UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Resource characterisation of quantum entanglement and nonlocality in multipartite settings

Caracterización del entrelazamiento y no localidad cuánticos como recursos en sistemas multipartitos

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Madrid

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Acknowledgments

I am indebted to lots of people for making this thesis possible through their support, inspiration and encouragement. Here's a (certainly incomplete) tribute:

I probably wouldn't have become interested in maths or physics had it not been for Mrs Robinson and Mr Combley. Their dedication and their determination to inspire curiosity was something I admired class after class, and which has stayed with me until today. Soon after, Roman Schubert, John Hannay, Sandu Popescu, Jasper van Wezel, Tony Short and Noah Linden all contributed to making quantum physics and quantum information tangible, and gave me a glimpse of this fascinating field that made me want to explore it further.

Sin mis directores, Carlos y Julio, esta tesis no se habría hecho realidad. De los dos he aprendido a leer, desarrollar y escribir matemáticas casi desde cero. Julio, gracias especialmente por escuchar y apoyarme cuando lo he necesitado. My collaborators for Chapter 5, Gianni, Alek, Adam and Piero, introduced me to a whole new field. Thank you all for engaging in so many divergent discussions, and converging into an actual paper!

Several people have been of great help during the development of the results. In particular, thanks go to Aleksander M. Kubicki and Elie Wolfe for enlightening discussions towards Chapter 3, Andreas Winter for a helpful communication about Chapter 4, and Eduardo Zambrano, Elie Wolfe, Valerio Scarani, Alex Pozas-Kerstjens, Miguel Navascués, Serge Fehr, Peter van Emde Boas and Ronald de Wolf for their comments and help towards Chapter 5.

Gianni deserves a special mention, for introducing some much needed serenity at key points in the process. Helen provided immense encouragement by praising achievements of mine such as opening a box or building a block tower with a very sincere "well done!".

Mamá y Papá, vuestro apoyo incondicional, estos años y siempre, hace que todo sea un poquito bastante más fácil. Papá, me has servido de gran inspiración en el camino. Mamá, gracias por todas esas conversaciones hasta las tantas. Antonio, gracias por tu entusiasmo con todo. ¡Nunca sospeché que hablaríamos de matemáticas en tanto detalle!

Thanks also go to the mathQI group, past and present, for many an interesting chat. Sobre todo, gracias al despacho 251, por tantas conversaciones con mayor o menor sentido y sobre todo muchas risas. La fuerza del equipo BYMAT ha sido tremendamente inspiradora. Ágata y Laura, gracias por abrirme una ventana al mundo y por bailar el corro de la patata con tanto convencimiento.

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Abstract

Quantum technologies are enjoying an unprecedented popularity, and some applications are already in the market. This thesis studies two phenomena that are behind a lot of quantum technologies: entanglement and nonlocality. We focus on multipartite systems, and ask what configurations of those systems are more useful than others. 'Usefulness' takes on different meanings depending on the context, but, roughly speaking, we aim for more entanglement or more nonlocality.

Chapter 2 is motivated by an important issue with traditional resource theories of multipartite entanglement: they give rise to isolated states and inequivalent forms of entanglement. We propose two new resource theories that do not give rise to these problems: the resource theory of non-full-separability under full separability-preserving operations, and the resource theory of genuine multipartite entanglement (GME) under biseparability-preserving operations. Further, the latter theory gives rise to a unique maximally GME state.

Chapters 3 and 4 focus on quantum networks, that is, configurations where pairs of parties share entangled states, and parties are bipartitely entangled to one or more of the others. First, we assume all shared states are pure. It is known that all connected networks of bipartite pure entangled states are GME (which is a necessary requirement for being nonlocal) so we ask what networks give rise to genuine multipartite nonlocality (GMNL). Surprisingly, they all do: any connected network of bipartite pure entangled states is GMNL. Next, we allow for the presence of noise, and study networks of mixed states taking isotropic states as a noise model. Not even GME is guaranteed in these networks, so our first task is to find out what networks, in terms of both noise and geometry, give rise to GME. We find that, unlike in the case of pure states, topology plays a crucial role: for any non-zero noise, tree networks and polygonal networks become biseparable if the number of parties is large enough. In sharp contrast, a completely connected network of isotropic states is GME for any number of parties as long as the noise is below a threshold. We further deduce that, while non-steerability of the shared

states can compromise GMNL or even render the networks fully local, taking many copies of bilocal networks can restore GMNL. Thus, we obtain, to our knowledge, the first example of superactivation of GMNL.

The thesis so far assumes that quantum theory is an apt description of Nature. While there are good reasons to believe so, it is possible that Nature allows for correlations that are stronger than those predicted by quantum theory, and which we have not yet observed. In order to find out whether Nature is quantum, one possibility is to devise physical principles that act as constraints that can rule out post-quantum theories which are consistent with experimentally observed results. In a departure from the main ideas explored in previous chapters, Chapter 5 is devoted to developing one such principle. The principle is inspired in a seminal result in epistemics, which is the formal study of knowledge and beliefs. We derive two notions of disagreement in agents' observations of perfectly correlated events: common certainty of disagreement, and singular disagreement. Both notions are impossible in classical and quantum agents. Thus, we contend, the principle of no disagreement must hold for any theory of Nature.

This thesis provides different ways of classifying multipartite systems in terms of their entanglement and nonlocality. As such, it opens up new questions, including the possibility of measuring multipartite entanglement in a unique way, whether it is possible to generate multipartite nonlocality from a single noiseless entangled state, and which network topologies are needed to obtain multipartite nonlocality when the shared states contain noise. Further, the task of constraining the possible theories of Nature via external principles is a complex one. We have proposed one step in this direction that, in addition, provides a way to link quantum information and epistemics more closely. A natural question that arises is to confirm whether our principle is intrinsic to Nature, or whether correlations can be found experimentally which do not satisfy the principle. Alternatively, a strong operational grounding for the best current description of Nature would be achieved by completing the list of physical principles that might characterise quantum theory.

Resumen

Las tecnologías cuánticas gozan actualmente de una popularidad sin precedentes, y ya tienen aplicaciones en el mercado. Esta tesis estudia dos fenómenos que están detrás de muchas de estas tecnologías: el entrelazamiento y la no localidad. Nos centramos en sistemas multipartitos, y tratamos de averiguar qué configuraciones de estos sistemas son más útiles. La noción de utilidad varía según el contexto pero, en términos generales, aspiramos a conseguir más entrelazamiento o más no localidad.

El capítulo 2 viene motivado por un problema importante en las teorías de recursos de entrelazamiento multipartito tradicionales: dan lugar a estados aislados y a formas de entrelazamiento no equivalentes. En este capítulo proponemos dos nuevas teorías de recursos que no generan estos problemas: la teoría de recursos de no-separabilidad-completa bajo operaciones que preservan separabilidad completa, y la teoría de recursos de entrelazamiento multipartito genuino (GME, por sus siglas en inglés) bajo operaciones que preservan biseparabilidad. Además, esta última teoría da lugar a un estado máximamente GME único.

Los capítulos 3 y 4 se centran en redes cuánticas, esto es, configuraciones donde se comparten estados entrelazados entre pares de agentes, y cada agente está conectado de esta manera a uno o varios más. Primero asumimos que todos los estados que se comparten son puros. Se sabe que todas las redes conexas de estados puros bipartitos entrelazados son GME (condición necesaria para ser no locales), con lo que nos preguntamos qué redes dan lugar a la no localidad multipartita genuina (GMNL, por sus siglas en inglés). Sorprendentemente, esto ocurre para todas las redes: cualquier red conexa de estados puros bipartitos entrelazados es GMNL. A continuación, estudiamos las redes de estados mezcla para analizar los efectos del ruido. Empleamos los estados isotrópicos como modelo de ruido. Ni siquiera está garantizado que estas redes sean GME, así que la primera tarea es investigar qué redes (a nivel tanto de ruido como de geometría) dan lugar a GME. Al contrario que en las redes de estados puros, vemos que la topología juega un papel fundamental: para cualquier nivel de ruido (distinto de

cero), cualquier red en forma de árbol o de polígono se vuelve biseparable si el número de nodos es lo suficientemente grande. En el otro extremo, una red totalmente conexa de estados isotrópicos es GME para cualquier número de nodos si el ruido está por debajo de un umbral. Deducimos además que, si los estados compartidos no son direccionables (*steerable*), la red puede volverse bilocal o incluso completamente local. Sin embargo, la GMNL se puede recuperar tomando muchas copias de una red bilocal.

Hasta ahora, la tesis asume que la teoría cuántica es una descripción válida de la naturaleza. Hay razones muy convincentes para creer que esto es así, aunque sigue siendo posible que en la naturaleza se puedan dar correlaciones que sean más fuertes de lo que predice la teoría cuántica, y que aún no se hayan observado. Para investigar si la naturaleza es cuántica, una posibilidad es desarrollar principios físicos que actúen de restricciones para eliminar teorías poscuánticas que pudieran describir los resultados experimentales. Saliéndonos de las ideas principales de los capítulos anteriores, el capítulo 5 desarrolla uno de estos principios. El principio está inspirado en un resultado muy influyente en el estudio científico del conocimiento. Desarrollamos dos nociones de desacuerdo que se aplican a las observaciones de eventos perfectamente correlacionados por parte de dos agentes: certeza común de desacuerdo, y desacuerdo singular. Ni los agentes clásicos, ni tampoco los cuánticos, son susceptibles a estos tipos de desacuerdo. Por eso argumentamos que el principio de no desacuerdo debería darse en cualquier teoría de la naturaleza.

Esta tesis ofrece diferentes maneras de clasificar los sistemas multipartitos de acuerdo a su entrelazamiento y no localidad. Sin duda da lugar a nuevas preguntas, como la posibilidad de medir el entrelazamiento multipartito de una manera única, si es posible generar no localidad multipartita genuina utilizando un único estado entrelazado sin ruido, y qué topologías se necesitan para obtener no localidad multipartita a partir de redes de estados con ruido. Además, la tarea de restringir las posibles teorías de la naturaleza a través de principios externos es compleja. Hemos propuesto un paso en esta dirección que, además, da lugar a conexiones más estrechas entre la información cuántica y el estudio científico del conocimiento. Surge naturalmente la pregunta de confirmar si este principio es intrínseco a la naturaleza, o si es posible generar correlaciones experimentales que no lo satisfacen. Por otro lado, si se completara la lista de principios físicos que puedan caracterizar la teoría cuántica, esto dotaría de una base operacional muy sólida a la mejor descripción de la naturaleza que tenemos actualmente.

This thesis is based on the following works:

Chapter 2: Patricia Contreras-Tejada, Carlos Palazuelos, and Julio I. de Vicente, *Resource Theory of Entanglement with a Unique Multipartite Maximally Entangled State*, Phys. Rev. Lett., **122** (2019), 120503.

Proposition 2.3 and the final observations are new results.

Chapter 3: Patricia Contreras-Tejada, Carlos Palazuelos, and Julio I. de Vicente, *Genuine Multipartite Nonlocality Is Intrinsic to Quantum Networks*, Phys. Rev. Lett., **126** (2021), 040501.

The final observation is a new result.

Chapter 4: Patricia Contreras-Tejada, Carlos Palazuelos, and Julio I. de Vicente, in preparation.

Chapter 5: Patricia Contreras-Tejada, Giannicola Scarpa, Aleksander M. Kubicki, Adam Brandenburger, and Pierfrancesco La Mura, *Agreement between observers: A physical principle?* (2021), arXiv:2102.08966. Accepted at the 18th International Conference on Quantum Physics and Logic. Submitted for publication.

Funding from the Spanish MINECO through grants SEV-2015-0554-16-3 and MTM2017-88385-P, and from the Comunidad de Madrid through grant QUITEMAD+CMP2018/TCS-4342, is gratefully acknowledged.

Notation

Here are some symbols that appear frequently throughout the text. These are their meanings, unless stated otherwise:

n	number of parties
d	local dimension, i.e., dimension of the Hilbert space pertaining to each party
$[n]$	$1, ..., n$, the set of n parties
$\succcurlyeq$	positive semidefinite
ρ, σ, τ	usually denote mixed states (σ is usually separable)
ψ (ϕ)	$\vert\psi\rangle\langle\psi\vert$ ($\vert\phi\rangle\langle\phi\vert$), pure states
$\vert\phi_d^+\rangle$ ($\vert\phi^+\rangle$)	maximally entangled state in dimension d (2)
$\mathbb{1}$	identity operator of local dimension d
$\tilde{\mathbb{1}}$	$= \mathbb{1}/d^2$, normalised identity state of local dimension d
$\mathcal{H}$	Hilbert space
$\mathcal{B}(\mathcal{H})$	the set of bounded operators on $\mathcal{H}$
A, B, A_i, B_i	parties called Alice, Bob, Alice$_i$, Bob$_i$ respectively
$M\vert\overline{M}$	a bipartition $\{M, \overline{M}\}$ of the n parties, where $M \cap \overline{M} = \emptyset$ and $M \cup \overline{M} = [n]$
$E_{a\vert x}$ ($F_{b\vert y}$)	POVM element with output a and input x (output b and input y)

$$\left.\begin{array}{c} P \\ p \\ \mathsf{P} \end{array}\right\} \quad \text{probability distributions of} \begin{cases} \text{inputs and outputs} \\ \text{hidden variables} \\ \text{states of the world} \end{cases}$$

Chapter 1

Introduction

Quantum technologies are enjoying an unprecedented popularity. Companies and governments alike are investing large amounts of money in quantum research, with high hopes that they will revolutionise our already highly technified lives. Long-term uses range from making artificial intelligence more powerful to establishing a quantum internet to communicate securely, synchronise clocks, perform faster computations or combine distant telescopes to form much more powerful ones. More modest, but still noteworthy, applications are already in the market: they include quantum random number generators, quantum key distribution for very secure cryptography, or even the GPS.

These applications are only made possible by our ability to control quantum systems. The special phenomena that are needed for quantum technologies to work require the physical apparatus used to be extremely isolated from its surroundings, something which gets harder the bigger the system is. Otherwise, the apparatus reverts to behaving classically. Hence the applications that require larger systems are still out of our reach. However, studying systems of many quantum particles from a theoretical standpoint is crucial to developing these applications: we need to know the theory well, in order to know how to exploit it. And, no less importantly, it is interesting in itself, as it deepens our understanding of Nature.

This thesis will study two phenomena that are behind a lot of quantum technologies: entanglement, which is a property of quantum states, and nonlocality, which is a property of the correlations in the classical information that we are able to extract from those states. Neither phenomenon is present in classical systems, hence they sometimes appear unintuitive since they challenge assumptions that are naturally (and strongly) held by macroscopic beings such as humans. We will focus on multipartite systems, and ask what configurations of those systems are more useful than others. 'Usefulness' will take on different meanings depending on the context, but, roughly speaking, we aim for more entanglement or more nonlocality. In Chapter 2, we come up with a way of ordering multipartite states according to the amount of entanglement they contain. Chapters 3 and 4 focus on quantum networks, that is, configurations where pairs of parties share entangled states, and parties are bipartitely entangled to one or more of the others. First, we assume all shared states are pure (that is, free of noise). It is known that all connected networks of bipartite pure entangled states are entangled, so we ask what networks give rise to nonlocality. Surprisingly, they all do: any connected network of bipartite pure entangled states is nonlocal. Next, we allow for the presence of noise, and study networks of a particular type of noisy states. Not even entanglement is guaranteed in these networks, so our first task is to find out what networks, in terms of both noise and geometry, give rise to entanglement. We find that, unlike in the case of pure states, topology plays a crucial role, as does the amount of noise in the bipartite states. We further deduce that some of these entangled networks display nonlocality too.

Evidently, quantum technologies rely on the world being quantum. That is, they take quantum theory at face value, like we have done so far, and exploit its properties. The fact that more and more quantum effects are being confirmed in the laboratory to very high precision, and even made into commercial apparatuses, is a very good indication that quantum theory is an apt description of Nature. And indeed, it is the best theory we have so far. However, experimental evidence is inevitably limited. For example, generating nonlocal correlations experimentally implies that Nature is not classical. However, it does not guarantee that it is quantum, since there could be a different theory underlying the experimentally observed correlations. Indeed, it is possible that Nature allows for correlations that are stronger than those predicted by quantum theory, and which we have not yet observed. So, how can we find out whether Nature is quantum? One possibility is to devise physical principles that act as constraints that can rule out post-quantum theories which are consistent with experimentally observed results. In a departure from the main ideas explored in previous chapters, Chapter 5 is devoted to developing one such principle. The principle is inspired in a seminal result in epistemics, which is the formal study of knowledge and beliefs. We derive a notion of agreement in agents' observations that holds for both classical and quantum agents, and, we contend, must hold for any theory of Nature.

The rest of this chapter introduces the main technical notions that will be used throughout this thesis, as well as fixing the notation. It also motivates and summarises the main results presented in this thesis.

1.1 Quantum formalism

We present the main notions of quantum theory that are used in this thesis. Ref. [NC00] is by now a classic textbook of quantum information where the interested reader can find a much more complete treatment of the ideas in this and the next section.

According to the first postulate of quantum mechanics, the state of a system is described by a density operator $\rho \in \mathcal{B}(\mathcal{H})$, the set of bounded operators on a Hilbert space $\mathcal{H}$. Vectors on $\mathcal{H}$ are denoted by $|\cdot\rangle$, while $\langle\cdot|$ denotes their duals. States ρ must be positive semidefinite, $\rho \succcurlyeq 0$, and have unit trace, $\operatorname{tr}(\rho) = 1$. If a density operator ρ is a rank-1 projector, i.e. $\rho^2 = \rho$ or, equivalently, $\operatorname{tr}\rho^2 = 1$, the system is said to be in a *pure* state. In this case, the unit vector $|\psi\rangle \in \mathcal{H}$ such that $\rho = |\psi\rangle\langle\psi|$ is sufficient to describe the state of the quantum system, and vectors are often used in place of projective operators. We use $\psi \equiv |\psi\rangle\langle\psi|$ whenever a state is specified as pure. Otherwise, the *mixed* state ρ can be written as a probabilistic mixture of pure states,

$$\rho = \sum_{i\in I} p_i \left|\psi_i\rangle\langle\psi_i\right|, \tag{1.1}$$

where I is an index set, often omitted from the notation, and $\{p_i\}_{i\in I}$ is a probability distribution, i.e., $0 \leq p_i \leq 1$ for all $i \in I$, and $\sum_{i\in I} p_i = 1$.

States are often written in terms of the *computational basis*, which, for $\dim\mathcal{H} = d$ (so that $\mathcal{H} \cong \mathbb{C}^d$), is represented symbolically as $\{|0\rangle, |1\rangle, ..., |d-1\rangle\}$. Each $|i\rangle, i = 0, ..., d-1$, is a column vector with a 1 in the $(i+1)$th position and 0 on the rest.

One of the themes of this thesis is to analyse quantum states that are shared by multiple parties. It is useful to assign names to the parties, and, following convention, Alice and Bob will be our protagonists, while Charlie will make an occasional appearance. We refer to parties, agents, particles, and subsystems interchangeably, denoting them by name or initial. We will start by defining relevant notions for bipartite systems, before turning to their multipartite analogues.

The Hilbert space of a joint system is the tensor product of the Hilbert spaces of its components. Therefore, if $\rho_A \in \mathcal{B}(\mathcal{H}_A)$, $\rho_B \in \mathcal{B}(\mathcal{H}_B)$ are the states of systems A, B respectively, and the systems are independent, then the state of the joint system AB is $\rho_{AB} := \rho_A \otimes \rho_B \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$. However, the state of a joint system is not always the tensor product of the component states. Indeed, a pure state such as

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B) \tag{1.2}$$

cannot be written as

$$|a\rangle_A \otimes |b\rangle_B \tag{1.3}$$

for any $|a\rangle_A$, $|b\rangle_B$, as can be easily shown by writing $|a\rangle_A$, $|b\rangle_B$ in the basis $\{|0\rangle, |1\rangle\}$. For vectors $|\cdot\rangle$, it is customary to omit the tensor product symbol: $|a\rangle_A \otimes |b\rangle_B \equiv |a\rangle_A |b\rangle_B \equiv |ab\rangle_{AB}$. Moreover, a balance between clarity and readability is sought for when including or omitting subscripts denoting subsystems.

Given the state of the joint system AB, the state of a subsystem, say A, can be found via the partial trace, denoted tr_A, tr_B, etc., where the subscript refers to the subsystem(s) to be traced out:

$$\mathrm{tr}_B(\rho) = \sum_{k=0}^{d-1} (\mathbb{1}_A \otimes \langle k|_B)\, \rho\, (\mathbb{1}_A \otimes |k\rangle_B)\,, \tag{1.4}$$

and similarly for $\mathrm{tr}_A(\rho)$. As evident from the mathematical characterisation, the partial trace can only be applied to density matrices, not vectors. The state of a subsystem obtained by applying the partial trace on all other subsystems is called the *reduced state* of that subsystem. For example, if Alice and Bob share the state $|\psi\rangle_{AB}$ in equation (1.2), then Alice's reduced state is

$$\begin{aligned} \mathrm{tr}_B(\psi_{AB}) =& (\mathbb{1}_A \otimes \langle 0|_B) \left[\frac{1}{2}(|01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB})\right] (\mathbb{1}_A \otimes |0\rangle_B) \\ &+ (\mathbb{1}_A \otimes \langle 1|_B) \left[\frac{1}{2}(|01\rangle\langle 01|_{AB} + |10\rangle\langle 10|_{AB})\right] (\mathbb{1}_A \otimes |1\rangle_B) \\ =& \frac{1}{2}\left(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A\right). \end{aligned} \tag{1.5}$$

This state is called the *maximally mixed state*, or the *identity state* (since it corresponds to a normalised identity on $\mathcal{H}_A$). Similarly,

$$\mathrm{tr}_A(\psi_{AB}) = \frac{1}{2}\left(|0\rangle\langle 0|_B + |1\rangle\langle 1|_B\right). \tag{1.6}$$

And yet, taking the tensor product of each party's reduced states does not give back the original state:

$$\mathrm{tr}_B(\psi_{AB}) \otimes \mathrm{tr}_A(\psi_{AB}) \neq \psi_{AB}. \tag{1.7}$$

This would happen if and only if ψ_{AB} could be written as a tensor product $|a\rangle_A \otimes |b\rangle_B$. This is the key observation behind the concept of entanglement, which we will review in detail shortly.

The second postulate of quantum mechanics characterises the time evolution of quantum states. We consider closed quantum systems first, which are those which do not interact with their environment. The evolution of closed quantum systems is described by unitary operators (sometimes called 'gates' in computational contexts), that is, operators U such that $U^\dagger U = \mathbb{1}$. If the state ρ evolves according to a unitary U, the evolved state ρ' is

$$\rho' = U\rho U^\dagger = \sum_i p_i U \left|\psi_i\right\rangle\!\left\langle\psi_i\right| U^\dagger. \tag{1.8}$$

That is, if ρ is a mixture of pure states $\left|\psi_i\right\rangle$, the evolved state ρ' under U is the mixture of the evolved states $U\left|\psi_i\right\rangle$ with the same weights. Unitarity ensures that $U\left|\psi_i\right\rangle$ and ρ' are still well-defined quantum states.

More generally, the evolution of a quantum system takes into account the presence of an environment. The system under study together with its environment do form a closed system, whose evolution must be unitary. Then, by Stinespring's dilation theorem [Sti55], the evolution of the system under study must be described by maps Λ that are required to be completely positive and trace-preserving. Complete positivity ensures that, if $\rho_{S,E}$ is a positive operator (e.g. describing the joint state of the system S under study and the environment E), then

$$(\Lambda \otimes \mathbb{1})(\rho_{S,E}) \tag{1.9}$$

is still positive. Added to trace preservation, which makes $\mathrm{tr}[\Lambda(\cdot)] = \mathrm{tr}(\cdot)$, the evolved state of the system $\rho' = \mathrm{tr}_E[(\Lambda \otimes \mathbb{1})(\rho_{S,E})]$ is guaranteed to be a well-defined quantum state.

Finally, the third postulate concerns measurements. In order to extract classical information out of quantum states, we can measure them. Measurement in quantum mechanics is defined by Positive Operator-Valued Measures, or POVMs, which are sets $\{E_a\}_{a\in\mathcal{A}}$ of operators E_a acting on $\mathcal{H}$ such that $E_a \succcurlyeq 0$ for all $a \in \mathcal{A}$ and $\sum_{a\in\mathcal{A}} E_a = \mathbb{1}_A$, where $\mathcal{A}$ is the set of possible *outcomes* or *outputs* of the measurement $\{E_a\}_{a\in\mathcal{A}}$. The probability that state $\rho \in \mathcal{B}(\mathcal{H})$ will give outcome a on being measured according to $\{E_a\}_{a\in\mathcal{A}}$ is given by the Born rule:

$$P(a) = \mathrm{tr}(E_a\rho). \tag{1.10}$$

If the POVM elements E_a are projectors, then the measurement is termed *projective.* A particular projective measurement can be constructed by choosing the rank-1 projectors onto a basis of the Hilbert space, which are thus positive and add to the identity. In this case, we sometimes speak of measuring *onto* a basis. A common choice is the computational basis measurement, $E_a = |a\rangle\langle a|$, for $a = 0, ..., d-1$.

We often consider measurements performed separately on two parts of a system, so that Alice performs $\{E_a\}_{a\in\mathcal{A}}$ and Bob performs a different POVM $\{F_b\}_{b\in\mathcal{B}}$. Then, the set of tensor-product operators $\{E_a \otimes F_b\}_{(a,b)\in\mathcal{A}\times\mathcal{B}}$ is a POVM too (or a projective measurement, if both $\{E_a\}_{a\in\mathcal{A}}$ and $\{F_b\}_{b\in\mathcal{B}}$ are), as can be straightforwardly shown.

It is commonly assumed that, upon measurement, states evolve non-unitarily and 'collapse' onto another state that depends on the outcome obtained.[1] This evolution can be defined for POVMs and projective measurements alike, but we will only need the projective case. If ρ is measured according to a projective measurement $\{E_a\}_{a\in\mathcal{A}}$ and outcome a is obtained, the post-measurement state ρ'_a is

$$\rho'_a = \frac{E_a \rho E_a}{\mathrm{tr}(E_a \rho)}. \tag{1.11}$$

A very useful tool to study composite systems in quantum mechanics is the *Schmidt decomposition.* We have seen that, at least outside of a measurement context, evolution of closed systems is unitary. In fact, applying a unitary operator amounts only to changing the basis in which the quantum state is represented. Indeed, equivalence classes of states that are equal up to local unitaries are often considered. The Schmidt decomposition of a state is a standard form for these equivalence classes.

Theorem 1.1. *For any pure, bipartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal bases $\{|i\rangle_A\}_i \subset \mathcal{H}_A$, $\{|i\rangle_B\}_i \subset \mathcal{H}_B$ such that*

$$|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\, |i\rangle_A \, |i\rangle_B \,, \tag{1.12}$$

where the Schmidt coefficients *λ_i are real numbers satisfying $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$, and $d = \min\{d_A, d_B\}$ where d_A, d_B is the dimension of $\mathcal{H}_A, \mathcal{H}_B$ respectively. The basis $\{|i\rangle_A |i\rangle_B\}_i \subset \mathcal{H}_A \otimes \mathcal{H}_B$ is called the* Schmidt basis *of the state $|\psi\rangle$.*

[1] Taking collapse too literally poses a multitude of problems, both physical and conceptual. To abate them, it is best to view a measurement as an operation that entangles the state being measured with the measurement apparatus, and in fact with its whole environment. By considering the joint system of the state and the environment, the evolution becomes unitary, and environment-induced decoherence makes sure that the parts of the state corresponding to different outcomes do not interact with each other. In particular, macroscopic observers such as humans never see states corresponding to more than one outcome in the same measurement process. Then, collapse can be understood as the effect of this decohering process, which causes an observer to perceive evolution non-unitarily. This is only a first step towards tackling the conceptual issues surrounding measurement in quantum mechanics, which lie outside of the scope of this thesis.

This theorem first appeared in Ref. [Sch07], and can be proven by arranging the coefficients of $|\psi\rangle$, written in terms of any basis, in a $d_A \times d_B$ matrix, and finding its singular value decomposition.

1.1.1 Entanglement in bipartite systems

Entanglement is a key notion in quantum mechanics, as there is no classical analogue. Conceptually, entanglement is a property of composite systems whereby a full description of each of the subsystems does not provide the full information about the system as a whole. In the early days of quantum mechanics, this was one of the most puzzling phenomena that gave quantum mechanics an aura of being difficult, weird or unintuitive. Faced with an entangled system, Einstein, Podolsky and Rosen famously concluded that quantum theory could not be complete [EPR35]. The fact that there could be more information in the system than that contained in its subsystems challenged the previously held assumption that subsystems have inherent properties that can be measured.

Mathematically, entanglement is a consequence of the tensor product structure. As anticipated above, a pure state is said to be *entangled* if it cannot be written as the tensor product of the states of its subsystems. Otherwise, it is *separable.* For example, $|\psi\rangle_{AB}$ in equation (1.2) above is entangled. More generally, a mixed state is entangled if it cannot be decomposed into pure, separable states:

Definition 1.1. A state $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is *separable* if there exist states $|\eta_i\rangle_A \in \mathcal{H}_A$, $|\chi_i\rangle_B \in \mathcal{H}_B$ and probability distribution $\{p_i\}_i$ such that

$$\rho = \sum_i p_i \, |\eta_i\rangle\!\langle\eta_i|_A \otimes |\chi_i\rangle\!\langle\chi_i|_B \, . \tag{1.13}$$

Otherwise, ρ is *entangled.*

To show that a state is separable, it is sufficient to find states η_i, χ_i to decompose it, according to Definition 1.1. Still, this problem is NP-hard [Gur03, Gha10]. Showing that a state is entangled is clearly not as straightforward in principle. However, a class of operators known as *entanglement witnesses* [HHH96, Ter00], proves very useful for this purpose:

Definition 1.2. An operator W acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is an *entanglement witness* (or simply a *witness*) if, for all separable states σ,

$$\mathrm{tr}(W\sigma) \geq 0. \tag{1.14}$$

If a state ρ is such that

$$\operatorname{tr}(W\rho) < 0, \tag{1.15}$$

the witness W is said to *detect* ρ.

Often, witnesses are required not to be positive operators, otherwise they are useless as they do not detect any state. The existence of witnesses follows from the Hahn-Banach theorem [Hah27, Ban29a, Ban29b]: since the set of separable states is convex and compact, there exists a hyperplane separating it from any entangled state, which is a point outside of the separable set.

Another useful criterion to detect entanglement is the PPT criterion, or Peres-Horodecki criterion [Per96, HHH96]. It is based on the partial transpose, an operation defined on composite states that transposes the part corresponding to one party while leaving the other untouched. Let $|\psi\rangle$ be a pure state, with Schmidt decomposition $|\psi\rangle = \sum_i \sqrt{\lambda_i}\,|ii\rangle_{AB}$. Then, its partial transpose with respect to system A is

$$\begin{aligned}
\psi^{\Gamma_A} &= \sum_{i,j} \sqrt{\lambda_i \lambda_j}\,(|i\rangle\langle j|_A)^T\,(|i\rangle\langle j|_B) \\
&= \sum_{i,j} \sqrt{\lambda_i \lambda_j}\,(|j\rangle\langle i|_A)\,(|i\rangle\langle j|_B) \\
&\equiv \sum_{i,j} \sqrt{\lambda_i \lambda_j}\,|ji\rangle\langle ij|_{AB}\,.
\end{aligned} \tag{1.16}$$

Since the Schmidt coefficients λ_i are real, we have $\psi^{\Gamma_A} = \psi^{\Gamma_B} \equiv \psi^{\Gamma}$. This definition extends linearly to mixed states. The transpose is not a completely positive operation, hence ρ^{Γ} may not be a positive operator although any quantum state ρ is. In fact, if the partial transpose of a quantum state (with respect to either party) is not a positive operator, then the state is entangled. The converse is only true when the dimensions of the local Hilbert spaces are 2×2 or 2×3. A state with a positive partial transpose is termed *PPT*.

This criterion can be used to find witnesses that are positive for all PPT states. While some entangled states will escape detection, these witnesses come in useful in many situations: they can be found with a semi-definite program, and it is simple to prove that an operator is a PPT witness: an operator W is a PPT witness if and only if it is decomposable, i.e., if there exist positive operators P, Q such that $W = P + Q^{\Gamma}$ [LKCH00].

Entangled states are interesting from a fundamental point of view since there is no analogue in classical systems. But further, from a practical point of view, they are a useful asset to own, since they help perform some communication-related tasks more efficiently. For example, they can be used to encode information in fewer bits than are needed without the help of entanglement, to generate a key for cryptographic protocols, or even as channels to transmit quantum information.

One prime example of the usefulness of entangled states is the *teleportation protocol* [BBC$^+$93], which uses a maximally entangled state together with classical communication in order to send an unknown state from one place to another (thus transmitting quantum information). Of course, if Alice knew the description of the state she wanted to transmit, she might be able to send it to Bob via a classical message, in order for Bob to recreate the state in his laboratory. However, teleportation still works if Alice does not know what state she has, and even if Alice's particle is entangled to some other particle: teleportation will preserve that entanglement.

In more detail, suppose that Alice wants to teleport a 2-dimensional state $|\psi\rangle_{A'} = \alpha\,|0\rangle_{A'} + \beta\,|1\rangle_{A'}$, where $|\alpha|^2 + |\beta|^2 = 1$, to Bob. We assume $|\psi\rangle$ is pure for the sake of simplicity, but the extension to mixed states is immediate by linearity. The protocol requires that Alice and Bob share a state $\left|\phi^+_{AB}\right\rangle := (|00\rangle_{AB} + |11\rangle_{AB})/\sqrt{2}$ (this state is the maximally entangled state in dimension 2, as we shall see later). Therefore, the joint state of Alice's and Bob's particles is

$$|\psi\rangle_{A'}\left|\phi^+\right\rangle_{AB} = \frac{1}{\sqrt{2}}\left(\alpha\,|0\rangle_{A'}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) + \beta\,|1\rangle_{A'}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)\right). \quad (1.17)$$

First, Alice applies a CNOT gate to her particles. This is a Controlled-NOT gate, meaning that it flips the state of the second qubit (from $|0\rangle$ to $|1\rangle$ and vice versa) if and only if the first qubit is in state $|1\rangle$. Therefore, the state becomes

$$\frac{1}{\sqrt{2}}\left(\alpha\,|0\rangle_{A'}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) + \beta\,|1\rangle_{A'}\left(|10\rangle_{AB} + |01\rangle_{AB}\right)\right). \quad (1.18)$$

Next, Alice applies a Hadamard gate to her first qubit. A Hadamard gate H is the unique unitary gate that maps $|0\rangle$ to $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle$ to $|-\rangle := (|0\rangle - |1\rangle)/\sqrt{2}$. Therefore, the joint state is now

$$\frac{1}{2}\left(\alpha(|0\rangle_{A'} + |1\rangle_{A'})(|00\rangle_{AB} + |11\rangle_{AB}) + \beta(|0\rangle_{A'} - |1\rangle_{A'})(|10\rangle_{AB} + |01\rangle_{AB})\right). \quad (1.19)$$

But, by regrouping the states of the particles that Alice holds, this state can be rewritten as

$$\begin{aligned} &\frac{1}{2}\left[|00\rangle_{A'A}\left(\alpha\,|0\rangle_B+\beta\,|1\rangle_B\right)+|01\rangle_{A'A}\left(\alpha\,|1\rangle_B+\beta\,|0\rangle_B\right)\right.\\ &\quad\left.+\,|10\rangle_{A'A}\left(\alpha\,|0\rangle_B-\beta\,|1\rangle_B\right)+|11\rangle_{A'A}\left(\alpha\,|1\rangle_B-\beta\,|0\rangle_B\right)\right]. \end{aligned} \tag{1.20}$$

Now, Alice can measure her qubits in the computational basis and thus find out what state Bob's particle is in: if she obtains 00, Bob's particle is in the original state $|\psi\rangle$ that Alice held. If she obtains 01, 10, or 11, Bob can recover the state $|\psi\rangle$ by performing a local unitary on his particle. Therefore, when Alice communicates her measurement outcome to Bob (which requires 2 bits of classical communication), he can recover the state that she originally held, even if the values of α and β are not known to either party. By the end of the protocol, the state ϕ^+ is no longer available to the parties: the entanglement they shared has been consumed.

Indeed, entanglement is consumed after other protocols too. Further, entangled states cannot be generated by acting separately on each system—not even with the help of classical communication. This is why they are a precious resource which will be the object of study of a large part of this thesis. Protocols such as teleportation, as well as superdense coding, quantum key distribution, and many others, all of which rely on entanglement, act on entangled states by applying operations locally on each subsystem, and communicating the results of some of these operations via classical channels. This is the paradigm of allowed operations that is often assumed when studying entangled states: Local Operations and Classical Communication, or LOCC. While its mathematical description is somewhat involved in general (see, e.g., [CLM+14,HES+21]), since local operations may depend on prior measurement results and communication rounds, conceptually it is a very useful tool. As hinted above, LOCC operations cannot generate entanglement. Moreover, while most well-known LOCC protocols consume the entanglement completely, in general they might only degrade it, and indeed LOCC cannot increase the amount of entanglement contained in a state (a precise way of measuring the entanglement of quantum states will be introduced shortly).

This is the basic intuition behind the notion of a resource theory. The aim of a resource theory is to order states according to their usefulness for practical tasks. While the framework of resource theories can be applied to many different scenarios [CG19], including coherence [SAP17], reference frame alignment [BRS07,GS08], noncontextuality [ACCA18, DA18], thermodynamics [BHO+13, GMN+15], nonlocality [de 14, GA17], steering [GA15], and many more, in this work we shall be concerned only with the resource theory of entanglement. In this context, LOCC is often taken as the set of *free operations*, i.e., those which are accessible to the agents at no cost. Hence, the set of *free* *states* contains those states that can be prepared using only the free operations, namely, the set of separable states. Entangled states (which cannot be prepared by LOCC) become a resource, and the free operations determine their relative usefulness: if a state ϕ can be converted by a free operation into another state ψ, then ϕ is at least as useful as ψ. Indeed, if ψ is needed for some task, but only ϕ is available, free operations can be used to convert ϕ into ψ before performing the task, while the converse need not be true. Any entangled state can be converted into a separable state using LOCC (a simple protocol is to ignore the input state and generate the desired separable state), therefore any entangled state is more useful than all separable states.

A resource theory considers all the possible conversions between resource states so as to obtain the induced partial order on this set. The possible LOCC interconversions among pure bipartite states where characterised by Nielsen [Nie99], who showed that the LOCC ordering reduces to majorisation [MOA11]: ϕ can be converted into ψ if and only if the Schmidt coefficients λ_i, μ_i of ϕ, ψ respectively are such that, for each $k = 0, ..., d-1$,

$$\sum_{i=0}^{k-1} \lambda_i^{\downarrow} \leq \sum_{i=0}^{k-1} \mu_i^{\downarrow}, \tag{1.21}$$

where the superscript indicates that the coefficients are taken in descending order. There are some entangled states that are incomparable under LOCC, since neither set of coefficients majorises the other, and hence none can be converted into the other using these free operations. However, this ordering gives rise to a *maximally entangled state* for every dimension d,

$$|\phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle\,, \tag{1.22}$$

which can be converted to any other state of the same dimension using LOCC. It is taken as the gold standard to measure entanglement and, unsurprisingly, this state is also the most useful one in some common protocols such as teleportation or superdense coding. The 2-dimensional maximally entangled state is denoted $|\phi^+\rangle$.

The order that a resource theory induces on entangled states can be used to quantify their entanglement. *Entanglement measures* are mappings $E : \mathcal{B}(\mathcal{H}) \to \mathbb{R}^+$ from density operators into non-negative real numbers such that $E(\rho) = 0$ if ρ is separable. Moreover, E must not increase under LOCC operations $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ performed on ρ:

$$E[\Lambda(\rho)] \leq E(\rho). \tag{1.23}$$

This entails that, if ρ is at least as entangled as τ according to the LOCC resourcetheoretic ordering, then $E(\rho) \geq E(\tau)$.

One commonly used entanglement measure is the *relative entropy of entanglement* [VPRK97, VP98], which, for pure states, is the von Neumann entropy H of the single-party reduced state: letting $(\psi)_A := \mathrm{tr}_B(\psi)$,

$$E(\psi) = H[(\psi)_A] = -\,\mathrm{tr}[(\psi)_A \log(\psi)_A]. \tag{1.24}$$

The generalisation to mixed states is called the *entanglement of formation,* and it is done via the convex roof construction, i.e., by minimising over all possible decompositions of a mixed state into pure states:

$$E(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i E(\psi_i). \tag{1.25}$$

In fact, it is often required that other entanglement measures reduce to the entropy of entanglement when considering pure states [PV07]. The d-dimensional maximally entangled state (equation (1.22)) has $E(\phi_d^+) = \log d$, which is the maximum value of E (see equation (1.24)). This is also a common normalisation requirement for entanglement measures.

In addition, the *robustness* R [VT99] intuitively captures the distance of a state ρ from the set of separable states. More specifically, the robustness of ρ quantifies the weight needed to mix ρ with the best choice of separable state, in order to obtain a separable state:

$$R(\rho) = \min_{\sigma \in \mathcal{S}} R(\rho||\sigma), \tag{1.26}$$

where

$$R(\rho||\sigma) = \min \left\{ s : \frac{\rho + s\sigma}{1+s} \in \mathcal{S} \right\}, \tag{1.27}$$

and $\mathcal{S}$ is the set of separable states. If the state σ is not required to be separable, the corresponding measure is termed *generalised robustness.*

Another useful measure is the *geometric measure of entanglement* G [WG03]. For pure states, it is related to the maximum overlap of the state with a separable state:

$$G(\psi) = 1 - \max_{\sigma \in \mathcal{S}} \mathrm{tr}(\psi\sigma), \tag{1.28}$$

where $\mathcal{S}$ is the set of separable states (note that σ can be assumed pure, without loss of generality). To cover mixed states, we use the convex roof construction, so that

$$G(\rho) = \min_{\{p_i,\psi_i\}} \sum_i p_i G(\psi_i), \tag{1.29}$$

where $\rho = \sum_i p_i \psi_i$. This ensures that G is a well-defined measure.

Many more entanglement measures are known (see, e.g., Ref. [PV07]), but we shall only be concerned with the above ones in this work.

A related concept is the *fidelity* between two quantum states, which is a measure of their closeness. The fidelity F is defined as

$$F(\rho,\sigma) = \mathrm{tr}^2 \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}, \tag{1.30}$$

and it is symmetric (i.e., $F(\rho,\sigma) = F(\sigma,\rho)$). If one of the states is pure (say, if $\sigma = |\psi\rangle\langle\psi|$), it takes a simpler form:

$$F(\rho,\psi) = \mathrm{tr}(\rho\psi) = \langle\psi|\,\rho\,|\psi\rangle\,. \tag{1.31}$$

(The fidelity is sometimes defined as the square root of the quantity in equation (1.30), but the choice made here means the F in equation (1.30) is linear in each argument when the other is fixed.)

While LOCC cannot be used to transform a less entangled state into a more entangled one, it is possible to transform *many* less entangled states into *few* more entangled ones. This process is called *distillation of entanglement,* and usually refers to transforming many copies of any input state into few copies of the maximally entangled state. In practice, perfect maximally entangled states cannot be achieved, but rather, the goal is to obtain states that are close (in terms of their fidelity) to the maximally entangled state. As the number of copies of the input state grows unboundedly, the fidelity can be made arbitrarily close to 1. The *rate* of a distillation protocol is the ratio between the number of copies of the input state and the number of copies of the output state. The best achievable rate for a given input state is termed the *distillable entanglement* of that state, which happens to be a measure of entanglement (in fact, for pure states it is equal to the relative entropy of entanglement), although it will not be used in this work.

1.1.2 Entanglement in multipartite systems

When considering more than two systems, the definition of entanglement becomes ambiguous: indeed, we can have Alice, Bob and Charlie holding three subsystems in tensor product, or Alice and Bob sharing an entangled state which is separable from Charlie's, or all three sharing a truly tripartite entangled state, and these cases do not fit Definition 1.1. Instead, for a system of n parties we define:

Definition 1.3. A state $\rho \in \mathcal{B}(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$, where $n \in \mathbb{N}$, is *fully separable* if there exist $|\psi_{i,1}\rangle \in \mathcal{H}_1$, ..., $|\psi_{i,n}\rangle \in \mathcal{H}_n$ and a probability distribution $\{p_i\}_i$ such that

$$\rho = \sum_i p_i \psi_{i,1} \otimes \cdots \otimes \psi_{i,n}. \tag{1.32}$$

If ρ is not fully separable, it is *entangled.*

The state ρ is *biseparable* if there exist $|\psi_{i,M}\rangle \in \bigotimes_{j\in M} \mathcal{H}_j$, $\left|\psi_{i,\overline{M}}\right\rangle \in \bigotimes_{j\in\overline{M}} \mathcal{H}_j$ for bipartitions $\{M, \overline{M}\}$ of $[n]$ and a probability distribution $\{p_i\}_i$ such that

$$\rho = \sum_{i,M} p_{i,M} \psi_{i,M} \otimes \psi_{i,\overline{M}}. \tag{1.33}$$

If ρ is not biseparable, it is *genuine multipartite entangled (GME).*

Finer-grained versions of this definition can be found by considering partitions of $[n]$ of different numbers $k \in [n]$ of elements, giving rise to the concept of k-separability, but we shall not be concerned with this here.

Two well-known GME states are the W state,

$$|W\rangle = \frac{1}{\sqrt{2}}(|001\rangle_{ABC} + |010\rangle_{ABC} + |100\rangle_{ABC}), \tag{1.34}$$

and the Greenberger-Horne-Zeilinger (GHZ) state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle_{ABC} + |111\rangle_{ABC}), \tag{1.35}$$

which can be generalised to any dimension d and number of parties n as

$$|GHZ(n,d)\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^{\otimes n}. \tag{1.36}$$

A consequence of the definition of biseparability that will be exploited in this work is that it is not closed under tensor products: for example, the tensor product of the states $|\phi^+\rangle_{AB} \otimes |0\rangle_C$ and $|\phi^+\rangle_{AC} \otimes |0\rangle_B$, which are both biseparable, is

$$\frac{1}{2}\left(|00,00,00\rangle_{ABC} + |01,00,01\rangle_{ABC} + |10,10,00\rangle_{ABC} + |11,10,01\rangle_{ABC}\right), \tag{1.37}$$

which cannot be written as $|\psi_M\rangle \otimes |\psi_{\overline{M}}\rangle$ for any bipartition $M|\overline{M}$ of A, B, C. Since the state in equation (1.37) is pure, this means it is GME.

Since the sets of fully separable and biseparable states are convex, witnesses can be used to detect entanglement in multipartite systems as well as bipartite ones. An operator which has positive trace with all fully separable states detects states that contain, at least, bipartite entanglement; a witness which detects GME has positive trace with all biseparable states. ($\mathcal{W}$ or other letters will be used to denote multipartite witnesses if there is risk of confusion with the state W.)

Similarly to the bipartite case, to study GME states it is sometimes useful to consider a slightly larger set of states, namely, the set of PPT mixtures [JMG11]. This set contains states that can be written as a mixture of PPT states, possibly in different bipartitions. Just like bipartite separability implies PPT, but not vice versa, the set of PPT mixtures strictly contains that of biseparable states. Further, operators which have positive trace with all PPT mixtures are, in particular, GME witnesses. These witnesses W can be written as $W = P_M + Q_M^{\Gamma_M}$, where P_M, Q_M are positive operators for each bipartition $M|\overline{M}$.

