of groups, is there a topological space X such that $\mathcal{E}(X) \simeq H$, $Aut(X) \simeq G$ and $\tau = f$?*

A positive answer to this question is obtained in Section 3.7. Even though this result generalizes the previous one, for the sake of clarity we prefer to present the results in this order.

Finally, a more general realization problem involving all the groups mentioned before can be stated as follows.

Problem 3.1.6. *Let H be a finite group, let G be a group and let $\{F_i\}_{i\in I}$ be a sequence of finitely generated groups, where $I \subseteq \mathbb{N}$ is a finite set. Is there a topological space X such that $Aut(X) \simeq G$, $\mathcal{E}(X) \simeq H$ and $H_i(X) \simeq F_i$ for every $i \in I$?*

Similarly, we also have the following problem for homotopy groups.

Problem 3.1.7. *Let H be a finite group, let G be a group and let T be a finitely presented (abelian) group (if $n > 1$). Is there a topological space X such that $Aut(X) \simeq G$, $\mathcal{E}(X) \simeq H$ and $\pi_i(X) \simeq F_i$?*

Both problems are solved in Section 3.9. In fact, they can be seen as corollaries of a more general result. Given a compact polyhedron K, a finite group H and a group G, there exists a finite topological space X satisfying that $Aut(X) \simeq G$, $\mathcal{E}(X) \simeq H$ and X is weak homotopy equivalent to K. To get this result, we obtain in Section 3.8 a family of asymmetric topological spaces having the weak homotopy type of a point.

A positive answer to the previous problems is telling us that there is no relation between the groups considered throughout this chapter. In contrast, for the category $HPol$, the situation is completely different since $\mathcal{E}(X)$ contains normal subgroups that are nilpotent. For instance, given a topological space X, $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) denotes the set of self-homotopy equivalences of X that induce the identity map in homotopy (homology), it is trivial to check that $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) is a normal subgroup of $\mathcal{E}(X)$. If X is a finite CW-complex, then $\mathcal{E}_{\#}(X)$ ($\mathcal{E}_*(X)$) is a nilpotent group. See [36] for more details. Using the construction obtained in Theorem 3.9.2, we find topological spaces that do not satisfy the previous property. This implies that the techniques used to study the group of self-homotopy equivalences for CW-complexes cannot be adapted in a natural way to general topological spaces.

3.2 Realization problem in Top

In [16], given a finite group G, J.A. Barmak and E.G. Minian obtained a finite T_0 topological space X_G such that $Aut(X_G) \simeq G$. We generalize their construction to non-finite groups using Alexandroff spaces. The proof of the result is similar to the one given by them.

Let G be a group. We consider a non-trivial generating set S' for G, i.e., the identity element does not belong to S'. We get a topological space X_G satisfying $Aut(X_G) \simeq G$.

Construction of X_G. By the well-ordering principle, we can consider a well-order on the set S' such that S' has a maximum if and only if S' is a finite set. Suppose that S' is not finite. Then consider a non-finite countable subset $T \subset S'$. We consider the well-order on T given by the well-order of the natural numbers. By the well-ordering principle there is a well-order on $S' \setminus T$. We consider the sum of the two ordered sets $S' \setminus T$ and T, that is, for any $x, x \in S' \setminus T \cup T$, $x \leq y$ if and only if one of the following holds:

- $x, y \in S' \setminus T$ and x is smaller than y with the order defined on $S' \setminus T$.
- $x, y \in T$ and x is smaller than y with the order defined on T.

- $x \in S' \setminus T$ and $y \in T$.

This is a well-order on S' without a maximum.

We consider $S = S' \cup \{-1, 0\}$, where we are assuming that $-1, 0 \notin S$. We extend the well-order defined on S' to S. For every $\alpha \in S'$ we declare $-1 < 0 < \alpha$. Finally, we consider $X_G = G \times S$ with the following relations:

- $(g, \beta) < (g, \gamma)$ if $-1 \leq \beta < \gamma$ where $g \in G$ and $\beta, \gamma \in S$.
- $(g\beta, -1) < (g, \gamma)$ if $0 < \beta \leq \gamma$ where $g \in G$ and $\beta, \gamma \in S$.

It is easy to show that X_G with the previous relations is a partially ordered set.

Remark 3.2.1. *If G is a finite group, then the construction of X_G is the same than the one given in [16]. At the same time, this construction can also be seen as a sort of generalization of the Cayley graph. The idea is to get a directed acylic graph from the Cayley graph with the same group of automorphisms. In a Cayley graph, there are directed closed paths, see Figure 3.2.1 where we have the Cayley graph of the cyclic group of two elements. Thus, for each vertex we add a column with the length of the number of generators of the group plus 1. Each vertex of the column represents a copy of the original vertex. Hence, the colors of the Cayley graph are codified somehow with the levels of the columns, where there is one extra level (0 level) that arises for technical reasons. In Figure 3.2.1, if the level 0 is not considered, then the group of automorphisms clearly changes because it is the Klein four-group. From this, we obtain a finite poset satisfying*

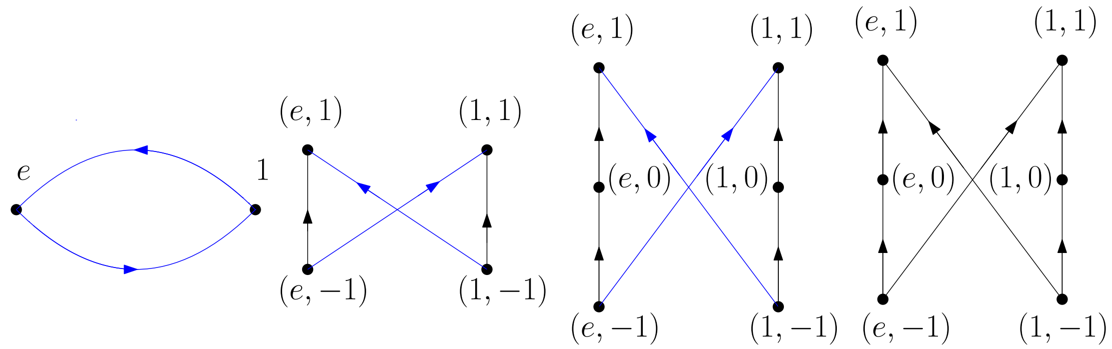

Figure 3.2.1: Schematic description: from the Cayley graph (left) to a directed acylic graph having the same group of automorphism.

the desired property of realization of groups. Furthermore, let $x, y \in X_G$. We say that $x \sim y$ if and only if the first coordinates of x and y are equal. It is easily seen that $\sim$ is an equivalence relation. The quotient space $X_G/\sim$ is a finite set with a preorder given by the relations from X_G. We can draw a directed graph related to $X_G/\sim$. The vertices are the points of $X_G/\sim$, the edges are given by the preorder obtained, that is, we have a directed edge from a to b if and only if $a \prec b$. The new graph is the Cayley graph of G without the colors.

We present a couple of examples in order to introduce the construction of X_G for a general group G. The first one is an example for a finite group, while the second one is an example for a non-finite group.

Example 3.2.2. Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the Klein four-group. We consider $S' = \{a, b\}$, where $a = (1, 0)$, $b = (0, 1)$ and we declare $a < b$. The Hasse diagram of $X_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ can be seen in Figure 3.2.2.

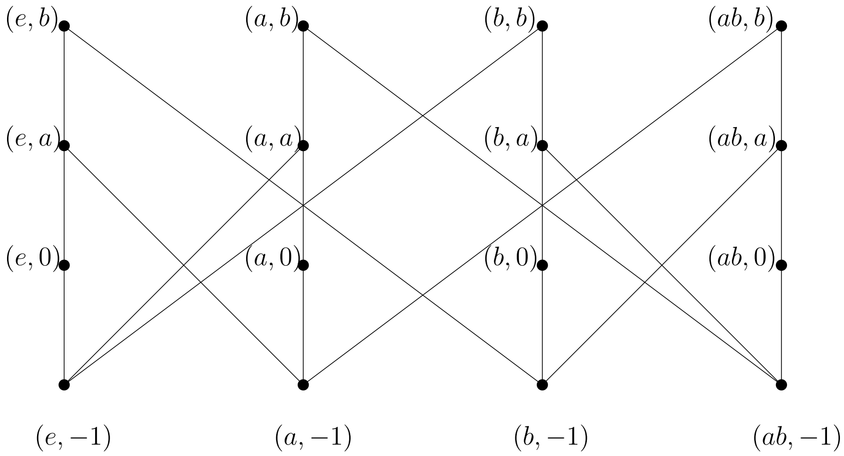

Figure 3.2.2: Hasse diagram of $X_{\mathbb{Z}_2\oplus\mathbb{Z}_2}$

If we use the equivalence relation introduced in Remark 3.2.1, that is, $(x,y)\sim(\overline{x},\overline{y})$ if and only if $x=\overline{x}$, then we get that $X_{\mathbb{Z}_2\oplus\mathbb{Z}_2}/\sim$ is a preordered set. The graph associated to this preordered set can be seen in Figure 3.2.3, where we also have the Cayley graph of $\mathbb{Z}_2\oplus\mathbb{Z}_2$ taking as generators a and b. Again, we observe the relations of this construction with the Cayley graph.

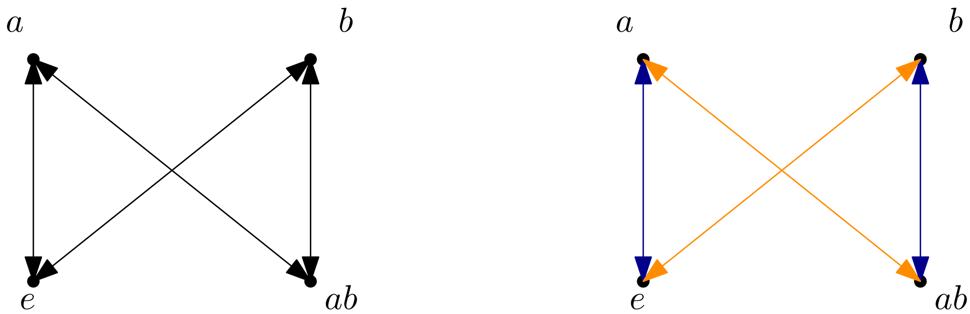

Figure 3.2.3: Graph associated to $X_{\mathbb{Z}_2\oplus\mathbb{Z}_2}/\sim$ and Cayley graph of $\mathbb{Z}_2\oplus\mathbb{Z}_2$.

Example 3.2.3. Let $\mathbb{Z}$ be the group of integer numbers. We consider $S'=\{1\}$. It is a simpler matter to prove that $X_{\mathbb{Z}}$ is a locally finite Alexandroff space. Since there is only one generator considered in the generating set, it follows that he maximum height that can have a point of $X_{\mathbb{Z}}$ is 2. In Figure 3.2.4 we present the Hasse diagram of $X_{\mathbb{Z}}$.

Proposition 3.2.4. *If $f: X_G\to X_G$ is a homeomorphism, then $f(G\times\{\beta\})=G\times\{\beta\}$ for every $\beta\in S$.*

Proof. We consider $A=X_G\backslash\{(g,-1)\in X_G|g\in G\}$. We have that A is clearly the product of a discrete poset with a well-ordered set. Firstly, we show that $f(A)=A$. The task is now to prove that $f(G\times\{-1\})=G\times\{-1\}$. We argue by contradiction. Suppose that there exists $(g,-1)\in X_G$ with $f(g,-1)=(h,\beta)$ for some $h\in G$ and $\beta>-1$. We have $f^{-1}(h,\beta)=(g,-1)$ so $f^{-1}(h,-1)\leq f^{-1}(h,\beta)$ since $(h,-1)<(h,\beta)$. By the minimality of $(g,-1)$, $f^{-1}(h,-1)=(g,-1)$, which leads to a contradiction with the injectivity of f^{-1}. We deduce that $f(G\times\{-1\})=G\times\{-1\}$ and $f^{-1}(G\times\{-1\})=G\times\{-1\}$. Thus, the restriction $f_{|A}: A\to A$ is a homeomorphism. We get that A has $|G|$ connected components, which are $A_g=\{(g,\beta)|\beta\geq 0\}$ for $g\in G$. If $(g,\beta)\in A$ satisfies that

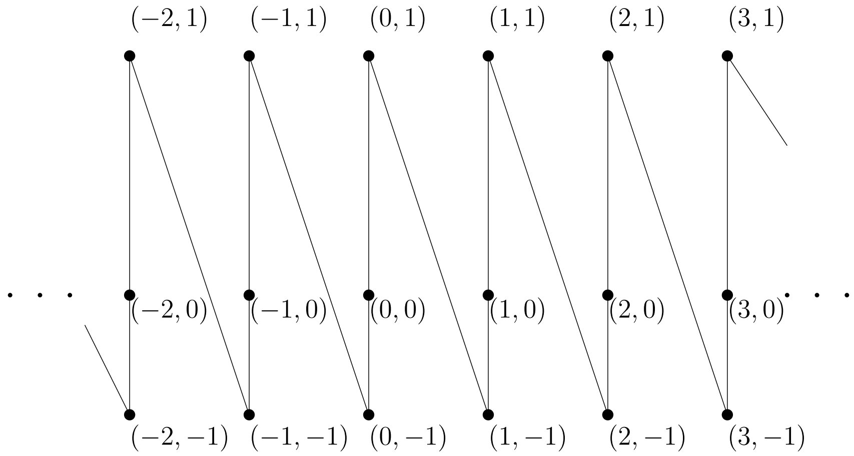

Figure 3.2.4: Hasse diagram of $X_{\mathbb{Z}}$

$f(g,\beta) = (h,\gamma)$ for some $h \in G$ and $\gamma \geq 0$, then we get by the continuity of f that $f(A_g) \subseteq A_h$. If there exists $y \in A_h \setminus f(A_g)$, then we can find an element $x \in f(A_g)$ such that $x < y$ or $x > y$. This can be done because A_h is a well-ordered set. Hence, we know that $f^{-1}(x) \in A_g$. By the continuity of f^{-1}, we get that $y \in f(A_g)$, which leads to a contradiction. Therefore, $f(A_g) = A_h$. The only isomorphism from a well-ordered set to itself is the identity, so $\gamma = \beta$. From here, we deduce the desired result. □

Remark 3.2.5. *From the proof of Proposition 3.2.4, it can be deduced that any homeomorphism* $f : X_G \setminus \{(g,-1) \in X_G | g \in G\} \to X_G \setminus \{(g,-1) \in X_G | g \in G\}$ *satisfies* $f(G \times \beta) = G \times \beta$, *where* $\beta \in S \setminus \{-1\}$.

Theorem 3.2.6. *Let G be a group. Then the group of automorphisms of X_G in Top is isomorphic to G.*

Proof. Given a group G, we consider the Alexandroff space X_G constructed before. We define $\varphi : G \to Aut(X_G)$ given by $\varphi(g)(s,\beta) = (gs,\beta)$, where $(s,\beta) \in X_G$. We need to show that φ is an isomorphism of groups. First, we check that φ is well-defined. We have that $\varphi(g) : X_G \to X_G$ is clearly continuous because it preserves the order of X_G. By construction, $\varphi(g)$ is also bijective. The inverse of $\varphi(g)$ is $\varphi(g^{-1})$, which is also continuous. It is straightforward to get that φ is a homomorphism of groups.

We prove that φ is a monomorphism of groups. Suppose that $\varphi(g) = id$, where $id : X_G \to X_G$ denotes the identity. Then, $(ge,-1) = \varphi(g)(e,-1) = (e,-1)$, where e denotes the identity element of the group G. This implies that $g = e$.

We verify that φ is an epimorphism of groups. Let $f \in Aut(X_G)$. By Proposition 3.2.4, $f(e,-1) = (h,-1)$ for some $h \in G$. We also have that $\varphi(h)(e,-1) = (h,-1)$. We consider $Y = \{x \in X_G | f(x) = \varphi(h)(x)\}$. If Y is open, Y is closed and X_G is a connected space, then $Y = X_G$ because Y is non-empty ($(e,-1) \in Y$).

We prove that Y is open. Let $(g,\beta) \in Y$. We have $f_{|U_{(g,\beta)}}, \varphi(h)_{|U_{(g,\beta)}} : U_{(g,\beta)} \to U_{f(g,\beta)}$ and $f(g,\beta) = (hg,\beta) = \varphi(h)(g,\beta)$. On the other hand, there is only one element for each γ with $0 \leq \gamma \leq \beta$ in $U_{(g,\beta)}$ and $U_{f(g,\beta)}$, (g,γ) and (hg,γ) respectively. Concretely, $U_{(g,\beta)}$ consists of points (g,γ) with $-1 \leq \gamma \leq \beta$ and points of the form $(g\gamma,-1)$ with $0 < \gamma \leq \beta$, the description of $U_{f(g,\beta)}$ is similar. Hence, by Proposition 3.2.4, we deduce that $f(g,\gamma) = (hg,\gamma) = \varphi(h)(g,\gamma)$ for every $0 \leq \gamma \leq \beta$. It only remains to show that $f(g,-1) = \varphi(h)(g,-1)$ and $f(g\gamma,-1) = \varphi(h)(g\gamma,-1)$ for every $0 < \gamma \leq \beta$. By the construction of X_G, $(g\gamma,-1) \prec (g,\gamma)$ for every $0 < \gamma$. By Proposition 1.1.76, $f(g\gamma,-1) \prec f(g,\gamma)$. If

$x \in U_{f(g,\beta)}$ satisfies $x \prec f(g,\gamma) = (hg,\gamma)$, then x can only be of the form (hg,α) for some $0 \leq \alpha < \gamma$ or $(hg\gamma,-1)$. Suppose that $f(g\gamma,-1) = (hg,\alpha)$ for some $0 \leq \alpha < \gamma$. We get a contradiction with Proposition 3.2.4. Therefore, $f(g\gamma,-1) = (hg\gamma,-1) = \varphi(h)(g\gamma,-1)$. Since $(g,-1) \prec (g,0)$, it follows that $f(g,-1) \prec f(g,0) = (hg,0)$. From this, $f(g,-1) = (hg,-1) = \varphi(h)(g,-1)$. Thus, $\varphi(h)(x) = f(x)$ for every $x \in U_{(g,\beta)}$, that is, $U_{(g,\beta)} \subset Y$.

We prove that Y is closed. We consider $(k,\beta) \in X_G \setminus Y$. By Proposition 3.2.4, $f(k,\beta) = (g,\beta) = \varphi(gk^{-1})(k,\beta)$ for some $g \in G$. Furthermore, $g \neq hk$ because otherwise we would get

$$f(k,\beta) = (g,\beta) = (hk,\beta) = \varphi(hkk^{-1})(k,\beta) = \varphi(h)(k,\beta),$$

which leads to a contradiction with $(k,\beta) \in X_G \backslash Y$. We can repeat the same argument used before to get that $f_{|U_{(k,\beta)}} = \varphi(gk^{-1})_{|U_{(k,\beta)}}$, but $gk^{-1} \neq h$, so $f(y) = \varphi(gk^{-1})(y) \neq \varphi(h)(y)$ for every $y \in U_{(k,\beta)}$, i.e., $U_{(k,\beta)} \cap Y = \emptyset$.

We prove that X_G is connected showing that it is path-connected. We need to show that for every $x, y \in X_G$ there is a sequence $x = x_0, x_1, ..., x_n = y$ with x_i comparable to x_{i+1} for every $i = 0, ..., n-1$. It suffices to check the existence of that sequence between points of the form $(g,-1), (h,-1)$ with $g, h \in G$ and $g \neq h$. This is due to the first relation of the partial order given in X_G. We prove that for every $g \in G$ there is a sequence of comparable points from $(g,-1)$ to $(e,-1)$. By hypothesis, S' is a generating set so $g = \alpha_1^{d_1}\alpha_2^{d_2}...\alpha_m^{d_m}$, where $\alpha_j \in S'$ and $d_j = 1$ or -1 with $j = 1, ..., m$. If $d_m = 1$, then

$$(\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_m, -1) \prec (\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_{m-1}^{d_{m-1}}, \alpha_m) > (\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_{m-1}^{d_{m-1}}, -1).$$

If $d_m = -1$, then

$$(\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_m^{-1}, -1) < (\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_m^{-1}, \alpha_m) \succ (\alpha_1^{d_1}\alpha_2^{d_2}...\alpha_{m-1}^{d_{m-1}}, -1).$$

We need to combine these two steps inductively to obtain a sequence of comparable points from $(g,-1)$ to $(e,-1)$. From here, get the desired result.

□

Remark 3.2.7. *By the proof of Theorem 3.2.6, the following property can be deduced: if $f, g \in Aut(X_G)$ and there exists $x \in X_G$ satisfying $f(x) = g(x)$, then $f = g$.*

3.3 Realization problem in $HTop$

In this section we solve Problem 3.1.1 for the homotopical category $HTop$ using Alexandroff spaces. To do this, we update the construction obtained in Section 3.2. Given a group G, if X_G is locally a core, then we get that $Aut(X_G)$ is isomorphic to $\mathcal{E}(X_G)$ by Corollary 1.1.33. Therefore, it suffices to show that $Aut(X_G)$ is isomorphic to G to obtain that every group can be realized as the group of homotopy classes of self-homotopy equivalences of an Alexandroff space. But X_G is far from being locally a core because it contains beat points, see Remark 1.1.28. Concretely, every point of the form (g,β), where $g \in G$ and $0 \leq \beta < max(S)$, is a beat point. $F_{(g,\beta)} \setminus \{(g,\beta)\}$ can be seen as a subset of S if we ignore the first coordinate ($F_{(g,\beta)} \setminus \{(g,\beta)\} = \{(g,\alpha) | \alpha > \beta\}$). Then, there exists a minimum in $F_{(g,\beta)} \setminus \{(g,\beta)\}$, which means that (g,β) is an up beat point. We need to add points to X_G to get a good candidate $\overline{X}_G$ to be locally a core. But we also want to modify X_G keeping the property that $Aut(\overline{X}_G)$ is isomorphic to $Aut(X_G)$. By Corollary 1.1.33 and showing that $Aut(X_G)$ is isomorphic to G, we would get the desired result. For the finite case, i.e., G is a finite group, the naive idea of studying the core of X_G expecting that $Aut(X_G) \simeq \mathcal{E}(X_G) \simeq G$ does not work. We illustrate this with the following example.

Example 3.3.1. We consider the cyclic group of two elements $\mathbb{Z}_2$. In Figure 3.3.1, we present the Hasse diagram of $X_{\mathbb{Z}_2}$ and the Hasse diagram of $X_{\mathbb{Z}_2}$ after removing beat points one by one. The group of homeomorphisms has changed, the group of homeomorphisms after removing the beat points is the Klein four-group while $Aut(X_{\mathbb{Z}_2}) \simeq \mathbb{Z}_2$. Then, $\mathcal{E}(X_{\mathbb{Z}_2})$ is not isomorphic to $Aut(X_{\mathbb{Z}_2})$.

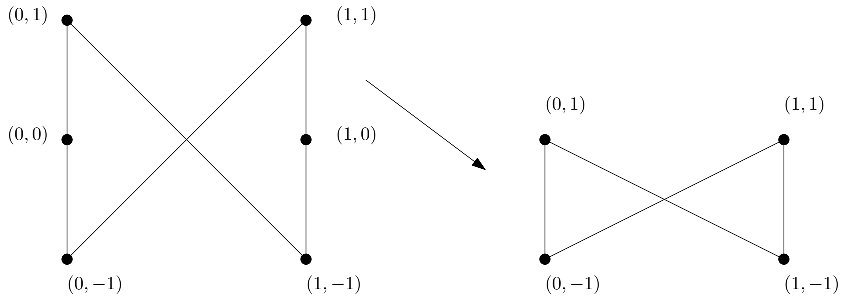

Figure 3.3.1: Hasse diagram of $X_{\mathbb{Z}_2}$ (left) and Hasse diagram of the core of $X_{\mathbb{Z}_2}$(right).

Hence, the idea to get $\overline{X}_G$ is to change the state of the beat points (g, β) of X_G. If we add the point $B_{(g,\beta)}$ to X_G with the relation $B_{(g,\beta)} > (g, \beta)$, then we get easily that (g, β) is not a beat point. On the other hand, the new point added is now a beat point. To solve this situation, we can add the point $C_{(g,\beta)}$ with the relation $C_{(g,\beta)} < B_{(g,\beta)}$. We have that $B_{(g,\beta)}$ is not a beat point but $C_{(g,\beta)}$ is a beat point. We argue following the previous method so as to get $S_{(g,\beta)}$, i.e., we add $A_{(g,\beta)}$ and $D_{(g,\beta)}$ with $C_{(g,\beta)} < A_{(g,\beta)} > D_{(g,\beta)} < (g, \beta)$. Now, the new points added and (g, β) are not beat points.

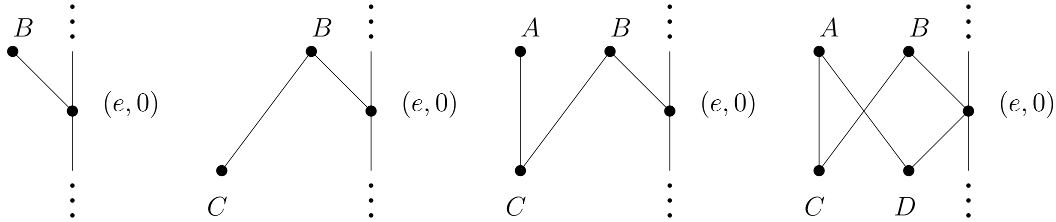

Figure 3.3.2: Intuitive idea for the construction of $S_{(e,0)}$.

We construct $T_{(g,\beta)}$ following the idea of not introducing beat points, see Figure 3.3.3. It is also important to get that $T_{(g,\beta)}$ is not homeomorphic to $S_{(g,\beta)}$. If $S_{(g,\beta)}$ is homeomorphic to $T_{(g,\beta)}$, then we cannot expect to obtain that $Aut(X_G)$ is isomorphic to $Aut(\overline{X}_G)$. This happens because we introduce more homeomorphisms. In particular, the homeomorphism f that satisfies $f(T_{(g,\beta)}) = S_{(g,\beta)}$, $f(S_{(g,\beta)}) = T_{(g,\beta)}$ and keeps the remaining points fixed. In addition, the group of homeomorphisms of $S_{(g,\beta)} \cup T_{(g,\beta)}$ is trivial as we will see in Lemma 3.3.3. Now, it is easy to find the set A_x required in Definition 1.1.29 for $x = (g, \beta)$, where $0 \leq \beta < max(S)$, $A_x = S_{(g,\beta)} \cup T_{(g,\beta)}$. It is also easy to find A_x for the remaining points as we will see in the proof of Lemma 3.3.8. However, if we do not consider the space $T_{(g,\beta)}$ in the construction of $\overline{X}_G$, then the existence of a set A_x satisfying the conditions required in Definition 1.1.29 is not guaranteed. Suppose G is a group and S' a non-finite countable generating set of G. We denote the elements of S' by s_i with $i \in \mathbb{N}$. We define a well-order on S' given by $s_i < s_j$ if and only if $i < j$. If we consider $X = X_G \cup (\bigcup_{\alpha \in S \setminus \{-1\}} S_{(g,\alpha)})$ and the point $x = (g, 0)$ for some $g \in G$, then a set A_x satisfying the conditions of Definition 1.1.29 does not exist. If it exists, then $(g, s_i) \in A_x$ for every $i \in \mathbb{N}$, which means that A_x cannot be a finite set.

We prove the last assertion. By construction, for every $j \in \mathbb{N}$ we have that (g, s_j) is not a maximal point. Furthermore, $minimal(\{z \in X | z > (g, 0)\}) = \{(g, s_1), B_{(g,0)}\}$. Then, $(g, s_1), B_{(g,0)} \in A_x$ because it is needed that $|A_x \cap minimal(\{z \in X | z > (g, 0)\})| \geq 2$. Moreover, $minimal(\{z \in X | z > (g, s_1)\}) = \{(g, s_2), B_{(g,s_1)}\}$, so $(g, s_2), B_{(g,s_1)} \in A_x$. In general, it can be argued inductively that $(g, s_{n+1}), B_{(g,s_n)} \in A_x$ for every $n \in \mathbb{N}$ because $minimal(\{z \in X | z > (g, s_n)\}) = \{(g, s_{n+1}), B_{(g,s_n)}\}$.

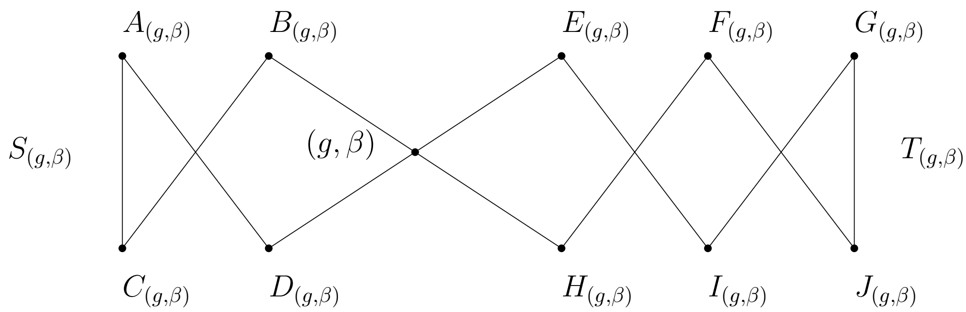

Figure 3.3.3: Hasse diagram of $S_{(g,\beta)} \cup T_{(g,\beta)}$.

Construction of $\overline{X}_G$. Given a group G, we consider the topological space X_G constructed in Section 3.2. For every $(g, \beta) \in X_G$, where $0 \leq \beta < max(S)$ and $g \in G$, we consider $S_{(g,\beta)}$ and $T_{(g,\beta)}$ in the following way: $S_{(g,\beta)} = \{A_{(g,\beta)}, B_{(g,\beta)}, C_{(g,\beta)}, D_{(g,\beta)}, (g, \beta)\}$ and $T_{(g,\beta)} = \{E_{(g,\beta)}, F_{(g,\beta)}, G_{(g,\beta)}, H_{(g,\beta)}, I_{(g,\beta)}, J_{(g,\beta)}, (g, \beta)\}$. Finally, we consider

$$\overline{X}_G = X_G \cup \Big(\bigcup_{\substack{(g,\beta) \in G \times S \\ 0 \leq \beta < max(S)}} (S_{(g,\beta)} \cup T_{(g,\beta)}) \Big)$$

with the following relations:

1. $(g, \beta) < (g, \gamma)$ if $-1 \leq \beta < \gamma$, where $g \in G$ and $\beta, \gamma \in S$.
2. $(g\beta, -1) < (g, \gamma)$ if $0 < \beta \leq \gamma$, where $g \in G$ and $\beta, \gamma \in S$.
3. $(g, \beta) > D_{(g,\gamma)}, H_{(g,\gamma)}$ if $-1 < \gamma < \beta$, where $g \in G$ and $\beta, \gamma \in S$.
4. $(g, \beta) > D_{(g,\beta)}, H_{(g,\beta)}$ if $0 \leq \beta < max(S)$, where $g \in G$ and $\beta \in S$.
5. $(g, \beta) < E_{(g,\gamma)}, B_{(g,\gamma)}$ if $-1 \leq \beta < \gamma < max(S)$, where $g \in G$ and $\beta, \gamma \in S$.
6. $(g, \beta) < E_{(g,\beta)}, B_{(g,\beta)}$ if $0 \leq \beta < max(S)$, where $g \in G$ and $\beta \in S$.
7. $A_{(g,\beta)} > C_{(g,\beta)}, D_{(g,\beta)}$ and $B_{(g,\beta)} > C_{(g,\beta)}$ if $0 \leq \beta < max(S)$, where $g \in G$ and $\beta \in S$.
8. $B_{(g,\beta)} > D_{(g,\gamma)}, H_{(g,\gamma)}$ if $-1 < \gamma \leq \beta < max(S)$, where $g \in G$ and $\beta, \gamma \in S$.
9. $E_{(g,\beta)} > D_{(g,\gamma)}, H_{(g,\gamma)}$ if $-1 < \gamma \leq \beta < max(S)$, where $g \in G$ and $\beta, \gamma \in S$.
10. $G_{(g,\beta)} > I_{(g,\beta)}, J_{(g,\beta)}$ if $0 \leq \beta < max(S)$, where $g \in G$ and $\beta \in S$.
11. $F_{(g,\beta)} > H_{(g,\beta)}, J_{(g,\beta)}$ and $E_{(g,\beta)} > I_{(g,\beta)}$ if $0 \leq \beta < max(S)$, where $g \in G$ and $\beta \in S$.

Remark 3.3.2. *If G is a finite group, then the structure T_x can be omitted in $\overline{X}_G$. For the finite case, $\overline{X}_G$ is a finite topological space. By construction, it is obvious that $\overline{X}_G$ is a minimal finite topological space, which implies that $\mathcal{E}(\overline{X}_G) \simeq Aut(\overline{X}_G)$. Therefore, it suffices to show that $Aut(\overline{X}_G) \simeq G$.*

Lemma 3.3.3. *Given $g \in G$, $0 \le \beta < max(S)$ and $S_{(g,\beta)} \cup T_{(g,\beta)}$ as a subspace of $\overline{X}_G$. The group of homeomorphisms of $S_{(g,\beta)} \cup T_{(g,\beta)}$ is trivial.*

Proof. We can deduce that (g, β) is a fixed point for every homeomorphism since (g, β) is not a maximal or minimal point. From here, it is easy to obtain the desired result. □

Again, we present a couple of examples to illustrate this construction. In fact, we update the examples introduced in Section 3.2.

Example 3.3.4. Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the Klein four-group. We consider $S' = \{a, b\}$ and we declare $a < b$, where $a = (1, 0)$ and $b = (0, 1)$. The Hasse diagram of $\overline{X}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ can be seen in Figure 3.3.4. We have used only the structure S_x since the Klein four-group is a finite group, see Remark 3.3.2.

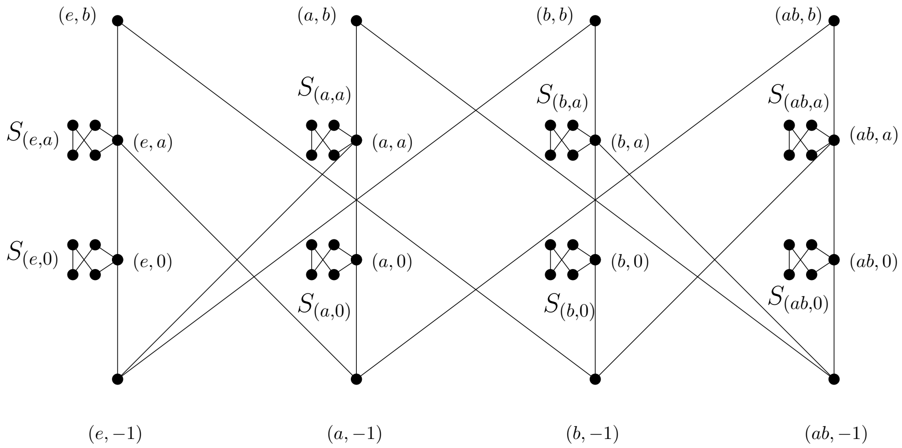

Figure 3.3.4: Hasse diagram of $\overline{X}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ without T_x.

Example 3.3.5. Let $\mathbb{Z}$ be the group of integer numbers. We consider $S' = \{1\}$. The Hasse diagram of $\overline{X}_{\mathbb{Z}}$ can be seen in Figure 3.3.5.

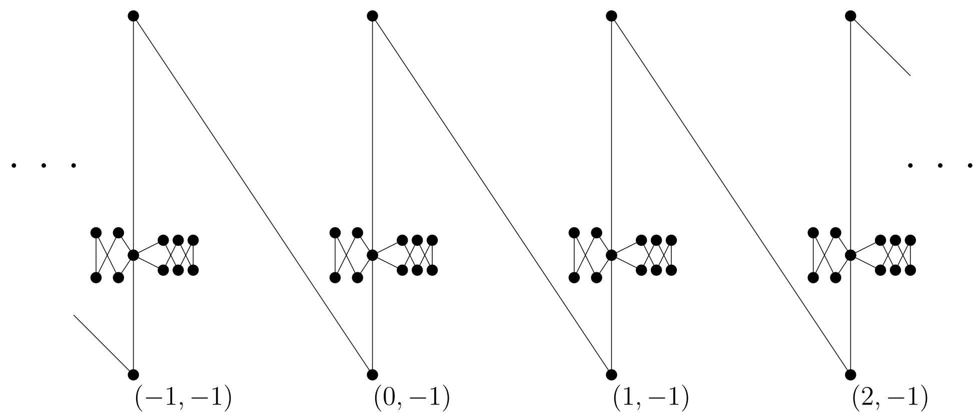

Figure 3.3.5: Hasse diagram of $\overline{X}_{\mathbb{Z}}$.

We want to prove that for every group G, there exists a topological space satisfying that its group of self-homotopy equivalences is isomorphic to G. Our candidate to satisfy this property is $\overline{X}_G$. We split the proof of the result in some technical lemmas.

Lemma 3.3.6. *Given a non-trivial group G, $Aut(X_G)$ is isomorphic to $Aut(\overline{X}_G)$.*

Proof. Firstly, we prove that every $f \in Aut(\overline{X}_G)$ satisfies $f(X_G) = X_G$, that is, $f_{|X_G} \in Aut(X_G)$. If $f \in Aut(\overline{X}_G)$, then we have trivially that $f(M) = M$ and $f(N) = N$, where M and N denote the sets of maximal and minimal elements of $\overline{X}_G$. From this, we get $f(\overline{X}_G \setminus (M \cup N)) = \overline{X}_G \setminus (M \cup N)$. Notice that $\overline{X}_G \setminus (M \cup N)$ is precisely X_G minus the maximal and minimal points of X_G. For the finite case, $\overline{X}_G \setminus (M \cup N) = X_G \setminus (\bigcup_{g\in G}\{(g,-1),(g,max(S))\})$. For the non-finite case, $\overline{X}_G \setminus (M \cup N) = X_G \setminus (\bigcup_{g\in G}\{(g,-1)\})$

We prove that $f(G \times \{-1\}) = G \times \{-1\}$. We argue by contradiction. Suppose there exists $g \in G$ such that $f(g,-1) \in N \setminus (G \times \{-1\})$. If $f(g,-1) = x$ with $x \in \{C_{(h,\alpha)}, D_{(h,\alpha)}, H_{(h,\alpha)}, I_{(h,\alpha)}, J_{(h,\alpha)}\}$ for some $h \in G$ and $0 \leq \alpha < max(S)$, then the set $Z = \{y \in \overline{X}_G | x \prec y\}$ has cardinality two and it contains at least one element in M. The set $Y = \{y \in \overline{X}_G | (g,-1) \prec y\}$ has cardinality at least two. If $|S'| \geq 2$, then $(g,0)$ and $(g\beta^{-1},\beta)$ with $\beta < max(S)$ are points in $Y \setminus (M \cup N)$. By Proposition 1.1.76 and the injectivity of f, $f(Y \setminus (M \cup N)) = Z$ but it contradicts $f(\overline{X}_G \setminus (M \cup N)) = \overline{X}_G \setminus (M \cup N)$. If $|S'| = 1$, then we know that $f(G \times \{0\}) = G \times \{0\}$ since the points of the form $(k,0)$ for some $k \in G$ are the only points that are neither maximal nor minimal points. If $f(g,-1) = x$ with $x \in \{C_{(h,0)}, I_{(h,0)}, J_{(h,0)}\}$ for some $h \in G$, then we get a contradiction studying the image by f of $(g,-1) \prec (g,0) \prec (g,max(S))$. If $f(g,-1) = x$ with $x \in \{D_{(h,0)}, H_{(h,0)}\}$ for some $h \in G$, then we get a contradiction studying the image by f of $(g,-1) \prec (g\,max(S)^{-1}, max(S)) \succ (g\,max(S)^{-1}, 0)$. We have shown that $f(G\times\{-1\}) \subseteq G\times\{-1\}$. The equality follows because f is a homeomorphism.

If S is finite, then we prove that $f(G \times \{max(S)\}) = G \times \{max(S)\}$. We argue by contradiction. Suppose there exists $g \in G$ such that $f(g,max(S)) = x$ with $x \in \{A_{(h,\beta)}$, $B_{(h,\beta)}$, $E_{(h,\beta)}$, $F_{(h,\beta)}$, $G_{(h,\beta)}\}$ for some $h \in G$ and $0 \leq \beta < max(S)$. We repeat the previous argument. The set $Z = \{y \in \overline{X}_G | x \succ y\}$ has cardinality two and it is a subset of $\overline{X}_G \setminus (G \times \{-1\})$. The set $Y = \{y \in \overline{X}_G | (g,max(S)) \succ y\}$ has cardinality two, its elements are (g,α) and $(g\,max(S),-1)$, where α is an immediate predecessor of $max(S)$. By Proposition 1.1.76 and the injectivity of f, $f(Y) = Z$ but it contradicts $f(G\times\{-1\}) = G \times \{-1\}$. Again, $f(G \times \{max(S)\}) = G \times \{max(S)\}$ follows from the fact that f is a homeomorphism.

By the previous arguments, it can be deduced that $f(X_G) = X_G$. Moreover, by Proposition 3.2.4, $f(G\times\{\beta\}) = G\times\{\beta\}$ for every $\beta \in S$. By Lemma 3.3.3 and Proposition 1.1.76, for every (g,β) with $g \in G$ and $0 \leq \beta < max(S)$ we get $f(S_{(g,\beta)}) = S_{f(g,\beta)}$ and $f(T_{(g,\beta)}) = T_{f(g,\beta)}$.

We define $\varphi : Aut(\overline{X}_G) \to Aut(X_G)$ given by $\varphi(f) = f_{|X_G}$. It is easy to show that φ is a homomorphism of groups and is well-defined. We prove that φ is a monomorphism of groups. If $f, h \in Aut(\overline{X}_G)$ with $f \neq h$, then it means that there exists a point $x \in \overline{X}_G$ such that $f(x) \neq h(x)$. Suppose $x \in (S_{(g,\beta)} \cup T_{(g,\beta)}) \setminus \{(g,\beta)\}$ for some $(g,\beta) \in X_G$. By the previous comments, $f(g,\beta) \neq h(g,\beta)$. Hence, we find $y \in X_G$ such that $f_{|X_G}(y) \neq h_{|X_G}(y)$. Now, we prove that φ is an epimorphism of groups. If $f \in Aut(X_G)$, then we extend $f : X_G \to X_G$ to $f' : \overline{X}_G \to \overline{X}_G$ just declaring that $f'(x) = f(x)$ for every $x \in X_G$ and $f'(S_{(g,\beta)}) = S_{f(g,\beta)}$, $f'(T_{(g,\beta)}) = T_{f(g,\beta)}$ for every $g \in G$ and $0 \leq \beta < max(S)$. It is easily seen that $f' \in Aut(\overline{X}_G)$. Thus, $\varphi(f') = f$. □

Lemma 3.3.7. *If X is an Alexandroff space, $a \in X$ is a maximal (resp. minimal) element such that $U_a \setminus \{a\}$ (resp. $F_a \setminus \{a\}$) is not connected and $r : X \to X$ is a comparative retraction, then $r(a) = a$.*

Proof. We argue by contradiction. Suppose that $r(a) < a$, the case $r(a) > a$ is not possible since a is a maximal element. By hypothesis, $U_a \setminus \{a\} = V \cup W$, where V and W are disjoint non-empty open sets. We can suppose that $x = r(a) \in V$, then $U_x \subseteq V$. We take $y \in W$ so $U_y \subseteq W$. By the continuity of r, we get $r(y) \leq r(a)$. We also have that r is a comparative retraction so $r(y) \leq y$ or $r(y) \geq y$. Suppose that $r(y) \leq y$. We have that $r(y) \leq x$. Therefore, $r(y) \in U_x \cap U_y \subseteq V \cap W$, which entails a contradiction. If $r(y) \geq y$, then $x \geq r(y) \geq y$ and $y \in U_x$, which leads to a contradiction.

