

Entropy decay for Davies semigroups of a one dimensional quantum lattice

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Abstract

Given a finite-range, translation-invariant commuting system Hamiltonians on a spin chain, we show that the Davies semigroup describing the reduced dynamics resulting from the joint Hamiltonian evolution of a spin chain weakly coupled to a large heat bath thermalizes rapidly at any temperature. More precisely, we prove that the relative entropy between any evolved state and the equilibrium Gibbs state contracts exponentially fast with an exponent that scales logarithmically with the length of the chain. Our theorem extends a seminal result of Holley and Stroock [40] to the quantum setting, up to a logarithmic overhead, as well as provides an exponential improvement over the non-closure of the gap proved by Brandao and Kastoryano [43]. This has wide-ranging applications to the study of many-body in and out-of-equilibrium quantum systems. Our proof relies upon a recently derived strong decay of correlations for Gibbs states of one dimensional, translation-invariant local Hamiltonians, and tools from the theory of operator spaces.

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1 Introduction

The mitigation of errors arising from noise represents a key challenge in the development of large-scale quantum architectures [57]. Unfortunately, noise adversely affects all stages of quantum computation, from the storage of quantum information in quantum memories to its manipulation by means of quantum gates and quantum channels. In the future, error correcting codes will be used to suppress noise and achieve fully fault-tolerant computation. However, full fault-tolerance requires an unattainable overhead for current and near-term hardware. This observation leads to the need for a better understanding and characterization of the underlying noise dynamics in current quantum devices, both in order to inform hardware design, and for the preparation of noise-dependent optimized error correction and mitigation protocols.

Davies generators [29, 30, 31] model the thermal dynamics that emerge for a system weakly interacting with a large thermal reservoir. They are the natural candidates for the description of the thermal noise occurring on self-correcting quantum memories [3, 2, 46]. The weak coupling between the system and its environment allows to consider only the reduced dynamics on the system alone, which can be modeled by a quantum Markov semigroup, that is a semigroup of quantum channels. Given a spin chain $\Lambda = \llbracket 1, n \rrbracket$ made of n qudits, of corresponding quantum system Hilbert space $\mathcal{H}_\Lambda := \bigotimes_{k \in \Lambda} \mathcal{H}_k$, $\mathcal{H}_k \equiv \mathbb{C}^d$, the closed evolution of the system in absence of environmental noise is described by a local Hamiltonian H_Λ acting on \mathcal{H}_Λ and of the form:

$$H_\Lambda := \sum_{A \subset \Lambda} h_A,$$

where for each subregion $A \subset \Lambda$, the local self-adjoint operator h_A acts non-trivially on A . In this article, we further impose that the Hamiltonian H_Λ is commuting, which means that for any two regions $A, A' \subset \Lambda$, $[h_A, h_{A'}] = h_A h_{A'} - h_{A'} h_A = 0$; and translation-invariant, i.e. for any $A \subset \mathbb{Z}$ and $k \in \mathbb{Z}$, $h_{A+k} = h_A$, where $A+k := \{x+k \mid x \in A\}$. We also assume that H_Λ has finite-range r and bounded interaction strength $J := \max_{A \subset \Lambda} \|h_A\|_\infty$ independently of the system size n . The former means that for any subregion A such that $\text{diam}(A) := \max\{k \mid k \in A\} - \min\{k \mid k \in A\} > r$, $h_A = 0$. Under some standard conditions on the generator $\mathcal{L}_{\Lambda^*}^D$ acting on the algebra $\mathcal{B}(\mathcal{H}_\Lambda)$ of linear operators on \mathcal{H}_Λ (see e.g. [62]), the Davies semigroup $(e^{\mathcal{L}_{\Lambda^*}^D})_{t \geq 0}$ admits the following asymptotic behavior: for all initial states $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

$$e^{t\mathcal{L}_{\Lambda^*}^D}(\rho) \xrightarrow[t \rightarrow \infty]{} \sigma^\Lambda, \quad \text{where} \quad \sigma^\Lambda := \frac{e^{-\beta H_\Lambda}}{\text{tr}[e^{-\beta H_\Lambda}]} \quad (1)$$

denotes the Gibbs state of the Hamiltonian H_Λ at the inverse temperature $\beta > 0$ of the reservoir.

A natural question emerging from the above discussion is that of the speed at which the thermalization (1) occurs. In order to study the latter, we introduce the *mixing time* of the semigroup: for any $\epsilon > 0$,

$$t_{\text{mix}}(\epsilon) := \inf \{t \geq 0 \mid \forall \rho \in \mathcal{D}(\mathcal{H}_\Lambda), \|e^{t\mathcal{L}_{\Lambda^*}^D}(\rho) - \sigma^\Lambda\|_1 \leq \epsilon\}, \quad (2)$$

where $\|\rho - \sigma\|_1 := \text{tr}|\rho - \sigma|$ denotes the trace distance between the states ρ and σ . In [43], the authors proved that the spectral gap of $\mathcal{L}_{\Lambda^*}^D$ is uniformly lower bounded by a constant $\lambda_0 > 0$ independent of the system size n . This can be proven to imply the following bound on the mixing time [44]:

$$t_{\text{mix}}(\epsilon) = \frac{1}{\lambda_0} \mathcal{O}\left(\ln\left(\frac{1}{\epsilon}\right) + n\right). \quad (3)$$

However, by analogy with the classical setting of Glauber dynamics, one would expect a logarithmic scaling with system size [40, 63]. In our main result, we answer this conjecture in the positive. In fact, we prove the following stronger result:

Theorem 1.1 (Informal formulation). *For any reservoir inverse temperature $\beta > 0$ and all states $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$:*

$$D(e^{t\mathcal{L}_{\Lambda^*}^D}(\rho)\|\sigma^\Lambda) \leq e^{-4\alpha_n t} D(\rho\|\sigma^\Lambda), \quad (4)$$

where $\alpha_n = \Omega(\ln(n)^{-1})$, and where $D(\rho\|\sigma^\Lambda) := \text{tr}(\rho \ln \rho) - \text{tr}(\rho \ln \sigma^\Lambda)$ denotes the relative entropy between a state ρ and σ^Λ . Moreover, the constant α_n scales exponentially with β .

Note that this theorem is indeed stronger than a logarithmic scaling of the mixing time of Equation (2) as a consequence of Pinsker's inequality and Equation (4). The inequality (4) is referred to as a *modified logarithmic Sobolev inequality* (MLSI) [44] and the constant α_n satisfying this bound is called the *MLSI constant*. The problem of determining whether a quantum Markov semigroup satisfies a MLSI has been addressed in various settings in the last years. Some examples appear in [50, 49], where it was shown that the MLSI constant of the depolarizing channel can be lower bounded by $1/2$. This was subsequently extended to the generalized depolarizing channel in [12, 20]. [9] constitutes the first attempt to prove the inequality in the setting of spin systems, where the heat-bath generator in 1D was shown to satisfy a MLSI under two conditions of decay of correlations on the Gibbs state. More recently, [21] proved the modified logarithmic Sobolev inequality for a family of Gibbs samplers of nearest neighbor commuting Hamiltonians above a certain threshold temperature. However, the drawback of [21] stems from the lack of physicality of the generator considered.

Although the proof of Theorem 1.1 relies on a notion of clustering of correlations similar in spirit to that used in [21], it also requires advanced techniques from the operator space structure of amalgamated L_p spaces [42]. To our knowledge, this is the first time that such techniques are being used to solve a problem from many-body quantum systems. It is also worth noticing that our result holds for any inverse temperature β .

1.1 Applications

As mentioned above, Equation (4) implies the following corollary, as a consequence of Pinsker's inequality:

Corollary 1.2. *For any $\beta > 0$, the semigroup $(e^{t\mathcal{L}_{\Lambda^*}^D})_{t \geq 0}$ satisfies the following rapid mixing property:*

$$t_{\text{mix}}(\epsilon) \leq \frac{1}{\alpha_n} \mathcal{O}\left(\ln\left(\frac{1}{\epsilon}\right) + \ln(n)\right).$$

Combining Corollary 1.2 with the main result in [26], we can conclude that local observables and correlation functions are stable against local perturbations in the generator of Davies semigroups corresponding to a one dimensional, short-range, translation-invariant commuting Hamiltonian at any inverse temperature. Moreover, entropic decays of the form of that of Theorem 1.1 are known to imply a strengthening of Pinsker's inequality [58, 52, 32], given the following

quantum generalization of the Lipschitz constant corresponding to the Hamming distance: for any $H = H^* \in \mathcal{B}(\mathcal{H}_\Lambda)$,

$$\|H\|_L := 2 \max_{i \in [n]} \min \left\{ \|H - H^{(i)}\|_\infty \mid H^{(i)} = (H^{(i)})^* \text{ does not act on } i\text{-th qudit} \right\},$$

we define the corresponding quantum Wasserstein distance between two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_\Lambda)$ as

$$W_1(\rho, \sigma) = \max \left\{ \text{tr}[(\rho - \sigma) H] \mid H = H^*, \|H\|_L \leq 1 \right\}. \quad (5)$$

Next, a state $\sigma \in \mathcal{D}(\mathcal{H}_\Lambda)$ satisfies a transportation-cost inequality with constant $C > 0$ if for any $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

$$W_1(\rho, \sigma) \leq \sqrt{C D(\rho \parallel \sigma)}. \quad (6)$$

It was shown in [32, Section 5] that fixed points of local quantum Markov semigroups satisfy a transportation-cost inequality with constant C proportional to n/α , where α is the MLSI constant of the semigroup. Combining this with Equation (4), we get

Proposition 1.3. *Gibbs states of finite-range, commuting, one dimensional Hamiltonians of bounded interaction strength at the inverse temperature $\beta > 0$ satisfy a transportation-cost inequality with constant $C = \mathcal{O}(n \text{ polylog}(n))$.*

Transportation-cost inequalities in the form of (6) were recently used to produce sample efficient Gibbs state learning algorithms in [59]. They also imply Gaussian concentration inequalities for the statistics of quasi-local observables in the state σ and can be used to prove an exponential improvement over the weak Eigenstate Thermalization Hypothesis [52].

Finally, we remark that Theorem 1.1 has implications in the context of Symmetry Protected Topological (SPT) phases [24], i.e. in systems protected against perturbations that keep the symmetry of the system. There has been a recent increase on the interest of SPT in open quantum systems [25, 47, 48], partially motivated by their connection with Majorana modes and their potential use in quantum information theory. In the weak coupling limit between the system and the reservoir, it is shown in [48] that, for unitary symmetries, the decoherence time scales exponentially with the inverse temperature β , even in the case in which all interactions involved in the problem preserve the symmetry that gives the topological protection to the (closed) system. Therefore, the decoherence time increases exponentially with the decrease of the temperature. However, there is in principle no analysis on the dependence of the decoherence time on the system size.

The particular case of the 1D cluster state [17] has a commuting Hamiltonian and it is a non-trivial SPT phase under a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry [61]. Thus, as a consequence of Theorem 1.1 for this state we obtain an example of a non-trivial SPT phase with decoherence time scaling only logarithmically on the size of the system in the presence of thermal noise, under the assumption that all relevant interactions preserve the symmetry. A previous result in this direction was already presented in [25] for on-site depolarizing noise, for which a similar behavior was shown. For further information on this, see our companion paper [8].

1.2 Outline

In Section 2, we introduce some notations and preliminary notions which are necessary for the rest of the paper. The main result is presented in Section 3, where we also include a complete scheme of its proof, leaving the specifics of the most technical parts to Sections 4 and 5 for sake of clarity.

2 Preliminary facts and definitions

2.1 Basic notations

Let \mathcal{H} be a finite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{T}_1(\mathcal{H})$) be the bounded operators (resp. trace class operators) on \mathcal{H} . We denote tr for the standard matrix trace, $\langle \cdot, \cdot \rangle_{\text{HS}}$ for the trace inner product and $\|\cdot\|_2$ for the Hilbert-Schmidt norm. The corresponding Hilbert-Schmidt space is denoted by $\mathcal{T}_2(\mathcal{H})$. Operators will be sometimes denoted by capital letters, and sometimes by lowercase letters, e.g. in order to emphasize their belonging to a subalgebra. We write A^\dagger for the adjoint of an operator $A \in \mathcal{B}(\mathcal{H})$, and Φ^* (or Φ_*) for the adjoint (or preadjoint) of a map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. The identity operator on \mathcal{H} is denoted as $\mathbb{1}_{\mathcal{H}}$ and the identity map on a von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is $\text{id}_{\mathcal{M}}$. We also denote the dimension of \mathcal{H} by $d_{\mathcal{H}} = \dim(\mathcal{H})$. Given two maps $\Phi, \Psi : \mathcal{M} \rightarrow \mathcal{M}$ on a von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, we say that $\Phi \leq_{\text{cp}} \Psi$ if $\Psi - \Phi$ is completely positive.

We say that an operator ρ is a state (or density operator) if $\rho \geq 0$ and $\text{tr}(\rho) = 1$. We denote by $\mathcal{D}(\mathcal{H})$ the set of states on \mathcal{H} . Given a system A , we write ρ_A to emphasize that $\rho_A \in \mathcal{D}(\mathcal{H}_A)$, though we will drop this subindex whenever it is clear from the context. With a slight abuse of notation, we write $\Psi(\rho) := (\Psi \otimes \text{id})(\rho)$ for a bipartite state $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathbb{C}^n)$ and a quantum channel $\Psi : \mathcal{M}_* \rightarrow \mathcal{M}_*$. For two states ρ and σ , their relative entropy is defined as

$$D(\rho \parallel \sigma) = \begin{cases} \text{tr}(\rho \ln \rho - \rho \ln \sigma), & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\text{supp}(\rho)$ (resp. $\text{supp}(\sigma)$) is the support projection of ρ (resp. σ). Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be two von Neumann subalgebras. Recall that a conditional expectation onto \mathcal{N} is a completely positive unital map $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ satisfying

- i) for all $X \in \mathcal{N}$, $E_{\mathcal{N}}(X) = X$
- ii) for all $a, b \in \mathcal{N}$, $X \in \mathcal{B}(\mathcal{H})$, $E_{\mathcal{N}}(aXb) = aE_{\mathcal{N}}(X)b$.

We denote by $E_{\mathcal{N}^*}$ its adjoint map with respect to the trace inner product, i.e.

$$\text{tr}(E_{\mathcal{N}^*}(X)Y) = \text{tr}(XE_{\mathcal{N}}(Y)).$$

A quantum Markov semigroup (QMS) $(\mathcal{P}_t)_{t \geq 0}$ is a semigroup of completely positive, unital maps over the bounded operators $\mathcal{B}(\mathcal{H})$ of a finite dimensional Hilbert space \mathcal{H} . By the semigroup property, the QMS is generated by a map $\mathcal{L} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ that is referred to as a Lindbladian. The QMS is said to be GNS-symmetric with respect to a state $\sigma \in \mathcal{D}(\mathcal{H})$ if for any two $X, Y \in \mathcal{B}(\mathcal{H})$,

$$\text{tr}[\sigma \mathcal{L}(X)Y] = \text{tr}[\sigma X \mathcal{L}(Y)]. \quad (7)$$

Whenever the state σ is full-rank, there exists a conditional expectation E onto the kernel of \mathcal{L} such that for any $X \in \mathcal{B}(\mathcal{H})$ [34]

$$e^{t\mathcal{L}}(X) \xrightarrow[t \rightarrow \infty]{} E[X]. \quad (8)$$

2.2 Hamiltonians and Davies local generators

In this paper, we consider a spin chain $\Lambda := \llbracket 1, n \rrbracket$ made of n sites, where at each site $k \in \Lambda$ lies a qudit system $\mathcal{H}_k \equiv \mathbb{C}^d$, and denote by $\mathcal{H}_\Lambda := \bigotimes_{k \in \Lambda} \mathcal{H}_k$ the Hilbert space of the whole system. At thermal equilibrium, the system is in a Gibbs state $\sigma^\Lambda := e^{-\beta H_\Lambda} / \text{tr} e^{-\beta H_\Lambda}$ at inverse temperature $\beta > 0$, for some fixed Hamiltonian H_Λ . Here, we further assume H_Λ to be finite-range and denote by r its interaction range. This means that there exist self-adjoint operators h_A supported on regions $A \subset \Lambda$ such that $H_\Lambda = \sum_{A \subset \Lambda} h_A \otimes \mathbb{1}_{A^c}$ and $h_A = 0$ whenever $\text{diam}(A) > r$, where $\text{diam}(A)$ denotes the diameter of A . The interaction strength of H_Λ is defined as $J := \max_{A \subset \Lambda} \|h_A\|_\infty$. In this paper, we assume that neither r , nor J depend on the system size Λ .