1.2 Probability distributions

As mentioned in the previous section, classical information (i.e., information that us humans can access) can be obtained out of quantum states by measuring them. It turns out that measurements on some quantum states give results that could not have arisen out of any classical system.

While POVMs are sets of operators indexed by a classical variable, one can also consider a dependence on a classical input. Thus, if a composite system is measured locally, i.e., Alice and Bob each measure their particle with inputs x, y and outputs a, b respectively, their results will be distributed according to a probability $P(a, b|x, y)$ of the outputs given the inputs. Such probability distributions can be correlated much more strongly if the state shared by Alice and Bob is entangled than if it is separable, a phenomenon known as *nonlocality*. This is one piece of classical evidence that can be searched for to confirm whether Nature is post-classical. And indeed, several experiments

[Asp76, BMM92, WJS+98, RKM+01, HBD+15, GVW+15, AAA+18], each more reliable than the last, have revealed correlations that could not have been generated by any purely classical system.

In this thesis we shall only be concerned with the theoretical aspects of nonlocality. To analyse them, we now define the most relevant notions: we use $E_{a|x}, F_{b|y}$ to denote the POVM elements with output a, b and input x, y respectively, and let $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ be the sets of Alice's and Bob's outputs and inputs respectively. These sets will often be left implicit. If these POVMs act on a quantum state ρ, the Born rule (equation (1.10)) yields a conditional probability $P(a, b|x, y)$. Conversely, probabilities of this form are said to be quantum if they could have arisen from a quantum state and measurements:

Definition 1.4. A probability distribution $\{P(a, b|x, y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ is *quantum* if it can be written in the form

$$P(a, b|x, y) = \operatorname{tr}\bigl(E_{a|x} \otimes F_{b|y}\rho\bigr) \tag{1.38}$$

for some quantum state ρ and POVMs $\left\{E_{a|x}\right\}_{a\in\mathcal{A}}, \left\{F_{b|y}\right\}_{b\in\mathcal{B}}$ for each $x \in \mathcal{X}, y \in \mathcal{Y}$.

Note that, strictly speaking, $\{P(a, b|x, y)\}_{(a,b)\in\mathcal{A}\times\mathcal{B}}$ is a probability distribution for each $(x, y) \in \mathcal{X} \times \mathcal{Y}$. However, throughout this work we refer to $\{P(a, b|x, y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ as a probability distribution.

Quantum distributions arising from some entangled states are especially interesting, since they cannot be generated using only classical resources. In fact, classical resources can only give rise to *local* probability distributions, which we now define.

Definition 1.5. A probability distribution $\{P(a, b|x, y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ is *local* if it can be written in the form

$$P(a, b|x, y) = \sum_{\lambda\in\Lambda} p(\lambda) P_A(a|x, \lambda) P_B(b|y, \lambda), \tag{1.39}$$

for some distributions $\{P_A(a|x, \lambda)\}_{(a,x,\lambda)\in\mathcal{A}\times\mathcal{X}\times\Lambda}, \{P_B(b|y, \lambda)\}_{(b,y,\lambda)\in\mathcal{B}\times\mathcal{Y}\times\Lambda}$, and where λ is a 'hidden variable' taking values in some set Λ and distributed according to $\{p(\lambda)\}_{\lambda\in\Lambda}$. Equation (1.39) is a *Local Hidden-Variable (LHV)* model for the probability distribution P. More parties can be accounted for by adding more distributions, correlated only by the hidden variable λ. A quantum state is local if, for any POVMs, it can only give rise to local distributions.

(Unfortunately for our purposes, λ is the most common choice for both Schmidt coefficients and hidden variables—context will determine what λ stands for throughout this work).

It is simple to show that separable states can only give rise to local distributions, i.e., all separable states are local. Suppose Alice and Bob share a state $\rho = \sum_i p_i \left|\eta_i\right\rangle\!\left\langle\eta_i\right|_A \otimes \left|\chi_i\right\rangle\!\left\langle\chi_i\right|_B$, and apply measurements given by $E_{a|x}, F_{b|y}$. Then,

$$\begin{aligned} P(a,b|x,y) &= \operatorname{tr}\left[\left(E_{a|x} \otimes F_{b|y}\right)\left(\sum_i p_i \left|\eta_i\right\rangle\!\left\langle\eta_i\right|_A \otimes \left|\chi_i\right\rangle\!\left\langle\chi_i\right|_B\right)\right] \\ &= \sum_i p_i \operatorname{tr}\left(E_{a|x} \left|\eta_i\right\rangle\!\left\langle\eta_i\right|_A\right) \operatorname{tr}\left(F_{b|y} \left|\chi_i\right\rangle\!\left\langle\chi_i\right|_B\right), \end{aligned} \tag{1.40}$$

which is of the form of equation (1.39) by letting

$$\begin{aligned} P_A(a|x,\lambda) &= \operatorname{tr}\left(E_{a|x} \left|\eta_i\right\rangle\!\left\langle\eta_i\right|_A\right) \\ P_B(b|y,\lambda) &= \operatorname{tr}\left(F_{b|y} \left|\chi_i\right\rangle\!\left\langle\chi_i\right|_B\right) \\ p(\lambda) &= p_i. \end{aligned} \tag{1.41}$$

In fact, every local distribution can be written as a quantum distribution arising from a separable state and measurements, that is, the set of local distributions is included in the set of quantum distributions. This inclusion is strict: there are entangled states and measurements that give rise to *nonlocal* distributions, i.e., those which cannot be expressed like equation (1.39), and this is the content of Bell's seminal theorem [Bel64].

While all pure entangled states can display nonlocality (for the right choice of measurements) [Gis91], entanglement is not equivalent to nonlocality. One prime example of this phenomenon is given by isotropic states [HH99]

$$\rho_p = p\phi_d^+ + (1-p)\frac{\mathbb{1}}{d^2}, \tag{1.42}$$

whose parameter p is termed the *visibility*. These states are entangled if and only if $p > 1/(d+1)$ [HH99], where d is the local dimension, and local if $p > p_L$. While the exact value of p_L is not known, the bounds $\Theta(3/\mathrm{e}d) \leq p_L \leq C \log^2 d/d$, where C is a constant, were given in Refs. [APB$^+$07, Pal14], and imply that, for $1/(d+1) < p < \Theta(3/\mathrm{e}d)$, isotropic states are entangled but cannot give rise to any nonlocal distributions. Werner states may also be entangled and local [Wer89, Bar02]. Still, taking many copies of an entangled, local mixed state sometimes yields nonlocality, a phenomenon termed *superactivation* of nonlocality [Pal12].

Nonlocal distributions have a wide variety of applications including cryptography

[GRTZ02, PAB+20], randomness extraction, amplication and certification [AM16], communication complexity reduction [BCMd10], etc. Indeed, they are one of the reasons why some entangled states are useful for communication-related tasks. Nonlocal distributions are also useful as a means of certification of quantum entanglement. For this, as well as for establishing sufficient conditions on quantum states to give rise to nonlocality, it is necessary to have a reliable way of knowing when a distribution is nonlocal, that depends only on the distribution itself. The main tool for this purpose is *Bell inequalities*.

A Bell inequality is a functional I that acts on distributions $\{P(a,b|x,y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ by assigning coefficients $c_{a,b,x,y}$ to each element $P(a,b|x,y)$, and is such that, for all local P,

$$\langle I,P\rangle \equiv \sum_{a,b,x,y} c_{a,b,x,y} P(a,b|x,y) \leq c_0 \tag{1.43}$$

for some $c_0 \in \mathbb{R}$. If a distribution P is such that

$$\langle I,P\rangle > c_0, \tag{1.44}$$

then P is said to *violate* the inequality I. Bell inequalities play the role of nonlocality witnesses, in analogy to entanglement witnesses. In fact, they are also hyperplanes separating a nonlocal point from the (convex) set of local distributions.

In some cases, Bell inequalities can be given a physical meaning by viewing them as *nonlocal games*. In a nonlocal game played cooperatively by Alice and Bob, a referee picks questions x, y, drawn from some alphabets $\mathcal{X}, \mathcal{Y}$, for Alice and Bob respectively, with a probability $\pi(x,y)$. Without knowing each other's questions, the players must each return an answer a, b, drawn from some alphabets $\mathcal{A}, \mathcal{B}$ respectively. The referee then decides whether the players win or lose according to a publicly known verification function $V(a,b,x,y)$ which depends on the questions and answers, and is equal to 1 if they win and to 0 otherwise. In this setting, the distribution $\{P(a,b|x,y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ captures the probability of the players answering a, b given questions x, y, and thus encodes their strategy. Then, the overall winning probability can be straightforwardly calculated as

$$\sum_{x,y} \pi(x,y) \sum_{a,b} V(a,b,x,y) P(a,b|x,y). \tag{1.45}$$

By associating the coefficients

$$G_{ab|xy} = \pi(x,y)V(a,b,x,y) \tag{1.46}$$

for each a, b, x, y, games can be viewed as functionals. Then, the winning probability is the action $\langle G, P\rangle$ of the game functional on the probability distribution. In this sense, games are a particular type of Bell inequalities, which have non-negative coefficients (since $\pi(x,y), V(a,b,x,y) \geq 0$ for all a, b, x, y).

One well-known nonlocal game is the CHSH game [CHSH69], where inputs and outputs take values 0 or 1, the probability distribution of the questions, $\pi(x,y)$, is uniform, and the verification function $V(a,b,x,y) = 1$ if and only if $a \oplus b = xy$. That is, Alice and Bob win if they provide equal outputs whenever x or y are 0, and different outputs if $x = y = 1$. Local strategies give a maximum winning probability of 3/4, while the maximally entangled state and certain measurements can be used to achieve a winning probability of $(2+\sqrt{2})/4 \simeq 0.85$. In fact, this is the maximum winning probability achievable by a quantum strategy, as Tsirelson's seminal result showed [Cir80].

Aside from inequalities, nonlocality can also be detected by giving a series of conditions that a distribution P can only meet if it is nonlocal. An example of this is Hardy's paradox [Har92, Har93]:

$$P(0,0|0,0) > 0 = P(0,1|0,1) = P(1,0|1,0) = P(0,0|1,1). \tag{1.47}$$

It is not difficult to prove that local distributions cannot satisfy Hardy's paradox, and, in fact, nor can maximally entangled states. However, all other pure entangled states can satisfy the paradox.

It can be shown that distributions of one input or output are always local, so the study of nonlocality only takes off for larger distributions. Hence, the simplest Bell inequalities are those of two inputs and two outputs. To decide whether distributions of more inputs and outputs (and, as will become important later, parties) are nonlocal, Bell inequalities that account for this greater complexity can be devised. Alternatively, one can apply local manipulations to the distribution of more inputs, outputs or parties in order to achieve an effective distribution of two inputs, outputs and parties, that is thus local if the original distribution is. More generally, any distribution P where inputs and outputs take values on $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ (and similar sets for any extra parties) can be mapped to an effective distribution $\tilde{P}$ shared by Alice and Bob where inputs and outputs take values on $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}, \tilde{\mathcal{X}}, \tilde{\mathcal{Y}}$. Any inputs in $\mathcal{X}$ that are not in $\tilde{\mathcal{X}}$ can be simply ignored, and similarly for $\mathcal{Y}$. Outputs in $\mathcal{A}$ can be grouped into $|\tilde{\mathcal{A}}|$ sets, so that effective probabilities $\tilde{P}$ are the sum of some probabilities $P(a, b|x, y)$ over certain values a, and similarly for $\mathcal{B}$. Finally, one can restrict attention to a particular input and output for any extra parties who are not Alice and Bob. It is not difficult to show that all of these transformations are local; that is, if P is local, then so is $\tilde{P}$. Conversely, if $\tilde{P}$ is obtained from P in the above way, and $\tilde{P}$ violates a Bell inequality, then P is nonlocal.

Equivalently, one can transform Bell inequalities, to make them account for more inputs, outputs and parties. To do this, it is useful to view them geometrically. For a given number of inputs and outputs (i.e., when $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ have fixed size), the set of local distributions forms a polytope. Each of these polytopes is completely characterised by a finite set of Bell inequalities that corresponds to one of its facets (that is, faces of maximal dimension). Thus, facet inequalities characterise the border between the local and nonlocal regions. It turns out that the facet inequalities of one local polytope provide necessary conditions that larger polytopes (i.e. for more inputs, outputs and parties) must meet, and hence it is possible to derive facet Bell inequalities for larger polytopes starting from facet Bell inequalities of smaller polytopes. This process is called *lifting* Bell inequalities, and is detailed in Ref. [Pir05]. The lifting process preserves the local bound as long as the original Bell inequality has a local bound of zero. However, any Bell inequality can be written so that this is the case, by expressing any constant as $\sum_{a,b} P(a, b|x, y)$ times the constant, for any fixed x, y.

Starting from a facet inequality with a local bound of zero that holds for the local polytope defined by $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$, more inputs can be accounted for by assigning coefficient 0 to any inputs outside of $\mathcal{X}, \mathcal{Y}$ (which corresponds to ignoring the relevant inputs in the distribution that the inequality acts on). Further, each new output outside of $\mathcal{A}, \mathcal{B}$ gets assigned the same coefficient as one of the outputs in the smaller polytope (so that, effectively, outputs are grouped into sets). Finally, to account for more parties, the coefficients pertaining to the smaller polytope are made equal to those of the larger polytope for some fixed input and output of the extra parties, and all other inputs and outputs get assigned coefficient 0 (corresponding to fixing the input and output of all extra parties). Transforming a Bell inequality for a smaller polytope in this way gives rise to a facet inequality for the larger polytope. These transformations of Bell inequalities are proven in Ref. [Pir05].

As well as generating nonlocal distributions, entangled states may also exhibit the property of quantum steering [WJD07, CS16, UCNG20]. Suppose Alice prepares a bipartite quantum state and sends one of the particles to Bob. They both measure their respective particles and communicate classically. Suppose they repeat this many times. Can Alice convince Bob that the state she prepares is entangled? The answer is yes only if the state is *steerable*, that is, if there is no *Local Hidden-State (LHS)* model describing it.

Definition 1.6. A state ρ has a Local Hidden-State model if, for any POVM $\left\{E_{a|x}\right\}_{a\in\mathcal{A}}$, the distribution arising from measuring the state can be written as

$$\mathrm{tr}_A(E_{a|x}\otimes\mathbb{1}\rho)=\sum_\lambda P_A(a|x,\lambda)p(\lambda)\sigma_\lambda, \tag{1.48}$$

where σ_λ is a single-party state which depends on the hidden variable λ.

Definition 1.7. If a state ρ has an LHS model of the form of equation (1.48), then ρ is non-steerable from Alice to Bob. Otherwise, ρ is steerable from Alice to Bob.

If ρ is non-steerable, Alice can simply prepare an ensemble of $\{\sigma_\lambda\}_\lambda$ which she sends to Bob, and deliver output a for input x with probability $P_A(a|x,\lambda)$. Thus, she is only able to convince Bob that the state she prepared is entangled if it is steerable. By considering $\mathrm{tr}\big(E_{a|x}\otimes F_{b|y}\rho\big)=\mathrm{tr}_B(\mathrm{tr}_A(E_{a|x}\otimes\mathbb{1}\rho)F_{b|y})$ and denoting $P_B(b|y,\lambda)=\mathrm{tr}\big(F_{b|y}\sigma_\lambda\big)$, it is evident that all non-steerable states are local, but the converse is not true.

This notion of steerability concerns the case where Alice can steer Bob's state. One can consider the analogous concept interchanging Alice and Bob. In fact, steerability is an asymmetric notion: there are states that are steerable from Alice to Bob, but not from Bob to Alice [BVQB14].

We have seen that local probability distributions can be generated using only classical resources, while quantum distributions can be generated using only quantum resources. It is possible in principle to consider larger, post-quantum sets of distributions, and one such set turns out to be very useful: the set of *nonsignalling* distributions.

Definition 1.8. A probability distribution $\{P(a,b|x,y)\}_{(a,b,x,y)\in\mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}}$ is nonsignalling if it is such that

$$\begin{aligned}\sum_a P(a,b|x,y)&=\sum_a P(a,b|x',y),\\ \sum_b P(a,b|x,y)&=\sum_b P(a,b|x,y'),\end{aligned} \tag{1.49}$$

for all $x'\neq x$, $y'\neq y$.

These conditions ensure that the marginal distributions $P(a|x)$, $P(b|y)$ are well-defined. Also, it follows from the definition that nonsignalling distributions cannot be used by Alice and Bob to communicate information faster than the speed of light: indeed, unless Alice is allowed to send messages to Bob, he cannot know her input, and vice versa. Further, all quantum distributions are nonsignalling, a condition ensured by the normalisation of the POVMs. In fact, the set of quantum distributions is strictly included in the set of nonsignalling distributions.

Nonsignalling distributions are often depicted as a *nonsignalling box*, which is a ficticious device that Alice and Bob can share while spatially separated, admits inputs x, y for each party respectively, and gives outputs a, b to each party respectively. By extension, one can abstract away from the physical realisation of quantum and local distributions, and imagine ficticious devices that generated them. Thus, we often talk about quantum and local boxes, to mean special cases of nonsignalling boxes whose underlying probability distributions are quantum or local, respectively.

The nonsignalling conditions are an example of a minimal requirement that a probability distribution should meet in order to be physical. Indeed, while quantum theory is widely accepted as a very reliable description of Nature, and is confirmed by experiment to a great level of precision, it is not the *only* possible theory compatible with observations so far. Nonsignalling conditions were first introduced as a physical principle that any physical theory should obey [PR94]. As well as gaining a better understanding of Nature, properties of nonsignalling distributions establish necessary conditions on quantum distributions. Since the nonsignalling set forms a polytope, unlike the quantum set (which is convex, but does not have a finite number of extremal points), it is often simpler to analyse. In light of this, it is helpful to have a means of knowing when a nonsignalling box is quantum. Two main tools will be relevant for this work: Tsirelson's theorem and quantum voids. They will be defined in Chapter 5.

We have seen that a distribution is local if it admits a Local Hidden-Variable model. These models are often given a physical interpretation in the context of epistemics and game theory, where the hidden variables represent the possible 'states of the world'. Thus, the model is defined by the probability space $(\Omega, \mathcal{E}, \mathsf{P})$, where Ω is the set of possible states of the world (which are commonly denoted by ω instead of λ in this context), $\mathcal{E}$ is the power set of Ω, i.e., the set of events, and P is a probability measure on Ω. Agents do not know which is the true state of the world ω^*, but they have limited information about it: Alice and Bob partition the state space according to $\mathcal{P}_A$, $\mathcal{P}_B$ respectively, and they know which partition element contains the true state of the world. From this information, they can each calculate the conditional probability of any event (which is a set of states of the world) given the element $\mathcal{P}_{A,B}(\omega^*)$ of their respective partitions that contains the true state of the world.

As mentioned above, Bell's theorem prevents the extension of this model to nonlocal settings, so, in particular, this model does not apply to parties who share quantum devices. However, this model can be made valid for all nonsignalling distributions via a simple relaxation: allowing P, the distribution of the states of the world, to be a *quasi-probability measure*, i.e., to take negative values as well as non-negative ones, with $\sum_{\omega\in\Omega}\mathsf{P}(\omega) = 1$. These constructions are referred to in the literature as *ontological models*, which are a quasi-probability space $(\Omega, \mathcal{E}, \mathsf{P})$ together with a set of partitions. In the literature (see, e.g., [Fer11]), they often include a set of preparations underlying the distribution over the state space, and the partitions are usually phrased in terms of measurements and outcomes. However, we consider preparations implicit and use the language of partitions to bridge the gap between the fields of epistemics and quantum information more smoothly. Thus, any nonsignalling box can be associated to an ontological model, and vice versa. This was derived in Refs. [AB11, AB14] from sheaf-theoretic concepts, but in Chapter 5 we provide a much more direct proof that is more suitable for the purposes of this work.

In the following, distributions p or P will always be assumed to take non-negative values, unless otherwise mentioned.

1.2.1 Multipartite nonlocality

Just like in the case of entanglement, the definitions of local, quantum and nonsignalling distributions can be extended in more than one way to multipartite settings. As hinted above, a natural way of extending the definition of local distributions to more than two parties is to add single-party distributions to equation (1.39), correlated only by the hidden variable. However, one can also imagine distributions that group parties into two sets, and are local only across this bipartition, or convex mixtures of these distributions. Analogously to entanglement, we define fully local, bilocal and genuine multipartite nonlocal distributions:

Definition 1.9. A probability distribution $\{P(a_1, ..., a_n|x_1, ..., x_n)\}_{(a_i,x_i)\in\mathcal{A}_i\times\mathcal{X}_i, i\in[n]}$, for some sets $\mathcal{A}_i, \mathcal{X}_i$ for each $i \in [n]$, where $n \in \mathbb{N}$, is *fully local* if it can be written in the form

$$P(a_1, ..., a_n|x_1, ..., x_n) = \sum_{\lambda} p(\lambda) \prod_{i\in[n]} P_i(a_i|x_i, \lambda) \tag{1.50}$$

for some probability distributions $\{P_i(a_i|x_i, \lambda)\}_{(a_i,x_i,\lambda)\in\mathcal{A}_i\times\mathcal{X}_i\times\Lambda}$ for each $i \in [n]$, and $\{p(\lambda)\}_{\lambda\in\Lambda}$.

It is *bilocal* if it can be written in the form

$$P(a_1, ..., a_n|x_1, ..., x_n) = \sum_{M \subsetneq [n]} \sum_{\lambda} p_M(\lambda) P_M(\{a_i\}_{i\in M}|\{x_i\}_{i\in M}, \lambda) P_{\overline{M}}(\{a_i\}_{i\in \overline{M}}|\{x_i\}_{i\in \overline{M}}, \lambda), \tag{1.51}$$

for some distributions $\{P_M(\{a_i\}_{i\in M}|\{x_i\}_{i\in M}, \lambda)\}_{(a_i,x_i,\lambda)\in\mathcal{A}_i\times\mathcal{X}_i\times\Lambda, i\in M}$, $\{P_{\overline{M}}(\{a_i\}_{i\in\overline{M}}|\{x_i\}_{i\in\overline{M}}, \lambda)\}_{(a_i,x_i,\lambda)\in\mathcal{A}_i\times\mathcal{X}_i\times\Lambda, i\in\overline{M}}$ which are nonsignalling, for each bipartition $M|\overline{M}$ of the parties, and where $p_M(\lambda) \geq 0$, $\sum_{M,\lambda} p_M(\lambda) = 1$.

Otherwise, it is *genuine multipartite nonlocal (GMNL)*.

Bilocal distributions are sometimes termed "hybrid" or "mixed". This is because the term "bilocal" has alternative definitions in the literature, notably regarding hidden variable models where variables correlate only pairs of parties [TRGB17, RWB$^+$19, RBB$^+$19, GBC$^+$20, KCC$^+$20]. Since there will be no ambiguity in this respect, in this work we use the term "bilocal" in analogy to "biseparable".

The requirement that the distributions $P_M, P_{\overline{M}}$ of the bipartitions are nonsignalling is an added subtlety that the definition of biseparability does not give rise to. Indeed, the original definition of bilocality, given by Svetlichny in Ref. [Sve87], left these distributions unrestricted; however, this has been shown to lead to operational problems [Bus12, GWAN12, BBGP13, GP14, de 14, GA17]. Hence, like most recent works on the topic, we assume these distributions are nonsignalling, which captures most physical situations better [SRB20, WSS$^+$20].

Contrary to locality, the nonsignalling conditions extend unambiguously to the multipartite setting. It is sufficient to assume that the marginal distribution of all but one party is independent of the input of this party, and that this holds for all parties, in order to conclude that the marginal of any subset of parties is independent of the inputs outside this subset.

Definition 1.10. A probability distribution $\{P(a_1, ..., a_n|x_1, ..., x_n)\}_{(a_i,x_i)\in\mathcal{A}_i\times\mathcal{X}_i, i\in[n]}$, for some sets $\mathcal{A}_i, \mathcal{X}_i$ for each $i \in [n]$, where $n \in \mathbb{N}$, is *nonsignalling* if, for any party $i \in [n]$,

$$\sum_{a_i} P(a_1, ..., a_n|x_1, ..., x_n) \tag{1.52}$$

is independent of x_i.

It is also worth remarking that alternative definitions of GME and GMNL are currently being proposed. Refs. [SRB20, WSS$^+$20] consider nonlocality (among other quantum phenomena) from the point of view of a resource theory whose free operations are Local Operations and Shared Randomness (LOSR). They also lay the ground for Ref. [NWRP20] to propose the notion of *genuine network entanglement*, a stricter notion than GME which rules out states which are tensor products of biseparable states, and one could imagine a similar notion of network nonlocality. However, we are interested mainly in quantum networks where pairs of parties share entangled states, so these notions will not help us determine which network configurations are more useful than others.

GMNL distributions, defined as in Definition 1.9, have numerous applications, for example in multiparty cryptography [AGCA12], the understanding of condensed matter physics [TAS+14, TDA+17], and the development of quantum networks [CASA11, GMT+17, ŠSC17, TRGB17, RBB+19, KCC+20], particularly for quantum computation [CEHM99, HV12, HWVE14] and correlating particles which never interacted [BGP10, BRGP12].

It is easy to see that biseparable states can only give rise to bilocal distributions. Indeed, a pure biseparable state is separable along a bipartition $M|\overline{M}$ of the parties, therefore, as shown above for the bipartite case, it can only give rise to a distribution that is local along that bipartition. Moreover, the distributions $P_M, P_{\overline{M}}$ are nonsignalling, since they arose from a quantum state and measurements. Biseparable mixed states thus give rise to convex combinations of bilocal distributions, which is still bilocal. Like in the bipartite case, GME is not sufficient for nonlocality, as there exist GME mixed states that are bilocal [ADTA15, ADT18] or even fully local [BFF+16]. While pure GME states are never fully local [PR92, GG17], it is not known whether all pure GME states can give rise to GMNL.

In general, distributions arising from the multipartite state ρ, and POVMs $\{E_{a_i|x_i}\}_{a_i\in\mathcal{A}_i}$ for each x_i, for each party $i\in[n]$, are of the form

$$P(a_1,...,a_n|x_1,...,x_n) = \mathrm{tr}\left(\bigotimes_{i=1}^{n} E_{a_i|x_i}\rho\right). \tag{1.53}$$

To find out which of these distributions are GMNL, one can use *GMNL inequalities*, which, in analogy to Bell inequalities for bipartite systems, are functionals that are bounded when acting on bilocal distributions:

$$\langle I,P\rangle \equiv \sum_{\substack{a_i,x_i\\ i\in[n]}} c_{a_1,...,a_n,x_1,...,x_n} P(a_1,...,a_n|x_1,...,x_n) \leq c_0 \tag{1.54}$$

for all bilocal P.

1.3 Our contribution

Having reviewed the main technical tools that will be used throughout this thesis, we now give an outline of the problems that we tackle and the main results we develop. Chapters 2-4 focus on multipartite systems, studying their entanglement properties and how to obtain nonlocality from network states. Chapter 5 introduces a principle that we contend should be satisfied by all physical theories.

1.3.1 A nontrivial resource theory of multipartite entanglement

As advanced above, entanglement is a striking feature of quantum theory with no classical analogue. Although initially studied to address foundational issues [Sch35, EPR35], the development of quantum information theory [NC00] in the last few decades has elevated it to a resource that allows the implementation of tasks which are impossible in classical systems. The resource theory of entanglement [PV07, HHHH09] aims at providing a rigorous framework to qualify and quantify entanglement and, ultimately, to understand fully its capabilities and limitations within the realm of quantum technologies. However, this theory is much more firmly developed for bipartite than multipartite systems. In fact, although a few applications have been proposed within the latter setting such as secret sharing [HBB99, Got00], the one-way quantum computer [RB01] and metrology [GLM11, TA14], a deeper understanding of the complex structure of multipartite entangled states might inspire further protocols in quantum information science and better tools for the study of condensed-matter systems.

As already reviewed in Section 1.1.1, the wide applicability of the formulation of entanglement theory as a resource theory has motivated an active line of work that studies different quantum effects from this point of view [CG19]. Although it was introduced above in the context of entanglement under LOCC operations, the framework of resource theories can be expressed independently of the object of study: the main question a resource theory addresses is to order the set of states and provide means to quantify their nature as a resource. The free operations are crucial to this task. This is a subset of transformations, which the given scenario dictates can be implemented at no cost. Thus, all states that can be prepared with these operations are free states. Conversely, non-free states acquire the status of a resource: granted such states, the limitations of the corresponding scenario might be overcome. Moreover, the concept of free operations allows an order relation to be defined. If a state ρ can be transformed into σ by some free operation, then ρ cannot be less resourceful than σ since any task achievable by σ is also achievable by ρ as the corresponding transformation can be freely implemented. However, the converse is not necessarily true. Furthermore, one can introduce resource quantifiers as functionals that preserve this order.

Since entanglement is a property of systems with many constituents which may be far away, the natural choice for free operations in this resource theory is local operations and classical communication (LOCC). Indeed, parties bound to LOCC can only prepare separable states, and entangled states become a resource to overcome the constraints imposed by LOCC manipulation. In pure bipartite states, the ordering induced by LOCC reduces to majorisation [Nie99, MOA11], and there is a unique maximally entangled state for fixed local dimension. This is because this state can be transformed by LOCC into any other state of that dimension but no other state of that dimension can be transformed into it.

Importantly, the situation changes drastically in the multipartite case. Here, Ref. [DVC00] and subsequent work [VDDMV02, BLTV04] have shown that there exist inequivalent forms of entanglement: the state space is divided into classes, the so-called stochastic LOCC (SLOCC) classes, of states which can be interconverted with non-zero probability by LOCC but cannot be transformed outside the class by LOCC, even probabilistically. This in particular shows that no maximally entangled state can exist for multipartite states. Still, one could in principle study the ordering induced by LOCC within each SLOCC class. Recent work [dSK13, SdK16, HSK16, SdSK17, dSSK17, GKW17] in this direction has revealed, however, an extreme feature that culminates with the result of Ref. [SWGK18]: almost all pure states of more than three parties are *isolated*, i.e. they cannot be obtained from nor transformed to another inequivalent pure state of the same local dimensions by LOCC. This means that almost all pure states are incomparable by LOCC, inducing a trivial ordering and a meaningless arbitrariness in the construction of entanglement measures. In this sense, one may say that the resource theory of multipartite entanglement with LOCC is generically trivial.

We believe this calls for a critical reexamination of the resource theory of entanglement and, in particular, for the notion of LOCC as the ordering-defining relation. Indeed, although LOCC transformations have a clear operational interpretation, this is not, in fact, the most general class of transformations that maps the set of separable states into itself. In other words, LOCC is strictly included in the class of non-entangling operations. Thus, from the abstract point of view of resource theories other consistent theories of entanglement (i.e. with separable states being the free states) are possible where the set of free operations is larger than LOCC. Hence, in principle, these could give a more meaningful ordering and revealing structure in the set of multipartite entangled states. To study such possibility is precisely the goal of Chapter 2. A similar approach has been taken to address other unsatisfying features of the resource theory of entanglement under LOCC such as irreversibility of state transformations for an arbitrarily large number of copies [VC01]. Remarkably, Ref. [BP08,BP10] has shown that shifting the paradigm from LOCC to asymptotic non-entangling operations provides a reversible theory of asymptotic entanglement interconversion with a unique entanglement measure and this result has been extended in [BG15] to arbitrary resource theories under asymptotic resource-non-generating operations [CG19]. Also, in the absence of a clear set of physical constraints determining the free operations, certain quantum resource theories have been constructed by first defining the set of free states and then considering classes of operations that preserve this set. This is the case of the resource theory of coherence [BCP14a], which has been found useful in e.g. metrology applications [BDW17] and quantum channel discrimination [NBC+16] and which has subsequently given rise to a fruitful research line considering an operational interpretation for the set of free operations (see [CG16, SAP17] and references therein).

Since we seek whether a non-trivial theory is at all possible for single-copy manipulations, here we consider the resource theory of entanglement under the largest possible class of free operations in this regime: strictly non-entangling operations. However, multipartite entanglement comes in two different forms, as seen in Definition 1.3. Thus, one can formulate two theories: one in which entangled states are considered a resource and where the free operations are full separability-preserving (FSP), and the analogous with GME states and biseparability-preserving (BSP) operations. Interestingly, our first result is that both formalisms lead to non-trivial theories: no resource state is isolated in any of these scenarios. Moreover, we show that there are no inequivalent forms of entanglement. Then, we consider whether there exists a unique multipartite maximally entangled state in these theories like in the bipartite case. While we find a negative answer (at least in the simplest non-trivial case of 3-qubit states) for FSP operations, our main result is that the question is answered affirmatively in the resource theory of GME under BSP operations. The maximally GME state turns out to be the generalised Greenberger-Horne-Zeilinger (GHZ) state.

1.3.2 Pure pair-entangled network states

As shown in Section 1.2, correlations between quantum particles may be much stronger than those between classical particles. Their applications are manifold: cryptography [GRTZ02, PAB+20], randomness extraction, amplification and certification [AM16], communication complexity reduction [BCMd10], etc., and the study of these nonlocal correlations has led to the growing field of device-independent quantum information processing [MY98, ABG+07, Col11] (see also Ref. [BCP+14b]).

While bipartite nonlocality has been well researched in the past three decades, much less is known about the multipartite case. Still, correlations in quantum multicomponent systems have gained increasing attention recently, with applications in multiparty cryptography [AGCA12], the understanding of condensed matter physics [TAS+14, TDA+17], and the development of quantum networks [CASA11, GMT+17, ŠSC17, TRGB17, RBB+19, GBC+20, KCC+20], particularly for quantum computation [CEHM99, HV12, HWVE14] and correlating particles which never interacted [BGP10, BRGP12].

A necessary condition for nonlocality is quantum entanglement. Indeed, this is one reason why entangled states are useful for communication-related tasks. However, not all entangled states are nonlocal: some bipartite entangled states only yield local distributions [Wer89, Bar02]. Still, for pure bipartite states, entanglement *is* sufficient for nonlocality, which is the content of Gisin's theorem [Gis91, GP92], and multipartite entangled pure states are never fully local [PR92, GG17]. Interestingly, distributing certain bipartite entangled states in certain multipartite networks yields nonlocality even if the involved states are individually local [SSB+05, CASA11, CRS12, ŠSC17, Luo18, Luo19].

Multipartite nonlocality is in principle harder to generate than bipartite nonlocality. By exploring the relationship between entanglement and nonlocality in the multipartite regime, in Chapter 3 we show that *pair-entangled network states* simplify the job considerably: distributing arbitrarily low node-to-node entanglement is sufficient to observe truly multipartite nonlocal effects involving all parties in the network independently of its geometry. Added to its practical consequences for applications, this fact points to a deep property of quantum networks.

We show that the nonlocality arising from networks of bipartite pure entangled states is a generic property and manifests in its strongest form, GMNL. Specifically, we obtain that any connected network of bipartite pure entangled states is GMNL. It was already known that a star network of maximally entangled states is GMNL [CASA11], but we provide a full, qualitative generalisation of this result by making it independent of both the amount of entanglement shared and the network topology. Thus, we show GMNL is an intrinsic property of networks of pure bipartite entangled states.

Further, there are known mixed GME states that are bilocal [ADTA15, ADT18]—some are even fully local [BFF+16]. Still, it is not known whether Gisin's theorem extends to the genuine multipartite regime. Recent results show that, for pure n-qubit symmetric states [CYZ+14] and all pure 3-qubit states [YO13], GME implies GMNL (at the single-copy level).[2] Our result above shows that all pure GME states that have a network structure are GMNL; interestingly, we further apply this property to establish a second result: all pure GME states are GMNL in the sense that measurements can be found on finitely many copies of any GME state to yield a GMNL behaviour. We thus tighten the relationship between multipartite entanglement and nonlocality.

Our construction exploits the fact that the set of bilocal states is not closed under tensor products. That is, GME can be superactivated by taking tensor products of states that are unentangled across different bipartitions. Thus, GME can be achieved by distributing bipartite entangled states among different pairs of parties. To obtain our results, we extend the superactivation property [NV11, Pal12, CMT15] from the level of states to that of probability distributions, i.e. GMNL can be superactivated by taking Cartesian products of probability distributions that are local across different bipartitions. In fact, when considering copies of quantum states, we only consider local measurements performed on each copy separately, thus pointing at a stronger notion of superactivation to achieve GMNL.

1.3.3 Mixed pair-entangled network states

As advanced in the previous section, quantum networks make it possible to generate GMNL using only bipartite entanglement. Since bipartite entanglement is in principle easier to distribute than truly multipartite entanglement such as GHZ-state entanglement, this makes networks a very useful tool for the wide variety of applications that require GME and GMNL. Added to the operational motivation, their conceptual simplicity and well-defined mathematical properties makes them a good platform in which to explore the relationship between entanglement and nonlocality in many-body systems. Indeed, it is experimentally much simpler to distribute bipartite entanglement between different nodes of a network than to establish genuine network entanglement [NWRP20] between the nodes. Therefore, understanding the behaviour of mixed pair-entangled network states is crucial to gauge the full potential of near-term quantum technologies. Quantum networks are widely studied, for example, as a means to achieve long-range entanglement starting from smaller entanglement links, as well as to entangle more than two parties [PKT+19, VGNT19]. Applications such as cryptography [GRTZ02], quantum error correction [BDSW96], quantum metrology [GLM11], quantum sensor networks [RH12, EFG+18, KBDGL19, QBB+20], multi-party

[2] Hidden GMNL for three parties beyond qubits can be shown if some form of preprocessing is allowed.

quantum communication [HBB99, ZXP15, MGKB20], or computation [RB01, DP06], all require multipartite entanglement and thus drive the need to devise ways of supplying end-to-end entanglement to nodes who request them. The main theoretical and practical challenges in this respect are ensuring high entanglement generation rates, high fidelity and long coherence times, but proposals for optimal ways of generating entanglement between two end nodes continue to be put forward [Cal17, CRDW19, SQ19, DPW20, BAKE20, LLLC21]. Further, realistic implementations of the quantum internet will rely on existing infrastructure on which to build a quantum network [RCAW20], so it is important to find the best ways of distributing entanglement in a given network configuration, in the line of the recent work [BCO21].

The results in Chapter 3 show that networks of pure states exhibit a very simple behaviour since, as long as they are connected, they yield GMNL independently of their topology and the amount of entanglement contained in the states on the edges. In particular, this means that all connected networks of pure entangled states are GME, and thus are sufficient to implement the above applications. However, to derive those results, we use properties that are exclusive of pure bipartite states: all pure entangled states are nonlocal, and, moreover, they can satisfy Hardy's paradox [Har92, Har93] (if they are not maximally entangled) and exhibit full nonlocality [EPR92, BKP06] (if they are). However, mixed states are very different to pure states in terms of the interplay between entanglement and nonlocality, even in the bipartite case: there exist mixed states, such as some isotropic states [HH99] and some Werner states [Wer89, Bar02], which are entangled and local. Multipartite mixed states may also be GME and bilocal [ADTA15, ADT18], or even fully local [BFF$^+$16].

In experimental settings, noise is unavoidable, and thus pure states are out of experimental reach. Hence, studying networks of bipartite mixed states is essential if they are to be used for applications. While, by continuity, the results about pure states obtained in Chapter 3 must be robust to some noise, in general not even GME is guaranteed for mixed-state networks (although connected networks of entangled states are never fully separable). We focus mainly on isotropic states as a noise model. These are the only states that are invariant under the action of $U \otimes U^*$ for any unitary U whose complex conjugate is U^*. Thus, in addition of representing a standard noise model in which a maximally entangled state is mixed with white noise, their symmetry properties make them a convenient object to study theoretically. For qubits, any state with a negative partial transpose (which is thus entangled) can be transformed into an entangled isotropic state by *twirling*, an LOCC operation consisting on averaging over all unitaries U [HHH98]. (For larger dimensions, this only happens if the fidelity with the maximally entangled state is large enough.) Indeed, the first step of many protocols such as distillation protocols is transforming the input state into an isotropic state. Twirling is an LOCC operation, therefore biseparability is closed under twirling. For this reason, if a given network of isotropic states is GME, substituting some or all of the states for NPT states that can be transformed into them preserves GME. In the particular case of qubits, all networks of entangled states would be GME if and only if all networks of isotropic states were.

We show that networks of mixed states exhibit very different properties to those of pure states. First, node-to-node entanglement does not necessarily imply that the network is GME. We give two examples of tripartite networks with entangled isotropic states on the edges, but which are nevertheless biseparable. Further, by studying networks of three parties we can already show a dependence of the entanglement properties on the topology of the network, unlike in the case of pure states.

Then, we find that, in the case of larger networks, this dependence manifests itself in the most extreme form. We show that all networks in the form of a tree graph (i.e., a graph which contains no cycles) or a polygon become biseparable for a sufficiently large number of edges, as long as the visibility of the states on the edges is strictly smaller than 1 (i.e., the noise parameter is strictly positive). Thus, any given experimental limitation to the preparation of pure-state entanglement prevents the observation of GME for these network configurations if the number of parties is large enough: too few connections in a network compromise entanglement.

Remarkably, GME depends crucially on the geometry of the network. We show, in contrast to the above results, that a completely connected network of isotropic states (i.e., a network where all vertices are connected to all others) remains GME for any number of parties for all visibilities above a threshold. As a consequence, GME in the completely connected network holds for any number of parties as long as the visibility is large enough. Since GME is a necessary condition for GMNL, we show that distributing nonlocal states is not sufficient to generate GMNL and, in particular, that the GMNL of networks of mixed states can depend on the topology and the amount of entanglement present in the network.

We also explore the nonlocality properties of some networks of isotropic states. Beyond practical applications, the symmetry properties of isotropic states and the fact that they can be entangled while local makes this family of states particularly interesting for theoretical study. We find that non-steerability is the main factor compromising GMNL in these networks, and we find that a star network with a non-steerable state on one edge and a maximally entangled state on the rest is GME but bilocal. And yet, steerability does not guarantee GMNL. As a consequence of the previous result, we provide an example of a steerable state which, when distributed in a star network, is bilocal. Further, we show that a star network of non-steerable states is fully local. Still, by taking many copies of the bilocal network, we obtain, to our knowledge, the first example of superactivation of GMNL from bilocality. In fact, our construction can be used to obtain more examples of superactivation of GMNL from many copies of bilocal networks.

1.3.4 A physical principle from observers' agreement

So far, we have focussed on analysing what resources can be extracted from different configurations of quantum states. We have classified the resource of multipartite entanglement, finding that there is a way to obtain a maximally resourceful state, and understood what network states give rise to GME and GMNL. All of these results take quantum theory as a given, and their applicability depends on quantum theory being an apt description of Nature. This is a reasonable assumption to make, as numerous quantum effects have been confirmed by experiments to very high precision. However, in principle, other post-classical theories are possible, and the fact that post-quantum correlations have not been observed so far does not mean that Nature does not allow for them. In a departure from the multipartite considerations that are the focus of the first part of the thesis, Chapter 5 aims to rule out at least some post-quantum theories as possible descriptions of Nature. In particular, we postulate a principle external to quantum theory, but satisfied by it, and contend that it should hold in all reasonable theories of Nature.

Quantum mechanics famously made its creators uncomfortable. Its differences with classical physics are so structural that the theory seems highly counterintuitive even today. Almost a century after its introduction, it still sparks much conceptual and philosophical discussion. Indeed, an active line of research in quantum foundations deals with the problem of singling out quantum theory from other post-classical physical theories. This field is a delicate balance between proposals for new theories that are 'tidier' than quantum mechanics [Spe07, Lar12] and proposals for desirable physical principles that such theories should obey [PR94, CBH03, PPK+09, SZY19, Yan13].

In Chapter 5 we propose a new principle inspired by a famous result in epistemics, which is the formal study of knowledge and beliefs. In the domain of classical probability theory, Aumann proved that Bayesian agents cannot agree to disagree [Aum76]. A slightly more general restatement of Aumann's theorem, which we will refer to as the *classical agreement theorem*, states that, if Alice and Bob, based on their partial information, assign probabilities q_A, q_B, respectively, to perfectly correlated events, and these probabilities are common certainty between them, then $q_A = q_B$. "Certainty" means assigning probability 1, and "common certainty" means that Alice is certain about q_B; Bob is certain about q_A; Alice is certain about Bob being certain about q_A; Bob is certain about Alice being certain about q_A; and so on infinitely.

This result is considered a basic requirement in classical epistemics, and we contend it should apply to all physical theories. The classical agreement theorem has been used to show that two risk-neutral agents, starting from a common prior, cannot agree to bet with each other [SG83], to prove "no-trade" theorems for efficient markets [MS82], and to establish epistemic conditions for Nash equilibrium [AB95]. These applications are all external to physics. Of course, the theorem holds equally in the physical domain, provided that classical probability theory applies.

But in the quantum domain, the classical model does not apply, and so we cannot assume that the same facts about agreement and disagreement between Bayesian agents hold when they observe quantum phenomena. In particular, a fundamental result of quantum mechanics is that no local hidden-variable theory can model the results of all quantum experiments [Bel64]. This implies that the classical Bayesian model does not apply, so the classical agreement theorem need not hold. The question then arises: can observers of quantum mechanical phenomena agree to disagree? We address this question by exploring it in the broader nonsignalling setting.

First, we establish that, in general, nonsignalling agents *can* agree to disagree about perfectly correlated events, and we give explicit examples of disagreeing nonsignalling distributions. In the particular case of two inputs and two outputs, we characterise the distributions that give rise to common certainty of disagreement. One might think that the fact that nonsignalling agents can agree to disagree is a direct consequence of the multitude of uncertainty relations in quantum mechanics, all of which put a limit on the precision with which the values of incompatible observables can be measured and have even been linked to epistemic inconsistencies in quantum mechanics [FR18]. Somewhat surprisingly, our next finding shows that this is not the case. We find that disagreeing nonsignalling distributions of two inputs and outputs cannot be quantum—i.e., the agreement theorem holds for quantum agents in this setting. Then, we go beyond this restriction and show that any disagreeing nonsignalling distribution with more than two inputs or outputs induces a disagreeing distribution with two inputs and outputs. Since the agreement theorem holds for quantum agents sharing distributions of two inputs and outputs, it does so for larger distributions too. Thus, even if quantum mechanics features uncertainty relations, this does not apply to observers' estimations of perfectly correlated events.

Next, we ask if nonsignalling and quantum agents can disagree in other ways. We define a new notion of disagreement, which we call *singular disagreement*, by removing the requirement of common certainty and, instead, imposing $q_A = 1$, $q_B = 0$, and we ask whether it holds for classical, quantum and nonsignalling agents. We find the same pattern: singular disagreement does not hold for classical or quantum agents, but can occur in nonsignalling settings, where we characterise the distributions that feature it. We then put our two characterisations together and search for distributions that satisfy both common certainty of disagreement and singular disagreement: we find that the PR box [PR94] is of this kind—i.e., it displays extremal disagreement in the above sense. This is neat, as the PR box is known to exhibit the most extreme form of nonsignalling correlations [BLM$^+$05].

Finally, we contend that agreement between observers could be a convenient principle for testing the consistency of new postquantum theories. Our results exhibit a clear parametrisation of the set of the probability distributions that allow observer disagreement. This set is easy to work with, thanks to its restriction to two observations and two outcomes per observer. If a new theory can be used to generate such a distribution, this might raise a red flag, as this theory violates a reasonable and intuitive and, importantly, testable property that quantum mechanics satisfies.

