Regular left-orders on groups

Yago Antolín, Cristóbal Rivas, Hang Lu Su

April 12, 2021

Abstract

A regular left-order on finitely generated group a group G is a total, left-multiplication invariant order on G whose corresponding positive cone is the image of a regular language over the generating set of the group under the evaluation map. We show that admitting regular left-orders is stable under extensions and wreath products and give a classification of the groups all whose left-orders are regular left-orders. In addition, we prove that solvable Baumslag-Solitar groups $B(1,n)$ admits a regular left-order if and only if $n \geq -1$. Finally, Hermiller and Šunić showed that no free product admits a regular left-order, however we show that if A and B are groups with regular left-orders, then $(A * B) \times \mathbb{Z}$ admits a regular left-order.

MSC 2020 classification: 06F15, 20F60, 68Q45

Contents

1 Introduction **2**

2 Preliminaries about languages and left-orderability **5**
2.1 Review of abstract families of languages . . . 5
2.2 Closure properties of families of languages . . . 8
2.3 Subsets of groups described by languages . . . 9
2.4 Left-orders and complexity classes . . . 10
2.5 Relative left-orders . . . 11

3 Lexicographic left-orders of group extensions **12**
3.1 Lexicographic left-orders where the quotient leads . . . 12
3.1.1 Polycyclic groups . . . 13
3.1.2 Solvable Baumslag-Solitar groups . . . 14
3.2 Lexicographic left-orders where the kernel leads . . . 18
3.2.1 Wreath products . . . 18

4 Groups where all positive cones are regular **21**
4.1 Order-convex subgroups and language-convex subgroups . . . 21
4.2 Groups all whose orders are regular . . . 23

5 Ordering quasi-morphisms **24**
5.1 Ordering quasi-morphism 25
5.2 Ordering quasi-morphism computable through rational transducers 26
5.3 Embedding theorem 30

A Appendix: pre-images of positive cones **34**

1 Introduction

A group G is *left-orderable* if there exists a total order $\prec$ on the elements of G which is invariant under left-multiplication, that is

$$g \prec h \iff fg \prec fh, \qquad \forall g, h, f \in G.$$

In this case, the relation $\prec$ is called a *left-order*. It is sometimes easier to understand left-orders in terms of sets of elements which are greater than the identity. Such sets are called *positive cones* (of the order) and positive cones encode completely the associated left-order. Equivalently, a positive cone is a subset $P \subset G$ which is closed under multiplication $PP \subseteq P$, and partitions G as $G = P \sqcup P^{-1} \sqcup \{1_G\}$ where the union is disjoint.

This paper discusses the computational complexity of left-orders (or equivalently positive cones). This is a topic that has gained interest in the recent years: Darbiyan [10] and Harrison-Trainor [14] have constructed groups with solvable word problems but no computable left-orders. Rourke and Weist have studied the complexity of left-orders on certain mapping class groups [28]. Šunić [31, 32] proved the existence of one-counter left-orders and later, with Hermiller [16], proved that Šunić's left-orders on free groups are the simplest possible by proving that no positive cone on a free product admits a regular left-order.

We are interested in finding groups that admit a computationally simple left-order and in describing such orders. Specifically, we investigate which finitely generated groups admit a *regular* left-orders. Recall that a positive cone $P \subseteq G$ is called regular if there is a finite generating set X of G and a regular formal language $\mathcal{L} \subseteq X^*$ which evaluates onto P. By a *formal language* we mean a subset of a finitely free monoid X^*. A formal language $\mathcal{L} \subseteq X^*$ is *regular* if there is a directed X-edge-labelled finite graph, and $\mathcal{L}$ consists on the set of labels of paths from a fixed initial vertex to a vertex in a fixed subset of vertices called accepted subsets. Such a graph is called a *finite state automaton*. Positive cones which are represented by a regular language are the simplest ones computationally, at least from the point of view of the Chomsky hierarchy of formal languages [9]. Section 2 will be devoted to reviewing all necessary material about formal languages and left-orders.

In previous works, the authors have given examples of groups not admitting regular left-orders [1, 30]. In this paper instead we focus on constructing regular left-orders. To this end, we first discuss constructions that preserve left-orderability: passing to subgroups, extensions, wreath products and free products.

It is clear that subgroups of left-ordered groups are left-ordered. Moreover, if $(N, \prec_N)$ is normal left-ordered subgroup of G and $(Q \cong G/N, \prec_Q)$ is left-ordered, we can can construct a left-order $\prec_G$ in G as follows: given $g_1, g_2 \in G$, we have that $g_1 \prec_G g_2$ if $g_1N \prec_Q g_2N$ or ($g_1N = g_2N$ and $1 \prec_N g_1^{-1}g_2$). We call such order *lexicographic* leaded by the quotient.

We start studying how these constructions preserve regularity of positive cones as well.

Theorem 1.1 (Propositions 2.18, 3.3 and 3.20)**.** *The class of finitely generated groups admitting regular positive cones is closed under passing to finite index subgroups, extensions and wreath products.*

The inheritance under passing to finite index subgroups is due to Su [30]. The other two closure properties are proved in Section 3 where we study left-orders associated to group extensions. There we note that starting from the fact that $\mathbb{Z}$ admits regular positive cones, the closure under group extension gives the following family of examples.

Corollary 1.2 (Proposition 3.7)**.** *Left-orderable virtually polycyclic groups admits regular positive cones.*

For the case of wreath products $G = N \wr Q$, we have an exact sequence $1 \to \oplus_{q\in Q} N \to G \to Q \to \{1\}$ but here $\oplus_{q\in Q} N$ does not admit a regular positive cone as, in general, it is not finitely generated. Therefore, the regular left-orders constructed for wreath products differs from the ones for the polycyclic groups in that when identifying G with the set $\oplus N \times Q$, our leading factor on the lexicographic order will be $\oplus N$ instead of the quotient group.

One cannot expect to generalize Corollary 1.2 for solvable non-polycyclic subgroups. Indeed, Darbiyan [10, Theorem 2] showed that there is a two-generated recursively presented left-orderable solvable group of derived length 3 with undecidable word problem (in fact their order turns out to be two-sided invariant). As we will see in the Appendix, this implies that no left-order on this group is computable (i.e. described by a language that can be recognized by a Turing machine).

Here, we are able to decide when a solvable Baumslag-Solitar group admits a regular positive cone or not. Recall that $BS(1,q) = \langle a, b \mid aba^{-1} = b^q \rangle$ and it is isomorphic to $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$. Note that $q \in \mathbb{Z}$ and for $q = 0$ we adopt the convention that $BS(1,0) = \mathbb{Z}$.

Theorem 1.3 (Theorem 3.14)**.** *All solvable Baumslag-Solitar groups $BS(1,q) = \langle a, b \mid aba^{-1} = b^q \rangle$ admit one-counter left-orders, and $BS(1,q)$ admits a regular left-order if and only if $-1 \leq q$.*

The class of one-counter languages generalizes the class of regular languages, and are languages that can be recognized by finite state automaton (the same machine as for regular languages) equipped with a very simple memory which consists on a stack that contains copies of the same letter (the counter). The machine, depending on the input and whether the stack is empty or not, can modify the stack by adding or removing a counter from the stack. See Section 2 for a formal definition.

From this theorem, we observe that while regularity of left-orders passes to finite index subgroups (Theorem 1.1), it is not true that it passes to finite-index overgroups (even if the overgroup is left-orderable). Indeed, $BS(1,-2)$ is a left-orderable group that does not admit regular left-orders but contains $BS(1,4)$ as an index 2 subgroup that does admit regular left-orders. Therefore, admitting a regular order is not a property preserved by commensurability among left-orderable groups.

The reason why $BS(1,-2)$ does not admit regular positive cones is a combination of an algebraic argument that says that left-orders on $BS(1,-2)$ must be lexicographic associated to the extension $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$ with $\mathbb{Z}$ as a leading factor (an argument that goes back to Tararin [33], see also [27]), together with a geometric argument that explain that such left-orders cannot be regular. Details are given in Section 3.1.2, see also the discussion before Theorem 1.7.

We will also use Tararin's classification of groups admitting finitely many left-orders to describe the groups admitting only regular left-orders. More precisely, in Section 4 we prove the following theorem.

Theorem 1.4 (Theorem 4.9)**.** *A finitely generated group G admits only regular left-orders if and only if it is Tararin poly-$\mathbb{Z}$, that is, it admits a unique subnormal series*

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

where $G_i/G_{i+1} \cong \mathbb{Z}$ and $G_i/G_{i+2} \cong K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$.

Finally, the last construction that preserves left-orderability that we will discuss are free products [25]. We know from the work of Hermiller and Šunić that no non-trivial free product (in particular free groups) admits a regular positive cone. One might think that then it is helpless to pursue this route, however we have the following.

Theorem 1.5 (Corollary 5.15)**.** *Suppose that A and B are groups admitting regular left-orders. Then $(A * B) \times \mathbb{Z}$ admits a regular left-order.*

This sort of phenomenon was already observed in [30], where H.L. Su found a finitely generated positive cone for $F_2 \times \mathbb{Z}$ (and hence a regular positive cone). In Section 5 we prove the previous theorem and give an explanation of the role of the factor $\mathbb{Z}$. For that, we will define the concept of an *ordering quasi-morphism.* A quasi-morphism on G is a map $\phi\colon G \to \mathbb{R}$ that is close to a homomorphism in the sense that there is a constant R such that $|\phi(g) + \phi(h) - \phi(gh)| \leq R$. We call a quasi-morphism ϕ an ordering quasi-morphism if $C = \phi^{-1}(\{0\})$ is a subgroup and $\{g \in| \phi(g) > 0\}$ is a positive cone relative to C (i.e. gives a left-order on the cosets G/C).

While we coin the name of ordering quasi-morphism in this paper, the original idea comes from the work of Šunić [31, 32] ordering of free groups, Dicks and Šunić ordering of free products [11] and Antolín, Dicks and Šunić [3] ordering of certain fundamental groups of graph of groups. When the ordering-quasimorphism can be computed with a transducer (another type of machine similar to a finite state automaton, see Definition 2.8) the corresponding left-order is one-counter. When embedding G in $G \times \mathbb{Z}$, we use the $\mathbb{Z}$ factor to compensate the ϕ increments/decrements in the first coordinate so that we no longer need a stack to keep track of the value of ϕ on the elements of G.

One of the motivations to study left-orders on free products is to get a better understanding of the influence of negative curvature on the complexity of positive cones. In particular, we are interested in the following.

Question 1.6. *Is there any non-elementary hyperbolic group admitting a regular positive cone?*

This question has some history. Hermiller and Šunić [16] proved that no non-abelian free groups admit a regular positive cone. Su [30] proved that acylindrically hyperbolic groups do not have positive cones that are quasi-geodesic and regular. This formalized and strengthened an argument sketched by Calegari [8]. Finally, Alonso, Antolín, Brum and Rivas [1], observed that coarse connectivity (see Definition 3.9) of a positive cone is a necessary condition for the regularity of it, and proved that non-abelian limit groups (in particular free groups and surface groups) does not admit coarsely connected positive cones.

Nevertheless, if G is an hyperbolic (finitely generated free)-by-cyclic group, then the Bieri-Neumann-Strebel theory guarantees that the lexicographic left-orders where the quotient group leads are coarsely connected (see discussion before Lemma 3.11). Thus, one might initially think that those left-orders on hyperbolic (f.g. free)-by-$\mathbb{Z}$ groups might be regular. However, as a consequence of the results developed to prove Theorem 1.4 we get that this is not the case.

Theorem 1.7 (Theorem 5.17)**.** *Let G be (finitely generated free)-by-$\mathbb{Z}$ group. Then $G \times \mathbb{Z}$ admits a regular positive cone. Moreover, such left-order restricted to G is a lexicographic left-order where the quotient factor leads, but this order is not regular.*

In Section 5 we also show that certain Artin groups admit regular positive cones.

We close the paper with an appendix. Let X be finitely generating set of a group G and $\pi\colon X^* \to G$ be a monoid surjection. If P is a positive cone, one can ask about the complexity of $\pi^{-1}(P)$ as a formal language, which is in more direct analogy with studying formal languages with the Word Problem. We explore this question for regular and context-free languages (the lowest ones in the Chomsky hierarchy), and show that there are extremely few examples of such left-orders (for instance, it is never the case that the language of preimages of a positive cone is regular). This gives partial justification as to why we have focus on our definition of regular positive cones.

2 Preliminaries about languages and left-orderability

In this section we review some formal language theory needed in the paper, focusing on the classes of regular and context-free languages. We also review the definition of left-orderability and we explain what we mean by a regular/context-free left-order on a group. Most of the content of this section is standard or well-known and the main purpose is to set the notation and the definitions for the rest of the paper.

2.1 Review of abstract families of languages

By a *formal language* we refer to a subset of $\mathcal{L}$ of X^*, where X is a finite set and X^* denotes the set of all finite sequences over X, also called *words*. We denote by ϵ the empty word. Note that X^* is a monoid with concatenation. We denote by X^+ the set $X^* \setminus \{\epsilon\}$, which is a semi-group with concatenation.

A *non-deterministic finite state automaton* is a 5-tuple $\mathbb{M} = (\mathcal{S}, X, \delta, s_0, \mathcal{A})$, where $\mathcal{S}$ is a finite set whose elements are called *states*, $\mathcal{A}$ is a subset of $\mathcal{S}$ of whose states are called *accepting states*, s_0 is a distinguished element of $\mathcal{S}$ called *initial state*, X is a finite set called the *input alphabet*, and $\delta\colon \mathcal{S} \times (X \sqcup \{\epsilon\}) \to \text{Subsets}(\mathcal{S})$ is a function called the *transition function.*

We extend δ to a function $\delta\colon \mathcal{S} \times (X \sqcup \{\epsilon\})^* \to \text{Subsets}(\mathcal{S})$ recursively, by setting $\delta(s, wx) = \cup_{\sigma \in \delta(s,w)} \delta(\sigma, x)$ where $w \in X^*$, $x \in X$ and $s \in \mathcal{S}$. A language $\mathcal{L}$ is *regular* if there is a non-deterministic finite state automaton $\mathbb{M}$ such that

$$\mathcal{L} = |\mathbb{M}| := \{w \in X^* \mid \delta(s_0, w) \cap \mathcal{A} \neq \emptyset\}.$$

We denote by Reg the class of all regular languages.

The outputs of δ under input ϵ should be thought as the machine acting spontaneously. Images with input ϵ are usually called *ϵ-moves* or *ϵ-transitions*. Informally, the machine $\mathbb{M}$ at a certain step is at a state $s \in \mathcal{S}$ and can change to another state $s' \in \mathcal{S}$ either because it reads some $x \in X$, and then $s' \in \delta(s, x)$ or spontaneously, meaning that $s' \in \delta(s, \epsilon)$.

Remark 2.1. The class Reg coincides with the class of languages accepted by non-deterministic finite state automata without ϵ-moves, and even with the class of languages accepted by *deterministic finite state automata* (See [18, Chapter 2]). Deterministic automata do not have ϵ-moves and in that setting $\delta(s, x)$ is always a singleton. Depending on our needs, we will choose to allow ϵ-moves or not (therefore choosing whether we impose determinism).

Remark 2.2. A finite state automaton $\mathbb{M}$ can be regarded as labelled directed graph. The vertices are the states of the automaton, and edges are given by the transition function δ. There is an edge from $s \in \mathcal{S}$ to $s' \in \mathcal{L}$ labeled by $x \in X \sqcup \{\epsilon\}$ if and only if $s' \in \delta(s, x)$. A word w is accepted by $\mathbb{M}$ if it is a label of a path from s_0 to some accepting state $\mathcal{A}$.

Notation 2.3. Given a word $w \in X^*$, and $x \in X$, we use $\sharp_x(w)$ to denote the number of times the letter x appears in the word w.

Example 2.4. Let $\mathbb{M}$ be the finite state automaton given in Figure 1. The automaton is described as a graph following Remark 2.2.

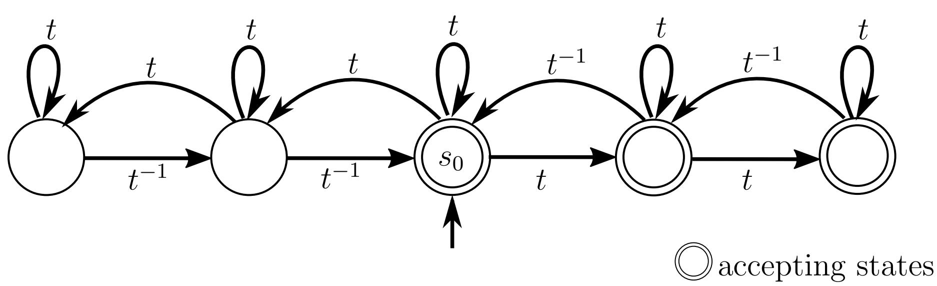

Figure 1: Finite state automaton accepting the language of Example 2.4.

The language accepted by $\mathbb{M}$ in Figure 1, consists on all the w in $\{w \in \{t^{-1}, t\}^* \mid \sharp_t(w) > \sharp_{t^{-1}}(w)\}$ such that for all subword u of w, $\sharp_t(u) - \sharp_{t^{-1}}(u) \geq -2$ holds.

A *(non-deterministic) pushdown automaton* is a 7-tuple $\mathbb{M} = (\mathcal{S}, X, \Sigma, \delta, s_0, u_0, \mathcal{A})$, where $\mathcal{S}$ is a finite set whose elements are called *states*, $\mathcal{A}$ is a subset of $\mathcal{S}$ whose states are called *accepting states*, $s_0 \in \mathcal{S}$ is a distinguished element called *initial state*, X is a finite set called the *input alphabet*, Σ is a finite set called the *stack alphabet*, u_0 is a distinguished word over Σ called the *initial stack*, and δ is a *non-deterministic transition function*

$$\delta\colon \mathcal{S} \times (X \sqcup \{\epsilon\}) \times (\Sigma \sqcup \{\epsilon\}) \to \text{Finite subsets of } (\mathcal{S} \times \Sigma^*).$$

The pushdown automaton $\mathbb{M}$ at each stage of the computation is at a certain state and contains a certain word over Σ which we call *stack word*. At the start stage, the state is s_0 and the stack word is u_0. To describe how the stack words change, it is convenient to extend δ so we allow stack words (and not only stack letters) as inputs. The behavior of $\mathbb{M}$ will depend only on the last letter σ of the stack word (if the stack word is empty we take $\sigma = \epsilon$). Precisely, we extend δ to a function $\delta\colon \mathcal{S} \times (X \sqcup \{\epsilon\}) \times \Sigma^* \to$ Finite subsets of $(\mathcal{S} \times \Sigma^*)$, where if we write $w = u\sigma$ with $\sigma \in \Sigma$, then $\delta(s, x, w) = \{(s', uu') \mid (s', u') \in \delta(s, x, \sigma)\}$. Note that δ on input word $w = u\sigma \in \Sigma^*$ operates by deleting σ and appending u' to the end to get the word uu' as output.