The thermalization of the system due to its interaction with a heat-bath modeled by the Hamiltonian H^{HB} can be described as follows: assume a set of chain-bath interaction operators $\{S_{\alpha,k} \otimes B_{\alpha,k}\}$, where α labels operators $S_{\alpha,k}$ and $B_{\alpha,k}$ associated to the site $k \in \Lambda$. For sake of simplicity, we assume that the operators $S_{\alpha,k}$ form an orthonormal basis of self-adjoint operators in \mathcal{H}_Λ with respect to the Hilbert-Schmidt inner product (e.g. qubit Pauli matrices). The Hamiltonian of the universe composed of the spin chain and its heat-bath is thus given by

$$H = H_\Lambda + H^{\text{HB}} + \sum_{\alpha,k \in \Lambda} S_{\alpha,k} \otimes B_{\alpha,k}. \quad (9)$$

Assuming that the bath is in a Gibbs state, by a standard argument (e.g. weak coupling limit, see [62]), the evolution on the system can be approximated by a quantum Markov semigroup whose generator is of the form

$$\mathcal{L}_{\Lambda^*}^D(\rho) := -i[H_\Lambda, \rho] + \sum_{k \in \Lambda} \mathcal{L}_{k^*}^D(\rho), \quad (10)$$

for some local generators $\mathcal{L}_{k^*}^D$ in GKLS form

$$\mathcal{L}_{k^*}^D(\rho) = \sum_{\omega, \alpha} \chi_{\alpha,k}^{\beta, \omega} \left(S_{\alpha,k}^\omega \rho S_{\alpha,k}^{\omega, \dagger} - \frac{1}{2} \{ \rho, S_{\alpha,k}^{\omega, \dagger} S_{\alpha,k}^\omega \} \right). \quad (11)$$

The sum in (11) ranges over the index α of the local basis $\{S_{\alpha,k}\}$ as well as the Bohr frequencies ω of the Hamiltonian H_Λ . Similarly, we denote the generator $\mathcal{L}_{A^*}^D$ by restricting the sum in (10) to the subregion A in the interior of Λ . Note that $\mathcal{L}_{A^*}^D$ acts non-trivially on $A\partial := \{k \in \Lambda : \text{dist}(k, A) \leq r\}$. Above, the Fourier coefficients of the two-point correlation functions of the environment $\chi_{\alpha,k}^{\beta, \omega}$, which we assume to be uniformly upper and lower bounded along the spin chain, $0 < \chi_{\min}^\beta \leq \chi_{\alpha,k}^{\beta, -\omega} \leq \chi_{\max}^\beta$, satisfy the KMS condition $\chi_{\alpha,k}^{\beta, -\omega} = e^{-\beta\omega} \chi_{\alpha,k}^{\beta, \omega}$. The operators $S_{\alpha,k}^\omega$ are the Fourier coefficients of the system couplings $S_{\alpha,k}$, which means that they satisfy the following defining equation for any $t \in \mathbb{R}$:

$$e^{-itH_\Lambda} S_{\alpha,k} e^{itH_\Lambda} = \sum_{\omega} e^{it\omega} S_{\alpha,k}^\omega. \quad (12)$$

2.3 Modified logarithmic Sobolev inequality

In this section, we consider the generator \mathcal{L} of a quantum Markov semigroup $(\mathcal{P}_t)_{t \geq 0}$ over the finite dimensional algebra $\mathcal{B}(\mathcal{H})$, which we assume GNS-symmetric with respect to a full-rank invariant state σ , and we denote by $E : \mathcal{B}(\mathcal{H}) \rightarrow \text{Ker}(\mathcal{L})$ the conditional expectation onto the kernel of \mathcal{L} such that $\mathcal{P}_t \rightarrow E$ as $t \rightarrow \infty$. The *entropy production* of \mathcal{L} is defined for any other state $\rho \in \mathcal{D}(\mathcal{H})$ by

$$\text{EP}_{\mathcal{L}}(\rho) := - \left. \frac{dD(e^{t\mathcal{L}_*}(\rho) \| E_*(\rho))}{dt} \right|_{t=0}.$$

The entropy production is always non-negative, by the monotonicity of the relative entropy under quantum channels. Moreover, it satisfies the following useful property:

Lemma 2.1. *For any other full-rank invariant state ω :*

$$\text{EP}_{\mathcal{L}}(\rho) = -\text{tr}[\mathcal{L}_*(\rho)(\ln(\rho) - \ln(\omega))].$$

Proof. This simply follows from the fact that the difference of the logarithms of the two invariant states $\ln(\omega) - \ln(E_*[\rho])$ belongs to the fixed point algebra $\mathcal{F}(\mathcal{L})$ (see for instance the structure of invariant states in Equation (2.10) of [11]). Therefore

$$\text{tr}[\mathcal{L}_*(\rho)(\ln(E[\rho]) - \ln \omega)] = \text{tr}[\rho \mathcal{L}(\ln(E[\rho]) - \ln \omega)] = 0.$$

□

Definition 2.2 ([44, 22, 7, 35]). *The quantum Markov semigroup $(\mathcal{P}_t)_{t \geq 0}$ is said to satisfy a (non-primitive) modified logarithmic Sobolev inequality (MLSI) if there exists a constant $\alpha > 0$ such that, for all $\rho \in \mathcal{D}(\mathcal{H})$:*

$$4\alpha D(\rho \| E_*(\rho)) \leq \text{EP}_{\mathcal{L}}(\rho). \tag{MLSI}$$

The best constant satisfying (MLSI) is called the modified logarithmic Sobolev constant and denoted by $\alpha(\mathcal{L})$. Moreover, the semigroup satisfies a complete modified logarithmic Sobolev inequality (CMLSI) if, for any reference system \mathcal{H}_R , the semigroup $(e^{t\mathcal{L}} \otimes \text{id}_R)_{t \geq 0}$ satisfies a modified logarithmic Sobolev inequality with a constant α independent of R . In this case, the best constant satisfying CMLSI is called the complete modified logarithmic Sobolev constant and is denoted by $\alpha_c(\mathcal{L})$.

The reason for the introduction of the complete modified logarithmic Sobolev constant is due to its tensorization property:

Lemma 2.3 ([16]). *Let \mathcal{L} and \mathcal{K} be two generators of KMS-symmetric quantum Markov semigroups, and denote by $E_{\mathcal{L}}$, resp. by $E_{\mathcal{K}}$, their corresponding conditional expectations. Moreover, assume that $[\mathcal{L}, \mathcal{K}] = 0$. Then*

$$\alpha_c(\mathcal{L} + \mathcal{K}) \geq \min\{\alpha_c(\mathcal{L}), \alpha_c(\mathcal{K})\}.$$

The following result was recently proved in [38] (see also [37] in the tracial setting):

Theorem 2.4 ([38]). *For any GNS-symmetric QMS $(e^{t\mathcal{L}})_{t \geq 0}$ over the algebra $\mathcal{B}(\mathcal{H})$ of linear operators on a finite dimensional Hilbert space \mathcal{H} , $\alpha_c(\mathcal{L}) > 0$.*

By Grönwall's inequality, the (complete) modified logarithmic Sobolev inequality is directly related to the exponential convergence of the evolution towards its equilibrium, as measured in relative entropy:

$$D(e^{t\mathcal{L}^*}(\rho) \| E_*(\rho)) \leq e^{-4\alpha(\mathcal{L})t} D(\rho \| E_*(\rho)).$$

In this paper, we consider the MLSI constants $\alpha(\mathcal{L}_{[1,n]}^D)$ of the family $\{\mathcal{L}_{[1,n]}^D\}_{n \in \mathbb{N}}$ of Davies Lindbladians defined in Section 2.2.

2.4 Operator space theory

In this section, we briefly recall some basic definitions from operator space theory, and refer the interested reader to the standard books [33, 56] for further details. Given a complex vector space V and an integer n , we denote by $\mathbb{M}_n(V)$ the space of V -valued $n \times n$ matrices. We also denote by $\mathbb{M}_{m,n}$ the space of $m \times n$ complex valued matrices, and write $\mathbb{M}_{n,n} \equiv \mathbb{M}_n$. An (abstract) operator space V is a complex vector space equipped with a sequence of norms on the spaces $\mathbb{M}_n(V)$, which satisfy Ruan's axioms:

- (i) For all integers m, n and two elements $v \in \mathbb{M}_m(V), w \in \mathbb{M}_n(V)$,

$$\|v \oplus w\|_{\mathbb{M}_{m+n}(V)} = \max\{\|v\|_{\mathbb{M}_m(V)}, \|w\|_{\mathbb{M}_n(V)}\}; \quad (13)$$

- (ii) For all integers m, n and $v \in \mathbb{M}_m(V), A \in \mathbb{M}_{n,m}, B \in \mathbb{M}_{m,n}$,

$$\|A \cdot v \cdot B\|_{\mathbb{M}_n(V)} \leq \|A\|_{\mathbb{M}_{n,m}} \|v\|_{\mathbb{M}_m(V)} \|B\|_{\mathbb{M}_{m,n}}, \quad (14)$$

where $v \oplus w$ denote the $(m+n) \times (m+n)$ diagonal matrix $\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix}$, the product $A \cdot v \cdot B$ is the multiplication on matrix coefficients and $\|\cdot\|_{\mathbb{M}_{m,n}}$ denotes the usual matrix norm. With this definition, $\mathbb{M}_n(V)$ is again an operator space with the identification $\mathbb{M}_m(\mathbb{M}_n(V)) \cong \mathbb{M}_{mn}(V)$ in the canonical way.

Given two operator spaces V, W and a linear mapping $\varphi : V \rightarrow W$, we denote for each $n \in \mathbb{N}$ the map $\varphi_n \equiv \text{id}_n \otimes \varphi : \mathbb{M}_n(V) \rightarrow \mathbb{M}_n(W)$ by simply mapping each matrix component of $v \in \mathbb{M}_n(V)$ to its image under the map φ , i.e. $[\varphi_n(v)]_{i,j} := [\varphi(v_{ij})]_{i,j}$. The *completely bounded norm* (in short CB-norm) of φ is defined as

$$\|\varphi\|_{\text{cb}} := \sup_{n \in \mathbb{N}} \|\text{id}_n \otimes \varphi : \mathbb{M}_n(V) \rightarrow \mathbb{M}_n(W)\|,$$

given the above supremum is finite. Here, for each n , $\|\text{id}_n \otimes \varphi : \mathbb{M}_n(V) \rightarrow \mathbb{M}_n(W)\|$ is taken with respect to the norm structures of $\mathbb{M}_n(V)$ and $\mathbb{M}_n(W)$. A map φ is said to be *completely isometric* if for all $n \in \mathbb{N}$, $\text{id}_n \otimes \varphi : \mathbb{M}_n(V) \rightarrow \mathbb{M}_n(W)$ is isometric. Two operator spaces V and W are said to be completely isometric, which we denote by $V \cong W$, if there exists a complete isometry $\varphi : V \rightarrow W$. We denote by $\text{CB}(V, W)$ the space of completely bounded maps $\varphi : V \rightarrow W$ equipped with the CB-norm $\|\cdot\|_{\text{cb}}$. The space $\text{CB}(V, W)$ itself is an operator space by the identification

$$\mathbb{M}_n(\text{CB}(V, W)) \cong \text{CB}(V, \mathbb{M}_n(W)).$$

Let V^* be the Banach space dual of V . The dual operator space structure of V^* is then given by $V^* \cong \text{CB}(V, \mathbb{C})$. This induces a norm on each $\mathbb{M}_n(V^*)$, which can be easily seen to verify condition (i) and (ii) above. For example, $\mathcal{T}_1(\mathcal{H})^* = \mathcal{B}(\mathcal{H})$ as an operator space's dual space.

A concrete operator space V is a norm-closed subspace of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} . The natural operator space structure is given by the embedding $\mathbb{M}_n(V) \subset \mathbb{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$. It was proved by Ruan [60] that every abstract operator space is a concrete operator space. We also need the notion of *operator space minimal tensor product* (also known as *operator space injective tensor product*, or *spatial tensor product*). Let $V \subset \mathcal{B}(\mathcal{H})$ and $W \subset \mathcal{B}(\mathcal{K})$ be two (concrete) operator spaces. The operator space minimal tensor product $V \otimes_{\min} W$ is given by the inclusion

$$V \otimes_{\min} W \subset \mathcal{B}(\mathcal{H} \otimes \mathcal{K}).$$

Note that this definition is independent of the choice of embedding $V \subset \mathcal{B}(\mathcal{H})$ and $W \subset \mathcal{B}(\mathcal{K})$. More precisely,

$$\begin{aligned} & \|u\|_{\min, n} \\ & := \sup\{\|(f \otimes g)(u)\|_n : f : V \rightarrow \mathbb{M}_p, g : W \rightarrow \mathbb{M}_q, \|f : V \rightarrow \mathbb{M}_p\|_{\text{cb}}, \|g : W \rightarrow \mathbb{M}_q\|_{\text{cb}} \leq 1\}, \end{aligned}$$

where the supremum is over any pair of integers q, p . Here, the norm $\|\cdot\|_n$ is taken with respect to the natural operator space structure on $\mathbb{M}_p \otimes \mathbb{M}_q$ obtained when identifying $\mathbb{M}_n(\mathbb{M}_p \otimes \mathbb{M}_q)$ with $\mathcal{B}((\mathbb{C}^p \otimes \mathbb{C}^q)^{\oplus n}, (\mathbb{C}^p \otimes \mathbb{C}^q)^{\oplus n})$.

In this manuscript, we are exclusively interested in the operator space structure of a non-commutative amalgamated L_p space, which is discussed in Section 5.1. We will use the following proposition (see e.g. [33, Proposition 8.1.2, Corollary 8.1.3]):

Proposition 2.5. *For any two operator spaces V and W , the natural embedding $\theta : V^* \otimes_{\min} W \hookrightarrow \text{CB}(V, W)$ defined by*

$$\theta(f \otimes w)(v) = f(v)w, \quad \forall f \in V^*, w \in W, v \in V, \quad (15)$$

is completely isometric. In particular, for any operator space W and $n \in \mathbb{N}$, $\mathbb{M}_n(W) \cong \mathbb{M}_n \otimes_{\min} W$.

2.5 Conditional expectations and subalgebra indices

Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be two von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. Recall that a conditional expectation onto \mathcal{N} is a completely positive unital map $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ satisfying

- i) for all $X \in \mathcal{N}$, $E_{\mathcal{N}}(X) = X$,
- ii) for all $a, b \in \mathcal{N}$, $X \in \mathcal{M}$, $E_{\mathcal{N}}(aXb) = aE_{\mathcal{N}}(X)b$.

We denote by $E_{\mathcal{N}*}$ its adjoint map with respect to the trace inner product, i.e.

$$\text{tr}(E_{\mathcal{N}*}(X)Y) = \text{tr}(XE_{\mathcal{N}}(Y)).$$

For a state ρ , the relative entropy with respect to \mathcal{N} is defined as follows

$$D(\rho \|\mathcal{N}) := D(\rho \|\mathcal{E}_{\mathcal{N}*}(\rho)) = \inf_{E_{\mathcal{N}*}(\sigma) = \rho} D(\rho \|\sigma),$$

where the infimum is always attained by $E_{\mathcal{N}^*}(\rho)$. Indeed, for any σ satisfying $E_{\mathcal{N}^*}(\sigma) = \sigma$, we have the identity (see [41, Lemma 3.4])

$$D(\rho\|\sigma) = D(\rho\|E_{\mathcal{N}^*}(\rho)) + D(E_{\mathcal{N}^*}(\rho)\|\sigma).$$

Hence the infimum is attained if and only if $D(E_{\mathcal{N}^*}(\rho)\|\sigma) = 0$. More explicitly, a finite dimensional von Neumann (sub)algebra is always given by a direct sum of matrix algebras with multiplicity, i.e.

$$\mathcal{N} = \bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{C}\mathbb{1}_{\mathcal{K}_i}, \quad \mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i \otimes \mathcal{K}_i.$$

Denote P_i as the projection onto $\mathcal{H}_i \otimes \mathcal{K}_i$. There exists a family of density operators $\tau_i \in \mathcal{D}(\mathcal{K}_i)$ such that

$$E_{\mathcal{N}}(X) = \bigoplus_{i=1}^n \text{tr}_{\mathcal{K}_i}(P_i X P_i (\mathbb{1}_{\mathcal{K}_i} \otimes \tau_i)) \otimes \mathbb{1}_{\mathcal{K}_i}, \quad E_{\mathcal{N}^*}(\rho) = \bigoplus_{i=1}^n \text{tr}_{\mathcal{K}_i}(P_i \rho P_i) \otimes \tau_i, \quad (16)$$

where $\text{tr}_{\mathcal{K}_i}$ is the partial trace with respect to \mathcal{K}_i . A state σ satisfies $E_{\mathcal{N}^*}(\sigma) = \sigma$ if and only if

$$\sigma = \bigoplus_{i=1}^n p_i \sigma_i \otimes \tau_i$$

for some density operators $\sigma_i \in \mathcal{D}(\mathcal{H}_i)$ and a probability distribution $\{p_i\}_{i=1}^n$. Denote $\mathcal{D}(E_{\mathcal{N}}) := \{\sigma \in \mathcal{D}(\mathcal{H}) \mid \sigma = E_{\mathcal{N}^*}(\sigma)\}$ as the subset of states that are invariant under $E_{\mathcal{N}^*}$. For any $\sigma \in \mathcal{D}(E_{\mathcal{N}})$,

$$E_{\mathcal{N}^*}(\sigma^{\frac{1}{2}} X \sigma^{\frac{1}{2}}) = \sigma^{\frac{1}{2}} E_{\mathcal{N}}(X) \sigma^{\frac{1}{2}}.$$