Aumann's theorem has appeared elsewhere in the physics literature. However, it has been examined in a different context [Khr15, KB14], where agents are assumed to use Born's rule as their probability update rule. The authors conclude that Aumann's theorem does not hold for this type of agents. Instead, our setting assumes that the agents are macroscopic and merely share a quantum state or a nonsignalling box. Our setting is appealing in quantum information for its applications to communication complexity, cryptography, teleportation, and many other scenarios. In turn, Ref. [AC19] introduces a different notion of disagreement in a nonsignalling context. The disagreement in that work concerns pieces of information about some variables, and agreement refers to consistency in the information provided about the variables. Hence, it is unrelated to the epistemic notion of disagreement that Aumann's theorem defines, and that the present work revisits from a nonsignalling perspective.

Chapter 2

A nontrivial resource theory of multipartite entanglement

Entanglement theory is formulated as a quantum resource theory in which the free operations are local operations and classical communication (LOCC). This defines a partial order among bipartite pure states that makes it possible to identify a maximally entangled state, which turns out to be the most relevant state in applications. However, the situation changes drastically in the multipartite regime. Not only do there exist inequivalent forms of entanglement forbidding the existence of a unique maximally entangled state, but recent results have shown that LOCC induces a trivial ordering: almost all pure entangled multipartite states are incomparable (i.e. LOCC transformations among them are almost never possible). In order to cope with this problem we consider alternative resource theories in which we relax the class of LOCC to operations that do not create entanglement. We consider two possible theories depending on whether resources correspond to multipartite entangled or genuinely multipartite entangled (GME) states and we show that they are both non-trivial: no inequivalent forms of entanglement exist in them and they induce a meaningful partial order (i.e. every pure state is transformable to more weakly entangled pure states). Moreover, we prove that the resource theory of GME that we formulate here has a unique maximally entangled state, the generalised GHZ state, which can be transformed to any other state by the allowed free operations.

2.1 Definitions and preliminaries

We will consider n-partite systems with local dimension d, i.e. states in the Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n = (\mathbb{C}^d)^{\otimes n}$. Given a subset M of $[n] = \{1, \ldots, n\}$ and its complement $\bar{M}$, we denote by $\mathcal{H}_M$ the tensor product of the Hilbert spaces corresponding to the parties in M and analogously with $\mathcal{H}_{\bar{M}}$. Reviewing Definition 1.3 for pure states, we say $|\psi\rangle \in \mathcal{H}$ is FS (otherwise entangled) if $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$ for some states $|\psi_i\rangle \in \mathcal{H}_i$ $\forall i$, while it is BS (otherwise GME) if $|\psi\rangle = |\psi_M\rangle \otimes |\psi_{\bar{M}}\rangle$ for some states $|\psi_M\rangle \in \mathcal{H}_M$ and $|\psi_{\bar{M}}\rangle \in \mathcal{H}_{\bar{M}}$ and $M \subsetneq [n]$. These notions are extended to mixed states by the convex hull and we define the sets of FS and BS states by

$$\mathcal{FS} = \mathrm{conv}\{\psi : |\psi\rangle \text{ is FS}\},\ \mathcal{BS} = \mathrm{conv}\{\psi : |\psi\rangle \text{ is BS}\}. \tag{2.1}$$

Definition 2.1. A completely positive and trace preserving (CPTP) map $\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is *full separability-preserving (FSP)* if $\Lambda(\rho) \in \mathcal{FS}$ $\forall \rho \in \mathcal{FS}$. It is *biseparability-preserving (BSP)* if $\Lambda(\rho) \in \mathcal{BS}$ $\forall \rho \in \mathcal{BS}$.

We will say that a functional E taking operators on $\mathcal{H}$ to non-negative real numbers is an FSP measure (BSP measure) if $E(\rho) \geq E(\Lambda(\rho))$ for every state ρ and FSP (BSP) map Λ. This is completely analogous to entanglement measures, which are required to be non-increasing under LOCC maps. Although LOCC is a strict subset of the FSP and BSP maps, some well-known entanglement measures are still FSP or BSP measures and this will play an important role in assessing which transformations are possible within the two formalisms that we consider here. Indeed, measures of the form

$$E_{\mathcal{X}}(\rho) = \inf_{\sigma \in \mathcal{X}} E(\rho||\sigma), \tag{2.2}$$

where $\mathcal{X}$ stands for either $\mathcal{FS}$ or $\mathcal{BS}$, have the corresponding monotonicity property as long as the distinguishability measure $E(\rho||\sigma)$ is *contractive*, i.e. $E(\Lambda(\rho)||\Lambda(\sigma)) \leq E(\rho||\sigma)$ for every CPTP map Λ. This includes the relative entropy of entanglement [VPRK97, VP98] for $E(\rho||\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma)$ and the robustness ($R_{\mathcal{X}}$) [VT99] for

$$E(\rho||\sigma) = R(\rho||\sigma) = \min\left\{ s : \frac{\rho + s\sigma}{1+s} \in \mathcal{X} \right\}. \tag{2.3}$$

If one uses the fidelity, $E(\rho||\sigma) = 1 - F(\rho||\sigma) = 1 - \mathrm{tr}^2 \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$, in the case of pure states equation (2.2) boils down to the geometric measure [WG03], which we will denote by $G_{\mathcal{X}}$ and which is then seen to be a measure under maps that preserve $\mathcal{X}$. We will only need to consider $G_{\mathcal{X}}$ for pure states:

$$G_{\mathcal{X}}(\cdot) = 1 - \left(\max_{|\phi\rangle \in \mathcal{X}} |\langle \phi | \cdot \rangle| \right)^2 \tag{2.4}$$

(see also the definitions on pages 22-23). Notice, however, that, as has been recently shown in the bipartite case in [CdGG20], not all LOCC measures remain monotonic under non-entangling maps since the latter formalism allows state conversions that the former does not. In the following, in order to understand the ordering of resources induced by these theories, we study which transformations are possible among pure states under FSP and BSP maps. However, first one should point out that whenever there exist maps Λ and Λ' in the corresponding class of free operations such that $\Lambda(\psi) = \phi$ and $\Lambda'(\phi) = \psi$, then the states ψ and ϕ are equally resourceful and should be regarded as equivalent in the corresponding theory. This is moreover necessary so as to have a well-defined partial order. Hence, although for simplicity we will talk about properties of states, one should have in mind that one is actually speaking about equivalence classes. Specifically, it is known that two pure states are interconvertible by LOCC if and only if they are related by local unitary transformations [Gin02]. Interestingly, we will see that the equivalence classes are wider in the resource theory of GME under BSP. It should be stressed that, to our knowledge, this is the first time that a resource theory of GME is formulated. Notice that the restriction to LOCC can only have FS states as free states. Furthermore, allowing a strict subset of parties to act jointly and classical communication does not fit the bill either as $\mathcal{BS}$ is not closed under these operations.

Throughout the proofs of the results we will use repeatedly that, if ρ_1 and ρ_2 are density matrices, the map

$$\Lambda(\rho) = \mathrm{tr}(A\rho)\rho_1 + \mathrm{tr}[(\mathbb{1} - A)\rho]\rho_2 \tag{2.5}$$

is CPTP if $0 \leq A \leq \mathbb{1}$ (see e.g. [CdGG20]).

2.2 Non-triviality of the theories

Our first two results are valid in both the FSP and BSP regimes. Thus, following the notation above, the two possible classes of maps will be referred to as $\mathcal{X}$-preserving.

Theorem 2.1 (Collapse of the SLOCC classes)**.** *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, all resource states are interconvertible with non-zero probability, i.e. given any pure $\psi_1, \psi_2 \notin \mathcal{X}$, there exists a completely* *positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = p\psi_2$ with $p \in (0,1]$.*

Proof. The proof is based on explicitly constructing a completely positive and trace non-increasing $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = p\psi_2$, for some p that can be ensured to be strictly larger than 0.

Notice that, since $\psi_1\,, \psi_2 \notin \mathcal{X}$ and both the geometric measure and the robustness are faithful measures [WG03, VT99], $R_{\mathcal{X}}(\psi_2)\,,\ G_{\mathcal{X}}(\psi_1) > 0$. Also, $G_{\mathcal{X}}(\psi_1) < 1$ because the fully (bi-)separable states span the whole Hilbert space. Pick $p \in]0,1]$ such that

$$p \leq \frac{1}{R_{\mathcal{X}}(\psi_2)} \frac{G_{\mathcal{X}}(\psi_1)}{1 - G_{\mathcal{X}}(\psi_1)} \tag{2.6}$$

and let

$$\Lambda(\eta) = p\,\mathrm{tr}(\psi_1\eta)\psi_2 + \mathrm{tr}\left[(\mathbb{1} - \psi_1)\eta\right]\rho_{\mathcal{X}}\,. \tag{2.7}$$

Here $\rho_{\mathcal{X}} \in \mathcal{X}$ is the state which gives the corresponding robustness of ψ_2, i.e., $R_{\mathcal{X}}(\psi_2) = R(\psi_2||\rho_{\mathcal{X}})$—cf. equation (2.3). (Note that Λ can be completed to a CPTP $\mathcal{X}$-preserving map by adding a term of the form $\Lambda'(\eta) = (1-p)\,\mathrm{tr}(\psi_1\eta)\rho_{\mathcal{X}}$.) Then $\Lambda(\psi_1) = p\psi_2$ and it remains to be shown that Λ is $\mathcal{X}$-preserving. Let $\sigma \in \mathcal{X}$. Then

$$\Lambda(\sigma) \propto \psi_2 + \frac{1}{p}\left(\frac{1}{\mathrm{tr}(\psi_1\sigma)} - 1\right)\rho_{\mathcal{X}}\,, \tag{2.8}$$

so $\Lambda(\sigma)/\,\mathrm{tr}(\Lambda(\sigma)) \in \mathcal{X}$ iff $\frac{1}{p}\left(\frac{1}{\mathrm{tr}(\psi_1\sigma)} - 1\right) \geq R_{\mathcal{X}}(\psi_2)$. But this holds from equation (2.6) and using $\mathrm{tr}(\psi_1\sigma) \leq 1 - G_{\mathcal{X}}(\psi_1)\ \forall\sigma \in \mathcal{X}$. $\square$

Theorem 2.2 (No isolation)**.** *In a resource theory of entanglement where the free operations are $\mathcal{X}$-preserving maps, no resource state is isolated, i.e. given any pure $\psi_1 \notin \mathcal{X}$ on $\mathcal{H}$, there exists an inequivalent pure $\psi_2 \notin \mathcal{X}$ on $\mathcal{H}$ and a CPTP $\mathcal{X}$-preserving map Λ such that $\Lambda(\psi_1) = \psi_2$.*

Proof. This result arises as a corollary of Theorem 2.1. Given any $\psi_1 \notin \mathcal{X}$, continuity arguments show that there always exists an inequivalent $\psi_2 \notin \mathcal{X}$ with $R_{\mathcal{X}}(\psi_2)$ small enough so that one can take $p = 1$ in equation (2.6) and construct a CPTP map.

Consider the map (2.7) from the proof of Theorem 2.1. This map can be made deterministic if $R_{\mathcal{X}}(\psi_2)$ is sufficiently smaller than $G_{\mathcal{X}}(\psi_1)$. Indeed, if

$$\frac{1}{R_{\mathcal{X}}(\psi_2)} \frac{G_{\mathcal{X}}(\psi_1)}{1 - G_{\mathcal{X}}(\psi_1)} > 1\,, \tag{2.9}$$

then we can pick $p = 1$ in the map (2.7) so Λ is CPTP (see equation (2.5)). Since robustness is a continuous function of the input state [VT99], it can be arbitrarily close to zero and so there exists ψ_2 such that the above condition is fulfilled for any ψ_1. Further, ψ_1, ψ_2 are inequivalent if they have different robustness, but $R(\psi_2)$ can always be picked to be different from $R(\psi_1)$ and still satisfying equation (2.9). □

Theorem 2.1 proves that in our case there are no inequivalent forms of entanglement. This is in sharp contrast to LOCC where, leaving aside the case $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3}$, the state space splits into a cumbersome zoology of infinitely many different SLOCC classes of unrelated entangled states. Theorem 2.2 provides the non-triviality of our theories. While almost all states turn out to be isolated under LOCC [SWGK18], our classes of free operations induce a meaningful partial order structure where, as in the case of bipartite entanglement, every pure state can be transformed into a more weakly entangled pure state. It is important to mention that the result of [SWGK18] proves generic isolation when transformations are restricted among GME states with the rank of all n single-particle reduced density matrices equal to d. However, Theorem 2.2 still holds under this restriction.

Theorems 2.1 and 2.2 show that limitations of the resource theory of multipartite entanglement under LOCC can be overcome if one considers FSP or BSP operations instead. These positive results raise the question of whether the induced structure is powerful enough to have a unique multipartite maximally entangled state. If this were so, our theories would point to a relevant class of states that should be at the heart of the applications of multipartite entanglement in a similar fashion to the maximally entangled state in the bipartite case. In order to answer this question, we first provide an unambiguous definition of a maximally resourceful state which, on the analogy of the bipartite case, depends on the number of parties n and local dimension d: a state ψ on $\mathcal{H}$ is the maximally resourceful state on $\mathcal{H}$ if it can be transformed by means of the free operations into any other state on $\mathcal{H}$.[1]

2.3 Existence of a maximally entangled state

2.3.1 FSP regime

We analyse the case of FSP operations first, where we find no maximally resourceful state exists.

[1]Notice that this already implies that there exists no free operation that transforms an inequivalent state on $\mathcal{H}$ into ψ.

Theorem 2.3. *In the resource theory of entanglement where the free operations are FSP maps, there exists no maximally entangled state on* $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3}$.

To prove this result, we use that, if a maximally entangled state in this case existed, it would need to be the W state $|W\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. This is because it has been shown in [CXZ10] that the W state is the unique state in this Hilbert space that achieves the maximal possible value of $G_{\mathcal{FS}}$, which we have shown above to be an FSP measure. Thus, if there existed a maximally entangled state, it would be necessary that the W state could be transformed by FSP into any other state. However, we show that there exists no FSP map transforming the W state into the GHZ state

$$|GHZ(3,2)\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{2.10}$$

To verify this last claim, it suffices to find an FSP measure E such that $E(GHZ) > E(W)$. However, as discussed above, not many FSP measures are known and, as with the geometric measure, it is also known that the relative entropy of entanglement of the W is larger than that of the GHZ state [MMV07]. This leaves us then with the robustness measure $R_{\mathcal{FS}}$, for which we are able to show that $R_{\mathcal{FS}}(W) = R_{\mathcal{FS}}(GHZ) = 2$. This alone does not forbid that $W \to_{FSP} GHZ$ but, from the insight developed in computing these quantities, an obstruction to such transformation can be found even though they are equally robust. It is worth mentioning that, to our knowledge, this is the first time that the robustness is computed for multipartite states and we have reasons to conjecture that the W and GHZ states attain its maximal value on $\mathcal{H}$, and they are the only states that do so.

To prove this theorem, it is useful to introduce the following two lemmas in order to compute the robustness of the W and GHZ states.

Lemma 2.1. $R_{\mathcal{FS}}(GHZ) = 2\,.$

Proof. The robustness can be bounded from above from the definition (equations (2.2), (2.3)), as any fully separable state which is a convex combination of the GHZ state with a fully separable state will give an upper bound to the robustness. Ref. [Bra05] provides a dual characterisation in terms of entanglement witnesses which we use to bound the robustness from below:

$$R_{\mathcal{FS}}(\rho) = \max\left\{0, -\min_{\mathcal{W}\in\mathcal{M}} \mathrm{tr}(\mathcal{W}\rho)\right\}\,. \tag{2.11}$$

A witness for a state ρ is an operator $\mathcal{W}$ such that $\mathrm{tr}(\mathcal{W}\sigma) \geq 0$ for all $\sigma \in \mathcal{FS}$ and

$\mathrm{tr}(\mathcal{W}\rho) < 0$. If the witness also satisfies $\mathrm{tr}(\mathcal{W}\sigma) \leq 1$ for all $\sigma \in \mathcal{FS}$ (which defines the set $\mathcal{M}$ above), then $-\,\mathrm{tr}(\mathcal{W}\rho)$ is a lower bound to the robustness.

First, we show $R_{\mathcal{FS}}(GHZ) \leq 2$. We will use the following notation as a means to characterise full separability of certain states (this is a simplified version of the separability criterion in [DCT99, §2.1]): a state of the form

$$\rho(\lambda^+, \lambda^-, \lambda) = \lambda^+ GHZ + \lambda^- GHZ_- + \frac{\lambda}{6} \sum_{i=001}^{110} |i\rangle\langle i| \,, \tag{2.12}$$

where $|GHZ_-\rangle = (|000\rangle - |111\rangle)/\sqrt{2}$ and the summation index i ranges from 001 to 110 in binary, is fully separable iff

$$|\lambda^+ - \lambda^-| \leq \lambda/3 \,. \tag{2.13}$$

We must also have $\lambda^+ + \lambda^- + \lambda = 1$ for normalisation, and $\lambda^\pm, \lambda \geq 0$ for $\rho(\lambda^+, \lambda^-, \lambda)$ to be positive. Thus, the set of fully separable states of the form (2.12) is a polytope, and this property will be used later.

Consider the following state:

$$\frac{1}{3}\left(GHZ + 2\rho\left(0, \frac{1}{4}, \frac{3}{4}\right)\right) = \rho\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right) \,, \tag{2.14}$$

It is straightforward to check that both $\rho\left(0, \frac{1}{4}, \frac{3}{4}\right)$ and $\rho\left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right)$ satisfy (2.13) with equality, so $R_{\mathcal{FS}}(GHZ) \leq 2$.

Next, we show $R_{\mathcal{FS}}(GHZ) \geq 2$. Let

$$\mathcal{W} = \frac{2}{3}\mathbb{1} - \frac{8}{3}GHZ + \frac{4}{3}GHZ_- \tag{2.15}$$

be a candidate witness for this purpose. To show $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ for all fully separable states σ, it is enough to restrict to states σ of the form (2.12), as can be shown by considering the twirling map T_{GHZ} onto the GHZ-symmetric subspace. This map is defined in [HMM$^+$06], but we will only need the following properties: it is FSP and self-dual, it maps all states onto states of the form (2.12), i.e.

$$T_{GHZ}(\tau) = \rho(\lambda^+, \lambda^-, \lambda) \tag{2.16}$$

for every state τ on $\mathcal{H}$ and for some $\lambda^\pm, \lambda$ and, moreover, these states are fixed points: $T_{GHZ}(\rho(\lambda^+, \lambda^-, \lambda)) = \rho(\lambda^+, \lambda^-, \lambda)$ for all $\lambda^\pm, \lambda$. In particular, $T_{GHZ}(GHZ) = GHZ$

and the witness $\mathcal{W}$ in equation (2.15) is such that $T_{GHZ}(\mathcal{W}) = \mathcal{W}$, and so

$$\mathrm{tr}(\mathcal{W}\sigma) = \mathrm{tr}(T_{GHZ}(\mathcal{W})\,\sigma) = \mathrm{tr}(\mathcal{W}\,T_{GHZ}(\sigma)) \tag{2.17}$$

holds for any state σ. Therefore, if $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ holds for all $\sigma \in \mathcal{FS}$ such that $T_{GHZ}(\sigma) = \sigma$, i.e. those of the form (2.12) where (2.13) holds [ES12, ES13], then it is guaranteed to hold for any $\sigma \in \mathcal{FS}$.

As the space of fully separable GHZ-symmetric states is a polytope, it is enough to show that $0 \leq \mathrm{tr}(\mathcal{W}\sigma) \leq 1$ at the vertices of the polytope, which are (cf. [ES12, ES13]):

$$\begin{aligned} \sigma_1 &= \rho(0,0,1) \\ \sigma_2 &= \rho\left(0, \frac{1}{4}, \frac{3}{4}\right) \\ \sigma_3 &= \rho\left(\frac{1}{2}, \frac{1}{2}, 0\right) \\ \sigma_4 &= \rho\left(\frac{1}{4}, 0, \frac{3}{4}\right) . \end{aligned} \tag{2.18}$$

It is straightforward to check that $0 \leq \mathrm{tr}(\mathcal{W}\sigma_j) \leq 1$ for all $j = 1, ..., 4$. Since $\mathrm{tr}(\mathcal{W}\,GHZ) = -2 < 0$, $\mathcal{W}$ is a witness for the GHZ-state that meets the required condition and so $R_{\mathcal{FS}}(GHZ) \geq 2$. □

Lemma 2.2. $R_{\mathcal{FS}}(W) = 2$.

Proof. The strategy is similar to the proof of Lemma 2.1. First, we prove $R_{\mathcal{FS}}(W) \leq 2$. We will show that

$$\eta = \frac{1}{3}(W + 2\tau), \tag{2.19}$$

where

$$\eta = \frac{9}{16}\,|000\rangle\langle 000| + \frac{3}{16}\,|111\rangle\langle 111| + \frac{1}{16}W + \frac{3}{16}\overline{W} \tag{2.20}$$

and

$$\tau = \frac{3}{8}\,|000\rangle\langle 000| + \frac{1}{8}\,|111\rangle\langle 111| + \frac{3}{8}W + \frac{1}{8}\overline{W} \tag{2.21}$$

are both fully separable. Here and in what follows, $\overline{W}$ denotes the qubit-flipped version of the W-state,

$$\left|\overline{W}\right\rangle = \frac{1}{\sqrt{3}}\left(|110\rangle + |101\rangle + |011\rangle\right) . \tag{2.22}$$

As shown in Theorem 6.2 of [ESBL02], if a symmetric 3-qubit state remains positive after partial transposition (PPT), then it is FS. Since both η and τ are symmetric 3-

qubit states, it is enough to check that they are PPT, which is readily done, to conclude that they are fully separable.

Another way to see this is by writing η and τ as a convex combination of fully separable states using a result from [HMM+08]. Observe that

$$\eta = \frac{5}{9}\,|000\rangle\langle 000| + \frac{4}{9}\left(\frac{1}{2^6}\,|000\rangle\langle 000| + \frac{27}{2^6}\,|111\rangle\langle 111| + \frac{9}{2^6}W + \frac{27}{2^6}\overline{W}\right) \tag{2.23}$$

and

$$\tau = \frac{1}{9}\,|111\rangle\langle 111| + \frac{8}{9}\left(\frac{27}{2^6}\,|000\rangle\langle 000| + \frac{1}{2^6}\,|111\rangle\langle 111| + \frac{27}{2^6}W + \frac{9}{2^6}\overline{W}\right) \tag{2.24}$$

where, in each case, the first term is clearly fully separable. As we shall see, the second term is of the form

$$\begin{aligned} &\operatorname{tr}(\phi^{\otimes 3}\,|000\rangle\langle 000|)\,|000\rangle\langle 000| + \operatorname{tr}\big(\phi^{\otimes 3}\,|111\rangle\langle 111|\big)\,|111\rangle\langle 111| \\ &+ \operatorname{tr}\big(\phi^{\otimes 3}\,W\big)\,W + \operatorname{tr}\big(\phi^{\otimes 3}\,\overline{W}\big)\,\overline{W} \end{aligned} \tag{2.25}$$

for some qubit state ϕ. Ref. [HMM+08] shows that all states of this form are fully separable. Writing

$$|\phi\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle\ . \tag{2.26}$$

and inserting it into equation (2.25), the parameter β cancels in all terms and the state in equation (2.25) can be written in terms of α alone with $\alpha = \pi/3$ for η and $\alpha = \pi/6$ for τ.

Next, we prove $R_{\mathcal{FS}}(W) \geq 2$. We will show that

$$A = |000\rangle\langle 000| - 3\,W + |001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100| + 3\,\overline{W} \tag{2.27}$$

is a witness for the state $|W\rangle\langle W|$ such that

$$\operatorname{tr}(AW) = -2 \tag{2.28}$$

and

$$0 \leq \operatorname{tr}(A\sigma) \leq 1 \tag{2.29}$$

for all $\sigma \in \mathcal{FS}$.

Let $\sigma \in \mathcal{FS}$. Without loss of generality, to prove (2.29) we can assume $\sigma = |\psi\rangle\langle\psi|$ is pure. So we want to show

$$0 \leq \mathrm{tr}(A\,|\psi\rangle\!\langle\psi|) \leq 1\,. \tag{2.30}$$

Notice that A is permutationally invariant, and that we can express A in the basis of Pauli matrices as

$$A = \sum_{ijk\in x,y,z} \lambda_{ijk}\sigma_i \otimes \sigma_j \otimes \sigma_k + \frac{\mathbb{1}_8}{2} \tag{2.31}$$

for some $\lambda_{ijk} \in \mathbb{R}$ and where $\mathbb{1}_d$ is the d-dimensional identity, so that

$$A' = A - \frac{\mathbb{1}_8}{2} \tag{2.32}$$

has no identity component in the basis of Pauli matrices. That is, A' contains only full correlation terms, and it is still permutationally invariant so it satisfies the conditions of Corollary 5 (ii) in [HKW+09]. In particular, A' can be viewed as a symmetric three-linear form acting on $\mathbb{R}^3$. This means that

$$\max_{|\psi\rangle\in\mathcal{FS}} \left|\mathrm{tr}\big(A'\,|\psi\rangle\!\langle\psi|\big)\right| \tag{2.33}$$

can be attained by a symmetric state $|\psi\rangle = |a\rangle\,|a\rangle\,|a\rangle \equiv |aaa\rangle$. The qubit $|a\rangle$ can be expressed in terms of two real parameters as

$$|a\rangle = \cos\alpha\,|0\rangle + \mathrm{e}^{\mathrm{i}\beta}\sin\alpha\,|1\rangle \tag{2.34}$$

and so

$$\left|\mathrm{tr}\big(A'\,|aaa\rangle\!\langle aaa|\big)\right| = \frac{1}{2}\,|\cos 6\alpha| \leq \frac{1}{2}\,. \tag{2.35}$$

But this completes the proof, since, by linearity, to show

$$-\frac{1}{2} \leq \mathrm{tr}\big(A'\,|\psi\rangle\!\langle\psi|\big) \leq \frac{1}{2} \tag{2.36}$$

(which is equivalent to (2.30)) it suffices to show

$$\max_{|\psi\rangle\in\mathcal{FS}} \left|\mathrm{tr}\big(A'\,|\psi\rangle\!\langle\psi|\big)\right| \leq \frac{1}{2}\,. \tag{2.37}$$

This can be seen by viewing $\mathrm{tr}(A'\,|\psi\rangle\!\langle\psi|)$ as a symmetric three-linear form in $\mathbb{R}^3$. If the maximum absolute value is attained by some state $|a^*\rangle$, then the state $|\tilde{a}^*\rangle$ which flips the sign of the vector which the three-linear form acts on will give a minimum of the expression equal to minus the maximum. Hence,

$$\begin{aligned}\max_{|\psi\rangle\in\mathcal{FS}}\left|\operatorname{tr}\left(A'\,|\psi\rangle\!\langle\psi|\right)\right| &= \max_{|\psi\rangle\in\mathcal{FS}}\operatorname{tr}\left(A'\,|\psi\rangle\!\langle\psi|\right)\\ &= -\min_{|\psi\rangle\in\mathcal{FS}}\operatorname{tr}\left(A'\,|\psi\rangle\!\langle\psi|\right).\end{aligned}\tag{2.38}$$

Therefore (2.30) holds true and hence the witness A gives the stated lower bound for the FS robustness of W. □

We note that the values obtained for the robustness $R_{\mathcal{FS}}$ of the W and GHZ states show that, unlike in the bipartite case, the robustness can be strictly larger than the generalised robustness. The generalised robustness, $R_G(\cdot)$, is defined as

$$R_G(\cdot) = \min_{\tau\in\mathcal{H}} R(\cdot||\tau) \tag{2.39}$$

where, this time, τ may be separable or entangled. Hence $R_G(\cdot) \leq R(\cdot)$ but, in addition, it was shown in [HN03] that $R_G(\cdot) = R(\cdot)$ for bipartite pure states. However, the generalised robustness of the W state has been computed in [HMM$^+$08] to be 5/4, and that of the GHZ state was shown to be 1 in [HMM$^+$06], so they are both strictly less than the robustness of these states. To the best of our knowledge, this is the first time that states such that $R_G(\cdot) < R(\cdot)$ have been found.

We are now ready to prove Theorem 2.3.

Proof. As we outlined in the main text, the only candidate for a maximally entangled state of three qubits is the W state, as it is the unique state on $\mathcal{H} = \left(\mathbb{C}^2\right)^{\otimes 3}$ that achieves the maximum value of the FSP-measure $G_{\mathcal{FS}}$ (among both pure and mixed states, since the convex-roof extension of $G_{\mathcal{FS}}$ to mixed states ensures that the maximum value will always be achieved by a pure state). So, if there existed a maximally entangled state, it would need to be possible that the W state be transformed into any other state via an FSP map. We will assume that there exists an FSP map Λ such that $\Lambda(W) = GHZ$, and will arrive at a contradiction by showing that there exists a state $\eta \in \mathcal{FS}$ such that $\Lambda(\eta) \notin \mathcal{FS}$.

Let Λ be an FSP map such that $\Lambda(W) = GHZ$ and let

$$\eta = \frac{1}{3}W + \frac{2}{3}\tau \in \mathcal{FS}\,, \tag{2.40}$$

where $\tau, \eta \in \mathcal{FS}$, be the convex combination that gives the upper bound to $R_{\mathcal{FS}}(W)$ in equations (2.19)-(2.21). Let T_{GHZ} be the twirling map onto the GHZ-symmetric subspace (defined in [HMM+08]; see also the proof of Lemma 2.1). Then,

$$\begin{aligned}\eta' =& T_{GHZ}\left(\Lambda\left(\frac{1}{3}W + \frac{2}{3}\tau\right)\right)\\ =& \frac{1}{3}GHZ + \frac{2}{3}T_{GHZ}(\Lambda(\tau))\,.\end{aligned} \tag{2.41}$$

Since both T_{GHZ} and Λ are full separability-preserving, it is the case that $\eta', \Lambda(\tau)$, $T_{GHZ}(\Lambda(\tau)) \in \mathcal{FS}$. Now, recall that τ has a non-zero W component:

$$\tau = p\,|W\rangle\langle W| + (1-p)\xi$$

for some $p \in (0,1)$ and some state ξ, so that

$$\eta' = \frac{1}{3}GHZ + \frac{2}{3}\left[p\,GHZ + (1-p)T_{GHZ}(\Lambda(\xi))\right]\,. \tag{2.42}$$

But, as we shall now show, the FS GHZ-symmetric state υ such that

$$\frac{1}{3}GHZ + \frac{2}{3}\upsilon \in \mathcal{FS} \tag{2.43}$$

is unique, i.e. if equation (2.43) holds then necessarily $\upsilon = \rho(0, 1/4, 3/4)$ as in equation (2.14). However, the state appearing in equation (2.42) is not υ (since $\mathrm{tr}(\upsilon\, GHZ) = 0$) hence, contrary to our assumption, η' cannot be FS.

Recall, from the proof of Lemma 2.1 (equation (2.12)), that all GHZ-symmetric states are of the form

$$\rho(\lambda^+, \lambda^-, \lambda) = \lambda^+ GHZ\, + \lambda^- GHZ_- + \frac{\lambda}{6}\sum_{i=001}^{110} |i\rangle\langle i| \tag{2.44}$$

so that equation (2.43) can be expressed in terms of the λ parameters as

$$\frac{1}{3}GHZ + \frac{2}{3}\rho\left(\lambda^+, \lambda^-, \lambda\right) = \rho\left(\frac{1}{3} + \frac{2}{3}\lambda^+, \frac{2}{3}\lambda^-, \frac{2}{3}\lambda\right)\,. \tag{2.45}$$

States of the form (2.44) are fully separable iff

$$|\lambda^+ - \lambda^-| \leq \lambda/3\,. \tag{2.46}$$

Since this condition must hold for both states $\rho(\cdot,\cdot,\cdot)$ in equation (2.45), we must also have

$$\left|\frac{1}{3}+\frac{2}{3}\lambda^{+}-\frac{2}{3}\lambda^{-}\right|\leq\frac{2}{9}\lambda \tag{2.47}$$

and, for normalisation, we need

$$\lambda^{+}+\lambda^{-}+\lambda=1\,. \tag{2.48}$$

It is straightforward to check that these three conditions hold only if

$$\lambda^{+}=0;\ \lambda^{-}=1/4;\ \lambda=3/4\,, \tag{2.49}$$

which corresponds to the state υ as claimed above.

Therefore η in equation (2.40) is fully separable, yet $\eta'=T_{GHZ}(\Lambda(\eta))$ is not fully separable. So Λ is not FSP and hence the theorem is proven. $\square$

Theorem 2.3 forbids the existence of a multipartite maximally entangled state under FSP in the simplest case of $\mathcal{H}=(\mathbb{C}^2)^{\otimes 3}$. However, it is instructive to compare with the LOCC scenario since these values of n and d make up the only case where no state is isolated in the latter formalism (aside from the bipartite case). Whenever no single maximally entangled state exists one needs to consider a maximally entangled set (MES) [dSK13], defined as the minimal set of states on $\mathcal{H}$ such that any state on $\mathcal{H}$ can be obtained by means of the free operations from a state in this set. The MES under LOCC for $n=3$ and $d=2$ has been characterised in [dSK13], and it is found to be relatively small in the sense that it has measure zero on $\mathcal{H}$ (in contrast, for other values of n and d the fact that isolation is generic imposes that the MES has full measure on $\mathcal{H}$). However, interestingly, the MES under FSP is smaller even in this case, given that it is strictly included in the MES under LOCC. This is because, as we will now show, the W and GHZ states can be transformed by FSP operations into inequivalent states that are in the MES under LOCC. It is worth mentioning that the target states may be chosen to lie in different SLOCC classes with respect to the initial states, and so this gives an explicit example of deterministic FSP conversions among states in different SLOCC classes.

Let ψ^{+}_{GHZ} denote states of the form

$$\left|\psi^{+}_{GHZ}\right\rangle=\sqrt{K}\left(|000\rangle+|\phi_A\phi_B\phi_C\rangle\right) \tag{2.50}$$

where

$$\begin{aligned} |\phi_A\rangle &= \cos\alpha\,|0\rangle + \sin\alpha\,|1\rangle\ , \\ |\phi_B\rangle &= \cos\beta\,|0\rangle + \sin\beta\,|1\rangle\ , \\ |\phi_C\rangle &= \cos\gamma\,|0\rangle + \sin\gamma\,|1\rangle\ , \end{aligned} \tag{2.51}$$

$\alpha, \beta, \gamma \in (0, \pi/2]$ and $K = (2(1 + \cos\alpha\cos\beta\cos\gamma))^{-1}$ is a normalisation factor. States of the form ψ^+_{GHZ} are in the MES under LOCC, since they cannot be reached by any LOCC map regardless of the input state on $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3}$ [DVC00, TGP10, dSK13]. So the following proposition does not hold in the LOCC regime.

Proposition 2.1. *There exists an FSP map Λ such that $\Lambda(W) = \psi^+_{GHZ}$ for some state of the form ψ^+_{GHZ}.*

Proof. Let

$$\Lambda(\eta) = \mathrm{tr}(W\eta)\psi^+_{GHZ} + \mathrm{tr}\,[(\mathbb{1} - W)\eta]\,\tau_{\mathcal{FS}}, \tag{2.52}$$

where $\tau_{\mathcal{FS}} \in \mathcal{FS}$ is the state that gives the robustness of the state ψ^+_{GHZ}. Clearly, $\Lambda(W) = \psi^+_{GHZ}$ and it remains to be shown that Λ is FSP. As argued in Theorems 2.1 and 2.2, this happens when

$$R_{\mathcal{FS}}(\psi^+_{GHZ}) \leq \frac{G_{\mathcal{FS}}(W)}{1 - G_{\mathcal{FS}}(W)} = \frac{5}{4}\,. \tag{2.53}$$

But, by continuity of the robustness, such a state ψ^+_{GHZ} can always be found by picking the parameters α, β, γ sufficiently close to zero since in this case the states ψ^+_{GHZ} approach the set of FS states.

Anyway, for the sake of completeness, we provide an explicit quantitative upper bound in what follows. Consider the invertible local operations

$$\begin{aligned} A &= \begin{pmatrix} 1 & \cos\alpha \\ 0 & \sin\alpha \end{pmatrix}, \\ B &= \begin{pmatrix} 1 & \cos\beta \\ 0 & \sin\beta \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & \cos\gamma \\ 0 & \sin\gamma \end{pmatrix}. \end{aligned} \tag{2.54}$$

Applying these to the FS states in equation (2.14) used to bound the robustness of the GHZ state,

$$A \otimes B \otimes C \left(\frac{1}{3} GHZ + \frac{2}{3} \upsilon \right) A^\dagger \otimes B^\dagger \otimes C^\dagger \,, \tag{2.55}$$

gives a state proportional to

$$\frac{1}{3}\left(1 + \cos\alpha \cos\beta \cos\gamma\right) \psi^+_{GHZ} + \frac{2}{3} \frac{4 - \cos\alpha \cos\beta \cos\gamma}{4} \upsilon' \,, \tag{2.56}$$

where $\upsilon' = A \otimes B \otimes C \upsilon A^\dagger \otimes B^\dagger \otimes C^\dagger$ is still fully separable since local operations cannot create entanglement. For the same reason, the state in equation (2.56) is fully separable, and hence the robustness of the state ψ^+_{GHZ} cannot exceed[2]

$$R_{\mathcal{FS}}(\psi^+_{GHZ}) \leq \frac{4 - \cos\alpha \cos\beta \cos\gamma}{2(1 + \cos\alpha \cos\beta \cos\gamma)} \,. \tag{2.57}$$

Clearly, there exist $\alpha, \beta, \gamma \in (0, \pi/2]$ such that this bound is lower than or equal to $5/4$, as required. For an example, take $\alpha = \beta = \pi/2$ and γ such that $\cos\gamma \geq 6/7$. □

We will now show the converse result: there are FSP maps which take the GHZ-state to states in the W-class which are in the MES under LOCC. Such states are of the form

$$|\psi_W\rangle = \sqrt{x_1}\,|001\rangle + \sqrt{x_2}\,|010\rangle + \sqrt{x_3}\,|100\rangle \tag{2.58}$$

where $x_1 + x_2 + x_3 = 1$. They are in the MES under LOCC, as no LOCC map can reach these states for any input state on $\mathcal{H} = (\mathbb{C}^2)^{\otimes 3}$ [DVC00, KT10, dSK13], but (as we will now prove) not under FSP.

Proposition 2.2. *There exists an FSP map Λ such that $\Lambda(GHZ) = \psi_W$ for some state of the form ψ_W.*

Proof. Since $G_{\mathcal{FS}}(GHZ) = 1/2$, it suffices to find a state ψ_W such that $R_{\mathcal{FS}}(\psi_W) \leq 1$, which can be done since the robustness is continuous and there are states ψ_W arbitrarily close to the set of FS states. Then,

$$\Lambda(\eta) = \mathrm{tr}(GHZ\eta)\psi_W + \mathrm{tr}\left[(\mathbb{1} - GHZ)\eta\right] \tau_{\mathcal{FS}}, \tag{2.59}$$

where $\tau_{\mathcal{FS}} \in \mathcal{FS}$ is the state such that that $R_{\mathcal{FS}}(\psi_W) = R(\psi_W || \tau_{\mathcal{FS}})$, is the required map. □

[2] This shows, in particular, that the robustness of all ψ^+_{GHZ} states is always less than or equal to 2.

2.3.2 BSP regime

Finally, we study the resource theory under BSP operations where, remarkably, we find a unique maximally GME state for any value of n and d, given by the generalised GHZ state

$$|GHZ(n,d)\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle^{\otimes n}. \tag{2.60}$$

Theorem 2.4. *In the resource theory of entanglement where the free operations are BSP maps, there exists a maximally GME state on every $\mathcal{H}$. Namely, $\forall|\psi\rangle\in(\mathbb{C}^d)^{\otimes n}$, there exists a CPTP BSP map Λ such that $\Lambda(GHZ(n,d))=\psi$.*

Proof. The main idea behind the proof is to use the construction of the proof of Theorems 2.1 and 2.2 again, which shows that there is a CPTP BSP map Λ that converts $GHZ(n,d)$ into ψ if the robustness of ψ is bounded above by an expression involving the geometric measure of $GHZ(n,d)$. However, unlike for the FS case, $G_{\mathcal{BS}}$ is straightforward to compute. Finally, a simple estimate shows that $R_{\mathcal{BS}}(\psi)\leq d-1$ $\forall|\psi\rangle\in(\mathbb{C}^d)^{\otimes n}$, which leads to the desired result.

For every given $|\psi\rangle\in\left(\mathbb{C}^d\right)^{\otimes n}$, let

$$\Lambda(\eta) = \mathrm{tr}(\eta\, GHZ(n,d))\,\psi\, + \mathrm{tr}\left[(\mathbb{1}-GHZ(n,d))\eta\right]\rho_{\mathcal{BS}} \tag{2.61}$$

where $\rho_{\mathcal{BS}}\in\mathcal{BS}$ is the state which gives the (biseparable) robustness of ψ (i.e. $R_{\mathcal{BS}}(\psi)=R(\psi||\rho_{\mathcal{BS}})$). Then, $\Lambda(GHZ(n,d))=\psi$ and it remains to be shown that Λ is BSP. As argued in the proofs of Theorems 2.1 and 2.2, this happens iff

$$R_{\mathcal{BS}}(\psi)\leq\frac{G_{\mathcal{BS}}(GHZ(n,d))}{1-G_{\mathcal{BS}}(GHZ(n,d))}\,. \tag{2.62}$$

However, unlike for the FS case, $G_{\mathcal{BS}}$ is straightforward to compute [BPSS14] in terms of the Schmidt decomposition across every possible bipartite splitting of the parties $M|\bar{M}$ (i.e., $|\psi\rangle=\sum_i\sqrt{\lambda_i^{M|\bar{M}}}|i\rangle_M|i\rangle_{\bar{M}}$ for each M) as

$$G_{\mathcal{BS}}(\psi)=1-\max_{M\subsetneq[n]}\lambda_1^{M|\bar{M}}, \tag{2.63}$$

where $\lambda_1^{M|\bar{M}}$ is the largest Schmidt coefficient of ψ in the corresponding splitting. This immediately shows that the generalised GHZ state has maximal value of the geometric measure, $G_{\mathcal{BS}}(GHZ(n,d))=(d-1)/d$. Therefore, Λ is BSP iff $R_{\mathcal{BS}}(\psi)\leq d-1$. It is shown in [VT99] that for every bipartite pure state $\psi_{A|B}$ with Schmidt decomposition

$$\psi_{A|B} = \sum_i \sqrt{\lambda_i^{A|B}}\, |i\rangle_A\, |i\rangle_B \tag{2.64}$$

it holds that

$$R_{\mathcal{BS}}(\psi) = \left(\sum_i \sqrt{\lambda_i^{A|B}} \right)^2 - 1\,. \tag{2.65}$$

Thus

$$\begin{aligned} R_{\mathcal{BS}}(\psi) &\leq \min_{M \subsetneq [n]} \left(\sum_i \sqrt{\lambda_i^{M|\bar{M}}} \right)^2 - 1 \\ &\leq d - 1, \end{aligned} \tag{2.66}$$

where the latter inequality follows from considering the state with all eigenvalues $\lambda_i = 1/d$. Hence, $\forall\, |\psi\rangle \in \left(\mathbb{C}^d\right)^{\otimes n}$ there exists a BSP map Λ such that $\Lambda(GHZ) = \psi$. □

It follows from the proof that it suffices to have maximal $G_{\mathcal{BS}}$ to be convertible to any other state by BSP operations. Thus, any state fulfilling that $G_{\mathcal{BS}} = (d-1)/d$ must automatically maximise any other BSP measure. More importantly, this also shows that any two states achieving this value of the geometric measure are deterministically interconvertible by BSP operations and, therefore, belong to the same GME-equivalence class despite potentially not being related by local unitary transformations. An example of such class when $d = 2$ are GME graph states for which it is known that $G_{\mathcal{BS}} = 1/2$ [TG05]. Hence, all graph states including the generalised GHZ state are in the equivalence class of the maximally GME state in this theory. It is remarkable to find that this very relevant family of states [HDE+06] in quantum computation and error correction has this feature in a resource theory of GME and we believe this is worth further research. Another previously considered family of states that belongs to this equivalence class is that of absolutely maximally entangled (AME) states [HCL+12], which is defined as those states for which all reduced density matrices are proportional to the identity in the maximum possible dimensions. It follows from equation (2.63) that $G_{\mathcal{BS}} = (d-1)/d$ holds for all AME states (for those values of n and d for which they exist). Equation (2.63) also tells us that a necessary condition for a state to be in the equivalence class of the maximally GME state is that all single-particle reduced density matrices must be proportional to the d-dimensional identity. However, this condition is not sufficient: the state in $(\mathbb{C}^2)^{\otimes 4}$ $|\phi\rangle = \sqrt{p}|\phi^+\rangle_{12}|\phi^+\rangle_{34} + \sqrt{1-p}|\phi^-\rangle_{12}|\phi^-\rangle_{34}$ ($|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$) is a GME state (if $p \neq 0, 1$) with this property but $G_{\mathcal{BS}}(\phi) < 1/2$ (if $p \neq 1/2$).

2.4 Comparison between the regimes

The fact that the set of FS states is (strictly) contained in the set of BS states might lead us to believe that the set of FSP operations is contained in the set of BSP operations. Evidently, if such an inclusion were to hold, it would be strict. Indeed, a BSP operation such as the one that transforms the GHZ state into, say, the W state, is manifestly not FSP. Otherwise, the results in Section 2.3.1 would not hold.

We show now that the reverse inclusion does not hold either: there exist FSP operations that are not BSP.

Proposition 2.3. *The set of FSP operations is not included in the set of BSP operations.*

Proof. We provide an example of an FSP map that is not BSP. Consider the GHZ-symmetric states, which were introduced in the proof of Lemma 2.1. Denoting them by ρ, they can be alternatively parametrised [ES12] in terms of two parameters x, y, where

$$\begin{aligned} x(\rho) =& \frac{1}{2}\left[\langle GHZ_+|\,\rho\,|GHZ_+\rangle - \langle GHZ_-|\,\rho\,|GHZ_-\rangle\right], \\ y(\rho) =& \frac{1}{\sqrt{3}}\left[\langle GHZ_+|\,\rho\,|GHZ_+\rangle + \langle GHZ_-|\,\rho\,|GHZ_-\rangle - \frac{1}{4}\right]. \end{aligned} \tag{2.67}$$

We denote a GHZ-symmetric state ρ with parameters x, y as $\rho(x, y)$. This set forms a triangle on the x, y plane, and the sets $\mathcal{FS}, \mathcal{BS}$ are well characterised in terms of lines on the plane, which correspond to witnesses. In particular, the witness which detects FS states is

$$\mathcal{W}_{FS}(x, y) = -x - \frac{\sqrt{3}}{6}y + \frac{1}{8} \geq 0, \tag{2.68}$$

while the one which detects BS states is

$$\mathcal{W}_{BS}(x, y) = -x - \frac{\sqrt{3}}{2}y + \frac{3}{8} \geq 0. \tag{2.69}$$

The candidate for our proof is the map Λ defined by

$$\Lambda(\eta) = \operatorname{tr}(W\eta)\rho\left(\frac{5}{16}, \frac{\sqrt{3}}{4}\right) + \operatorname{tr}((\mathbb{1} - W)\eta)\rho\left(-\frac{1}{8}, 0\right). \tag{2.70}$$

Its possible outputs are

$$\sigma(p) = p\rho\left(\frac{5}{16}, \frac{\sqrt{3}}{4}\right) + (1 - p)\rho\left(-\frac{1}{8}, 0\right), \tag{2.71}$$

where p is the trace of the input state X with W. Now, $\mathrm{tr}(WX)$ achieves a maximum of 4/9 for FS states X, as implied by the geometric measure of W [CXZ10]. The state that gives such a maximum is mapped under Λ to

$$\sigma\left(\frac{4}{9}\right) = \rho\left(\frac{5}{72}, \frac{\sqrt{3}}{9}\right), \tag{2.72}$$

which is FS since

$$\mathcal{W}_{FS}\left(\frac{5}{72}, \frac{\sqrt{3}}{9}\right) = 0. \tag{2.73}$$

Therefore, Λ is FSP.