As in the case of finite state automata, we extend δ to a function $\delta\colon \mathcal{S} \times (X \sqcup \{\epsilon\})^* \times \Sigma^* \to$ Finite subsets of $(\mathcal{S} \times \Sigma^*)$ recursively.

A language $\mathcal{L}$ is *context-free* if there is a non-deterministic pushdown automaton such that

$$\mathcal{L} = |\mathbb{M}| := \{w \in X^* \mid \exists (s, u) \in \delta(s_0, w, u_0) \text{ with } s \in \mathcal{A}\}.$$

We denote by CF the class of all context-free languages.

A language $\mathcal{L}$ is *one-counter* if there is a non-deterministic pushdown automaton with $|\Sigma| = 1$ and such that $\mathcal{L} = |\mathbb{M}|$. The class of one-counter languages will be denoted by $\mathsf{1C}$.

Remark 2.5. In many computer science books, they use a model of pushdown automaton that has access to a tape where they write the stack word and read the last letter from it. In that model, the stack alphabet must include an extra symbol to denote the blank space on the tape.

We clearly have

$$\mathsf{Reg} \subseteq \mathsf{1C} \subseteq \mathsf{CF}.$$

Note that Reg and CF are the simplest complexities in the Chomsky hierarchy [9].

Example 2.6. Consider the language $\mathcal{L} = \{w \in \{t^{-1}, t\}^* \mid \sharp_t(w) > \sharp_{t^{-1}}(w)\}$. This language is in the class $\mathsf{1C}$ and will appear many times in the paper. For completeness, we give in Figure 2 a one-counter push-down automaton for $\mathcal{L}$ with the graphical notation found in Hopcroft, Motwani and Ullman [18, Section 6.1.3]. The vertices of the graph represent the states of the automaton. An arrow from state s to state s' has label $x, \alpha/\beta$, where $x \in X = \{t, t^{-1}\}$, $\alpha \in \Sigma \cup \{\epsilon\}$ and $\beta \in \Sigma^*$, if and only if $(s', \beta) \in \delta(s, x, \alpha)$. That is, the label tells what input is used and gives the old and new suffixes of the stack word.

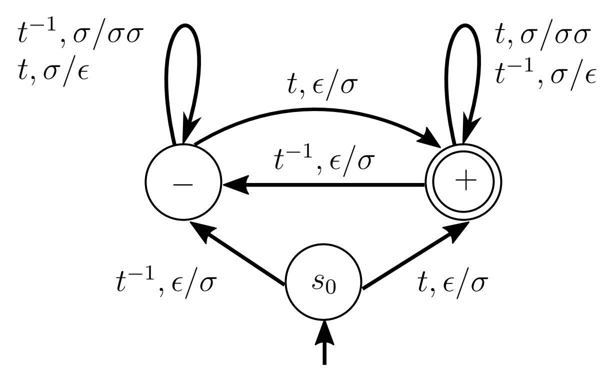

Figure 2: 1-counter pushdown automaton accepting $\mathcal{L} = \{w \in \{t^{-1}, t\}^* \mid \sharp_t(w) > \sharp_{t^{-1}}(w)\}$.

We note that this language is not regular. This follows easily from the pumping lemma for regular languages (see [18, Theorem 4.1]). However, the reader not familiar with the pumping lemma will find a proof that $\mathcal{L}$ is not regular in Lemma 4.4.

2.2 Closure properties of families of languages

The set X^* equipped with concatenation has the structure of a free monoid over the set X.

Let $\mathcal{C}$ be a class of languages. We say that the class $\mathcal{C}$ is *closed under homomorphism* if for every $\mathcal{L} \subseteq X^*$ with $\mathcal{L} \in \mathcal{C}$ and every monoid homomorphism $f \colon X^* \to Y^*$ we have that $f(\mathcal{L}) \in \mathcal{C}$. Similarly, the class is *closed under inverse homomorphism* if for every $\mathcal{L} \subseteq X^*$ with $\mathcal{L} \in \mathcal{C}$ and every monoid homomorphism $f \colon Y^* \to X^*$ we have that preimage of $\mathcal{L}$ under f, here denoted $f^{-1}(\mathcal{L})$, belongs to $\mathcal{C}$. Finally, the *reversal or mirror image* of $\mathcal{L} \subseteq X^*$ is the language $\mathcal{L}^R := \{x_n x_{n-1} \dots x_1 \mid x_1 \dots x_n \in \mathcal{L}\}$.

A class of languages is called *full abstract family of languages* (or *full AFL* for short) if it is closed under homomorphism, inverse homomorphism, intersection with regular languages, union, concatenation and concatenation closure.

Proposition 2.7. *The classes* $\mathsf{Reg}, \mathsf{1C}$ *and* CF *languages are full AFL.*

The classes Reg *and* CF *languages are closed under reversal.*

The class Reg *is closed under complement, but* CF *is not.*

Proof. All the claims are contained in either [26, Page 30] for all classes but $\mathsf{1C}$. For $\mathsf{1C}$ the claims are contained in [4, Chapter VII Theorem 4.4]. □

We introduce a last type of machine that will be play an important role in the paper.

Definition 2.8. A *rational transducer* is a finite state automaton which can output a finite number of symbols for each input symbol. Formally, a rational transducer is a six-tuple $\mathbb{T} = (\mathcal{S}, X, Y, \delta, s_0, \mathcal{A})$ where $\mathcal{S}$, X, and Y are finite sets called the states, input alphabet, and output alphabet, respectively. The function δ is a map from $\mathcal{S} \times (X \sqcup \{\epsilon\}) \to$ Finite Subsets$(\mathcal{S} \times Y^*)$. The element $s_0 \in \mathcal{S}$ is the initial state and $\mathcal{A} \subseteq \mathcal{S}$ are the accept states. The interpretation of $(r, w) \in \delta(s, x)$ is that $\mathbb{T}$ in state s with input symbol x may, as one possible choice of move, enter state r and output the string $w \in Y^*$.

As usual, we extend recursively δ to a function $\mathcal{S} \times X^* \to$ Finite Subsets$(\mathcal{S} \times Y^*)$. We have that $(r, v) \in \delta(s, ux)$ for some $x \in X$ and $u \in X^*$ if there is $(t, v') \in \delta(s, u)$ and $(r, w) \in \delta(t, x)$ such that $v = vw$.

For $u \in X^*$ let $\mathbb{T}(u) \coloneqq \{v \mid (a, v) \in \delta(s_0, u) \text{ and } a \in \mathcal{A}\}$. With this, we can define the image of $\mathcal{L} \subseteq X^*$ under $\mathbb{T}$ as $\mathbb{T}(\mathcal{L}) \coloneqq \{\mathbb{T}(u) \mid u \in \mathcal{L}\}$. Finally, for $\mathcal{L} \subseteq Y^*$, we can define the inverse image of $\mathcal{L}$ under $\mathbb{T}$ as the set $\mathbb{T}^{-1}(\mathcal{L}) \coloneqq \{u \in X^* \mid (a, v) \in \delta(s_0, u) \text{ with } a \in \mathcal{A} \text{ and } v \in \mathcal{L}\}$.

Remark 2.9. We will also use a graph representation of a transducer. This is like the graph associated to a finite state automaton, with the decorations on the edges indicating the output of reading a certain imput. If $\mathbb{T} = (\mathcal{S}, X, Y, \delta, s_0, \mathcal{A})$ is a rational transducer, we consider a graph with vertices $\mathcal{S}$. There is a directed edge from $s \in \mathcal{S}$ to $s' \in \mathcal{S}$ if $(s', u) \in \delta(s, x)$ for some $x \in X \sqcup \{\epsilon\}$ and $u \in Y^*$. In that case, we put a label x/u on that edge. It means that we can move along that edge from s to s' if we have imput x and on doing so we ouput the word u. An example of a graphical representation can be found in Figure 7.

Proposition 2.10. *Full AFL classes are closed under rational transducers and inverse rational transducers.*

Proof. Inverse rational transducers are rational transducers (See [4, III.4 and III.Theorem 6.1]). Nivat's theorem says that if $\mathbb{T}$ is a rational transducer with input alphabet X, output alphabet Y and $\mathcal{L} \subseteq X^*$ is a language, then there is a finite alphabet Z, a regular language $\mathcal{R} \subseteq Z^*$ and homomorphisms $f\colon Z^* \to X^*$ and $h\colon Z^* \to Y^*$ such that $\mathbb{T}(\mathcal{L}) = h(f^{-1}(\mathcal{L}) \cap \mathcal{R})$. See [4, III.Theorem 4.1 and III.Theorem 6.1]). Now it follows that full AFL classes are closed under rational transducers. □

2.3 Subsets of groups described by languages

Let G be a group. A *(monoid) generating set for* G is a set X together with a surjective monoid homomorphism map $\pi\colon X^* \to G$. If π is understood (for example X is a subset of G) we often just say that X is a generating set. We refer to images under π as *evaluations*.

We will use $\ell(w)$ to denote the length of a word $w \in X^*$. For an element $g \in G$, we set $|g|_X = \min\{\ell(w) \mid \pi(w) = g\}$. For $g, h \in G$, the *word distance between* g *and* h, is equal to $|g^{-1}h|_X$ and is denoted by $\mathrm{d}_X(g,h)$. We might drop the subscript $_X$ if the generating set is clear from the context.

The next lemma is standard and the proof is omitted. It will be used to show that the complexity of the languages that we are interested in are independent of the generating set.

Lemma 2.11. *Let $\mathcal{C}$ be a class of languages closed under homomorphisms and inverse homomorphism. Let (X, π_X) and (Y, π_Y) be two finite generating sets of a group G. Let $S \subseteq G$ be any subset.*

1. *There is a language $\mathcal{L}_X \subseteq X^*$ in the class $\mathcal{C}$ such that $\pi_X(\mathcal{L}_X) = S$ if and only if there is a language $\mathcal{L}_Y \subseteq Y^*$ in the class $\mathcal{C}$ such that $\pi_Y(\mathcal{L}_Y) = S$.*

2. *There is a language $\mathcal{L}_X \subseteq X^*$ in the class $\mathcal{C}$ such that $\mathcal{L}_X = \pi_X^{-1}(S)$ if and only if there is a language $\mathcal{L}_Y \subseteq Y^*$ in the class $\mathcal{C}$ such that $\mathcal{L}_Y = \pi_Y^{-1}(S)$.*

The following observation will be useful.

Lemma 2.12. *Let (X, π) be a finite generating set of G and $P \subset G$. Let $\mathcal{C}$ be a language closed by homomorphism, inverse homomorphism and reversal.*

(i) there is a language $\mathcal{L} \in \mathcal{C}$ such that $\pi(\mathcal{L}) = P$ if and only there is a language $\mathcal{L}' \in \mathcal{C}$ such that $\pi(\mathcal{L}') = P^{-1}$

(ii) $\pi^{-1}(P) \subseteq X^*$ is in the class $\mathcal{C}$ if and only if $\pi^{-1}(P^{-1}) \subseteq X^*$ is in the class $\mathcal{C}$.

Proof. Since $\mathcal{C}$ is closed by homomorphism and inverse homorphism, Lemma 2.11 implies that the properties claimed in (i) and (ii) are independent of the generating set. So we will assume that $X \subseteq G$ is a finite generating set closed under taking inverses.

Let $f\colon X^* \to X^*$ be the map sending $x_1 \dots x_n \mapsto x_n^{-1} \dots x_1^{-1}$. Then f is a composition of the homomorphism map sending $x \mapsto x^{-1}$ and the reversal map. Moreover

$f^2 = id\colon X^* \to X^*$. In particular, $\mathcal{L} \subseteq X^*$ is in $\mathcal{C}$ if and only if $f(\mathcal{L})$ is in $\mathcal{C}$.

Note that we have that $P^{-1} = \pi(f(\mathcal{L}))$ and so (i) follows. Also $\pi^{-1}(P^{-1}) = f^{-1}(\pi^{-1}(P))$ and (ii) follows. □

We will now recall a property that allows us to pass the complexity of sets to subsets.

Definition 2.13. Let (X, π) be a finite generating set of G. Let $\mathcal{L}$ be a regular language over X. A subset $H \subseteq G$ is *language-convex with respect to* $\mathcal{L}$ (or $\mathcal{L}$*-convex* for short) if there exists an $R \geq 0$ such that for each $w \in \mathcal{L}$ with $\pi(w) \in H$, where $w = x_1 \dots x_n$ and $w_i = x_1 \dots x_i$, all prefixes w_i of w satisfy that $\mathrm{d}(\pi(w_i), \pi(\mathcal{L})) \leq R$.

The following result was proved by the second author in [30, Corollary 4.4].

Proposition 2.14. *Let $\mathcal{L}$ be a regular language, let (X, π) be a finite generating set of a group G and let H be a subgroup of G. If H is language-convex with respect to $\mathcal{L}$, then there exists a regular language $\mathcal{L}' \subseteq X^*$ that evaluates onto $H \cap \pi(\mathcal{L})$.*

2.4 Left-orders and complexity classes

A *left-order* $\prec$ on a group G is a total order on G that is invariant under left G-multiplication. That is, for all $a, b, c \in G$, $a \prec b$ holds if and only if $ca \prec cb$ holds. If a left-order exists on G we say that G *is left-orderable*. A *positive cone* of G is a subsemigroup P of G with the property that G is the disjoint union of P, P^{-1} and $\{1_G\}$. A positive cone P defines a left-order $\prec_P$ on G by $a \prec_P b \Leftrightarrow a^{-1}b \in P$. Conversely, a left-order $\prec$ on G defines a positive cone $P_\prec = \{g \in G \mid 1_G \prec g\}$.

A group G is *bi-orderable* if there is a left-order $\prec$ on G such that it is also invariant under right G-multiplication. In that case, $\prec$ is a *bi-order*.

We now link positive cones with formal languages.

Definition 2.15. Let $\mathcal{C}$ be a class of languages. Let (X, π) be a generating set of G. Let $\prec$ a left-order. We say that $\prec$ is a $\mathcal{C}$*-left-order*, or equivalently $P_\prec$ *is a* $\mathcal{C}$*-positive cone*, if there exists a language $\mathcal{L} \subseteq X^*$ in the class $\mathcal{C}$ such that $\pi(\mathcal{L}) = P_\prec$.

Example 2.16. An infinite cyclic group $G = \langle t \mid \; \rangle$ is left-orderable. Consider the generating set $\{t, t^{-1}\}$ of G. The set $\{t^n \mid n > 0\} \subseteq G$ is a positive cone and it is easy to check that $\langle t \rangle^+ = \{t^n \mid n > 0\} \subseteq \{t, t^{-1}\}^*$ is a regular language. Thus $\mathbb{Z}$ admits Reg-left-orders.

In Example 2.6, we saw that $\pi^{-1}(\mathbb{Z}_{\geq 1}) = \{w \in \{t, t^{-1}\}^* \mid \sharp_t(w) > \sharp_{t^{-1}}(w)\}$ is $\mathsf{1C}$ but not regular. Taking the full pre-image of a positive cone under π gives a language that represents the positive cone but it might not be of minimal language complexity. This will be discussed in more detail in the Appendix.

We now get some applications from the lemmata we proved for the complexity of subsets in the previous subsection. By Lemma 2.11, we get the following result.

Corollary 2.17. *Let $\mathcal{C}$ be a class of languages closed under homomorphism (eg. $\mathcal{C} = \mathsf{Reg}$ or $\mathcal{C} = \mathsf{CF}$). Admitting a $\mathcal{C}$-left-order is a group property, independent of the generating set.*

Left-orders restrict to left-orders on subgroups. It is natural to ask if their complexity is preserved. This is a subtle question. For example $F_2 \times \mathbb{Z}$ has a Reg-left-order (Theorem 1.5 or [30]), however F_2 does not have Reg-left-orders [16].

A sufficient condition for a left-order to inherit the complexity class when passing to a subgroups is given in terms of language-convexity. For instance, since finite index subgroups of a finitely generated groups are always language-convex, we get the following consequence of Proposition 2.14 (which was stated as [30, Theorem 1.1]).

Proposition 2.18. *A Reg-left-order on a finitely generated group G restricts to a Reg-left-order on any finite index subgroup.*

2.5 Relative left-orders

Definition 2.19. A subsemigroup $P \subseteq G$ is a *positive cone relative to $H \leqslant G$* if $G = P \sqcup H \sqcup P^{-1}$.

Suppose that P is a positive cone relative to H. Then both $P \cdot H \subseteq P$ and $H \cdot P \subseteq P$ hold. Indeed, suppose not. Then $P \cdot H$ will have a non-trivial intersection with $P^{-1} \cup H$. Suppose that $P \cdot H \cap P^{-1} \neq \emptyset$. Then there exists $p_1, p_2 \in P$ such that $p_1 h = p_2^{-1}$, meaning that $h = p_1^{-1} p_2^{-1}$, so H is not disjoint from P^{-1}, a contradiction. Similarly, if $P \cdot H \cap H$ is non-empty, then there exists $h_1, h_2 \in H$ such that $ph_1 = h_2$, meaning that $p = h_2 h_1^{-1}$, meaning that P and H are not disjoint, a contradiction.

With the previous remark, we see that if P is a positive cone relative to H then we can define a total G-invariant order on G/H by setting $g_1 H \prec g_2 H \Leftrightarrow g_1^{-1} g_2 \in P$. This is well-defined, since if we pick different coset representatitives such that $g_1' = g_1 h$ and $g_2' = g_2 h'$ then $g_1^{-1} g_2 \in P$ implies that $h^{-1} g_1^{-1} g_2 h' \in P$. The fact that $\prec$ is a total left-invariant order on G/H follows now easily.

Definition 2.20. Let $\mathcal{C}$ be a class of languages. Let (X, π) be a generating set of G. Let $\prec$ a left-order. We say that $P \subseteq G$ is *$\mathcal{C}$-positive cone relative to $H \leqslant G$*, if P is a positive cone relative to H and there exists a language $\mathcal{L} \subseteq X^*$ in the class $\mathcal{C}$ such that $\pi(\mathcal{L}) = P_\prec$.

The following argument will be used several times:

Lemma 2.21. *Let G be a group and H a subgroup. Let P_{rel} be a positive cone relative to H and P_H a positive cone for H. Then $P = P_{rel} \cup P_H$ is a positive cone for G.*

Moreover, if P_{rel} is a $\mathcal{C}$-positive cone relative to H and P_H is a $\mathcal{C}$-positive cone for H, and $\mathcal{C}$ is a class of languages $\mathcal{C}$ closed under union, then P is a $\mathcal{C}$-positive cone.

Proof. We have that $G = P_{rel} \sqcup H \sqcup P_{rel}^{-1} = (P_{rel} \sqcup P_H) \sqcup \{1\} \sqcup (P_H^{-1} \sqcup P_{rel}^{-1}) = P \sqcup \{1\} \sqcup P^{-1}$. To see that P is a semigroup, note that we have $P_{rel} P_{rel} \subseteq P_{rel}$ and $P_H P_H \subseteq P_H$ by assumption, and that we observed previously that $P_{rel} H \subseteq P_{rel}$ and $H P_{rel} \subseteq P_{rel}$, therefore $(P_{rel} \cup P_H)(P_{rel} \cup P_H) \subseteq (P_{rel} \cup P_H)$.

