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Topologies on groups related to the theory of shape and generalized coverings

Topologías en grupos asociados con la teoría de la forma y recubridores generalizados

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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TOPOLOGIES ON GROUPS RELATED TO THE THEORY OF SHAPE AND GENERALIZED COVERINGS

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Memoria presentada para optar al grado de
Doctor en Ciencias Matemáticas por

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"In these days the angel of topology and
the devil of abstract algebra fight for the soul
of every individual discipline of mathematics."
Hermann Weyl.

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Contents

Introduction

The present work consists of two different parts: the main one (Chapters 1 and 2) is devoted to introduce additional structure on groups which arise naturally in the theory of shape. In the second part (Chapter 3), some generalizations of the theory of covering spaces are studied and, in particular, one is proposed according to the spirit of the theory of shape.

The starting point of this doctoral thesis are the works [25, 26, 68, 69, 70] in which the authors introduced and exploited some ultrametrics in the set of shape morphisms between two (pointed) topological spaces. In particular, if the domain space is particularized in $(S^1, 1)$, the construction made in [68] allows to give an ultrametric on the shape group $\check{\pi}_1(X, x_0)$ of a compact metric space X, as it was observed in [69] and detailed [80]. If the space X is non-compact metric, the construction leads to a generalized ultrametric, in the sense of Priess-Crampe and Ribenboim [78, 79].

In [7], D. K. Biss introduced the idea of topologizing the fundamental group of a topological space, in such a way that the topology on $\pi_1(X, x_0)$ was a group topology which allows to detect the (non) existence of universal covering for X. The approach consists just in taking on $\pi_1(X, x_0)$ the quotient topology from the compact-open topology on the loop space of X, $\Omega(X, x_0)$.

However, there are some errors in the referred paper, specifically, the error related with our work is revealed by P. Fabel in [33] showing that, in general, the group operation on $\pi_1(X, x_0)$ with the quotient topology is not continuous. Using a similar point of view, different authors tried to endow the fundamental group with a topology such that $\pi_1(X, x_0)$ is a topological group and the projection $q : \Omega(X, x_0) \to \pi_1(X, x_0)$ is continuous.

The idea of introducing a topology on $\pi_1(X, x_0)$ is not new at all. Earliest works around this idea seem to be by W. Hurewicz [46] and specially by J. Dugundji [27]. After the paper of Biss, some other works have appeared in which the fundamental group is endowed with different topologies. The most relevant are, among others, [36] where the so-called whisker topology is used, [19, 20] where the lasso topology is introduced, and [13] where the author works with a slight modification of the quotient topology.

Another relevant tool in algebraic topology are the Čech homology groups. One of the key points for the construction of the ultrametric on $\check{\pi}_1(X, x_0)$ is the fact that this group is obtained as an inverse limit. The mentioned homology groups $\check{H}_*(X; \mathbb{Z})$ are constructed as inverse limits. The classical construction of these groups does not come from the theory of shape, but it is also recovered via shape expansions. However, the inverse structure of these groups has not been used to obtain a topology of them as it has been done for shape groups.

In relation with covering spaces, it is well-known that if X is path-connected, locally path-connected and semilocally simply connected, then there exists the universal covering of X, i.e., there is a path connected, locally path-connected and simply connected topological space $\widetilde{X}$, and a continuous and surjective map $\pi : \widetilde{X} \to X$ with the unique lifting property of certain maps. In this case, $\widetilde{X}$ can be identified with homotopy classes of paths emanating from a prefixed base point $x_0 \in X$. In addition, there is a correspondence between intermediate (normal) coverings of X and normal subgroups of $\pi_1(X, x_0)$ (see e.g. [83]).

However, if the previous hypothesis on X are not required, the theory is not as satisfactory. In particular, there exists compact metric spaces without universal covering. Some authors had tried to obtain a theory which, including the classical point of view, gives a nice framework for a wider class of spaces.

There are some early works on this ideas, such as [58]. Specially remarkable, in the framework of the theory of shape, is the concept of overlay introduced in [37] by Fox. Most recent are [4, 8, 36, 19, 20, 16] and several papers from the authors of [66]. Some of the mentioned works try to adapt the idea of what a covering is by slight modifications of the requirements for a covering, while others take [8] as starting point: Based on the classical theory, Bogley and Sieradski consider in that paper the so-called universal path space, that is, the set of homotopy classes of paths emanating from the base-point $x_0 \in X$ as the natural candidate to be the universal covering. Endowing the universal path space with different topologies (e.g. whisker, lasso, quotient) it is possible to study what properties from the classical theory still hold.

Objectives

The present dissertation have three main objectives.

Our first goal is to prove the existence of a group topology on the fundamental group such that the projection from the loop space is continuous, and to relate it with the inverse limit topology of the shape group. We also would like to connect this constructions with classical results of Spanier, Hurewicz and Dugundji.

The second objective is to exploit the inverse limit structure of the Čech homology groups to construct a complete ultrametric on $\check{H}_n(X)$. The adaptation of the construction of this homology seems needed in order to apply methods used for shape morphisms.

Finally, we are interested in the generalization of the theory of covering spaces. For this part, we would like to give a new candidate for a generalized universal covering in the sense of Fischer and Zastrow. In addition, we will search for a new topology on the Universal Path Space, comparing it with the topologies already existent on it.

Outline and results

In Chapter 1 we use similar ideas as in [68] to construct a pseudoultrametric in $\pi_1(X, x_0)$ which generates a topology satisfying the above mentioned requisites (Theorems 1.2.2, 1.2.7 and 1.2.12). This work was done in [80] for the compact metric case, and in the present work it is shown that this topology coincides in fact with the pull-back of the inverse limit topology via the canonical homomorphism $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ (Corollary 1.2.25). The interest in this topology on $\pi_1(X, x_0)$ has been recovered in [14] (see section 3.4), and also in [71] where the same results are re-obtained. In the case of φ being a monomorphism, the pseudoultrametric on $\pi_1(X, x_0)$ is in fact an ultrametric. For completeness, the construction for the non-compact metric case is indicated here also (Proposition 1.2.22) and some examples of how the topology and algebra interact are provided.

As it is shown in this first chapter, some of the topologies already appeared in the literature are intimately related with relevant subgroups of $\pi_1(X, x_0)$, and more precisely, subgroups contained in $Ker\varphi$ which correspond to different equivalence relations on $\pi_1(X, x_0)$ (Section 1.3). Here we study the relative position of the correspondent equivalence classes, quotient groups and topologies induced on $\pi_1(X, x_0)$, showing which ones are contained in the others (Proposition 1.4.1 and Theorems 1.4.2 and 1.4.6) and giving counterexamples for the relations which do not hold. This work of comparison of the topologies continues [87] and it has also been done independently by J. Brazas an P. Fabel in [18].

It seems that the previous comparison depends on some local properties of the space X. In particular, the local path connectivity of X provides an important class of spaces in which some of the above topologies coincide (Corollary 1.4.14). Moreover, we extend here these results to the class of approximative absolute neighbourhood retracts in the sense of Clapp ($AANR_C$), which is a wider class of spaces not necessarily locally-path connected (Corollary 1.4.29).

In Chapter 2 we adapt the techniques of [68] for shape morphisms to Čech homology groups. It has been necessary to redefine the Čech homology groups in terms of what we have called approximative homology (Definitions 2.1.3). This homology is just the reinterpretation of homology groups using approximative sequences of cycles, as Borsuk introduced shape groups using approximative maps [10]. From this point on, it is possible to follow the outline of [68] to obtain an ultrametric on $\check{H}_*(X; \mathbb{Z})$ for any compact metric space X (Theorem 2.1.18). Furthermore, this ultrametric on $\check{H}_*(X; \mathbb{Z})$ generates a group topology (Proposition 2.1.24) and the structure obtained is shape invariant (Theorem 2.1.32).

The point of view of the preceding constructions recovers the fact that the information of X, when X is a polyhedron (Proposition 2.1.22), is completely encoded in the algebraic structure of the group, in the sense that the topology obtained is uniformly discrete for X (while it is not discrete in general). Also, the construction for arbitrary X is described here, obtaining again a generalized ultrametric in the sense of Priess-Crampe and Ribenboim (Theorem 2.1.43). We enunciate here the correspondent topological versions of the classical Hurewicz homomorphism (Theorem 2.2.2).

On the other hand, if we pay attention to the Čech cohomology groups $\check{H}^*(X;\mathbb{Z})$, it is not possible to obtain this kind of additional structure. If we want to follow the idea that the algebra is enough when X is a polyhedron, the topology obtained must be necessarily discrete. However, it is still possible to find connections between homology and cohomology if we use S^1 as group of coefficients for cohomology, instead of the integers $\mathbb{Z}$. Using the theory of Pontryagin duality, we stablish that Čech cohomology groups $\check{H}^*(X;S^1)$ are the Pontryagin duals of Čech homology groups $\check{H}_*(X,\mathbb{Z})$ (Theorem 2.3.6). Hence, the universal coefficient theorem for $H_*(X;\mathbb{Z})$ remains valid in this case, just replacing the homomorphisms functor $Hom(-,S^1)$ by the continuous homomorphisms functor $CHom(-,S^1)$. This allows us to reinterpret the Čech cohomology groups as Pontryagin duals of ultrametric abelian groups.

In Chapter 3, we focus on theories generalizing the theory of covering spaces. In [36], Fischer and Zastrow introduce the concept of generalized universal covering and, following the line of [8], they consider the so-called whisker topology on the Universal Path Space. The main result of this paper is that if X is shape injective, then the Universal Path Space with the whisker topology is a generalized universal covering.

Here we propose a slight modification of this construction: instead of taking homotopy classes of paths, we take shape classes of paths emanating from the base-point. Thus, we obtain a space $\widehat{X}$ and a map $\widehat{\pi}:\widehat{X}\to X$ satisfying the same properties as a generalized covering in the sense of Fischer and Zastrow, except that $\widehat{X}$ is not simply connected (Propositions from 3.2.5 to 3.2.11). We obtain $Ker\varphi$ as its fundamental group, which represents the neutral element in shape, instead of the neutral element in homotopy. As a consequence (Corollary 3.2.12), when X is shape injective, $\widehat{X}$ coincides with $\widetilde{X}$ and we recover all the results of [36].

As a part of a collaboration with A. Zastrow [81], we extend to the universal path space the topology on $\pi_1(X,x_0)$ given in the first part of the thesis. This is done with the lifted system (Subsection 3.3.2), that is, the inverse system of covering spaces of an inverse system which defines the shape of X. The universal path space is mapped into the inverse limit of this lifted system in an analogous way as $\pi_1(X,x_0)$ is mapped to $\check{\pi}_1(X,x_0)$. In [87], the authors compare the whisker, the lasso and the quotient topologies on the universal path space. We complete this comparison, establishing the relation of this topology induced by the shape with the other ones (Section 3.4). Again, the local-path-connectivity appears as condition for the coincidence of the lasso and shape-induced topologies (Theorem 3.4.6).

At this point, the relation of this second part with the first one is unveiled. In the first part, we constructed an ultrametric in $\check{\pi}_1(X,x_0)$, which led to a pseudoultrametric in $\pi_1(X,x_0)$. Classically, it is possible to identify the fibre $\pi^{-1}(x_0)$ of the covering projection $\pi:\widetilde{X}\to X$ with the fundamental group $\pi_1(X,x_0)$. The topological counterpart is then reflected here, since $\pi^{-1}(x_0)$ inherits a topology from $\widetilde{X}$, and it coincide with the topology generated by the pseudoultrametric on $\pi_1(X,x_0)$ (Proposition 3.5.3). If we look at the inverse limit of the lifted system, we can identify $\check{\pi}_1(X,x_0)$ with a fibre of the limit of the projections, and the topology inherited coincides with the topology generated by the ultrametric defined on the first part.

Conclusions

In this doctoral thesis we have answered positively to the question of the existence of a topology on the fundamental group which makes $\pi_1(X, x_0)$ into a topological group and such that the projection from the loop space is continuous. We have compared this topology with others appeared in the classical literature, and we have showed that some ideas from the theory of shape were already implicitly in works of Spanier, Hurewicz and Dugundji.

We have been able to construct a complete ultrametric on Čech homology groups which, in addition, has allowed us to establish connections with the cohomology groups via the Pontryagin duality. This construction has lead to a shape invariant.

Finally, we have extended some known results about the theory of generalized universal coverings. The topology introduced on the Universal Path Space is the correspondent generalization of the topology on the fundamental group given by the pseudoultrametric and this topologies have been satisfactorily compared with others topologies used in the framework of generalized coverings.

Introducción

El presente trabajo consiste en dos partes diferenciadas: la principal de ellas (Capítulos 1 y 2) está dedicada a introducir estructura adicional en grupos que aparecen de manera natural en el contexto de la teoría de la forma. En la segunda parte (Capítulo 3), se plantea cómo generalizar la teoría de espacios recubridores y, en particular, se propone una línea de trabajo relacionada con la teoría de la forma.

El punto de partida de esta tesis doctoral son los trabajos [25, 26, 68, 69, 70] en los que los autores introducen y utilizan algunas ultramétricas en el conjunto de los morfismos shape entre dos espacios topológicos punteados. En particular, si el dominio es $(S^1, 1)$, la construcción realizada en [68] permite explicitar una ultramétrica en el grupo shape $\check{\pi}_1(X, x_0)$ de un espacio métrico compacto X, como ya fue observado en [69] y [80]. Si el espacio no es métrico compacto, la construcción nos lleva a utilizar el concepto de ultramétrica generalizada, en el sentido de Priess-Crampe y Ribenboim [78, 79].

En [7], D. K. Biss introduce la idea de topologizar el grupo fundamental de un espacio, de forma que la topología en $\pi_1(X, x_0)$ sea una topología de grupo que permita detectar la (no) existencia de un recubridor universal para X. La forma de proceder sugerida es tomar en $\pi_1(X, x_0)$ la toplogía cociente inducida por la topología compacto-abierta en el espacio de lazos $\Omega(X, x_0)$.

Sin embargo, hay algunos errores en el artículo mencionado: en concreto, el error relacionado con el presente trabajo fue puesto de manifiesto por P. Fabel en [33], mostrando que, en general, la operación de grupo en $\pi_1(X, x_0)$ con la topología cociente no es continua. Utilizando un punto de vista similar, varios autores han tratado de dotar al grupo fundamental con una topología, de forma que $\pi_1(X, x_0)$ sea un grupo topológico y la proyección $q : \Omega(X, x_0) \to \pi_1(X, x_0)$ sea continua.

La idea de introducir un topología en $\pi_1(X, x_0)$ no es nueva del todo. Los primeros trabajos alrededor de esta idea parecen ser de W. Hurewicz [46] y, especialmente, de J. Dugundji [27]. Tras el artículo de Biss, han ido apareciendo algunos otros trabajos en los que el grupo fundamental es dotado de diferentes topologías: uno de los más relevantes, entre otros, es [36] en el que se utiliza la denominada topología whisker. En [19, 20] se introduce la topología lasso y en [13] el autor trabaja con una modificación de la topología cociente.

Otra herramienta relevante en topología algebraica son los grupos de homología de Čech. Uno de los puntos claves de la construcción de la ultramétrica en $\check{\pi}_1(X, x_0)$ es el hecho de que este grupo se obtiene como un límite inverso. Los mencionados grupos de homología $\check{H}_*(X;\mathbb{Z})$ también se construyen como límite inverso. La construcción clásica de estos grupos no proviene de la teoría de la forma, pero se puede recuperar utilizando expansiones. Sin embargo, la estructura de límite inverso de estos grupos no ha sido utilizada para obtener una topología tal y como se hizo para los grupos shape.

En relación con los espacios recubridores, es bien conocido que si X es conexo por caminos, localmente conexo por caminos y semilocalmente simplemente conexo, entonces existe el recubridor universal de X, es decir, existe una aplicación sobreyectiva $\pi : \widetilde{X} \to X$ con la propiedad de elevación única para ciertas aplicaciones. En este caso, $\widetilde{X}$ se puede identificar con las clases de homotopía que emanan desde un punto base $x_0 \in X$ prefijado. Es más, existe una correspondencia entre recubridores intermedios de X y subgrupos normales de $\pi_1(X, x_0)$ (ver [83]).

Sin embargo, si las hipótesis anteriores sobre X no se cumplen, la teoría no resulta ser del todo satisfactoria. En particular, existen espacios compactos métricos que no tienen recubridor universal en el sentido anterior. Algunos autores han tratado de obtener una teoría que, incluyendo el punto de vista clásico, diera cabida a una clase más amplia de espacios.

Existen algunos trabajos alrededor de estas ideas, tales como [58]. Especialmente reseñable en el marco de la teoría de la forma , es el concepro de overlay, introducido por Fox en [37]. Más recientes son [4, 8, 36, 19, 20, 16] y varios artículos de los autores de [66]. Algunos de los trabajos mencionados intentan adaptar la idea de lo que se entiende por espacio recubridor, mediante ligeras modificaciones de las propiedades que se les requieren mientras que otros toman [8] como punto de partida: Basados en la teoría clásica, Bogley y Sieradski consideran en ese artículo al espacio universal de caminos, esto es, el conjunto de clases de homotopía de caminos emanando de un punto base $x_0 \in X$, como el candidato natural a ser el recubridor universal. Equipando al espacio universal de caminos con diferentes topologías (por ejemplo, whisker, lasso o cociente) es posible estudiar qué propiedades de la teoría clásica se siguen conservando.

Objetivos

La presente memoria tiene tres objetivos principales.

Nuestro primer objetivo es demostrar la existencia de una topología de grupo sobre el grupo fundamental que haga que la proyección desde el espacio de lazos sea continua, relacionando ésta construcción con la topología de límite inverso del grupo shape. Nos planteamos también conectar estas construcciones con resultados clásicos de Spanier, Hurewicz y Dugundji.

En segundo lugar, queremos sacar partido de la estructura de límite inverso de los grupos de homología de Čech para construir una ultramétrica completa en $\check{H}_n(X)$. La adaptación de la construcción de esta homología es necesaria para poder aplicar los métodos utilizados para morfismos shape.

Finalmente, nos interesamos en la generalización de la teoría de espacios recubridores. Para esta parte, queremos dar un nuevo candidato a recubridor universal generalizado en el sentido de Fischer y Zastrow. Además, queremos introducir una nueva topología en el espacio universal de caminos y situarla en relación con otras topologías utilizadas en él.

Estructura y resultados

En el Capítulo 1 utilizamos ideas similares a las empleadas en [68], para contruir un pseudoultramétrica en $\pi_1(X, x_0)$ que genera una topología que satisface los requisitos mencionados anteriormente (Teoremas 1.2.2, 1.2.7 y 1.2.12). Este trabajo fue hecho en [80] para el caso compacto métrico, y en el presente trabajo se muetra que esta topología coincide, de hecho, con la topología inicial asociada a la topología de límite inverso y al homomorfismo canónico $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ (Corolario 1.2.25). El interés en esta topología sobre $\pi_1(X, x_0)$ se ha recuperado [14] (sección 3.4), y también en [71] donde se obtienen resultados análogos. En el caso en el que φ es un monomorfismo, la pseudoultramétrica en $\pi_1(X, x_0)$ es, de hecho, una ultramétrica. Por completitud, indicamos aquí también la construcción para el caso no compacto métrico (Proposición 1.2.22) y se proporcionan diferentes ejemplos en los que la topología y el álgebra interactúan.

Como se muestra en el primer capítulo, varias de estas topologías están íntimamente relacionadas con subgrupos de $\pi_1(X, x_0)$ y, más precisamente, con subgrupos contenidos en $Ker\varphi$ y que corresponden con diferentes relaciones de equivalencia en $\pi_1(X, x_0)$ (Sección 1.3). Estudiamos aquí la posición relativa de las correspondientes clases de equivalencia, los grupos cocientes y las topologías inducidas sobre $\pi_1(X, x_0)$, mostrando cuáles están contenidas en otras (Proposición 1.4.1 y Teoremas 1.4.2 y 1.4.6) y aportando contraejemplos para las relaciones que no se satisfacen. Este trabajo de comparación continúa el iniciado en [87] y ha aparecido también, de forma independiente, en [18].

Parece que esta comparación depende de las propiedades locales del espacio X. En particular, la conexión local por caminos aporta una clase importante de espacios en los que algunas de las topologías estudiadas coinciden (Corolario 1.4.14). Más aún, extendemos aquí estos resultados a la clase de los retractos aproximativos en el sentido de Clapp ($AANR_C$), que es una clase más amplia de espacios no necessariamente localmente conexos por caminos (Corolario 1.4.29).

En el Capítulo 2 adaptamos las técnicas de [68] en morfismos shape para los grupos de homología de Čech. Es necesario redefinir los grupos de homología de Čech en términos de lo que hemos llamos homología aproximativa (Definiciones 2.1.3). Esta homología es precisamente la reinterpretación de los grupos de homología mediante sucesiones aproximativas de ciclos, de la misma forma que Borsuk introdujo los grupos shape utilizando aplicaciones aproximativas [10]. Desde este punto, y siguiendo el guión establecido en [68] se obtiene una ultramétrica en $\check{H}_*(X;\mathbb{Z})$ para cualquier espacio métrico compacto X (Teorema 2.1.18). Más aún, esta ultramétrica sobre $\check{H}_*(X;\mathbb{Z})$ genera una topología de grupo (Proposición 2.1.24) y la estructura obtenida resulta ser un invariante del tipo de forma (Teorema 2.1.32).

El punto de vista de las construcciones anteriores recupera el hecho de que la información de X está totalmente codificada en la estructura algebraica, cuando X es un poliedro (Proposición 2.1.22), en el sentido de que la topología que se obtiene es uniformemente discreta (mientras que, en general, no es discreta). También hacemos aquí la construcción para un espacio X arbitrario, obteniendo también una ultramétrica generalizada en el sentido de Priess-Crampe y Ribenboim (Teorema 2.1.43). Mostramos aquí también la versión correspondiente al homomorfismo clásico de Hurewicz (Teorema 2.2.2).

Por otro lado, si prestamos atención a los grupos de cohomología de Čech $\check{H}^*(X;\mathbb{Z})$, no es posible obtener este tipo de estructura adicional. Si queremos reproducir la idea de que el álgebra es suficiente cuando X es un poliedro, la topología obtenida debe ser necesariamente discreta. Sin embargo, es posible encontrar conexiones entre la homología y la cohomología utilizando S^1 como grupo de coeficientes para la cohomología, en lugar de $\mathbb{Z}$. Utilizando la teoría de dualidad de Pontryagin, establecemos que los grupos de cohomología de Čech $\check{H}^*(X;S^1)$ son los duales de Pontryagin de los grupos de homología de Čech (Teorema 2.3.6). Así, el teorema de coeficientes universales para $H_*(X;\mathbb{Z})$ sigue siendo válido en este caso, pero reemplazando el functor de homomorfismos $Hom(-,S^1)$ por el de homomorfismos continuos $CHom(-,S^1)$. Esto permite reinterpretar los grupos de cohomología de Čech como duales de Pontryagin de grupos abelianos ultramétricos.

En el Capítulo 3, nos centramos en teorías que generalizan la teoría de espacios recubridores. En [36], Fischer y Zastrow introducen el concepto de recubridor universal generalizado, y siguiendo la línea de [8], consideran la topología whisker en el espacio universal de caminos. El resultado principal de ese artículo es que si X es shape-inyectivo, entonces el espacio universal de caminos con la topología whisker resulta ser un recubridor universal generalizado.

Proponemos aquí una ligera variación de esta construcción: en lugar de tomar clases de homotopía de caminos, utilizamos clases shape de caminos que emanan del puntos base. Así, obtenemos un espacio $\widehat{X}$ y una aplicación $\widehat{\pi}:\widehat{X}\to X$ que satisface las mismas propiedades que un recubridor universal generalizado en el sentido de Fischer y Zastrow, salvo que $\widehat{X}$ no es simplemente conexo (Proposiciones desde 3.2.5 a 3.2.11). Obtenemos que su grupo fundamental es $Ker\varphi$, que representa el elemento neutro en shape, en lugar de en homotopía. Como consecuencia (Corolario 3.2.12), cuando X es shape-inyectivo, se obtiene que $\widehat{X}$ coincide con $\widetilde{X}$ y se recuperan los resultados de [36].

Como parte de la colaboración con A. Zastrow [81], se extiende al espacio universal de caminos la topología de $\pi_1(X,x_0)$ que se estudia en el primer capítulo. La construcción se realiza a través del sistema elevado (Susección 3.3.2), esto es, el sistema inverso de espacios recubridores que se obtiene de un sistema inverso que define la forma de X. Existe una aplicación del espacio universal de caminos en el límite inverso de este sistema elevado, que es análoga al homomorfismo que existe entre $\pi_1(X,x_0)$ y $\check{\pi}_1(X,x_0)$. En [87], los autores comparan las topologías whisker, lasso y cociente sobre el espacio universal de caminos. Aquí completamos esta comparativa, estableciendo la relación de esta topología inducida por la forma con respecto a las otras (Sección 3.4). Nuevamente, la conexión local por caminos aparece como condición para la igualdad de las topologías lasso e inducida por la forma (Teorema 3.4.6).

En este punto, aparece la relación de la segunda parte con la primera. En la primera parte, se construye una ultramétrica en $\check{\pi}_1(X, x_0)$ que lleva a una pseudoultramétrica en $\pi_1(X, x_0)$. De forma clásica, se puede identificar la fibra $\pi^{-1}(x_0)$ de la proyección recubridora $\pi : \widetilde{X} \to X$ con el grupo fundamental $\pi_1(X, x_0)$. La construcción topológica se refleja aquí, en el hecho que $\pi^{-1}(x_0)$ recibe una topología de $\widetilde{X}$, que coincide con la topología generada por la pseudoultramétrica sobre $\pi_1(X, x_0)$ (Proposición 3.5.3). Si atendemos al límite inverso del sistema elevado, podemos identificar igualmente $\check{\pi}_1(X, x_0)$ con la fibra del límite de las proyecciones, y la topología recibida coincide con la topología generada por la ultramétrica definida en la primera parte.

Conclusiones

En esta tesis doctoral hemos respondido afirmativamente a la cuestión planteada de existencia de una topología en el grupo fundamental que hiciera de $\pi_1(X, x_0)$ un grupo topológico y tal que la proyección desde el espacio de lazos fuera continua. Hemos comparado esta topología con otras aparecidas en la literatura clásica y hemos mostrado cómo algunas ideas de la teoría de la forma estaban ya implícitas en trabajos de Spanier, Hurewicz y Dugundji.

Hemos contruido una ultramétrica completa sobre los grupos de homología de Čech que, además, ha permitido establecer conexiones con los grupos de cohomología a través de la dualidad de Pontryagin. Esta construcción ha dado lugar a un invariante del tipo de forma.

Finalmente, hemos extendido algunos resultados sobre la teoría de recubridores universales generalizados. La topología introducida en el espacio universal de caminos es la correspondiente generalización de la topología sobre el grupo fundamental dada por la pseudoultramétrica y estas topologías han sido comparadas satisfactoriamente con otras topologías ya utilizadas en el marco de los recubridores en sentido generalizado.

Preliminaries

We review here the basics of different concepts that we shall use all along the present work, and also remind some important results that we shall need at different places of our exposition.

Algebraic topology

Fundamental group

Let X be a topological space. If $x, y \in X$, a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$ is called a *path from x to y*. The points x, y are the *end-points* of the path α. We say that α starts at (or emanates from) x. If $x = y$, then α is called a *loop*.

We shall denote by $\overline{\alpha}$ the *reverse path* of α. In other words, $\overline{\alpha}$ is a path with the same trace as α, but in the opposite direction. If $\alpha : [0,1] \to X$, then $\overline{\alpha}(t) = \alpha(1-t)$.

If $x_0 \in X$, the *path space of X based at x_0* is the collection of all paths starting at x_0. It shall be denoted by $P(X, x_0)$. Analogously, the *loop space of* (X, x_0) is denoted by $\Omega(X, x_0)$ and is the collection of all loops in X based at x_0. The *concatenation of α and β* is denoted by $\alpha * \beta$ and it is defined as a path such that

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \\ \beta(2t-1) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Given two paths $\alpha, \beta : [0,1] \to X$ with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, we shall say that a continuous map

$$H : [0,1] \times [0,1] \to X$$

is an *homotopy between α and β* if $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$ and H satisfies $H(0,t) = \alpha(t)$, $H(1,t) = \beta(t)$, $H(s,0) = \alpha(0) = \beta(0)$ and $H(s,1) = \alpha(1) = \beta(1)$ for each $s \in [0,1]$.

On $P(X, x_0)$ it is defined the relation

$$\alpha \simeq \beta \Leftrightarrow \text{ there exists an homotopy } H \text{ between } \alpha \text{ and } \beta$$

which is an equivalence relation. The equivalence class of an element shall be denoted by $[\alpha]$ and we call it the *homotopy class of the path α*. Analogously, the same relation is defined by restriction to $\Omega(X, x_0)$.

The quotient space of $\Omega(X, x_0)$ by the relation of homotopy of loops is known as *the fundamental group of* (X, x_0), denoted by $\pi_1(X, x_0)$. It is well-known that $\pi_1(X, x_0)$ is a group under the operation of concatenation of classes $[\alpha] * [\beta] = [\alpha * \beta]$.

Inverse and direct systems and limits

Let Λ be a set:

- A relation $\lambda < \lambda'$ in Λ it is called a *preorder* if it satisfies reflexive and transitive properties. Then Λ is called a *preordered set.* If in addition the antisymmetric property is satisfied, Λ is called an *ordered set.*
- Λ is a *directed set* if Λ is preordered and also for each $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 < \lambda_3$ and $\lambda_2 < \lambda_3$.
- Let Λ, M be directed sets. A correspondence $\phi : \Lambda \to M$ is an *order-preserving map* if $\lambda < \lambda'$ in Λ implies $\phi(\lambda) < \phi(\lambda')$ in M.

Given a category $\mathcal{C}$ we can consider its procategory, that is, the family of inverse systems of objects and arrows of the category with a certain equivalence class of maps of systems. Along this work we shall restrict ourselves to categories of topological spaces or compact metric spaces (and continuous maps), groups (and homomorphisms) or topological groups (with continuous homomorphisms as arrows). In addition, the definitions and results that we recall in this preliminaries are not general at all. For deeper explanations, see [30] and also [65].

An *inverse system* in a category $\mathcal{C}$ over a directed set Λ is a collection

$$\mathbf{X} = \{X_\lambda, p_{\lambda\,\lambda'}, \Lambda\}$$

where for each $\lambda \in \Lambda$, X_λ is an object of $\mathcal{C}$ and for $\lambda, \lambda' \in \Lambda$ with $\lambda < \lambda'$ there is an arrow of $\mathcal{C}$

$$p_{\lambda\,\lambda'} : X_{\lambda'} \to X_\lambda$$

such that:

i) $p_{\lambda\,\lambda}$ is the identity of X_λ;

ii) $p_{\lambda\,\lambda'} \circ p_{\lambda'\,\lambda''} = p_{\lambda\,\lambda''}$ in $\mathcal{C}$ for each $\lambda, \lambda', \lambda'' \in \Lambda$ with $\lambda'' \geq \lambda' \geq \lambda$.

The objects X_λ are called *terms* and the arrows $p_{\lambda\,\lambda'}$ are called *bonding morphisms.* In the special case $\Lambda = \mathbb{N}$, we refer to an inverse system as *inverse sequence.* In that case, only the morphisms $p_{n\,n+1}$ are needed, since $p_{n\,m}$ are defined by composition.

Let $\mathbf{X} = \{X_\lambda, p_{\lambda\,\lambda'}, \Lambda\}$ and $\mathbf{Y} = \{Y_\mu, q_{\mu\,\mu'}, M\}$ be two inverse systems. A *map of inverse systems* is given by a map $\phi : M \to \Lambda$ (which can be supposed ordering-preserving, without loss of generality) such that for each $\mu \in M$ there exists a map $f_\mu : X_{\phi(\mu)} \to Y_\mu$ and the diagram

$$\begin{array}{ccc} X_{\phi(\mu)} & \xleftarrow{p_{\phi(\mu)\,\phi(\mu')}} & X_{\phi(\mu')} \\ {\scriptstyle f_\mu}\downarrow & & \downarrow{\scriptstyle f_{\mu'}} \\ Y_\mu & \xleftarrow[q_{\mu\,\mu'}]{} & Y_{\mu'} \end{array}$$

is commutative for $\mu < \mu'$. We shall denote by $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ a morphism of inverse systems.

For $\mathbf{Z} = \{Z_\nu, r_{\nu\,\nu'}, N\}$, it is possible to define a composition of morphisms of systems $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$ and $(g_\nu, \psi) : \mathbf{Y} \to \mathbf{Z}$ in the obvious way: it is enough to consider the composition

$$\phi \circ \psi : N \to \Lambda$$

and the maps

$$h_\nu = g_\nu \circ f_{\psi(\nu)} : X_{\phi\circ\psi(\nu)} \to Z_\nu.$$

In some special cases (in particular for compact metric spaces) the *inverse limit* of the inverse system $\mathbf{X} = \{X_\lambda, p_{\lambda\,\lambda'}, \Lambda\}$ can be defined as

$$X_\infty = \varinjlim \mathbf{X} = \{(x_\lambda) \in \prod_{\lambda\in\Lambda} \mid p_{\lambda\,\lambda'}(x'_\lambda) = x_\lambda\},$$

joint with *projections*

$$p_\lambda : X_\infty \to X_\lambda$$

such that $p_{\lambda\,\lambda'} \circ p'_\lambda = p_\lambda$.

If we have a map of inverse systems $(f_\mu, \phi) : \mathbf{X} \to \mathbf{Y}$, then an unique map between the inverse limits is induced. We shall denote it by $f_\infty : X_\infty \to Y_\infty$ or by $\lim f : \lim X \to \lim Y$. This map satisfies:

$$f_\mu \circ p_{\phi(\mu)} = q_\mu \circ f_\infty$$

for every $\mu \in M$.

Similarly, a *direct system* in a category $\mathcal{C}$ over a directed set Λ is a collection

$$\mathbf{X} = \{X_\lambda, p_{\lambda\,\lambda'}, \Lambda\}$$

where for each $\lambda \in \Lambda$, X_λ is an object of $\mathcal{C}$ and for $\lambda, \lambda' \in \Lambda$ with $\lambda < \lambda'$ there is an arrow of $\mathcal{C}$

$$p_{\lambda\,\lambda'} : X_\lambda \to X_{\lambda'}$$

such that:

i) $p_{\lambda\,\lambda}$ is the identity of X_λ;

ii) $p_{\lambda'\,\lambda''} \circ p_{\lambda\,\lambda'} = p_{\lambda\,\lambda''}$ in $\mathcal{C}$ for each $\lambda, \lambda', \lambda'' \in \Lambda$ with $\lambda'' \geq \lambda' \geq \lambda$.

Let x_λ and $x_{\lambda'}$ be elements of X_λ and $X_{\lambda'}$ respectively. This two elements are identified, $x_\lambda \sim x_{\lambda'}$, if and only if $p_{\lambda\lambda''}(x_\lambda) = p_{\lambda'\lambda''}(x_{\lambda'})$ for some $\lambda'' \in \Lambda$ with $\lambda'' \geq \lambda, \lambda'$. Then $X^\infty = \sqcup_{\lambda\in\Lambda} X_\lambda / \sim$. An equivalent formulation is that an element is identified with all its images under the bonding maps of the directed system, that is, $x_\lambda \sim p_{\lambda\lambda'}(x_\lambda)$.

An special case in which we shall use direct limits is in groups. For a direct system $\{G_\lambda, p_{\lambda\lambda'}, \Lambda\}$, we consider the collection of elements $g_\lambda \in G_\lambda$ identified with the previous relation, and denote it by G^∞. The operation in G^∞ is defined as the class generated by the sum of representatives of each class in a common factor G_λ. Alternatively, G^∞ can be regarded as a suitable quotient of the direct sum $\oplus_{\lambda\in\Lambda} G_\lambda$. The elements $g_\lambda - p_{\lambda\lambda'}(g_\lambda)$ (for $\lambda' > \lambda$) generate a subgroup H of $\oplus_{\lambda\in\Lambda} G_\lambda$. Then, $G^\infty = \oplus_{\lambda\in\Lambda} G_\lambda / H$.

The space X^∞ (resp. the group G^∞) is called *direct limit* of the direct system. In addition, there are maps $p_\lambda : X_\lambda \to X^\infty$ called *injections* of X_λ into X^∞.

Analogous expositions are valid for inverse and direct systems and limits of pairs (in particular, for pointed spaces), so we omit it for brevity. The details of definitions, constructions and results can be found in [57].

Singular homology

Let $\Delta^n = [a_0, ..., a_n]$ be the geometric n-simplex in $\mathbb{R}^{n+1}$ and let G be an Abelian group. We usually take $G = \mathbb{Z}$ if no other group is specified. An *n-simplex in X* is a continuous function $\sigma : \Delta^n \to X$. An *n-chain in X* is a (finite) formal sum of the form

$$\sum_{i=1}^{k} n_i \sigma_i = n_1\sigma_1 + n_2\sigma_2 + \ldots + n_k\sigma_k$$

where each σ_i is an n-simplex in X and n_i takes values in G for all i. Given two n-chains σ and τ their sum $\sigma + \tau$ is defined as the formal sum of the correspondents addends of each chain. We will denote by $\mathcal{C}_n(X, G)$ the group of all n-chains in X with coefficients in G, or shortly $\mathcal{C}_n(X)$ if $G = \mathbb{Z}$.

Given an n-simplex $\sigma : \Delta^n \to X$, the *boundary of* σ is an (n-1)-chain defined as follows:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[a_0,...,\hat{a}_i,...,a_n]}$$

where $a_0, ..., a_n$ are the vertices of Δ^n and, as usual, $[a_0, ..., \hat{a}_i, ..., a_n]$ is the proper face of Δ^n that not contains the vertex a_i.

This boundary defined over n-simplexes, is extended over all n-chains acting separately in each simplex of the chain, i. e. if $\sigma = \sum_{i=1}^{k} n_i\sigma_i$ is a n-chain, its boundary is

$$\partial\sigma = \partial\Big(\sum_{i=1}^{k} n_i\sigma_i\Big) = \sum_{i=1}^{k} n_i \partial\sigma_i.$$

Hence, we have a boundary map

$$\partial : \mathcal{C}_n(X,G) \to \mathcal{C}_{n-1}(X,G).$$

If an n-chain σ satisfies that $\partial\sigma = 0$, we say that σ is an *n-cycle* and if there exists a (n+1)-chain γ such that $\partial\gamma = \sigma$ we say that σ is a *n-boundary.* We denote by $Z_n(X,G)$ and $B_n(X,G)$ the groups of cycles and boundaries respectively. It is obvious that $Z_n(X,G) = Ker\partial$ and $B_n(X,G) = Im\partial$ (in the corresponding dimensions). Hence, we have arrived to the classical definition of the singular homology group of X,

$$H_n(X,G) = \frac{Ker\partial}{Im\partial},$$

It is also well-known that every continuous map $f : X \to Y$ induces a *chain map* $f_\# : \mathcal{C}_n(X) \to \mathcal{C}_n(Y)$ such that $f_\#\partial = \partial f_\#$. So also an homomorphism $f_* : H_n(X) \to H_n(Y)$ between singular homology groups is induced.

Čech homology and cohomology

With every topological space X one can associate an inverse system $\mathbf{C}(X) = \{X_\mathcal{U}, p_{\mathcal{U}\mathcal{U}'}, \Lambda\}$ called *the Čech system of* X. As we shall briefly recall here, this inverse system is in the category **HPol** of polyhedra (and continuous maps) up to homotopy.

The indexing set Λ is the set of all normal coverings $\mathcal{U}$ of X ordered by the relation of refinement of coverings ($\mathcal{U} \leq \mathcal{U}'$ if and only if $\mathcal{U}'$ refines $\mathcal{U}$). Recall that a *normal covering* is an open covering $\mathcal{U}$ of X which admits a partition of unity subordinated to $\mathcal{U}$ (this is the case for normal spaces, in particular for paracompact and Hausdorff spaces).

Each term $X_\mathcal{U}$ is the nerve $|N(\mathcal{U})|$ associated to the covering $\mathcal{U}$. In $|N(\mathcal{U})|$ there exists a vertex corresponding to each set $U \in \mathcal{U}$ and $\{U_1, ..., U_n\}$ expand an n-simplex if and only if $U_1 \cap \cdots \cap U_n \neq \emptyset$.

Finally, for $\mathcal{U} \leq \mathcal{U}'$, let $p_{\mathcal{U}\mathcal{U}'}$ be the simplicial projection $p_{\mathcal{U}\mathcal{U}'} : |N(\mathcal{U})| \to |N(\mathcal{U}')|$ sending a vertex $U' \in \mathcal{U}'$ to a vertex $U \in \mathcal{U}$ with $U' \subseteq U$. This map is not uniquely determined, but any other projection between the nerves is homotopic to this. Hence, the projection is unique up to homotopy.

Similarly, for $\mathcal{U} \in \Lambda$, $p_\mathcal{U} : X \to X_\mathcal{U}$ is a canonical projection uniquely determined up to homotopy. Furthermore, $p_{\mathcal{U}\mathcal{U}'} \circ p_{\mathcal{U}'} \simeq p_\mathcal{U}$.

If we apply the simplicial homology functor to the Čech inverse system, we get an inverse system of groups:

$$\mathbf{H_n}(\mathbf{X}) = \{H_n(X_\lambda), p_{\lambda\lambda'*}, \Lambda\}$$

where $p_{\lambda\lambda'*}$ is the correspondent induced map in homology. The inverse limit of this system

$$\check{H}_n(X) = \lim \mathbf{H}_n(X) = \varprojlim\{H_n(X_\lambda), p_{\lambda\lambda'*}, \Lambda\}$$

is called *Čech homology group* of X. This is also a functor which, for a continuous map $f : X \to Y$, induces an homomorphism $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.

Similarly, for cohomology we obtain a direct system of groups:

$$\mathbf{H}^n(X) = \{H^n(X_\lambda), p^*_{\lambda\lambda}, \Lambda\}$$

where $p^*_{\lambda\lambda'}$ is the induced map in cohomology. The direct limit of this system

$$\check{H}^n(X) = \lim \mathbf{H}^n(X) = \varinjlim\{H^n(X_\lambda), p^*_{\lambda\lambda'}, \Lambda\}$$

is called *Čech cohomology group* of X.

Theory of shape

The theory of shape is a branch of topology that provides a coarser classification of topological spaces than homotopy theory does. Necessary concepts shall be introduced in the following section but, roughly speaking, the idea is to approximate the given topological space X by an inverse system of nice spaces (e. g. by polyhedra), in which the classical homotopy theory works as expected, and then pass to the limit. For a systematic definition of shape, see first chapters of [65] (see also [11] for foundations of shape for compact metric spaces).

Along this work, we are going to use two different approaches to the the theory of shape. The first one is the classical way as Borsuk introduced it for compact metric spaces at the very beginning of this theory. The second one is the categorical point of view given by Mardesic and Segal, using inverse systems. We shall use one approach or the other, depending on the purposes of each section of the work.

Borsuk's theory of shape

Let X be a compact metric space. It is a well-known fact that X can be embedded as a closed subset in (Q, ρ), the Hilbert Cube with a prefixed metric. For convenience, we usually shall take a convex copy of this space:

$$Q = \left\{(a_n) \in \mathbb{R}^{\mathbb{N}} \mid a_n \in \left[-\frac{1}{n}, \frac{1}{n}\right]\right\}$$

with the ℓ_2 norm.

Recall that, given X in Q and $\varepsilon > 0$,

$$B(X, \varepsilon) = \{y \in Q \,|\, \rho(y, X) = \inf\{\rho(y, x) \,|\, x \in X\} < \varepsilon\}$$

is the *ball of radius ε centered at X*.

A *fundamental sequence* $\{f_n\} : X \to Y$ is a sequence of maps $f_n : Q \to Q$ such that for every neighbourhood V of Y there exists a neighbourhood U of X and an integer $n_0 \in \mathbb{N}$ such that

$$f_n|_U \simeq f_{n+1}|U \text{ in } V \text{ for every } n \geq n_0.$$

It is possible to compose fundamental sequence, and the definition of the identity fundamental sequence $\{1_X\}$ is also clear. Now, it is possible to define a equivalence relation between fundamental sequences as $\{f_n\} \simeq \{g_n\}$ if and only if for every neighbourhood V of Y there exists a neighbourhood U of X and an integer $n_0 \in \mathbb{N}$ such that

$$f_n|_U \simeq g_n|_U \text{ in } V \text{ for every } n \geq n_0.$$

It is clear that this relation is compatible with the composition of fundamental sequences, it is associative and the class of the identity fundamental sequence is the identity class. This classes of fundamental sequences shall be called *shape morphisms.* Two compact metric spaces X and Y are said of *the same shape* if there exists fundamental sequences $\{f_n\} : X \to Y$ and $\{g_n\} : Y \to X$ such that $\{f_n \circ g_n\} \simeq \{1_Y\}$ and $\{g_n \circ f_n\} \simeq \{1_X\}$.

Inverse system approach to shape

We can also describe the shape of a space in terms of inverse limits. Let **HPol** be the category whose objects are polyhedra and continuous maps between them as arrows, up to homotopy. Let **HTop** be the category of homotopy topological spaces and continuous maps between them. Since **HPol** is dense in **HTop** (see [65] for basic definitions) there exists an HPol-expansion of any topological pointed space (X, x_0). Assume that

$$\underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\}$$

is the inverse system in **HPol**, such that $p_{\lambda,\lambda'} : (X_\lambda, x_\lambda) \longrightarrow (X_{\lambda', x_{\lambda'}})$ (we can suppose that $(\Lambda, \leq)$ is a directed set) and

$$\mathbf{p} = \{p_\lambda\}_{\lambda \in \Lambda} : (X, x_0) \longrightarrow \underline{(\mathbf{X}, \mathbf{x_0})}$$

the *HPol*-expansion of X. The maps satisfy that $p_{\lambda,\lambda'} \circ p_\lambda = p_{\lambda'}$ for all $\lambda' \leq \lambda$ and they must respect the base points.

The connection with the Borsuk's theory of shape is the following: it is well-known that there exists a sequence X_n of polyhedra and maps $f_{n\,n+1} : X_{n+1} \to X_n$ such that the inverse system $\{X_n, f_{n\,n+1} : n \in \mathbb{N}\}$ has X as its inverse limit[1]. More specifically, given a compactum X in the Hilbert cube, and for any $\varepsilon > 0$ there exists a polyhedron $\mathcal{P}$ which is a neighbourhood of X in Q such that $X \subseteq \mathcal{P} \subseteq B(X, \varepsilon)$. Consequently, there exists a sequence of positive real numbers $\{\varepsilon_n\}$ converging to zero such that for each $n \in \mathbb{N}$ there exists a polyhedron $\mathcal{P}_n$ (which is again a neighbourhood of X in Q) and

$$Q \supseteq B(X, \varepsilon_1) \supseteq \mathcal{P}_1 \supseteq B(X, \varepsilon_2) \supseteq \mathcal{P}_2 \supseteq \cdots \supseteq X.$$

[1]This is, essentially, a result of Alexandroff and Freudenthal which can be found in Ch. I, §5.2 of [65]. There is proved that every compact metric space is the inverse of some inverse sequence of compact polyhedra.

We consider inclusion maps $i_{n\,n+1} : B(X,\varepsilon_{n+1}) \to B(X,\varepsilon_n)$ or also $p_{n\,n+1} : \mathcal{P}_{n+1} \to \mathcal{P}_n$ and, clearly, the inverse limit is X for both inverse sequences $\{B(X,\varepsilon_n), i_{n\,n+1}\}$ and $\{\mathcal{P}_n, p_{n\,n+1}\}$, since $\cap B(X,\varepsilon_n) = X$. We shall refer to this inverse sequence as *Borsuk's inverse sequence* or *neighbourhood inverse sequence.* See appendix of [65] for the equivalence of this two approaches to shape.

For the pointed and pairs cases the definitions are completely analogous. We have already introduced a very special expansion of a topological space, which is no other than the Čech inverse system. The algebraic invariants of the theory of shape can be constructed with any expansion of the space X and, in particular, the Čech system justifies the name given to the inverse limit $\check{H}_n(X)$ of the inverse system

$$\mathbf{H}_n(X) = \{H_n(X_\lambda), p_{\lambda\,\lambda'*}, \Lambda\}$$

for any inverse system associated to X.

Similar construction can be done with homotopy groups. Let

$$\underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\}$$

be an inverse system such that

$$\{p_\lambda\} : (X, x_0) \to \underline{(\mathbf{X}, \mathbf{x_0})}$$

is an $\mathbf{HPol}_*$-expansion of a pointed space (X, x_0). By applying the homotopy groups functor $\pi_n(-)$ to each term, we get the inverse system

$$\pi_{\mathbf{n}}(\mathbf{X}, \mathbf{x_0}) = \{\pi_n(X_\lambda, x_\lambda), p_{\lambda,\lambda'*}, \Lambda\}$$

where $p_{\lambda\,\lambda'*}$ is the map induced in homotopy. The inverse limit of this inverse system

$$\check{\pi}_n(X, x_0) = \varprojlim \pi_{\mathbf{n}}(\mathbf{X}, \mathbf{x_0})$$

is called *n-shape group* of (X, x_0). The most important case is for $n = 1$, in which the *shape group* in the theory of shape plays the same roll as the fundamental group in homotopy theory.