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a finite dimensional von Neumann algebra and $\mathcal{N} \subset \mathcal{M}$ be a subalgebra. The trace preserving conditional expectation $E_{\mathcal{N},\text{tr}} : \mathcal{M} \rightarrow \mathcal{N}$ is defined so that for any $X \in \mathcal{M}$ and $Y \in \mathcal{N}$,

$$\text{tr}(XY) = \text{tr}(E_{\mathcal{N},\text{tr}}(X)Y).$$

$E_{\mathcal{N},\text{tr}}$ is self-adjoint and corresponds to taking $\tau_i = d_{\mathcal{K}_i}^{-1} \mathbb{1}_{\mathcal{K}_i}$ in (16). We recall the definition of the index associated to the algebra inclusion $\mathcal{N} \subset \mathcal{M}$,

$$C(\mathcal{M} : \mathcal{N}) = \inf\{c > 0 \mid \rho \leq c E_{\mathcal{N},\text{tr}}(\rho) \text{ for all states } \rho \in \mathcal{M}\},$$

$$C_{\text{cb}}(\mathcal{M} : \mathcal{N}) = \sup_{n \in \mathbb{N}} C(\mathcal{M} \otimes \mathbb{M}_n : \mathcal{N} \otimes \mathbb{M}_n),$$

where the supremum in $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$ is taken over all finite dimensional matrix algebras \mathbb{M}_n . The index $C(\mathcal{M} : \mathcal{N})$ was first introduced by Pimsner and Popa in [54] for the connection to subfactor index and Connes entropy, and the completely bounded version $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$ was studied in [36]. These indices are closely related to the notion of maximal relative entropy. Recall that for two states ρ, ω , their maximal relative entropy is [28]

$$D_{\text{max}}(\rho\|\omega) = \ln \inf\{c > 0 \mid \rho \leq c\omega\}.$$

Indeed,

$$\ln C(\mathcal{M} : \mathcal{N}) = \sup_{\rho \in \mathcal{D}(E_{\mathcal{M},\text{tr}})} D_{\text{max}}(\rho\|E_{\mathcal{N},\text{tr}}(\rho)).$$

For all finite dimensional inclusion $\mathcal{N} \subset \mathcal{M}$, the index $C(\mathcal{M} : \mathcal{N})$ is explicitly calculated in [54, Theorem 6.1] (hence also $C_{\text{cb}}(\mathcal{M} : \mathcal{N})$). In particular, for $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{N} = \bigoplus_{i=1}^n \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{C}\mathbf{1}_{\mathcal{K}_i}$,

$$C(\mathcal{B}(\mathcal{H}) : \mathcal{N}) = \sum_{i=1}^n \min\{d_{\mathcal{H}_i}, d_{\mathcal{K}_i}\} d_{\mathcal{K}_i}, \quad C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathcal{N}) = \sum_{i=1}^n d_{\mathcal{K}_i}^2. \quad (17)$$

For example, if we take $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ to be the subalgebra of diagonal matrices and \mathbb{C} as the multiple of identity

$$\begin{aligned} C(\mathcal{B}(\mathcal{H}) : \mathcal{D}) &= C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathcal{D}) = d_{\mathcal{H}}, \\ C(\mathcal{B}(\mathcal{H}) : \mathbb{C}) &= d_{\mathcal{H}}, \quad C_{\text{cb}}(\mathcal{B}(\mathcal{H}) : \mathbb{C}) = d_{\mathcal{H}}^2. \end{aligned} \quad (18)$$

In [38], the authors considered a generalization of these indices for a general conditional expectation $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$. We recall that here \mathbb{M}_n is the n -dimensional matrix algebra and $E_{\mathcal{N}} \otimes \text{id}_{\mathbb{M}_n} \equiv E_{\mathcal{N}} \otimes \text{id}_n$ is a conditional expectation from $\mathcal{M} \otimes \mathbb{M}_n \rightarrow \mathcal{N} \otimes \mathbb{M}_n$. Denote

$$\tau = \bigoplus_{i=1}^n \mathbf{1}_{\mathcal{H}_i} \otimes \tau_i. \quad (19)$$

Note that $E_{\mathcal{N}}$ and $E_{\mathcal{N}^*}$ are uniquely defined by τ as follows,

$$E_{\mathcal{N}}(X) = E_{\mathcal{N}, \text{tr}}(\tau^{\frac{1}{2}} X \tau^{\frac{1}{2}}), \quad E_{\mathcal{N}^*}(\rho) = \tau^{\frac{1}{2}} E_{\mathcal{N}, \text{tr}}(\rho) \tau^{\frac{1}{2}}. \quad (20)$$

In particular, $E_{\mathcal{N}}$ is faithful if and only if τ is. Next, we define

$$\begin{aligned} C_{\tau}(\mathcal{M} : \mathcal{N}) &:= \inf_c \{c > 0 \mid \rho \leq c E_{\mathcal{N}^*}(\rho) \text{ for all states } \rho \in \mathcal{M}\} \\ C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}) &:= \sup_{n \in \mathbb{N}} C_{\tau \otimes \mathbf{1}_n}(\mathcal{M} \otimes \mathbb{M}_n : \mathcal{N} \otimes \mathbb{M}_n). \end{aligned} \quad (21)$$

Since τ commutes with \mathcal{N} ,

$$C_{\tau}(\mathcal{M} : \mathcal{N}) \leq \mu_{\min}(\tau)^{-1} C(\mathcal{M} : \mathcal{N}), \quad C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}) \leq \mu_{\min}(\tau)^{-1} C_{\text{cb}}(\mathcal{M} : \mathcal{N}) \quad (22)$$

where $\mu_{\min}(\tau) = \min_i \mu_{\min}(\tau_i)$ is the minimal eigenvalue of τ . Combined with (17), this implies $C_{\tau}(\mathcal{M} : \mathcal{N})$ and $C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})$ are finite iff τ is faithful. Moreover, for any invariant state $\sigma \in \mathcal{D}(E_{\mathcal{N}})$, by the obvious bound $\sigma \leq \tau$, we also have

$$C_{\tau}(\mathcal{M} : \mathcal{N}) \leq \mu_{\min}(\sigma)^{-1} C(\mathcal{M} : \mathcal{N}), \quad C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}) \leq \mu_{\min}(\sigma)^{-1} C_{\text{cb}}(\mathcal{M} : \mathcal{N}). \quad (23)$$

2.6 Bimodule maps and module Choi operators

Let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be a conditional expectation onto \mathcal{N} . Recall that there exists a module basis $\{\xi_i\}_{i=1}^n \in \mathcal{M}$ satisfying ([53, Theorem 3.15], see also [6, Conséquence 1.8]):

$$E_{\mathcal{N}}(\xi_i^{\dagger} \xi_j) = \delta_{ij} p_i, \quad (24)$$

where $p_i \in \mathcal{N}$ are some projections. Also recall that $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is a \mathcal{N} -bimodule map if $\Phi(aXb) = a\Phi(X)b$ for all $a, b \in \mathcal{N}$ and $X \in \mathcal{M}$. In particular, this implies $\Phi \circ E_{\mathcal{N}} = E_{\mathcal{N}}$ if Φ is unital. Next, we define the *module Choi operator* of a bimodule map Φ as

$$\chi_{\Phi} = \sum_{i,j=1}^n |i\rangle \langle j| \otimes \Phi(\xi_i^{\dagger} \xi_j) \in \mathcal{B}(l_2^n) \otimes \mathcal{M},$$

where l_2^n denotes the space of n -dimensional vectors and $\{|i\rangle\}_{i=1}^n$ is a fixed orthonormal basis in l_2^n . Thus Φ and χ_{Φ} determine each other because for each $x \in \mathcal{M}$, we have a unique decomposition $x = \sum_i \xi_i x_i$ with $x_i \in \mathcal{N}$ satisfying $p_i x_i = x_i$. Indeed, we have $x_i = E_{\mathcal{N}}(\xi_i^{\dagger} x)$. Moreover, Φ is completely positive if and only if χ_{Φ} is a positive operator in $\mathcal{B}(l_2^n) \otimes \mathcal{M}$. Indeed, for any finite family $y_1, \dots, y_m \in \mathcal{M}$, we assume the decomposition $y_j = \sum_l \xi_l x_{jl}$ with $x_{jl} \in \mathcal{N}$. Then

$$\begin{aligned} (\text{id} \otimes \Phi) \left(\sum_{i,j} |i\rangle \langle j| \otimes y_i^{\dagger} y_j \right) &= \sum_{i,j} |i\rangle \langle j| \otimes \Phi(y_i^{\dagger} y_j) \\ &= \sum_{i,j,k,l} |i\rangle \langle j| \otimes x_{ik}^{\dagger} \Phi(\xi_k^{\dagger} \xi_l) x_{jl} \\ &= \left(\sum_{i,k} |i\rangle \langle k| \otimes x_{ik}^{\dagger} \right) \chi_{\Phi} \left(\sum_{j,l} |l\rangle \langle j| \otimes x_{jl} \right), \end{aligned}$$

from which the equivalence claimed directly follows. We remark that when $\mathcal{N} = \mathbb{C}1$, χ_{Φ} is the standard Choi matrix (up to a unitary equivalence).

3 The main result

In this section, we present the main result of the paper, namely the existence of a positive MLSI constant for any Davies generator in 1D converging to the Gibbs state of a finite-range, translation-invariant and commuting Hamiltonian, at any temperature, with a logarithmic dependence with the system size. We leave the specifics of the more technical parts of the proof, Lemma 3.3 and Lemma 3.5 below, to Sections 4 and 5, respectively, for sake of clarity.

Theorem 3.1. *Let $\Lambda = \llbracket 1, n \rrbracket$. For any $\beta > 0$, we denote by $\sigma \equiv \sigma^{\beta}$ the Gibbs state of a finite-range, translation-invariant, commuting Hamiltonian at inverse temperature $\beta > 0$. Consider $\mathcal{L}_{\Lambda^*}^D$ the Davies generator of a quantum Markov semigroup $\{e^{t\mathcal{L}_{\Lambda^*}^D}\}_{t \geq 0}$ with unique fixed point σ . Then, there exists $\alpha_n = \Omega(\ln(n)^{-1})$ such that, for all $\rho \in \mathcal{D}(\mathcal{H}_{\Lambda})$ and all $t \geq 0$,*

$$D(\rho_t \| \sigma) \leq e^{-\alpha_n t} D(\rho \| \sigma), \quad (25)$$

where $\rho_t := e^{t\mathcal{L}_{\Lambda^*}^D}(\rho)$. Moreover, $\alpha_n = e^{-\mathcal{O}(\beta)}$ as a function of β .

Remark 3.2. The scaling of the MLSI constant with the inverse temperature is optimal, as proved in [51].

Note that the previous result is equivalent to the existence of a positive MLSI constant $\alpha_n = \Omega(\ln(n)^{-1})$ for $\mathcal{L}_{\Lambda^*}^D$ (cf. Section 2.3): Indeed, (25) holds if, and only if, for all $\rho \in \mathcal{D}(\mathcal{H}_{\Lambda})$ and all $t \geq 0$,

$$\alpha_n D(\rho \| \sigma) \leq -\text{tr}[\mathcal{L}_{\Lambda^*}^D(\rho)(\ln \rho - \ln \sigma)] =: \text{EP}(\rho), \quad (26)$$

where $\alpha_n = \mathcal{O}(\ln(n))$ and the entropy production in the right-hand side of the inequality is obtained by

$$\text{EP}(\rho) = - \left. \frac{d}{dt} \right|_{t=0} D(\rho_t \parallel \sigma).$$

One way to prove (26), and therefore entropic convergence of the Davies semigroup $(e^{t\mathcal{L}_{\Lambda^*}^D})_{t \geq 0}$ as claimed in Theorem 3.1, is by reducing the MLSI constant in Λ to the complete MLSI constants in smaller regions $A, B \subset \Lambda$. This procedure was introduced in [23] and [27] to simplify the traditional strategy for proving positivity of MLSI constants for classical spin systems. In the past few years, a similar idea has been explored in a number of works regarding quantum spin lattice systems [20, 9, 18, 10, 21, 38], in which a strategy to prove positivity of a MLSI constant via results of quasi-factorization (a.k.a. approximate tensorization) of the relative entropy has been conceived. The proof of Theorem 3.1 also follows this direction.

In the next few lines, we provide an intuition for the reduction from MLSI constant in Λ to complete MLSI constants in $A, B \subset \Lambda$. Given any sub-region $A \subseteq \Lambda$, we denote the projection onto the fixed points of \mathcal{L}_A^D by E_A . Its adjoint map E_{A^*} is a quantum channel that corresponds to the infinite time limit of the semigroup generated by $\mathcal{L}_{A^*}^D$, i.e. $e^{t\mathcal{L}_{A^*}^D} \rightarrow E_{A^*}$ as $t \rightarrow \infty$. Then, $e^{t\mathcal{L}_{A^*}^D}$ satisfies a positive complete MLSI if there exists $\alpha_A > 0$ such that for any $m \in \mathbb{N}$ and all states $\rho \in \mathcal{D}(\mathcal{H}_\Lambda \otimes \mathbb{C}^m)$,

$$\alpha_A D(\rho \parallel (E_{A^*} \otimes \text{id}_m)(\rho)) \leq \text{EP}_A(\rho), \quad (27)$$

where $\text{EP}_A(\rho)$ denotes the entropy production of ρ in A and is defined as

$$\text{EP}_A(\rho) := - \left. \frac{d}{dt} \right|_{t=0} D(\rho_t^A \parallel (E_{A^*} \otimes \text{id}_m)(\rho))$$

for $\rho_t^A := (e^{t\mathcal{L}_{A^*}^D} \otimes \text{id}_m)(\rho)$. Indeed, (27) is equivalent to the existence of a uniform MLSI constant for the semigroup $e^{t\mathcal{L}_{A^*}^D}$ coupled with environment of all dimensions. By a straightforward derivation, the entropy production in A can be shown to be equal to

$$\text{EP}_A(\rho) = -\text{tr} \left[\mathcal{L}_{A^*}^D(\rho) (\ln(\rho) - \ln(\sigma \otimes \text{tr}_\Lambda(\rho))) \right]$$

where $\text{tr}_\Lambda(\rho)$ is the reduced density on the environment system. Note that, in particular, for any two non-overlapping regions $A, B \subseteq \Lambda$,

$$\text{EP}_{A \cup B}(\rho) = \text{EP}_A(\rho) + \text{EP}_B(\rho). \quad (28)$$

This simple yet important observation regarding the linearity of the entropy production (inherited by the linearity of the generator) allows us to reduce the right-hand side of Equation (26) from a chain Λ to two sub-regions A, B such that $A \cup B = \Lambda$. The remaining part of the reduction procedure concerns the splitting of the left-hand side, namely the relative entropy. For that, we need the following two-steps strategy:

- (i) **Global-to-local reduction** (Section 3.1): We first prove the following *approximate tensorization* of the relative entropy that for any state $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

$$D(\rho \parallel \sigma) \leq \mathcal{C} \sum_i D(\rho \parallel E_{A_i^*}(\rho)), \quad (29)$$

for some universal constant $\mathcal{C} \geq 1$ and a covering $\{A_i\}_i$ of the chain Λ by intervals A_i of size $|A_i| = O(\ln(n))$. This turns out to be a consequence of a generic approximate tensorization for quantum states introduced in [20, 19] (see also [9]) and a recently proven mixing property of 1D quantum Gibbs states at any temperature [15].

- (ii) **Local control of the constant** (Section 3.2): We then prove that the complete MLSI (27) holds for the local subregions A_i with a constant $\alpha_A := \min_i \alpha_{A_i} = \min_i \Omega(|A_i|^{-1})$. In order to prove this second step, we will in fact resort to another approach of approximate tensorization for conditional expectations recently studied in [45, 38].

The next two subsections detail the plan drawn above.

3.1 Global-to-local reduction

The first step in the proof of (29) consists in the following abstract approximate tensorization for the relative entropy proved in [20, 19] (see also [9]). For any finite lattice $\Lambda \subset \mathbb{Z}^d$ in any dimension d , consider any two regions $A, B \subset \Lambda$ with non-overlapping complements and such that $A \cup B = \Lambda$, as in the following picture:

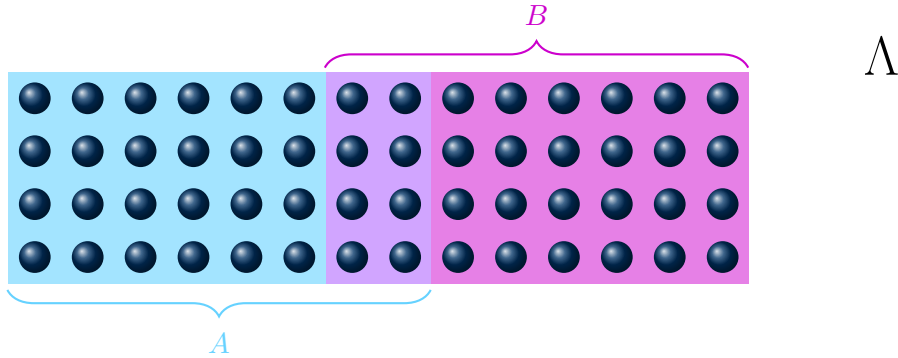


Figure 1: Possible splitting of a lattice Λ into two subregions A, B with non-overlapping complements and such that $\Lambda = A \cup B$.