The case when a is a minimal element follows from the previous case. If $r : X \to X$ is a comparative retraction, then it is also a comparative retraction when we consider the opposite order in X. With the opposite order we are in the previous conditions, that is, a is now a maximal element and $U_a \setminus \{a\}$ is not connected. Thus, we get that $r(a) = a$. $\square$

Lemma 3.3.8. *Given a non-trivial group G, $\mathcal{E}(\overline{X}_G)$ is isomorphic to $Aut(\overline{X}_G)$.*

Proof. We need to prove that $\overline{X}_G$ is locally a core. Then, applying Lemma 1.1.33, we get the desired result. Firstly, we show that $\overline{X}_G$ is a $\mathcal{C}$-core. Hence, it suffices to verify that the only comparative retraction of X is the identity.

Let r be a comparative retraction. If $x \in (S_{(g,\beta)} \cup T_{(g,\beta)}) \setminus \{(g,\beta)\}$ or $x \in \{(g,-1), (g, max(S))\}$, where $g \in G$ and $0 \leq \beta < max(S)$, then we are trivially in the hypothesis of Lemma 3.3.7 so $r(x) = x$.

It only remains to study the points of the form (g,β) with $0 \leq \beta < max(S)$ and $g \in G$. We argue by contradiction. Suppose that $r(g,\beta) \neq (g,\beta)$. Then, $r(g,\beta) < (g,\beta)$ or $r(g,\beta) > (g,\beta)$. Suppose $r(g,\beta) < (g,\beta)$. If $r(g,\beta) \neq D_{(g,\beta)}$, then we get $D_{(g,\beta)} = r(D_{(g,\beta)}) < r(g,\beta) < (g,\beta)$, which leads to a contradiction because $D_{(g,\beta)} \prec (g,\beta)$. If $r(g,\beta) = D_{(g,\beta)}$, then we get a contradiction because we get $H_{(g,\beta)} = r(H_{(g,\beta)}) \leq r(g,\beta) = D_{(g,\beta)}$. Suppose $r(g,\beta) > (g,\beta)$. We can repeat the same arguments used before. If $r(g,\beta) \neq B_{(g,\beta)}$, then we have $B_{(g,\beta)} = r(B_{(g,\beta)}) > r(g,\beta) > (g,\beta)$, which leads to a contradiction with $B_{(g,\beta)} \succ (g,\beta)$. If $r(g,\beta) = B_{(g,\beta)}$, then we get $E_{(g,\beta)} = r(E_{(g,\beta)}) \geq r(g,\beta) = B_{(g,\beta)}$, which entails a contradiction.

We have shown that $\overline{X}_G$ is a $\mathcal{C}$-core. Now, we prove that $\overline{X}_G$ is locally a core. For every $x \in S_{(g,\beta)} \cup T_{(g,\beta)}$, where $0 \leq \beta < max(S)$ and $g \in G$, we consider $A_x = S_{(g,\beta)} \cup T_{(g,\beta)}$. Suppose $x = (g,-1)$ with $g \in G$. Then, $(g,-1) \prec (g\gamma^{-1},\gamma)$ for some $\gamma \in S'$. If $\gamma \neq max(S)$, then we consider $A_x = S_{(g,0)} \cup T_{(g,0)} \cup S_{(g\gamma^{-1},\gamma)} \cup T_{(g\gamma^{-1},\gamma)} \cup \{(g,-1)\}$. If S is finite and $\gamma = max(S)$, then we consider $A_x = S_{(g,0)} \cup T_{(g,0)} \cup S_{(g\gamma^{-1},\alpha)} \cup T_{(g\gamma^{-1},\alpha)} \cup \{(g,-1),(g\gamma^{-1},\gamma)\}$, where $(g\gamma^{-1},\alpha) \prec (g\gamma^{-1},\gamma)$. If S is finite and $x = (g, max(S))$ with $ginG$, then there exists $\alpha \in S$ such that $(g,\alpha) \prec (g, max(S))$ and we consider $A_x = S_{(g,\alpha)} \cup T_{(g,\alpha)} \cup S_{(g\ max(S),0)} \cup T_{(g\ max(S),0)} \cup \{x,(g\ max(S),-1)\}$. It follows easily that A_x satisfies the property required in Definition 1.1.29 for every $x \in \overline{X}_G$. $\square$

In general, for an arbitrary Alexandroff space we cannot expect to obtain an isomorphism of groups between its group of homeomorphisms and its group of homotopy classes of self-homotopy equivalences.

Example 3.3.9. Let $A = \{a,b,c,d,e\}$, we use the topology associated to the following partial order: $a, b < d, c, e$ and $c < e$. Firstly, we study $Aut(A)$. A homeomorphism preserves the order and therefore should send maximal chains to maximal chains. In A, there are two maximal chains, which are $a < c < e$ and $b < c < e$. From here, it is easy to deduce that e, d and c are fixed points for every homeomorphism and then $Aut(A) \simeq \mathbb{Z}_2$. On the other hand, $A^c = \{a,b,c,d\}$ is the core of A because e is clearly a down beat point and A^c does not contain beat points. Hence, $\mathcal{E}(A) \simeq \mathcal{E}(A^c)$. Since A^c is a core, it follows that $Aut(A^c) \simeq \mathcal{E}(A^c)$. From this, it is obvious that $\mathcal{E}(A^c)$ is the Klein four-group. We describe the two generators f and g of $\mathbb{Z}_2 \times \mathbb{Z}_2 \simeq \mathcal{E}(A)$. We have that f is given by $f(a) =$

$b, f(b) = a, f(c) = c, f(d) = d$ and g is given by $g(c) = d, g(d) = c, g(a) = a, g(b) = b$. A schematic situation in the Hasse diagrams can be seen in Figure 3.3.6.

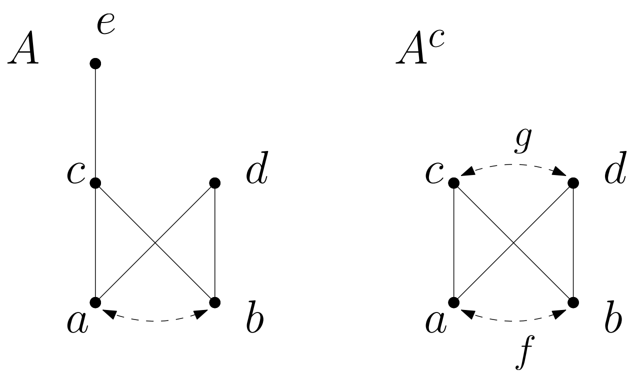

Figure 3.3.6: Hasse diagram of A and A^c.

Remark 3.3.10. *From the proofs of Remark 3.2.7, Lemma 3.3.6 and Lemma 3.3.8, it can be deduced that the set* $L_{x,y} = \{f : (\overline{X}_G, x) \to (\overline{X}_G, y) | f(x) = y$ *and* $f \in \mathcal{E}(\overline{X}_G)\}$ *has cardinality at most* 1.

Theorem 3.3.11. *Let* G *be a group. Then there exists a topological space* $\overline{X}_G$ *such that its group of homotopy classes of self-homotopy equivalences is isomorphic to* G. *Concretely,* $Aut(\overline{X}_G) \simeq \mathcal{E}(\overline{X}_G) \simeq G$.

Proof. Given a non-trivial group G, we consider X_G and $\overline{X}_G$. By the proof of Theorem 3.2.6, $G \simeq Aut(X_G)$. By Lemma 3.3.6 and Lemma 3.3.8, we get $\mathcal{E}(\overline{X}_G) \simeq Aut(X_G)$. If G is the trivial group, then we only need to consider the Alexandroff space given by one point. □

3.4 Realization problem in $HTop_*$ and Top_*

Given a group G, the topological space $\overline{X}_G$ introduced in Section 3.3 is far from satisfying that their group of pointed homotopy classes of pointed self-homotopy equivalences is isomorphic to G due to Remark 3.2.7. We deduce that $\{f : (\overline{X}_G, x) \to (\overline{X}_G, y) | x, y \in \overline{X}_G\}$ has cardinality at most 1. The same situation also holds if we consider X_G and $Aut_*(X_G)$. Therefore, we need to modify $\overline{X}_G$. The idea is to add one extra point that will play the role of a fixed point for every self-homotopy equivalence or autohomeomorphism.

Construction of $\overline{X}_{G*}$. Given a group G, we consider $\overline{X}_G$ and define $\overline{X}_{G*}$ as $\overline{X}_G \cup \{*\}$, where we keep the relations defined on $\overline{X}_G$ and we declare $* > (g, -1)$ for every $g \in G$.

We update previous examples to illustrate the new modification.

Example 3.4.1. Let $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the Klein four-group. We consider $S' = \{a, b\}$ and we declare $a < b$, where $a = (0, 1)$ and $b = (0, 1)$. The Hasse diagram of $\overline{X}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2 *}$ can be seen in Figure 3.4.1.

Example 3.4.2. Let $\mathbb{Z}$ be the group of integer numbers. We consider the following generating set $S' = \{1\}$. The Hasse diagram of $\overline{X}_{\mathbb{Z}*}$ can be seen in Figure 3.4.2.

Theorem 3.4.3. *Given a group* G, $Aut(\overline{X}_{G*}) \simeq \mathcal{E}_*(\overline{X}_{G*}) \simeq G$.

Proof. Suppose G is not a trivial group. We show that $\overline{X}_{G*}$ is a $\mathcal{C}$-core. We need to verify that the only comparative retraction $r : \overline{X}_{G*} \to \overline{X}_{G*}$ is the identity. If $x \in$

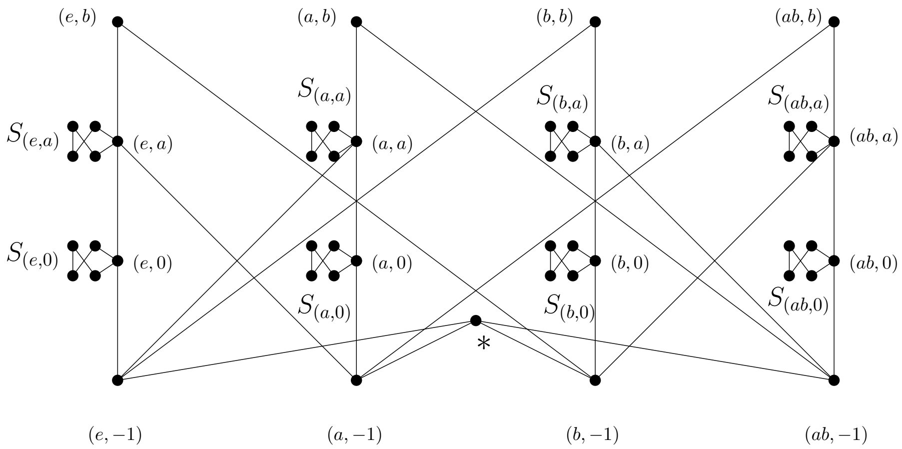

Figure 3.4.1: Hasse diagram of $\overline{X}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2 *}$ without T_x.

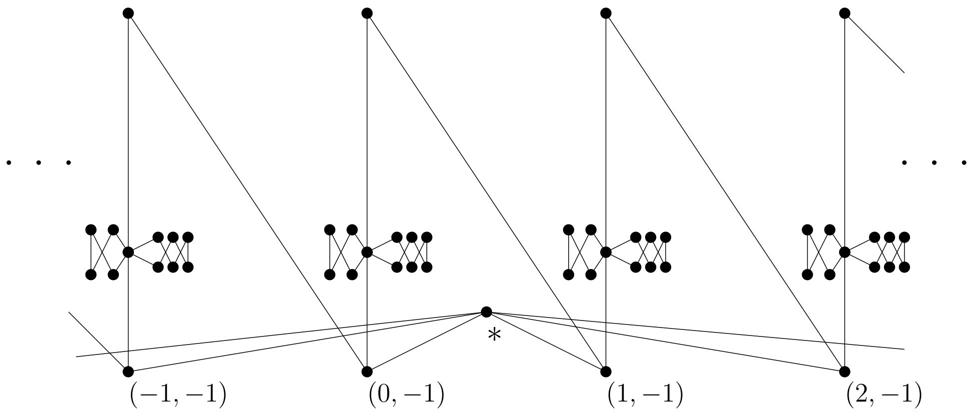

Figure 3.4.2: Hasse diagram of $X_{\mathbb{Z}*}$

$(S_{(g,\beta)} \cup T^n_{(g,\beta)}) \setminus \{(g,\beta)\}$ or $x \in \{(g,-1), *, (g, max(S))\}$, where $0 \leq \beta < max(S)$ and $g \in G$, then x satisfies the hypothesis of Lemma 3.3.7. Hence, we need to show that $r(x) = x$ for the points of the form (g,β), where $0 \leq \beta < max(S)$ and $g \in G$. It suffices to repeat the same arguments used in the proof of Lemma 3.3.8 for that sort of points to conclude.

We prove that $\overline{X}_{G*}$ is locally a core. If $x \in S_{(g,\beta)} \cup T^n_{(g,\beta)}$, where $g \in G$ and $0 \leq \beta < max(S)$, then we consider $A_x = S_{(g,\beta)} \cup T_{(g,\beta)}$. Suppose $x = (g,-1)$ with $g \in G$. Therefore, $(g,-1) \prec (g\gamma^{-1}, \gamma)$ for some $\gamma \in S'$. If $\gamma \neq max(S)$, then we consider $A_x = S_{(g,0)} \cup T_{(g,0)} \cup S_{(g\gamma^{-1},\gamma)} \cup T_{(g\gamma^{-1},\gamma)} \cup \{(g,-1)\}$. If S is finite and $\gamma = max(S)$, then we consider $A_x = S_{(g,0)} \cup T_{(g,0)} \cup S_{(g\gamma^{-1},\alpha)} \cup T_{(g\gamma^{-1},\alpha)} \cup \{(g,-1), (g\gamma^{-1}, \gamma)\}$, where $(g\gamma^{-1}, \alpha) \prec (g\gamma^{-1}, \gamma)$. If S is finite and $x = (g, max(S))$ for some $g \in G$, then there exists $\alpha \in S$ such that $(g,\alpha) \prec (g, max(S))$ and we consider $A_x = S_{(g,\alpha)} \cup T_{(g,\alpha)} \cup S_{(g\ max(S),0)} \cup T^n_{(g\ max(S),0)} \cup \{x, (g\ max(S), -1)\}$. Finally, if $x = *$, then we consider the sets $A_{(g,-1)}$ and $A_{(h,-1)}$ defined before, where $g, h \in G$ and $g \neq h$. Therefore, $A_* = \{*\} \cup A_{(g,-1)} \cup A_{(h,-1)}$. For every $x \in \overline{X}_{G*}$ we have that A_x satisfies the property required in Definition 1.1.29. Thus, $\overline{X}_{G*}$ is locally a core. By Lemma 1.1.33, $\mathcal{E}(\overline{X}_{G*})$ is isomorphic to $Aut(\overline{X}_{G*})$.

We prove that $Aut(\overline{X}_G)$ is isomorphic to $Aut(\overline{X}_{G*})$. To do that, we verify that $*$ is a fixed point for every $f \in Aut(\overline{X}_{G*})$. We argue by contradiction. Suppose $*$ is not a fixed point for $f \in Aut(\overline{X}_{G*})$. We study cases. By construction, $*$ is a maximal element. Then, $f(*) = x$ with x being a maximal element of $\overline{X}_G$, that is, x is a maximal element in $S_{(g,\beta)} \cup T_{(g,\beta)}$ or is equal to $(g, max(S))$, where $g \in G$ and $0 \leq \beta < max(S)$. We consider $Y_z = \{y \in \overline{X}_{G*} | y \prec z\}$. We have $|Y_*| = |G|$ and $|Y_x| = 2$. By Proposition 1.1.76, $f(Y_*) \subseteq Y_x$. If $|G| > 2$, then we get a contradiction with the injectivity of f. If $|G| = 2$, then G is isomorphic to $\mathbb{Z}_2$. Since $*$ is a maximal element, it follows that $f(*)$ is a maximal element. Again, we study cases. If $f(*)$ is a maximal element in $S_{(i,0)} \cup T_{(i,0)}$, where $i \in \{0,1\}$, then we get a contradiction studying the image by f of $* \succ (0,-1) \prec (0,0) \prec (0,1)$. If $f(*) = (i,1)$, where $i \in \{0,1\}$, then we get a contradiction studying the image by f^{-1} of $(i,1) \succ (i,0) \succ (i,-1)$. Thus, the only possibility is $f(*) = *$, which means that $*$ is a fixed point for every $f \in Aut(\overline{X}_{G*})$. It follows that $Aut(\overline{X}_G)$ is isomorphic to $Aut(\overline{X}_{G*})$.

We consider the group of pointed homotopy classes of pointed self-homotopy equivalences of the pointed space $(\overline{X}_{G*}, *)$. We also denote that group by $\mathcal{E}((\overline{X}_{G*}, *))$. By the proof of Lemma 1.1.33 and Remark 1.1.34, it can be deduced that $*$ is a fixed point for every self-homotopy equivalence $f \in \mathcal{E}(\overline{X}_{G*})$. Then, we get $\mathcal{E}(\overline{X}_{G*}) = \mathcal{E}((\overline{X}_{G*}, *))$. By the proof of Theorem 3.3.11, we obtain $Aut(\overline{X}_{G*}) \simeq G$. From here, the result is obtained.

If G is the trivial group, we only need to consider $\{(S_{(e,0)} \cup T_{(e,0)}, (e,0))\}_{n \in \mathbb{N}}$, where we have that $(e,0)$ is a fixed point for every homeomorphism. From here, it can be deduced the result. □

3.5 Infinitely many spaces realizing groups in Top, $HTop$, Top_* and $HTop_*$ and some properties of $\overline{X}_G$

Given a group G, we produce infinitely many Alexandroff spaces ($\overline{X}^n_G$ for $n \in \mathbb{N}$) satisfying that their groups of automorphisms in Top, $HTop$, Top_* and $HTop_*$ are isomorphic to G. In addition, these spaces are pairwise distinct in Top and $HTop$, this gives a negative answer to Problem 3.1.2 also for the homotopical category.

Construction of $\overline{X}^n_{G*}$. Let G be a non-trivial group and let $\overline{X}_{G*}$ be the space constructed in Section 3.4. For every $(g,\beta) \in X_G$, where $0 \leq \beta < max(S)$, we change $T_{(g,\beta)}$ by $T^n_{(g,\beta)}$ so as to get $\overline{X}^n_G$. We have that $T^n_{(g,\beta)}$ consists of $2n+5$ points, concretely, $T^n_{(g,\beta)} = \{x_1, ..., x_{2+n}, y_1, ..., y_{2+n}, (g,\beta)\}$, where x_i denotes a maximal element and y_i denotes a minimal element for $i = 1, ..., 2+n$. The relations are given by the following formulas:

$$(g,\beta) < x_1 > y_2 < x_3 > ... < x_{1+n} > y_{2+n} < x_{2+n} > y_{1+n} < x_n > ... < x_2 > y_1 < (g,\beta), \tag{3.1}$$

$$(g,\beta) < x_1 > y_2 < x_3 > ... > y_{1+n} < x_{2+n} > y_{2+n} < x_{1+n} > y_n < ... < x_2 > y_1 < (g,\beta), \tag{3.2}$$

where the first case is considered when n is even and the second case when n is odd. An example of the Hasse diagrams can be seen in Figure 3.5.1. It is clear that $T^1_{(g,\beta)} = T_{(g,\beta)}$ and $\overline{X}^1_{G*} = \overline{X}_{G*}$. We consider

$$\overline{X}^n_{G*} = X_G \cup \Big(\bigcup_{\substack{(g,\beta) \in G \times S \\ 0 \leq \beta < max(S)}} (S_{(g,\beta)} \cup T^n_{(g,\beta)}) \Big) \cup \{*\},$$

where we extend the partial order of $X_G \cup (\bigcup_{\substack{(g,\beta) \in G \times S \\ 0 \leq \beta < max(S)}} S_{(g,\beta)}) \cup \{*\}$ as a subspace of $\overline{X}_G$ to $\overline{X}^n_{G*}$. Let $x_1, y_1 \in T^n_{(g,\beta)}$, we identify them with $E_{(g,\beta)}$ and $H_{(g,\beta)}$ respectively. Then,

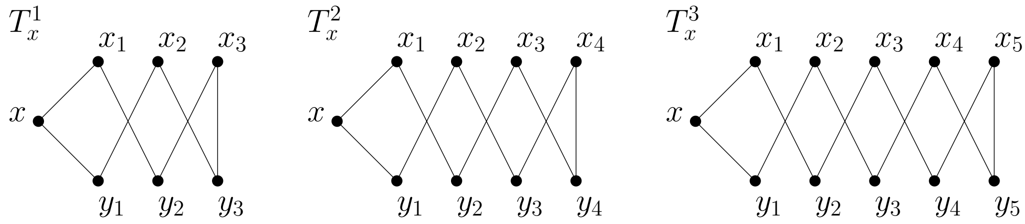

Figure 3.5.1: Hasse diagrams of T_x^1, T_x^2 and T_x^4

we consider the relations 3. 5. 8. and 9. of $\overline{X}_G$. Finally, we consider the relations given in Formula (3.1) and Formula (3.2).

Theorem 3.5.1. *Given a group G, $Aut(\overline{X}^n_{G*}) \simeq \mathcal{E}(\overline{X}^n_{G*}) \simeq Aut_*(\overline{X}^n_{G*}) \simeq \mathcal{E}_*(\overline{X}^n_{G*}) \simeq G$ for every $n \in \mathbb{N}$. Furthermore, $\overline{X}^n_G$ is homotopy equivalent to $\overline{X}^m_G$ if and only if $m = n$.*

Proof. We assume that G is a non-trivial group. The first part of the result is similar to the proof of Theorem 3.4.3. Then, we prove the second part.

If $\overline{X}^n_{G*}$ is homotopy equivalent to $\overline{X}^m_{G*}$, where $m \neq n$, then there exist two continuous functions $f : \overline{X}^n_{G*} \to \overline{X}^m_{G*}$, $g : \overline{X}^m_{G*} \to \overline{X}^n_{G*}$ such that $f \circ g \simeq id_{\overline{X}^m_{G*}}$ and $g \circ f \simeq id_{\overline{X}^n_{G*}}$. Without loss of generality we assume that $n > m$. By Theorem 1.1.31, $f \circ g = id_{\overline{X}^m_{G*}}$ and $g \circ f = id_{\overline{X}^n_{G*}}$. Then, f is a homeomorphism and g is the inverse of f. Therefore, $f(G \times \{\beta\}) = G \times \{\beta\}$ with $0 \leq \beta < max(S)$. To prove the last assertion we need to argue as we did in the proof of Lemma 3.3.6. If M^s and N^s denote the set of maximal and minimal points of $\overline{X}^s_{G*}$, where $s = n, m$, then we have $f(\overline{X}^n_{G*} \setminus (M^n \cup N^n)) = \overline{X}^m_{G*} \setminus (M^m \cup N^m)$, where $\overline{X}^s_{G*} \setminus (M^s \cup N^s)$ is the product of a discrete poset with a well-ordered set for $s = n, m$. By the proof of Proposition 3.2.4, it can be deduced the desired assertion. By Proposition 1.1.76 and the previous assertion, for every (g, β), where $g \in G$ and $0 \leq \beta < max(S)$, we deduce that $f(S_{(g,\beta)} \cup T^n_{(g,\beta)}) \subseteq S_{(g,\beta)} \cup T^m_{(g,\beta)}$, which leads to a contradiction with the injectivity of f.

If G is the trivial group, we only need to consider $\{S_{(e,0)} \cup T^n_{(e,0)}\}_{n \in \mathbb{N}}$. We apply the same techniques and a generalization of Lemma 3.3.3 to prove the desired result. ☐

We study some properties of $\overline{X}_G$. From now on in this section, G denotes a countable group, i.e., it is countable as a set or, equivalently, it has a countable set of generators. Let S' be a set of non-trivial generators of G. If S' is countable, we can denote the elements of S' by s_i with $i \in I \subseteq \mathbb{N}$. If S' is finite, then I can be taken as $I = \{1, 2, ..., |S'|\}$. We declare $s_i < s_j$ if and only if $i < j$. We get that S' with the previous relation is a well-ordered set. In addition, for every $x \in S'$, U_x is a finite set. Thus, taking S' and the previous relation in the construction of $\overline{X}_G$, it can be deduced that $\overline{X}_G$ is a locally finite Alexandroff space.

We define the undirected graph $H_u(\overline{X}_G)$ given by the Hasse diagram of $\overline{X}_G$, the set of vertices are the points of $\overline{X}_G$ and there is an edge between two vertices x and y if and only if $x \prec y$ ($x \succ y$). We have that $H_u(\overline{X}_G)$ can be seen as a one-dimensional CW-complex.

Proposition 3.5.2. *If G is a countable infinite group, then $|\mathcal{K}(\overline{X}_G)|$ is homotopy equivalent to $\bigvee_{|\mathbb{N}|} S^1$. If G is a finite group, then $|\mathcal{K}(\overline{X}_G)|$ is homotopy equivalent to the wedge sum of $3nr - n + 1$ copies of S^1, where $|G| = n$ and $|S'| = r$.*

Proof. We prove that $H_u(\overline{X}_G)$ and $|\mathcal{K}(\overline{X}_G)|$ have the same homotopy type. The idea of the proof is to show that the natural inclusion $i : H_u(\overline{X}_G) \to |\mathcal{K}(\overline{X}_G)|$ is a weak homotopy equivalence between two CW-complexes. Then, by a well-known theorem of Whitehead, we would get that $H_u(\overline{X}_G)$ is homotopy equivalent to $|\mathcal{K}(\overline{X}_G)|$.

By the construction of the McCord complex, $H_u(\overline{X}_G)$ is a subcomplex of $\mathcal{K}(\overline{X}_G)$ so i is well-defined and continuous. By Theorem 1.1.43, we know that there is a weak homotopy equivalence $f : |\mathcal{K}(\overline{X}_G)| \to \overline{X}_G$. Furthermore, by the proof of [75, Lemma 3], $f^{-1}(U_x) = \bigcup_{y \in U_x} star(y)$, where $star(y)$ denotes the union of all open simplices from $\mathcal{K}(\overline{X}_G)$ containing y as a vertex. By the proof of [100, Corollary 11, Chapter 3], we get that $|\mathcal{K}(U_x)|$ is a strong deformation retract of $f^{-1}(U_x)$ because $\mathcal{K}(\overline{X}_G) \setminus f^{-1}(U_x)$ is the largest subcomplex of $\mathcal{K}(\overline{X}_G)$ disjoint from $\mathcal{K}(U_x)$ and $\mathcal{K}(U_x)$ is a full subcomplex of $\mathcal{K}(\overline{X}_G)$. Finally, U_x is contractible because it has a maximum. We get that $|\mathcal{K}(U_x)|$ is also contractible.

Let $\mathcal{U}$ denote the basis-like open cover given by $\{U_x\}_{x \in \overline{X}_G}$. It is straightforward to check that $f^{-1}(\mathcal{U}) = \{f^{-1}(U_x) | U_x \in \mathcal{U}\}$ is a basis-like open cover for $|\mathcal{K}(\overline{X}_G)|$.

We denote $B_x = \bigcup_{y \in U_x \subset H_u(U_x)} star(y)$, here, $star(y)$ denotes the union of all open simplices from $H_u(\overline{X}_G)$ containing y as a vertex. We have that $i^{-1}(f^{-1}(U_x)) = B_x$. It is trivial to show that $H_u(U_x)$ is a full subcomplex of $H_u(\overline{X}_G)$. We can repeat the previous arguments to prove that $H_u(U_x)$ is a strong deformation retract of B_x, i.e., we use the proof of [100, Corollary 11, Chapter 3].

On the other hand, $H_u(U_x)$ is contractible for every $x \in \overline{X}_G$ since $H_u(U_x)$ is a tree. The vertices of $H_u(U_x)$ are the points of $\overline{X}_G$ that are smaller or equal to x. In Figure 3.5.2, we have the different graphs that can appear when we consider $H_u(U_x)$. If $x \in \{A_{(g,\beta)}, F_{(g,\beta)}, G_{(g,\beta)}\}$, where $g \in G$ and $0 \leq \beta < max(S)$, then $H_u(U_x)$ is isomorphic to graph A). If $x \in \{C_{(g,\beta)}, D_{(g,\beta)}, H_{(g,\beta)}, I_{(g,\beta)}, J_{(g,\beta)}, (g,-1)\}$, where $g \in G$ and $0 \leq \beta < max(S)$, then $H_u(U_x)$ is isomorphic to graph B). If $x = (g, 0)$, for some $g \in G$, then $H_u(U_x)$ is isomorphic to graph C). If $x \in \{B_{(g,\beta)}, E_{(g,\beta)}\}$, where $g \in G$ and $0 \leq \beta < max(S)$, then $H_u(U_x)$ is isomorphic to graph D), which is the graph in blue and black. If $x = (g, s_n)$ for some $g \in G$ and $n \in I \subseteq \mathbb{N}$, then $H_u(U_x)$ is isomorphic to graph E). If $x = (g, max(S))$ for some $g \in G$, then $H_u(U_x)$ is isomorphic to graph F). Thus, we are in the hypothesis of Theorem 1.1.46 and we get that i is a weak homotopy equivalence.

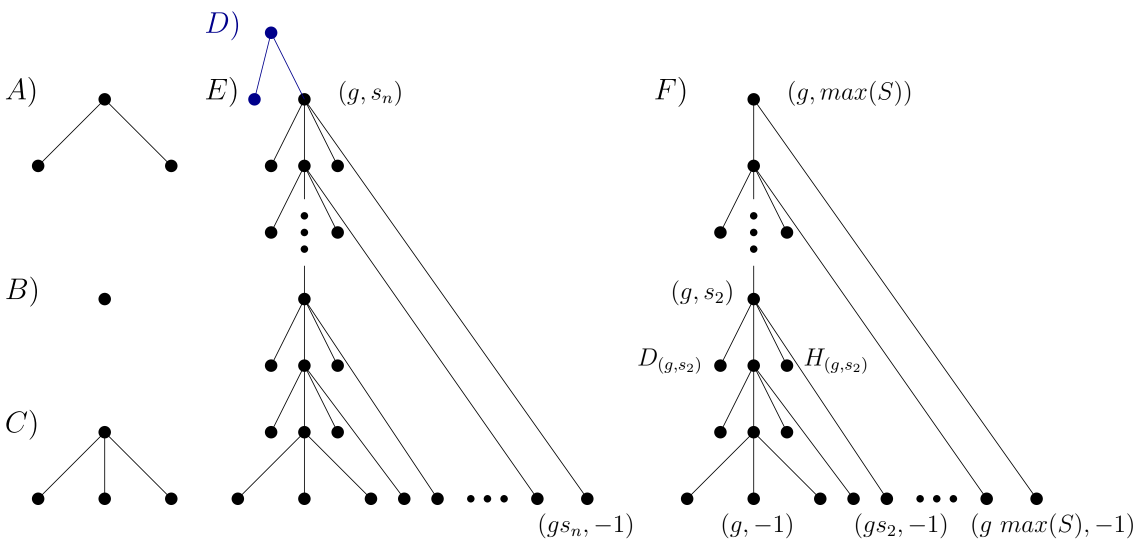

Figure 3.5.2: Graph isomorphic to $H_u(U_x)$ for different cases of $x \in \overline{X}_G$.

We prove that $H_u(\overline{X}_G)$ is a connected CW-complex. We have $(g, -1) \prec (g, 0) \prec (g, s_1) \prec ... \prec (g, s_n) \prec ...$ for every $g \in G$. Repeating the same arguments used in the last part of the proof of Theorem 3.2.6, we deduce the result. Therefore, $H_u(\overline{X}_G)$ is a one-dimensional connected CW-complex, so it is homotopy equivalent to a wedge sum of circles. It remains to determine the number of circles.

Suppose G is a non-finite set. By [60, Proposition 1A.1] or [107, Theorem 8.4.7], $H_u(\overline{X}_G)$ contains a maximal tree. Let W_g denote the subcomplex of $H_u(\overline{X}_G)$ given by $(g,-1) \prec (g,0) \prec ... \prec (g,s_n) \prec ....$ We consider a maximal tree T in $H_u(\overline{X}_G)$ containing the subcomplex $\bigcup_{g\in G} W_g$. It is clear that there is one edge of $S_{(g,\beta)}$ and another one of $T_{(g,\beta)}$ not contained in T for every (g,β), where $g \in G$ and $max(S) > \beta \geq 0$. Therefore, we have that the wedge sum of circles is at least of $2|S'||G|$ circles. But using the arithmetic of infinite cardinals, we have that $2|S'||G| = |G|$. There are $|G||S'|$ edges in $H_u(\overline{X}_G)$ not contained in $(\bigcup_{g\in G, 0\leq\beta<max(S)} H_u(S_{(g,\beta)} \cup T_{(g,\beta)})) \cup (\bigcup_{g\in G} W_g)$. Then, there are at most $2|G||S'| + |G||S'|$ edges not contained in T. Thus, $H_u(\overline{X}_G)$ is homotopy equivalent to $\bigvee_{|\mathbb{N}|} S^1$.

Suppose G is finite. We know that $H_u(\overline{X}_G)$ has the homotopy type of a wedge sum of a finite number of circles. Therefore, we only need to use the Euler characteristic to determine the number of circles. The number of vertices is $v = n(r+2) + 10nr$, where the first term corresponds to the number of vertices of $H_u(X_G) \subset H_u(\overline{X}_G)$ and the second one to the vertices of $H_u((S_{(g,\beta)} \cup T_{(g,\beta)}) \setminus \{(g,\beta)\})$, where $g \in G$ and $0 \leq \beta < max(S)$. The number of edges is $e = n(r+1) + nr + 12nr$, where the two first terms correspond to the number of edges of $H_u(X_G)$ and the third term corresponds to the edges of $\bigcup_{g\in G, 0\leq\beta<max(S)} H_u(S_{(g,\beta)} \cup T_{(g,\beta)})$. We get that the Euler characteristic of $H_u(X_G)$ is $n-3nr$, which means that $H_u(X_G)$ is homotopy equivalent to the wedge sum of $3nr-n+1$ copies of S^1. □

Remark 3.5.3. *Given a countable group G, the proof of Proposition 3.5.2 can be adapted to $\overline{X}^n_G$ for every $n \in \mathbb{N}$. Then for every countable group G we get that $\overline{X}^n_G$ is weak homotopy equivalent to $\overline{X}_G$ for every $n \in \mathbb{N}$.*

Proposition 3.5.4. *Given a group G, the McCord functor induces a natural monomorphism of groups $K : \mathcal{E}(\overline{X}_G) \to \mathcal{E}(|\mathcal{K}(\overline{X}_G)|)$.*

Proof. If $[f] \in \mathcal{E}(\overline{X}_G)$, then we get by Remark 1.1.34 that $[f] = f \in Aut(\overline{X}_G)$. We have that $\mathcal{K}(f) \in Aut(|\mathcal{K}(\overline{X}_G)|)$, where $\mathcal{K}(f)$ denotes the induced map between the geometric realization of $\mathcal{K}(\overline{X}_G)$. It is also clear that $\mathcal{K}(f)$ defines a homotopy class such that $[\mathcal{K}(f)] \in \mathcal{E}(|\mathcal{K}(\overline{X}_G)|)$. It is trivial to check that $K : \mathcal{E}(\overline{X}_G) \to \mathcal{E}(|\mathcal{K}(\overline{X}_G)|)$ given by $K(f) = [\mathcal{K}(f)]$ is a well-defined homomorphism of groups. We prove the injectivity of K. If $f, g \in \mathcal{E}(\overline{X}_G)$ with $f \neq g$, then there exists $(h,\beta) \in \overline{X}_G$ with $f(h,\beta) \neq g(h,\beta)$. Therefore, $f(S_{(h,\beta)}) = S_{f(h,\beta)} \neq S_{g(h,\beta)} = g(S_{(h,\beta)})$ and $\mathcal{K}(f)(|\mathcal{K}(S_{(h,\beta)})|) \neq \mathcal{K}(g)(|\mathcal{K}(S_{(h,\beta)})|)$. On the other hand, $|\mathcal{K}(S_{(h,\beta)})|$ is homotopy equivalent to S^1. Therefore, $\mathcal{K}(f)$ and $\mathcal{K}(g)$ send the same copy of S^1 to different copies of S^1 in $|\mathcal{K}(\overline{X}_G)|$. Using Proposition 3.5.2, it can be deduced that $\mathcal{K}(f)$ is not homotopic to $\mathcal{K}(g)$ so $K(f) \neq K(g)$. We argue by contradiction. Suppose that $\mathcal{K}(f)$ is homotopic to $\mathcal{K}(g)$. We know that —$\mathcal{K}(\overline{X}_G)|$ is homotopy equivalent to a wedge sum of circles. We proved earlier that $\mathcal{K}(f)$ and $\mathcal{K}(g)$ send the same copy of S^1 to different copies of S^1 in $|\mathcal{K}(\overline{X}_G)|$. Thus, $\mathcal{K}(f)$ and $\mathcal{K}(g)$ induce different morphisms in homology, which entails a contradiction. □

Remark 3.5.5. *It is not difficult to check that the monomorphism of groups defined in the proof of Proposition 3.5.4 is not an isomorphism of groups. For instance, consider the continuous function that interchanges $H_u(S_{(g,\beta)})$ with $H_u(T_{(g,\beta)})$ in $H_u(\overline{X}_G)$ for some (g,β) with $g \in G$ and $0 \leq \beta < max(S)$. In other words, consider the symmetry through (g,β) of $H_u(S_{(g,\beta)})$ and $H_u(T_{(g,\beta)})$ in $H_u(\overline{X}_G)$ that fixes the remaining points.*

For a general Alexandroff space A the homomorphism of groups $K : \mathcal{E}(A) \to \mathcal{E}(|\mathcal{K}(A)|)$ given in Proposition 3.5.4 is not necessarily a monomorphism of groups.

Example 3.5.6. Let us consider the Alexandroff space A^c considered in Example 3.3.9. The McCord complex $\mathcal{K}(A^c)$ of A^c is a triangulation of S^1. Then, $\mathcal{E}(|\mathcal{K}(A^c)|) \simeq \mathcal{E}(S^1) \simeq \mathbb{Z}_2$, while $Aut(A^c) \simeq \mathcal{E}(A^c) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Remark 3.5.7. *In general, the image of the monomorphism of Proposition 3.5.4 is not a normal subgroup of $\mathcal{E}(|\mathcal{K}(\overline{X}_G)|)$. We consider the cyclic group of three elements $\mathbb{Z}_3$. By Proposition 3.5.2, $|\mathcal{K}(\overline{X}_{\mathbb{Z}_3})|$ is homotopy equivalent to $\bigvee_{i=1}^7 S_i^1$, so $\mathcal{E}(|\mathcal{K}(\overline{X}_{\mathbb{Z}_3})|) \simeq \mathcal{E}(\bigvee_{i=1}^7 S_i^1)$. Consider $f \in \mathcal{E}(\overline{X}_{\mathbb{Z}_3}) \simeq \mathbb{Z}_3$ of order 3. Consider $\rho \in \mathcal{E}(\bigvee_{i=1}^7 S^1)$ given by $\rho(S_i^1) = S_{i+1}^1$ and $\rho(S_7^1) = S_1^1$, where $i = 1, ..., 6$. It can be deduced that $K(f)$ viewed as an element of $\mathcal{E}(\bigvee_{i=1}^7 S^1)$ satisfies that $K(f)(S_1^1) = S_3^1$, $K(f)(S_3^1) = S_5^1$, $K(f)(S_5^1) = S_1^1$, $K(f)(S_2^1) = S_4^1$, $K(f)(S_4^1) = S_6^1$, $K(f)(S_6^2) = S_2^1$ and $K(f)(S_7^1) = S_7^1$. Then, it is easy to check that $\rho K(f)\rho^{-1} \notin L$, where L denotes $K(\mathcal{E}(\overline{X}_{\mathbb{Z}_3}))$ considered as a subgroup of $\mathcal{E}(\bigvee_{i=1}^7 S_i^1)$. In Figure 3.5.3, we have the undirected Hasse diagram of $\overline{X}_{\mathbb{Z}_3}$.*

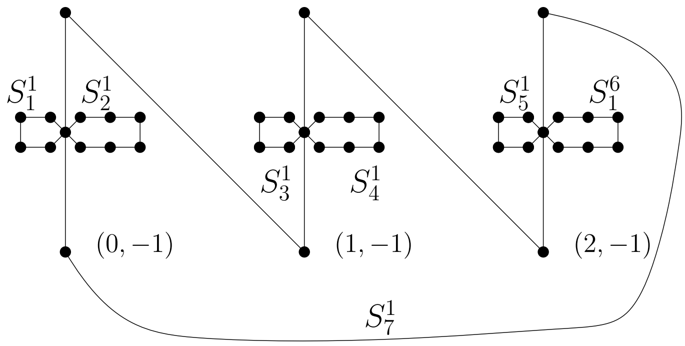

Figure 3.5.3: Undirected Hasse diagram of $\overline{X}_{\mathbb{Z}_3}$.

3.6 Realization problem in Top and $HTop$

Given two groups G and H, we construct a topological space X_H^G such that $Aut(X_H^G) \simeq G$ and $\mathcal{E}(X_H^G) \simeq H$. Before giving the proof of this result we present an example to motivate the main ideas.