By definition of inverse limit, the elements of $\check{\pi}_1(X, x_0)$ are collections of homotopy classes $\langle\alpha\rangle = \{[\alpha_\lambda]\}$ where $[\alpha_\lambda] \in \pi_1(X_\lambda, x_\lambda)$ and such that $p_{\lambda\,\lambda'*}([\alpha_{\lambda'}]) = [\alpha_\lambda]$. Moreover, there exists a natural map from the fundamental to the shape group, that assigns to each homotopy class $[\alpha]$ in $\pi_1(X, x_0)$ the sequence generated by $[\alpha]$, which is $\langle[\alpha]\rangle = \{[p_\lambda \circ \alpha]\}$

$$\begin{array}{rccc} \varphi : & \pi_1(X, x_0) & \longrightarrow & \check{\pi}_1(X, x_0) \\ & [\alpha] & \longmapsto & <[\alpha]> \end{array}$$

This canonical morphism is neither injective nor surjective in general.

Theory of covering spaces

We resume here the classical theory of covering spaces. For details, see [83] or [39].

A *covering space* os a space X is a space $\widetilde{X}$ together with a map $\pi : \widetilde{X} \to X$ satisfying the *evenly covering condition:* there exists an open cover $\{U_\lambda\}$ of X such that for each λ, $\pi^{-1}(U_\lambda)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped by π homeomorphically onto U_λ.

From the definition, some properties about lifting are derived. Recall that a *lift* of a map $f : Y \to X$ is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $\pi \circ \widetilde{f} = f$. In particular, it is deduced from the definition the *homotopy lifting property* which also implies the path lifting property as a special case. Moreover, we have the following lifting criterion:

Proposition 0.0.1 *If $\pi : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering space, and $f : (Y, y_0) \to (X, x_0)$ is a map with Y path-connected and locally path-connected, then there exists an unique lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f if and only if $f_*(\pi_1(Y, y_0)) \subseteq \pi_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

If the covering $\widetilde{X}$ is path-connected and satisfies that $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{1\}$, then it is called the *universal covering* of X. It is easy to check that this covering is unique if it exists. In fact:

Proposition 0.0.2 *Let X be a path-connected and locally path-connected topological space. Then, there exists its universal covering $\widetilde{X}$ if and only if X is semilocally simply-connected.*

Given a path-connected, locally path-connected and semilocally simply-connected space X with a base point $x_0 \in X$, we can define the set

$$\widetilde{X} = \{[\alpha] \mid \alpha \text{ is a path in } X \text{ starting at } x_0\}$$

where $[\alpha]$ denotes the class of the path α with respect homotopies of paths (that is, relatives to end-points). The map $\pi : \widetilde{X} \to X$ defined as $\pi([\alpha]) = \alpha(1)$ is well-defined and plays the roll of a the projection. This map receives the name of *end-point projection.*

To define the topology on $\widetilde{X}$, it is enough to consider open sets U of X such that $\pi_1(U) \to \pi_1(X)$ is trivial (being this map induced by the inclusion $U \hookrightarrow X$). For such an open set U and a path α with $\alpha(1) \in U$, let

$$B([\alpha], U) = \{[\alpha * \gamma] \mid \gamma \text{ is a path in } U \text{ with } \gamma(0) = \alpha(1)\}.$$

The family of sets

$$\{B([\alpha], U) \mid [\alpha] \in \widetilde{X}, U \subseteq X \text{ with } \pi_1(U) \to \pi_1(X) \text{ trivial}\}$$

is a base for a topology on $\widetilde{X}$ which in fact satisfies the properties of the universal covering.

Miscellanea of topics

Finally we give some others definitions, results and facts that we would need along the present work.

Ultrametrics and generalized ultrametrics

The definition of what a metric is stays clear. If we replace the condition of the triangle inequality by a stronger condition we obtain the definition of an ultrametric. Let X be a set and $d : X \times X \to \mathbb{R}$ a map. d is an *ultrametric* if for every $x, y, z \in X$, the next properties are satisfied:

i) $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

ii) $d(x,y) = d(y,x)$;

iii) $d(x,y) \leq \max\{d(x,z), d(z,x)\}$ (strong triangle inequality).

If d is an ultrametric, the pair (X,d) is called *ultrametric* (or *non-Archimedean*) *space.* Also note that the third property implies the usual triangle inequality $d(x,y) \leq d(x,z)+d(z,y)$ so every ultrametric space is, in particular, a metric space.

There are different generalizations of the concept of ultrametric. In particular, the definition of generalized ultrametric we are going to use is due to S. Priess-Crampe and P. Ribenboim:

Definition 0.0.3 Let X be a set and $(\Gamma, \leq)$ be a partial ordered set with a least element 0. A *generalized ultrametric* on X is a map $d : X \times X \longrightarrow \Gamma$ such that for $x, y, z \in X$ and $\gamma \in \Gamma$, it satisfies:

1) $d(x,y) = 0$ if and only if $x = y$.

2) $d(x,y) = d(y,x)$.

3) If $d(x,y) \leq \gamma$ and $d(y,z) \leq \gamma$, then $d(x,y) \leq \gamma$.

If only properties 2) and 3) hold, it is said that d is a *pseudoultrametric.*

If we delete the second part of the first property on the previous definition, we obtain a *generalized pseudoultrametric.*

Topological groups

Recall that a *topological group* is a triple $(G, *, \mathcal{T})$ where $(G, *)$ is a group and $\mathcal{T}$ is a topology on G such that the maps $g \mapsto g^{-1}$ and $(g, h) \mapsto g * h$ are continuous. Such a $\mathcal{T}$ is called a *group topology on G*.

Given a continuous group homomorphism $h : G \to H$ between topological groups, if h is simultaneously isomorphism and (uniform) homeomorphism, h is called *(uniform) topological isomorphism* and G and H are said to be *(uniformly) topologically isomorphic.*

Polyhedra, ANR's and CW-complexes

In the preliminaries above, we have dealt with polyhedra. But also we could change our exposition with ANR's or CW's from the following:

Theorem 0.0.4 *Every CW-complex X has the homotopy type of a polyhedron P. In addition, for a space X the following conditions are equivalent:*

i) X has the homotopy type of a compact polyhedron.

ii) X has the homotopy type of a compact ANR.

Recall that if X, Y are topological spaces, and $\mathcal{U}$ is an open covering of Y, it is said that two maps $f, g : X \to Y$ are *$\mathcal{U}$-near* if for every $x \in X$ there exists an open set $U \in \mathcal{U}$ such that $f(x), g(x) \in U$.

Similarly, if $H : X \times [0,1] \to Y$ is an homotopy, we say that H is an *$\mathcal{U}$-homotopy* if for every $x \in X$ there exists an open set $U \in \mathcal{U}$ such that $H(x,t) \in U$ for every $t \in [0,1]$. We shall say that two continuous maps f, g are *$\mathcal{U}$-homotopic* if there exists an $\mathcal{U}$-homotopy between them.

Proposition 0.0.5 *Let X be a metrizable space, Y a polyhedron and $\mathcal{U} = \{U_\lambda \mid \lambda \in \Lambda\}$ an open covering of Y. Then there exists an open covering $\mathcal{V}$ of Y which refines $\mathcal{U}$ such that for every pair of continuous maps $f, g : X \to Y$ which are $\mathcal{V}$-near and for every $\mathcal{V}$-homotopy $J : A \times [0,1] \to Y$ from a closed set $A \subset X$ with $J(x,0) = f|_A$ and $J(x,1) = g|_A$, there exists an $\mathcal{U}$-homotopy $H : X \times [0,1] \to Y$ which is an extension of J, that is, $H(x,0) = f(x)$, $H(x,1) = g(x)$ and $H|_{A\times[0,1]} = J$.*

The corollary of the proposition 0.0.5 which we are interested in, is the following:

Corollary 0.0.6 *If Y is a polyhedron, then there exists an open covering $\mathcal{V}$ such that any two paths with the same end-points which are $\mathcal{V}$-near are path-homotopic.*

Proof. Taking $\mathcal{U}$ as the trivial covering $\mathcal{U} = \{Y\}$, we obtain the corresponding covering $\mathcal{V}$ given by Proposition 0.0.5 above.

Now, let us take two $\mathcal{V}$-near paths $f, g : [0,1] \to Y$ with $f(0) = g(0) = y_0$ and $f(1) = g(1) = y_1$. For the closed set $A = \{0,1\}$ of $[0,1]$, we can consider the trivial homotopy $J : A \times [0,1] \to Y$ defined by $J(0,t) = y_0$ and $J(1,t) = y_1$.

Again by 0.0.5, we obtain an $\mathcal{U}$-homotopy H which extends J. Thus, H actually is an homotopy relative to $\{0,1\}$, so f and g are homotopic as paths. □

Observe that in the case of compact metric spaces, the previous results can be reformulated in terms of metrics and $\varepsilon - \delta$ expressions.

Chapter 1

The shape group and the fundamental group

In this first chapter we develop the techniques which shall allow us to construct ultrametrics on different groups related to the theory of shape. In particular, we first introduce a ultrametric on the shape groups of a compact metric space. After that, we also give a way to generalize that construction to the arbitrary case.

We reproduce those constructions in the case of the fundamental group, obtaining a pseudoultrametric in the compact metric case, and we also give the generalization to the arbitrary case. Finally, we relate this additional structure with other topologies on the fundamental group previously appeared in the literature.

1.1 Ultrametric on the shape group

The works [25, 26, 68, 69] are the starting point of this section. In particular, in the introduction of [69] the following remark about a particular case of [68] is made: the techniques used in [68] remain valid in order to obtain a complete ultrametric on the shape group of a (pointed) compact metric space X. This observation was developed in [69] and also in [80].

After recalling the main steps of the construction of the ultrametric on $\check{\pi}_1(X, x_0)$ for a compact metric space X, we shall give an appropriate generalization of the construction for an arbitrary topological space X. This part also follows the line of [25, 26].

Compact metric case

Let (X, x_0) be a compact metric space with a fixed base point $x_0 \in X$, and consider it embedded as closed subspace of the Hilbert Cube (Q, ρ). Given two loops $f, g : [0,1] \to Q$, we can define

$$F(f,g) = \inf\{\varepsilon > 0 \mid f \simeq g \text{ as loops in } B(X,\varepsilon)\}.$$

The definition of $F(f,g)$ for the pointed case requires f and g to be loops based at the same point, and also f and g to be homotopic as loops. We are interested in the case when the loops in Q are based in $x_0 \in X$.

Definition 1.1.1 A sequence $\{f_n\}$ of loops in Q is said to be F-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $F(f_m, f_n) < \varepsilon$ for all $m, n \geq n_0$. We say that two F-Cauchy sequences of loops in Q, $\{f_n\}$ and $\{g_n\}$, are F-related if the sequence $f_1, g_1, f_2, g_2, f_3, ...$ is again F-Cauchy. We shall denote this relation by $\{f_n\}F\{g_n\}$.

The same definitions remain to be valid for the pointed case, using sequences $\{f_n\}$ of loops all of them based at $x_0 \in X$. The proofs of the following results are omitted, since they can be found at the mentioned paper [68]. Also, similar proofs but in the context of homology shall be done in chapter 2.

Proposition 1.1.2 *i) The F-relation is an equivalence relation.*

ii) For every pair $\{\alpha_n\}, \{\beta_n\}$ of F-Cauchy sequences, there exists $\lim_{n\to\infty} F(\alpha_n, \beta_n)$.

iii) Let $\{\alpha_n\}, \{\beta_n\}, \{\alpha'_n\}, \{\beta'_n\}$ be F-Cauchy sequences such that $\{\alpha_n\}F\{\alpha'_n\}$ and $\{\beta_n\}F\{\beta'_n\}$. Then $\lim_{n\to\infty} F(\alpha_n, \beta_n) = \lim_{n\to\infty} F(\alpha'_n, \beta'_n)$.

An element of the shape group $\check{\pi}_1(X, x_0)$ is a shape class of a (pointed) approximative map from $(S^1, 1)$ to (X, x_0).

Proposition 1.1.3 *Let $\{\alpha_n\}$ be a sequence of loops $\alpha_n : (S^1, 1) \to (Q, x_0)$. Then:*

i) $\{\alpha_n\}$ is an approximative sequence if and only if it is F-Cauchy.

ii) Let β_n another F-Cauchy sequence of loops based at x_0. Then, $\{\alpha_n\}$ is related with $\{\beta_n\}$ as approximative sequences if and only if they are F-related.

Hence, it is possible to identify each element of $\check{\pi}_1(X, x_0)$ with a class of F-Cauchy sequences. Then, given $\langle\alpha\rangle, \langle\beta\rangle \in \check{\pi}_1(X, x_0)$, we can define:

$$d(\langle\alpha\rangle, \langle\beta\rangle) = \lim_{n\to\infty} F(\alpha_n, \beta_n)$$

where $\langle\alpha\rangle = \{\alpha_n\}$ and $\langle\beta\rangle = \{\beta_n\}$.

Theorem 1.1.4 *The pair $(\check{\pi}_1(X, x_0), d)$ is a complete ultrametric space.*

Some of the properties of this ultrametric space are summarize in the following:

Proposition 1.1.5 *If (X, x_0) is a compact metric space, then:*

a) If X is an ANR, then $\check{\pi}_1(X,x_0)$ is uniformly discrete.

b) If (Y,y_0) is another pointed compact metric space, every continuous map $f:(X,x_0)\to(Y,y_0)$ induces a continuous homomorphism $f_*:\check{\pi}_1(X,x_0)\to\check{\pi}_1(Y,y_0)$ of topological groups.

c) If (X,x_0) and (Y,y_0) have the same (pointed) shape, then $\check{\pi}_1(X,x_0)$ and $\check{\pi}_1(Y,y_0)$ are topologically isomorphic.

d) There exists a norm $\|\cdot\|$ on $\check{\pi}_1(X,x_0)$, which is both left and right invariant. It is enough to take $\|\alpha\|=d(\alpha,1)$.

e) $\check{\pi}_1(X,x_0)$ is separable. Hence, if X is an ANR, then card$(\check{\pi}_1(X,x_0))\leq\aleph_0$.

f) $\check{\pi}_1(X,x_0)$ has a base of clopen neighbourhoods of 1. Hence, it is zero-dimensional.

g) $\check{\pi}_1(X,x_0)$ is countable if and only if it is discrete.

As a consequence of item c) in the previous proposition, the topology induced by the ultrametric on $\check{\pi}_1(X,x_0)$ does not depend on the chosen sequence. In particular, let us consider the inverse sequence given by the neighbourhoods of X in Q:

$$X_n = B(X,\varepsilon_n) \qquad \text{for each } n\in\mathbb{N}$$

and the inclusion maps $i_{n\,n+1}:X_{n+1}\to X_n$. If the sequence $\{\varepsilon_n\}$ is chosen as $\varepsilon_n\to 0$, then the inverse limit of this inverse sequence is $X_\infty=\bigcap X_n = X$.

As recalled in the preliminaries, we can assume to have an inverse sequence $\{\mathcal{P}_n,p_{n\,n+1}\}$ of polyhedra, such that

$$Q\supseteq\mathcal{P}_1\supseteq B(X,\varepsilon_1)\supseteq\mathcal{P}_2\supseteq\cdots\supseteq\mathcal{P}_{n-1}\supseteq B(X,\varepsilon_n)\supseteq\mathcal{P}_n\supseteq\dots$$

for a suitable sequence of positive real numbers $\{\varepsilon_n\}$ converging to zero.

Since $X\subseteq X_n$ for each $n\in\mathbb{N}$, the same holds for $\mathcal{P}_n$. Then, we can take the point $x_0\in X$ as the base point of all of the polyhedra of the inverse sequence in order to get an inverse sequence $\{(\mathcal{P}_n,x_0),i_{n\,n+1}\}$ with (X,x_0) as its inverse limit. Then:

$$\check{\pi}_1(X,x_0)=\varprojlim\{\pi_1(\mathcal{P}_n,x_0),(i_{n\,n+1})_*\}$$

Now, it is easy to represent an approximative map $\langle\alpha\rangle:(S^1,1)\to(X,x_0)$ as a sequence $\{\alpha_n\}$ of loops $\alpha_n:(S^1,1)\to(\mathcal{P}_n,x_0)$, and we arrive to a reformulation of the ultrametric on the shape group.

Proposition 1.1.6 *There exists a complete ultrametric*

$$\begin{array}{rccc} d: & \check{\pi}_1(X,x_0)\times\check{\pi}_1(X,x_0) & \longrightarrow & \check{\pi}_1(X,x_0)\\ & (\langle\alpha\rangle,\langle\beta\rangle) & \longmapsto & d(\langle\alpha\rangle,\langle\beta\rangle)\end{array}$$

which agrees with the inverse limit topology on $\check{\pi}_1(X, x_0)$. *In fact, d is defined as*

$$d(\langle\alpha\rangle, \langle\beta\rangle) = \begin{cases} 0 & \textit{if } \alpha_n \simeq \beta_n \textit{ in } \mathcal{P}_n \textit{ for every } n \in \mathbb{N} \\ \frac{1}{2^m} & \textit{if } m = \max\{n \in \mathbb{N} \mid \alpha_n \simeq \beta_n \textit{ in } \mathcal{P}_n\} \end{cases}$$

Corollary 1.1.7 *Each complete ultrametric on* $\check{\pi}_1(X, x_0)$ *which induces the inverse limit topology is uniformly equivalent to the ultrametric defined in Proposition 1.1.6.*

Example 1.1.8 We can illustrate the above taking as compact metric space $X = \mathcal{W}$ to be the Warsaw circle.

This space is the graph of the topologist's sinus curve with its closure, with a simple arc joining this two pieces (see picture 1.1).

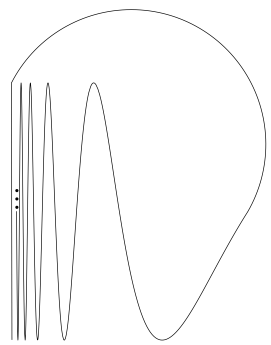


Figure 1.1: An sketch of the Warsaw circle $\mathcal{W}$.

Using the Borsuk's neighbourhood system, we obtain an inverse sequence $\{(X_n, x_n), p_{n\,n+1}\}$ where each X_n is an annulus (all of them with the same base-point) and $p_{n\,n+1}$ is the correspondent inclusion from X_{n+1} into X_n.

If we consider the induced sequence in the fundamental groups $\{\pi_1(X_n, x_n), (p_{n\,n+1})_*\}$, we obtain:

$$\mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z} \longleftarrow \cdots$$

where the homomorphisms are the identity (since the inclusions $p_{n\,n+1}$ are orientation preserving). Obviously, this inverse sequence has $\mathbb{Z}$ as inverse limit. Hence, the shape group of the Warsaw circle is $\check{\pi}_1(\mathcal{W}, x_0) = \mathbb{Z}$.

Now, given two shape classes $\langle\alpha\rangle, \langle\beta\rangle \in \check{\pi}_1(\mathcal{W}, x_0)$ respectively represented by $\{[\alpha_n]\}$ and $\{[\beta_n]\}$ where each $[\alpha_n], [\beta_n]$ are homotopy classes of loops in X_n, it is enough to set

$$d(\langle\alpha\rangle, \langle\beta\rangle) \leq \frac{1}{n_0} \Leftrightarrow [\alpha_n] = [\beta_n] \text{ in } X_n \text{ for all } n \geq n_0$$

which is equivalent (compare with 1.1.6) to the ultrametric

$$d(\langle\alpha\rangle,\langle\beta\rangle)=\frac{1}{n}$$

where n is the first place where $[\alpha_n]\neq[\beta_n]$.

The following example shows that, in general, the topology on $\check{\pi}_1(X,x_0)$ is non-discrete.

Example 1.1.9 Let us consider the *Hawaiian Earrings space* $X=HE=\bigcup_{n\in\mathbb{N}}C_n$, where C_n is the circle with center $(0,1/n)$ and radio $1/n$. Then, $x_0=(0,0)$ is the tangency point of all circles and we shall consider it as the base-point of X. We denote by l_n the loop, based at x_0, which runs counterclockwise along C_n.

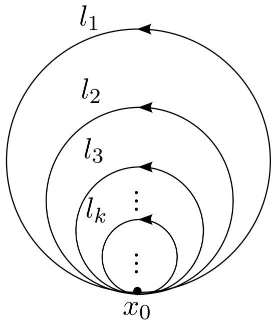


Figure 1.2: The Hawaiian Earrings space.

For some $n\in\mathbb{N}$, let $\langle l_n\rangle$ be the shape class of the loop l_n. It is clear that $l_n\simeq c_{x_0}$ in $B(HE,\frac{1}{m})$ if and only if $m>n$. Hence,

$$d(\langle l_n\rangle,\langle c_{x_0}\rangle)=\frac{1}{n}$$

and the sequence $\{\langle l_n\rangle\}$ of shape classes generated by the loops of HE satisfies

$$\langle l_n\rangle\longrightarrow\langle c_{x_0}\rangle$$

for the ultrametric d.

Generalization to arbitrary topological spaces X.

We outline here the generalization to the arbitrary case, in the spirit of [25]. Let (X,x_0) be an arbitrary pointed topological space, and let

$$\underline{\mathbf{p}}=\{p_\lambda\}:(X,x_0)\to\underline{(\mathbf{X},\mathbf{x_0})}=\{(X_\lambda,x_\lambda),p_{\lambda\lambda'},\Lambda\}$$

be an $\mathbf{HPol}_*$-expansion of (X,x_0).

For a directed set $(\Lambda, \leq)$, denote by $\mathcal{L}(\Lambda)$ the *set of all lower classes* in Λ ordered by inclusion. A *lower class* in Λ is a subset $\Delta \subset \Lambda$ such that if $\delta \in \Delta$ and $\lambda \in \Lambda$ with $\lambda \leq \delta$, then $\lambda \in \Delta$. From now on we consider the empty set $\varnothing$ as a lower class. Moreover, it is defined the reverse inclusion order in $\mathcal{L}(\Lambda)$, in such a way that $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1 \supseteq \Delta_2$.

Remark 1.1.10 $(\mathcal{L}(\Lambda), \leq)$ is a partially ordered set with a least element which shall be denoted by 0. Moreover, $\mathcal{L}(\Lambda)^* = \mathcal{L}(\Lambda) \setminus \{0\}$ is downward directed.

Indeed, the properties of a partial order (reflexive, antisimetric and transitive) are easy to check. The least element is just the lower class $0 = \Lambda$.

To show that $\mathcal{L}(\Lambda)^*$ is downward directed, let $\Delta_1, \Delta_2 \in \mathcal{L}(\Lambda)^*$. Consider $\Delta_3 = \Delta_1 \cup \Delta 2$. Of course, $\Delta_3 \leq \Delta_1, \Delta_2$. Let us check that $\Delta_3 \in \mathcal{L}(\Lambda)^*$.

Note that Δ_3 is a lower class: if $\delta \in \Delta_3$, then $\delta \in \Delta_1$ or $\delta \in \Delta_2$. In any case, if $\lambda \in \Lambda$ such that $\lambda \leq \delta$ it is $\lambda \in \Delta_1$ or $\lambda \in \Delta_2$, since Δ_1, Δ_2 are lower classes. Consequently $\lambda \in \Delta_3$.

Moreover, $\Delta_3 \neq \Lambda$. Suppose, on the contrary, that $\Lambda = \Delta_3 = \Delta_1 \cup \Delta_2$. Since $\Delta_1, \Delta_2 \in \mathcal{L}(\Lambda)^*$, it follows that $\Delta_1 \neq \Delta_1 \cap \Delta_2 \neq \Delta_2$. Take two elements $\delta \in \Delta_1 \setminus \Delta_2$ and $\delta' \in \Delta_2 \setminus \Delta_1$. Since Λ is a directed set, there exists an element $\delta'' \in \Lambda$ such that $\delta'' \geq \delta, \delta'$. As $\delta'' \in \Delta_1$ or $\delta'' \in \Delta_2$, it follows that $\delta' \in \Delta_1$ or $\delta \in \Delta_2$, giving a contradiction. So $\Delta_3 \in \mathcal{L}(\Lambda)^*$.

Theorem 1.1.11 *Let (X, x_0) be an arbitrary pointed topological space, and let*

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \to \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda\}$$

*be an **HPol***$_*$*-expansion of (X, x_0). Let $\check{\pi}_1(X, x_0)$ be the shape group of (X, x_0). For $\langle\alpha\rangle, \langle\beta\rangle \in \check{\pi}_1(X, x_0)$, the formula*

$$d(\langle\alpha\rangle, \langle\beta\rangle) = \{\lambda \in \Lambda \mid p_\lambda(\langle\alpha\rangle) \simeq p_\lambda(\langle\beta\rangle) \ \text{ in } \ X_\lambda\}$$

defines a generalized ultrametric

$$d : \check{\pi}_1(X, x_0) \times \check{\pi}_1(X, x_0) \to (\mathcal{L}(\Lambda), \leq)$$

The connection with the compact metric case is evident from the following corollary.

Corollary 1.1.12 *Let (Q, ρ) be the Hilbert cube, and assume $X \subset Q$ as a closed subset, with a fixed base point $x_0 \in X$. For each $\varepsilon > 0$, consider $X_\varepsilon = B(X, \varepsilon)$, pointed in x_0 too. Also, for $\varepsilon' > \varepsilon > 0$, let $p_{\varepsilon\varepsilon'} : X_\varepsilon \to X_{\varepsilon'}$ and $p_\varepsilon : X \to X_\varepsilon$ the corresponding (pointed) inclusions. Finally, let us denote $(\mathbb{R}^+)^{-1}$ the set of non-negative real numbers with the reverse usual order. Then, the generalized ultrametric d constructed in Theorem 1.1.11 for the **HPol***$_*$*-expansion*

$$\underline{\mathbf{p}} = \{p_\varepsilon\} : (X, x_0) \to \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\varepsilon, x_0), p_{\varepsilon\varepsilon'}, (\mathbb{R}^+)^{-1}\}$$

coincides with the ultrametric constructed in the previous subsection.

Once we have an ultrametric, it is immediate to construct a topology from that. Using the ideas of [41], we have the following description of a base of a topology on $\check{\pi}_1(X,x_0)$ from the ultrametric:

Given a lower class $\Delta \in \mathcal{L}(\Lambda)^*$ and an element $\langle\alpha\rangle \in \check{\pi}_1(X,x_0)$, we can consider

$$B_\Delta(\langle\alpha\rangle) = \{\langle\beta\rangle \in \check{\pi}_1(X,x_0) \mid d(\langle\alpha\rangle,\langle\beta\rangle) < \Delta\}.$$

It is easy to check that the family

$$\{B_\Delta(\langle\alpha\rangle) \mid \langle\alpha\rangle \in \check{\pi}_1(X,x_0), \Delta \in \mathcal{L}(\Lambda)\}$$

is a base for a topology in $\check{\pi}_1(X,x_0)$ which is completely regular, Hausdorff and zero-dimensional. It shall be called *the canonical topology induced by the ultrametric d.*

However, this topology is not useful in order to obtain information related to the theory of shape, since it depends on the particular $\mathbf{HPol}_*$-expansion used for the construction, and not on the shape of the involved space. This limitation can be illustrated with the following example.

Example 1.1.13 Given any pointed topological space (X,x_0) and an $\mathbf{HPol}_*$-expansion

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X,x_0) \to \underline{(\mathbf{X},\mathbf{x_0})} = \{(X_\lambda,x_\lambda), p_{\lambda\lambda'}, \Lambda\}$$

of (X,x_0), consider a new inverse system

$$\underline{(\mathbf{X}',\mathbf{x}'_0)} = \{(X_{(\lambda,\gamma)},x_{(\lambda,\gamma)}), p_{(\lambda,\gamma)\,(\lambda',\gamma')}, \Lambda\times\Lambda\}$$

where $(X_{(\lambda,\gamma)},x_{(\lambda,\gamma)}) = (X_\lambda,x_\lambda)$, $p_{(\lambda,\gamma)\,(\lambda',\gamma')} = p_{\lambda\lambda'}$ and $\Lambda\times\Lambda$ is ordered with the usual product order. Then,

$$\underline{\mathbf{p}'} = \{p'_{(\lambda,\gamma)}\} : (X,x_0) \to \underline{(\mathbf{X}',\mathbf{x}'_0)}$$

is an $\mathbf{HPol}_*$-expansion, where $p'_{(\lambda,\gamma)} = p_\lambda$.

Using the previous $\mathbf{HPol}_*$-expansion, the canonical topology induced by the metric d (for this new expansion) is the discrete one (a general proof for shape morphisms can be found in [25], it shall be done in the following section for the fundamental group of pointed topological spaces, and we also shall give similar arguments in the context of homology in chapter 2 of this work). Since the topology of $\check{\pi}_1(X,x_0)$ is non-discrete in general for a compact metric space X, the previous paragraph implies that the topology depends on the chosen expansion.

Previous example motivates an alternative topology also related with the ultrametric d, but slightly different from the canonical topology.

Let $(\Lambda,\leq)$ be a directed set and $(\mathcal{L}(\Lambda),\leq)$ be the corresponding ordered set of lower classes of Λ. A particular kind of lower classes are those generated by an element. That is, for $\lambda\in\Lambda$, let us denote by (λ) the *lower class generated by* λ which is the set

$$(\lambda) = \{\lambda' \in \Lambda \mid \lambda' \leq \lambda\}.$$

We have an assignation

$$\begin{array}{rccc} \phi : & (\Lambda, \leq) & \longrightarrow & (\mathcal{L}(\Lambda), \leq) \\ & \lambda & \longmapsto & (\lambda) \end{array}$$

which needs not to be injective, but if $\lambda \leq \lambda'$ then $(\lambda) \geq (\lambda')$. In particular, $(\phi(\Lambda), \leq)$ is a partially ordered set, while $(\Lambda, \leq)$ does not need to be. Moreover, $(\phi(\Lambda), \leq)$ is downward directed in $\mathcal{L}(\Lambda)$.

Let (X, x_0) be an arbitrary pointed topological space, and

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \to (\underline{\mathbf{X}, \mathbf{x_0}}) = \{(X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda\}$$

an $\mathbf{HPol}_*$-expansion of (X, x_0). Given $\langle\alpha\rangle \in \check{\pi}_1(X, x_0)$, we consider only balls of radius a lower class generated by an element, that is, $B_\Delta(\langle\alpha\rangle)$ with $\Delta = (\lambda)$ for some $\lambda \in \Lambda$. Then the family

$$\{B_{(\lambda)}(\langle\alpha\rangle) \mid \langle\alpha\rangle \in \check{\pi}_1(X, x_0), \lambda \in \Lambda\}$$

is a basis for a topology on $\check{\pi}_1(X, x_0)$, and we call it the *intrinsic topology.* The main reason to use this topology instead of the canonical one, is the following.

Proposition 1.1.14 *The intrinsic topology on $\check{\pi}_1(X, x_0)$ is independent on the fixed **HPol$_*$**-expansion of (X, x_0) and it coincides with the topology given by the ultrametric for compact metric spaces (X, x_0).*

Remark 1.1.15 Equivalent arguments are valid for all shape groups $\check{\pi}_k(X, x_0)$, $k \geq 1$.

We shall detail and expand the arguments for the general case in the following section for the fundamental group, and also in chapter 2, but in the context of homology.

1.2 Pseudoultrametric on the fundamental group

In this section we give a topology on the fundamental group which is natural in the framework of the theory of shape. This topology can be expressed in terms of a pseudoultrametric (or a generalized pseudoultrametric, for an arbitrary X) and reflects shape properties of the space. Moreover, it is somehow induced from the topology explained for the shape groups in the previous section, via the relation between homotopy and shape groups.

Compact metric case

Let X be a compact metric space embedded in (Q, ρ). Each path in X can be seen as a path in Q via this embedding. In particular, if a point $x_0 \in X$ is chosen as a base point, every loop $\alpha : (S^1, 1) \to (X, x_0)$ with $\alpha(1) = x_0$ can be regarded as a loop in Q, that is, each loop $\alpha \in \Omega(X, x_0)$ is a loop of $\Omega(Q, x_0)$.

Given α, β in $\Omega(X, x_0)$, let us consider

$$\varepsilon(\alpha, \beta) = \inf\{\varepsilon > 0 : \alpha \simeq \beta \text{ in } B(X, \varepsilon)\}$$

Observe that the set $\{\varepsilon > 0 : \alpha \simeq \beta \text{ in } B(X, \varepsilon)\}$ is bounded. The Hilbert cube is contractible, so every pair of loops α, β in $\Omega(X, x_0)$ are homotopic loops in Q, that is $\alpha \simeq \beta$ in Q. Moreover, if $\varepsilon > diam(Q)$ then $B(X, \varepsilon) = Q$, so it is clear that

$$0 \leq \varepsilon(\alpha, \beta) \leq diam(Q)$$

Now, if $[\alpha], [\beta]$ are in $\pi_1(X, x_0)$, we define

$$d([\alpha], [\beta]) = \varepsilon(\alpha, \beta)$$

If α, β in $\Omega(X, x_0)$ are such that $[\alpha] = [\beta]$ in $\pi_1(X, x_0)$, it is easy to check that $d([\alpha], [\beta]) = \varepsilon(\alpha, \beta) = 0$, since $\alpha \simeq \beta$ in $B(X, \varepsilon)$ for all $\varepsilon > 0$.

Remark 1.2.1 d is well-defined.

Proof. Let $[\alpha], [\alpha'], [\beta]$ and $[\beta']$ in $\pi_1(X, x_0)$ such that $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$. Since $\alpha \simeq \alpha'$ in X, then $\alpha \simeq \alpha'$ in $B(X, \varepsilon)$, and the same holds for β and β'. Equivalently, $d([\alpha], [\alpha']) = 0 = d([\beta], [\beta'])$.

Suppose now that $\alpha \simeq \beta$ in $B(X, \varepsilon)$ for some $\varepsilon > 0$. By the symmetric and transitive properties of the path-homotopy relation,

$$\alpha' \simeq \alpha \simeq \beta \simeq \beta'$$

in $B(X, \varepsilon)$. From that, and by definition of $d([\alpha], [\beta])$ through $\varepsilon(\alpha, \beta)$, we obtain

$$d([\alpha], [\beta]) = \varepsilon(\alpha, \beta) = \varepsilon(\alpha', \beta') = d([\alpha'], [\beta']),$$

as desired. □

Theorem 1.2.2 *d is a pseudoultrametric in $\pi_1(X, x_0)$.*

Proof.

i) Non-negativity. Given $[\alpha], [\beta]$ in $\pi_1(X, x_0)$

$$d([\alpha], [\beta]) = \varepsilon(\alpha, \beta) = \inf\{\varepsilon > 0 : \alpha \simeq \beta \text{ in } B(X, \varepsilon)\} \geq 0.$$

Moreover, it is clear that $d([\alpha], [\alpha]) = 0$ for every $[\alpha]$ in $\pi_1(X, x_0)$.

ii) Symmetry. It is an easy consequence of the symmetric property of the path-homotpy relation:

$$d([\alpha],[\beta]) = \varepsilon(\alpha,\beta) = \inf\{\varepsilon > 0 : \alpha \simeq \beta \text{ in } B(X,\varepsilon)\} = \\ = \inf\{\varepsilon > 0 : \beta \simeq \alpha \text{ in } B(X,\varepsilon)\} = \varepsilon(\beta,\alpha) = d([\beta],[\alpha]).$$

iii) Strong triangle inequality. Let $[\alpha],[\beta],[\gamma]$ be in $\pi_1(X,x_0)$ and

$$\varepsilon_1 = \varepsilon(\alpha,\gamma) = \inf\{\varepsilon > 0 : \alpha \simeq \gamma \text{ in } B(X,\varepsilon)\},$$

$$\varepsilon_2 = \varepsilon(\gamma,\beta) = \inf\{\varepsilon > 0 : \gamma \simeq \beta \text{ in } B(X,\varepsilon)\}.$$

Let $\varepsilon_0 > \max\{\varepsilon_1,\varepsilon_2\}$. Again, by transitivity of the path-homotopy relation, since $\alpha \simeq \gamma$ and $\gamma \simeq \beta$ in $B(X,\varepsilon_0)$ then $\alpha \simeq \beta$ in $B(X,\varepsilon_0)$ hence,

$$d([\alpha],[\beta]) = \varepsilon(\alpha,\beta) = \inf\{\varepsilon > 0 : \alpha \simeq \beta \text{ in } B(X,\varepsilon)\} \leq \varepsilon_0$$

Since this holds for any $\varepsilon_0 > \max\{\varepsilon_1,\varepsilon_2\}$, we have

$$d([\alpha],[\beta]) \leq \max\{\varepsilon_1,\varepsilon_2\} = \max\{d([\alpha],[\gamma]), d([\gamma],[\beta])\}.$$

□

Definition 1.2.3 Given a pointed compact metric space (X,x_0), we shall refer as *the shape topology* to the topology generated by the pseudoultrametric d, and we denote by $\pi_1^{Sh}(X,x_0)$ the fundamental group endowed with this topology.

Example 1.2.4 Let us consider again the *Hawaiian Earrings space* $X = HE = \bigcup_{n\in\mathbb{N}} C_n$, from example 1.1.9. Considering X inside the Hilbert cube, it should be clear that

$$d([l_n],[c_{x_0}]) = \frac{1}{n}.$$

Alternatively, we can define X as an inverse limit in the following way. Let X_n be the union of the first n circles, i.e.,

$$X_n = \bigcup_{m=1}^{n} C_m$$

and let $p_{n\,n+1} : X_{n+1} \to X_n$ be a map such that coincides with 1_{X_n} over X_n, and $p_{n\,n+1}(C_{n+1}) = \{x_0\}$. Then, we have an inverse sequence $\{X_n, p_{n\,n+1}\}$ whose inverse limit is $X_\infty = HE$. Now, the assertion

$$d([l_n],[c_{x_0}]) = \frac{1}{n}$$

is obvious.

Remark 1.2.5 Of course, the pseudometric constructed above only contains useful information for compact metric spaces with non-trivial fundamental group. This is not the case, for example, of the Warsaw circle $\mathcal{W}$ (see example 1.1.8). It is well-known that $\pi_1(\mathcal{W}, x_0) = \{1\}$ (there is no non-trivial loops lying in this space), so the construction of the pseudoultrametric is trivial for $\mathcal{W}$.

As we have observed above, if α, β in $\Omega(X, x_0)$ are such that $[\alpha] = [\beta]$ in $\pi_1(X, x_0)$, then $d([\alpha], [\beta]) = 0$. However, the converse is not always true (so d is not always a metric), as shown in the following example.

Example 1.2.6 Let us consider the Griffiths' space $\mathcal{G}$, constructed as follows: Let $Y_1 = Y_2 = HE$ be two copies of the Hawaiian Earrings space, and take two points to construct X_1, X_2, the respective cones over Y_1, Y_2. Consider $\mathcal{G} = X_1 \vee X_2$, the wedge product obtained identifying the tangency points of the base Hawaiian Earrings of both spaces, which will be the base point x_0.

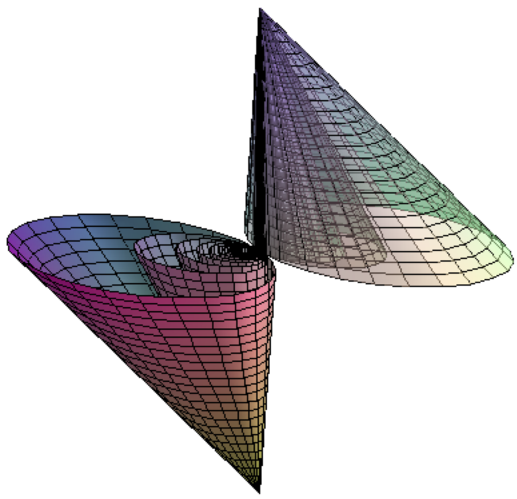


Figure 1.3: The Griffiths' space.

Denote by α_n the counterclockwise loop along the n-th circle of Y_1 and β_n the corresponding clockwise loop in Y_2. Using the usual concatenation of paths, take $\gamma_i = (\alpha_i * \beta_i)$ and $\gamma = *_{n\in\mathbb{N}}\gamma_i$. Observe that γ is a loop in $\mathcal{G}$ with base point x_0 since the sequence of radii of the circles is decreasing to zero.

Consider $\mathcal{G}$ embedded in the Hilbert cube and $\varepsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n(t)$ and $\beta_n(t)$ are in $B(x_0, \varepsilon)$ for all t in S^1 and for all $n \geq n_0$. Thus, it is only necessary to make the homotopy between γ_n and the constant map for finitely many n, and this clearly exists through the vertices of the cones. The remaining small loops of the path can be connected through a homotopy via line segments.

Therefore, $\gamma \simeq c_{x_0}$ in $B(\mathcal{G}, \varepsilon)$ for all $\varepsilon > 0$ and hence $d([\gamma], [c_{x_0}]) = 0$. On the other hand, it is clear that $[\gamma] \neq [c_{x_0}]$.

In [7], the author proposed a topology on the fundamental group, and he wrongly asserted that this topology turned the fundamental into a topological group with continuous projection from the loop space $\Omega(X, x_0)$. We show now that the topology introduced above answers in a possitive way to the question of the existence of a topology (different, in general, from the trivial and discrete ones) in $\pi_1(X, x_0)$ that makes it into a topological group. Moreover, it shows the existence of an invariant of homotopy type. We prove that this topology also respects the continuity of the usual quotient map $q : \Omega(X, x_0) \to \pi_1(X, x_0)$ and, at the end of this section, a special case of such pseudometric gives, in fact, a uniform discrete metric.

Theorem 1.2.7 *$\pi_1^{Sh}(X, x_0)$ is a topological group.*

Proof. Consider first the map

$$\begin{array}{ccc} \pi_1^{Sh}(X, x_0) & \xrightarrow{r} & \pi_1^{Sh}(X, x_0) \\ [\alpha] & \longmapsto & [\alpha]^{-1} = [\overline{\alpha}] \end{array}$$

If two loops α and β are homotopic with $H(s,t)$ as an homotopy between α and β, then $\overline{\alpha}$ and $\overline{\beta}$ are also homotopic using $G(s,t) = H(1-s,t)$. Hence, if $d([\alpha],[\beta]) = d_0$, then $d([\alpha]^{-1},[\beta]^{-1}) = d_0$. Consequently, r is an isometry.

Therefore, given $[\alpha] \in \pi_1(X, x_0)$ and $\varepsilon > 0$, we have

$$r^{-1}(B([\alpha]^{-1}, \varepsilon)) = B([\alpha], \varepsilon)$$

so r is continuous.

Let us now verify that the map

$$\begin{array}{ccc} \pi_1^{Sh}(X, x_0) \times \pi_1^{Sh}(X, x_0) & \xrightarrow{m} & \pi_1^{Sh}(X, x_0) \\ ([\alpha],[\beta]) & \longmapsto & [\alpha] * [\beta] \end{array}$$

is also continuous.

In order to do that, let $[\alpha], [\beta]$ be in $\pi_1^{Sh}(X, x_0)$ and consider a ball $B([\alpha] * [\beta], \varepsilon)$, for $\varepsilon > 0$. In this case, it is sufficient to verify that the open set $V = B([\alpha], \varepsilon) \times B([\beta], \varepsilon)$ satisfies that $m(V) \subset B([\alpha] * [\beta], \varepsilon)$.

Again, by the properties of the path-homotopy relation this fact is true: Let $[\gamma], [\delta]$ be in $\pi_1^{Sh}(X, x_0)$ such that $d([\alpha],[\gamma]) < \varepsilon$ and $d([\beta],[\delta]) < \varepsilon$. So $\alpha \simeq \gamma$ and $\beta \simeq \delta$ in some ball $B(X, \eta)$ with $\eta < \varepsilon$. Hence, $\alpha * \beta \simeq \gamma * \delta$ in $B(X, \eta)$ hence $d([\alpha] * [\beta], [\gamma] * [\delta]) < \varepsilon$ and $[\gamma] * [\delta]$ is in $B([\alpha] * [\beta], \varepsilon)$.

Since the maps r and m are continuous, $\pi_1^{Sh}(X, x_0)$ is a topological group. ◻

Lemma 1.2.8 *Let $f : (X, x_0) \to (Y, y_0)$ be a continuous map between two pointed compact metric spaces, such that $f(x_0) = y_0$. Then, the induced map between fundamental groups $f_* : \pi_1^{Sh}(X, x_0) \to \pi_1^{Sh}(Y, y_0)$ is uniformly continuous.*

Proof. As usual, we consider X, Y as closed subsets of the Hilbert cube Q. Since Q is an absolute extensor for metric spaces, there exists a continuous extension $F : (Q, x_0) \to (Q, y_0)$ of f, i.e., a map such that $F|_X = f$. Hence, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $F(B(X,\delta)) \subset B(Y,\varepsilon)$.

On the other hand, the induced map f_* is defined by

$$\begin{array}{rccc} f_*: & \pi_1^{Sh}(X,x_0) & \longrightarrow & \pi_1^{Sh}(Y,y_0) \\ & [\alpha] & \longmapsto & [f\circ\alpha] \end{array}$$

Let $[\alpha],[\beta]$ be in $\pi_1^{Sh}(X,x_0)$ such that $d([\alpha],[\beta]) < \delta$, which means $\alpha \simeq \beta$ in $B(X,\delta)$. Applying F (that coincides with f over X) we obtain $f\circ\alpha = F\circ\alpha \simeq F\circ\beta = f\circ\beta$ in $B(Y,\varepsilon)$, so $d(f_*([\alpha]), f_*([\beta])) < \varepsilon$, hence f_* is uniformly continuous. □

Theorem 1.2.9 *Let (X,x_0) and (Y,y_0) be two compact metric spaces with the same pointed homotopy type. Then, $\pi_1^{Sh}(X,x_0)$ and $\pi_1^{Sh}(Y,y_0)$ are uniformly homeomorphic.*

Proof. Let $f : (X,x_0) \to (Y,y_0)$ and $g : (Y,y_0) \to (X,x_0)$ be inverse maps in homotopy. Hence, $g_*\circ f_* = 1_{\pi_1^{Sh}(X,x_0)}$ and $f_*\circ g_* = 1_{\pi_1^{Sh}(Y,y_0)}$ and, by the previous lemma, f_* and g_* are uniformly continuous. □

Corollary 1.2.10 *Let X, Y be to compact metric spaces. If there exists an homeomorphism $h : X \to Y$ then $\pi_1^{Sh}(X,x_0)$ and $\pi_1^{Sh}(Y,h(x_0))$ are uniformly homeomorphic.*

Proof. Immediate from theorem 1.2.9. □

Remark 1.2.11 Theorem 1.2.9 allows us to ensure that the construction of the pseudoultrametric does not depend on the embedding of X in the Hilbert cube Q.

Finally, we prove that the pseudoultrametric on $\pi_1(X,x_0)$ answers affirmatively the question of the existence of a group topology making the quotient map continuous.

Theorem 1.2.12 *Let X be a compact metric space and let x_0 be a point in X. The map*

$$\begin{array}{rccc} q: & \Omega(X,x_0) & \longrightarrow & \pi_1^{Sh}(X,x_0) \\ & \alpha & \longmapsto & [\alpha] \end{array}$$

is uniformly continuous.

Proof. Since X is metric (with μ as its metric) and S^1 is compact, let us consider in the space of loops $\Omega(X, x_0)$ the metric $\eta(\alpha,\beta) = \max\{\mu(\alpha(t),\beta(t)) : t \in S^1\}$.

Given $\varepsilon > 0$, there exists a polyhedron P such that $X \subset P \subset B(X,\varepsilon)$ and P is a neighbourhood of X. Since P is an ANR, we can apply the result 0.0.5 in order to obtain $\delta > 0$ such that for every pair of maps $f, g : S^1 \to P$, and any δ-homotopy between $f|_A, g|_A$ where A is a closed subset of S^1 there exists an ε-homotopy between f and g which is an extension of the δ-homotopy.

In particular, let α, β be two loops such that $\mu(\alpha,\beta) < \delta$ and consider α and β as loops in the polyhedron P via the inclusion $(X, x_0) \hookrightarrow (P, x_0)$. Take $A = \{1\}$ as a closed subset of S^1 and the obvious homotopy between $\alpha|_A$ and $\beta|_A$, $J : \{1\} \times [0,1] \to P$ such that $J(1,t) = x_0$ for every t in $[0,1]$. Then there exists a homotopy $H : S^1 \times [0,1] \to P$ such that $H(s,0) = \alpha(s)$ and $H(s,1) = \beta(s)$. In addition, $H(1,t) = x_0$ for every t in $[0,1]$.

Since $P \subset B(X,\varepsilon)$, the homotopy H in P can be viewed also as a homotopy in $B(X,\varepsilon)$, hence $d([\alpha],[\beta]) < \varepsilon$ and q is uniformly continuous. □

We have already remarked that $\pi_1^{Sh}(X, x_0)$ is a pseudometric space that, in general, is not a metric space. We show a relevant case in which $\pi_1^{Sh}(X, x_0)$ is a metric space.

Theorem 1.2.13 *Let X be an ANR space. Then $\pi_1^{Sh}(X, x_0)$ is a uniform discrete metric space. That is, there exists $\varepsilon_0 > 0$ such that if $d([\alpha],[\beta]) < \varepsilon_0$ then $[\alpha] = [\beta]$.*

Proof. Since $X \subset Q$ is an ANR space, there exists an open neighborhood U such that $X \subset U \subset Q$ and there exists a retraction map $r : U \to X$.