The moreover part is clear by definition of P. $\square$

3 Lexicographic left-orders of group extensions

In this section we study left-orders of groups constructed from group extensions. First recall that one naturally constructs a total order on a product of totally ordered sets.

Definition 3.1 (Lexicographic order on direct products of totally ordered sets)**.** Let $(N, \prec_N)$ and $(Q, \prec_Q)$ be two totally ordered sets. The *lexicographic order* $\prec_{lex}$ *on* $N \times Q$ *with leading factor* N is given by $(n, q) \prec_{lex} (n', q')$ if and only if $n \prec_N n'$ or $n = n'$ and $q \prec_Q q'$.

Suppose now that that N and Q are groups and that G is *an extension of* N *by* Q. That is, there is a short exact sequence $1 \to N \to G \xrightarrow{f} Q \to 1$. Fixing a right inverse s of f (i.e. a section of f), with $s(1_Q) = 1_G$, we obtain the bijection $G \to N \times Q$, $g \mapsto (gs(f(g))^{-1}, f(g))$. In the next two subsections, we will find condition on orders on N and Q so that the lexicographic order on the underlying set $N \times Q$ induces a left-order on the group G.

3.1 Lexicographic left-orders where the quotient leads

Lemma 3.2. *Let* $f \colon G \to Q$ *a group epimorpishm with kernel* N*. Let* P_Q *and* P_N *be positive cones on* Q *and* N *respectively. Then* $P_{\prec_{lex}} := f^{-1}(P_Q) \cup P_N$ *is a positive cone of* G*.*

We call to the left-order on G corresponding to $P_{\prec_{lex}}$ *lexicographic with leading factor* Q.

Proof. Clearly $f^{-1}(P_Q)$ is a subsemigroup as it is the pre-image of a semi-group under a homomorphism. Also, as $Q = P_Q^{-1} \sqcup \{1_Q\} \sqcup P_Q$ we get that $G = f^{-1}(P_Q)^{-1} \sqcup N \sqcup f^{-1}(P_Q)$ and hence $f^{-1}(P_Q)$ is a positive cone relative to N. The result now follows from Lemma 2.21. □

Let $(N, \prec_N)$ and $(Q, \prec_Q)$ be left-ordered with the left-orders coming from P_N and P_Q respectively. It is easy to check that indeed, the left-order on the previous lemma on G coincides to the lexicographic order on the set $Q \times N$ (which is naturally in bijection with G) with leading factor Q.

Proposition 3.3. *Let* $\mathcal{C}$ *be a class of languages closed under union and inverse homomorphisms. Let* N *and* Q *be finitely generated groups and* G *an extension of* N *by* Q*. Then,* $\mathcal{C}$*-left-orders on* N *and* Q *extend to a lexicographic* $\mathcal{C}$*-left-order on* G *with leading factor* Q*.*

Proof. Fix finite generating sets (X, π_N) and (Y, π_Q) for N and Q. Let P_N and P_Q be $\mathcal{C}$-left-positive cones in N and Q, and let $\mathcal{L}_N \subseteq X^*$ and $\mathcal{L}_Q \subseteq Y^*$ be in $\mathcal{C}$ such that $\pi_N(\mathcal{L}_N) = P_N$ and $\pi_Q(\mathcal{L}_Q) = P_Q$.

Denote by f the epimorphism of G onto Q, i.e. $f \colon G \to Q$. We can define a generating set $(X \sqcup Y, \pi)$ for G such that $\pi_N(x) = \pi(x)$ for $x \in X$ and $\pi_Q(y) = f(\pi(y))$ for $y \in Y$.

Let $\widetilde{\mathcal{L}_Q}$ be the preimage of $\mathcal{L}_Q$ under the monoid morphism $\pi_{X \sqcup Y \to Y} \colon (X \sqcup Y)^* \to Y^*$ that is the identity on Y and sends elements of X to the empty word. Note that $f(\pi(\widetilde{\mathcal{L}_Q})) = P_Q$ and hence $\pi(\widetilde{\mathcal{L}_Q}) \subseteq f^{-1}(P_Q)$.

To see that $\pi(\widetilde{\mathcal{L}_Q}) \supseteq f^{-1}(P_Q)$, let $g \in G$ such that $f(g) \in P_Q$. Then, there is $w \in \mathcal{L}_Q$ such that $\pi_Q(w) = f(\pi(w))$. There is $\tilde{w} \in \widetilde{\mathcal{L}_Q}$ such that $\pi_{X \sqcup Y \to Y}(\tilde{w}) = w$ and therefore if

$\tilde{g} = \pi(\tilde{w})$ we get that $f(g) = f(\tilde{g})$ and hence $g\tilde{g}^{-1} \in N$. There is $u \in X^*$ such that $\pi(u) = \pi_N(u) = g\tilde{g}^{-1}$. Note that $\pi_{X \sqcup Y \to Y}(u\tilde{w}) = w$, therefore $u\tilde{w} \in \widetilde{\mathcal{L}_Q}$ and $\pi(u\tilde{w}) = g\tilde{g}^{-1}\tilde{g} = g$.

As $\mathcal{C}$ is closed under inverse homomomorphism, $\widetilde{\mathcal{L}_Q} \in \mathcal{C}$ and $f^{-1}(P_Q)$ is $\mathcal{C}$-positive cone relative to N. By Lemma 2.21, $P_G = f^{-1}(P_Q) \cup P_N$ is a $\mathcal{C}$-positive cone. □

3.1.1 Polycyclic groups

We use Proposition 3.3 to show that left-orderable virtually polycyclic groups admit Reg-left-orders. Let us recall some definitions.

Definition 3.4. Let G be a group. A *subnormal series for G* is a decreasing sequence of subgroups of G

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

such that G_{i+1} is normal in G_i for $0 \leq i < n$. The quotients G_i/G_{i+1} are called *factors*.

Definition 3.5. A group G is *polycyclic* (resp. *poly-$\mathbb{Z}$*) if there is a finite subnormal series for G with cyclic (resp. infinite cyclic) factors.

As a consequence of Proposition 3.3 we have the following.

Corollary 3.6. *Poly-$\mathbb{Z}$ groups have* Reg*-left-orders.*

Proof. The corollary follows by induction on the length of the subnormal series. The base case is $\mathbb{Z}$ and Example 2.16 shows that $\mathbb{Z}$ has Reg positive cones. Proposition 3.3 allows the inductive argument. □

Finally, we use a theorem of Morris [23] to show the following.

Proposition 3.7. *Left-orderable virtually polycyclic groups are poly-$\mathbb{Z}$. In particular, they have* Reg*-left-orders.*

Proof. Recall that if G is polycyclic, the number of $\mathbb{Z}$-factors in a subnormal series of cyclic factors is well-defined and called the *Hirsch length* and denoted $h(G)$. For a virtually polycyclic group G, we can define $h(G)$ to be the Hirsch length of any finite index polycyclic subgroup, and it is still well-defined (see for example [29]). Moreover, if H is a normal subgroup in G then both H and G/H are virtually polycyclic and $h(G) = h(H) + h(G/H)$.

We argue by induction on the Hirsch length that a virtually polycyclic left-orderable group is poly-$\mathbb{Z}$. If $h(G) = 0$, then G left-orderable and finite, so G is trivial. Suppose that $h(G) > 0$, and recall that Morris' theorem says that finitely generated, left-orderable amenable groups have infinite abelianization [23]. Thus, there is a normal subgroup $N \trianglelefteq G$ such that $G/N \cong \mathbb{Z}$. Since N is virtually polycyclic, left-orderable and $h(N) = h(G) - 1$, we get by hypothesis that N is poly-$\mathbb{Z}$, and so is G. □

3.1.2 Solvable Baumslag-Solitar groups

The result about regularity of left-orders on polycyclic groups cannot be promoted to the case of solvable groups [10]. Here we give the complete picture for when a solvable Baumslag-Solitar groups $BS(1,q)$, $q \in \mathbb{Z}-\{0\}$ admits a regular positive cone. These groups are defined by the presentation

$$BS(1,q) = \langle a, b \,|\, aba^{-1} = b^q \rangle,$$

and can be viewed as a semidirect product $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$, where a is the generator for the $\mathbb{Z}$-factor and acts on $\mathbb{Z}[1/q]$ by multiplying by $1/q$. The element b can be identified with the unity of the ring $\mathbb{Z}[1/q]$ (the minimal sub-ring of $\mathbb{Q}$ containing $\mathbb{Z}$ and $1/q$). Therefore, an element $r/q^s \in \mathbb{Z}[1/q]$ where $s \geq 0$ and $r \in \mathbb{Z}-\{0\}$, can be written as $a^{-s}b^{\varepsilon r}a^s$ where $\varepsilon \in \{-1, 1\}$ depends on the parity of s and the sign of r and q as in the following table.

sign(q)	s	ε
$+$	even/odd	sign(r)
$-$	even	sign(r)
$-$	odd	$-$sign(r)

In particular, for solvable Baumslag-Solitar groups, every group element can be written in the form $a^n(a^{-m}b^k a^m)$ with $m \geq 0$, $n, k \in \mathbb{Z}$.

Let $q > 1$. From the point of view of orders, $BS(1,q)$ and $BS(1,-q)$ behave quite differently. On one hand, $BS(1,-q)$ falls into the classification of groups admitting only finitely many left-orders obtained by Tararin [33] (see also [19, 13], or Subsection 4): actually it admits only four left-orders and all of them are lexicographic coming from the semi-direct product decomposition $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$, where the leading factor is $\mathbb{Z}$. On the other hand $BS(1,q)$ admits uncountably many left-orders, four of them coming from its semi-direct product decomposition and the rest appearing as induced orders from the (order-preserving) affine action of $BS(1,q)$ on the real line given by $a\colon x \mapsto qx$, $b\colon x \mapsto x+1$. This is explicitly show in [27] only for the case of $BS(1,2)$ but the same proof works for general $q > 1$.

We first deal with $BS(1,-q)$. The four positive cones of $BS(1,-q)$ are given by the sets

$$P_1 = \{(r/q^s, n) \in \mathbb{Z}[1/q] \rtimes \mathbb{Z} \mid n > 0 \text{ or } (n = 0 \text{ and } r/q^s > 0)\},$$

$$P_2 = \{(r/q^s, n) \in \mathbb{Z}[1/q] \rtimes \mathbb{Z} \mid n < 0 \text{ or } (n = 0 \text{ and } r/q^s > 0)\},$$

$P_3 := P_1^{-1}$ and $P_4 := P_2^{-1}$. Note that these are also positive cones for $BS(1,q)$.

Proposition 3.8. *The following two languages*

$$\mathcal{L}_1 = \{a^n(a^{-m}b^k a^m) \mid n > 0 \text{ or } [n = 0 \text{ and } (m \text{ odd and } k < 0) \text{ or } (m \text{ even and } k > 0)]\}$$

$$\mathcal{L}_2 = \{a^n(a^{-m}b^k a^m) \mid n < 0 \text{ or } [n = 0 \text{ and } (m \text{ odd and } k < 0) \text{ or } (m \text{ even and } k > 0)]\}$$

where we assume $m \geq 0$ and $k \in \mathbb{Z}$, are one-counter languages and evaluate onto the positive cones P_1 and P_2 of $BS(1,-q)$ respectively, $q \geq 2$. Moreover, none of the positive cones P_1, P_2, P_3, P_4 is regular.

To show Proposition 3.8 we need to recall some facts, the first one being that regular sets are coarsely connected.

Definition 3.9. Let $(\mathcal{M}, \mathrm{d})$ be a metric space. A subset $Y \subseteq \mathcal{M}$ is *coarsely connected* if there is $R > 0$ such that $\{p \in \mathcal{M} \mid \mathrm{d}(p, Y) \leq R\}$, the R-neighborhood of Y, is connected.

The following is [1, Proposition 7.2], in view of Lemma 2.12.

Lemma 3.10. *Let G be a finitely generated group. If P is a* Reg*-positive cone of G, then P and P^{-1} are coarsely connected subsets of the Cayley graph of G.*

For the second fact recall that given a finitely generated group G, a non-trivial homomorphism $\phi\colon G \to \mathbb{R}$ belongs to $\Sigma^1(G)$, the Bieri-Neumann-Strebel invariant (BNS invariant for short), if and only if $\phi^{-1}((0, \infty))$ is coarsely connected. Moreover, the kernel of ϕ is finitely generated if and only if both ϕ and $-\phi$ belong to $\Sigma^1(G)$ (see [7]). Therefore we have the following.

Lemma 3.11. *Suppose that G is an extension of N by $\mathbb{Z}$. If N is not finitely generated, then no lexicographic orders on G where $\mathbb{Z}$ leads is* Reg.

Proof. Suppose that $f\colon G \to \mathbb{Z}$ is a homomorphism with kernel N. Without loss of generality, we can assume that a lexicographic order where $\mathbb{Z}$ leads has a positive cone of the form $P_G \coloneqq f^{-1}(\mathbb{Z}_{\geq 1}) \cup P_N$ where P_N is a positive cone for N.

Since $f(P_N) = 0$, and there is a group generator mapped under f to a positive integer, we have that $f^{-1}(\mathbb{Z}_{\geq 1})$ is coarsely connected if and only if $f^{-1}(\mathbb{Z}_{\geq 1}) \cup P_N$ is coarsely connected. Therefore, $f \in \Sigma^1(G)$ if and only if $f^{-1}(\mathbb{Z}_{\geq 1}) \cup P_N$ is coarsely connected. Similarly, $-f \in \Sigma^1(G)$ if and only if $f^{-1}(\mathbb{Z}_{\leq -1}) \cup P_N^{-1}$ is coarsely connected.

Assume that N is not finitely generated and P_G is a Reg-positive cone. From BNS theory, either $P_G = f^{-1}(\mathbb{Z}_{\geq 1}) \cup P_N$ or $P_G^{-1} = f^{-1}(\mathbb{Z}_{\leq -1}) \cup P_N^{-1}$ is not coarsely connected. By Lemma 3.10 if the left-order given by P_G is Reg, then both cones P_G and P_G^{-1} would be coarsely connected. We have a contradiction. □

We are ready to prove the proposition.

Proof of Proposition 3.8. From Tararin's classification (it is reviewed in Section 4), every left-order on $BS(1, -q) \simeq \mathbb{Z}[1/q] \rtimes \mathbb{Z}$ is a lexicographic extension with leading factor $\mathbb{Z}$. It follows from Lemma 3.11 and the fact that $\mathbb{Z}[1/q]$ is not finitely generated, that none of these orders can be regular.

The fact that $\mathcal{L}_1$ and $\mathcal{L}_2$ evaluate to P_1 and P_2 follows easily from our description of normal forms at the beginning of the subsection. By Lemma 2.12, it is enough to show that $\mathcal{L}_1$ and $\mathcal{L}_2$ are one-counter, to deduce that $P_3 = P_1^{-1}$ and $P_4 = P_2^{-1}$ are also one-counter.

We see that $\mathcal{L}_1 = \{a\}^+ \cdot \mathcal{L}_{quot} \cup \mathcal{L}_{ker}$ and $\mathcal{L}_2 = \{a^{-1}\}^+ \cdot \mathcal{L}_{quot} \cup \mathcal{L}_{ker}$ where

$$\mathcal{L}_{quot} = \{a^{-m} b^k a^m \mid m \geq 0, k \in \mathbb{Z}\}$$

and

$$\mathcal{L}_{ker} = \{a^{-m} b^k a^m \mid m \geq 0, k \in \mathbb{Z}, (m \text{ odd and } k < 0) \text{ or } (m \text{ even and } k > 0)\}.$$

By the closure properties of the class $\mathsf{1C}$, to see that $\mathcal{L}_1$ and $\mathcal{L}_2$ are one-counter languages, it is enough to see that $\mathcal{L}_{quot}$ and $\mathcal{L}_{ker}$ are in $\mathsf{1C}$. Moreover,

$$\begin{aligned}\mathcal{L}_{ker} = {} & \mathcal{L}_{quot} \cap (\{a^{-1}a^{-1}\}^* \cdot \{b\}^* \cdot \{aa\}^*) \\ & \cup \mathcal{L}_{quot} \cap (\{a^{-1}\} \cdot \{a^{-1}a^{-1}\}^* \cdot \{b^{-1}\}^* \cdot \{aa\}^* \cdot \{a\})\end{aligned}$$

Again, since the class 1C is closed under union and intersection with regular languages, we see that it is enough to show that $\mathcal{L}_{quot}$ is one-counter. We illustrate a pushdown automaton accepting $\mathcal{L}_{quot}$ in Figure 3 and leave the rest of the proof to the reader.

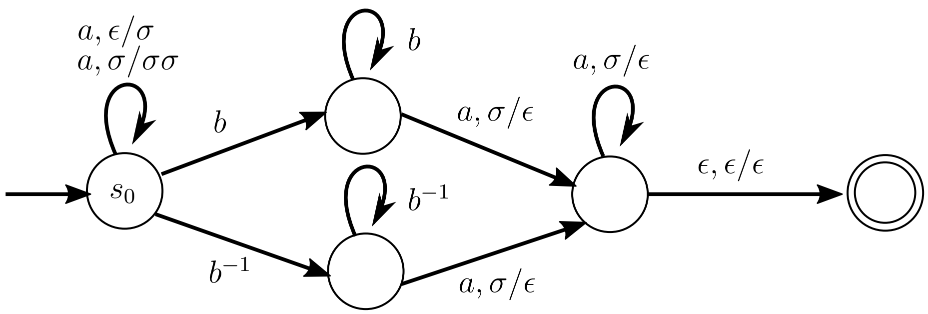

Figure 3: Pushdown automaton accepting $\mathcal{L}_{quot}$. Here $\{\sigma\}$ is the stack alphabet (i.e the counter symbol).

□

The proof of the following proposition is analogous to the previous one, with a simpler automaton.

Proposition 3.12. *The following language*

$$\mathcal{L} = \{a^n(a^{-m}b^k a^m) \mid n > 0 \text{ or } [n = 0 \text{ and } k > 0]\}$$

is a one-counter languages and evaluate onto the positive cones P_1 of $BS(1,q)$ for $q \geq 2$. Moreover, P_1 is not a regular positive cone.

The situation drastically changes for $BS(1,q)$, which admits Reg-left-orders.

Lemma 3.13. *Let $BS(1,q) = \langle a, b \mid aba^{-1} = b^q \rangle$ with $q \geq 2$. Then*

$$P = \{a^n(a^{-m}b^k a^m) \mid k > 0, m, n \in \mathbb{Z}\}$$

is a Reg*-positive cone relative to $\langle a \rangle$. In particular, $P_0 = \langle a \rangle^+ \cup P$ is* Reg*-positive cone.*

Proof. Consider the action give by $a\colon x \mapsto qx$ and $b\colon x \mapsto x+1$. If $g \in BS(1,q)$ is an element with normal form $g = a^n(a^{-m}b^k a^m)$, then g maps x to $q^n x + \frac{k}{q^m}$, which implies that this affine representation is faithful.