Example 3.6.1. Let us consider the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, where we denote $g_1 = (0,0), g_2 = (1,0), g_3 = (0,1)$ and $g_4 = (1,1)$, and the cyclic group of two elements $\mathbb{Z}_2$, where we denote $h_1 = 0$ and $h_2 = 1$. We also denote $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H = \mathbb{Z}_2$ for simplicity. Moreover, we consider $S_G = \{g_2, g_3\}$ and $S_H = \{h_2\}$ as generating sets of G and H respectively. We declare $g_2 < g_3$. The goal is to construct a finite T_0 topological space X_H^G such that $Aut(X_H^G) \simeq G$ and $\mathcal{E}(X_H^G) \simeq H$.

By the results obtained in Section 3.2, there exists a finite T_0 topological space X^G with $Aut(X^G)$ isomorphic to G. In Figure 3.6.1 we represent in blue the Hasse diagram of X^G. It is clear that adding to X_G a minimum $*$, i.e., $X_*^G = X^G \cup \{*\}$ with $* < x$ for every $x \in X_G$, we get that $\mathcal{E}(X_*^G)$ is trivial because X_*^G is contractible. Moreover, if $f \in Aut(X_*^G)$, then we get that $f(*) = *$. From this we obtain that $Aut(X_*^G)$ is isomorphic to $Aut(X^G)$. Now, we want to construct a topological space X_H^* satisfying that $Aut(X_H^*)$ is trivial and $\mathcal{E}(X_H^*)$ is isomorphic to H. By the results obtained in Section 3.4, there exists a finite T_0 topological space X_H with $\mathcal{E}(X_H) \simeq Aut(X_H) \simeq H$. In figure 3.6.1, the Hasse diagram of X_H corresponds to the red and black parts of the diagram on the right. We need to modify X_H to reduce the number of self-homeomorphisms without changing the number of self-homotopy equivalences. Therefore, we add to X_H some points. We consider $X_H^* = X_H \cup \{w_{h_1}, w_{h_2}, a\}$, where we have the following relations: $A_{(h_1,0)} \prec w_{h_1}$ and $A_{(h_2,0)} \prec w_{h_2} \prec a$. The Hasse diagram of X_H^* can be seen on the right in Figure 3.6.1, where we have in orange the new points. It is simple to check that $\mathcal{E}(X_H^*) \simeq \mathcal{E}(X_H) \simeq H$. The new points are all of them beat points, so we can remove them without changing the homotopy type of X_H^*. Hence, X_H^* and X_H have the same homotopy type. In addition, $Aut(X_H^*)$ is trivial. We prove the last assertion. If $f \in Aut(X_H^*)$, then $f(M) = M$ and $f(N) = N$, where M and N denote the set of maximal and minimal elements of X_H^* respectively. From here, using Proposition 1.1.76 and the fact that f preserves heights, it can be deduced that f is the identity.

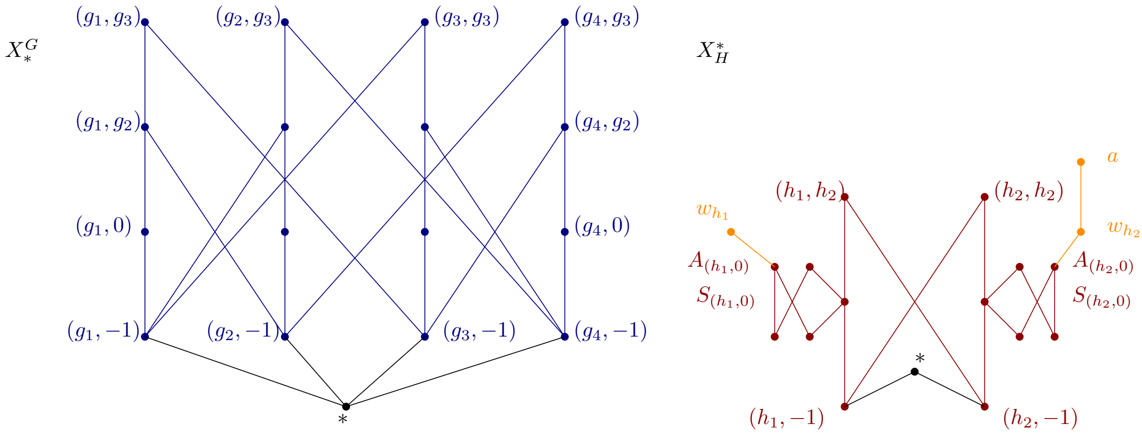

Figure 3.6.1: Hasse diagrams of X_*^G and X_H^*.

Finally, we combine properly the spaces constructed before to get X_H^G. We identify the point $*$ of X_H^* and X_*^G and extend the partial order of the two previous posets using transitivity, that is, if $x \in X_H^*$ and $y \in X_*^G$, then $x < y$ if and only if $x < * < y$. It is not difficult to check that X_H^G satisfies the property required at the beginning. Since we can collapse X_*^G to $*$, it follows that X_H^G and X_H have the same homotopy type. This implies that $\mathcal{E}(X_H^G) \simeq \mathcal{E}(X_H) \simeq H$. It is easily seen that $*$ is a fixed point for every $f \in Aut(X_H^G)$ because $*$ is the only point with height 1 and $P_* = (2,4)$. By the continuity of f, it can be deduced that $f(X^G) = X^G$ and $f(X_H^*) = X_H^*$. This is because $x > *$ for every $x \in X_*^G$. From here, we obtain that $Aut(X_H^G)$ is isomorphic to G.

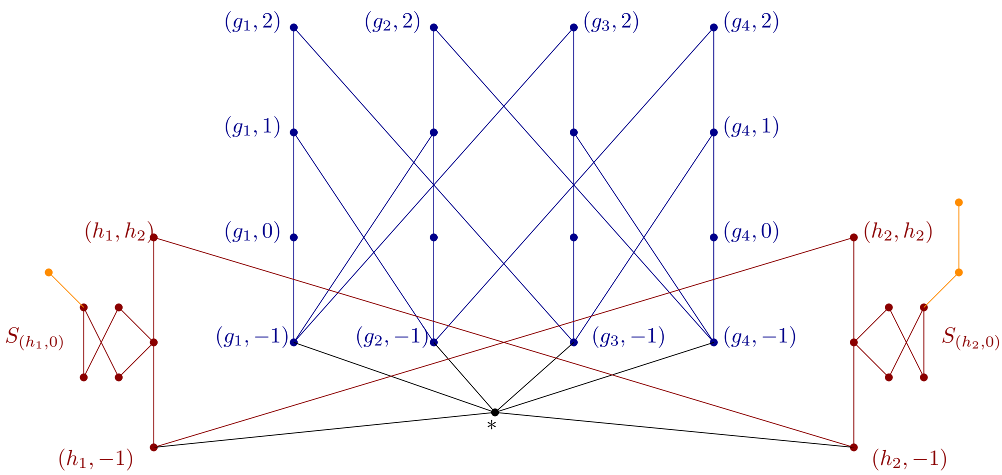

Figure 3.6.2: Hasse diagram of X_H^G.

We can also consider what we call the dual case, i.e., X_G^H, where we have that $Aut(X_G^H) \simeq H$ and $\mathcal{E}(X_G^H) \simeq G$. We argue as we did in the previous case, we construct X_G^* and X_*^H, see Figure 3.6.3.

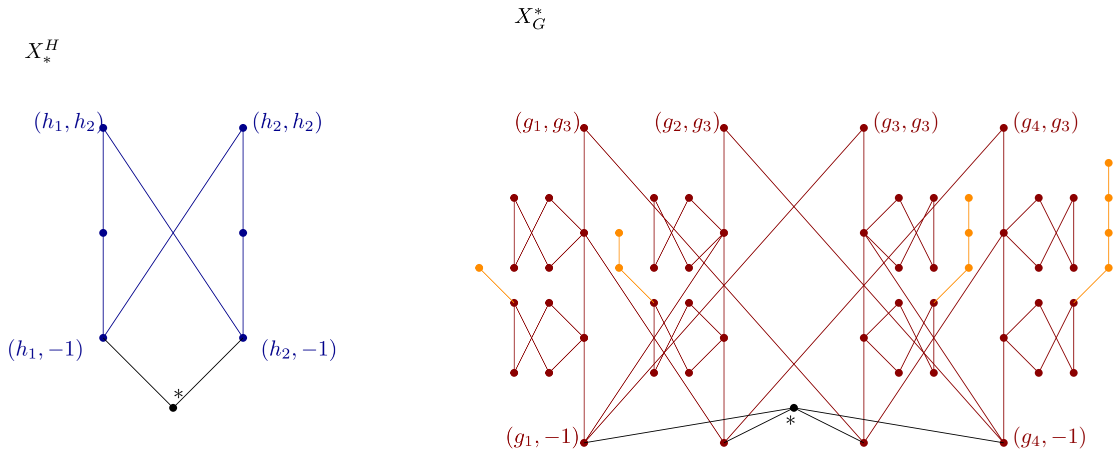

Figure 3.6.3: Hasse diagrams of X_*^H and X_G^*.

Now, we put together the previous constructions to obtain X_G^H, i.e., the finite topological space given by the Hasse diagram of Figure 3.6.4. It is trivial to verify that X_G^H is not homeomorphic to X_H^G because of their different cardinality. Furthermore, X_H^G and X_G^H are not homotopy equivalent. After removing one by one the beat points of X_H^G we get X_H; after removing one by one the beat points of X_G^H we get X_G. We have that X_G is not homeomorphic to X_H because of their different cardinality. By Theorem 1.1.21, we deduce that X_H^G and X_G^H are not homotopy equivalent. Moreover, studying the McCord complexes it can be deduced that X_H^G and X_G^H are not weak homotopy equivalent.

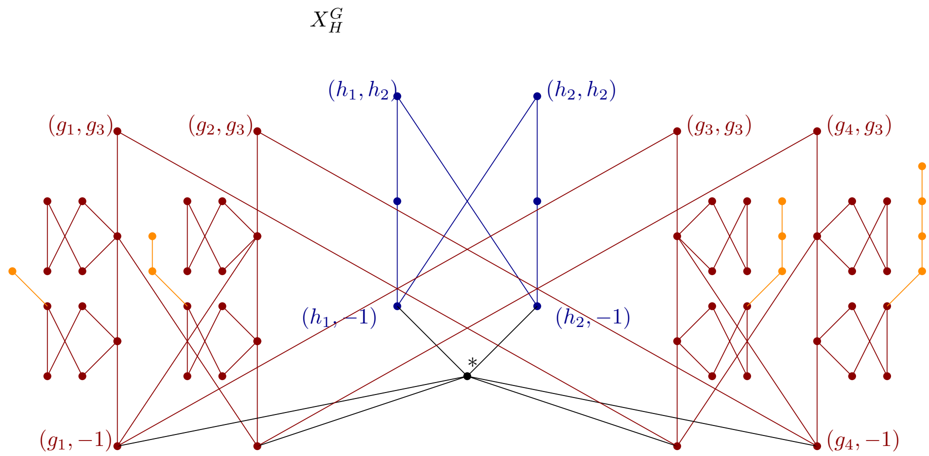

Figure 3.6.4: Hasse diagram of X_G^H.

Theorem 3.6.2. *Let G and H be groups. Then there exists a topological space X_H^G such that $Aut(X_H^G)$ is isomorphic to G and $\mathcal{E}(X_H^G)$ is isomorphic to H.*

Proof. The idea of the proof is to follow the same strategy of Example 3.6.1. Find a topological space X_H^* such that $\mathcal{E}(X_H^*) \simeq H$ and $Aut(X_H^*) = 0$ and a topological space X_*^G such that $Aut(X_*^G) \simeq G$ and $\mathcal{E}(X_*^G) = 0$, where 0 is denoting the trivial group. Then, combine properly both topological spaces to obtain a topological space X_H^G with $Aut(X_H^G) \simeq G$ and $\mathcal{E}(X_H^G) \simeq H$.

Suppose G and H are not trivial groups. The trivial case will be proved later.

Construction of X_*^G and properties. Let S'_G denote a set of non-trivial generators for G. Without loss of generality we can consider a well-order on S'_G such that if $max(S'_G)$ exists, then S'_G is a finite set. We consider $S_G = S'_G \cup \{0, -1\}$, where we assume that $-1, 0 \notin S'_G$, and extend the well-order defined on S'_G to S_G as follows: $-1 < 0 < \alpha$ for every $\alpha \in S'_G$. We consider

$$X_*^G = (G \times S_G) \cup \{*\},$$

where we have the following relations:

- $(g, \alpha) < (g, \delta)$ if $1 \leq \alpha < \delta$, where $g \in G$ and $\alpha, \delta \in S_G$.
- $(g\alpha, -1) \prec (g, \alpha)$, where $g \in G$ and $\alpha \in S_G \setminus \{-1, 0\}$.
- $* \prec (g, -1)$, where $g \in G$.

The remaining relations can be deduced from the above relations using transitivity. It is easy to check that X_*^G is a partially ordered set.

We prove that $Aut(X_*^G) \simeq G$ and $\mathcal{E}(X_*^G)$ is the trivial group. We have that $\mathcal{E}(X_*^G)$ is the trivial group because X_*^G is contractible to $*$, which is a minimum. Since $*$ is a minimum, it follows that every self-homeomorphism must fix this point. From this we deduce that $Aut(X_*^G) \simeq Aut(X_*^G \setminus \{*\})$. In addition, $X_*^G \setminus \{*\}$ is the same topological space considered in Section 3.2 and denoted by X_G. Hence, we know that $Aut(X_*^G) \simeq Aut(X_G) \simeq G$, where $\varphi : G \to Aut(X_G)$ is given by $\varphi(s)(g, \alpha) = (sg, \alpha)$ and is an isomorphism of groups.

Construction of X_H^* and properties. We consider a set of non-trivial generators S'_H for H. There is no loss of generality in assuming that there exists a well-order on S'_H satisfying that if $max(S'_H)$ exists, then S'_H is a finite set. We repeat the same construction made before, that is, we consider $S_H = S'_H \cup \{0, -1\}$, where we assume that $-1, 0 \notin S'_H$, and extend the well-order defined on S'_H to S_H as follows: $-1 < 0 < \beta$ for every $\beta \in S'_H$.

For every $h \in H$ we take a well-ordered non-empty set W_h such that W_h is isomorphic to W_t if and only if $h = t$. For every $h \in H$ let $w_h \in W_h$ denote the first element or minimum of W_h. We consider

$$X_H^* = (H \times S_H) \cup \Big(\bigcup_{\substack{(h,\beta) \in G \times S_H \\ 0 \leq \beta < max(S_H)}} (S_{(h,\beta)} \cup T_{(h,\beta)}) \cup \big(\bigcup_{h \in H} W_h\big)\Big) \cup \{*\},$$

where

$$S_{(h,\beta)} = \{A_{(h,\beta)}, B_{(h,\beta)}, C_{(h,\beta)}, D_{(h,\beta)}\}, T_{(h,\beta)} = \{E_{(h,\beta)}, F_{(h,\beta)}, G_{(h,\beta)}, H_{(h,\beta)}, I_{(h,\beta)}, J_{(h,\beta)}\},$$

and we have the following relations:

1. $(h, \beta) < (h, \gamma)$ if $-1 \leq \alpha < \gamma$, where $h \in H$ and $\beta, \gamma \in S_H$.
2. $(h\beta, -1) \prec (h, \beta)$, where $h \in H$ and $\beta \in S_H \setminus \{-1, 0\}$.
3. $A_{(h,\beta)} \succ C_{(h,\beta)}, D_{(h,\beta)}$; $B_{(h,\beta)} \succ (h, \beta), C_{(h,\beta)}$ and $(h, \beta) \succ D_{(h,\beta)}$, where $h \in H$ and $\beta \in S_H \setminus \{-1\}$.
4. $E_{(h,\beta)} \succ (h, \beta), I_{(h,\beta)}$; $F_{(h,\beta)} \succ H_{(h,\beta)}, J_{(h,\beta)}$; $G_{(h,\beta)} \succ I_{(h,\beta)}, J_{(h,\beta)}$ and $(h, \beta) \succ H_{(h,\beta)}$, where $h \in H$ and $\beta \in S_H \setminus \{-1\}$.
5. $* \succ (h, -1)$, where $h \in H$.

6. We extend the partial order defined on W_h to X_H declaring that $A_{(h,0)} \prec w_h$, where $h \in H$.

The remaining relations can be deduced from the above using transitivity. It is routine to verify that X^*_H with the previous relations is a partially ordered set.

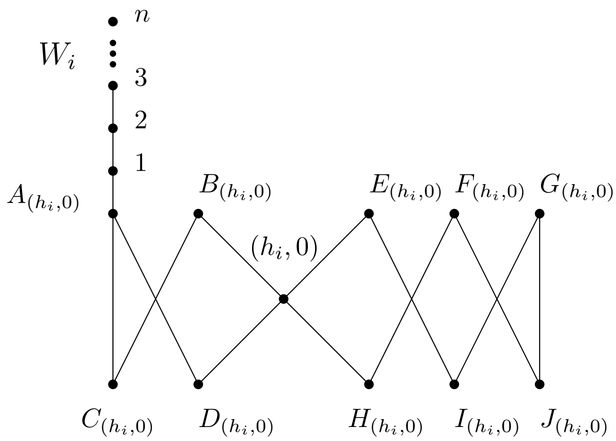

Figure 3.6.5: Hasse diagram of $S_{(h,0)} \cup T_{(h,0)} \cup W_h$, where W_h is a finite well-ordered set.

We proceed to show that $Aut(X^*_H)$ is the trivial group and $\mathcal{E}(X^*_H) \simeq H$. It is clear that X^*_H and $X^*_H \setminus \{W_h | h \in H\}$ have the same homotopy type. We define $r : X^*_H \to X^*_H \setminus \{W_h | h \in H\}$ given by

$$r(x) = \begin{cases} A_{(h,0)} & x \in W_h \\ x & x \in X^*_H \setminus \{W_h | h \in H\}. \end{cases}$$

It is trivial to show that r is continuous and satisfies that $r(x) \leq id(x)$ for every $x \in X^*_H$, where $id : X^*_H \to X^*_H$ denotes the identity map. This implies that $X^*_H \setminus \{W_h | h \in H\}$ is a strong deformation retract of X^*_H. On the other hand, $X^*_H \setminus \{W_h | h \in H\}$ is the same topological space considered in Section 3.3 and denoted by $\overline{X}^*_H$. Therefore we know that $\mathcal{E}(X^*_H) \simeq \mathcal{E}(\overline{X}^*_H) \simeq Aut(\overline{X}^*_H) \simeq H$, where $\phi : H \to Aut(\overline{X}^*_H)$ given by $\phi(t)(h, \beta) = (th, \beta)$ and $\phi(t)(S_{(h,\beta)} \cup T_{(h,\beta)}) = S_{(th,\beta)} \cup T_{(th,\beta)}$ is an isomorphism of groups.

The task is now to prove that $Aut(X^*_H)$ is the trivial group. Let us take $f \in Aut(X^*_H)$. We consider $A_{(h,0)}$ for some $h \in H$. Since $F_{A_{(h,0)}} \setminus \{A_{(h,0)}\}$ has a minimum w_h, it follows that $A_{(h,0)}$ is an up beat point. By Proposition 1.1.76, we know that $f(A_{(h_i,0)})$ is also an up beat point. Therefore, $f(A_{(h_i,0)})$ is of the form $A_{(t,0)}$ for some $t \in H$. By Proposition 1.1.76 we get that $f(w_h) = w_t$. It follows from the continuity of f that $f(W_h) \subseteq W_t$. Since f is a homeomorphism, we have that $f_{|W_h}$ is also a homeomorphism. Therefore, we get that $h = t$; otherwise we would get a contradiction since W_h is homeomorphic to W_t if and only if $h = t$. Using Proposition 1.1.76 it is easy to verify that f fixes $S_{(h,0)}$ for every $h \in H$. From Remark 3.2.7 we get that if a homeomorphism $g : X^*_H \setminus \{W_h | h \in H\} \to X^*_H \setminus \{W_h | h \in H\}$ coincides at one point with the identity map, then g is the identity map. Thus, f is the identity map and $Aut(X^*_H)$ is the trivial group.

Construction of X^G_H. We consider $X^G_H = X^*_H \cup X^G_*$, where we are identifying the point $*$ of both topological spaces, i.e., the partial order of X^G_H preserves the relations defined on X^*_H and X^G_*:

- If $x \in X^*_H$ and $y \in X^G_*$, then x is smaller than y if and only if $x \leq *$ and $* \leq y$.

- If $x, y \in X_H^*$, then x is smaller (greater) than y if and only if x is smaller (greater) than y with the partial order defined on X_H^*.
- If $x, y \in X_*^G$, then x is smaller (greater) than y if and only if x is smaller (greater) than y with the partial order defined on X_*^G.

It is evident that $\mathcal{E}(X_H^G)$ is isomorphic to H because X_*^G is contractible to $*$ and X_H^* is homotopy equivalent to $\overline{X}_H^*$. It suffices to show that $Aut(X_H^G)$ is isomorphic to G. We verify that every $f \in Aut(X_H^G)$ satisfies that $f(x) \in X_H^*$ for every $x \in X_H^*$ and $f(x) \in X_*^G$ for every $x \in X_*^G$. Firstly, we show that $*$ is a fixed point for every homeomorphism f. We have $ht(*) = 1$. Since for every $x \in X_*^G \setminus \{*\}$ the height of x is at least 2 or different from 1, it follows that $f(*) \notin X_*^G \setminus \{*\} \subset X_H^G$. The only elements of X_H^* that have height one are of the form $*$ or $(h, 0)$ or $A_{(s,\alpha)}$, $F_{(s,\alpha)}$, $E_{(s,\alpha)}$ for some $h, s \in H$ and $\alpha \in S_H \setminus \{-1\}$. We can discard the maximal elements, otherwise f^{-1} would send a maximal element to a non-maximal element. If $f(*) = A_{(h,0)}$, then we get a contradiction since $A_{(h,0)}$ is an up beat point. If $f(*) = (h, 0)$ for some $h \in H$, then we get that $f^{-1}(E_{(h,0)}) \succ f^{-1}(h, 0) = *$ by Proposition 1.1.76. We have that $f^{-1}(E_{(h,0)}) \neq (g, -1)$ for every $g \in G$ because $E_{(h,0)}$ is not a down beat point. Hence, the only possibility is $f(*) = *$. Finally, by the continuity of f, we get that $f(x) \in X_*^G \subset X_H^G$ for every $x \in X_*^G \subset X_H^G$. This implies that $Aut(X_H^G)$ is isomorphic to G.

We prove the remaining case. If G is the trivial group, then it suffices to consider X_H^* to conclude. If H is the trivial group, then X_*^G satisfies the desired properties. □

Remark 3.6.3. *If $f, k \in Aut(X_H^G)$ are such that there exists $x \in X_*^G \setminus \{*\}$ satisfying $f(x) = k(x)$, then $f = k$. This is an immediate consequence of the isomorphism of groups φ given in the proof of Theorem 3.6.2. Similarly, if $[f], [k] \in \mathcal{E}(X_H^G)$ are such that there exists $x \in X_H^G \setminus (\{X_G\} \cup \{W_h | h \in H\})$ satisfying that $f(x) = k(x)$, where $f \in [f]$ and $k \in [k]$, then $[f] = [k]$. It is an immediate consequence of the construction of X_H^G and ϕ.*

Proposition 3.6.4. *Let G and H be groups. The Alexandroff space X_H^G constructed in the proof of Theorem 3.6.2 has the weak homotopy type of the wedge sum of $3|H||S_H|$ circles when H is a finite group and the wedge sum of $|\mathbb{N}|$ circles when H is a non-finite countable set.*

Proof. We have that X_H^G has the same homotopy type of X_H^*. Therefore, repeating same arguments used in Section 3.5, we can obtain the desired result. □

Remark 3.6.5. *If G and H are finite groups, then we can remove $\{T_{(h,\beta)} | h \in H, \beta \in S_H \setminus \{-1, 0\}\}$ in the construction of X_H^G and Theorem 3.6.2 also holds true. Thus, we have a finite topological space with $|G|(|S_G|+2) + |H|(|S_H|+2) + 4|S_H||H| + \frac{|H|(|H|+1)}{2} + 1$ points. The first term corresponds to $X_*^G \setminus \{*\}$, the second term corresponds to $X_H^* \setminus \{*\}$, the third term corresponds to the sets $S_{(h,\beta)}$, the fourth term corresponds to the points of the sets W_h and the last term corresponds to the point $*$.*

We can change the sets $T_{(h,\beta)}$ from the proof of Theorem 3.6.2 by $T^n_{(h,\beta)}$ as it was done in Section 3.5, where $T^n_{(h,\beta)} := \{x_1, x_2, ...x_{n+3}, y_1, y_2, ..., y_{n+3}\}$ with the following relations:

$$(h, \beta) < x_1 > y_2 < ... < x_{n+2} > y_{n+3} < x_{n+3} > y_{n+2} < x_{n+1} < ... < x_2 > y_1 < (h, \beta), \tag{3.3}$$

$$(h, \beta) < x_1 > y_2 < ... > y_{n+2} < x_{n+3} > y_{n+3} < x_{n+2} > y_{n+1} < ... < x_2 > y_1 < (h, \beta). \tag{3.4}$$

We consider the first relation for n odd and the second relation for n even. We denote this poset by X^G_{Hn}.

Corollary 3.6.6. *Given two finite groups G and H, there are infinitely many (non-homotopy-equivalent) topological spaces $\{X^G_{Hn}\}_{n\in\mathbb{N}}$ such that $Aut(X^G_{Hn})$ is isomorphic to G and $\mathcal{E}(X^G_{Hn})$ is isomorphic to H for every $n \in \mathbb{N}$.*

Proof. The proof is analogous to the proof of Theorem 3.6.2. By Theorem 1.1.21, we have that the topological spaces are not homotopy equivalent due to their different cardinality after removing all the beat points one by one. □

3.7 Realizing homomorphisms of groups

Given a homomorphism of groups $f : G \to H$, we modify slightly the topological space X^*_H constructed in the proof of Theorem 3.6.2. The new topological space obtained X_f is similar to the one considered in Theorem 3.6.2, but new relations are added in order to control the homomorphisms of groups $\tau : Aut(X_f) \to \mathcal{E}(X_f)$ given by $\tau(f) = [f]$, as we wanted.

Example 3.7.1. Let us consider the cyclic group of two elements $\mathbb{Z}_2$ and the group of integer numbers $\mathbb{Z}$. We consider the homomorphism of groups $f : \mathbb{Z} \to \mathbb{Z}_2$ given by $f(n) = n \ mod\ 2$. We consider the topological space $X^{\mathbb{Z}}_{\mathbb{Z}_2}$ constructed in the proof of Theorem 3.6.2 and remove W_0 and W_1 from it. The resulting poset X_f corresponds to the Hasse diagram in black in Figure 3.7.1.

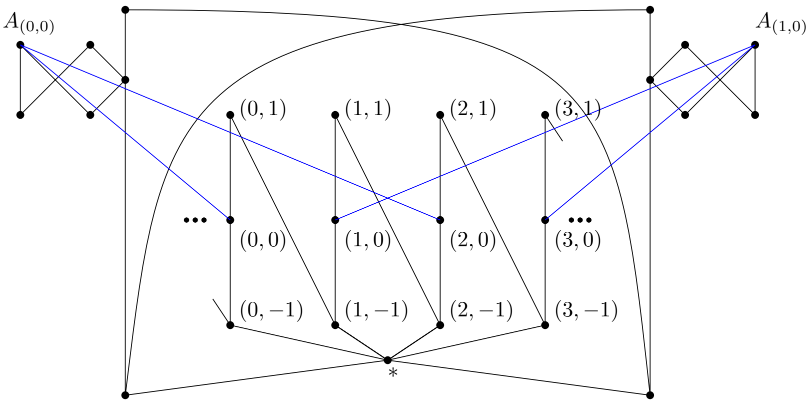

Figure 3.7.1: Hasse diagram of X_f.

Now, we add the following relations to X_f: $(n, 0) \prec A_{(1,0)}$ if $f(n) = 1$ and $(n, 0) \prec A_{(0,0)}$ if $f(n) = 0$, where $n \in \mathbb{Z}$. These relations are represented in blue. It is easy to check that X_f satisfies that $Aut(X_f) = G$, $\mathcal{E}(X_f) = H$ and $f = \tau$, where $\tau : Aut(X_f) \to \mathcal{E}(X_f)$ is given by $\tau(f) = [f]$.

Example 3.7.2. Let $f : \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be the homomorphism of groups given by $f(0) = (0, 0)$ and $f(1) = (1, 0)$. In Figure 3.7.2 we present the Hasse diagram of X_f, where we keep the same notation introduced in Example 3.6.1. Again, we have in black the Hasse diagram of $X^{\mathbb{Z}_2}_{\mathbb{Z}_2\oplus\mathbb{Z}_2} \setminus \{W_{g_1}, W_{g_2}, W_{g_3}, W_{g_4}\}$. We add to the previous poset the following relations: $(h_i, 0) \prec A_{(f(h_i),0)}$ with $i = 1, 2$ and $* \prec A_{(g_j,0)}$ with $j = 3, 4$. These relations are represented in blue. It is easy to get that $Aut(X_f) = \mathbb{Z}_2$, $\mathcal{E}(X_f) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\tau = f$.

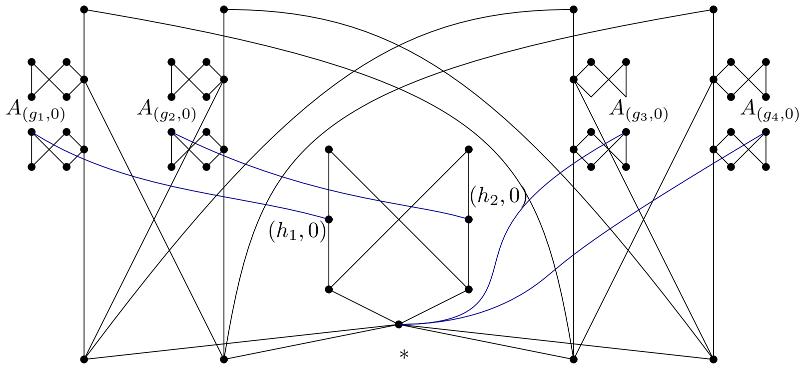

Figure 3.7.2: Hasse diagram of X_f.

Theorem 3.7.3. *Let $f : G \to H$ be a homomorphism of groups. Then there exists a topological space X_f such that $Aut(X_f) = G$, $\mathcal{E}(X_f) = H$ and $\tau = f$.*

Proof. Suppose G and H are not trivial groups. If G is the trivial group or H is the trivial group, the result follows from Theorem 3.6.2. We consider the topological space X_H^G constructed in the proof of Theorem 3.6.2 and we define $X_f = X_H^G \setminus \{W_h | h \in H\}$. We keep the same relations defined on X_f as a subspace of X_H^G and we add the following relations.

- $A_{(h,0)} \succ (g, 0)$ if $f(g) = h$, where $h \in H$ and $g \in G$.
- $A_{(h,0)} \succ *$ if $h \notin f(G)$.

It is easy to check that X_f is a partially ordered set with the above relations. Firstly, we prove that $\mathcal{E}(X_f) \simeq H$. We consider $r : X_f \to X_f$ given by

$$r(x) = \begin{cases} * & x \in X_*^G \\ x & x \in X_f \setminus \{X_*^G\}. \end{cases}$$

We have that r preserves the order, so it is a continuous map. It is simple to check that $r(x) \leq id(x)$ for every $x \in X_f$, where id denotes the identity map. From here, we can deduce that X_f is homotopy equivalent to to $r(X_f) = X_H^* \subset X_f$. On the other hand, repeating the same arguments used in previous sections, it can be proved that X_H^* is locally a core [67] or a minimal finite space for the case H is a finite group, which implies that $\mathcal{E}(X_f) \simeq \mathcal{E}(X_H^*) = Aut(X_H^*)$. Since $P_* = (|H|, |H|)$ and $ht(*) = 1$, we get that every homeomorphism $T : X_H^* \to X_H^*$ must fix $*$ and then the group of homeomorphisms of X_H^* as a subspace of X_f is isomorphic to the group of homeomorphisms of X_H^* as the topological space considered in the proof of Theorem 3.6.2. Thus, $\mathcal{E}(X_f) \simeq H$.

We show that $Aut(X_f) \simeq G$. We consider the following auxiliary sets $Col_h = \{(h, \beta) | \beta \in S_H\} \cup \{S_{(h,\beta)} \cup T_{(h,\beta)} | \beta \in S_H \setminus \{-1, max(S_H)\}\}$, where $h \in H$, and $Col^g = \{(g, \alpha) | \alpha \in S_G\}$, where $g \in G$. If $x \in X_*^G \subset X_f$, then every homeomorphism $T : X_f \to X_f$ satisfies that $T(x) \in X_*^G$. We prove the last assertion. We know that $X_f \setminus \{X_*^G\}$ does not contain beat points. Moreover, $(g, -1)$ is a beat point of height 2 for every $g \in G$. Using Proposition 1.1.76 and continuity, we can deduce that for every $T \in Aut(X_f)$ and (g, α), where $g \in G$ and $\alpha \in S_G$, we get $T(g, \alpha) = (g', \alpha)$ for some $g' \in G$.

We consider $\varphi : G \to Aut(X_f)$ given by $\varphi(g)(g', \alpha) = (gg', \alpha)$ if $g' \in G$ and $\alpha \in S_G$, $\varphi(g)(h, \beta) = (f(g)h, \beta)$ if $h \in H$ and $\beta \in S_H$, $\varphi(g)(S_{(h,\beta)} \cup T_{(h,\beta)}) = S_{(f(g)h,\beta)} \cup T_{(f(g)h,\beta)}$ defined in the natural way if $h \in H$ and $\beta \in S_H \setminus \{-1, max(S_H)\}$ and $\varphi(g)(*) = *$. We prove that φ is well-defined. We check the continuity of $\varphi(g)$, where $g \in G$. Suppose $(g', 0) \prec A_{(h,0)}$ for some $g' \in G$ and $h \in H$. By hypothesis, $f(g') = h$. Then,

$$\varphi(g)(g', 0) = (gg', 0) \prec A_{(f(gg'),0)} = A_{(f(g)f(g'),0)} = A_{(f(g)h,0)} = \varphi(g)A_{(h,0)}.$$

It is simple to check that $\varphi(g)$ preserves the rest of the relations. The inverse of $\varphi(g)$ is given by $\varphi(g^{-1})$. Hence, φ is well-defined. By construction, φ is a monomorphism of groups. Suppose $T \in Aut(X_f)$. By Proposition 1.1.76, Remark 3.6.3 and the fact that $T_{|X_*^G} \in Aut(X_*^G)$, it can be deduced that every $T \in Aut(X_f)$ satisfies that $T(Col_h) = Col_{h'}$ and $T(Col^g) = Col_{g'}$ for some $g' \in G$ and $h' \in H$, where $g \in G$ and $h \in H$. We consider $(g, 0) \prec A_{(h,0)}$ for some $g \in G$ and $h \in H$. We get $T(Col^g) = Col^{g'}$ for some $g' \in G$ and $T(Col_h) = Col_{h'}$ for some $h' \in H$. Since $T_{|X_G^*} \in Aut(X_*^G)$, by Remark 3.6.3 and the proof of Theorem 3.6.2, there exists $t \in G$ such that $T(Col^s) = Col^{ts}$, where $s \in G$. Hence, $g' = tg$. By Proposition 1.1.76, $T(A_{(h,0)}) = A_{(h',0)} \succ (tg, 0) = T(g, 0)$, we have $h' = f(tg) = f(t)f(g) = f(t)h$. Thus, $T = \varphi(t)$ because of Remark 3.6.3 and the fact that $T_{|X_f \setminus \{X_*^G\}} \in Aut(X_H^* \setminus \{*\})$. By construction, $\tau(g) = f(g)$ for every $g \in G$. Since every $T \in Aut(X_f)$ can be seen as $T = \varphi(g)$ for some g, we have that $\tau(T) = f(g)$, where $f(g) = \varphi(g)_{|X_H^*} \in \mathcal{E}(X_f)$. $\square$

Remark 3.7.4. *Theorem 3.7.3 generalizes Theorem 3.6.2 and the results of realization obtained in previous sections. In fact, given two groups G and H and using Theorem 3.7.3, we obtain a family of topological spaces $\{X_f\}_{f:G\to H}$ satisfying that $Aut(X_f) \simeq G$ and $\mathcal{E}(X_f) \simeq H$. Hence, if we consider the trivial map $c : G \to H$ that sends each element of G to the identity element of H, we get another proof of Theorem 3.6.2*

Example 3.7.5. Let us consider the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the cyclic group of two elements $\mathbb{Z}_2$. We keep the same notation and generating sets considered in Example 3.6.1. We consider $c : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ given by $c(g_i) = h_1$ for every $i \in \{1, 2, 3, 4\}$. In Figure 3.7.3, we have the Hasse diagram of X_c.

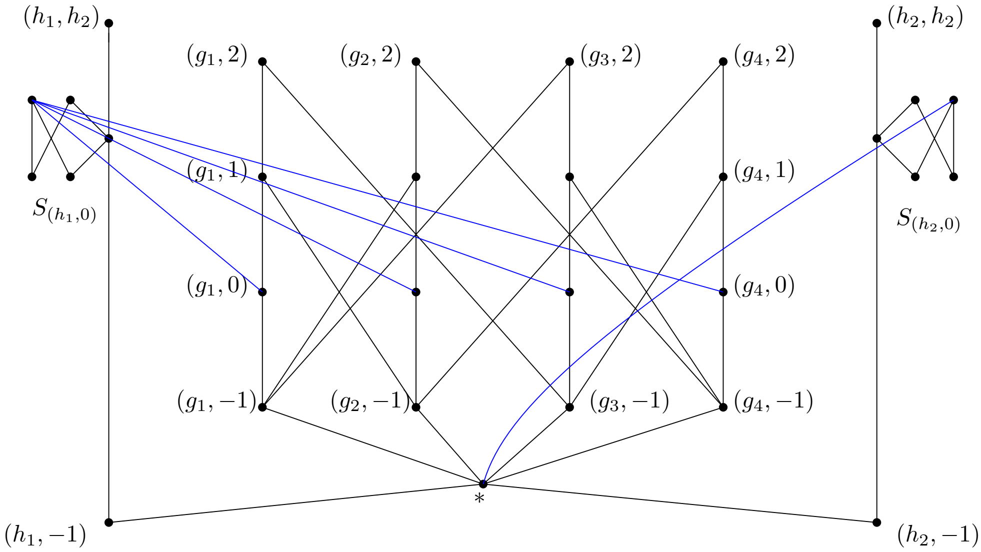

Figure 3.7.3: Hasse diagram of X_c.

With the following propositions we explore the relations between the homomorphisms from G to H and the topological spaces associated.

Proposition 3.7.6. *Let G and H be groups. If $g, f : G \to H$ are homomorphisms of groups, then X_f is homotopy equivalent to X_g.*

Proof. The result is an immediate consequence of the construction. X_f is homotopy equivalent to X_H^* for every homomorphism of groups $f : G \to H$. Therefore, the homotopy type of the topological space obtained in the proof of Theorem 3.7.3 does not depend on the homomorphism. From here, we deduce the desired result. $\square$

Proposition 3.7.7. *Let G and H be groups and let $f, g : G \to H$ be homomorphisms of groups. Then $f = g$ if and only if X_f is homeomorphic to X_g.*

Proof. One of the implications is trivial. Hence, we only need to show that if X_f is homeomorphic to X_g, then $f = g$. Since X_f is homeomorphic to X_g, there exists a homeomorphism $T' : X_f \to X_g$. By the construction of X_f and X_g in the proof of Theorem 3.7.3, it can be deduced easily that $T'_{|X_*^G} \in Aut(X_*^G) \simeq G$ and $T'_{|X_H^*} \in Aut(X_H^*) \simeq H$. This is due to the fact that X_*^G contains beat points while X_H^* does not have beat points. Therefore, $T'_{|X_*^G}$ can be related with the action of an element $T \in G$ and $T'_{|X_H^*}$ can be related with the action of an element $\overline{T} \in H$. We have $(e, 1) \prec A_{(f(e),0)}$, where e denotes the identity element in G, we also have

$$T'(e, 1) = (T, 1) \prec A_{(\overline{T}f(e),0)} = T'(A_{(f(e),0)}),$$

which implies that $g(T) = \overline{T}f(e)$. Thus, $g(T) = \overline{T}$ because f is a homomorphism of groups. In addition, for every $h \in G$, we know that there exists a relation in X_f of the following form $(h, 1) \prec A_{(f(h),0)}$. We have

$$T'(h, 1) = (Th, 1) \prec A_{(\overline{T}f(h),0)} = T'(A_{(f(h),0)}).$$

By the construction of X_g, we get $g(Th) = g(T)g(h) = \overline{T}f(h)$. Previously, we proved that $g(T) = \overline{T}$, which implies that $g(h) = f(h)$ for every $h \in G$. $\square$

3.8 An asymmetric family of topological spaces

We construct a family of topological spaces $\{W_n\}_{n\in\mathbb{N}}$ satisfying that for each member the following groups are trivial: the group of homeomorphisms, the group of self-homotopy equivalences, all the homotopy and homology groups. With the following construction we show that the groups mentioned above do not determine neither the homotopy type nor the topological type of a topological space X in general. On the other hand, if the topological space X satisfies some properties, e.g., X is compact and a locally Euclidean manifold with or without boundary, then the group of homeomorphisms determines the topological type of X, see [113] or [94] for more information.

Construction of W_n. Let us consider the Alexandroff space W_2 given by the Hasse diagram of Figure 3.8.1. It is the union of $L_1 = \{x_i\}_{i=1,\ldots,9}$ and $L_2 = \{x_j\}_{j=9,\ldots 17}$, where we are identifying the point x_9 of L_i for $i = 1, 2$. For simplicity, W_1 denotes L_1. The topological space W_1 was introduced in [93, Figure 2] and has the weak homotopy type of a point. In [30], it is proved that W_1 is the smallest finite topological space having the same weak homotopy type of a point but not contractible. It is clear that W_2 does not have beat points, so $Aut(W_2)$ is isomorphic to $\mathcal{E}(W_2)$. On the other hand, it is easy to show that $Aut(W_2)$ is the trivial group. Since $P_x = (3, 5)$ only for $x \in \{x_5, x_{13}\}$ and the heights of these points are different, we can deduce that every homeomorphism must fix

x_5 and x_{13}. Using Proposition 1.1.76, it can be deduced the desired result. Furthermore, the homotopy and singular homology groups of W_2 are trivial. We can study the weak homotopy type of W_2 studying the McCord complex $\mathcal{K}(W_2)$ or removing beat and weak beat points. We know that x_{16} is a weak beat point. After removing this point, x_{12} and x_{14} are up beat points. If we remove them, then the remaining space is homotopy equivalent to the space given by the points $\{x_i\}_{i=1,\dots,9}$. We can repeat the same process. We have that x_8 is a weak beat point. After removing this point, x_7 and x_9 are down beat points. Thus, X has the same weak homotopy type of a point. We get that W_2 is a topological space satisfying that $Aut(W_2) \simeq \mathcal{E}(W_2) \simeq \pi_n(W_2) \simeq H_n(W_2) \simeq 0$ for every $n > 0$ but it does not have neither the topological type nor the homotopy type of a point. We can generalize this topological space taking more copies of the topological space introduced in [93]. For instance, we can define W_3 just as $L_3 \cup W_2$, where $L_3 = \{x_i\}_{i=17,\dots,25}$ and we are identifying the point x_{17} of L_3 and W_2. It is easy to check that W_n has the weak homotopy type of a point for every $n \in \mathbb{N}$ because $x_i \in W_n$ with $i \equiv 0 \pmod 8$ is a weak beat point. Again, it is easy to show that the group of homeomorphisms and the group of self-homotopy equivalences of W_3 are trivial.