Let us consider $\varepsilon_0 > 0$ such that $B(X, \varepsilon_0) \subset U$ and let $[\alpha], [\beta]$ be in $\pi_1^{Sh}(X, x_0)$ such that $d([\alpha],[\beta]) < \varepsilon_0$. We shall check that $[\alpha] = [\beta]$.

Since $d([\alpha],[\beta]) < \varepsilon_0$, there exists a homotopy $H : S^1 \times [0,1] \to B(X, \varepsilon_0) \hookrightarrow U$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$ and $H(1,t) = x_0$ for all t in $[0,1]$. Composing the homotopy with the retraction, we obtain $G = r \circ H : S^1 \times [0,1] \to X$ such that $G(s,0) = r(\alpha(s)) = \alpha(s)$, $G(s,1) = r(\beta(s)) = \beta(s)$ and $G(1,t) = r(x_0) = x_0$ for all t in $[0,1]$. Hence, G is a loop-homotopy in X between α and β so $[\alpha] = [\beta]$.

In particular, if $d([\alpha],[\beta]) = 0$ then $[\alpha] = [\beta]$ so $\pi_1^{Sh}(X, x_0)$ is a metric space. □

The pseudoultrametric is trivially an ultrametric when the topology is discrete, such as in the case of ANR's (as shown above). A less trivial case in which it is an ultrametric is in the following class of spaces.

Definition 1.2.14 A compact metric pointed space (X, x_0) is said to be *shape-injective* if the canonical homomorphism $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ is a monomorphism.

Proposition 1.2.15 *If (X, x_0) is shape-injective, then d is an ultrametric.*

Proof. We only need to verify that if $d([\alpha],[\beta]) = 0$, then $[\alpha] = [\beta]$.

If $d([\alpha],[\beta]) = 0$, then $\alpha \simeq \beta$ in $B(X,\varepsilon)$ for every $\varepsilon > 0$. That is, $\langle\alpha\rangle = \langle\beta\rangle$, so $\varphi([\alpha]) = \varphi([\beta])$. Using the shape-injectivity, then $[\alpha] = [\beta]$. □

Corollary 1.2.16 *Let* (X, x_0) *be a pointed compact of* $\mathbb{R}^2$*. Then* d *is an ultrametric on* $\pi_1(X, x_0)$.

Proof. As a consequence of the main result of [35], X is shape-injective. Thus, the result is obtained by applying Proposition 1.2.15 above. □

Generalization to arbitrary topological spaces X

If we pretend to generalize methods (and to obtain analogous results) as above for spaces X not necessarily compact metric, we need to use the notion of generalized pseudoultrametric.

Let

$$\underline{\mathbf{p}} = \{p_\lambda\}_{\lambda\in\Lambda} : (X, x_0) \longrightarrow \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\}$$

be an $\mathbf{HPol}_*$-expansion of (X, x_0). Given a loop $\alpha \in \Omega(X, x_0)$, for each $\lambda \in \Lambda$ we shall denote $\alpha_\lambda = p_\lambda \circ \alpha$ the correspondent loop in the polyhedron X_λ with base point x_λ. Let us consider the usual relation of homotopy of loops in each $\Omega(X_\lambda, x_\lambda)$.

If $\alpha, \beta \in \Omega(X, x_0)$, let

$$d(\alpha, \beta) = \{\lambda \in \Lambda : \alpha_\lambda \simeq \beta_\lambda \text{ in } X_\lambda\}.$$

Remark 1.2.17 It is obvious that $d(\alpha, \beta) \in \mathcal{P}(\Lambda)$. But we also have the relation that if $\alpha_\lambda \simeq \beta_\lambda$ in X_λ for some $\lambda \in \Lambda$, then $\alpha_{\lambda'} \simeq \beta_{\lambda'}$ in $X_{\lambda'}$ for every $\lambda' \in \Lambda$ such that $\lambda' \leq \lambda$.

We have

$$p_\lambda \circ \alpha = \alpha_\lambda \simeq \beta_\lambda = p_\lambda \circ \beta$$

and since $p_{\lambda,\lambda'} \circ p_\lambda = p_{\lambda'}$ for $\lambda' \leq \lambda$,

$$\alpha_{\lambda'} = p_{\lambda'} \circ \alpha = p_{\lambda,\lambda'} \circ p_\lambda \circ \alpha \simeq p_{\lambda,\lambda'} \circ p_\lambda \circ \beta = p_{\lambda'} \circ \beta = \beta_{\lambda'}$$

This remark justifies the use of lower classes to define a topology on $\pi_1(X, x_0)$.

Theorem 1.2.18 *Let* (X, x_0) *be a pointed topological space, and*

$$\boldsymbol{p} = \{p_\lambda\}_{\lambda\in\Lambda} : (X, x_0) \longrightarrow \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\}$$

be an $\boldsymbol{HPol}_*$*-expansion of* (X, x_0)*. For each* $[\alpha], [\beta] \in \pi_1(X, x_0)$*, the map*

$$d : \pi_1(X, x_0) \times \pi_1(X, x_0) \longrightarrow (\mathcal{L}(\Lambda), \leq)$$

given by the formula

$$d(\alpha,\beta) = \{\lambda \in \Lambda : \alpha_\lambda \simeq \beta_\lambda \text{ in } X_\lambda\}$$

defines a generalized pseudoultrametric on X.

Proof. First of all, let us show that the formula is well-defined. Let $\alpha', \beta' \in \Omega(X, x_0)$ such that $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$. Since $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ in X, it follows that

$$\alpha'_\lambda = p_\lambda \circ \alpha' \simeq p_\lambda \circ \alpha = \alpha_\lambda$$

and

$$\beta'_\lambda = p_\lambda \circ \beta' \simeq p_\lambda \circ \beta = \beta_\lambda$$

for all $\lambda \in \Lambda$. Using the transitive property of the homotopy relation we obtain $d(\alpha,\beta) = d(\alpha',\beta')$.

Using again the properties of the homotopy relation, it is very easy to check the properties of the pseudoultrametric required in definition 0.0.3. □

Theorem 1.2.19 *Given a pointed topological space (X, x_0) and an **HPol**$_*$-expansion*

$$\mathbf{p} = \{p_\lambda\}_{\lambda\in\Lambda} : (X, x_0) \longrightarrow \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\},$$

of (X, x_0), there is a topology T_d in $\pi_1(X, x_0)$ such that $(\pi_1(X, x_0), T_d)$ is a topological group. Specifically, this T_d is the topology induced by the pseudoultrametric d, and has as basis the family

$$\mathcal{B} = \{B_\Delta([\alpha]) : [\alpha] \in \pi_1(X, x_0), \Delta \in \mathcal{L}(\Lambda)^*\}$$

where $B_\Delta([\alpha]) = \{[\beta] \in \pi_1(X, x_0) : d(\alpha,\beta) \leq \Delta\}$.

Proof. Let us check that the family $\mathcal{B}$ is a base of a topology in $\pi_1(X, x_0)$. It is obvious that every $[\alpha] \in \pi_1(X, x_0)$ is in some element of $\mathcal{B}$. Take $B_{\Delta_1}([\alpha]), B_{\Delta_2}([\beta]) \in \mathcal{B}$ with nonempty intersection. Then there exists some $[\gamma] \in \pi_1(X, x_0)$ such that $d(\alpha,\gamma) \leq \Delta_1$ and $d(\beta,\gamma) \leq \Delta_2$. Since $\mathcal{L}(\Lambda)^*$ is downward directed, there exists $\Delta_3 \neq \Lambda$ such that $\Delta_3 \leq \Delta_1, \Delta_2$. Let us show that $B_{\Delta_3}([\gamma]) \subseteq B_{\Delta_1}([\alpha]) \cap B_{\Delta_2}([\beta])$.

Take $[\delta] \in B_{\Delta_3}([\gamma])$. Then, $d(\delta,\gamma) \leq \Delta_3 \leq \Delta_1$. Since $d(\alpha,\gamma) \leq \Delta_1$ and d is a pseudoultrametric (property 3) we obtain that $d(\delta,\alpha) \leq \Delta_1$, so $[\delta] \in B_{\Delta_1}([\alpha])$. Analogously, $[\delta] \in B_{\Delta_2}([\beta])$.

We denote by T_d the topology induced by the base $\mathcal{B}$. Let us show that $(\pi_1(X, x_0), T_d)$ is a topological group. Recall that in order to do that, we should prove that the inversion and the multiplication maps are continuous.

Let us start by showing that the inversion map

$$\begin{array}{rccc} r: & \pi_1(X, x_0) & \longrightarrow & \pi_1(X, x_0) \\ & [\alpha] & \longmapsto & [\alpha]^{-1} = [\overline{\alpha}] \end{array}$$

is continuous. Take any $[\alpha] \in \pi_1(X, x_0)$. We show that

$$r(B_\Delta([\alpha])) \subseteq B_\Delta([\alpha]^{-1}).$$

Take $[\beta] \in B_\Delta([\alpha])$, that is $d(\alpha, \beta) \le \Delta$. We assert that $d(\alpha, \beta) = d(\overline{\alpha}, \overline{\beta})$, hence $d(\overline{\alpha}, \overline{\beta}) \le \Delta$ and $[\beta]^{-1} \in B_\Delta([\alpha]^{-1})$.

To prove the assertion, first observe that

$$\overline{\alpha}_\lambda(t) = p_\lambda(\overline{\alpha}(t)) = p_\lambda(\alpha(1-t)) = \overline{(p_\lambda \circ \alpha)}(t)$$

for any $\lambda \in d(\alpha, \beta)$. Thus, we have

$$\alpha_\lambda = p_\lambda \circ \alpha \simeq p_\lambda \circ \beta = \beta_\lambda \text{ in } X_\lambda$$

so

$$\overline{\alpha}_\lambda = p_\lambda \circ \overline{\alpha} = \overline{p_\lambda \circ \alpha} \simeq \overline{p_\lambda \circ \beta} = p_\lambda \circ \overline{\beta} = \overline{\beta}_\lambda \text{ in } X_\lambda.$$

It follows immediately that $d(\alpha, \beta) = d(\overline{\alpha}, \overline{\beta})$, as we wanted to show.

For the continuity of the multiplication map

$$\begin{array}{rccc} m: & \pi_1(X, x_0) \times \pi_1(X, x_0) & \longrightarrow & \pi_1(X, x_0) \\ & ([\alpha], [\beta]) & \longmapsto & [\alpha] * [\beta] \end{array}$$

take $[\alpha], [\beta] \in \pi_1(X, x_0)$. Let us show that

$$m(B_\Delta([\alpha]) \times B_\Delta([\beta])) \subseteq B_\Delta([\alpha] * [\beta]).$$

Take $([\gamma], [\delta]) \in B_\Delta([\alpha]) \times B_\Delta([\beta])$, let us check that $[\gamma] * [\delta] \in B_\Delta([\alpha] * [\beta])$. For each $\lambda \in \Delta$, since $d(\alpha, \gamma) \le \Delta$ and $d(\beta, \delta) \le \Delta$ we have that

$$p_\lambda \circ \alpha \simeq p_\lambda \circ \gamma \text{ and } p_\lambda \circ \beta \simeq p_\lambda \circ \delta \text{ in } X_\lambda$$

It follows that, in X_λ,

$$p_\lambda \circ (\alpha * \beta) = (p_\lambda \circ \alpha) * (p_\lambda \circ \beta) \simeq (p_\lambda \circ \gamma) * (p_\lambda \circ \delta) = p_\lambda \circ (\gamma * \delta)$$

so $\lambda \in d(\alpha * \beta, \gamma * \delta)$ and therefore $d(\alpha * \beta, \gamma * \delta) \le \Delta$.

Because of the definition of the concatenation of loops, is not difficult to check the fact we used before that $p_\lambda \circ (\alpha * \beta) = (p_\lambda \circ \alpha) * (p_\lambda \circ \beta)$. □

Remark 1.2.20 If (X, x_0) is a compact metric space, the canonical topology induced by d coincides with the topology constructed above. It is enough to consider again the $\mathbf{HPol}_*$-expansion

$$\underline{\mathbf{p}} = \{p_\varepsilon\} : (X, x_0) \to (\underline{\mathbf{X}, \mathbf{x_0}}) = \{(X_\varepsilon, x_0), p_{\varepsilon\varepsilon'}, (\mathbb{R}^+)^{-1}\}$$

where $(\mathbb{R}^+)^{-1}$ is the set of non-negative real numbers with the reverse usual order, $X_\varepsilon = B(X, \varepsilon)$ (also pointed in x_0) for each $\varepsilon > 0$, and $p_\varepsilon : X \to X_\varepsilon$ and $p_{\varepsilon\varepsilon'} : X_\varepsilon \to X_{\varepsilon'}$ for $\varepsilon' > \varepsilon > 0$ are the corresponding (pointed) inclusions.

As in the case of the shape group, the main drawback of this topology is that it depends on the chosen **HPol**$_*$-expansion, as shown in the following example.

Example 1.2.21 Let (X, x_0) be a pointed topological space and

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \longrightarrow \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda,\lambda'}, \Lambda\}$$

be an **HPol**$_*$-expansion. Consider the usual order product in $\Lambda \times \Lambda$ and define the inverse system

$$(\mathbf{X}', \mathbf{x}'_0) = \{(X_{(\lambda_1,\lambda_2)}, x_{(\lambda_1,\lambda_2)}), p_{(\lambda_1,\lambda_2),(\lambda'_1,\lambda'_2)}, \Lambda \times \Lambda\}$$

where $(X_{(\lambda_1,\lambda_2)}, x_{(\lambda_1,\lambda_2)}) = (X_{\lambda_1}, x_{\lambda_1})$, $p_{(\lambda_1,\lambda_2),(\lambda'_1,\lambda'_2)} = p_{\lambda_1,\lambda'_1}$. Then

$$\underline{\mathbf{p}}' = \{p'_{(\lambda_1,\lambda_2)}\} : (X, x_0) \longrightarrow \underline{(\mathbf{X}', \mathbf{x}'_0)}$$

is an **HPol**$_*$-expansion, where $p'_{\lambda_1,\lambda_2} = p_{\lambda_1}$. Moreover, for every pointed topological space (X, x_0), the topology induced on $\pi_1(X, x_0)$ by the pseudoultrametric associated to the **HPol**$_*$-expansion $\mathbf{p}'$ is the discrete one.

In order to see that, fix a non maximal element $\lambda_0 \in \Lambda$ and take

$$D = \{(\lambda, \lambda_0) \mid \lambda \in \Lambda\}$$

and let $\Delta_D \in \mathcal{L}(\Lambda \times \Lambda)^*$ be the minimal lower class containing D, i.e.,

$$\Delta_D = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda : \lambda_2 \leq \lambda_0\}.$$

Now, if $[\alpha] \in \pi_1(Y, y_0)$, then $B_{\Delta_D}([\alpha]) = \{[\alpha]\}$.

Because of this defect, we must introduce a slightly modified base for a topology which is again a group topology but independent on the selected expansion. As in the previous section, we consider again the lower class (λ) generated by an element $\lambda \in \Lambda$.

Let (X, x_0) be an arbitrary pointed topological space, and

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \to \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda\}$$

an **HPol**$_*$-expansion of (X, x_0). Given $[\alpha] \in \pi_1(X, x_0)$, we consider balls

$$B_{(\lambda)}([\alpha]) = \{[\beta] \in \pi_1(X, x_0) \mid p_\lambda \circ \alpha \simeq p_\lambda \circ \beta \text{ in } X_\lambda\}.$$

Then the family

$$\{B_{(\lambda)}([\alpha]) \mid [\alpha] \in \pi_1(X, x_0), \lambda \in \Lambda\}$$

is a base for a topology on $\pi_1(X, x_0)$, and we call it the *intrinsic topology.* Indeed, given $B_{(\lambda_1)}([\alpha])$ and $B_{(\lambda_2)}([\beta])$ with non-empty intersection, it is enough to take a common element $[\gamma]$ of this two balls, and $\lambda_3 \geq \lambda_1, \lambda_2$ (this λ_3 exists because Λ is a directed set) to get

$$B_{(\lambda_3)}([\gamma]) \subseteq B_{(\lambda_1)}([\alpha]) \cap B_{(\lambda_2)}([\beta]).$$

The main reason to use this topology instead of the canonical one, is the following:

Proposition 1.2.22 *The intrinsic topology on $\pi_1(X, x_0)$ is independent on the fixed **HPol**$_*$-expansion of (X, x_0) and it coincides with the topology given by the pseudoultrametric for compact metric spaces (X, x_0).*

Proof. Let

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \to \underline{(\mathbf{X}, \mathbf{x_0})} = \{(X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda\}$$

and

$$\underline{\mathbf{p}}' = \{p'_\lambda\} : (X, x_0) \to \underline{(\mathbf{X}', \mathbf{x}'_0)} = \{(X'_\mu, x'_\mu), p'_{\mu\mu'}, M\}$$

be two **HPol**$_*$-expansions of (X, x_0). Then, there exists a map of inverse systems (induced by the identity $i : (X, x_0) \to (X, x_0)$) in **HPol**$_*$

$$\underline{\mathbf{i}} = (i_\mu, \phi) : \underline{(\mathbf{X}, \mathbf{x_0})} \to \underline{(\mathbf{X}', \mathbf{x}'_0)}$$

with $\phi : M \to \Lambda$ and $i_\mu : X_{\phi(\mu)} \to X'_\mu$ such that

$$i_\mu \circ p_{\phi(\mu)} \simeq p'_\mu.$$

Given a ball $B_{(\mu)}([\alpha])$ for the intrinsic topology induced by the **HPol**$_*$-expansion $\underline{(\mathbf{X}', \mathbf{x}'_0)}$, take some element $[\beta] \in B_{(\mu)}([\alpha])$ and $\lambda = \phi(\mu)$. Hence,

$$B_{(\phi(\mu))}([\beta]) \subseteq B_{(\mu)}([\alpha]).$$

Indeed, for any $[\gamma] \in B_{(\phi(\mu))}([\beta])$:

$$p'_\mu \circ \gamma \simeq i_\mu \circ p_{\phi(\mu)} \circ \gamma \simeq i_\mu \circ p_{\phi(\mu)} \circ \beta \simeq p'_\mu \circ \beta \simeq p'_\mu \circ \alpha$$

and consequently $B_{(\mu)}([\alpha])$ is open for the intrinsic topology induced by the **HPol**$_*$-expansion $\underline{(\mathbf{X}, \mathbf{x_0})}$.

The converse is analogous reasoning with the map of inverse systems

$$\underline{\mathbf{j}} = (j_\lambda, \psi) : \underline{(\mathbf{X}', \mathbf{x}'_0)} \to \underline{(\mathbf{X}, \mathbf{x_0})}$$

with $\psi : \Lambda \to M$ and $j_\lambda : X'_{\psi(\lambda)} \to X_\lambda$ such that

$$j_\lambda \circ p'_{\psi(\lambda)} \simeq p_\lambda.$$

In conclusion, the intrinsic topology induced by two different expansions are equivalent. The last assertion of the proposition is immediate from the fact that in a compact metric case it is also possible to obtain expansion represented by inverse sequences (that is, indexed by the naturals). □

Remark 1.2.23 All the arguments are also valid for higher homotopy groups $\pi_k(X, x_0)$, $k \geq 2$, but we have restricted ourselves to the fundamental group.

The topology on $\pi_1(X, x_0)$ is induced by the topology on $\check{\pi}_1(X, x_0)$

In this last part of the section, we show that the topology that we have already defined on the fundamental group (either the topology induced by the pseudoultrametric in the compact metric case, or the intrinsic topology in the arbitrary case) is, in some sense, just the reflect of the corresponding topology on the shape group, introduced in the first section of this chapter.

Let us consider the canonical morphism which relates the fundamental and the shape groups of a pointed space (X, x_0) :

$$\begin{array}{rccc} \varphi : & \pi_1(X, x_0) & \longrightarrow & \check{\pi}_1(X, x_0) \\ & [\alpha] & \longmapsto & \langle \alpha \rangle \end{array}$$

which is neither injective nor surjective. Recall that, given an **HPol**$_*$-expansion

$$\underline{\mathbf{p}} = \{p_\lambda\} : (X, x_0) \to (\underline{\mathbf{X}, \mathbf{x_0}}) = \{(X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda\},$$

$\langle \alpha \rangle$ can be represented by a Λ-sequence $\{[\alpha_\lambda]\}$ where each α_λ is a loop on X_λ. Moreover, $\varphi([\alpha]) = \{[p_\lambda \circ \alpha]\}$. From the construction of the topologies, the following result holds.

Proposition 1.2.24 *The intrinsic topology on $\pi_1(X, x_0)$ is the initial topology on $\pi_1(X, x_0)$ induced by φ and the intrinsic topology on $\check{\pi}_1(X, x_0)$.*

Proof. By definition of initial topology, the topology on $\pi_1(X, x_0)$ induced by φ is generated by sets $\varphi^{-1}(G)$ where G is open of the intrinsic topology on $\check{\pi}_1(X, x_0)$. Let us fix a ball $B_{(\lambda)}(\langle \alpha \rangle)$. Then:

$$\varphi^{-1}(B_{(\lambda)}(\langle \alpha \rangle)) = \{[\beta] \in \pi_1(X, x_0) \mid p_\lambda \circ \beta \simeq \alpha_\lambda \text{ in } X_\lambda\}.$$

Hence, if such a $[\beta]$ does not exist, then $\varphi^{-1}(B_{(\lambda)}(\langle \alpha \rangle)) = \varnothing$. If there is at least one $[\beta]$ as above, then it is clear that $\varphi^{-1}(B_{(\lambda)}(\langle \alpha \rangle)) = B_{(\lambda)}([\beta])$.

Moreover, if $\langle \alpha \rangle = \varphi([\alpha])$, then $\varphi^{-1}(B_{(\lambda)}(\langle \alpha \rangle)) = B_{(\lambda)}([\alpha])$. □

Corollary 1.2.25 *If (X, x_0) is a pointed compact metric space, the topology associated to the pseudoultrametric on $\pi_1(X, x_0)$ is just the initial topology on $\pi_1(X, x_0)$ induced by the topology associated to the ultrametric on $\check{\pi}_1(X, x_0)$.*

Remark 1.2.26 As we established in definition 1.2.3 for the compact metric case, we shall also refer to the intrinsic topology on $\pi_1(X, x_0)$ as the *shape topology* on the fundamental group. We also denote it as $\pi_1^{Sh}(X, x_0)$, either in the compact metric or in the general case.

1.3 Other topologies on $\pi_1(X, x_0)$

We exhibit in this section other topologies already defined in the literature, and explore their relations with the shape topology defined on the fundamental group.

All the definitions in this section remain valid in the arbitrary case, but we shall restrict ourselves to the compact metric case. Sometimes we shall make some remarks about the definitions or results in the general framework. Hence, from now on we deal with X compact metric, unless we explicit other conditions.

1.3.1 Relevant subgroups and equivalence relations on $\pi_1(X, x_0)$

Here we reveal some important subgroups of $\pi_1(X, x_0)$ and some different equivalence relations on the space of loops (or between homotopy classes of loops), all of them playing a role in the literature.

Equivalent loops up to shape

We consider again the canonical homomorphism of groups:

$$\begin{array}{rccc} \varphi: & \pi_1(X, x_0) & \longrightarrow & \check{\pi}_1(X, x_0) \\ & [\alpha] & \longmapsto & \langle \alpha \rangle \end{array}$$

and focus on the kernel of this homomorphism $Ker\varphi$, which is normal in $\pi_1(X, x_0)$.

Definition 1.3.1 Loops belonging to $Ker\varphi$ will be called *null-shape loops* and the quotient

$$\pi_1(X, x_0)/Ker\varphi \cong \varphi(\pi_1(X, x_0))$$

is the group of *shape classes generated by loops* in X.

From the definition of φ, a loop η is a null-shape loop if and only if $i_n \circ \eta$ is a null-homotopic loop in $B(X, \varepsilon_n)$ for every $n \in \mathbb{N}$, where $i_n : X \hookrightarrow B(X, \varepsilon_n)$ and $\{\varepsilon_n\}$ is a decreasing sequence converging to zero.

An equivalent way to define this group is to consider, on $\Omega(X, x_0)$, the relation

$$\alpha R_{Sh} \beta \Leftrightarrow \alpha * \overline{\beta} \text{ is null-shape}$$

which is routinely checked to be an equivalent relation. The quotient set

$$\Omega(X, x_0)/R_{Sh}$$

is a group (under the concatenation of classes of loops as the class of the concatenation) and it coincides with $\varphi(\pi_1(X, x_0))$.

Hurewicz-Dugundji group

In [46], Hurewicz already used the notion of what we could name *discrete homotopy,* but he dealt only with metric spaces. For the general case, we follow the definitions given by Dugundji in [27].

Definitions 1.3.2 Let X be a topological space and let $\mathcal{U}$ be an open covering of X. Let $\alpha, \beta : [0,1] \to X$ be two paths with the same endpoints:

i) α and β are said to be *$\mathcal{U}$-near* if for every $t \in [0,1] \to X$ there exists $U \in \mathcal{U}$ such that $\alpha(t), \beta(t)$ belongs to U.

ii) α is *$\mathcal{U}$-homotopic* to β if there exists an *$\mathcal{U}$-chain* of paths, that is, a finite sequence of paths $\alpha_0 = \alpha, \alpha_1, ..., \alpha_{n-1}, \alpha_n = \beta$ all of them with the same endpoints and such that α_i and α_{i+1} are $\mathcal{U}$-near for every $i = 0, ..., n-1$.

iii) α is *homotopic in the sense of Dugundji* or *D-homotopic* to β if they are $\mathcal{U}$-homotopic for every open covering $\mathcal{U}$ of X.

We have to mention here that other authors have dealt with similar definitions. In particular, the next definitions are introduced in [18]:

Definitions 1.3.3 Let X be a topological space and let $\mathcal{U}$ be an open covering of X. Let $\alpha, \beta : [0,1] \to X$ be two paths with the same endpoints:

i) α and β are *$\mathcal{U}$-close* if there exist partitions $0 = a_0 < a_1 < ... < a_n = 1$ and $b_0 = 0 < b_1 < ... < b_n = 1$ of the unit interval $[0,1]$ and there exist $U_1, ..., U_n$ elements of $\mathcal{U}$ such that $\alpha([a_{i-1}, a_i]), \beta([b_{i-1}, b_i]) \subset U_i$.

ii) α is $\mathcal{U}$-related to β if there exists a finite sequence of paths $\alpha_0 = \alpha, \alpha_1, ..., \alpha_{n-1}, \alpha_n = \beta$ all of them with the same endpoints and such that α_i and α_{i+1} are $\mathcal{U}$-close for every $i = 0, ..., n-1$.

iii) α is *homotopic in the sense of Brazas and Fabel* or *BF-homotopic* to β if they are $\mathcal{U}$-related for every open covering $\mathcal{U}$ of X.

Remark 1.3.4 The authors in [18] introduce definitions 1.3.3 and warned about them are similar to those in 1.3.2. We observe that the correspondent definitions *iii*) of 1.3.2 and 1.3.3 are in fact equivalent, despite definitions *i*) are not. Next lemmas are devoted to prove this.

Lemma 1.3.5 *Let (X, x_0) be a pointed topological space. Then $[\alpha] \subseteq D(\alpha)$ for any $\alpha \in \Omega(X, x_0)$. In other words, if two paths α and β are homotopic in X, then they are D-homotopic, i.e., they are $\mathcal{U}$-homotopic for every open covering $\mathcal{U}$.*

Proof. Let $\alpha, \beta : [0,1] \to X$ be two homotopic paths in X. Then, $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$ and there exists an homotopy $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = \alpha(s)$, $H(s,1) = \beta(s)$, $H(0,t) = \alpha(0) = \beta(0)$ and $H(1,t) = \alpha(1) = \beta(1)$ for all $s, t \in [0,1]$.

Let $\mathcal{U}$ be an open covering of X. Let us consider $\mathcal{V} = \{H^{-1}(U) | U \in \mathcal{U}\}$. So $\mathcal{V}$ is an open cover of the compact set $[0,1] \times [0,1]$. Let $\varepsilon_0 > 0$ be the Lebesgue number for $\mathcal{V}$, so that for every $A \subset [0,1] \times [0,1]$ with $diam(A) < \varepsilon_0$ there exists $V \in \mathcal{V}$ such that $A \subset V$.

In particular, let us fix $\varepsilon < \varepsilon_0$. For every point $(a,b) \in [0,1] \times [0,1]$ there exists $V \in \mathcal{V}$ such that $B((a,b), \varepsilon/2) \subset V$.

Take $N \in \mathbb{N}$ such that $\frac{1}{2^{N+1}} < \frac{\varepsilon}{2}$ (equivalently, $\frac{1}{2^N} < \varepsilon$) and consider the paths which are defined by the homotopy H in each level $\frac{i}{2^{N+2}}$ for $i = 0, 1, ..., 2^{N+2}$:

$$\alpha_i(s) = H\left(s, \frac{i}{2^{N+2}}\right)$$

In order to complete the argument, it is enough to show that α_i with $i = 0, 1, ..., 2^{N+2}$ is an $\mathcal{U}$-chain. Let $s \in [0,1]$, then:

$$\left(s, \frac{i}{2^{N+2}}\right), \left(s, \frac{i+1}{2^{N+2}}\right) \in B\left(\left(s, \frac{i}{2^{N+2}}\right), \frac{1}{2^{N+1}}\right) \subset B\left(\left(s, \frac{i}{2^{N+2}}\right), \frac{\varepsilon}{2}\right) \subset V$$

for some $V \in \mathcal{V}$. But also $V = H^{-1}(U)$ for some $U \in \mathcal{U}$. Hence, $H(s, \frac{i}{2^{N+2}}), H(s, \frac{i+1}{2^{N+2}}) \in U$, so $\alpha_i(s), \alpha_{i+1}(s) \in U$ for some $U \in \mathcal{U}$.

Since the argument shown is valid for every open covering, we infer that α and β are D-homotopic. □

The analogous statement to the previous one which asserts that homotopic paths in X are also BF-homotopic paths is stated and proved in [18] and it can be proved with a similar argument to one shown in the previous result.

Lemma 1.3.6 *Let X be a topological space. If two paths α and β are homotopic in X, then they are BF-homotopic. In particular, they are $\mathcal{U}$-related for every open covering $\mathcal{U}$.*

Proof. See [18]. □

Lemma 1.3.7 *Let α, β be two paths in a topological space X and fix $\mathcal{U}$ an open covering of X. If the paths are $\mathcal{U}$-near then they are $\mathcal{U}$-close.*

Proof. Since α and β are $\mathcal{U}$-near, for every $t \in [0,1]$ there exists $U_t \in \mathcal{U}$ such that $\alpha(t), \beta(t) \in U_t$. Paths are, in particular, continuous functions, so $\alpha^{-1}(U_t)$ and $\beta^{-1}(U_t)$ are open sets in $[0,1]$, so also $V_t = \alpha^{-1}(U_t) \cap \beta^{-1}(U_t)$ is open in $[0,1]$.

The family $\mathcal{V} = \{V_t | t \in [0,1]\}$ is an open covering of the compact space $[0,1]$. We can choose a finite ordered refinement $\mathcal{W}$ of $\mathcal{V}$, that is $\mathcal{W} = \{W_1, ..., W_n\}$ with $w_j \in W_j \cap W_{j+1}$ and $w_j < w_{j+1}$ for $j = 1, ..., n-1$.

So $0 = w_0 < w_1 < ... < w_n < w_{n+1} = 1$ is a partition of $[0,1]$. We use this partition both for α and β in order to satisfy the definition of being $\mathcal{U}$-close:

$$\alpha([w_{k-1}, w_k]) \subset \alpha(W_k) \subset \alpha(V_{t_k}) \subset U_{t_k}$$

and

$$\beta([w_{k-1}, w_k]) \subset \beta(W_k) \subset \beta(V_{t_k}) \subset U_{t_k}$$

because $W_k \subset V_{t_k}$ for some $t_k \in [0,1]$ and $V_{t_k} = \alpha^{-1}(U_{t_k}) \cap \beta^{-1}(U_{t_k})$.

Then, $a_j = b_j = w_j$ for $j = 0, ...n+1$ and U_{t_j} for $j = 1, ..., n+1$ are the partitions and open sets required. Hence, α and β are $\mathcal{U}$-close. □

Proposition 1.3.8 *Let X be a topological space and $\alpha, \beta : [0,1] \to X$ two paths in X with the same endpoints. Then, α and β are D-homotopic if and only if they are BF-homotopic.*

Proof. Let us suppose first that α and β are D-homotopic and let $\mathcal{U}$ be a open covering of X. Since α and β are $\mathcal{U}$-near, there exists an $\mathcal{U}$-chain $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = \beta$. Since each α_i, α_{i+1} are $\mathcal{U}$-near, by lemma 1.3.7 we obtain that α_i and α_{i+1} are $\mathcal{U}$-close, so the same chain is valid in order to obtain that α and β are $\mathcal{U}$-related. Hence, α and β are BF-homotopic.

On the other hand, let us suppose now that α and β are BF-homotopic. Let $\mathcal{U}$ be an open covering of X :

Case 1 (Particular case): α and β are $\mathcal{U}$-related by a trivial chain $\alpha_0 = \alpha, \alpha_1 = \beta$. That is equivalent to α and β being $\mathcal{U}$-close.

From the definition of being $\mathcal{U}$-close, there exist partitions $0 = a_0 < a_1 < ... < a_n = 1$ and $0 = b_0 < b_1 < ... < b_n = 1$ of $[0,1]$ and there exist $U_1, ..., U_n \in \mathcal{U}$ such that $\alpha([a_{i-1}, a_i]), \beta([b_{i-1}, b_i]) \subset U_i$ for every $i = 1, ..., n$.

Let β' be a reparametrization of β such that $\beta'|_{[a_{i-1}, a_i]} = \beta|_{[b_{i-1}, b_i]}$. As β and β' are the same path up to a reparametrization, then they are homotopic. By lemma 1.3.5, we obtain that β' is $\mathcal{U}$-homotopic to β. Hence, there exists an $\mathcal{U}$-chain $\beta_0 = \beta', \beta_1, ..., \beta_m = \beta$ with β_i, β_{i+1} $\mathcal{U}$-near for $i = 0, 1, ..., m-1$.

Observe also that for every $t \in [0,1]$ there exists $i_0 \in \{1, ..., n\}$ such that $t \in [i_0 - 1, i_0]$, so $\alpha(t), \beta'(t) \in U_{i_0}$. Therefore, α and β' are $\mathcal{U}$-near.

Consequently, letting $\alpha_0 = \alpha$ and $\alpha_i = \beta_{i-1}$ for $i = 1, ..., m$ we obtain an $\mathcal{U}$-chain between α and β, so they are D-homotopic.

Case 2 (General case): α and β are $\mathcal{U}$-related by a finite sequence of paths $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = \beta$. Each two consecutive paths α_i, α_{i+1} are $\mathcal{U}$-close, so we can apply the previous case to them, for $i = 0, ..., m-1$. We obtain a finite collection of $\mathcal{U}$-chains, so the whole collection is itself an $\mathcal{U}$-chain. Hence, α and β are D-homotopic. □

Remark 1.3.9 In the compact metric case (X,d), which is the main case throughout this work, equivalent definitions of 1.3.2 in terms of the metric had already been given by Hurewicz in [46]: Given $\varepsilon > 0$, two paths α, β are ε-near if $d(\alpha(t), \beta(t)) < \varepsilon$ for all $t \in [0,1]$. The paths are ε-homotopic if there exists a ε-chain joining them, that is, a finite sequence of paths $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = \beta$ where two consecutive paths are ε-near. Finally, α and β are D-homotopic if they are ε-homotopic for all $\varepsilon > 0$.

It is a straightforward exercise to check that the D-homotopy relation is in fact an equivalent relation over the space of paths. In view of 1.3.8, also BF-homotopy relation is an equivalence relation with the same classes of equivalence.

Moreover, if we restrict ourselves to loops in the pointed space (X, x_0), definition iii) of 1.3.2 induces an equivalent relation over $\Omega(X, x_0)$, which we denote by $\simeq_D$. We shall denote by $D(\alpha)$ the D-homotopy class generated by a loop α.

Definition 1.3.10 The quotient space $\Omega(X, x_0)/ \simeq_D$ with the operation $D(\alpha) * D(\beta) = D(\alpha * \beta)$ is the *group of Dugundji* of (X, x_0). We shall denote it by $\chi(X, x_0)$.

Of course, the first thing which has to be shown is that $\chi(X, x_0)$ is in fact a group:

Proposition 1.3.11 *Let (X, x_0) be a pointed topological space. Then, $(\chi(X, x_0), *)$ is a group.*

Proof. The concatenation is well-defined over the D-classes: Let $\alpha, \alpha', \beta, \beta' \in \Omega(X, x_0)$ be such that $D(\alpha) = D(\alpha')$ and $D(\beta) = D(\beta')$. Let us check that $D(\alpha) * D(\beta) = D(\alpha') * D(\beta')$.

For any open cover $\mathcal{U}$, there exist $\mathcal{U}$-chains $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = \alpha'$ and $\beta_0 = \beta, \beta_1, ..., \beta_m = \beta'$. Without loss of generality, we can suppose $n > m$ and define $\gamma_i = \alpha_i * \beta_i$ for $0 \le i \le m$ and $\gamma_i = \alpha_i * \beta'$ for $m+1 \le i \le n$. This leads us to a $\mathcal{U}$-chain between $\alpha * \beta$ and $\alpha' * \beta'$, so they are $\mathcal{U}$-homotopic. Since $\mathcal{U}$ is arbitrary:

$$D(\alpha) * D(\beta) = D(\alpha * \beta) = D(\alpha' * \beta') = D(\alpha') * D(\beta').$$

For associative property, let α, β and γ be loops in (X, x_0). The path $\alpha * (\beta * \gamma)$ is not the same path as $(\alpha * \beta) * \gamma$, but they are the same up to a reparametrization. Hence, they are homotopic and applying lemma 1.3.5, they in fact are D-homotopic. Consequently:

$$\begin{aligned} D(\alpha) * (D(\beta) * D(\gamma)) =& D(\alpha) * D(\beta * \gamma) = D(\alpha * (\beta * \gamma)) = \\ =& D((\alpha * \beta) * \gamma) = D(\alpha * \beta) * D(\gamma) = (D(\alpha) * D(\beta)) * D(\gamma). \end{aligned}$$

Finally, it is clear that if c_{x_0} denotes the constant path at x_0, the D-homotopy class $D(c_{x_0})$ is the identity element and the D-homotopy class $D(\overline{\alpha})$ is the opposite of $D(\alpha)$, where $\overline{\alpha}(t) = \alpha(1-t)$. □

Remark 1.3.12 By lemma 1.3.5, $[\alpha] \subseteq D(\alpha)$ (as sets) for every $\alpha \in \Omega(X, x_0)$. This observation allows to define the next homomorphism of groups:

$$\begin{array}{rccc} \Psi: & \pi_1(X, x_0) & \longrightarrow & \chi(X, x_0) \\ & [\alpha] & \longmapsto & D(\alpha) \end{array}$$

Definition 1.3.13 The elements of $\pi_1(X, x_0)$ which are trivial as D-classes shall be called *null-Dugundji loops,* and we denote by $\nu(X, x_0)$ the set of these elements.

The set $\nu(X, x_0)$ is clearly a normal subgroup with the operation of concatenation of D-homotopy classes. That is:

$$\nu(X, x_0) = \{[\alpha] \in \pi_1(X, x_0) | D(\alpha) = D(c_{x_0})\} \trianglelefteq \pi_1(X, x_0).$$

Equivalently, $\nu(X, x_0)$ is formed by the D-homotopy classes of loops α which are $\mathcal{U}$-homotopic to c_{x_0} for every open cover $\mathcal{U}$ of X.

In [18], the authors denote by $\nu(\mathcal{U}, x_0)$ the subgroup of the elements of $\pi_1(X, x_0)$ which are $\mathcal{U}$-related to the constant path. In view of the precedent lemmas:

$$\nu(X, x_0) = \bigcap_{\mathcal{U} \in Cov(X)} \nu(\mathcal{U}, x_0)$$

For the compact metric case, Hurewicz in [46] denotes this subgroups as $\pi_\varepsilon(X, x_0)$, so that

$$\nu(X, x_0) = \bigcap_{\varepsilon > 0} \pi_\varepsilon(X, x_0).$$

Moreover, it is not hard to see that $Ker\Psi = \nu(X, x_0)$. Since Ψ is obviously onto, by applying the first isomorphism theorem we get to the original definition of the Dugundji group which Hurewicz made in the compact metric case:

$$\chi(X, x_0) = \frac{\pi_1(X, x_0)}{\nu(X, x_0)}.$$

Original definitions of Hurewicz and Dugundji show that shape-flavour concepts were already in literature. In fact, these are weaker than shape morphisms induced by a path but as we will show, sometimes they coincide.

Lemma 1.3.14 *Let (X, x_0) be a pointed topological space. Then $D(\alpha) \subseteq \langle \alpha \rangle$ for every loop $\alpha \in \Omega(X, x_0)$.*

Proof. We make use of the Čech expansion of a topological space here. Let us fix $\alpha \in \Omega(X, x_0)$ and take a loop β such that $\beta \in D(\alpha)$. For an arbitrary normal covering $\mathcal{U}$ of X, there exists an $\mathcal{U}$-chain $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = \beta$ joining α and β. Considering the projections of the paths of this chain, we obtain a finite sequence of paths $p_{\mathcal{U}}(\alpha_i)$. Since for two consecutive paths α_i and α_{i+1}, their projections are contiguous maps in $X_{\mathcal{U}}$, they are homotopic.

So we get $p_{\mathcal{U}}(\alpha_0) \simeq p_{\mathcal{U}}(\alpha_1) \simeq ... \simeq p_{\mathcal{U}}(\alpha_n)$. By the transitivity of the homotopy relation, we obtain:

$$p_{\mathcal{U}}(\alpha) = p_{\mathcal{U}}(\alpha_0) \simeq p_{\mathcal{U}}(\alpha_n) = p_{\mathcal{U}}(\beta)$$

which means $\beta \in \langle\alpha\rangle$ since $\mathcal{U}$ was arbitrary. □

Remark 1.3.15 As it is well-known, not every shape morphism is generated by a map in the space lying in the space X, so the equality above is not probably to be satisfied. For this reason, the lemma above should be read as: if α and β are two D-homotopic loops, then their shape classes coincide.

Example 1.3.16 Let us consider Griffiths' space $\mathcal{G}$ from previous examples and the loop γ which runs through the circles alternating the copies of the base spaces HE. This loop satisfies $D(\gamma) = D(c_{x_0})$ but γ is not homotopically trivial, so $[\gamma] \subsetneq D(\gamma)$.

This example also shows that despite the map Ψ is always onto, it does not need to be injective.

Example 1.3.17 Let us consider X as the *rotated topologist's sine curve* about the vertical axis in order to obtain an outsider cylinder generated by the revolution of the accumulation of $\sin(\frac{1}{x})$. Moreover, to obtain a path-connected space, let us link the two pieces with a simple arc (see the figure 1.4). Constructions in this way receive the name of *sombrero spaces,* and they were studied, for example, in [34].

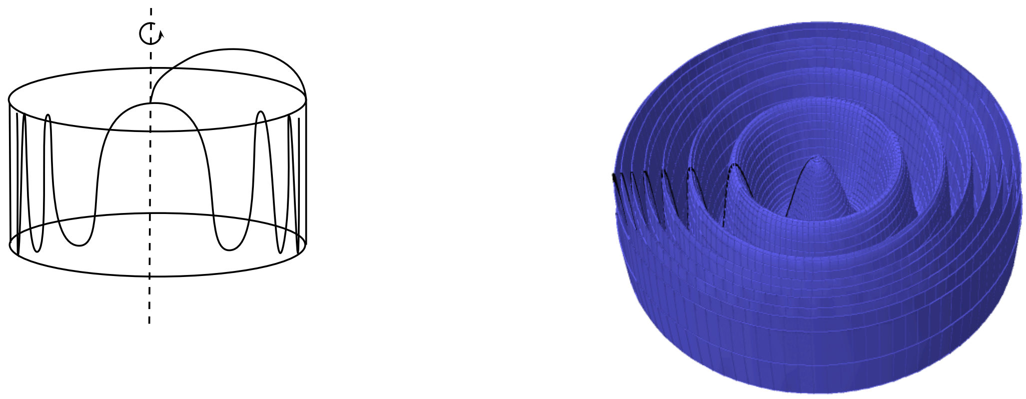


Figure 1.4: The sketched sombrero space, and its surface portion.

This space X is compact, metric and path-connected (but it is not locally path-connected). Let x_0 be a base-point located in the surface portion of X, and let us consider a loop α at x_0 running through the arc, circles around the exterior wall and comes back to x_0.

Using Borsuk's neighbourhood system, $i_n \circ \alpha$ is null-homotopic in $B(X, \varepsilon_n)$ for any $n \in \mathbb{N}$. Hence, $\langle \alpha \rangle = \langle c_{x_0} \rangle$. On the other hand, it should be clear that there is not any finite sequence $\alpha_0 = c_{x_0}, \alpha_1, \ldots, \alpha_n = \alpha$ of $\mathcal{U}$-near loops for a suitable open covering $\mathcal{U}$ with small enough open sets (see [34] for the details of the argument). Thus, $D(\alpha) \neq D(c_{x_0})$.

The canonical homomorphism φ between the fundamental and shape groups factorizes via the homomorphism Ψ defined above:

$$\begin{array}{ccc} \pi_1(X, x_0) & & \\ {\scriptstyle\Psi}\downarrow & \searrow^{\varphi} & \\ \chi(X, x_0) & \xrightarrow[\Phi]{} & \check{\pi}_1(X, x_0) \end{array} \tag{1.1}$$

since $[\alpha] \subseteq D(\alpha) \subseteq Sh(\alpha)$ for any $\alpha \in \Omega(X, x_0)$, from lemmas 1.3.5 and 1.3.14.

Remark 1.3.18 The map Φ is not always injective, in view of example 1.3.17. Let us recall how injectivity of this homomorphism is read:

Φ is injective if and only if $Sh(\alpha) = Sh(\beta)$ implies $D(\alpha) = D(\beta)$ for every $\alpha, \beta \in \Omega(X, x_0)$.

The latter means that when Ψ is injective, the D-homotopy class of a loop is exactly the same as its shape morphism. So in this cases, different representatives of a shape morphism can be described as homotopic paths by internal discrete homotopies.

Small loops: the Spanier group

The following definitions already appeared in [83] and several authors had used them before.

Definition 1.3.19 Let $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ be a open covering of X. The *Spanier group of X with respect to $\mathcal{U}$* is the subgroup of $\pi_1(X, x_0)$ generated by the elements of the form $[\gamma * \alpha * \overline{\gamma}]$ where $\gamma : [0,1] \to X$ is a path starting in $\gamma(0) = x_0$ and $\alpha : (S^1, 1) \to (U, \gamma(1))$ is a loop in $\gamma(1)$ contained in some element $U \in \mathcal{U}$. This subgroup will be denoted by $\pi^{Sp}(\mathcal{U}, x_0)$.

The Spanier group of (X, x_0), denoted by $\pi^{Sp}(X, x_0)$, is

$$\pi^{Sp}(X, x_0) = \bigcap_{\mathcal{U} \in Cov(X)} \pi^{Sp}(\mathcal{U}, x_0)$$

Remark 1.3.20 There are different ways to adapt this definitions to the compact metric case. We can use coverings of balls of radius ε, or coverings by open sets of diameter less than ε. We could even forget the coverings, and just the require diameter of the image of α to be less than ε, for concatenations $\gamma * \alpha * \overline{\gamma}$.

Loops α which are concatenations of the form $\gamma * \alpha * \overline{\gamma}$ are also denominated as *$\mathcal{U}$-lassos.* Concatenation of paths is not an associative operation since, as maps, the path $\alpha * (\beta * \gamma)$

is different from the path $(\alpha * \beta) * \gamma$. However, these concatenations are the same up to homotopy.

Lemma 1.3.21 If $[\alpha] \in \pi^{Sp}(\mathcal{U}, x_0)$ for some open covering $\mathcal{U}$, then $[\eta * \alpha * \overline{\eta}] \in \pi^{Sp}(\mathcal{U}, x_0)$ for any path $\eta \in P(X, x_0)$.

Proof. Let

$$[\alpha] = \left[\prod_{i=1}^{n} (\gamma_i * \alpha_i * \overline{\gamma}_i)\right]$$

where each α_i is a loop contained in some $U_i \in \mathcal{U}$. Then:

$$\left[\eta * \left(\prod_{i=1}^{n} \gamma_i * \alpha_i * \overline{\gamma}_i\right) * \overline{\eta}\right] = \left[\prod_{i=1}^{n} \eta * (\gamma_i * \alpha_i * \overline{\gamma}_i) * \overline{\eta}\right] = \left[\prod_{i=1}^{n} (\eta * \gamma_i) * \alpha_i * (\overline{\eta * \gamma_i})\right]$$

□

Also the following lemma can be routinely checked:

Lemma 1.3.22 *Let* $\mathcal{V} \prec \mathcal{U}$, *then* $\pi^{Sp}(\mathcal{V}, x_0) \leq \pi^{Sp}(\mathcal{U}, x_0)$. *Moreover,* $\pi^{Sp}(\mathcal{U}, x_0) \trianglelefteq \pi_1(X, x_0)$ *for every open covering* $\mathcal{U}$, *and* $\pi^{Sp}(X, x_0) \trianglelefteq \pi_1(X, x_0)$.