Given $\sigma \in \mathcal{D}(\mathcal{H}_\Lambda)$, let us denote

$$h(\sigma_{A^c B^c}) := \left(\sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \right) \sigma_{A^c B^c} \left(\sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \right) - \mathbb{1}_{A^c B^c}. \quad (30)$$

Then, for all states $\rho, \sigma \in \mathcal{D}(\mathcal{H}_\Lambda)$ such that $\|h(\sigma_{A^c B^c})\|_\infty < 1/2$, the following holds [20]:

$$D(\rho \|\sigma) \leq \frac{1}{1 - 2 \|h(\sigma_{A^c B^c})\|_\infty} [D_A(\rho \|\sigma) + D_B(\rho \|\sigma)], \quad (31)$$

where for any region $C \subseteq \Lambda$, we define the *conditional relative entropy between ρ and σ on C* as

$$D_C(\rho \|\sigma) := D(\rho \|\sigma) - D(\rho_{C^c} \|\sigma_{C^c}).$$

The operator $h(\sigma_{A^c B^c})$, which constitutes a natural quantum generalization of the mixing condition of [27], is a measure of independence of the regions A and B as measured in the Gibbs

state σ . Now, we focus on a unidimensional $\Lambda \subset \mathbb{Z}$ and, following the lines of [9], we construct A, B as two non-connected regions. Indeed, we choose the regions A and B to be unions of small intervals $\cup_{i=1}^m A_i$ and $\cup_{j=1}^m B_j$ respectively, such that subregions A_i (resp. B_j) do not overlap with each other, and $A \cup B$ covers the spin chain Λ . More specifically, for a certain $\ell \in \mathbb{N}$ to be determined later, we assume the following conditions on $\{A_i\}_{i=1}^m$ and $\{B_j\}_{j=1}^m$:

- $|A_i| = |B_j| = 2(r + \ell) - 1$ for every $1 \leq i, j \leq m$, where r is the range of the interaction of the Hamiltonian.
- $|A_i \cap B_i| = |B_i \cap A_{i+1}| = \ell$ for every $1 \leq i \leq m - 1$.

For a better intuition on this construction, see Figure 2:

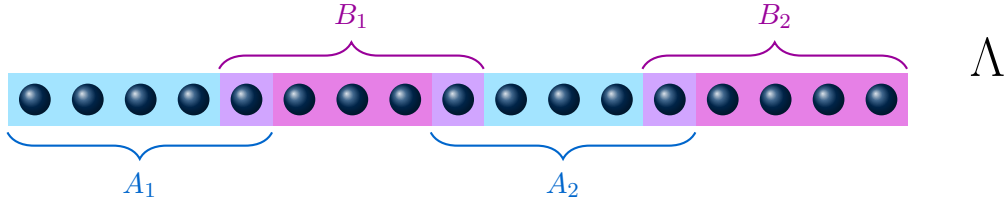


Figure 2: Representation of an interval Λ split into two regions A and B , which are the union of small intervals $\cup_{i=1}^m A_i$ and $\cup_{j=1}^m B_j$, such that $A_i \cap B_i \neq \emptyset \neq B_i \cap A_{i+1}$ for all $i = 1, \dots, m - 1$ and subregions $A_i \cap A_j = \emptyset = B_i \cap B_j$ for all $i \neq j$. In the picture, $m = 2$, $r = 2$ and $\ell = 1$.

Next, using the geometry of the chain that we have just introduced, we are able to further upper bound the conditional relative entropies at the right-hand side of (31). Indeed, since the A_i 's and B_j 's have been defined so that their boundaries do not overlap, if we consider the splitting of the chain as $A_i \leftrightarrow \partial(A_i) \leftrightarrow (A_i \partial)^c$ for any $1 \leq i \leq m$ (and analogously for B_j), the Gibbs state σ is, in particular, a quantum Markov chain between these three regions, as shown in Figure 3:

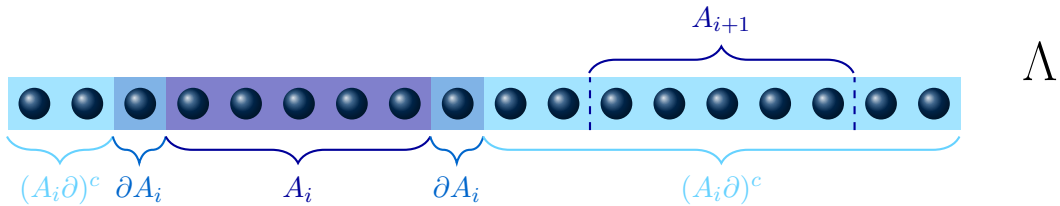


Figure 3: Splitting of Λ into the three regions $A_i \leftrightarrow \partial(A_i) \leftrightarrow (A_i \partial)^c$. Note that, in particular, any other A_j satisfies $A_j \subset (A_i \partial)^c$.

This property yields a privileged structural decomposition of σ of the following form:

$$\sigma = \bigoplus_j q_j \sigma_{A_i \partial(A_i)_j^L} \otimes \sigma_{\partial(A_i)_j^R (A_i \partial)^c} \cdot \quad (32)$$

This decomposition allowed some of the authors to show in [9, Step 2 of Theorem 7] the following inequality for such a state σ :

$$D(\rho\|\sigma) \leq \frac{1}{1 - 2\|h(\sigma_{A^c B^c})\|_\infty} \sum_{i=1}^m [D_{A_i}(\rho\|\sigma) + D_{B_i}(\rho\|\sigma)]. \quad (33)$$

Now, we aim at estimating the multiplicative term appearing in (28) (and in (33)) given this geometry. Intuitively, the quantity $\|h(\sigma_{A^c B^c})\|_\infty$ should decrease with the size of the overlap between A and B , since in this case their complements get more separated. This intuition was recently given a rigorous justification in [15, Proposition 8.1]. There, building up on the seminal result of Araki [5], the authors prove the following *mixing condition*: Consider any three convex regions $X, Y, Z \subset \mathbb{Z}$ with Y shielding X away from Z as in Figure 4,

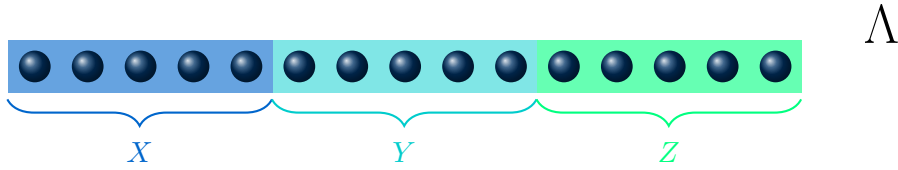


Figure 4: Splitting of a finite interval Λ into three subintervals X, Y, Z with Y shielding X away from Z .

Then, given the Gibbs state of a possibly non-commuting, finite-range, translation-invariant 1D Hamiltonian, for any inverse temperature $\beta > 0$ there exist constants $\mathcal{K} \geq 0$ and $\gamma > 0$ independent of n such that,

$$\|\sigma_{XZ}(\sigma_X \otimes \sigma_Z)^{-1} - \mathbb{1}_{XZ}\|_\infty \leq \mathcal{K} e^{-\gamma|Y|}, \quad (34)$$

Moreover, note that by [14, Proposition IX.1.1], we have for any $P > 0$ and any real observable Q that

$$\|P^{1/2}QP^{1/2} - \mathbb{1}\| \leq \|QP - \mathbb{1}\|. \quad (35)$$

Therefore, putting both inequalities together, we have

$$\|h(\sigma_{XZ})\|_\infty \leq \mathcal{K} e^{-\gamma|Y|} \quad (36)$$

With the geometry $\Lambda = A \cup B$ described above, this bound can be used in cascade in order to yield a good enough control over $\|h(\sigma_{A^c B^c})\|_\infty$. This is the content of the following Lemma, whose proof we defer to Section 4.

Lemma 3.3. *Let $\{A_i, B_i\}_{i=1}^m$ be a covering of Λ with $m = \Omega(n/\ln(n))$ intervals of size $|A_i| = |B_i| = \mathcal{O}(\ln(n))$ for every $1 \leq i \leq m$. For any inverse temperature β , there exists a constant \mathcal{C} independent of $|\Lambda| \equiv n$ such that for any state $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,*

$$D(\rho\|\sigma) \leq \mathcal{C} \sum_{i=1}^m [D_{A_i}(\rho\|\sigma) + D_{B_i}(\rho\|\sigma)],$$

To conclude the proof of (29), we recall [10, Proposition 5], in which it was proven that the conditional relative entropy between ρ and σ in a region $A \subset \Lambda$ is upper bounded by the relative entropy between ρ and its conditional expectation onto A , i.e.

$$D_A(\rho\|\sigma) \leq D(\rho\|E_{A^*}(\rho)). \quad (37)$$

This, together with Lemma 3.3, allows us to conclude the following approximate tensorization:

Lemma 3.4. *In the notations of Lemma 3.3, we have that for any state $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,*

$$D(\rho\|\sigma) \leq \mathcal{C} \sum_{i=1}^m [D(\rho\|E_{A_i^*}(\rho)) + D(\rho\|E_{B_i^*}(\rho))] . \quad (38)$$

Proof. The result follows directly from Lemma 3.3 and (37). \square

3.2 Quasi-local control of the constant

In the next step of the proof, we need to reduce the conditional expectations in the relative entropies on the right-hand side of (38) to single-site conditional expectations in each of the sites composing the region where the latter was conditioning. Inspired by the work of [45], some of the authors provide in [38, Corollary 5.5] that, in the case of tracial conditional expectations $\{\tilde{E}_B\}_{B \subseteq \Lambda}$ and for any $A \subset \Lambda$,

$$D(\rho\|\tilde{E}_{A^*}(\rho)) \leq 4k_A \sum_{i \in A} D(\rho\|\tilde{E}_{i^*}(\rho)), \quad (39)$$

whenever $k_A \in \mathbb{N}$ satisfies

$$\frac{1}{2} \tilde{E}_A \leq_{\text{cp}} \left(\prod_{i \in A} \tilde{E}_i \right)^{k_A} \leq_{\text{cp}} \frac{3}{2} \tilde{E}_A, \quad (40)$$

where $\prod_{i \in A} \tilde{E}_i$ is a product of conditional expectations in an arbitrary ordering, and \leq_{cp} stands for the completely positive partial order. In Section 5, we make use of operator space theory methods together with spectral gap estimates to (i) extend (39) to non-tracial conditional expectations like the ones corresponding to the infinite time limit of local Davies semigroups, and (ii) further control the integer k_A appearing in (39) (see Lemma 5.1 for technical details).

Lemma 3.5. *The approximate tensorization (39) is satisfied for*

$$k_A = \left\lceil \frac{\ln(2d^{2|A\partial|} e^{2\beta|A\partial|J})}{-\ln \lambda} \right\rceil, \quad (41)$$

where J is the interaction strength of H_Λ and

$$\lambda := \left\| \prod_{i \in A} E_i - E_A : L_2(\sigma) \rightarrow L_2(\sigma) \right\| < 1.$$

is a constant independent of the system size $n = |\Lambda|$.

For $|A| = \mathcal{O}(\ln(n))$, the constant k_A in (39) scales logarithmically with the system size n if the constant λ is independent of n . As we show in Section 5, this is a direct consequence of the non-closure of the spectral gap proved for 1D commuting Gibbs samplers in [43] and the detectability lemma, which precisely relates the gap of a commuting, finite-range Gibbs sampler to λ [1, 4, 43, 38]. Moreover, the exponential dependence of k_A on the inverse temperature β causes the scaling of $\alpha_n = e^{-\mathcal{O}(\beta)}$ stated in Theorem 3.1.

3.3 Merging global and quasi-local analysis

Theorem 3.1 is now a simple consequence of the reasoning provided in the last two subsections, and of Lemmas 3.4 and 3.5 in particular. Indeed, putting both lemmas together, we have proven that

$$D(\rho\|\sigma) \leq \mathcal{O}(\ln(n)) \sum_{i \in \Lambda} D(\rho\|E_{i^*}(\rho)). \quad (42)$$

Moreover, it was proved in [38, Theorem 3.3] that the local generators \mathcal{L}_k^D always satisfy a complete modified logarithmic Sobolev inequality (cf. (27)). That is, there exists a constant $\alpha_0 > 0$ such that for all $i \in \Lambda$ and any state $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

$$\alpha_0 D(\rho\|E_{i^*}(\rho)) \leq \text{EP}_i(\rho). \quad (43)$$

Combining the bounds (43) and (42) together with the additivity of the entropy production (28), we conclude the existence of a constant $\alpha_n = \alpha_0 \Omega(\ln(n)^{-1}) = \Omega(\ln(n)^{-1})$ such that

$$\forall \rho \in \mathcal{D}(\mathcal{H}_\Lambda), \alpha_n D(\rho\|\sigma) \leq \text{EP}_\Lambda(\rho) \quad \Rightarrow \quad \forall \rho \in \mathcal{D}(\mathcal{H}_\Lambda), D(e^{t\mathcal{L}_{\Lambda^*}^D}(\rho)\|\sigma) \leq e^{-\alpha_n t} D(\rho\|\sigma).$$

This concludes the proof of Theorem 3.1. □

4 Proof of the mixing condition (Lemma 3.3)

This section is devoted to the proof of Lemma 3.3. Let us first recall the construction devised in Section 3.1 for the covering of $\Lambda \subset \mathbb{Z}$. We consider two regions A and B composed of small intervals, $A := \cup_{i=1}^m A_i$, $B := \cup_{j=1}^m B_j$ such that $A \cup B$ covers the spin chain Λ and the following holds for a certain $\ell \in \mathbb{N}$:

- $A_i \cap A_j = B_i \cap B_j = \emptyset$ for all $1 \leq i, j \leq m$.
- $|A_i| = |B_j| = 2(r + \ell) - 1$ for every $1 \leq i, j \leq m$, where r is the range of the interaction of the Hamiltonian.
- $|A_i \cap B_i| = |B_i \cap A_{i+1}| = \ell$ for every $1 \leq i \leq m - 1$.

Next, let us write $C := B^c$ and $D := A^c$. Note that both of them are composed of m disjoint segments, namely $C := \cup_{i=1}^m C_i$ and $D := \cup_{i=1}^m D_i$, respectively. Moreover, for every $i = 1, \dots, m$ (resp. $i = 1, \dots, m - 1$) let us denote by E_i , resp. F_i , the connected set that separates C_i from D_i , resp. D_i from C_{i+1} . See this construction in Figure 5.

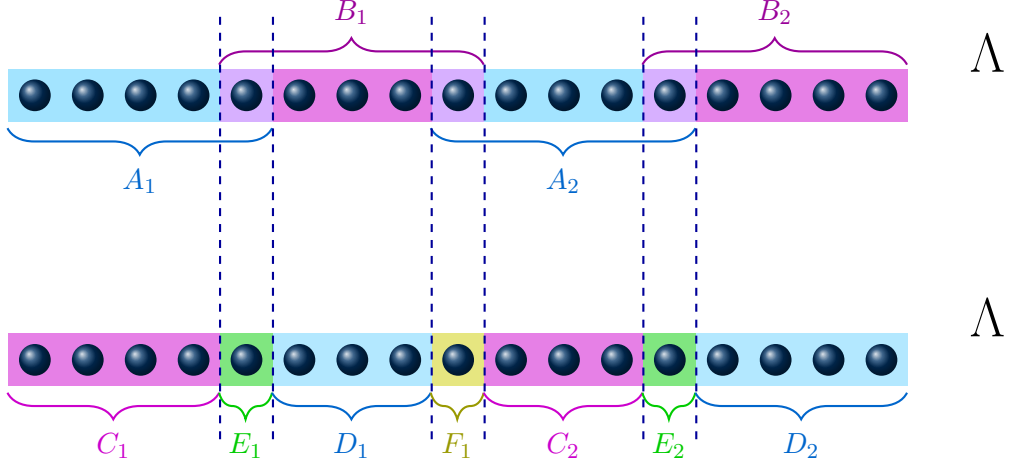


Figure 5: Notation introduced for the splitting of Λ into non-overlapping regions $\{C_i\}_{i=1}^m, \{D_i\}_{i=1}^m, \{E_i\}_{i=1}^m$ and $\{F_i\}_{i=1}^{m-1}$. Here we are taking $m = 2$ for simplicity.

Because of the definition of A and B , it is clear that

- $|C_i| = |D_i| = 2r - 1$ for every $1 \leq i \leq m$.
- $|E_i| = |F_j| = \ell$, for every $1 \leq i \leq m, 1 \leq j \leq m - 1$.

Note that, with this new notation, we aim to prove the following inequality

$$\left\| \left(\sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \right) \sigma_{CD} \left(\sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \right) - \mathbb{1}_{CD} \right\|_{\infty} \leq \mathcal{C} < 1/2, \quad (44)$$

for \mathcal{C} independent of Λ , whenever $d(C, D) = \mathcal{O}(\log n)$. The proof of this result follows from a repeated use of the following estimate:

$$\left\| \sigma_{XZ} (\sigma_X^{-1} \otimes \sigma_Z^{-1}) - \mathbb{1}_{XZ} \right\|_{\infty} \leq \mathcal{K} e^{-\gamma \ell}, \quad (45)$$

for finite intervals XYZ with Y shielding X from Z and $|Y| = \ell$. Let us denote $\eta(\ell) := \mathcal{K} e^{-\gamma \ell}$ hereafter for simplicity. As the aforementioned inequality only holds for constructions such as the one presented in Figure 4 (i.e. Λ consisting of three connected parts X, Y and Z such that Y shields X from Z and the given estimate scales with the size of Y) while the construction devised in Figure 5 consists of a more complex structure, we need to use (45) recursively to prove the statement of the proposition. We divide the proof into two parts:

1. **Splitting step:** In this part, we use (45) repeatedly to approximate

$$\sigma_{CD} \sim \bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}).$$

2. **Joining step:** Now, we use (45) again to entangle σ in C and D separately, namely

$$\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \sim \sigma_C \otimes \sigma_D.$$

Note that the approximations stated above are in the sense that the distance of the product of one of the terms by the inverse of the other in operator norm decays exponentially with the distance between C and D (i.e. the size of the overlap between A and B , denoted by ℓ). Therefore, these two steps together allow to conclude the proof of the lemma.