However, if X is BS, then $\mathrm{tr}(WX)$ can be as high as 2/3 (take, for example, $|\psi\rangle = (|001\rangle + |010\rangle)/\sqrt{2}$). For this input state, we have

$$\Lambda(\psi) = \sigma\left(\frac{2}{3}\right) = \rho\left(\frac{1}{6}, \frac{\sqrt{3}}{6}\right). \tag{2.74}$$

But its overlap with the GME witness in equation (2.69) is negative:

$$\mathcal{W}_{BS}\left(\frac{1}{6}, \frac{\sqrt{3}}{6}\right) = -\frac{1}{24} < 0, \tag{2.75}$$

meaning that Λ is not BSP. □

2.5 Looking beyond

While the resource theory of GME leads to a unique maximally entangled state, the set of free states (i.e., biseparable states) is not closed under tensor products. Indeed, as mentioned in Chapter 1, there are many examples of biseparable states which, when tensored, lead to a GME state. This means that, if taking copies is allowed, biseparable states can be a resource. In particular, bipartite entangled states distributed in a network are potential resources to obtain GME. This observation will be key in the following chapters, where we will study which pair-entangled network states lead to GME and GMNL (we will often identify networks with graphs, which should not be confused with the graph states mentioned above). In fact, some such networks are maximally GME in the resource theory where the free operations are BSP operations:

Observation. *Consider a regular graph such that the number of edge-disjoint paths* *between every pair of vertices is equal to the degree. Any such graph where each pair of parties shares a maximally entangled state in dimension d is maximally GME in the sense of Theorem 2.4.*

Proof. Let ρ denote the state in the statement, whose local dimension is d^k where k is the degree of the graph. It is sufficient to show that $G_{\mathcal{BS}}(\rho) = (d^k - 1)/d^k$, i.e., that the largest Schmidt coefficient in each bipartition is $1/d^k$. To show this, it is enough to check that one can obtain a d^k-dimensional maximally entangled state between any two parties by LOCC. But such a maximally entangled state is equivalent to k copies of a d-dimensional maximally entangled state. Then, the LOCC protocol is as follows: let A, B be any two parties. The idea is to use each of the other parties that are intermediate in the paths connecting A and B, who already share a maximally entangled state with A, as a bridge. Each of these other parties can teleport the particle they hold to B, using the maximally entangled state they share with B as a channel. This achieves the goal. □

Less ambitiously, if we only require GME, having a connected network of entangled pure states is sufficient.

Observation. *Any connected network of entangled pure states is GME.*

Proof. The observation holds because the partial trace over any strict subset of parties yields a mixed state: let

$$\rho = \bigotimes_{k=1}^{K} \psi_k \tag{2.76}$$

be the state of the network, where $k = 1, ..., K$ labels the edges, and the parties are left implicit. Then, taking the partial trace over one party, say, A_i, amounts to taking the partial trace of each ψ_k that is incident to A_i, while the remaining ψ_k are left untouched:

$$\mathrm{tr}_{A_i}(\rho) = \mathrm{tr}_{A_i} \bigotimes_{k=1}^{K} \psi_k = \bigotimes_{k \in I} \mathrm{tr}_{A_i^k} \psi_k \otimes \bigotimes_{k \notin I} \psi_k, \tag{2.77}$$

where I is the set of edges k incident to party A_i, and A_i^k is the particle held by party A_i corresponding to edge k. For all i, $k \in I$, the state $\mathrm{tr}_{A_i^k} \psi_k$ is mixed, since ψ_k is entangled. Therefore, the state $\mathrm{tr}_{A_i}(\rho)$ is mixed. This does not change if the partial trace is taken over more parties. Therefore, ρ is not biseparable in any bipartition, which is a sufficient condition for GME in the case of pure states. □

Networks of states like the ones just considered will be studied in Chapters 3 and 4. As we shall see, all connected networks of pure states are not only genuine multipartite entangled, but also genuine multipartite nonlocal, while allowing for noise on the shared states can compromise even their entanglement properties. However, like in the result above, entanglement is stronger if the network is completely connected. Indeed, we find that these networks are always GME if the noise in the shared states is below a threshold.

Chapter 3

Pure pair-entangled network states

Quantum entanglement and nonlocality are inextricably linked. However, while entanglement is necessary for nonlocality, it is not always sufficient in the standard Bell scenario. We derive sufficient conditions for entanglement to give rise to genuine multipartite nonlocality in networks. We find that any network where the parties are connected by bipartite pure entangled states is genuine multipartite nonlocal, independently of the amount of entanglement in the shared states and of the topology of the network. As an application of this result, we also show that all pure genuine multipartite entangled states are genuine multipartite nonlocal in the sense that measurements can be found on finitely many copies of any genuine multipartite entangled state to yield a genuine multipartite nonlocal behaviour. Our results pave the way towards feasible manners of generating genuine multipartite nonlocality using any connected network.

3.1 Definitions and preliminaries

We consider distributions arising from GME states, and ask whether they are bilocal (see Definitions 1.3 and 1.9). The set of bilocal distributions is a polytope: indeed, the set of local distributions across each bipartition $M|\overline{M}$ is a polytope, and convex combinations preserve that structure. We call this n-partite polytope $\mathcal{B}_n$.

In particular, we consider distributions arising from networks where each party measures individually on each particle they hold. Therefore, we reserve the usual notation for inputs and outputs, $x, y; a, b$ respectively, for those corresponding to *particles*, while we denote inputs and outputs for each *party* as $\chi, \upsilon; \alpha, \beta$ respectively.

We use results from Ref. [Pir05] to lift inequalities to account for more parties, inputs and outputs. They consider the fully local polytope $\mathcal{L}$, which only includes distributions

$$P(\alpha\beta|\chi\upsilon) = \sum_{\lambda} p(\lambda) P_A(\alpha|\chi, \lambda) P_B(\beta|\upsilon, \lambda) \tag{3.1}$$

where each party may have different numbers of inputs and outputs (more parties may be considered by adding more distributions correlated only by λ). Polytope $\mathcal{B}_n$ includes convex combinations of distributions that are local across different bipartitions $M|\overline{M}$ of the parties, but the lifting results in [Pir05] still hold. Indeed, to check an inequality holds for a polytope, it is sufficient by convexity to check the extremal points. As all extremal points in $\mathcal{B}_n$ are contained in some polytope $\mathcal{L}$ (splitting the parties in two as per the bipartition $M|\overline{M}$), lifting results for $\mathcal{L}$ can be straightforwardly extended to $\mathcal{B}_n$.

We also use the EPR2 decomposition [EPR92]: any bipartite distribution P can be expressed (nonuniquely) as

$$P(\alpha\beta|\chi\upsilon) = p_L P_L(\alpha\beta|\chi\upsilon) + (1 - p_L) P_{NS}(\alpha\beta|\chi\upsilon) \tag{3.2}$$

for some $0 \leq p_L \leq 1$, where P_L is local (i.e. satisfies equation (3.1)) and P_{NS} is nonsignalling (since so is P). P is nonlocal if all such decompositions have $p_L < 1$, and fully nonlocal[1] if all such decompositions have $p_L = 0$. A quantum state ρ is fully nonlocal if, for all $\varepsilon > 0$, there exist local measurements giving rise to a distribution P such that any decomposition (3.2) has $p_L < \varepsilon$.

The EPR2 decomposition can be extended to the multipartite case [ACSA10] as

$$\begin{aligned} P(\alpha_1 ... \alpha_n|\chi_1 ... \chi_n) = \sum_{M \subsetneq [n]} & p_L^M P_L^M(\alpha_1 ... \alpha_n|\chi_1 ... \chi_n) \\ & + p_{NS} P_{NS}(\alpha_1 ... \alpha_n|\chi_1 ... \chi_n) \end{aligned} \tag{3.3}$$

where $p_L^M \geq 0$ for every M, $p_{NS} \geq 0$ and

$$\sum_M p_L^M + p_{NS} = 1, \tag{3.4}$$

P_L^M is local across the bipartition $M|\overline{M}$ (i.e. satisfies equation (3.1)), and P_{NS} is nonsignalling. We are interested in decompositions which maximise the local EPR2

[1] Not to be confused with "nonfully local", which is the opposite of "fully local". "fully nonlocal" is a particular case of "nonfully local".

components, in order to deduce properties about the distributions. For a distribution P, we define

$$EPR2(P) = \max \left\{ \sum_M p_L^M : P = \sum_M p_L^M P_L^M + p_{NS} P_{NS}, \sum_M p_L^M + p_{NS} = 1 \right\} \tag{3.5}$$

and, for a state ρ, we define (with a slight abuse of notation)

$$EPR2(\rho) = \inf \left\{ EPR2(P) : P = \mathrm{tr} \left(\bigotimes_{i=1}^n E^i_{\alpha_i|\chi_i} \rho \right) \right\}, \tag{3.6}$$

where the infimum is taken over local measurements $E^i_{\alpha_i|\chi_i}$ on each particle such that

$$E^i_{\alpha_i|\chi_i} \succcurlyeq 0 \,\forall \alpha_i, \chi_i, \; \sum_{\alpha_i} E^i_{\alpha_i|\chi_i} = \mathbb{1} \,\forall \chi_i, \; \forall i \in [n], \tag{3.7}$$

with any number of inputs and outputs. Then, a distribution P or a state ρ are GMNL if $EPR2(\cdot) < 1$, while they are fully GMNL if $EPR2(\cdot) = 0$. Notice that the optimisation for probability distributions yields a maximum since the number of inputs and outputs is fixed. Instead, the optimisation for a state may involve measurements with an arbitrarily large number of inputs or outputs, as is the case for the maximally entangled state [BKP06]. In this work, the number of inputs and outputs is always finite, and this will become relevant when bounding the EPR2 components of distributions arising from maximally entangled states in Theorems 3.1 and 3.2.

3.2 GMNL from bipartite entanglement

Our first result shows that any connected network of pure bipartite entanglement (see Figure 3.1) is GMNL.

Theorem 3.1. *Any connected network of bipartite pure entangled states is GMNL.*

We first establish the main ideas of the proof by outlining it for a tripartite network, before turning to the general case. Since it is sufficient to consider tree graphs, i.e., graphs without cycles, we consider a Lambda network where A_1 is entangled to each of A_2 and A_3.

Proof for the tripartite case. Since it turns out to be sufficient to measure individually on each party's different particles (see Figure 3.1 for the n-partite structure), the shared

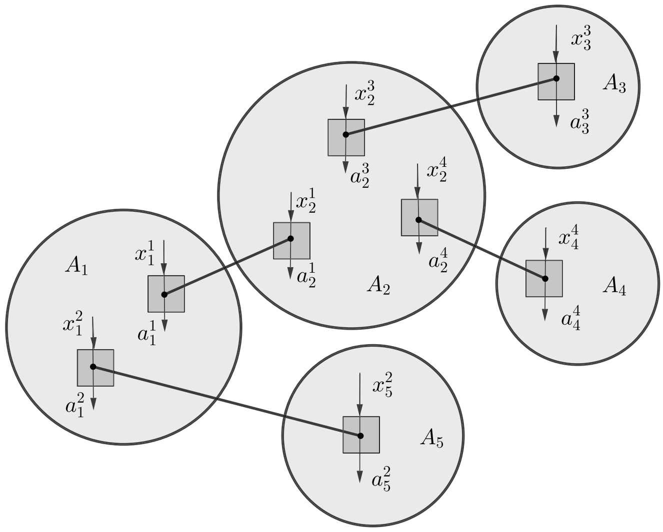

Figure 3.1: Connected network of bipartite entanglement. For each $i \in [n]$, party A_i has input x_i^k and output a_i^k on the particle at edge k. Particles connected by an edge are entangled.

distribution $P(a_1^1 a_1^2, a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2)$ takes the form

$$P_1(a_1^1 a_2^1 | x_1^1 x_2^1) P_2(a_1^2 a_3^2 | x_1^2 x_3^2), \tag{3.8}$$

where parties A_i, A_j are connected by edge k (we label vertices and edges independently), and $P_k(a_i^k a_j^k | x_i^k x_j^k)$ is the distribution arising from the state at edge k.

We consider three cases, depending on whether none, one or both of the shared states are maximally entangled. If none are, we devise inequalities to detect bipartite nonlocality at each edge of the network, and combine them to form a multipartite inequality. Then, we find measurements on the shared states to violate it. If both states are maximally entangled, existing results show the network is fully GMNL [CASA11, ACSA10]. Combining these two cases for a heterogeneous network completes the proof.

To prove the first case, we take bipartite inequalities between A_1 and each other party, lift them to three parties and combine them using Refs. [Pir05, CAA19], to obtain the following GMNL inequality:

$$\begin{aligned} I_3 = &I^1 + I^2 + P(00,0,0|00,0,0) \\ &- \sum_{a_1^2=0,1} P(0a_1^2,0,0|00,0,0) - \sum_{a_1^1=0,1} P(a_1^1 0,0,0|00,0,0) \le 0. \end{aligned} \tag{3.9}$$

Here,

$$\begin{aligned} I^1 = \sum_{a_1^2=0,1} \big[&P(0a_1^2,0,0|00,0,0) - P(0a_1^2,1,0|00,1,0) \\ &- P(1a_1^2,0,0|10,0,0) - P(0a_1^2,0,0|10,1,0)\big] \leq 0; \end{aligned} \tag{3.10}$$

$$\begin{aligned} I^2 = \sum_{a_1^1=0,1} \big[&P(a_1^1 0,0,0|00,0,0) - P(a_1^1 0,0,1|00,0,1) \\ &- P(a_1^1 1,0,0|01,0,0) - P(a_1^1 0,0,0|01,0,1)\big] \leq 0 \end{aligned} \tag{3.11}$$

are liftings of

$$I = P(00|00) - P(01|01) - P(10|10) - P(00|11) \leq 0 \tag{3.12}$$

to three parties with A_1 having 4 inputs and 4 outputs. Inequality (3.12) is equivalent to the CHSH inequality [CHSH69] for nonsignalling distributions [CAA19]. Thus, inequalities (3.10), (3.11) are satisfied by distributions that are local across $A_1|A_2$ and $A_1|A_3$ respectively. To see that equation (3.9) is a GMNL inequality it is sufficient to check it holds for distributions that are local across some bipartition. This is straightforwardly done by observing the cancellations that occur when I^1 or I^2 are ≤ 0.

Since both states are less-than-maximally entangled, A_1 can satisfy Hardy's paradox [Har92, Har93] with each other party, achieving

$$P_k(00|00) > 0 = P_k(01|01) = P_k(10|10) = P_k(00|11) \tag{3.13}$$

for both k (the proof for qubits in Refs. [Har92,Har93] is extended to qudits by measuring on a two-dimensional subspace, see Proposition 3.1). Then, each negative term in I^1 and I^2 is zero, as

$$\sum_{a_1^2=0,1} P(0a_1^2,1,0|00,1,0) = P_1(01|01) \sum_{a_1^2=0,1} P_2(a_1^2 0|00) \tag{3.14}$$

and similarly for the others. Hence, only

$$P(00,0,0|00,0,0) = P_1(00|00)P_2(00|00) > 0 \tag{3.15}$$

survives, violating the inequality.

If, instead, A_1A_2 share a maximally entangled state, and A_2A_3 share a less-than-maximally entangled state, then A_1A_3 can measure so that P_2 satisfies Hardy's paradox;

hence $\exists\,\varepsilon > 0$ such that its local component in any EPR2 decomposition satisfies

$$p_{L,2} \leq 1-\varepsilon. \tag{3.16}$$

Since the maximally entangled state is fully nonlocal [BKP06], for this ε, A_1A_2 can measure such that any EPR2 decomposition of P_1 satisfies

$$p_{L,1} < \varepsilon. \tag{3.17}$$

Then, we assume for a contradiction that $P(a_1^1 a_1^2, a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2)$ is bilocal and decompose it in its bipartite splittings,

$$\begin{aligned} P(a_1^1 a_1^2,& a_2^1, a_3^2 | x_1^1 x_1^2, x_2^1, x_3^2) \\ &= \sum_\lambda \big[p_L(\lambda) P_{A_1A_2}(a_1^1 a_1^2, a_2^1 | x_1^1 x_1^2, x_2^1, \lambda) P_{A_3}(a_3^2 | x_3^2, \lambda) \\ &\qquad + q_L(\lambda) P_{A_1A_3}(a_1^1 a_1^2, a_3^2 | x_1^1 x_1^2, x_3^2, \lambda) P_{A_2}(a_2^1 | x_2^1, \lambda) \\ &\qquad + r_L(\lambda) P_{A_1}(a_1^1 a_1^2 | x_1^1 x_1^2, \lambda) P_{A_2A_3}(a_2^1, a_3^2 | x_2^1, x_3^2, \lambda)\big] \end{aligned} \tag{3.18}$$

where $\sum_\lambda [p_L(\lambda) + q_L(\lambda) + r_L(\lambda)] = 1$.

Summing equation (3.18) over a_1^2, a_3^2 and using equation (3.8), we get an EPR2 decomposition of P_1 with local components q_L, r_L. By equation (3.17), this entails $\sum_\lambda [q_L(\lambda) + r_L(\lambda)] < \varepsilon$, so

$$\sum_\lambda p_L(\lambda) > 1-\varepsilon. \tag{3.19}$$

Summing, instead, equation (3.18) over a_1^1, a_2^1, we obtain an EPR2 decomposition of P_2 whose only nonnegligible component, $\sum_\lambda p_L(\lambda)$, is local in $A_1|A_3$, contradicting equation (3.16). Therefore, P must be GMNL. □

Proof of Theorem 3.1. Turning to the fully general case, we consider the network as a connected graph where vertices are parties and edges are states. The graph is such that, at each vertex, there is one particle for every incident edge.[2] We label the edges as $k = 1, ..., K$ (where K is the number of edges of the graph) and the parties as $A_1, ..., A_n$. Since it will be enough to consider individual measurements on each particle, we denote the input and output of party A_i at edge k as x_i^k, a_i^k respectively. We group the inputs and outputs of each party as $\chi_i = \{x_i^k\}_{k\in E_i}$, $\alpha_i = \{a_i^k\}_{k\in E_i}$ where E_i is the set of edges

[2]Throughout the proof we assume $k \geq 2$. If $k = 1$, there are only two parties sharing bipartite pure entangled states, so the network is nonlocal by Refs. [Gis91, GP92].

incident to vertex i. Then, the shared distribution is of the form

$$P(\alpha_1, ..., \alpha_n|\chi_1, ..., \chi_n) = \prod_{k=1}^{K} P_k(a_i^k a_j^k|x_i^k x_j^k), \tag{3.20}$$

where parties A_i, A_j are connected by edge k (notice that we label vertices and edges independently), and $P_k(a_i^k a_j^k|x_i^k x_j^k)$ is the distribution arising from the state at edge k. It will be sufficient to consider tree graphs, i.e., graphs such that every pair of vertices (parties) is connected by exactly one path of edges. If the given graph is not a tree, any extra edges can be ignored.

Depending on the nature of the shared states, we consider three cases:

(i) every shared state is less-than-maximally entangled;

(ii) every shared state is maximally entangled;

(iii) some shared states are maximally entangled, some are not.

Case (i): if all states are less-than-maximally entangled, we prove the result by deriving an inequality that detects GMNL and finding measurements on the shared states to violate it. To derive the inequality, we will find bipartite inequalities that can be violated by the state at each edge k, lift them to more inputs, outputs and parties using the techniques in Ref. [Pir05] and combine them to obtain a GMNL inequality using tools in Ref. [CAA19]. We will consider 2-input 2-output measurements on each particle. Thus, the global distribution will have $2^{|E_i|}$ inputs and outputs for each party A_i.

We start from the inequality

$$I = P(00|00) - P(01|01) - P(10|10) - P(00|11) \leq 0, \tag{3.21}$$

which is a facet inequality equivalent to the CHSH inequality [CHSH69] for nonsignalling distributions [CAA19]. This inequality detects any bipartite nonlocality present in any bipartition that splits the parties connected by edge k [CAA19]. To lift it to n parties, each with $2^{|E_i|}$ inputs and outputs (see Ref. [Pir05]), we must set the inputs and outputs of the parties that are not connected by edge k to a fixed value (0, wlog). For the parties i that are connected by edge k, any extra inputs other than $x_i^k = 0_i^k, 1_i^k$ can be ignored. Outputs must be grouped, by summing over some of their digits, in order to get an effective 2-output distribution. It will be convenient to add over the output components $a_i^{\bar{k}}$ that do not correspond to edge k, varying only the digit $a_i^k = 0_i^k, 1_i^k$. Thus, we obtain the following n-partite inequality at each edge k :

$$\begin{aligned} I^k = \sum_{\vec{a}_i^{\bar{k}}, \vec{a}_j^{\bar{k}}} \Big[& P\left(0_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \\ & - P\left(0_i^k \vec{a}_i^{\bar{k}}, 1_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 1_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \\ & - P\left(1_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 1_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \\ & - P\left(0_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 1_i^k 0_i^{\bar{k}}, 1_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \Big] \leq 0, \end{aligned} \tag{3.22}$$

where the sum is over each binary digit $a_i^{\bar{k}}, a_j^{\bar{k}}$ of the outputs of parties i, j (which are connected by edge k), except digits a_i^k, a_j^k which are fixed to 0 or 1 in each term. The term $\vec{0}_{\bar{i},\bar{j}}$ denotes input or output 0 for all components of all parties that are not i, j. Thus, each inequality I^k detects the bipartite nonlocality present in the distribution P across any bipartition that splits the parties connected by edge k. In the particular case of the distribution (3.20), it tells whether the component P_k is nonlocal.

Now, we can combine the inequalities I^k to form a GMNL inequality:

$$I_n = \sum_{k=1}^{K} I^k + P(\vec{0}, \vec{0} | \vec{0}, \vec{0}) - \sum_{k=1}^{K} \sum_{\vec{a}_i^{\bar{k}}, \vec{a}_j^{\bar{k}}} P\left(0_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \leq 0\,. \tag{3.23}$$

To show that this is indeed a GMNL inequality, we must show that it holds for any distribution P that is local across some bipartition. A bipartition of the network defines a cut of the graph. Because the graph is assumed connected, for every cut there exists an edge k_0 which crosses the cut. Therefore, if P is local across a bipartition which is crossed by edge k_0, then by Ref. [CAA19] we have

$$I^{k_0} \leq 0. \tag{3.24}$$

Hence,

$$I_n \leq \sum_{\substack{k=1 \\ k \neq k_0}}^{K} I^k + P(\vec{0}, \vec{0} | \vec{0}, \vec{0}) - \sum_{k=1}^{K} \sum_{\vec{a}_i^{\bar{k}}, \vec{a}_j^{\bar{k}}} P\left(0_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right). \tag{3.25}$$

For each $k \neq k_0$, the only nonnegative term gets subtracted in the final summation. The term $P(\vec{0}, \vec{0} | \vec{0}, \vec{0})$ then cancels out with the first term in the final summation for $k = k_0$, leaving only negative terms in the expression as required.

To complete the proof, we find local measurements for each party to violate inequality (3.23). Since all shared states are nonseparable and less-than-maximally entangled, the parties can choose local measurements on each particle such that all resulting distributions satisfy Hardy's paradox [Har92, Har93]:

$$P_k(00|00) > 0 = P_k(01|01) = P_k(10|10) = P_k(00|11) \tag{3.26}$$

for each $k = 1, ..., K$. This was proven for qubits in Refs. [Har92, Har93], and we show the extension to any local dimension in Proposition 3.1 below. Because the distribution is of the form (3.20), each term in each inequality (3.22) simplifies significantly. For example, the second term gives

$$\begin{aligned}
&\sum_{\vec{a}_i^{\bar{k}}, \vec{a}_j^{\bar{k}}} P\left(0_i^k \vec{a}_i^{\bar{k}}, 1_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 1_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right) \\
&\quad = P_k(0_i^k 1_j^k | 0_i^k 1_j^k) \prod_{\ell} \sum_{a_i^\ell} P_\ell(a_i^\ell 0_{j'}^\ell | 0_i^\ell 0_{j'}^\ell) \prod_{\ell'} \sum_{a_j^{\ell'}} P_\ell(0_{i'}^{\ell'} a_j^{\ell'} | 0_{i'}^{\ell'} 0_j^{\ell'}) \prod_m P_m(0_{i'}^m 0_{j'}^m | 0_{i'}^m 0_{j'}^m) \\
&\quad = P_k(0_i^k 1_j^k | 0_i^k 1_j^k)\, p_k \,,
\end{aligned} \tag{3.27}$$

where edges ℓ connect party i to party $j' \neq j$,, edges ℓ' connect party j to party $i' \neq i$, and edges m connect parties i' and j' where $i', j' \neq i, j$. (Depending on the structure of the graph, there may be no edges ℓ, ℓ' or m for a given pair of parties i, j, but that does not affect the proof.)

The product of the terms P_ℓ, $P_{\ell'}$ and P_m will give a number p_k. This is similar for the third and fourth terms, which factorise to

$$\begin{aligned}
&P_k(1_i^k 0_j^k | 1_i^k 0_j^k)\, p_k, \\
&P_k(0_i^k 0_j^k | 1_i^k 1_j^k)\, p_k
\end{aligned} \tag{3.28}$$

respectively. The first term of each I^k cancels out with the last summation in I_n, and the only term that remains is

$$P(\vec{0}, \vec{0} | \vec{0}, \vec{0}) = \prod_{k=1}^{K} P_k(0_i^k 0_j^k | 0_i^k 0_j^k). \tag{3.29}$$

Since P_k satisfies Hardy's paradox for every k, then the components of each P_k appearing in equations (3.27), (3.28) are all zero, while the only surviving term, $P(\vec{0}, \vec{0} | \vec{0}, \vec{0})$, is strictly greater than zero. Thus, the inequality I_n is violated, showing that P is GMNL.

Case (ii): for every bipartition, there is an edge that crosses the corresponding cut, and each of these edges already contains a maximally entangled state. Therefore, the present network meets the requirements of Theorem 2 in [ACSA10], so the network is GMNL—in fact it is fully GMNL.

Case (iii): assume wlog that each edge $k = 1, ..., K_0$ contains a less-than-maximally entangled state, while each edge $k = K_0 + 1, ..., K$ contains a maximally entangled state. Let

$$P = P_H P_+ \tag{3.30}$$

where

$$\begin{aligned} P_H(\{a_i^k\}_{k\leq K_0, i\in[n]} | \{x_i^k\}_{k\leq K_0, i\in[n]}) &= \prod_{k=1}^{K_0} P_k(a_i^k a_j^k | x_i^k x_j^k), \\ P_+(\{a_i^k\}_{k> K_0, i\in[n]} | \{x_i^k\}_{k> K_0, i\in[n]}) &= \prod_{k=K_0+1}^{K} P_k(a_i^k a_j^k | x_i^k x_j^k) \end{aligned} \tag{3.31}$$

where, on the right-hand side, parties i, j are connected by edge k. For $k = 1, ..., K_0$, terms P_k satisfy Hardy's paradox (equation (3.26)), as they arise from the measurements performed in Case (i). For $k = K_0 + 1, ..., K$, the terms P_k arise from measurements on the maximally entangled state to be specified later. We now classify bipartitions depending on whether or not they are crossed by an edge $k \leq K_0$ or $k > K_0$: let $S_{\leq K_0}$ be the set of bipartitions $M|\overline{M}$ (indexed by M) which are crossed by an edge $k \leq K_0$, and $T_{\leq K_0}$ be its complement, i.e. the set of bipartitions which are *not* crossed by an edge $k \leq K_0$. Similarly, $S_{>K_0}$ (respectively, $T_{>K_0}$) is the set of bipartitions which are (not) crossed by an edge $k > K_0$.

Let I_H^k be an inequality detecting nonlocality on edge k, for the distribution P_H. That is, I_H^k is as in equation (3.22) but where the sum over $\overrightarrow{a}_i^{\bar{k}}, \overrightarrow{a}_j^{\bar{k}}$ concerns only the components of parties A_i, A_j that belong only to edges $k' \leq K_0, k' \neq k$. Then, consider the following functional acting on distributions of the form of P_H :

$$I_H = \sum_{k=1}^{K_0} I_H^k + P(\overrightarrow{0}, \overrightarrow{0} | \overrightarrow{0}, \overrightarrow{0}) - \sum_{k=1}^{K_0} \sum_{\overrightarrow{a}_i^{\bar{k}}, \overrightarrow{a}_j^{\bar{k}}} P\left(0_i^k \overrightarrow{a}_i^{\bar{k}}, 0_j^k \overrightarrow{a}_j^{\bar{k}}, \overrightarrow{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \overrightarrow{0}_{\bar{i},\bar{j}}\right). \tag{3.32}$$

Again, the summation in the last term concerns only components that belong to edges $k' \leq K_0, k' \neq k$. We claim that the functional I_H is non-positive for any distribution P that is local across a bipartition of type $S_{\leq K_0}$, i.e. one that is crossed by an edge $k_0 \leq K_0$. The reasoning is similar to that in Case (i): if P is local across a bipartition crossed by an edge $k_0 \leq K_0$, then $I_H^{k_0} \leq 0$ will be satisfied, and so

$$I_H \leq \sum_{\substack{k=1\\k\neq k_0}}^{K_0} I_H^k + P(\vec{0},\vec{0}|\vec{0},\vec{0}) - \sum_{k=1}^{K_0} \sum_{\vec{a}_i^{\bar{k}},\vec{a}_j^{\bar{k}}} P\left(0_i^k \vec{a}_i^{\bar{k}}, 0_j^k \vec{a}_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}} \middle| 0_i^k 0_i^{\bar{k}}, 0_j^k 0_j^{\bar{k}}, \vec{0}_{\bar{i},\bar{j}}\right). \tag{3.33}$$

Now, for each $k \neq k_0$, the only nonnegative term gets subtracted in the final summation. The term $P(\vec{0},\vec{0}|\vec{0},\vec{0})$ then cancels out with the first term in the final summation for $k = k_0$, leaving only negative terms in the expression as required.

We now show that, for $P = P_H$, we have $I_H > 0$. Indeed, the terms in I_H^k simplify in a similar manner to Case (i). Then, since each P_k, $k \leq K_0$ satisfies Hardy's paradox, the second, third and fourth terms in each I_H^k are zero, the first cancels out with the last summation, and the only surviving term is

$$P(\vec{0},\vec{0}|\vec{0},\vec{0}) = \prod_{k=1}^{K_0} P_k(0_i^k 0_j^k | 0_i^k 0_j^k) > 0. \tag{3.34}$$

This means that there exists an $\varepsilon > 0$ such that, for any EPR2 decomposition of P_H,

$$P_H = \sum_M p_{L,H}^M P_{L,H}^M + p_{NS,H} P_{NS,H}, \tag{3.35}$$

we have that the terms where $P_{L,H}^M$ is local across a bipartition such that $M \in S_{\leq K_0}$ satisfy

$$\sum_{M \in S_{\leq K_0}} p_{L,H}^M \leq 1 - \varepsilon. \tag{3.36}$$

Also, it can be deduced from Ref. [ACSA10] that, given the ε above, the parties can choose suitable measurements such that P_+ is fully nonlocal across all bipartitions $S_{>K_0}$. That is, any multipartite EPR2 decomposition of P_+,

$$P_+ = \sum_M p_{L,+}^M P_{L,+}^M + p_{NS,+} P_{NS,+}, \tag{3.37}$$

is such that the terms where $P_{L,+}^M$ is local across a bipartition such that $M \in S_{>K_0}$ satisfy

$$\sum_{M \in S_{>K_0}} p_{L,+}^M < \varepsilon. \tag{3.38}$$

To prove that the global distribution P is GMNL, as is our goal, we assume the converse, and we derive a contradiction from the nonlocality properties of P_H and P_+.

Assuming P is bilocal, we can express the distribution as

$$P = \sum_{\lambda,M} p_L^M(\lambda) P_M(\{\alpha_i\}_{i\in M}|\{\chi_i\}_{i\in M},\lambda) P_{\overline{M}}(\{\alpha_i\}_{i\in\overline{M}}|\{\chi_i\}_{i\in\overline{M}},\lambda), \tag{3.39}$$

where $p_L^M(\lambda)$ are nonnegative numbers for every M, λ such that

$$\sum_{\lambda,M} p_L^M(\lambda) = 1, \tag{3.40}$$

for each $\alpha_i, \chi_i, i = 1, ..., n$.

Then, summing over the output components a_i^k for all $k \leq K_0$ and all i, we get P_+ on the left-hand side, from equation (3.30). On the right-hand side, we get two types of terms (depending on the type of bipartition) that turn out to form an EPR2 decomposition of P_+.[3] Indeed, the local terms are given by bipartitions such that $M \in S_{>K_0}$, while the nonlocal terms are given by bipartitions such that $M \in T_{>K_0}$ (since all terms are nonsignalling). By equation (3.38), the choice of measurements on the particles involved in P_+ ensures that

$$\sum_{\lambda, M\in S_{>K_0}} p_L^M(\lambda) < \varepsilon, \tag{3.41}$$

while

$$\sum_{\lambda, M\in T_{>K_0}} p_L^M(\lambda) > 1-\varepsilon. \tag{3.42}$$

If, instead, we sum over the output components a_i^k for all $k > K_0$ and all i, we get P_H on the left-hand side, from equation (3.30). On the right-hand side, by similar reasoning we find an EPR2 decomposition of P_H. This time, $S_{\leq K_0}$ will give the local terms and $T_{\leq K_0}$ will give the nonlocal terms. By equation (3.36), we have

$$\sum_{\lambda, M\in S_{\leq K_0}} p_L^M(\lambda) \leq 1-\varepsilon. \tag{3.43}$$

Now, since the graph is connected, if a bipartition is not crossed by an edge $k > K_0$, then it must be crossed by an edge $k \leq K_0$. That is, $T_{>K_0} \subseteq S_{\leq K_0}$. This means that equation (3.43) also holds if the sum is over $T_{>K_0}$, but this contradicts equation (3.42).

[3]Note that, while each of the terms on the right-hand side may depend on the whole of each party's input χ_i, the left-hand side does not, because the distribution is of the form (3.30). That is, the resulting EPR2 decomposition of P_+ holds for any fixed value of the inputs $\{x_i^k\}_{k\leq K}$ on the left- and right-hand sides.

Therefore, the distribution P must be GMNL. $\square$

In Theorem 3.1 we assumed that all less-than-maximally entangled states satisfy Hardy's paradox. This is shown for qubits in [Har93], and we now extend the proof to any dimension.

Proposition 3.1. *Let* $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \cong \left(\mathbb{C}^d\right)^{\otimes 2}$ *be a nonseparable and less-than-maximally entangled pure state. Then,* $|\psi\rangle$ *satisfies Hardy's paradox.*

Proof. Let $|\psi\rangle$ be as in the statement of the Proposition. We present 2-input, 2-output measurements for $|\psi\rangle$ to generate a distribution which satisfies Hardy's paradox [Har92, Har93] using tools from Ref. [CAA19].

Consider the Schmidt decomposition

$$|\psi\rangle = \sum_{i=0}^{d-1} \lambda_i^{1/2} |ii\rangle \tag{3.44}$$

and assume the coefficients are ordered such that $0 \neq \lambda_0 \neq \lambda_1 \neq 0$, which is always possible if the state is nonseparable and less-than-maximally entangled. Wlog assume the Schmidt basis of the state is the canonical basis. Let $\alpha \in]0, \pi/2[$ and $\delta \in \mathbb{R}$ and consider the dual vectors

$$\begin{aligned}
\left\langle e_{0|0}\right| &= \cos\alpha \langle 0| + \mathrm{e}^{\mathrm{i}\delta} \sin\alpha \langle 1| \\
\left\langle e_{1|1}\right| &= \lambda_0 \cos\alpha \langle 0| + \lambda_1 \mathrm{e}^{\mathrm{i}\delta} \sin\alpha \langle 1| \\
\left\langle f_{0|0}\right| &= \lambda_1^{3/2} \mathrm{e}^{\mathrm{i}\delta} \sin\alpha \langle 0| - \lambda_0^{3/2} \cos\alpha \langle 1| \\
\left\langle f_{1|1}\right| &= \lambda_1^{1/2} \mathrm{e}^{\mathrm{i}\delta} \sin\alpha \langle 0| - \lambda_0^{1/2} \cos\alpha \langle 1|
\end{aligned} \tag{3.45}$$

(one can write the projectors in the Schmidt basis of the state instead of assuming the state decomposes into the canonical basis). Define the measurements $E_{a|x}$ for Alice, with input x and output a, and $F_{b|y}$ for Bob, with input y and output b, given by

$$\begin{aligned}
E_{0|0} &= \left|e_{0|0}\middle\rangle\!\middle\langle e_{0|0}\right| \\
E_{1|0} &\propto \left|e_{0|0}\middle\rangle\!\middle\langle e_{0|0}\right|^{\perp} \oplus \mathbb{1}_{2,\ldots,d-1} \\
E_{0|1} &\propto \left|e_{1|1}\middle\rangle\!\middle\langle e_{1|1}\right|^{\perp} \\
E_{1|1} &\propto \left|e_{1|1}\middle\rangle\!\middle\langle e_{1|1}\right| \oplus \mathbb{1}_{2,\ldots,d-1} \\
F_{0|0} &\propto \left|f_{0|0}\middle\rangle\!\middle\langle f_{0|0}\right| \\
F_{1|0} &\propto \left|f_{0|0}\middle\rangle\!\middle\langle f_{0|0}\right|^{\perp} \oplus \mathbb{1}_{2,\ldots,d-1}
\end{aligned} \tag{3.46}$$

$$F_{0|1} \propto \left|f_{1|1}\middle\rangle\middle\langle f_{1|1}\right|^{\perp} \oplus \mathbb{1}_{2,\dots,d-1}$$
$$F_{1|1} \propto \left|f_{1|1}\middle\rangle\middle\langle f_{1|1}\right|$$

where $\left|e_{0|0}\middle\rangle\middle\langle e_{0|0}\right|^{\perp}$ denotes the density matrix corresponding to the vector orthogonal to $\left|e_{0|0}\right\rangle$ when restricted to the subspace spanned by $\{|0\rangle, |1\rangle\}$, and $\mathbb{1}_{2,\dots,d-1}$ is the identity operator on the subspace spanned by $\{|i\rangle\}_{i=2}^{d-1}$, for either Alice or Bob. Note that, since we are only interested in whether some probabilities are equal or different from zero, normalisation will not play a role.

We now show that the distribution given by

$$P(ab|xy) = \operatorname{tr}\big(E_{a|x} \otimes F_{b|y} |\psi\rangle\langle\psi|\big) \tag{3.47}$$

satisfies Hardy's paradox. Indeed, because of the probabilities considered and the form of the measurements, only the terms in $i = 0, 1$ contribute to the probabilities that appear in Hardy's paradox, therefore

$$\begin{aligned} P(01|01) &\propto \left|\sum_{i=0}^{1} \lambda_i^{1/2} \left(\left\langle e_{0|0}\right| \otimes \left\langle f_{1|1}\right|\right) |ii\rangle\right|^2 = 0 \\ P(10|10) &\propto \left|\sum_{i=0}^{1} \lambda_i^{1/2} \left(\left\langle e_{1|1}\right| \otimes \left\langle f_{0|0}\right|\right) |ii\rangle\right|^2 = 0 \\ P(00|11) &\propto \left|\sum_{i=0}^{1} \lambda_i^{1/2} \left(\left\langle e_{0|1}\right| \otimes \left\langle f_{0|1}\right|\right) |ii\rangle\right|^2 = 0. \end{aligned} \tag{3.48}$$

For $P(00|00)$, we find

$$\begin{aligned} P(00|00) &\propto \left|\sum_{i=0}^{1} \lambda_i^{1/2} \left(\left\langle e_{0|0}\right| \otimes \left\langle f_{0|0}\right|\right) |ii\rangle\right|^2 \\ &= \left|\mathrm{e}^{\mathrm{i}\delta} \sin\alpha\cos\alpha\, \lambda_0^{1/2}\, \lambda_1^{1/2} \left(\lambda_1 - \lambda_0\right)\right|^2, \end{aligned} \tag{3.49}$$

which is strictly greater than zero when $\alpha \in]0, \pi/2[$ and $0 \neq \lambda_0 \neq \lambda_1 \neq 0$, like we assumed. This proves the claim. □

3.3 GMNL from GME

By Theorem 3.1, a star network whose central node shares pure-state entanglement with all others is GMNL. We now ask whether all GME states are GMNL (i.e. the genuine

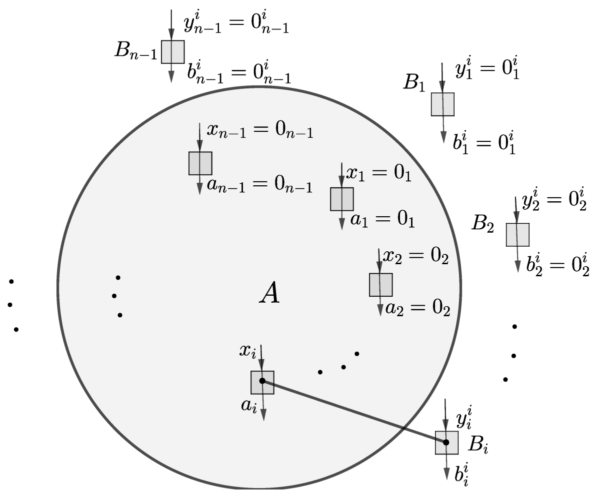

Figure 3.2: Element $i \in [n-1]$ of the star network of bipartite entanglement created from a GME state $|\Psi\rangle$. Parties $\{B_j\}_{j\in[n-1],j\neq i}$ have already measured $|\Psi\rangle$ and are left unentangled. Alice and party B_i share a pure bipartite entangled state. Alice has input x_i and output a_i while each party B_j, $j \in [n-1]$, has input y^i_j and output b^i_j.

multipartite extension of Gisin's theorem). We show $(n-1)$ copies of any pure GME n-partite state suffice to generate n-partite GMNL. We do this by generating a distribution from these copies that mimics the star network configuration.

We fix some notation that we will use in Theorem 3.2 below. The result considers a GME state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} ... \otimes \mathcal{H}_{B_{n-1}} \cong (\mathbb{C}^d)^{\otimes n}$, $n-1$ copies of which are shared between n parties $A, B_1, ..., B_{n-1}$. Each party measures locally on each particle, like in Theorem 3.1. We denote Alice's input and output, respectively, as $\chi \equiv x_1...x_{n-1}$, $\alpha \equiv a_1...a_{n-1}$ in terms of the digits x_i, a_i corresponding to each particle $i \in [n-1]$. We let the measurement made by party B_j on copy i have input y^i_j and output b^i_j , where $i, j = 1, ..., n-1$, and for each j we denote $\upsilon_j = y^1_j ... y^{n-1}_j$ and $\beta_j = b^1_j ... b^{n-1}_j$ digit-wise. Then, after measurement, the parties share a distribution

$$\{P(\alpha\beta_1...\beta_{n-1}|\chi\upsilon_1...\upsilon_{n-1})\}_{\substack{\alpha,\beta_1...\beta_{n-1}\\ \chi,\upsilon_1...\upsilon_{n-1}}}. \tag{3.50}$$

Because we are considering local measurements made on each particle, this distribution is of the form

$$P(\alpha\beta_1...\beta_{n-1}|\chi\upsilon_1...\upsilon_{n-1}) = \prod_{i=1}^{n-1} P_i(a_i b^i_1 ... b^i_{n-1}|x_i y^i_1 ... y^i_{n-1}), \tag{3.51}$$

where each P_i is the distribution arising from copy i of the state $|\Psi\rangle$. Each copy i of the state $|\Psi\rangle$ will give an edge of a star network connecting Alice and party B_i. Because of the structure of this particular network, we can simplify the notation with respect to Theorem 3.1 and identify the index of each party B_i with its corresponding edge i.

Theorem 3.2. *Any GME state* $|\Psi\rangle \in \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n \cong (\mathbb{C}^d)^{\otimes n}$ *is such that* $|\Psi\rangle^{\otimes(n-1)}$ *is GMNL.*

We first outline the proof for the tripartite case, and then extend it to the general case.

Proof for the tripartite case. Since $n = 3$, we consider two copies of the state. For each copy, we derive measurements for Bob1 and Bob2 that leave Alice bipartitely entangled with Bob2 and Bob1 respectively. This yields a network as in equation (3.8) but postselected on the inputs and outputs of these measurements. We generalise Theorem 3.1 to show this network is also GMNL.

For $i, j = 1, 2$, on copy i, B_j's measurements have input y_j^i and output b_j^i and Alice's measurement has input x_i and output a_i. We denote B_j's inputs and outputs in terms of their digits as $\upsilon_j = y_j^1 y_j^2$ and $\beta_j = b_j^1 b_j^2$. Then, after measurement, the parties share a distribution

$$\begin{aligned} &P(\alpha\beta_1\beta_2|\chi\upsilon_1\upsilon_2) \\ &\quad = P_1(a_1, b_1^1 b_2^1|x_1, y_1^1 y_2^1) P_2(a_2, b_1^2 b_2^2|x_2, y_1^2 y_2^2)\,. \end{aligned} \tag{3.52}$$

For each $i, j = 1, 2$, $i \neq j$, we assume B_j uses input 0_j^i and output 0_j^i to project the ith copy of $|\Psi\rangle$ onto $|\phi_i\rangle_{AB_i}$, as shown in Figure 3.2 for n parties. Then, Refs. [PR92, GG17] and a continuity argument serve to show we only have two possibilities for each i: either there exists an input and output per party such that $|\phi_i\rangle_{AB_i}$ is less-than-maximally entangled, or there exists an input per party such that, for all outputs, $|\phi_i\rangle_{AB_i}$ is maximally entangled. In each case we generalise the proof in Theorem 3.1 to show $|\Psi\rangle^{\otimes 2}$ is GMNL.

If both $|\phi_i\rangle_{AB_i}$, $i = 1, 2$ are less-than-maximally entangled, we use the following expression, which is a GMNL inequality by the same reasoning as in Theorem 3.1:

$$\begin{aligned} I_3 = &\sum_{i=1}^{2} I^i + P(00, 00, 00|00, 00, 00) \\ &- \sum_{i=1}^{2} \sum_{\substack{a_j, b_i^j, \\ b_j^j = 0,1, \\ j \neq i}} P(0_i a_j\,, 0_i^i b_i^j\,, 0_j^i b_j^j\,|0_i 0_j\,, 0_i^i 0_i^j\,, 0_j^i 0_j^j) \leq 0, \end{aligned} \tag{3.53}$$

where

$$I^i = \sum_{\substack{a_j,b_i^j,b_j^j=0,1,\\ j\neq i}} \Big[P(0_ia_j\,,0_i^ib_i^j\,,0_j^ib_j^j\,|0_i0_j\,,0_i^i0_i^j\,,0_j^i0_j^j) - P(0_ia_j\,,1_i^ib_i^j\,,0_j^ib_j^j\,|0_i0_j\,,1_i^i0_i^j\,,0_j^i0_j^j)$$
$$-P(1_ia_j\,,0_i^ib_i^j\,,0_j^ib_j^j\,|1_i0_j\,,0_i^i0_i^j\,,0_j^i0_j^j) - P(0_ia_j\,,0_i^ib_i^j\,,0_j^ib_j^j\,|1_i0_j\,,1_i^i0_i^j\,,0_j^i0_j^j)\Big]\,. \tag{3.54}$$

Evaluating the inequality on the distribution (3.52), we find again that all negative terms in each I^i can be sent to zero. For each i we get, for example,

$$\begin{aligned}\sum_{\substack{a_j,b_i^j,b_j^j\\=0,1}} &P(0_ia_j\,,1_i^ib_i^j\,,0_j^ib_j^j\,|0_i0_j\,,1_i^i0_i^j\,,0_j^i0_j^j)\\ &= P_i(0_i1_i^i0_j^i|0_i1_i^i0_j^i)\end{aligned} \tag{3.55}$$

as the sum over P_j is 1. But, conditioned on B_j's input and output being 0_j^i, parties AB_i can measure so P_i satisfies Hardy's paradox, hence this term is zero, and similarly for the other two negative terms. This means all terms in I_3 are zero except $P(00,00,00|00,00,00) > 0$, violating the inequality. Therefore, $|\Psi\rangle^{\otimes 2}$ is GMNL.