Proof. Take

$$[\alpha] = \left[\prod_{i=1}^{n} (\gamma_i * \alpha_i * \overline{\gamma}_i)\right],$$

For each $i = 1, ..., n$ α_i is contained in some $V_i \in \mathcal{V}$, and there exists $U_i \in \mathcal{U}$ such that $V_i \subseteq U_i$. Hence, each α_i is a loop contained in U_i and, therefore, $[\alpha] \in \pi^{Sp}(\mathcal{U}, x_0)$.

The second part is a consequence of the previous lemma 1.3.21. □

It is easy to see that we can also define the Spanier group as an inverse limit, which is a viewpoint more suitable for our interests. From the above lemma, we can take inclusions as homomorphisms of groups $i_{\mathcal{U}\mathcal{V}} : \pi^{Sp}(\mathcal{V}, x_0) \to \pi^{Sp}(\mathcal{U}, x_0)$ for $\mathcal{V} \prec \mathcal{U}$ and we obtain an inverse system of groups $\{\pi^{Sp}(\mathcal{U}, x_0), i_{\mathcal{U}\mathcal{V}}, Cov(X)\}$, so its inverse limit is

$$\varprojlim \{\pi^{Sp}(\mathcal{U}, x_0), i_{\mathcal{U}\mathcal{V}}, Cov(X)\} = \bigcap_{\mathcal{U} \in \mathcal{C}(\mathcal{X})} \pi^{Sp}(\mathcal{U}, x_0) = \pi^{Sp}(X, x_0).$$

This subgroup is preserved under the action of the homomorphism between fundamental groups induced by a map.

Lemma 1.3.23 *If* $f : (X, x_0) \to (Y, y_0)$ *is a continuous map between pointed topological spaces, then* $f_*(\pi^{Sp}(X, x_0)) \subset \pi^{Sp}(Y, y_0)$, *where* $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ *is the homomorphism induced by* f.

Proof. Let $[\alpha] \in \pi^{Sp}(X, x_0)$, we must show $f_*([\alpha]) = [f \circ \alpha] \in \pi^{Sp}(Y, y_0)$.

Given an open cover $\mathcal{U}$ of Y, $\mathcal{V} = f^{-1}(\mathcal{U}) = \{f^{-1}(U) | U \in \mathcal{U}\}$ is an open cover of X. Since $[\alpha] \in \pi^{Sp}(X, x_0)$, in particular $[\alpha] \in \pi^{Sp}(\mathcal{V}, x_0)$ and

$$[\alpha] = \Big[\prod_{i=1}^{n} (\gamma_i * \beta_i * \overline{\gamma_i}\Big]$$

where $\gamma_i : [0,1] \to X$ are paths with $\gamma_i(0) = x_0$ and $\beta_i : (S^1, 1) \to (V_i, \gamma(1))$ are loops in V_i for some open sets $V_i \in \mathcal{V}$, $i = 1, ..., n$.

Therefore,

$$f_*([\alpha]) = f_*\Big(\Big[\prod_{i=1}^{n} \gamma_i * \beta_i * \overline{\gamma_i}\Big]\Big) = \Big[\prod_{i=1}^{n} f \circ \gamma_i * f \circ \beta_i * f \circ \overline{\gamma_i}\Big].$$

Observe that for each $i = 1, ..., n : f \circ \beta_i : (S^1, 1) \to U_i$ since $V_i = f^{U_i}$ for some $U_i \in \mathcal{U}$ and β_i is a loop in V_i; moreover, $f \circ \gamma_i : [0,1] \to Y$ and $f \circ \gamma_i(0) = y_0$. Hence, $f_*([\alpha]) \in \pi(\mathcal{U}, y_0)$. Since $\mathcal{U}$ was arbitrary, then $f_*([\alpha]) \in \pi^{Sp}(Y, y_0)$. □

As in the case of Dugundji group, we have a well-defined homomorphism

$$\Theta : \quad \pi_1(X, x_0) \quad \longrightarrow \quad \frac{\pi_1(X, x_0)}{\pi^{Sp}(X, x_0)}$$

with $\pi^{Sp}(X, x_0)$ as kernel. In addition, the homomorphism φ from the fundamental group to the shape group factorizes via the quotient group $\pi_1(X, x_0)/\pi^{Sp}(X, x_0)$ which we denote by $Sp(X, x_0)$.

$$\begin{array}{ccc} \pi_1(X, x_0) & & \\ \Theta \downarrow & \overset{\varphi}{\searrow} & \\ Sp(X, x_0) = \dfrac{\pi_1(X, x_0)}{\pi^{Sp}(X, x_0)} & \xrightarrow[\Upsilon]{} & \check{\pi}_1(X, x_0) \end{array} \tag{1.2}$$

The analogous of Remark 1.3.18 is valid here. We shall denote by $Sp(\alpha)$ the class of a loop in the quotient $Sp(X, x_0)$.

Lemma 1.3.24 *$Sp(\alpha) \subset D(\alpha)$ for every loop α in a pointed topological space (X, x_0). Consequently, $Sp(\alpha) \subseteq \langle\alpha\rangle$.*

Proof. For the first assertion, it is enough to observe that if $\mathcal{U}$ is an open covering of X, then a loop $\gamma_i * \eta_i * \overline{\gamma_i}$ with η_i contained in some $U \in \mathcal{U}$ is $\mathcal{U}$-near to $\gamma_i * c_{\gamma(1)} * \overline{\gamma_i}$. The second part is a consequence of the first part jointly with 1.3.14. □

1.3.2 Topology induced by shape

We briefly recall the construction of the shape topology and its basic neighbourhoods on $\pi_1(X, x_0)$, because this is the motivation and guideline for the topologies on the fundamental group from the Hurewicz-Dugundji and Spanier groups.

Let us consider the pointed space (X, x_0) as an inverse limit of an inverse sequence of pointed polyhedra:

$$(X, x_0) = \varprojlim\{(X_n, x_n), p_{n\,n+1}\}$$

Then,

$$\check{\pi}_1(X, x_0) = \varprojlim\{\pi_1(X_n, x_n), (p_{n\,n+1})_*\}$$

is endowed with the topology induced by the ultrametric of section 1, and this topology coincides with the inverse limit topology. From results obtained in sections 1 and 2, the fundamental group $\pi_1(X_n, x_n)$ of compact polyhedra is uniformly discrete (result 1.2.11). Consequently, $\check{\pi}_1(X, x_0)$ is the inverse limit of discrete topological spaces and it is metrizable (compare with result 1.1.6).

Using the homomorphism $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ we can explicit a base of the initial topology on the fundamental group. Equivalently, using the Borsuk's neighbourhood system, we have already seen in the process of construction of the pseudoultrametric, that we can use the balls

$$B([\alpha], \varepsilon) = \{[\beta] \in \pi_1(X, x_0) \mid \alpha \simeq \beta \text{ in } B(X, \varepsilon)\}$$

to obtain a family which is a base of a topology on $\pi_1(X, x_0)$. This topology is just the shape topology introduced in section 1.2 above.

As it was proved in 1.2.24, the shape topology is the initial topology on $\pi_1(X, x_0)$ induced by φ for the ultrametric on $\check{\pi}_1(X, x_0)$. The following result is a trivial consequence.

Proposition 1.3.25 *The homomorphism* $\varphi : \pi_1^{Sp}(X, x_0) \to \check{\pi}_1(X, x_0)$ *is continuous.*

As we have already exhibited, the pseudoultrametric d is not always a metric in $\pi_1^{Sh}(X, x_0)$. Hence, the closure of a point does not coincide with itself, which means that $\pi_1^{Sh}(X, x_0)$ is not T_1 in general. First we need an easy fact for the next result.

Lemma 1.3.26 *In* $\pi_1^{Sh}(X, x_0))$, *a class* $[\beta]$ *is in* $\overline{\{[\alpha]\}}$ *if and only if* $d([\alpha], [\beta]) = 0$.

Proof. It is a general fact that, in a pseudometric space, the closure of a point coincides with those elements whose distance to the element is zero.

□

As a consequence, we obtain the following characterization of the closure of an element on the shape topology:

Theorem 1.3.27 *The closure of a class in homotopy is the set of loops that generates the same class in shape, i.e.,*

$$[\beta] \in \overline{\{[\alpha]\}} \Leftrightarrow \langle\alpha\rangle = \langle\beta\rangle.$$

Proof. For some element $\langle\alpha\rangle = \varphi([\alpha])$ in $\pi_1^{Sh}(X, x_0)$:

$$\begin{aligned}\overline{\{[\alpha]\}} =& \{[\beta] \in \pi_1^{Sh}(X, x) \mid d([\alpha],[\beta]) = 0\} = \\ =& \{[\beta] \in \pi_1^{Sh}(X, x_0) \mid \alpha \simeq \beta \text{ in } B(X, \varepsilon) \text{ for all } \varepsilon > 0\} = \\ =& \{[\beta] \in \pi_1^{Sh}(X, x_0) \mid \alpha \simeq \beta \text{ in } X_n \text{ for all } n \in \mathbb{N}\} = \\ =& \{[\beta] \in \pi_1^{Sh}(X, x_0) \mid \langle\alpha\rangle = \langle\beta\rangle\}\end{aligned}$$

□

1.3.3 Topology associated to the Hurewicz-Dugundji group

Given a pointed topological space (X, x_0), a topology is introduced in [27] for the group $\chi(X, x_0)$. The neighbourhoods of the basis are defined as

$$N(D(\alpha), \mathcal{U}) = \{D(\beta) \in \chi(X, x_0) \mid \alpha \text{ is } \mathcal{U}\text{-homotopic to } \beta\}.$$

In the compact metric case, this neighbourhoods can be rewritten as

$$N(D(\alpha), \varepsilon) = \{D(\beta) \in \chi(X, x_0) \mid \alpha \text{ is } \varepsilon\text{-homotopic to } \beta\},$$

and the topology can be recovered by setting

$$\eta(D(\alpha), D(\beta)) = \inf\{\varepsilon > 0 \mid \alpha \text{ is } \varepsilon\text{-homotopic to } \beta\},$$

which can be shown to be an ultrametric in a straightforward way.

With the homomorphism

$$\Psi : \pi_1(X, x_0) \longrightarrow \chi(X, x_0)$$

we can define the initial topology on $\pi_1(X, x_0)$ induced by Ψ and the Hurewicz-Dugundji topology. Following the same reasoning as in the shape topology, a base for this topology on the fundamental group is given by the sets

$$N([\alpha], \mathcal{U}) = \{[\beta] \in \pi_1(X, x_0) \mid \alpha \text{ is } \mathcal{U}\text{-homotopic to } \beta\}$$

or in the compact metric case as

$$N([\alpha], \varepsilon) = \{[\beta] \in \pi_1(X, x_0) \mid \alpha \text{ is } \varepsilon\text{-homotopic to } \beta\}.$$

This basic open sets correspond to open balls associated with a pseudoultrametric

$$\eta([\alpha],[\beta]) = \inf\{\varepsilon > 0 \mid \alpha \text{ is } \varepsilon\text{-homotopic to } \beta\}.$$

In the homomorphisms of diagram (1.1), we consider $\check{\pi}_1(X,x_0)$ with the topology induced by the ultrametric of section 1.1, $\pi_1(X,x_0)$ endowed with the topology generated by the pseudoultrametric described in section 1.2 and, in $\chi(X,x_0)$ let us consider the Dugundji's topology which has been described above. This topologies give us more information about the relation of homotopy, Dugundji and shape classes of a loop:

Proposition 1.3.28 *If Ψ is continuous, then Φ is injective.*

Proof. Let us suppose that Ψ is continuous. Let $D(\alpha), D(\beta) \in \chi(X,x_0)$ such that $\Phi(D(\alpha)) = Sh(\alpha) = Sh(\beta) = \Phi(D(\beta))$. Observe that, $D(\alpha) = \Psi([\alpha])$ and $D(\beta) = \Psi([\beta])$.

Let $N(D(\alpha),\mathcal{U})$ be a basic neighbourhood where $\mathcal{U}$ is an open covering of X. Since Ψ is continuous, there exists $B([\alpha],\mathcal{V})$ satisfying $\Psi(B([\alpha],\mathcal{V})) \subseteq N(D(\alpha),\mathcal{U})$ where $\mathcal{V}$ is a normal covering $\mathcal{V}$ of X. Since $Sh(\alpha) = Sh(\beta)$, then $[\beta] \in B([\alpha],\mathcal{V})$. Hence, $D(\beta) \in N(D(\alpha),\mathcal{U})$. Consequently, α and β are $\mathcal{U}$-homotopic for any arbitrary open covering $\mathcal{U}$, so $D(\alpha) = D(\beta)$. □

1.3.4 Topology associated to the Spanier group

We describe now the Spanier group and remake an analogous reasoning as in the case of the null-Dugundji loops, and analogous results can be obtained. Declare as basics neighbourhoods the sets

$$M(Sp(\alpha),\mathcal{U}) = \{Sp(\beta) \in Sp(X,x_0) \mid \alpha \simeq \ell * \beta \text{ with } \ell \in \pi^{Sp}(\mathcal{U},x_0)\}$$

and consider the topology generated on $Sp(X,x_0)$ by the family

$$\{M(Sp(\alpha),\mathcal{U}) \mid Sp(\alpha) \in Sp(X,x_0), \mathcal{U} \text{ open covering of } X\}.$$

With the homomorphism

$$\Theta : \pi_1(X,x_0) \longrightarrow Sp(X,x_0)$$

we can define another topology on $\pi_1(X,x_0)$.

Definition 1.3.29 The *Spanier topology* on $\pi_1(X,x_0)$ is the initial topology induced by Θ.

Consequently, a base for the Spanier topology on the fundamental group is given by sets of the form

$$M([\alpha],\mathcal{U}) = \{[\beta] \in \pi_1(X,x_0) \mid \beta \simeq \ell * \alpha \text{ with } \ell \in \pi^{Sp}(\mathcal{U},x_0)\}.$$

1.3.5 Quotient topology

Consider in $\Omega(X, x_0)$ the compact-open topology, and endow $\pi_1(X, x_0)$ with the quotient topology associated to the (natural) quotient map

$$\begin{array}{rccc} q: & \Omega(X,x_0) & \longrightarrow & \pi_1^{Sh}(X,x_0) \\ & \alpha & \longmapsto & [\alpha] \end{array}$$

Definition 1.3.30 The *quotient topology* from *the compact-open topology* on $\pi_1(X, x_0)$ is the final topology associated to the projection q from the loop space $\Omega(X, x_0)$ when the compact-open topology is considered. We shall denote $\pi_1^{QCO}(X, x_0)$ to the fundamental group endowed with this topology.

Remark 1.3.31 In [7] this group is named as *topological fundamental group.* Since we deal with many topologized versions of the fundamental groups, we feel that this is not an appropriate nomenclature. Moreover, in the mentioned work is wrongly proved that this topology makes the fundamental group into a topological group. This is not true in general, as it can be shown with a counterexample in [33] with the Hawaiian Earrings (we shall use this example in chapter 3 of this work). However, the fundamental group with the quotient topology is an homogeneous quasi-topological group (see [21] or [13]).

1.3.6 Brazas topology

In [13] it is introduced a modification of the quotient topology, in order to obtain a group topology. The philosophy of this tricky categorical construction is just to detect the open sets of the quotient topology which prevent the continuity of the group operation in $\pi_1(X, x_0)$, and remove them.

This process stabilizes at some point, and a topology (in general not trivial nor discrete) is obtained. See the mentioned paper for the details of the construction.

Definition 1.3.32 The topology constructed in [13] shall be called *Brazas topology* and we denote $\pi_1^B(X, x_0)$ to the fundamental group endowed with this topology.

The main property of this topology is the following (see [13] for the proof):

Proposition 1.3.33 *The Brazas topology is the finest group topology on $\pi_1(X, x_0)$ which makes continuous the projection q from $\Omega(X, x_0)$.*

However, it is not known any separation property for this topology, not even the closure of a point. In [14], relations of this problems with other open problems in topology are established, giving an idea of the difficulty to deal with this topology and to obtain satisfactory results.

1.4 Comparison of groups and topologies on $\pi_1(X, x_0)$

Here we study the relation between the groups and topologies presented in the previous sections. First we give a comparison between the groups of null loops relative to different equivalence relations, which shall allow to relate the correspondent topologies.

Proposition 1.4.1 *Let (X, x_0) be a topological space. Then,*

$$\{1\} \trianglelefteq \pi^{Sp}(X, x_0) \trianglelefteq \nu(X, x_0) \trianglelefteq Ker\varphi \trianglelefteq \pi_1(X, x_0).$$

Proof. Let us see at first the subsets relations:

- $\{1\} \subseteq \pi^{Sp}(X, x_0)$ is obvious.
- $\pi^{Sp}(X, x_0) \subseteq \nu(X, x_0)$.

  Let $\mathcal{U}$ be an open cover of X and take $[\alpha] \in \pi^{Sp}(X, x_0)$, i.e.,

  $$\alpha \simeq \prod_{i=1}^{n} \gamma_i * \beta_i * \overline{\gamma_i},$$

  where for $i = 1, ..., n$ $\gamma_i : [0,1] \to X$ is a path with $\gamma_i(0) = x_0$ and $\beta_i : (S^1, 1) \to (U_i, \gamma_i(1))$ is a loop for some $U_i \in \mathcal{U}$.

  By lemma 1.3.5, α and $\prod_{i=1}^{n} \gamma_i * \beta_i * \overline{\gamma_i}$ are $\mathcal{U}$-homotopic. On the other hand, $\prod_{i=1}^{n} \gamma_i * \beta_i * \overline{\gamma_i}$, is $\mathcal{U}$-homotopic to $\prod_{i=1}^{n} \gamma_i * c_{\gamma_i(1)} * \overline{\gamma_i}$ and $\prod_{i=1}^{n} \gamma_i * c_{\gamma_i(1)} * \overline{\gamma_i}$ is homotopic to c_{x_0}. Applying again 1.3.5, $\prod_{i=1}^{n} \gamma_i * c_{\gamma_i(1)} * \overline{\gamma_i}$ and c_{x_0} are $\mathcal{U}$-homotopic. Putting all together, we obtain that α is $\mathcal{U}$-homotopic to c_{x_0}, and $[\alpha] \in \nu(X, x_0)$.

- $\nu(X, x_0) \subseteq Ker\varphi$.

  Let $[\alpha] \in \nu(X, x_0)$. Let $\mathcal{U}$ be a normal covering of X, by hypothesis α is $\mathcal{U}$-homotopic to c_{x_0}, so there exists an $\mathcal{U}$-chain $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = c_{x_0}$ between these paths.

  Let us consider now the projections $p_{\mathcal{U}}(\alpha_i)$ of the paths. Since two consecutive paths α_i and α_{i+1} are $\mathcal{U}$-near, then its projections are contiguous maps in $X_{\mathcal{U}}$ thus, homotopic. Hence, we obtain $p_{\mathcal{U}}(\alpha_0) \simeq p_{\mathcal{U}}(\alpha_1) \simeq ... \simeq p_{\mathcal{U}}(\alpha_n)$. By the transitivity of the homotopy relation, $p_{\mathcal{U}}(\alpha) = p_{\mathcal{U}}(\alpha_0) \simeq p_{\mathcal{U}}(\alpha_n) = p_{\mathcal{U}}(c_{x_0})$. Since $\mathcal{U}$ was an arbitrary normal covering, the latter means that $\langle \alpha \rangle = \langle c_{x_0} \rangle$, so $[\alpha] \in Ker\varphi$.

- $Ker\varphi \subseteq \pi_1(X, x_0)$ just by definition of φ.

Now let us prove that each group is normal in the next.

- $\{1\} \trianglelefteq \pi^{Sp}(X, x_0)$ is again obvious.

- $\pi^{Sp}(X,x_0) \trianglelefteq \nu(X,x_0)$.

  This is an immediate consequence of remark 1.3.21: taking $[\alpha] \in \pi^{Sp}(X,x_0)$ and $[\beta] \in \nu(X,x_0)$, the element $[\beta]*[\alpha]*[\beta]^{-1}$ is a conjugate of an $\mathcal{U}$-lasso, so it is itself an $\mathcal{U}$-lasso for an arbitrary open covering $\mathcal{U}$ of X. Hence, $[\beta]*[\alpha]*[\beta]^{-1} \in \pi^{Sp}(X,x_0)$.

- $\nu(X,x_0) \trianglelefteq Ker\varphi$

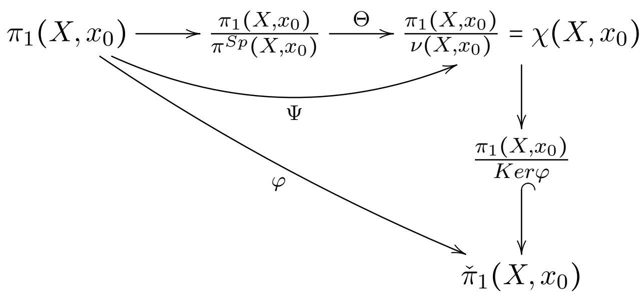
  Let us take $[\alpha] \in \nu(X,x_0)$ and $\beta \in Ker\varphi$ and let us show that $[\beta]*[\alpha]*[\beta]^{-1} \in \nu(X,x_0)$.

  Given an arbitrary normal covering $\mathcal{U}$ of X, there exists an $\mathcal{U}$-chain $\alpha_0 = \alpha, \alpha_1, ..., \alpha_n = c_{x_0}$ between α and c_{x_0}. Clearly, the finite sequence $\beta*\alpha_0*\overline{\beta} = \beta*\alpha*\overline{\beta}, \beta*\alpha_1*\overline{\beta}, ..., \beta*\alpha_n*\overline{\beta} = \beta*c_{x_0}*\overline{\beta}$ is an $\mathcal{U}$-chain between $\beta*\alpha*\overline{\beta}$ and $\beta*c_{x_0}*\overline{\beta}$. In addition, $\beta*c_{x_0}*\overline{\beta}$ is homotopic to c_{x_0}, so they are $\mathcal{U}$-homotopic by lemma 1.3.5. Consequently, $[\beta]*[\alpha]*[\beta]^{-1} = [\beta*\alpha*\overline{\beta}] \in \nu(X,x_0)$.

- $Ker\varphi \trianglelefteq \pi_1(X,x_0)$ by definition of the kernel of an homomorphism.

□

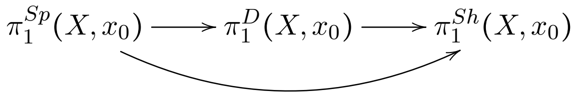
As a consequence of this chain of relations between groups, we have the next well-defined chain of homomorphism, relating the fundamental, Spanier, Dugundji and shape groups of a pointed space:



1.4.1 Arbitrary case

From the diagram above, and using the definition of shape, Dugundji and Spanier topologies as initial topologies on $\pi_1(x,x_0)$, we arrive to the next result.

Theorem 1.4.2 *On $\pi_1(X,x_0)$, the shape topology τ^{Sh} is coarser than Dugundji topology τ^D and this is coarser than Spanier topology τ^{Sp}. Equivalently, the identity maps on the diagram*



are continuous.

Proof. It is an immediate consequence of 1.3.14, 1.3.24 and 1.4.1. □

From Proposition 1.3.28 we have a clue of the deeper relation between shape and discrete homotopies:

Proposition 1.4.3 $\Psi : \pi_1^{Sh}(X, x_0) \to \chi(X, x_0)$ *is continuous if and only* $Id : \pi_1^{Sh}(X, x_0) \to \pi_1^{D}(X, x_0)$ *is continuous.*

Proof. The result should be obvious from the diagram

$$\pi_1^{Sh}(X, x_0) \xrightarrow{Id} \pi_1^{D}(X, x_0) \xrightarrow{\Psi} \chi(X, x_0)$$
$$\pi_1^{Sh}(X, x_0) \xrightarrow{\Psi} \chi(X, x_0)$$

and the properties of Dugundji topology as an initial topology. □

Corollary 1.4.4 *If shape and Dugundji topologies agree, then* $\varphi(\pi_1(X, x_0)) = \chi(X, x_0)$.

Remark 1.4.5 In terms of shape, the last result says that $\chi(X, x_0)$ is the subgroup of $\check{\pi}_1(X, x_0)$ generated by internal elements of X. Consequently, we would get that shape classes generated by loops in X are completely captured by discrete homotopies. This situation is not true in general as it was shown in example 1.4.

We have the following result of comparison of the topologies.

Proposition 1.4.6 *The shape topology* τ^{Sh} *is coarser than the Brazas topology* τ^B*, and this is coarser than the quotient from the compact open topology* τ^{QCO} *on* $\pi_1(X, x_0)$.

Proof. Since τ^B is obtained from τ^{QCO} just by removing some open sets, it is obvious that $\tau^B \subseteq \tau^{QCO}$. Moreover, τ^B is the finest topology on $\pi_1(X, x_0)$ which makes it into a topological group and $q : \Omega(X, x_0) \to \pi_1(X, x_0)$ continuous. Then, $\tau^{Sh} \subseteq \tau^B$ from 1.2.7 and 1.2.12. □

Corollary 1.4.7 *If* X *is an ANR, then the topological spaces* $\pi_1^D(X, x_0)$, $\pi_1^{\ell}(X, x_0)$, $\pi_1^{QCO}(X, x_0)$, $\pi_1^B(X, x_0)$ *and* $\pi_1^{Sh}(X, x_0)$ *are all uniformly discrete.*

Proof. Combine 1.2.13 and 1.4.6. □

1.4.2 Locally path-connected case

If the space X is locally path-connected, some of the topologies exhibited above agree. Moreover, the definition of equivalent relations agree. First, we need a technical (but important) lemma. We also shall use this lemma in the third chapter of this work. The situation that we would like to detect is depicted in the figure: If we have two $\mathcal{U}$-near paths for some open covering $\mathcal{U}$, they could not differ in lassos, despite to have homotopic projections into the nerve of the covering.

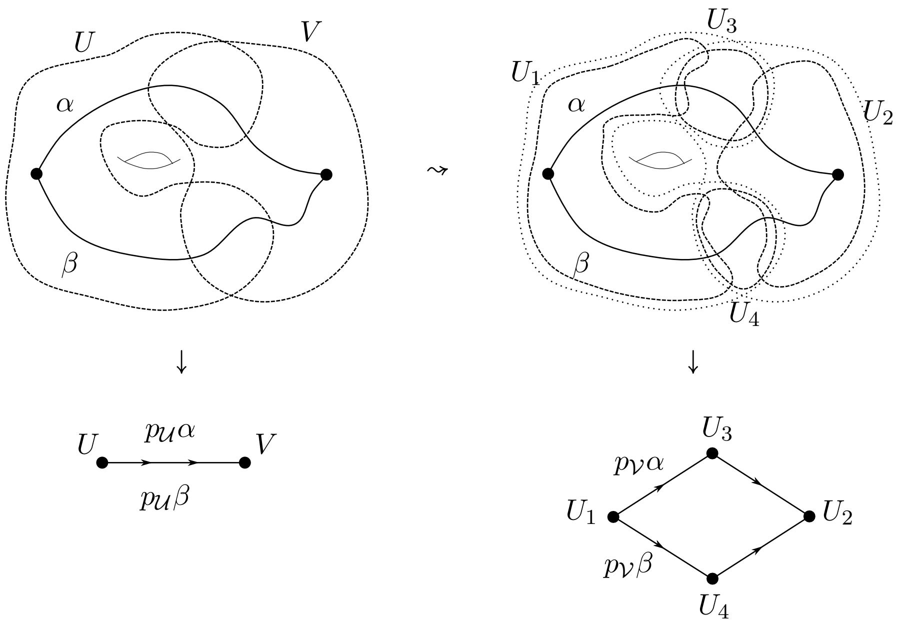


Figure 1.5: A covering $\mathcal{V}$ finer than $\mathcal{U}$ detects that two paths do not differ only in lassos.

When the situation is adequate, the idea of how to obtain one path from another by adding lassos is contained in the following lemma.

Lemma 1.4.8 *Let $\alpha, \beta : [0,1] \to X$ be two paths with $\alpha(0) = \beta(0) = x_0$, and let $\mathcal{U}$ be a covering of X by path-connected sets such that the intersection of any two covering sets is path-connected. If α, β are $\mathcal{U}$-near, then $\beta \simeq \ell * \alpha * \gamma$, where ℓ is a concatenation of $\mathcal{U}$-lassos and $\gamma : [0,1] \to U$ for some $U \in \mathcal{U}$ containing $\alpha(1)$.*

Proof. Given two paths as in the statement of the lemma, it is possible to find a finite collection of open sets $U_1, ..., U_n \in \mathcal{U}$ and a partition $t_0 = 0 < t_1 < ... < t_{n-1} < t_n = 1$ with $\alpha(t_i), \beta(t_i) \in U_i \cap U_{i+1}$ for $1 \le i \le n-1$, $\alpha(t_0) = x_0 = \beta(t_0) \in U_1$ and $\alpha(t_n) = \beta(t_n) \in U_n$ such that the images of $\alpha|_{[t_{i-1}, t_i]}$ and $\beta|_{[t_{i-1}, t_i]}$ are contained in U_i.

Since $U_i \cap U_{i+1}$ is path connected, there exists a path $\gamma_i : [0,1] \to U_i \cap U_{i+1}$ joining $\alpha(t_i)$ with $\beta(t_i)$. Denote $\alpha_i = \alpha|_{[t_{i-1}, t_i]}$ and $\beta_i = \beta|_{[t_{i-1}, t_i]}$, and observe the following:

- The concatenation $\ell_0 = \beta_1 * \overline{\gamma}_1 * \overline{\alpha}_1$ is contained in U_1, so it is an $\mathcal{U}$-lasso.
- For each $i = 1, ..., n-1$ the concatenation $\ell_i = \alpha|_{[0,a_i]} * \gamma_i * \beta_{i+1} * \overline{\gamma}_{i+1}\overline{\alpha}_{i+1} * \overline{\alpha}|_{[0,a_i]}$ is an $\mathcal{U}$-lasso, since $\gamma_i * \beta_{i+1} * \overline{\gamma}_{i+1} * \overline{\alpha}_{i+1}$ is contained in U_i.

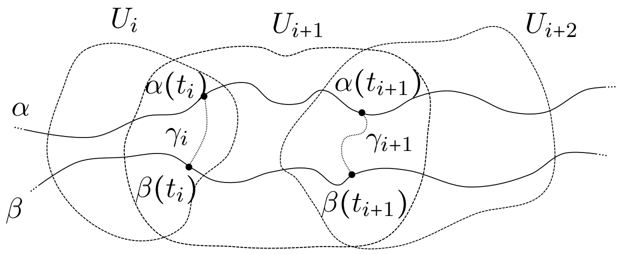


Figure 1.6: Adding small paths to obtain a lasso.

Moreover, for the end-points, $\alpha(1), \beta(1) \in U_n$ and there exists $\gamma : [0,1] \to U_n$ with $\gamma(0) = \alpha(1)$ and $\gamma(1) = \beta(1)$.

Finally, we have arrived to

$$\beta \simeq \ell_0 * ... * \ell_{n-1} * \alpha * \gamma$$

where $\ell_0 * ... * \ell_{n-1} \in \pi^{Sp}(\mathcal{U}, x_0)$. ◻

To detect when two paths differ only in lassos, we need a technical lemma[1] of refinement of coverings in order to avoid non-path connected intersections between open sets of a covering. As said above, the goal of this part is to show that, given two paths with homotopic projections into a nerve, they just only differ in lassos.

Lemma 1.4.9 *Let X be a paracompact, Hausdorff, path-connected and locally path-connected topological space. For every locally finite covering $\mathcal{U}$, there exists an open covering $\mathcal{W}$ of path-connected sets satisfying the following properties:*

i) $\mathcal{W}$ is a refinement of $\mathcal{U}$,

ii) For any finite collection $\{W_1, ..., W_n\} \subset \mathcal{W}$, if $W_1 \cap ... \cap W_n \neq \varnothing$ then $W_1 \cup ... \cup W_n \subseteq U$ for some $U \in \mathcal{U}$.

Proof. Let $\mathcal{U} = \{U_i \,|\, i \in I\}$ be a locally finite open covering of X.

Since paracompact and Hausdorff properties imply that X is a normal space, it is possible to construct locally finite coverings (with the same indexing set) $\mathcal{U}_1 = \{U_i^1 \,|\, i \in I\}$ and $\mathcal{U}_2 =$

[1] It must be said that this lemma was originally developed in a joint work [81] with A. Zastrow for the comparison of topologies on the universal path space (chapter 3 of the present work). After that, we observed that the same result applies to the fundamental group. However, the original lemma appear in the first chapter due to the order of exposition of this work.

$\{U_i^2 \,|\, i \in I\}$ refining $\mathcal{U}$, such that $\overline{U_i^1} \subseteq U_i^2$ for each $i \in I$ (consequently, also $\overline{U_i^1} \subseteq U_i^2 \subseteq U_i$). Applying iteratively this construction, we obtain countably many coverings $\mathcal{U}_n = \{U_i^n \,|\, i \in I\}$ with the property that $\overline{U_i^n} \subseteq U_i^{n+1}$.

Hence, for each index $i \in I$, we have a nested sequence of open sets, where each set belongs to a determined covering. We shall keep the notation U_i^n, where the lower index $i \in I$ denotes the nested sequence which corresponds to the i-th element of $\mathcal{U}$ and the upper index $n \in \mathbb{N}$ denotes the position of the set in the nested sequence, being $n = 1$ the innermost open set and so on. In addition, we shall denote by U_i^∞ the corresponding elements of the initial locally finite covering.

For a fixed $i_0 \in I$, we shall also refer to $U_{i_0}^n \smallsetminus U_{i_0}^{n-1}$ as the *n-th ring or n-th annulus.* According to this, the ∞-ring or ∞-annulus would be $U_{i_0}^\infty \smallsetminus (\cup_{n\in\mathbb{N}} U_{i_0}^n)$ (which could be possibly empty, if $U_i^\infty = \cup_{n\in\mathbb{N}} U_i^n$) for each $i \in I$.

Given a finite subset of indexes $\{i_1, ..., i_j\} \in \mathcal{P}_F(I)$, let us define the set

$$U_{\{i_1,\dots,i_j\}} = \left(U_{i_1}^{2j-1} \cap \dots \cap U_{i_j}^{2j-1}\right) \smallsetminus \Big(\bigcup_{i \notin \{i_1,\dots,i_j\}} \overline{U_i^{2j}}\Big) \tag{$*$}$$

which is open because of the locally finiteness. For fixing notation, any point x in a set $U_{\{i_1,\dots,i_j\}}$ that is defined by the formula $(*)$ has a neighbourhood U^x which only intersects finitely many open sets of the covering $\mathcal{U}$. Of course, $U_{i_1}^{2j-1}, \dots, U_{i_j}^{2j-1}$ appear as sets with non empty intersection with U^x. On the other hand, only some others indices $i = i_{j+1}, \dots, i_k$ satisfies $U_i^{2j} \cap U^x \neq \varnothing$. Consequently,

$$U^x \cap \left(U_{i_1}^{2j-1} \cap \dots \cap U_{i_j}^{2j-1}\right) \smallsetminus \left(\cup_{i \notin \{i_1,\dots,i_j\}} \overline{U_i^{2j}}\right) = U^x \cap \left(U_{i_1}^{2j-1} \cap \dots \cap U_{i_j}^{2j-1}\right) \smallsetminus \left(\overline{U_{i_{j+1}}^{2j}} \cup \dots \cup \overline{U_{i_k}^{2j}}\right)$$

is an open set contained in $U_{\{i_1,\dots,i_j\}}$, hence this last set is open.

Moreover, the collection of $U_{\{i_1,\dots,i_j\}}$ is a covering of the space X, when the subset of indexes runs all over finite parts of the original index set I. In order to see this, let us introduce a sequence of natural numbers $(a_k)_{k\in\mathbb{N}}$ that we shall refer as *the characteristic tuple of a point* $x \in X$.

A formula for the definition of each a_k is as follows: fixed a point $x \in X$, the entry a_k indicates the number of lower indexes for which the point x lies in a set of upper index k but not before. In other words, a_k gives the number of nested sequences having the point x in the k-th annulus, that is, inside the k-th open set of the sequence, but not in the $(k-1)$-th open set. For the construction of the characteristic tuple, we can forget about nested sequences having x only in its ∞-annulus.

Let us observe the following: given a point $x \in X$, it only could lie in finitely many sets of $\mathcal{U}$, again by the locally finiteness. Hence, only finitely many lower indexes appear, so $a_k \neq 0$ only for finitely many $k \in \mathbb{N}$. Furthermore, since the first refinement of nested sets is still a covering, for some lower index i_0, the point x must be in some open set with upper index $n = 1$. That is, $x \in U_{i_0}^1$. Consequently, $a_1 \geq 1$ in the characteristic tuple.

Now, we have to decide a combination of finitely many indexes of I, in order to construct an open set of the form $(*)$ above containing the point x. In order to do that, let us define a new variable $j = j(i)$ as the number of nested sequences having x in its first i levels. That is, $j(i) = \sum_{k=1}^{i} a_k$ (observe that $j(i)$ is monotone increasing). We need to compare the growth of the values of the variables $j, 2j-1$ and $2j$ in comparison with the levels $i \in \mathbb{N}$.

We define *the overtake position* $OP(x)$ as the first place $i_0 \in \mathbb{N}$ satisfying $i_0 = j_0 - 1$ (where $j_0 = j(i_0)$) and $a_{i_0+1} = 0$:

$$OP(x) = min\{i \in \mathbb{N} \mid 2j(i) = i \text{ and } a_{i+1} = 0\}$$

The overtake position always occurs: the worst case is in that the condition for $OP(x)$ is never satisfied. However, it was observed above that the characteristic tuple of a point is stabilizes to zero, i. e. $a_k = 0$ for $k \geq k_0$ for some place k_0. From this place, $2j-1$ and $2j$ are constant, so at some point the value of i, it shall overtake them and of course $a_{k+1} = 0$.

Note also that if $i_0 = OP(x)$, then $a_{i_0} = 0$. Let us suppose that $a_{i_0} > 0$. Thus:

$$i_0 = 2j_0 = 2j(i_0) = 2\sum_{k=1}^{i_0} a_k = 2\sum_{k=1}^{i_0-1} a_k + 2a_{i_0} \geq 2j(i_0-1) + 2 \quad \Rightarrow \quad i_0 - 2 \geq 2j(i_0-1).$$

At this point we have two different cases:

1) If $2j(i_0-2) \geq i_0 - 2$. Then:

$$2j(i_0-2) \geq i_0 - 2 \geq 2j(i_0-1) \geq 2j(i_0-2)$$

   so $2j(i_0-2) = i_0 - 2 = 2j(i_0-1)$. If $a_{i_0-1} = 0$, it contradicts the definition of $OP(x)$. On the other hand, if $a_{i_0-1} \neq 0$, then $j_{i_0-1} > j_{i_0-2}$ which is impossible in this case.

2) If $2j(i_0-2) < i_0-2$. Then either $2j(i_0-4) = i_0-4$ or still $2j(i_0-4) < i_0-4$. For the first option, we can reasoning as in case 1 to conclude that either the definition of $OP(x)$, or the assumption of case 2 is contradicted. Iterating this reasoning, if the second option holds, take two positions before, and compare $2j(i_0-2m)$ and i_0-2m. The equality must occur for some m, since for $i = 1$ is $2j(1) \geq 2 > 1$. At this point, reasoning as above, we arrive to a contradiction.

Summing up, in every case we arrive to a contradiction, so it must be $a_{i_0} = 0$.

In addition, at the level $i_0 = 2j_0$ of the characteristic tuple we have exactly j_0 nested sequences containing the point x up to its $2j_0-1$ first levels. Indeed, for $i = i_0$, we have j_0 nested sequences with x in its $2j_0$ first levels. Since $a_{i_0} = 0$, this j_0 sequences already contain x in its $2j_0-1$ first levels.

Finally, let us check that if $i_0 = OP(x)$, then the formula $(*)$ works, being $\{i_1, ..., i_{j_0}\}$ the j_0 indexes representing the nested sequences which contain the point x : as remarked in the paragraph above, $x \in U_{i_k}^{2j_0-1}$ for each $k = 1, ..., j_0$. Since $a_{i_0} = 0$, then $x \notin U_i^{2j_0}$ for $i \neq i_1, ..., i_{j_0}$. Moreover, $x \notin \overline{U_i^{2j_0}}$ since $a_{i_0+1} = 0$ is required for $OP(x)$. Hence,

$$x \in U_{i_1,\dots,i_{j_0}} = U_{i_1}^{2j_0-1} \cap \dots \cap U_{i_{j_0}}^{2j_0-1} \setminus \Big(\bigcup_{i \notin \{i_1,\dots,i_{j_0}\}} \overline{U_i^{2j_0}} \Big).$$

For the last part of the lemma, we shall show that the union of any two sets with non-empty intersection lies in one set of the covering $\mathcal{U}$, that is, with upper index ∞. It is enough to prove the following two facts:

i) If $i_0 \in \{i_1,\dots,i_j\} \cap \{i'_1,\dots,i'_{j'}\} \neq \varnothing$, then $U_{\{i_1,\dots,i_j\}} \cup U_{\{i'_1,\dots,i'_{j'}\}} \subseteq U_{i_0}^{\infty}$.

ii) If $\{i_1,\dots,i_j\} \cap \{i'_1,\dots,i'_{j'}\} = \varnothing$, then $U_{\{i_1,\dots,i_j\}} \cap U_{\{i'_1,\dots,i'_{j'}\}} = \varnothing$.

The first statement is obvious from the way that formula $(*)$ works. For the second one, recall that the point is in some set with upper index $n = 1$, so $x \in U_{i_0}^1$ for some $i_0 \in I$. Hence, this index has to appear in the finite subset of indexes defining $U_{\{i_1,\dots,i_j\}}$. Otherwise, the point x would not have any chance to lie in the set $U_{\{i_1,\dots,i_j\}}$. So, if the defining sets $\{i_1,\dots,i_j\}$ and $\{i'_1,\dots,i'_{j'}\}$ are disjoint, then also $U_{\{i_1,\dots,i_j\}}$ and $U_{\{i'_1,\dots,i'_{j'}\}}$ must be disjoint.

Finally, the desired covering $\mathcal{W}$ consists of the path-components of the open sets $U_{\{i_1,\dots,i_j\}}$ given by $(*)$. From the locally-path connectedness, each path-component correspondent to each $U_{\{i_1,\dots,i_j\}}$ is open. Then, it is obvious that $\mathcal{W}$ is still a covering by open sets of X, and $\mathcal{W}$ verifies the conditions stated in the lemma, because of the corresponding properties for the sets $U_{\{i_1,\dots,i_j\}}$ already checked above. □

Remark 1.4.10 Previous lemma has a geometrical meaning, in the following sense: the original sets are made smaller in order to avoid the original intersections. It should be now clear that intersections $U_{i_1} \cap U_{i_2} \neq \varnothing$ between original sets of $\mathcal{U}$ have been replaced by $U_{\{i_1,i_2\}}$ (or by its path-connected components) and now $U_{\{i_1\}} \cap U_{\{i_2\}} = \varnothing$, and similarly for any general finite subset $\{i_1,\dots,i_j\} \subseteq I$. This explain the motivational figure 1.5. The case for a covering of three sets are drawn in the figure below, showing how the formula $(*)$ of the lemma works.

With the previous technical lemma, we can state one of the main results of this work:

Theorem 1.4.11 *Let (X, x_0) be a paracompact, Hausdorff, path-connected and locally path-connected pointed space. If $\alpha, \beta \in \Omega(X, x_0)$ with $\langle\alpha\rangle = \langle\beta\rangle$, then $Sp(\alpha) = Sp(\beta)$.*

Proof. We already know, from 1.3.24, that $Sp(\alpha) = Sp(\beta)$ implies $\langle\alpha\rangle = \langle\beta\rangle$. For the converse, let us take an open covering $\mathcal{U}$. Without loss of generality, we can assume that each set in $\mathcal{U}$ is path-connected (if not, take the refinement of $\mathcal{U}$ consisting of the path-components of sets in $\mathcal{V}$, which are open from the locally-path connectedness). Since X is paracompact, then there exists a locally finite open covering $\mathcal{V}$ which refines $\mathcal{U}$. Applying lemma 1.4.9, we obtain an open covering $\mathcal{W}$ with the properties stated in the referred lemma.

Since $\langle\alpha\rangle = \langle\beta\rangle$, in particular $p_{\mathcal{W}} \circ \alpha \simeq p_{\mathcal{W}} \circ \beta$ in $|N(\mathcal{W})|$. Using the simplicial approximation theorem, we can find paths $f, g : [0,1] \to sk_1(|N(\mathcal{W})|)$ satisfying the following:

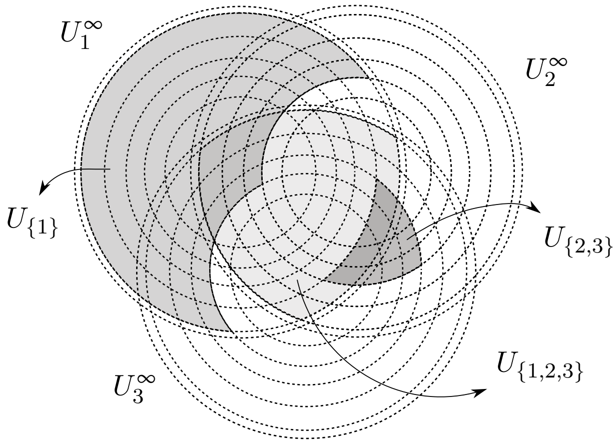


Figure 1.7: The effect of formula ($*$) for the case of three covering sets.

- f is contiguous to $(p_{\mathcal{W}} \circ \alpha)$ and g is contiguous to $p_{\mathcal{W}} \circ \beta$ (hence, f and g are homotopic).

- The paths f and g are affine: there exist partitions $0 = t_0 < t_1 < ... < t_{m-1} < t_m = 1$ and $0 = s_0 < s_1 < ... < s_{n-1} < s_n = 1$ of $[0,1]$ such that $f(t_i)$ and $g(s_j)$ are vertexes of $|N(\mathcal{W})|$, $f([t_i, t_{i+1}]) = [f(t_i), f(t_{i+1})]$ and $g([s_j, s_{j+1}]) = [g(s_j), g(s_{j+1})]$.

- In addition, the homotopy H between f and g is affine: there exists a triangulation $\mathcal{T}$ of $[0,1] \times [0,1]$ such that if $\Delta \in \mathcal{T}$ is a triangle of it, then $H(\Delta)$ is a face of $|N(\mathcal{W})|$ of dimension less or equal than 2.

Let us suppose that the triangulation has l triangles. For each $1 \le j \le l$, there is a triangle $\Delta_j \in \mathcal{T}$, and let us denote by d_0^j, d_1^j, d_2^j its vertexes. All of the images of this vertexes correspond, respectively, to open sets W_k^j $(k = 0,1,2)$ of $\mathcal{W}$. Moreover, we consider this sets as labelled, by choosing a distinguished point $x_k^j \in W_k^j$, $j = 1, ..., l$ and $k = 0,1,2$.

Now, we can obtain a sequence $f_0 = f, ..., f_l = g$ of paths in $|N(\mathcal{W})|$ in the following way: starting with $f_0 = f$, let f_j be constructed from f_{j-1} by adding just one triangle Δ_j of the triangulation of the unitary square (sharing at least an edge with the previous ones). The trace of f_j shall be the same as the trace of f_{j-1}, except in the interval correspondent to $[d_0^j, d_1^j]$ in which f_j runs along the image of $[d_0^j, d_2^j] + [d_2^j, d_1^j]$ (see the picture below).

Here two cases appear: if we add one or two edges to complete a triangle to obtain the $(j+1)$-th path from the j-th. We shall consider only the case in which two edges are added, the other is analogous.

Let us state $\alpha_0 = \alpha$. For $j = 1, ..., l$ we construct a collection of paths $\alpha_j : [0,1] \to X$ such that $p_{\mathcal{W}} \circ \alpha_j \simeq f_j$. The path α_j is defined as α_{j-1}, except for the interval correspondent to $[d_0^j, d_1^j]$ in which it is defined as follows:

- If $H(\Delta_j)$ is a vertex, then $W_0^j = W_1^j = W_2^j$. We can declare
$$\alpha_j|_{[d_0^j,d_2^j]+[d_2^j,d_1^j]} \equiv x_k^j,$$
and no matter if $k = 0,1,2$. (Observe that this case is only possible if α_{j-1} was also constant in $[d_0^j, d_1^j]$, since H is affine. Thus, $\alpha_j = \alpha_{j-1}$).

- If $H(\Delta_j)$ is an edge, a similar reasoning as above allows to take $\alpha_j = \alpha_{j-1}$.