4.1 Splitting step

Given the construction introduced above, let us further denote, for $j = 1, \dots, 2m-1$, the intervals obtained by considering the convex hull of joining the first j segments from the set $\{C_i, D_i\}_{i=1}^m$, namely

$$X_j := \left(\bigcup_{k=1}^{\frac{j-1}{2}} C_k \cup E_k \cup D_k \cup F_k \right) \cup C_{\frac{j+1}{2}} \text{ and } Z_j := D_{\frac{j+1}{2}} \cup \left(\bigcup_{k=\frac{j+3}{2}}^n C_k \cup E_k \cup D_k \cup F_{k-1} \right), \quad (46)$$

if j is odd, and

$$X_j := \left(\bigcup_{k=1}^{\frac{j}{2}} C_k \cup E_k \cup D_k \cup F_k \right) \setminus F_{\frac{j}{2}} \text{ and } Z_j := \left(\bigcup_{k=\frac{j}{2}+1}^n C_k \cup E_k \cup D_k \cup F_{k-1} \right) \setminus F_{\frac{j}{2}}, \quad (47)$$

if j is even. Note that both constructions, as shown in Figure 6 below, are in the form of Figure 4, where the role of Y is played in each case by the unique E_i or F_i in between the corresponding X_j and Z_j .

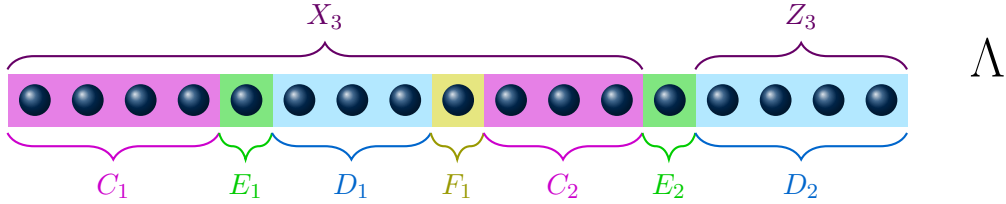


Figure 6: Definition of X_j and Z_j from C_i , D_i , E_i and F_i .

Moreover, note that in both cases we are in position to apply (45) to X_j and Z_j . Then, the following holds for every $j = 1, \dots, 2m-1$:

$$\left\| \sigma_{X_j Z_j} (\sigma_{X_j} \otimes \sigma_{Z_j})^{-1} - \mathbf{1}_{X_j Z_j} \right\|_{\infty} \leq \eta(\ell). \quad (48)$$

In particular, the previous expression also holds when restricting X_j and Z_j to $C \cup D$, i.e.

$$\left\| \sigma_{CD} \left(\sigma_{(C \cup D) \cap X_j} \otimes \sigma_{(C \cup D) \cap Z_j} \right)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} \leq \eta(\ell). \quad (49)$$

Let us further define for simplicity, for $j = 1, \dots, 2m$, the intervals R_j from the set $\{C_i, D_i\}_{i=1}^m$ from left to right, such that

$$R_j := C_{(j+1)/2} \quad \text{if } j \text{ is odd, and} \quad R_j := D_{j/2} \quad \text{if } j \text{ is even.} \quad (50)$$

Furthermore, let us write, for $1 \leq k \leq 2m$, the union of the of all the segments $\{R_j\}_{j=1}^{2m}$ except for the first $k-1$, namely:

$$R^{(k)} := \bigcup_{i=k}^{2m} R_i. \quad (51)$$

Then, (51) jointly with (49) imply the following inequality for each $1 \leq k \leq 2m-1$:

$$\left\| \left(\left(\bigotimes_{i=1}^{k-1} \sigma_{R_i} \right) \otimes \sigma_{R^{(k)}} \right) \left(\left(\bigotimes_{i=1}^k \sigma_{R_i} \right) \otimes \sigma_{R^{(k+1)}} \right)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} \leq \eta(\ell), \quad (52)$$

Let us write for every $2 \leq k \leq 2m-1$

$$\xi_k := \left(\left(\bigotimes_{i=1}^{k-1} \sigma_{R_i} \right) \otimes \sigma_{R^{(k)}} \right) \left(\left(\bigotimes_{i=1}^k \sigma_{R_i} \right) \otimes \sigma_{R^{(k+1)}} \right)^{-1}, \quad (53)$$

and $\xi_1 := \sigma_{CD} (\sigma_{R_1} \otimes \sigma_{R^{(1)}})^{-1}$. Hence, it is clear that

$$\begin{aligned} \left\| \sigma_{CD} \left(\bigotimes_{k=1}^{2m} \sigma_{R_k} \right)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} &= \left\| \sigma_{CD} (\sigma_{R_1} \otimes \sigma_{R^{(1)}})^{-1} (\sigma_{R_1} \otimes \sigma_{R^{(1)}}) \dots \left(\bigotimes_{k=1}^{2m} \sigma_{R_k} \right)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} \\ &= \left\| \prod_{k=1}^{2m-1} \xi_k - \mathbf{1}_{CD} \right\|_{\infty}. \end{aligned}$$

Now, note that

$$\begin{aligned} \left\| \prod_{k=1}^{2m-1} \xi_k - \mathbf{1}_{CD} \right\|_{\infty} &= \left\| (\xi_1 - \mathbf{1}_{CD}) \prod_{k=2}^{2m-1} \xi_k + \prod_{k=2}^{2m-1} \xi_k - \mathbf{1}_{CD} \right\|_{\infty} \\ &\leq \|\xi_1 - \mathbf{1}_{R_1}\|_{\infty} \left\| \prod_{k=2}^{2m-1} \xi_k \right\|_{\infty} + \left\| \prod_{k=2}^{2m-1} \xi_k - \mathbf{1}_{R^{(1)}} \right\|_{\infty} \\ &\leq \eta(\ell) \prod_{k=2}^{2m-1} \|\xi_k\|_{\infty} + \left\| \prod_{k=2}^{2m-1} \xi_k - \mathbf{1}_{R^{(1)}} \right\|_{\infty}, \end{aligned}$$

where we have used (52) in the last inequality, as well as triangle inequality and submultiplicativity of Schatten norms repeatedly. We further estimate each of the terms in the product of the first summand of the right-hand side by:

$$\|\xi_k\|_{\infty} \leq \|\xi_k - \mathbf{1}_{R^{(k)}}\|_{\infty} + 1 \leq \eta(\ell) + 1. \quad (54)$$

Therefore,

$$\left\| \prod_{k=1}^{2m-1} \xi_k - \mathbf{1}_{CD} \right\|_{\infty} \leq \eta(\ell) (\eta(\ell) + 1)^{2m-2} + \left\| \prod_{k=2}^{2m-1} \xi_k - \mathbf{1}_{R^{(1)}} \right\|_{\infty}.$$

Repeating the same procedure $2m-2$ times on the last term in the right-hand side, we obtain

$$\begin{aligned} \left\| \prod_{k=1}^{2m-1} \xi_k - \mathbf{1}_{CD} \right\|_{\infty} &\leq \eta(\ell) \sum_{k=0}^{2m-2} (\eta(\ell) + 1)^k \\ &= (\eta(\ell) + 1)^{2m-1} - 1. \end{aligned} \quad (55)$$

4.2 Joining step

In the first part of the proof, we have provided an estimate for the distance of σ_{CD} from being a tensor product between all the segments of the form $\{C_i, D_i\}_{i=1}^m$. Now, we need to approximate:

$$\bigotimes_{i=1}^m \sigma_{C_i} \sim \sigma_C, \quad \bigotimes_{i=1}^m \sigma_{D_i} \sim \sigma_D.$$

The idea followed here is similar to that of the previous step, but in a reversed direction. Let us denote, for every $1 \leq k \leq m$,

$$C^{(k)} := \bigcup_{i=1}^k C_i, \quad D^{(k)} := \bigcup_{i=1}^k D_i. \quad (56)$$

Similarly to (52), it is clear that the following holds for every $1 \leq k \leq m-1$,

$$\left\| \left(\sigma_{C^{(k-1)}} \otimes \left(\bigotimes_{i=k}^m \sigma_{C_i} \right) \right) \left(\sigma_{C^{(k)}} \otimes \left(\bigotimes_{i=k+1}^m \sigma_{C_i} \right) \right)^{-1} - \mathbf{1}_C \right\|_{\infty} \leq \eta(\ell), \quad (57)$$

and analogously for D . Note that, in this case, we actually have $\eta(2(r+\ell)-1)$, since that is the distance between every two segments C_i and C_{i+1} . However, since η is monotonically decreasing, we just consider $\eta(\ell)$ as an upper bound for the previous norm for simplicity. Let us denote

$$\xi_k^C := \left(\sigma_{C^{(k-1)}} \otimes \left(\bigotimes_{i=k}^m \sigma_{C_i} \right) \right) \left(\sigma_{C^{(k)}} \otimes \left(\bigotimes_{i=k+1}^m \sigma_{C_i} \right) \right)^{-1}, \quad (58)$$

$$\xi_k^D := \left(\sigma_{D^{(k-1)}} \otimes \left(\bigotimes_{i=k}^m \sigma_{D_i} \right) \right) \left(\sigma_{D^{(k)}} \otimes \left(\bigotimes_{i=k+1}^m \sigma_{D_i} \right) \right)^{-1}. \quad (59)$$

Then, we have

$$\begin{aligned} \left\| \left(\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \right) (\sigma_C \otimes \sigma_D)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} &= \left\| \left(\left(\bigotimes_{i=1}^m \sigma_{C_i} \right) \sigma_C^{-1} \right) \otimes \left(\left(\bigotimes_{i=1}^m \sigma_{D_i} \right) \sigma_D^{-1} \right) - \mathbf{1}_{CD} \right\|_{\infty} \\ &= \left\| \left(\prod_{k=1}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{CD} \right\|_{\infty} \end{aligned}$$

Next, we need to separate the norms of the difference between each of the terms $\xi_k^{C,D}$ and identity from the others, as we did in the previous step. The main difference now lies in the fact that we need to do it for both C and D . We first fix the terms in D and work on the terms with support

on C :

$$\begin{aligned}
& \left\| \left(\prod_{k=1}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{CD} \right\|_{\infty} \\
&= \left\| (\xi_1^C - \mathbf{1}_{CD}) \left(\prod_{k=2}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) + \left(\prod_{k=2}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{CD} \right\|_{\infty} \\
&\leq \|\xi_1^C - \mathbf{1}_{C^{(1)}}\|_{\infty} \left\| \prod_{k=2}^{m-1} \xi_k^C \right\|_{\infty} \left\| \prod_{k=1}^{m-1} \xi_k^D \right\|_{\infty} + \left\| \left(\prod_{k=2}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{\Lambda \setminus C^{(1)}} \right\|_{\infty} \\
&\leq \eta(\ell) \prod_{k=2}^{m-1} \|\xi_k^C\|_{\infty} \prod_{k=1}^{m-1} \|\xi_k^D\|_{\infty} + \left\| \left(\prod_{k=2}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{\Lambda \setminus C^{(1)}} \right\|_{\infty},
\end{aligned}$$

where we have used (57) as well as triangle inequality and submultiplicativity for the operator norm. Therefore, using now an analogue of (54) for $\xi_k^{C,D}$, we obtain

$$\left\| \left(\prod_{k=1}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{CD} \right\|_{\infty} \leq \eta(\ell) (\eta(\ell) + 1)^{2m-3} + \left\| \left(\prod_{k=2}^{m-1} \xi_k^C \right) \otimes \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_{\Lambda \setminus C^{(1)}} \right\|_{\infty}.$$

By repeating the same procedure $m - 2$ times on the terms with support on C , we get

$$\left\| \left(\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \right) (\sigma_C \otimes \sigma_D)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} \leq \eta(\ell) \sum_{k=m-1}^{2m-3} (\eta(\ell) + 1)^k + \left\| \left(\prod_{k=1}^{m-1} \xi_k^D \right) - \mathbf{1}_D \right\|_{\infty}.$$

Now, following the same idea for the terms on D , we can clearly conclude:

$$\begin{aligned}
\left\| \left(\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \right) (\sigma_C \otimes \sigma_D)^{-1} - \mathbf{1}_{CD} \right\|_{\infty} &\leq \eta(\ell) \sum_{k=m-1}^{2m-3} (\eta(\ell) + 1)^k + \eta(\ell) \sum_{k=0}^{m-2} (\eta(\ell) + 1)^k \\
&= \eta(\ell) \sum_{k=0}^{2m-3} (\eta(\ell) + 1)^k \\
&= (\eta(\ell) + 1)^{2m-2} - 1.
\end{aligned} \tag{60}$$

4.3 Merging both steps

To conclude the proof of Lemma 3.3, we need to combine (55) and (60). We also need the following estimate on the norm of the difference of a product of observables X_1, X_2 and the identity:

$$\|X_1 X_2 - \mathbf{1}\|_{\infty} \leq \|X_1 - \mathbf{1}\|_{\infty} \|X_2 - \mathbf{1}\|_{\infty} + \|X_1 - \mathbf{1}\|_{\infty} + \|X_2 - \mathbf{1}\|_{\infty}. \tag{61}$$

With this at hand, we can prove

$$\begin{aligned}
& \left\| \sigma_{CD}(\sigma_C^{-1} \otimes \sigma_D^{-1}) - \mathbb{1}_{CD} \right\|_{\infty} \\
&= \left\| \underbrace{\sigma_{CD} \left(\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \right)}_{=:\Xi_1} \underbrace{\left(\bigotimes_{i=1}^m (\sigma_{C_i} \otimes \sigma_{D_i}) \right)^{-1}}_{=:\Xi_2} (\sigma_C^{-1} \otimes \sigma_D^{-1}) - \mathbb{1}_{CD} \right\|_{\infty} \\
&\leq \|\Xi_1 - \mathbb{1}_{CD}\|_{\infty} \|\Xi_2 - \mathbb{1}_{CD}\|_{\infty} + \|\Xi_1 - \mathbb{1}_{CD}\|_{\infty} + \|\Xi_2 - \mathbb{1}_{CD}\|_{\infty} \\
&\leq \left[(\eta(\ell) + 1)^{2m-1} - 1 \right] \left[(\eta(\ell) + 1)^{2m-2} - 1 \right] + \left[(\eta(\ell) + 1)^{2m-1} - 1 \right] + \left[(\eta(\ell) + 1)^{2m-2} - 1 \right] \\
&\leq (\eta(\ell) + 1)^{4m-3} - 1 .
\end{aligned}$$

Finally, let us recall that $\eta(\ell) = \mathcal{K} e^{-\gamma\ell}$ for some constants $\mathcal{K}, \gamma > 0$ determined by (34). Thus, choosing $\ell = \mathcal{O}(\ln(n))$ for $n = |\Lambda|$, we clearly have $|A_i| = |B_i| = \mathcal{O}(\ln(n))$ for every $i = 1, \dots, m$, due to the explicit form of the covering considered. Moreover, the number of small sub-regions considered is $m = \mathcal{O}(n/\ln(n))$. Hence, we can control the above upper bound. Indeed, note that, in such a case, the limit of the bound above with m tending to infinity is a constant, that we can make smaller than $1/2$ by choosing properly the specific value of ℓ .

Therefore, we have proven that (31) holds with $\|h(\sigma_{A^c B^c})\|_{\infty} \leq \mathcal{C}$ independent of n as long as the regions A_i and B_j grow logarithmically with n , since those regions depend linearly on ℓ .

5 Proof of local control of the constant (Lemma 3.5)

This section is devoted to proving the other main technical tool in the proof of Theorem 3.1, namely Lemma 3.5. We first state a slightly more general formulation of Lemma 3.5 before proving the latter. Recall that χ_{Φ} denotes the module Choi operator for a bimodule map Φ .