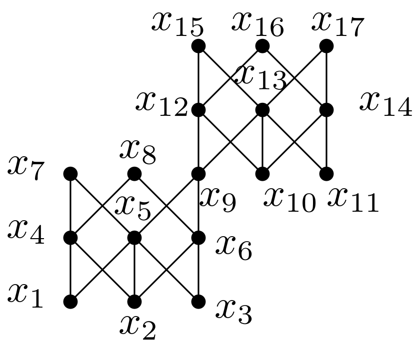

Figure 3.8.1: Hasse diagram of W_2.

One possible consequence of the previous construction is the following result.

Proposition 3.8.1. *Let G and H be finite groups. Then there exists a topological space X such that $Aut(X)$ is isomorphic to G, $\mathcal{E}(X)$ is isomorphic to H and X is weak homotopy equivalent to a point.*

Proof. We consider the topological space X_H^G constructed in the proof of Theorem 3.6.2 and the finite topological space W_2 constructed before. Let X denote $X_H^G \circledast W_2$. By Proposition 1.1.65 and the Proof of Theorem 3.6.2, $Aut(X) \simeq Aut(X_H^G) \times Aut(W_2) \simeq Aut(X_H^G) \simeq G$. We have that X is homotopy equivalent to $X_H^* \circledast W_2$, we only need to remove one by one the beat points. Since $X_H^* \circledast W_2$ does not contain beat points we get that $\mathcal{E}(X_H^* \circledast W_2) \simeq Aut(X_H^* \circledast W_2) \simeq Aut(X_H^*) \simeq H$. In addition, since W_2 is collapsible, we can deduce that X has the weak homotopy type of a point, see Remark 1.1.64. $\square$

Example 3.8.2. Let us consider the Klein four-group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and the cyclic group of two elements $\mathbb{Z}_2$, where we keep the same notation and generating sets considered in Example 3.6.1. Let $X_{\mathbb{Z}_2}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ denote also the same topological space considered in Example 3.6.1. In Figure 3.8.2 we present the Hasse diagram of $X_{\mathbb{Z}_2}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \circledast W_2$, which is a topological space having the same weak homotopy type of a point, $Aut(X_{\mathbb{Z}_2}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \circledast W_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathcal{E}(X_{\mathbb{Z}_2}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} \circledast W_2) = \mathbb{Z}_2$.

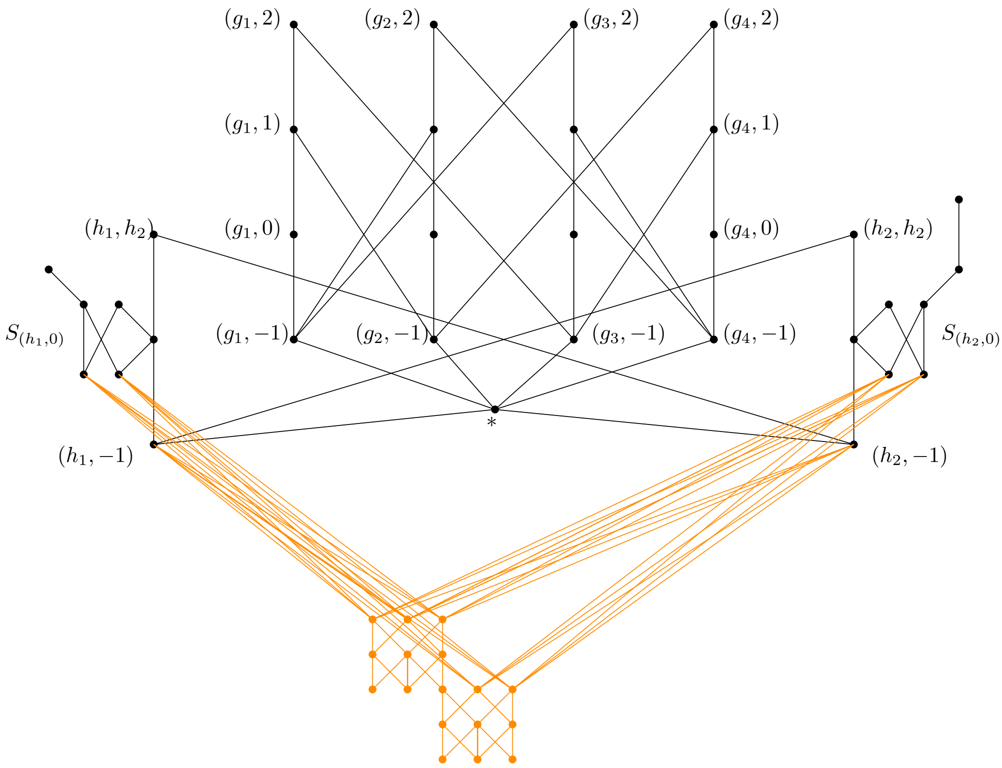

Figure 3.8.2: Hasse diagram of $X_{\mathbb{Z}_2}^{\mathbb{Z}_2\oplus\mathbb{Z}_2} \circledast W_2$.

3.9 General realization problems

Firstly, we introduce an example to illustrates the techniques that will be used to get Theorem 3.9.2.

Example 3.9.1. Let us consider $G = \mathbb{Z}_3$ and $H = \mathbb{Z}_2$. We consider the minimal finite model of the 2-dimensional sphere X, that is, $X = \{A, B, C, D, E, F\}$, where $A, B > C, D, E, F$ and $C, D > E, F$. Then, $|\mathcal{K}(X)|$ is homeomorphic to S^2.

We want to find a finite T_0 topological space $\overline{X}_H^G$ that is weak homotopy equivalent to a 2-dimensional sphere, $Aut(\overline{X}_H^G)$ is isomorphic to $\mathbb{Z}_3$ and $\mathcal{E}(\overline{X}_H^G)$ is isomorphic to $\mathbb{Z}_2$. The idea is to modify X in order to get a new space satisfying that its group of homeomorphisms is trivial. We enumerate the points of X. For each $i \in X$ with $i = 1, ..., |X|$, we add W_i to X, see Section 3.8. The Hasse diagram of the new topological space, that is denoted by X', can be seen in Figure 3.9.1. In black, we have the Hasse diagram of X, in blue and purple the new part added. In purple, we have the weak beat points that are not beat points. It is clear that X' does not have beat points, so $Aut(X')$ is isomorphic to $\mathcal{E}(X')$. A homeomorphism f sends weak beat points to weak beat points by Proposition 1.1.76. From here, it is easy to deduce that $Aut(X')$ is the trivial group. On the other hand, the new structure added can be removed without changing the weak homotopy type of the space. Then, we have that X' is weak homotopy equivalent to a X, which is weak homotopy equivalent to a 2-dimensional sphere.

Finally, we add a new point t that connects X' to $X_H^G \circledast W_2$, where X_H^G is the space constructed in the proof of Theorem 3.6.2 for finite groups. In Figure 3.9.1, we have the Hasse diagram of the new topological space $\overline{X}_G^H$. In green, it can be observed the relations with the point t. In red and orange, it can be seen the Hasse diagram of $X_H^G \circledast W_2$. It is easy to check that $\overline{X}^G_H$ satisfies the desired properties.

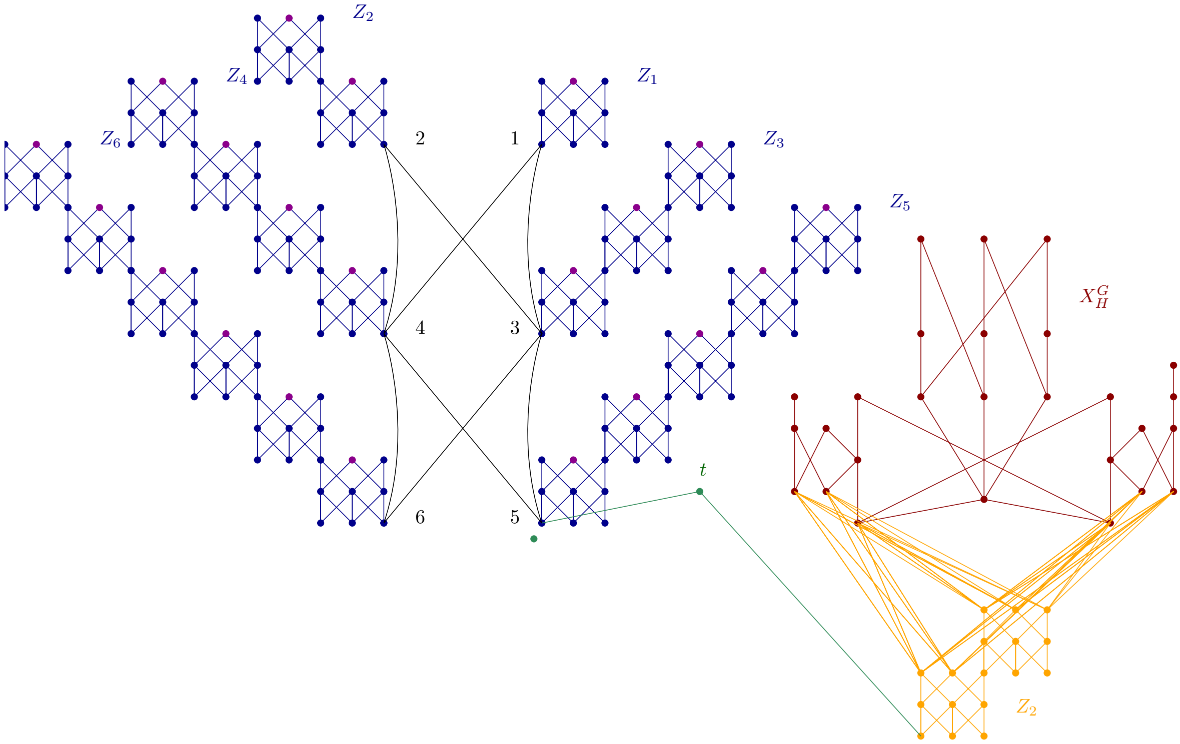

Figure 3.9.1: Hasse diagram of $\overline{X}^G_H$

Theorem 3.9.2. *Let G be a group, let H be a finite group and let X be a topological space with the homotopy type of a compact CW-complex. Then there exists an Alexandroff space $\overline{X}^G_H$ such that $Aut(\overline{X}^G_H)$ is isomorphic to G, $\mathcal{E}(\overline{X}^G_H)$ is isomorphic to H and it is weak homotopy equivalent to X, which implies that $H_n(\overline{X}^G_H)$ is isomorphic to $H_n(X)$ and $\pi_n(\overline{X}^G_H)$ is isomorphic to $\pi_n(X)$ for every $n \in \mathbb{N}$.*

Proof. If X has the same homotopy type of a point, then the result can be deduced from Proposition 3.8.1. Therefore, we assume that X does not have the same homotopy type of a point. The idea of the proof is to follow the same techniques used in the proof of Theorem 3.6.2. By the simplicial approximation to CW-complexes, [60, Theorem 2C.5.], we get that there exists a finite simplicial complex, that will be also denoted by X, homotopy equivalent to X. We apply the McCord functor $\mathcal{X}$ to X in order to obtain a finite T_0 topological space $\mathcal{X}(X)$ such that $\mathcal{X}(X)$ is weak homotopy equivalent to X. We can suppose that $\mathcal{X}(X)$ does not have beat points or weak beat points, otherwise we remove them one by one until there are none left. We denote $n = |\mathcal{X}(X)|$ and label the points in $\mathcal{X}(X)$, that is, $\mathcal{X}(X) = \{y_i\}_{i=1...n}$. For each $y_i \in \mathcal{X}(X)$ we consider W_i, where W_i is the topological space constructed in Section 3.8. We consider $Z = \mathcal{X}(X) \cup \bigcup_{i=1,...,n} W_i$, where we are identifying the point y_i with $x_1 \in W_i$ for every $i = 1, ..., n$. We define the partial order on Z extending the previous partial orders. To do that we use transitivity, i.e., suppose $x, y \in Z$, we write $x \geq y$ if and only if one of the following conditions is satisfied:

- $x, y \in \mathcal{X}(X)$ and x is greater than y with the partial order defined on $\mathcal{X}(X)$.
- $x, y \in W_i$ for some i and x is greater than y with the partial order defined on W_i.

- $y \in \mathcal{X}(X)$, $x \in W_i$ for some i and $x \geq y_i (= x_1) \geq y$.

Now, we consider $\overline{X}^G_H = Z \cup X^G_H \circledast W_2 \cup \{t\}$, where X^G_H is the space constructed in the proof of Theorem 3.6.2 and W_2 is the space constructed in Section 3.8. We extend the partial order defined on Z and $X^G_H \circledast W_2$ to $\overline{X}^G_H$ declaring that $y_j < t > x_1$ for some $h \in H$, where y_j is a minimal point in $\mathcal{X}(X)$ and $x_1 \in W_2$. We prove that $f \in Aut(\overline{X}^G_H)$ restricted to Z is the identity. In W_i, there are i weak beat points, that we will denote by z^i_j with $j = 1, ..., i$. In fact, we have that the only weak beat points that are not beat points or do not have a bigger beat point are in Z. Hence, if $z^i_j \in W_i$ is a weak beat point, then $f(z^i_j) = z^k_l \in W_k$ for some $l \leq k \leq n$. By Proposition 1.1.76, it can be deduced that $f(W_i) = W_k$. But W_i is homeomorphic to W_k if and only if $i = k$. From the continuity of f we get that $f(W_i) = id(W_i)$, so $f_{|Z} = id_Z$. It is simple to check that t is also a fixed point for every homeomorphism since $y_j \prec t$ and $f(y_j) = y_j$. We get $Aut(\overline{X}^G_H) \simeq Aut(X^G_H \circledast W_2) \simeq Aut(X^G_H) \times Aut(W_2)$ by Proposition 1.1.65. By Section 3.8 and the Proof of Theorem 3.6.2, $Aut(\overline{X}^G_H) \simeq Aut(X^G_H) \simeq Aut(X^G_*) \simeq G$. Moreover, $\mathcal{E}(\overline{X}^G_H) \simeq \mathcal{E}(Z \cup \{t\} \cup X^*_H \circledast W_2)$, but $Z \cup \{t\} \cup X^*_H \circledast W_2$ does not contain beat points. Therefore, $\mathcal{E}(Z \cup \{t\} \cup X^*_H \circledast W_2) \simeq Aut(Z \cup \{t\} \cup X^*_H \circledast W_2)$. From here, repeating similar arguments than the ones used before, it can be deduced that $Aut(Z \cup \{t\} \cup X^*_H \circledast W_2) \simeq Aut(X^*_H) \simeq H$.

Finally, $|\mathcal{K}(\overline{X}^G_H)|$ is clearly the wedge sum of $|\mathcal{K}(Z)|$ and $|\mathcal{K}(X^G_H \circledast W_2)|$. By Proposition 1.1.64, we deduce that $\mathcal{K}(X^G_H \circledast W_2)$ is homotopy equivalent to a point since W_2 is collapsible, which implies that $\mathcal{K}(W_2)$ is also collapsible, and X^G_H is homotopy equivalent to X^*_H. We also get that $|\mathcal{K}(Z)|$ is homotopy equivalent to X because every W_i can be removed without changing the weak homotopy type of Z. Therefore, for every $n \in \mathbb{N}$ we have $\pi_n(X) \simeq \pi_n(Z)$ and $H_n(X) \simeq H_n(Z)$. $\square$

From this, it is easy to deduce two results about the realization of groups in topological spaces.

Corollary 3.9.3. *Let H be a finite group, let G be a group and let $\{F_i\}_{i \in I}$ be a sequence of finitely generated abelian groups, where $I \subset \mathbb{N}$ is a finite set. Then there exists a topological space X such that $Aut(X)$ is isomorphic to G, $\mathcal{E}(X)$ is isomorphic to H and $H_i(X)$ is isomorphic to F_i for every $i \in I$.*

Proof. It is an immediate consequence of Theorem 3.9.2. We only need to consider the wedge sum of Moore spaces and then apply Theorem 3.9.2. $\square$

Corollary 3.9.4. *Let H be a finite group, let G be a group, let $n \in \mathbb{N}$ and let T be a finitely presented (abelian) group (if $n > 1$). Then there exists a topological space X such that $Aut(X)$ is isomorphic G, $\mathcal{E}(X)$ is isomorphic to H and $\pi_n(X)$ is isomorphic to T.*

Proof. We only need to use the beginning of the construction of Eilenberg-Maclane spaces to obtain a compact CW-complex X with $\pi_n(X) \simeq H$ and possibly non-trivial higher homotopy groups. Then, the result is an immediate consequence of Theorem 3.9.2. $\square$

Remark 3.9.5. *It could be possible to get a more general result for arbitrary groups. The idea is to generalize the constructions of this section and use the theory developed in [67].*

We see how these results are related with realizations problems in CW-complexes. For a compact CW-complex X, $\mathcal{E}_*(X)$ and $\mathcal{E}_\#(X)$ are nilpotent groups, see for instance [36, Section 4]. We get that $\mathcal{E}_\#(X)$ ($\mathcal{E}_*(X)$) can be seen as the kernel of a homomorphism of groups. We consider the functor π (H_*) between $HPol$ and the category of groups given by $\pi(X) = \bigoplus_{i=1}^{dim(X)} \pi_i(X)$ ($H_*(X) = \bigoplus_{i=1}^{\infty} H_i(X)$). It is easy to check that π (H_*) induces a homomorphisms of groups $\overline{\pi} : \mathcal{E}(X) \to Aut(\pi(X))$ ($\overline{H}_* : \mathcal{E}(X) \to Aut(H_*(X))$) sending each self-homotopy equivalence to its induced morphism in the homotopy groups (homology groups), where $Aut(\cdot)$ denotes here the group of automorphisms of a group in the category of groups. Then, $\mathcal{E}_\#(X)$ ($\mathcal{E}_*(X)$) can be seen as the kernel of $\overline{\pi}$ ($\overline{H}_*$) and are normal subgroups of $\mathcal{E}(X)$. With the following example we prove that for a general topological space we cannot expect the same result.

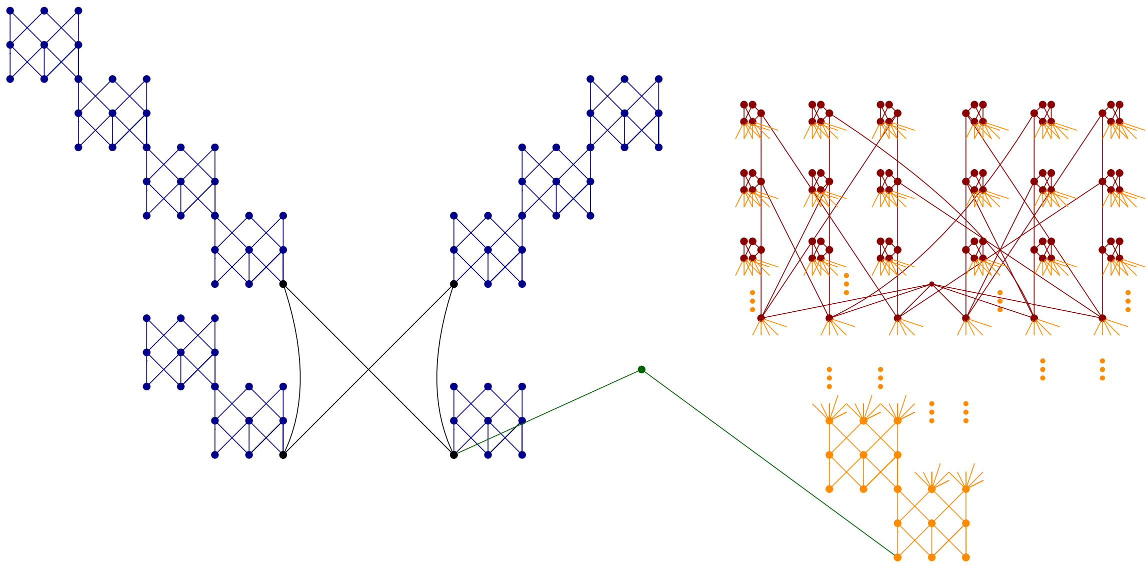

Figure 3.9.2: Hasse diagram of X.

Example 3.9.6. Applying the construction obtained in the proof of Theorem 3.9.2, we can obtain a topological space X such that $Aut(X)$ is trivial, $\mathcal{E}(X) = S_3$ and X is weak homotopy equivalent to a circle, where S_3 denotes the symmetric group on a set of 3 elements. In Figure 3.9.2 we present the Hasse diagram of X. By construction, every self-homotopy equivalence of X fixes the blue, black, green and orange part of the Hasse diagram. The only part that contributes to the homotopy groups or homology groups is the black part of the Hasse diagram. Therefore, we can deduce that $\mathcal{E}_\#(X) = \mathcal{E}_*(X) = \mathcal{E}(X) = S_3$, which implies that $\mathcal{E}_\#(X)$ and $\mathcal{E}_*(X)$ are not nilpotent groups.

Remark 3.9.7. *It is not difficult to show that the topological space X obtained in the proof of Theorem 3.9.2 satisfies that $\mathcal{E}_*(X) = \mathcal{E}_\#(X) = \mathcal{E}(X)$.*

Chapter 4

Fixed point theory for finite topological spaces, a first approach to the approximation of dynamical systems

In this chapter, we develop fixed point and coincidence theorems for finite topological spaces. This is a first approach to develop a theory of approximation to dynamical systems. We present an application to discrete dynamical systems defined on simplicial complexes. Finally, we introduce a category enclosing the previous theory. This category generalizes the usual notions of homotopy, simple homotopy and other homotopies for finite spaces. Consequently, we give some remarks regarding to the topological degree for finite spaces.

4.1 Introduction

A dynamical system for a topological space X consists of a triad $(\mathbb{T}, X, \varphi)$, where $\mathbb{T}$ is usually $\mathbb{Z}$ (discrete dynamical system) or $\mathbb{R}$ (continuous dynamical system) and $\varphi : \mathbb{T} \times X \to X$ is a continuous function satisfying the following: $\varphi(0, x) = x$ for every $x \in X$ and $\varphi(t + s, x) = \varphi(t, \varphi(s, x))$ for all $s, t \in \mathbb{T}$ and $x \in X$. Let $t \in \mathbb{T}$, we denote by $\varphi_t : X \to X$ the map $\varphi(t, \cdot) : X \to X$ for simplicity. Dynamical systems arise in a natural manner in different areas of mathematics and physics. For instance, every autonomous differential equation generates a dynamical system.

In Chapter 2, given a compact metric space X, we studied techniques to reconstruct the homotopy type of X using finite topological spaces. Concretely, we associate to X an inverse sequence of finite topological spaces such that its inverse limit contains a homeomorphic copy of X that is a strong deformation retract. Now, given a dynamical system $(\mathbb{T}, X, \varphi)$, it is natural to try to define a dynamical system for finite topological combining the previous results to get a theory of approximation for $(\mathbb{T}, X, \varphi)$. In Figure 4.1.1 we have a schematic drawing of this idea for a discrete dynamical system. The bigger the value of n is, the better the approximation of the dynamical system is.

However, the classical definition of dynamical system for an Alexandroff space seems uninteresting.

Proposition 4.1.1. *If A is a T_0 Alexandroff space, the only continuous dynamical system $\varphi : \mathbb{R} \times A \to A$ is the trivial one, i.e., $\varphi_t : A \to A$ is the identity map for every $t \in \mathbb{R}$.*

Proof. We will treat A as a poset $(A, \leq)$ with the opposite order, that is, we consider the upper sets in Theorem 1.1.7. For every $x \in A$ we have that $F_x = \{y \in A | y \geq x\}$ is an open set. We argue by contradiction. Suppose that φ_t is not the identity map for every $t \in \mathbb{R}$. Then, there exists $s \in \mathbb{R}$ with $\varphi_s(x) \neq x$ for some $x \in A$. Since φ is continuous

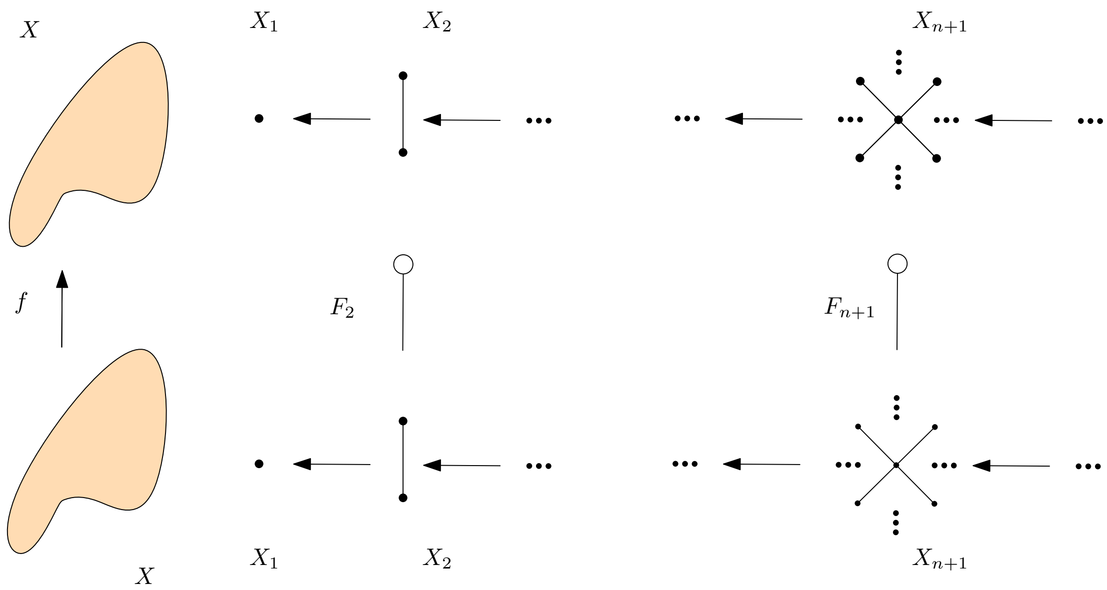

Figure 4.1.1: Schematic drawing.

at $(0, x)$, there exists $\epsilon > 0$ such that $\varphi((-\epsilon, \epsilon) \times F_x) \subseteq F_x$. For every $t \in (-\epsilon, \epsilon)$ we get that $\varphi_t : F_x \to F_x$ is a homeomorphism. We prove that $\varphi_t(F_x) = F_x$. Suppose there exists $z \in F_x \setminus \varphi_t(F_x)$. Since $-t \in (-\epsilon, \epsilon)$, it follows that $\varphi_{-t}(z) \in F_x$, which implies the contradiction. From here, it is immediate to deduce the desired assertion.

Now, we prove that s can be considered in $(-\epsilon, \epsilon)$. We argue by contradiction. Suppose that for every $t \in (-\epsilon, \epsilon)$ and $t > 0$ we have $\varphi(t, x) = x$ (the case $t < 0$ is analogue). We can take $\tau \in \mathbb{R}$ with $0 < \tau < \frac{\epsilon}{2}$, so $\mathbb{R}^+ = \bigcup_{n \in \mathbb{N}} (n\tau, (n+2)\tau)$, where $\mathbb{R}^+$ denotes the set of positive real values. Therefore, $s \in (n\tau, (n+2)\tau)$ for some $n \in \mathbb{N}$. We also have that $s - n\tau \in (0, 2\tau) \subset (0, \epsilon)$. By hypothesis, $\varphi(s - n\tau, x) = x$. Then,

$$x \neq \varphi(s, x) = \varphi(s - n\tau + n\tau, x) = \varphi((n-1)\tau + s - n\tau, \varphi(\tau, x)) = \varphi((n-1)\tau + s - n\tau, x) = \\ = \varphi((n-2)\tau + s - n\tau, \varphi(\tau, x)) = \varphi((n-2)\tau + s - n\tau, x) = \ldots = \varphi(s - n\tau, x) = x,$$

which entails the contradiction. Hence, we can assume that $s \in (-\epsilon, \epsilon)$ and $\varphi(s, x) = y \in F_x$ with $y \neq x$. We know that $\varphi_s : F_x \to F_x$ is a homeomorphism. Thus, there exists $z \in F_x$ such that $\varphi_s(z) = x$. By continuity φ_s preserves the order, so $x = \varphi_s(z) > \varphi_s(x) = y$, but $y > x$. □

Furthermore, if X is a finite T_0 topological space and we have a discrete dynamical system $\varphi : \mathbb{Z} \times X \to X$, then we get that $\varphi_1 = f$ is a homeomorphism. Hence, it is easy to check that there exists a natural number n such that $f^n = id$, where id denotes the identity map.

By the previous arguments, it seems natural to consider multivalued maps to be able to establish new definitions of dynamical systems in this context. Moreover, multivalued maps have been also considered for the classical setting of dynamical systems to develop computational methods. Some applications and the theoretical development can be seen in [55]. The Conley index has been adapted also for multivalued maps [63]. For the finite setting, a first approach to start can be to develop a proper fixed point theory. In [20], J.A. Barmak, M. Mrozek and T. Wanner provide a Lefschetz fixed point theorem for multivalued maps and finite T_0 topological spaces. Before recalling the definitions and results obtained in [20], we recall the classical Lefschetz fixed point theorem.

Given a finite polyhedron X and a continuous map $f : X \to X$. The Lefschetz number of f is defined as follows:

$$\Lambda(f) = \sum_{i=0}(-1)^i tr(f_* : H_i(X) \to H_i(X)),$$

where f_* denotes the linear map induced by f on the torsion-free part of the homology groups of X and tr denotes the trace. This definition can be also extended for general topological spaces with finitely generated rational homology groups. Concretely, for finite T_0 topological spaces. Moreover, if f_* is a linear map on the torsion-free part of the homology groups of a topological space X that is not induced by a continuous map, we will denote also by $\Lambda(f_*)$ the alternate sum of the traces of f_*. We denote by $Fix(f)$ the subspace of X given by the fixed points of f.

Theorem 4.1.2 (Lefschetz fixed point theorem). *Given a finite polyhedron X and a continuous function $f : X \to X$. If $\Lambda(f) \neq 0$, then there exists a point $x \in X$ such that $x = f(x)$.*

A finite version of this theorem can be found in [11], where it is proved the following result.

Theorem 4.1.3. *Let P be a finite poset and let $f : P \to P$ be an order-preserving map. Then $\Lambda(f)$ is the Euler characteristic of $Fix(f)$. In particular, if $\Lambda(f) \neq 0$, then $Fix(f) \neq \emptyset$.*

The Lefschetz fixed point theorem for finite T_0 topological spaces provides more information about the structure of the fixed points set of a continuous map.

Definition 4.1.4. *A topological space X is acyclic if the homology groups of X are isomorphic to the homology groups of a point.*

If X is an acyclic finite polyhedron and $f : X \to X$ is a continuous map, then f has a fixed point. This says that The Lefschetz fixed point theorem can be seen as a generalization of the Brouwer fixed point theorem.

A generalization of the Lefschetz fixed point theorem is the so-called coincidence theorem, which can be found in [43]. Given two continuous maps $f, g : X \to Y$, it is said that f and g have a coincidence point if there exists $x \in X$ such that $f(x) = g(x)$. Before enunciating the coincidence theorem, we recall a result of [108] that was generalized in [22]. For simplicity, the results will be only stated for compact polyhedra.

Definition 4.1.5. *Let X and Y be two compact polyhedra. A continuous map $f : X \to Y$ is a Vietoris map if $f^{-1}(y)$ is acyclic for every $y \in Y$.*

Theorem 4.1.6 (Vietoris-Begle mapping theorem). *Let X and Y be two compact polyhedra. If $f : X \to Y$ is a Vietoris map, then f induces isomorphisms in the homology groups.*

Theorem 4.1.7 (Coincidence theorem). *Let X and Y be two compact polyhedra. If $f, g : X \to Y$ are continuous maps, where g is a Vietoris map, then $\Lambda(f_* \circ g_*^{-1})$ is defined, and if $\Lambda(f_* \circ g_*^{-1}) \neq 0$, then there exists a point $x \in X$ such that $f(x) = g(x)$.*

We recall some definitions and results of [20]. We will take as definitions some characterizations instead of the original ones to simplify.

Definition 4.1.8. *[20, Lemma 3.2] Let $F : X \multimap Y$ denote an arbitrary multivalued map between two finite T_0 topological spaces. It is said that F is upper semicontinuous (lower semicontinuous) if for all $x_1, x_2 \in X$ with $x_1 \leq x_2$ ($x_1 \geq x_2$) and for all $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$ such that $y_1 \leq y_2$ ($y_1 \geq y_2$).*

The notion of upper semicontinuity is defined for general topological spaces. Concretely, a multivalued map $F : X \multimap Y$ between two topological spaces is an upper semicontinuous multivalued map if for each point $x \in X$ and for each neighborhood V of $F(x)$ in Y there exists a neighborhood U of x in X such that $F(U) = \bigcup_{x \in X} F(x)$ is contained in V. Definition 4.1.8 is a particularization of the previous one to finite topological spaces.

Definition 4.1.9. *[20, Lemma 3.4] Let $F : X \multimap Y$ denote an arbitrary multivalued map between two finite T_0 topological spaces. It is said that F is strongly upper semicontinuous (strongly lower semicontinuous) or susc (slsc) if for all $x_1, x_2 \in X$ with $x_1 \leq x_2$ ($x_1 \geq x_2$), $F(x_1) \subseteq F(x_2)$ ($F(x_1) \supseteq F(x_2)$).*

From now on, if $F : X \multimap Y$ is a multivalued map between two finite T_0 topological spaces, then we will denote by $\Gamma(F)$ the graph of F, which is given by $\Gamma(F) = \{(x, y) | y \in F(x)\}$. Hence, $\Gamma(F)$ can also be seen as a finite T_0 topological space, where we are taking the product order on $\Gamma(F) \subseteq X \times Y$. We denote by $p : \Gamma(F) \to X$ and $q : \Gamma(F) \to Y$ the projections of the first and the second coordinate of $\Gamma(F)$ respectively. If $F : X \multimap X$ is a multivalued map, then $x \in X$ is called fixed point if $x \in F(x)$.

In [20, Lemma 4.1], it is shown that if $F : X \multimap Y$ is a susc (slsc) multivalued map with acyclic values between two finite T_0 topological spaces, then $p : \Gamma(F) \to X$ induces isomorphism in homology. Therefore, $F_* : H_n(X) \to H_n(Y)$ is defined in [20, Definition 4.2] as $F_* = q_* \circ p_*^{-1}$. From here, it is proved a Lefschetz fixed point theorem.

Theorem 4.1.10. *[20, Theorem 5.3] Let X be an arbitrary finite T_0 space and let $F : X \multimap X$ be a susc or slsc multivalued map with acyclic values. If $\Lambda(F) \neq 0$, then F has a fixed point.*

Now, we state three of the main results of this chapter. These results rely on the notion of Vietoris-like maps and multivalued maps, which play a central role herein. This notion generalizes the multivalued maps considered in [20], i.e., every susc multivalued map $F : X \multimap Y$ between finite T_0 topological spaces with acyclic values is a Vietoris-like multivalued map, see Proposition 4.2.17. The concept of Vietoris-like map is a finite analogous of the classical definition of Vietoris map. Indeed, it also satisfies that induces isomorphisms in homology groups. Then, we can obtain a coincidence theorem, as in the classical setting.

Theorem 4.1.11 (Coincidence theorem)**.** *Let $f, g : X \to Y$ be continuous maps between finite T_0 topological spaces, where f is a Vietoris-like map, then $\Lambda(g_* \circ f_*^{-1})$ is defined and if $\Lambda(g_* \circ f_*^{-1}) \neq 0$, then there exists $x \in X$ such that $f(x) = g(x)$.*

From here, the classical Lefschetz fixed point theorem for finite topological spaces and single valued maps can be deduced since the identity map is a Vietoris-like map. We generalize the notion of Vietoris-like map to multivalued maps to obtain a new Lefschetz fixed point theorem. One advantage of this notion is the fact that it is very flexible. In fact, there is no kind of continuity required in the definition of a Vietoris-like multivalued map. Despite the previous fact, Vietoris-like multivalued maps induce morphisms in homology groups. Then, we obtain a generalization of Theorem 4.1.10.

Theorem 4.1.12 (Lefschetz fixed point theorem for multivalued maps)**.** *Let X be a finite T_0 topological space. If $F : X \multimap X$ is a Vietoris-like multivalued map and $\Lambda(F_*) \neq 0$, then there exists $x \in X$ with $x \in F(x)$.*

In general, the composition of two Vietoris-like multivalued maps is not a Vietoris-like multivalued map. However, the composition of Vietoris-like multivalued maps presents a good behavior in terms of the Lefschetz fixed point theorem. Specifically,

Theorem 4.1.13. *Let X be a finite T_0 topological space and let $F : X \multimap X$ be a multivalued map. Suppose that $F = G_n \circ \cdots \circ G_0$, where $G_i : Y_i \multimap Y_{i+1}$, $Y_0 = Y_{n+1} = X$, Y_i is a finite T_0 topological space and G_i is a Vietoris-like multivalued map. If $\Lambda(G_{n*} \circ \cdots \circ G_{0*}) \neq 0$, then there exists a point $x \in X$ such that $x \in F(x)$.*

In Section 4.2 we introduce the notion of Vietoris-like for single valued maps and multivalued maps and present examples. In Section 4.3 a coincidence theorem for finite topological spaces is obtained, that is, Theorem 4.1.11. Then, Lefschetz fixed point theorems are deduced, e.g., Theorem 4.1.12. Moreover, it is introduced the notion of continuous selector for a multivalued map. An existence result of selectors for a certain class of multivalued maps is given. Finally, we get coincidence theorems for multivalued maps. In Section 4.4 a hypothesis regarding to the image of the multivalued maps considered in previous sections is relaxed. A Lefschetz fixed point theorem is obtained for this new class of multivalued maps, Theorem 4.1.13. In Section 4.5 we prove that the fixed point property for multivalued maps between Alexandroff spaces does not hold. In Section 4.6 we propose a method to approximate discrete dynamical systems by finite topological spaces.

Hence, it is important to get a category enclosing Vietoris-like multivalued maps. Furthermore, if we want to adapt some classical invariants of dynamical systems like the Conley index, then we need a proper category. The Conley index can be seen as a normal functor and be described for abstract categories [105]. In Section 4.7 we adapt the notion of diagram that was introduced for multivalued maps and non-finite spaces in [55]. Our notion of diagram contains as a particular case the continuous maps and also the Vietoris-like multivalued maps between finite topological spaces. To conclude this section, we prove a Lefschetz fixed point theorem for diagrams. Since diagrams with the composition given in Section 4.7 do not form a category, we introduce in Section 4.8 an equivalence relation between diagrams to define the category of morphisms. A Lefschetz fixed point theorem is also obtained for this category. In Section 4.9 we define homotopy for diagrams and we consider the homotopical category of diagrams HD. In Section 4.10 we show that the definition of homotopy given in the previous section generalizes the usual homotopy for the topological category restricted to finite spaces and the homotopy introduced in [20] for a special class of multivalued maps. Some natural results that hold for polyhedra also hold in this category. For the case of simplicial complexes, the barycentric subdivision of a simplicial complex do not modify neither the homotopy type nor the topological type. Nevertheless, if X is a finite topological space, then the finite barycentric subdivision of X is not homotopy equivalent to X with the usual notion of homotopy. In the homotopical category of diagrams every finite topological space is isomorphic to its finite barycentric subdivision. We also have that if X and Y have the same homotopy type or have the same simple homotopy type, then X is isomorphic to Y in HD. Thus, the homotopical category of diagrams seems more suitable for the study of finite topological. Finally, in Section 4.11, some remarks regarding to the topological degree for finite spaces are given.

4.2 Vietoris-like maps and multivalued maps

In this section we introduce the notion of Vietoris-like for single valued maps and multivalued maps, we also present some examples and properties.

Definition 4.2.1. *Let $f : X \to Y$ be a continuous map between two finite T_0 topological spaces, we say that f is a Vietoris-like map if for every chain $y_1 < y_2 < ... < y_n$ in Y, we have that $\bigcup_{i=1}^{n} f^{-1}(y_i)$ is acyclic.*

Remark 4.2.2. *Definition 4.2.1 implies the surjectivity of the maps considered. Therefore, $f : X \to X$ is a Vietoris-like map if and only if f is a homeomorphism. Moreover, if $f : X \to Y$ is a Vietoris-like map, then f is also a Vietoris-like map when it is considered the other possible partial order on X and Y.*

Theorem 4.2.3. *If $f : X \to Y$ is a Vietoris-like map, then f induces isomorphisms in all homology groups.*

Proof. The idea of the proof is to use [13, Corollary 6.5], which says that if $\varphi : X \to Y$ is a continuous map between finite T_0 topological spaces satisfying that $\mathcal{K}(\varphi^{-1}(U_y))$ (or equivalently $\varphi^{-1}(U_y)$) is acyclic for every $y \in Y$, then $\mathcal{K}(\varphi)$ (or φ) induces isomorphism in all homology groups. On the one hand, we have $\mathcal{K}(f)^{-1}(\overline{\sigma}) = \mathcal{K}(\bigcup_{i=1}^{n} f^{-1}(y_i))$, where σ is a simplex given by some chain $y_1 < ... < y_n$ in Y and $\overline{\sigma}$ denotes the subcomplex of $\mathcal{K}(Y)$ given by σ and all its faces, i.e., all the possible subchains of $y_1 < ... < y_n$. We prove the last assertion. If $\tau \in \mathcal{K}(f)^{-1}(\overline{\sigma})$, then $\mathcal{K}(f)(\tau) \subseteq \overline{\sigma}$. Hence, $\mathcal{K}(f)(\tau)$ is given by a subchain of $y_1 < ... < y_n$, which implies that $\tau \in \mathcal{K}(\bigcup_{i=1}^{n} f^{-1}(y_i))$. We prove the other content. If $\tau \in \mathcal{K}(\bigcup_{i=1}^{n} f^{-1}(y_i))$, then τ is given by a chain $x_1 < ... < x_m$, where $x_i \in f^{-1}(y_j)$ for some $j = 1, ..., n$ and $i = 1, ..., m$. We get that $\mathcal{K}(f)(\tau)$ is given by the chain $f(x_1) \leq ... \leq f(x_n)$, which is a subchain of $y_1 < ... < y_n$. Therefore, $\mathcal{K}(f)(\tau) \in \overline{\sigma}$.