- If $H(\Delta_j)$ is properly a triangle, let W_0^j, W_1^j, W_2^j be the sets associated to $H(d_0^j), H(d_1^j)$ and $H(d_2^j)$, respectively. Then, there exist points
$$y_0^j \in W_0^j \cap W_1^j,$$
$$y_1^j \in W_1^j \cap W_2^j$$
and
$$y_2^j \in W_0^j \cap W_2^j,$$
because $[H(d_0^j), H(d_1^j)], [H(d_1^j), H(d_2^j)]$ and $[H(d_0^j), H(d_2^j)]$ are 1-simplexes of $|N(\mathcal{W})|$. Since W_0^j, W_1^j, W_2^j are path-connected, there exist paths
$$\xi_k^j, \zeta_k^j : [0,1] \to W_k^j$$
such that
$$\xi_k^j(0) = \zeta_k^j(0) = x_k^j, \quad \xi_k^j(1) = y_k^j,$$
$$\zeta_0^j(1) = y_2^j, \quad \zeta_1^j(1) = y_0^j \text{ and } \zeta_2^j(1) = y_1^j.$$
Let us observe that the concatenation $\xi_0^j * \overline{\zeta}_1^j$ was already chosen from the construction of β_{j-1}. The path α_j in the interval correspondent to $[d_0^j, d_2^j] + [d_2^j, d_1^j]$ is defined as $\zeta_0^j * \overline{\xi}_2^j * \zeta_2^j * \overline{\xi}_1^j$.

Moreover,
$$W_0^j \cap W_1^j \cap W_2^j \neq \varnothing$$
since $H(\Delta_j)$ is a triangle. From the properties of $\mathcal{W}$ stated in lemma 1.4.9, there exists an open set $V_j \in \mathcal{V}$ such that
$$W_0^j \cup W_1^j \cup W_2^j \subseteq V_j.$$

Finally, since $\mathcal{V}$ was a refinement of $\mathcal{U}$, there exists $U_j \in \mathcal{U}$ such that $V_j \subseteq U_j$. Thus, the concatenation $\xi_0^j * \overline{\zeta}_1^j * \xi_1^j * \overline{\zeta}_2^j * \xi_2^j * \overline{\zeta}_0^j$ is a loop contained in U_j, so we have reconstructed $\mathcal{U}$-lassos with concatenations, analogously than in lemma 1.4.8.

The last step of this finite process is $\alpha_l = \beta$. Hence, $\beta \simeq \ell * \alpha$, where $\ell \in \pi^{Sp}(\mathcal{U}, x_0)$. Since $\mathcal{U}$ was arbitrary, then $Sp(\alpha) = Sp(\beta)$, as desired. □

This result has some consequences because of the definitions of topologies and subgroups:

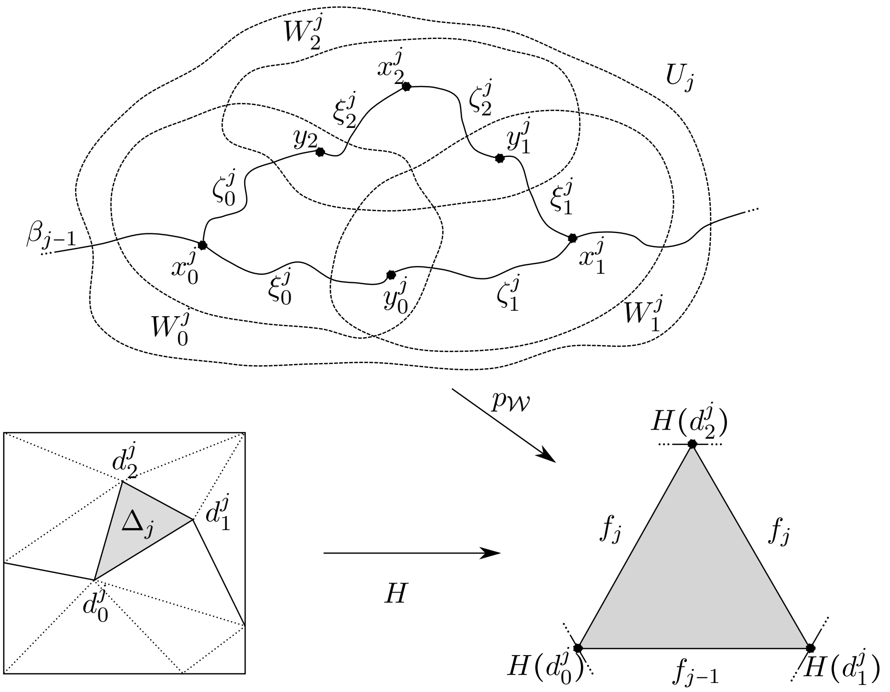


Figure 1.8: How to obtain lassos from the nerve following the homotopy

Corollary 1.4.12 *If (X, x_0) is a paracompact, Hausdorff, path-connected and locally path connected pointed space, then $Sp(\alpha) = D(\alpha) = \langle\alpha\rangle$ for each $\alpha \in \Omega(X, x_0)$.*

Proof. It has been already proved $Sp(\alpha) \subseteq D(\alpha) \subseteq \langle\alpha\rangle$. In addition, by 1.4.11 $Sp(\alpha) = \langle\alpha\rangle$. □

This corollary immediately leads to:

Corollary 1.4.13 [2] *If (X, x_0) is a paracompact, Hausdorff, path-connected and locally path connected pointed space, then $Ker\varphi = \nu(X, x_0) = \pi^{Sp}(X, x_0)$.*

We point out that, in particular, for the compact metric case which is the main part of this work, the above result implies the next.

Corollary 1.4.14 *If X is a Peano continuum, then $Ker\varphi = \nu(X, x_0) = \pi^{Sp}(X, x_0)$.*

Corollary 1.4.15 *If (X, x_0) is shape-injective, then $Ker\varphi = \nu(X, x_0) = \pi^{Sp}(X, x_0) = \{1\}$*

[2]The same theorem was obtained independently by J. Brazas and P. Fabel in [18]. There, the authors introduce an intermediate group, which they call *thick Spanier group*. Our proof is completely different and we are not going to need this concept here, so we obviate that definition.

1.4.3 Extensions of results to a wider class: $AANR_{C*}$

We restrict again only to compact metric spaces.

Remark 1.4.16 In general, the corollary 1.4.14 is not true if X is not locally path-connected, which also shows that previous relations between groups can be strict ones. Coming back to the sombrero space of example 1.3.17, it is easy to check that there we have that $\pi^{Sp}(X,x_0) = \nu(X,x_0)$ but $\pi^{Sp}(X,x_0) \neq Ker\varphi$.

The aim of the following is to extend this corollary to a bigger class of spaces.

Proposition 1.4.17 *If (Y,y_0) is pointed homotopy dominated by (X,x_0) and $\pi^{Sp}(X,x_0) = Ker\varphi_X$, then $\pi^{Sp}(Y,y_0) = Ker\varphi_Y$. In particular, if (X,x_0) and (Y,y_0) are homotopy equivalent as pointed topological spaces and $\pi^{Sp}(X,x_0) = Ker\varphi_X$, then $\pi^{Sp}(Y,y_0) = Ker\varphi_Y$.*

Proof. By hypothesis, there exist maps $f:(X,x_0)\to(Y,y_0)$ and $g:(Y,y_0)\to(X,x_0)$ such that $f\circ g\simeq 1_Y$. Since the next square is commutative:

$$\begin{array}{ccc} \pi_1(Y,y_0) & \xrightarrow{g_*} & \pi_1(X,x_0) \\ {\scriptstyle\varphi_Y}\downarrow & & \downarrow{\scriptstyle\varphi_X} \\ \check{\pi}_1(Y,y_0) & \xrightarrow[\check{g}]{} & \check{\pi}_1(X,x_0) \end{array}$$

we claim that $g_*(Ker\varphi_Y)\subseteq Ker\varphi_X$. Actually, let $[\alpha]$ be in $\pi_1(Y,y_0)$ such that $\varphi_Y([\alpha])=0$. Then

$$\varphi_X\circ g_*([\alpha]) = \check{g}\circ\varphi_Y([\alpha]) = \check{g}(0) = 0,$$

as desired. In addition, because of $f_*\circ g_* = 1_{\pi_1(Y,y_0)}$, we obtain

$$Ker\varphi_Y = f_*g_*(Ker\varphi_Y)\subseteq f_*(Ker\varphi_X) = f_*(\pi^{Sp}(X,x_0))\subseteq\pi^{Sp}(Y)\subseteq Ker\varphi_Y$$

where lemma 1.3.23 was applied. □

Corollary 1.4.18 *If (X,x_0) is pointed homotopy dominated by a locally path connected space, then $\pi^{Sp}(X,x_0) = Ker\varphi_X$.*

Proof. It follows by combining 1.4.14 and 1.4.17. □

A very particular case in which Corollary 1.4.14 holds, is for ANR's. This suggest us to try to analyse the situation in a bigger class of spaces. Such spaces are approximative absolute neighbourhood retracts in the sense of Clapp, introduced in [22] and denoted by

$\mathrm{AANR_C}$, which are a natural generalization of ANR's since they are limits of polyhedra in the metric of continuity in the hyperspace of the Hilbert cube.

Although the unpointed version of $\mathrm{AANR_C}$ seems to be well-known, there is no definition made for the pointed version. Since we are dealing with pointed spaces, we first introduce the analogous definition for pointed $\mathrm{AANR_C}$ and the pointed metric of continuity.

Definition 1.4.19 Given two pointed compact metric spaces (X,x_0) and (Y,y_0) in the Hilbert cube (Q,ρ), we shall say that $d_{c*}((X,x_0),(Y,y_0)) < \varepsilon$ if and only if there exist continuous maps $f:(X,x_0)\to(Y,y_0)$ and $g:(Y,y_0)\to(X,x_0)$ such that $\rho(x,f(x))<\varepsilon$ for every $x\in X$ and $\rho(y,g(y))<\varepsilon$ for every $y\in Y$. It is clear that d_{c*} is a metric, and we shall call it *the pointed metric of continuity.*

Remark 1.4.20 If d_c and d_H denote, respectively, the metric of continuity and the Hausdorff metric in 2^Q, then $d_{c*}\geq d_c\geq d_H$.

Proposition 1.4.21 *Let X be an ANR, and $\{x_n\}_{n=1}^{\infty}\subseteq X$ a sequence of points such that $x_n\to x_0$, $x_0\in X$. Then, $(X,x_n)\to(X,x_0)$ in d_{c*}.*

Proof. Let X be embedded in $Q=\prod_{n\in\mathbb{N}}[-\frac{1}{n},\frac{1}{n}]$ and let $\varepsilon>0$. There exist a polyhedron $P\subseteq B(X,\varepsilon)$ which is a neighbourhood of X in Q and a (uniform) retraction $r:P\to X$, so there is δ_1 such that if $\rho(a,b)<\delta_1$, then $\rho(r(a),r(b))<\varepsilon$.

Let us take $\delta_2<\delta_1$, such that $B(X,\delta_2)\subseteq P$ and let $\delta>0$ with $\delta<\delta_2$, in order to verify $\overline{B}(x_0,\delta)\subseteq B(x_0,\delta_2)$.

Since $x_n\to x_0$, there exists n_0 big enough such that $x_n\in B(x_0,\delta)$ for every $n\geq n_0$.

Fix $N\geq n_0$, and consider the map $\tilde f:Q\to Q$ given by

$$\tilde f(x)=\begin{cases} x+\frac{\delta-\rho(x_0,x)}{\delta}(x_N-x_0) & if \quad x\in\overline{B}(x_0,\delta)\\ x & if \quad x\in Q\setminus B(x_0,\delta)\end{cases}$$

which is continuous. Moreover,

$$\begin{aligned}\rho(x,\tilde f(x)) &= \|x+\frac{\delta-\rho(x_0,x)}{\delta}(x_N-x_0)-x\| = \frac{\delta-\rho(x_0,x)}{\delta}\|x_N-x_0\| = \\ &= \frac{\|x_N-x_0\|}{\delta}(\delta-\rho(x_0,x)) < \delta-\rho(x_0,x)<\delta<\delta_2<\delta_1.\end{aligned}$$

Then,

$$f=\tilde f|_X : X\to B(X,\delta_2)$$

and, in addition, $r:P\to X$ and $B(X,\delta_2)\subseteq P$. Therefore,

$$F=r\circ f:X\to X$$

satisfies $F(x_0) = r(f(x_0)) = r(x_N) = x_N$ and $\rho(x, F(x)) = \rho(r(x), r(f(x))) < \varepsilon$, because $\rho(x, f(x)) < \delta_1$.

By an analogous reasoning but with $B(x_N, \delta)$, we obtain a map $G : X \to X$ such that $G(x_N) = x_0$ and $\rho(x, G(x)) < \varepsilon$. Hence, $d_{c*}((X, x_n), (X, x_0)) < \varepsilon$ for all $n \geq n_0$. Consequently, $(X, x_n) \to (X, x_0)$ in d_{c*}. □

Definition 1.4.22 A pointed space (X, x_0) is called a *pointed absolute approximative neighbourhood retract in the sense of Clapp,* $\mathrm{AANR}_{\mathrm{C}^*}$ for short, if and only if for every $\varepsilon > 0$ there exists a neighbourhood $(U_\varepsilon, x_0) \supset (X, x_0)$ and a ε-retraction, that is, a map $r_\varepsilon : (U_\varepsilon, x_0) \to (X, x_0)$ such that $\rho(r_\varepsilon(x), x) < \varepsilon$ for every $x \in X$.

Remark 1.4.23 Following [22], the previous definition is equivalent to the existence of a nested sequence $X_1 \supset X_2 \supset ...$ of ANR's such that $(X_n, x_0) \to (X, x_0)$ in d_{C*}.

In addition, it is clear that every $\mathrm{AANR}_{\mathrm{C}^*}$ is, in particular, an $\mathrm{AANR}_{\mathrm{C}}$.

Example 1.4.24 The Hawaiian Earring HE is an $\mathrm{AANR}_{\mathrm{C}^*}$. It is clear that $(\vee_n S^1, x_0) \to (HE, x_0)$ in d_{C*}. Just consider inclusions $i_n : \vee_n S^1$ and the maps $p_n : HE \to \vee_n S^1$ which is the identity in the first n circles of HE and collapses the rest to the tangency point x_0.

Example 1.4.25 The Warsaw circle is not an $\mathrm{AANR}_{\mathrm{C}^*}$. If it would be an $\mathrm{AANR}_{\mathrm{C}^*}$, since it is not an $\mathrm{AANR}_{\mathrm{C}}$.

The following examples show that there exist $\mathrm{AANR}_{\mathrm{C}^*}$ which are not a Peano continua, so the class of $\mathrm{AANR}_{\mathrm{C}^*}$ spaces is not the same as that of Peano continua. Neither the second class is contained in the first, since there exist Peano continua which are not $\mathrm{AANR}_{\mathrm{C}^*}$.

Example 1.4.26 The closed comb (see picture 1.9) is an $\mathrm{AANR}_{\mathrm{C}^*}$, but not a Peano continuum.

Similar than in the Hawaiian Earrings example, it is enough to consider maps $f_n : C \to C_n$ where C is the closed comb space and C_n is the same space deleting the vertical lines from the n-th line onwards.

Example 1.4.27 The continuum of Borsuk (described in [12]) consisting on a perforated pyramid by cylinders is a Peano continuum which is not movable. Thus, it can not be an $\mathrm{AANR}_{\mathrm{C}}$, so neither an $\mathrm{AANR}_{\mathrm{C}^*}$.

The behaviour of $\mathrm{AANR}_{\mathrm{C}^*}$ spaces in the pointed metric of continuity is as follows.

Proposition 1.4.28 *The class of spaces in* 2^Q *which satisfies* $\nu(X, x_0) = Ker\varphi$ *is closed in* $(2^Q, d_{c*})$.

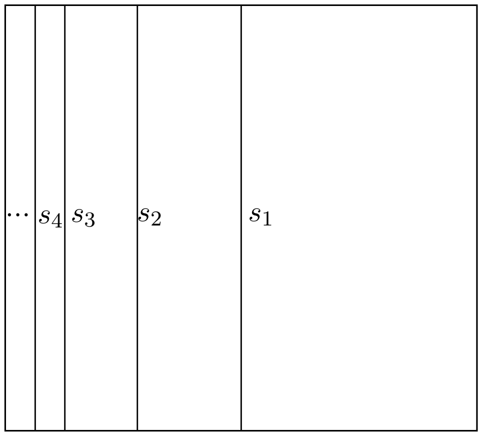


Figure 1.9: The closed comb space.

Proof. Let us take a convergent sequence of pointed compact metric spaces, $(X_n, x_n) \to (X, x_0)$ in d_{c*} such that $\nu(X_n, x_n) = Ker\varphi_{X_n}$, where $\varphi_{X_n} : \pi_1(X_n, x_n) \to \check{\pi}_1(X_n, x_n)$. We have to prove that also $\nu(X, x_0) = Ker\varphi_X$.

Only $Ker\varphi_X \subseteq \nu(X, x_0)$ needs to be shown, because the other inclusion holds in general, from 1.4.1. Let $[\alpha]$ be in $\pi_1(X, x_0)$ such that $\varphi_X([\alpha]) = Sh(c_{x_0})$, and let $\varepsilon > 0$. We have to construct an ε-chain between α and c_{x_0}.

Since $(X_n, x_n) \to (X, x_0)$, there exists $N \in \mathbb{N}$ such that $d_{c*}((X_n, x_n), (X, x_0)) < \varepsilon/2$, that is, there exist $f^N : (X, x_0) \to (X_N, x_N)$ and $g^N : (X_N, x_N) \to (X, x_0)$ such that $\rho(x, f^N(x)) < \varepsilon/2$ for all $x \in X$ and $\rho(y, g^N(y)) < \varepsilon/2$ for all $y \in X_N$.

Consider now the induced maps $f_*^N, g_*^N, \check{f}^N$ and $\check{g}^N$ induced by f and g over the fundamental and shape groups, respectively. We claim that $[f^N \circ \alpha] \in \nu(X_N, x_n)$. In fact,

$$\varphi_{X_N}([f^N \circ \alpha]) = \varphi_{X_N} \circ f_*^N([\alpha]) = \check{f}^N \circ \varphi_X([\alpha]) = 0$$

since $[\alpha] \in Ker\varphi_X$. Hence, $[f^N \circ \alpha] \in Ker\varphi_{X_N}$. In addition, we have $\nu(X_N, x_N) = Ker\varphi_{X_N}$ by hypothesis.

The map g^N is uniformly continuous, so there exists $\delta > 0$ such that $\rho(a, b) < \delta$ implies $\rho(g^N(a), g^N(b)) < \varepsilon$.

Let $\gamma_0 = f^N \circ \alpha, \gamma_1, ..., \gamma_m = c_{x_N}$ be a δ-chain in (X_N, x_N) joining $f^N \circ \alpha$ with c_{x_N}. Then, $g^N \circ \gamma_0 = g^N \circ f^N \circ \alpha, g^N \circ \gamma_1, ..., g^N \circ \gamma_m = g^N \circ c_{x_N} = c_{x_0}$ is an ε-chain joining $g^N \circ f^N \circ \alpha$ and c_{x_0}. Moreover:

$$\rho(\alpha(t)), g^N \circ f^N \circ \alpha(t) \leq \rho(\alpha(t), f^N(\alpha(t))) + \rho(f^N(\alpha(t)), g^N \circ f^N \circ \alpha(t)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\alpha_0 = \alpha, \alpha_1 = g^N \circ f^N \circ \alpha, \alpha_2 = g^N \circ \gamma_1, ..., \alpha_{m+1} = g^N \circ \gamma_m = c_{x_0}$ is an ε-chain between α and c_{x_0}. Consequently, $[\alpha] \in \nu(X, x_0)$. □

Corollary 1.4.29 *If* (X, x_0) *is an* $AANR_{C^*}$*, then* $\nu(X, x_0) = Ker\varphi$.

Example 1.4.30 Consider again the sombrero space of example 1.3.17. We see there $\pi^{Sp}(X, x_0) = \nu(X, x_0) = \{1\}$, while $Ker\varphi = \pi(X, x_0)$. Hence, $(X, x_0) \notin \overline{\{Peano_*\}}$ in $(2^Q, d_{c*})$.

Chapter 2

Čech homology and cohomology groups

The crucial idea of the ultrametric constructed in [68] and [69] is that shape morphisms have a structure of inverse limit. As we have shown in the first chapter of the present work, this ideas give in particular an ultrametric for the shape groups $\check{\pi}_k(X, x_0)$ and also a pseudoultrametric for the fundamental group $\pi_1(X, x_0)$ (and in higher homotopy groups $\pi_k(X, x_0)$) of a pointed compact metric space (X, x_0). In this chapter, we shall exhibit how the same techniques are also useful in order to define an ultrametric in Čech homology groups, since the way of constructing these groups is by means of inverse limits.

We need to adapt Čech homology to our purposes. After that, we define the correspondent ultrametric in $\check{H}_*(X)$ and give some results. Also, we give the topological version of the Hurewicz homomorphism, and its analogous homomorphism in the theory of shape.

Finally, we show that is not possible to obtain a similar construction on the Čech cohomology groups. Considering S^1 as group of coefficients for cohomology, it is shown how to reinterpret $\check{H}^*(X; S^1)$ as the dual group of $\check{H}_*(X; \mathbb{Z})$ in the sense of the theory of Pontryagin duality.

2.1 Ultrametric on the Čech homology groups

We have already seen in the case of the shape groups that the key point is to reinterpret the notion of fundamental sequence, given by Borsuk, as a certain Cauchy sequence. Then, it is necessary to define the analogous concepts of fundamental sequences for cycles and a relation of homology between them. Actually, here we shall make use of the expansion associated to a compact metric space X by its neighbourhoods in the Hilbert cube Q. We refer to the chapter of preliminaries for the terminology and notation used for homology.

2.1.1 Approximative homology

Let us introduce a suitable way to read the Čech homology, similar to Borsuk's construction of shape groups, using the neighbourhoods of X.

Definition 2.1.1 Let $Z_n(Q)$ be the set of all n-dimensional cycles in Q. Given $A \subseteq Q$, we say that σ, σ' in $Z_n(Q)$ are *homologous in* A if there exists γ in $\mathcal{C}_{n+1}(A)$ such that $\partial\gamma = \sigma - \sigma'$. We shall denote it by $\sigma \sim \sigma'$ in A.

Remark 2.1.2 The previous definition implies that each simplex of the $(n+1)$-chain γ lies in A. In particular, the simplexes of σ and σ' must lie in A. Otherwise, it is impossible that σ and σ' would be homologous in A. Also note that every pair of cycles are homologous in Q, because $H_n(Q, G) = 0$ (since Q is contractible).

As we have already mentioned, we follow the construction of Borsuk for the shape groups in order to obtain a more geometrical interpretation of the Čech homology groups.

Definitions 2.1.3 Let X be a compact metric space in Q.

a) A sequence $\sigma = \{\sigma_i\}_{i\in\mathbb{N}}$ in $\mathcal{C}_n(Q)$ is said to be an *infinite n-chain (in Q) for X* if there exists a sequence $\{\varepsilon_i\} \to 0$ such that σ_i is an n-chain in $B(X, \varepsilon_i)$ for each $i \in \mathbb{N}$. (If it is clear, we shall omit the dimension n of the chains in the sequel).

b) The *boundary of an infinite chain* is defined as the classical boundary acting in each chain of the sequence, i.e.,
$$\partial\sigma = \partial\{\sigma_i\} = \{\partial\sigma_i\}$$
for an infinite chain σ. An infinite chain σ is called *infinite cycle* if $\partial\sigma = 0$, or equivalently, if each chain of the sequence is a cycle.

c) An infinite cycle $\sigma = \{\sigma_i\}$ is said to be an *approximative cycle* if for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i+1}$ in $B(X, \varepsilon)$ for all $i \geq i_0$, or equivalently, if there exists an infinite chain $\gamma = \{\gamma_i\}$ such that $\partial\gamma_i = \sigma_i - \sigma_{i+1}$. We shall denote by $\mathcal{Z}_n^{\mathcal{A}}(X)$ the set of approximative cycles of dimension n.

d) Given two approximative cycles $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$, a sum $\sigma + \tau$ is defined as the (usual) sum of singular chains in each level, i.e.,
$$\sigma + \tau = \{\sigma_i\} + \{\tau_i\} = \{\sigma_i + \tau_i\}.$$
Immediately, $\mathcal{Z}_n^{\mathcal{A}}(X)$ with this operation is a group.

e) Two approximative cycles $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$ are said to be *homologous approximative cycles* if for all $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X, \varepsilon)$ for all $i \geq i_0$, or equivalently, if there exists an infinite chain $\gamma = \{\gamma_i\}$ such that $\partial\gamma = \sigma - \tau$. If an approximative cycle σ is homologous to zero, $\sigma \sim 0$, i. e. if there exists an infinite chain γ such that $\partial\gamma = \sigma$, we said that σ is an *approximative boundary.* We shall denote by $\mathcal{B}_n^{\mathcal{A}}(X)$ the subgroup of $\mathcal{Z}_n^{\mathcal{A}}(X)$ of approximative boundaries of dimension n.

f) The *n-dimensional approximative homology group of* X is defined as the quotient group

$$H_n^{\mathcal{A}}(X) = \frac{\mathcal{Z}_n^{\mathcal{A}}(X)}{\mathcal{B}_n^{\mathcal{A}}(X)}.$$

It is obvious how to define an operation between homology classes of approximative cycles, just as the sum of its representatives,

$$[\sigma] + [\tau] = [\{\sigma_i\}] + [\{\tau_i\}] = [\{\sigma_i\} + \{\tau_i\}] = [\{\sigma_i + \tau_i\}] = [\sigma + \tau]$$

and the following it is straightforward obtained.

Proposition 2.1.4 $(H_n^{\mathcal{A}}(X), +)$ *is an Abelian group.*

Proof. It is immediately checked that the operation is well-defined over the quotient $\mathcal{Z}_n^{\mathcal{A}}(X)/\mathcal{B}_n^{\mathcal{A}}(X)$. Because of the properties of the singular homology (applied to each term of the sequences that define the approximative cycles), the properties of a commutative group are satisfied by $(\check{H}_n(X), +)$, as it is shown in the next.

- Associativity:

$$\begin{aligned}\Big([\{\sigma_i\}] + [\{\tau_i\}]\Big) + [\{\eta_i\}] =& [\{\sigma_i + \tau_i\}] + [\{\eta_i\}] = \\ =& [\{(\sigma_i + \tau_i) + \eta_i\}] = [\{\sigma_i + (\tau_i + \eta_i)\}] = \\ =& [\{\sigma_i\}] + [\{\tau_i + \eta_i\}] = [\{\sigma_i\}] + \Big([\{\tau_i\}] + [\{\eta_i\}]\Big).\end{aligned}$$

- The identity element is $0 = [\{0\}]$, the class generated by the sequence in which each term is the (formal) zero element of the corresponding chain groups.
- Given $[\sigma] = [\{\sigma_i\}]$, its inverse $-[\sigma]$ is given by the sequence of the inverse element of each term of the sequence σ, i. e., $-[\sigma] = [\{-\sigma_i\}]$.
- Commutativity:

$$[\{\sigma_i\}] + [\{\tau_i\}] = [\{\sigma_i\} + \{\tau_i\}] = [\{\tau_i + \sigma_i\}] = [\{\tau_i\} + \{\sigma_i\}] = [\{\tau_i\}] + [\{\sigma_i\}].$$

□

Remark 2.1.5 Every singular cycle of X gives an approximative cycle: if $s = \sum_{i=1}^{k} n_i s_i$ is a cycle where $s_i : \Delta^n \to X$, put $\sigma_i = s$ for every $i \in \mathbb{N}$ in order to obtain a constant sequence $\sigma = \{\sigma_i\}$ which obviously is an approximative cycle. In this case, we say that σ *is generated by the singular chain* s *of* X.

Moreover, if $s \sim t$ as n-cycles in X, then the corresponding approximative cycles σ and τ satisfies $\sigma \sim \tau$ as approximative cycles. In that way, for each $n \geq 0$, it is obtained an assignation

$$\varphi^{\mathcal{A}} : H_n(X) \longrightarrow H_n^{\mathcal{A}}(X)$$

such that $\varphi^{\mathcal{A}}([\sigma])$ is a well-defined approximative homology class.

In fact, it is obvious that $\varphi^{\mathcal{A}}$ is a homomorphism of Abelian groups.

The functorial properties of the approximative homology are resumed in the following results. For this, we follow the description of Borsuk for the (Vietoris) homology properties of shape, but in the framework of approximative cycles.

Lemma 2.1.6 *If $\{f_k, X, Y\}$ is a fundamental sequence between compact metric spaces, then for every approximative cycle $\sigma = \{\sigma_i\}$ for X, there exists an increasing sequence $\{i_k\}$ of indices such that for every sequence of indices $\{j_k\}$ such that $j_k \geq i_k$ for $k \in \mathbb{N}$, the sequence $\{f_{k_\#}(\sigma_{j_k})\}$ is an approximative cycle for Y, where $f_{k\#}$ is the induced map by each f_k in the group of chains of Q.*

Proof. By definition of fundamental sequence, for every neighbourhood V of Y there exists a neighbourhood U of X such that $f_k|_U \simeq f_{k+1}|_U$ in V for almost k. So for any sequence $\{\eta_k\}$ (without loss of generality it can be supposed that $\eta_k = \frac{diamQ}{k}$) of positive numbers tending to zero, there exists a sequence $\{\delta_k\}$, also of positive numbers converging to zero, such that

$$\text{if } \sigma \text{ is a cycle in } B(X, \delta_k) \text{ then } f_{k_\#}(\sigma) \sim f_{k+1_\#}(\sigma) \text{ as cycles in } B(Y, \varepsilon_k) \text{ for } k \in \mathbb{N},$$

and

$$\text{if } \sigma \sim \sigma' \text{ as cycles in } B(X, \delta_k) \text{ then } f_{k_\#}(\sigma) \sim f_{k+1_\#}(\sigma') \text{ as cycles in } B(Y, \varepsilon_k) \text{ for } k \in \mathbb{N}.$$

Let $\sigma = \{\sigma_i\}$ be an approximative cycle and take $\{\varepsilon_i\}$ such that σ_i is a cycle in $B(X, \varepsilon_i)$. This sequence $\{\varepsilon_i\}$ converges to zero because of the definition of approximative cycle. Fix a sequence of indices $\{i_k\}$ such that $\varepsilon_j \leq \delta_k$ for every $j \geq i_k$. Then, if a sequence of indices $\{j_k\}$ satisfies $j_k \geq i_k$ for $k \in \mathbb{N}$, then $\sigma_{j_k}, \sigma_{j_{k+1}}$ are homologous cycles in $B(X, \delta_k)$, so

$$f_{k_\#}(\sigma_{j_{k+1}}) \sim f_{k+1_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k)$$

and

$$f_{k_\#}(\sigma_{j_k}) \sim f_{k_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k).$$

Therefore,

$$f_{k_\#}(\sigma_{j_k}) \sim f_{k+1_\#}(\sigma_{j_{k+1}}) \text{ in } B(Y, \eta_k)$$

and it follows that $\{f_{k_\#}(\sigma_{j_k})\}$ is an approximative cycle for Y. □

Remark 2.1.7 From the previous proof, if $\sigma' = \{\sigma'_i\}$ is another approximative cycle homologous to σ, then the sequence of indices $\{i_k\}$ can be selected so that the inequality $j_k \geq i_k$ for $k \in \mathbb{N}$ implies that $\{f_{k\#}(\sigma_{j_k})\}$ and $\{f_{k\#}(\sigma'_{j_k})\}$ are homologous approximative cycles. The approximative homology class of the approximative cycle $\{f_{k\#}(\sigma_{j_k})\}$ is independent from the choice of the subsequence.

With the lemma above, we get the next.

Proposition 2.1.8 *Every continuous map $f : X \to Y$ between compact metric spaces induces, for each $n \geq 0$, a homomorphism $f_* : H_n^{\mathcal{A}}(X) \to H_n^{\mathcal{A}}(Y)$ between its approximative homology groups. If $g : Y \to Z$ is another continuous map to a compact metric space Z, then $(g \circ f)_* = g_* \circ f_*$. Moreover, the identity map $i : X \to X$ induces the identity homomorphism $i_* : H_n^{\mathcal{A}}(X) \to H_n^{\mathcal{A}}(X)$.*

Proof. Given an approximative class in $H_n^{\mathcal{A}}(X)$ with representative $\{\sigma_i\}$, let us assign the approximative class in $H_n^{\mathcal{A}}(Y)$ with representative $\{f_{k\#}(\sigma_{j_k})\}$ given by 2.1.6. In that way, we obtain a map between the correspondent approximative homology groups, that clearly is an homomorphism of groups[1].

The remaining properties are straightforward checked. □

The viewpoint of the homology from approximative cycles agrees with singular homology in the case of being X an ANR. We need to use the following assertion (see lemma 3.8 of [11]).

Lemma 2.1.9 *If $r : U \to X$ is a retraction of a neighbourhood U of X (in Q), then for every neighbourhood V of X (in Q) there is a neighbourhood V' of X (in Q) and a homotopy $h : V' \times [0,1] \to V$ such that $h(x,0) = x$ and $h(x,1) = r(x)$ for every point $x \in V'$.*

Lemma 2.1.10 *Let X be an ANR in Q. Every approximative cycle σ is homologous to an approximative cycle τ generated by a singular chain of X.*

Proof. Let $\sigma = \{\sigma_i\} \subset Z_n(Q)$ be an approximative cycle. Since X is an ANR, there exists a neighbourhood U of X in Q and a retraction $r : U \to X$. Let us take $\varepsilon_0 > 0$ such that $B(X, \varepsilon_0) \subseteq U$.

For this ε_0, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i+1}$ in $B(X, \varepsilon_0)$ for all $i \geq i_0$. Hence, $\sigma_{i_0} \sim \sigma_i$ in $B(X, \varepsilon_0)$ for all $i \geq i_0$.

The retraction r induces $r_\# : \mathcal{C}_n(U) \to \mathcal{C}_n(X)$ by

$$r_\#(\sum_{i=1}^{k} n_i \gamma_i) = \sum_{i=1}^{k} n_i (r \circ \gamma_i).$$

[1] Notice also that homotopic fundamental sequences induce the same homomorphism in homology. We do not need that result and, in fact, it shall be deduced from the isomorphism of the approximative homology to the Čech homology.

Let us consider $r_{\#}(\sigma_{i_0})$, which is a singular cycle in X, and put $\tau_i = r_{\#}(\sigma_{i_0})$ for all $i \in \mathbb{N}$. Thus we have an approximative cycle $\tau = \{\tau_i\}$ generated by the singular chain $r_{\#}(\sigma_{i_0})$.

The induced infinite chain $r_{\#}(\sigma) = \{r_{\#}(\sigma_i)\}$ is again an approximative cycle such that $r_{\#}(\sigma) \sim \tau$. Indeed:

The relation

$$\sigma_{i_0} \sim \sigma_i \text{ in } B(X, \varepsilon_0) \text{ for all } i \geq i_0$$

implies

$$r_{\#}(\sigma_{i_0}) \sim r_{\#}(\sigma_i) \text{ in } X \text{ for all } i \geq i_0,$$

but $r_{\#}(\sigma_{i_0}) = \tau_i$, so

$$\tau_i \sim r_{\#}(\sigma_i) \text{ in } X \text{ for all } i \geq i_0,$$

thus, $\tau \sim r_{\#}(\sigma)$.

On the other hand, using 2.1.9, for $\varepsilon > 0$ let $\delta > 0$ (we can assume $0 < \delta < \varepsilon < \varepsilon_0$) be such that there exists an homotopy

$$h : B(X, \delta) \times [0, 1] \to B(X, \varepsilon)$$

such that $h(x, 0) = x$ and $h(x, 1) = r(x)$ for every point $x \in B(X, \delta)$. For this δ, there exists $i_1 \in \mathbb{N}$ such that σ_i lies in $B(X, \delta)$ for every $i \geq i_1$ (and also in $B(X, \varepsilon)$). If we denote

$$h_0 = h|_{B(X,\delta)\times\{0\}} \quad \text{and} \quad h_1 = h|_{B(X,\delta)\times\{1\}},$$

then

$$\sigma_i = id_{\#}(\sigma_i) = h_{0\#}(\sigma_i) \sim h_{1\#}(\sigma_i) = r_{\#}(\sigma_i)$$

in $B(X, \varepsilon)$. Thus, $r_{\#}(\sigma) \sim \sigma$ and consequently, $\sigma \sim \tau$. □

Proposition 2.1.11 *Let X be an ANR in Q. Then, $H_n^{\mathcal{A}}(X)$ is isomorphic to $H_n(X)$, for every $n \geq 0$.*

Proof. As it was pointed out in 2.1.5, every singular homology class generates an approximative one. Moreover, if two cycles are representatives of the same singular class, they are also in the same approximative class. Conversely, by the preceding lemma, every approximative class is generated by a well-defined singular class, so $\psi^{\mathcal{A}}$ is a bijective homomorphism. □

In fact, the groups defined in 2.1.3 are the same as classical Čech homology groups, as the next proposition shows.

Proposition 2.1.12 *Let X be a compact metric space in Q. Then $H_n^{\mathcal{A}}(X)$ is isomorphic to $\check{H}_n(X)$, for each $n \geq 0$.*

Proof. Consider Borsuk's inverse system for a decreasing sequence of real numbers $\{\varepsilon_m\}$. That is, for each $m \in \mathbb{N}$ take $X_m = B(X, \varepsilon_m)$, i_{mm+1} as the inclusion of X_{m+1} in X_m, and polyhedra P_m such that

$$B(X, \varepsilon_1) \supseteq P_1 \supseteq B(X, \varepsilon_2) \supseteq P_2 \supset \cdots \supset X.$$

The inverse limit of this inverse sequence is $\bigcap_{m \in \mathbb{N}} X_m = X$, and therefore the same inverse limit is valid for the sequence of polyhedra and inclusions $p_{m\,m+1}$. Hence, $\check{H}_n(X)$ is the inverse limit of the inverse sequence of groups $\{H_n(P_m), (p_{m\,m+1})_*\}$.

By definition of inverse limit, the elements of $\check{H}_n(X)$ are sequences $\{\sigma_m\}$ such that each σ_m lies in $P_m \subseteq B(X, \varepsilon_m)$ and $(p_{mm+1})_*(\sigma_{m+1}) = \sigma_m$. That is, $\sigma_{m+1} \sim \sigma_m$ in $B(X, \varepsilon_m)$ for every $m \in \mathbb{N}$. Therefore, it is clear that every Čech homology class is an approximative one.

On the other hand, let $\sigma = \{\sigma_i\}$ be an approximative cycle. We can choose a subsequence $\sigma' = \sigma_{i_m}$ with σ_{i_m} lying in $B(X, \varepsilon_m)$. Obviously, σ' is again an approximative cycle and $\sigma \sim \sigma'$. From that, $\sigma_{i_m} \sim \sigma_{i_{m+1}}$ in $B(X, \varepsilon_m)$. Consequently, σ' is an element of $\check{H}_n(X)$.

Moreover, it is clear that the group operations of $\check{H}_n(X)$ and $H^{\mathcal{A}}(X)$ are equivalent.

□

Corollary 2.1.13 *Let X be a compact metric space, and $\{X_n, p_{n\,n+1}\}$ an inverse sequence of ANR's such that $X = lim_{\leftarrow}\{X_n, p_{n\,n+1}\}$. Then,*

$$H_k^{\mathcal{A}}(X) = \lim_{\leftarrow}\{H_k^{\mathcal{A}}(X_n), (p_{n\,n+1})_*\}$$

for each $k \geq 0$.

Proof. From 2.1.11 and 2.1.12:

$$H_k^{\mathcal{A}}(X) = \check{H}_k(X) = \lim_{\leftarrow}\{H_k(X_n), (p_{n\,n+1})_*\} = \lim_{\leftarrow}\{H_k^{\mathcal{A}}(X_n), (p_{n\,n+1})_*\}.$$

□

2.1.2 Cantor completion process for an ultrametric on $\check{H}_n(X, \mathbb{Z})$

Over $Z_n(Q)$, we define a map $F : Z_n(Q) \times Z_n(Q) \to \mathbb{R}$ as

$$F(\sigma, \tau) = \inf\{\varepsilon > 0 \,|\, \sigma \sim \tau \text{ in } B(X, \varepsilon)\}.$$

This map is well-defined since the singular homology of the Hilbert cube is zero in any dimension. This implies that if $\varepsilon > diam Q$, then $B(X, \varepsilon) = Q$ and $\sigma \sim \tau$ in Q for every σ, τ in $Z_n(Q)$. Thus,

$$0 \leq F(\sigma, \tau) \leq diam Q$$

for any pair of cycles σ, τ in Q.

It can be easily checked that the map F enjoys the following properties for every σ, τ, η in $Z_n(Q)$:

i) $F(\sigma, \tau) \geq 0$;

ii) $F(\sigma, \tau) = F(\tau, \sigma)$;

iii) $F(\sigma, \tau) \leq \max\{F(\sigma, \eta), F(\eta, \tau)\}$ (in particular, this implies the triangle inequality).

Remark 2.1.14 In spite of the previous properties, F is not a metric, nor even a pseudometric in $Z_n(Q)$. There are cycles such that $F(\sigma, \sigma) \neq 0$. If we restrict ourselves to $Z_n(X)$ (cycles formed by simplexes lying in X) we obtain a pseudometric.

Definitions 2.1.15 a) A sequence $\{\sigma_i\} \subset Z_n(Q)$ is said to be *F-Cauchy* if for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \sigma_{i'}) < \varepsilon$ for all $i, i' \geq i_0$.

b) Two F-Cauchy sequences $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$ are said to be *F-related,* denoted by $\{\sigma_i\}F\{\tau_i\}$, if the sequence $\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3 ...$ is again F-Cauchy.

Proposition 2.1.16 *a) The F-relation is an equivalence relation.*

b) For every pair $\{\sigma_i\}, \{\tau_i\}$ of F-Cauchy sequences, there exists $\lim_{i\to\infty} F(\sigma_i, \tau_i)$.

c) Let $\{\sigma_i\}, \{\sigma'_i\}, \{\tau_i\}, \{\tau'_i\}$ be F-Cauchy sequences such that $\{\sigma_i\}F\{\sigma'_i\}$ and $\{\tau_i\}F\{\tau'_i\}$. Then $\lim_{i\to\infty} F(\sigma_i, \tau_i) = \lim_{i\to\infty} F(\sigma'_i, \tau'_i)$.

Proof.

a) Reflexive and symmetric properties are obvious. For transitive property, given $\{\sigma_i\}F\{\tau_i\}$ and $\{\tau_i\}F\{\omega_i\}$, let us take $\varepsilon > 0$. Then, there exists i_1 and i_2 realizing $\{\sigma_i\}F\{\tau_i\}$ and $\{\tau_i\}F\{\omega_i\}$, respectively. Also, there exists i_3, i_4 and i_5 for the fact that $\{\sigma_i\}, \{\tau_i\}$ and $\{\omega_i\}$ are F-Cauchy sequences. Let us take $i_0 = max\{i_1, i_2, i_3, i_4, i_5\}$. Hence, for $i, i' \geq i_0$, we have:

$$F(\sigma_i, \sigma_{i'}) < \varepsilon$$

$$F(\omega_i, \omega_{i'}) < \varepsilon$$

$$F(\sigma_i, \omega_{i'}) \leq max\{F(\sigma_i, \tau_i), F(\tau_i, \tau_{i'}), F(\tau_{i'}, \omega_{i'})\} < \varepsilon.$$

Thus, $\sigma_1, \omega_1, \sigma_2, \omega_2, ...$ is F-Cauchy, so $\{\sigma_i\}F\{\omega_i\}$.

b) We claim that $|F(\sigma_i, \tau_i) - F(\sigma_{i'}, \tau_{i'})| \leq F(\sigma_i, \sigma_{i'}) + F(\tau_i, \tau_{i'})$. Assuming this, the sequence $\{F(\sigma_i, \tau_i)\}$ is a Cauchy sequence in $\mathbb{R}$, so it is convergent.

In order to prove the claim, we distinguish two cases:

Case 1: $F(\sigma_i,\tau_i) \geq F(\sigma_{i'},\tau_{i'})$.

$$\begin{aligned}|F(\sigma_i,\tau_i) - F(\sigma_{i'},\tau_{i'})| =& F(\sigma_i,\tau_i) - F(\sigma_{i'},\tau_{i'}) \leq \\ \leq& F(\sigma_i,\sigma_{i'}) + F(\sigma_{i'},\tau_{i'}) + F(\tau_{i'},\tau_i) - F(\sigma_{i'},\tau_{i'}) = \\ =& F(\sigma_i,\sigma_{i'}) + F(\tau_i,\tau_{i'}).\end{aligned}$$

Case 2: $F(\sigma_i,\tau_i) \leq F(\sigma_{i'},\tau_{i'})$.

$$\begin{aligned}|F(\sigma_i,\tau_i) - F(\sigma_{i'},\tau_{i'})| =& F(\sigma_{i'},\tau_{i'}) - F(\sigma_i,\tau_i) \leq \\ \leq& F(\sigma_{i'},\sigma_i) + F(\sigma_i,\tau_i) + F(\tau_i,\tau_{i'}) - F(\sigma_i,\tau_i) = \\ =& F(\sigma_i,\sigma_{i'}) + F(\tau_i,\tau_{i'}).\end{aligned}$$

c) It is sufficient to apply b) to the sequences $\{a_i\}$ and $\{b_i\}$ formed respectively by $\sigma_1, \sigma_1', \sigma_2, \sigma_2'...$ and $\tau_1, \tau_1', \tau_2, \tau_2'...$, that are F-Cauchy by hypothesis. $\{F(\sigma_i,\tau_i)\}$ and $\{F(\sigma_i',\tau_i')\}$ are subsequences of $\{F(a_i,b_i)\}$, which has limit by b), so $\lim_{i\to\infty} F(\sigma_i,\tau_i) = \lim_{i\to\infty} F(\sigma_i',\tau_i')$.

□

The next result is easily stated, and it connects with the definitions established in the precedent section.

Proposition 2.1.17 *a) A sequence $\{\sigma_i\} \subset Z_n(Q)$ is F-Cauchy if and only if it is an approximative cycle.*

b) Two F-Cauchy sequences $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$ are F-related if and only if $\{\sigma_i\} \sim \{\tau_i\}$ as approximative cycles.

Proof.

a) Let $\{\sigma_i\} \subset Z_n(Q)$ be F-Cauchy and $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i,\sigma_{i'}) < \varepsilon'$ for every $i, i' \geq i_0$. This means that, for each $\delta > \varepsilon'$, $\sigma_i \sim \sigma_{i'}$ in $B(X,\delta)$ for all $i, i' \geq i_0$. In particular, $\sigma_i \sim \sigma_{i+1}$ in $B(X,\varepsilon)$ for every $i \geq i_0$, so $\{\sigma_i\}$ is an approximative cycle.

Conversely, let $\{\sigma_i\}$ be an approximative cycle and $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \sigma_{i'}$ in $B(X,\varepsilon')$ for all $i, i' \geq i_0$.

Let $i, i' \geq i_0$. Without loss of generality, suppose that $i' = i + j$. By the transitivity of the homology relation we obtain

$$\sigma_i \sim \sigma_{i+1} \sim ... \sim \sigma_{i+j-1} \sim \sigma_{i+j} = \sigma_{i'}$$

in $B(X, \varepsilon')$. Hence,

$$F(\sigma_i, \sigma_{i'}) = \inf\{\delta > 0 \mid \sigma_i \sim \sigma_{i'} \text{ in } B(X, \delta)\} \leq \varepsilon' < \varepsilon$$

so $\{\sigma_i\}$ is F-Cauchy.

b) Let $\{\sigma_i\}, \{\tau_i\} \subset Z_n(Q)$be F-Cauchy sequences such that $\{\sigma_i\}F\{\tau_i\}$. Let $\varepsilon > 0$ and take $\varepsilon' < \varepsilon$. Then, there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \tau_i) < \varepsilon'$ for all $i \geq i_0$. That is, for each $\delta > \varepsilon'$, $\sigma_i \sim \tau_i$ in $B(X, \delta)$ for every $i \geq i_0$. In particular, $\sigma_i \sim \tau_i$ in $B(X, \varepsilon)$ for each $i \geq i_0$.

   On the other hand, if $\{\sigma_i\}, \{\tau_i\}$ are homologous approximative cycles, they are F-Cauchy sequences by the previous point. We check now that the sequences are F-related. Let $\varepsilon > 0$. For $\varepsilon' < \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X, \varepsilon')$ for every $i \geq i_0$. Hence, $F(\sigma_i, \tau_i) \leq \varepsilon' < \varepsilon$ for all $i \geq i_0$. Therefore, the sequence $\sigma_1, \tau_1, \sigma_2, \tau_2 ...$ is F-Cauchy, so $\{\sigma_i\}F\{\tau_i\}$.

□

The main result of this section is the following.

Theorem 2.1.18 *Let $\check{H}_n(X)$ be the n-dimensional Čech homology group of a compact metric space X. Given two homology classes in $\alpha, \beta \in \check{H}_n(X)$, define*

$$d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i),$$

where $\{\sigma_i\}$ and $\{\tau_i\}$ are F-Cauchy sequences corresponding to α and β respectively. Then, $(\check{H}_n(X), d_X)$ is a complete ultrametric space.