Lemma 5.1. *Let $\mathcal{N} \subset \mathcal{M}$ be finite-dimensional von Neumann subalgebras and let $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be a conditional expectation. Suppose $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ is a \mathcal{N} -bimodule map.*

i) *If $\|\chi_{\Phi} - \chi_{E_{\mathcal{N}}}\|_{\mathcal{B}(\ell_2^{\mathcal{N}}) \otimes \mathcal{M}} \leq \epsilon \leq 1$, then*

$$(1 - \epsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi \leq_{\text{cp}} (1 + \epsilon)E_{\mathcal{N}} .$$

where the order $\Phi \leq_{\text{cp}} \Psi$ means $\Psi - \Phi$ is completely positive.

ii) *Assume that Φ is unital and self-adjoint with respect to the KMS inner product $\langle \cdot, \cdot \rangle_{\sigma}$ for an (arbitrary) invertible invariant state $\sigma = E_{\mathcal{N}^*}(\sigma)$ and $\lambda := \|\Phi - E_{\mathcal{N}} : L_2(\sigma) \rightarrow L_2(\sigma)\| < 1$. Then for $k > \frac{\ln C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N})}{-\ln \lambda}$,*

$$(1 - \epsilon)E_{\mathcal{N}} \leq_{\text{cp}} \Phi^k \leq_{\text{cp}} (1 + \epsilon)E_{\mathcal{N}}$$

for $\epsilon = \lambda^k C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}) < 1$. In particular, one can choose k such that $\epsilon = \lambda^k \mu_{\min}(\sigma)^{-1} C_{\text{cb}}(\mathcal{M} : \mathcal{N}) < 1$ for any invertible invariant state $\sigma = E_{\mathcal{N}^}(\sigma)$.*

The proof of Lemma 5.1 is postponed to Section 5.2. For now, we show how it implies Lemma 3.5:

Proof of Lemma 3.5. Specializing Lemma 5.1 to the spin chain setting, we choose the map Φ to be $\prod_{i \in A} E_i$ and $E_{\mathcal{N}} := E_A$ for a given region $A \subseteq \Lambda$. Then it suffices to control the constants $\mu_{\min}(\sigma^{(A\partial)})$ and $C_{\text{cb}}(\mathcal{B}(\mathcal{H}_{A\partial}) : \mathcal{N}_A)$, where $\sigma^{(A\partial)}$ is taken as the Gibbs state on region $A\partial$, which is a fixed point of E_{A^*} . First of all

$$\sigma^{(A\partial)} = e^{-\beta H_{A\partial}} / \text{tr} e^{-\beta H_{A\partial}} \geq e^{-\beta \|H_{A\partial}\|_{\infty}} / \text{tr} e^{\beta \|H_{A\partial}\|_{\infty}} \geq d^{-|A\partial|} e^{-2\beta |A\partial| J},$$

where J denotes the interaction strength of H . Moreover, $C_{\text{cb}}(\mathcal{B}(\mathcal{H}_{A\partial}) : \mathcal{N}_A) \leq C_{\text{cb}}(\mathcal{B}(\mathcal{H}_{A\partial}) : \mathbb{C}\mathbb{1}_{A\partial}) \leq d^{|A\partial|}$. Then, the conclusion of Lemma 5.1 holds true for $\epsilon = \lambda^k d^{2|A\partial|} e^{2\beta |A\partial| J} < 1$. Equation (41) follows after taking $\epsilon = \frac{1}{2}$ and solving for k .

That λ is independent of n is a simple consequence of the uniform positivity of the gap:

$$\lambda(\mathcal{L}_{\Lambda}^D) := \inf_{\Lambda} \inf_{X \subset \Lambda} \frac{-\langle X, \mathcal{L}_{\Lambda}^D(X) \rangle_{\sigma}}{\|X - \text{tr}(\sigma X)\mathbb{1}\|_{L_2(\sigma)}^2} > 0$$

for 1D Davies generators associated to commuting Hamiltonians [43, Proposition 29], together with the detectability lemma [1, 4] which asserts that:

$$\lambda^2 \leq \frac{1}{\lambda(\mathcal{L}_{\Lambda}^D)/g^2 + 1} < 1,$$

where g denotes the maximum number of conditional expectations E_i which do not commute with any given E_j . In the present framework of a finite-range commuting Hamiltonian, the conditional expectations E_j are supported on finite regions of locality controlled by the range r of H . Therefore, g is finite and thus $\lambda < 1$ independently of the system size. \square

In order to prove Lemma 5.1, we first need to derive some technical results on amalgamated L_p spaces which might be of independent interest. These are gathered in Section 5.1.

5.1 Amalgamated L_p spaces

In this section, we briefly review the operator space structure of the weighted amalgamated L_p spaces introduced in [42] (see also [11] for a recent account of these spaces in finite dimensions) and connect them to the completely bounded subalgebra indices introduced in Section 2.5. These spaces are generalizations of the non-commutative vector-valued L_p spaces introduced by Pisier in [55]. Given a full-rank state $\sigma \in \mathcal{D}(\mathcal{H})$ and $p \geq 1$, we define the weighted $L_p(\sigma)$ space by the norm

$$\|x\|_{L_p(\sigma)} := \left(\text{tr} \left| \sigma^{\frac{1}{2p}} x \sigma^{\frac{1}{2p}} \right|^p \right)^{\frac{1}{p}}. \quad (62)$$

As expected, for $p = 2$, $L_2(\sigma)$ is a Hilbert space associated with the so-called σ -KMS inner product:

$$\langle x, y \rangle_{\sigma} := \text{tr} \left(x^{\dagger} \sigma^{\frac{1}{2}} y \sigma^{\frac{1}{2}} \right). \quad (63)$$

Let \mathcal{M} be a finite dimensional von Neumann algebra equipped with trace tr . Let $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ be a subalgebra and let $E_{\mathcal{N}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}$ be a conditional expectation. We define the invariant state

$$\sigma_{\text{tr}} := E_{\mathcal{N}^*} \left(\frac{\mathbb{1}_{\mathcal{M}}}{\text{tr}(\mathbb{1}_{\mathcal{M}})} \right).$$

Note that σ_{tr} is explicit from (16), and σ_{tr} and $E_{\mathcal{N}}$ determine each other. Moreover $\sigma_{\text{tr}} \in \mathcal{N}'$ because for any $x \in \mathcal{N}, y \in \mathcal{M}$

$$\begin{aligned} \text{tr}(\sigma_{\text{tr}}xy) &= \text{tr}\left(E_{\mathcal{N}^*}\left(\frac{\mathbb{1}}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\right)xy\right) = \frac{1}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\text{tr}\left(E_{\mathcal{N}}(xy)\right) = \frac{1}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\text{tr}\left(xE_{\mathcal{N}}(y)\right) \\ &= \frac{1}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\text{tr}\left(E_{\mathcal{N}}(y)x\right) = \frac{1}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\text{tr}\left(E_{\mathcal{N}}(yx)\right) = \text{tr}\left(E_{\mathcal{N}^*}\left(\frac{\mathbb{1}}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\right)yx\right) \\ &= \text{tr}\left(xE_{\mathcal{N}^*}\left(\frac{\mathbb{1}}{\text{tr}(\mathbb{1}_{\mathcal{M}})}\right)y\right) = \text{tr}\left(x\sigma_{\text{tr}}y\right). \end{aligned}$$

In other words, σ_{tr} restricted to \mathcal{N} is a trace. Let $1 \leq q, p \leq \infty$ and fix $\frac{1}{r} = |\frac{1}{q} - \frac{1}{p}|$. The amalgamated L_p spaces are defined via the following norms: for $p \geq q$,

$$\|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} := \inf_{a, b \in \mathcal{N}, y \in \mathcal{M}, x = ayb} \|a\|_{L_{2r}(\sigma_{\text{tr}})} \|b\|_{L_{2r}(\sigma_{\text{tr}})} \|y\|_{L_p(\sigma_{\text{tr}})}, \quad (64)$$

where the infimum is over all factorizations $x = ayb$ for $a, b \in \mathcal{N}$ and $y \in \mathcal{M}$. For $p \leq q$,

$$\|y\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} := \sup_{a, b \in \mathcal{N}} \frac{\|ayb\|_{L_p(\sigma_{\text{tr}})}}{\|a\|_{L_{2r}(\sigma_{\text{tr}})} \|b\|_{L_{2r}(\sigma_{\text{tr}})}} \quad (65)$$

where the supremum is over all $a, b \in \mathcal{N}$. For any $1 \leq q, p \leq \infty$, we denote by $L_q^p(\mathcal{N} \subset \mathcal{M})$ the space \mathcal{M} equipped with the above norms. We gather some basic properties of amalgamated L_p -norms before discussing their operator space structure. In the following, we fix p' and q' to be the Hölder conjugate of p and q respectively such as $\frac{1}{p} + \frac{1}{p'} = 1$.

Proposition 5.2. *Let $\mathcal{N} \subset \mathcal{M}$ be finite dimensional von Neumann algebras. Let $1 \leq q, p \leq \infty$. Then*

(i) *Hölder's inequality: for any $x \in L_p^q(\mathcal{N} \subset \mathcal{M})$ and $y \in L_{p'}^{q'}(\mathcal{N} \subset \mathcal{M})$,*

$$|\langle x, y \rangle_{\sigma_{\text{tr}}}| \leq \|x\|_{L_p^q(\mathcal{N} \subset \mathcal{M})} \|y\|_{L_{p'}^{q'}(\mathcal{N} \subset \mathcal{M})}. \quad (66)$$

(ii) *Duality: For any $x \in L_q^p(\mathcal{N} \subset \mathcal{M})$,*

$$\|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} = \sup \left\{ |\langle x, y \rangle_{\sigma_{\text{tr}}}| : \|y\|_{L_{q'}^{p'}(\mathcal{N} \subset \mathcal{M})} = 1 \right\}. \quad (67)$$

(iii) *The infimum (resp. supremum) in the definition (64) (resp. (65)) can be restricted to the set of positive semidefinite operators $a, b \geq 0$. Furthermore, for all positive semidefinite x ,*

$$\|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} = \inf_{\substack{a \in \mathcal{N}, a > 0 \\ \|a\|_{L_1(\sigma_{\text{tr}})} = 1}} \left\| a^{-\frac{1}{2r}} x a^{-\frac{1}{2r}} \right\|_{L_p(\sigma_{\text{tr}})} \quad \text{if } p \geq q \quad (68)$$

$$\|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} = \sup_{\substack{a \in \mathcal{N}, a > 0 \\ \|a\|_{L_1(\sigma_{\text{tr}})} = 1}} \left\| a^{\frac{1}{2r}} x a^{\frac{1}{2r}} \right\|_{L_q(\sigma_{\text{tr}})} \quad \text{if } p \leq q \quad (69)$$

(iv) The following complex interpolation relation holds (see. e.g. [13] for an introduction to interpolation spaces):

$$L_q^p(\mathcal{N} \subset \mathcal{M}) \cong [L_{q_0}^{p_0}(\mathcal{N} \subset \mathcal{M}), L_{q_1}^{p_1}(\mathcal{N} \subset \mathcal{M})]_\theta, \quad (70)$$

where $0 \leq \theta \leq 1$, $(1-\theta)/p_0 + \theta/p_1 = 1/p$, $(1-\theta)/q_0 + \theta/q_1 = 1/q$ and $(p_1 - q_1)(p_0 - q_0) \geq 0$.

(v) Relation with $L_p(\sigma_{\text{tr}})$ norms: if $q \leq p$, then for any $x \in L_p(\sigma_{\text{tr}})$,

$$\|x\|_{L_q(\sigma_{\text{tr}})} \leq \|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} \leq \|x\|_{L_p(\sigma_{\text{tr}})}, \quad (71)$$

$$\|x\|_{L_q(\sigma_{\text{tr}})} \leq \|x\|_{L_q^p(\mathcal{N} \subset \mathcal{M})} \leq \|x\|_{L_p(\sigma_{\text{tr}})}. \quad (72)$$

In particular, the equality holds for $p = q$, which is referred as Fubini's Theorem.

Proof. The property (i-iii) and (v) are proved in [11, Proposition 3.1]. The complex interpolation (iv) is proved in [42, Theorem 3.2 & 4.6] (for the reduction from the Haagerup L_p -norm to the Kosaki L_p -norm we used in (62), see [39, Proposition 2.4]). For completeness, we include the proof of (68) $p = \infty$, which will be sufficient for our discussion. We show that when $x = x^\dagger$ is self-adjoint, it suffices to consider $a = b > 0$ in the infimum of the definition of the L_q^∞ -norm (64). Without losing generality, we assume $\|x\|_{L_q^\infty(\mathcal{N} \subset \mathcal{M})} < 1$ with $x = ayb$ such that

$$\|a\|_{L_{2q}(\sigma_{\text{tr}})} = \|b\|_{L_{2q}(\sigma_{\text{tr}})} < 1, \quad \|y\|_\infty < 1.$$

By polar decomposition, we can further assume $a, b \geq 0$. Denote $Q = (\frac{1}{2}a^2 + \frac{1}{2}b^2 + \delta\mathbf{1})^{1/2} \in \mathcal{N}$ for some small $\delta > 0$. Since $x = \frac{1}{2}(x + x^\dagger) = \frac{1}{2}(ayb + by^\dagger a)$, we have $x = QYQ$ for

$$Y = \frac{1}{2}Q^{-1}(ayb + by^\dagger a)Q^{-1} = \frac{1}{2}Q^{-1} \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} y & 0 \\ 0 & y^\dagger \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} Q^{-1}.$$

Note that

$$\|Q\|_{L_{2q}(\sigma_{\text{tr}})} = \left\| \frac{1}{2}a^2 + \frac{1}{2}b^2 + \delta\mathbf{1} \right\|_{L_q(\sigma_{\text{tr}})}^{1/2} \leq \left(\frac{1}{2}\|a^2\|_{L_q(\sigma_{\text{tr}})} + \frac{1}{2}\|b^2\|_{L_q(\sigma_{\text{tr}})} + \delta \right)^{1/2} = (1 + \delta)^{1/2}$$

and

$$\begin{aligned} \left\| \begin{bmatrix} Q^{-1}a & Q^{-1}b \end{bmatrix} \right\|_\infty &= \|Q^{-1}(a^2 + b^2)Q^{-1}\|_\infty \\ &= \left\| (a^2 + b^2)^{1/2} \left(\frac{1}{2}a^2 + \frac{1}{2}b^2 + \delta\mathbf{1} \right)^{-1} (a^2 + b^2)^{1/2} \right\|_\infty \\ &\leq \left\| (a^2 + b^2)^{1/2} \left(\frac{1}{2}a^2 + \frac{1}{2}b^2 \right)^{-1} (a^2 + b^2)^{1/2} \right\|_\infty \leq \sqrt{2}. \end{aligned}$$

Thus

$$\|Y\|_\infty \leq \frac{1}{2} \left\| \begin{bmatrix} Q^{-1}a & Q^{-1}b \end{bmatrix} \right\|_\infty \left\| \begin{bmatrix} y & 0 \\ 0 & y^\dagger \end{bmatrix} \right\|_\infty \left\| \begin{bmatrix} aQ^{-1} \\ bQ^{-1} \end{bmatrix} \right\|_\infty \leq \|y\|_\infty = 1.$$

Take $A = \frac{1}{\|Q\|_{L_{2q}(\sigma_{\text{tr}})}^2} Q^{2q}$. We have $A > 0$, $\|A\|_{L_1(\sigma_{\text{tr}})} = 1$ and

$$\left\| A^{-1/2} x A^{-1/2} \right\|_\infty \leq \|Q\|_{L_{2q}(\sigma_{\text{tr}})}^2 \|Y\|_\infty \leq 1 + \delta$$

Since $\delta > 0$ is arbitrary, this proves (68) for $p = \infty$. □

We will also need the following two factorization Lemmas. The first one is dual form of [42, Theorem 3.19] at $p = 1$.

Lemma 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be finite dimensional subalgebras and σ_{tr} be defined as above. Then for any $x \in \mathcal{M}$,*

$$\|x\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})} = \inf_{x=yz} \left\| yy^\dagger \right\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}^{1/2} \left\| z^\dagger z \right\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}^{1/2} \quad (73)$$

where the infimum is over all factorization $x = yz$ with $y, z \in \mathcal{M}$.

Proof. In the following proof, we use the short norm notation $\|\cdot\| := \|\cdot\|_\infty$, $\|\cdot\|_p := \|\cdot\|_{L_p(\sigma_{\text{tr}})}$ and $\|\cdot\|_{L_1^\infty} := \|\cdot\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}$. First, for any factorization $x = yz$,

$$\begin{aligned} \|x\|_{L_1^\infty} &= \inf_{A, B > 0, \|A\|_1 = \|B\|_1 = 1} \left\| A^{-1/2} x B^{-1/2} \right\| \\ &\leq \inf_{A, B > 0, \|A\|_1 = \|B\|_1 = 1} \left\| A^{-1/2} y \right\| \left\| z B^{-1/2} \right\| \\ &= \inf_{A > 0, \|A\|_1 = 1} \left\| A^{-1/2} yy^\dagger A^{-1/2} \right\| \inf_{B > 0, \|B\|_1 = 1} \left\| B^{-1/2} z^\dagger z B^{-1/2} \right\| \\ &= \left\| yy^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z^\dagger z \right\|_{L_1^\infty}^{1/2}, \end{aligned}$$

where in the last step we used the property (68) for $yy^\dagger, z^\dagger z \geq 0$.