By hypothesis, $\mathcal{K}(f)^{-1}(\overline{\sigma})$ is acyclic due to the equality that we proved above. We have $\mathcal{X}(\mathcal{K}(f)^{-1}(\overline{\sigma})) = \mathcal{X}(\mathcal{K}(f))^{-1}(U_\sigma)$, where U_σ is the minimal open set of $\sigma \in \mathcal{X}(\mathcal{K}(Y))$, that is, U_σ consists of all subsimplices of σ. Therefore, we are in the hypothesis of [13, Corollary 6.5] because for every $\sigma \in \mathcal{X}(\mathcal{K}(Y))$ we get that $\mathcal{X}(\mathcal{K}(f))^{-1}(U_\sigma)$ is acyclic. Thus, $\mathcal{X}(\mathcal{K}(f))$ induces isomorphism in all homology groups. By Theorem 1.1.43 and Theorem 1.1.49, it can be deduced that f induces isomorphisms in all homology groups. □

Theorem 4.2.3 is a finite analogue of the Vietoris-Begle mapping theorem. It also justifies Definition 4.2.1. In the following example we show that it is not possible to obtain an analogue of a Vietoris-Begle mapping theorem (Theorem 4.1.6) for finite topological spaces if we use an analogue of the classical definition of Vietoris map (Definition 4.1.5).

Example 4.2.4. We consider $X = \{A, B, C, D, E, F\}$ and the following relations: $A > C, D, E, F$; $B > C, D, E, F$; $C >, E, F$ and $D >, E, F$, i.e., X is the minimal finite model of the 2-dimensional sphere [17]. We consider $Y = \{M, N\}$ and the relation $M < N$. We define f as follows: $f(A) = f(B) = f(C) = N$ and $f(D) = f(E) = f(F) = M$. Clearly, f is a continuous surjective function, we also have that $f^{-1}(M)$ and $f^{-1}(N)$ are contractible. But f does not induce isomorphism in homology, X is weak homotopy equivalent to a 2-dimensional sphere and Y is weak homotopy equivalent to a point. In Figure 4.2.1 we present a schematic description.

Lemma 4.2.5. *If $f : X \to Y$ and $g : Y \to Z$ are two Vietoris-like maps between finite T_0 topological spaces, then the composition $g \circ f$ is also a Vietoris-like map.*

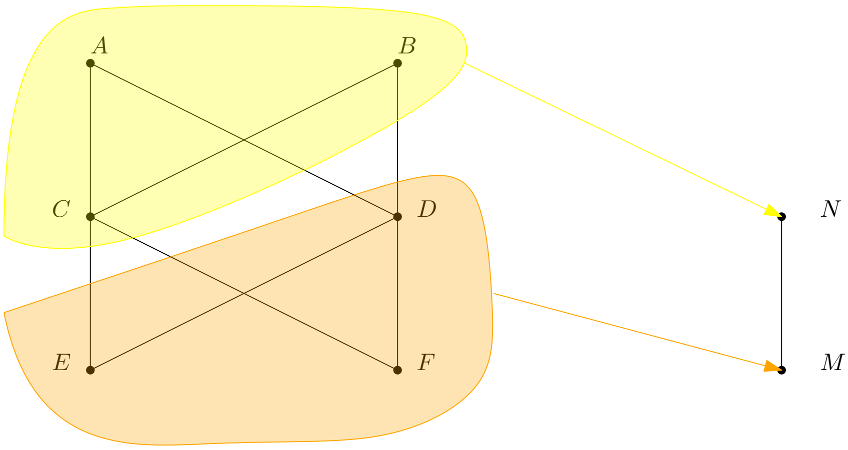

Figure 4.2.1: Schematic description of f on the Hasse diagrams of X and Y.

Proof. The continuity of $g \circ f$ is trivial. Let us take a chain $z_1 < ... < z_n$ in Z. We have that $\bigcup_{i=1}^{n} g^{-1}(z_i)$ is acyclic by hypothesis. Let A denote $\bigcup_{i=1}^{n} g^{-1}(z_i)$ for simplicity. Then $f_{|f^{-1}(A)} : f^{-1}(A) \to A$ is a Vietoris-like map trivially. Therefore, by Theorem 4.2.3, we get that $f^{-1}(A)$ and A have the same homology groups, which implies that $f^{-1}(A)$ is acyclic. $\square$

Lemma 4.2.6. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps between finite T_0 topological spaces. If f and $g \circ f$ are Vietoris-like maps, then g is also a Vietoris-like map.*

Proof. Let h denote $g \circ f$. We take a chain $z_1 < z_2 < ... < z_n$ in Z. By hypothesis, $\bigcup_{i=1}^{n} h^{-1}(z_i)$ is acyclic, we denote $A = \bigcup_{i=1}^{n} g^{-1}(z_i)$. We get that $f_{|f^{-1}(A)} : f^{-1}(A) \to A$ induces isomorphisms in all homology groups because f is a Vietoris-like map. Then, A is acyclic and g is also a Vietoris-like map. $\square$

Despite Lemma 4.2.5 and Lemma 4.2.6, the 2-out-of-3 property does not hold for Vietoris-like maps in general as we show with the following example.

Example 4.2.7. We consider $X = \{A, B, C\}$ with the following partial order: $A, B < C$. We also consider $Y = \{D, E\}$, where we declare $D < E$. Finally, $Z = \{F\}$. We define $f : X \to Y$ given by $f(A) = D, f(B) = D$ and $f(C) = E$. We consider $g : Y \to Z$ as the constant map. It is trivial to check that g and $g \circ f$ are Vietoris-like maps. However, $f^{-1}(D) = \{A, B\}$ has the weak homotopy type of the disjoint union of two points. Then, f is not a Vietoris-like map.

In addition, if $f : X \to Y$ is a Vietoris-like map and $g : X \to Y$ is a weak homotopy equivalence that is homotopic to f, then g is not necessarily a Vietoris-like map.

Example 4.2.8. We consider $X = \{A, B\}$, where we declare $A > B$, and $W = \{C, D, E, F, G, H, I, J, K\}$, where we declare $C > F, G, I, J, K$; $D > F, H, I, J, K$; $E > G, H, I, J, K$; $F > I, J$; $G > I, J, K$ and $H > J, K$. Since W is the finite T_0 topological space introduced in [93, Figure 2], it follows that W is weak homotopy equivalent to a point but it is not contractible. We consider $f : W \to X$ given by $f(C) = f(E) = f(G) = A$ and $f(D) = f(F) = f(H) = f(I) = f(J) = f(K) = B$ and $g : W \to X$ given by $g(C) = g(E) = A$ and $g(W \setminus \{C, E\}) = B$. It is easy to check that f and g are continuous maps. We get that f is a Vietoris-like map since $f^{-1}(A)$ is the minimal closed set containing G (F_G) and $f^{-1}(B)$ is the minimal open set containing D (U_D), which means that $f^{-1}(A)$ and

$f^{-1}(B)$ are contractible. Furthermore, g is homotopic to f because $g \leq f$. Nevertheless, g is not a Vietoris-like map, $g^{-1}(A)$ does not have the same weak homotopy type of a point because it is not connected. In Figure 4.2.2 we have represented the Hasse diagrams of X and W.

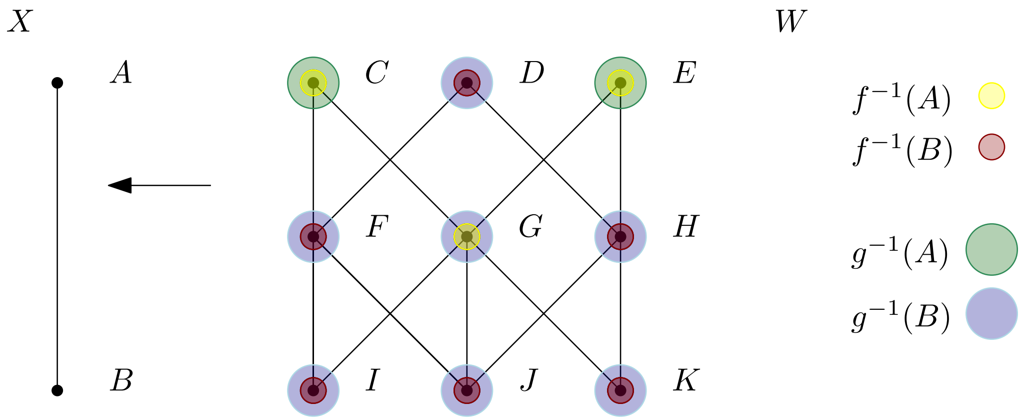

Figure 4.2.2: Hasse diagrams of W and X and schematic representations of $f^{-1}(A)$, $f^{-1}(B)$, $g^{-1}(A)$ and $g^{-1}(B)$.

Now, we provide one of the main definitions.

Definition 4.2.9. *Let $F : X \multimap Y$ be a multivalued map between finite T_0 topological spaces, F is a Vietoris-like multivalued map if the projection p onto the first coordinate from the graph of $\Gamma(F)$ is a Vietoris-like map.*

It is important to observe that we do not require any notion of continuity in the multivalued maps of Definition 4.2.9.

Example 4.2.10. We consider $X = \{A, B\}$ with $A < B$ and $Y = \{C, D, E, F\}$ satisfying that $C < E > D < F$. We define $G : X \multimap Y$ given by $G(A) = \{F, D\}$ and $G(B) = \{E, C\}$. It is easy to check that G is not a usc multivalued map because there is no $z \in G(B)$ with $z > F$, see Definition 4.1.8. Since $p^{-1}(A)$, $p^{-1}(B)$ and $p^{-1}(A) \cup p^{-1}(B)$ are contractible spaces, G is a Vietoris-like multivalued map. In Figure 4.2.3 we have a schematic representation of the situation described above.

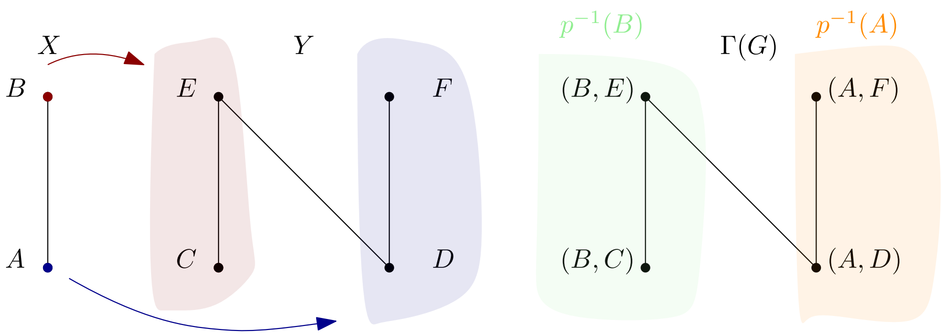

Figure 4.2.3: Hasse diagram of X,Y and $\Gamma(G)$ and schematic representation of G.

Remark 4.2.11. *If $F : X \multimap Y$ is a Vietoris-like multivalued map between finite T_0 topological spaces, then $F(x)$ is acyclic for every $x \in X$. This is due to the fact that $p^{-1}(x) = x \times F(x)$.*

Furthermore, the compositions of Vietoris-like multivalued maps are not in general Vietoris-like multivalued maps as we show in the following example. Before the example, we recall the definition of the composition of multivalued maps. If $F : X \multimap Y$ and $G : Y \to Z$ are multivalued maps, then $G \circ F$ is given by $G(F(x)) = \bigcup_{y \in F(x)} G(y)$.

Example 4.2.12. We consider $X = \{A, B, C, D\}$ with the following partial: $A > C, D$ and $B >, C, D$, i.e., X is the minimal finite model of the circle. We define $F : X \multimap X$ given by $F(A) = \{A, C, D\}$, $F(B) = \{B, C, D\}$, $F(C) = C$ and $F(D) = D$. We define $G : X \multimap X$ given by $G(A) = A$, $G(B) = B$, $G(C) = \{C, A, B\}$ and $G(D) = \{D, A, B\}$. It is easy to check that F and G are Vietoris-like multivalued maps. Nevertheless, $G \circ F : X \multimap X$ is not a Vietoris-like multivalued map since $G(F(A)) = X$ is not acyclic, which implies a contradiction with Remark 4.2.11. In addition, G is another example of a Vietoris-like multivalued map which is not usc.

Lemma 4.2.13. *Let X, Y and Z be finite T_0 topological spaces. If $f : X \to Y$ is a continuous map and $G : Y \multimap Z$ is a Vietoris-like multivalued map, then $G \circ f : X \multimap Z$ is a Vietoris-like multivalued map.*

Proof. By hypothesis, G is a Vietoris-like multivalued map. Then, p_G is a Vietoris-like map, where p_G denotes the projection onto the first coordinate of the graph of G. We consider a chain $x_1 < ... < x_n$ in X, by the continuity of f, $f(x_1) \leq ... \leq f(x_n)$ is a chain in Y. Hence, $A = \bigcup_{i=1}^{n} p_G^{-1}(f(x_i))$ is acyclic. One point in A is of the form $(f(x_i), y)$, where $y \in G(f(x_i))$. We will prove that $p_{G \circ f}$ is a Vietoris-like map, where $p_{G \circ f}$ denotes the projection onto the first coordinate of the graph of $G \circ f$. Then, we need to show that $B = \bigcup_{i=1}^{n} p_{G \circ f}^{-1}(x_i)$ is acyclic. To do that, we verify that A and B are homotopy equivalent. One point in B is of the form (x_i, y), where $y \in G(f(x_i))$. We define $L : B \to A$ given by $L(x_i, y) = (f(x_i), y)$. We have that L is trivially well-defined and continuous. We consider $R : A \to B$ given by $R(z, y) = (f_{min}^{-1}(z), y)$, where $f_{min}^{-1}(z)$ denotes the minimum of the intersection of $f^{-1}(z)$ with $\{x_1, ..., x_n\}$. We get that R is well-defined since $f_{min}^{-1}(z)$ is a non-empty subset of a totally ordered set $(x_1 < ... < x_n)$. If $(z, y), (z', y') \in A$ with $(z, y) \leq (z', y')$, then $R(z, y) = (f_{min}^{-1}(z), y) \leq (f_{min}^{-1}(z'), y') = R(z', y')$. We argue by contradiction. Suppose $f_{min}^{-1}(z) > f_{min}^{-1}(z')$. Therefore $z = f(f_{min}^{-1}(z)) \geq f(f_{min}^{-1}(z')) = z'$ but $z \leq z'$. From this, we have that the only possibility is $z = z'$ and we get $f_{min}^{-1}(z) = f_{min}^{-1}(z')$, which entails a contradiction. It is easy to check that $L \circ R$ is the identity map in A. Finally, if $(x_i, y) \in B$, then $R(L(x_i, y)) = R(f(x_i), y) = (min(f^{-1}(f(x_i)) \cap \{x_1, ..., x_n\}), y) \leq (x_i, y)$, which means that $R \circ L$ is homotopic to the identity map in B. Thus, A and B are homotopy equivalent. □

If $f : Y \to Z$ is a continuous map between finite T_0 topological spaces and $F : X \multimap Y$ is a Vietoris-like multivalued map between finite T_0 topological spaces, then the composition $f \circ F$ is defined as follows:

$$f(F(x)) = \bigcup_{y \in F(x)} f(y).$$

In the previous conditions, it is not possible to get an analogue of Lemma 4.2.13 as we prove in the following example.

Example 4.2.14. We consider the finite topological space of one point, $X = \{A\}$. We consider $Y = \{B, C, D, E\}$ with the following relations: $B < C > D < E$ and we consider $Z = \{F, G, H, I\}$ with the following relations: $F > H, I$ and $G > H, I$. We define $F : X \multimap Y$ given by $F(A) = Y$. Then, F is clearly a Vietoris-like multivalued map. We define $f : Y \to Z$ given by $f(B) = H$, $f(D) = I$, $f(C) = F$ and $f(E) = G$. It is immediate to get that f is a continuous map. We have $f(F(A)) = Z$, which implies that $f(F(A))$ is weak homotopy equivalent to a circle because Z is the minimal finite model of the circle. If we suppose that $f \circ F : X \multimap Z$ is a Vietoris-like multivalued map, then we get a contradiction with Remark 4.2.11. In Figure 4.2.4 we present a schematic description of the above situation.

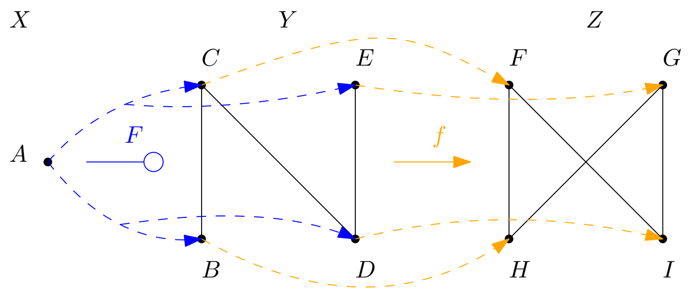

Figure 4.2.4: Schematic description of F and f on the Hasse diagrams of X, Y and Z.

Remark 4.2.15. *If $f : X \to Y$ is a continuous map between finite T_0 topological spaces, then the projection of the graph of f onto the first coordinate p is a Vietoris-like map. In fact, p is a homeomorphism. Since every continuous map can be seen as a multivalued map, it follows that every continuous map is a Vietoris-like multivalued map. It is simple to show that $f_* = q_* \circ p_*^{-1}$ because $f \circ p = q$, where $q : \Gamma(f) \to Y$ denotes the projection onto the second coordinate.*

Let $F : X \multimap Y$ be a Vietoris-like multivalued map between finite T_0 topological spaces. We denote by $F_* : H_*(X) \to H_*(Y)$ the induced morphism in homology given by $F_* = q_* \circ p_*^{-1}$. Since p induces isomorphisms in all homology groups, it follows that F_* is well-defined. If p is not a Vietoris-like map but induces isomorphisms in all homology groups, then F_* is also considered as $F_* = q_* \circ p_*^{-1}$.

Example 4.2.16. We consider $X = \{A, B, C, D, E\}$, where $A, B, C < D, E$, and $F : X \multimap X$ is given by $F(A) = \{D, B, A\}$, $F(B) = B$, $F(C) = C, F(D) = D$, $F(E) = \{C, E, D\}$. It is immediate that F is a usc multivalued map. Moreover, it is easy to prove that F is not a Vietoris-like multivalued map since $p^{-1}(A) \cup p^{-1}(E)$ is weak homotopy equivalent to a wedge sum of two circles. Then, the property of being usc does not imply the property of being a Vietoris-like multivalued map.

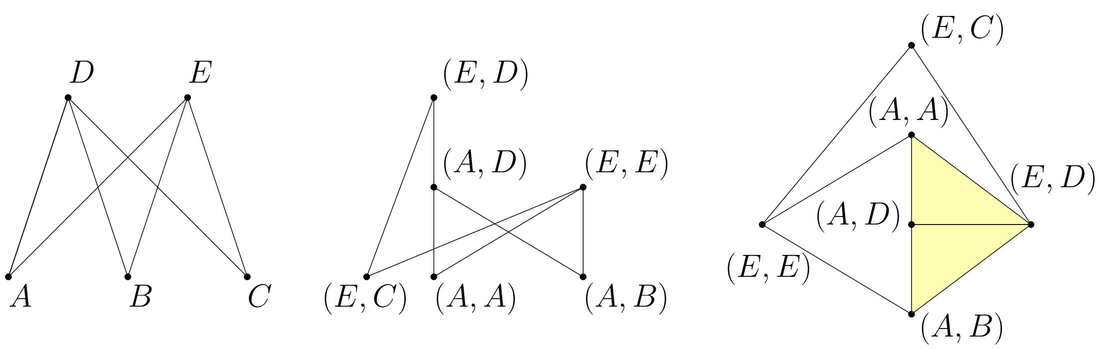

Figure 4.2.5: From left to right, Hasse diagrams of X and $p^{-1}(A) \cup p^{-1}(E)$ and McCord complex of $p^{-1}(A) \cup p^{-1}(E)$.

Important examples of Vietoris-like multivalued maps are the multivalued maps considered in [20]. In terms of continuity, Vietoris-like multivalued maps are more flexible than the ones mentioned above since the properties of being susc or slsc are not considered to prove Theorem 4.1.12 (Lefschetz fixed point theorem for multivalued maps) in Section 4.3. It is easy to find Vietoris-like multivalued maps that are not susc with acyclic values. For instance, every continuous map $f : X \to Y$ can be seen as a Vietoris-like multivalued map, Remark 4.2.15. For more examples, see the multivalued map G considered in Example 4.2.12, Example 4.2.10 or the following propositions. The multivalued map considered in Proposition 4.2.18 cannot be clearly a susc multivalued map unless it is constant.

Proposition 4.2.17. *If $F : X \multimap Y$ is a multivalued map susc (slsc) such that $F(x)$ is acyclic for every $x \in X$, then F is a Vietoris-like multivalued map.*

Proof. The continuity of p is trivial to show. We follow similar techniques than the ones used in [20, Lemma 4.1] so as to get that p is a Vietoris-like map. Let us take a chain $x_1 < ... < x_n$ in X, we denote $A = \bigcup_{i=1}^n p^{-1}(x_i)$. The idea is to show that A has the same homotopy type of $F(x_n)$. We define $i : F(x_n) \to A$ given by $i(z) = (x_n, z)$, clearly, i is well-defined and continuous. We also consider $r : A \to F(x_n)$ given by $r(x_i, y) = y$, we get by hypothesis that $x_i \leq x_n$ implies $F(x_i) \subseteq F(x_n)$, so r is well-defined. The continuity of r follows trivially. It is immediate that $r \circ i = id_{F(x_n)}$. On the other hand, $i \circ r \simeq id_A$ because for every $(x_i, y) \in A$ we get that $i(r(x_i, y)) = i(y) = (x_n, y) \geq (x_i, y) = id_A(x_i, y)$. Thus, A is homotopy equivalent to $F(x_n)$, which implies that F is a Vietoris-like multivalued map.

The second case, the one in parenthesis, is analogous. If we change the partial order on X and Y, then we are in the previous hypothesis and the result follows immediately. □

Proposition 4.2.18. *If $f : X \to Y$ is a Vietoris-like map, then $F : Y \multimap X$ given by $F(y) = f^{-1}(y)$ is a Vietoris-like multivalued map.*

Proof. We need to show that the projection of the graph of F onto the first coordinate is a Vietoris-like map. We take a chain $y_1 < ... < y_n$ in Y. We denote $A = \bigcup_{i=1}^n p^{-1}(y_i)$. We define $g : A \to \bigcup_{i=1}^n f^{-1}(y_i)$ given by $g(y_i, z_i) = z_i$, where we have that $z_i \in f^{-1}(y_i)$ for some y_i, so g is well-defined. The continuity of g follows trivially. Now, we prove that g is injective. We take $(y_i, z), (y_j, w) \in A$ satisfying that $(y_i, z) \neq (y_j, w)$. We have two options. The first one is $y_i \neq y_j$, then, it is clear that $z \neq w$ because $f(z) = y_i$ and $f(w) = y_j$. We deduce that $g(y_i, z) \neq g(y_j, w)$. The second option, $y_i = y_j$ and $z \neq w$, implies $z = g(y_i, z) \neq g(y_j, w) = w$. We define $t : \bigcup_{i=1}^n f^{-1}(y_i) \to A$ as follows: $t(z) = (y_i, z)$, where $z \in f^{-1}(y_i)$ for some y_i in the chain $y_1 < ... < y_n$. We have that t is well-defined since f is a map. Suppose $z \leq w$, where $z, w \in \bigcup_{i=1}^n f^{-1}(y_i)$. Then $t(z) = (y_i, z)$ and $t(w) = (y_j, w)$ for some y_i and y_j. The chain $y_1 < ... < y_n$ is a totally ordered set so $y_i \leq y_j$ or $y_i > y_j$. Suppose that $y_i > y_j$ holds. By the continuity of f we obtain a contradiction because $y_i = f(z) \leq f(w) = y_j$. Therefore, the only possibility is $y_i \leq y_j$, which implies that $t(z) = (y_i, z) \leq (y_j, w) = t(w)$. Hence, t is a continuous map. It is easy to check that $t \circ g$ and $g \circ t$ are the identity map in A and $\bigcup_{i=1}^n f^{-1}(y_i)$ respectively. Thus, g is indeed a homeomorphism. From here, we get that A is acyclic and p is a Vietoris-like map. □

Proposition 4.2.19. *If $F : X \multimap Y$ is a usc (resp. slc) multivalued map with $F(x)$ containing a maximum (resp. minimum) for every $x \in X$, then F is a Vietoris-like multivalued map.*

Proof. We prove the result for the first case. We take a chain $x_1 < ... < x_n$ in X and we denote $A = \bigcup_{i=1}^n p^{-1}(x_i)$. We argue as we did in the proof of Proposition 4.2.17. We denote by $\overline{x}_i$ the maximum of $F(x_i)$ for $i = 1, ..., n$. We define $f : A \to A$ as follows $f(x_i, z) = (x_n, \overline{x}_n)$. We have that f is trivially a continuous map. We check that $f \geq id_A$. By hypothesis, F is usc, therefore, for every $z \in F(x_i)$ there exists $w \in F(x_n)$ such that $z \leq w$. We know that $F(x_n)$ contains a maximum, so $z \leq w \leq \overline{x}_n$. From this, we deduce the desired result.

For the second case, we only need to change the partial order on X and Y. Then, we are in the hypothesis of the first case. □

With the following example, we show that if $F : X \multimap Y$ is a usc multivalued map such that $F(x)$ contains a minimum for every $x \in X$, then F is not necessarily a Vietoris-like multivalued map.

Example 4.2.20. We consider $X = \{H, I\}$ with $H < I$ and $Y = \{A, B, C, D\}$ with $A > C, D$ and $B > C, D$, that is, a finite model of the unit interval and the circle. We define $F : X \multimap Y$ given by $F(H) = \{D\}$ and $F(I) = \{A, B, C\}$. We get that F is clearly usc and satisfies that for every $x \in X$ there exists a minimum in $F(x)$. The finite topological space associated to the graph of F is given by $\Gamma(F) = \{(I, A), (I, B), (I, C), (H, D)\}$ with the following partial order: $(I, A) > (I, C), (H, D)$ and $(I, B) > (I, C), (H, D)$. Hence, $\Gamma(F)$ is a finite model of the circle. Thus, the projection onto the first coordinate of the graph of F is not a Vietoris-like map.

Proposition 4.2.21. *If $f : X \to Y$ is a continuous map between finite T_0 topological spaces, then the second projection of the graph of $F : f(X) \multimap X$ given by $F(y) = f^{-1}(y)$ is a Vietoris-like map.*

Proof. We denote by $q : \Gamma(F) \to X$ the projection onto the second coordinate. We have trivially that q is surjective, for every $x \in X$ we get $(f(x), x) \in \Gamma(F)$. If $z, t \in q^{-1}(x)$ for some $x \in X$, then $f(x) = z$ and $f(x) = t$, which implies $z = t$. Therefore, the cardinality of $q^{-1}(x)$ is one. Let us take a chain $x_1 < x_2 < ... < x_n$ in X. Hence, we need to show that $A = \bigcup_{i=1}^n q^{-1}(x_i)$ is acyclic. We check that $(t_n, x_n) = q^{-1}(x_n) \in A$ is a maximum. If $(t_i, x_i) = q^{-1}(x_i)$ with $i < n$, then $f(x_i) = t_i$ and $f(x_n) = t_n$. By the continuity of f, we deduce that $t_i \leq t_n$, which implies $q^{-1}(x_n) \geq q^{-1}(x_i)$. Thus, A is contractible. □

4.3 A coincidence theorem for finite topological spaces

In this section, we prove a coincidence theorem for finite topological spaces. Consequently, we will obtain some versions of the Lefschetz fixed point theorem.

Proposition 4.3.1. *Given two finite T_0 topological spaces X, Y and two continuous maps $f, g : X \to Y$. If there exists $x \in |\mathcal{K}(X)|$ such that $|\mathcal{K}(f)|(x) = |\mathcal{K}(g)|(x)$, then there exists $y \in X$ with $f(y) = g(y)$.*

Proof. We denote $f' = |\mathcal{K}(f)|$ and $g' = |\mathcal{K}(g)|$ for simplicity. By [75, Theorem 2], we have the following relations, $f_Y \circ f' = f \circ f_X$ and $f_Y \circ g' = g \circ f_X$, where $f_X : |\mathcal{K}(X)| \to X$ and $f_Y : |\mathcal{K}(Y)| \to Y$ are weak homotopy equivalences. Let us take $x \in |\mathcal{K}(X)|$ such that $f'(x) = g'(x)$. Using the previous relations, we get

$$f(f_X(x)) = f_Y(f'(x)) = f_Y(g'(x)) = g(f_X(x)).$$

Therefore, $y = f_X(x) \in X$ satisfies that $f(y) = g(y)$, as we wanted. □

Lemma 4.3.2. *Let $g, f : X \to Y$ be continuous maps between finite topological spaces where f is a Vietoris-like map. Then $\Lambda(\mathcal{K}(g)_* \circ \mathcal{K}(f)_*^{-1}) = \Lambda(g_* \circ f_*^{-1})$.*

Proof. By Theorem 1.1.43, we have $f_* \circ f_{X*} = f_{Y*} \circ \mathcal{K}(f)_*$ and $g_* \circ f_{X*} = f_{Y*} \circ \mathcal{K}(g)_*$. Since f_*, $\mathcal{K}(f)_*$, f_{X*} and f_{Y*} are isomorphisms, we get $f_*^{-1} = f_{X*} \circ \mathcal{K}(f)_*^{-1} \circ f_{Y*}^{-1}$ and $g_* = f_{Y*} \circ \mathcal{K}(g)_* \circ f_{X*}^{-1}$. Then, $g_* \circ f_*^{-1} = f_{Y*} \circ \mathcal{K}(g)_* \circ \mathcal{K}(f)_*^{-1} \circ f_{Y*}^{-1}$. By the properties of the trace, we get the desired equality. □

Now, we prove one of the main results.

Proof of Theorem 4.1.11. By Theorem 4.2.3, $\Lambda(g_* \circ f_*^{-1})$ is well-defined. For the second part, we argue by contradiction. Suppose that f and g do not have coincidence points, i.e., for every $x \in X$ we have $f(x) \neq g(x)$. We define an acyclic carrier $\Phi : \mathcal{K}(Y) \to \mathcal{K}(X)$, that is, Φ is a function which assigns an acyclic subcomplex of $\mathcal{K}(X)$ for each simplex $\sigma \in \mathcal{K}(Y)$ and satisfies that if $\sigma \subseteq \tau$, then $\Phi(\sigma) \subseteq \Phi(\tau)$. For every simplex $\sigma \in \mathcal{K}(Y)$, where σ is given by a chain $y_1 < ... < y_n$ of Y, we define $\Phi(\sigma) = \mathcal{K}(\bigcup_{i=1,...,n} f^{-1}(y_i))$. By hypothesis, f is a Vietoris-like map, so $\Phi(\sigma)$ is acyclic. Furthermore, by construction, it is clear that if $\tau \subseteq \sigma$, we have $\Phi(\tau) \subseteq \Phi(\sigma)$. By the acyclic carrier theorem (see for instance [87, Theorem 13.3]), there exists a chain-map $\phi : C_*(\mathcal{K}(Y)) \to C_*(\mathcal{K}(X))$, which is carried by Φ, i.e., for every n-simplex σ in $\mathcal{K}(Y)$, $\phi(\sigma)$ is a linear combination of simplices of $\Phi(\sigma)$. We will show inductively that $\mathcal{K}(f)_\#(\phi(\sigma)) = \sigma$ for every $\sigma \in \mathcal{K}(Y)$, where $\mathcal{K}(f)_\#$ denotes the chain-map induced by the simplicial map $\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(Y)$. We have that $\mathcal{K}(f)_\#(\phi(v)) = v$ for every vertex in $\mathcal{K}(Y)$ since $\phi(v)$ is a linear combination of vertices in $\mathcal{K}(f)^{-1}(v)$ with augmentation 1, see [87, Proof of Theorem 13.3]. If e is an edge, then $\partial(\mathcal{K}(f)_\#(\phi(e))) = \partial e$. We also know that $\phi(e)$ is a linear combination of edges of $\Phi(e)$, that is, $\phi(e) = \sum k_i \tau_i$, where k_i is a coefficient and τ_i is an edge in $\Phi(e)$ for every i. We have that $\mathcal{K}(f)$ is a simplicial map, which implies that $\mathcal{K}(f)_\#$ sends τ_i to zero or e or $-e$. Therefore, $\mathcal{K}(f)_\#(\phi(e)) = \sum_j (-1)^{a_j} k_j e$, where $a_j = 0$ if $\mathcal{K}(f)_\#$ sends τ_j to e and $a_j = 1$ if $\mathcal{K}(f)_\#$ sends τ_j to $-e$. Since $\partial(\mathcal{K}(f)_\#(\phi(e))) = \partial e$ we get that $\sum_j (-1)^{a_j} k_j = 1$ and $\mathcal{K}(f)_\#(\phi(e)) = e$. We can follow inductively to prove the result. From here, We get that $\mathcal{K}(f)_\# \circ \phi$ is the identity chain-map $id_\# : C_*(\mathcal{K}(Y)) \to C_*(\mathcal{K}(Y))$. Then, ϕ corresponds in homology to $\mathcal{K}(f)_*^{-1}$.

We consider $\mu = \mathcal{K}(g)_\# \circ \phi : C_*(\mathcal{K}(Y)) \to C_*(\mathcal{K}(Y))$. We denote by μ_* the induced homomorphism in homology. We also have

$$\sum_{i=0} (-1)^i \mathrm{trace}(\mu_i) = \sum_{i=0} (-1)^i \mathrm{trace}(\mu_* : H_i(\mathcal{K}(Y)) \to H_i(\mathcal{K}(Y))).$$

See for instance [60, Proof of Theorem 2C.3].

If the trace of μ_i is not zero for some i, then there exists a simplex $\sigma \in C_i(\mathcal{K}(Y))$ with $\mu_i(\sigma) = k\sigma + ...$, where k is a non-zero coefficient. Therefore, there exists γ in $\Phi(\sigma)$ with $\mathcal{K}(g)_\#(\gamma) = \sigma$. We prove the last assertion. We have $\phi(\sigma) = \sum k_j \tau_j$, where $\tau_j \in C_i(\Phi(\sigma))$ is a simplex and k_j is a coefficient for every j. Since $\mathcal{K}(g)$ is a simplicial map, $\mathcal{K}(g)_\#$ sends τ_j to an i-simplex or zero, which implies the desired assertion. In fact, $|\mathcal{K}(g)|$ restricted to $\overline{\gamma}$ is a homeomorphism, where $\overline{\gamma} \subset |\mathcal{K}(X)|$ denotes the closed simplex given by the simplex $\gamma \in \mathcal{K}(X)$. We get that $|\mathcal{K}(g)|(\overline{\gamma}) \subseteq \overline{\sigma}$. Then, by the Brouwer fixed point theorem, $|\mathcal{K}(f)| \circ |\mathcal{K}(g)|_{|\overline{\gamma}}^{-1} : \overline{\sigma} \to \overline{\sigma}$ has a fixed point t. Therefore, $|\mathcal{K}(g)|_{|\overline{\gamma}}^{-1}(t)$ is a coincidence point for $|\mathcal{K}(f)|$ and $|\mathcal{K}(g)|$. By Proposition 4.3.1 we get that f and g have a coincidence point, which entails a contradiction.

Earlier we proved that $\mathcal{K}(f)_*^{-1} = \phi_*$. Then, $\mu_* = \mathcal{K}(g)_* \circ \mathcal{K}(f)_*^{-1}$, which implies that $\sum_i (-1)^i \mathrm{trace}(\mathcal{K}(g)_* \circ \mathcal{K}(f)_*^{-1}) = 0$. By Lemma 4.3.2, we have $\Lambda(g_* \circ f_*^{-1}) = 0$, which entails a contradiction. Thus, there must exists a point $x \in X$ such that $f(x) = g(x)$.

□

Suppose $f, g : X \to Y$ are continuous maps, where X and Y are finite T_0 topological spaces and f is a Vietoris-like map. If $\Lambda(g_* \circ f_*^{-1}) \neq 0$ and $g' : X \to Y$ is a continuous map homotopic to g, then g' and f have at least one coincidence point. The opposite result does not hold, that is, if $f' : X \to Y$ is a continuous map homotopic to f, then f' and g do not have necessarily a coincidence point.

Example 4.3.3. We consider $X = \{A, B, C\}$, where $C > A, B$, $f : X \to X$ given by $f(C) = C$, $f(A) = B$ and $f(B) = A$, $g : X \to X$ given by $g(y) = A$ for every $y \in X$ and $f' : X \to X$ given by $f'(y) = B$ for every $y \in X$. It is easy to deduce that f and f' are homotopic. Moreover, f is a Vietoris-like map and $\Lambda(g_* \circ f_*^{-1}) \neq 0$. But f' and g do not have a coincidence point.

We can also obtain coincidence theorems for multivalued maps as corollaries of Theorem 4.1.11.

Corollary 4.3.4. *Let $f : X \to Y$ be a continuous map between finite T_0 topological spaces such that f is a Vietoris-like map and let $F : X \multimap Y$ be a Vietoris-like multivalued map. Then, $\Lambda(F_* \circ f_*^{-1})$ is defined and if $\Lambda(F_* \circ f_*^{-1}) \neq 0$, then there exists $x \in X$ such that $f(x) \in F(x)$.*

Proof. We have the following diagram,

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \Gamma(F) \arrow[u, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

By hypothesis, p is a Vietoris-like map. Then, $f \circ p$ is a Vietoris-like map due to Lemma 4.2.5, so we are in the hypothesis of Theorem 4.1.11. Therefore, there exists $(x, y) \in \Gamma(F)$ such that $f(p(x, y)) = q(x, y)$, so $f(x) = y \in F(x)$. □

Corollary 4.3.5. *Let $f : X \to Y$ be a continuous map between finite T_0 topological spaces and let $F : X \multimap Y$ be a multivalued map such that the second projection, q, is a Vietoris-like map. Then, $\Lambda(f_* \circ F_*^{-1})$ is defined, where $F_*^{-1} = p_* \circ q_*^{-1}$, and if $\Lambda(f_* \circ F_*^{-1}) \neq 0$, then there exists $x \in X$ such that $f(x) \in F(x)$.*

Proof. We have the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \Gamma(F) \arrow[u, "p"] \arrow[ur, "q"'] & \end{tikzcd}$$

Where F_*^{-1} is well-defined since q induces isomorphisms in all homology groups, see Theorem 4.2.3. We are in the hypothesis of Theorem 4.1.11. Thus, there exists a coincidence point $(x, y) \in \Gamma(F)$ with $y = q(x, y) = f(p(x, y)) = f(x)$ so $f(x) \in F(x)$. □

From the theory developed previously, it is easy to get an analogue of the Lefschetz fixed point theorem for Vietoris-like multivalued maps.

Proof of Theorem 4.1.12. We get that $\Lambda(F_*)$ is well-defined because F is a Vietoris-like multivalued map. Then, we are in the hypothesis of Corollary 4.3.4, where we are considering the identity map as the single valued map. $\square$

We can also obtain the classical Lefschetz fixed point theorem for finite spaces using the previous techniques.

Corollary 4.3.6 (Lefschetz fixed point theorem)**.** *Let $f : X \to X$ be a continuous function and let X be a finite T_0 topological space. If $\Lambda(f) \neq 0$, there exists a fixed point. Furthermore, $\Lambda(f)$ is the Euler characteristic of $Fix(f)$.*

Proof. By Remark 4.2.15, f can be seen as a Vietoris-like multivalued map. Then, we are in the hypothesis of Theorem 4.1.12.

The second part follows easily from the fact that p is a homeomorphism. Then, we have $q \circ p^{-1} : X \to X$ and the following relations:

$$\begin{aligned}\Lambda(f_*) = \Lambda(q_* \circ p_*^{-1}) =& \sum_{i=0}(-1)^i \text{trace}(q_* \circ p_*^{-1}) = \sum_{i=0}(-1)^i |\{\sigma \in \mathcal{K}(X)_i | q(p^{-1}(\sigma)) = \sigma\}| = \\ =& \sum_{i=0}(-1)^i |\{v_0 < ... < v_i | q(p^{-1}(v_0 < ... < v_i)) = v_0 < ... < v_i\}| = \\ =& \sum_{i=0}(-1)^i |\{v_0 < ... < v_i | q(p^{-1}(v_j)) = v_j \text{ where } \ j \in \{0, ..., i\}\}| = \\ =& \chi(Fix(f)).\end{aligned}$$

The previous relations are the same used in [11, Theorem 1.1]. $\square$

Using the multivalued maps mentioned in Proposition 4.2.17, it can be obtained a stronger version of the Lefschetz fixed point theorem than the one obtained in [20], which is a generalization of [11, Theorem 1.1] for multivalued maps.