Proof. As it has been proved in 2.1.12, for every Čech homology class α, there exists one and only one class of approximative cycles with representative σ associated to α.

Let $\sigma = \{\sigma_i\}$ and $\tau = \{\tau_i\}$ be approximative cycles representatives of the class of α and β respectively. d_X is well-defined since the limit

$$d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i)$$

always exists by 2.1.16.b) and others representatives σ', τ' in the classes of α and β give us the same limit by 2.1.16.c). Moreover, the properties of an ultrametric can be checked for d_X, as shown in the next.

i) $d_X(\alpha, \beta) \geq 0$ since $F(\cdot, \cdot)$ is always non-negative. In addition, $d_X(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.

   Suppose first that $d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i) = 0$. That is, for every $\varepsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\sigma_i \sim \tau_i$ in $B(X, \varepsilon)$ for all $i \geq i_0$. But that means that σ and τ are homologous as approximative cycles, so that $\alpha = \beta$.

   Conversely, if $\alpha = \beta$ then $\{\sigma_i\} \sim \{\tau_i\}$ hence, by 2.1.17, $\{\sigma_i\}F\{\tau_i\}$, and

$$d_X(\alpha, \beta) = \lim_{i \to \infty} F(\sigma_i, \tau_i) = 0.$$

ii) Since

$$d_X(\alpha,\beta) = \lim_{i\to\infty} F(\sigma_i,\tau_i) = \lim_{i\to\infty} F(\tau_i,\sigma_i) = d_X(\beta,\alpha),$$

then F is symmetric.

iii) Let $\eta = \{\eta_i\}$ be an approximative cycle representing another Čech homology class γ. From the property iii) of the map F,

$$\begin{aligned} d_X(\alpha,\beta) &= \lim_{i\to\infty} F(\sigma_i,\tau_i) \leq \\ &\leq \lim_{i\to\infty} \max\{F(\sigma_i,\eta_i), F(\eta_i,\tau_i)\} = \\ &= \max\{\lim_{i\to\infty} F(\sigma_i,\eta_i), \lim_{i\to\infty} F(\eta_i,\tau_i)\} = \\ &= \max\{d_X(\alpha,\gamma), d_X(\gamma,\beta)\}. \end{aligned}$$

Finally, the completeness of $(\check{H}_n(X), d_X)$ is obtained in the same way as the completion of $\mathbb{R}$ from $\mathbb{Q}$ via Cauchy sequences. □

The equivalence between the approximative homology and Čech homology, allows to give an easy interpretation of the meaning of the the ultramtric that we have defined.

Theorem 2.1.19 *Let X be a compact metric space, and suppose it embedded in the Hilbert cube (Q,ρ) as a closed subset. If $\alpha,\beta \in \check{H}_n(X)$ are two homology classes, and $\varepsilon > 0$, then $d(\alpha,\beta) < \varepsilon$ if and only if $i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta)$ as homology classes in $H_n(B(X,\varepsilon))$, where $i_{\varepsilon *}$ is the induced homomorphism in homology by the inclusion map $i_\varepsilon : X \hookrightarrow B(X,\varepsilon)$.*

Proof. Let $\alpha,\beta \in \check{H}_n(X)$ be two homology classes. Using 2.1.12, α and β are represented by approximative cycles $\{\alpha_k\}$ and $\{\beta_k\}$ respectively. If we consider a sequence $\{\varepsilon_k\} \to 0$ (e.g. $\varepsilon_k = \frac{\text{diam}Q}{k}$), without loss of generality, we can suppose that α_k,β_k are cycles lying in $B(X,\varepsilon_k)$ for each $k \in \mathbb{N}$ (if it would not be the case, pass to a subsequence).

First, if $d(\alpha,\beta) < \varepsilon$, then $\lim_{k\to\infty} F(\alpha_k,\beta_k) = l < \varepsilon$. Then, for any l' such that $l < l' < \varepsilon$, there exists $k_0 \in \mathbb{N}$ satisfying $F(\alpha_k,\beta_k) < l'$ for $k \geq k_0$. Thus, $\alpha_k \sim \beta_k$ in $B(X,l') \subset B(X,\varepsilon)$.

On the other hand, we have $\alpha_k = i_{\varepsilon_k *}(\alpha)$ and $\beta_k = i_{\varepsilon_k *}(\beta)$. Hence,

$$i_{\varepsilon *}(\alpha) = i_{\varepsilon\varepsilon_k *} \circ i_{\varepsilon_k *}(\alpha) = i_{\varepsilon\varepsilon_n *} \circ i_{\varepsilon_k *}(\beta) = i_{\varepsilon *}(\beta)$$

in $H_n(B(X,\varepsilon))$.

For the converse, let us suppose $i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta)$. Using 2.1.17, let us take $k_0 \in \mathbb{N}$ such that

$$F(\alpha_k,\alpha_{k+1}) < \frac{\varepsilon}{2} \text{ and } F(\beta_k,\beta_{k+1}) < \frac{\varepsilon}{2} \text{ for each } k \geq k_0.$$

Hence, for every $k \geq k_0$ we have $\alpha_k \sim \alpha_{k+1}$ as cycles in $B(X, \varepsilon/2)$, so there exist $(n+1)$-chains γ_k such that $\gamma_k = \alpha_{k+1} - \alpha_k$, and all simplexes of γ_k lie in $B(X, \varepsilon/2)$. Analogously, there exist $(n+1)$-chains γ'_k lying in $B(X, \varepsilon/2)$ such that $\gamma'_k = \beta_{k+1} - \beta_k$.

In addition,

$$\alpha_{k_0} = i_{\varepsilon *}(\alpha) = i_{\varepsilon *}(\beta) = \beta_{k_0}$$

as homology classes in $H_n(B(X, \varepsilon))$, using the hypothesis and because $\alpha_{k_0} = i_{\varepsilon_{k_0} *}(\alpha)$ is a cycle homologous to $i_\varepsilon(\alpha)$ in $B(X, \varepsilon)$ (resp. for β_k).

Let γ be an $(n+1)$-chain lying in $B(X, \varepsilon)$ such that $\gamma = \alpha_{k_0} - \beta_{k_0}$. Recall that γ is finite formal sum $\sum a_i \gamma_i$ where $a_i \in \mathbb{Z}$ and $\gamma_i : \Delta^n \to B(X, \varepsilon)$ is continuous. Hence, $\bigcup \gamma_i(\Delta^n)$ is compact, so there exists $0 < \delta < \varepsilon$ such that γ lies in $B(X, \delta) \subset B(X, \varepsilon)$.

Putting all together, we obtain $F(\alpha_k, \beta_k) \leq \max\{\delta, \varepsilon/2\}$ for $k \geq k_0$, so

$$d(\alpha, \beta) = \lim_{k \to \infty} F(\alpha_k, \beta_k) < \varepsilon,$$

as desired. □

Corollary 2.1.20 *Let X be a compact metric space. Then:*

a) Every ball (open or closed) is a clopen set in $(\check{H}_n(X), d)$.

b) $dim(\check{H}_n(X), d) = 0$ for every $n \in \mathbb{N}$.

Proof. Both properties are consequence of well-known facts about ultrametrics. It is immediate to check that $\overline{B(\alpha, \varepsilon)} = B(\alpha, \varepsilon)$. Indeed, let us take $\beta \in \overline{B(\alpha, \varepsilon)}$.

Since

$$B(\alpha, \varepsilon) \cap B(\beta, \delta) \neq \varnothing$$

for any $\delta > 0$ with $\delta < \varepsilon$, we can take some γ such that

$$d(\alpha, \gamma) = d_1 < \varepsilon \quad \text{and} \quad d(\beta, \gamma) = d_2 < \delta$$

Hence,

$$d(\alpha, \beta) \leq \max\{d_1, d_2\} < \varepsilon$$

so $\beta \in B(\alpha, \varepsilon)$.

Consequently,

$$\{B(\alpha, \varepsilon) \mid \varepsilon > 0\}$$

is a base of neighbourhoods of clopen sets for α. Thus, $\check{H}_n(X)$ is 0-dimensional. □

Example 2.1.21 Let $\mathbb{P}^2$ be the real projective plane. It is well-known that $\check{H}_1(\mathbb{P}^2) = H_1(\mathbb{P}^2) = \mathbb{Z}_2$. Let $0, \alpha$ be the two elements of this group. Without loss of generality, we can suppose that the metric in the group is given by

$$d(0,0) = 0, \qquad d(0,\alpha) = 1, \qquad d(\alpha,\alpha) = 0.$$

In this case $B(0,1) = \{0\}$, so $\overline{B(0,1)} = \{0\}$, while $\overline{B}(0,1) = \{0,\alpha\} = \mathbb{Z}_2$.

Previous example is just a particular case of what happens with any polyhedron, since the additional metric structure that we have introduced in the groups gets reduced only to the algebraic information in the case of nice spaces as ANR's. We show that in the next.

Proposition 2.1.22 *If X is a compact metric ANR (in Q), then $(\check{H}_n(X), d)$ is uniformly discrete.*

Proof. Since X is an ANR, there exists a neighbourhood U of X in Q and a retraction $r : U \to X$. Take $\varepsilon_0 > 0$ such that $B(X, \varepsilon_0) \subseteq U$.

Let $\alpha, \beta \in \check{H}_n(X) = H_n(X)$ such that $d(\alpha,\beta) < \varepsilon_0$. By 2.1.10 and 2.1.11, there exists two singular cycles lying in X such that the approximative cycles generated by them are representatives of the classes α and β respectively. Then,

$$d(\alpha,\beta) = \lim_{i\to\infty} F(\sigma_i, \tau_i) = F(\sigma,\tau) = l < \varepsilon_0$$

which means in particular that $\sigma \sim \tau$ in $B(X, \varepsilon_0.)$ Hence, there exists a singular chain γ lying in $B(X,\varepsilon_0)$ such that $\partial\gamma = \sigma - \tau$.

The retraction $s = r|_{B(X,\varepsilon_0)} : B(X,\varepsilon_0) \to X$ induces a map $s_\#$ over the chain groups such that $s_\#\partial = \partial s_\#$. Hence, $s_\#(\gamma)$ is chain lying in X such that

$$\partial s_\#(\gamma) = s_\#(\partial\gamma) = s_\#(\sigma - \tau) = s_\#(\sigma) - s_\#(\tau) = \sigma - \tau,$$

thus $\sigma \sim \tau$ in X. Consequently, $\alpha = \beta$. □

Example 2.1.23 The situation is completely different for a general compact metric space. Let us consider again the Hawaiian Earrings space, HE. An inverse sequence of polyhedra for this space was detailed in Example 1.1.9. The induced sequence in homology is given by

$$\begin{array}{ccccccc} \mathbb{Z} & \longleftarrow & \mathbb{Z}\times\mathbb{Z} & \longleftarrow & \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} & \longleftarrow & \cdots \\ a & \longmapsfrom & (a,b) & \longmapsfrom & (a,b,c) & \longmapsfrom & \cdots \end{array}$$

which has $\prod \mathbb{Z} = \mathbb{Z}^{\mathbb{N}}$ as inverse limit. The ultrametric on $\check{H}_1(HE) = \mathbb{Z}^{\mathbb{N}}$ is just

$$d(\alpha,\beta) = \frac{1}{n_0}$$

where $n_0 = min\{n \in \mathbb{N} \mid \alpha_n = \beta_n \text{ in } H_1(X_n)\}$, where α_n, β_n are cycles representing homology classes in $H_1(X_n)$. The topology generated by this ultrametric is no other than the usual product topology on $\mathbb{Z}^{\mathbb{N}}$.

The ultrametric structure mixes properly with the algebraic one:

Proposition 2.1.24 *$(\check{H}_n(X), d)$ is a topological group.*

Proof. By 2.1.12, it is enough to check the continuity of the maps

$$r: \begin{array}{ccc} H_n^{\mathcal{A}}(X) & \longrightarrow & H_n^{\mathcal{A}} \\ [\{\sigma_i\}] & \longmapsto & -[\{\sigma_i\}] \end{array} \qquad s: \begin{array}{ccc} H_n^{\mathcal{A}} \times H_n^{\mathcal{A}}(X) & \longrightarrow & H_n^{\mathcal{A}} \\ ([\{\sigma_i\}],[\{\tau_i\}]) & \longmapsto & [\{\sigma_i\}] + [\{\tau_i\}] \end{array}$$

with the metric $d([\{\sigma_i\}],[\{\tau_i\}]) = \lim_{i\to\infty} F(\sigma_i, \tau_i)$.

If $d([\{\sigma_i\}],[\{\tau_i\}]) = l$, then

$$d([\{\sigma_i\}],[\{\tau_i\}]) < l + \varepsilon \text{ for every } \varepsilon > 0.$$

Then there exists $i_0 \in \mathbb{N}$ such that $F(\sigma_i, \tau_i) < l + \varepsilon$ for every $i \geq i_0$, which means

$$\sigma_i \sim \tau_i \text{ in } B(X, l + \varepsilon) \text{ for every } i \geq i_0.$$

Hence, $-\sigma_i \sim -\tau_i$ in $B(X, l+\varepsilon)$, and therefore, $F(-\sigma_i, -\tau_i) < l + \varepsilon$. Thus,

$$d(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq l + \varepsilon,$$

which implies $d(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq l$.

This gives the inequality $d(-[\{\sigma_i\}], -[\{\tau_i\}]) \leq d([\{\sigma_i\}],[\{\tau_i\}])$.

By symmetry, the same argument give us the other inequality. So the equality

$$d([\{\sigma_i\}],[\{\tau_i\}]) = d(-[\{\sigma_i\}], -[\{\tau_i\}])$$

holds. Then, r is an isometry. In particular, r is continuous.

A similar argument is valid to see that

$$s(B([\{\sigma_i\}], \delta) \times B([\{\tau_i\}], \delta)) \subset B([\{\sigma_i\}] + [\{\tau_i\}], \varepsilon),$$

taking $\delta = \varepsilon$, for any $\varepsilon > 0$ given. Then s and r are continuous, and $H_n^{\mathcal{A}}(X)$ (or $\check{H}_n(X)$) is a topological group. □

Remark 2.1.25 As in the proof above, it can be checked that the inequality

$$d([\sigma] + [\mu], [\tau] + [\eta]) \leq \max\{d([\sigma],[\tau]), d([\mu],[\eta])\}$$

is valid for every $[\sigma],[\tau],[\mu],[\eta] \in H_n^{\mathcal{A}}(X)$.

Corollary 2.1.26 *The translations in $H_n^{\mathcal{A}}(X)$ (hence in $\check{H}_n(X)$) are isometries.*

Proof. From the last remark,

$$d([\sigma]+[\mu],[\tau]+[\mu]) \leq \max\{d([\sigma],[\tau]),d([\mu],[\mu])\} = \max\{d([\sigma],[\tau]),0\} = d([\sigma],[\tau])$$

Then,

$$d([\sigma],[\tau]) = d([\sigma]+[\mu]-[\mu],[\tau]+[\mu]-[\mu]) \leq d([\sigma]+[\mu],[\tau]+[\mu]) \leq d([\sigma],[\tau])$$

so $d([\sigma]+[\mu],[\tau]+[\mu]) = d([\sigma],[\tau])$ in $H_n^{\mathcal{A}}(X)$.
Hence,

$$d(\alpha+\gamma,\beta+\gamma) = d(\alpha,\beta) \text{ in } \check{H}_n(X).$$

□

The last result allows us to give define a norm on Čech homology groups.

Definition 2.1.27 Given $\alpha \in \check{H}_n(X)$, we define *the norm of* α as

$$\|\alpha\| = d(\alpha,0).$$

From 2.1.26, this norm recovers the definition of the metric d by means of

$$d(\alpha,\beta) = \|\alpha-\beta\|.$$

Corollary 2.1.28 *For any $n \in \mathbb{Z}$, $\|n\sigma\| \leq \|\sigma\| \leq diamQ$.*

Proof. For $\alpha \in \check{H}_n(X)$,

$$\|\alpha+\alpha\| = d(\alpha+\alpha,0) = d(\alpha,-\alpha) \leq \max\{d(\alpha,0),d(-\alpha,0)\} = d(\alpha,0) = \|\alpha\|$$

and the statement holds by induction. □

Some consequences of d being an ultrametric are resumed in the next result.

Proposition 2.1.29 *Let X be a compact metric space. For any $\varepsilon > 0$, let us denote $H_\varepsilon = \{\alpha \in \check{H}_n(X) \mid \|\alpha\| < \varepsilon\}$. Then, H_ε is a clopen normal subgroup and a neighbourhood of the neutral element 0 of $\check{H}_n(X)$. In particular, $\{H_{\frac{1}{m}}\}_{m\in\mathbb{N}}$ is a numerable basis of neighbourhoods of the identity in $\check{H}_n(X)$ formed by clopen normal subgroups.*

Proof. Let us fix $\varepsilon > 0$ and let α, β be in H_ε. Then,

$$\|\alpha + \beta\| \leq \max\{\|\alpha\|, \|\beta\|\} < \varepsilon,$$

so $\alpha+\beta$ is in H_ε. This subgroup is normal because $\check{H}_n(X)$ is Abelian. Moreover, $H_\varepsilon = B(0,\varepsilon)$ is clopen by 2.1.20. □

The metric d_X has functorial properties with respect to homomorphisms induced by continuous maps between compact metric spaces or, more generally, by shape morphisms. These properties are immediate from the functoriality of approximative homology and its identification with Čech homology groups. By 2.1.12, we obtain also a homomorphism between $\check{H}_n(X)$ and $\check{H}_n(Y)$ which is in fact the classical homomorphism induced by a shape morphism, with its usual functorial properties.

Proposition 2.1.30 *Let $\{f_k, X, Y\}$ be a shape morphism between compact metric spaces in Q. Then $\{f_k\}$ induces an uniformly continuous homomorphism of topological groups $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.*

Proof. As it has been recalled before, every shape morphism induces a homomorphism between the correspondent Čech homology groups. It remains to verify the uniform continuity of this homomorphism.

Let $\varepsilon > 0$. Since $\{f_k\}$ is a fundamental sequence, for $0 < \varepsilon' < \varepsilon$ take $\delta > 0$ such that

$$f_k|_{B(X,\delta)} \simeq f_{k+1}|_{B(X,\delta)} \text{ in } B(Y,\varepsilon') \text{ for almost } k.$$

Let α, β be two $\check{C}$ech homology classes and $\sigma = \{\sigma_i\}, \tau = \{\tau_i\}$ be two approximative cycles for X representing α and β respectively such that $d_X(\alpha,\beta) = \lim_{i\to\infty} F(\sigma_i,\tau_i) < \delta$.

From the proof of 2.1.6, we can take a subsequence $\{i_k\}$ of indices such that the inequality $j_k \geq i_k$ implies that both $\{f_{k_\#}(\sigma_{i_k})\}, \{f_{k_\#}(\tau_{i_k})\}$ are approximative cycles for Y. The subsequences $\{\sigma_{i_k}\}$ and $\{\tau_{i_k}\}$ are again approximative cycles and obviously $\{\sigma_i\}F\{\sigma_{i_k}\}$ and $\{\tau_i\}F\{\tau_{i_k}\}$, which implies $\lim_{k\to\infty} F(\sigma_{i_k},\tau_{i_k}) = \lim_{i\to\infty} F(\sigma_i,\tau_i) < \delta$.

Therefore, we have that

$$\sigma_{i_k} \sim \tau_{i_k} \text{ in } B(X,\delta)$$

for $k \in \mathbb{N}$ large enough, and consequently

$$f_{k_\#}(\sigma_{i_k}) \sim f_{k_\#}(\tau_{i_k}) \text{ in } B(Y,\varepsilon')$$

for k sufficiently large.

As it has been remarked, the image of an approximative cycle is independent of the chosen subsequence, thus

$$d_Y(f_*(\alpha), f_*(\beta)) = \lim_{k\to\infty} F(f_{k_\#}(\sigma_{i_k}), f_{k_\#}(\tau_{i_k})) \leq \varepsilon' < \varepsilon,$$

which finishes the proof. □

Corollary 2.1.31 *Let X, Y be two compact metric spaces and let $f : X \to Y$ be a map between them. Then f induces a uniformly continuous homomorphism of topological groups $f_* : \check{H}_n(X) \to \check{H}_n(Y)$.*

Proof. Since Q is an absolute extensor for metric spaces and Y is in Q, then there exists an extension map $\tilde{f} : Q \to Q$ of f. Setting $f_k = \tilde{f}$ for every $k \in \mathbb{N}$, we obtain a fundamental sequence generated by f. Now it is sufficient to apply 2.1.30 to that fundamental sequence. □

With this functorial properties, we can show now the invariance up to shape of the topology induced by the ultrametric.

Theorem 2.1.32 *Let X, Y be compact metric spaces. If X, Y are of the same shape, then $\check{H}_n(X)$ and $\check{H}_n(Y)$ are uniformly topologically isomorphic.*

Proof. Since X, Y are of the same shape, there exist fundamental sequences $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ such that $\{g_k f_k\} \simeq \{id_X\}$ and $\{f_k g_k\} \simeq \{id_Y\}$. Hence,

$$g_* f_* = id_{\check{H}_n(X)} \text{ and } f_* g_* = id_{\check{H}_n(Y)}$$

which implies the statement of the theorem because of 2.1.30. □

Corollary 2.1.33 *If two compact metric spaces X, Y have the same homotopy type, then $\check{H}_n(X)$ and $\check{H}_n(Y)$ are uniformly topologically isomorphic.*

Proof. Having the same homotopy type implies that, in particular, X, Y have the same shape and the result comes from the previous theorem. □

Corollary 2.1.34 *Let X, Y be compact metric spaces such that X is shape dominated by Y. Then $\check{H}_n(X)$ injects in $\check{H}_n(Y)$. Moreover, if $\check{H}_n(Y)$ is discrete, then so is $\check{H}_n(X)$. In particular, if X is dominated in shape by an ANR, then $\check{H}_n(X)$ is uniformly discrete.*

Proof. By hypothesis, there exist fundamental sequences $\{f_k, X, Y\}$ and $\{g_k, Y, X\}$ such that $\{g_k f_k\} \simeq \{id_X\}$. So $g_* f_* = id_{\check{H}_n(X)}$ which in particular implies that $\check{H}_n(X)$ injects in $\check{H}_n(Y)$ via f_*. From that, it is obvious that if $\check{H}_n(Y)$ is discrete, then so is $\check{H}_n(X)$.

If in addition Y is an ANR, $\check{H}_n(Y)$ is uniformly discrete by 2.1.22, so $\check{H}_n(X)$ must also be uniformly discrete. □

A stronger version of this result is the following.

Proposition 2.1.35 *Let X and Y be two compact metric spaces. If $Sh(X) \leq Sh(Y)$, then $\check{H}_n(X)$ is a factor subgroup of $\check{H}_n(Y)$ (for each $n \geq 0$). Moreover, there exists a continuous retraction from $\check{H}_n(Y)$ to $\check{H}_n(X)$.*

Proof. Let $F : X \to Y$ and $G : Y \to X$ two shape morphisms such that $G \circ F = 1_X$ in Sh.

The already known part can be sketched as follows: The induced maps on homology $F_* : \check{H}_n(X) \to \check{H}_n(Y)$ and $G_* : \check{H}_n(Y) \to \check{H}_n(X)$ satisfy $G_* \circ F_* = (G \circ F)_* = 1_{\check{H}_n(X)}$. Then, F_* is injective and G_* is onto. In addition, $\check{H}_n(Y) = \check{H}_n(X) \oplus KerG_*$ where, by abuse of notation, we have identified $\check{H}_n(X)$ with its image $F_*(\check{H}_n(X))$ in $\check{H}_n(Y)$.

To prove the topological part, observe that the induced map on homology

$$G_* : \check{H}_n(Y) \to \check{H}_n(X)$$

is continuous (by 2.1.30). Using the previous decomposition $\check{H}_n(Y) \cong \check{H}_n(X) \oplus KerG_*$, we have that every element γ of $\check{H}_n(Y)$ can be decomposed (in a unique way) as $\gamma = \gamma_1 + \gamma_2$ with γ_1 in $F_*(\check{H}_n(X))$ and γ_2 in $KerG_*$. Furthermore, $\gamma_1 = F_*(\gamma_1')$ for (a unique) γ_1' of $\check{H}_n(X)$, and γ_1 is the same element, under identification, as γ_1'. Hence,

$$G_*(\gamma) = G_*(\gamma_1) = G_*(F_*(\gamma_1')) = \gamma_1' = \gamma_1.$$

In particular, if $\gamma_2 = 0$ (i. e. if γ is in $\check{H}_n(X)$), then $G_*(\gamma) = \gamma$, hence G_* is a continuous retraction from $\check{H}_n(Y)$ onto $\check{H}_n(X)$. □

Corollary 2.1.36 *If X is a FANR, then $\check{H}_n(X)$ is uniformly discrete.*

Proof. Since X is a FANR, there exists a neighbourhood U of X which is an ANR and there exists a fundamental sequence $\{r_k, U, X\}$ which is a fundamental retraction. This means that $\{r_k i_k\} \simeq \{id_X\}$, where $\{i_k, X, U\}$ is the fundamental sequence generated by the inclusion of X in U. In particular, X is shape dominated by U, so we can apply 2.1.34 joint with 2.1.22 in order to obtain the desired result. □

Example 2.1.37 Let us consider $X = \mathcal{W}$ the Warsaw circle. As we have already seen, an inverse sequence $\{X_n, p_{n\,n+1}\}$ which represents the shape of this space is given by annuli $X_n = B(X, \frac{1}{n})$ and inclusions $p_{n\,n+1} = X_{n+1} \hookrightarrow X_n$. Thus, the induced inverse sequence in homology is given by $\{\mathbb{Z}, id_{\mathbb{Z}}\}$ which obviously has $\mathbb{Z}$ with discrete topology as inverse limit. Hence, $\check{H}_1(\mathcal{W}) = \mathbb{Z}$ and the ultramtric is just the discrete metric. This fact is a consequence of being $Sh(\mathcal{W}) = Sh(S^1)$.

Some properties that the metric space that we have constructed enjoys are resumed in the following.

Theorem 2.1.38 $(\check{H}_n(X), d_X)$ *is a second-countable space (hence, separable), homeomorphic to a closed subset of the irrationals.*

Proof. Let $\{X_m\}$ be a basis of neighbourhoods of X in Q formed by compact ANR's such that $X_m \supset X_{m+1}$. By 2.1.11 and 2.1.12, for each $m \in \mathbb{N}$

$$H_n^{\mathcal{A}}(X_m) = H_n(X_m) = \check{H}_n(X_m)$$

and by 2.1.22 $H_n(X_m)$ is discrete.

Given any compact ANR, A, there exists a compact polyhedron P which dominates A in shape, so $H_n(A)$ injects in $H_n(P)$ by applying 2.1.34. In addition, P has only a finite number of simplexes in dimension n, so that $H_n(P)$ is finitely generated. Hence $H_n(P)$ has countable many elements, and then so has $H_n(A)$.

Last argument shows that card$(H_n(X_m))\leq \aleph_0$, so it is trivially separable. Numerable products of separable metric spaces is again a separable metric space, thus

$$\prod_{m\in\mathbb{N}} H_n(X_m) \leq \mathbb{Z}^{\mathbb{N}} \approx \mathbb{R} \smallsetminus \mathbb{Q}$$

is separable metric and, hence, second numerable (which is an hereditary property). Since

$$\check{H}_n(X) = \lim_{\leftarrow}\{H_n(X_m), i_{mm+1*}\} \leq \prod_{m\in\mathbb{N}} H_n(X_m)$$

is closed, the result follows. □

Using this topological information, we have the following characterization of discreteness on $\check{H}_n(X)$.

Proposition 2.1.39 *If X is a compact metric space, then the topology generated by the ultrametric d is discrete if and only if $\check{H}_n(X)$ is countable.*

Proof. Obviously, if $\check{H}_n(X)$ is discrete and separable (by 2.1.38), it must be countable.

On the other hand, if $\check{H}_n(X)$ is countable, we can put

$$\check{H}_n(X) = \bigcup_{\alpha\in\check{H}_n(X)} \{\alpha\}$$

where $\{\alpha\}$ is closed for each $\alpha \in \check{H}_n(X)$. By 2.1.38, $\check{H}_n(X)$ is complete (and d is a metric), so we can apply the Baire's theorem. Thus, there exists an element α_0 such that $int(\{\alpha_0\}) \neq \varnothing$ which implies $int(\{\alpha_0\}) = \{\alpha_0\}$ so $\{\alpha_0\}$ open. Since $\check{H}_n(X)$ is a topological group, this implies that every point is open and, therefore, $\check{H}_n(X)$ is discrete. □

We can also state the following.

Proposition 2.1.40 *Let $F : X \to Y$ be a shape morphism which induces an isomorphism $F_* : \check{H}_n(X) \to \check{H}_n(Y)$ for each $n \in \mathbb{N}$. Then F_* is a uniform topological isomorphism.*

Proof. Without loss of generality, we can suppose that F is given by a fundamental sequence, and apply 2.1.30 in order to obtain that F_* is continuous. Now, the Banach open mapping theorem for separable and completely metrizable topological groups stays that F_* is open, and this is equivalent to say that the inverse homomorphism of F_* is continuous. Furthermore, since the ultrametric d is both-sides invariant, any continuous homomorphism is uniformly continuous. Thus, F_* is a uniform topological isomorphism. □

Example 2.1.41 Let $X = HE$ the Hawaiian Earrings space and $Y = \prod_{k\in\mathbb{N}} S^1$. There exists a uniform topological isomorphism between $\check{H}_1(X)$ and $\check{H}_1(Y)$, despite X and Y are not of the same shape.

Let us consider X, Y embedded in the Hilbert cube Q in such a way that there exists a sequence $\{\varepsilon_n\}$ of positive real numbers converging to zero such that

$$B(HE, \varepsilon_n) \cong \bigvee_n S^1 \qquad \text{and} \qquad B(\prod_{k\in\mathbb{N}} S^1) \cong \prod_{k=1}^{n} S^1.$$

Then,

$$H_1(B(HE, \varepsilon_n)) \cong H_1(B(\prod_{k\in\mathbb{N}} S^1, \varepsilon_n)) \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$$

and let us denote f_n the correspondent isomorphism of groups. This induces an isomorphism between the inverse limits

$$f = \lim_{\leftarrow} f_n : \check{H}_1(HE) \to \check{H}_1(\prod_{k\in\mathbb{N}} S^1).$$

It is enough to compare distances to the neutral element $0 \in \prod_{k\in\mathbb{N}} \mathbb{Z}$. Recall that a class $\alpha \in \check{H}_1(X)$ is represented by an approximative cycle $\{\sigma_n\}$ (by 2.1.12). By definition of the ultrametric d for HE,

$$d(0, \alpha) < \varepsilon \Leftrightarrow \lim_{n\to\infty} F(0, \sigma_n) < \varepsilon$$

for an approximative cycle $\{\sigma_n\}$ associated to the class α. Moreover, each cycle σ_n lies in $B(HE, \varepsilon_n)$ and, by definition of approximative cycle,

$$\exists n_0 \in \mathbb{N} \text{ such that } \sigma_n \sim 0 \text{ in } B(HE, \varepsilon_n) \text{ for } n \geq n_0 (\text{and } \varepsilon_n < \varepsilon).$$

This is equivalent to

$$f_n(\sigma_n) \sim 0 \text{ in } B(\prod_{k\in\mathbb{N}} S^1, \varepsilon_n) \text{ for } n \geq n_0 (\text{and } \varepsilon_n < \varepsilon)$$

and, hence

$$\lim_{n\to\infty} F(0, \{f_n(\sigma_n)\}) < \varepsilon,$$

thus $d(0, f(\alpha)) < \varepsilon$ for the ultrametric d for $\prod_{k\in\mathbb{N}} S^1$.

Now we have a sufficient condition for the compactness of $\check{H}_n(X)$.

Proposition 2.1.42 *If X is the inverse limit of an inverse sequence of compact polyhedron $\{X_k, p_{kk+1}\}$ all of them with finite homology groups, then $\check{H}_n(X)$ is compact.*

Proof. Let $\{X_k, p_{kk+1}\}$ an inverse sequence of polyhedra such that X is its inverse limit. Suppose, as in the statement, that $H_n(X_k)$ is finite (so compact). Then, $\check{H}_n(X)$ is a closed subgroup of the compact space $\prod_{k\in\mathbb{N}} H_n(X_k)$, thus $\check{H}_n(X)$ is compact itself. □

2.1.3 Generalization to arbitrary topological spaces X

The guideline of this section is the same as exposed in chapter 1 for the ultrametric on the shape group and for the pseudoultrametric on the fundamental group.

Let us consider an arbitrary topological space X and let

$$\underline{\mathbf{p}} = \{p_\lambda\}_{\lambda\in\Lambda} : X \longrightarrow \underline{\mathbf{X}} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

be an HPol-expansion of (X, x_0). This induces an inverse system in homology such that

$$\check{H}_n(X) = \varprojlim\{H_n(X_\lambda), (p_{\lambda\lambda'})_*, \Lambda\}.$$

Given an homology class $\alpha \in \check{H}_n(X)$, for each $\lambda \in \Lambda$, we shall denote $\alpha_\lambda = p_{\lambda *}(\alpha)$ the correspondent homology class in $H_n(X_\lambda)$.

The existence of a generalized ultrametric also on the homology groups is given in the following.

Theorem 2.1.43 *Let X be an arbitrary topological space, and let*

$$\underline{\mathbf{p}} = \{p_\lambda\} : X \to \underline{\mathbf{X}} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

*be an **HPol**-expansion of X. Given two Čech homology classes $\alpha, \beta \in \check{H}_n(X)$, set*

$$d(\alpha, \beta) = \{\lambda \in \Lambda \mid p_{\lambda *}(\alpha) = p_{\lambda *}(\beta) \text{ in } H_k(X_\lambda)\}.$$

The formula above defines a generalized ultrametric

$$d : \check{H}_n(X) \times \check{H}_n(X) \to (\mathcal{L}(\Lambda), \leq).$$

Proof. It is clear that the formula of d is well-defined. Moreover, $d(\alpha, \beta)$ is a lower class: If $\lambda \in d(\alpha, \beta)$ let us take $\lambda' \in \Lambda$ with $\lambda' < \lambda$. Then

$$p_{\lambda *}(\alpha) = p_{\lambda *}(\beta)$$

implies

$$p_{\lambda' *}(\alpha) = p_{\lambda' \lambda *}(p_{\lambda *}(\alpha)) = p_{\lambda' \lambda *}(p_{\lambda *}(\beta)) = p_{\lambda' *}(\beta)$$

and $\lambda' \in d(\alpha, \beta)$.

The properties of a generalized ultrametric are easy to check. Let $\alpha = \{\alpha_\lambda\}, \beta = \{\beta_\lambda\} \in \check{H}_n(X)$.

- If $\alpha = \beta$, then
$$\alpha_\lambda = p_{\lambda *}(\alpha) = p_{\lambda *}(\beta) = \beta_\lambda$$
for every $\lambda \in \Lambda$. Let us recall that Λ is the lower class which we have denoted by 0 in $\mathcal{L}(\Lambda)$, so $d(\alpha, \beta) = 0$.
Conversely, if $d(\alpha, \beta) = 0$, then $\alpha_\lambda = \beta_\lambda$ in $H_n(X_\lambda)$. Thus, $\alpha = \beta$.

- Obviously, for each $\lambda \in d(\alpha, \beta)$,
$$\alpha_\lambda = p_{\lambda *}(\alpha) = p_{\lambda *}(\beta) = \beta_\lambda$$
in $H_n(X_\lambda)$, so $\beta_\lambda = \alpha_\lambda$. Hence, $\lambda \in d(\beta, \alpha)$. The other inequality is completely analogous, thus $d(\alpha, \beta) = d(\beta, \alpha)$.

- Let $\gamma = \{\gamma_\lambda\}$ be another element in $\check{H}_n(X)$, such that $d(\alpha, \gamma) \leq \Delta$ and $d(\beta, \gamma) \leq \Delta$ for some $\Delta \in \mathcal{L}(\Lambda)$. For $\lambda \in \Delta$, we have
$$\alpha_\lambda = p_{\lambda *}(\alpha) = p_{\lambda *}(\gamma) = \gamma_\lambda$$
and
$$\beta_\lambda = p_{\lambda *}(\beta) = p_{\lambda *}(\gamma) = \gamma_\lambda$$
in $H_n(X_\lambda)$. The transitivity of the homology of cycles ensures then that
$$\alpha_\lambda = p_{\lambda *}(\alpha) = p_{\lambda *}(\beta) = \beta_\lambda$$
in $H_n(X_\lambda)$, so $\lambda \in d(\alpha, \beta)$ and we have checked that $d(\alpha, \beta) \leq \Delta$.

□

The connection with the compact metric case is explicit in the next.

Corollary 2.1.44 *Let (Q, ρ) be the Hilbert cube, and assume $X \subset Q$ as a closed subset. For each $\varepsilon > 0$, consider $X_\varepsilon = B(X, \varepsilon)$. Also, for $\varepsilon' > \varepsilon > 0$, let $p_{\varepsilon \varepsilon'} : X_\varepsilon \to X_{\varepsilon'}$ and $p_\varepsilon : X \to X_\varepsilon$ the corresponding inclusions. Finally, let us denote by $(\mathbb{R}^+)^{-1}$ the set of non-negative real numbers with the reverse usual order. Then, the generalized ultrametric d constructed in Theorem 2.1.43 for the* ***HPol****-expansion*

$$\underline{\mathbf{p}} = \{p_\varepsilon\} : X \to (\underline{\mathbf{X}} = \{X_\varepsilon, p_{\varepsilon \varepsilon'}, (\mathbb{R}^+)^{-1}\}$$

coincides with the ultrametric constructed in the previous section.

The topology from the generalized ultrametric has a description in terms of a base on $\check{H}_n(X)$. We can consider

$$B_\Delta(\alpha) = \{\beta \in \check{H}_n(X) \mid d(\alpha,\beta) \leq \Delta\},$$

for each lower class $\Delta \in \mathcal{L}(\Lambda)^*$ and an element $\alpha \in \check{H}_n(X)$.

Proposition 2.1.45 *The family*

$$\mathcal{B} = \{B_\Delta(\alpha) \mid \alpha \in \check{H}_n(X), \Delta \in \mathcal{L}(\Lambda)\}$$

is a base for a topology in $\check{H}_n(X)$ which is completely regular, Hausdorff and zero-dimensional. It shall be called the canonical topology induced by the ultrametric d.

Proof. It is obvious that every element $\alpha \in \check{H}_n(X)$ is in some set of $\mathcal{B}$. Let $B_{\Delta_1}(\alpha)$ and $B_{\Delta_2}(\beta)$ two sets of $\mathcal{B}$ such that $B_{\Delta_1}(\alpha) \cap B_{\Delta_2}(\beta) \neq \varnothing$.

For some $\gamma \in B_{\Delta_1}(\alpha) \cap B_{\Delta_2}(\beta)$, let us take $\Delta_3 = \Delta_1 \cup \Delta_2$, which is in $\mathcal{L}(\Lambda) \smallsetminus \{\Lambda\}$ and $\Delta_3 \leq \Delta_1, \Delta_2$. Then, $B_{\Delta_3}(\gamma) \subseteq B_{\Delta_1}(\alpha)$. Indeed, for $\eta \in B_{\Delta_3}(\gamma)$

$$d(\eta,\gamma) < \Delta_3 < \Delta_1$$

and

$$d(\gamma,\alpha) < \Delta_1$$

hence

$$d(\eta,\alpha) < \Delta_1.$$

Consequently, $\eta \in B_{\Delta_1}(\alpha)$. The reasoning is completely analogous to obtain $B_{\Delta_3}(\gamma) \subseteq B_{\Delta_2}(\beta)$. □

As it is expected, the topology obtained on $\check{H}_n(X)$ is a group topology.

Proposition 2.1.46 $(\check{H}_n(X), +)$ *with the canonical topology induced by d is a topological group.*

Proof. The proof is analogous as in the case of the fundamental group. The map

$$\begin{array}{rccc} r: & \check{H}_n(X) & \longrightarrow & \check{H}_n(X) \\ & \alpha & \longmapsto & \alpha^{-1} \end{array}$$

is continuous since

$$r(B_\Delta(\alpha)) \subseteq B_\Delta(\alpha^{-1})$$

for any $\alpha \in \check{H}_n(X)$.

On the other side, the map

$$
\begin{array}{rccc}
s: & \check{H}_n(X) \times \check{H}_n(X) & \longrightarrow & \check{H}_n(X) \\
 & (\alpha, \beta) & \longmapsto & \alpha + \beta
\end{array}
$$

is continuous because

$$s(B_\Delta(\alpha) \times B_\Delta(\beta)) \subseteq B_\Delta(\alpha + \beta)$$

for $\alpha, \beta \in \check{H}_n(X)$. □

Once again, this topology has the trouble of dependence on the particular **HPol**-expansion used for the construction, and not on the shape of the involved space. This defect can be shown in a similar way as in the case of the fundamental group. If X is a topological space, and

$$\underline{\mathbf{p}} = \{p_\lambda\} : X \to \underline{\mathbf{X}} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

is an **HPol**-expansion of X, consider the new inverse system

$$\underline{\mathbf{X'}} = \{X_{(\lambda,\gamma)}, p_{(\lambda,\gamma)\,(\lambda',\gamma'), \Lambda\times\Lambda}\}$$

where $X_{(\lambda,\gamma)} = X_\lambda$, $p_{(\lambda,\gamma)\,(\lambda',\gamma')} = p_{\lambda\lambda'}$ and $\Lambda \times \Lambda$ is ordered with the usual product order. Then,

$$\underline{\mathbf{p'}} = \{p'_{(\lambda,\gamma)}\} : X \to \underline{\mathbf{X'}}$$

is an **HPol**-expansion, where $p'_{(\lambda,\gamma)} = p_\lambda$.

The canonical topology induced by the metric d for this last expansion is the discrete one. As in the case of the fundamental group (or for the shape group or, more in general, for shape morphisms in [25]), observe that

$$B_{\Delta_D}(\alpha) = \{\alpha\}$$

for the lower class

$$\Delta_D = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda : \lambda_2 \leq \lambda_0\}$$

which is the minimal lower class containing the set

$$D = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda : \lambda_2 = \lambda_0\}$$

for a fixed non maximal element $\lambda_0 \in \Lambda$.

Since the topology of $\check{H}_n(X)$ is non-discrete in general for a compact metric space X, the previous paragraph implies that the topology depends on the chosen expansion. This motivates a slight modification of the canonical topology. We make use again of lower classes (λ) generated by a fixed element $\lambda \in \Lambda$.

Proposition 2.1.47 *Let X be an arbitrary topological space, and*

$$\underline{\mathbf{p}} = \{p_\lambda\} : X \to \underline{\mathbf{X}} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

*an **HPol**-expansion of X. The family*

$$\{B_{(\lambda)}(\alpha) \mid \alpha \in \check{H}_n(X), \lambda \in \Lambda\}$$

is a base for a topology on $\check{H}_n(X)$, *Proof.* The idea of the proof is basically the same as for the canonical topology. It is enough to observe that

$$B_{(\lambda_3)}(\gamma) \subseteq B_{(\lambda_1)}(\alpha) \cap B_{(\lambda_2)}(\beta)$$

for some $\gamma \in B_{(\lambda_1)}(\alpha) \cap B_{(\lambda_2)}(\beta)$ and $\lambda_3 \geq \lambda_1, \lambda_2$ (which exists since Λ is a directed set). □

Definition 2.1.48 We call the *intrinsic topology* to the topology on $\check{H}_n(X)$ described above.

Proposition 2.1.49 *The intrinsic topology on $\check{H}_n(X)$ is independent on the fixed **HPol**-expansion of (X, x_0) and it coincides with the topology given by the ultrametric for compact metric spaces (X, x_0).*

Proof. Let

$$\underline{\mathbf{p}} = \{p_\lambda\} : X \to \underline{\mathbf{X}} = \{X_\lambda, p_{\lambda\lambda'}, \Lambda\}$$

and

$$\underline{\mathbf{p}}' = \{p'_\lambda\} : X \to \underline{\mathbf{X}}' = \{X'_\mu, p'_{\mu\mu'}, M\}$$

be two **HPol**-expansions of X. Then, there exists a map of inverse systems (induced by the identity $i : X \to X$) in **HPol**

$$\underline{\mathbf{i}} = (i_\mu, \phi) : \underline{\mathbf{X}} \to \underline{\mathbf{X}}'$$

with $\phi : M \to \Lambda$ and $i_\mu : X_{\phi(\mu)} \to X'_\mu$ such that

$$i_\mu \circ p_{\phi(\mu)} \simeq p'_\mu,$$

so the induced maps in homology coincide

$$i_{\mu*} \circ p_{\phi(\mu)*} = (i_\mu \circ p_{\phi(\mu)})_* = p'_{\mu*}.$$

Given a ball $B_{(\mu)}(\alpha)$ for the intrinsic topology induced by the **HPol**-expansion $\underline{\mathbf{X}}'$, take some element $\beta \in B_{(\mu)}(\alpha)$ and $\lambda = \phi(\mu)$. Hence,

$$B_{(\phi(\mu))}(\beta) \subseteq B_{(\mu)}(\alpha).$$

Indeed, for any $\gamma \in B_{(\phi(\mu))}(\beta)$:

$$\gamma_\mu = p'_{\mu*}(\gamma) = i_{\mu*}(p_{\phi(\mu)*}(\gamma)) = i_{\mu*}(p_{\phi(\mu)*})(\beta) = p'_{\mu*}(\beta) = p'_{\mu*}(\alpha)$$

in $\check{H}_n(X)$. Consequently, $B_{(\mu)}(\alpha)$ is open for the intrinsic topology induced by the **HPol**-expansion $\underline{\mathbf{X}}$.

For the converse, the same reasoning is valid but using the map of inverse systems

$$\underline{\mathbf{j}} = (j_\lambda, \psi) : \underline{\mathbf{X}}' \to \underline{\mathbf{X}}$$

with $\psi : \Lambda \to M$ and $j_\lambda : X'_{\psi(\lambda)} \to X_\lambda$ such that

$$j_\lambda \circ p'_{\psi(\lambda)} \simeq p_\lambda.$$

In conclusion, the intrinsic topologies induced by two different expansions are equivalent. The last assertion of the proposition is immediate from the fact that in a compact metric case it is also possible to obtain expansion represented by inverse sequences (that is, indexed by the naturals). □

Remark 2.1.50 Let us observe that, again, if the space is a polyhedron, the new structure reduces to the algebraic one because $\check{H}_n(X)$ is discrete in that case.

Out of the compact metric hypothesis, it is easy to produce examples on which the topology on the groups distinguishes topological spaces, while the group structure alone cannot.

Example 2.1.51 Let us consider $X = HE$ as the Hawaiian Earrings space, which has $\check{H}_1(X) = \mathbb{Z}^{\mathbb{N}}$ with its usual product topology (which is non-discrete). On the other hand, if we take $Y = K(\mathbb{Z}^{\mathbb{N}}, 1)$ as the Eilenberg-MacLane space of the discrete group $\mathbb{Z}^{\mathbb{N}}$, we obtain that Y is a CW-complex also with $\check{H}_1(Y) = \mathbb{Z}^{\mathbb{N}}$. However, this group would inherit the discrete topology, since Y is of the homotopy type of a polyhedron. Hence, HE and $K(\mathbb{Z}^{\mathbb{N}}, 1)$ are not homotopy equivalent, nor of the same shape.

We finish this section, making a brief mention about the singular homology groups. The fundamental group is related with the shape group, and singular (also simplicial) homology groups are related with Čech homology groups as well in a similar way.

If X is compact metric (considered as a closed subset of the Hilbert cube Q) and s is a singular cycle, we have described in 2.1.5 an homomorphism

$$\varphi^{\mathcal{A}} : H_n(X) \to H_n^{\mathcal{A}}(X)$$

from the singular homology to the approximative homology. Using 2.1.12, this homomorphism leads to

$$\varphi : H_n(X) \to \check{H}_n(X)$$

which is in fact the canonical homomorphism relating singular and Čech homology groups.

On the other hand, it was observed that the map F (used to define de ultrametric d) was not a metric. But if we restrict the domain of F to cycles entirely lying on X we get a pseudoultrametric (from the properties of F). This pseudoultrametric generates a topology on $H_n(X)$ and it coincides with the initial topology on $H_n(X)$ induced by ψ and the intrinsic topology on $\check{H}_n(X)$ (this is the analogous result on the fundamental group with respect to the shape group). In the general case, the map φ is easy to construct via the induced maps in homology of the projections of an expansion, and the same result is obtained.

2.2 Topological Hurewicz homomorphism

Homotopy and homology groups are related via the well-known *Hurewicz homomorphism* (see [83]), which for any pointed space (X, x_0) and each $n \in \mathbb{N}$ is a group homomorphism

$$\varphi : \pi_n(X, x_0) \longrightarrow H_n(X)$$

such that $\varphi([\alpha]) = \alpha_*(Z_n)$ where $\alpha : (I^n, \partial I^n) \to (X, x_0)$ represents an element of $\pi_1(X, x_0)$ and Z_n is the canonical generator of $H_n(I^n, \partial I^n)$.

It is natural to consider the analogous homomorphism but in the case of shape groups and Čech homology groups instead of homotopy groups and singular homology groups.