We take the left-order $\prec_0$ induced from 0 out of this affine action, that is $1_{BS(1,q)} \prec_0 g$ if and only if $g(0) > 0$, which are all elements of the form $g = a^n(a^{-m}b^k a^m)$ where $k > 0$. This coincides with the set P of the statement, and therefore P is positive cone relative to $Stab_{BS(1,q)}(0) = \langle a \rangle \cong \mathbb{Z}$. Indeed, since $q > 1$, the affine action preserves the order on the line and hence it is easy to see that P is a semi-group of $BS(1,q)$. Also, one has that $BS(1,q) = \{g \in BS(1,q) : g(0) < 0\} \sqcup Stab_{BS(1,q)}(0) \sqcup \{g \in BS(1,q) : g(0) > 0\} = P^{-1} \sqcup \langle a \rangle \sqcup P$.

From Lemma 2.21, the set P_0 given by

$$P_0 = \{a^n \mid n > 0\} \cup \{a^n(a^{-m}b^k a^m) \mid k > 0, m, n \in \mathbb{Z}\},$$

is a positive cone for $BS(1, q)$.

Viewing the elements of P_0 as words in $\{a, b, a^{-1}, b^{-1}\}^*$, P_0 can be represented as a language accepted by the automaton of Figure 4. One can construct an automaton that gives a regular language for P from the automaton of Figure 4 by just changing the state right to s_0 to a non-accepting state.

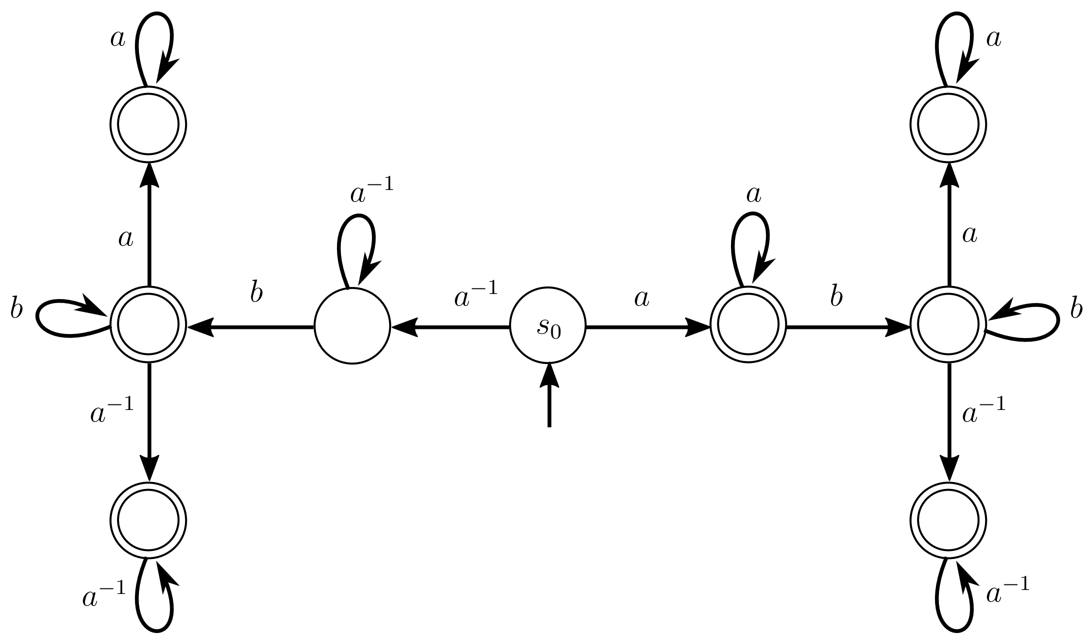

Figure 4: Finite state automaton accepting a positive cone language for $BS(1, q)$ with $q > 1$.

Remark that this is a subgraph of the automaton-graph describing the language of normal forms in $BS(1, q)$.

□

Theorem 3.14. *All solvable Baumslag-Solitar groups $BS(1, q) = \langle a, b \mid aba^{-1} = b^q \rangle$ admit one-counter left-orders, and $BS(1, q)$ admits a regular left-order if and only if $-1 \leq q$. Moreover, for $q \geq 2$ all* Reg*-left-orders on $BS(1, q)$ are induced by affine actions on $\mathbb{R}$.*

Proof. From Propositions 3.8 and 3.12 all solvable Baumslag-Solitar groups admit a one-counter left-order. By Proposition 3.8 $BS(1, -q)$ with $q > 1$ do not admit regular orders.

If $|q| \leq 1$, then then $BS(1, q)$ is poly-$\mathbb{Z}$ and then has regular left-orders (our convention is that $BS(1, 0) = \mathbb{Z}$).

Finally, suppose that $q \geq 2$. From [27], a left-order on G is either one induced from an affine action on $\mathbb{R}$ or a lexicographic order with leader factor $\mathbb{Z}$ coming from the semidirect product $\mathbb{Z}[1/q] \rtimes \mathbb{Z}$. We saw in Lemma 3.11 that the lexicographic left-orders on $BS(1, q)$ with leading factor $\mathbb{Z}$ cannot be regular. We saw in Lemma 3.13 that $BS(1, q)$ admits regular orders and thus must be induced by affine actions (as it is shown in the proof of the lemma). □

3.2 Lexicographic left-orders where the kernel leads

In general, if G is an N by Q extension of left-orderable groups, the lexicographic order on $N \times Q$ with leading factor N is not G-left-invariant. However, in some special situation it is. The following lemma will be used to produce left-orders on extensions where the kernel leads.

Lemma 3.15. *Let G be a semidirect product of the subgroups $N \trianglelefteq G$ and $Q \leqslant G$. Let P_N and P_Q be positive cones of N and Q respectively, and assume that $qP_Nq^{-1} = P_N$ for all $q \in Q$. Then, the lexicographic order on the underlying set $N \times Q$ with leading factor N is G-left-invariant. In particular, it is a left-order on G.*

Proof. Since G is a semidirect product of N and Q, we have that $N \cap Q = \{1_G\}$ and $NQ = G$. There is a natural bijection between $N \times Q$ to $G = NQ$, and under this bijection, the elements of $N \times Q$ that are lexicographical greater to $(1_N, 1_Q)$ correspond to the subset $P := P_N Q \cup P_Q$ of G. Note that $P^{-1} = Q^{-1}P_N^{-1} \cup P_Q^{-1}$ and since $qP_Nq^{-1} = P_N$ for all $q \in Q$, we get that $P^{-1} = P_N^{-1}Q \cup P_Q^{-1}$. Therefore $G = P \sqcup P^{-1} \sqcup \{1_G\}$.

To show that P is a subsemigroup, let $nq, n'q' \in P$, with $n, n' \in N$ and $q, q' \in Q$. Recall that n and n' either are trivial or belong to P_N. Then $nqn'q' = n(qn'q^{-1})qq'$ and we see that if $n \neq 1_G$ or $n' \neq 1_G$ then $n(qn'q^{-1}) \in P_N$ and $nqn'q' \in P_N Q \subseteq P$. If $n = 1_G = n'$, then $q, q' \in P_Q$ and $nqn'q' = qq' \in P_Q \subseteq P$. □

This lemma will be helpful to construct regular left-orders on extensions when the kernel is not finitely generated (and thus the kernel alone cannot support a regular left-order). Our main example are wreath products, however we have already observed this phenomenon in Baumslag-Solitar groups.

Example 3.16 (Lexicographic left-orders on $BS(1,q)$ where the kernel leads)*.* For $q > 0$ we have already seen that $BS(1,q)$ has regular orders, constructed through an affine action on the real line. Viewing $BS(1,q) = \langle a, b \mid aba^{-1} = b^q \rangle \cong \mathbb{Z}[1/q] \rtimes \mathbb{Z}$, the left-order of the Figure 4 is lexicographic where the factor $\mathbb{Z}[1/q]$ leads. Indeed, a positive cone $P_{\mathbb{Z}[1/q]}$ for $\mathbb{Z}[1/q]$ is the set of elements greater than 0. Since $\langle a \rangle \cong \mathbb{Z}$ acts on $\mathbb{Z}[1/q]$ by multiplying by q, we have that $a^n P_{\mathbb{Z}[1/q]} a^{-n} = P_{\mathbb{Z}[1/q]}$ for all $n \in \mathbb{Z}$. The previous lemma tell us that $\langle a \rangle \cdot P_{\mathbb{Z}[1/q]} \cup \{a\}^+$ is a positive cone for $BS(1,q)$. This is exactly the left-order of the automaton of Figure 4.

3.2.1 Wreath products

The *wreath product* $N \wr Q$ of groups N and Q is the semidirect product of $\mathbf{N} := \oplus_{q \in Q} N$ by Q, where the conjugation action of Q on $\mathbf{N}$ is given by the left-multiplication action on the indexes of the copies of N. That is, if $\mathbf{n} = (n_q)_{q \in Q} \in \mathbf{N}$ and $q' \in Q$, we have $q'\mathbf{n}q'^{-1} = (n_{q'q})_{q \in G}$.

We begin with an example that will illustrate the construction we develop in this section.

Proposition 3.17. *The group $\mathbb{Z} \wr \mathbb{Z}$ has* Reg*-left-orders.*

Proof. The group $\mathbb{Z} \wr \mathbb{Z}$ is isomorphic to $\mathbb{Z}[X, X^{-1}] \rtimes C_\infty$ where $C_\infty = \langle t \rangle$ acts by multiplying by X on $\mathbb{Z}[X, X^{-1}]$. An element of $\mathbb{Z} \wr \mathbb{Z}$ is therefore uniquely written as of the form $(p = a_{i_0}X^{i_0} + a_{i_1}X^{i_1} + \dots a_{i_k}X^{i_k}, t^s)$ with $i_0 > i_1 > i_2 > \dots > i_k \in \mathbb{Z}$ and $s \in \mathbb{Z}$. We say that a_{i_0} is the leading coefficient of p, and define $\mathsf{leadcoef}(p) := a_{i_0}$. It is easy to check that $P = \{(p, t^s) \mid \mathsf{leadcoef}(p) > 0\} \cup \{(0, t^s) \mid s > 0\}$ is a positive cone.

The group is generated by the constant polynomial $c = 1$ and t, since $t^k c^m t^{-k}$ represents the polynomial mX^k. Therefore a word representing the element given by $(a_{i_0}X^{i_0} + a_{i_1}X^{i_1} + \dots a_{i_k}X^{i_k}, t^s)$ is $t^{i_0}c^{a_{i_0}}t^{i_1-i_0}c^{a_{i_1}}t^{i_2-i_1}\cdots c^{a_{i_k}}t^{-i_k}t^s$. Since $i_0 > i_1 > i_2 > \dots > i_k$, we have that $i_1 - i_0, i_2 - i_1, \dots, i_k - i_{k-1}$ are all negative. A language for this positive cone is

$$\{t^n c^m t^{n_1} c^{m_1} t^{n_2} c^{m_2} \dots c^{m_k} t^l \mid m > 0, n_i < 0, m_i \in \mathbb{Z}, k \geq 0, l \in \mathbb{Z}\} \cup \{t^s \mid s > 0\}.$$

This language is recognized by the finite state automaton of Figure 5. □

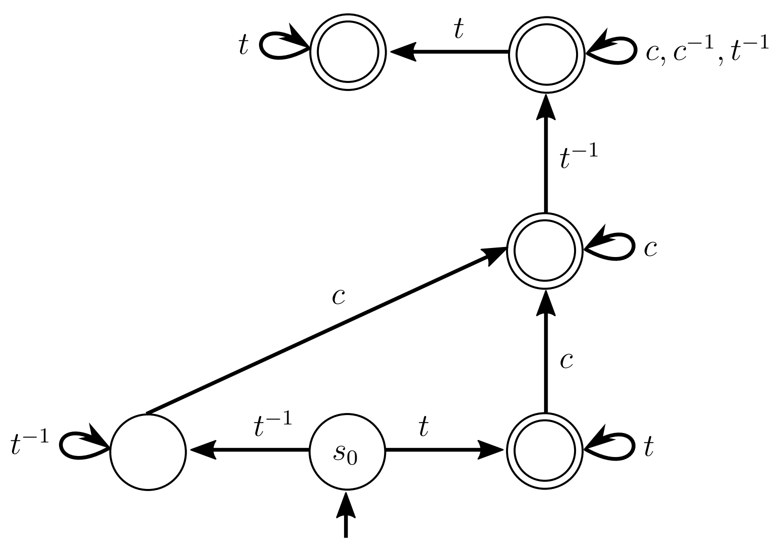

Figure 5: Finite state automaton accepting a positive cone language for $\mathbb{Z} \wr \mathbb{Z}$.

The previous strategy also work in more general wreath products $N \wr Q$. Let $\prec_N$ and $\prec_Q$ be left-orders on N and Q respectively, with corresponding positive cones P_N and P_Q. We can construct a lexicographic order $\prec_{\mathbf{N}}$ on $\mathbf{N} = \oplus_{q \in Q} N$ as follows. Given $\mathbf{n} = (n_q)_{q \in Q}, \mathbf{n}' = (n'_q)_{q \in Q} \in \mathbf{N}$ we put $\mathbf{n} \prec \mathbf{n}'$ if $\mathbf{n} \neq \mathbf{n}'$ and for $q' = \max_{\prec_Q}\{q \in Q \mid n_q \neq n'_q\}$ we have that $n_{q'} \prec_N n'_{q'}$.

Lemma 3.18. *The lexicographic order on $N \wr Q = \mathbf{N} \rtimes Q$ extending $\prec_{\mathbf{N}}$ and $\prec_Q$ is an $N \wr Q$-left-invariant lexicographic order with leading factor $\mathbf{N}$.*

Proof of Lemma 3.18. Observe that $P_{\mathbf{N}}$, the positive cone $\mathbf{N}$ associated to $\prec_{\mathbf{N}}$, is equal to

$$\{(n_q)_{q \in G} \in \mathbf{N} \setminus \{1_{\mathbf{N}}\} \mid n_q \in P_N \text{ where } q' = \max\{q \in G \mid n_q \neq 1_N\}\}.$$

Now, given $\mathbf{n} \in P_{\mathbf{N}}$ and $q' = \max\{q \in Q \mid n_q \neq 1_N\}$, and given $q \in Q$, we set $\mathbf{n}' = q\mathbf{n}q^{-1} = (n'_q)_{q \in Q}$, and we see that $qq' = \max_{\prec_Q}\{p \in Q \mid n'_p \neq 1_N\}$ since $\prec_Q$ is left Q-invariant. Therefore $n'_{qq'} = n_{q'} \in P_N$ and thus $\mathbf{n}' \in P_{\mathbf{N}}$. We have showed that $qP_{\mathbf{N}}q^{-1} \subseteq P_{\mathbf{N}}$ for all $q \in Q$, which implies $qP_{\mathbf{N}}q^{-1} = P_{\mathbf{N}}$ for all $q \in Q$. The result then follows from Lemma 3.15. □

Let X and Y be generating sets of N and Q respectively. Then the set $X \cup Y$ generates set of $N \wr Q$ since the q-th copy of N in $\oplus_{q\in Q} N$ is identified with qNq^{-1} and thus it can be generated by qXq^{-1} (the conjugates of X by q) and each element of qXq^{-1} can be expressed in terms of X and Y.

An element $(\mathbf{n} = \{n_q\}_{q\in Q}, p) \in N \wr Q$ can be written as $(\prod_{q\in Q} qn_qq^{-1})p$, and we can use the $\prec_Q$-order to order to write this uniquely as

$$(q_1n_1q_1^{-1})(q_2n_2q_2^{-1})\cdots(q_nn_mq_m^{-1})p$$

with the property that $q_1 \succ_Q q_2 \succ_Q q_3 \succ_Q \cdots \succ_Q q_m$.

Thus, with this unique way of writing the elements of $N \wr Q$, the lexicographic positive cone of $N \wr Q$ is

$$P = \left\{ (q_1n_1q_1^{-1})(q_2n_2q_2^{-1})\cdots(q_mn_mq_m^{-1})p \;\middle|\; \begin{array}{c} q_1 \succ_Q q_2 \succ_Q \cdots \succ_Q q_m \\ (n_1 \in P_N \text{ and } p \in Q) \text{ or } (m = 0 \text{ and } p \in P_Q) \end{array} \right\}. \tag{1}$$

We will now define a positive cone language for wreath products of groups in terms of the positive (and negative) cone languages for N and Q.

Proposition 3.19. *Let $\mathcal{L}_N \subseteq X^*$ and $\mathcal{L}_Q, \mathcal{M}_Q \subseteq Y^*$ be languages such that $\pi_N(\mathcal{L}_N) = P_N$ and $\pi_Q(\mathcal{L}_Q) = P_Q$ are positive cones for N and Q respectively, and $\pi_Q(\mathcal{M}_Q) = P_Q^{-1}$. Then, a language that evaluates to P as in Equation* (1) *is given by*

$$\mathcal{L} \coloneqq Y^*\mathcal{L}_N\mathcal{M}_Q(X^*\mathcal{M}_Q)^*Y^* \cup \mathcal{L}_Q. \tag{2}$$

Proof. First observe that

$$\mathcal{L} = \left\{ vu_1w_1u_2w_2\dots u_mw_mz \;\middle|\; \begin{array}{c} v, z \in Y^*, \\ u_1 \in \mathcal{L}_N \text{ or } (m = 0, v = \varepsilon \text{ and } z \in \mathcal{L}_Q), \\ u_i \in X^*, w_i \in \mathcal{M}_Q \end{array} \right\}. \tag{3}$$

Let P be the positive cone described in (1).

Let us first prove that $P \subseteq \pi(\mathcal{L})$. Let $g \in P$, and assume that $g = (q_1n_1q_1^{-1})(q_2n_2q_2^{-1})\cdots(q_mn_mq_m^{-1})p$ with q_i, n_i, p as in (1). Let $z \in Y^*$ such that $\pi(z) = p$. If $m = 0$, then $g = p \in P_Q$. We can assume that $z \in \mathcal{L}_Q$. If $m > 0$, there is $v \in Y^*$ such that $\pi(v) = q_1$, $u_1 \in \mathcal{L}_N$ such that $n_1 \in P_N$, and $w_i \in \mathcal{M}_Q$ such that $\pi(w_i) = q_i^{-1}q_{i+1} \in P_Q^{-1}$, $u_i \in X^*$, such that $\pi(u_i) = n_i$ for $2 \leq i \leq m$. We see that $g = \pi(vu_1w_2u_2\dots u_mw_mz)$.