If, for both $i = 1,2$, there exists a local measurement for party B_j, $j \neq i$ such that, for *all* outputs, $|\Psi\rangle$ is projected onto a maximally entangled state $|\phi_i\rangle_{AB_i}$, then $|\Psi\rangle$ satisfies Theorem 2 in Ref. [ACSA10], so $|\Psi\rangle$ itself is GMNL. Therefore so is $|\Psi\rangle^{\otimes 2}$.

Finally, if $|\phi_1\rangle_{AB_1}$ is maximally entangled for all of B_2's outputs, and $|\phi_2\rangle_{AB_2}$ is less-than-maximally entangled, using Refs. [PR92, ACSA10] we deduce that the bipartite EPR2 components of $P_{1,2}$ across $A|B_{1,2}$ respectively are bounded like in Theorem 3.1. That is, $\exists\,\varepsilon > 0$ such that the local component of any EPR2 decomposition across $A|B_2$ satisfies

$$p_{L,2}^{A|B_2} \leq 1-\varepsilon \tag{3.56}$$

and, given this ε, parties AB_1 can measure locally such that all bipartite EPR2 decompositions across $A|B_1$ have a local component

$$p_{L,1}^{A|B_1} < \varepsilon. \tag{3.57}$$

Then, we assume $P(\alpha\beta_1\beta_2|\chi\upsilon_1\upsilon_2)$ is bilocal and decompose it in local terms across different bipartitions, like in equation (3.18) in Theorem 3.1. Summing over a_2, b_j^2, $j = 1,2$ gives an EPR2 decomposition of P_1 whose local components can be bounded using equation (3.57). Summing over a_1, b_j^1, $j = 1,2$ instead gives an EPR2 decomposition of

P_2. But the bound on the local component of P_1 entails a bound on that of P_2 which contradicts equation (3.56), proving P is GMNL. $\square$

Proof of Theorem 3.2. We turn now to the case of n parties, for any $n \in \mathbb{N}$. For each copy $i = 1, ..., n-1$ of the state $|\Psi\rangle$, we will find measurements for parties $\{B_j\}_{j\neq i}$ that leave Alice and party B_i with a bipartite entangled state. This will yield a network in a similar configuration to Theorem 3.1 for a star network, but conditionalised on the inputs and outputs of these measurements. We will generalise the result of Theorem 3.1 as it applies to a star network to show that this network is also GMNL.

Let $i \in [n-1]$ and consider the ith copy of $|\Psi\rangle$. Suppose each party B_j, $j \neq i$, performs a local, projective measurement onto a basis $\{|b_j\rangle\}_{b_j=0}^{d-1}$. We pick the computational basis on each party's Hilbert space to be such that the measurement performed by the parties B_j, $j \neq i$, leave Alice and B_i in state $|\phi_{\overrightarrow{b}}\rangle_{AB_i}$, where $\overrightarrow{b} = b_1...b_{i-1}b_{i+1}...b_{n-1}$ denotes the output obtained by the parties B_j, $j \neq i$ (we briefly omit the script i referring to the copy of the state, for readability). This means that we can write the state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{\overrightarrow{b}} \lambda_{\overrightarrow{b}} \left|\phi_{\overrightarrow{b}}\right\rangle_{AB_i} \left|\overrightarrow{b}\right\rangle_{B_1...B_{i-1}B_{i+1}...B_{n-1}}. \tag{3.58}$$

Ref. [PR92], whose proof was completed in Ref. [GG17], showed that there always exist measurements (i.e. bases) $\{|b_j\rangle\}_{b_j=0}^{d-1}$ such that $\left|\phi_{\overrightarrow{b}}\right\rangle_{AB_i}$ is entangled for a certain output $\overrightarrow{b}$. We now show that this opens up only two possibilities for each i: either there exists an output such that $\left|\phi_{\overrightarrow{b}}\right\rangle_{AB_i}$ is less-than-maximally entangled, or for all outputs $\overrightarrow{b}$, $\left|\phi_{\overrightarrow{b}}\right\rangle_{AB_i}$ is maximally entangled. Indeed, the only option left to discard is one where, for some $\overrightarrow{b} = \overrightarrow{b^*}$, $\left|\phi_{\overrightarrow{b^*}}\right\rangle_{AB_i}$ is maximally entangled, and for some other $\overrightarrow{b} = \overrightarrow{b^{**}}$, $\left|\phi_{\overrightarrow{b^{**}}}\right\rangle_{AB_i}$ is separable. But it is easy to see, by using a continuity argument, that in this case the bases $\{|b_j\rangle\}_{b_j=0}^{d-1}$ can be modified so that there exists one output for which AB_i are projected onto a less-than-maximally entangled state: it suffices to consider one (normalised) element of the measurement basis to be $c_0 \left|b_j^*\right\rangle + c_1 \left|b_j^{**}\right\rangle$ for some values $c_0, c_1 \in \mathbb{C}$, for each j.

Therefore, we consider the following cases:

(i) for all $i \in [n-1]$, there exists an input and output for each $B_j, j \neq i$ such that $|\phi_i\rangle_{AB_i}$ is less-than-maximally entangled;

(ii) for all $i \in [n-1]$, there exists an input for each $B_j, j \neq i$ such that $|\phi_i\rangle_{AB_i}$ is maximally entangled for all outputs;

(iii) there exist $i, k \in [n-1]$ such that $|\phi_i\rangle_{AB_i}$ is as in Case (ii) and $|\phi_k\rangle_{AB_k}$ is as in Case (i).

Case (i): let $i \in [n-1]$. Suppose parties $\{B_j\}_{j\neq i}$ perform the measurements explained above that leave Alice and B_i less-than-maximally entangled. Then, Alice and B_i can perform local measurements on the resulting state to satisfy Hardy's paradox. We will modify the inequality in Theorem 3.1 and show that these measurements on $|\Psi\rangle^{\otimes(n-1)}$ give a distribution which violates the inequality.

To modify the inequality in Theorem 3.1, we import the same strategy to lift inequality (3.21) to n parties, each with 2^{n-1} inputs and outputs. We want I^{AB_i} to detect bipartite nonlocality between Alice's ith particle and B_i's ith particle, that is, nonlocality in $a_i b_i^i | x_i y_i^i$. Therefore, for each i we now need to fix all other inputs x_j, y_i^j, y_j^j and add over all other outputs a_j, b_i^j, b_j^j, $j \neq i$, so that

$$I^{AB_i} = \sum_{a_{\bar{i}}, b_i^{\bar{i}}, b_{\bar{i}}^{\bar{i}}=0,1} \Big[P(0_i a_{\bar{i}}, 0_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 0_i 0_{\bar{i}}, 0_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) - P(0_i a_{\bar{i}}, 1_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 0_i 0_{\bar{i}}, 1_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}})$$
$$- P(1_i a_{\bar{i}}, 0_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 1_i 0_{\bar{i}}, 0_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) - P(0_i a_{\bar{i}}, 0_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 1_i 0_{\bar{i}}, 1_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) \Big], \tag{3.59}$$

where the outputs in the first term are denoted as follows: $0_i a_{\bar{i}}$ denotes output $\alpha = a_1 ... 0_i ... a_{n-1}$, $0_i^i b_i^{\bar{i}}$ denotes output $\beta_i = b_i^1 ... 0_i^i ... b_i^{n-1}$, and $0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}}$ denotes output $\beta_j = b_j^1 ... 0_j^i ... b_j^{n-1}$ for *all* $j \neq i$. Inputs are denoted similarly, and the notation is similar for the other three terms. Then, the inequality

$$I_n = \sum_{i=1}^{n-1} I^{AB_i} + P(\vec{0}, \vec{0} | \vec{0}, \vec{0}) - \sum_{i=1}^{n-1} \sum_{a_{\bar{i}}, b_i^{\bar{i}}, b_{\bar{i}}^{\bar{i}}=0,1} P(0_i a_{\bar{i}}, 0_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 0_i 0_{\bar{i}}, 0_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) \leq 0 \tag{3.60}$$

is a GMNL inequality, by the same reasoning as in Theorem 3.1.

Evaluating the inequality on the distribution (3.51), we find again that each term simplifies. For each i we get, for example,

$$\sum_{a_{\bar{i}}, b_i^{\bar{i}}, b_{\bar{i}}^{\bar{i}}=0,1} P(0_i a_{\bar{i}}, 1_i^i b_i^{\bar{i}}, 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} | 0_i 0_{\bar{i}}, 1_i^i 0_i^{\bar{i}}, 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) \tag{3.61}$$

$$= P_i(0_i 1_i^i 0_{\bar{i}}^i | 0_i 1_i^i 0_{\bar{i}}^i) \prod_{\substack{j=1 \\ j\neq i}}^{n-1} \sum_{\substack{a_j, b_k^j = 0,1 \\ k \neq j}} P_j(a_j b_1^j ... b_{j-1}^j b_{j+1}^j ... b_{n-1}^j \, | 0_j 0_1^j ... 0_{j-1}^j 0_{j+1}^j ... 0_{n-1}^j)$$

$$= P_i(0_i 1_i^i 0_{\bar{i}}^i | 0_i 1_i^i 0_{\bar{i}}^i)$$

and, similarly,

$$\begin{aligned} \sum_{a_{\bar{i}}, b_i^{\bar{i}}, b_{\bar{i}}^{\bar{i}} = 0,1} P(1_i a_{\bar{i}} \, , 0_i^i b_i^{\bar{i}} \, , 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} \, | 1_i 0_{\bar{i}} \, , 0_i^i 0_i^{\bar{i}} \, , 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) &= P_i(1_i 0_i^i 0_{\bar{i}}^i | 1_i 0_i^i 0_{\bar{i}}^i) \, ; \\ \sum_{a_{\bar{i}}, b_i^{\bar{i}}, b_{\bar{i}}^{\bar{i}} = 0,1} P(0_i a_{\bar{i}} \, , 0_i^i b_i^{\bar{i}} \, , 0_{\bar{i}}^i b_{\bar{i}}^{\bar{i}} \, | 1_i 0_{\bar{i}} \, , 1_i^i 0_i^{\bar{i}} \, , 0_{\bar{i}}^i 0_{\bar{i}}^{\bar{i}}) &= P_i(0_i 0_i^i 0_{\bar{i}}^i | 1_i 1_i^i 0_{\bar{i}}^i). \end{aligned} \tag{3.62}$$

Also,

$$P(\overrightarrow{0}\, , \overrightarrow{0} \, | \, \overrightarrow{0}\, , \overrightarrow{0}\,) = \prod_{i=1}^{n-1} P_i(0_i 0_i^i 0_{\bar{i}}^i | 0_i 0_i^i 0_{\bar{i}}^i) \, . \tag{3.63}$$

Now each P_i in equation (3.51) arises from measurements by $\{B_j\}_{j\neq i}$ to create a less-than-maximally entangled state between Alice and B_i, who can then choose measurements to satisfy Hardy's paradox. Hence all terms are zero except $P(\overrightarrow{0}\, , \overrightarrow{0} \, | \, \overrightarrow{0}\, , \overrightarrow{0}\,) > 0$, and so the inequality is violated. Therefore, $|\Psi\rangle^{\otimes(n-1)}$ is GMNL.

Case (ii): we assumed that, for all $i \in [n-1]$, there exist local measurements on $|\Psi\rangle$ for parties $\{B_j\}_{j\neq i}$ that, for *all* outcomes, create a maximally entangled state $|\phi_i\rangle_{AB_i}$ shared between Alice and B_i. Since all bipartitions can be expressed as $A|B_i$ for some i, we find that $|\Psi\rangle$ meets the requirements of Theorem 2 in [ACSA10], and so $|\Psi\rangle$ is GMNL. That is, one copy of the shared state $|\Psi\rangle$ is already GMNL, and therefore so is $|\Psi\rangle^{\otimes(n-1)}$.

Case (iii): assume wlog that the state $|\phi_i\rangle_{AB_i}$ is less-than-maximally entangled for $i = 1, ..., K_0$ and maximally entangled for $i = K_0 + 1, ..., n-1$. We will show that $|\Psi\rangle^{\otimes(K_0+1)}$ is GMNL, which implies that $|\Psi\rangle^{\otimes(n-1)}$ is so too.

It will be useful to classify bipartitions $M|\overline{M}$ like in Theorem 3.1. We will always assume that Alice belongs to M in order not to duplicate the bipartitions. Let $S_{\leq K_0}$ be the set of bipartitions $M|\overline{M}$ (indexed by M) which are crossed by an edge $j \leq K_0$, i.e., where $\overline{M}$ contains at least one index $j \in \{1, ..., K_0\}$, and $T_{\leq K_0}$ be its complement, i.e. the set of bipartitions where $\overline{M}$ contains only indices $j \in \{K_0 + 1, ..., n-1\}$. Similarly, $S_{>K_0}$ (respectively, $T_{>K_0}$) is the set of bipartitions which are (not) crossed by an edge $j > K_0$. That is, in $S_{>K_0}$, there is some $j \in \{K_0 + 1, ..., n-1\}$ which belongs to $\overline{M}$, while in $T_{>K_0}$, $\overline{M}$ contains only indices $j \in \{1, ..., K_0\}$.

For each $i = 1, ..., K_0$, parties AB_i can perform measurements on their shared state $|\phi_i\rangle_{AB_i}$ which, together with the measurements of parties $\{B_j\}_{j\neq i}$ that projected $|\Psi\rangle$ onto $|\phi_i\rangle_{AB_i}$, give rise to a distribution

$$P_i(a_i b_1^i ... b_{n-1}^i | x_i y_1^i ... y_{n-1}^i) \tag{3.64}$$

which satisfies Hardy's paradox when post-selected on the inputs and outputs of parties $\{B_j\}_{j\neq i}$. Then, the distribution arising from the first K_0 copies of $|\Psi\rangle$ is

$$P_H(\{a_i\}_{i\leq K_0}\{b_j^i\}_{i\leq K_0, j\in[n-1]}|\{x_i\}_{i\leq K_0}\{y_j^i\}_{i\leq K_0, j\in[n-1]}) = \prod_{i=1}^{K_0} P_i(a_i b_1^i ... b_{n-1}^i | x_i y_1^i ... y_{n-1}^i), \tag{3.65}$$

with P_i as in equation (3.64). This distribution is similar to that in Case (i) when post-selected on the inputs and outputs of parties $\{B_j\}_{j>K_0}$. More precisely, by the nonsignalling condition, we have

$$\begin{aligned}
&P_H(\{a_i\}_{i\leq K_0}\{b_j^i\}_{i\leq K_0, j\leq K_0}\{b_j^i = 0_j^i\}_{i\leq K_0, j>K_0}|\{x_i\}_{i\leq K_0}\{y_j^i\}_{i\leq K_0, j\leq K_0}\{y_j^i = 0_j^i\}_{i\leq K_0, j>K_0}) \\
&\quad = P_{AB_1...B_{K_0}}(\{a_i\}_{i\leq K_0}\{b_j^i\}_{i\leq K_0, j\leq K_0} \\
&\qquad\qquad\qquad |\{x_i\}_{i\leq K_0}\{y_j^i\}_{i\leq K_0, j\leq K_0}, \{b_j^i = 0_j^i\}_{i\leq K_0, j>K_0}, \{y_j^i = 0_j^i\}_{i\leq K_0, j>K_0}) \\
&\qquad \times P_{B_{K_0+1}...B_{n-1}}(\{b_j^i = 0_j^i\}_{i\leq K_0, j>K_0}|\{y_j^i = 0_j^i\}_{i\leq K_0, j>K_0}),
\end{aligned} \tag{3.66}$$

where by Case (i) we know that $P_{AB_1...B_{K_0}}$ is GMNL in its parties. Then, P_H must be (K_0+1)-way nonlocal (i.e., GMNL when restricted to parties $A, B_1, ..., B_{K_0}$). Indeed, if this were not the case, by equation (3.66) we could obtain a bilocal decomposition for $P_{AB_1...B_{K_0}}$, which would contradict the fact that this distribution is GMNL.

Therefore, there exists an $\varepsilon > 0$ such that any EPR2 decomposition of P_H as

$$P_H = \sum_M p_{L,H}^M P_{L,H}^M + p_{NS,H} P_{NS,H} \tag{3.67}$$

we have that the terms where $P_{L,H}^M$ is local across a bipartition such that $M \in S_{\leq K_0}$ satisfy

$$\sum_{M\in S_{\leq K_0}} p_{L,H}^M \leq 1 - \varepsilon. \tag{3.68}$$

On the other hand, $|\Psi\rangle$ satisfies Theorem 1 in Ref. [ACSA10] for all bipartitions $A|B_i$ for $i = K_0+1, ..., n-1$, hence it is fully nonlocal across all such bipartitions. This means that, for any $\delta_i > 0$, there exist local measurements on $|\Psi\rangle$ (which depend on i)

that lead to a distribution

$$P_+(ab_1...b_{n-1}|xy_1...y_{n-1}) \tag{3.69}$$

such that any bipartite EPR2 decomposition across a bipartition $A|B_i$, for $i = K_0 + 1, ..., n-1$,

$$P_+ = p_{L,+}^{A|B_i} P_{L,+}^{A|B_i} + (1 - p_{L,+}^{A|B_i}) P_{NS,+}^{A|B_i} \tag{3.70}$$

satisfies

$$p_{L,+}^{A|B_i} < \delta_i. \tag{3.71}$$

Thus, considering the possibility of implementing all the above measurements for each i leads to a distribution of the form (3.69) in which equation (3.71) holds for every $i = K_0 + 1, ..., n-1$.

Therefore, given the ε above, the parties can choose suitable δ_i to bound the bipartitely local components and hence ensure that any multipartite EPR2 decomposition of P_+,

$$P_+ = \sum_M p_{L,+}^M P_{L,+}^M + p_{NS,+} P_{NS,+} \tag{3.72}$$

is such that the terms where $P_{L,+}^M$ is local across a bipartition such that $M \in S_{>K_0}$ satisfy

$$\sum_{M \in S_{>K_0}} p_{L,+}^M < \varepsilon. \tag{3.73}$$

Since we only need to consider $(K_0 + 1)$ copies of the state, we denote the inputs and outputs of Alice and each party B_j, $j \in [n-1]$ by $\chi = x_1...x_{K_0+1}$, $\upsilon_j = y_j^1...y_j^{K_0+1}$; $\alpha = a_1...a_{K_0+1}$, $\beta_j = b_j^1...b_j^{K_0+1}$ respectively. Then, the global distribution obtained from $|\Psi\rangle^{\otimes(K_0+1)}$ is

$$\begin{aligned} P(\alpha\beta_1...\beta_{n-1}|\chi\upsilon_1...\upsilon_{n-1}) = \\ & P_H(\{a_i\}_{i \le K_0}\{b_j^i\}_{i \le K_0, j \in [n-1]}|\{x_i\}_{i \le K_0}\{y_j^i\}_{i \le K_0, j \in [n-1]}) \\ & \times P_+(a_{K_0+1}b_1^{K_0+1}...b_{n-1}^{K_0+1}|x_{K_0+1}y_1^{K_0+1}...y_{n-1}^{K_0+1}), \end{aligned} \tag{3.74}$$

where P_H comes from equation (3.65) and the EPR2 components of P_H, P_+ are as per equations (3.68), (3.73).

We now follow a similar strategy to that in Theorem 3.1. To prove that the global distribution P is GMNL, as is our goal, we assume the converse, and we derive a contradiction from the nonlocality properties of P_H and P_+. Assuming P is bilocal, we can express the distribution as

$$P(\alpha\beta_1...\beta_{n-1}|\chi\upsilon_1...\upsilon_{n-1}) \\ = \sum_{\lambda,M} p_L^M(\lambda) P_M(\alpha\{\beta_j\}_{j\in M}|\chi\{\upsilon_j\}_{j\in M},\lambda) P_{\overline{M}}(\{\beta_j\}_{j\in\overline{M}}|\{\upsilon_j\}_{j\in\overline{M}},\lambda), \tag{3.75}$$

where

$$\sum_{\lambda,M} p_L^M(\lambda) = 1, \tag{3.76}$$

for *each* $\alpha, \beta_j, \chi, \upsilon_j, j = 1, ..., n-1$, where we recall that each $\beta_j = b_j^1 ... b_j^{K_0+1}$ and similarly for υ_j.

Now, if we sum equation (3.75) over a_i, b_j^i for $i = 1, ..., K_0$ and $j = 1, ..., n-1$ (that is, we sum over the ith digit, $i \leq K_0$, of Alice and all parties B_j), we obtain P_+ on the left-hand side, from equation (3.74). On the right-hand side, we obtain, for each M,[4]

$$\sum_{\lambda} p_L^M(\lambda) P_M(a_{K+1}\{b_j^{K_0+1}\}_{j\in M}|\chi\{\upsilon_j\}_{j\in M},\lambda) P_{\overline{M}}(\{b_j^{K_0+1}\}_{j\in\overline{M}}|\{\upsilon_j\}_{j\in\overline{M}},\lambda)\,, \tag{3.77}$$

whose sum turns out to form an EPR2 decomposition of P_+. Indeed, local terms are given by bipartitions such that $M \in S_{>K_0}$, as in these terms there is some digit $b_j^{K_0+1}$ with $j > K_0$ appearing in $P_{\overline{M}}$, thus they are local across $A|B_j$ for some $j > K_0$. The nonlocal terms are given by bipartitions such that $M \in T_{>K_0}$ (since all terms are nonsignalling). Therefore, the choice of measurements which generated P_+ ensures (by equation (3.73)) that

$$\sum_{\lambda, M\in S_{>K_0}} p_L^M(\lambda) < \varepsilon \tag{3.78}$$

and hence

$$\sum_{\lambda, M\in T_{>K_0}} p_L^M(\lambda) > 1 - \varepsilon. \tag{3.79}$$

Going back now to equation (3.75), we sum over $a_{K_0+1}, b_j^{K_0+1}$ for $j = 1, ..., n-1$ (that is, we sum over the (K_0+1)th digit of Alice and all parties B_j). Then, we obtain P_H on the left-hand side, from equation (3.74). On the right-hand side, we obtain for

[4]Note that, once more, the distribution obtained by summing over only some of the digits of a party's output still depends on the whole input as it may be signalling in the different digits of the party's input. However, as in Theorem 3.1, these extra inputs can be fixed to an arbitrary value as the left-hand side is independent of them.

each M,

$$\sum_{\lambda} p_L^M(\lambda) P_M(\{a_i\}_{i\leq K_0}\{b_j^i\}_{i\leq K_0, j\in M}|\chi\{\upsilon_j\}_{j\in M},\lambda) P_{\overline{M}}(\{b_j^i\}_{i\leq K_0, j\in\overline{M}}|\{\upsilon_j\}_{j\in\overline{M}},\lambda)\,, \tag{3.80}$$

whose sum over M gives an EPR2 decomposition of P_H. This time, $S_{\leq K_0}$ will give the local terms, as $P_{\overline{M}}$ will contain at least some digit b_j^j for $j \leq K_0$, while $T_{\leq K_0}$ will give the nonlocal terms. By equation (3.68), our choice of ε implies that

$$\sum_{\lambda, M\in S_{\leq K_0}} p_L^M(\lambda) \leq 1-\varepsilon. \tag{3.81}$$

Now, any bipartition in $T_{>K_0}$ is such that all $j \in \{K_0+1, ..., n-1\}$ are in M. Hence, there must be some $j \leq K_0$ in $\overline{M}$, otherwise $\overline{M}$ would be empty. Therefore, $P_{\overline{M}}$ always contains at least one digit b_j^j for some $j \leq K_0$, and so terms where $M \in T_{>K_0}$ are local across the bipartition $A|B_j$ for some $j \leq K_0$. That is, $T_{>K_0} \subseteq S_{\leq K_0}$.

This means that equation (3.81) also holds if the sum is over $T_{>K_0}$, but this is in contradiction with equation (3.79). □

3.4 Looking beyond

While continuity ensures that Theorem 3.1 is robust to some noise, other extensions of this result to mixed states might be considered. In fact, a very simple construction can be used to show that Theorem 3.1 extends, at least, to networks of some mixed states:

Observation. *There exist bipartite mixed states which, distributed in any connected network, yield GMNL independently of the noise parameter.*

Proof. Consider a connected network of pure bipartite entangled states with K edges, where, at each edge k, the less-than-maximally entangled state ρ_k gets measured with POVMs $E_{a_i^k|x_i^k}$, $F_{a_j^k|x_j^k}$ (which depend on k, but this is omitted from the notation for readability) and gives rise to a distribution

$$P_k(a_i^k, a_j^k|x_i^k, x_j^k) = \mathrm{tr}\Big(E_{a_i^k|x_i^k} \otimes F_{a_j^k|x_j^k}\rho_k\Big) \tag{3.82}$$

which violates the Bell inequality I^k, which is ≤ 0 for all local distributions, with value $\omega_P > 0$. Consider another network of the same topology, but with less-than-maximally entangled states σ_k and measurements $G_{a_i^k|x_i^k}$, $H_{a_j^k|x_j^k}$ (which also depend on k), giving rise to a distribution

$$Q_k(a_i^k, a_j^k | x_i^k, x_j^k) = \mathrm{tr}\Big(G_{a_i^k|x_i^k} \otimes H_{a_j^k|x_j^k} \sigma_k\Big) \tag{3.83}$$

at each edge k. Suppose Q_k violates I^k with value $\omega_Q > 0$. Then, for any $p \in (0,1)$, placing the mixed state $p\rho_k \oplus (1-p)\sigma_k$ and measurements $E_{a_i^k|x_i^k} \oplus G_{a_i^k|x_i^k}$, $F_{a_j^k|x_j^k} \oplus H_{a_j^k|x_j^k}$ at each edge, one gets the distribution

$$\begin{aligned} R_k(a_i^k, a_j^k | x_i^k, x_j^k) &= \mathrm{tr}\left((E_{a_i^k|x_i^k} \oplus G_{a_i^k|x_i^k}) \otimes (F_{a_j^k|x_j^k} \oplus H_{a_j^k|x_j^k})(p\rho_k \oplus (1-p)\sigma_k)\right) \\ &= pP_k(a_i^k, a_j^k | x_i^k, x_j^k) + (1-p)Q_k(a_i^k, a_j^k | x_i^k, x_j^k), \end{aligned} \tag{3.84}$$

which is also a quantum distribution. By convexity, the value of each R_k on inequality I^k is

$$I^k(R_k) = pI^k(P_k) + (1-p)I^k(Q_k) = p\omega_P + (1-p)\omega_Q > 0, \tag{3.85}$$

which constitutes a violation. Therefore, by Theorem 3.1, any connected network with the mixed states $p\rho_k \oplus (1-p)\sigma_k$ at each edge is GMNL. □

However, this construction cannot be used to show GMNL in networks of states that are a mixture of an entangled state with separable noise. In fact, in such networks, not even GME is guaranteed in principle. In Chapter 4 we will study networks of mixed states, and find that even their entanglement properties depend on their topology, as well as on the level of noise contained in the states.

Chapter 4

Mixed pair-entangled network states

Pair-entangled networks of mixed states are widely studied as an experimentally feasible way of achieving genuine multipartite quantum effects. Moreover, they provide a good platform in which to explore the relationship between entanglement and nonlocality in many-body systems. We focus on networks where pairs of parties share isotropic states. First, we provide bounds on the noise parameter needed to guarantee biseparability or GME in tripartite networks. Next, we obtain no-go results which show that tree networks and polygonal networks cannot be GME if the number of parties is large enough. In sharp contrast, completely connected networks are always GME if the visibility of the shared states is above a threshold. Still, GME in any connected network of entangled isotropic states can be recovered by taking many copies. In addition, we find that sharing non-steerable states can compromise the GMNL of a network, or even render it fully local. This leads us to provide constructions of networks that are GME but not GMNL. However, these limitations to the obtention of nonlocality can be overcome: surprisingly, taking many copies of some bilocal networks make it possible to restore the GMNL. Thus, genuine multipartite effects can be obtained from some networks if enough copies of the pair-entangled states are available. This result constitutes, to our knowledge, the first example of superactivation of GMNL from bilocality.

4.1 Entanglement in mixed-state networks

We will consider networks where pairs of parties share isotropic states, and analyse whether or not the network is biseparable. For notational convenience, unless otherwise specified we will group the Hilbert spaces in terms of the shared states (the edges of the network), although the bipartitions considered for biseparability will always refer to parties (the vertices, or nodes, of the network). So, for example, a Lambda network where Alice shares isotropic states with each of Bob1 and Bob2 will be denoted as

$$\rho_{p_1,p_2}^{\otimes 2} = \rho_{p_1,A_1B} \otimes \rho_{p_2,A_2C}, \tag{4.1}$$

with or without the subscripts referring to parties, where the isotropic state ρ_p is given in equation (1.42).

We will also use the flip, or swap, operator,

$$\Pi = \sum_{i,j=0}^{d-1} |ij\rangle \langle ji| , \tag{4.2}$$

in any dimension d, which swaps the states of the particles in a bipartite system.

We begin by exploring the tripartite setting, where the only two connected networks are a Lambda network, where Alice shares bipartite states with Bob and Charlie, and a triangle network, where each pair of Alice, Bob and Charlie share bipartite states. We find that, in contrast to the case of pure states, where any amount of entanglement yields GME, networks of isotropic states are only GME if the visibility of the states on the edges is large enough. Remarkably, this implies that entanglement can be *deactivated* in mixed-state networks. In each case, we provide bounds on the visibilities needed to achieve GME or biseparability.

For tripartite networks, we denote the parties as A, B, C, with subindices wherever a party holds more than one particle. These results were motivated by the numerical techniques in Ref. [JMG11].

Theorem 4.1. *A Lambda network where Alice shares a 2-dimensional isotropic state ρ_{p_1} with Bob, and another ρ_{p_2} with Charlie, is GME when $p_i > 1/3$, $p_j > 1/(3p_i)$, $i, j = 1, 2$.*

Proof. We will show that the operator

$$W = \mathbb{1} \otimes \mathbb{1} + 2\mathbb{1} \otimes \phi^+ + 2\phi^+ \otimes \mathbb{1} - 8\phi^+ \otimes \phi^+ \tag{4.3}$$

is a GME witness and detects $\rho_{p_1} \otimes \rho_{p_2}$ for the stated bounds.

To show W is a witness, it suffices to show that $\mathrm{tr}(W\rho) \geq 0$ for every ρ that is a PPT mixture. In turn, for this it is enough to see that there exist $P_M, Q_M \succcurlyeq 0$ such that $W = P_M + Q_M^{\Gamma_M}$ for $M = A, B, C$ [JMG11]. It is straightforward to verify that this is indeed the case (we only need to use that the partial transpose of the flip operator Π is twice the maximally entangled state: $\Pi^\Gamma = 2\phi^+, \phi^+ := |\phi^+\rangle\langle\phi^+|$) if:

$$P_A = 2\phi^+ \otimes (\mathbb{1} - \phi^+) + 2(\mathbb{1} - \phi^+) \otimes \phi^+, \tag{4.4}$$

$$Q_A = \frac{1}{2}[(\mathbb{1} - \Pi) \otimes (\mathbb{1} + \Pi) + (\mathbb{1} + \Pi) \otimes (\mathbb{1} - \Pi)], \tag{4.5}$$

$$P_B = 0, \quad Q_B = (\mathbb{1} + \Pi) \otimes (\mathbb{1} - \phi^+) + 3(\mathbb{1} - \Pi) \otimes \phi^+, \tag{4.6}$$

$$P_C = 0, \quad Q_C = (\mathbb{1} - \phi^+) \otimes (\mathbb{1} + \Pi) + 3\phi^+ \otimes (\mathbb{1} - \Pi), \tag{4.7}$$

where we have used that $\phi^+, \mathbb{1} - \phi^+, \mathbb{1} \pm \Pi, \succcurlyeq 0$ and that the sum and tensor product of positive semidefinite matrices is positive semidefinite.

Next, we have

$$\mathrm{tr}(W \rho_{p_1} \otimes \rho_{p_2}) = \frac{3}{2}(1 - 3p_1p_2), \tag{4.8}$$

which is strictly smaller than zero whenever $p_1p_2 > 1/3$. Since we must also have $p_1, p_2 \leq 1$, the stated bounds follow. □

Theorem 4.2. *A Lambda network of d-dimensional isotropic states ρ_p is biseparable for $p \leq \left[(1 + \sqrt{2})d - 1\right]/(d^2 + 2d - 1)$.*

Proof. Consider a tripartite network where Alice and each of Bob and Charlie share an entangled ρ_p, i.e., let $p > 1/(d+1)$. The state of the network is

$$\begin{aligned}&\rho_{p,A_1B} \otimes \rho_{p,A_2C} \\ &\quad = p^2 \phi^+_{A_1B} \otimes \phi^+_{A_2C} + p(1-p)\phi^+_{A_1B} \otimes \tilde{\mathbb{1}}_{A_2C} + p(1-p)\tilde{\mathbb{1}}_{A_1B} \otimes \phi^+_{A_2C} + (1-p)^2 \tilde{\mathbb{1}}_{A_1B} \otimes \tilde{\mathbb{1}}_{A_2C},\end{aligned} \tag{4.9}$$

where $\tilde{\mathbb{1}} = \mathbb{1}/4$ is the normalised identity, and which can be rewritten as

$$\begin{aligned}\rho_{p,A_1B} \otimes \rho_{p,A_2C} &= (1-q)p^2 \phi^+_{A_1B} \otimes \phi^+_{A_2C} + (1-p)^2 \tilde{\mathbb{1}}_{A_1B} \otimes \tilde{\mathbb{1}}_{A_2C} \\ &+ \phi^+_{A_1B} \otimes \left(\frac{qp^2}{2}\phi^+_{A_2C} + p(1-p)\tilde{\mathbb{1}}_{A_2C}\right) + \left(\frac{qp^2}{2}\phi^+_{A_1B} + p(1-p)\tilde{\mathbb{1}}_{A_1B}\right) \otimes \phi^+_{A_2C}\end{aligned} \tag{4.10}$$

for any $q \in [0, 1]$. Now, the first line can be seen as an (unnormalised) isotropic state in parties $A|BC$ (with dimension d^2), while the second line contains isotropic states in $A_2|C$ and $A_1|B$ respectively (with dimension d). Showing that each of these states is separable will entail the result.

Denote these isotropic states by $\sigma_0, \sigma_1, \sigma_2$ in the order that they appear in equation (4.10). Normalising σ_0, we find

$$\sigma_0 = \frac{(1-q)p^2\phi^+_{A_1B} \otimes \phi^+_{A_2C} + (1-p)^2\tilde{\mathbb{1}}_{A_1B} \otimes \tilde{\mathbb{1}}_{A_2C}}{(1-q)p^2 + (1-p)^2}, \tag{4.11}$$

meaning it is separable in $A|BC$ if

$$\frac{(1-q)p^2}{(1-q)p^2 + (1-p)^2} \leq \frac{1}{d^2+1}, \tag{4.12}$$

i.e., if

$$q \geq 1 - \frac{(1-p)^2}{p^2d^2}. \tag{4.13}$$

Normalising σ_1, we obtain

$$\sigma_1 = \frac{qp^2\phi^+_{A_2C} + 2p(1-p)\tilde{\mathbb{1}}_{A_2C}}{qp^2 + 2p(1-p)}, \tag{4.14}$$

which is separable if

$$\frac{qp^2}{qp^2 + 2p(1-p)} \leq \frac{1}{d+1}. \tag{4.15}$$

Simplifying, this entails that

$$q \leq \frac{2-2p}{pd}. \tag{4.16}$$

Reasoning symmetically, the separability of σ_2 gives the same bound.

Both bounds on q together entail that

$$1 - \frac{(1-p)^2}{p^2d^2} \leq \frac{2-2p}{pd} \tag{4.17}$$

and, solving for p, we find that $\rho_{p,A_1B} \otimes \rho_{p,A_2C}$ is biseparable for

$$p \leq \frac{(1+\sqrt{2})d - 1}{d^2 + 2d - 1}. \tag{4.18}$$

In particular, for any d there exists p such that

$$\frac{1}{d+1} < p \leq \frac{(1+\sqrt{2})d - 1}{d^2 + 2d - 1}, \tag{4.19}$$

showing that entanglement can be deactivated in a Lambda network. □

In the particular case where the states on the edges of a Lambda network are the same, $\rho_{p_1} = \rho_{p_2} = \rho_p$, and for $d = 2$, Theorem 4.1 implies that the network is GME for $p > 1/\sqrt{3} \simeq 0.577$, while Theorem 4.2 implies that the network is biseparable for $p \leq (1+2\sqrt{2})/7 \simeq 0.547$. Adding an extra edge to the network so that it forms a triangle makes it possible to achieve GME with less entanglement on the edges, as we now show.

Theorem 4.3. *A triangle network of 2-dimensional states ρ_p is GME for $p > (2\sqrt{5} - 3)/3 \simeq 0.491$.*

Proof. We will show the operator

$$\begin{aligned} W =& \mathbb{1} \otimes \mathbb{1} \otimes \phi^+ + \mathbb{1} \otimes \phi^+ \otimes \mathbb{1} + \phi^+ \otimes \mathbb{1} \otimes \mathbb{1} \\ & - \mathbb{1} \otimes \phi^+ \otimes \phi^+ - \phi^+ \otimes \phi^+ \otimes \mathbb{1} - \phi^+ \otimes \mathbb{1} \otimes \phi^+ - 3\phi^+ \otimes \phi^+ \otimes \phi^+, \end{aligned} \tag{4.20}$$

(where the Hilbert spaces are ordered as $A_1B_1A_2C_1B_2C_2$) is a witness and detects the triangle with ρ_p at each edge, for all $p > (2\sqrt{5} - 3)/3$. To show W is a witness, it is sufficient to show that it can be decomposed as

$$W = P_M + Q_M^{\Gamma_M} \tag{4.21}$$

for each bipartition $M = A, B, C$, where $P_M, Q_M \succcurlyeq 0$ for all M. Indeed, we have

$$\begin{aligned} P_A &= \mathbb{1} \otimes \phi^+ \otimes (\mathbb{1} - \phi^+) + \phi^+ \otimes (\mathbb{1} - \phi^+) \otimes (\mathbb{1} - \phi^+) \\ Q_A &= \frac{1}{2}[(\mathbb{1} - \Pi) \otimes (\mathbb{1} + \Pi) + (\mathbb{1} + \Pi) \otimes (\mathbb{1} - \Pi)] \otimes \phi^+ \\ P_B &= \mathbb{1} \otimes (\mathbb{1} - \phi^+) \otimes \phi^+ + \phi^+ \otimes (\mathbb{1} - \phi^+) \otimes (\mathbb{1} - \phi^+) \\ Q_B &= \frac{1}{2}[(\mathbb{1} - \Pi) \otimes (\mathbb{1} + \Pi) + (\mathbb{1} + \Pi) \otimes (\mathbb{1} - \Pi)]_{A_1B_1B_2C_2} \otimes \phi^+_{A_2C_1} \\ P_C &= (\mathbb{1} - \phi^+) \otimes \phi^+ \otimes \mathbb{1} + (\mathbb{1} - \phi^+) \otimes (\mathbb{1} - \phi^+) \otimes \phi^+ \\ Q_C &= \phi^+ \otimes \frac{1}{2}[(\mathbb{1} - \Pi) \otimes (\mathbb{1} + \Pi) + (\mathbb{1} + \Pi) \otimes (\mathbb{1} - \Pi)], \end{aligned} \tag{4.22}$$

where the Hilbert spaces of all operators are ordered as $A_1B_1A_2C_1B_2C_2$, except Q_B, and where we use that $\mathbb{1}, \phi^+, \mathbb{1} - \phi^+, \mathbb{1} \pm \Pi \succcurlyeq 0$.

Then,

$$\operatorname{tr}\big(W\rho_p^{\otimes 3}\big) = \frac{3}{64}(11 + 15p - 63p^2 - 27p^3), \tag{4.23}$$

which is strictly smaller than zero when $p > (2\sqrt{5} - 3)/3$. □

Since the visibilities required for biseparability or GME are different in the case of a

Lambda network and a triangle network, we find that, unlike in the case of pure states, topology does influence the entanglement of mixed-state networks. Indeed, in the case of $d = 2$, a Lambda network where $p_1 = p_2 =: p$ is biseparable for $p \leq (1+2\sqrt{2})/7 \simeq 0.547$ (by Theorem 4.2), while a triangle network is GME for $p > (2\sqrt{5}-3)/3 \simeq 0.491$ (by Theorem 4.3). This means that adding extra connections in a network has an impact on its entanglement. If, instead, we hold the number of shared states fixed, but distribute them in networks of different numbers of parties (as long as the networks are kept connected), the presence of cycles makes it possible to achieve GME with less visibility on the edges. Indeed, adding an extra party to a Lambda network such that the one in Theorem 4.2, so that AB, BC and CD each share an isotropic state, does not change the biseparability bound. This is because a biseparable state can be decomposed into states that are separable along a given bipartition. So the biseparable decomposition of the Lambda network works also for the network with an added party, substituting C for CD in each term.

In a similar pattern, a triangle network is biseparable for a smaller range of visibilities than the triangle network, as we now show.

Theorem 4.4. *A triangle network with a d-dimensional isotropic state ρ_p at each edge is biseparable for $p \leq 3/(3+2d)$.*

Proof. We show that the state of the triangle network can be decomposed into four matrices, three of which are separable along one bipartition each, and the fourth of which is fully separable. Let $\rho_p^{\otimes 3}$ be the state of the network. Then,

$$\rho_p^{\otimes 3} = \frac{p(p^2-3p+3)}{3}(\sigma_1+\sigma_2+\sigma_3) + (1-p)^3\tilde{\mathbb{1}}\otimes\tilde{\mathbb{1}}\otimes\tilde{\mathbb{1}}, \tag{4.24}$$

where

$$\begin{aligned}
&\sigma_1 = \\
&\frac{p^2\phi^+\otimes\phi^+\otimes\phi^+ + (3p(1-p)/2)\left(\phi^+\otimes\tilde{\mathbb{1}}\otimes\phi^+ + \tilde{\mathbb{1}}\otimes\phi^+\otimes\phi^+\right) + 3(1-p)^2\tilde{\mathbb{1}}\otimes\tilde{\mathbb{1}}\otimes\phi^+}{p^2-3p+3} \\
&\sigma_2 = \\
&\frac{p^2\phi^+\otimes\phi^+\otimes\phi^+ + (3p(1-p)/2)\left(\phi^+\otimes\phi^+\otimes\tilde{\mathbb{1}} + \tilde{\mathbb{1}}\otimes\phi^+\otimes\phi^+\right) + 3(1-p)^2\tilde{\mathbb{1}}\otimes\phi^+\otimes\tilde{\mathbb{1}}}{p^2-3p+3} \\
&\sigma_3 = \\
&\frac{p^2\phi^+\otimes\phi^+\otimes\phi^+ + (3p(1-p)/2)\left(\phi^+\otimes\tilde{\mathbb{1}}\otimes\phi^+ + \phi^+\otimes\phi^+\otimes\tilde{\mathbb{1}}\right) + 3(1-p)^2\phi^+\otimes\tilde{\mathbb{1}}\otimes\tilde{\mathbb{1}}}{p^2-3p+3}.
\end{aligned} \tag{4.25}$$

Clearly, the matrix $\tilde{\mathbb{1}} \otimes \tilde{\mathbb{1}} \otimes \tilde{\mathbb{1}}$ is fully separable. We will show that σ_1 is separable in $A|BC$, and separability of σ_2 and σ_3 in $B|AC$ and $C|AB$ respectively will follow by symmetry. We have that

$$\sigma_1 = \tau_1 \otimes \phi^+_{B_2C_2}, \tag{4.26}$$

where showing that

$$\begin{aligned}\tau_1 = &\\ &\frac{p^2\phi^+_{A_1B_1} \otimes \phi^+_{A_2C_1} + (3p(1-p)/2)\,(\phi^+_{A_1B_1} \otimes \tilde{\mathbb{1}}_{A_2C_1} + \tilde{\mathbb{1}}_{A_1B_1} \otimes \phi^+_{A_2C_1})}{p^2-3p+3} \\ &+ \frac{3(1-p)^2\tilde{\mathbb{1}}_{A_1B_1} \otimes \tilde{\mathbb{1}}_{A_2C_1}}{p^2-3p+3}\end{aligned} \tag{4.27}$$

is separable in $A_1A_2|B_1C_1$ is sufficient to show that σ_1 is separable in $A|BC$. Indeed, we can write

$$\begin{aligned}\tau_1 =& \frac{(3-p)^2}{4(3-3p+p^2)}\left(\frac{2p}{3-p}\phi^+_{A_1B_1} + \frac{3(1-p)}{3-p}\tilde{\mathbb{1}}_{A_1B_1}\right) \otimes \left(\frac{2p}{3-p}\phi^+_{A_2C_1} + \frac{3(1-p)}{3-p}\tilde{\mathbb{1}}_{A_2C_1}\right) \\ &+ \frac{3(1-p)^2}{4(3-3p+p^2)}\tilde{\mathbb{1}}_{A_1B_1} \otimes \tilde{\mathbb{1}}_{A_2C_1}.\end{aligned} \tag{4.28}$$

The isotropic state

$$\frac{2p}{3-p}\phi^+_{A_1B_1} + \frac{3(1-p)}{3-p}\tilde{\mathbb{1}}_{A_1B_1} \tag{4.29}$$

is separable whenever $2p/(3-p) \leq 1/(d+1)$, i.e., $p \leq 3/(3+2d)$, therefore τ_1 is fully separable in $A_1|A_2|B_1|C_1$ which guarantees the required separability of σ_1. Reasoning symmetrically, separability of σ_2 in $B|AC$ and of σ_3 in $C|AB$ follows for the same values of p, hence $\rho_p^{\otimes 3}$ is biseparable for the stated bounds. □

The results of Theorems 4.1-4.4 are summarised in Table 4.1.

	biseparable for $p \leq$	GME for $p >$
Λ	$(1+2\sqrt{2})/7 \simeq 0.547$	$1/\sqrt{3} \simeq 0.577$
$\triangle$	$3/7 \simeq 0.429$	$(2\sqrt{5}-3)/3 \simeq 0.491$

Table 4.1: Bounds for biseparability and GME in a Lambda network (Λ) and a triangle network ($\triangle$) where the shared states are isotropic states with visibility p in dimension 2.

We now explore the generalisation of these results to the case of larger networks, where we find the dependency on the topology is most extreme. Tree networks are those which contain no cycles, and are thus the networks with fewest edges, $n-1$, for a fixed number of parties n. Polygonal networks, i.e., networks in the form of a closed chain, have only one more edge. We find that, if the number of parties is large enough, distributing isotropic states on either of these networks renders them biseparable no matter the visibility (as long as it is < 1). In sharp contrast, if the network is completely connected (i.e., every pair of parties shares an isotropic state), it remains GME for any number of parties for all visibilities above a threshold.