Theorem 4.3.7. *Let $F : X \multimap X$ be a multivalued map and let X be a finite T_0 topological space. If F is a slsc (susc) multivalued map such that $F(x)$ is acyclic for every $x \in X$, then $\Lambda(F_*)$ is the Euler characteristic of $Fix(F)$.*

Proof. We follow a similar argument than the one used in the proof of Theorem 4.1.11. We consider $\Phi : \mathcal{K}(X) \to \mathcal{K}(\Gamma(F))$ given by $\Phi(\sigma) = \mathcal{K}(\bigcup_{i=0}^n p^{-1}(x_i))$, where σ is given by a chain $x_0 < x_1 < ... < x_n$ in X. Then, Φ is an acyclic carrier trivially. We construct explicitly a chain-map $\phi : C_*(\mathcal{K}(X)) \to C_*(\mathcal{K}(\Gamma(F)))$ carried by Φ. We consider the augmented chain complexes of $C_*(\mathcal{K}(X))$ and $C_*(\mathcal{K}(\Gamma(F)))$, where an augmented chain complex of a chain complex C_* is given as C_* adjoining $\mathbb{Z}$ in dimension -1 and using an epimorphism $\epsilon : C_0 \to \mathbb{Z}$, which satisfies $\epsilon \circ \partial = 0$, as boundary operator. The homology groups of the augmented chain complexes are the reduced homology groups. We will construct a chain-map on the augmented chain complexes. Firstly, we consider $id : \mathbb{Z} \to \mathbb{Z}$. Suppose v is a vertex in $\mathcal{K}(X)$, i.e., a point in X. If v satisfies that $(v, v) \in \Gamma(F)$, then we define $\phi(v) = (v, v)$. If v does not satisfy that $(v, v) \in \Gamma(F)$, then we define $\phi(v)$ choosing a vertex in $\mathcal{K}(p^{-1}(v))$, that is, a point in $p^{-1}(v)$. We extend ϕ to $C_0(\mathcal{K}(X))$ using linearity. Finally, we declare $\epsilon(\phi(v)) = 1$ for every vertex v. Hence, we get $\epsilon(\partial(e)) = 0$ and $\epsilon = \epsilon \circ \phi$. In addition, since $\phi(v)$ is a vertex for every vertex v, we have $\partial(\phi(v)) = 0 = \phi(\partial(v))$. By construction, for every $c \in C_0(\mathcal{K}(X))$, $\phi(c)$ is carried by $\Phi(c)$. Suppose e is an edge in $\mathcal{K}(X)$, that is, a chain $v_0 < v_1$ in X. If e satisfies that $(v_0, v_0), (v_1, v_1) \in \Gamma(F)$, then we define $\phi(e)$ as the edge in $\mathcal{K}(\Gamma(F))$ given by $(v_0, v_0) < (v_1, v_1)$. By construction, it is immediate that $\partial(\phi(e)) = \phi(\partial(e))$. Suppose

e does not satisfy that $(v_0, v_0), (v_1, v_1) \in \Gamma(F)$. We know that $\phi(\partial e)$ is defined. Since $\Phi(e)$ is acyclic, it follows that the reduced homology of $\Phi(e)$ is trivial and there exists $c \in C_1(\Phi(e))$ such that $\partial c = \phi(\partial(e))$. Then, we define $\phi(e) = c$. It is easily seen that $\partial(\phi(e)) = \phi(\partial(e))$. We extend ϕ to $C_1(\mathcal{K}(X))$ using linearity. Therefore, if $c \in C_1(\mathcal{K}(X))$, then $\phi(c)$ is carried by $\Phi(c)$ and $\partial(\phi(c)) = \phi(\partial(c))$. We can proceed inductively so as to get a chain-map ϕ, which is carried by Φ. The chain map constructed can be restricted to the non-augmented chain complexes, i.e., we only consider ϕ and we omit $id : \mathbb{Z} \to \mathbb{Z}$. We have that ϕ is clearly a chain-map. It can also be proved that $\mathcal{K}(p)_\#(\phi(\sigma)) = \sigma$ for every $\sigma \in \mathcal{K}(X)$, as in the proof of Theorem 4.1.11. We have chosen a chain-map that is carried by Φ, but this choice is not relevant because two chain-maps carried by the same acyclic carrier are chain-homotopic, see [87, Theorem 13.3]. We consider the chain-map $\mu = \mathcal{K}(q)_\# \circ \phi : C_*(\mathcal{K}(X)) \to C_*(\mathcal{K}(X))$. Repeating the same arguments used in the proof of Theorem 4.1.11, we get the following relation:

$$\sum_{i=0} (-1)^i \text{trace}(\mu_i) = \Lambda(F_*).$$

Now, we prove that trace(μ_i) is the cardinal of $\{\sigma \in \mathcal{K}(X)_i | \mathcal{K}(q)_\#(\phi(\sigma)) = \sigma\}$, where $\mathcal{K}(X)_i$ denotes the set of i-simplices of $\mathcal{K}(X)$. Suppose that $\sigma \in \mathcal{K}(X)_i$ satisfies $\mathcal{K}(q)_\#(\phi(\sigma)) = t\sigma + ...$, where σ is given by a chain $v_0 < ... < v_i$ in X and t is a non-zero coefficient. Since $\mathcal{K}(q)$ is a simplicial map, there exists a simplex τ in the linear combination $\phi(\sigma)$ such that $\mathcal{K}(q)_\#(\tau) = \sigma$. Since $\phi(\sigma)$ is carried by $\Phi(\sigma) = \mathcal{K}(\bigcup_{j=0}^{i} p(v_j)^{-1})$, τ is a simplex given by a chain $(v_{j_0}, v_0) < (v_{j_1}, v_1) < ... < (v_{j_i}, v_i)$ in $\Gamma(F)$, where $v_{j_k} \leq v_{j_l}$ if $k \leq l$ and $j_k \in \{0, 1, ..., i\}$ for every $k = 0, ..., i$. Suppose that $v_{j_0} > v_0$. Since F is a slsc multivalued map, we have $F(v_{j_0}) \subseteq F(v_0)$. By the construction of the graph of F and the previous observation, we get $v_0 \in F(v_0)$. Then, we can deduce that the simplex $\tau^{(0)}$ in $\mathcal{K}(\Gamma(F))$ given by the chain $(v_0, v_0) < (v_{j_1}, v_1) < ... < (v_{j_i}, v_i)$ is a simplex in $\Phi(\sigma)$. We can proceed inductively until we get that $\tau^{(i)}$ given by $(v_0, v_0) < ... < (v_i, v_i)$ is a simplex of $\Phi(\sigma)$. By the construction of ϕ we know that if $w_0 < ... < w_m$ is a chain in X (a simplex in $\mathcal{K}(X)$) where $w_i \in F(w_i)$ for every $i = 1, ..., m$, then $\phi(w_0 < ... < w_m) = (w_0, w_0) < ... < (w_m, w_m)$. From this, we get that $\phi(\sigma) = \tau^{(i)}$ and $\mathcal{K}(q)_\#(\phi(\sigma)) = \sigma$. Thus, for every $j \in \{0, ..., i\}$ we get $(v_j, v_j) \in \Gamma(F)$, which means that $v_j \in Fix(F)$. From here, it is easy to deduce that the Euler characteristic of $Fix(F)$ is $\Lambda(F_*)$. If F is susc, then we consider the other possible partial order on X. Now, F is slsc and we can apply the previous arguments to get the desired result. □

Problem 4.3.8. *It remains to study the general case of the previous result. Given a Vietoris-like multivalued map $F : X \multimap X$, do we have that $\Lambda(F_*)$ is the Euler characteristic of $Fix(F)$?*

Definition 4.3.9. *Let X and Y be finite T_0 topological spaces. A continuous selector for a multivalued map $F : X \multimap Y$ is a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$.*

In [20], it is proved that if $F : X \multimap X$ is a susc (slsc) multivalued map with acyclic values and f is a continuous selector, then $\Lambda(f) = \Lambda(F_*)$. With the following proposition, we extend the class of multivalued maps satisfying that $\Lambda(f) = \Lambda(F)$ for every continuous selector f of F.

Proposition 4.3.10. *Let X and Y be finite T_0 topological spaces. If $F : X \multimap Y$ is a multivalued map usc (lsc) such that $F(x)$ contains a maximum (minimum) $\overline{x}$ for every $x \in X$, then there exists a continuous selector f for F with $F_* = f_*$. Furthermore, if $X = Y$, then $\Lambda(F_*) = \Lambda(f)$. If g is another continuous selector for F, then $\Lambda(g) = \Lambda(F_*)$.*

Proof. By Proposition 4.2.19, F is a Vietoris-like multivalued map. Firstly, we construct a continuous selector f for F. We consider $f : X \to Y$ given by $f(x) = \overline{x} \in F(x)$. We prove the continuity of f. If $x \leq y$, then we know by the usc property that for every $x' \in F(x)$ there exists $y' \in F(y)$ with $x' \leq y'$. There exists $y' \in F(y)$ with $\overline{x} \leq y' \leq \overline{y}$, so $\overline{x} = f(x) \leq f(y) = \overline{y}$. We have the following diagram, where p and q denote the projection from the graph of F onto the first and second coordinates respectively.

$$\begin{tikzcd} & \Gamma(F) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

If $(x, z) \in \Gamma(F)$, then $z = q(x, z) \leq f(p(x, z)) = f(x) = \overline{x} \in F(x)$. Therefore, $f \circ p$ is homotopic to q. Concretely, we have $f_* = q_* \circ p_*^{-1} = F_*$ since p is a Vietoris-like map. Thus, if $X = Y$, then $\Lambda(F_*) = \Lambda(f_*)$.

Suppose there is another continuous selector g for F. We prove that $g \leq f$. If $x \in X$, then $g(x) \in F(x)$ and $f(x) \geq y$ for every $y \in F(x)$, which implies that $f(x) \geq g(x)$. Thus, $\Lambda(g) = \Lambda(f) = \Lambda(F_*)$.

The result in parenthesis follows directly from the previous one. Suppose F is a multivalued map lsc such that $F(x)$ contains a minimum for every $x \in X$. Then, taking the other possible partial order on X and Y, we are in the previous conditions. □

Example 4.3.11. We consider $X = \{A, B, C, D\}$ with the partial order given as follows: $A > C, D$ and $B > C, D$, that is, X is the minimal finite model of the circle. We consider $Y = \{E, F, G, H, I, J, K, L\}$ with the following relations: $E > L < H > D < G > B < F > C < E$. We have that Y is also a finite model of the circle. We define $T : X \multimap Y$ as follows: $T(C) = I$, $T(D) = K$, $T(A) = \{E, H, L\}$ and $T(B) = \{F, G, J\}$. It is trivial to check that T is usc and the image of every point has a minimum. Furthermore, T does not have any continuous selector. Arguing by contradiction this result follows easily.

Let $F, G : X \multimap Y$ be multivalued maps, we say that F and G have a coincidence point if there exists $x \in X$ such that $G(x) \cap F(x)$ is non-empty. From Theorem 4.1.7, Corollary 4.3.4, Corollary 4.3.5 and the notion of continuous selector, it is easy to deduce the following result.

Theorem 4.3.12. *Let $F : X \multimap Y$ and $G : X \to Y$ be multivalued maps between finite T_0 topological spaces.*

1. *If F is a Vietoris-like multivalued map and G admits a continuous selector g that is also a Vietoris-like map, then $\Lambda(F_* \circ g_*^{-1})$ is defined and if $\Lambda(F_* \circ g_*^{-1}) \neq 0$, then F and G have a coincidence point.*

2. *If the projection onto the second coordinate of the graph of F is a Vietoris-like map and G admits a continuous selector g, then $\Lambda(g_* \circ F_*^{-1})$ is defined and if $\Lambda(g_* \circ F_*^{-1}) \neq 0$, then F and G have a coincidence point.*

3. *If G admits a continuous selector g that is a Vietoris-like map, F admits a continuous selector f and $\Lambda(f_* \circ g_*^{-1}) \neq 0$, then F and G have a coincidence point.*

4.4 Lefschetz fixed point theorem for non-acyclic multivalued maps

In Section 4.3, given a multivalued map $F : X \multimap X$, a hypothesis regarding to the image of every point $x \in X$ is required to obtain a version of the classical Lefschetz fixed point theorem. We will show that a Lefschetz fixed point theorem can be obtained for some special multivalued maps such that the image of every point is not acyclic.

Lemma 4.4.1. *Let X, Y and Z be finite T_0 topological spaces, let $f : X \to Y$ be a continuous map and let $G : Y \multimap Z$ be a multivalued map such that the projection of its graph onto the first coordinate induces isomorphisms in homology. If $H = G \circ f$ satisfies that the projection of its graph onto the first coordinate induces isomorphisms in homology, then $H_* = G_* \circ f_*$.*

Proof. We only need to follow the same ideas for the analogue result obtained in [90, Theorem 3.8]. We denote by $\Gamma(f), \Gamma(G)$ and $\Gamma(H)$ the finite T_0 topological spaces given by the graphs of f, G and H respectively. Moreover, p_f, q_f, p_G, q_G, p_H and q_H denote their respective projections, where p denotes the projection onto the first coordinate and q denotes the projection onto the second coordinate. We also define an auxiliary map from $\Gamma(H)$ to Y, $\overline{F} : \Gamma(H) \to Y$ is given by $\overline{F}(x,z) = f(x)$. The continuity of $\overline{F}$ is trivial since $\overline{F}$ is the composition of continuous maps, p_H and f. Again, $\Gamma(\overline{F})$ denotes the finite T_0 topological space given by the graph of $\overline{F}$, we denote by ϕ_1 and ϕ_2 their respective projections onto the first and second coordinates. Finally, we define two extra auxiliary maps from $\Gamma(\overline{F})$, $\phi_3 : \Gamma(\overline{F}) \to \Gamma(f)$ is given by $\phi_3((x,z),y) = (x,y)$ and $\phi_4 : \Gamma(\overline{F}) \to \Gamma(G)$ is given by $\phi_4((x,z),y) = (y,z)$. It is clear that ϕ_3 and ϕ_4 are continuous maps because they preserve the order. It is easy to check that the following diagram of continuous maps is commutative.

$$\begin{tikzcd}
X & \Gamma(f) \arrow[l, "p_f"'] \arrow[r, "q_f"] & Y & \Gamma(G) \arrow[l, "p_G"'] \arrow[r, "q_G"] & Z \\
 & & \Gamma(\overline{F}) \arrow[ul, "\phi_3"] \arrow[u, "\phi_2"] \arrow[ur, "\phi_4"'] \arrow[d, "\phi_1"] & & \\
 & & \Gamma(H) \arrow[uull, "p_H"] \arrow[uurr, "q_H"'] & &
\end{tikzcd}$$

By the commutativity of the diagram, we can obtain the following equalities:

$$p_H \circ \phi_1 = p_f \circ \phi_3 \tag{4.1}$$

$$\phi_2 = q_f \circ \phi_3 \tag{4.2}$$

$$q_H \circ \phi_1 = q_G \circ \phi_4 \tag{4.3}$$

$$\phi_2 = p_G \circ \phi_4. \tag{4.4}$$

In addition, p_f, p_G, p_H, ϕ_1 induce isomorphism in homology, so we can take its inverses after applying the homological functor to the previous diagram. Therefore, from (4.1) and (4.2) we deduce the following equality.

$$q_{f*} \circ p_{f*}^{-1} = \phi_{2*} \circ \phi_{1*}^{-1} \circ p_{H*}^{-1} \tag{4.5}$$

Combining (4.3) with (4.4) we obtain the following relation:

$$q_{H*} = q_{G*} \circ p_{G*}^{-1} \circ \phi_{2*} \circ \phi_{1*}^{-1}. \tag{4.6}$$

To conclude, we only need to combine (4.5) with (4.6) to obtain the desired result.

$$H_* = q_{H*} \circ p_{H*}^{-1} = q_{G*} \circ p_{G*}^{-1} \circ \phi_{2*} \circ \phi_{1*}^{-1} \circ p_{H*}^{-1} = q_{G*} \circ p_{G*}^{-1} \circ q_{f*} \circ p_{f*}^{-1} = G_* \circ f_* \tag{4.7}$$

$\square$

Now, using the previous lemmas, we can prove the generalization of Theorem 4.1.12.

Proof of Theorem 4.1.13. We show the result for the case $F = G_1 \circ G_0$, that is, $n = 1$. For a general $n \in \mathbb{N}$, the proof is a generalization of the case $n = 1$, we will indicate how to do it for the case $n = 2$ and the general case.

We have that $G_0 : X \multimap Y$ and $G_1 : Y \multimap X$ are Vietoris-like multivalued maps. $\Gamma(G_0)$ denotes the graph of G_0, where we have the natural projections $p_0 : \Gamma(G_0) \to X$ and $q_0 : \Gamma(G_0) \to Y$. By hypothesis, we know that p_0 is a Vietoris-like map. We consider the following composition, $G_1 \circ q_0 : \Gamma(G_0) \multimap X$, by Lemma 4.2.13, $G_1 \circ q_0$ is a Vietoris-like multivalued map. Again, $\Gamma(G_1 \circ q_0)$ denotes the finite T_0 topological space given by the graph of $G_1 \circ q_0$, where we also have the natural projections $p_1 : \Gamma(G_1 \circ q_0) \to G_0$ and $q_1 : \Gamma(G_1 \circ q_0) \to X$. Again, p_1 is a Vietoris-like map. We also have that $p_0 \circ p_1$ is a Vietoris-like map by Lemma 4.2.5. We get the following diagram.

$$\begin{tikzcd}
\Gamma(G_1 \circ q_0) \arrow[rr, "q_1"] \arrow[d, "p_1"'] & & X \\
\Gamma(G_0) \arrow[rr, "q_0"] \arrow[dr, "p_0"'] & & Y \arrow[u, "G_1"'] \\
& X \arrow[ur, "G_0"'] &
\end{tikzcd}$$

It is clear that $q_{0*} = G_{0*} \circ p_{0*}$ because G_{0*} is by construction $q_{0*} \circ p_{0*}^{-1}$. By Lemma 4.4.1, $G_{1*} \circ q_{0*} = (G_1 \circ q_0)_* = q_{1*} \circ p_{1*}^{-1}$ and then $G_{1*} \circ q_{0*} \circ p_{1*} = q_{1*}$. From here, we can deduce

$$G_{1*} \circ G_{0*} = G_{1*} \circ q_{0*} \circ p_{0*}^{-1} = q_{1*} \circ p_{1*}^{-1} \circ p_{0*}^{-1} = q_{1*} \circ (p_0 \circ p_1)_*^{-1}. \tag{4.8}$$

Therefore, $\Lambda(G_{1*} \circ G_{0*}) = \Lambda(q_{1*} \circ (p_0 \circ p_1)_*^{-1}) \neq 0$, where $q_1, p_0 \circ p_1 : \Gamma(G_1 \circ q_1) \to X$ are continuous maps between finite T_0 topological spaces. Thus, we are in the hypothesis of Theorem 4.1.11, so there is a coincidence point for q_1 and $p_0 \circ p_1$, let us denote that point by $((x, y), z) \in \Gamma(G_1 \circ q_0)$, where $(x, y) \in \Gamma(G_0)$. We have that $q_1((x, y), z) = z = p_0(p_1((x, y), z)) = p_0(x, y) = x$. Hence, $x = z$ and $x \in G_1(q_0(x, y)) = G_1(y)$. But $y \in G_0(x)$, so $x \in G_1(G_0(x)) = F(x)$.

Now, suppose that $F = G_2 \circ G_1 \circ G_0$, where $G_0 : X \to Y_1$, $G_1 : Y_1 \to Y_2$ and $G_2 : Y_2 \to X$ are Vietoris-like multivalued maps and Y_1, Y_2 are finite T_0 topological spaces. We argue as we did before. We consider $G_1 \circ q_0 : \Gamma(G_0) \to Y_2$, which is a Vietoris-like multivalued map. Repeating the arguments used before, it can be obtained that the square and the bottom triangle of the following diagram commute after applying the homological functor. We consider $G_2 \circ q_1 : \Gamma(G_1 \circ q_0) \to X$, which is a Vietoris-like multivalued map by Lemma 4.2.13. Let $\Gamma(G_2 \circ q_1)$ denote the graph of $G_2 \circ q_1$, where p_2 and q_2 are the projections onto the first and second coordinates respectively.

$$\begin{tikzcd}
& \Gamma(G_2 \circ q_1) \arrow[dl, "p_2"'] \arrow[drr, "q_2"] & & \\
\Gamma(G_1 \circ q_0) \arrow[rr, "q_1"] \arrow[d, "p_1"'] & & Y_2 \arrow[r, "G_2"'] & X \\
\Gamma(G_0) \arrow[rr, "q_0"] \arrow[dr, "p_0"'] & & Y_1 \arrow[u, "G_1"'] & \\
& X \arrow[ur, "G_0"'] & &
\end{tikzcd}$$

By Lemma 4.2.5, $p_0 \circ p_1 \circ p_2$ is a Vietoris-like map. Using Lemma 4.4.1, it can be proved that $\Lambda(G_{2*} \circ G_{1*} \circ G_{0*}) = \Lambda(q_{2*} \circ (p_0 \circ p_1 \circ p_2)_*^{-1})$. By Theorem 4.1.11, there exists a coincidence point for $p_0 \circ p_1 \circ p_2$ and q_2. Let us denote that point by $(((x, y), z), t) \in \Gamma(G_2 \circ q_1)$. Then, $p_0(p_1(p_2((((x, y), z), t)) = p_0(p_1((x, y), z)) = p_0(x, y) = x = t = q_2((((x, y), z), t))$. By construction, $t \in F(x)$, so there exists a fixed point for F.

For a general n, we only need to use the same arguments described before. Keeping the same notation introduced, we have $0 \neq \Lambda(G_{n*} \circ \cdots \circ G_{0*}) = \Lambda(q_n \circ (p_0 \circ \cdots p_n)_*^{-1})$. We get that $p_0 \circ \cdots p_n$ is a Vietoris-like map since it is the composition of Vietoris-like maps, see Lemma 4.2.5. Then $(p_0 \circ \cdots \circ p_n)_*^{-1}$ is well-defined. By Lemma 4.4.1, we get

$$\begin{aligned}
&G_{n*} \circ \cdots \circ G_1 * \circ G_{0*} = G_{n*} \circ \cdots \circ G_1 * \circ q_{0*} \circ p_{0*}^{-1} = G_{n*} \circ \cdots \circ (G_1 \circ q_0)_* \circ p_{0*}^{-1} = \\
&= G_{n*} \circ \cdots \circ G_{2*} \circ q_{1*} \circ p_{1*}^{-1} \circ p_{0*}^{-1} = G_{n*} \circ \cdots \circ (G_2 \circ q_1)_* \circ p_{1*}^{-1} \circ p_{0*}^{-1} = \\
&= G_{n*} \circ \cdots \circ q_{2*} \circ p_{2*}^{-1} \circ p_{1*}^{-1} \circ p_{0*}^{-1} = \cdots = q_{n*} \circ p_{n*}^{-1} \circ \cdots p_{0*}^{-1}.
\end{aligned}$$

By Theorem 4.1.11, there is a coincidence point for q_n and $p_0 \circ \cdots p_n$, which implies that there is a fixed point for F. □

In general, we cannot expect to obtain every multivalued map as a composition of Vietoris-like multivalued maps.

Example 4.4.2. Let us consider $X = \{A, B, C\}$ with $A, B < C$ and $F : X \multimap X$ given by $F(A) = B$,$F(B) = A$ and $F(C) = \{A, B\}$. We get that F is clearly a susc multivalued map with images that are not weak homotopy equivalent to a point because $F(C)$ is not connected. Suppose that there exists a sequence of spaces and Vietoris-like multivalued maps such that $F = G_n \circ \cdots \circ G_0$, where $G_i : Y_i \multimap Y_{i+1}$ and $Y_0 = Y_{n+1} = X$. It is trivial to show that $\Lambda(G_{n*} \circ \cdots G_{0*}) \neq 0$ due to the fact that $H_i(X) = 0$ for every $i > 0$. By Theorem 4.1.13, we should have a fixed point for F but this is not true. Then, F cannot be expressed as the previous decomposition of multivalued maps.

Example 4.4.3. Let X be the finite T_0 topological space given by the Hasse diagram of Figure 4.4.1. We consider $F : X \multimap X$ defined by Table 4.4.1.

Table 4.4.1: $F : X \multimap X$

x	A	B	C	D	E
$F(x)$	$\{A, B, C\}$	$\{A, B, D\}$	$\{A, B, C, D\}$	$\{A, B, C, D\}$	$\{A, B, C, D\}$

Table 4.4.2: $G_0 : X \multimap X$

x	A	B	C	D	E
$G_0(x)$	$\{C\}$	$\{D\}$	$\{C, D, B\}$	$\{C, D, B\}$	$\{C, D, B\}$

It is easy to check that F is a susc multivalued map. But $F(C)$ is weak homotopy equivalent to S^1 because $F(C)$ is indeed a finite model of S^1, which implies that F is not a Vietoris-like multivalued map. We have that $F = G_1 \circ G_0$, where $G_0 : X \multimap X$ and $G_1 : X \multimap X$ are given by Table 4.4.2 and Table 4.4.3 respectively. It is simple to prove that G_0 and G_1 are susc multivalued maps with contractible images, that is, they are Vietoris-like multivalued maps.

Table 4.4.3: $G_1 : X \multimap X$

x	A	B	C	D	E
$G_1(x)$	$\{B\}$	$\{A\}$	$\{A, B, C\}$	$\{A, B, D\}$	X

We know that X is contractible because it has a maximum. Therefore, we get that $\Lambda(G_{1*} \circ G_{0*}) \neq 0$. By Theorem 4.1.13, there exists a fixed point. Looking at F it is easy to verify that C and D are fixed points. In Figure 4.4.1, we have a schematic description of G_0 and G_1.

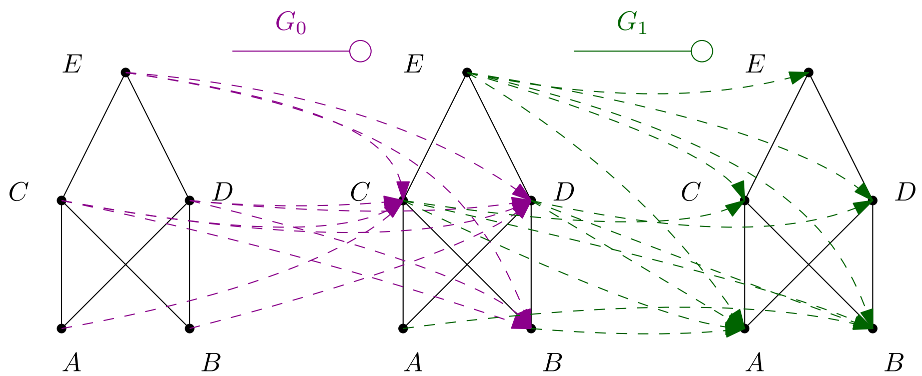

Figure 4.4.1: Schematic description of G_0 and G_1 on the Hasse diagram of X.

Remark 4.4.4. *In Example 4.4.3, we have another example of a composition of Vietoris-like multivalued maps that is not a Vietoris-like multivalued map. In addition, it can be proved that the graph of F is not contractible, it has the same weak homotopy type of a circle.*

4.5 Fixed point property

In this short section we prove that the fixed point property for susc multivalued maps with acyclic values is not a homotopy invariant for the category of Alexandroff spaces. A topological space X is said to have the fixed point property if for every map $f : X \to X$, there exists a fixed point. In the classical setting, it can be seen that the fixed point property is not a homotopy invariant. In [69], two CW-complexes are given satisfying that they are homotopy equivalent but one of them has the fixed point property while the other does not. In [65], a compact metric space is given satisfying that it is contractible but does not have the fixed point property. For the category of finite T_0 topological spaces, the fixed point property is a homotopy invariant, see for example [12, Chapter 10].

The fixed point property can be adapted to multivalued maps for the category of finite T_0 topological spaces. If X is an acylic finite T_0 topological space, then X has the fixed point property for Vietoris-like multivalued maps applying Theorem 4.1.11 because $\Lambda(F_*) \neq 0$. In [20], a study of the fixed point property for susc multivalued maps with acyclic values is done. It is described an example [20, Example 6.3] of a non-acyclic finite topological space having the fixed point property for susc multivalued maps. The example is the face poset of a simplicial complex that was first considered in [97, Corollary 2.7]. This simplicial complex is an example of a simplicial complex having the fixed point property and Euler characteristic 2.

Example 4.5.1. Let X denote the topological space considered in [20, Example 6.3]. We define $Y = X \times \mathbb{N}$, where $\mathbb{N}$ is the totally ordered set of the natural numbers with the opposite order $<_o$, that is, $n+1 <_o n$ for every $n \in \mathbb{N}$. We consider the product order on Y. It is clear that Y is homotopy equivalent to X. We consider $r : Y \to X$ given by $r(x, n) = x$ and $i : X \to Y$ given by $i(x) = (x, 0)$. We have that $r \circ i = id_X$ and $i \circ r$ is homotopic to id_Y. Given a susc multivalued map with acyclic values $F : X \multimap X$. We define $\overline{F} : Y \to Y$ given by $\overline{F}(x, n) = F(X) \times [n+1, \infty)$, where $[n+1, \infty) = \{m \in \mathbb{N} | m \leq_o n+1\}$. It is clear that $\overline{F}(x, n)$ is acyclic for every $(x, n) \in Y$ since $F(X) \times [n+1, \infty)$ is homotopy equivalent to $F(X)$. It is also easy to get that $\overline{F}$ is susc using Definition 4.1.9. If $(x, n) \leq (y, m)$, then $x \leq y$ and $n \leq_o m$. Hence,

$$\overline{F}(x, n) = F(x) \times [n + 1, \infty) \subseteq F(y) \times [n + 1, \infty) \subseteq F(y) \times [m + 1, \infty) = \overline{F}(y).$$

It is trivial to check that $\overline{F}$ does not have fixed points. But Y is homotopy equivalent to a topological space X which has the fixed point property for susc multivalued maps with acyclic values. In Figure 4.5.1, it can be seen a schematic draw of the situation described above.

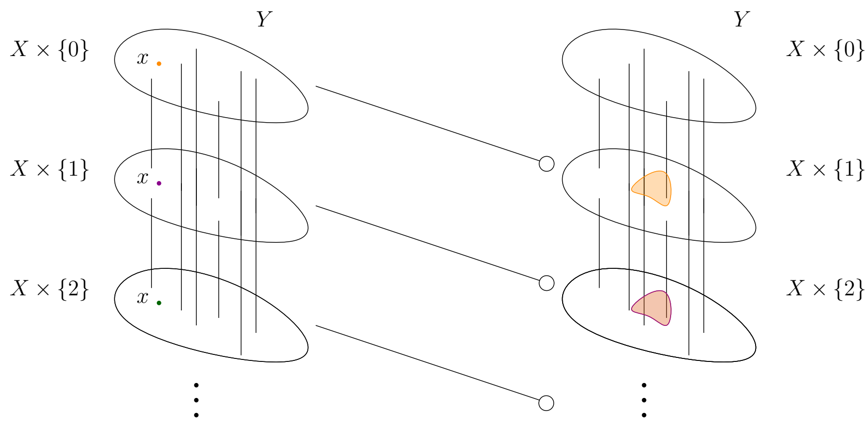

Figure 4.5.1: Schematic Hasse diagram of $\overline{F} : Y \multimap Y$.

The construction obtained in Example 4.5.1 can be generalized to general finite topological spaces in order to get easily the following result for Vietoris-like multivalued maps.

Proposition 4.5.2. *Let X be a finite topological space having the fixed point property for Vietoris-like multivalued maps. Then, there exists an Alexandroff space Y such that Y is homotopy equivalent to X and Y does not have the fixed point property for Vietoris-like multivalued maps.*

The construction of Example 4.5.1 also serves to illustrate that it is not possible to get a direct generalization of the Lefschetz fixed point theorem for non-finite Alexandroff spaces.

Example 4.5.3. We consider the natural numbers $\mathbb{N}$ with the usual order and $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$. It is clear that f is continuous and does not have fixed points. In addition, since $\mathbb{N}$ is contractible because it contains a minimum, we get that $\Lambda(f) \neq 0$.

Thus, to generalize this theory to general Alexandroff spaces, we need to impose more hypotheses. For instance, it is reasonable to require that $F(X)$ is a compact subset of Y as in the classical setting, where $F : X \multimap Y$ is a multivalued map between Alexandroff spaces.

4.6 Approximation of dynamical systems

In this section the finite barycentric subdivision of a finite T_0 topological space X will be denoted by X'. If it is applied n-times, then the n-th finite barycentric subdivision of X will be denoted by X^n. If K is a simplicial complex, then we will also keep the same notation for its barycentric subdivisions, i.e., the n-th barycentric subdivision of K is denoted by K^n.

There is a natural map between a finite T_0 topological space X and its finite barycentric subdivision X' which is a weak homotopy equivalence. This map is denoted by $h : X' \to X$, see Subsection 1.1.4 for its definition and [74] for more information.

Remark 4.6.1. *It is easy to show that $|\mathcal{K}(h)|$ is simplicially close to the identity map on $|\mathcal{K}(X)|$. Then, $h : X' \to X$ induces the identity in homology.*

Proposition 4.6.2. *Let X be a finite T_0 topological space. Then $h : X' \to X$ is a Vietoris-like map.*

Proof. We take a chain $x_1 < ... < x_n$ in X and denote $A = \bigcup_{i=1}^{n} h^{-1}(x_i)$. We define $f : A \to A$ given by $f(y) = y \cap \{x_j\}_{j=1}^{n}$, where we denote by $y \cap \{x_j\}_{j=1}^{n}$ the subchain of y given just by elements of $\{x_j\}_{j=1}^{n}$. We prove the continuity of f. If $y \leq z$, then y is a subchain of z. By the construction of f we get easily that $f(y) \leq f(z)$. In addition, $f : A \to f(A)$ is a retraction. We also have that $f \leq id$. If $y \in A$, then $f(y)$ is a subchain of y so $f(y) \leq y = id(y)$. From here, it is easy to deduce that $f(A)$ is a strong deformation retract of A. We get that $f(A)$ contains a maximum, which is $x_1 < x_2 < ... < x_{n-1} < x_n$. Thus, $f(A)$ is contractible. □

Given a finite T_0 topological space X, we denote by $h_{n,n+1} : X^{n+1} \to X^n$ the natural weak homotopy equivalence described before. If $m \geq n$, where $n, m \in \mathbb{N}$, then $h_{n,m} : X^m \to X^n$ is given by $h_{n,n+1} \circ \cdots \circ h_{m-2,m-1} \circ h_{m-1,m}$.

Example 4.6.3. Let X be a finite T_0 topological space that has the weak homotopy type of a point. Let us denote $X^0 = X$. By Proposition 4.6.2, Lemma 4.2.5 and Corollary 4.3.4, for every Vietoris-like multivalued map $F : X^m \multimap X^n$ and $m > n$ there exists a point $x \in X^m$ such that $h_{n,m}(x) \in F(x)$.

We consider the multivalued map $H : X \multimap X'$ given by $H(x) = h^{-1}(x)$. It is important to observe that $H(x)$ consists of chains containing x as a maximum element. More generally, we can consider $H_{n,m} : X^n \multimap X^m$ given by $H_{n,m}(x) = h_{n,m}^{-1}(x)$ for every $m \geq n$ and $m, n \in \mathbb{N}$

We have that H is an example of a usc multivalued map such that $H(x)$ contains a minimum for every $x \in X$ and is a Vietoris-like multivalued map. The chain that consists of a single element $x \in X$ is the minimum of $H(x)$ for every $x \in X$. Now, we prove that H is usc. We consider $x, y \in X$ such that $x \leq y$. If $x' \in H(x)$, then x' is a chain of X containing x as a maximum element. We can extend this chain to a new chain containing y as a maximum element since $x \leq y$. But the new chain is an element of $H(y)$. Therefore, H is a usc multivalued map, see Definition 4.1.8.

Proposition 4.6.4. *Let X be a finite T_0 topological space and $n, m \in \mathbb{N}$ are such that $m \geq n$. Then $H_{n,m} : X^n \multimap X^m$ is a Vietoris-like multivalued map.*

Proof. By Lemma 4.2.5 and Proposition 4.6.2, $h_{n,m}$ is a Vietoris-like map. Finally, by Proposition 4.2.18, we obtain that $H_{n,m}$ is a Vietoris-like multivalued map. $\square$

Remark 4.6.5. *It is easy to show that H induces the identity in homology, it is an immediate consequence of Remark 4.6.1 and the commutativity of the following diagram, where p and q denote the projection onto the first and second coordinates respectively from the graph of H.*

$$\begin{tikzcd} & \Gamma(H) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X' \arrow[ll, "h"] \end{tikzcd}$$

It is clear that q is a weak homotopy equivalence by the 2-out-of-3 property, but it can be obtained that q is indeed a Vietoris-like map by Proposition 4.2.21.

Now, we will use the concept of inverse sequence and inverse limit, see Subsections 1.2.5 and 1.2.5. For a complete exposition of these notions, see for instance [73].

Given a simplicial complex K. Let X^n denote the n-th finite barycentric subdivision of $\mathcal{X}(K)$, where $\mathcal{X}(K)$ is considered with the opposite order. Let $h_{n,n+1} : X^{n+1} \to X^n$ denote the weak homotopy equivalence defined before. In [31], it is proved that the inverse limit of $(X^n, h_{n,n+1})$ contains a homeomorphic copy of K which is a strong deformation retract. Then, the inverse limit of this inverse sequence reconstructs the homotopy type of K.

If $f : |K| \to |L|$ is a continuous map between the geometric realizations of two simplicial complexes, then there is a natural induced morphism by f over the inverse sequences related to K and L. Let $(X^n, h_{n,n+1})$ denote the inverse sequence of finite T_0 topological spaces associated to K and let $(Y^n, h_{n,n+1})$ denote the inverse sequence of finite T_0 topological spaces associated to Y, where we are denoting the bonding maps of the two inverses sequences with the same notation for simplicity. By the simplicial approximation theorem, there exists $n_0 \in \mathbb{N}$ such that for every $m \geq n_0$ there exists a simplicial map from the m-th barycentric subdivision of K to L which is simplicially close to f in $|L|$, that is, there exists $f(0) = n_0$ and a simplicial map $f_0 : K^{f(0)} \to L$ simplicially close to f. Using Theorem 1.1.43, we can obtain the finite version of the previous result, i.e., $\mathcal{X}(f_0) : X^{f(0)} \to Y^0$. Now, we can consider the barycentric subdivision of L, denoted by L^1, and repeat the same arguments so as to obtain $f(1) \geq f(0)$ and $f_1 : K^{f(1)} \to L^1$. Thus, $\mathcal{X}(f_1) : X^{f(1)} \to Y^1$. We can follow inductively until we get an increasing map $f : \mathbb{N} \to \mathbb{N}$ and $\mathcal{X}(f_n) : X^{f(n)} \to Y^n$ continuous for every $n \in \mathbb{N}$. We also denote by f_n the map $\mathcal{X}(f_n)$ for simplicity. Moreover, the following diagram commutes after applying a homological functor for every $m \geq n$.

$$\begin{tikzcd}
X^{f(n)} \arrow[rr, "f_n"] & & Y^n \\
X^{f(m)} \arrow[u, "h_{f(n),f(m)}"] \arrow[rr, "f_m"] & & Y^m \arrow[u, "h_{n,m}"']
\end{tikzcd}$$

Hence, every continuous map between the geometric realization of simplicial complexes induces a morphism between the inverse sequences associated to them.

If $f : |K| \to |K|$ is a continuous map between the geometric realization of a simplicial complex K, then we have an inverse sequence $(X^n, h_{n,n+1})$ and a morphism $\{f_n\}$ from this inverse sequence to itself. We only need to repeat the previous arguments. We construct a new inverse sequence using as bonding maps continuous maps of $\{f_n\}$. We start with $f_0 : X^{f(0)} \to X^0$, we rename $f_{0,1} = f_0$ and $X^1 = X^{f(0)}$. We have $f(1) > f(0)$, so we can consider $f_{1,2} = f_{f(f(0))} : X^{f(f(0))} \to X^1$, we rename $X^{f(f(0))}$ just by X^2 and continue this process. Then, we obtain an inverse sequence $(X^n, f_{n,n+1})$, we also have an inverse sequence isomorphic to $(X^n, h_{n,n+1})$ with the same terms of $(X^n, f_{n,n+1})$ and bonding maps of $(X^n, h_{n,n+1})$ just relabeling and using cofinality. This new inverse sequence will also be denoted by $(X^n, h_{n,n+1})$. It is also trivial to check that $(X^n, h_{n,n+1})$ preserves the same good properties of reconstruction in its inverse limit. An inverse sequence obtained from a geometric realization $|K|$ of a simplicial complex K as we did before will be called finite approximative sequence for f. Now, we can compare all the bonding maps in a direct way because for every $m > n$ we have $f_{n,m}, h_{n,m} : X^m \to X^n$, where $h_{n,m}$ is a Vietoris-like map. We define $\Lambda_{n,m}(f)$ as $\Lambda(f_{n,m*} \circ h_{n,m*}^{-1})$. By Theorem 4.1.11, if $\Lambda_{n,m}(f) \neq 0$, there exists a coincidence point for $f_{n,m}$ and $h_{n,m}$. But $|\mathcal{K}(h_{n,m})|$ is homotopic to the identity map and $|\mathcal{K}(f_{n,m})|$ is homotopic to f, so $\Lambda_{n,m}(f) = \Lambda(f_{n,m*} \circ h_{n,m*}^{-1}) = \Lambda(f)$.

We define a multivalued map for each level of the inverse sequence, we only need to consider the multivalued map $H_{n,m}$ induced by $h_{n,m}$, where $m \geq n$, that is, $H_{n,m}(x) = h_{n,m}^{-1}(x)$ for every $x \in X^n$. Thus, $F_{n+1} : X^{n+1} \multimap X^{n+1}$ is given by $F_{n+1} = H_{n,n+1} \circ f_{n,n+1}$ and we have the following diagram.

$$\begin{tikzcd}
X^0 & X^1 \arrow[l, "h_{0,1}"] \arrow[d, dash, "F_1"] & \dots \arrow[l, "h_{1,2}"] & X^n \arrow[l, "h_{n-1,n}"] \arrow[d, dash, "F_n"] & X^{n+1} \arrow[l, "h_{n,n+1}"] \arrow[d, dash, "F_{n+1}"] & X^{n+2} \arrow[l, "h_{n+1,n+2}"] \arrow[d, dash, "F_{n+2}"] & \arrow[l, "h_{n+2,n+3}"] \\
X^0 & X^1 \arrow[l, "f_{0,1}"] & \dots \arrow[l, "f_{1,2}"] & X^n \arrow[l, "f_{n-1,n}"] & X^{n+1} \arrow[l, "f_{n,n+1}"] & X^{n+2} \arrow[l, "f_{n+1,n+2}"] & \arrow[l, "f_{n+2,n+3}"]
\end{tikzcd}$$

Proposition 4.6.6. $F_{n+1} : X^{n+1} \multimap X^{n+1}$ *is a Vietoris-like multivalued map such that* $F_{n+1*} = H_{n,n+1*} \circ f_{n+1,n*}$ *for every* $n \in \mathbb{N}$.

Proof. By Proposition 4.6.4 we have that $H_{n,n+1}$ is a Vietoris-like multivalued map. By Lemma 4.2.13, F_{n+1} is a Vietoris-like multivalued map. The last part is an immediate consequence of Lemma 4.4.1. □

Proposition 4.6.7. *If* $\Lambda(F_{n+1*}) \neq 0$*, then there exists a point* $x \in X_{n+1}$ *such that* $x \in F_{n+1}(x)$*, where* $n \in \mathbb{N}$.

Proof. By Proposition 4.6.6, $\Lambda(F_{n+1}) = \Lambda(H_{n,n+1*} \circ f_{n,n+1*})$. We prove that $\Lambda(H_{n,n+1*} \circ f_{n,n+1*}) = \Lambda(f_{n,n+1*} \circ h_{n,n+1*}^{-1})$. We know that $h_{n,n+1*}$ is the identity, see Remark 4.6.1. We also have that $H_{n,n+1*}$ is also the identity by Remark 4.6.5. Hence, $\Lambda(f_{n,n+1*} \circ h_{n,n+1*}^{-1}) \neq 0$.