To prove that $\pi(\mathcal{L}) \subseteq P$, let $\mathbf{w} = vu_1w_1u_2w_2\dots u_mw_mz \in \mathcal{L}$ as in the description in (3). If $m = 0$, then $\mathbf{w} = z$ and $z \in \mathcal{L}_Q$. Thus $\pi(\mathbf{w}) \in P$. If $m > 0$, let $q_1 = \pi(v)$ and for $i > 1$, $q_i = q_{i-1}\pi(w_i)$, thus $\pi(w_i) = q_{i-1}^{-1}q_i$. For $i = 1, \dots, m$ let $n_i = \pi(u_i)$ and $p = \pi(z)$. Therefore $\pi(\mathbf{w}) = (q_1u_1q_1^{-1})(q_2u_2q_2^{-1})\cdots(q_mu_mq_m^{-1})p$. Note that since $w_i \in \mathcal{M}_Q$, we have that $q_i \prec_Q q_{i-1}$. It follows that $\pi(\mathbf{w}) \in P$. □

We now state a generalization of Proposition 3.17.

Proposition 3.20. *Let $\mathcal{C}$ be a full AFL closed under reversal. Let N and Q be finitely generated groups. Suppose that N and Q have a $\mathcal{C}$-left-order represented by $\mathcal{L}_N$ and $\mathcal{L}_Q$ respectively. Then, the wreath product of $N \wr Q$ admits a $\mathcal{C}$-left-order.*

In particular admitting Reg*-left-orders is closed under wreath products.*

Proof. Assume that N is generated by a finite set X, Q is generated by a finite set Q and we will construct a language over the generating set $X \sqcup Y$. Let $\mathcal{M}_Q = \mathcal{L}_Q^{-1}$ be the negative cone language associated to $\mathcal{L}_Q$ obtained by reversal and sending each letter $x \mapsto x^{-1}$. Let $\mathcal{L}$ be the language of Equation (2). By Proposition 3.19, $\mathcal{L}$ is a positive cone language for $N \wr Q$. Since a class full AFL is closed by concatenation, concatenation closure and union, we see that $\mathcal{L}$ is in $\mathcal{C}$. □

4 Groups where all positive cones are regular

In this section we classify the groups that only admit Reg-left-orders.

4.1 Order-convex subgroups and language-convex subgroups

Definition 4.1. Let G be a group with left-order $\prec$. A subgroup $H \leqslant G$ is called $\prec$*-convex* if for all $h_1, h_2 \in H$ and $g \in G$ satisfying $h_1 \prec g \prec h_2$ we have that $g \in H$.

Left-order convexity relates to relative left-orders introduced in Section 2.5 in the following way.

Lemma 4.2. *If H is $\prec$-convex for a left-order $\prec$ on G, then $P = \{g \in G \mid H \prec g\}$ is a positive cone relative to H.*

Proof. First, we observe that for all $g \notin H$, $g \prec h$ or $h \prec g$ for all $h \in H$, as otherwise we would have that $h \prec g \prec h'$ for $h, h' \in H$, which by $\prec$-convexity of on H implies that $g \in H$. Therefore, for all $g \in G - H$ either $g \prec H$ or $H \prec g$. Now, if $H \prec g$, then $g^{-1} \prec 1 \in H$. As $g^{-1} \notin H$ this means that $g^{-1} \prec H$. This shows that $G = P \sqcup P^{-1} \sqcup H$. Finally, to show that P is a semigroup, notice that since $1 \in H$, we have that $1 \prec g_1$ and $1 \prec g_2$ for $g_1, g_2 \in P$. This implies that $H \prec g_1 \prec g_1 g_2$. □

Thanks to the previous lemma and the discussion in Section 2.5, we can conclude that if H is $\prec$-convex, then for every $g_1 \prec g_2$, $g_1, g_2 \in G$ one has that $g_1 h_1 \preceq g_2 h_2$ for all $h_1, h_2 \in H$. Moreover, if H is $\prec$-convex, then $\prec$ induces a left-order on the coset space G/H.

We start proving a key ingredient, that says that for certain subgroups being $\prec$-convex implies being language-convex.

Proposition 4.3. *Let $G = H \rtimes \mathbb{Z}$ be finitely generated by (X, π). Let $\prec$ be a lexicographical* Reg*-left-order on G leaded by $\mathbb{Z}$, with $\mathcal{L} \subseteq X^*$, a regular positive cone language. If H is finitely generated, then H is language-convex with respect to $\mathcal{L}$.*

In particular, the restriction of $\prec$ to H is a Reg*-left-order.*

Recall that we set in Notation 2.3, that given word $w \in X^*$, and $x \in X$, we use $\sharp_x(w)$ to denote the number of times the letter x appears in the word w.

Before proving the Proposition 4.3 we need describe all regular positive cones on $\mathbb{Z}$ over the alphabet $\{t, t^{-1}\}$. Given a word $w = x_1 \dots x_n \in \{t, t^{-1}\}^*$, with $x_i \in \{t, t^{-1}\}$ we define a function $f_w \colon \{0, 1, \dots, n\} \to \mathbb{Z}$ by $f_w(i) = \sharp_t(x_1 \dots x_i) - \sharp_{t^{-1}}(x_1 \dots x_i)$.

Lemma 4.4. *Let $\mathcal{L} \subseteq \{w \in \{t,t^{-1}\}^* \mid \sharp_t(w) - \sharp_{t^{-1}}(w) \geq 0\}$ be a regular language. Then every f_w is coarsely non-decreasing in the following sense: there is a constant $K \geq 0$ such that for all $w \in \mathcal{L}$ and for all $i,j \in \{0,1\dots,\ell(w)\}$, if $j > i$ then $f_w(j) > f_w(i) - K$.*

Proof. Let $\mathbb{M}$ be an automaton accepting $\mathcal{L}$. We will consider $\mathbb{M}$ to be without ϵ-moves and the image of δ being singletons (see Remark 2.1). We will think of this automaton as a directed graph as explained in Remark 2.2. Every edge has a label from X, and from every vertex there is at most one outgoing edge with label $x \in X$. Hence, every $w \in \mathcal{L}$ correspond to a unique a path in $\mathbb{M}$.

Every $w \in \mathcal{L}$ can be decomposed as $w = xyz$, where y is a (possibly trivial) loop in $\mathbb{M}$. Then, $w \in \mathcal{L}$ implies that $xy^nz \in \mathcal{L}$ for any $n \in \{0,1,2,\dots\}$. In other words, for each word w accepted in the automaton for $\mathcal{L}$, we may remove or insert words y representing loops in $\mathbb{M}$ and still get an accepted word xy^nz.

Let $w \in \mathcal{L}$. Let $g(w) := \#_t(w) - \#_{t^{-1}}(w)$. Write $w = xyz$ where y is a (possibly) trivial loop. Observe that $g(y) \geq 0$, for otherwise $xy^iz \in \mathcal{L}$ for any i, as we may pick i large enough such that $g(xy^iz) < 0$, contradicting our assumption about $\mathcal{L}$. In other words, for any loop y in $w = xyz$,

$$g(xz) \leq g(xyz) \tag{4}$$

and $xz \in \mathcal{L}$ since we have only removed a loop y.

Let n be the number of states of $\mathbb{M}$. For any subword u of $w \in \mathcal{L}$, decompose $u = x_1y_1x_2y_2\dots x_{k-1}y_{k-1}x_k$, where the y_i are loops and the length of the word $x_1x_2\dots x_k$ is minimal. We allow the subwords x_i to be empty. Viewing u as a subpath of w in $\mathbb{M}$, we construct a new path whose label is $u' = x_1\dots x_k$ that consist of removing all loops from u. In particular, by pigeonhole principle $\ell(u') \leq n$ as otherwise, the path associated to u', would go through the same vertex in $\mathbb{M}$ twice, thus we have a factorization of u with a loop, that removing it produces a word shorter than u'. By (4) we have that $g(u) \geq g(u')$, and as $\ell(u') \leq n$, this implies that

$$g(u) \geq -n \text{ for all subword } u \text{ of } w \in \mathcal{L}, \tag{5}$$

since each transition can only contribute one t or t^{-1} and $\ell(u') \leq n$.

Let $1 \leq i < j \leq \ell(w)$. We need to show that there is a constant $K \geq 0$ such that $f_w(j) - f_w(i) \geq -K$. But $f_w(j) - f_w(i) = g(u)$ for u equal to the subword of w consisting of taking the prefix of w of length j and removing to it the prefix of length i. Now the result follows from (5) and taking $K = n$. □

Now we can prove the proposition.

Proof of Proposition 4.3. Let (X, π_H) a generating set for H, and $\{t, t^{-1}\}$ a generating set for $\mathbb{Z}$. We combine them to make $(X' = X \sqcup \{t,t^{-1}\}, \pi)$ a generating set for G with evaluation map π. Let $\mathcal{L} \subseteq (X')^*$ be a regular language such that $\pi(\mathcal{L})$ is a lexicographic positive cone with the quotient being the leading factor.

Let $\phi\colon (X')^* \to \{t,t^{-1}\}^*$ consisting on deleting the letters of X. This is monoid morphism, and hence $\phi(\mathcal{L})$ is regular. Since $\mathcal{L}$ is the language of a lexicographic order, $\phi(\mathcal{L})$ is contained in $\{w \in \{t,t^{-1}\}^* \mid \sharp_t(w) - \sharp_{t^{-1}}(w) \geq 0\}$. By Lemma 4.4, we get that there is a $K \geq 0$ such that $f_{\phi(w)}(j) > f_{\phi(w)}(i) - K$ for all $w \in \mathcal{L}$ and for all $j > i$.

Now, to see that H is language-convex with respect to $\mathcal{L}$, let $w \in \mathcal{L}$, with $\pi(w) \in H$. We get that $\sharp_t\phi(w) - \sharp_{t^{-1}}\phi(w) = 0$. Then, $0 = f_{\phi(w)}(\ell(w)) \geq \max_i f_{\phi(w)}(i) - K$, so $\max_i f_{\phi(w)}(i) \leq K$. Also $\min_i f_{\phi(w)}(i) > f_{\phi(w)}(0) - K = -K$. It follows that for every prefix u of w, $|\sharp_t\phi(u) - \sharp_{t^{-1}}\phi(u)| \leq K$. Therefore

$$\mathrm{d}_G(\pi(u), \pi(ut^{-\sharp_t\phi(u)+\sharp_{t^{-1}}\phi(u)})) \leq K.$$

Observe that $\pi(ut^{-\sharp_t\phi(u)+\sharp_{t^{-1}}\phi(u)}) \in H$ since the exponent $-\sharp_t\phi(u) + \sharp_{t^{-1}}\phi(u)$ cancels the t's in u. This shows that H is language-convex.

By Proposition 2.14, we get that the restriction of the left-order to H is regular. □

4.2 Groups all whose orders are regular

Now we will characterize groups whose orders are all regular. Note that the set of Turing machines is countable. Therefore a left-orderable group can have at most a countable number of computable left-orders. The following result of Linnell [22] implies that if all the left-orders are computable, then there should be finitely many of them.

Theorem 4.5. *[22] If a group admits infinitely many left-orders, then it admits uncountably many.*

The case when a group admits finitely many left-orders was classified by Tararin [33] (see also [19, 13]) and such groups are called Tararin groups.

Recall that a torsion-free abelian group has *rank 1* if for any two non-identity elements a and b there is a non-trivial relation between them over the integers: $na+mb = 0$. Torsion-free abelian groups of rank 1 are, up to isomorphism, subgroups of $\mathbb{Q}$.

A group G is a *Tararin group* if admits a unique subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

such that all the factors are torsion-free abelian groups of rank 1 and such that no quotient G_i/G_{i+2} is bi-orderable.

A group admit finitely many left-orders if and only if it is Tararin group (see [13, Theorem 2.2.13]). Moreover, for any left-order on Tararin group with the subnormal series as above, the unique proper order-convex subgroups are the groups $G_1, G_2, \ldots, G_n$. It follows that there are 2^n left-orders on G and each of them is completely determined by the sign of a non-trivial element of G_i/G_{i+1} for $i = 1, \ldots, n$. More concretely, all left-orders on G_i are lexicographic orders associated to the extension $G_{i+1} \to G_i \to G_i/G_{i+1}$ where the quotient leads.

Lemma 4.6. *All left-orders on a poly-$\mathbb{Z}$ Tararin group are regular.*

Proof. The proof is by induction on the Hirsch length. If $h(G) = 0$, then $G \cong \{1\}$ and the lemma holds.

Now assume that $h(G) > 0$. Let $G_1 \trianglelefteq G$ such that G/G_1 is infinite cyclic. Since G is a Tararin group G_1 is $\prec$-convex in G for any left-order $\prec$. Thus, any order on G is a lexicographic order associated to an extension G_1-by-$\mathbb{Z}$ in which the quotient group is the leading lexicographic factor. Since G_1 is a poly-$\mathbb{Z}$ Tararin group with $h(G_1) < h(G)$, we get by induction that all the left-orders on G_1 are regular. Recall from Example 2.16, that the two left-orders that $\mathbb{Z}$ admits are regular. Therefore, all the left-orders of G are lexicographic extensions of a regular order on G_1 and a regular order on $\mathbb{Z}$ and, by Proposition 3.3, all left-orders of G are regular. □

Lemma 4.7. *Suppose that G is a finitely generated Tararin group with all the left-orders being regular. Then G is is poly-$\mathbb{Z}$.*

Proof. Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

be the unique subnormal series of G with all the factors are torsion-free abelian groups of rank 1. We have to show that all factors are cyclic. We argue by induction on the length of the series. If the length is 0, $G \cong \{1\}$.

Suppose that the length is > 0. Since G is finitely generated, we get that G_0/G_1 is a finitely generated subgroup of $\mathbb{Q}$ and hence $G_0/G_1 \cong \mathbb{Z}$. Thus, $G = G_1 \rtimes \mathbb{Z}$.

Suppose that G_1 is finitely generated. Then by Proposition 4.3 all the induced left-orders on G_1 are regular. Since G is a Tararin group, all left-orders on G_1 are restrictions of left-orders in G. Therefore, all left-orders in G_1 are regular, and by induction G_1 is poly-$\mathbb{Z}$ and so is G .

The remaining case is that G_1 is not finitely generated. Now, it follows from Lemma 3.11 that the lexicographic orders can not be regular. □

Remark 4.8. A group G is poly-$\mathbb{Z}$ Tararin if and only if there exists a unique subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = \{1\}$$

such that for all i, $G_i/G_{i+1} \cong \mathbb{Z}$ and $G_i/G_{i+2} \cong K$ where $K = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ is the Klein bottle (fundamental) group.

Theorem 4.9. *A group only admits regular left-orders if and only if it is Tararin poly-$\mathbb{Z}$.*

Proof. By the previous discussion a group that admits only regular left-orders must admit only a countable number of left-orders and therefore it must be a Tararin group.

So we need to show that a Tararin group only admits regular left-orders if and only if it is poly-$\mathbb{Z}$. The result follows from the two previous lemmata. □

5 Ordering quasi-morphisms

On this section, where we prove one of the main theorems of the paper, we discuss some type of one-counter positive cones. These one-counter positive cones will be constructed through a quasi-morphism, that we will call *ordering quasi-morphism*, that is computable with a transducer. The interesting fact of these one-counter positive cones, is that if G has one of such cones, then $G \times \mathbb{Z}$ will have a regular positive cone.

5.1 Ordering quasi-morphism

A *quasi-morphism* $\phi\colon G \to \mathbb{R}$ is a function that is at bounded distance from a group homomorphism, i.e. there is a constant D such that $|\phi(g)+\phi(h)-\phi(gh)| \leq D$. A quasi-morphism naturally assigns a sign to elements of G. Dicks and Šunić [11], stated sufficient conditions for defining a positive cone as group elements mapped to positive numbers under a quasi-morphism.

The following is essentially [11, Lemma 4] with slightly more generality.

Lemma 5.1. *Let G be a group and $\tau\colon G \to \mathbb{Z}$ be a function with the following properties. For all $g, h \in G$,*

(i) $C = \{g \in G \mid \tau(g) = 0\}$ *is a subgroup of G,*

(ii) $\tau(g) = -\tau(g^{-1})$,

(iii) $\tau(g) + \tau(h) + \tau((gh)^{-1}) \leq 1$.

Let $P = \{g \in G \mid \tau(g) > 0\}$. Then P is a positive cone relative to C.

Proof. We will first prove that P^{-1} is disjoint from P.

$$\begin{aligned} P^{-1} &= \{g \in G \mid g^{-1} \in P\} \\ &= \{g \in G \mid \tau(g^{-1}) > 0\} \\ &= \{g \in G \mid -\tau(g) > 0\} \\ &= \{g \in G \mid \tau(g) < 0\}. \end{aligned}$$

Thus $P \cap P^{-1} = \emptyset$. Since $\tau(g) = 0 \implies g \in C$, $G = P \sqcup P^{-1} \sqcup C$.

As for the semigroup closure property, suppose that $g, h \in P$. Recall that $\tau(g) + \tau(h) + \tau((gh)^{-1}) - 1 \leq 0$ and that $\tau(g), \tau(h) \geq 1$. Then,

$$\begin{aligned} \tau(gh) &\geq \tau(gh) + \tau(g) + \tau(h) + \tau((gh)^{-1}) - 1 \\ &= \tau(g) + \tau(h) - 1 \geq 1. \end{aligned}$$

This shows that $gh \in P$. □

Definition 5.2. A quasi-morphism $\tau\colon G \to \mathbb{Z}$ satisfying (i), (ii), (iii) of the Lemma above, will be called an *ordering quasi-morphism*. The subgroup C of (i) is called the *kernel* of τ.

Example 5.3 (Ordering quasi-morphism on free products). Dicks and Šunić proved in [11, Proposition 13] that if $G = *_{i\in I} G_i$ is a free product of left-ordered groups $(G_i, \prec_i)$ and I is has total order $\prec_I$, then one can define a ordering quasi-morphism on G as follows.

Given $g \in G$, write $g = g_1 g_2 \dots g_k$ in normal form, where each g_i (called *syllable*) is in some $G_j \setminus \{1\}$ and two consecutive syllables g_i, g_{i+1} lie in different free factors. The normal form of an element $g \in G$ is unique. For a syllable g_i, write $\mathrm{factor}(g_i) = j$, to indicate that $g_i \in G_j$. A syllable g_i is positive if $1 \prec_{\mathrm{factor}(g_i)} g_i$ and negative if $1 \succ_{\mathrm{factor}(g_i)} g_i$. An *index jump* in $g_1 \dots g_n$ is a pair of consecutive syllables $g_i g_{i+1}$ such that $\mathrm{factor}(g_i) \prec_I \mathrm{factor}(g_{i+1})$. Similarly, an *index* *drop* in $g_1 \dots g_n$ is a pair of consecutive syllables $g_i g_{i+1}$ such that $\mathrm{factor}(g_i) \succ_I \mathrm{factor}(g_{i+1})$. Define

$$\tau(g) = \sharp(\text{positive syllables in } g) - \sharp(\text{negative syllables in } g) \\ + \sharp(\text{index jumps in } g) - \sharp(\text{index drops in } g).$$

As mentioned, [11, Proposition 13] claims that previous function is an ordering quasi-morphism on G with trivial kernel.