Given (X, x_0) a pointed compact metric space, let $\{(X_m, x_m), p_{mm+1}\}$ be an inverse sequence of pointed spaces such that its inverse limit is (X, x_0). In this case, taking the Hurewicz homomorphism (with $n \in \mathbb{N}$ fixed) in each level, we obtain a sequence of homomorphisms $\{\varphi_m\}$ such that all squares of the form

$$\begin{array}{ccc} \pi_n(X_{m+1}, x_{m+1}) & \xrightarrow{\varphi_{m+1}} & H_n(X_{m+1}) \\ \downarrow & \circlearrowright & \downarrow \\ \pi_n(X_m, x_m) & \xrightarrow{\varphi_m} & H_n(X_m) \end{array}$$

are commutative (where vertical maps are the correspondent homomorphisms induced in homotopy and homology by p_{mm+1}). The correspondent limit homomorphism of $\{\varphi_m\}$

$$\check{\varphi} : \check{\pi}_n(X, x_0) \longrightarrow \check{H}_n(X)$$

is a homomorphism between the n-dimensional shape and Čech homology groups. It is also called the *Hurewicz homomorphism.*

It seems to be a natural question what happens when we endow these groups with the ultrametrics for $\check{\pi}_n(X, x_0)$ and $\check{H}_n(X)$, respectively. In order to answer that question, it is more convenient to consider again Borsuk's viewpoint of shape, considering elements of the shape group as approximative maps (see [10]) from $(I^n, \partial I^n)$ to (X, x_0).

Recall that if (X, x_0) is a pointed compact metric space in Q, a shape morphism α in $\check{\pi}_n(X, x_0)$ is represented by a sequence $\{\alpha_m\}$ of maps such that $\alpha_m : (I^n, \partial I^n) \to (Q, x_0)$ satisfies that for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\alpha_m \simeq \alpha_{m+1}$ in $B(X, \varepsilon)$ for every $m \geq m_0$. The definition of homotopy of approximative maps is defined in a similar way than homotopy of fundamental sequences. Given $\{\alpha_m\}$ and $\{\beta_m\}$ two approximative maps, they are homotopic provided for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\alpha_m \simeq \beta_m$ in $B(X, \varepsilon)$ for every $m \geq m_0$.

With this definitions in mind, Hurewicz homomorphism can be rewritten as

$$\begin{array}{rccc} \check{\varphi} : & \check{\pi}_n(X, x_0) & \longrightarrow & \check{H}_n(X) \\ & \langle \alpha \rangle = \{[\alpha_m]\} & \longmapsto & \check{\varphi}(\langle \alpha \rangle) = \{\alpha_{m*}(Z_n)\} \end{array}$$

where $\{\alpha_m\}$ is an approximative map representing the shape class $\langle\alpha\rangle$ in $\check{\pi}_n(X,x_0)$, and α_{m*} is the induced map in singular homology between $H_n(I^n,\partial I^n)$ and $H_n(Q,x_0)$, being Z_n the canonical generator of $H_n(I^n,\partial I^n)$.

Remark 2.2.1 The sequence of cycles $\{\alpha_{m*}(Z_n)\}$ is in fact an approximative cycle: given $\varepsilon > 0$, the approximative sequence $\{\alpha_m\}$ gives an index $m_0 \in \mathbb{N}$ such that

$$\alpha_m \simeq \alpha_{m+1} \quad \text{in } B(X,\varepsilon) \quad \text{for every } m \geq m_0$$

and, consequently,

$$\alpha_{m*}(Z_n) \sim \alpha_{m+1*}(Z_n) \quad \text{in } B(X,\varepsilon) \quad \text{for every } m \geq m_0,$$

since homotopic maps induce the same homomorphism in homology.

Hence, $\check{\varphi}(\langle\alpha\rangle)$ belongs to $H_n^{\mathcal{A}}(X)$, which is isomorphic to $\check{H}_n(X)$ by 2.1.12.

Theorem 2.2.2 *The Hurewicz homomorphism*

$$\check{\varphi} : \check{\pi}_n(X,x_0) \longrightarrow \check{H}_n(X)$$

is an uniformly continuous homomorphism between topological groups. In fact, $\check{\varphi}$ is a non-expanding homomorphism.

Proof. The fact that $\check{\varphi}$ is an homomorphism is in [65].

Let $\varepsilon > 0$ and let $\langle\alpha\rangle, \langle\beta\rangle$ be in $\check{\pi}_n(X,x_0)$ respectively represented by approximative maps $\{\alpha_k\}$ and $\{\beta_k\}$. Let us assume that $d(\alpha,\beta) < \varepsilon$, that is, $\lim_{k\to\infty} F(\alpha_k,\beta_k) = l < \varepsilon$ and take $l < \varepsilon' < \varepsilon$. There exist $k_0 \in \mathbb{N}$ such that

$$\alpha_k \simeq \beta_k \text{ in } B(X,\varepsilon') \text{ for } k \geq k_0$$

so α_{k*} and β_{k*} induce the same homomorphism in homology for $k \geq m_0$.

Therefore,

$$\alpha_{k*}(Z_n) \sim \beta_{k*}(Z_n) \quad \text{in } B(X,\varepsilon') \quad \text{for every } k \geq k_0,$$

so $\check{\varphi}(\alpha) = \{\alpha_{k*}(Z_n)\}$ and $\check{\varphi}(\beta) = \{\beta_{k*}(Z_n)\}$ are approximative cycles such that

$$d(\check{\varphi}(\alpha),\check{\varphi}(\beta)) = \lim_{k\to\infty} F(\alpha_{k*}(Z_n),\beta_{k*}(Z_n)) \leq \varepsilon' < \varepsilon,$$

as desired. □

2.3 Čech cohomology groups and Pontryagin duality

Given an Abelian group G, cohomology groups $H^n(X;G)$ are developed dualizing the construction of the homology groups, using the $Hom(-,G)$ functor. In spite of this initial idea, this assertion is no further true at the end of the construction process of the cohomology. Fortunately, it is possible to understand cohomology as the dual of homology but in a different way, using Pontryagin duality theory. In addition, this duality leads us to a non-trivial topology in the Čech cohomology groups over S^1, while such a topology would lack of interest in the case $G = \mathbb{Z}$.

2.3.1 Group coefficients $G = \mathbb{Z}$

If we would like to define a topology over the Čech cohomology groups $\check{H}^n(X;\mathbb{Z})$ analogous those defined over the shape groups $\check{\pi}_n(X,x_0)$ and over the Čech homology groups $\check{H}_n(X;Z)$, we should ask it to have similar properties. In particular, the following two properties should be asked:

a) If the space X is an ANR, all the data should be contained in the algebraic structure, so the topology should be the discrete one.

b) This topology should agree with the direct limit topology.

Moreover, property a) is easily deduced from b), taking the trivial expansion of X. However, if we impose this two elementary properties to such a topology over $\check{H}^n(X;\mathbb{Z})$, it is easy to see that the following result holds.

Proposition 2.3.1 *Let X be a compact metric space, and let τ be any topology over $\check{H}^n(X;\mathbb{Z})$ satisfying the conditions* a) and b) *above. Then, τ is the discrete topology.*

Proof. Let $\mathbf{X} = \{X_k, p_{kk+1}\}$ be an inverse sequence of ANRs which defines X as an inverse limit. Taking singular cohomology over each level, we obtain the direct sequence in cohomology $H^n(\mathbf{X},\mathbb{Z}) = \{H^n(X_k,\mathbb{Z}), p_{kk+1_*}\}$. Hence, the Čech cohomology of X is

$$\check{H}^n(X,\mathbb{Z}) = \lim_{\rightarrow} H^n(\mathbf{X},\mathbb{Z}).$$

Since *a)* is satisfied, then each $H^n(X_k,\mathbb{Z})$ is discrete. Moreover, from property *b),* the topology over $\check{H}^n(X,\mathbb{Z})$ is the direct limit topology. But the direct limit of discrete spaces is itself discrete, so τ must be the discrete topology. □

By the result above, such a topological structure on Čech cohomology groups over $\mathbb{Z}$ lacks of any interest. The case over S^1 is interesting because it links cohomology with Pontryagin duality theory in a natural way.

Remark 2.3.2 Historically, there have been some works connecting shape theory and Pontryagin duality theory. In particular, J. Keesling pointed out different links between cohomology and duality, obtaining satisfactory results in the theory of shape. As it was pointed out above, the group $\check{H}^k(X,\mathbb{Z})$ would be naturally discrete, so its dual is compact. In fact, in [47] it is proved that $(\check{H}^k(X,\mathbb{Z}))^\wedge = \check{H}_k(X,S^1)$, and this relation is used by Keesling to obtain his results.

As we shall see in the following section, an analogous statement also holds for homology, but as it is usually remarked in some classical books, there is no a natural way to topologize the cohomology groups. Of course, this assertion is reasonable if one wants to keep the compactness, but if not, the Pontryagin duality for direct and inverse limits respect this topologies in an admissible way.

2.3.2 Group coefficients $G = S^1$

The construction which we propose in this section is the dual version of the preceding: Starting with the homology groups over the integers of some space, we arrive to the cohomology groups with S^1 coefficients.

The singular cohomology over an abelian group G is defined starting with the singular chain complex

$$\cdots \to \mathcal{C}_{n+1}(X) \overset{\partial}{\to} \mathcal{C}_n(X) \overset{\partial}{\to} \mathcal{C}_{n-1}(X) \to \cdots$$

and applying the $Hom(-,G)$ functor to it. Chain groups are replaced with cochain groups $\mathcal{C}^n(X;G) = Hom(\mathcal{C}_n, G)$ and so is the boundary operator ∂ with the coboundary $\delta = Hom(\partial, G)$. Thus, we obtain the chain complex

$$\cdots \leftarrow \mathcal{C}^{n+1}(X;G) \overset{\delta}{\leftarrow} \mathcal{C}^n(X;G) \overset{\delta}{\leftarrow} \mathcal{C}^{n-1}(X;G) \leftarrow \cdots$$

where $\delta\varphi = \varphi\partial$ for $\varphi \in \mathcal{C}^n(X;G)$.

Finally, cohomology groups $H^n(X;G)$ are defined as the quotient group $Ker\delta/Im\delta$. However, the final result is not exactly de dual object of the homology, as it is stated in the following well-known theorem.

Theorem 2.3.3 (The universal coefficient theorem) *Let $\mathcal{C}$ be a chain complex of free abelian groups with homology groups $H_n(C)$. Then, the cohomology groups $H_n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split exact sequences*

$$0 \longrightarrow Ext(H_{n-1}(C), G) \longrightarrow H^n(C;G) \overset{h}{\longrightarrow} Hom(H_n(C), G) \longrightarrow 0$$

In particular, this theorem can be applied in the case of the singular chain complex of a CW-complex X. Recall that for an abelian group H, $Ext(H,G)$ is just the cohomology group $H^1(F;G)$ being F a free resolution $0 \to F_1 \to F_0 \to H \to 0$ of H (see §3.1 of [39]). Here we only need some properties of the functor $Ext(-,G)$ and not its rigorous definition. For finitely generated H, it is not difficult to compute $Ext(H,G)$, since it satisfies the next properties:

- $Ext(H \oplus H', G) \approx Ext(H,G) \oplus Ext(H',G)$.
- $Ext(H,G) = 0$ if H is free.
- $Ext(\mathbb{Z}_n, G) \approx G/nG$.

From the universal coefficient theorem, we obtain a split exact sequence

$$0 \longrightarrow Ext(H_{n-1}(X),G) \longrightarrow H^n(X;G) \xrightarrow{h} Hom(H_n(X),G) \longrightarrow 0$$

for each $n \geq 0$.

If now X is a compact metric space, there exists an inverse sequence $\mathbf{X} = \{X_k, p_{kk+1}\}$ where each X_k is an ANR and $X = \lim_{\leftarrow} \mathbf{X}$. Applying the universal coefficient theorem to each level, we obtain split exact sequences related by the maps induced by p_{nn+1} as shown in the next commutative diagrams:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ext(H_{n-1}(X_{k-1}),G) & \longrightarrow & H^n(X_{k-1};G) & \xrightarrow{h_{k-1}} & Hom(H_n(X_{k-1}),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ext(H_{n-1}(X_k),G) & \longrightarrow & H^n(X_k;G) & \xrightarrow{h_k} & Hom(H_n(X_k),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ext(H_{n-1}(X_{k+1}),G) & \longrightarrow & H^n(X_{k+1};G) & \xrightarrow{h_{k+1}} & Hom(H_n(X_{k+1}),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Shortly, this diagram are level morphisms between the correspondent inverse sequences induced by $Ext(H_{n-1}(-),G), H^n(-;G)$ and $Hom(H_n(-),G)$ respectively. Consequently, this level morphism defines an exact short sequence in the direct limits:

$$0 \longrightarrow \lim_{\rightarrow} Ext(H_{n-1}(X_k),G) \longrightarrow \lim_{\rightarrow} H^n(X_k;G) \xrightarrow{\lim_{\rightarrow} h_k} \lim_{\rightarrow} Hom(H_n(X_k),G) \longrightarrow 0 \qquad (2.1)$$

Again, $\check{H}^n(X;G) = \lim_{\rightarrow} H^n(X_k;G)$. But the computation of the rest of direct limits above are not easy in general. Dydak proved in [29] the next result for integer coefficients.

Theorem 2.3.4 *Let X be a continuum. Then $\check{H}^n(X;\mathbb{Z})/Tor\check{H}^n(X;\mathbb{Z})$ is isomorphic to* $\lim_{\to} Hom(H_n(X_k),G)$ *and $Tor\check{H}^n(X;\mathbb{Z})$ is isomorphic to* $\lim_{\to} Ext(H_{n-1}(X_k),G)$ *for each* $n>0$.

This result states that, as in the case of singular cohomology over $\mathbb{Z}$, the Čech cohomology is not exactly the dual of the Čech homology. In order to obtain a genuine duality, we take $G=S^1$. In this case $\check{H}^n(X;S^1)$ coincides with the Pontryagin dual of $\check{H}_n(X;S^1)$. We refer to the preliminaries of this text for a briefly introduction to the needed terminology and results about this duality. For deeper results, see [3] or [76].

Let us come back to the exact sequence (2.1), having in mind the Pontryagin duals of the Čech homology groups of any space. In our particular case, if X is a compact ANR, the homology groups of X are finitely generated (X is homotopy equivalent to a polyhedron and such polyhedron must have a finite number of simplexes in each dimension). Then, for a fixed natural number k, we have a decomposition in a finite direct sum of the form:

$$H_k(X)\approx\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus\mathbb{Z}_{n_1}\oplus\cdots\oplus\mathbb{Z}_{n_m}.$$

Applying the above properties of Ext for $G=S^1$, we get

$$\begin{aligned}Ext(H_n(X),S^1)\approx&Ext(\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}\oplus\mathbb{Z}_{n_1}\oplus\cdots\oplus\mathbb{Z}_{n_m},S^1)\approx\\ \approx&Ext(\mathbb{Z}\oplus\cdots\oplus\mathbb{Z},S^1)\oplus Ext(\mathbb{Z}_{n_1},S^1)\oplus\cdots\oplus Ext(\mathbb{Z}_{n_m},S^1)\approx\\ \approx&S^1/n_1S^1\oplus\cdots\oplus S^1/n_mS^1=0\end{aligned}$$

This allows us to give a topology on the cohomology groups $\check{H}^n(X;S^1)$, when X is an ANR. This topology has as open sets (in the cohomology groups) the corresponding open sets of $(\check{H}_n(X))^\wedge$ via the isomorphism. The well-known duality between compact and discrete spaces of Pontryagin theory, immediately implies the following result.

Lemma 2.3.5 *Let X be a compact ANR. Then $H^n(X;S^1)$ can be endowed with a compact topology.*

Proof. For a compact ANR we already know that $H_n(X)$ is discrete. Hence,

$$Hom(H_n(X),S^1)=CHom(H_n(X),S^1)=(H_n(X))^\wedge.$$

From the theory of duality, it is well-known that the dual of a discrete space is compact. Finally, the algebraic isomorphism

$$h:H^n(X,S^1)\to Hom(H_n(X),S^1)$$

obtained from the remarks above, is also a topological homeomorphism by declaring $A\subseteq H^n(X;S^1)$ open if and only if $A=f^{-1}(U)$ for $U\subseteq(H_n(X))^\wedge$ open. Therefore, $H^n(X;S^1)$ is compact. □

If now X is a compact metric space, we can repeat the above reasoning for each ANR X_k, obtaining $Ext(H_n(X_k); G) = 0$ for each $k \in \mathbb{N}$ and h_k is then an isomorphism. Consequently, in (2.1) $\lim\limits_{\rightarrow} Ext(H_{n-1}(X_k), G) = 0$ and the map $h = \lim_{\rightarrow} h_k$ is an isomorphism, and also an homeomorphism with the correspondent direct limit topologies.

Theorem 2.3.6 *If X is a compact metric space, then the Čech cohomology group $\check{H}^n(X; S^1)$ is homeomorphically isomorphic to the Pontryagin dual of the Čech homology group $\check{H}_n(X)$.*

Proof. From the exact sequence (2.1) and the previous argument, the homomorphism h is both an isomorphism and a homeomorphism between $\check{H}^n(X; S^1)$ and $\lim\limits_{\rightarrow} Hom(H_n(X_k), S^1)$.

On the other hand,

$$Hom(H_n(X_k), S^1) = CHom(H_n(X_k), S^1) = (H_n(X_k))^\wedge,$$

since $H_n(X_k)$ is discrete. We can apply now the result about reflexivity of inverse and direct limits of locally compact spaces of [48]. For that, we need all $H_n(X_k)$ to be locally compact, which in fact they are, since they are all discrete groups. Then, we obtain

$$\lim_{\rightarrow} Hom(H_n(X_k), S^1) = \lim_{\rightarrow} (H_n(X_k))^\wedge = (\lim_{\leftarrow} H_n(X_k))^\wedge = (\check{H}_n(X))^\wedge.$$

Hence, $\check{H}^n(X; S^1)$ and $(\check{H}_n(X))^\wedge$ are isomorphic as groups and homeomorphic as topological spaces. □

As we pointed out in the first paragraph of this section, the process of construction of cohomology as a dual object from homology is not accurate, since Ext functor appears. However, using continuous homomorphisms, we can understand the construction of cohomology as a dualization from homology if we re-state the idea of what we understand for dualizing and we replace $Hom(-, S^1)$ by $CHom(-, S^1)$, obtaining a duality in the sense of Pontryagin. Let us recall that this is a slight change, since in fact it does not modify anything in the classical theory for polyhedra.

Remark 2.3.7 As we have tried to show up, Pontryagin duality is intimately related with cohomology with coefficients in S^1. Coming back to the works of Keesling (see e.g. [49]) and our exposition, shape and Pontryagin duality interacts in both ways. In the mentioned paper, the author uses fruitfully a functor from Sh to CTG (the category of compact topological groups) which assigns to the Pontryagin dual of the cohomology over integer coefficients, $(\check{H}^k(X, \mathbb{Z}))^\wedge$.

In the case of a polyhedron P, the homology group of P can be factorized as $H_k(P, \mathbb{Z}) = \mathbb{Z}^{\beta_k} \oplus Tor(H_k(P, \mathbb{Z}))$, where β_k is the $k^t h$ Betti number of the space. The universal coefficient theorem yields $H^k(P,\mathbb{Z}) = \mathbb{Z}^{\beta_k} \oplus Tor(H_{k-1}(P,\mathbb{Z}))$, and, in agreement with the way proposed by Keesling, its dual is $(S^1)^{\beta_k} \oplus Tor(H_{k-1}(P,\mathbb{Z}))$.

On the other hand, the approach that we propose seems more natural in the sense that it only depends on the homology at the k-level, since the dual of $H_k(P,\mathbb{Z}) = \mathbb{Z}^{\beta_k} \oplus Tor(H_k(P,\mathbb{Z}))$, is $(S^1)^{\beta_k} \oplus Tor(H_k(P,\mathbb{Z}))$.

From 2.3.6, the theory of Pontryagin duality allows to state some properties about Čech cohomology with S^1 coefficients.

Corollary 2.3.8 *Let X be a compact metric space. Then, $\check{H}^n(X;S^1)$ is hemicompact and separable.*

Corollary 2.3.9 *If $\check{H}_n(X)$ is discrete (e.g. for X ANR or FANR) then $\check{H}^n(X;S^1)$ is compact.*

Corollary 2.3.10 *If $\check{H}_n(X)$ is compact, then $\check{H}^n(X;S^1)$ is discrete.*

Example 2.3.11 Let us consider $X = \prod_{n\in\mathbb{N}} \mathbb{R}P^2$, the countable product of real projective planes. This space can be viewed as an inverse system $\{X_n, p_{nn+1}\}$ with $X_n = \prod_{i=1}^n \mathbb{R}P^2$ and $p_{nn+1} : X_{n+1} \to X_n$ as the projection deleting the $n+1$ coordinate:

$$\begin{array}{ccccccccc} \mathbb{R}P^2 & \longleftarrow & \mathbb{R}P^2 \times \mathbb{R}P^2 & \longleftarrow & \mathbb{R}P^2 \times \mathbb{R}P^2 \times \mathbb{R}P^2 & \longleftarrow & \cdots & \longleftarrow & \prod_{n\in\mathbb{N}} \mathbb{R}P^2 \\ x_1 & \longleftarrow\!\shortmid & (x_1,x_2) & \longleftarrow\!\shortmid & (x_1,x_2,x_3) & \longleftarrow\!\shortmid & \cdots & \longleftarrow\!\shortmid & (x_i)_{i\in\mathbb{N}} \end{array}$$

The induced inverse system in homology is

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \longleftarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longleftarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longleftarrow & \cdots \\ \alpha_1 & \longleftarrow\!\shortmid & (\alpha_1,\alpha_2) & \longleftarrow\!\shortmid & (\alpha_1,\alpha_2,\alpha_3) & \longleftarrow\!\shortmid & \cdots \end{array}$$

which has as inverse limit the Čech homology group

$$\check{H}_1(X) = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$$

On the other hand, the direct limit induced in cohomology with coefficients in S^1 is the dual chain of the previous one,

$$\begin{array}{ccccccc} \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow & \cdots \\ \alpha_1 & \longmapsto & (\alpha_1,0) & & & & \\ & & (\alpha_1,\alpha_2) & \longmapsto & (\alpha_1,\alpha_2,0) & \longmapsto & \cdots \end{array}$$

and therefore

$$\check{H}^1(X;S^1) = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2 = \left(\prod_{n\in\mathbb{N}} \mathbb{Z}_2\right)^{\wedge} = \left(\check{H}_1(X)\right)^{\wedge}$$

as direct limit. Observe that the Čech cohomology of X is the dual of an ultrametric group which is not the dual of any discrete group.

Chapter 3

Generalized coverings theories

In the classical theory of covering spaces (see the preliminaries or Chapter 2 in [83]) for a given topological space X, a *covering space of* X is a pair $(\widetilde{X}, \pi)$, where $\widetilde{X}$ is a topological space and $\pi : \widetilde{X} \to X$ is the so-called *covering map,* satisfying the property that every point of X has a neighbourhood evenly covered by π. If $\widetilde{X}$ is path-connected, locally path connected and simply connected, then it is called *the universal covering* of X, since each other covering of X is also covered by $\widetilde{X}$, in a similar fashion.

When the base space X is path-connected, locally path-connected and semilocally simply connected, it is easy to prove the existence and uniqueness of the universal covering space. In fact, such universal covering can be identified with a collection of homotopy classes of paths emanating from a fixed point, with a certain topology. However, out of the framework of these spaces, the classical theory of coverings is not as nice as it should be, and the way to face the problem of the generalization of coverings leads to several and different approaches, as in [4, 5, 6, 19, 36, 58]. Also in [42] the relations between groups and different concepts of coverings are studied (see the introduction of [36] and also of [87] for some historical evolution of the problem). In particular, H. Fischer and A. Zastrow give in [36] a definition of generalized covering spaces which enlarges the classical one.

In this chapter we give a reconstruction of this classical theory but using shape concepts, changing some definitions appropriately. In this way, we obtain coverings which are, in most cases, generalized coverings in the sense of Fischer and Zastrow. This construction gives also another way to topologize the natural candidate for being a covering space of some topological space X (namely, the universal path space), in the spirit of the developing of the theory of shape through inverse systems. We also compare this new topology with others already defined, related with different theories, to generalize the classical covering theory to badly behaved spaces.

3.1 On the concept of covering space

We follow here the spirit of [19, 36, 87] to treat the problem of enlarging the definition of covering. The starting point in those works is to take the natural candidate to be the universal covering space and make some suitable choice of the topology, changing the requirements of what a covering means. For this situation, the construction of the topology is not completely intuitive and some different choices appear, depending on the properties of coverings that one desires to retain.

The main work that we follow here is the mentioned [36] in which the authors make an interesting digression about the properties that a suitable covering space would enjoy. In particular, they argue that every attempt to generalize the notion of universal covering space is linked to a list of properties which are demanded as indispensable and another properties which must be discarded. Moreover, Fischer and Zastrow point out that most of the applications and usefulness of a simply-connected covering space do not lie in the evenly covered neighborhoods, but rather in a list of consequences of this definition.

In conclusion, it seems adequate to modify the definition of what we understand by a covering in order to enlarge the class of spaces for which the theory applies. We take from [36] the following definition.

Definition 3.1.1 Let (X, x_0) be a pointed topological space. A *generalized universal covering* of X is a topological space $\widetilde{X}$ joint with a map $\pi : \widetilde{X} \to X$ satisfying the following properties:

U1) The space $\widetilde{X}$ is path-connected, locally path-connected and simply connected.

U2) The map $\pi : \widetilde{X} \to X$ is a continuous surjection.

U3) For every path-connecetd and locally path-connected topological space Y, for every continuous function $f : (Y, y_0) \to (X, x_0)$ with $f_*(\pi_1(Y, y_0)) = 1$, and for every $\widetilde{x} \in \widetilde{X}$ with $\pi(\widetilde{x}) = x_0$, there exists a unique continuous lift $g : (Y, y_0) \to (\widetilde{X}, \widetilde{x})$ with $\pi \circ g = f$.

3.1.1 The Universal Path Space

In classical theory of covering spaces, it is well-known the following result on the existence of universal covering (see e.g. [39]).

Theorem 3.1.2 *Let (X, x_0) be a pointed topological space. (X, x_0) has universal covering if and only if X is path-connected, locally path-connected and semilocally simply-connected.*

From the definition of universal covering, it is not difficult to prove its uniqueness. Furthermore, in the above hypothesis it is possible to identify the universal covering of a space with the set of homotopy classes of paths emanating from the base-point. This motivates the next definition, which follows the nomenclature from [8].

Definition 3.1.3 Let (X, x_0) be a pointed topological space. The *path space of* (X, x_0) is defined as

$$P(X, x_0) = \{\alpha : [0,1] \to X, \,|\, \alpha(0) = x_0\}.$$

Let us denote $\simeq$ the equivalence relation of homotopy of paths relative to the endpoints. We call

$$\widetilde{X} = P(X, x_0)/\simeq = \{[\alpha] \mid \alpha \text{ is a path with } \alpha(0) = x_0\}$$

the *universal path space of* X.

When X is path-connected, locally path-connected and semilocally simply-connected, there is no doubt about what topology should be given to this set, and neither it is difficult to explicitly define this topology (see section 1.3 in [39]).

Taking the end-point projection $\pi : \widetilde{X} \to X$ such that $\pi([\alpha]) = \alpha(1)$, and all previous considerations, $\pi : \widetilde{X} \to X$ is the universal covering of X. The question, posed by Fischer and Zastrow in [36], is how far this candidate remains valid as a generalized universal covering.

3.1.2 Existence of generalized universal coverings

Let X be a path-connected space, and $x_0 \in X$. The theory developed by Fischer and Zastrow considers the universal path space of X endowed with a certain topology (which is a modification of the usual given in the hypothesis of existence of classical universal covering).

Definition 3.1.4 The family

$$\{B([\alpha], U) \mid [\alpha] \in \widetilde{X} \text{ and } U \subseteq X \text{ open}\}$$

where

$$B([\alpha], U) = \{[\beta] \in \widetilde{X} \mid \exists \gamma : [0,1] \to U, \gamma(0) = \alpha(1) \text{ such that } [\beta] = [\alpha * \gamma]\},$$

is a base for a topology on $\widetilde{X}$. This topology shall be called *whisker topology.*

Remark 3.1.5 For a path-connected, locally path-connected and semilocally simply connected topological space, the whisker topology coincides with the topology given in the classical case on $\widetilde{X}$. This follows from Lemma 2.1 in [36]. In particular, this two topologies agree in the case of polyhedra (or ANR's or CW complexes).

Consider again the projection map as the end-point projection. Then we can explore if $\pi : \widetilde{X} \to X$ is a generalized universal covering of X or not. The answer is contained in Part a) of Theorem 4.10 in [36], which is the main result of the mentioned paper.

Theorem 3.1.6 *Suppose X is shape-injective. Then the map $\pi : \widetilde{X} \to X$ is a generalized universal covering.*

In the following section, we propose a construction of a sort of covering, but changing homotopy by shape.

3.2 Generalized coverings adapted to the theory of shape

Using notions from shape theory, and following the guideline of the article of Fischer and Zastrow, we construct a slightly more general covering which in particular recovers the theory developed in [36] in the case of X being shape-injective. Recall that we have a canonical homomorphism $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ mapping the homotopy class $[\alpha]$ of a loop in X to its shape class $\langle\alpha\rangle$, and X is called *shape-injective* if φ is injective, that is, if $Ker(\varphi) = \{1\}$.

As in previous chapters, we shall restrict to compact metric spaces to simplify argumentations and gain geometrical intuition. We shall make brief comments about the general situation.

Let (X, x_0) be a pointed compact metric space (assume it embedded as a closed set of the Hilbert cube). For any two paths $\alpha, \beta : [0,1] \to X$ with $\alpha(0) = \beta(0) = x_0$, let us consider the relation

$$\alpha R \beta \Leftrightarrow \alpha(1) = \beta(1) \text{ and } \beta \simeq \eta * \alpha, \text{ where } \gamma \text{ is trivial in shape.}$$

Equivalently, $\alpha R \beta$ if and only if $\varphi([\alpha * \overline{\beta}]) = \langle c_{x_0}\rangle$, that is, if $\alpha * \overline{\beta}$ is trivial in shape.

It is easy to check that this relation is in fact an equivalence relation on the path space $P(X, x_0)$. Let us denote

$$\widehat{X} = \{\langle\alpha\rangle \mid \alpha : [0,1] \to X, \alpha(0) = x_0\}$$

the set of classes of this equivalence relation. This relation is compatible with the relation of homotopy of paths, so it induces a well-defined equivalence relation in $\widetilde{X}$, defined in a similar fashion,

$$[\alpha] R [\beta] \Leftrightarrow \langle\alpha\rangle = \langle\beta\rangle.$$

Note that, with this reformulation, it is clear $\widehat{X} = \widetilde{X}/Ker\varphi$.

In particular, we have a map

$$\widehat{p} : \widetilde{X} \to \widehat{X}$$

which shall be automatically continuous if we endow $\widehat{X}$ with the quotient topology from the whisker topology on $\widetilde{X}$. A basis for the topology considered on $\widehat{X}$ is given by the basic sets

$$B(\langle\alpha\rangle, U) = \{\langle\beta\rangle \mid \beta \simeq \eta * \alpha * \gamma \text{ such that } \eta \in Ker\varphi \, \gamma : [0,1] \to U\}.$$

Under compacity and metrizability assumptions, we can consider $U = B_X(\alpha(1), \varepsilon)$, where the subindex denotes that the ball is referred to the original space X (not to the Hilbert cube).

Remark 3.2.1 There is another way to topologize the set $\widehat{X}$ using shape ideas, which seems to be more concordant with the development of shape by means of inverse systems. This topology shall appear naturally in the following section.

The following observation justifies the notation used for the classes of the relation above.

Remark 3.2.2 *$\widehat{p}$ is obviously onto. Moreover, $\widehat{p}|_{\pi_1(X,x_0)} = \varphi$.*

Lemma 3.2.3 *$\widehat{p}$ bijective if and only if φ is injective.*

Proof. Let $[\alpha] \in \pi_1(X,x_0)$ such that $\varphi([\alpha]) = \langle c_{x_0}\rangle$. By the remark above,

$$\widehat{p}([c_{x_0}]) = \langle c_{x_0}\rangle = \varphi([\alpha]) = \widehat{p}([\alpha])$$

which implies $[\alpha] = [c_{x_0}]$, using that $\widehat{p}$ is injective. This shows that φ is injective.

On the other hand, let us take $[\alpha],[\beta] \in \widetilde{X}$ with $\widehat{p}([\alpha]) = \widehat{p}([\beta])$. Then, $\langle\alpha\rangle = \langle\beta\rangle$ or equivalently, $\varphi([\alpha * \overline{\beta}]) = \langle\alpha * \overline{\beta}\rangle = \langle c_{x_0}\rangle$. Thus, $[\alpha * \overline{\beta}] = [c_{x_0}]$, since φ is injective. This implies $[\alpha] = [\beta]$, which shows that $\widehat{p}$ is injective (consequently, $\widehat{p}$ is bijective, because it is always onto). □

If we are in the hypothesis for the existence of classical universal covering, this construction coincides with the classical one.

Proposition 3.2.4 *If X is path-connected, locally path-connected and semilocally simply-connected, then $\widehat{X} = \widetilde{X}_{Ker\varphi}$. Moreover, in this case φ is an isomorphism, so $\widehat{\pi} : \widehat{X} \to X$ coincides with the universal covering of X.*

Proof. In the construction of intermediate coverings (see [83] or [39]), let us take $H = Ker\varphi$. Then, $[\alpha]R[\beta]$ if and only if $[\alpha*\overline{\beta}] \in H$. Consequently, if $\pi : \widetilde{X} \to X$ is the (classical) universal covering of X, then $\widetilde{X}_{Ker\varphi} = \widetilde{X}/Ker\varphi = \widehat{X}$.

In addition, $\varphi : \pi_1(X,x_0) \to \check{\pi}_1(X,x_0)$ is an isomorphism (see [52, 54] or [36]). In particular, $Ker\varphi = 1$. Hence, $\widetilde{X} = \widehat{X}$ and, obviously, the topologies coincide. □

In the following, we shall see that in the general settings, we recover analogous results to those of Fischer and Zastrow when $\widehat{X}$ is endowed with the topology given by the basic sets $B(\langle\alpha\rangle, U)$. In that way, we obtain a sort of covering but replacing the neutral element in homotopy by $Ker\varphi$. This subgroup can be naturally considered as neutral when we operate with shape classes.

The following result is the property $(U2)$ required for a generalized universal covering.

Proposition 3.2.5 *The projection map $\widehat{\pi} : \widehat{X} \to X$ given by $\widehat{\pi}(\langle\alpha\rangle) = \alpha(1)$ is continuous and onto.*

Proof. $\widehat{\pi}$ is onto since X is path-connected. Let us observe that $\widehat{\pi} \circ \widehat{p} = \pi$. Since π is continuous and $\widehat{p}$ is a quotient map, then $\widehat{\pi}$ is continuous.

However, it is possible to show the continuity of $\widehat{\pi}$ independently from the continuity of π. If $\langle\alpha\rangle \in \widehat{X}$ and U is an open set of X with $\alpha(1) \in U$, then

$$\widehat{\pi}(B(\langle\alpha\rangle, U)) \subseteq U.$$

Indeed, for $\langle\beta\rangle \in B(\langle\alpha\rangle, U)$, there exists a path γ such that

$$\beta \simeq \alpha * \gamma \text{ in } B(X,\varepsilon) \text{ for each } \varepsilon > 0 \text{ and } \gamma([0,1]) \subseteq U.$$

Hence, $\widehat{\pi}(\langle\beta\rangle) = \beta(1) = \gamma(1)$. In particular, $\gamma(1) \in U$ and $\beta(1) \in U$. □

The first part of property $(U1)$ is contained in the following result.

Proposition 3.2.6 *$\widehat{X}$ is path-connected and locally path-connected.*

Proof. It is enough to check that $B(\langle\alpha\rangle, U)$ is path-connected for each U with $\alpha(1) \in U$. Let us take $\langle\beta\rangle \in B(\langle\alpha\rangle, U)$, that is, $\beta \simeq \eta * \alpha * \gamma$.

Consider $r_t : [0,1] \to [0,t]$ given by $r_t(s) = st$, and for each $t \in [0,1]$ denote $\gamma_t = \gamma \circ r_t$. If $\widehat{x} = \langle\alpha\rangle$ and $\widehat{y} = \langle\beta\rangle$, then $\widehat{f}(t) = \langle\alpha * \gamma_t\rangle$ gives a path in $\widehat{X}$ such that $\widehat{f}(0) = \widehat{x}$ and $\widehat{f}(1) = \widehat{y}$.

Note that $\widehat{f}$ is continuous by the continuity of γ, because

$$\widehat{f}([t-\varepsilon, t+\varepsilon]) \subseteq B(\langle\alpha * \gamma_t\rangle, V)$$

for some open set $V \subseteq X$ with $\gamma_t(1) = \gamma(t) \in V$. □

In the definition of a generalized universal covering, we need to replace the trivial element in homotopy by the group of trivial elements in shape, $Ker\varphi$. With this assumption, the property $(U3)$ is rewritten as follows.

Proposition 3.2.7 *For every Y path-connected and locally path-connected space, $f : (Y,y) \to (X,x_0)$ with $f_*(\pi_1(Y,y)) \subseteq Ker\varphi$, and for each $\widehat{x} \in \widehat{X}$ with $\widehat{\pi}(\widehat{x}) = x_0$, there exists a lift $\widehat{f} : (Y,y) \to (\widehat{X},\widehat{x})$ such that $\widehat{\pi} \circ \widehat{f} = f$.*

Proof. Let us fix $\widehat{x} = \langle\alpha\rangle$, and take $w \in Y$. Let $p : [0,1] \to Y$ be a path such that $p(0) = y$ and $p(1) = w$, and define $\widehat{f}(w) = \langle\alpha * (f \circ p)\rangle$.

Then, $\widehat{f}$ is well-defined: If p_1 and p_2 are two paths with $p_1(0) = p_2(0) = y$ and $p_1(1) = p_2(1) = w$, then $p_1 * \overline{p}_2 \in \pi_1(Y,y)$, so $f \circ (p_1 * \overline{p}_2) \in Ker\varphi$. Hence, $\langle\alpha * (f \circ p_1)\rangle = \langle\alpha * (f \circ p_2)\rangle$.

Using the continuity of f, we get that $\widehat{f}$ is continuous, because

$$\widehat{f}(V^w) \subseteq B(\widehat{f}(w), U).$$

Moreover,

$$\widehat{\pi} \circ \widehat{f}(w) = \widehat{\pi}(\langle\alpha * (f \circ p)\rangle) = \alpha * (f \circ p)(1) = f \circ p(1) = f(w),$$

thus $\widehat{\pi} \circ \widehat{f} = f$. □

The uniqueness of the lift is implied by the unique path lifting property.

Definition 3.2.8 We say that $\widehat{\pi} : \widehat{X} \to X$ has the *unique path lifting property* if for two paths $\gamma_1, \gamma_2 : [0,1] \to \widehat{X}$ such that $\widehat{\pi} \circ \gamma_1 = \widehat{\pi} \circ \gamma_2$ and $\gamma_1(0) = \gamma_2(0)$, then $\gamma_1 = \gamma_2$.

Remark 3.2.9 *Let us suppose that the map $\widehat{\pi} : \widehat{X} \to X$ has the unique path lifting property, take a map $f : (Y, y) \to (X, x_0)$ from a path-connected and locally path-connected space (Y, y_0) and fix $\widehat{x} \in \widehat{X}$, such that $\widehat{\pi}(\widehat{x}) = x_0$. If f has a lift $\widehat{f} : (Y, y) \to (\widehat{X}, \widehat{x})$, then it is unique.*

In addition, the last part of property $(U1)$ is intimately related with the unique path lifting property.

Proposition 3.2.10 *$\pi_1(\widehat{X}, \widehat{x}) = Ker\varphi$ if and only if $\widehat{\pi} : \widehat{X} \to X$ has the unique path lifting property.*

Proof. Let $\widehat{f}_1, \widehat{f}_2 : [0,1] \to (\widehat{X}, \widehat{x})$ two paths such that $\widehat{\pi} \circ \widehat{f}_1 = \widehat{\pi} \circ \widehat{f}_2 = f$, and $\widehat{f}_1(0) = \widehat{f}_2(0)$.

Denote $\langle\beta\rangle = \widehat{f}_1(1)$ and $\langle\gamma\rangle = \widehat{f}_2(1)$. Let $\widehat{\beta}$ and $\widehat{\gamma}$ be lifts of β and γ respectively. Observe that $\widehat{\beta}(1) = \langle\beta\rangle = \widehat{f}(1)$ and $\widehat{\gamma}(1) = \langle\gamma\rangle = \widehat{f}_2(1)$. Using that $\pi_1(\widehat{X}, \widehat{x}) = Ker\varphi$, the homotopy classes $[\widehat{\beta} * \overline{\widehat{f}_1}]$ and $[\widehat{\gamma} * \overline{\widehat{f}_2}]$ are mapped to the trivial shape class by φ, thus,

$$\langle\widehat{\beta} * \overline{\widehat{f}_1}\rangle = \langle\widehat{\gamma} * \overline{\widehat{f}_2}\rangle$$

Then,

$$\widehat{\pi}_*(\langle\widehat{\beta} * \overline{\widehat{f}_1}\rangle) = \widehat{\pi}_*(\langle\widehat{\gamma} * \overline{\widehat{f}_2}\rangle) \quad \Rightarrow \quad \langle(\widehat{\pi} \circ \widehat{\beta}) * (\widehat{\pi} \circ \overline{\widehat{f}_1})\rangle = \langle(\widehat{\pi} \circ \widehat{\gamma}) * (\widehat{\pi} \circ \overline{\widehat{f}_2})\rangle \quad \Rightarrow$$
$$\Rightarrow \quad \langle\beta * \overline{f}\rangle = \langle\gamma * \overline{f}\rangle \quad \Rightarrow \quad \langle\beta\rangle = \langle\gamma\rangle \quad \Rightarrow \quad \widehat{f}_1(1) = \widehat{f}_2(1).$$

Reasoning as above for each $t \in [0,1]$, we obtain $\widehat{f}_1 = \widehat{f}_2$.

Conversely, consider $\widehat{\pi}_* : \pi_1(\widehat{X}, \widehat{x}_0) \to \pi_1(x, x_0)$, being $\widehat{x}_0 = \langle c_{x_0}\rangle$. We show that the map $\widehat{\pi}_*$ is injective.

Let us take $[\widehat{f}] \in \pi_1(\widehat{X}, \widehat{x}_0)$, where $\widehat{f} : [0,1] \to \widehat{X}$ such that $\widehat{\pi}_*([\widehat{f}]) = [\widehat{\pi} \circ \widehat{f}] = [c_{x_0}]$.

Then, there exists an homotopy $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = \widehat{\pi} \circ \widehat{f}(s)$, $H(s,1) = x_0$ and $H(0,t) = H(1,t) = x_0$. Let us denote $f_t(\cdot) = H(\cdot, t)$ and let $\widehat{f_t}$ be its lift. This is equivalent to saying that the homotopy H lifts to an homotopy between $\widehat{f_0} = \widehat{f}$ and $\widehat{f_1} = \widehat{x}_0$. Consequently, $[\widehat{f}] = 1$.

In addition, $Im(\widehat{\pi}_*) = Ker\varphi$. First, let us observe that $Im(\widehat{\pi}_*)$ is characterized as those loops which its lifting is also a loop. Now, let us take a loop $\alpha : [0,1] \to X$, such that

$\widehat{\alpha}:[0,1]\to\widehat{X}$ is its lift (that is, $\widehat{\pi}\circ\widehat{\alpha}=\alpha$.) Then, $\widehat{\alpha}(0)=\widehat{x}_0=\langle c_{x_0}\rangle$ and $\widehat{\alpha}(1)=\langle\alpha\rangle$. So, $\widehat{\alpha}$ is a loop if and only if $\langle\alpha\rangle=\langle c_{x_0}\rangle$, i.e., $\varphi([\alpha])=1$ and, equivalently, $[\alpha]\in Ker\varphi$.

Since $\widehat{\pi}_*$ is injective and $Im(\widehat{\pi}_*)=Ker\varphi$, we obtain that $\pi_1(\widehat{X},\widehat{x}_0)=Ker\varphi$. □

The existence of the covering $\widehat{\pi}:\widehat{X}\to X$ is always guaranteed by the following result.

Proposition 3.2.11 *If $f:[0,1]\to X$ is a path with $f(0)=x_0$, then there exists a unique path $\widehat{f}:[0,1]\to\widehat{X}$ with $\widehat{f}(0)=\widehat{x}_0$ such that $\widehat{\pi}\circ\widehat{f}=f$.*

Proof. The existence part is a particular case of 3.2.7.

We now check the unique path lifting property. Let us suppose that there exist $\widehat{f_1},\widehat{f_2}:[0,1]\to\widehat{X}$ with $\widehat{f_1}(0)=\widehat{f_2}(0)=\widehat{x}_0$ (where $\widehat{x}_0$ is such that $\widehat{\pi}(\widehat{x}_0)=x_0$) and $\widehat{\pi}\circ\widehat{f_1}=f=\widehat{\pi}\circ\widehat{f_2}$. We need to show that $\widehat{f_1}=\widehat{f_2}$.

Let us denote $\langle\alpha_t\rangle=\widehat{f_1}(t)$ and $\langle\beta_t\rangle=\widehat{f_2}(t)$, for each $t\in[0,1]$. By hypothesis, $\alpha_t(1)=\beta_t(1)$ for every $t\in[0,1]$.

Suppose that there exists some $r>0$ such that $\widehat{f_1}(r)\neq\widehat{f_2}(r)$. That means $\langle\alpha_r\rangle\neq\langle\beta_r\rangle$. Hence, there exists $\varepsilon>0$ such that $\alpha_r\simeq\beta_r$ as paths in $B(X,\varepsilon)$, since $\alpha_r(1)=\beta_r(1)=x$.

Denote

$$t_0=sup\{t\in[0,r]\mid\alpha_t\simeq\beta_t\ \text{ in } B(X,\varepsilon)\}$$

and choose $U=B_X(x,\varepsilon)=B_Q(x,\varepsilon)\cap X$. By continuity of $\widehat{f_1}$ and $\widehat{f_2}$, there exists $\delta>0$ such that

$$\widehat{f_1}(t)=\langle\alpha_t\rangle\in B(\langle\alpha_{t_0}\rangle,U)$$

and

$$\widehat{f_2}(t)=\langle\beta_t\rangle\in B(\langle\beta_{t_0}\rangle,U)$$

for every $t\in(t_0-\delta,t_0+\delta)$.

Let us suppose first that $\alpha_{t_0}\simeq\beta_{t_0}$ in $B(X,\varepsilon)$. Then, $0\leq t_0<r$. Take $t\in(t_0,r)$ with $t-t_0<\delta$. By definition of the basic sets on $\widehat{X}$, there exists $\eta_1,\eta_2\in Ker\varphi$ and $\gamma_1,\gamma_2:[0,1]\to U$ such that

$$\alpha_t\simeq\eta_1*\alpha_{t_0}*\gamma_1$$

$$\beta_t\simeq\eta_2*\beta_{t_0}*\gamma_2$$

in X. Moreover, $\gamma_1*\overline{\gamma}_2$ is a loop in U, since $\alpha_t(1)=\beta_t(1)$. This loop is contractible in $B_Q(x,\varepsilon)$ and, consequently,

$$\alpha_t\simeq\eta_1*\alpha_{t_0}*\gamma_1\simeq\eta_1*\beta_{t_0}*\gamma_1\simeq\eta_1*\overline{\eta}_2*\beta_t*\overline{\gamma_2}*\gamma_1\simeq\beta_t$$

in $B(X,\varepsilon)$. This contradicts the choice of t_0.

On the other hand, suppose $\alpha_{t_0}\not\simeq\beta_{t_0}$ in $B(X,\varepsilon)$. Then, $0<t_0\leq r$, and a similar reasoning gives $\alpha_t\not\simeq\beta_t$ in $B(X,\varepsilon)$ for $t\in(0,t_0)$ with $t_0-t<\delta$. Such a t is an upper bound which again contradicts the definition of t_0.

Therefore, $\widehat{f_1}(t) = \widehat{f_2}(t)$ for every $t \in [0,1]$. □

We deduce now that the main result of Fischer and Zastrow about the existence of generalized universal covering is a consequence of the above.

Corollary 3.2.12 *If X is shape-injective, then $\widehat{\pi} : \widehat{X} \to X$ is the generalized universal covering of X.*

Proof. We have that $\pi_1(\widehat{X}, \widehat{x}_0) = Ker\varphi = 1$. This fact, together with 3.2.6 implies the property $(U1)$. Property $(U2)$ was proved in 3.2.5. The existence part of $(U3)$ is in 3.2.7, and the uniqueness is deduced combining remark 3.2.9 and proposition 3.2.11 above.

In fact, $\widehat{X} = \widetilde{X}$, since $\widehat{p}$ is bijective under the assumption of shape-injectivity. □

This construction can be applied to spaces which are not shape-injective. In particular, the following result ensures that this construction is maximal in some sense.