In what follows, notation for parties will depend on the geometry of the network. In general, parties will be denoted by A_i, but we will also use B_i whenever parties can be naturally divided into two types (for example, in the case of a star network, with the central node vs all others).

Theorem 4.5. *For any $p \in [0, 1)$, there exists $K \in \mathbb{N}$ such that distributing an isotropic state with visibility p in any tree network of K edges yields a biseparable state.*

Proof. The main idea is to write the state of the network as a tensor product of the state of the edges, and observe that, since each bipartition is crossed by exactly one edge, all terms with $\tilde{\mathbb{1}}$ on at least one edge are biseparable. Then, the only GME term is the one where all edges contain ϕ_d^+. Distributing this term among the terms with exactly one $\tilde{\mathbb{1}}$ gives terms with ϕ_d^+ on all but one edge, and a mixture of $\tilde{\mathbb{1}}$ and ϕ_d^+ on the remaining edge where the weight of $\tilde{\mathbb{1}}$ is inversely proportional to the number of edges. Thus, for a sufficiently large number of edges, this term can be made biseparable too, proving the result.

Consider a network in the form of a tree graph where each edge has a copy of the state ρ_p. Let K be the number of edges in the network, where the edges are indexed by i, and denote the state of the network as $\rho_p^{\otimes K}$. Expanding the tensor product, we find

$$\rho_p^{\otimes K} = p^K \bigotimes_{i=1}^{K} \phi_{d,i}^+ + p^{K-1}(1-p) \sum_{i=1}^{K} \tilde{\mathbb{1}}_i \otimes \bigotimes_{j \neq i} \phi_{d,j}^+ + \dots , \tag{4.30}$$

where the omitted terms are all separable along at least one bipartition, since each bipartition is crossed by exactly one edge and at least one edge in each term contains $\tilde{\mathbb{1}}$. Showing that the above expression is biseparable for some K is sufficient to prove the claim. But we can rewrite the above as

$$\begin{aligned}\rho_p^{\otimes K} &= \frac{p^K}{K}\sum_{i=1}^{K}\bigotimes_{j=1}^{K}\phi_{d,j}^+ + p^{K-1}(1-p)\sum_{i=1}^{K}\tilde{\mathbb{1}}_i \otimes \bigotimes_{j\neq i}\phi_{d,j}^+ + \dots \\ &= \sum_{i=1}^{K}\left(\frac{p^K}{K}\phi_{d,i}^+ + p^{K-1}(1-p)\tilde{\mathbb{1}}_i\right)\otimes\bigotimes_{j\neq i}\phi_{d,j}^+ + \dots .\end{aligned} \tag{4.31}$$

Here, each bracket has ϕ_d^+ on $K-1$ edges, and the state

$$\frac{(p^K/K)\phi_d^+ + p^{(K-1)}(1-p)\tilde{\mathbb{1}}}{p^K/K + p^{(K-1)}(1-p)} = \frac{(p/K)\phi_d^+ + (1-p)\tilde{\mathbb{1}}}{p/K + 1 - p} \tag{4.32}$$

on the rest. But this is an isotropic state with visibility $(p/K)/(p/K+1-p)$, which is thus guaranteed to become separable when the visibility is smaller than or equal to $1/(d+1)$. For fixed p, this can be achieved by choosing $K \geq dp/(1-p)$. This bound, however, is not optimal, as lower K could be achieved by distributing the term $\bigotimes_{i=1}^{K}\phi_{d,i}^+$ among some or all of the omitted terms in equation (4.30) as well, as in the proof of Theorem 4.2. □

In fact, Theorem 4.5 holds for a more general class of states, namely, convex mixtures of an entangled state and a separable state that is not on the boundary of the set of separable states, since the proof also holds for such states.

Theorem 4.6. *For any $p \in [0,1)$, there exists $K \in \mathbb{N}$ such that distributing an isotropic state with visibility p in a polygonal network of K edges yields a biseparable state.*

Proof. The proof is very similar to that of Theorem 4.5. This time, terms with $\tilde{\mathbb{1}}$ on only one edge are not biseparable, but those with $\tilde{\mathbb{1}}$ on two or more edges are. Therefore, we distribute the term where all edges contain ϕ_d^+ among the terms where two edges contain $\tilde{\mathbb{1}}$, and the terms with one $\tilde{\mathbb{1}}$ among the terms with three or more. Like in the case of a tree network, the weight of ϕ_d^+ in each case decreases with the number of edges in the network, proving the claim.

Consider a polygonal network where each edge has a copy of the state $\rho_p = p\phi_d^+ + (1-p)\tilde{\mathbb{1}}$. Let A_i denote the parties, for each $i \in [K]$. Let i also index the edge to the right of party A_i, as well as the state on that edge. Denote the state of the network as

$\rho_p^{\otimes K}$. Expanding the tensor product, we find

$$\rho_p^{\otimes K} = p^K \bigotimes_{i=1}^{K} \phi_{d,i}^+ + p^{K-1}(1-p) \sum_{i=1}^{K} \tilde{\mathbb{1}}_i \otimes \bigotimes_{j\neq i} \phi_{d,j}^+ + p^{K-2}(1-p)^2 \sum_{\substack{i,j=1 \\ i\neq j}}^{K} \tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_j \otimes \bigotimes_{k\neq i,j} \phi_{d,k}^+ + \dots , \tag{4.33}$$

where all terms with two or more edges containing $\tilde{\mathbb{1}}$ are separable along at least one bipartition, since each bipartition of the polygon is crossed by exactly two edges. We will show that the terms where fewer than two edges contain $\tilde{\mathbb{1}}$ can be paired with separable terms in order to write $\rho_p^{\otimes K}$ as a convex mixture of separable states. Out of the terms containing $\tilde{\mathbb{1}}$ on two edges, there are K such terms containing $\tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_{i+1}$ (identifying $K+1 \equiv 1$) for some $i \in K$, i.e., where $\tilde{\mathbb{1}}$ lies on two adjacent edges. This means that the term is separable in $A_{i+1}|\{A_j\}_{j\neq i+1}$. Pairing these K terms with $p^K \bigotimes_{i=1}^{K} \phi_{d,i}^+$, we can write a fragment of $\rho_p^{\otimes K}$ as

$$\begin{aligned} &p^K \bigotimes_{i=1}^{K} \phi_{d,i}^+ + p^{K-2}(1-p)^2 \left(\sum_{i=1}^{K} \tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_{i+1} \otimes \bigotimes_{j\neq i,i+1} \phi_{d,j}^+ \right) \\ &= p^{K-2} \sum_{i=1}^{K} \left[\left(\frac{p^2}{K} \phi_{d,i}^+ \otimes \phi_{d,i+1}^+ + (1-p)^2 \tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_{i+1} \right) \otimes \bigotimes_{j\neq i,i+1} \phi_{d,j}^+ \right] . \end{aligned} \tag{4.34}$$

If, for each i, the state

$$\frac{p^2}{K} \phi_{d,i}^+ \otimes \phi_{d,i+1}^+ + (1-p)^2 \tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_{i+1}, \tag{4.35}$$

once normalised, is separable in $A_{i+1}|\{A_j\}_{j\neq i+1}$, then the fragment of $\rho_p^{\otimes K}$ in equation (4.34) will be biseparable.

Now, the state in equation (4.35) is a convex mixture of $\phi_{d,i}^+ \otimes \phi_{d,i+1}^+$ and $\tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_{i+1}$. Therefore, the state (4.35) is guaranteed to become separable when $(1-p)^2/(p^2/K + (1-p)^2)$ is close enough to 1. For fixed p, this can be achieved by choosing a large enough K.

Using a similar strategy, the terms containing $\tilde{\mathbb{1}}$ on one edge can be combined with some of those containing $\tilde{\mathbb{1}}$ on three appropriately chosen edges. Thus, assuming that

$K > 4$, another fragment of $\rho_p^{\otimes K}$ can be written as

$$p^{K-1}(1-p)\tilde{\mathbb{1}}_i \otimes \bigotimes_{j\neq i} \phi^+_{d,j} + p^{K-3}(1-p)^3 \sum_{\substack{j\neq i,i\pm 1,\\ i-2}} \tilde{\mathbb{1}}_i \otimes \tilde{\mathbb{1}}_j \otimes \tilde{\mathbb{1}}_{j+1} \otimes \bigotimes_{k\neq i,j,j+1} \phi^+_{d,k}$$

$$= p^{K-3}(1-p) \sum_{\substack{j\neq i,i\pm 1,\\ i-2}} \left[\tilde{\mathbb{1}}_i \otimes \bigotimes_{k\neq i,j,j+1} \phi^+_{d,k} \otimes \left(\frac{p^2}{K-4} \phi^+_{d,j} \otimes \phi^+_{d,j+1} + (1-p)^2 \tilde{\mathbb{1}}_j \otimes \tilde{\mathbb{1}}_{j+1} \right) \right]. \tag{4.36}$$

Hence, it is sufficient to show that the state

$$\frac{p^2}{K-4} \phi^+_{d,j} \otimes \phi^+_{d,j+1} + (1-p)^2 \tilde{\mathbb{1}}_j \otimes \tilde{\mathbb{1}}_{j+1}, \tag{4.37}$$

once normalised, is separable in $A_{j+1}|\{A_k\}_{k\neq j+1}$ to deduce that the fragment in equation (4.36) is separable. Again, for fixed p, this is guaranteed for large enough K. Since every term that does not appear in fragments (4.34) and (4.36) is already biseparable, the claim follows. $\square$

At the other extreme of connected networks, we explore completely connected networks, which are those where every pair of parties shares a bipartite state. It is to be expected that their genuine multipartite entanglement is more robust to noise than in the case of tree or polygonal networks. In fact, the contrast with tree and polygonal networks is as sharp as it can be: we find that a completely connected network of isotropic states remains GME for any number of parties for all visibilities above a threshold. We first show the result for all visibilities above a threshold if the number of parties is sufficiently large. As a consequence, we show that, by considering high enough visibilities, GME can be achieved for completely connected networks of any number of edges.

Theorem 4.7. *There exists $p_1 < 1$ such that, for all $n \in \mathbb{N}$, a completely connected network of n parties where each pair of parties shares an isotropic state ρ_p of any dimension d is GME for all $p_1 < p \leq 1$.*

The proof strategy is to show that, in the limit of large n, a completely connected network of n parties where each pair of parties shares an isotropic state ρ_p can be used to distill maximally entangled states between any pair of parties with a fidelity unachievable by any biseparable state of n parties.

We first consider an LOCC protocol acting on a biseparable state of n parties, which distills maximally entangled states between any given pair of parties. In Lemma 4.1, we find an upper bound for the fidelity of any such protocol, added over all distinct pairs of parties. Next, to prove the Theorem, we consider a specific LOCC protocol that uses the state of the network to distill maximally entangled states between any pair of parties, and lower bound its fidelity added over all pairs of parties. We find the bound corresponding to the state of the network is strictly larger than that of any biseparable state, proving the claim in the limit of large n. Once GME is ensured to persist for a large number of parties, it follows that, for a fixed, large enough visibility, GME can be guaranteed for completely connected networks of any size.

Lemma 4.1. *Consider an LOCC transformation that maps an n-partite biseparable state of any dimension d to a 2-qubit maximally entangled state shared by any two parties. The achievable fidelity of this transformation, added over all distinct pairs of parties, is bounded above by $(n-1)^2/2$.*

Proof. Let χ be an n-partite biseparable state. Consider an LOCC protocol Λ_{ij} : $\mathcal{B}(\bigotimes_{i=1}^n \mathcal{H}_i) \to \mathcal{B}(\mathcal{H}_i) \otimes \mathcal{B}(\mathcal{H}_j)$ that maps n-partite states to bipartite states shared between parties A_i, A_j, where $i < j \in [n]$. Since χ is biseparable, we have

$$\chi = \sum_M p_M \chi_M, \tag{4.38}$$

where each χ_M is separable across the bipartition $M|\overline{M}$ and $\sum_M p_M = 1$. Let M_{ij} be the set of bipartitions that split parties A_i and A_j. Then, we can write the evolution of χ under the protocol Λ_{ij} as

$$\Lambda_{ij}(\chi) = \sum_{M \notin M_{ij}} p_M \tau_M(i,j) + \sum_{M \in M_{ij}} p_M \sigma_M(i,j), \tag{4.39}$$

where $\tau_M(i,j)$ and $\sigma_M(i,j)$ are bipartite states of parties A_i, A_j. The state $\tau_M(i,j)$ is in principle unrestricted, so it can have up to unit fidelity with the 2-qubit maximally entangled state ϕ_{ij}^+. However, since LOCC operations cannot create entanglement, $\sigma_M(i,j)$ must be separable, therefore its fidelity with ϕ_{ij}^+ cannot be larger than 1/2. Therefore, for any LOCC protocol Λ_{ij}, the fidelity F of $\Lambda_{ij}(\chi)$ with ϕ_{ij}^+ is bounded above:

$$F(\Lambda_{ij}(\chi), \phi_{ij}^+) \leq \sum_{M \notin M_{ij}} p_M + \frac{1}{2} \sum_{M \in M_{ij}} p_M = 1 - \frac{1}{2} \sum_{M \in M_{ij}} p_M. \tag{4.40}$$

Summing over all parties $i < j$, we obtain

$$\sum_{i<j} F(\Lambda_{ij}(\chi), \phi_{ij}^+) \leq \frac{n(n-1)}{2} - \frac{n-1}{2} = \frac{(n-1)^2}{2}. \tag{4.41}$$

The first term is the number of distinct pairs $i < j$, while the second comes from the following observation: the sums over $i < j$ and $M \in M_{ij}$ run through all bipartitions M, more than once. In fact, the number of times each bipartition is counted is equal to the number of times each bipartition is crossed by the edge connecting i and j. In turn, this number is equal to the number of edges crossing each bipartition. A bipartition splitting k parties from the remaining $(n-k)$ is crossed by $k(n-k)$ edges, which is smallest when $k = 1$. That is, each bipartition $M \in M_{ij}$ appears at least $(n-1)$ times, and so the sum $\sum_M p_M$, running over *all* M, appears at least $(n-1)$ times. That is,

$$\sum_{i<j} \sum_{M \in M_{ij}} p_M \geq n - 1. \tag{4.42}$$

□

Proof of Theorem 4.7. Let ρ be the state of the complete graph of n parties, with isotropic states ρ_p on each edge. In the complete graph, for all i, j, each party $A_k, k \neq i, j$ shares a copy of ρ_p with party A_i, which we denote $\rho_p(i,k)$, and another with party A_j, denoted $\rho_p(j,k)$. The protocol starts with each A_k, $k \neq i, j$, teleporting their half of $\rho_p(i,k)$ to A_j by using the channel $\rho_p(j,k)$. This is a noisy version of the standard teleportation protocol [BBC+93]. The teleported state will be a mixture of four terms, namely the four combinations of teleporting half of a maximally entangled or maximally mixed state along a maximally entangled or a maximally mixed channel. The first term, with weight p^2, will give a maximally entangled state, the other three turn out to give a maximally mixed state (as can be simply checked by performing the calculations in the standard protocol, replacing the teleported state and/or the channel by an identity in each case). Therefore, the teleportation protocol yields

$$\rho_{p^2}(i,j) = p^2 \phi_d^+ + (1 - p^2)\tilde{\mathbb{1}}. \tag{4.43}$$

In fact, parties A_i, A_j end up sharing $(n-2)$ copies of this state, one coming from each party $A_k, k \neq i, j$. Parties A_i, A_j can now apply a distillation protocol D_{ij} to obtain something close to a maximally entangled state, whose fidelity approaches 1 in the limit of large n. More specifically,

$$F(\Lambda_{ij}(\rho), \phi_{ij}^+) = F(D_{ij}(\rho_{p^2}^{\otimes(n-2)}), \phi_{ij}^+) \geq 1 - \varepsilon_n, \tag{4.44}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. To compare to equation (4.41) in Lemma 4.1, there only remains to show that $\varepsilon_n \to 0$ sufficiently fast as n grows.

Ref. [BDSW96] showed that a one-way distillation protocol acting on isotropic states and having rate R is equivalent to a quantum error-correcting code on a depolarising channel with the same rate R. In turn, Ref. [Ham02] proved a lower bound on the fidelity F of a d-dimensional quantum error-correcting code of rate R acting on a certain class of memoryless channels, which, in particular, include depolarising channels. After n uses of the channel, the lower bound[1] is

$$F \geq 1 - 2(n+1)^{2(d^2-1)} d^{-nE}. \tag{4.45}$$

Here, E is a function of the rate R and the noise parameter of the depolarising channel, which, in turn, corresponds to a function of the noise parameter p of the isotropic state. It holds that $E > 0$ when the rate is strictly below the maximum achievable rate of the channel. By the correspondence with distillation, this entails that ε_n in equation (4.44) goes to zero exponentially fast for one-way distillable isotropic states if the rate is suboptimal. Since, for our protocol Λ_{ij}, we are only interested in obtaining one copy of ϕ^+, we can achieve this exponential decay. Therefore,

$$\sum_{i<j} F(\Lambda_{ij}(\rho), \phi^+_{ij}) \geq \frac{n(n-1)}{2}(1 - \varepsilon_n), \tag{4.46}$$

which, by Lemma 4.1, is strictly larger than the fidelity achievable by any biseparable state, if n is large enough.

It is now possible to bound the visibility p of the isotropic states for which the statement holds if n is large enough. The one-way distillation protocol [DW05] requires that the states η to be distilled are such that

$$H(\mathrm{tr}_A(\eta)) - H(\eta) > 0, \tag{4.47}$$

where $H(\cdot)$ is the von Neumann entropy. It can be seen that isotropic states $\eta = \rho_{p^2}$ of any dimension satisfy this inequality if p is large enough.

In the particular case of $d = 2$, a bound for p can be calculated explicitly. Equation (4.47) reduces to

$$\frac{3(1-p^2)}{4} \log_2(1-p^2) + \frac{1+3p^2}{4} \log_2(1+3p^2) > 1, \tag{4.48}$$

which holds when $p_0 < p \leq 1$ for $p_0 \simeq 0.865$.

[1] Notice that the definition of the fidelity F used in Ref. [Ham02] is the square root of the one used in this work (see equation (1.30), hence the factor of 2 appears here in front of $(n+1)$.

Therefore, there exists $n_0 \in \mathbb{N}$ such that the statement is true for all $n \geq n_0$, for $p > p_0$. Now, the claim must hold as well for all $n < n_0$ because, for every fixed such n, the complete graph with $p = 1$ is GME (since, in this case, the edges are maximally entangled) and, since the set of GME states is open, the state must remain GME for all $p > p^*(n)$, where $p^*(n) < 1$. Thus, the statement of the Theorem holds by picking

$$p_1 = \max\{p_0, \max_{n<n_0} p^*(n)\}. \tag{4.49}$$

□

In fact, p_1 can be estimated as a function of n_0. Consider a network of 2-dimensional isotropic states. Consider also an LOCC protocol described by maps Λ_{ij} (like in Theorem 4.7) acting on the network state ρ, which traces out all particles except those corresponding to the isotropic state ρ_p shared by parties A_i, A_j. Then, the fidelity with the maximally entangled state is

$$F(\Lambda_{ij}(\rho), \phi_{ij}^+) = \frac{1+3p}{4} \tag{4.50}$$

for all $i < j$. Comparing to the biseparability bound in equation (4.41), the complete graph of n vertices is GME if $p > 1 - 4/3n$. Hence, taking $p_1 = 1 - 4/3n_0$ we have that the network is GME for all $n < n_0$ for all $p > p_1$.

The limitations to the obtention of GME from certain networks shown in Theorems 4.5 and 4.6 can be overcome by taking many copies of such networks. Indeed, exploiting the lack of closure under tensor products of the set of biseparable states, taking copies makes it possible to obtain GME from any connected network of entangled isotropic states, as we now show.

Theorem 4.8. *The state corresponding to many copies of any connected network where the nodes share arbitrary entangled isotropic states is GME if the number of copies is large enough.*

Proof. Many copies of the network in the statement are equivalent to a network where the nodes share many copies of arbitrary entangled isotropic states. Since all entangled isotropic states are distillable [HH99], this means that there exists a bipartite LOCC protocol that brings sufficiently many copies of an isotropic state as close as desired to at least one copy of a maximally entangled state. Performing these LOCC protocols for all edges of the network is in itself an LOCC protocol for the parties in the network. With this they can obtain a state arbitrarily close to a connected network of maximally entanged states, which is GME (this can be shown in many ways; for example, the results of Chapter 3 show that such a network is GMNL, which is a sufficient condition for GME) and, since the set of biseparable states is closed, the output state of this protocol can be GME as well. Since the set of biseparable states is closed under LOCC, this entails that the original network must also have been GME. □

4.2 Locality in mixed pair-entangled networks

The results above imply that, in contrast to networks of pure states, distributing nonlocal mixed states in any network does not necessarily lead to GME, let alone GMNL. Further, while GME is a necessary condition for GMNL, it is not sufficient, and there are known examples of states that are GME and bilocal [ADTA15, ADT18] or even fully local [BFF+16]. We find that networks provide a particularly good setting to find more examples of this kind, and non-steerability of the shared states can compromise the obtention of GMNL. We focus on star networks, and find that one non-steerable edge is enough to render the network bilocal while still being GME. Moreover, if all shared states in a star network are non-steerable, then the network is fully local. While steerability in these settings is necessary for GMNL, we show it is not sufficient: we provide an example of a steerable state which, when distributed in a star network, is bilocal. Still, GMNL can be recovered by taking many copies of these networks: we find, to our knowledge, the first example of superactivation of GMNL from bilocality.

Our bilocality result for a star network with one non-steerable edge actually applies to a slightly larger class of networks, as shown below:

Theorem 4.9. *Consider a network such that party A_1 is connected to the rest of the network only by one state which is non-steerable from A_1 to A_2. Then, the network is local across that bipartition. In particular, it is bilocal.*

Proof. We show that any probability distribution arising from local POVMs on the network state can be rewritten as a bipartite distribution arising from POVMs acting on the non-steerable state. Since this state is local, the bipartite distribution is local in the bipartition that the state crosses, which, by assumption, splits one party from the rest. The LHS model of the non-steerable state is then used to show that the term corresponding to the rest of the parties is nonsignalling, proving the Theorem.

We consider the network as a connected graph where vertices are parties and edges are states. We label the nodes as A_i for $i = 1, ..., n$, and the edges as $k = 1, ..., K$, where K is the number of edges in the graph. Thus, the Hilbert space corresponding to each node i is $\mathcal{H}_{A_i} = \bigotimes_k \mathcal{H}_{A_i^k}$, where the tensor product runs over all the edges incident to node i. On each edge k lies a quantum state ρ_k (the parties that edge k connects are left implicit). We will label the parties such that the non-steerable state lies on edge 1 (which connects parties A_1 and A_2) and we will call this state σ_1. We will show that the network is local across the bipartition $A_1|A_2...A_n$.

If each party A_i measures according to the POVM $\left\{E^i_{a_i|x_i}\right\}_{a_i}$ with outputs a_i and inputs x_i, they generate a probability distribution of the form

$$P(a_1...a_n|x_1...x_n) = \operatorname{tr}\left(\bigotimes_{i=1}^{n} E^i_{a_i|x_i} \bigotimes_{k\neq 1} \rho_k \otimes \sigma_1\right). \tag{4.51}$$

This distribution can be rewritten as

$$P(a_1...a_n|x_1...x_n) = \operatorname{tr}_{A_1A_2^1}\left[E^1_{a_1|x_1} \otimes \operatorname{tr}_{A_2^{k\neq 1}...A_n}\left[\bigotimes_{i=2}^{n} E^i_{a_i|x_i}\left(\mathbb{1}_{A_2^1} \otimes \bigotimes_{k\neq 1} \rho_k\right)\right]\sigma_1\right] \tag{4.52}$$

where we will show that

$$F_{a_2...a_n|x_2...x_n} := \operatorname{tr}_{A_2^{k\neq 1}...A_n}\left[\bigotimes_{i=2}^{n} E^i_{a_i|x_i}\left(\mathbb{1}_{A_2^1} \otimes \bigotimes_{k\neq 1} \rho_k\right)\right] \tag{4.53}$$

is a POVM element acting on $\mathcal{H}_{A_2^1}$. By denoting $A_2^1 =: A$, $A_2^{k\neq 1}...A_n =: B$, $a_2...a_n =: a$, $x_2...x_n =: x$, $\bigotimes_{i=2}^n E^i_{a_i|x_i} =: E^{AB}_{a|x}$, and $\bigotimes_{k\neq 1} \rho_k =: \tau_B$, $a_2...a_n =: a$, and $x_2...x_n =: x$, we can rewrite this as

$$F_{a|x} = \operatorname{tr}_B\left(E^{AB}_{a|x}(\mathbb{1}_A \otimes \tau_B)\right). \tag{4.54}$$

To show positivity, we notice that τ_B is a quantum state, and thus can be written as a convex combination of pure states $|\psi\rangle\langle\psi|$. Therefore, to show positivity of $F_{a|x}$ we can assume

$$F_{a|x} = \operatorname{tr}_B\left(E^{AB}_{a|x}(\mathbb{1}_A \otimes |\psi\rangle\langle\psi|)\right) \equiv \langle\psi| E^{AB}_{a|x} |\psi\rangle. \tag{4.55}$$

If $F_{a|x}$ were not positive, there would exist $|x\rangle \in \mathcal{H}_A$ such that

$$\langle x| F_{a|x} |x\rangle < 0, \tag{4.56}$$

which would imply that

$$\langle\psi| \langle x| E^{AB}_{a|x} |x\rangle |\psi\rangle < 0, \tag{4.57}$$

and hence that $E^{AB}_{a|x}$ would not be positive. But this is false, as $E^{AB}_{a|x}$ is a POVM element. Therefore, $F_{a|x} \succcurlyeq 0$.

Normalisation of $F_{a|x}$ is guaranteed by the normalisation of $E^{AB}_{a|x}$, as

$$\begin{aligned}\sum_a F_{a|x} &= \mathrm{tr}_B\left(\sum_a E^{AB}_{a|x}(\mathbb{1}_A \otimes \tau_B)\right) \\ &= \mathrm{tr}_B\left(\mathbb{1}_A \otimes \tau_B\right) = \mathbb{1}_A.\end{aligned} \tag{4.58}$$

Therefore, since equation (4.52) expresses the distribution achieved by the parties in the network as two POVM elements acting on the non-steerable state σ_1 (which is, in particular, local), the network is local across the bipartition $A_1|A_2...A_n$. Indeed, the distribution can be written as

$$P(a_1...a_n|x_1...x_n) = \sum_\lambda p_\lambda P_1(a_1|x_1\lambda)P_2(a_2...a_n|x_2...x_n\lambda). \tag{4.59}$$

This is sufficient to prove the claim according to Svetlichny's definition of bilocality [Sve87], where the probability distributions of the bipartition elements are not required to be nonsignalling. In fact, the proof so far applies to any local state σ_1 lying on any edge of the network. However, the operational definition that is used throughout this work requires, in addition, that the local components P_1 and P_2 be nonsignalling. To this end, we use that, since σ_1 is a non-steerable state, it has an LHS model [WJD07]. This means that any distribution arising from measuring σ_1 with the POVMs $E^1_{a_1|x_1}$, $F_{a_2...a_n|x_2...x_n}$(of the form of equation (4.53)) can be written as

$$P(a_1...a_n|x_1...x_n) = \sum_\lambda p_\lambda P_1(a_1|x_1\lambda)\,\mathrm{tr}\big(F_{a_2...a_n|x_2...x_n}\eta_\lambda\big). \tag{4.60}$$

Since P_2 is now of the form

$$P_2(a_2...a_n|x_2...x_n\lambda) = \mathrm{tr}_{A_2^{k\neq 1}...A_n}\left[\bigotimes_{i=2}^n E^i_{a_i|x_i}\left(\mathbb{1}_{A_2^1} \otimes \bigotimes_{k\neq 1}\rho_k\right)\Bigg)\eta_\lambda\right], \tag{4.61}$$

the normalisation of each $E^i_{a_i|x_i}$ ensures that this expression is nonsignalling. Also, P_1 is a single-party distribution, so the concept of signalling is mute on this side, and hence both distributions are nonsignalling, as required. $\square$

Theorem 4.9 can be used to construct a network which is GME and not GMNL for any number of parties, as we now show. The non-steerable state must be entangled, otherwise the network would be biseparable along the bipartition $A_1|A_2...A_n$. If, in addition, it is an isotropic state, and the network takes the form of a star where all other edges contain maximally entangled states, we obtain the desired construction:

Theorem 4.10. *A star network where Alice and Bob1 share an entangled isotropic state, and Alice and all other Bobs share maximally entangled states, is GME.*

Proof. We give a local operator that transforms the given state into a state which is GME, as detected by a witness. Since local operators cannot generate entanglement, this proves the Theorem.

Consider a star network with Alice in the central node and parties $B_1, ..., B_{n-1}$ in the rest. Let Alice and B_1 share an isotropic state

$$\rho_p = p\phi_d^+ + (1-p)\tilde{\mathbb{1}}, \tag{4.62}$$

where $\tilde{\mathbb{1}}$ is the normalised identity and ϕ_d^+ is the maximally entangled state in dimension d, with $p > 1/(d+1)$. Let Alice and each of $B_2, ..., B_n$ share a maximally entangled state (for ease of notation, for the remainder of this proof we omit the subscript d from ϕ_d^+). Then, the state of the network is

$$\begin{aligned} \tau &= \rho_{p,AB_1} \otimes \phi^+_{AB_2} \otimes \cdots \otimes \phi^+_{AB_{n-1}} \\ &= p\phi^+_{AB_1} \otimes \phi^+_{AB_2} \otimes \cdots \otimes \phi^+_{AB_{n-1}} + (1-p)\tilde{\mathbb{1}}_{AB_1} \otimes \phi^+_{AB_2} \otimes \cdots \otimes \phi^+_{AB_{n-1}}. \end{aligned} \tag{4.63}$$

We will show that, if Alice applies

$$A = \sum_{i=0}^{d-1} |i\rangle \langle i|^{\otimes n-1}, \tag{4.64}$$

and the Bobs apply the identity, the resulting state is GME. Since local operators preserve biseparability, this will mean that τ is GME too. Let

$$\tilde{\tau}_f = \left(A_A \otimes \mathbb{1}^{\otimes n-1}_{B_1...B_{n-1}}\right) \tau \left(A^\dagger_A \otimes \mathbb{1}^{\otimes n-1}_{B_1...B_{n-1}}\right) \tag{4.65}$$

be the unnormalised state after the parties apply their operations. We can write the components of τ as

$$\begin{aligned}\phi^+_{AB_1} \otimes \phi^+_{AB_2} \otimes \cdots \otimes \phi^+_{AB_{n-1}} &= \frac{1}{d^{n-1}} \sum_{i,j=0}^{d^{n-1}-1} |i\rangle_A |i\rangle_{B_1 \ldots B_{n-1}} \langle j|_A \langle j|_{B_1 \ldots B_{n-1}} \\ \tilde{\mathbb{1}}_{AB_1} \otimes \phi^+_{AB_2} \otimes \cdots \otimes \phi^+_{AB_{n-1}} &= \frac{1}{d^n} \sum_{i,j=0}^{d-1} \sum_{k,\ell=0}^{d^{n-2}-1} |ik\rangle_A |jk\rangle_{B_1 \ldots B_{n-1}} \langle i\ell|_A \langle j\ell|_{B_1 \ldots B_{n-1}} .\end{aligned} \tag{4.66}$$

Applying A to each $|i\rangle_A$, where $i = 0, ..., d^{n-1} - 1$, picks out the terms where all digits of i are equal, and thus gives simply $|i\rangle_A$ with $i = 0, ..., d-1$. Then, the digits of $|i\rangle_{B_1 \ldots B_{n-1}}$ must also be equal. Similarly, the action of A on $|ik\rangle_A$, where $i = 0, ..., d-1$ and $k = 0, ..., d^{n-2} - 1$, makes $i = k = 0, ..., d-1$, with the corresponding effect on the k index of $|jk\rangle_{B_1 \ldots B_{n-1}}$. Therefore, we obtain

$$\begin{aligned}\tilde{\tau}_f =& \frac{p}{d^{n-1}} \sum_{i,j=0}^{d-1} |i\rangle_A |i\rangle^{\otimes n-1}_{B_1 \ldots B_{n-1}} \langle j|_A \langle j|^{\otimes n-1}_{B_1 \ldots B_{n-1}} \\ &+ \frac{1-p}{d^n} \sum_{i,j=0}^{d-1} |i\rangle_A |j\rangle_{B_1} |i\rangle^{\otimes n-2}_{B_2 \ldots B_n} \langle i|_A \langle j|_{B_1} \langle i|^{\otimes n-2}_{B_2 \ldots B_n} .\end{aligned} \tag{4.67}$$

Hence, the normalised state after the transformation is

$$\tau_f = \frac{\tilde{\tau}_f}{\operatorname{tr} \tilde{\tau}_f} = p \, |GHZ\rangle\!\langle GHZ| + \frac{1-p}{d^2} \sum_{i,j=0}^{d-1} |i\rangle_A |j\rangle_{B_1} |i\rangle^{\otimes n-2}_{B_2 \ldots B_n} \langle i|_A \langle j|_{B_1} \langle i|^{\otimes n-2}_{B_2 \ldots B_n} . \tag{4.68}$$

To show that this state is GME, it is sufficient to find a witness that detects it. The operator

$$W = \frac{1}{d} \mathbb{1} - |GHZ\rangle\!\langle GHZ| \tag{4.69}$$

fits the bill: since the maximum overlap of the GHZ state with a biseparable state is $1/d$ [BPSS14], we have that

$$\operatorname{tr}(W\sigma) \geq 0 \tag{4.70}$$

for all biseparable states σ. Moreover,

$$\operatorname{tr}(W\tau_f) = \frac{d - 1 - p(d^2 - 1)}{d^2}, \tag{4.71}$$

which is strictly smaller than zero for all values of p such that ρ_p is entangled, i.e., for all

$$p > \frac{1}{d+1}. \tag{4.72}$$

□

Theorems 4.9 and 4.10 imply that, if Alice and Bob1 share an entangled, non-steerable state ρ_p, while Alice and all other Bobs share maximally entangled states, then the star network is GME and bilocal. Taking into account the bounds for separability and steerability given in Refs. [HH99, APB+07] respectively, this occurs if $1/(d+1) < p \leq (3d-1)(d-1)^{d-1}/[(d+1)d^d]$.

We have seen that a non-steerable state on one edge of a star network compromises its GMNL. In fact, this behaviour is more extreme if *all* edges are made non-steerable, in which case the network becomes fully local, as we show in Theorem 4.11. However, steerability does not guarantee GMNL. We show this in Theorem 4.12 by finding a steerable state which, when distributed in a star network, makes the network bilocal.

Theorem 4.11. *Any star network of states which are non-steerable from each external node to the centre node is fully local.*

Proof. We iterate the ideas used to prove Theorem 4.9. We start by rewriting the probability distribution arising from the n-partite network state as an $(n-1)$-partite distribution arising from POVMs acting on all but one of the non-steerable states. Then, the LHS model of the remaining state is used to show that the distribution is local in the bipartition crossed by that state. Since all such bipartitions split one party from the rest (as the network takes the form of a star), the local model contains a single-party distribution corresponding to one of the external nodes. Iterating this process for each of the states of the network completes the proof.

Let Alice share a non-steerable state σ_i acting on $\mathcal{H}_{AB_i}$ with each of Bob$_i$, $i = 1, ..., n-1$. Let Alice apply a POVM $\{E_{a|x}\}_a$ for each input x, and each Bob$_i$ apply a POVM $\{F^i_{b_i|y_i}\}_{b_i}$ for each input y_i. Then, the distribution obtained is

$$P(a, b_1, ..., b_{n-1}|x, y_1, ..., y_{n-1}) = \mathrm{tr}\left[\left(E_{a|x} \otimes \bigotimes_{i=1}^{n-1} F^i_{b_i|y_i}\right) \bigotimes_{i=1}^{n-1} \sigma_i\right]. \tag{4.73}$$

Using similar ideas to the proof of Theorem 4.9, we can rewrite this distribution as

$$P(a, b_1, ..., b_{n-1}|x, y_1, ..., y_{n-1}) \\ = \mathrm{tr}_{A\backslash A_1, B\backslash B_1}\left[\left(\mathrm{tr}_{A_1B_1}\left[\left(E_{a|x}\otimes F^1_{b_1|y_1}\right)\left(\sigma_1\otimes \mathbb{1}_{A\backslash A_1}\right)\right]\otimes\bigotimes_{i\neq 1} F^i_{b_i|y_i}\right)\bigotimes_{i\neq 1}\sigma_i\right]. \tag{4.74}$$

Now, denoting $A\backslash A_1 =: A'$, we have

$$\mathrm{tr}_{A_1B_1}\left[\left(E_{a|x}\otimes F^1_{b_1|y_1}\right)\left(\sigma_1\otimes \mathbb{1}_{A\backslash A_1}\right)\right] = \mathrm{tr}_{A_1}\left[E_{a|x}\left(\mathrm{tr}_{B_1}\left[\left(\mathbb{1}_{A_1}\otimes F^1_{b_1|y_1}\right)\sigma_1\right]\otimes \mathbb{1}_{A'}\right)\right]. \tag{4.75}$$

Since σ_1 is non-steerable from B_1 to A_1, it has an LHS model, therefore

$$\mathrm{tr}_{B_1}\left[\left(\mathbb{1}_{A_1}\otimes F^1_{b_1|y_1}\right)\sigma_1\right] = \sum_\lambda p_\lambda \eta_\lambda P_{B_1}(b_1|y_1,\lambda), \tag{4.76}$$

where η_λ is a state that depends on the hidden variable λ, which is distributed according to $\{p_\lambda\}_\lambda$. Therefore, equation (4.75) can be rewritten as

$$\sum_\lambda p_\lambda P_{B_1}(b_1|y_1,\lambda)\,\mathrm{tr}_{A_1}\left[E_{a|x}\left(\eta_\lambda\otimes \mathbb{1}_{A'}\right)\right]. \tag{4.77}$$

We want to show that, for all x, λ,

$$\mathrm{tr}_{A_1}\left[E_{a|x}\left(\eta_\lambda\otimes \mathbb{1}_{A'}\right)\right] =: \tilde{E}^\lambda_{a|x} \tag{4.78}$$

is a POVM element in A', following a similar strategy to the proof of Theorem 4.9. Since η_λ is a state, it can be written as a convex combination of pure states. Therefore, to show positivity of $\tilde{E}^\lambda_{a|x}$ we can assume

$$\tilde{E}^\lambda_{a|x} = \mathrm{tr}_{A_1}\left[E_{a|x}\left(|\psi_\lambda\rangle\langle\psi_\lambda|\otimes \mathbb{1}_{A'}\right)\right] \equiv \langle\psi_\lambda|\, E_{a|x}\,|\psi_\lambda\rangle\,. \tag{4.79}$$

If $\tilde{E}^\lambda_{a|x}$ were not positive, there would exist $|x\rangle \in \mathcal{H}_{A'}$ such that

$$\langle x|\,\tilde{E}^\lambda_{a|x}\,|x\rangle < 0, \tag{4.80}$$

which would imply that

$$\langle\psi_\lambda|\,\langle x|\,E_{a|x}\,|x\rangle\,|\psi_\lambda\rangle < 0, \tag{4.81}$$

and hence that $\tilde{E}^\lambda_{a|x}$ would not be positive. But this is false, as $E_{a|x}$ is a POVM element.

Therefore, $\tilde{E}^{\lambda}_{a|x} \succcurlyeq 0$.

Normalisation of $\tilde{E}^{\lambda}_{a|x}$ follows from that of $E_{a|x}$, as

$$\sum_a \tilde{E}^{\lambda}_{a|x} = \mathrm{tr}_{A_1}\left[\sum_a E_{a|x}\left(\eta_\lambda \otimes \mathbb{1}_{A'}\right)\right] = \mathrm{tr}_{A_1}\left[\left(\eta_\lambda \otimes \mathbb{1}_{A'}\right)\right] = \mathbb{1}_{A'}. \tag{4.82}$$

Therefore, from equation (4.74) we deduce

$$\begin{aligned} &P(a, b_1, ..., b_{n-1}|x, y_1, ..., y_{n-1}) \\ &\qquad = \sum_\lambda p_\lambda \, \mathrm{tr}_{A\setminus A_1, B\setminus B_1}\left[\left(\tilde{E}^{\lambda}_{a|x} \otimes \bigotimes_{i\neq 1} F^i_{b_i|y_i}\right) \bigotimes_{i\neq 1} \sigma_i\right] P_{B_1}(b_1|y_1, \lambda). \end{aligned} \tag{4.83}$$

Denoting $\lambda =: \lambda_1$, and $p_\lambda =: p^1_{\lambda_1}$, this argument can now be iterated for $i = 2, ..., n-1$, obtaining new POVMs of the form of $\tilde{E}^{\lambda_1}_{a|x}$ which depend on more hidden variables $\lambda_1, ..., \lambda_{n-1}$, so that

$$P(a, b_1, ..., b_{n-1}|x, y_1, ..., y_{n-1}) = \sum_{\lambda_1, ..., \lambda_{n-1}} \prod_{i=1}^{n-1} p^i_{\lambda_i} P_{B_i}(b_i|y_i\lambda_i) P_A(a|x, \lambda_{n-1}). \tag{4.84}$$

By unifying the hidden variables $\lambda_1, ..., \lambda_{n-1}$ into a single variable λ, such that $p_\lambda := \prod_{i=1}^{n-1} p^i_{\lambda_i}$, equation (4.84) shows that P is fully local. □

We note that the previous proof still holds if one of the σ_i is merely local, but not necessarily non-steerable: labelling the parties such that σ_{n-1} is steerable but local, while σ_i for $i \in [n-2]$ are non-steerable, we have that the first $n-2$ iterations of the argument lead to a decomposition including P_{B_i}, for $i \in [n-2]$, and a POVM element of the form of $\tilde{E}^{\lambda_1}_{a|x}$ but depending on all hidden variables $\lambda_1, ..., \lambda_{n-2}$ and acting on $\mathcal{H}_{A_{n-1}}$. Then, the action of this POVM and $F^{n-1}_{b_{n-1}|y_{n-1}}$ on σ_{n-1} gives a distribution $\sum_{\lambda_{n-1}} p^{n-1}_{\lambda_{n-1}} P_{B_{n-1}}(b_{n-1}|y_{n-1}, \lambda_{n-1}) P_A(a|x, \lambda_{n-1})$, which is local.

While non-steerability of a state in certain networks makes the network bilocal, steerability does not always guarantee GMNL: a star network of steerable states may be bilocal, as we now show. In fact, the existence of a steerable state which makes a star network bilocal is a consequence of Theorem 4.9, as a star network of steerable, isotropic states is a convex mixture of star networks which satisfy that Theorem.

Theorem 4.12. *There exists a steerable state which, when distributed in a star network of any number of parties, is bilocal.*

Proof. We write the network state as a convex mixture of terms that contain a non-steerable state on at least one edge of the star network. By Theorem 4.9, each such term is bilocal, and, since bilocality is closed under convex mixtures, this completes the proof.

Consider a star network where Alice shares the isotropic state

$$\rho_p = p\phi^+ + (1-p)\tilde{\mathbb{1}}, \tag{4.85}$$

with $p = p_0$ to be defined later, with each of Bob_k, $k = 1, ..., K$, which we will denote as $\rho_{p_0}^{\otimes K}$. Let p_S be the steerability threshold of the isotropic state, i.e., ρ_p is non-steerable for $p \leq p_S$ and steerable otherwise. We will show that, for any $K \in \mathbb{N}$, there exists $p_0 > p_S$ such that $\rho_{p_0}^{\otimes K}$ is bilocal.

Expanding the tensor product, we find

$$\begin{aligned}\rho_{p_0}^{\otimes K} =& p_0^K \phi^{+\otimes K} \\ &+ p_0^{K-1}(1-p_0)\left(\phi^{+\otimes(K-1)} \otimes \tilde{\mathbb{1}} + \phi^{+\otimes(K-2)} \otimes \tilde{\mathbb{1}} \otimes \phi^+ + \cdots + \tilde{\mathbb{1}} \otimes \phi^{+\otimes(K-1)}\right) + \ldots,\end{aligned} \tag{4.86}$$

where the omitted terms all contain at least one term $\tilde{\mathbb{1}}$ acting on $\mathcal{H}_{AB_k}$ for some k, hence they are local (in fact, separable) across at least one bipartition $A\{B_{\bar{k}}\}_{\bar{k}\neq k}|B_k$. We can rewrite the above as

$$\begin{aligned}\rho_{p_0}^{\otimes K} =& \left(\frac{p_0^K}{K}\phi^{+\otimes K} + p_0^{K-1}(1-p_0)\phi^{+\otimes(K-1)} \otimes \tilde{\mathbb{1}}\right) \\ &+ \left(\frac{p_0^K}{K}\phi^{+\otimes K} + p_0^{K-1}(1-p_0)\phi^{+\otimes(K-2)} \otimes \tilde{\mathbb{1}} \otimes \phi^+\right) \\ &+ \cdots + \left(\frac{p_0^K}{K}\phi^{+\otimes K} + p_0^{K-1}(1-p_0)\tilde{\mathbb{1}} \otimes \phi^{+\otimes(K-1)}\right) + \ldots .\end{aligned} \tag{4.87}$$

Here, each bracket has ϕ^+ on $K-1$ branches, and the normalised state

$$\sigma = \frac{(p_0/K)\phi^+ + (1-p_0)\tilde{\mathbb{1}}}{p_0/K + 1 - p_0} \tag{4.88}$$

on the remaining one. The state σ is itself isotropic, and hence non-steerable if the coefficient of ϕ^+ is equal to p_S, that is, if

$$p_0 = \frac{Kp_S}{1+(K-1)p_S}. \tag{4.89}$$

Since $p_S < 1$, we have

$$p_0 = \frac{p_S + (K-1)p_S}{1+(K-1)p_S} > \frac{p_S + (K-1)p_S^2}{1+(K-1)p_S} = p_S, \tag{4.90}$$

showing that there exists $p_0 > p_S$ such that the isotropic states σ are non-steerable and the states ρ_{p_0} in the star network are steerable.

This means that we can write $\rho_{p_0}^{\otimes K}$ as a convex mixture of star networks where the state in at least one branch is non-steerable: either an isotropic state with parameter p_S, or the identity. By Theorem 4.9, each term in the mixture is local across $A\{B_{\bar{k}}\}_{\bar{k}\neq k}|B_k$. Further, each such bipartition has one element containing a single party B_k, and both the isotropic state and the identity have an LHS model. Therefore, the network is bilocal. □

In fact, the same argument can be applied to any tree network of nonlocal, isotropic states to prove bilocality for Svetlichny's definition [Sve87], where the probability distributions of the bipartition elements are not required to be nonsignalling. A tree network can be decomposed into a convex combination of terms, each of which has an edge k containing either an isotropic state or an identity. The isotropic state is local for a certain value of the parameter p_0, therefore each term is local across the bipartition crossed by the edge k. However, not all bipartitions will have an element containing a single party, therefore the argument for nonsignalling can't be imported.

4.3 Superactivation of GMNL in networks

We have seen that placing non-steerable states on either one or all edges of a star network renders the network bilocal or even fully local, by Theorems 4.9 and 4.11 respectively. We now show that the bilocal network displays superactivation of GMNL. Indeed, taking many copies of this network makes it possible to win a generalisation of the Khot-Vishnoi game with a much higher probability than with bilocal resources. We first introduce the game and its extension to the star network via a Lemma, and then prove a second Lemma to bound the probability of winning with a bilocal strategy, before showing the superactivation result.