Example 5.4 (Ordering quasi-morphism on amalgamated free products). In [3, Theorem 17], the first author together with Dicks and Šunić described how to generalize the previous ordering quasi-morphism to more general groups acting on trees. We describe here the case of amalgamated free products. Suppose for example that G is the free product of G_i, $i \in I$ amalgamated along a common subgroup $C \leqslant G_i$ for all i. For each i, assume that P_i is a positive cone relative to C (i.e. $G_i = P_i \sqcup C \sqcup P_i^{-1}$ and P_i is a sub-semigroup of G_i.) Then, any element $g \in G$ can be written (uniquely fixing transversals) as $g = g_1 g_2 \dots g_k c$ with $c \in C$ and $g_i \in P_{j_i} \cup P_{j_i}^{-1}$, for $i \geq 0$. Define

$$\tau(g) = \tau(g_1 \dots g_k) = \sharp(\text{positive syllables in } g_1 \dots g_k) - \sharp(\text{negative syllables in } g_1 \dots g_k) \\ + \sharp(\text{index jumps in } g_1 \dots g_k) - \sharp(\text{index drops in } g_1 \dots g_k).$$

Thus, from [3, Theorem 17], τ is an ordering-quasimorphism on the free product of the G_i amalagamated over C with kernel C.

5.2 Ordering quasi-morphism computable through rational transducers

We are interested in the situation when the ordering quasi-morphism τ can be computed through a rational transducer.

Definition 5.5. Let G be a group finitely generated by (X, π). Let $\mathbb{Z}$ be generated by $(\{t^{-1}, t\}, \pi_{\mathbb{Z}})$. Le $\tau\colon G \to \mathbb{Z}$ be an ordering quasi-morphism.

A *τ-transducer* is a rational transducer $\mathbb{T}$ with input alphabet X and output alphabet $\{t^{-1}, t\}$ such that

1. $G = \pi(\mathbb{T}^{-1}(\{t^{-1}, t\}^*))$,
2. for every $w \in \mathbb{T}^{-1}(\{t^{-1}, t\}^*)$, one has that $\tau(\pi(w)) = \pi_{\mathbb{Z}}(\mathbb{T}(w))$.

In the following we show that groups admitting ordering quasi-morphism computable with a τ-transducer have one-counter positive cones.

Proposition 5.6. *Let G be a group finitely generated by (X, π) and $\tau\colon G \to \mathbb{Z}$ an ordering quasi-morphism with kernel C. If a τ-transducer exists, then $P_\tau = \{g \in G \mid \tau(g) > 0\}$ is a* 1C*-positive cone relative to C.*

In particular, if there is a 1C *language $\mathcal{L}_C$ such that $\pi(\mathcal{L}_C) = P_C$ is a positive cone for C, then $P_\tau \cup P_C$ is a* 1C *positive cone for G.*

Proof. The language $\mathcal{L} = \{w \in \{t^{-1}, t\} \mid \sharp_t(w) - \sharp_{t^{-1}}(w) > 0\}$ is a one-counter language (See Example 2.6). Let $\mathbb{T}$ be a τ-transducer. Since the class of one-counter languages is closed under inverse transducers (Proposition 2.10), we have that $\tilde{\mathcal{L}} = \mathbb{T}^{-1}(\mathcal{L})$ is a one-counter language. Now, observe that from the Definition 5.5, we get that $\pi(\tilde{\mathcal{L}}) = P_\tau$.

Finally, suppose that $\mathcal{L}_C$ is a $\mathsf{1C}$ language such that $\pi(\mathcal{L}_C)$ is a positive cone for C. Since one-counter languages are closed under union, we get that $\tilde{\mathcal{L}} \cup \mathcal{L}_C$ is a one-counter language representing $P_\tau \cup P_C$. □

Let τ be an ordering quasi-morphism on a free product constructed as in Example 5.4. Our objective now is to construct τ-transducer when the free factors have regular positive cones relative to C.

We start by observing that we can use a single finite state automaton to decide whether a word belongs to the union of the positive and negative cones.

Lemma 5.7. *Let (X, π) be a finite generating set of G. Suppose that P is a* Reg*-positive cone relative to $C \leqslant G$.*

Then, there exists a non-deterministic finite state automaton $\mathbb{M} = (\mathcal{S}, X, \delta, s_0, \mathcal{A})$ without ϵ-moves, where $\mathcal{A}$ the set of accepting states is a disjoint union $\mathcal{A} = \mathcal{A}^- \sqcup \mathcal{A}^+$ such that the language $\mathcal{L}^-$ accepted by $\mathbb{M}^- = (\mathcal{S}, X, \delta, s_0, \mathcal{A}^-)$ and the language $\mathcal{L}^+$ accepted by $\mathbb{M}^+ = (\mathcal{S}, X, \delta, s_0, \mathcal{A}^+)$ satisfy that $\pi(\mathcal{L}^-) = P^{-1}$ and $\pi(\mathcal{L}^+) = P$.

Proof. By hypothesis, there is a regular language $\mathcal{L}^+ = \mathcal{L} \subseteq X^*$ that evaluates to P. Moreover, by Lemma 2.12, there is a regular language $\mathcal{L}^- \subseteq X^*$ that evaluates to P^{-1}. Now there are non-deterministic finite state automata $\mathbb{M}^-$ and $\mathbb{M}^+$ without ϵ-moves accepting $\mathcal{L}^-$ and $\mathcal{L}^+$ respectively (Remark 2.1). Viewing this automata as directed graphs, we obtain the desired automaton by identifying the start vertex of $\mathbb{M}^-$ with the start vertex of $\mathbb{M}^+$ and designing that vertex to be s_0, the start vertex of $\mathbb{M}$. In Figure 6 we see an example of this construction. □

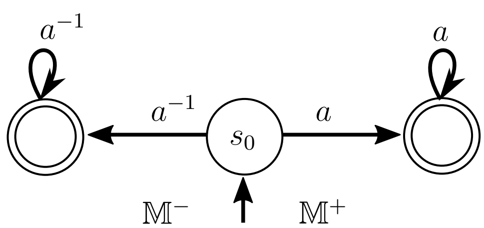

Figure 6: Example of the construction of Lemma 5.7, which gives an automaton that have states for recognizing the positive cone and states for recognizing the negative cone.

Proposition 5.8. *Let I be a finite set. For each $i \in I$, let G_i be a group finitely generated by (X_i, π_i) and assume that all G_i have a common finitely generated subgroup C. Let G denote the free product of the G_i amalgamated over C.*

Let (Y, π_C) be a finite generating set for C and $(X = Y \sqcup \bigsqcup X_i, \pi)$ be a generating set for G where for $x \in X_i$ we have that $\pi(x) = \pi_i(x)$ and for $y \in Y$ we have that $\pi(y) = \pi_C(y)$.

For each $i \in I$, assume that P_i is a positive cone relative to C. Suppose that there is a regular language $\mathcal{L}_i \subseteq X_i^*$ such that $\pi_i(\mathcal{L}_i) = P_i$.

Then G has an ordering quasi-morphism $\tau\colon G \to 2\mathbb{Z}+1 \cup \{0\}$ with kernel C admitting a τ-transducer $\mathbb{T}$. Moreover, the following language

$$\mathbb{T}^{-1}(\{w \in \{t, t^{-1}\}^* : \sharp_t(w) - \sharp_{t^{-1}}(w) > 0\})$$

evaluates onto a positive cone relative to C and it is equal to

$$\{w = w_{i_1} \dots w_{i_m} z \in X^* \mid i_t \in \{1, \dots, n\}, i_j \neq i_{j+1}, w_{i_j} \in \mathcal{L}^+_{i_j} \cup \mathcal{L}^-_{i_j}, z \in Y^*, \tau(\pi(w)) > 0\}.$$

Proof. By the previous Lemma 5.7, there are finite state automata $\mathbb{M}_i = (\mathcal{S}_i, X_i, \delta_i, s_{0i}, \mathcal{A}^-_i \sqcup \mathcal{A}^+_i)$ for $i = 1, \dots, n$, such that the words accepted by $\mathbb{M}_i$ on a state from $\mathcal{A}^+_i$ form a regular language $\mathcal{L}^+_i$ evaluating to P_i and the words accepted by $\mathbb{M}_i$ on a state from $\mathcal{A}^-_i$ form a regular language $\mathcal{L}^-_i$ evaluating to P_i^{-1}.

We first construct a non-deterministic finite state automaton $\mathbb{M}$ accepting the language

$$\mathcal{L} := \{w = w_{i_1} \dots w_{i_m} z \in X^* \mid i_t \in \{1, \dots, n\}, i_j \neq i_{j+1}, w_{i_j} \in \mathcal{L}^+_{i_j} \cup \mathcal{L}^-_{i_j}, z \in Y^*\}.$$

Then we will modify this automaton $\mathbb{M}$ to produce a τ-transducer. An example of this construction can be found in Figure 7 and might help the reader check the example as one reads the proof.

Note that $G - C = \pi(\mathcal{L})$. Note that by Example 5.4, $\{w \in \mathcal{L} \cdot Y^* \mid \tau(\pi(w)) > 0\}$ evaluates onto a positive cone relative to C of G.

We construct $\mathbb{M}$ taking the union of the automata $\mathbb{M}_i$, $i = 1, 2, \dots, n$ with their transitions and we add an extra start state s_0 and a final state f and the following ϵ-moves:

(I) there is an ϵ-move from s_0 to every start state each $\mathbb{M}_i$.

(II) there is an ϵ-move from each accepting states of $\mathbb{M}_i$ to the start state of $\mathbb{M}_j$ with $i \neq j$.

(III) there is an ϵ-move from each accepting states of $\mathbb{M}_i$ to f.

The state f is the only accepting state of $\mathbb{M}$. We add loops on f with label y, for each $y \in Y$. It is easy to see that $\mathcal{L}$ is accepted by $\mathbb{M}$.

Now we construct a τ-transducer $\mathbb{T}$ from $\mathbb{M}$ by adding some outputs on $T = \{t^{-1}, t\}$ to the ϵ-moves. It will be clear from the construction that the output is always an odd number of t and t^{-1} unless the input is a word in Y.

The ϵ-moves of type I do not output any word.

The ϵ-moves of type II output t^2 if they start on some $\mathcal{A}^+_i$ $i = 1, \dots, j$ and go to the start state of $\mathbb{M}_j$ with $i <_I j$. The ϵ-moves of type II output t^{-2} if they start on some $\mathcal{A}^-_i$ $i = 1, \dots, j$ and go to the start state of $\mathbb{M}_j$ with $i >_I j$. The other ϵ-moves of type II do not output any word.

The ϵ-moves of type III output a t if they start on some vertex of $\mathcal{A}^+_i$ and outputs a t^{-1} if they start on some vertex of $\mathcal{M}^-_i$.

It is easy to see that this gives a τ-transducer for τ. □

Propositions 5.6 and 5.8, we get the following.

Corollary 5.9. *Let A, B be groups admitting* Reg*-left-orders. Then $A * B$ admits a* 1C*-left-order.*

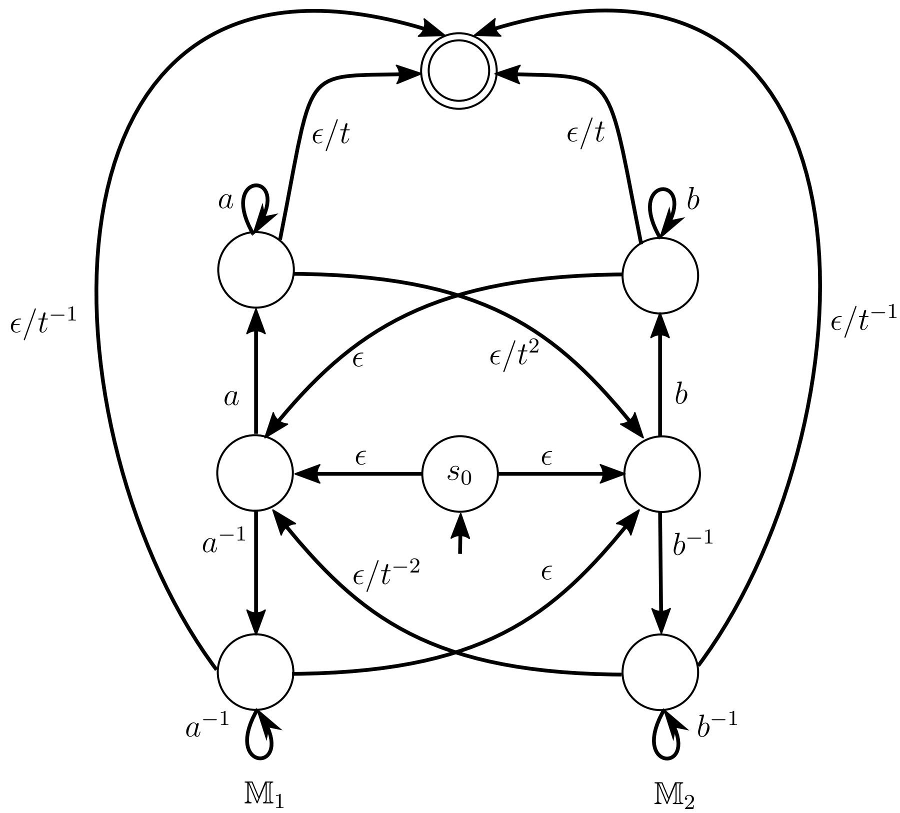

Figure 7: The τ-transducer for $F_2 = \langle a \rangle * \langle b \rangle$ constructed following the proof of Proposition 5.8. Observe that the three vertices on the left, and the three vertices of the right are copies of the automaton of Figure 6.

This corollary is optimal in the view of a result of Hermiller and Šunić [16] that says that free products do not admit regular positive cones. We also note that the first and last author, together with J. Alonso and J. Brum [1, Theorem 1.6], proved that certain free products with amalgamation also do not admit regular positive cones.

To introduce an interesting example[1] of an amalgamated free product, let n, m be two positive integers and consider the group

$$\begin{aligned} BS(1, m; 1, n) &\coloneqq \langle a, b, c \mid aba^{-1} = b^m,\, aca^{-1} = c^n \rangle \\ &\cong \langle a, b \mid aba^{-1} = b^m \rangle *_{\langle a \rangle} \langle a, c \mid aca^{-1} = c^n \rangle. \end{aligned}$$

By Lemma 3.13, $BS(1, n)$ has Reg-positive cones relative to $\langle a \rangle$. Therefore by Proposition 5.8, $BS(1, m; 1, n)$ admits a $\mathsf{1C}$-positive cone relative to $\langle a \rangle$ and by Proposition 5.6 we get the following.

Corollary 5.10. *For $n, m \geq 1$, the group $BS(1, m; 1, n)$ has a $\mathsf{1C}$-positive cone.*

[1]The actions of $BS(1, m; 1, n)$ on the closed interval $[0, 1]$ have been studied in [6], which showed that no smooth action exists.

5.3 Embedding theorem

The main result of this section is a construction of regular orders on $G \times \mathbb{Z}$ from a ordering quasi-morphism $\tau \colon G \to \mathbb{Z}$. We start by describing a positive cone for $G \times \mathbb{Z}$, and then we will show that it is regular (Theorem 5.12).

Proposition 5.11. *Let G be group, C be a subgroup of G, and $\tau \colon G \to 2\mathbb{Z}+1 \cup \{0\}$ be an ordering quasi-morphism with kernel C. Let*

$$P = \{(g,n) \in G \times \mathbb{Z} \mid \tau(g) + 2n > 0\}.$$

The set P is a positive cone relative to $C \times \{0\}$ for $G \times \mathbb{Z}$.

Proof. We define a map $\tau' \colon G \times \mathbb{Z} \to \mathbb{Z}$ as $\tau'((g,n)) = \tau(g) + 2n$ and show that it satisfies the conditions of Lemma 5.1. For (i), since $\tau(g)$ is odd for every $g \in G - C$, we get that $\tau'((g,n)) = 0$ if and only if $g \in C$ and $n = 0$. For (ii), observe that $-\tau'((g,n)) = -\tau(g) - 2n = \tau(g^{-1}) - 2n = \tau'((g^{-1}, -n))$. Finally, for (iii), observe that $\tau'((g,n)) + \tau'((h,m)) - \tau'((g^{-1}h^{-1}, -n-m)) = \tau(g) + \tau(h) - \tau(g^{-1}h^{-1}) \leq 1$. □

Our main theorem is the following.

Theorem 5.12. *Let G and τ be as in Proposition 5.11. If a τ-transducer exists, then the set $P = \{(g,n) \in G \times \mathbb{Z} \mid \tau(g) + 2n > 0\}$ can be represented by a regular language.*

In particular, by Proposition 5.11 this means that P is a regular positive cone on $G \times \mathbb{Z}$ when C is the trivial group. Also, implicit on the hypothesis of the existence of a τ-transducer is that G is finitely generated.

Remark 5.13. In the proof of Theorem 5.12 we will make use of the graph descriptions of finite state automata (Remark 2.2) and transducers (Remark 2.9). In the case of finite state automatons we will allow ourselves to label edges by words in the input alphabet and not just letters. To get a proper automaton, one has to change that edge by a path of the length of the label, so that the label on the path coincides with the label on the original edge. This will allow us to simplify the presentation of the proof.

Proof of Theorem 5.12. We will construct a finite state automaton accepting a language representing P. Let X be a finite generating set for G and let $\mathbb{T} = (\mathcal{S}, X, T = \{t, t^{-1}\}, \delta_{\mathbb{T}}, s_0, \mathcal{A})$ be a τ-transducer. We will modify $\mathbb{T}$ to construct a finite state automaton $\mathbb{M}$ accepting a language $\mathcal{L}$ that will evaluate onto P. The alphabet of the finite state automaton will be $(X \cup Z, \pi)$, where the elements of $Z = \{z, z^{-1}\}$ evaluates to 1 and -1 on $\mathbb{Z}$ respectively. Let $\tau' \colon G \times \mathbb{Z} \to \mathbb{Z}$ be defined by $\tau'((g,n)) = \tau(g) + 2n$, so $P = \{(g,n) \mid \tau'((g,n)) > 0\}$.

We first exemplify the idea, with the case where τ only takes even values. We would like for the $\mathbb{Z}$ factor of $G \times \mathbb{Z}$ to 'absorb' the output of the τ-transducer on G, such that each arrow of the form x/t^{2n} and x/t^{-2n} for $x \in X$ and $t \in T$ in $\mathbb{T}$ is replaced by an arrow of the form xz^{-n} and xz^{n} in $\mathbb{M}$ respectively. The desired result is that to a word $w \in X^*$ for which the transducer $\mathbb{T}$ outputs a word on $\{t^2, t^{-2}\}^*$ will correspond a word $w' \in (X \cup Z)^*$ accepted in $\mathbb{M}$ by the same path as in $\mathbb{T}$ such that $\tau'(\pi(w'))) = 0$. Note that w' is obtained from w by inserting z and z^{-1} so that $\tau'(\pi(w'))) = 0$. Then $\mathbb{M}$ will accept words of the form $w'z^{\alpha}$ with $\alpha > 0$ so that $\tau'((\pi(w'z^{\alpha})) > 0$.