By Theorem 4.1.7, there is a coincidence point for $f_{n,n+1}$ and $h_{n,n+1}$, i.e., there exists $t \in X^{n+1}$ such that $f_{n,n+1}(t) = h_{n,n+1}(t)$. From here, it is easy to deduce that $t \in F_{n+1*}(t)$. $\square$

Remark 4.6.8. *By the proof of Proposition 4.6.7, it can be deduced that $x \in X^{n+1}$ is a fixed point for F_{n+1} if and only if x is a coincidence point for $h_{n,n+1}$ and $f_{n,n+1}$.*

Corollary 4.6.9. *If $\Lambda(f) \neq 0$, then there exists a point $x_{n+1} \in X^{n+1}$ such that $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n \in \mathbb{N}$.*

Proof. By the proof of Proposition 4.6.7, $\Lambda(f) = \Lambda(f_{n,n+1}) = \Lambda(f_{n,n+1*} \circ h^{-1}_{n,n+1*}) = \Lambda(H_{n,n+1*} \circ f_{n,n+1*}) = \Lambda(F_{n+1})$. $\square$

Theorem 4.6.10. *Let K be a compact simplicial complex. If $f : |K| \to |K|$ is a continuous map, then f has a fixed point if and only if there exist a finite approximative sequence for f, denoted by $(X^n, h_{n,n+1})$ and obtained from a triangulation of $|K|$, a sequence $\{x_{n+1}\}_{n\in\mathbb{N}}$ and $m \in \mathbb{N}$ such that $x_{n+1} \in X^{n+1}$, $x_{n+1} = h_{n,n+1}(x_{n+1})$ for every $n \in \mathbb{N}$ and $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n + 1 \geq m$.*

Proof. Firstly, we assume that there exist a finite approximative sequence for f, denoted by $(X^n, h_{n,n+1})$, and a sequence $\{x_{n+1}\}_{n\in\mathbb{N}}$ such that $x_{n+1} \in X_{n+1}$, $x_{n+1} = h_{n,n+1}(x_{n+1})$ for every $n \in \mathbb{N}$ and $x_{n+1} \in F_{n+1}(x_{n+1})$ for every $n + 1 \geq m$. By the proof of Proposition 4.6.7, $h_{n,n+1}(x_{n+1}) = f_{n,n+1}(x_{n+1})$ for every $n \in \mathbb{N}$. We have that $|\mathcal{K}(h_{n,n+1})|$ is simplicially close to the identity map. We also know that $|\mathcal{K}(f_{n,n+1})|$ is simplicially close to $|f_{n,n+1}| : |K^{n+1}| \to |K^n|$ and $|f_{n,n+1}|$ is simplicially close to $|f|$. The maximum diameter for a closed simplex in $|K^n|$ is denoted by ϵ_n for every $n \in \mathbb{N}$. If x_{n+1} is viewed as a point of $|\mathcal{K}(X_{n+1})| = |K^{n+2}| = |K|$, then by the triangle inequality $d(x_{n+1}, f(x_{n+1})) \leq d(x_{n+1}, |\mathcal{K}(h_{n,n+1})|(x_{n+1})) + d(|\mathcal{K}(h_{n,n+1})|(x_{n+1}), |f_{n,n+1}|(x_{n+1})) + d(|f_{n,n+1}|(x_{n+1}), f(x_{n+1})) \leq 3\epsilon_n$, where we also know that $lim_{n\to\infty} 3\epsilon_n = 0$ due to the fact that after a barycentric subdivision the diameters of the new simplices are smaller, see for example [60, 100]. In addition, every $x_{n+1} \in X^{n+1}$ can be seen a closed simplex of $|K|$ since X^{n+1} is the face poset of K^{n+1}, where have that $x_{n+1} \subset x_n$ for every $n \in \mathbb{N}$. Hence, $\bigcap_{n\in\mathbb{N}} x_{n+1}$ is non-empty because it is the intersection of a nested sequence of compact sets. In fact, the diameter of $\bigcap_{n\in\mathbb{N}} x_n$ is zero, which implies that it is a point $x*$. We show that $f(x*) = x*$. The sequence $\{x_{n+1}\}_{n\in\mathbb{N}}$, where we are treating now x_n as a point of $|K|$, is convergent to $x*$. From this, we get the desired result.

If f has a fixed point denoted by t, then we only need to consider a triangulation that has t as a vertex. From here, we construct the desired inverse sequence. $\square$

Example 4.6.11. We consider the unit circle centered at the origin in $\mathbb{R}^2$ and the symmetry $f : S^1 \to S^1$ given by $f(x, y) = (x, -y)$. In Figure 4.6.1 we have a triangulation K of S^1. We denote by $0, 1$ and 2 the vertices of K, where $0 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, $1 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $2 = (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. It is clear that f is not a simplicial map since $f(2)$ is not a vertex of K. We can apply the simplicial approximation theorem. In this case, the first bayrcentric subdivision is enough to get a simplicial map $f_1 : K^1 \to K$ satisfying that it is simplicially close to f. Let $\{i, i + 1\}$ denote the vertex of K^1 that lies between i and $i + 1$, where i and $i + 1$ are considered modulo 3. Then, f_1 can be defined by Table 4.6.1. Applying the theory of [75], we get a continuous map $\mathcal{X}(f_1) : X^1 \to X^0$, where $X^0 = \mathcal{X}(K)$ and $X^1 = \mathcal{X}(K^1) = \mathcal{X}(\mathcal{K}(X^0))$. We have the natural Vietoris-like map $h_{0,1} : X^1 \to X^0$ and Vietoris-like multivalued map $H_{0,1} : X^0 \multimap X^1$. We obtain that $H_{0,1}$ is given by Table 4.6.2. Now, we can consider $F_1 = H_{0,1} \circ \mathcal{X}(f_1)$. Then, a explicit description of F_1 can be seen in Table 4.6.3.

Table 4.6.1: $f_1 : X^1 \to X^0$.

x	0	1	2	$\{0,1\}$	$\{1,2\}$	$\{2,0\}$
$f_1(x)$	1	0	0	0	0	2

Table 4.6.2: $H_{0,1} : X^0 \multimap X^1$.

x	$H_{0,1}(X)$
0	0
1	1
2	2

x	$H_{0,1}(X)$
$\{0,1\}$	$\{0,1\}$, $\{0,\{0,1\}\}$, $\{1,\{0,1\}\}$
$\{1,2\}$	$\{1,2\}$, $\{1,\{1,2\}\}$, $\{2,\{1,2\}\}$
$\{2,0\}$	$\{2,0\}$,$\{0,\{2,0\}\}$, $\{2,\{2,0\}\}$

It is easy to check that F_1 is a Vietoris-like multivalued map by Lemma 4.2.13 and Proposition 4.6.4. In addition, it can be computed that $\Lambda(f) = 2 = \Lambda(F_{1*})$. Therefore, there must exists at least one fixed point for F_1 by Theorem 4.1.12. Indeed, it is immediate to check that F_1 has only two fixed points that are $\{2,\{2,0\}\}$ and $\{0,\{0,1\}\}$. It is easily seen that f has also two fixed points that are $(1,0)$ and $(-1,0)$. In this step or approximation, we can say that we have an approximation of the localization of the fixed points of f.

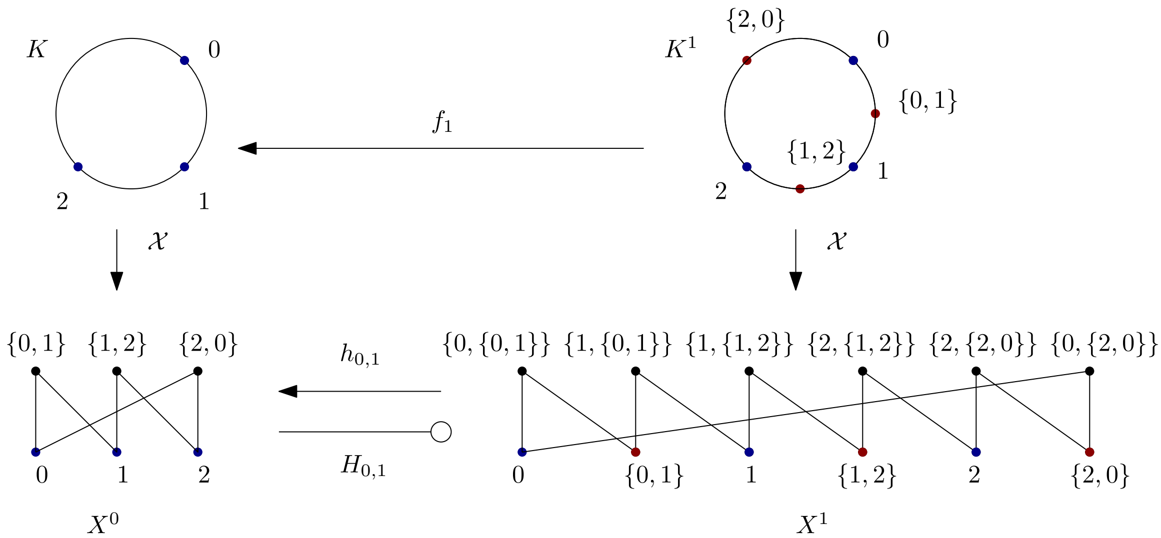

Figure 4.6.1: $f_1 : K^1 \to K$, $h_{0,1} : X^1 \to X^0$ and $H_{0,1} : X^0 \multimap X^1$.

We can continue this method to obtain an approximative sequence for $f : S^1 \to S^1$. We compute the following step. Again, $f : K^1 \to K^1$ is not a simplicial map due to the fact that $f(\{2,3\})$ is not a vertex of K^1. We can apply the simplicial approximation theorem to get a simplicial map f_2 which is simplicially close to f. In this case, it is enough to consider the first barycentric subdivision of K^1, K^2. We define $f_2 : K^2 \to K^1$ given by Table 4.6.4. In Figure 4.6.2 we present K^1, K^2 and the Hasse diagrams of $X^1 = \mathcal{X}(K^1)$ and $X^2 = \mathcal{X}(K^2)$.

We define $F_2 : X^2 \multimap X^2$ given by $H_{1,2} \circ \mathcal{X}(f_2)$, where $X^2 = \mathcal{X}(K^2) = \mathcal{X}(\mathcal{K}(X^1))$. Again, from Lemma 4.2.13 and Proposition 4.6.4 we get that F_2 is a Vietoris-like multivalued map. In addition, we know that $\Lambda(F_{2*}) = 2 = \Lambda(f)$. By Theorem 4.1.12, F_2 has

Table 4.6.3: $F_1 : X^1 \multimap X^1$.

x	$F_1(X)$
0	1
1	0
2	0
$\{0,1\}$	0
$\{1,2\}$	0
$\{2,0\}$	2

x	$F_1(X)$
$\{0,\{0,1\}\}$	$\{0,\{0,1\}\}$, $\{1,\{0,1\}\}$, $\{0,1\}$
$\{1,\{0,1\}\}$	0
$\{1,\{1,2\}\}$	0
$\{2,\{1,2\}\}$	0
$\{0,\{0,2\}\}$	$\{1,\{1,2\}\}$, $\{2,\{1,2\}\}$, $\{1,2\}$
$\{2,\{0,2\}\}$	$\{0,\{0,2\}\}$, $\{2,\{0,2\}\}$, $\{0,2\}$

Table 4.6.4: $f_2 : K^2 \to K^1$.

x	$f_2(X)$
0	1
$\{0,\{0,1\}\}$	$\{0,1\}$
$\{0,1\}$	$\{0,1\}$
$\{1,\{0,1\}\}$	$\{0,1\}$
1	0
$\{1,\{1,2\}\}$	0

x	$f_2(X)$
$\{1,2\}$	0
$\{2,\{1,2\}\}$	$\{2,0\}$
2	$\{2,0\}$
$\{2,\{2,0\}\}$	$\{2,0\}$
$\{2,0\}$	2
$\{0,\{2,0\}\}$	$\{1,2\}$

at least one fixed point. We can describe F_2 explicitly, F_2 is given by Table 4.6.5 and Table 4.6.6. Then, it is trivial to verify that F_2 has only two fixed points that are $\{0,1\}$ and $\{\{2,0\},\{2,\{2,0\}\}\}$. We have a situation similar than in the previous step. However, in this step, we have obtained a real fixed point of S^1, that is, the point $\{0,1\}$ that corresponds to the point $(1,0)$ in S^1. Furthermore, $h_{1,2}(\{\{2,0\},\{2,\{2,0\}\}\}) = \{2,\{2,0\}\}$ is one of the fixed points of F_1.

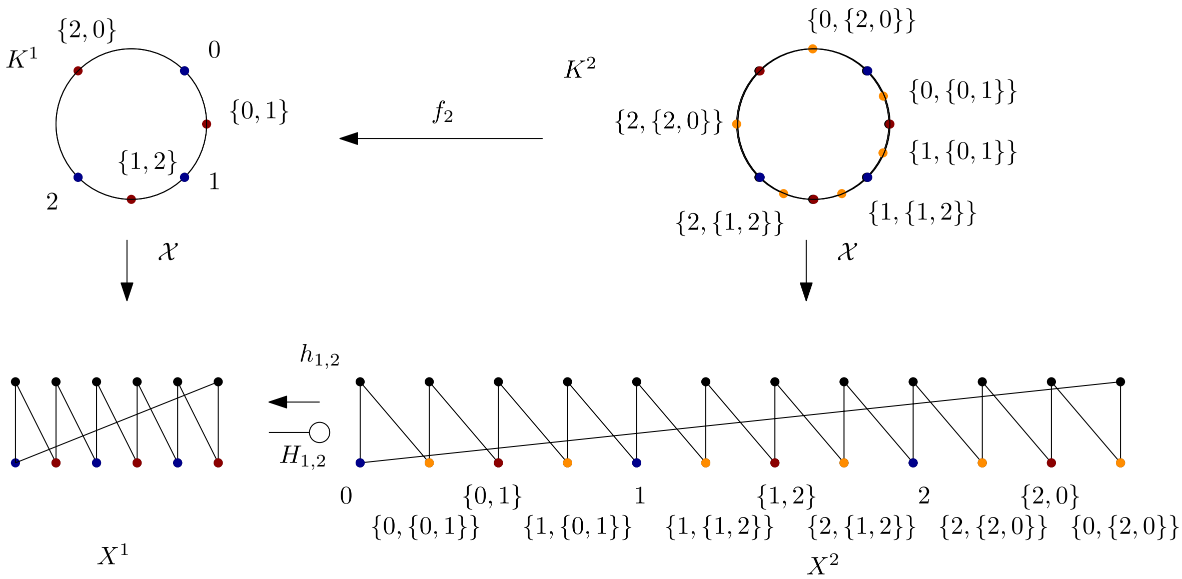

Figure 4.6.2: $f_2 : K^2 \to K^1$, $h_{1,2} : X^2 \to X^1$ and $H_{1,2} : X^1 \multimap X^2$.

If we proceed inductively, we can obtain a sequence of multivalued maps $F_{n,+1}$:

Table 4.6.5: $F_2 : X^2 \multimap X^2$, minimal points of X^2.

x	$F_2(X)$
0	1
$\{0, \{0, 1\}\}$	$\{0, 1\}$
$\{0, 1\}$	$\{0, 1\}$
$\{1, \{0, 1\}\}$	$\{0, 1\}$
1	0
$\{1, \{1, 2\}\}$	0

x	$F_2(X)$
$\{1, 2\}$	0
$\{2, \{1, 2\}\}$	$\{2, 0\}$
2	$\{2, 0\}$
$\{2, \{2, 0\}\}$	$\{2, 0\}$
$\{2, 0\}$	2
$\{0, \{2, 0\}\}$	$\{1, 2\}$

Table 4.6.6: $F_2 : X^2 \multimap X^2$, maximal points of X^2.

x	$F_2(X)$
$\{0, \{0, \{0, 1\}\}\}$	$\{1, \{0, 1\}\}$, $\{\{0, 1\}, \{1, \{0, 1\}\}\}$, $\{1, \{1, \{0, 1\}\}\}$
$\{\{0, 1\}, \{0, \{0, 1\}\}\}$	$\{0, 1\}$
$\{\{0, 1\}, \{1, \{0, 1\}\}\}$	$\{0, 1\}$
$\{1, \{1, \{0, 1\}\}\}$	$\{0, \{0, 1\}\}$, $\{0, \{0, \{0, 1\}\}\}$, $\{\{0, 1\}, \{0, \{0, 1\}\}\}$
$\{1, \{1, \{1, 2\}\}\}$	0
$\{\{1, 2\}, \{1, \{1, 2\}\}\}$	0

x	$F_2(X)$
$\{\{1, 2\}, \{2, \{1, 2\}\}\}$	$\{0, \{2, 0\}\}$, $\{\{2, 0\}, \{0, \{2, 0\}\}\}$, $\{0, \{0, \{2, 0\}\}\}$
$\{2, \{2, \{1, 2\}\}\}$	$\{2, 0\}$
$\{2, \{2, \{2, 0\}\}\}$	$\{2, 0\}$
$\{\{2, 0\}, \{2, \{2, 0\}\}\}$	$\{2, \{2, 0\}\}$, $\{2, \{2, \{2, 0\}\}\}$, $\{\{2, 0\}, \{2, \{2, 0\}\}\}$
$\{\{2, 0\}, \{0, \{2, 0\}\}\}$	$\{2, \{1, 2\}\}$, $\{2, \{2, \{1, 2\}\}\}$, $\{\{1, 2\}, \{2, \{1, 2\}\}\}$
$\{0, \{0, \{2, 0\}\}\}$	$\{1, \{1, 2\}\}$, $\{1, \{1, \{1, 2\}\}\}$, $\{\{1, 2\}, \{1, \{1, 2\}\}\}$

$X^{n+1} \multimap X^{n+1}$. This sequence of multivalued maps is approximating the continuous map f. In Figure 4.6.3 we have a schematic representation of this method, where we have an inverse sequence $(X^n, h_{n,n+1})$ that reconstructs the homotopy type of S^1 in its inverse limit. In addition, we know that for each level of the approximation there exists at least one fixed point by Corollary 4.6.9, that is, for every $n \in \mathbb{N}$ there is a fixed point in $F_{n+1} : X^{n+1} \multimap X^{n+1}$.

Remark 4.6.12. *In Example 4.6.11 we obtained two fixed points in the first and second step (F_1 and F_2), which is precisely the number of fixed points of the map we wanted to approximate. In general, this situation does not always hold true due to a poor approximation or the complexity of the continuous map that generates the dynamical system. If we obtain a better approximation (F_{n+1} for a high value of n), then we could localize the areas where there are candidates to be fixed points.*

4.7 Diagrams

In this section we adapt the notion of diagram that can be seen in [55]. Throughout this section and subsequent sections, every topological space that will be considered is a

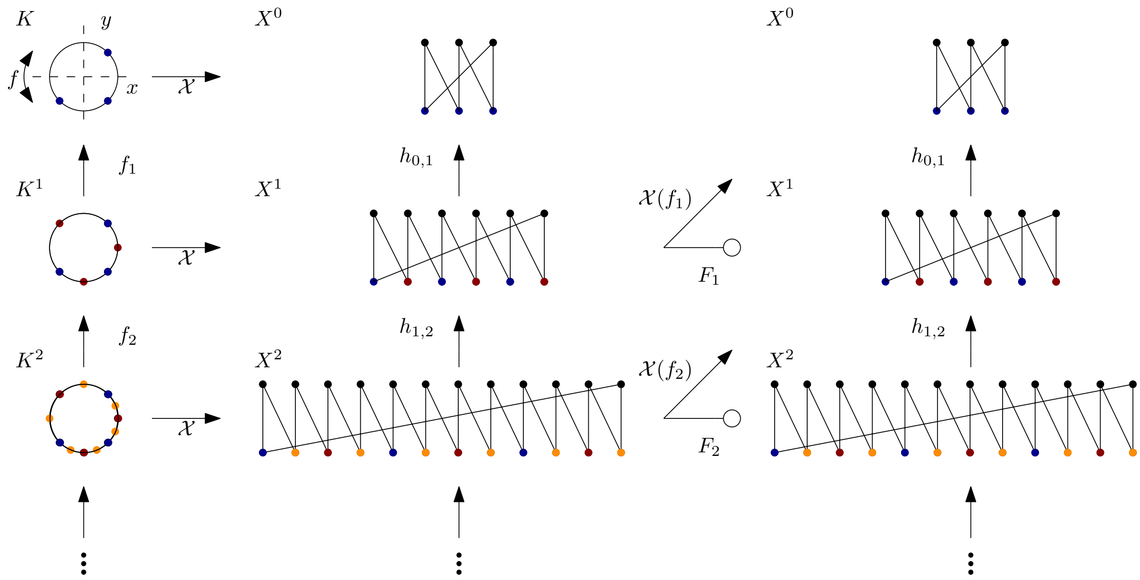

Figure 4.6.3: Approximative sequence for $f : S^1 \to S^1$.

finite T_0 topological space. Let X and Y be finite T_0 topological spaces. We consider $D(X,Y) = \{(p,q,\Gamma) | p : \Gamma \to X$ is a Vietoris-like map, $q : \Gamma \to Y$ is a continuous map and Γ is a finite T_0 topological space $\}$. Hence, $(p,q,\Gamma) \in D(X,Y)$ is just a diagram as follows:

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

In addition, given a finite topological space X, the identity diagram for X is given as follows: (id, id, X), where id denotes the identity map, that is, we have the following diagram:

$$\begin{tikzcd} & X \arrow[dl, "id"'] \arrow[dr, "id"] & \\ X & & X \end{tikzcd}$$

Before giving the definition for composing diagrams, we see some basic examples.

Proposition 4.7.1. *If $F : X \multimap Y$ is a Vietoris-like multivalued map, then $(p_F, q_F, \Gamma(F)) \in D(X,Y)$.*

Proof. By hypothesis, p_F is a Vietoris-like map and the graph of F is also a finite T_0 topological space. $\square$

Since every continuous map $f : X \to Y$ can be seen as a Vietoris-like multivalued map, we have $(p_f, q_f, \Gamma(f)) \in D(X,Y)$, where p_f is more than a Vietoris-like map because it is indeed a homeomorphism.

We introduce some technical notions to define the composition of diagrams.

Definition 4.7.2. *We consider the following diagram:*

$$X_1 \xrightarrow{\quad q \quad} Y \xleftarrow{\quad p \quad} X_2,$$

where $q : X_1 \to Y$ and $p : X_2 \to Y$ are continuous maps between finite T_0 topological spaces. The fibre product of the previous diagram is a map $f : X_1 \boxtimes_Y X_2 \to Y$ given by $f(x_1, x_2) = q(x_1)$, where $X_1 \boxtimes_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 | q(x_1) = p(x_2)\}$ is a finite T_0 topological space because it is a subposet of $X_1 \times X_2$ with the product order. The pull-back of the previous diagram is the following diagram:

$$X_1 \xleftarrow{\overline{p}} X_1 \boxtimes_Y X_2 \xrightarrow{\overline{q}} X_2,$$

where $\overline{p}(x_1, x_2) = x_1$ and $\overline{q}(x_1, x_2) = x_2$ for every $(x_1, x_2) \in X_1 \boxtimes_Y X_2$.

Definition 4.7.2, can be given in a more general way, see [55] for more information.

Lemma 4.7.3. *If we have the following diagram:*

$$X_1 \xrightarrow{q} Y \xleftarrow{p} X_2,$$

where p is a Vietoris-like map, then $\overline{p} : X_1 \boxtimes X_2 \to X_1$ given by $\overline{p}(x_1, x_2) = x_1$ in the pull-back of the previous diagram is a Vietoris-like map.

Proof. We need to prove that $\overline{p}$ is a Vietoris-like map, where we have the following diagram:

$$\begin{tikzcd} & X_1 \boxtimes_Y X_2 \arrow[dl, "\overline{p}"'] \arrow[dr, "\overline{q}"] & \\ X_1 \arrow[r, "q"'] & Y & X_2. \arrow[l, "p"] \end{tikzcd}$$

Let $x_1 < ... < x_n$ be a chain in X_1. We will prove that $A = \bigcup_{i=1}^n \overline{p}^{-1}(x_i)$ is acyclic. We consider $B = \bigcup_{i=1}^n p^{-1}(q(x_i))$, which is acyclic by hypothesis. Then, the idea of the proof is to show that B is homotopy equivalent to A.

We define $T : B \to A$ given by $T(z) = (x_z, z)$, where $x_z = \min_{i=1,...,n}\{x_i | q(x_i) = p(z)\}$. It is clear that for every z we have that x_z is well-defined since $x_1 < ... < x_n$ is a finite totally ordered set. By construction, it follows easily that $(x_z, z) \in X_1 \boxtimes_Y X_2$. Hence, T is well-defined. Now, we verify that T is continuous. Let us consider $y \leq z$, where $y, z \in A$. We argue by contradiction. Suppose that $(x_y, y) = T(y) \not\leq T(z) = (x_z, z)$. Since x_y and x_z are elements of a totally ordered finite set, it follows that $x_y > x_z$. But $q(x_y) = p(y) \leq p(z) = q(x_z)$, which implies that $q(x_y) = p(y) = p(z) = q(x_z)$. Thus, we have $x_z = x_y$, which entails a contradiction.

We consider $S : A \to B$ given by $S(x_i, y) = y$. By construction, S is well-defined because $p(y) = q(x_i)$, so $y \in p^{-1}(q(x_i))$. It is trivial to show the continuity of S.

Finally, we have $S \circ T = id_B : B \to B$. For every $(x_i, y) \in A$ we get $T(S(x_i, y)) = T(y) = (x_y, y) \leq (x_i, y)$, which implies that $T \circ S \leq id_A$. Thus, A is homotopy equivalent to B. $\square$

Now, we can define the composition of diagrams, as we wanted.

Definition 4.7.4. *Given $(p, q, \Gamma) \in D(X, Y)$ and $(p', q', \Gamma') \in D(Y, Z)$. The composition $(p', q', \Gamma') \circ (p, q, \Gamma)$ is given by the pull-back of the following diagram*

$$\Gamma \xrightarrow{q} Y \xleftarrow{p'} \Gamma',$$

that is, $(p', q', \Gamma') \circ (p, q, \Gamma) = (p \circ \overline{p}', q' \circ \overline{q}, \Gamma \boxtimes \Gamma'_Y)$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
& & \Gamma \boxtimes_Y \Gamma' \arrow[dl, "\overline{p}'"'] \arrow[dr, "\overline{q}"] & & \\
& \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & & \Gamma' \arrow[dl, "p'"'] \arrow[dr, "q'"] & \\
X & & Y & & Z
\end{tikzcd}$$

Remark 4.7.5. *Since the composition of Vietoris-like maps is again a Vietoris-like map, it follows that the composition of diagrams is again a diagram by Lemma 4.7.3*

Let X, Y and Z be finite topological spaces. For simplicity, if $f : X \to Y$ is a continuous map, i.e., $(p_f, q_f, \Gamma(f)) \in D(X, Y)$, and $(p, q, \Gamma) \in D(Y, Z)$, then $(p, q, \Gamma) \circ (p_f, q_f, \Gamma(f))$ will be denoted just by $(p, q, \Gamma) \circ f$.

Remark 4.7.6. *Notice that diagrams do not form a category. One of the reasons is the following. We consider a diagram $(p, q, \Gamma) \in D(X, Y)$ and (id, id, Y). It is immediate that $(id, id, Y) \circ (p, q, \Gamma) = (p \circ \overline{id}, id \circ \overline{q}, \Gamma \boxtimes_Y Y) \neq (p, q, \Gamma)$. It is simple to check that there exists a homeomorphism T making the following diagram commutative.*

$$\begin{tikzcd}
& \Gamma \boxtimes_Y Y \arrow[dl, "p\circ\overline{id}"'] \arrow[dr, "id\circ\overline{q}"] & \\
X & & Y \\
& \Gamma \arrow[ul, "p"] \arrow[ur, "q"'] \arrow[uu, "T"] &
\end{tikzcd}$$

We define $T : \Gamma \to \Gamma \boxtimes_Y Y$ given by $T(x, y) = ((x, y), (y, y))$. We get that T is clearly well-defined and continuous. The bijectivity of T follows from the construction of $\Gamma \boxtimes_Y Y$. In addition, every point $z \in \Gamma \boxtimes_Y Y$ is of the form $((x, y), (y, y))$ for some $x \in X$ and $y \in Y$. Therefore, $T^{-1}((x, y), (y, y)) = (x, y)$ is trivially continuous and the inverse of T.

The composition with the identity fails due to the definition of composition that was given. If we want to solve this problem to get a category, then a new equivalence relation between diagrams must be given. This relation is the one that will be given in Section 4.8.

Let X and Y be finite topological spaces. Every diagram $(p, q, \Gamma) \in D(X, Y)$ induces morphisms in homology. We get that $H_*(p, q, \Gamma) = q_* \circ p_*^{-1} : H_*(X) \to H_*(Y)$ is well-defined because p is a Vietoris-like map. For clarity, we will also denote $H_*(p, q, \Gamma)$ just by $H_*(p, q)$. Moreover, the homology behaves properly for the composition of diagrams.

Proposition 4.7.7. *If $(p, q, \Gamma) \in D(X, Y)$ and $(p', q', \Gamma') \in D(Y, Z)$, then $H_*(p', q') \circ H_*(p, q) = H_*((p', q', \Gamma') \circ (p, q, \Gamma))$*

Proof. By Definition 4.7.4, we have that $(p', q') \circ (p, q) = (p \circ \overline{p}', q' \circ \overline{q})$ and $q \circ \overline{p}' = p' \circ \overline{q}$, where $\overline{p}' : \Gamma \boxtimes_Y \Gamma' \to \Gamma$ is given by $\overline{p}'(x, y) = x$ and $\overline{q} : \Gamma \boxtimes_Y \Gamma' \to \Gamma'$ is given by $\overline{q}(x, y) = y$. Therefore,

$$H_*((p', q') \circ (p, q)) = (q \circ \overline{q})_* \circ (p \circ \overline{p}')_*^{-1} = q_* \circ \overline{q}_* \circ \overline{p}_*'^{-1} \circ p_*^{-1} =$$

$$= q'_* \circ p'^{-1}_* \circ q_* \circ p_*^{-1} = H_*(p', q') \circ H_*(p, q).$$

□

Definition 4.7.8. *A fixed point for a diagram $(p, q, \Gamma) \in D(X, X)$ is a point $x \in X$ satisfying that $x \in q(p^{-1}(x))$.*

Given a diagram $(p, q, \Gamma) \in D(X, Y)$, we define the Lefschetz number of the diagram as follows:

$$\Lambda(p, q, \Gamma) = \sum_{i=0} (-1)^n trace(q_* \circ p_*^{-1} : H_i(X) \to H_i(X)),$$

where *trace* denotes the trace of a linear map. Using the coincidence theorems for finite topological spaces obtained in Section 4.3, it can be deduced a Lefschetz fixed point theorem for diagrams.

Theorem 4.7.9 (Lefschetz fixed point theorem for diagrams)**.** *Given a diagram $(p, q, \Gamma) \in D(X, X)$. If $\Lambda(p, q, \Gamma) \neq 0$, then there exists a fixed point x for (p, q, Γ).*

Proof. We are in the hypothesis of Theorem 4.1.11, which implies the desired result. □

4.8 Category of morphisms

Let X and Y be finite topological spaces. We introduce a relation in $D(X, Y)$. Given $(p, q, \Gamma), (p', q', \Gamma') \in D(X, Y)$, we write $(p, q, \Gamma) \sim (p', q', \Gamma')$ if and only if there exists a homeomorphism $h : \Gamma \to \Gamma'$ satisfying that the following diagram commutes.

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\
X & & Y \\
 & \Gamma' \arrow[ul, "p'"] \arrow[ur, "q'"'] &
\end{tikzcd}$$

Lemma 4.8.1. *Let X and Y be finite topological spaces. The relation $\sim$ in $D(X, Y)$ is an equivalence relation*

Proof. Reflexivity. It is clear that $(p, q, \Gamma) \sim (p, q, \Gamma)$, we only need to consider the identity map $id : \Gamma \to \Gamma$.

Symmetry. If $(p, q, \Gamma) \sim (p', q', \Gamma')$, then there exists a homeomorphism $h : \Gamma \to \Gamma'$ such that $p = p' \circ h$ and $q = q' \circ h$. Therefore, if we consider h^{-1} we get that $(p', q', \Gamma') \sim (p, q, \Gamma)$ since $q' = q \circ h^{-1}$ and $p' = p \circ h^{-1}$.

Transitivity. If $(p, q, \Gamma) \sim (p', q', \Gamma')$ and $(p', q', \Gamma') \sim (\overline{p}, \overline{q}, \overline{\Gamma})$, then there exist homeomorphisms $h : \Gamma \to \Gamma'$ and $h' : \Gamma' \to \overline{\Gamma}$ such that $p = p' \circ h$, $q = q' \circ h$, $p' = \overline{p} \circ h'$ and $q' = \overline{q} \circ h'$. Then, $h' \circ h : \Gamma \to \overline{\Gamma}$ is a homeomorphism such that $p = \overline{p} \circ h' \circ h$ and $q = \overline{q} \circ h' \circ h$, which implies that $(p, q, \Gamma) \sim (\overline{p}, \overline{q}, \overline{\Gamma})$. □

Lemma 4.8.2. *Given $(p, q, \Gamma), (p*, q*, \Gamma*) \in D(X, Y)$ and $(p', q', \Gamma'), (p^+, q^+, \Gamma^+) \in D(Y, Z)$. If $(p, q, \Gamma) \sim (p*, q*, \Gamma*)$ and $(p', q', \Gamma') \sim (p^+, q^+, \Gamma^+)$, then $(p, q, \Gamma) \circ (p', q', \Gamma') \sim (p*, q*, \Gamma*) \circ (p^+, q^+, \Gamma^+)$.*

Proof. By hypothesis, there exist homeomorphisms $h : \Gamma \to \Gamma*$ and $h' : \Gamma' \to \Gamma^+$ such that the following diagram is commutative.

$$\begin{tikzcd}
& \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & & \Gamma' \arrow[dl, "p'"'] \arrow[dr, "q'"] \arrow[dd, "h'"] & \\
X & & Y & & Z \\
& \Gamma* \arrow[ul, "p*"] \arrow[ur, "q*"'] & & \Gamma^+ \arrow[ul, "p^+"] \arrow[ur, "q^+"'] &
\end{tikzcd}$$

We consider $H = h \times h' : \Gamma \times \Gamma' \to \Gamma * \times \Gamma^+$ given by $H(s,t) = (h(s), h'(t))$, where we consider the product order in $\Gamma \times \Gamma'$ and $\Gamma * \times \Gamma^+$. It is clear that H is well-defined and bijective. The continuity of H follows trivially. If $(t,s) \leq (u,v)$, then $H(t,s) = (h(t), h'(s)) \leq (h(u), h'(v)) = H(u,v)$ by the continuity of h and h'. The inverse of H is given by $H^{-1} = h^{-1} \times h'^{-1}$, which is also continuous and bijective. Therefore, H is a homeomorphism. If we restrict the domain of H to $\Gamma \boxtimes_Y \Gamma' \subseteq \Gamma \times \Gamma'$, then we get the desired homeomorphism. For simplicity, we denote $\underline{h} = H_{|\Gamma \boxtimes_Y \Gamma'}$. We show that $\underline{h}$ is well-defined. Let us consider $(s,t) \in \Gamma \boxtimes \Gamma'$, we need to show that $\underline{h}(s,t) \in \Gamma * \boxtimes_Y \Gamma^+$, i.e., $p^+(h'(t)) = q * (h(s))$. We know by the commutativity of the diagram above that $p = p * \circ h$, $q = q * \circ h$, $p' = p^+ \circ h'$ and $q' = q^+ \circ h'$. Therefore, $q * (h(s)) = q(s) = p'(t) = p^+(h'(t))$. From here, it is easy to deduce that $\underline{h}$ is the desired homeomorphism to prove that $(p,q,\Gamma) \circ (p',q',\Gamma') \sim (p*, q*, \Gamma*) \circ (p^+, q^+, \Gamma^+)$. □

We define the category of morphisms $\mathcal{M}$ for finite topological spaces. The objects of $\mathcal{M}$ are the finite topological spaces. Let X and Y be finite topological spaces. Then the set of morphisms from X to Y is given by $\mathcal{M}(X,Y) = D(X,Y)/\sim$.

Theorem 4.8.3. *$\mathcal{M}$ is a category.*

Proof. We prove the associativity. By Lemma 4.8.2, the representatives are independent. Therefore, we only need to check that $((p,q,\Gamma) \circ (p',q',\Gamma')) \circ (\overline{p}, \overline{q}, \overline{\Gamma}) \sim (p,q,\Gamma) \circ ((p',q',\Gamma') \circ (\overline{p}, \overline{q}, \overline{\Gamma}))$, where $(p,q,\Gamma) \in D(Z,W)$, $(p',q',\Gamma') \in D(Y,Z)$ and $(\overline{p}, \overline{q}, \overline{\Gamma}) \in D(X,Y)$. We consider $T : (\overline{\Gamma} \boxtimes_Y \Gamma') \boxtimes_Z \Gamma \to \overline{\Gamma} \boxtimes_Y (\Gamma' \boxtimes_Z \Gamma)$ given by $T(((x,y),(y',z),(z',w)) = ((x,y),((y',z),(z',w)))$. It is routine to check that T is the desired homeomorphism that proves the result.

The properties regarding to the identity morphism can be deduced from Remark 4.7.6. □

Remark 4.8.4. *If $f : X \to Y$ and $g : X \to Y$ are single-valued continuous maps, then Definition 4.7.4 corresponds with the usual notion of composition. We get that f can be seen as $(p_f, q_f, \Gamma(f))$ and g can be seen as $(p_g, q_g, \Gamma(g))$. Similarly, $g \circ f$ can be seen as $(p_{g \circ f}, q_{g \circ f}, \Gamma(g \circ f))$. Therefore, we have that $(p_g, q_g, \Gamma(g)) \circ (p_f, q_f, \Gamma(f)) \sim (p_{g \circ f}, q_{g \circ f}, \Gamma(g \circ f))$. To prove the last assertion we only need to consider $T : \Gamma(g \circ f) \to \Gamma(g) \boxtimes_Y \Gamma(f)$ given by $T(x,y) = ((x, f(x)), (f(x), y))$. It is easy to prove that T is a homeomorphism* *making the following diagram commutative.*

$$\begin{tikzcd}
& \Gamma(f) \boxtimes_Y \Gamma(g) \arrow[dl, "p_f \circ \overline{p_g}"'] \arrow[dr, "q_g \circ \overline{q_f}"] & \\
X & & Z \\
& \Gamma(g \circ f) \arrow[ul, "p_{g \circ f}"] \arrow[ur, "q_{g \circ f}"'] \arrow[uu, "T"'] &
\end{tikzcd}$$

We prove that the homological functor H_* can be applied properly to $\mathcal{M}$. To do that, we show a technical lemma.

Lemma 4.8.5. *Let X and Y be finite topological spaces. If $(p, q, \Gamma), (p', q', \Gamma') \in D(X, Y)$ satisfy that $(p, q, \Gamma) \sim (p', q', \Gamma')$, then $H_*(p, q) = H_*(p', q')$.*

Proof. By hypothesis, there exists a homeomorphism $h : \Gamma \to \Gamma'$ satisfying that $p = p' \circ h$ and $q = q' \circ h$. Therefore, applying H_* and using the previous relations, we get

$$H_*(p, q) = q_* \circ p_*^{-1} = q'_* \circ h_* \circ (p' \circ h)_*^{-1} = q'_* \circ h_* \circ h_*^{-1} \circ p'^{-1}_* = q'_* \circ p'^{-1}_* = H_*(p', q').$$

□

Proposition 4.8.6. *If $\mathcal{A}$ denotes the category of abelian groups, then the homological functor $H_* : \mathcal{M} \to \mathcal{A}$ is well-defined.*

Proof. It is an immediate consequence of Lemma 4.8.2 and Lemma 4.8.5. □

Again, we can obtain a Lefschetz fixed point theorem. Before stating this theorem, we prove an auxiliary lemma.

Lemma 4.8.7. *If x is a fixed point for $(p, q, \Gamma) \in D(X, X)$ and $(p', q', \Gamma') \in D(X, X)$ satisfies that $(p, q, \Gamma) \sim (p', q', \Gamma')$, then x is a fixed point for (p', q', Γ').*

Proof. Since $(p, q, \Gamma) \sim (p', q', \Gamma')$, it follows that there exists a homeomorphism $h : \Gamma \to \Gamma'$ such that $p = p' \circ h$ and $q = q' \circ h$. If x is a fixed point for (p, q, Γ), there exists $y \in p^{-1}(x)$ satisfying that $p(y) = x = q(y)$. Hence, we only need to consider $h(y)$ to get the desired result. We have $h(y) \in p'^{-1}(x)$ because $x = p(y) = p'(h(y))$ and $q'(h(y)) = x$ because $x = q(y) = q'(h(y))$. Therefore, we get $x \in q'(p'^{-1}(x))$. □

The previous lemma proves that the notion of fixed point is well-defined for morphisms. By Lemma 4.8.5, we get that the Lefschetz number that was defined for diagrams in Section 4.7 is well-defined for morphisms, i.e., the Lefschetz number does not depend on the representative chosen.

Theorem 4.8.8 (Lefschetz fixed point theorem for morphisms)**.** *If $[(p, q, \Gamma)] \in \mathcal{M}(X, X)$ satisfies that $\Lambda([(p, q, \Gamma)]) \neq 0$, then there exists a fixed point for $[(p, q, \Gamma)]$*

Proof. It is an immediate consequence of Lemma 4.8.7, Lemma 4.8.5 and Theorem 4.7.9. □

4.9 Homotopical category of diagrams

We introduce a notion of homotopy for diagrams, the idea is to generalize the notion of homotopy introduced in [20] for strongly upper semicontinuous (susc) multivalued maps with acyclic values. We recall this notion of homotopy.

Definition 4.9.1. *Let X and Y be finite topological spaces. Two susc multivalued maps $F, G : X \multimap Y$ with acyclic values are said to be homotopic, if there exists a susc multivalued map $H : X \times [0,1] \multimap Y$ with acyclic values which satisfies $H(x,0) = F(x)$ and $H(x,1) = G(x)$ for every $x \in X$.*

In [20], the following combinatorial characterization of Definition 4.9.1 is obtained.

Proposition 4.9.2. *Let X and Y be two finite topological spaces. Then two susc multivalued maps $F, G : X \multimap Y$ with acyclic values are homotopic if and only if there exists a sequence of multivalued maps $F_i : X \multimap Y$ which are susc and have acyclic values, and which satisfy $F(x) = F_0(x) \subseteq F_1(x) \supseteq F_2(x) \subseteq \ldots F_n(x) = G(x)$ for every $x \in X$.*

Now, we give our notion of homotopy.