For the other direction, let us denote

$$\|x\|_h := \inf_{x=yz} \left\| yy^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z^\dagger z \right\|_{L_1^\infty}^{1/2}$$

where the infimum takes over all factorization $x = yz$. We first show $\|\cdot\|_h$ is a norm. To verify the triangle inequality, it suffices to show that for any $x_1 = y_1 z_1$, $x_2 = y_2 z_2$, and $\delta > 0$, there exist y_0, z_0 such that $x_1 + x_2 = y_0 z_0$ and

$$\left\| y_0 y_0^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z_0^\dagger z_0 \right\|_{L_1^\infty}^{1/2} \leq \left\| y_1 y_1^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z_1^\dagger z_1 \right\|_{L_1^\infty}^{1/2} + \left\| y_2 y_2^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z_2^\dagger z_2 \right\|_{L_1^\infty}^{1/2} + \delta.$$

By rescaling $x_1 = t y_1 \cdot t^{-1} z_1$, we can assume

$$\begin{aligned} &\left\| y_1 y_1^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z_1^\dagger z_1 \right\|_{L_1^\infty}^{1/2} + \left\| y_2 y_2^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z_2^\dagger z_2 \right\|_{L_1^\infty}^{1/2} \\ &= \left(\left\| y_1 y_1^\dagger \right\|_{L_1^\infty} + \left\| y_2 y_2^\dagger \right\|_{L_1^\infty} \right)^{1/2} \left(\left\| z_1^\dagger z_1 \right\|_{L_1^\infty} + \left\| z_2^\dagger z_2 \right\|_{L_1^\infty} \right)^{1/2} \end{aligned} \quad (74)$$

Take $\delta > 0$ and

$$\begin{aligned} y &= (y_1 y_1^\dagger + y_2 y_2^\dagger + \delta \mathbf{1})^{1/2}, \quad z = (z_1^\dagger z_1 + z_2^\dagger z_2 + \delta \mathbf{1})^{1/2}, \\ X &= y^{-1} (y_1 z_1 + y_2 z_2) z^{-1} = y^{-1} (x_1 + x_2) z^{-1}. \end{aligned}$$

We have $\left\| \begin{bmatrix} y^{-1}y_1 & y^{-1}y_2 \\ 0 & 0 \end{bmatrix} \right\| \leq 1$ and $\left\| \begin{bmatrix} z_1z^{-1} & 0 \\ z_2z^{-1} & 0 \end{bmatrix} \right\| \leq 1$. Thus

$$\|X\| = \left\| \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y^{-1}y_1 & y^{-1}y_2 \\ 0 & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} z_1z^{-1} & 0 \\ z_2z^{-1} & 0 \end{bmatrix} \right\| \leq 1.$$

Then we have $x = x_1 + x_2 = yXz$ and

$$\begin{aligned} & \left\| yy^\dagger \right\|_{L_1^\infty} \left\| z^\dagger X^\dagger X z \right\|_{L_1^\infty} \\ & \leq \left\| yy^\dagger \right\|_{L_1^\infty} \left\| z^\dagger z \right\|_{L_1^\infty} \\ & \leq \left\| y_1 y_1^\dagger + y_2 y_2^\dagger + \delta \right\|_{L_1^\infty} \left\| z_1^\dagger z_1 + z_2^\dagger z_2 + \delta \right\|_{L_1^\infty} \\ & \leq \left(\left\| y_1 y_1^\dagger \right\|_{L_1^\infty} + \left\| y_2 y_2^\dagger \right\|_{L_1^\infty} + \delta \right) \left(\left\| z_1^\dagger z_1 \right\|_{L_1^\infty} + \left\| z_2^\dagger z_2 \right\|_{L_1^\infty} + \delta \right) \end{aligned}$$

Since $\delta > 0$ is arbitrary, we have by (74)

$$\begin{aligned} & \left\| yy^\dagger \right\|_{L_1^\infty}^{1/2} \left\| z^\dagger X^\dagger X z \right\|_{L_1^\infty}^{1/2} \\ & \leq \left\| y_1 y_1^\dagger \right\|_{L_1^\infty}^{1/2} \left\| y_2 y_2^\dagger \right\|_{L_1^\infty}^{1/2} + \left\| z_1^\dagger z_1 \right\|_{L_1^\infty}^{1/2} \left\| z_2^\dagger z_2 \right\|_{L_1^\infty}^{1/2} \end{aligned}$$

This proves the triangular inequality and also

$$\|x\|_h = \inf_{x = \sum_i y_i z_i} \left\| \sum_i y_i y_i^\dagger \right\|_{L_1^\infty}^{1/2} \left\| \sum_i z_i^\dagger z_i \right\|_{L_1^\infty}^{1/2},$$

where the supremum is over all finite families $\{y_i\}$ and $\{z_i\}$ such that $\sum_{i=1}^k y_i z_i = x$. We now use a standard Grothendieck-Pietsch factorization to show $\|\cdot\|_h = \|\cdot\|_{L_1^\infty}$. Suppose $\|x\|_h = 1$. By Hahn-Banach Theorem, there exists a linear functional $\phi : \mathcal{M} \rightarrow \mathbb{C}$ such that $\phi(x) = \|x\|_h = 1$ and for any finite families $\{y_i\}$ and $\{z_i\}$,

$$\left| \sum_{i=1}^k \phi(y_i z_i) \right| \leq \sup_{\|a\|_{L_1^\infty}=1} \langle a, \sum_i y_i y_i^\dagger \rangle_{\sigma_{\text{tr}}}^{1/2} \sup_{\|b\|_{L_1^\infty}=1} \langle b, \sum_i z_i^\dagger z_i \rangle_{\sigma_{\text{tr}}}^{1/2}.$$

Here we use the duality $L_1^\infty(\mathcal{N} \subset \mathcal{M})^* = L_1^1(\mathcal{N} \subset \mathcal{M})$ and for positive $Y \geq 0$,

$$\|Y\|_{L_1^\infty} = \sup_{\|a\|_{L_1^\infty}=1} |\langle Y, a \rangle_{\sigma_{\text{tr}}}| = \sup_{a \geq 0, \|a\|_{L_1^\infty}=1} \langle Y, a \rangle_{\sigma_{\text{tr}}}.$$

By modifying the phase factor and arithmetic-geometric mean inequality, we have

$$\sup_{a \geq 0, \|a\|_{L_1^\infty}=1} \langle a, \sum_i y_i y_i^\dagger \rangle_{\sigma_{\text{tr}}} + \sup_{b \geq 0, \|b\|_{L_1^\infty}=1} \langle b, \sum_i z_i^\dagger z_i \rangle_{\sigma_{\text{tr}}} - 2 \sum_{i=1}^k \text{Re } \phi(y_i z_i) \geq 0. \quad (75)$$

Denote B_+ as the positive unit ball of $L_\infty^1(\mathcal{N} \subset \mathcal{M})$ and $C(B_+ \times B_+)$ as the real continuous function space. For each pair of finite families $\mathbf{y} = \{y_i\}$ and $\mathbf{z} = \{z_i\}$, we define the function

$$f_{\mathbf{y}, \mathbf{z}} : B_+ \times B_+ \rightarrow \mathbb{R}, f_{\mathbf{y}, \mathbf{z}}(a, b) = \langle a, \sum_i y_i y_i^\dagger \rangle_{\sigma_{\text{tr}}} + \langle b, \sum_i z_i^\dagger z_i \rangle_{\sigma_{\text{tr}}} - \sum_{i=1}^k \text{Re} \phi(y_i z_i).$$

We define the cones in $C(B_+ \times B_+)$

$$\begin{aligned} C &= \{f_{\mathbf{y}, \mathbf{z}} \mid \{y_i\}, \{z_i\} \subset \mathcal{M}\} \\ C_- &= \{f \in C(B_+ \times B_+, \mathbb{R}) \mid \sup f < 0\}. \end{aligned}$$

Note that both C and C_- are convex and C_- is open. Moreover, $C \cap C_- = \emptyset$ because of (75). By Hahn-Banach separation Theorem, there exists a linear functional $\psi : C(B_+ \times B_+) \rightarrow \mathbb{R}$ such that

$$\psi(f_-) \leq \lambda \leq \psi(f_{\mathbf{y}, \mathbf{z}})$$

for any $f_- \in C_-$ and $f_{\mathbf{y}, \mathbf{z}} \in C$. Since C_- is a cone, $\lambda \geq 0$ and hence ψ is a positive linear function. Up to normalization, there exists a probability measure μ on $B_+ \times B_+$ such that $\psi(f) = \int_{B_+ \times B_+} f(a, b) d\mu(a, b)$. Take

$$a_0 = \int_{B_+ \times B_+} a d\mu(a, b), \quad b_0 = \int_{B_+ \times B_+} b d\mu(a, b).$$

By convexity of B_+ , we have $a_0, b_0 \in B_+$ and moreover for every $y, z \in \mathcal{M}$,

$$\begin{aligned} \psi(f_{\{y\}, \{z\}}) &= \int_{B_+ \times B_+} f(a, b) d\mu(a, b) \\ &= \int_{B_+ \times B_+} \langle a, y y^\dagger \rangle_{\sigma_{\text{tr}}} d\mu(a, b) + \int_{B_+ \times B_+} \langle b z^\dagger z \rangle_{\sigma_{\text{tr}}} d\mu(a, b) - 2 \text{Re} \phi(yz) \\ &= \langle a_0, y y^\dagger \rangle_{\sigma_{\text{tr}}} + \langle b_0, z^\dagger z \rangle_{\sigma_{\text{tr}}} - \text{Re} \phi(yz) \geq 0. \end{aligned}$$

Rescaling y_i and z_i again, we have

$$|\phi(yz)| \leq \text{tr}(a_0 \sigma_{\text{tr}}^{1/2} y y^\dagger \sigma_{\text{tr}}^{1/2})^{1/2} \text{tr}(b_0 \sigma_{\text{tr}}^{1/2} z^\dagger z \sigma_{\text{tr}}^{1/2})^{1/2} = \left\| a_0^{1/2} \sigma_{\text{tr}}^{1/2} y \right\|_{2, \text{tr}} \left\| z \sigma_{\text{tr}}^{1/2} b_0^{1/2} \right\|_{2, \text{tr}},$$

where $\|\cdot\|_{2, \text{tr}}$ denotes the Hilbert Schmidt norm. One can further find invertible $a_1, b_1 \in (1 + \epsilon)B_+$ such that

$$|\phi(yz)| \leq \left\| a_1^{1/2} \sigma_{\text{tr}}^{1/2} y \right\|_{2, \text{tr}} \left\| z \sigma_{\text{tr}}^{1/2} b_1^{1/2} \right\|_{2, \text{tr}}$$

Because of the invertibility of a_1, b_1 , there exists a contraction u such that

$$\phi(yz) = \text{tr}(u a_1^{1/2} \sigma_{\text{tr}}^{1/2} y z \sigma_{\text{tr}}^{1/2} b_1^{1/2}) = \langle a_1^{1/2} u^\dagger b_1^{1/2}, yz \rangle_{\sigma_{\text{tr}}}$$

Note that by Hölder inequality,

$$\begin{aligned}
\left\| a_1^{1/2} u^\dagger b_1^{1/2} \right\|_{L_\infty^1} &= \sup_{\|X\|_2 = \|Y\|_2 = 1} \left\| X a_1^{1/2} u^\dagger b_1^{1/2} Y \right\|_1 \\
&= \sup_{X, Y} \left\| \sigma_{\text{tr}}^{1/2} X a_1^{1/2} \right\|_{2, \text{tr}} \left\| u^\dagger b_1^{1/2} Y \sigma_{\text{tr}}^{1/2} \right\|_{2, \text{tr}} \\
&= \sup_{X, Y} \text{tr}(\sigma_{\text{tr}} X a_1 X^\dagger)^{1/2} \text{tr}(Y^\dagger b_1^{1/2} u u^\dagger b_1^{1/2} Y \sigma_{\text{tr}}) \\
&\leq \sup_{X, Y} \left\| X a_1 X^\dagger \right\|_1^{1/2} \left\| Y^\dagger b_1 Y \right\|_1^{1/2} \leq \|a_1\|_{L_\infty^1}^{1/2} \|b_1\|_{L_\infty^1}^{1/2} = 1 + \epsilon.
\end{aligned}$$

Therefore, for any factorization $x = yz$,

$$\|x\|_h = \phi(x) = \langle a_1^{1/2} u^\dagger b_1^{1/2}, x \rangle_{\sigma_{\text{tr}}} \leq \left\| a_1^{1/2} u^\dagger b_1^{1/2} \right\|_{L_\infty^1} \|x\|_{L_1^\infty} \leq (1 + \epsilon) \|x\|_{L_1^\infty}.$$

Since ϵ is arbitrary, that concludes the proof. \square

The second factorization lemma is generalization of [55, Theorem 1.5] for subalgebra.

Lemma 5.4. *For any $x \in L_p(\sigma_{\text{tr}})$,*

$$\|x\|_{L_p(\sigma_{\text{tr}})} = \inf_{\substack{x=ayb \\ a, b \in \mathcal{N}}} \|a\|_{L_{2p}(\sigma_{\text{tr}})} \|y\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M})} \|b\|_{L_{2p}(\sigma_{\text{tr}})}. \quad (76)$$

where the infimum is over all factorizations $x = ayb$ with $a, b \in \mathcal{N}$ and $y \in \mathcal{M}$.

Proof. Once again, we use the shorter notations $\|\cdot\| := \|\cdot\|_\infty$, $\|\cdot\|_p := \|\cdot\|_{L_p(\sigma_{\text{tr}})}$ and $\|\cdot\|_{L_p^q} := \|\cdot\|_{L_p^q(\mathcal{N} \subset \mathcal{M})}$. Denote

$$\|x\|_h := \inf_{x=ayb} \|a\|_{2p} \|y\|_{L_\infty^p} \|b\|_{2p}$$

where the infimum is over all factorizations $x = ayb$ with $a, b \in \mathcal{N}$ and $y \in \mathcal{M}$. We first show that $\|\cdot\|_h$ is a norm. Let $x_1 = a_1 y_1 b_1$ and $x_2 = a_2 y_2 b_2$ with $\|y_1\|_{L_\infty^p} = \|y_2\|_{L_\infty^p} = 1$. Take small $\delta > 0$ and denote $a = (a_1 a_1^\dagger + a_2 a_2^\dagger + \delta \mathbf{1})^{1/2}$, $b = (b_1^\dagger b_1 + b_2^\dagger b_2 + \delta \mathbf{1})^{1/2}$. We have $x_1 + x_2 = ayb$ for

$$y = a^{-1}(x_1 + x_2)b^{-1} = a^{-1}(a_1 y_1 b_1 + a_2 y_2 b_2)b^{-1}$$

We show that

$$\|y\|_{L_\infty^p} \leq \max\{\|y_1\|_{L_\infty^p}, \|y_2\|_{L_\infty^p}\} \leq 1$$

Indeed, we have

$$\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a^{-1} a_1 & a^{-1} a_2 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} b_1 b^{-1} & 0 \\ b_2 b^{-1} & 0 \end{bmatrix}$$

Denote $A = \begin{bmatrix} a^{-1} a_1 & a^{-1} a_2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 b^{-1} & 0 \\ b_2 b^{-1} & 0 \end{bmatrix}$. We see that $A, B \in \mathbb{M}_2(\mathcal{N})$ and

$$\begin{aligned}
\|A\|_\infty &= \left\| a^{-1}(a_1 a_1^\dagger + a_2 a_2^\dagger) a^{-1} \right\|_\infty \leq 1 \\
\|B\|_\infty &= \left\| b^{-1}(b_1^\dagger b_1 + b_2^\dagger b_2) b^{-1} \right\|_\infty \leq 1
\end{aligned}$$

Then by the operator space structure of L_∞^p ,

$$\begin{aligned}
\|y\|_{L_\infty^p} &= \left\| \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix} \right\|_{L_\infty^p(\mathbb{M}_2(\mathcal{N}) \subset \mathbb{M}_2(\mathcal{M}))} \\
&= \left\| A \cdot \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \cdot B \right\|_{L_\infty^p(\mathbb{M}_2(\mathcal{N}) \subset \mathbb{M}_2(\mathcal{M}))} \\
&\leq \|A\|_\infty \cdot \left\| \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} \right\|_{L_\infty^p(\mathbb{M}_2(\mathcal{N}) \subset \mathbb{M}_2(\mathcal{M}))} \cdot \|B\|_\infty \\
&\leq \max\{\|y_1\|_{L_\infty^p}, \|y_2\|_{L_\infty^p}\} \leq 1.
\end{aligned}$$

Therefore, we have $x_1 + x_2 = ayb$ for

$$\begin{aligned}
\|a\|_{2p} &= \left\| a_1 a_1^\dagger + a_2 a_2^\dagger + \delta \mathbb{1} \right\|_p^{1/2} \leq \left(\|a_1 a_1^\dagger\|_p + \|a_2 a_2^\dagger\|_p + \delta \right)^{1/2} \\
\|b\|_{2p} &= \left\| b_1^\dagger b_1 + b_2^\dagger b_2 + \delta \mathbb{1} \right\|_p^{1/2} \leq \left(\|b_1^\dagger b_1\|_p + \|b_2^\dagger b_2\|_p + \delta \right)^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_1 + x_2\|_h &\leq \|a\|_{2p} \|y\|_{L_\infty^p} \|b\|_{2p} \\
&\leq \left(\|a_1 a_1^\dagger\|_p + \|a_2 a_2^\dagger\|_p + \delta \right)^{1/2} \left(\|b_1^\dagger b_1\|_p + \|b_2^\dagger b_2\|_p + \delta \right)^{1/2}.
\end{aligned}$$

Taking $\delta \rightarrow 0$ and rescaling a_1, b_1 , we obtain

$$\|x_1 + x_2\|_h \leq \|a_1\|_{2p} \|b_1\|_{2p} + \|a_2\|_{2p} \|b_2\|_{2p}$$

This proves the triangle inequality.