Since τ really takes values in odd integers and zero, a problem arises when we have arrows with labels x/t^{2n+1} or x/t^{-2n-1} in $\mathbb{T}$ since we cannot replace them with arrows labeled by $xz^{-n-\frac{1}{2}}$ or $xz^{n+\frac{1}{2}}$ in $\mathbb{M}$. We remedy the problem as follows. We will modify the labels of $\mathbb{T}$ in a similar fashion to produce the automaton $\mathbb{M}$, such that if a given path of $\mathbb{T}$ originally processed $w \in X^*$ and $\mathbb{T}(w)$ 'absorbed' output $\tau(\pi(w))$, now the path with modified labels in $\mathbb{M}$ will accept $w' \in (X \cup Z)^*$ that consists on inserting appropriately z and z^{-1} to w such that $|\tau'(\pi(w'))| \leq 1$. To achieve this, we will need to increase the number of states so we can recall how far away is τ' on the processed word from 0. To construct the states of $\mathbb{M}$ we will multiply each state in $\mathbb{T}$ by $\{-1, 0, +1\}$. With the appropriate transitions that will have that for $w' \in (X \cup Z)^*$ such that when read by $\mathbb{M}$ we end up at state $(b, \dagger)$ with $\dagger \in \{-1, 0, +1\}$ (and this state is different from an special final state), then $\tau'(\pi(w)) = \dagger$.

We now formalize our construction of $\mathbb{M}$. We remark that Figure 8 exemplifies our construction starting from the transducer of Figure 7. The reader might find helpful to check these examples while following the construction.

The set of states $\mathcal{S}_{\mathbb{M}}$ of $\mathbb{M}$ contains $\mathcal{S} \times \{-1, 0, +1\}$ where $\mathcal{S}$ are the states of $\mathbb{T}$. The initial state of $\mathbb{M}$ is $(s_0, 0)$. The accepting states of $\mathbb{M}$ are the states $\{(\alpha, +1) \mid \alpha \in \mathcal{A}\} \cup \{f\}$ where f is a new state.

The transitions that go to f are as follows. First, $\delta_{\mathbb{M}}(f, z) = f$ (i.e. if a word is accepted, we can keep reading z's). Second $\delta_{\mathbb{M}}((s_0, 0), z) = f$ (i.e. all words of $\{z\}^+$ are accepted). We have that $\delta_{\mathbb{M}}((\alpha, \dagger), z) = f$ for all $\dagger \in \{-1, 0, 1\}$ and $\alpha \in \mathcal{A}$. There are no more transitions from the state f or going to the state f. Assuming that $(\alpha, \dagger) \in \delta((s_0, 0), w)$ implies that $\tau'(\pi(w)) = \dagger$, we see that $\mathbb{M}$ only accepts words that represent elements of P.

We will define the other transitions as to support the assumption. Recall that $\delta_{\mathbb{T}}$, the transition function of $\mathbb{T}$, is of the form $\delta_{\mathbb{T}} \colon \mathcal{S} \times X \to \text{Finite Subsets}(\mathcal{S} \times T^*)$. For each $\dagger \in \{-1, 0, +1\}$, we define the transition $\delta_{\mathbb{M}}((s, \dagger), x)$ in bijection with $\delta_{\mathbb{T}}(s, x)$ such that the transition function $\delta_{\mathbb{M}}$ behaves as $\delta_{\mathbb{T}}$ on the situations that $\mathbb{T}$ does not change τ of the output. That is, if $(r, u) \in \delta_{\mathbb{T}}(s, x)$ for $s \in \mathcal{S}$ and $x \in X$ and some $u \in T^*$ with $\pi_T(u) = 0$, then $(r, \dagger) \in \delta_{\mathbb{M}}((s, \dagger), x)$ for all $\dagger \in \{-1, 0, 1\}$. Here π_T denotes the evaluation from $T^* = \{t, t^{-1}\}^*$ to $\mathbb{Z}$.

Now, suppose that $(r, u) \in \delta_{\mathbb{T}}(s, x)$ with $u \in T^*$ and $n = \pi_T(u) \neq 0$. That is in $\mathbb{T}$ there is an edge from s to r with label x/u. Corresponding to this edge in $\mathbb{T}$ we construct the following edges in $\mathbb{M}$ from $(s, \dagger)$ to $(r, \dagger')$ with label xv where $v \in Z^*$ as follows:

- If n is even, then $\dagger' = \dagger$ and $v = z^{-n/2}$.
- If n is odd, and $\dagger = 0$ then let k such that $|n + 2k| = 1$ and set $\dagger' = n + 2k$ and $v = z^k$.
- If n is odd, and $\dagger = +1$ or $\dagger = -1$ then let k such that $n + 2k + \dagger = 0$ and set $\dagger' = 0$ and $v = z^k$.

There are no more states or transitions (i.e the transitions not describe go to the emptyset) and this completes the description of $\mathbb{M}$.

Let $\mathcal{L}$ be the language accepted by $\mathbb{M}$. We can observe that by induction, that if $\delta_{\mathbb{M}}((s_0, 0), w) = (s, \dagger)$ for some $s \in \mathcal{S}$ and for $w \in (X \sqcup Z)^*$ then there exists $u \in T^*$ and a factorization of w as $w_1 v_1 w_2 v_2 \dots w_k v_k$ such that $w_i \in X^*$, $v_i \in Z^*$ and $(s, u) \in \delta_{\mathbb{T}}(s_0, w_1 w_2 \dots w_k)$ with $\pi_T(u) + \dagger + 2\pi_Z(v_1 \dots v_k) = 0$. And from that, we can observe that if w is accepted by $\mathbb{M}$, then $\pi(w) \in P$. Thus $\pi(\mathcal{L}) \subseteq P$.

On the other hand, to see that $P \subseteq \pi(\mathcal{L})$ observe first that given $(g, n) \in P$ there is some $w \in \mathbb{T}^{-1}(T^*)$ such that $\pi_X(w) = g$. Suppose that a possible way for $\mathbb{T}$ to process $w = x_1 \dots x_n$ is a sequence of states of $\mathbb{T}$ with outputs in T^* as follows

$$s_0,\ (s_1, u_1) \in \delta_{\mathbb{T}}(s_0, x_1),\ (s_2, u_2) \in \delta_{\mathbb{T}}(s_2, x_2), \dots,\ (s_n, u_n) \in \delta_{\mathbb{T}}(s_{n-1}, u_{n-1})$$

with $s_i \in \mathcal{S}$. Consider the word $w' = x_1 z^{\nu_1} x_2 z^{\nu_2} \dots x_n z^{\nu_n}$ where the ν_i's are determined by the following two conditions:

(i) $|\pi_T(u_1 \dots u_i) - 2\sum_{j=1}^{i} \nu_j| \leq 1$,

(ii) if $\pi_T(u_1 \dots u_i)$ is odd, then it has the same sign as $\pi_T(u_1 \dots u_i) - 2\sum_{j=1}^{i} \nu_j$.

Then, it can be checked that, $(s_n, \dagger) \in \delta_{\mathbb{M}}((s_0, 0), w')$ with $\dagger = \pi_T(u_1 \dots u_i) - 2\sum_{j=1}^{i} \nu_j$ and thus, $w' z^\alpha$ with $\alpha = n - \sum_{j=1}^{i} \nu_j$ is accepted by $\mathbb{M}$. □

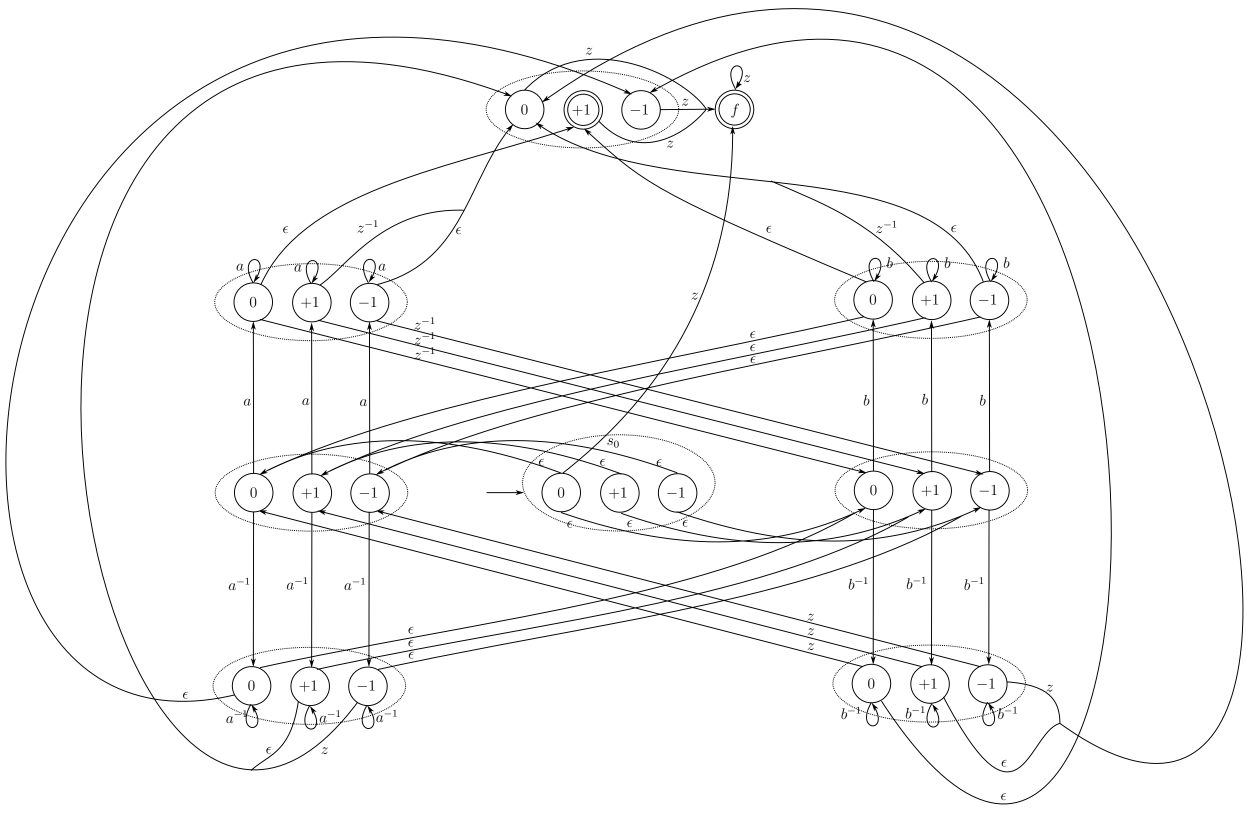

Figure 8: A finite state automaton for $F_2 \times \mathbb{Z}$. It looks similar to Figure 7, with each state tripled and extra arrows added. Due to the large number of arrows, some have been merged together.

Theorem 5.14. *Let $G_1, G_2, \dots G_n$ be finitely generated groups with a common subgroup C such that each G_i admits a* Reg*-positive cone relative to C and C admits a* Reg*-positive cone. Let G be the free product of the G_i's amalgamated over C. Then $G \times \mathbb{Z}$ admits a* Reg*-left-order.*

Proof. By Proposition 5.8, G admits an ordering quasi-morphism $\tau\colon G \to 2\mathbb{Z}+1 \cup \{0\}$ with kernel C and such that τ is computable though a rational transducer. Now by Theorem 5.12, $G\times\mathbb{Z}$ has Reg-positive relative to $C\times\{0\}$. Finally, since $C \cong C\times\{0\}$ has Reg-positive cones, we get from Lemma 2.21 that $G\times\mathbb{Z}$ has relative positive cones. □

Corollary 5.15. *Let A, B be groups admitting* Reg-*left-orders. Then $(A * B)\times\mathbb{Z}$ admits* Reg-*left-orders.*

Also from the previous theorem, and by the discussion prior to Corollary 5.10 we have

Corollary 5.16. *For $n, m \geq 1$, the group $BS(1, m; 1, n)\times\mathbb{Z}$ has a* Reg-*positive cone.*

Another interesting application of the results of this paper, is the following.

Theorem 5.17. *Suppose that G is a (non-abelian finitely generated free)-by-$\mathbb{Z}$ group. Then, no lexicographic left-order $\prec$ on G where $\mathbb{Z}$ leads is regular.*

However, there is a lexicographic left-order on G where $\mathbb{Z}$ leads that is one-counter and extends to regular left-order on $G\times\mathbb{Z}$.

Proof. From Proposition 4.3, if G admits a regular lexicographic left-order where $\mathbb{Z}$ leads, then there will be a regular order on a finitely generated free group, contradicting the theorem of Hermiller and Šunić [16] that says that no non-abelian free group admits a regular positive cone.

Suppose that $f\colon G \to \mathbb{Z}$ is a surjective homomorphism with kernel F_n, a free group of rank n. By Proposition 5.8, there is an ordering-quasimorphism $\tau\colon F_n \to 2\mathbb{Z}+1\cup\{0\}$ with trivial kernel, admitting a τ-transducer. By Proposition 5.6, F_n admits a one-counter positive cone, and by Proposition 3.3, since $\mathbb{Z}$ has regular orders, G has a one-counter positive cone.

By Theorem 5.14, we see that $F_n\times\mathbb{Z}$ has a regular positive cone. Note that $F_n\times\mathbb{Z}$ is the kernel of $\tilde{f}\colon G\times\mathbb{Z}\to\mathbb{Z}$ given by $(g, n)\mapsto f(g)$. As $\mathbb{Z}$ has regular positive cones, Proposition 3.3 guaranties that $G\times\mathbb{Z}$ has regular positive that is lexicographic with leading factor the quotient, when viewing $G\times\mathbb{Z}$ as a $(F_n\times\mathbb{Z})$-by-$\mathbb{Z}$ extension. The restriction of this order on G is still lexicographic. □

Some families of groups that are known to be (finitely-generated free)-by-cyclic are provided in the next corollary.

Definition 5.18 (Artin groups and defining graphs)**.** Let Γ be a finite simplicial graph with edges labeled by integers greater than one. We associate to Γ a group $A(\Gamma)$ whose presentation has generators corresponding to the vertices of Γ and relations

$$\underbrace{aba\ldots}_{n \text{ letters}} = \underbrace{bab\ldots}_{n \text{ letters}}$$

where $\{a, b\}$ is an edge of Γ labeled with n. The graph Γ is called the *defining graph* of the *Artin group* $A(\Gamma)$.

Corollary 5.19. *Let $A(\Gamma)$ be an Artin groups whose defining graph Γ is a tree. Then G has one-counter left-orders and $G\times\mathbb{Z}$ has regular orders.*

Proof. By a result of Hermiller and Meier [15], $A(\Gamma)$ admits a short exact sequence $1 \to F_m \to A \to \mathbb{Z} \to 1$, where $m = \sum_{e_i \in \Gamma}(n_i - 1)$, where n_i is the label of the edge e_i. □

Definition 5.20. A right-angled Artin group is an Artin group whose defining graph only has edges with label 2.

Corollary 5.21. *Let G be a right-angled Artin group based on a connected graph with no induced subgraph isomorphic to C_4 (the cycle with 4 edges) of L_3 (the line with 3 edges). Then G has a regular orders.*

Proof. The proof is by induction on the size of the defining graph Γ. If the graph has one vertex, then $G \cong \mathbb{Z}$ and we know that $\mathbb{Z}$ only has regular orders. Now suppose that the defining graph has more than one vertex. Droms [12, Lemma] observed that in that case there is a vertex connected to all other vertex of the graph Γ. That is $G \cong \mathbb{Z} \times H$ where H is right-angled Artin group based on a graph Γ' that contains no induced subgraph isomorphic to C_4 or L_3. Note that Γ' has fewer vertices that Γ. Then, by induction, each connected component of Γ' defines a subgroup of H that has regular left-orders. If Γ' is connected, then H has regular left-orders and so does G. If Γ' is not connected, then it is a free product of groups having regulars and by Proposition 5.8 and Theorem 5.12 we get that $H \times \mathbb{Z}$ has regular left-orders. □

Remark 5.22. It follows from [16, Theorem 4], that there is no positive cone P on a right-angled Artin group defined over a graph of diameter ≥ 3 such the set of all geodesic words in the standard generating set that represent elements of P form a regular language.

We remark that all the defining graphs of the previous Corollary have diameter at most two.

A Appendix: pre-images of positive cones

In this appendix, we explore the following definition.

Definition A.1. Let $\mathcal{C}$ be a class of languages. Let G be a finitely generated by (X, π) and $\prec$ a left-order. We say that $\prec$ is a *$\mathcal{C}$-preimage left-order* if the language $\pi^{-1}(P_\prec) \in \mathcal{C}$.

Note that the difference between a $\mathcal{C}$-left-order and a $\mathcal{C}$-preimage left-order is that given an element $g \in P_\prec$, a $\mathcal{C}$-left-order does not contain all words which map to g, but has at least one. For a given class $\mathcal{C}$, a left-order might be a $\mathcal{C}$-left-order but not a $\mathcal{C}$-preimage left-order. In fact, for the lower classes in the Chomsky hierarchy as Reg and CF, we will see that there are few examples of $\mathcal{C}$-preimage left-orders. On the other hand, for the class of recursively enumerable languages, the highest in the Chomsky hierarchy, we will see that admitting a $\mathcal{C}$-left-order and is equivalent to admitting $\mathcal{C}$-preimage left-orders if G is finitely presented.

The first observation follows from Lemma 2.11.

Lemma A.2. *If $\mathcal{C}$ is class of languages closed under inverse homomorphisms, then having a $\mathcal{C}$-preimage left-order is independent of generating set.*

One of the advantages of studying $\mathcal{C}$-preimage left-orders compared to $\mathcal{C}$-left-orders is that being a $\mathcal{C}$-preimage left-order is preserved when passing to subgroups.

Lemma A.3. *Let $\mathcal{C}$ a class of languages closed under inverse homomorphism and intersection with regular languages.*

Let H be a finitely generated subgroup of G. If $\prec$ is $\mathcal{C}$-preimage left-order on G then the induced order on H is also a $\mathcal{C}$-preimage left-order.

Proof. By the previous lemma, we can assume that we have a finite generating set (X, π) of G, and that there is a subset of Y of X such that $(Y, \pi_Y = \pi|_{Y^*})$ is a generating set for H. Thus, if P is a positive cone of G with $\pi^{-1}(P)$ in the class $\mathcal{C}$, we get that $\pi_Y^{-1}(H \cap P) = \pi^{-1}(P) \cap Y^*$ is in $\mathcal{C}$. □

Now we will see that the classes of Reg-preimage left-orders and CF-preimage orders are quite limited.

Let (X, π) be a finite generating set of a group G. Recall that the set $\mathrm{WP}(G, X) = \{w \in X^* : \pi(w) = 1\} = \pi^{-1}(\{1\})$ is called the *the word problem* of G (with respect to X). A classic result of Anisimov [2] states that $\mathrm{WP}(G, X)$ is regular if and only if G is finite.