Proposition 3.2.13 *Let X be a path-connected, locally path-connected compact metric space. If $x_0 \in X$ is such that $\pi_1(X, x_0) = Ker\varphi$, then $\widehat{X} = X$.*

Proof. From the proof of existence of lifting, consider the assignation $x \mapsto \langle\alpha\rangle$, being α a path from x_0 to x, which is the unique lift of the identity of X. It is well-defined, since another path β from x_0 to x would lead a loop $\alpha * \overline{\beta}$ and consequently $\langle\alpha\rangle = \langle\beta\rangle$. In addition, this map is easily checked to be an homeomorphism. □

Corollary 3.2.14 *Let $f : X \to Y$ be a continuous map between path connected compact metric spaces. Then, there exists a continuous map $\widehat{f} : \widehat{X} \to \widehat{Y}$ such that $\widehat{\pi} \circ \widehat{f} = f \circ \widehat{\pi}$.*

Proof. It is enough to consider the map $f \circ \widehat{\pi} : \widehat{X} \to Y$. Its lift to $\widehat{Y}$ is $\widehat{f}$. □

3.3 Topologies on the universal path space

From the construction of the previous section, we can observe that the object

$$\widehat{X} = \{\langle\alpha\rangle \mid \alpha : [0,1] \to X, \alpha(0) = x_0\}$$

is composed just of those paths emanating from a base-point identified under a shape equivalence relation. This reminds the idea of those classes on the shape group $\check{\pi}_1(X, x_0)$ generated by elements of the fundamental group $\pi_1(X, x_0)$.

In this section we shall show that it is possible to reinterpret $\widehat{X}$ from $\widetilde{X}$, as an image of a suitable assignation which relates homotopy and shape. It also allows us to give a topology on $\widetilde{X}$ which generalizes the shape topology on $\pi_1(X, x_0)$ from the ultrametric on $\check{\pi}_1(X, x_0)$, in a way that reproduces the construction as an inverse limit.

Finally, we relate this topology with other topologies already introduced on the universal path space, and show how the induced topologies on the fibre of a generalized universal covering can be identified with the topologies constructed in Chapter 1.

3.3.1 Whisker, lasso and quotient topologies

In order to address the question of how the properties of the covering affect $\pi_1(X, x_0)$, several authors tried to introduce topology on the fundamental group, which in particular would reflect the discreteness on $\pi_1(X, x_0)$ whenever the universal covering exists. The paper [7] can be considered as one of the initial works in this way. Despite of some gaps through the mentioned article, Biss proposed the quotient topology from the loop space (with the compact-open topology) as suitable for the fundamental group. This topology makes $\pi_1(X, x_0)$ into a quasi-topological group which is discrete if X is connected, locally path-connected and semilocally simply connected. Of course, the same guideline can be followed for a quotient topology on the universal path-space.

Throughout the literature we can find different topologies which are susceptible of being defined on the universal path space, some of them appeared related with various generalized covering spaces theories (as in [19] or in [36]) while some of them appear as a generalization of a topology initially proposed for groups (as in [7] or in [68]). In [87], the authors focused on the comparison of three topologies on the universal path space $\widetilde{X}$. We define this topologies below.

Definition 3.3.1 Let (X, x_0) be a pointed topological space. On the universal path space $\widetilde{X}$ the following topologies are defined:

i) *Whisker topology:* this topology is generated by the basis

$$\mathcal{B}_W = \{B([\alpha], U) \,|\, [\alpha] \in \widetilde{X}, U \text{ open set in } X \text{ with } \alpha(1) \in U\}$$

where

$$B([\alpha], U) = \{[\beta] \in \widetilde{X} \,|\, \beta \simeq \alpha * \delta \text{ with } \delta : [0,1] \to U,\ \delta(0) = \alpha(1)\}.$$

The path δ is called an *U-whisker of* α. We shall denote this topology by τ_W, and call *whisker neighbourhood of* $[\alpha]$ to the sets $B([\alpha], U)$.

ii) *Lasso topology:* it is generated by the basis

$$\mathcal{B}_\ell = \{B([\alpha], \mathcal{U}, U) \,|\, [\alpha] \in \widetilde{X}, \mathcal{U} \text{ open covering of } X \text{ and } U \in \mathcal{U} \text{ such that } \alpha(1) \in U\},$$

where

$$B([\alpha], \mathcal{U}, U) = \{[\beta] \in \widetilde{X} \,|\, \beta \simeq l * \alpha * \delta \text{ where } l \in \pi^{Sp}(\mathcal{U}, x_0),\ \delta \text{ an } U\text{-whisker}\}.$$

Recall that the group $\pi^{Sp}(\mathcal{U}, x_0)$ is known as *the Spanier group of* $\mathcal{U}$ and it is generated by the elements of the form $\eta * \beta * \eta^{-1}$, with η a path from x_0 and β lying in U_β for some $U_\beta \in \mathcal{U}$. The elements of this group are usually called *lassos,* and the lasso topology shall be denoted by τ_ℓ. We also call *lasso neighbourhood of* $[\alpha]$ to the sets $B([\alpha], \mathcal{U}, U)$.

iii) *Quotient from the compact-open topology:* if the path space $P(X, x_0)$ is endowed with the compact-open topology, $\widetilde{X}$ is then equipped with the quotient topology with respect to the quotient map $q : P(X, x_0) \to \widetilde{X}$. This topology shall be denoted by τ_{QCO}.

Remark 3.3.2 *It is immediate from the definition that the whisker topology is finer than the lasso topology, that is,* $\tau_\ell \subseteq \tau_W$.

The relations stated in [87] between the topologies on $\widetilde{X}$ defined above, are explained in the following diagram:

For an arbitrary space X			For a locally path-connected space X		
	Whisker			Whisker	
$\subsetneq$		$\supsetneq$	$\subsetneq$		$\supsetneq$
Quotient	$\not\subseteq$ $\not\supseteq$	Lasso	Quotient	$\supsetneq$	Lasso

Remark 3.3.3 Also in [87] it is made the observation that if X is a path-connected, locally path-connected and semilocally simply-connected space, then the three topologies coincide. In fact, all of them are equivalent to the usual topology on the universal covering space. This is the case, for example, for polyhedra.

However, none of these topologies above can be reinterpreted as an inverse limit topology, which seems to have been used implicitly at least in the case of the Hawaiian Earrings space. The fundamental group of this space injects in the inverse limit of free groups, so here the inverse limit topology naturally appears [67]. The philosophy of the inverse system approach to shape seems to be adequate for our purposes of regaining an inverse limit topology on the universal path space, which is the motivation of this section.

3.3.2 Inverse limit of coverings

The construction that we offer here is the generalization for the universal path space of the idea of considering inverse limits for groups. We show the construction of this new topology on the universal path space $\widetilde{X}$, which is the adaptation of the inverse limit topology (developed in chapter 1) inherited by $\pi_1(X, x_0)$ from $\check{\pi}_1(X, x_0)$ using the homomorphism $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$.

From now on, we shall always assume that (X, x_0) is a path-connected[1] compact metric space. Let us consider (X, x_0) as an inverse limit of an inverse sequence $(\mathbf{X}, \mathbf{x}) = \{(X_n, x_n), p_{n\,n+1}\}$, i.e.,

$$(X, x_0) = \varprojlim(\mathbf{X}, \mathbf{x}) = \varprojlim\{(X_n, x_n), p_{n\,n+1}\},$$

where each term X_n is a path-connected polyhedron and for $n \in \mathbb{N}$

$$p_n : (X, x_0) \to (X_n, x_n)$$

are the canonical projections.

Since each polyhedron is path-connected, locally path-connected and semilocally simply connected, then there exists its (classical) universal covering

$$\pi_n : \widetilde{X}_n \to X_n.$$

Moreover, $\widetilde{X}$ is again a polyhedron, because the universal covering of a polyhedron is itself a polyhedron. Since we can identify $\widetilde{X}_n$ with the set of homotopy classes of paths emanating from x_n, let us fix, for each $n \in \mathbb{N}$, the homotopy class of the path which is constant at x_0 as a base point $\widetilde{x}_n \in \widetilde{X}_n$. Then, $\pi_n(\widetilde{x}_n) = x_0$, and now we have

$$\pi_n : (\widetilde{X}_n, \widetilde{x}_n) \to (X_n, x_n).$$

Moreover, we have the diagram

$$\begin{array}{ccc}
(\widetilde{X}_n, \widetilde{x}_n) & \overset{\widetilde{p}_{n\,n+1}}{\dashleftarrow} & (\widetilde{X}_{n+1}, \widetilde{x}_{n+1}) \\
{\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_{n+1}} \\
(X_n, x_n) & \underset{p_{n\,n+1}}{\longleftarrow} & (X_{n+1}, x_{n+1})
\end{array}$$

which can be filled up with an upper arrow

$$\widetilde{p}_{n\,n+1} := p_n \widetilde{_{n+1} \circ \pi}_{n+1},$$

which is the lift of the map $p_{n\,n+1} \circ \pi_{n+1}$. Also the commutativity of the square holds, since

$$\pi_n \circ \widetilde{p}_{n\,n+1} = \pi_n \circ (p_n \widetilde{_{n+1} \circ \pi}_{n+1}) = p_{n\,n+1} \circ \pi_{n+1}.$$

In addition, for each $n \in \mathbb{N}$ we have

$$\pi_n \circ (\widetilde{p}_{n\,n+1} \circ \widetilde{p}_{n+1\,n+2}) = p_{n\,n+1} \circ \pi_{n+1} \circ \widetilde{p}_{n+1\,n+2} = p_{n\,n+1} \circ p_{n+1\,n+2} \circ \pi_{n+2} = p_{n\,n+2} \circ \pi_{n+2}.$$

[1] For the construction that we offer, it would be enough to consider X to be connected. However, we demand path-connectivity in order to have invariance changing the base point, since we shall use this construction for homotopy and not for shape purposes.

and, using the uniqueness of the lift we obtain

$$\widetilde{p}_{n\,n+2} = \widetilde{p}_{n\,n+1} \circ \widetilde{p}_{n+1\,n+2},$$

since $\widetilde{p}_{n\,n+2}$ is, by its definition, the unique map satisfying

$$\pi_n \circ \widetilde{p}_{n\,n+2} = p_{n\,n+2} \circ \pi_{n+2}.$$

Hence,

$$(\widetilde{\mathbf{X}}, \widetilde{\mathbf{x}}) = \{(\widetilde{X}_n, \widetilde{x}_n), \widetilde{p}_{n\,n+1}\}$$

is an inverse sequence of polyhedra.

Definition 3.3.4 Given an inverse sequence $\{(X_n, x_n), p_{n\,n+1}\}$, we call *the lifted system of* $\{(X_n, x_n), p_{n\,n+1}\}$ to the inverse sequence $\{(\widetilde{X}_n, \widetilde{x}_n), \widetilde{p}_{n\,n+1}\}$ constructed above.

Let us denote by $\widetilde{X}_\infty$ the inverse limit of the inverse sequence $\{(\widetilde{X}_n, \widetilde{x}_n), \widetilde{p}_{n\,n+1}\}$, by $\widetilde{p}_n : \widetilde{X}_\infty \to \widetilde{X}_n$ the correspondent projections and by $\langle\alpha\rangle$ the elements of $\widetilde{X}_\infty$.

Remark 3.3.5 As we have mentioned above, we made the identification

$$\widetilde{X}_n = \{[\alpha_n] : [0,1] \to X_n \mid \alpha_n(0) = x_n\}.$$

Hence, the elements of the inverse limit $\widetilde{X}_\infty$ are sequences of homotopy classes of paths, $\{[\alpha_\lambda]\}$, such that

$$\widetilde{p}_{n\,n+1}([\alpha_{n+1}]) = [\alpha_n]$$

for each $n \in \mathbb{N}$. In addition,

$$\widetilde{p}_{n\,n+1}([\alpha_{n+1}]) = [p_{n\,n+1} \circ \alpha_n],$$

that is, $\widetilde{p}_n(\langle\alpha\rangle)$ is precisely the n^{th}-coordinate of the sequence. We have arrived to

$$\widetilde{X}_\infty = \{\langle\alpha\rangle = \{[\alpha_n]\} \mid p_{n\,n+1} \circ \alpha_{n+1} \simeq \alpha_n\}.$$

Definition 3.3.6 Let $\{X_n, p_{n\,n+1}\}$ an inverse sequence of polyhedra. An *approximative path* for this inverse sequence is a sequence of paths in X_n, $\{\alpha_n\}$, such that $p_{n\,n+1} \circ \alpha_{n+1} \simeq \alpha_n$.

Observe that if we replace each path α_n by another homotopic path α'_n, the sequences are equivalent regarded as a map between inverse sequences. In fact, the relation of homotopy of maps between inverse systems gives an homotopy relation of approximative paths. The classes of approximative paths under this relation coincides then with elements $\langle\alpha\rangle$.

We recall here that approximative paths have already been used in the theory of shape. They were introduced by Krasinkiewicz and Minc in [53] under the form of approximative maps from $[0,1]$ to X considering the space as embedded in the Hilbert cube.

Then, the set $\widetilde{X}_\infty$ is just the set of shape classes of approximative paths emanating from $x_0 = \{x_n\}$. This reflects the analogy with the classical theory of coverings, using now shape classes instead of homotopy classes and also approximative paths for X instead of paths lying on X.

In the following lemma, we give a more explicit description of the inverse limit topology considered on $\widetilde{X}_\infty$. From the remark 3.3.3 above, we can consider any of those topologies over each $\widetilde{X}_n$ (whisker, lasso or quotient) for the definition of the inverse limit topology on $\widetilde{X}_\infty$. In particular, we use here the basis given by the whisker topology on each $\widetilde{X}_n$.

Lemma 3.3.7 *Let (X, x_0) be a pointed compact metric space, and let*

$$(\mathbf{X}, \mathbf{x_0}) = \{(X_n, x_n), p_{n\,n+1}\}$$

be an inverse sequence of path-connected polyhedra such that (X, x_0) is its inverse limit. Let

$$\mathbf{p} = \{p_n\} : (X, x_0) \to (\mathbf{X}, \mathbf{x_0}) = \{(X_n, x_n), p_{n\,n+1}\}$$

be the canonical projections. For a fixed index $n_0 \in \mathbb{N}$ and an open set U_{n_0} of X_{n_0} let us consider the set

$$B(\langle\alpha\rangle, U_{n_0}) = \{\langle\beta\rangle \in \widetilde{X}_\infty \mid \exists \gamma_{n_0} : [0,1] \to U_{n_0} \text{ such that } \beta_{n_0} \simeq \alpha_{n_0} * \gamma_{n_0}\}.$$

Then, the topology τ on $\widetilde{X}_\infty$, generated by the family

$$\mathcal{B} = \{B(\langle\alpha\rangle, U_n) \mid \langle\alpha\rangle \in \widetilde{X}_\infty, U_n \subseteq X_n \text{ open }, n \in \mathbb{N}\}$$

coincides with the inverse limit topology.

Proof. It is routine to check that $\mathcal{B}$ is a basis for a topology on $\widetilde{X}_\infty$, which shall be denoted by τ. From

$$\widetilde{X}_n \xleftarrow{p_{n\,n'}} \widetilde{X}_{n'} \xleftarrow{p_{n'\,n''}} \widetilde{X}_{n''}, \qquad \widetilde{X}_n \xleftarrow{p_{n\,n''}} \widetilde{X}_{n''}$$

it is immediate that

$$p_{n''} \circ \alpha(1) \in p_{n\,n''}^{-1}(U_n) \cap p_{n'\,n''}^{-1}(U_{n'})$$

for any $n, n', n''\mathbb{N}$ with $n'' \geq n, n'$.Thus, it is enough to take an open set

$$U_{n''} \subseteq p_{n\,n''}^{-1}(U_n) \cap p_{n'\,n''}^{-1}(U_{n'})$$

containing $p_{n''} \circ \alpha(1)$. Then,

$$B(\langle\alpha\rangle, U_{n''}) \subseteq B(\langle\alpha\rangle, U_n) \cap B(\langle\alpha\rangle, U_{n'}).$$

Let us now compare the topology τ with the inverse limit topology. Consider $\widetilde{U}$ a basic open set of $\widetilde{X}_\infty$ for the inverse limit topology. As usual, $\widetilde{U}$ is of the form

$$\widetilde{U} = (\prod_{n \in \mathbb{N}} \widetilde{U}_n) \cap \widetilde{X}_\infty$$

for open sets $\widetilde{U}_n$ of $\widetilde{X}_n$, where all but a finitely many of them are the whole factor $\widetilde{X}_n$. Let us denote by $F \subset \mathbb{N}$ this finite set of indexes, and take $\langle\alpha\rangle \in \widetilde{U}$. Without loss of generality, we can assume that $\widetilde{U}_n = B([\alpha_n], U_n)$, for some open sets U_n of X_n.

There exists an index n_0 with $n_0 \geq n$ for any other index $n \in F$, and we consider this index to construct the open set

$$U_{n_0} = \bigcap_{n \in F} p_{n\,n_0}^{-1}(U_n)$$

of X_{n_0}. Note also that U_{n_0} is non-empty, since $p_{n\,n_0} \circ \alpha_{n_0}(1) \in U_n$ for each $n \in F$. Then, $B(\langle\alpha\rangle, U_{n_0}) \subseteq \widetilde{U}$.

In fact, if $\langle\beta\rangle \in B(\langle\alpha\rangle, U_{n_0})$, then there exists a path $\gamma : [0,1] \to U_{n_0}$ such that

$$\beta_{n_0} = \widetilde{p}_{n_0}(\langle\beta\rangle) \simeq \widetilde{p}_{n_0}(\langle\alpha\rangle) * \gamma = \alpha_{n_0} * \gamma.$$

Consequently, for each $n \in F$,

$$\beta_n \simeq p_{n\,n_0} \circ \beta_{n_0} \simeq p_{n\,n_0} \circ (\alpha_{n_0} * \gamma) \simeq p_{n\,n_0} \circ \alpha_{n_0} * p_{n\,n_0} \circ \gamma \simeq \alpha_n * (p_{n\,n_0} \circ \gamma).$$

In addition $p_{n\,n_0} \circ \gamma$ lies in U_n, since γ lies in U_{n_0}. Hence, $\langle\beta\rangle \in B(\langle\alpha\rangle, U_n)$. This shows that $\langle\beta\rangle \in \widetilde{U}$.

Finally, any open set for τ is also open in the inverse limit topology, since

$$\widetilde{p}_n^{-1}(B([\alpha_{n_0}], U_{n_0})) = B(\langle\alpha\rangle, U_{n_0}).$$

□

Let us take now a path $\alpha : [0,1] \to X$ such that $\alpha(0) = x_0$. For each $n \in \mathbb{N}$, we can consider the path $\alpha_n = p_n \circ \alpha : [0,1] \to X_n$, which obviously satisfies $\alpha_n(0) = x_n$. In addition,

$$p_{n\,n+1} \circ \alpha_{n+1} = p_{n\,n+1} \circ p_n \circ \alpha = p_n \circ \alpha = \alpha_n.$$

Hence, the sequence $\{[\alpha_n]\}$ where $\alpha_n = p_n \circ \alpha$ is an approximative path, and $\widetilde{p}_{n\,n+1}([\alpha_{n+1}]) = [\alpha_n]$. In particular, it belongs to $\widetilde{X}_\infty$. We shall denote by $\langle[\alpha]\rangle$ the element $\{[p_n \circ \alpha]\}$ generated by $[\alpha]$. Observe that $\langle[\alpha]\rangle$ is well-defined, since another representative of the homotopy class of the path α would lead the same sequence.

In this way, we obtain an assignment

$$\widetilde{\varphi} : \widetilde{X} \to \widetilde{X}_\infty$$

which allows to pull-back to $\widetilde{X}$ the inverse limit topology of $\widetilde{X}_\infty$. Let us remark that this assignment is completely analogous to the homomorphism of groups $\varphi : \pi_1(X, x_0) \to \check{\pi}_1(X, x_0)$ between the fundamental group and the first shape group of the space (X, x_0). Indeed, $\widetilde{\varphi}$ can be restricted to the fundamental group, since $\pi_1(X, x_0)$ is identified as the subset of $\widetilde{X}$ formed by loops. Then,

$$\widetilde{\varphi}|_{\pi_1(X,x_0)} = \varphi.$$

Definition 3.3.8 Let X be a topological space. We call *shape-induced topology on* $\widetilde{X}$ to the initial topology on $\widetilde{X}$ associated to $\widetilde{\varphi}$, considering on $\widetilde{X}_\infty$ the inverse limit topology. We shall denote this topology by τ_{Sh}.

Given an element $[\alpha] \in \widetilde{X}$, we identify it with its image $\langle [\alpha] \rangle$ in $\widetilde{X}_\infty$ via $\widetilde{\varphi}$. In accordance with Lemma 3.3.7 above, the next result should be clear.

Lemma 3.3.9 *Let us consider the family of subsets of* $\widetilde{X}$

$$\mathcal{B} = \{B([\alpha], U_\lambda) \mid [\alpha] \in \widetilde{X}, U_\lambda \subseteq X_\lambda \text{ open }, \lambda \in \Lambda\}$$

where

$$B([\alpha], U_\lambda) = \{[\beta] \in \widetilde{X} \mid \exists \gamma : [0,1] \to U_\lambda \text{ and } \beta_\lambda \simeq \alpha_\lambda * \gamma \text{ in } X_\lambda\}.$$

Then, the family $\mathcal{B}$ *is a base for a topology in* $\widetilde{X}$ *and this topology coincides with* τ_{Sh}.

Proof. By definition of initial topology, τ_{Sh} on $\widetilde{X}$ is generated by the preimages of open sets of $\widetilde{X}_\infty$. But $\widetilde{\varphi}^{-1}(B(\langle \alpha \rangle, U_\lambda)) = B([\alpha], U_\lambda)$. □

Through the rest of the present section, we assume $\widetilde{X}$ endowed with the shape-induced topology without any other explicit mention. We summarize now the functorial properties of this construction:

Proposition 3.3.10 *Every continuous map* $f : (X, x_0) \to (Y, y_0)$ *between pointed compact metric spaces induces a continuous map* $\widetilde{f} : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{Y}, \widetilde{y}_0)$. *In addition, a composition* $g \circ f$ *induces* $\widetilde{g \circ f} = \widetilde{g} \circ \widetilde{f}$ *and the identity map* $1_{(X,x_0)}$ *induces the identity* $\widetilde{1}_{(X,x_0)} = 1_{(\widetilde{X},\widetilde{x}_0)}$. *Consequently, if two pointed path-connected compact metric spaces* (X, x_0) *and* (Y, y_0) *have the same homotopy type, then* $(\widetilde{X}, \widetilde{x}_0)$ *and* $(\widetilde{Y}, \widetilde{y}_0)$ *have the same homotopy type. In particular, the construction of the shape topology does not depend on the chosen sequence which defines* (X, x_0) *as an inverse limit.*

Proof. Given $f : (X, x_0) \to (Y, y_0)$, the induced map is described as

$$\begin{array}{rccc} \widetilde{f} : & (\widetilde{X}, \widetilde{x}_0) & \longrightarrow & (\widetilde{Y}, \widetilde{y}_0) \\ & [\alpha] & \longmapsto & \widetilde{f}([\alpha]) := [f \circ \alpha] \end{array}$$

Only the continuity of this map needs to be proved, the functorial properties should be clear. The argument relies on the fact that the map f induces a shape morphism F represented by a fundamental sequence $\mathbf{f}$. We can suppose that this fundamental sequence is given level-by-level

$$\mathbf{f} = (f_n) : (\mathbf{X}, \mathbf{x_0}) \to (\mathbf{Y}, \mathbf{y_0}),$$

between inverse sequences $(\mathbf{X}, \mathbf{x_0}) = \{(X_n, x_n), p_{n\,n+1}\}$ and $(\mathbf{Y}, \mathbf{y_0}) = \{(Y_n, y_n), q_{n\,n+1}\}$ which define (X, x_0) and (Y, y_0) respectively as inverse limits. Recall for each $n \in \mathbb{N}$, $f_n : X_n \to Y_n$ is such that

$$f_n \circ p_n = q_n \circ f.$$

Now, given an element $[\alpha] \in \widetilde{X}$ and $B(\widetilde{f}([\alpha]), V_n)$ in $\widetilde{Y}$, where V_n is open in Y_n, it is enough to consider the open set $U_n = f_n^{-1}(V_n)$, in order to obtain

$$\widetilde{f}(B([\alpha], U_n)) \subseteq B([f \circ \alpha], V_n).$$

Indeed, take $[\beta] \in B([\alpha], U_n)$. Then, there exists a path $\gamma : [0,1] \to U_n$ such that

$$p_n \circ \beta \simeq (p_n \circ \alpha) * \gamma$$

as paths in X_n. Hence,

$$\begin{aligned} q_n \circ (f \circ \beta) = f_n \circ p_n \circ \beta \simeq f_n \circ ((p_n \circ \alpha) * \gamma) \simeq \\ \simeq (f_n \circ p_n \circ \alpha) * (f_n \circ \gamma) = (q_n \circ (f \circ \alpha)) * (f_n \circ \gamma) \end{aligned}$$

and $f_n \circ \gamma$ lies in V_n, since γ was in $U_n = f_n^{-1}(V_n)$. Thus, $[f \circ \beta] \in B([f \circ \alpha], V_n)$, and $\widetilde{f}$ is continuous. □

Remark 3.3.11 In the compact metric case, we can use Borsuk's neighbourhood inverse sequence as a very special inverse sequence to define the shape topology on $\widetilde{X}$. Remind that this sequence assumes X as embedded in the Hilbert cube, and there exists a sequence of ε_n decreasing to 0 and of polyhedra $\mathcal{P}_n$ such that $X_n = B(X, \varepsilon_n)$ and

$$Q \supseteq X_1 \supseteq \mathcal{P}_1 \supseteq X_2 \supseteq \mathcal{P}_2 \supseteq \cdots \supseteq X$$

and, in addition, denote $p_{n\,n+1} : \mathcal{P}_{n+1} \to \mathcal{P}_n$ the correspondent inclusions. It is clear that

$$\varprojlim \widetilde{\mathcal{P}}_n = \widetilde{X}_\infty.$$

The topology on the set $\widetilde{X}_\infty$ of approximative paths of X can be redefined as follows: for $\langle\alpha\rangle \in \widehat{X}$ and $\varepsilon > 0$, let us consider

$$B(\langle\alpha\rangle, \varepsilon) = \{\langle\beta\rangle \in \widehat{X} \mid \text{ there exists } \gamma : [0,1] \to B_Q(\alpha(1), \varepsilon) \text{ such that } \beta \simeq \alpha * \gamma \text{ in } B(X, \varepsilon)\}$$

Pay attention to the double role of $\varepsilon > 0$. On one hand, it defines where the homotopy of paths must occur and, on the other hand, also restrict where the path γ lies. Also take in account the notation $B_Q(\alpha(1), \varepsilon)$ to denote that the ball is with respect to the metric of the Hilbert cube, meanwhile we shall denote by $B_X(\alpha(1), \varepsilon)$ when the ball is properly of X. The relation between them is

$$B_X(\alpha(1), \varepsilon) = B_Q(\alpha(1), \varepsilon) \cap X.$$

It is routinely checked that the family of sets

$$\{B(\langle\alpha\rangle,\varepsilon) \mid \langle\alpha\rangle \in \widehat{X}, \varepsilon > 0\}$$

is a base for a topology on $\widehat{X}$.

Example 3.3.12 In $\mathbb{R}^3$, let us consider $\mathcal{G}$ the Griffiths' space introduced in 1.2.6. For each n, let $C(X_n)$ be the cone over the first n circles which defines the Hawaiian Earring. The induced maps $C(p_{nn+1}) : C(X_{n+1}) \to C(X_n)$ defined over the correspondent cones (which is the identity over all the cones unless over the smallest one, which is collapsed to the common apothem of all the cones), defines $C(X)$ as inverse limit of the inverse sequence $\{C(X_n), C(p_{nn+1})\}$. The correspondent notation and construction applies to X'.

Let $G_n = C(X_n) \vee C(X'_n)$ be the one-point union of the correspondent cones. Also a map $g_n = C(p_{nn+1}) \vee C(p'_{nn+1})$ is induced in the obvious way. The inverse sequence $\{G_n, g_{nn+1}\}$ defines $\mathcal{G}$ as its inverse limit. The pointed version is analogous, taking $g_n = g_0$ in each space of the sequence.

It is clear that G_n is simply-connected and locally path connected for every $n \in \mathbb{N}$, so G_n is its own (classical) universal covering. Then, $\widetilde{G}_n = G_n$ and, clearly, also $\widetilde{g}_{nn+1} = g_{nn+1}$. Then, we can identify $\widetilde{\mathcal{G}}_\infty$ with $\mathcal{G}$. Since $\pi_1(G_n, g_n) = 0$ for every $n \in \mathbb{N}$, then the shape group $\check{\pi}_1(\mathcal{G}, g_0) = 0$, meanwhile $\pi_1(\mathcal{G}, g_0)$ is non-trivial.

On the other hand, let us consider $\widetilde{\mathcal{G}} = \{[\alpha] \,|\, \alpha : [0,1] \to \mathcal{G}, \alpha(0) = g_0\}$. Then, we have the map $\widetilde{\varphi} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}_\infty$. It is not difficult to see that this function maps any element $[\alpha] \in \widetilde{\mathcal{G}}$ to the null-sequence in $\widetilde{\mathcal{G}}_\infty$. Indeed, if $\alpha : [0,1] \to \mathcal{G}$, then the homotopy class of $g_n \circ \alpha$ in G_n is trivial. Hence, the sequence $\{[\alpha_n]\}$ is constantly the trivial sequence. Consequently, the topology originated in $\widetilde{\mathcal{G}}$ is the trivial one.

3.4 Comparison of the topologies on the universal path space

We compare the topology already defined with all those topologies mentioned in definition 3.3.1, giving examples which exhibit the non-coincidence of the topologies. This generalizes the comparison of topologies on $\pi_1(X, x_0)$ made in chapter 1. At the end of the section, we draw the completed version of the diagram of comparison of the topologies on the universal path space of X. Again through all this section, we restrict to path-connected compact metric spaces.

If we want to compare τ_{Sh} with those defined in 3.3.1, it seems that using of Čech expansion would be adequate, since it makes use of coverings. However, this expansion is not indexed by the natural numbers, so it does not allow to obtain the shape topology in the same terms as our definition. To fix this inconvenience, we use the relation between systems defining an space by means of expansion, to relate it with the neighbourhood inverse sequence of Borsuk.

In accordance with results and observations of previous section, we have that the shape-induced topology on $\widetilde{X}$ can be given by

$B([\alpha],\varepsilon) = \{\langle\beta\rangle \in \widehat{X} \mid$ there exists $\gamma : [0,1] \to B_Q(\alpha(1),\varepsilon)$ such that $\beta \simeq \alpha * \gamma$ in $B(X,\varepsilon)\}$.

By analogy with [87], there are two cases to discuss separately. First, the general case in which no further assumptions on the space X are made. Secondly, the case of locally path-connected spaces.

3.4.1 General case

Proposition 3.4.1 *The lasso topology is finer than the shape topology. That is,* $\tau_{Sh} \subseteq \tau_\ell$.

Proof. Let us start with a basic neighbourhood $B([\alpha],\varepsilon)$ of the shape-induced topology. Let us consider the open set $V = B_X(\alpha(1),\varepsilon)$ in X, and the covering of X given by $\mathcal{V} = \{B_X(x,\varepsilon) \mid x \in X\}$.

Now, if $[\beta] \in B([\alpha],\varepsilon)$ then $B([\beta],\mathcal{V},V) \subseteq B([\alpha],\varepsilon)$. Actually, there exists a path $\gamma : [0,1] \to B_Q(\alpha(1),\varepsilon)$ such that

$$\beta \simeq \alpha * \gamma$$

in $B(X,\varepsilon)$. On the other hand, for $[\eta] \in B([\beta],\mathcal{V},V)$, we have that

$$\eta \simeq \ell * \beta * \xi,$$

where $\xi : [0,1] \to V$ is a whisker of β and ℓ is a $\mathcal{V}$-lasso. Since a $\mathcal{V}$-lasso is contractible in the ball $B_Q(x,\varepsilon)$, we have

$$\eta \simeq \ell * \beta * \xi \simeq \ell * \alpha * \gamma * \xi \simeq \alpha * \gamma * \xi$$

in $B(X,\varepsilon)$.

Observe that $\gamma * \xi$ is, in fact, a path whose trace is in $B(\alpha(1),\varepsilon)$, so $[\eta] \in B([\alpha],\varepsilon)$. This proves that the set $B([\alpha],\varepsilon)$ is also open for the lasso topology, as desired. □

Corollary 3.4.2 *The whisker topology is finer than the shape topology,* $\tau_{Sh} \subseteq \tau_W$.

Proof. It is immediate from the previous proposition joint the remark 3.3.2 above. □

Example 3.4.3 The converse of 3.4.1 is not true. In order to show this, we can take a *sombrero space*. This example is in the spirit of [23] (see also [34]) and it was considered in example 1.3.17.

This space X is compact, metric and path-connected (but it is not locally path-connected). Let x_0 be the base-point of X, and let us consider the path α which, starting at x_0, circles around the exterior wall and comes back to x_0.

For a fixed open covering $\mathcal{U}$ of X and $U \in \mathcal{U}$ such that $\alpha(1) \in U$, let $B([\alpha],\mathcal{U},U)$ be a lasso-neighbourhood of $[\alpha]$. We shall show that it is not open in the shape topology. For instance, it should be clear that in general (see Proposition 3.4 for the space Z of [34]) the homotopy class of the constant path $[c_{x_0}]$, is not in $B([\alpha],\mathcal{U},U)$, for suitable $\mathcal{U}$ with enough small covering sets.

On the other hand, $[c_{x_0}] \in B([\alpha],\varepsilon)$ for any neighbourhood of $[\alpha]$ in the shape-induced topology. Moreover, for any $\varepsilon > 0$, it is clear that $i_\varepsilon \circ \alpha$ is homotopic to the constant path $i_\varepsilon \circ c_{x_0}$ inside $B(X,\varepsilon)$. Hence, $[c_{x_0}]$ belongs to any neighbourhood of $[\alpha]$ in the shape-induced topology.

This shows that, in general, the lasso topology τ_ℓ is not contained in the shape-induced topology τ_{Sh}.

For analogous results about the quotient topology, we shall need to invoke some facts on homotopy properties of polyhedra, which were recalled in the preliminaries of the present work (see also [45]). In particular, we make use of 0.0.6 to obtain the following.

Proposition 3.4.4 *The quotient topology is finer than the shape topology,* $\tau_{Sh} \subseteq \tau_{QCO}$.

Proof. Recall that if $P(X,x_0)$ is endowed with the compact open topology, we consider the quotient map $q : P(X,x_0) \to \widetilde{X}$ with the identification of homotopy classes of paths. The topology τ_{QCO} on $\widetilde{X}$ is characterized as the finest topology among all those which make the map q to be continuous. Consequently, if q is continuous endowing $\widetilde{X}$ with τ_{Sh}, then $\tau_{Sh} \subseteq \tau_{QCO}$.

Moreover, since X is metrizable, the compact-open topology corresponds with the metric

$$D(f,g) = \sup\{d(f(t),g(t)) \mid t \in [0,1]\}.$$

For a path $\alpha \in P(X,x_0)$, and a neighbourhood $B([\alpha],\varepsilon)$ in τ_{Sh}, let $\mathcal{P}$ be a polyhedron such that $X \subseteq \mathcal{P} \subseteq B(X,\varepsilon)$. For this $\mathcal{P}$, there exists $\delta > 0$ such that if two maps f,g are δ-near then they are homotopic in $\mathcal{P}$. It is enough to consider the set

$$B(\alpha,\delta) = \{\beta \in P(X,x_0) \mid D(\alpha,\beta) < \delta\}.$$

If $\beta \in B(\alpha,\delta)$, in particular, $d(\alpha(t),\beta(t)) < \delta$. Hence, α and β are δ-near and, consequently, homotopic in $\mathcal{P}$ and so in $B(X,\varepsilon)$. This shows

$$q(B(\alpha,\delta)) \subseteq B([\alpha],\varepsilon)$$

and q is continuous. Thus, $\tau_{Sh} \subseteq \tau_{QCO}$. □

Example 3.4.5 The converse of 3.4.4 is not true, even in the locally path-connected case: let HE be the Hawaiian Earrings space (see again example 1.1.9).

Let us denote by l_k the loop based at x_0 traversing the k-th circle of HE. Given $k, n \in \mathbb{N}$, with $k \geq 2$, let $\alpha(n,k)$ be the concatenation of $(l_n * l_k * l_n^{-1} * l_k^{-1})^{n+k}$ with $(l_1 * l_k * l_1^{-1} * l_k^{-1})^n$, and

$$A = \{[\alpha(n,k)] \in \widetilde{X} \,|\, n, k \in \mathbb{N},\, k \geq 2\}.$$

The set A is the same considered in the proof of Theorem 1 of [33], where it is shown that the homotopy class of the constant path $[c_{x_0}]$ is not in A, and A is closed for the quotient topology. Consequently, $[c_{x_0}] \notin \overline{A}$.

On the other hand, $[c_{x_0}] \in \overline{A}$ when the shape-induced topology is considered in $\widetilde{HE}$. Actually, for any $\varepsilon > 0$, there exist sufficiently large $n_0, k_0 \in \mathbb{N}$, such that the loops l_n and l_k are contained in $B_X(x_0, \varepsilon)$ for $n \geq n_0$ and $k \geq k_0$. Consequently, regarding this small loops in $B(X, \varepsilon)$, they are contractible to x_0 in $B_Q(x_0, \varepsilon)$. Hence,

$$\alpha(n,k) \simeq l_1 * l_1^{-1} \simeq c_{x_0} \text{ in } B(X, \varepsilon)$$

so $[\alpha(n,k)] \in B([c_{x_0}], U_m)$ (for any m with $\frac{1}{m} < \varepsilon$). This shows that the quotient topology τ_{QCO} is not contained in the shape-induced topology τ_{Sh}.

3.4.2 Locally path-connected case

In this subsection we shall prove the main results of this part. The lasso topology and the shape-induced topology coincide in the locally path-connected case. From example 3.4.5, a similar result cannot be expected for the quotient topology.

Hence, the goal of this subsection is to show that, given two paths homotopic in $B(X, \varepsilon)$, they just only differ in lassos (of appropriate size). The idea of this part is analogous to the comparison of topologies on the fundamental group, made in section 1.4.2, in which it was observed in Lemma 1.4.8 that near paths are homotopic if the intersection of covering sets were path-connected.

It was also pointed out that this lemma does not work in general, as it is shown in Figure 1.5. In order to generalize that situation for any covering, we need again Lemma 1.4.9 of refinement of coverings to avoid non-path connected intersections between open sets of a covering.

With the mentioned technical lemma, we can state the main result of this part, which generalizes result 1.4.11 obtained in Chapter 1 for the correspondent topologies on the fundamental group of a pointed space. The idea of the proof is quite similar.

Theorem 3.4.6 *If X is a path-connected and locally path-connected compact metric space, then the shape and lasso topologies on $\widetilde{X}$ coincide, $\tau_{Sh} = \tau_{\ell}$.*

Proof. We have already proved in 3.4.1 that $\tau_{Sh} \subseteq \tau_{\ell}$. For the converse, let us start with a lasso neighbourhood $B([\alpha], \mathcal{U}, U)$, where $[\alpha] \in \widetilde{X}$, $\mathcal{U}$ is an open covering of X and $U \in \mathcal{U}$ such that $\alpha(1) \in U$.

Without loss of generality, we can assume that each set in $\mathcal{U}$ is path-connected (if not, take the refinement of $\mathcal{U}$ consisting of the path-components of sets in $\mathcal{U}$, which are open from the locally-path connectedness). Since X is compact, then there exists a finite subcovering of $\mathcal{U}$. Applying lemma 1.4.9, we obtain an open covering $\mathcal{W}$ with the properties stated in the referred lemma (observe that the process of 1.4.9 also gives $\mathcal{W}$ finite if it is applied to a finite covering). As mentioned in the preliminaries, the Čech and Borsuk inverse systems are related by a map of inverse systems, so there exists $\varepsilon > 0$ and a map

$$f : B(X,\varepsilon) \longrightarrow |N(\mathcal{W})|$$

such that

$$f \circ i_\varepsilon \simeq p_{\mathcal{W}}$$

where $i_\varepsilon : X \to B(X,\varepsilon)$ is an inclusion and $p_{\mathcal{W}} : X \to |N(\mathcal{W})|$ is the projection of X into the nerve of the covering $\mathcal{W}$.

For some $[\beta] \in B([\alpha],\mathcal{U},U)$, let us show that

$$B([\beta],\varepsilon) \subseteq B([\alpha],\mathcal{U},U).$$

Since $[\beta] \in B([\alpha],\mathcal{U},U)$, then $\beta \simeq \ell * \alpha * \gamma$ where ℓ is an $\mathcal{U}$-lasso and γ is an U-whisker. Let $[\eta]$ be in $B([\beta],\varepsilon)$. Hence, there exists a path $\delta : [0,1] \to B(\beta(1),\varepsilon)$ such that

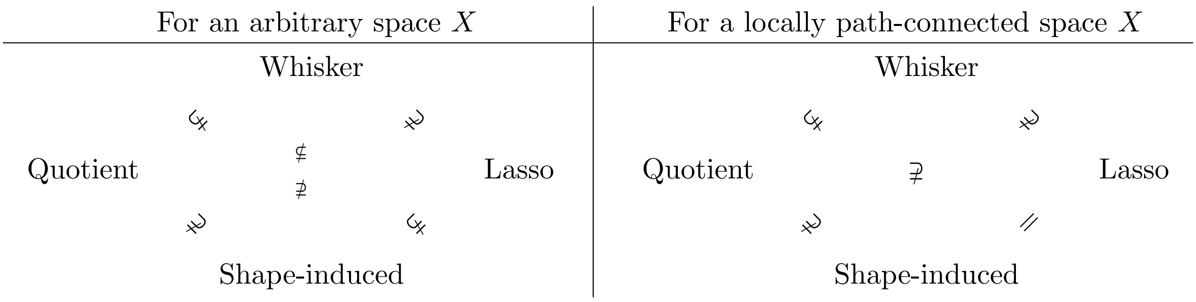
$$\eta \simeq \beta * \delta$$

in $B(X,\varepsilon)$. Applying the map f we obtain

$$f \circ \eta \simeq f \circ (\beta * \delta) = f \circ \beta * f \circ \delta$$

in $|N(\mathcal{W})|$. By a similar argument as in proof of 1.4.11 we obtain that $\eta \simeq \ell' * \beta * \gamma'$ so $\eta \simeq \ell' * \ell * \alpha * \gamma * \gamma'$ to deduce $[\eta] \in B([\alpha],\mathcal{U},U)$. □

The following diagram summarize the relations between the topologies on $\widetilde{X}$:



3.5 Topologies induced on the fibre of a covering

In the classical theory of covering spaces it is well-known that the fundamental group $\pi_1(X, x_0)$ can be identified with the fibre of the covering projection. Moreover, this fibre as a subset of the whole covering space, is discrete. At first glance, this identification implies the discreteness of the fundamental group, if it would exist a topology defined on it.

As we have shown in the first chapter of this work, there are different topologies defined on the fundamental group. The counterpart of these topologies on the universal path space has been studied on this third chapter. To finish this work, we show that these topologies induce topologies on the fibre which agree with topologies on the fundamental group.

Recall that $\pi_1(X, x_0) \subseteq \widetilde{X}$, since elements of $\pi_1(X, x_0)$ are homotopy class of loops. That is, paths emanating from x_0 which also ends at x_0. Hence, we can consider the different topologies on $\widetilde{X}$ and restrict them to the fundamental group.

It is obvious that the quotient topology associated to

$$q : P(X, x_0) \to \widetilde{X}$$

can be restricted to the correspondent quotient topology on the fundamental group associated to

$$q : \Omega(X, x_0) \to \pi_1(X, x_0).$$

The case of the lasso topology is immediate but nos as trivial as the case of the quotient topology.

Proposition 3.5.1 *The lasso topology on $\widetilde{X}$ induces the Spanier topology on $\pi_1(x, x_0)$.*

Proof. Let us consider a neighbourhood of the lasso topology $B([\alpha], \mathcal{U}, U)$ for some $[\alpha] \in \pi_1(X, x_0)$. If $[\beta] \in B([\alpha], \mathcal{U}, U)$, then

$$\beta \simeq \ell * \alpha * \gamma \simeq \ell * \alpha * \gamma * \overline{\alpha} * \alpha.$$

Observe that in this case, $\gamma(0) = \alpha(1) = x_0$ and $\gamma(1) = \beta(1) = x_0$ so γ is also a loop. Consequently, $\alpha * \gamma * \overline{\alpha}$ is a lasso and so $\ell * \alpha * \gamma * \overline{\alpha}$ is a lasso, since the concatenation of lassos is again a lasso. □

Using the lifted inverse system, we can show also a topology recovered on the shape group. Let us remind the diagram

$$\begin{array}{ccc} (\widetilde{X}_n, \widetilde{x}_n) & \overset{\widetilde{p}_{n\,n+1}}{\dashleftarrow} & (\widetilde{X}_{n+1}, \widetilde{x}_{n+1}) \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} \\ (X_n, x_n) & \underset{p_{n\,n+1}}{\longleftarrow} & (X_{n+1}, x_{n+1}) \end{array}$$

and the identification of the fundamental group $\pi_1(X_n, x_n)$ with the fibre $\pi_n^{-1}(x_n)$ of the projection map. Since $\check{\pi}_1(X, x_0) = \lim_{\leftarrow} \pi_1(X_n, x_n)$ and $\widetilde{X}_\infty = \lim_{\leftarrow} \widetilde{X}_n$, it should be clear that

$$\check{\pi}_1(X, x_0) \subseteq \widetilde{X}_\infty.$$

Remark 3.5.2 From the previous paragraph, it is clear that the inverse limit topology on $\widetilde{X}_\infty$ induces the inverse limit topology of $\check{\pi}_1(X, x_0)$ in a natural way. Hence, we can reconstruct the topology given by the ultrametric on the shape group.

The last considerations also allows us to relate the shape-induced topology on $\widetilde{X}$ with the topology given by the pseudoultrametric on the fundamental group in the first chapter:

Proposition 3.5.3 *The topology inherited on $\pi_1(X, x_0)$ from the shape-induced topology on $\widetilde{X}$ coincides with the topology generated by the pseudoultrametric*

$$d([\alpha], [\beta]) = \inf\{\varepsilon > 0 \mid \alpha \simeq \beta \text{ in } B(X, \varepsilon)\}.$$

Proof. It is enough to consider a neighbourhood $B([\alpha], \varepsilon)$ of the shape-induced topology, for some $[\alpha] \in \pi_1(X, x_0)$. If $[\beta] \in B([\alpha], \varepsilon)$, then

$$\beta \simeq \alpha * \gamma$$

in $B(X, \varepsilon)$ for some $\gamma : [0, 1] \to B_Q(\alpha(1), \varepsilon)$. Again, γ is a loop since $\alpha(1) = x_0 = \beta(1)$ and it is contractible to a point in the ball $B_Q(\alpha(1), \varepsilon)$. Hence, $\beta \simeq \alpha$ in $B(X, \varepsilon)$ and consequently $B([\alpha], \varepsilon)$ coincides with a ball for the pseudolultrametric defined on $\pi_1(X, x_0)$. □

Future lines of work

The present work leads to some questions and problems to continue working on.

In the first chapter we have shown some examples in which the relations

$$\pi^{Sp}(X, x_0) \trianglelefteq \nu(X, x_0) \trianglelefteq Ker\varphi$$

were strict. However, we do not know any example of a compact metric space in which the inequalities are simultaneously strict.

We also think that understanding the metric d_{c*} is important. For this purpose, we would like to determine $\overline{\{Peano_*\}}$ in $(2^Q, d_{c*})$. At least, we know that this closure contains all the pointed approximative absolute neighbourhood retracts in the sense of Clapp.

For the second chapter, it would be interesting to determine the homology groups of a majorant space X_0 for movable compact metric spaces, given by Spież in [84]. As a corollary of domination results, $\check{H}_n(X_0)$ would contain $\check{H}_n(X)$ as a factor subgroup and retracts onto it, for each movable compact metric space X.

In particular, $\check{H}_1(X_0)$ would contain a copy of $\prod_{i\in\mathbb{N}} \mathbb{Z}$ as a direct factor, since the Hawaiian Earrings space HE is a movable space with $\check{H}_1(HE) = \prod_{i\in\mathbb{N}} \mathbb{Z}$. Analogously, the group $\check{H}_n(X_0)$ must contain as (algebraic and topological) retract each homology group of polyhedra (since they are movable) and also all at most countable products of them. Hence, it is reasonable to think that this group should be of the form

$$\check{H}_n(X_0) = \left(\prod_{k\in\mathbb{N}} \mathbb{Z}\right) \times \left(\prod_{k\geq 2} \left(\prod_{l\in\mathbb{N}} \mathbb{Z}_k\right)\right).$$

In the third chapter, it is still necessary to define a notion of a covering (not necessarily the universal covering) in order to regain the correspondence between intermediate (generalized) coverings and subgroups. In particular, it seems that this extension should correspond with closed subgroups for the shape topology on the fundamental group.

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