The Khot-Vishnoi game [KV05, BRSd12] is parametrised by a number v, which is assumed to be a power of 2, and a noise parameter $\eta \in [0, 1/2]$. Consider the group $\{0,1\}^v$ of all v-bit strings, with operation $\oplus$ denoting bitwise modulo 2 addition, and the subgroup H of all Hadamard codewords. The subgroup H partitions the group

$\{0,1\}^v$ into $2^v/v$ cosets of v elements each. These cosets will act as questions, and answers will be elements of the question cosets. The referee chooses a uniformly random coset $[x]$, as well as a string $z \in \{0,1\}^v$ where each bit $z(i)$ is chosen independently and is 1 with probability η and 0 otherwise. Alice's question is the coset $[x]$, which can be thought of as $u \oplus H$ for a uniformly random $u \in \{0,1\}^v$, while Bob's is the coset $[x \oplus z]$, which can be thought of as $u \oplus z \oplus H$. The aim of the players is to guess the string z, and thus they must output $a \in [x]$ and $b \in [x \oplus z]$ such that $a \oplus b = z$. Ref. [BRSd12] showed that any local strategy for Alice and Bob, implemented by a distribution denoted by P_{local}, achieves a winning probability of

$$\langle G_{KV}, P_{\text{local}} \rangle \leq \frac{v}{v^{1/(1-\eta)}}. \tag{4.91}$$

A much higher winning probability can be obtained with a distribution $P_{\max}$ arising from certain projective measurements on the maximally entangled state:

$$\langle G_{KV}, P_{\max} \rangle \geq (1-2\eta)^2. \tag{4.92}$$

Picking the value $\eta = 1/2 - 1/\log v$ gives rise to the bounds

$$\langle G_{KV}, P_{\text{local}} \rangle \leq \frac{C}{v} \quad \langle G_{KV}, P_{\max} \rangle \geq D/\log^2 v, \tag{4.93}$$

for universal constants C, D.

Lemma 4.2. *The Khot-Vishnoi game can be extended to the star network by letting Alice and each of Bob*$_i$*, for* $i = 1, ..., K$*, play the bipartite Khot-Vishnoi game. This defines a game whose coefficients are normalised, i.e., satisfy*

$$\sum_{\substack{x_1,\ldots,x_K \\ y_1,\ldots,y_K}} \max_{\substack{a_1,\ldots,a_K \\ b_1,\ldots,b_K}} \widetilde{G}_{a_1\ldots a_K b_1 \ldots b_K | x_1 \ldots x_K y_1 \ldots y_K} \leq 1. \tag{4.94}$$

Proof. Consider a star network of K edges, each of which connect Alice to Bob$_i$ for $i = 1, ..., K$. Alice will play the bipartite Khot-Vishnoi game G_{KV} with each Bob$_i$. Denoting Alice's inputs and outputs as $x_1, ..., x_K$ and $a_1, ..., a_K$ respectively, and Bob$_i$'s input and output as y_i, b_i respectively, we denote the coefficients of each game as $G_{a_i b_i | x_i y_i}$. Thus, the $(K+1)$-partite game being played on the star network, which we denote by $\widetilde{G}$, has coefficients

$$\widetilde{G}_{a_1\ldots a_K b_1 \ldots b_K | x_1 \ldots x_K y_1 \ldots y_K} = \prod_{i=1}^{K} G_{a_i b_i | x_i y_i}. \tag{4.95}$$

Since G_{KV} is a game, so is $\widetilde{G}$, i.e. all of its coefficients are positive. Moreover, one can check that the coefficients $G_{a_ib_i|x_iy_i}$ satisfy the normalisation condition

$$\sum_{x_i,y_i} \max_{a_i,b_i} G_{a_ib_i|x_iy_i} \leq 1 \tag{4.96}$$

for all $i \in [K]$, and hence

$$\begin{aligned}
\sum_{\substack{x_1,\dots,x_K \\ y_1,\dots,y_K}} \max_{\substack{a_1,\dots,a_K \\ b_1,\dots,b_K}} \widetilde{G}_{a_1\dots a_K b_1\dots b_K|x_1\dots x_K y_1\dots y_K} &= \sum_{\substack{x_1,\dots,x_K \\ y_1,\dots,y_K}} \max_{\substack{a_1,\dots,a_K \\ b_1,\dots,b_K}} \prod_{i=1}^{K} G_{a_ib_i|x_iy_i} \\
&= \prod_{i=1}^{K} \sum_{x_i,y_i} \max_{a_i,b_i} G_{a_ib_i|x_iy_i} \\
&\leq 1.
\end{aligned} \tag{4.97}$$

In fact, this normalisation condition also holds if we take only a subset of games G_{KV}, i.e. the product of $G_{a_ib_i|x_iy_i}$ for i in some subset of $[K]$. □

Lemma 4.3. *The extension of the Khot-Vishnoi game to the star network is such that the winning probability using any bilocal strategy is bounded above by C/v, where v is the parameter of the game and C is a universal constant.*

Proof. Consider the game in Lemma 4.2. To bound the winning probability of bilocal strategies, we consider the two possible types of bilocal distributions P_{BL}: those local in a bipartition that separates Alice from all the Bobs, and those where some of the Bobs are in Alice's partition element. The first case can be upper bounded by considering the parallel repetition of the Khot-Vishnoi game [Raz98], and we prove an upper bound on the winning probability equal to that of local strategies. The second case uses the normalisation constraint in Lemma 4.2 to reduce to the parallel repetition case.

First, we take a distribution of the form

$$P_1(a_1, ..., a_K|x_1, ..., x_K)P_2(b_1, ..., b_K|y_1, ..., y_K) \tag{4.98}$$

for each a_i, b_i, x_i, y_i, $i \in [K]$. In a bilocal strategy, P_1, P_2 would need to be nonsignalling, however, an upper bound on the winning probability can be obtained more easily by removing this constraint. Using such a distribution to play $\widetilde{G}$, the parties are effectively playing the K-fold parallel repetition of G_{KV}, which we denote as $G_{KV}^{\otimes K}$. To bound their winning probability we will use the same techniques as in Refs. [APd20, BRSd12]. Recall that, for the bipartite game, the questions are cosets of H in the group $\{0,1\}^v$, which can be thought of as $u \oplus H$ for Alice and $u \oplus z \oplus H$ for Bob (where $u \in \{0,1\}^v$ is sampled uniformly and $z \in \{0,1\}^v$ is sampled bitwise independently with noise η), and the answers are elements of the question cosets. Without loss of generality, we can assume Alice and Bob's strategy is deterministic, and identify it with Boolean functions $A, B : \{0,1\}^v \to \{0,1\}$ which take the value 1 for exactly one element of each coset. That is, for each question, A, B respectively pick out Alice's and Bob's answer. Since the players win if and only if their answers a, b satisfy $a \oplus b = z$, we have that for all u, z,

$$\sum_{h \in H} A(u \oplus h) B(u \oplus z \oplus h) \tag{4.99}$$

is 1 if the players win on inputs $u \oplus H$, $u \oplus z \oplus H$, and 0 otherwise. Therefore, the winning probability is

$$\begin{aligned} \mathop{\mathbb{E}}_{u,z}\left[\sum_{h \in H} A(u \oplus h) B(u \oplus z \oplus h)\right] &= \sum_{h \in H} \mathop{\mathbb{E}}_{u,z}\left[A(u \oplus h) B(u \oplus z \oplus h)\right] \\ &= v \mathop{\mathbb{E}}_{u,z}\left[A(u) B(u \oplus z)\right], \end{aligned} \tag{4.100}$$

since for all h, the distribution of $u \oplus h$ is uniform.

For the parallel repetition $G_{KV}^{\otimes K}$, Alice and Bob must pick an answer for each copy of the game, so we can identify their strategy with some new Boolean functions $A, B : \{0,1\}^{vK} = \{0,1\}^v \times \cdots \times \{0,1\}^v \to \{0,1\}$ which, restricted to each set of K questions (i.e. K cosets), take the value 1 for exactly one element. Alice's question of $G_{KV}^{\otimes K}$ is given by $u = (u_1, ..., u_K)$ where each $u_i \in \{0,1\}^v$, $i \in [K]$ is sampled uniformly. But this is equivalent to sampling u uniformly in $\{0,1\}^{vK}$. Similarly, $z = (z_1, ..., z_K)$ is sampled bitwise independently, since each z_i is. Therefore, the winning probability is given by

$$\begin{aligned} &\mathop{\mathbb{E}}_{\substack{u_i, z_i \\ i \in [K]}}\left[\sum_{\substack{h_i \in H_i \\ i \in [K]}} A((u_1, ..., u_K) \oplus (h_1, ..., h_K)) B((u_1, ..., u_K) \oplus (z_1, ..., z_K) \oplus (h_1, ..., h_K))\right] \\ &\quad = \sum_{\substack{h_i \in H_i \\ i \in [K]}} \mathop{\mathbb{E}}_{u,z}\left[A(u \oplus (h_1, ..., h_K)) B(u \oplus z \oplus (h_1, ..., h_K))\right] \\ &\quad = v^K \mathop{\mathbb{E}}_{u,z}\left[A(u) B(u \oplus z)\right] \end{aligned} \tag{4.101}$$

since there are v^K choices of strings of the form $(h_1, ..., h_K) \in H_1 \times ... \times H_K$. To bound $\mathbb{E}_{u,z}\left[A(u) B(u \oplus z)\right]$, we follow the computation of Ref. [BRSd12, Theorem 4.1], which uses the Cauchy-Schwarz and hypercontractive inequalities, and obtain

$$\mathbb{E}_{u,z}\left[A(u)B(u\oplus z)\right] \leq \frac{1}{v^{K/(1-\eta)}}. \tag{4.102}$$

If, instead, the parties share a distribution of the form

$$P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k})P_2(\{b_i\}_{i>k}|\{y_i\}_{i>k}), \tag{4.103}$$

for each $\alpha \equiv (a_1, ..., a_K)$, $\chi \equiv (x_1, ..., x_K)$, b_i, y_i, $i \in [K]$, for some $k \in [K]$, then their winning probability is given by

$$\begin{aligned}
\left\langle \widetilde{G}, P_1P_2 \right\rangle &= \sum_{\substack{a_i,b_i,x_i,y_i\\ i\in[K]}} \prod_{i=1}^{K} G_{a_ib_i|x_iy_i} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k})P_2(\{b_i\}_{i>k}|\{y_i\}_{i>k}) \\
&= \sum_{\substack{a_i,b_i,x_i,y_i\\ k<i\leq K}} \prod_{i=k+1}^{K} G_{a_ib_i|x_iy_i} \left(\sum_{\substack{a_i,b_i,x_i,y_i\\ 1\leq i\leq k}} \prod_{i=1}^{k} G_{a_ib_i|x_iy_i} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k}) \right) \\
&\qquad \times P_2(\{b_i\}_{i>k}|\{y_i\}_{i>k}) \qquad (4.104) \\
&= \sum_{\substack{a_i,b_i,x_i,y_i\\ k<i\leq K}} \prod_{i=k+1}^{K} G_{a_ib_i|x_iy_i} f(\{a_i\}_{i>k}, \{x_i\}_{i>k}) P_2(\{b_i\}_{i>k}|\{y_i\}_{i>k}),
\end{aligned}$$

where we define

$$f(\{a_i\}_{i>k}, \{x_i\}_{i>k}) = \sum_{\substack{a_i,b_i,x_i,y_i\\ 1\leq i\leq k}} \prod_{i=1}^{k} G_{a_ib_i|x_iy_i} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k}). \tag{4.105}$$

We will now show that we can define a probability distribution $\tilde{P}(\{a_i\}_{i>k}|\{x_i\}_{i>k})$ all of whose components are greater than or equal to those of $f(\{a_i\}_{i>k}, \{x_i\}_{i>k})$, and use this, together with the previous parallel repetition result, to bound $\left\langle \widetilde{G}, P_1P_2 \right\rangle$. First, note that the function f is pointwise positive and such that

$$\sum_{a_{k+1},\cdots,a_K} f(\{a_i\}_{i>k}, \{x_i\}_{i>k}) \leq 1 \tag{4.106}$$

for all $x_{k+1}, ..., x_K$. Indeed, fixing $x_{k+1}, ..., x_K$, we have

$$\begin{aligned}
\sum_{a_{k+1},...,a_K} \sum_{\substack{a_i,b_i,x_i,y_i \\ 1\leq i\leq k}} \prod_{i=1}^{k} G_{a_ib_i|x_iy_i} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k}) \\
&= \sum_{\substack{a_i,b_i,x_i,y_i \\ 1\leq i\leq k}} \prod_{i=1}^{k} G_{a_ib_i|x_iy_i} \left(\sum_{a_{k+1},...,a_K} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k}) \right) \\
&\leq \sum_{\substack{x_i,y_i \\ 1\leq i\leq k}} \prod_{i=1}^{k} \max_{a_i,b_i} G_{a_ib_i|x_iy_i} \left(\sum_{\substack{a_i, i\in[K] \\ b_i, i\leq k}} P_1(\alpha, \{b_i\}_{i\leq k}|\chi, \{y_i\}_{i\leq k}) \right) \\
&= \sum_{\substack{x_i,y_i \\ 1\leq i\leq k}} \prod_{i=1}^{k} \max_{a_i,b_i} G_{a_ib_i|x_iy_i} \leq 1,
\end{aligned} \tag{4.107}$$

where the last inequality follows from equation (4.97). Thus, for each $x_{k+1}, ..., x_K$ we can define $\tilde{P}(\{a_i\}_{i>k}|\{x_i\}_{i>k})$ to have the same elements as f except when all $a_i = 0$:

$$\tilde{P}(\{a_i\}_{i>k}|\{x_i\}_{i>k}) = \begin{cases} f(\{a_i\}_{i>k}, \{x_i\}_{i>k}) & \text{if } a_i \neq 0 \text{ for some } i > k, \\ 1 - \sum_{\{a'_i\}_{i>k} \neq \vec{0}} f(\{a'_i\}_{i>k}, \{x_i\}_{i>k}) & \text{if } a_i = 0 \text{ for all } i > k. \end{cases} \tag{4.108}$$

Then, $\tilde{P}$ is a probability distribution, all of whose components are larger than or equal to those of f, and hence from equation (4.3) we deduce that

$$\left\langle \widetilde{G}, P_1P_2 \right\rangle \leq \sum_{\substack{a_i,b_i,x_i,y_i \\ k<i\leq K}} \prod_{i=k+1}^{K} G_{a_ib_i|x_iy_i} \tilde{P}(\{a_i\}_{i>k}|\{x_i\}_{i>k}) P_2(\{b_i\}_{i>k}|\{y_i\}_{i>k}). \tag{4.109}$$

But the right-hand side is the winning probability of the $(K-k)$-fold parallel repetition of G_{KV} using the bilocal distribution $\tilde{P}P_2$ which, repeating the calculation above, can be found to be bounded as

$$\left\langle \widetilde{G}, P_1P_2 \right\rangle \leq \frac{v^{K-k}}{v^{(K-k)/(1-\eta)}}. \tag{4.110}$$

This exhausts the local strategies available to the players. Comparing the bound just obtained to the one from the distribution in equation (4.98), we find $v^K \geq v^{K-k} \geq v$, since $v > 1$ and $K > k$. Hence,

$$\frac{v^K}{v^{K/(1-\eta)}} \leq \frac{v^{K-k}}{v^{(K-k)/(1-\eta)}} \leq \frac{v}{v^{/(1-\eta)}}. \tag{4.111}$$

Taking $\eta = 1/2 - 1/\log v$, we have that, for any bilocal distribution P_{BL},

$$\left\langle \widetilde{G}, P_{\mathrm{BL}} \right\rangle \leq \frac{C}{v} \tag{4.112}$$

for some constant C. □

We can now use the extension of the Khot-Vishnoi game to the star network to show that GMNL can be activated in networks. We take advantage of the large separation between the quantum and classical bounds obtained in [BRSd12] and used to prove superactivation of nonlocality in [Pal12]. We extend the superactivation property to the star network setting using similar techniques as Ref. [APd20] uses in the triangle network, with the construction in Ref. [CABV13]. This enables us to show that taking many copies of the network which was shown to be bilocal in Theorem 4.9 gives rise to GMNL.

Theorem 4.13. *A star network where Alice and Bob1 share any entangled isotropic state, and Alice and all other Bobs share maximally entangled states, gives rise to GMNL by taking many copies.*

Proof. Taking L copies of the star network in the statement of the Theorem is equivalent to taking a star network where Alice shares L copies of an isotropic state, $\rho_p^{\otimes L}$, with Bob_1, and L copies of a d-dimensional maximally entangled state, $\phi_d^{+\otimes L}$, with each Bob_i, $i = 2, ..., K$. The latter state is in turn equivalent to a d^L-dimensional maximally entangled state $\phi^+_{d^L}$. We will use the superactivation result first proved in Ref. [Pal12] and extended in Ref. [CABV13] to show that these states allow the parties to win the game $\widetilde{G}$ with a higher probability than if they use any bilocal strategy.

Given the structure of the game, the probability of winning $\widetilde{G}$ using the quantum state of the network is lower bounded by the product of the probabilities of winning each G_{KV} with the state at each edge i, since the players can play every game independently. On the maximally entangled edges, the probability of winning G_{KV} is bounded by equation (4.92). We will obtain a similar bound for the isotropic edge. Let ρ_p be a d-dimensional isotropic state with a visibility p such that the state is entangled and local. Its entanglement fraction [HH99] is $F = \left\langle \phi_d^+ \right| \rho_p \left| \phi_d^+ \right\rangle = p + (1-p)/d^2$, which we can use to write the isotropic state in the form

$$\rho_F = F\phi_d^+ + (1-F)\frac{\mathbb{1} - \phi_d^+}{d^2-1}. \tag{4.113}$$

We have that ρ_F is entangled if and only if $F > 1/d$. Expanding the tensor product in the state $\rho_F^{\otimes L}$, we can write it as

$$\rho_F^{\otimes L} = F^L \phi_d^{+\otimes L} + \cdots = F^L \phi_{d^L}^+ + \ldots, \tag{4.114}$$

where the omitted terms are tensor products of ϕ_d^+ and $(\mathbb{1} - \phi_d^+)/(d^2-1)$ with coefficients that are products of F and $(1-F)$. Acting on $\rho_F^{\otimes L}$ with the same projective measurements as above gives a probability distribution P_{iso} which is linear in the terms of equation (4.114), and the action of G_{KV} on this distribution is linear too. Since the coefficients $G_{a_ib_i|x_iy_i}$ are nonnegative, we have

$$\langle G_{KV}, P_{\mathrm{iso}} \rangle \geq F^L \langle G_{KV}, P_1 \rangle, \tag{4.115}$$

where P_1 is the probability distribution obtained from the projective measurements acting on $\phi_{d^L}^+$. By equation (4.92), we find

$$F^L \langle G_{KV}, P_1 \rangle \geq F^L (1-2\eta)^2. \tag{4.116}$$

Taking $\eta = 1/2 - 1/\log v$ like in Lemma 4.3, we find

$$F^L \langle G_{KV}, P_1 \rangle \geq F^L \frac{D}{\ln^2 v}, \tag{4.117}$$

where D is a universal constant. If d is a power of 2, we can choose $v = d^L$ to obtain

$$F^L \frac{D}{\ln^2 v} = F^L \frac{D}{L^2 \ln^2 d}. \tag{4.118}$$

Otherwise, the game can be modified like in Remark 1.1 of Ref. [Pal14], obtaining a similar bound on the quantum winning probability but with a different constant D. The bound on the classical winning probability (equation (4.112)) is unchanged.

Putting the bounds from both types of edges together, and denoting by P_Q the probability distribution obtained from the state of the whole network and the projective measurements that are performed on each edge, we obtain

$$\left\langle \widetilde{G}, P_Q \right\rangle \geq F^L \frac{D}{L^2 \ln^2 d} \left(\frac{D}{L^2 \ln^2 d} \right)^{K-1} = \frac{F^L D^K}{L^{2K} \ln^{2K} d}. \tag{4.119}$$

Finally, we use Lemma 4.3 to compare the local and quantum bounds. Using equations (4.112) and (4.119), we find

$$\frac{\left\langle \widetilde{G}, P_Q \right\rangle}{\sup_{P_{BL} \in \mathcal{BL}} \left\langle \widetilde{G}, P_{\mathrm{BL}} \right\rangle} \geq \frac{D^K}{C L^{2K} \ln^{2K} d} F^L d^L, \tag{4.120}$$

where $\mathcal{BL}$ is the set of bilocal distributions. Since $F > 1/d$ (as ρ_F is entangled), this expression tends to ∞ as L grows unbounded. In particular, the ratio is > 1, proving that GMNL is obtained. Since one copy of the star network is bilocal, there is superactivation of GMNL. $\square$

Chapter 5

A physical principle from observers' agreement

Is the world quantum? An active research line in quantum foundations is devoted to exploring what constraints can rule out the post-quantum theories that are consistent with experimentally observed results. We explore this question in the context of epistemics, and ask whether agreement between observers can serve as a physical principle that must hold for any theory of the world. Aumann's seminal Agreement Theorem states that two (classical) agents cannot agree to disagree. We propose an extension of this theorem to no-signaling settings. In particular, we establish an Agreement Theorem for quantum agents, while we construct examples of (post-quantum) no-signaling boxes where agents can agree to disagree. The PR box is an extremal instance of this phenomenon. These results make it plausible that agreement between observers might be a physical principle, while they also establish links between the fields of epistemics and quantum information that seem worthy of further exploration.

5.1 Classical agreement theorem

Aumann's theorem [Aum76] is formulated on a probability space, and partial information of the observers is represented by different partitions of the space. Each observer knows which of their partition elements obtains, and estimates the probability of an event of interest by Bayesian inference. We refer to a probability space, together with some given partitions, as a (classical) *ontological model*. Ontological models appearing in the literature (see, e.g., [Fer11]) also contain a set of preparations underlying the distribution over the state space, and the partitions are usually phrased in terms of measurements and outcomes. However, we consider preparations implicit and use the language of partitions to bridge the gap between classical probability spaces and nonsignalling boxes more smoothly.

For the sake of simplicity and following Aumann, we will restrict our analysis to two observers, Alice and Bob. Aumann's original theorem considers common *knowledge* about *one single* event of interest to both observers. We provide a slight generalisation with common *certainty* about *two perfectly correlated* events of interest, one for each observer. This allows us to jump into the framework of nonsignalling boxes that we will use later. We refer to this generalisation as the classical agreement theorem. This nomenclature can be further motivated by the fact that, for purely classical situations, both statements—the original Aumann's theorem and our formulation with perfectly correlated events—can be proven to be *equivalent* (as long as states of the world with null probability are ignored, as in Ref. [Aar05]).

Consider a probability space $(\Omega, \mathcal{E}, \mathsf{P})$ where Ω is a set of possible states of the world; $\mathcal{E}$ is its power set (i.e., the set of events); and P is a probability measure over Ω. We will consider two events $E_A, E_B \in \mathcal{E}$ of interest to Bob and Alice, respectively (the choice of subscripts will become clear later). We assume that they are *perfectly correlated*: $\mathsf{P}(E_A \backslash E_B) = \mathsf{P}(E_B \backslash E_A) = 0$.

Fix partitions $\mathcal{P}_A, \mathcal{P}_B$ of Ω for Alice and Bob, respectively. For convenience, assume that all members of the join (coarsest common refinement) of $\mathcal{P}_A$ and $\mathcal{P}_B$ are non-null. For a state $\omega \in \Omega$, $\mathcal{P}_{A,B}(\omega)$ is the partition element (of Alice's or Bob's, respectively) that contains ω. For each $n \in \mathbb{N}$, fix numbers $q_A, q_B \in [0,1]$ and consider the following sets:

$$\begin{aligned}
A_0 &= \{\omega \in \Omega : \mathsf{P}(E_B|\mathcal{P}_A(\omega)) = q_A\}\,, \\
B_0 &= \{\omega \in \Omega : \mathsf{P}(E_A|\mathcal{P}_B(\omega)) = q_B\}\,, \\
A_{n+1} &= \{\omega \in A_n : \mathsf{P}(B_n|\mathcal{P}_A(\omega)) = 1\}\,, \\
B_{n+1} &= \{\omega \in B_n : \mathsf{P}(A_n|\mathcal{P}_B(\omega)) = 1\}\,.
\end{aligned} \tag{5.1}$$

Here, the set A_0 is the set of states ω such that Alice assigns probability q_A to event E_B; the set B_1 is the set of states ω such that Bob assigns probability q_B to event E_A *and* probability 1 to the states in A_0—i.e., states where Bob assigns probability q_B to E_A and is certain that Alice assigned probability q_A to E_B; and so on, and similarly for B_0, A_1, etc.

It is common certainty at a state $\omega^* \in \Omega$ that Alice assigns probability q_A to E_B and that Bob assigns probability q_B to E_A if

$$\omega^* \in A_n \cap B_n \quad \forall n \in \mathbb{N}. \tag{5.2}$$

If equation (5.2) does not hold for all $n \in \mathbb{N}$, but, instead, only for $n \leq N$ for a certain $N \in \mathbb{N}$, then we talk about Nth-order mutual certainty.

We now state and prove the classical agreement theorem that will be the basis of our work:

Theorem 5.1. *Fix a probability space $(\Omega, \mathcal{E}, \mathsf{P})$, where E_A and E_B are perfectly correlated events. If it is common certainty at a state $\omega^* \in \Omega$ that Alice assigns probability q_A to E_B and Bob assigns probability q_B to E_A, then $q_A = q_B$.*

Proof. The main idea behind the proof is to notice that, since Ω is finite, there is a finite $N \in \mathbb{N}$ such that, for all $n \geq N$, $A_{n+1} = A_n$ and $B_{n+1} = B_n$. Using the definition of A_{N+1}, noticing that A_N is a union of Alice's partition elements, and using convex combination arguments together with the perfect correlations between E_A and E_B leads to

$$\mathsf{P}(E_A \cap E_B | A_N \cap B_N) = q_A. \tag{5.3}$$

Running the parallel argument for Bob entails that the same expression is equal to q_B, proving the claim.

In more detail, since Ω is finite, there is a finite $N \in \mathbb{N}$ such that, for all $n \geq N$, $A_{n+1} = A_n$ and $B_{n+1} = B_n$. From the definition of A_{N+1}, we have that

$$\mathsf{P}(B_N | \mathcal{P}_A(\omega)) = 1 \ \forall \omega \in A_N. \tag{5.4}$$

Now, A_N is a union of partition elements of $\mathcal{P}_A$, i.e., $A_N = \bigcup_{i \in I} \pi_i$ where each $\pi_i \in \mathcal{P}_A$ and I is a finite index set. From Equation (5.4), we have

$$\mathsf{P}(B_N | \pi_i) = 1 \ \forall i \in I. \tag{5.5}$$

Since $\mathsf{P}(B_N | A_N)$ is a convex combination of $\mathsf{P}(B_N | \pi_i)$ for $i \in I$, we must have

$$\mathsf{P}(B_N | A_N) = 1. \tag{5.6}$$

Now, since $A_N \subseteq A_0$, then $\mathsf{P}(E_B | \pi_i) = q_A$ for all $i \in I$ too. Using a convex combination argument once more, this entails that

$$\mathsf{P}(E_B|A_N) = q_A. \tag{5.7}$$

Equations (5.6) and (5.7) together imply that

$$\mathsf{P}(E_B|A_N \cap B_N) = q_A. \tag{5.8}$$

But events E_A and E_B are perfectly correlated, so that

$$\mathsf{P}(E_A \cap E_B|A_N \cap B_N) = q_A, \tag{5.9}$$

as well.

Running the parallel argument with A and B interchanged, we obtain

$$\mathsf{P}(E_A \cap E_B|A_N \cap B_N) = q_B, \tag{5.10}$$

which implies that $q_A = q_B$. □

5.2 Mapping agreement to nonsignalling boxes

We now map the classical agreement theorem into the nonsignalling framework, in order to explore its applicability beyond the classical realm.

We consider nonsignalling distributions, or boxes [PR94], as per Definition 1.8.

We now show that we can associate a nonsignalling box with any ontological model, and vice versa. Remarkably, this can be accomplished even in the case in which the nonsignalling box is nonlocal, obtaining an ontological model with a quasi-probability measure instead of standard positive probabilities [AB11]. (The appearance of quasi-probabilities here should not surprise the reader. In fact, one cannot hope to obtain ontological models with only non-negative probabilities for post-classical nonsignalling boxes, as this would provide local hidden-variable models that contradict, for instance, Bell's theorem. In any case, the use of this mathematical tool has been well rooted in the study of quantum mechanics since its origins—see [Fer11] for a nice review of this subject.) This makes it possible to translate results from one framework to the other, something that might be of interest in order to establish further connections between epistemics and quantum theory. However, once we establish these mappings, we focus on nonsignalling boxes and leave this digression aside.

Proposition 5.1. *Given any ontological model, there is a corresponding nonsignalling box which reproduces the observable statistics of the model. If the ontological model is classical, then the box is local.*

Proof. Let $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}$ be index sets. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, and, for each $x \in \mathcal{X}$, let $\mathsf{A}_{a|x}$ be a partition of the states $\omega \in \Omega$ where $a \in \mathcal{A}$ denotes the partition elements. Similarly, for each $y \in \mathcal{Y}$, let $\mathsf{B}_{b|y}$ be another partition of the states $\omega \in \Omega$, where $b \in \mathcal{B}$ denotes the partition elements. According to that, we can understand labels $x \in \mathcal{X}$, $y \in \mathcal{Y}$ as *inputs*—this information fixes what partition Alice and Bob look at—and $a \in \mathcal{A}$, $b \in \mathcal{B}$ as *outputs*—this is the information that the agents gain by observing their corresponding partitions.

With all the above, $\big\{(\Omega, \mathcal{F}, \mathsf{P}), \{\mathsf{A}_{a|x}, \mathsf{B}_{b|y}\}_{a,b,x,y}\big\}$ is an ontological model that we now want to associate to a nonsignalling box that reproduces its statistics. In this ontological model, given inputs $x \in \mathcal{X}$, $y \in \mathcal{Y}$, the probability of obtaining outputs $a \in \mathcal{A}$, $b \in \mathcal{B}$ is given by $\mathsf{P}(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y})$. This simple observation leads us to construct the nonsignalling box $\{P(a, b|x, y)\}_{a,b,x,y}$, where each probability is given by

$$P(a, b|x, y) := \mathsf{P}\big(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}\big), \qquad \forall (a, b, x, y) \in \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y}. \tag{5.11}$$

Clearly, the probability that Alice and Bob make observations according to the partitions given by x, y and conclude that they are in the partition element that corresponds to a, b respectively is the same as the probability that they input x, y in their nonsignalling box and obtain outputs a, b. That is, the nonsignalling box reproduces the statistics of the ontological model.

Further, while an ontological model with a quasi-probability distribution might be such that some states ω are such that $\mathsf{P}(\omega) < 0$, all partition elements must be observable, that is, $\mathsf{P}(\mathsf{A}_{a|x}), \mathsf{P}(\mathsf{B}_{b|y}) \geq 0$ for all a, b, x, y. In particular, the probabilities of all intersection of partition elements is non-negative, therefore so is P. Normalisation of P follows from the normalisation of P: since for all x, y we have $\bigcup_a \mathsf{A}_{a|x} = \bigcup_b \mathsf{B}_{b|y} = \Omega$, and $\mathsf{P}(\Omega) = 1$, then

$$\sum_{a,b} P(a, b|x, y) = \mathsf{P}(\bigcup_a \mathsf{A}_{a|x} \cap \bigcup_b \mathsf{B}_{b|y}) = \mathsf{P}(\Omega) = 1, \tag{5.12}$$

for all x, y.

The fact that P is nonsignalling follows from the same ideas: for all $x \neq x'$, we have

$$\sum_a P(a,b|x,y) = \mathsf{P}(\bigcup_a \mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}) = \mathsf{P}(\mathsf{B}_{b|y}) = \mathsf{P}(\bigcup_a \mathsf{A}_{a|x'} \cap \mathsf{B}_{b|y}) = \sum_a P(a,b|x',y), \tag{5.13}$$

and $\sum_b P(a,b|x,y) = \sum_b P(a,b|x,y')$ for all $y \neq y'$ follows similarly.

Finally, if the ontological model is classical, it is an LHV model for the box, where $\mathsf{P}(\omega) \equiv p(\lambda)$, $\mathsf{P}(\mathsf{A}_{a|x}) \equiv P(a|x,\lambda)$, and $\mathsf{P}(\mathsf{B}_{b|y}) \equiv P(b|y,\lambda)$. By Definition 1.5, such a box is local. □

Proposition 5.2. *Given any nonsignalling box, there is a (non-unique) corresponding ontological model whose probabilities assigned to the states of the world are not necessarily non-negative.*

This result was already derived in [AB11] from sheaf-theoretic concepts, however we provide a much more direct proof that is more suitable for the purposes of this work.

Proof. Let $\{P(ab|xy)\}_{a,b,x,y}$ be a nonsignalling box. We construct its associated ontological model $\big\{(\Omega,\mathcal{F},\mathsf{P}),\ \{\mathsf{A}_{a|x},\mathsf{B}_{b|y}\}_{a,b,x,y}\big\}$. We provide the proof for $a,b,x,y \in \{0,1\}$ for ease of notation, but the generalisation to more inputs and outputs is immediate.

To construct the ontological model, we postulate the existence of a set of states

$$\omega_{a_0a_1b_0b_1} \tag{5.14}$$

with quasi-probabilities

$$\mathsf{P}_{a_0a_1b_0b_1} \equiv \mathsf{P}(\omega_{a_0a_1b_0b_1}). \tag{5.15}$$

Each state corresponds to an *instruction set* [AB14], i.e., the state where Alice outputs a_0 on input $x=0$ and a_1 on input $x=1$, and Bob outputs b_0 on input $y=0$ and b_1 on input $y=1$. Then, each $\mathsf{P}_{a_0a_1b_0b_1}$ is the quasi-probability of the corresponding instruction set. Of course, if the given box is post-classical, not all of these quasi-probabilities will be non-negative. In fact, in principle it need not even be guaranteed that one can find a quasi-probability distribution over these states. But we will use the probability distribution of the inputs and outputs of the given nonsignalling box to derive a linear system of equations over the quasi-probabilities, and show that it does have a solution.

There are 16 states in total, as there are two possible outputs for each of the 4 inputs

($|\mathcal{A}|^{|\mathcal{X}|} \cdot |\mathcal{B}|^{|\mathcal{Y}|}$ in general). Then, each partition corresponds to a set of states as follows:

$$\begin{aligned} \mathsf{A}_{a|x} &= \{\omega_{a_0a_1b_0b_1} : a_x = a\} \\ \mathsf{B}_{b|y} &= \{\omega_{a_0a_1b_0b_1} : b_y = b\} \ . \end{aligned} \tag{5.16}$$

We associate the probabilities $P(ab|xy)$ of the nonsignalling box to the probabilities $\mathsf{P}(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y})$ of each intersection of partitions, for each input pair (x, y) and output pair (a, b). This gives rise to a set of equations for the probabilities $\mathsf{P}_{a_0a_1b_0b_1}$. Indeed, the probability of each intersection is given by

$$\mathsf{P}(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}) = \sum_{a_{\bar{x}}\ , b_{\bar{y}}} \mathsf{P}_{a_x a_{\bar{x}} b_y b_{\bar{y}}} \tag{5.17}$$

where we denote the output corresponding to the input that is not x as $a_{\bar{x}}$, and similarly for $b_{\bar{y}}$, and so we have, for each a, b, x, y,

$$\sum_{a_{\bar{x}}\ , b_{\bar{y}}} \mathsf{P}_{a_x a_{\bar{x}} b_y b_{\bar{y}}} = P(ab|xy)\,. \tag{5.18}$$

Since there are 16 values of $P(ab|xy)$ in the 2-input 2-output nonsignalling box, we arrive at 16 equations ($|\mathcal{A}| \times |\mathcal{B}| \times |\mathcal{X}| \times |\mathcal{Y}|$ in general). Of course, there are some linear dependencies between the equations, but we will show that the system still has a solution.

The system of equations can be expressed as

$$M\overline{\mathsf{P}} = C \tag{5.19}$$

where M is the matrix of coefficients, $\overline{\mathsf{P}}$ is the vector of probabilities $\mathsf{P}_{a_0a_1b_0b_1}$ and C is the vector of independent terms $P(ab|xy)$. The system has a solution (which is not necessarily unique) if and only if

$$\operatorname{rank}(M) = \operatorname{rank}(M|C)\,. \tag{5.20}$$

Since the rank of a matrix is the number of linearly independent rows, it is trivially true that

$$\operatorname{rank}(M) \leq \operatorname{rank}(M|C)\,, \tag{5.21}$$

as including the independent terms can only remove some relations of linear dependence, not add more. Equivalently, the number of relations of linear dependence of $M|C$ is always smaller than or equal to the number of relations of linear dependence of M. Therefore, to show that their ranks are equal, it is sufficient to show that every relation of linear dependence that we find in M still holds in $M|C$. That is, for every relation of linear dependence between the probabilities $\mathsf{P}_{a_0a_1b_0b_1}$ that is contained in M , it is sufficient to show that the relation still holds when the sums of probabilities are matched to the elements $P(ab|xy)$ of the nonsignalling box in order to show that the system of equations has a solution.

Observe that M contains only zeros and ones, as the equations (5.18) are just sums of probabilities. Moreover, each column of M corresponds to the probability of a state $\omega_{a_0a_1b_0b_1}$, while each row corresponds to an equation with independent term $P(ab|xy)$. Because the equations (5.18) correspond to intersections of partitions of the set of $\omega_{a_0a_1b_0b_1}$, we can observe that each row of M has a 1 in the column corresponding to the states $\omega_{a_0a_1b_0b_1}$ contained in the corresponding partition, and a 0 elsewhere. Put another way, in order to construct M one must first partition the set of $\omega_{a_0a_1b_0b_1}$ in four different ways, corresponding to

$$\begin{gathered}\left\{\mathsf{A}_{a|0}\right\}_a , \left\{\mathsf{A}_{a|1}\right\}_a , \\ \left\{\mathsf{B}_{b|0}\right\}_b , \left\{\mathsf{B}_{b|1}\right\}_b\end{gathered} \tag{5.22}$$

for Alice and Bob respectively. This gives partitions of the columns of M . Then, the 16 possible ways of intersecting partitions of Alice's with partitions of Bob's give the 16 equations with independent term $P(ab|xy)$. But notice now that the partition structure imposes a certain relation of linear dependence between the rows of M. Indeed, for each b, y, we have

$$\left\{\bigcup_a \left(\mathsf{A}_{a|0} \cap \mathsf{B}_{b|y}\right)\right\} = \left\{\bigcup_a \left(\mathsf{A}_{a|1} \cap \mathsf{B}_{b|y}\right)\right\} , \tag{5.23}$$

as

$$\bigcup_a \left(\mathsf{A}_{a|0} \cap \mathsf{B}_{b|y}\right) = \left(\bigcup_a \mathsf{A}_{a|0}\right) \cap \mathsf{B}_{b|y} = \Omega \cap \mathsf{B}_{b|y} = \left(\bigcup_a \mathsf{A}_{a|1}\right) \cap \mathsf{B}_{b|y} = \bigcup_a \left(\mathsf{A}_{a|1} \cap \mathsf{B}_{b|y}\right) , \tag{5.24}$$

and, similarly, for each a, x we have

$$\left\{\bigcup_b \left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|0}\right)\right\} = \left\{\bigcup_b \left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|1}\right)\right\} . \tag{5.25}$$

Using the correspondence of these partitions with the partitions of the columns of M

gives 8 relations of linear dependence between its rows. Now, noticing that a union of columns of M corresponds to a sum of probabilities $\mathsf{P}_{a_0a_1b_0b_1}$, we find that these relations correspond exactly to the nonsignalling conditions, as

$$\mathsf{P}\left(\bigcup_a \left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}\right)\right) = \sum_a \mathsf{P}\left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}\right) \tag{5.26}$$

so

$$\sum_a \mathsf{P}\left(\mathsf{A}_{a|0} \cap \mathsf{B}_{b|y}\right) = \sum_a \mathsf{P}\left(\mathsf{A}_{a|1} \cap \mathsf{B}_{b|y}\right) \tag{5.27}$$

and similarly for Bob. Of course, by definition of nonsignalling box, these relations hold for the independent terms $P(ab|xy)$ as well, since

$$\begin{aligned} \sum_a \mathsf{P}\left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}\right) &= \sum_a P(ab|xy)\,, \\ \sum_b \mathsf{P}\left(\mathsf{A}_{a|x} \cap \mathsf{B}_{b|y}\right) &= \sum_b P(ab|xy) \end{aligned} \tag{5.28}$$

by construction of the linear system (see equations (5.17) and (5.18)). Therefore, every relation of linear dependence between the rows of M holds also between the rows of $M|C$, as required.

Notice also that the implication goes both ways: the linear system has a solution *only if* the set of probabilities $P(ab|xy)$ is nonsignalling. The coefficient matrix M includes the nonsignalling conditions by construction of the states $\omega_{a_0a_1b_0b_1}$ with probabilities $\mathsf{P}_{a_0a_1b_0b_1}$. Therefore, if these conditions do not hold for the independent terms $P(ab|xy)$, then the rank of $M|C$ must be larger than that of M, as $M|C$ contains more linearly independent rows than M. □

With the mapping between ontological models and nonsignalling boxes in mind, we next define *common certainty of disagreement* for nonsignalling boxes. The idea is to reinterpret the definitions in Section 5.1 in this latter setting.

We first provide meaning for the events of interest (previously identified as E_A, E_B) in the present setting. Now, these events correspond to some set of outputs, given that the nonsignalling box was queried with some particular inputs. For the sake of concreteness, we fix these inputs to be $x = 1$, $y = 1$ and the *outputs of interest* to be $a = 1$, $b = 1$. This motivates us to consider the events $F_A = \{(1, b, 1, y)\}_{b\in\mathcal{B},y\in\mathcal{Y}}$ (on Alice's side) and $F_B = \{(a, 1, x, 1)\}_{a\in\mathcal{A},x\in\mathcal{X}}$ (on Bob's side). Then, we say that F_A and

F_B are perfectly correlated when

$$P(a,b|x=1,y=1) = 0 \text{ for all } a \neq b. \tag{5.29}$$

Given this, we assume that the agents will actually conduct their measurements according to some other partitions. Again, for concreteness, let us assume that those partitions are the ones associated with inputs $x=0$, $y=0$. These inputs take on the role of partitions $\mathcal{P}_A$, $\mathcal{P}_B$ in the ontological model picture. The outputs obtained from these measurements are the nonsignalling box analogue to the events $\mathcal{P}_A(\omega)$, $\mathcal{P}_B(\omega)$. In order to make the following expressions more concrete, we assume, when $x=0$, $y=0$ are inputted, that the outputs obtained are $a=0$ and $b=0$, respectively.

Therefore, given the perfectly correlated events F_A, F_B and numbers q_A, $q_B \in [0,1]$, we define the sets

$$\alpha_0 = \{a : P(b=1|a, x=0, y=1) = q_A\} \ , \tag{5.30}$$
$$\beta_0 = \{b : P(a=1|b, x=1, y=0) = q_B\} \ , \tag{5.31}$$

and, for all $n \geq 0$,

$$\alpha_{n+1} = \{a \in \alpha_n : P(B_n|a, x=0, y=0) = 1\} \ , \tag{5.32}$$
$$\beta_{n+1} = \{b \in \beta_n : P(A_n|b, x=0, y=0) = 1\} \ , \tag{5.33}$$

where

$$A_n = \alpha_n \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y} \, , \tag{5.34}$$
$$B_n = \mathcal{A} \times \beta_n \times \mathcal{X} \times \mathcal{Y} \, . \tag{5.35}$$

By analogy with the sets in equation (5.1), the set α_0 is the set of Alice's outcomes such that she assigns probability q_A to F_B, having input $x=0$. The set β_1 is the set of Bob's outcomes such that he is certain that Alice assigned probability q_A to F_B, and so on, and similarly for β_0, α_1, etc.

Suppose that Alice and Bob both input 0 and get output 0. Then, there is *common certainty of disagreement* about the event that Alice assigns probability q_A to F_B and Bob assigns probability q_B to F_A if $q_A \neq q_B$ and

$$(a=0, b=0, x=0, y=0) \in A_n \cap B_n \quad \forall n \in \mathbb{N}. \tag{5.36}$$

Notice the relationship between this definition and the previous one: the ω^* in equation (5.2), at which the disagreement occurred, fixed the partition elements that Alice and Bob observed. Here, disagreement occurs at the inputs and outputs ($a = 0, b = 0, x = 0, y = 0$) that the agents obtain.

We are now in a position to state and prove the classical agreement theorem in the nonsignalling language, i.e. for local boxes. We restrict to boxes of two inputs and two outputs since, by Theorem 5.4, any larger box exhibiting disagreement can be reduced to a 2-input 2-output box that also exhibits disagreement, while preserving its locality properties. With the mapping defined above, the following is now a corollary of Theorem 5.1, although we provide a standalone proof of the result in the interest of readers more familiarised with the language of nonsignalling boxes. Moreover, Theorem 5.1 and Corollary 5.1 can be shown to be equivalent.

Corollary 5.1. *Suppose Alice and Bob share a local nonsignalling box with underlying probability distribution P. Let $q_A, q_B \in [0,1]$, and let*

$$\begin{aligned} P(b=1|a=0,x=0,y=1) &= q_A, \\ P(a=1|b=0,x=1,y=0) &= q_B. \end{aligned} \tag{5.37}$$

If q_A and q_B are common certainty between the agents, then $q_A = q_B$.

Proof. By definition of q_A, q_B, and using the fact that the shared distribution is local and hence satisfies Definition 1.5, we have

$$\begin{aligned} q_A \sum_\lambda p(\lambda) P_A(0|0\lambda) &= \sum_\lambda p(\lambda) P_A(0|0\lambda) P_B(1|1\lambda) \\ q_B \sum_\lambda p(\lambda) P_B(0|0\lambda) &= \sum_\lambda p(\lambda) P_A(1|1\lambda) P_B(0|0\lambda)\,. \end{aligned} \tag{5.38}$$

In the proof of Theorem 5.2 we show that, if $1 \in \alpha_n$ or $1 \in \beta_n$ for all $n \in \mathbb{N}$ then there is no common certainty of disagreement for any nonsignalling distribution, and these encompass local distributions. Hence there only remains to prove the claim for $1 \notin \alpha_n$ and $1 \notin \beta_n$, for some $n \in \mathbb{N}$. This implies that

$$\begin{aligned} P(b=0|a=0,x=0,y=0) &= 1 \\ P(a=0|b=0,x=0,y=0) &= 1 \end{aligned} \tag{5.39}$$

and hence

$$
\begin{aligned}
\sum_\lambda p(\lambda)P_A(0|0\lambda) &= \sum_\lambda p(\lambda)P_A(0|0\lambda)P_B(0|0\lambda) \\
\sum_\lambda p(\lambda)P_B(0|0\lambda) &= \sum_\lambda p(\lambda)P_A(0|0\lambda)P_B(0|0\lambda),
\end{aligned}
\tag{5.40}
$$

which implies, on the one hand, that

$$
\sum_\lambda p(\lambda)P_A(0|0\lambda) = \sum_\lambda p(\lambda)P_B(0|0\lambda) \tag{5.41}
$$

and, on the other, that

$$
\sum_\lambda p(\lambda)P_A(0|0\lambda)P_B(1|0\lambda) = \sum_\lambda p(\lambda)P_A(1|0\lambda)P_B(0|0\lambda) = 0, \tag{5.42}
$$

that is,

$$
P_A(0|0\lambda)P_B(1|0\lambda) = P_A(1|0\lambda)P_B(0|0\lambda) = 0 \tag{5.43}
$$

for all λ. Therefore, there remains to prove only that

$$
\sum_\lambda p(\lambda)P_A(0|0\lambda)P_B(1|1\lambda) = \sum_\lambda p(\lambda)P_A(1|1\lambda)P_B(0|0\lambda). \tag{5.44}
$$

Because the outputs for inputs $x = 1, y = 1$ are perfectly correlated, we have

$$
P_A(0|1\lambda)P_B(1|1\lambda) = P_A(1|1\lambda)P_B(0|1\lambda) = 0 \tag{5.45}
$$

for all λ and, since $P_A(0|1\lambda) + P_A(1|1\lambda) = 1$ and similarly for P_B, this implies

$$
P_A(1|1\lambda) = P_B(1|1\lambda). \tag{5.46}
$$

Then we can prove (5.44) by simple manipulations of the probability distributions of each party: where we have used the fact that $\sum_{b\in\mathcal{B}} P_B(b|y\lambda) = 1$ for all y, λ in the first equality, (5.43) in the second and third, $\sum_{a\in\mathcal{A}} P_A(a|x\lambda) = 1$ for all x, λ in the fourth, (5.45) again in the fifth, and $P_A(1|1\lambda)^2 = P_A(1|1\lambda)$ for all λ (since $P_A(a|x\lambda)$ can be assumed to be either 1 or 0 for every a, x, λ) in the last. □

5.3 Nonsignalling agents can agree to disagree

Given the mapping exhibited above, as well as the restatement of the agreement theorem for local boxes, it is now natural to ask whether the agreement theorem holds when dropping the locality constraint. When we generalise the setting and allow the agents to share a generic nonsignalling box, we find that the agreement theorem does not hold. That is, nonsignalling observers *can* agree to disagree, and we characterise the distributions that give rise to common certainty of disagreement. Later, we find that no such distribution can be quantum—i.e., quantum observers *cannot* agree to disagree.