Proposition A.4. *A finitely generated group G has a* Reg*-preimage left-order if and only if it is trivial.*

Proof. Let (X, π) be finite generating set for G. If G is trivial, the positive cone is the empty set.

Suppose that P is a Reg-preimage positive cone. That is, $\pi^{-1}(P)$ is a regular language. By Lemma 2.12, we have that $\pi^{-1}(P^{-1})$ is also a regular language. Since $WP(G, X) = X^* \setminus (\pi^{-1}(P^{-1}) \cup \pi^{-1}(P))$, we get that $WP(G, X)$ is regular, and by Anisimov's theorem G must be finite. Finally, a finite left-orderable groups must be trivial. □

Now we move to the case of CF-preimage left-orders. We start showing that this class contains free groups.

Proposition A.5. *Finitely generated free groups have a* CF*-preimage left-orders.*

Proof. Let F be a finitely generated free group with generating set X. From Corollary 5.9 there exists a 1C-language $\mathcal{L} \subseteq X^*$ representing the positive cone P of F.

Let e be a symbol disjoint from X. Define $\phi : (X \cup \{e\})^* \to X^*$ to be the monoid homomorphism sending e to the empty word. Then $\phi^{-1}(\mathcal{L})$ is the language of words in $w \in \mathcal{L}$ with arbitrary insertions of symbols e in between its letters. Let s be a substitution (in the sense of Hopcroft, Motwani, and Ullman [18]) such that $s(e) = \mathrm{WP}(F)$, meaning we replace the symbol e with a word over X which is equal to the identity in F, and $s(x) = x$ for $x \in X$ otherwise.

We claim that $s(\phi^{-1}(\mathcal{L}))$ is a context-free language which is the preimage of P under π_X. The language is context-free by closure properties of context-free languages under substitution and inverse homomorphism (see for example [18, Theorem 7.23 and Theorem 7.30]). Moreover, $\pi_X(s(\phi^{-1}(\mathcal{L}))) = \pi_X(\mathcal{L})$ by construction, as each e-substitution is a word which is equal to the identity. Therefore, $s(\phi^{-1}(\mathcal{L})) \subseteq \pi_X^{-1}(P)$. For the converse, observe that for every positive element $g \in P$, there is a word in $w \in \mathcal{L}$ such that $\pi(w) = g$ and also observe that our representative w was given above in geodesic form by our choice of $\mathcal{L}_i$. The preimage of g under the evaluation map π_X is w with arbitrarily many insertion of words which are equal to the identity. Therefore, $\pi_X^{-1}(g) \in s(\phi^{-1}(\mathcal{L}))$, so $\pi^{-1}(P) \subseteq s(\phi^{-1}(\mathcal{L}))$. □

However, the class of CF-pre-image left-orders is still very limited. Indeed, all finitely generated abelian subgroups of a group admitting a CF-preimage positive cone must be cyclic.

Proposition A.6. *$\mathbb{Z}^2$ does not have a CF-preimage left-orders. In particular, if G has CF-preimage left-order, it does not contain $\mathbb{Z}^2$ as a subgroup.*

Proof. Positive cones of $\mathbb{Z}^2$ are very well understood. See for example [13, Section 1.2.1] and references therein. We view $\mathbb{Z}^2$ as subspace of $\mathbb{R}^2$. A positive cone in $\mathbb{Z}^2$ is determined by a line in $\mathbb{R}^2$ through the origin, and then selecting one of the half-space (excluding the origin) delimited by the line. If the slope of the line is rational, causing some elements of $\mathbb{Z}^2$ to lie on the line, then one has to choose half of the line (starting from but excluding the origin) to be in the positive cone as well.

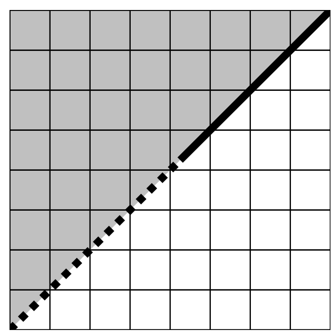

Figure 9: $\mathbb{Z}^2$ with rational line and choices of orientation

By Lemma A.2 we may choose any generating set of $\mathbb{Z}^2$ for this proof. Take a and b to be standard basis vectors, in the x and y direction respectively. We use A and B to denote a^{-1} and b^{-1} respectively.

Suppose that the half-space defining the positive cone is $y > \lambda x$, where $\lambda \in \mathbb{R}$. The case $y < \lambda x$ is analogous. If $\lambda \in \mathbb{Q}$, we choose $y = \lambda x$ $x > 0$ to be in the positive cone. Again, the case $x < 0$ is analogous.

Therefore, we can assume that words evaluating to the positive cone consists on

$$\left\{ w \in \{a, b, A, B\}^* \middle| \begin{array}{c} \text{either } (\sharp_b(w) - \sharp_B(w)) > \lambda(\sharp_a(w) - \sharp_A(w)) \text{ or} \\ ((\sharp_b(w) - \sharp_B(w)) = \lambda(\sharp_a(w) - \sharp_A(w)) \text{ and } (\sharp_a(w) - \sharp_A(w)) > 0) \end{array} \right\}$$

Suppose the previous set is a CF language. Then if we take the intersection with the regular language $a^*b^*A^*B^*$ we get that

$$\mathcal{L} = \{a^\alpha b^\beta A^\gamma B^\delta | (\beta - \delta) > \lambda(\alpha - \gamma) \text{ or } ((\beta - \delta) = \lambda(\alpha - \gamma) \text{ and } (\alpha - \gamma) > 0)\}$$

is CF. We now use the pumping lemma for CF-languages (see [18] for example) to derive a contradiction. Suppose that the pumping length is p and take a word $s = a^c b^d A^e B^f \in \mathcal{L}$ with $c, d, e, f > p$. The pumping lemma says that s can be written as $uvwxy$, with $\ell(vwx) \leq p$, and with at least v or x is non-empty, such that for all $i \geq 0$ uv^iwx^iy belongs to $\mathcal{L}$. Note that the points $\pi(uv^iwx^iy)$ for $i \geq 0$ lie on a line ℓ that goes through $\pi(s)$ and there are points at both sides of $\ell \setminus \{\pi(s)\}$. Note that to get points in both sides of the line we need the condition $c, d, e, f > p$.

Now take s such that $\pi(s)$ is very close to the line $y = \lambda x$ (or on the line if λ is rational) and with the property that for any line ℓ going through $\pi(s)$ there is a half-line of $\ell \setminus \{\pi(s)\}$ such that all rational points on this half-line are outside of P. If λ is not rational, then we can find such point $\pi(s)$ by taking a rational approximation of λ. If λ is rational, we take $\pi(s)$ to be the first point with integer coordinates on the set $\{(x, y) \mid y = \lambda x, x > 0\}$.

Then for that s, some choice of i on the pumping lemma will give an element not in $\mathcal{L}$. Indeed, evaluating pumped words with $i = 0$ and $i = 2$, we get two different points on $\mathbb{Z}^2$ that lie on a line going through $\pi(s)$, and moreover these points lie in different components of the line minus $\pi(s)$, thus some of them is in the negative cone. This gives the desired contradiction.

From Lemma A.3 we get that no group with CF-preimage left-orders can contain $\mathbb{Z}^2$. □

The celebrated result of Muller and Schupp [24] states that the word problem of a group is context-free if and only if G is virtually free. Unfortunately, the argument of Lemma A.4, does not follow directly for context-free languages, since this class is not closed under complement. Thus it is natural to ask

Problem A.7. *Is there a non-free group admitting a* CF*-preimage left-order?*

A simpler, but related question is the following.

Problem A.8. *Is there a non-cyclic group admitting a* $\mathsf{1C}$*-preimage left-order?*

Recall that the class of languages that are complements of a context-free languages are called co-context-free and we denote it by co-CF. The class of groups for which $WP(G, X)$ is co-CF was first studied by Holt, Rees, Röver, and Thomas [17]. We have the following easy observation.

Proposition A.9. *Let G be finitely generated group with* CF*-preimage left-orders. Then the word problem of G is in the class co-$\mathcal{CF}$.*

Proof. Since CF is closed under reversal and homomorphisms, we get that the full language of $\prec$-negative words is CF. Since the class CF is closed under union, $WP(G)$ is co-$\mathcal{CF}$. □

The class of groups of co-CF word problem is closed by taking finite direct products, taking restricted standard wreath products $\oplus H \rtimes Q$ with Q having context-free word problem, passing to finitely generated subgroups, and passing to finite index overgroups [17]. In [17, Theorem 13] it is also shown that a Baumslag-Solitar group has a co-CF word problem if and only if it is virtually abelian. Thus, we obtain the following.

Corollary A.10. *Baumslag-Solitar groups do not admit* CF*-preimage left-orders.*

Note that $\mathbb{Z}^2$ embeds into every Baumslag-Solitar group, except $BS(1, n)$ with $n \neq \pm 1$ [21, Proposition 7.11], thus, Corollary A.10 does not follow from Proposition A.6 on the solvable case.

Lehnert and Schweitzer [20] showed that Thompson's group V has co-CF word problem, and a conjecture of Lehnert can be formulated as all groups with co-CF word problem are finitely generated subgroups of V [5]. Thus, a potential place to look for examples of groups with CF-preimage left-orders are left-orderable subgroups of V that do not contain $\mathbb{Z}^2$.

Positive cones whose preimage is recursively enumerable. We now recall some definitions and results that we will need.

Definition A.11. A *recursively enumerable language* is a formal language for which there exists a Turing machine which enumerates all valid strings of the language. We denote the family of recursively enumerable languages by RE.

We will not give a formal definition of a Turing machine. It can be found in [18], for example. We will use that if $\mathcal{L}$ is RE is recursively enumerable, then there is an algorithm that lists all elements of $\mathcal{L}$. On the other hand, if there is an algorithm that decides if a given word w is in $\mathcal{L}$, then $\mathcal{L}$ is in RE. Note that for such algorithm, on an input $w \in \mathcal{L}$ always halts and decides correctly, where for $w \notin \mathcal{L}$ it never says that w lies in $\mathcal{L}$ but it might not halt (thus we can not deduce that w does not belong to $\mathcal{L}$ in that case).

The following is well-known. See [18].

Proposition A.12. *The class* RE *is a full AFL and it is closed under reversal.*

We now observe the following.

Lemma A.13. *Let G be a finitely presentable group. Then G admits* RE*-left-orders if and only if it admits* RE*-preimage left-orders.*

Proof. Let (X, π_X) be a finite generating set of a group G. If $\prec$ is a RE-preimage left-order, then $\prec$ is a RE left-order.

Assume that $\prec$ is a RE left-order and let P be the corresponding positive cone. Let $\mathcal{L} \subseteq X^*$ be in RE so that $\pi(\mathcal{L}) = P$. Since G is finitely presentable, the word problem $WP(G, X) := \pi^{-1}(\{1\})$, is in RE.

Let $w \in X^*$ such that $\pi(w) \in P$. Then there is $u \in \mathcal{L}$ such that $\pi(w) = \pi(u)$ and therefore, $wu^{-1} \in WP(G, X)$. This implies that $w =_{F(X)} vu$ with $v \in WP(G, X)$ and $u \in \mathcal{L}$, and the equality is as group elements of the free group, not as words.

Since set $WP(G, X) \cdot \mathcal{L}$ is recursively enumerable, and freely reducing words can be computed by a Turing machine, we can decide if $w \in \pi^{-1}(P)$ by checking if the freely reduced version of w is equal to a freely reduced version of some word in $WP(G, X) \cdot \mathcal{L}$ whose elements can be listed by an algorithm. $\square$

We record the following special case.

Corollary A.14. *If G is finitely presented and has a* RE*-left-order, then G has solvable word problem.*

Proof. Recall that having solvable word problem is independent of generating sets. Fix (X, π) a generating set for G. We need to show that if $\mathcal{L} \subseteq X$ is a RE language representing a positive cone, and G is finitely presented, then there is an algorithm that decides if given a word in the generators, this word represents the trivial element or not.

For a finitely presented group, $WP(G, X) = \pi^{-1}(\{1\})$ is RE.

Since the class of recursively enumerable languages are closed under reversal and homomorphism and union (see Proposition 2.7), using Lemma 2.12, it is easy to see that there is a recursively enumerable language over X, such that $\pi^{-1}(P \sqcup P^{-1})$ is RE.

Thus, given a word, we can check if it belongs to $\pi^{-1}(\{1\})$ or to $\pi^{-1}(G \setminus \{1\})$ since both are recursively enumerable languages, and hence the word problem is solvable. $\square$

We note that the converse of Corollary A.14 is false. First, not all left-orderable groups admit RE-left-orders. This has been recently proved by Harrison-Trainor [14] for left-ordererable groups and, in the bi-orderable case by Darbinyan [10]. Moreover, the lack of RE-left-orders is not related to the solvability of the word problem. Since [10, Corollary 2] says that there exists a finitely presentable left-orderable group with solvable word problem and without RE-left-orders.

Acknowledgments

Yago Antolín and Hang Lu Su acknowledge partial support from the Spanish Government through grants number MTM2017-82690-P, and through the "Severo Ochoa Programme for Centres of Excellence in R&D" (SEV-2015-0554) and (CEX2019-000904-S)

Hang Lu Su has received funding from "la Caixa" Foundation (ID 100010434) with fellowship code LCF/BQ/IN17/11620066, from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 713673.

Cristóbal Rivas acknowledge the support from Fondecyt 1181548 and Fondecyt 1210155.

References

[1] J. Alonso, Y. Antolin, J. Brum, C. Rivas, On the geometry of positive cones, *Preprint* (2020)

[2] A.V. Anisimov. The group languages. *Kibernetika* (Kiev) 1971, no. 4, 18–24.

[3] Y. Antolín, W. Dicks, Z. Šunić, Left relativity convex subgroups. Topological methods in group theory, 1–18, London Math. Soc. Lecture Note Ser., 451, Cambridge Univ. Press, Cambridge, 2018.

[4] J. Berstel, Transductions and context-free languages. Leitfäden der Angewandten Mathematik und Mechanik [Guides to Applied Mathematics and Mechanics], 38. B. G. Teubner, Stuttgart, 1979.

[5] C. Bleak, F. Matucci, M. Neunhöffer, Embeddings into Thompson's group V and coCF groups. *J. Lond. Math. Soc.* (2) 94 (2016), no. 2, 583–597.

[6] C. Bonatti, I. Monteverde, A. Navas, C. Rivas, Rigidity for C^1-actions on the interval arising from hyperbolicity I: solvable groups. *Math. Z.* (2017) 286:919–949.

[7] R. Bieri, W. D. Neumann, and R. Strebel. A geometric invariant of discrete groups. *Invent. Math.* **90** (1987), 451–477.

[8] D. Calegari. Problems in foliations and laminations of 3-manifolds. In Topology and geometry of manifolds (Athens, GA, 2001), volume 71 of Proc.Sympos. Pure Math., pages 297–335. Amer. Math. Soc., Providence, RI, 2003.

[9] N. Chomsky, . Three models for the description of language, *IRE Transactions on Information Theory* (2): 113–124(1956)

[10] Arman Darbinyan, Computability, orders, and solvable groups. *Jounal of Symbolic Logic* (to appear)

[11] W. Dicks, and Z. Šunić, Orders on trees and free products of left-ordered groups. *Canad. Math. Bull.* 63 (2020), no. 2, 335–347

[12] C. Droms. Subgroups of Graphs Groups, *Journal of Algebra*, 110 (2) (1987),519–522.

[13] B. Deroin, A. Navas, C. Rivas. Groups, Orders, and Dynamics. *Preprint* (2016), available on arXiv:1408.5805v2.

[14] M. Harrison-Trainor, Left-orderable computable groups. *J. Symbolic Logic* 83 (2018), no. 1, 237–255.

[15] S. Hermiller, J. Meier. Artin groups, rewriting systems and three-manifolds. *J. Pure Appl. Algebra* 136 (1999), no. 2, 141–156.

[16] S. Hermiller, Z. Šunić. No positive cone in a free product is regular, *Internat. J. Algebra Comput.* **27** (2017), 1113-1120.

[17] D. F. Holt, Sarah Rees, Claas E. Röver, and Richard M. Thomas. Groups with context-free co-word problem. *J. London Math. Soc.* (2), 71(3):643–657, 2005

[18] J. Hopcroft, R. Motwani and J. D. Ullman. *Introduction to Automata Theory, Languages and Computation*, Reading, MA: Addison-Wesley. 3rd Edition (2006).

[19] V. M. Kopytov, N. Y. Medv.edev *Right-ordered groups*, Consultants Bureau, New York, Siberian School of Algebra and Logic (1996).

[20] J. Lehnert, P. Schweitzer. The co-word problem for the Higman-Thompson group is context-free, *Bull. Lond. Math. Soc.* 39 (2007), no. 2, 235–241.

[21] G. Levitt, Quotients and subgroups of Baumslag-Solitar groups. *Journal of Group Theory* 18.1 (2015): 1–43.

[22] P. Linnell. The space of left-orders of a group is either finite or uncountable. *Bull. London Math. Society* 43 (2011), no. 1, 200–202.

[23] D. W. Morris. Amenable groups that act on the line. *Algebraic & Geometric Topology* **6** (2006), 2509–2518.

[24] D. E. Muller, P. E. Schupp, Groups, the theory of ends, and context-free languages, *J. computer and system science.* **26** (1983), 295–310.

[25] D. S. Passman. The algebraic structure of group rings. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1977.

[26] Edited by G. Rozenberg and A. Salomaa. *Handbook of formal languages. Vol. 1. Word, language, grammar.* Springer-Verlag, Berlin, 1997. xxiv+873 pp.

[27] C. Rivas. On the space of Conradian group orderings. *J. of Group Theory,* **13** (2010), 337–353.

[28] C. Rourke, B. Wiest. Order automatic mapping class groups. *Pacific J. Math.* 194 (2000), no. 1, 209–227.

[29] D. Segal. *Polycyclic groups.* Cambridge Tracts in Mathematics, 82. Cambridge University Press,1983.

[30] H.L. Su. Formal language convexity in left-orderable groups. *Internat. J. Algebra Comput.* Vol. 30, No. 07, pp. 1437–1456 (2020)

[31] Z. Šunić. Explicit left orders on free groups extending the lexicographic order on free monoids. *C. R. Math. Acad. Sci. Paris* 351 (2013), no. 13-14, 507–511.

[32] Z. Šunić. Orders on free groups induced by oriented words. Available athttp://arxiv.org/abs/1309.6070, 2013

[33] V. Tararin. On groups having a finite number of orders. Dep. Viniti (Report), Moscow (1991).

Yago Antolín,
Fac. Matemáticas, Universidad Complutense de Madrid and
Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM
Madrid, Spain
email: yago.anpi@gmail.com

Hang Lu Su
Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM and
Dpto. de Matemáticas Universidad Autónoma de Madrid
Madrid, Spain
email: homeowmorphism@gmail.com

Cristóbal Rivas
Dpto. de Matemáticas y C.C.
Universidad de Santiago de Chile
Alameda 3363, Estación Central
Santiago, Chile
email: cristobal.rivas@usach.cl