Definition 4.9.3. *Given two finite topological spaces X and Y and $(p,q,Z),(p',q',Z') \in D(X,Y)$. We say that (p,q,Z) is homotopic to (p',q',Z') if there exists $(p*,q*,Z*) \in D(X \times \{0,1\}, Y)$ and maps $f_0 : Z \to Z*$, $f_1 : Z' \to Z*$ such that the following diagram commutes.*

$$\begin{tikzcd}
X \arrow[d, "i_0"'] & Z \arrow[l, "p"'] \arrow[d, "f_0"'] \arrow[dr, "q"] & \\
X \times \{0,1\} & Z* \arrow[l, "p*"] \arrow[r, "q*"'] & Y \\
X \arrow[u, "i_1"] & Z' \arrow[l, "p'"] \arrow[u, "f_1"] \arrow[ur, "q'"'] &
\end{tikzcd}$$

Where $i_0(x) = (x,0)$, $i_1(x) = (x,1)$, $\{0,1\}$ is a poset $(0 < 1)$ and $X \times \{0,1\}$ is the product of posets, that is, $(x,t) \leq (y,s)$ if and only if $x \leq y$ and $t \leq s$. We say that $(p,q*,Z*)$ is the homotopy between (p,q,Z) and (p',q',Z').*

Remark 4.9.4. *Definition 4.9.3 is an adaptation of the definition given in [41] for the classical setting of multivalued maps. At the same time, the definition given in [41] arises as a sort of generalization of the one given in [55]. The need of the definition given in [41] relies on the fact that with this notion of homotopy, the homotopy relation is an equivalence relation.*

With the following lemma, a useful way to check if two diagrams are homotopic is given.

Lemma 4.9.5. *If (p_0,q_0,Z_0), $(p_1,q_1,Z_1) \in D(X,Y)$ and there exists a map $f : Z_0 \to Z_1$*

making the following diagram commutative

$$\begin{tikzcd} & Z_0 \arrow[dl, "p_0"'] \arrow[dd, "f"] \arrow[dr, "q_0"] & \\ X & & Y, \\ & Z_1 \arrow[ul, "p_1"] \arrow[ur, "q_1"'] & \end{tikzcd}$$

then (p_0, q_0, Z_0) is homotopic to (p_1, q_1, Z_1).

Proof. We consider the non-Hausdorff mapping cylinder of f, denoted by $Z*$, i.e., we consider the disjoint union of Z_0 and Z_1, where we have that for $y \in Z_1$ and $z \in Z_0$, $y > z$ if and only if $f(z) = y$. We define $p* : Z* \to X \times \{0,1\}$ given by $p*(z) = (p_0(z), 0)$ if $z \in Z_0$ and $p*(z) = (p_1(z), 1)$ if $z \in Z_1$. We prove the continuity of $p*$. If $y > z$, where $y \in Z_1$ and $z \in Z_0$, then $f(z) = y$ and $p*(y) = (p_1(y), 1) = (p_1(f(z)), 1) = (p_0(z), 1) \geq (p_0(z), 0) = p*(z)$. It is simple to check that $p*$ preserves the remaining relations. Now, we show that $p*$ is a Vietoris-like map. If $x_1 < ... < x_n$ is a chain in $X \times \{0,1\}$, then we need to prove that $A = \bigcup_{i=1}^n p*^{-1}(x_i)$ is acyclic. Suppose $A \cap Z_0 = \emptyset$ or $A \cap Z_1 = \emptyset$. Since p_0 and p_1 are Vietoris-like maps, we get the desired result. Suppose the opposite, i.e., $A \cap Z_0 \neq \emptyset$ and $A \cap Z_1 \neq \emptyset$. We consider $r : A \to Z_1 \cap A$ given by $r(z) = z$ if $z \in Z_1 \cap A$ and $r(z) = f(z)$ if $z \in Z_0 \cap A$. We prove that r is continuous. If $z \in Z_0$ and $y \in Z_1$ satisfy that $y > z$, then $r(y) = y = f(z) = r(z)$. It is trivial to prove that r is order-preserving for the remaining relations. Let $i : Z_1 \cap A \to A$ denote the inclusion map. We have that $r \circ i = id$ and $i \circ r \geq id$. Thus, $Z_1 \cap A$ is a strong deformation retract of A. Since p_1 is a Vietoris-like map, $Z_1 \cap A$ is acyclic. Similarly, we define $q* : Z* \to Y$, that is, $q*(z) = q_0(z)$ if $z \in Z_0$ and $q*(z) = q_1(z)$ if $z \in Z_1$. It is simple to check that $q*$ is continuous. If $z < y$, where $z \in Z_0$ and $y \in Z_1$, then $f(z) = y$ and we get $q*(y) = q_1(y) = q_1(f(z)) = q_0(z)$. We get that $q*$ is order-preserving for the remaining relations since q_0 and q_1 are continuous maps. We have the following diagram.

$$\begin{tikzcd} X \arrow[d, "i_0"'] & Z_0 \arrow[l, "p_0"] \arrow[d, "i_{Z_0}"'] \arrow[dr, "q_0"] & \\ X \times \{0,1\} & Z* \arrow[l, "p*"] \arrow[r, "q*"] & Y \\ X \arrow[u, "i_1"] & Z_1 \arrow[l, "p_2"] \arrow[u, "i_{Z_1}"] \arrow[ur, "q_1"'] & \end{tikzcd}$$

Where i_{Z_1} and i_{Z_0} denote the natural inclusions. By construction, it is easy to deduce that the diagram is commutative. $\square$

Remark 4.9.6. *Lemma 4.9.5 is also telling that if $(p, q, \Gamma), (p', q', \Gamma') \in D(X, Y)$ are such that $(p, q, \Gamma) \sim (p', q', \Gamma')$, then they are homotopic.*

The usual composition of continuous maps does not correspond exactly with the composition of its respective diagrams. But the graph of the usual composition and the domain of the fiber product are homeomorphic topological spaces. The last assertion is a trivial exercise. However, we can check that both compositions are homotopic.

Proposition 4.9.7. *Let X, Y and Z be finite topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then the usual composition $g \circ f$ as a diagram is homotopic to $(p_g, q_g, \Gamma(g)) \circ (p_f, q_f, \Gamma(f))$*

Proof. The idea of the proof is to use Lemma 4.9.5, that is, we need to find a continuous map $T : \Gamma(g \circ f) \to \Gamma(f) \boxtimes_Y \Gamma(g)$ satisfying that the following diagram commutes.

$$\begin{tikzcd}
 & \Gamma(f) \boxtimes_Y \Gamma(g) \arrow[dl, "p_f \circ \overline{p_g}"'] \arrow[dr, "q_g \circ \overline{q_f}"] & \\
X & & Z \\
 & \Gamma(g \circ f) \arrow[uu, "T"] \arrow[ul, "p_{g \circ f}"] \arrow[ur, "q_{g \circ f}"'] &
\end{tikzcd}$$

We define T given by $T(x, y) = ((x, f(x)), (f(x), y))$. It is clear that T is well-defined by construction. The continuity of T follows from the continuity of f. Finally, the diagram commutes trivially. $\square$

In our case, as it happens in [41], the homotopy relation is an equivalence relation.

Lemma 4.9.8. *The homotopy relation is an equivalence relation in $D(X, Y)$.*

Proof. We prove the transitive property. The reflexive and symmetric properties are trivial to show. Suppose that (p_0, q_0, Z_0) is homotopic to (p_1, q_1, Z_1) and (p_1, q_1, Z_1) is homotopic to (p_2, q_2, Z_2, Z_2), that is, there exist Z, $\overline{Z}$ and continuous maps f_0, f_1, f_2, f_3, p, q, $\overline{p}$, $\overline{q}$ such that p and $\overline{p}$ are Vietoris-like maps and the following two diagrams are commutative.

$$\begin{tikzcd}
X \arrow[d, "i_0"'] & Z_0 \arrow[l, "p_0"'] \arrow[d, "f_0"'] \arrow[dr, "q_0"] & \\
X \times \{0,1\} & Z \arrow[l, "p"] \arrow[r, "q"'] & Y \\
X \arrow[u, "i_1"] & Z_1 \arrow[l, "p_1"] \arrow[u, "f_1"] \arrow[ur, "q_1"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, "i_0"'] & Z_1 \arrow[l, "p_1"'] \arrow[d, "f_2"'] \arrow[dr, "q_1"] & \\
X \times \{0,1\} & \overline{Z} \arrow[l, "\overline{p}"] \arrow[r, "\overline{q}"'] & Y \\
X \arrow[u, "i_1"] & Z_2 \arrow[l, "p_2"] \arrow[u, "f_3"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

We consider $Z_1 \times \{0, 1\}$, where $0 < 1$. Hence, we have the product of two posets. We consider $B(f_1) = Z_1 \times \{0, 1\} \sqcup Z$, where $Z_1 \times \{0, 1\} \sqcup Z$ denotes the disjoint union of $Z_1 \times \{0, 1\}$ and Z and we have that $(x, 1) > y$ if and only if $f_1(x) = y$. Analogously, we consider $B(f_2) = Z_1 \times \{0, 1\} \sqcup \overline{Z}$, where $(x, 0) > y$ if and only if $f_2(x) = y$. We can say that $B(f_1)$ and $B(f_2)$ are a sort of non-Hausdorff mapping cylinders. Finally, we define $Z* = B(f_1) \cup B(f_2)$. In Figure 4.9.1 we present a schematic representation of $Z*$. We define $p* : Z* \to X \times \{0, 1\}$ given by

$$p * (s) = \begin{cases} (p_1(x), t), & \text{if } s = (x, t) \in Z_1 \times \{0, 1\} \\ p(y) & \text{if } s = y \in Z \\ \overline{p}(y) & \text{if } s = y \in \overline{Z}. \end{cases}$$

We need to show that $p*$ is a Vietoris-like map. Firstly, we show the continuity of $p*$. We study cases. If $z < s$, where $z, s \in Z_1 \times \{0, 1\}$ or $z, s \in Z$ or $z, s \in \overline{Z}$, then $p * (z) < p * (s)$

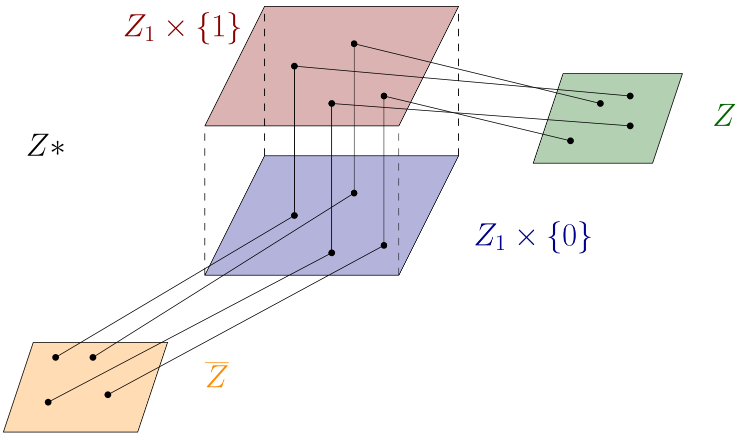

Figure 4.9.1: Schematic representation of $Z*$.

trivially. Suppose $(z,1) > y$, where $y \in Z$ and $(z,1) \in Z_1 \times \{0,1\}$. By construction, we know that $f_1(z) = y$. Using the commutativity of the first diagram, we get

$$p*(z,1) = (p_1(z),1) = i_1(p_1(z)) = p(f_1(z)) = p(y) = p*(y).$$

Similarly, if $(z,0) > y$, where $(z,0) \times Z_1 \times \{0,1\}$ and $y \in \overline{Z}$, then

$$p*(z,0) = (p_1(z),0) = i_0(p_1(z)) = \overline{p}(f_2(z)) = \overline{p}(y) = p*(y).$$

We consider a chain $x_1 < ... < x_n$ in $X \times \{0,1\}$ and denote $A = \bigcup_{i=1}^{n} p*^{-1}(x_i)$. It is clear that $A_1 = A \cap Z \subset Z*$ and $A_2 = A \cap \overline{Z} \subset Z*$ are acyclic since p and $\overline{p}$ are Vietoris-like maps.

We consider the closure of A_1 and A_2, i.e., $F_1 = \bigcup_{x \in A_1} F_x$ and $F_2 = \bigcup_{x \in A_2} F_x$, where F_x denotes the minimal closed set in A containing x. We prove that $F_1 \cup F_2 = A$. We only need to check that $A \cap Z_1 \times \{0,1\}$ is a subset of $F_1 \cup F_2$. We argue by contradiction. Suppose $z \in A \setminus \{F_1 \cup F_2\}$. Then there exists x_i in the chain $x_1 < ... < x_n$ such that $z \in p*^{-1}(x_i)$. If $x_i = (s,0)$ for some $s \in X$, then z can only be of the form $z = (y,0)$ for some $y \in Z_1$. Therefore, we have that $f_2(y) < z$ and $p*(f_2(y)) = x_i$ since $p*(f_2(y)) = \overline{p}(f_2(y)) = i_0(p_1(y)) = (p_1(y),0) = p*(z) = x_i$, which entails a contradiction. Similarly, if $x_i = (s,1)$ for some $s \in X$, we get that z can only be of the form $z = (y,1)$ for some $y \in Z_1$. We have $f_1(y) < z$ and $p*(f_1(y)) = p(f_1(y)) = i_1(p_1(y)) = (p_1(y),1) = x_i$. Therefore, $z \in F_1$ because $f_1(y) < z$, but we have supposed that $z \notin F_1$.

We prove that F_1 and F_2 are acyclic. We consider $r : F_1 \to A_1$ given by $r(y) = y$ if $y \in A_1$ and $r(y,1) = f_1(y)$ if $(y,1) \in F_1 \setminus A_1$. We prove the continuity of r. The only problematic relation that we can have is of the following type: $y < (z,1)$, where $y \in A_1$ and $(z,1) \in F_1 \setminus A_1$. By construction, $f_1(z) = y$, therefore, $r(y) = y = f_1(z) = r(z,1)$. In addition, $r \circ i = id$ and $i \circ r \geq id$, where id denotes the identity and i denotes the inclusion. From here, we get that A_1 is a strong deformation retract of F_1. Similarly, we can proceed with F_2. We consider $r : F_2 \to A_2$ given by $r(y) = y$ if $y \in A_2$ and $r(y,t) = f_2(y)$ if $(y,t) \in F_1 \setminus A_1$. Repeating the same arguments, it is easy to show that A_1 is a strong deformation retract of U_1. Since A_1 and A_2 are acyclic, it follows the desired assertion.

Finally, we prove that $F_1 \cap F_2$ is also acyclic. If $z \in F_1 \cap F_2$, then z is of the form $(y,1)$ for some $y \in Z_1$ such that $p*(y,1) = x_i$, where $i = 1,...,n$. By abuse of notation, the first coordinate of $x_i \in X \times \{0,1\}$ is also denoted by x_i. Hence, the idea is to prove that $B = \bigcup_{i=1}^{n} p_1^{-1}(x_i)$ is homeomorphic to $F_1 \cap F_2$. We define $T : F_1 \cap F_2 \to B$ given by $T(y,1) = y$. We have that T is well-defined by construction. The continuity and injectivity of T follow easily. If $z \in B$, then $p_1(z) = x_i$ for some $i = 1, ..., n$. It is clear that $(z, 1) > f_1(z), f_2(z)$. Then, we have that $(z, 1) \in F_1 \cap F_2$ and $T(z, 1) = z$. It is trivial to check that $S : B \to F_1 \cap F_2$ given by $S(z) = (z, 1)$ is a bijective and continuous map, which is the inverse of T. Thus, $F_1 \cap F_2$ is acyclic because B is acyclic.

By a Mayer-Vietoris argument, we can get that A is also acyclic.

$$\cdots \to H_n(F_1 \cap F_2) \to H_n(F_1) \oplus H_n(F_2) \to H_n(F_1 \cup F_2) \to \cdots$$

To conclude, we need to define continuous maps $q* : Z* \to Y$, $f_0' : Z_0 \to Z*$, $f_3' : Z_2 \to Z*$ making the following diagram commutative.

$$\begin{tikzcd}
X \arrow[d, "i_0"'] & Z_0 \arrow[l, "p_0"'] \arrow[d, "f_0'"'] \arrow[dr, "q_0"] & \\
X \times \{0,1\} & Z* \arrow[l, "p*"'] \arrow[r, "q*"'] & Y \\
X \arrow[u, "i_1"] & Z_2 \arrow[l, "p_2"] \arrow[u, "f_3'"] \arrow[ur, "q_2"'] &
\end{tikzcd}$$

We define $f_0' = i \circ f_0$ and $f_3' = \bar{i} \circ f_3$, where $i : Z \to Z*$ and $\bar{i} : \overline{Z} \to Z$ are the natural inclusions., it is clear that f_0' and f_3' are continuous maps. We define $q*$ given by

$$q*(s) = \begin{cases} q_1(x), & \text{if } s = (x, t) \in Z_1 \times \{0, 1\} \\ q(y) & \text{if } s = y \in Z \\ \overline{q}(y) & \text{if } s = y \in \overline{Z}. \end{cases}$$

We prove the continuity of $q*$. We study the problematic cases. If $(y, 1) > z$, where $z \in Z$, then $f_1(y) = z$ and we have $q*(y, 1) = q_1(y) = q(f_1(y)) = q(z) = q*(z)$. If $(y, 0) > z$, where $z \in \overline{Z}$, then $f_2(z) = y$ and we have $q*(y, 0) = q_1(y) = \overline{q}(f_2(z)) = \overline{q}(z) = q*(z)$. The commutativity of the above diagram follows by construction. $\square$

The following technical lemma proves that the composition of diagrams behaves as it is expected.

Lemma 4.9.9. *Given two finite topological spaces X and Y. If $(p_0, q_0, \Gamma_0), (p_0', q_0', \Gamma_0') \in D(X, Y)$ and $(p_1, q_1, \Gamma_1), (p_1', q_1', \Gamma_1') \in D(Y, W)$ satisfy that (p_0, q_0, Γ_0) is homotopic to (p_0', q_0', Γ_0') and (p_1, q_1, Γ_1) is homotopic to (p_1', q_1', Γ_1') , then $(p_1, q_1, \Gamma_1) \circ (p_0, q_0, \Gamma_0)$ is homotopic to $(p_1', q_1', \Gamma_1') \circ (p_0', q_0', \Gamma_0')$.*

Proof. We introduce a bit of notation, $(p_1, q_1, \Gamma_1) \circ (p_0, q_0, \Gamma_0) = (\overline{p_1}, \overline{q_0}, \Gamma_0 \boxtimes_y \Gamma_1)$, $(p_1', q_1', \Gamma_1') \circ (p_0', q_0', \Gamma_0') = (\overline{p_1'}, \overline{q_0'}, \Gamma_0' \boxtimes_y \Gamma_1')$ by Definition 4.7.4. By hypothesis, we have that there exist continuous maps and finite topological spaces $T_0 : Z_0 \to Z$, $T_0' : Z_0' \to Z$, $T_1 : Z_1 \to \overline{Z}$, $T_1' : Z_1' \to \overline{Z}$, $p : Z \to X$, $q : Z \to Y$, $\overline{p} : \overline{Z} \to X$ and $\overline{q} : \overline{Z} \to W$, where p and $\overline{p}$ are

Vietoris-like maps, satisfying that the following diagram commutes.

$$\begin{tikzcd}
 & & \Gamma_0 \boxtimes_Y \Gamma_1 \arrow[dl, "\overline{p_1}"'] \arrow[dr, "\overline{q_0}"] & & \\
 & \Gamma_0 \arrow[dl, "p_0"'] \arrow[d, "T_0"'] \arrow[dr, "q_0"] & & \Gamma_1 \arrow[dl, "p_1"'] \arrow[d, "T_1"'] \arrow[dr, "q_1"] & \\
X & Z \arrow[l, "p"'] \arrow[r, "q"] & Y & \overline{Z} \arrow[l, "\overline{p}"'] \arrow[r, "\overline{q}"] & W \\
 & \Gamma_0' \arrow[ul, "p_0'"] \arrow[u, "T_0'"] \arrow[ur, "q_0'"'] & & \Gamma_1' \arrow[ul, "p_1'"] \arrow[u, "T_1'"] \arrow[ur, "q_1'"'] & \\
 & & \Gamma_0' \boxtimes_Y \Gamma_1' \arrow[ul, "\overline{p_1'}"] \arrow[ur, "\overline{q_0'}"'] & &
\end{tikzcd}$$

We consider the pull-back of the following diagram.

$$\begin{tikzcd}
Z \arrow[r, "q"] & Y & \overline{Z} \arrow[l, "\overline{p}"']
\end{tikzcd}$$

Then, we have the following commutative diagram.

$$\begin{tikzcd}
 & Z \boxtimes_Y \overline{Z} \arrow[dl, "\overline{\overline{p}}"'] \arrow[dr, "\overline{q}"] & \\
Z \arrow[r, "q"'] & Y & \overline{Z} \arrow[l, "\overline{p}"]
\end{tikzcd}$$

We define $T_{0,1} : \Gamma_0 \boxtimes_Y \Gamma_1 \to Z \boxtimes \overline{Z}$ given by $T_{0,1} = T_0 \times T_1$ and $T'_{0,1} : \Gamma'_0 \boxtimes_Y \Gamma'_1 \to Z \boxtimes_Y \overline{Z}$ given by $T'_{0,1} = T'_0 \times T'_1$. By construction and the commutativity of the previous diagrams, it is easy to check the that the following diagram commutes, which implies the desired result using Lemma 4.9.5 and Lemma 4.9.8.

$$\begin{tikzcd}
 & \Gamma_0 \boxtimes_Y \Gamma_1 \arrow[dl, "p_0 \circ \overline{p_1}"'] \arrow[d, "T_{0,1}"'] \arrow[dr, "\overline{q_0} \circ q_1"] & \\
X & Z \boxtimes_Y \overline{Z} \arrow[l, "p \circ \overline{\overline{p}}"'] \arrow[r, "\overline{q} \circ \overline{q}"] & W \\
 & \Gamma_0' \boxtimes_Y \Gamma_1' \arrow[ul, "\overline{p_1}' \circ p_0'"] \arrow[u, "T'_{0,1}"] \arrow[ur, "\overline{q_0'} \circ q_1'"'] &
\end{tikzcd}$$

□

Even though diagrams do not form a category, we can consider the homotopy classes in order to get a category, that is, the homotopical category of diagrams HD. The objects of HD are the finite T_0 topological spaces. Given two finite topological spaces X and Y, $HD(X, Y)$ is given by the homotopy classes of diagrams in $D(X, Y)$.

Theorem 4.9.10. *HD is a category.*

Proof. Repeating the same arguments used in the proof Theorem 4.8.3 and using Lemma 4.9.9 the result follows easily. $\square$

We prove that the homotopy relation introduced in this section works properly for the homological functor.

Proposition 4.9.11. *If $(p, q, \Gamma), (p', q', \Gamma') \in D(X, Y)$ are homotopic diagrams, then (p, q, Γ) and (p', q', Γ') induce the same morphism in homology.*

Proof. Let $(p*, q*, \Gamma*) \in D(X \times \{0, 1\})$ denote the homotopy between (p, q, Γ) and (p', q', Γ'). By hypothesis, there exist f and f' such that the following diagram is commutative.

$$\begin{tikzcd}
X \arrow[d, "i_0"] & \Gamma \arrow[l, "p"'] \arrow[d, "f"] \arrow[dr, "q"] & \\
X \times \{0,1\} & \Gamma* \arrow[l, "p*"'] \arrow[r, "q*"] & Y \\
X \arrow[u, "i_1"'] & \Gamma' \arrow[l, "p'"'] \arrow[u, "f'"'] \arrow[ur, "q'"'] &
\end{tikzcd}$$

It is easy to deduce that (p, q, Γ) is homotopic to $(p*, q*, \Gamma*) \circ i_0$ and (p', q', Γ') is homotopic to $(p*, q*, \Gamma*) \circ i_1$. It is clear that $H_*(i_0) = H_*(i_1)$. From Proposition 4.7.7 we get the desired result. $\square$

As an immediate consequence of Proposition 4.9.11, Proposition 4.7.7 and Lemma 4.9.9, we obtain the following result.

Proposition 4.9.12. *If $\mathcal{A}$ denotes the category of abelian groups, then the homological functor $H_* : HD \to \mathcal{A}$ is well-defined.*

Remark 4.9.13. *The notion of homotopy described in this section can also be adapted to the category of morphisms. If $[(p, q, \Gamma)] \in \mathcal{M}(X, Y)$, the homotopy class of $[(p, q, \Gamma)]$ is exactly the homotopy class of a diagram $(p, q, \Gamma) \in [(p, q, \Gamma)]$. Therefore, the homotopical category of diagrams and a homotopical version of the category of morphisms can be seen as the same.*

4.10 Relations of the homotopy for diagrams with other notions of homotopy

As it was mentioned before, Definition 4.9.3 generalizes the notion of homotopy obtained in Proposition 4.9.2. In the following proposition we formalize the last assertion.

Proposition 4.10.1. *Let X and Y be finite topological spaces and let $G : X \multimap Y$ and $F : X \multimap Y$ be Vietoris-like multivalued maps such that $F(x) \subseteq G(x)$ for every $x \in X$. Then $(p_F, q_F, \Gamma(F))$ is homotopic to $(p_G, q_G, \Gamma(G))$.*

Proof. Since $F(x) \subseteq G(x)$ for every $x \in X$, it follows that $\Gamma(F) \subseteq \Gamma(G)$. We have the natural inclusion $i : \Gamma(F) \to \Gamma(G)$ and the following diagram.

$$\begin{tikzcd}
 & \Gamma(F) \arrow[dl, "p_F"'] \arrow[dd, "i"] \arrow[dr, "q_F"] & \\
X & & Y \\
 & \Gamma(G) \arrow[ul, "p_G"] \arrow[ur, "q_G"'] &
\end{tikzcd}$$

If $(x, y) \in \Gamma(F)$, then $p_F(x, y) = x = p_G(x, y) = p_G(i(x, y))$. Moreover, $q_F(x, y) = y = q_G(x, y) = q_G(i(x, y))$. Therefore, the above diagram is commutative. By Lemma 4.9.5 we get the desired result. $\square$

As an immediate consequence of this result, it can be obtained that Definition 4.9.3 also generalizes the usual notion of homotopy for single-valued maps.

Proposition 4.10.2. *Let X and Y be finite topological spaces. If $f, g : X \to Y$ are homotopic maps, then $(p_f, q_f, \Gamma(f))$ is homotopic to $(p_g, q_g, \Gamma(g))$.*

Proof. By Proposition 1.1.16, if f is homotopic to g, then there exists a finite sequence of continuous maps $g_i : X \to Y$, where $i = 1, ..., n$, such that $g_0 = f$,$g_n = g$ and $f \geq g_1 \leq ... \geq g_{n-1} \leq g$. Therefore, it suffices to prove that $f \geq g$ implies that $(p_f, q_f, \Gamma(f))$ is homotopic to $(p_g, q_g, \Gamma(g))$. We consider $F : X \multimap Y$ given by $F(x) = U_x$. It is clear that F is susc with acyclic values, which means that F is a Vietoris-like map by Proposition 4.2.17. By Proposition 4.10.1, we get that $(p_f, q_f, \Gamma(f))$ is homotopic to $(P_F, q_F, \Gamma(F))$. Again, $(p_g, q_g, \Gamma(g))$ is homotopic to $(P_F, q_F, \Gamma(F))$ by Proposition 4.10.1. By Lemma 4.9.8, we can conclude that $(p_f, q_f, \Gamma(f))$ is homotopic to $(p_g, q_g, \Gamma(g))$ $\square$

Remark 4.10.3. *It is also easy to check that if X is homotopy equivalent to Y in the usual sense, then X is isomorphic to Y in HD.*

This category also generalizes the notion of simple homotopy as we prove in the following proposition.

Proposition 4.10.4. *If x is a down (up) weak beat point of X, then $X \setminus \{x\}$ is isomorphic to X in HD.*

Proof. Without loss of generality, we assume that x is a down weak beat point. We have the natural inclusion $i : X \setminus \{x\} \to X$. We consider $R : X \multimap X \setminus \{x\}$ given by $R(y) = y$ if $y \neq x$ and $R(x) = U_x \setminus \{x\}$. We prove that R is a Vietoris-like multivalued map. Then, we verify that p_R is a Vietoris-like map. It is clear that for every chain $x_1 < ... < x_n$ in X not containing x we get that $A = \bigcup_{i=1}^{n} p_R^{-1}(x_i)$ is acyclic. Therefore, we check the case where $x_1 < ... < x_n$ is a chain in X containing x. If $x = x_j$ for some $j < n$, then A is contractible since it has a maximum, which is (x_n, x_n). Suppose that $x = x_n$. We consider $T : A \to x \times U_x \setminus \{x\}$ given by $T(x_i, x_i) = (x, x_i)$ if $i < n$ and $T(x_n, y) = (x_n, y)$ if $y \in U_x \setminus \{x\}$. Since $x_i < x_n = x$, it follows that T is well-defined. To prove the continuity of T, we check that T preserves the relations of the form $(x_i, x_i) < (x, y)$, where $i < n$ and $y \in U_x \setminus \{x\}$, because the remaining relations are preserved by T trivially. We have $T(x_i, x_i) = (x, x_i) < (x, y) = T(x, y)$, which implies the continuity of T. We have the natural inclusion $j : x \times U_x \setminus \{x\} \to A$. It is easily seen that $T \circ j = id$ and $j \circ T \geq id$. From this, we deduce that $x \times U_x \setminus \{x\}$ is a strong deformation retract of A.

By hypothesis, $x \times U_x \setminus \{x\}$ is contractible. Thus, p_R is a Vietoris-like map and R is a Vietoris-like multivalued map.

We know that $(p_R, q_R, \Gamma(R)) \circ (p_i, q_i, \Gamma(i)) = (p_i \circ \overline{p_R}, q_i \circ \overline{q_R}, \Gamma(i) \boxtimes_X \Gamma(R))$ and $(p_i, q_i, \Gamma(i)) \circ (p_R, q_R, \Gamma(R)) = (p_R \circ \overline{p_i}, q_i \circ \overline{q_R}, \Gamma(R) \boxtimes_{X \setminus \{x\}} \Gamma(i))$. Now, we prove that $(p_i \circ \overline{p_R}, q_i \circ \overline{q_R}, \Gamma(i) \boxtimes_X \Gamma(R))$ is homotopic to $(id_{X\setminus\{x\}}, id_{X\setminus\{x\}}, X \setminus \{x\})$ and $(p_R \circ \overline{p_i}, q_i \circ \overline{q_R}, \Gamma(R) \boxtimes_{X\setminus\{x\}} \Gamma(i))$ is homotopic to (id_X, id_X, X). Firstly, we show that id_X is homotopic to $U : X \multimap X$ given by $U(x) = U_x$. This is true since $id_X(y) \in U(y)$ for every $y \in X$ and U is a Vietoris-like map. We define $S : \Gamma(R) \boxtimes_{X\times\{x\}} \Gamma(i) \to \Gamma(U)$ given by $S(y, w, z, t) = (y, t)$. By construction, $q_R(y, w) = w = z = p_i(z, t)$, where $z \in R(y)$ and $t \in i(z)$, so $y = z \neq x$ and $z = t \in R(y) \subseteq U(y)$. Therefore, S is well-defined. The continuity of S follows trivially. By the construction of S, we have that the following diagram commutes.

$$\begin{tikzcd}
& \Gamma(R) \boxtimes_{X\times\{x\}} \Gamma(i) \arrow[dl, "p_R\circ\overline{p_i}"'] \arrow[dr, "q_i\circ\overline{q_R}"] \arrow[dd, "S"] & \\
X & & X \\
& \Gamma(U) \arrow[ul, "p_U"] \arrow[ur, "q_u"'] &
\end{tikzcd}$$

From here, applying Lemma 4.9.5, we obtain that $(p_R \circ \overline{p_i}, q_i \circ \overline{q_R}, \Gamma(R) \boxtimes_{X\setminus\{x\}} \Gamma(i))$ is homotopic to (id_X, id_X, X). We define $H : X \to \Gamma(i) \boxtimes_X \Gamma(R)$ given by $H(y) = (y, y, y, y)$. Since $y \in i(y) = y$ and $y = R(y)$ for every $y \in X \setminus \{y\}$, it follows that H is well-defined. The continuity of H follows trivially. Again, by Lemma 4.9.5, we deduce that $(p_i \circ \overline{p_R}, q_i \circ \overline{q_R}, \Gamma(i) \boxtimes_X \Gamma(R))$ is homotopic to $(id_{X\setminus\{x\}}, id_{X\setminus\{x\}}, X \setminus \{x\})$. $\square$

Remark 4.10.5. *If X and Y are simple homotopy equivalent, then X and Y are isomorphic in HD.*

We also have that finite barycentric subdivisions do not change the type of a topological space in HD.

Proposition 4.10.6. *Let X be a finite topological space. If X' denotes the finite barycentric subdivision of X, then X is isomorphic to X' in HD.*

Proof. Since X and X' are simple homotopy equivalent [12, Proposition 4.2.9], the desired result follows by Proposition 4.10.4. $\square$

Nevertheless, if two finite topological spaces are isomorphic in HD, they are not necessarily weak homotopy equivalent. We provide an example satisfying the previous assertion.

Example 4.10.7. Let us consider the topological space constructed in [60, Example 2.38]. We recall it. Let X be obtained from the wedge sum of two 1-dimensional spheres by attaching two 2-cells by the words a^5b^{-3} and $b^3(ab)^{-2}$. It can be shown that X is acyclic but it is not contractible. Indeed, the fundamental group of X has the presentation $\langle a, b | a^5b^{-3}, b^3(ab)^{-2}\rangle$. We consider a triangulation for X and then $\mathcal{X}(X)$. We define $f : \mathcal{X}(X) \to *$ given by $f(x) = *$ for every $x \in \mathcal{X}(X)$. It is clear that f is a Vietoris-like map. We define $F : * \to \mathcal{X}(X)$ given by $F(*) = \mathcal{X}(X)$. Again, it is simple to check that F is a Vietoris-like multivalued map. We prove that $*$ is isomorphic to $\mathcal{X}(X)$ in HD. To do this, we show that $(p_f, q_f, \Gamma(f)) \circ (p_F, q_F, \Gamma(F))$ is homotopic to $(id, id, *)$

and $(p_F, q_F, \Gamma(F)) \circ (p_f, q_f, \Gamma(f))$ is homotopic to $(id, id, \mathcal{X}(X))$. For the first case we define $T : \Gamma(F) \boxtimes_{\mathcal{X}(X)} \Gamma(f) \to *$ as the constant map. Using Lemma 4.9.5, we get the desired result. For the second case we define $S : \mathcal{X}(X) \to \Gamma(f) \boxtimes_* \Gamma(F)$ given by $S(x) = ((x, *), (*, x))$. Again, by Lemma 4.9.5, we get the desired result.

Remark 4.10.8. *Let X and Y be finite topological spaces. Let $X \overset{he}{\simeq} Y$ denote that X is homotopy equivalent to Y. Let $X \overset{we}{\simeq} Y$ denote that X is weak homotopy equivalent to Y. Let $X \overset{HD}{\simeq} Y$ denote that X is isomorphic to Y in HD. Let $X \overset{se}{\simeq} Y$ denote that X is simple homotopy equivalent to Y. We have the following schematic diagram.*

$$\begin{tikzcd}
X \overset{he}{\simeq} Y \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & X \overset{se}{\simeq} Y \arrow[r, Rightarrow] \arrow[d, Rightarrow] & X \overset{we}{\simeq} Y \\
& X \overset{HD}{\simeq} Y &
\end{tikzcd}$$

4.11 Some remarks about the topological degree for finite topological spaces

The classical notion of topological degree has been proved to be a very useful tool in topology. For example, the topological degree provides a complete classification of continuous maps from a compact connected oriented n-dimensional manifold X to the n-dimensional sphere S^n. Namely, given continuous maps $f, g : X \to S^n$, f is homotopic to g if and only if f and g have the same degree, that is, the Hopf theorem [62]. For a complete introduction to the topological degree see [89] or [79].

The notion of degree seems to be more promising for this category than the usual one for continuous maps and finite space. Indeed, in [23], the notion of topological degree is defined for continuous maps between finite models of spheres in the natural way, i.e., if X and Y are two finite models of a n-dimensional sphere and $f : X \to Y$ is a continuous map, then the degree of f is defined as $\mathcal{K}(f)_n([X]) = deg(f)[Y]$, where $[X]$ and $[Y]$ are the fundamental classes of $H_n(\mathcal{K}(X))$ and $H_n(\mathcal{K}(Y))$ respectively and $\mathcal{K}(f)_n : H_n(\mathcal{K}(X)) \to H_n(\mathcal{K}(Y))$. But it is shown that given two finite models of the 1-sphere and an integer number n, we cannot expect in general to have a continuous map f between them having $deg(f) = n$. However, for the category HD the previous result holds true in a more general way. We give first a definition of degree for a diagram.

Definition 4.11.1. *Let X and Y be finite models of closed connected oriented m-dimensional manifolds and let $(p, q, \Gamma) \in D(X, Y)$. The degree of (p, q, Γ) is given as follows $q_*(p_*^{-1}([X])) = deg(p, q)[Y]$, where $deg(p, q) \in \mathbb{Z}$.*

Remark 4.11.2. *If $f : X \to Y$ is a continuous map between two finite models of closed connected oriented m-dimensional manifolds, then $deg(f) = deg(p_f, q_f)$ by construction.*

Proposition 4.11.3. *If $(p, q, \Gamma), (p', q', \Gamma') \in D(X, Y)$ are homotopic diagrams, then $deg(p, q) = deg(p', q')$.*

Proof. It is an immediate consequence of Proposition 4.9.12. $\square$

Proposition 4.11.4. *If $(p, q, \Gamma) \in D(X, Y)$ and $(p', q', \Gamma') \in D(Y, Z)$, then $deg((p', q') \circ (p, q) = deg(p, q)deg(p', q')$.*

Proof. It is a consequence of Proposition 4.7.7. □

Proposition 4.11.5. *If n is a positive integer number, l is an integer number and X is a finite model of the n-dimensional sphere, then there exists a diagram $(p, q, \Gamma) \in D(X, X)$ such that its degree is l.*

Proof. Let us consider a continuous map $f' : S^n \to S^n$ such that $deg(f') = l$. We can consider a triangulation in S^n. By the simplicial approximation theorem, there exists a positive integer number m and a simplicial map $f : S^{n,m} \to S^n$ satisfying that $deg(f) = deg(f')$. Applying the McCord functor, we get $\mathcal{X}(f) : \mathcal{X}(S^{n,m}) \to \mathcal{X}(S^n)$. Since X is a finite model of S^n, it follows that there exists a positive integer t and a continuous map $g : X^t \to \mathcal{X}(S^{n,m})$ satisfying that g induces the identity in homology. Additionally, there exists a positive integer s and continuous map $h : \mathcal{X}(S^n)^s \to X^t$ such that h induces the identity in homology. Using the natural Vietoris-like maps and multivalued maps between finite barycentric subdivisions, we get

$$X \multimap X^t \xrightarrow{g} \mathcal{X}(S^{n,m}) \xrightarrow{\mathcal{X}(f)} \mathcal{X}(S^n) \multimap \mathcal{X}(S^n)^s \xrightarrow{h} X^t \to X.$$

In the above diagram, the only map that does not induce the identity in homology is $\mathcal{X}(f)$. Since the degree of a composition is the product of the degrees, we get the desired result. □

Using similar arguments as in the proof of Proposition 4.11.5, it can be obtained immediately the following result.

Proposition 4.11.6. *If n is a positive integer number, l is an integer number, X is a finite model of the n-dimensional sphere and Y is a finite model of a closed connected oriented n-dimensional manifold, then there exists a diagram $(p, q, \Gamma) \in D(Y, X)$ such that its degree is l.*

Proof. We know that there exists a continuous map $f : \mathcal{K}(X) \to S^n$ satisfying that its degree is 1. By the simplicial approximation theorem, there exists a barycentric subdivision $\mathcal{K}(X)^m$ and a simplicial map $f' : |\mathcal{K}(X)^m| \to S^n$ which is homotopic to f. Applying $\mathcal{X}$, we have $\mathcal{X}(f') : \mathcal{X}(\mathcal{K}(X)^m) \to \mathcal{X}(S^n)$. Using Proposition 4.11.5 and Proposition 4.10.6, we can deduce the desired result. □

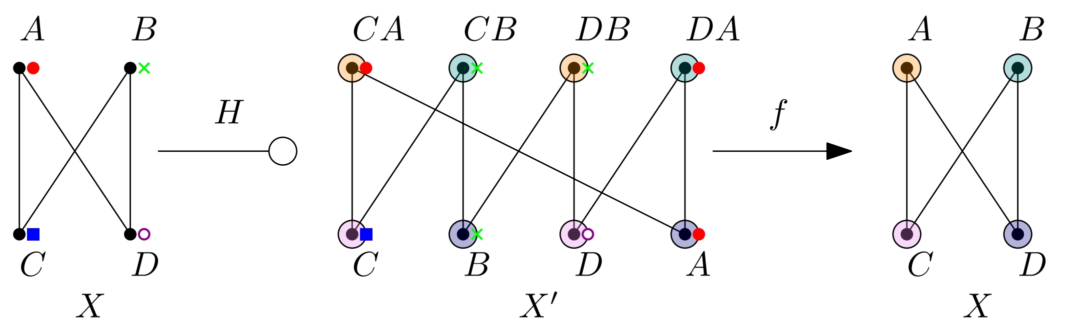

Figure 4.11.1: Schematic representation of H and f.

Example 4.11.7. We consider the minimal finite model of the 1-dimensional sphere, that is, $X = \{A, B, C, D\}$ satisfying that $A, B > C, D$. We will construct a diagram of degree 2. To do that, we consider the finite barycentric subdivision of X. $f : X' \to X$

is given by $f(CA) = A$, $f(CB) = B$, $f(DB) = A$, $f(DA) = B$, $f(C) = C$, $f(B) = D$, $f(D) = C$ and $f(A) = D$. In Figure 4.11.1, it can be seen a schematic representation using Hasse diagrams. It follows that $\mathcal{K}(f)$ has degree 2. Furthermore, we have the natural Vietoris-like multivalued map $H : X \multimap X$. Therefore, composing the diagrams associated to the previous maps and multivalued maps we get the desired result, that is, $(p_f, q_f, \Gamma(f)) \circ (p_H, q_H, \Gamma(H)) : X \to X$ is a diagram of degree 2.

Remark 4.11.8. *There are some interesting questions regarding to this topic. For example, is it true a finite version of the Hopf theorem? It could also very useful to get a combinatorial description of the degree and state a class of finite topological spaces for which it can be used.*

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