We first present the following theorem in which the nonsignalling box has two inputs and two outputs, but we will show in Theorem 5.4 that the result is fully general. In place of "common certainty of disagreement about the event that Alice assigns probability q_A to $F_B = \{(a, 1, x, 1)\}_{a\in\mathcal{A}, x\in\mathcal{X}}$ and Bob assigns probability q_B to $F_A = \{(1, b, 1, y)\}_{b\in\mathcal{B}, y\in\mathcal{Y}}$, at event $(0, 0, 0, 0)$," we simply say "common certainty of disagreement."

Theorem 5.2. *A 2-input 2-output nonsignalling box gives rise to common certainty of disagreement if and only if it takes the form of Table 5.1.*

$xy\backslash ab$	00	01	10	11
00	r	0	0	$1-r$
01	$r-s$	s	$-r+t+s$	$1-t-s$
10	$t-u$	u	$r-t+u$	$1-r-u$
11	t	0	0	$1-t$

Table 5.1: Parametrisation of 2-input 2-output nonsignalling boxes with common certainty of disagreement. Here, $r, s, t, u \in [0, 1]$ are such that all the entries of the box are non-negative, $r > 0$, and $s - u \neq r - t$.

We provide an outline of the proof before turning to the proof itself.

To prove the direct implication, we first consider the case in which, for some n onwards, α_n, β_n each contain only one output, $a = 0, b = 0$, respectively. By the definition of α_{n+1}, this implies that $P(B_n|a = 0, x = 0, y = 0) = 1$, and, thus, $P(01|00) = 0$. Similarly, the definition of β_{n+1} gives $P(10|00) = 0$. Perfect correlations in the inputs $x = 1, y = 1$ imply that $P(01|11) = P(10|11) = 0$, and the rest of the table is deduced in terms of parameters r, s, t, u by using nonsignalling and normalisation constraints. The condition $r > 0$ ensures that $P(00|00) > 0$, as per the input and output that the agents in fact obtained. Finally, $q_A \neq q_B$ if and only if $s - u \neq r - t$, which concludes the proof of this case.

If, for all n, one or both of α_n, β_n contain(s) both outputs, we find $q_A = q_B$, contradicting common certainty of disagreement.

The converse implication is proved by writing q_A, q_B in terms of the parameters of the box. If $\alpha_0 = \{a = 0\}$ and $\beta_0 = \{b = 0\}$, then we find $\alpha_1 = \alpha_0$ and $\beta_1 = \beta_0$, and common certainty of disagreement follows.

If the parameters of the box are such that $\alpha_0 = \{a = 0, a = 1\}$ but $\beta_0 = \{b = 0\}$, then the definition of β_1 implies $P(b = 0|a = 0, x = 0, y = 0) = 1$; therefore, $(0, 0, 0, 0) \in A_1$, and common certainty of disagreement follows. One can reason symmetrically if $\alpha_0 = \{a = 0\}$ but $\beta_0 = \{b = 0, b = 1\}$. Finally, if both α_0 and β_0 are the full set of outcomes, then $s - u = r - t$, contradicting the statement of the Theorem.

In the proof of Theorem 5.2, we will make use of the following Lemma:

Lemma 5.1. *Consider a nonsignalling box of 2 inputs and 2 outputs. Then,* $\alpha_0 = \{0, 1\}$ *if and only if* $q_A = P(b = 1|y = 1)$. *Analogously,* $\beta_0 = \{0, 1\}$ *if and only if* $q_B = P(a = 1|x = 1)$.

Proof. By hypothesis,

$$\begin{aligned} q_A = P(b = 1|a = 0, x = 0, y = 1) &= \frac{P(01|01)}{P(a = 0|x = 0)} \\ = P(b = 1|a = 1, x = 0, y = 1) &= \frac{P(11|01)}{P(a = 1|x = 0)}. \end{aligned}$$

But now, we can write

$$P(b = 1|y = 1) = P(01|01) + P(11|01) = P(a = 0|x = 0)q_A + P(a = 1|x = 0)q_A = q_A.$$

The reverse implication is trivial. The analogous statement can be proved by interchanging the roles of Alice and Bob. □

Proof of Theorem 5.2. We first prove that common certainty of disagreement imposes the claimed structure for the nonsignalling box. Therefore, we assume common certainty of disagreement, i.e.,

$$(0, 0, 0, 0) \in A_n \cap B_n \qquad \forall n \in \mathbb{N}. \tag{5.47}$$

In particular, we also assume that Alice and Bob input $x = y = 0$ and obtain $a = b = 0$. This implies

$$P(00|00) > 0. \tag{5.48}$$

We split the proof into three cases based on the contents of the sets A_n, B_n:

Case 1. $1 \notin \alpha_n$, $1 \notin \beta_n$ for some n.[1] From common certainty of disagreement

[1] This need not happen at the same stage, i.e., possibly $1 \notin \alpha_m$, for some $m < n$. However in this

(equation (5.47)), we have that

$$P(B_n|a=0,x=0,y=0)=1, \qquad P(A_n|b=0,x=0,y=0)=1,$$

which, together with $1 \notin \alpha_n$, $1 \notin \beta_n$, translates into:

$$P(01|00)=0, \qquad P(10|00)=0.$$

We also assumed that the agents in fact obtained outputs $a=0, b=0$ on inputs $x=0, y=0$, so we must have $P(00|00)>0$. The rest of the table is determined by nonsignalling constraints in terms of parameters r, s, t and u. Given the box in the statement of the theorem, $q_A \neq q_B$ if and only if $s-u \neq r-t$, which concludes the proof of this case.

Case 2. $\alpha_n=\{0,1\}$, for all $n \in \mathbb{N}$ while $1 \notin \beta_m$ for some m. We show that this case implies $q_A=q_B$, so it contradicts common certainty of disagreement. Indeed, the definition of α_{m+1} enforces the conditions:

$$P(b=0|a=0,x=0,y=0)=1=P(b=0|a=1,x=0,y=0).$$

This implies

$$\begin{aligned} P(b=1|a=0,x=0,y=0) &= \frac{P(01|00)}{P(a=0|x=0)}=0 \quad \Rightarrow \quad P(01|00)=0,\\ P(b=1|a=1,x=0,y=0) &= \frac{P(11|00)}{P(a=1|x=0)}=0 \quad \Rightarrow \quad P(11|00)=0. \end{aligned}$$

Adding nonsignalling conditions to these last equations, we also obtain

$$0=P(b=1|y=0)=P(01|10)+P(11|10), \tag{5.49}$$

and so

$$P(01|10)=0=P(11|10) \tag{5.50}$$

and

$$P(b=0|y=0)=1. \tag{5.51}$$

case, since the sets are nonempty by assumption, we have $\alpha_n=\alpha_m$.

This allows us to identify q_B with $P(a=1|x=1)$, since

$$\begin{aligned} q_B &= P(a=1|b=0,x=1,y=0) \\ &= \frac{P(10|10)}{P(b=0|y=0)} \\ &= P(10|10) \\ &= P(a=1|x=1) - P(11|10) \\ &= P(a=1|x=1), \end{aligned}$$

where the third and last equalities follow from equations (5.51) and (5.50) respectively. Now, taking into account Lemma 5.1 and perfect correlations, we have

$$q_A = P(b=1|y=1) = P(a=1|x=1),$$

which shows that $q_A = q_B$, as mentioned above.

Case 3. $\alpha_n = \{0,1\}$, $\beta_n = \{0,1\}$ for all $n \in \mathbb{N}$. We now show that this case also implies $q_A = q_B$, contradicting common certainty of disagreement. Using Lemma 5.1 we have

$$q_B = P(a=1|x=1) \qquad \text{as well as} \qquad q_A = P(b=1|y=1).$$

Now, perfect correlations impose that $P(a=1|x=1) = P(b=1|y=1)$, that is, $q_A = q_B$.

Next, we prove the converse implication of the theorem. We show that any nonsignalling box of the above form must exhibit common certainty of disagreement. Since $s-u \neq r-t$, we have that Alice and Bob assign different probabilities to output $a,b=1$ on input $x,y=1$:

$$\begin{aligned} q_A &= P(b=1|a=0,x=0,y=1) = s/r, \\ q_B &= P(a=1|b=0,x=1,y=0) = (r-t+u)/r\,. \end{aligned} \tag{5.52}$$

In the case that $1 \notin \alpha_0$, $1 \notin \beta_0$, we also have that $\alpha_1 = \alpha_0$ and $\beta_1 = \beta_0$, and common certainty of disagreement follows, because $(0,0,0,0)$ is in $A_n \cap B_n$ for all n.

If the parameters are such that

$$\frac{1-t-s}{1-r} = \frac{s}{r}, \tag{5.53}$$

but

$$\frac{1-r-u}{1-r} \neq \frac{r-t+u}{r}, \tag{5.54}$$

then

$$P(b=1|a=1,x=0,y=1) = q_A, \tag{5.55}$$

as well, but

$$P(a=1|b=1,x=1,y=0) \neq q_B, \tag{5.56}$$

and so $1 \in \alpha_0$, $1 \notin \beta_0$. Since we have

$$P(b=0|a=0,x=0,y=0) = 1, \tag{5.57}$$

we find $(0,0,0,0) \in A_1$,[2] and hence all A_n still contain $(0,0,0,0)$, yielding common certainty of disagreement.

Symmetric reasoning covers the case $1 \notin \alpha_0$, $1 \in \beta_0$, and only the case where $\alpha_0 = \{0,1\}$, $\beta_0 = \{0,1\}$ remains. This happens when

$$\begin{aligned} P(b=1|a=1,x=0,y=1) &= P(b=1|a=0,x=0,y=1), \\ P(a=1|b=0,x=1,y=0) &= P(a=1|b=1,x=1,y=0) \end{aligned} \tag{5.58}$$

which, in terms of the parameters, is equivalent to

$$\frac{1-t-s}{1-r} = \frac{s}{r}, \tag{5.59}$$

$$\frac{1-r-u}{1-r} = \frac{r-t+u}{r}. \tag{5.60}$$

However, these two conditions are satisfied simultaneously only when $s-u=r-t$, as we now show. From Equation (5.59) we get

$$s = r(1-t),$$

while from Equation (5.60) we obtain

$$u = t(1-r).$$

This means that if Equations (5.59) and (5.60) are both satisfied, then

$$s-u = r(1-t) - t(1-r) = r-t,$$

[2]Note $(1,0,0,0) \notin A_1$, though this does not affect the present proof.

which contradicts the statement of the Theorem. □

5.4 Quantum agents cannot agree to disagree

While some nonsignalling distributions can exhibit common certainty of disagreement, we find that probability distributions arising in quantum mechanics do satisfy the agreement theorem. This is surprising: it is well-known that a given measurement of a quantum system (say, that corresponding to the input $x,y=0$) need not offer any information about the outcome of an incompatible measurement on the same system (say, $x,y=1$). However, some consistency remains: common certainty of disagreement is impossible, even for incompatible measurements.

Theorem 5.3. *No 2-input 2-output quantum box can give rise to common certainty of disagreement.*

Proof. In order to give rise to common certainty of disagreement, the probability distribution that the state and measurements generate must be of the form of Table 5.1. Theorem 1 in Tsirelson's seminal paper [Cir80] implies that, if there is a quantum realisation of the box, then there exist real, unit vectors

$$|w_x\rangle\,, |v_y\rangle \tag{5.61}$$

such that the correlations

$$c_{xy} := P(a=b|xy) - P(a\neq b|xy) \tag{5.62}$$

satisfy

$$c_{xy} = \langle w_x|v_y\rangle \tag{5.63}$$

for each x, y. For the box in Theorem 5.2, this means, in particular, that

$$\begin{aligned}\langle w_0|v_0\rangle &= 1\,,\\ \langle w_1|v_1\rangle &= 1\,,\end{aligned} \tag{5.64}$$

and, since the vectors have unit norm, this implies that

$$\begin{aligned}|w_0\rangle &= |v_0\rangle\ ,\\ |w_1\rangle &= |v_1\rangle\ .\end{aligned} \tag{5.65}$$

Then, we are left with

$$\begin{aligned} c_{01} &= \langle w_0 | w_1 \rangle \ , \\ c_{10} &= \langle w_1 | w_0 \rangle \ . \end{aligned} \tag{5.66}$$

Since the vectors are real, we find

$$c_{01} = c_{10} \, , \tag{5.67}$$

but this implies that

$$s - u = r - t, \tag{5.68}$$

which implies that $q_A = q_B$ and, hence, impedes disagreement. $\square$

We have seen that no 2-input 2-output quantum box can give rise to common certainty of disagreement. We now lift the restriction on the number of inputs and outputs and show that no quantum box can give rise to common certainty of disagreement.

First, notice that the proof for 2 inputs and outputs did not require *common* certainty, but only first-order mutual certainty. Indeed, by observing the definitions of the sets α_n, β_n, one can see that $\alpha_n = \alpha_1$ and $\beta_n = \beta_1$ for all $n \geq 1$. This means that first-order mutual certainty *implies* common certainty, and, therefore, first-order certainty suffices to characterise the nonsignalling box that displays common certainty of disagreement.

As the number of outputs grows, first-order mutual certainty is no longer sufficient. However, since the number of outputs is always finite, there exists an $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ and $\beta_n = \beta_N$ for all $n \geq N$. Since $\alpha_{n+1} \subseteq \alpha_n \, \forall n$, and similarly for β, the sets α_N, β_N are the *smallest* sets of outputs for which the disagreement occurs. Because of this, any (a, b, x, y) that belongs to $A_N \cap B_N$ will also belong to $A_n \cap B_n$ for all n; that is, Nth-order mutual certainty implies common certainty. So, for any finite nonsignalling box, one needs only Nth-order mutual certainty to characterise it. As the number of outputs grows unboundedly, one needs *common* certainty to hold [GP82]. These observations will be relevant to extending Theorem 5.3 beyond two inputs and outputs.

Theorem 5.4. *No quantum box can give rise to common certainty of disagreement.*

We show that any nonsignalling box with common certainty of disagreement induces a 2-input 2-output nonsignalling box with the same property. Thus, if there existed a quantum system that could generate the bigger box, it could also generate the smaller box. Then, Theorem 5.2 implies that no quantum box can give rise to common certainty of disagreement.

To show the reduction of the box, we use the ideas presented in Chapter 1 about transforming probability distributions while preserving locality (which, as we shall see, preserve normalisation and nonsignalling too). Since the original, larger box exhibits common certainty of disagreement *at* event $(0,0,0,0)$ *about* event $(1,1,1,1)$, it is enough to consider inputs $0,1$ for each party, and any extra available inputs can be ignored. Grouping the outputs of the original box in two sets in order to map them to the effective box is not as straightforward, as we must ensure that the effective box also displays common certainty of disagreement. Recalling the discussion preceding Theorem 5.4, there exists an $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ and $\beta_n = \beta_N$ for all $n \geq N$. Outputs for each agent are then grouped according to whether or not they belong in each of these sets respectively. Because the transformations in the probabilities are local, the effective box is still normalised and nonsignalling. It is then possible to check that the effective box satisfies common certainty of disagreement if the original box did.

Proof. We define a mapping from a distribution $\{P(ab|xy)\}_{a\in\mathcal{A},b\in\mathcal{B},x\in\mathcal{X},y\in\mathcal{Y}}$ to an effective distribution $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}_{\tilde{a},\tilde{b},\tilde{x},\tilde{y}\in\{0,1\}}$ such that the following conditions hold:

(i) if $\{P(ab|xy)\}$ is quantum, then so is $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}$,

(ii) if $\{P(ab|xy)\}$ satisfies common certainty of disagreement, then so does $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}$.

First, notice that the number of inputs can be reduced to 2 without loss of generality, as common certainty of disagreement is always defined to be *at* an event (wlog, $(0,0,0,0)$) *about* another event (wlog, $(1,1,1,1)$). One can associate the inputs $x=0, y=0$ with $\tilde{x}=0, \tilde{y}=0$, respectively, and $x=1, y=1$ with $\tilde{x}=1, \tilde{y}=1$ respectively, and ignore all other possible inputs in $\mathcal{X},\mathcal{Y}$. The outputs, instead, must be grouped according to whether or not they belong in the sets α_n, β_n (for input 0) and whether or not they correspond to the event obtaining, i.e., whether or not they are equal to 1 (for input 1).

Since P satisfies common certainty of disagreement, we know that $(0,0,0,0) \in A_n \cap B_n$. Moreover, by the definitions of the sets α_n, β_n (and since we only consider finite sets $\mathcal{A},\mathcal{B},\mathcal{X},\mathcal{Y}$) there exists an $N \in \mathbb{N}$ such that $\alpha_n = \alpha_N$ and $\beta_n = \beta_N$ for all $n \geq N$. Take such N, and define the following indicator functions:

$$\begin{aligned} \chi^{\alpha}_{0|0}(a) &= \begin{cases} 0 & a \notin \alpha_N \\ 1 & a \in \alpha_N \end{cases} \\ \chi^{\beta}_{0|0}(b) &= \begin{cases} 0 & b \notin \beta_N \\ 1 & b \in \beta_N \end{cases} \\ \chi^{\alpha}_{0|1}(c) = \chi^{\beta}_{0|1}(c) &= \begin{cases} 0 & c = 1 \\ 1 & c \neq 1 \end{cases} \end{aligned} \tag{5.69}$$

(where c stands for output a, b for Alice and Bob, respectively), with

$$\begin{aligned} \chi^{\alpha}_{1|x}(a) &= 1 - \chi^{\alpha}_{0|x}(a) \\ \chi^{\beta}_{1|y}(b) &= 1 - \chi^{\beta}_{0|y}(b) \end{aligned} \tag{5.70}$$

for each a, b, x, y. Then, the mapping from P to $\tilde{P}$ is defined as follows:

$$\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) = \sum_{a,b} \delta_{x,\tilde{x}}\delta_{y,\tilde{y}}\chi^{\alpha}_{\tilde{a}|x}(a)\chi^{\beta}_{\tilde{b}|y}(b)P(ab|xy) \tag{5.71}$$

where

$$\delta_{s,t} = \begin{cases} 0 & s \neq t \\ 1 & s = t\,. \end{cases} \tag{5.72}$$

We note that the distribution $\tilde{P}$ is merely a local post-processing of P, and hence it is quantum if P is. Indeed, the function χ that defines $\tilde{P}$ only relates the inputs and outputs of each agent individually. Therefore, condition (i) holds, as, letting $E_{a|x}, F_{b|y}, \rho$ be the POVMs and state defining P, we have

$$\begin{aligned} \tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y}) &= \sum_{a,b} \delta_{x,\tilde{x}}\delta_{y,\tilde{y}}\chi^{\alpha}_{\tilde{a}|x}(a)\chi^{\beta}_{\tilde{b}|y}(b)\,\mathrm{tr}\big(E_{a|x} \otimes F_{b|y}\rho\big) \\ &= \mathrm{tr}\left[\left(\sum_{a} \delta_{x,\tilde{x}}\chi^{\alpha}_{\tilde{a}|x}(a)E_{a|x}\right) \otimes \left(\sum_{b} \delta_{y,\tilde{y}}\chi^{\beta}_{\tilde{b}|y}(b)F_{b|y}\right)\rho\right] \\ &= \mathrm{tr}\left[E_{\tilde{a}|\tilde{x}} \otimes F_{\tilde{b}|\tilde{y}}\rho\right], \end{aligned} \tag{5.73}$$

where

$$
\begin{aligned}
E_{\tilde{a}|\tilde{x}} &= \sum_a \delta_{x,\tilde{x}} \chi^{\alpha}_{\tilde{a}|x}(a) E_{a|x} \\
F_{\tilde{b}|\tilde{y}} &= \sum_b \delta_{y,\tilde{y}} \chi^{\beta}_{\tilde{b}|y}(b) F_{b|y}
\end{aligned}
\tag{5.74}
$$

for each $\tilde{a}, \tilde{b}, \tilde{x}, \tilde{y}, x, y$.

In particular, one can check that $\tilde{P}$ is normalised and nonsignalling provided that P is normalised and nonsignalling. Normalisation follows straightforwardly from the definition, since for each input, each output in P gets mapped to a unique output in $\tilde{P}$, and all of the outputs in P get mapped to some output in $\tilde{P}$ (i.e. the map from P to $\tilde{P}$ is a surjective function). Because the map is defined differently for each pair of inputs and outputs, the nonsignalling conditions need to be checked for each line. However, the computations all follow the same pattern, and we perform only one as an example:

$$
\begin{aligned}
\sum_{\tilde{a}} \tilde{P}(\tilde{a}0|00) &= \sum_{\substack{a \in \alpha_N \\ b \in \beta_N}} P(ab|00) + \sum_{\substack{a \notin \alpha_N \\ b \in \beta_N}} P(ab|00) \\
&= \sum_{\substack{a \in \mathcal{A} \\ b \in \beta_N}} P(ab|00) \\
&= \sum_{\substack{a \in \mathcal{A} \\ b \in \beta_N}} P(ab|10) \\
&= \sum_{\substack{a \neq 1 \\ b \in \beta_N}} P(ab|10) + \sum_{b \in \beta_N} P(1b|10) \\
&= \sum_{\tilde{a}} \tilde{P}(\tilde{a}0|10)
\end{aligned}
\tag{5.75}
$$

where we have used the nonsignalling property of P in the third line, and the rest follows from the definition of the map (5.71).

To check condition (ii), let N be as in the definition of the map (5.71) and let $a \in \alpha_N$. Then, by definition of the set α_{N+1}, we have

$$
P(\beta_N | a, x = 0, y = 0) = 1 \tag{5.76}
$$

and, therefore,

$$
\frac{\sum_{b \in \beta_N} P(ab|00)}{\sum_{b \in \mathcal{B}} P(ab|00)} = 1\,, \tag{5.77}
$$

which entails

$$\sum_{b \notin \beta_N} P(ab|00) = 0\,. \tag{5.78}$$

Summing over $a \in \alpha_N$, we get

$$\sum_{\substack{a \in \alpha_N \\ b \notin \beta_N}} P(ab|00) = \tilde{P}(01|00) = 0\,. \tag{5.79}$$

Similarly, we find $\tilde{P}(10|00) = 0$. Since P satisfies common certainty of disagreement, its outputs on input $x = 1, y = 1$ must be perfectly correlated. That is, $P(ab|11) = 0$ if $a \neq b$. Hence,

$$\tilde{P}(01|11) = \sum_{a \neq 1} P(a1|11) = 0 \tag{5.80}$$

and similarly for $\tilde{P}(10|11)$. So far, the nonsignalling box corresponding to $\tilde{P}$ has two zeros in the first row and another two in the last. Using normalisation and nonsignalling conditions to fill in the rest of the table, we find it is of the form of the nonsignalling box in Theorem 5.2. There remains to check for disagreement, i.e. that if

$$q_A = P(b = 1|a = 0, x = 0, y = 1) \neq P(a = 1|b = 0, x = 1, y = 0) = q_B \tag{5.81}$$

then

$$\tilde{P}(\tilde{b} = 1|\tilde{a}, \tilde{x} = 0, \tilde{y} = 1) \neq \tilde{P}(\tilde{a} = 1|\tilde{b}, \tilde{x} = 1, \tilde{y} = 0)\,. \tag{5.82}$$

Since $\alpha_N \subseteq \alpha_0$ and $\beta_N \subseteq \beta_0$, $P(b = 1|a^*, x = 0, y = 1) \neq P(a = 1|b^*, x = 1, y = 0)$ holds in particular for all $a^* \in \alpha_N, b^* \in \beta_N$. This means that, for $a^* \in \alpha_N, b^* \in \beta_N$,

$$\frac{P(a^*1|01)}{\sum_{b \in \mathcal{B}} P(a^*b|01)} \neq \frac{P(1b^*|10)}{\sum_{a \in \mathcal{A}} P(ab^*|10)} \tag{5.83}$$

and so

$$P(a^*1|01) \sum_{a \in \mathcal{A}} P(ab^*|10) \neq P(1b^*|10) \sum_{b \in \mathcal{B}} P(a^*b|01)\,. \tag{5.84}$$

Then, we can sum over α_N and β_N on both sides to find

$$\sum_{a^* \in \alpha_N} P(a^*1|01) \sum_{\substack{a \in \mathcal{A} \\ b^* \in \beta_N}} P(ab^*|10) \neq \sum_{b^* \in \beta_N} P(1b^*|10) \sum_{\substack{a^* \in \alpha_N \\ b \in \mathcal{B}}} P(a^*b|01)\,. \tag{5.85}$$

But in terms of $\tilde{P}$, this corresponds to

$$\tilde{P}(01|01) \sum_{\tilde{a} \in \{0,1\}} \tilde{P}(\tilde{a}0|10) \neq \tilde{P}(10|10) \sum_{\tilde{b} \in \{0,1\}} \tilde{P}(0\tilde{b}|01) \tag{5.86}$$

which implies

$$\tilde{P}(\tilde{b} = 1|\tilde{a} = 0, \tilde{x} = 0, \tilde{y} = 1) \neq \tilde{P}(\tilde{a} = 1|\tilde{b} = 0, \tilde{x} = 1, \tilde{y} = 0) \tag{5.87}$$

and hence the disagreement occurs for the $\tilde{P}$ distribution as well, which proves the result.

Notice that the sets $\tilde{\alpha}_0, \tilde{\beta}_0$ in the distribution $\tilde{P}$ (defined analogously to α_0, β_0 in the distribution P) will correspond to outputs $\tilde{a}, \tilde{b} = 0$, respectively. This is to be expected, as the map $P \to \tilde{P}$ gives rise to a nonsignalling box of the form of the one in Theorem 5.2, where the sets $\tilde{\alpha}_0, \tilde{\beta}_0$ contain a single element each. (In effect, this means we are ignoring the outputs $a^* \in \alpha_0 \backslash \alpha_N$ and $b^* \in \beta_0 \backslash \beta_N$, but those outputs lead to disagreement but not to common certainty of it, so they can be safely discarded.)

Thus, if there existed a quantum box with common certainty of disagreement, there would also exist a 2-input 2-output quantum box with the same property. By Theorem 5.2 implies that no quantum box can give rise to common certainty of disagreement. □

5.5 Quantum agents cannot disagree singularly

We explore other forms of disagreement that might arise about perfectly correlated events. Since common certainty is a strong requirement, we remove it and, instead, suppose that the agents assign probabilities that differ maximally. We find that this new notion of disagreement exhibits the same behaviour as common certainty of disagreement.

In a nonsignalling box, there is *singular disagreement* about the probabilities assigned by Alice and Bob to perfectly correlated events $F_A = \{(1, b, 1, y)\}_{b \in \mathcal{B}, y \in \mathcal{Y}}$ and $F_B = \{(a, 1, x, 1)\}_{a \in \mathcal{A}, x \in \mathcal{X}}$, respectively, at event $(0, 0, 0, 0)$ if it holds that

$$q_A = 1, \; q_B = 0. \tag{5.88}$$

This time, there is no notion of common certainty—we just require that Alice's and Bob's assignments differ maximally.

Similarly to the previous section, we refer to the above definition simply as "singular disagreement."

We restrict ourselves first to boxes of two inputs and outputs and show that local boxes cannot exhibit singular disagreement. Then, we characterise the nonsignalling boxes that do satisfy singular disagreement and show they cannot be quantum. Finally, we generalise to boxes of any number of inputs and outputs.

Theorem 5.5. *There is no local 2-input 2-output box that gives rise to singular disagreement.*

Proof. Assume Alice and Bob input $x = y = 0$ and obtain $a = b = 0$. This implies

$$P(00|00) > 0. \tag{5.89}$$

Alice assigns

$$P(b = 1|a = 0, x = 0, y = 1) = 1, \tag{5.90}$$

and Bob assigns

$$P(a = 1|b = 0, x = 1, y = 0) = 0\,. \tag{5.91}$$

Further, the outputs for input $(x, y) = (1, 1)$ are perfectly correlated, so, in particular,

$$P(01|11) = 0. \tag{5.92}$$

. Equations (5.90) and (5.91) imply, respectively,

$$P(00|01) = 0 \text{ and } P(10|10) = 0. \tag{5.93}$$

However, equations (5.89), (5.92) and (5.93) make up a form of Hardy's paradox [Har92], which is known not to hold for local distributions. □

We now lift the local restriction and characterise the nonsignalling boxes in which singular disagreement occurs.

Theorem 5.6. *A 2-input 2-output nonsignalling box gives rise to singular disagreement if and only if it takes the form of Table 5.2.*

Proof. First, we show that singular disagreement implies that the nonsignalling box must be of the above form. By construction, the inputs $x = y = 1$ have perfectly correlated outputs, so that

$$P(01|11) = P(10|11) = 0\,. \tag{5.94}$$

Also, singular disagreement requires

$xy\backslash ab$	00	01	10	11
00	s	t	$1-s-u-t$	u
01	0	$s+t$	r	$1-s-t-r$
10	$1-u-t$	$u+t+r-1$	0	$1-r$
11	r	0	0	$1-r$

Table 5.2: Parametrisation of 2-input 2-output nonsignalling boxes with singular disagreement. Here, r, s, t, u, $\in [0,1]$ are such that all the entries of the box are non-negative, $s > 0$, and $s+t \neq 0$ and $u+t \neq 1$.

$$P(b = 1|a = 0, x = 0, y = 1) \quad = 1, \tag{5.95}$$

$$P(a = 1|b = 0, x = 1, y = 0) \quad = 0. \tag{5.96}$$

Equation (5.95) implies that $P(00|01) = 0$ and $P(01|01) \neq 0$, while Equation (5.96) implies that $P(10|10) = 0$ and $P(00|10) \neq 0$. The rest of the entries follow from normalisation and nonsignalling conditions. The condition $s > 0$ ensures that $P(00|00) > 0$, as per the input and output that the agents in fact obtained. Therefore, any two-input two-output nonsignalling box that gives rise to singular disagreement must be of the above form.

Proving the converse is straightforward, as it suffices to check that equations (5.95) and (5.96) are satisfied for the parameters of the box. □

However, singular disagreement cannot arise in quantum systems. This is another way in which quantum mechanics provides some consistency between (possibly incompatible) measurements, just like in the case of common certainty of disagreement.

Theorem 5.7. *No 2-input 2-output quantum box can give rise to singular disagreement.*

Proof. Due to their form, the boxes in Theorem 5.6 are *quantum voids* [RDBC19]; i.e., they are either local or post-quantum. This can be seen by observing that the mapping

$$x \mapsto x \oplus 1\,, \tag{5.97}$$

which is a symmetry of the box, makes all four 0's lie in entries $P(ab|xy)$ such that $a \oplus b \oplus 1 = xy$. As stated in Sections III and V.B of Ref. [RDBC19], all boxes with four 0's in entries of the above form are quantum voids.

Therefore, the box in Table 5.2 is either local, in which case it does not lead to singular disagreement, or has no possible quantum realisation, proving the claim. □

Finally, the above results can be generalised to any finite box:

Theorem 5.8. *No quantum box can give rise to singular disagreement.*

Proof. Like in Theorem 5.4, we show that any nonsignalling box with singular disagreement induces a 2-input 2-output nonsignalling box with the same property, and rely on Theorem 5.7 to deduce that no quantum system can give rise to singular disagreement. The mapping from P to $\tilde{P}$ for inputs $x, y = 0$ is as in Theorem 5.4 but substitutes α_N, β_N for α_0, β_0 respectively.

Analogously to Theorem 5.4, to prove the Theorem for singular disagreement we define a mapping from a distribution $\{P(ab|xy)\}_{a\in\mathcal{A},b\in\mathcal{B},x\in\mathcal{X},y\in\mathcal{Y}}$ to an effective distribution $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}_{\tilde{a},\tilde{b},\tilde{x},\tilde{y}\in\{0,1\}}$ such that the following conditions hold:

(i) if $\{P(ab|xy)\}$ is quantum, then so is $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}$,

(ii) if $\{P(ab|xy)\}$ satisfies singular disagreement, then so does $\left\{\tilde{P}(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})\right\}$.

Again, the number of inputs can be reduced to 2 without loss of generality. To group the outputs, we notice that the sets A_0, B_0 also play a role in singular disagreement, as they group the outputs of each party which lead them to assign their respective probabilities to the event. Then, we group the outputs according to whether or not they belong in the sets α_0, β_0 (for input 0) and whether or not they correspond to the event obtaining, i.e. whether or not they are equal to 1 (for input 1). We obtain the same mapping (5.71) as before, substituting α_N for α_0 and β_N for β_0. With this replacement, condition (i) follows by the same argument as before. To check condition (ii), we know that, for all $a^* \in \alpha_0$,

$$P(b = 1|a^*, x = 0, y = 1) = 1 \tag{5.98}$$

and so

$$P(a^*1|01) = \sum_{b\in\mathcal{B}} P(a^*b|01). \tag{5.99}$$

Summing over $a^* \in \alpha_0$ and rewriting the expression in terms of $\tilde{P}$, we find

$$\tilde{P}(01|01) = \sum_{\tilde{b}\in\{0,1\}} \tilde{P}(0\tilde{b}|01) \tag{5.100}$$

which implies

$$\tilde{P}(\tilde{b} = 1|\tilde{a} = 0, \tilde{x} = 0, \tilde{y} = 1) = 1. \quad (5.101)$$

Similarly, for all $b^* \in \beta_0$ we have

$$P(a = 1|b^*, x = 1, y = 0) = 0, \quad (5.102)$$

hence

$$P(1b^*|10) = 0 \quad (5.103)$$

and so, by adding over $b^* \in \beta_0$ and mapping to $\tilde{P}$, we find

$$\tilde{P}(10|10) = 0 \quad (5.104)$$

as required. □

Chapter 6

Conclusions

6.1 Multipartite entanglement and nonlocality

In a time where some quantum technologies are already within our reach, the theoretical study of multicomponent systems is essential to develop more applications and implement the ones that have already been proposed. Entanglement and nonlocality are the main quantum effects behind these applications, hence they have been the main object of study of this thesis. First, we have ordered the set of multipartite entangled states via a resource theory, in order to understand which multipartite states are more useful than which others. Then, we have focused on a particular kind of multipartite states that are arguably the easiest to implement in practice: pair-entangled network states. We have shown that GMNL is intrinsic to these states if the pair-entanglement is pure, and given fundamental limitations to the obtention of GMNL and even GME from some of these states when the pair-entanglement is mixed. Further, we have shown several ways to overcome these limitations, including by adding more connections to the network or taking several copies of it (thus achieving superactivation).

In Chapter 2, we have addressed the problem of ordering the set of multipartite entangled states. While LOCC and its stochastic variant give rise to inequivalent forms of entanglement and isolated states which cannot be converted to or from any other state (hence rendering the resource theory trivial), we have shown that enlarging the set of free operations makes it possible to obtain non-trivial resource theories of entanglement without inequivalent classes. However, no resource theory of non-full-separability can have a maximally entangled state for 3-qubit states, since this is not possible under full separability-preserving transformations, the largest conceivable class of free operations. While we conjecture that this no-go result extends beyond 3 qubits, in future work it would be interesting to study whether it holds in full generality:

- Can there exist a resource theory of non-full-separability with a maximally entangled state for more than 3 parties, or local dimension larger than 2?

On the other hand, the biseparability-preserving paradigm induces a resource theory of GME with a maximally resourceful state. Given this positive result, it would be interesting to analyse further features of this theory. In particular,

- Can we find an operational grounding to the conceptually satisfying structure that biseparability-preserving operations induce?

Despite the fact that the resource theory of pure bipartite entanglement yields only a partial order for single-copy transformations [Nie99], asymptotic transformations give rise to a total order in terms of the entropy of entanglement by measuring the cost and distillation rates with respect to the maximally entangled state [BBPS96]. Thus, the existence of this state acts as a gold standard that leads to a unique measure of entanglement in the asymptotic setting. It is known that this is not possible for pure multipartite states. An asymptotically reversible theory in this case cannot exist with respect to a single reference state [BPR$^+$00], a non-surprising result perhaps given the lack of existence of a unique maximally multipartite entangled state under the LOCC paradigm. This has led to the search of a minimal reversible entanglement generating set (MREGS), which would at least enable to define a collection of reversible asymptotic rates with respect to the states in this set. However, progress in this problem has been scarce and it is believed that that the cardinality of the MREGS might be infinite. Our result that the single-copy resource theory of entanglement under BSP operations has a unique maximally GME state invites one to think that an asymptotically reversible theory of pure-state GME could be possible in this paradigm using this state as the reference state.

- Do BSP operations lead to an asymptotically reversible theory of pure-state GME in which the maximally GME state acts a gold standard to measure the cost and distillation rates? If so, what would be the corresponding unique measure of GME for pure states?

Regarding mixed states, the work of Refs. [BP08, BP10] shows that the resource theory of multipartite entanglement under FSP is not reversible, which includes the case of BSP for two parties. Nevertheless, Refs. [BP08, BP10] show that such a theory is possible by extending the set of FSP operations to asymptotically FSP operations. Furthermore,

[BG15] shows that this result remains true for any resource theory fulfilling some general postulates under asymptotically resource non generating operations . However, this does not extend to our case because GME dos not meet Postulate 1 therein (the set of biseparable states is not closed under tensor products).

- Is an asymptotically reversible theory of GME for general (mixed) states possible under BSP or asymptotically BSP operations?

In the realm of quantum networks, in Chapters 3 and 4 we have studied pair-entangled network states, i.e., multipartite states where each party shares a bipartite entangled state with one or more of the others. Strikingly, we have shown that GMNL is intrinsic to these networks if the bipartite entangled states are pure: GMNL can be obtained by distributing arbitrary pure bipartite entanglement in any connected topology. This paves the way towards feasible generation of GMNL from any network. In fact, our results imply that, given a set of nodes, distributing pure-state entanglement in the form of a tree is sufficient to observe GMNL.

Further, we have shown that a tensor product of finitely many GME states is always GMNL. However, our construction is not necessarily optimal in the number of copies, therefore we ask:

- What is the smallest number of copies of a pure GME state needed to obtain GMNL?

And, in particular, the multipartite analogue of Gisin's theorem remains open:

- Do all single-copy pure GME states give rise to GMNL?

On a different note, the assumption that the distributions P_M, $P_{\overline{M}}$ are nonsignalling in the GMNL definition is physically natural. Still, removing it raises the stakes to achieve nonlocality. Therefore,

- Is it possible to establish analogous results to those in Chapter 3 with the stronger definition of GMNL where the distributions P_M, $P_{\overline{M}}$ may be signalling?

Very recently, Ref. [NWRP20] proposed the concept of "genuine network entanglement", a stricter notion than GME which rules out states which are a tensor product of non-GME states. One might hope that states that are GME but not genuine network entangled might be detected device independently by not passing GMNL tests. However, our results show this will not work. Any distribution of pure bipartite states, even with arbitrarily weak entanglement, always displays GMNL as long as all parties are connected. This further motivates:

- Can an analogous concept of genuine network nonlocality be found, that may detect genuine network entanglement?

In practical applications, the entanglement shared by the nodes of a network would unavoidably degrade to mixed-state form. By continuity, the GMNL in the pure pair-entangled networks considered in Chapter 3 must be robust to some noise. However, as we showed in Chapter 4, topology plays a key role in the entanglement and nonlocality properties of general mixed pair-entangled networks. In particular, tree networks are not sufficient to establish GME between the nodes, even for arbitrarily low noise, if the networks are large enough. In sharp contrast, a completely connected network exhibits GME for any number of parties for all visibilities above a threshold. While distributing bipartite entanglement in the edges of a network is experimentally very feasible, adding edges to the network undoubtedly comes at a cost. A scheme in which a resourceful central lab prepares entangled states and sends them to the remaining less powerful parties, as in Ref. [VGNT19], is doomed to failure in any realistic scenario in which entanglement preparation and distribution is bound to a certain degree of noise. Such protocols can only give rise to genuine multipartite effects for a bounded number of parties. In fact, our study also shows that this does not work if all parties are able to distribute entanglement but with moderate capacities so as to lead to tree networks. Our results require that all parties are technologically fully capable to entangle themselves with all others. For this reason, it would be crucial for applications to establish a middle ground between our results. Understanding whether a square lattice can give rise to GME might be a good starting point, and, in more generality, we ask:

- What is the network with the lowest connectivity that leads to GME for non-zero noise in the shared states, when the network is large?

Conversely, we have provided new network states that are GME but not GMNL, and our constructions can be used to establish new such examples. Still, we have found that the main factor compromising the GMNL of a network is the non-steerability of the states in one or more of the nodes. Locality is a weaker condition than non-steerability, and the possibility of having local (possibly steerable) states forming a GMNL network remains open:

- Can a network of local states give rise to GMNL?

While relatively low noise on the edges can already compromise GMNL in the network, we have shown that taking many copies can restore the nonlocality. We have provided an example of superactivation of GMNL in networks, which to our knowledge constitutes a completely new result. Further, the ideas presented here go beyond this specific example, and can be used to construct more networks exhibiting this phenomenon.

The understanding of pair-entangled networks, in particular for applications, would be significantly advanced by answering:

- Which pair-entangled network topologies and noise tolerances can lead to GMNL?

Finally, our results show that GME is robust in the fully connected network as the number of parties grows. Extending this result to GMNL remains an open question:

- Is the fully connected network robust not only for GME, but also to GMNL, as the number of parties grows?

6.2 A physical principle from observers' agreement

In addition to the results on multipartite entanglement and nonlocality, in this thesis we have also questioned whether the quantum description of Nature is the best possible, or the only one possible. In order to constrain the set of theories that are physically 'reasonable', we have provided a principle that should be satisfied by all physical theories: the impossibility of disagreement. In Chapter 5 we have defined two notions of disagreement, common certainty of disagreement and singular disagreement, inspired by notions from epistemics. We have shown that nonsignalling boxes can be disagreeing in each of these senses, while quantum and local boxes cannot.

Additionally, both notions of disagreement induce an immediate test for new theories—namely, the tables in Theorems 5.2 and 5.6. These tests are very general, in the sense that they are based only on the capability of a theory to realise undesirable correlations between non-communicating parties. Also, both principles have their roots in epistemics, common certainty of disagreement being closer to Aumann's original idea, singular disagreement having a simpler description.

These two definitions are compatible, and it is indeed possible to find examples displaying both kinds of disagreement at once. Strikingly, a prime example of this is given by the Popescu-Rohrlich box [PR94], proving that it is not only an extremal resource as an extreme point of the polytope of nonsignalling distributions, but also as a disagreeing distribution in the strongest possible sense.

On a speculative note, it would be very interesting to explore the application of the notions introduced here to practical tasks in which consensus between parties plays a role, such as the coordination of the action of distributed agents or the verification of distributed computations. The impossibility of disagreement could be useful in order for distant agents to coordinate while not having access to each other's full information. Ref. [FHMV04] proposes some specific connections along these lines in the classical case, and it would be interesting to find such connections in the quantum realm:

- Can the impossibility of quantum disagreement be used to perform some practical information-processing task?

As hinted above, our results suggest that agreement can be used to design experiments to test the behaviour of Nature. In experimental settings, noise is unavoidable. Adding white noise to the boxes in Tables 5.1 and 5.2 (both of which are quantum voids) would mean the zeros in the boxes are now small but finite parameters. Ref. [RDBC19] claimed that there is strong numerical evidence of the robustness of quantum voids to this type of noise, which would imply the same kind of robustness in our results. Still, analytical confirmation of this phenomenon would be desirable:

- If a box with common certainty of disagreement or singular disagreement is mixed with white noise, is it still impossible for it to be quantum?

Alternatively, another future direction for continuing this work concerns defining approximate notions for disagreement. Several approaches are possible, and they are compatible:

- Is the impossibility of quantum disagreement preserved if the events which were perfectly correlated are now *approximately* perfectly correlated?
- Is the impossibility of quantum common certainty of disagreement preserved if the certainty is approximate? I.e., if the agents assign a probability bounded away from 1 to each other's outcomes?
- Is the impossibility of quantum singular disagreement preserved if the difference between the agents' estimations of probabilities is bounded away from 1?

The robustness of our results to different kinds of noise would make it possible to test our principle experimentally. Obtaining disagreeing correlations in an experiment would be groundbreaking for science, as it would imply both that disagreement is not a physical principle *and* that Nature is not quantum. While this seems to be a very challenging question to tackle, future work should move towards answering:

- Can disagreeing correlations be found in Nature?

A complementary approach is the study of disagreement in theories generalising quantum theory. For instance, almost quantum correlations [NGHA15] is a set of correlations strictly larger than those achievable by measuring quantum states but that were designed to satisfy all physical principles previously proposed in the literature. While their usefulness has been since questioned [SGAN18], almost quantum correlations are well characterised in terms of nonsignalling boxes, so a natural question is whether they display common certainty of disagreement or singular disagreement. First, a straightforward adjustment of the proof of Theorem 5.3 shows that common certainty of disagreement is *not* present in almost quantum correlations. As for singular disagreement, due to the simple characterisation of almost quantum correlations in terms of a semidefinite program [NPA08], it is possible numerically to search for almost quantum boxes displaying singular disagreement. Using this, we have found numerical evidence that singular disagreement is also *not* present in almost quantum correlations. Hence, our principles give more support to the claim that quantum theory need not be the best description of Nature: new theories giving rise to almost quantum correlations can be physically reasonable. In any case, it would be desirable to obtain an analytical proof that almost quantum correlations cannot display singular disagreement:

- Can we show analytically that almost quantum correlations cannot give rise to singular disagreement?

More importantly, the quest for physical principles external to quantum theory has given rise to several proposals, and continues to be a fruitful line of research. Understanding their compatibility, i.e., whether or not some are implied by, or equivalent to, others, is crucial if we are to use such principles to constrain the allowed correlations in Nature. However, the variety of ways in which such principles are phrased makes this a challenging task. In what concerns our work, an ambitious open question is:

- Does disagreement imply, or is it implied by, any of the other physical principles proposed so far?

Finally, this work is a very modest step towards characterising quantum theory in terms of constraints external to it. Possibly the main open question that this Chapter leaves is to complete this task:

- Can quantum theory be characterised in terms of external principles?

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