We now show that $\|\cdot\|_h = \|\cdot\|_p$ coincide. First, note that by definition of L_∞^p

$$\|y\|_{L_\infty^p} \geq \frac{\|ayb\|_p}{\|a\|_{2p} \|b\|_{2p}}$$

for any $a, b \in \mathcal{N}, y \in \mathcal{M}$. This implies $\|\cdot\|_h \geq \|\cdot\|_p$. To see the converse direction, we consider the

dual norm $\|\cdot\|_{h^*}$ given by the sesquilinear pairing $\langle \cdot, \cdot \rangle_{\sigma_{\text{tr}}}$. Then

$$\begin{aligned}
\|z\|_{h^*} &= \sup_{\|x\|_h=1} |\langle z, x \rangle_{\sigma_{\text{tr}}}| \\
&= \sup_{\substack{a, b \in \mathcal{N}, y \in \mathcal{M} \\ \|a\|_{2p} = \|b\|_{2p} = \|y\|_{L_\infty^p} = 1}} |\langle z, ayb \rangle_{\sigma_{\text{tr}}}| \\
&= \sup_{\substack{a, b \in \mathcal{N}, y \in \mathcal{M} \\ \|a\|_{2p} = \|b\|_{2p} = \|y\|_{L_\infty^p} = 1}} |\langle a^\dagger z b^\dagger, y \rangle_{\sigma_{\text{tr}}}| \\
&= \sup_{\substack{a, b \in \mathcal{N}, y \in \mathcal{M} \\ \|a\|_{2p} = \|b\|_{2p} = 1}} \left\| a^\dagger z b^\dagger \right\|_{L_1^{p'}} \\
&= \inf_{\substack{a_1, b_1 > 0, a_1, b_1 \in \mathcal{N} \\ \|a_1\|_{2p} = \|b_1\|_{2p} = 1}} \sup_{\substack{a, b \in \mathcal{N}, y \in \mathcal{M} \\ \|a\|_{2p} = \|b\|_{2p} = 1}} \left\| b_1^{-1} b^\dagger z a^\dagger a_1^{-1} \right\|_{p'} \\
&\geq \sup_{\substack{a, b \in \mathcal{N}, y \in \mathcal{M} \\ \|a\|_{2p} = \|b\|_{2p} = 1}} \inf_{\substack{a_1, b_1 > 0, a_1, b_1 \in \mathcal{N} \\ \|a_1\|_{2p} = \|b_1\|_{2p} = 1}} \left\| b_1^{-1} b^\dagger z a^\dagger a_1^{-1} \right\|_{p'} \\
&\geq \|z\|_{p'}
\end{aligned}$$

where the last inequality follows from choosing $a = a_1, b = b_1$. This proves $\|z\|_{h^*} \geq \|z\|_{p'}$ which by duality gives $\|x\|_h \leq \|x\|_p$. That completes the proof. \square

We now discuss the operator space structure of amalgamated L_p spaces (see the analogous treatment of the symmetric case in [36, Appendix A.2]). We first define the operator space structure of $L_\infty^1(\mathcal{N} \subset \mathcal{M})$, then extend the structure to $L_\infty^p(\mathcal{N} \subset \mathcal{M})$ for other p by interpolation. We define the matrix norm of $L_\infty^1(\mathcal{N} \subset \mathcal{M})$ via the following isometry,

$$\mathbb{M}_n(L_\infty^1(\mathcal{N} \subset \mathcal{M})) \cong L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M})), \quad (77)$$

where for each $n \in \mathbb{N}$ the enlarged amalgamated space on $L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))$ is defined with respect to the state $\sigma_{\text{tr}}^{(n)} := n^{-1} \mathbf{1}_n \otimes \sigma_{\text{tr}}$. We verify that these norms satisfy Ruan's axioms recalled in Section 2.4. Let e_1 (resp. e_2) be projection onto $\mathbb{M}_m(\mathcal{M})$ (resp. $\mathbb{M}_n(\mathcal{M})$). Given

$v \in \mathbb{M}_m(\mathcal{M})$ and $w \in \mathbb{M}_n(\mathcal{M})$, we have $x := v \oplus w$ where $v = e_1 x e_1$ and $x = e_2 x e_2$. Then,

$$\begin{aligned}
\|x\|_{L_\infty^1(\mathbb{M}_{m+n}(\mathcal{N}) \subset \mathbb{M}_{m+n}(\mathcal{M}))} &= \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1}} \|axb\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} \\
&\leq \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1}} \|ae_1 x e_1 b\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} + \|ae_2 x e_2 b\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} \\
&\leq \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1}} \|ae_1 v e_1 b\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} + \|ae_2 w e_2 b\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} \\
&\leq \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1}} \| |ae_1| v |(e_1 b)^\dagger \|_{L_1(\sigma_{\text{tr}}^{(m)})} + \| |ae_2| w |(e_2 b)^\dagger \|_{L_1(\sigma_{\text{tr}}^{(n)})}.
\end{aligned} \tag{78}$$

In (78) above, we used that

$$\begin{aligned}
\|ae_1 v e_1 b\|_{L_1(\sigma_{\text{tr}}^{(m+n)})} &\stackrel{(1)}{=} \text{tr} \left| (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} ae_1 v e_1 b (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} \right| \stackrel{(1)}{=} \text{tr} \left| (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} U |ae_1| v |(e_1 b)^\dagger| U' (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} \right| \\
&\stackrel{(2)}{=} \text{tr} \left| U (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} |ae_1| v |(e_1 b)^\dagger| (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} U' \right| = \| |ae_1| v |(e_1 b)^\dagger \|_{L_1(\sigma_{\text{tr}}^{(m+n)})}
\end{aligned}$$

where in (1), U and U' respectively are given by the polar decomposition $ae_1 := U|ae_1|$ and $(e_1 b)^\dagger = |(e_1 b)| (U')^\dagger$. In (2), we use the fact that both U' and U belong to $\mathbb{M}_{m+n}(\mathcal{N})$ and hence commute with $\sigma_{\text{tr}}^{(m+n)}$. The same trick applies to ae_2 and $e_2 b$. Moreover, by the fact that both $|ae_1|$ and $|(e_1 b)^\dagger|$ belong to $\mathbb{M}_m(\mathcal{N})$ (resp. $|ae_2|, |(e_2 b)^\dagger| \in \mathbb{M}_n(\mathcal{N})$), (78) becomes

$$\begin{aligned}
\|x\|_{L_\infty^1(\mathbb{M}_{m+n}(\mathcal{N}) \subset \mathbb{M}_{m+n}(\mathcal{M}))} &\leq \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})} = 1}} \| |ae_1| \|_{L_2(\sigma_{\text{tr}}^{(m)})} \|v\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))} \| |(e_1 b)^\dagger \|_{L_2(\sigma_{\text{tr}}^{(m)})} \\
&\quad + \| |ae_2| \|_{L_2(\sigma_{\text{tr}}^{(n)})} \|w\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))} \| |(e_2 b)^\dagger \|_{L_2(\sigma_{\text{tr}}^{(n)})}.
\end{aligned} \tag{79}$$

Now, we observe that

$$\begin{aligned}
\|a\|_{L_2(\sigma_{\text{tr}}^{(m+n)})}^2 &= \|a(e_1 + e_2)\|_{L_2(\sigma_{\text{tr}}^{(m+n)})}^2 \\
&= \text{tr} \left((\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} (e_1 + e_2)^\dagger a^\dagger (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} a (e_1 + e_2) \right) \\
&\stackrel{(1)}{=} \text{tr} \left((\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} (e_1 + e_2) (e_1 + e_2)^\dagger a^\dagger (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} a \right) \\
&= \text{tr} \left((\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} (e_1 + e_2) (e_1 + e_2)^\dagger a^\dagger (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} a \right) \\
&= \text{tr} \left((\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} e_1 a^\dagger (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} a \right) + \text{tr} \left((\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} e_2 a^\dagger (\sigma_{\text{tr}}^{(m+n)})^{\frac{1}{2}} a \right) \\
&\stackrel{(2)}{=} \|ae_1\|_{L_2(\sigma_{\text{tr}}^{(m+n)})}^2 + \|ae_2\|_{L_2(\sigma_{\text{tr}}^{(m+n)})}^2 \\
&= \| |ae_1| \|_{L_2(\sigma_{\text{tr}}^{(m)})}^2 + \| |ae_2| \|_{L_2(\sigma_{\text{tr}}^{(n)})}^2.
\end{aligned}$$

In (1) and (2), we used the tracial property and the fact that both projections e_1 and e_2 commute with $\sigma_{\text{tr}}^{(m+n)}$ since the e_i 's act trivially on \mathcal{M} whereas $\sigma_{\text{tr}}^{(m+n)}$ acts trivially on \mathbb{M}_{m+n} . Similarly,

$$\|b\|_{L_2(\sigma_{\text{tr}}^{(m+n)})}^2 = \|(e_1 b)^\dagger\|_{L_2(\sigma_{\text{tr}}^{(m)})}^2 + \|(e_2 b)^\dagger\|_{L_2(\sigma_{\text{tr}}^{(n)})}^2.$$

Applying Hölder's inequality to (79), we have that

$$\begin{aligned} & \|x\|_{L_\infty^1(\mathbb{M}_{m+n}(\mathcal{N}) \subset \mathbb{M}_{m+n}(\mathcal{M}))} \\ & \leq \left(\|ae_1\|_{L_2(\sigma_{\text{tr}}^{(m)})}^2 + \|ae_2\|_{L_2(\sigma_{\text{tr}}^{(n)})}^2 \right)^{\frac{1}{2}} \cdot \max\{\|v\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))}, \|w\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))}\} \\ & \quad \cdot \left(\|(e_1 b)^\dagger\|_{L_2(\sigma_{\text{tr}}^{(m)})}^2 + \|(e_2 b)^\dagger\|_{L_2(\sigma_{\text{tr}}^{(n)})}^2 \right)^{\frac{1}{2}} \\ & = \max\{\|v\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))}, \|w\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))}\}. \end{aligned}$$

This verifies (i) in Ruan's axioms. For the second axiom (ii), consider $x \in \mathbb{M}_m(\mathcal{M})$ and $A \in \mathbb{M}_{n,m}$, $B \in \mathbb{M}_{m,n}$. We have

$$\begin{aligned} \|AxB\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))} &= \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1}} \|aAxBb\|_{L_1(\sigma_{\text{tr}}^{(n)})} \\ &\stackrel{(1)}{\leq} \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1}} \|aA\|_{L_2(\sigma_{\text{tr}}^{(n)})} \|x\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))} \|Bb\|_{L_2(\sigma_{\text{tr}}^{(n)})} \\ &\stackrel{(2)}{\leq} \sup_{\substack{\|a\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1 \\ \|b\|_{L_2(\sigma_{\text{tr}}^{(n)})} = 1}} \|a\|_{L_2(\sigma_{\text{tr}}^{(n)})} \|A\|_\infty \|x\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))} \|B\|_\infty \|b\|_{L_2(\sigma_{\text{tr}}^{(n)})} \\ &= \|A\|_\infty \|x\|_{L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))} \|B\|_\infty. \end{aligned}$$

Here, equation (1) above follows again from the definition of $L_\infty^1(\mathbb{M}_m(\mathcal{N}) \subset \mathbb{M}_m(\mathcal{M}))$. In (2), we used Hölder's inequality and the fact that $\|A\|_\infty := \|A\|_{\mathbb{M}_{n,m}} = \|A \otimes \mathbf{1}_{\mathcal{M}}\|_{L_\infty(\sigma_{\text{tr}}^{(n)})}$, and similarly for B . This verifies Ruan's axiom (ii), which implies that Equation (77) indeed provides an operator space structure for $L_\infty^1(\mathcal{N} \subset \mathcal{M})$. As mentioned, this observation can be extended to the case of $L_\infty^p(\mathcal{N} \subset \mathcal{M})$ by complex interpolation:

Proposition 5.5. *For any $1 \leq p \leq \infty$, the identification*

$$\mathbb{M}_n(L_\infty^p(\mathcal{N} \subset \mathcal{M})) \cong L_\infty^p(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))$$

defines an operator space structure on $L_\infty^p(\mathcal{N} \subset \mathcal{M})$.

Proof. We recall that $L_\infty^\infty(\mathcal{N} \subset \mathcal{M}) \cong L_\infty(\mathcal{M})$ by definition and its operator space structure is naturally given by $\mathbb{M}_n(L_\infty(\mathcal{M})) \cong L_\infty(\mathbb{M}_n(\mathcal{M}))$. Then by the interpolation (70):

$$\begin{aligned} L_\infty^p(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M})) &\cong [L_\infty(\mathbb{M}_n(\mathcal{M})), L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))]_{\frac{1}{p}} \\ &\cong [\mathbb{M}_n(L_\infty(\mathcal{M})), \mathbb{M}_n(L_\infty^1(\mathcal{N} \subset \mathcal{M}))]_{\frac{1}{p}} \\ &\cong \mathbb{M}_n(L_\infty^p(\mathcal{N} \subset \mathcal{M})). \end{aligned}$$

□

Combining Proposition 5.5 with (76), we find (see also [35, Lemma 3.12] in the tracial case):

Proposition 5.6. *For any \mathcal{N} -bimodule map $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ and $1 \leq p, q, s \leq \infty$,*

$$\|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\| = \|\Phi : L_s^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_s^q(\mathcal{N} \subset \mathcal{M})\|. \quad (80)$$

In the case $p = 2$, assuming that Φ is self-adjoint with respect to some invariant state $\sigma = E_{\mathcal{N}^}(\sigma)$, we further have*

$$\begin{aligned} \|\Phi : L_2(\rho) \rightarrow L_2(\rho)\| &= \|\Phi : L_2(\sigma_{\text{tr}}) \rightarrow L_2(\sigma_{\text{tr}})\| \\ &= \|\Phi : L_\infty^2(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^2(\mathcal{N} \subset \mathcal{M})\|, \end{aligned} \quad (81)$$

for any other invariant state $\rho = E_{\mathcal{N}^}(\rho)$.*

Proof. We first prove the direction " \geq " in (80) for $s = 1$: let $x \in L_1^p(\mathcal{N} \subset \mathcal{M})$ with $\|x\|_{L_1^p(\mathcal{N} \subset \mathcal{M})} < 1$. By definition, there exists a decomposition $x = ayb$ of x with $a, b \in \mathcal{N}$ and $\|a\|_{L_{2p'}(\sigma_{\text{tr}})}, \|b\|_{L_{2p'}(\sigma_{\text{tr}})} < 1$ and $\|y\|_{L_p(\sigma_{\text{tr}})} \leq 1$, where p' is the Hölder conjugate of p . Moreover, by the factorization property (76), $y = czd$ with $c, d \in \mathcal{N}$, $\|c\|_{L_{2p}(\sigma_{\text{tr}})}, \|d\|_{L_{2p}(\sigma_{\text{tr}})} < 1$ and $\|z\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M})} < 1$. Therefore, $x = (ac)z(db)$ and by Hölder inequality

$$\|ac\|_{L_2(\sigma_{\text{tr}})} \leq \|a\|_{L_{2p'}(\sigma_{\text{tr}})} \|c\|_{L_{2p}(\sigma_{\text{tr}})} < 1 \quad (82)$$

where we used the fact σ_{tr} is a trace on \mathcal{N} . Similarly, we have $\|db\|_{L_2(\sigma_{\text{tr}})} < 1$. Therefore, one can find $a', b', c', d' \in \mathcal{N}$ such that $ac = a'c', db = d'b'$ with

$$\|a'\|_{L_{2q}(\sigma_{\text{tr}})}, \|c'\|_{L_{2q'}(\sigma_{\text{tr}})}, \|b'\|_{L_{2q'}(\sigma_{\text{tr}})}, \|d'\|_{L_{2q}(\sigma_{\text{tr}})} \leq 1.$$

Indeed, using the polar decomposition $ac = U|ac| = U|ac|^{1/q}|ac|^{1/q'}$, one can choose $a' = U|ac|^{1/q}$ and $c' = |ac|^{1/q'}$. We then use the module property of the map \mathcal{N} , so that $\Phi(x) = \Phi(a'c'zd'b') = a'c'\Phi(z)d'b'$. Since $\|z\|_{L_\infty^p(\mathcal{N} \subset \mathcal{M})} \leq 1$, we have

$$\|\Phi(z)\|_{L_\infty^q(\mathcal{N} \subset \mathcal{M})} \leq \|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\|.$$

By definition of the L_1^q and L_∞^q norms,

$$\begin{aligned} \|\Phi(x)\|_{L_1^q(\mathcal{N} \subset \mathcal{M})} &\leq \|c'\Phi(z)d'\|_{L_q(\mathcal{N} \subset \mathcal{M})} \leq \|c'\|_{L_{2q}(\sigma_{\text{tr}})} \|\Phi(z)\|_{L_\infty^q(\mathcal{N} \subset \mathcal{M})} \|d'\|_{L_{2q}(\sigma_{\text{tr}})} \\ &\leq \|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\|, \end{aligned}$$

Therefore

$$\|\Phi : L_1^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_1^q(\mathcal{N} \subset \mathcal{M})\| \leq \|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\|.$$

By complex interpolation [13], we can prove that this last claim holds true for all $1 \leq s \leq \infty$:

$$\|\Phi : L_s^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_s^q(\mathcal{N} \subset \mathcal{M})\| \leq \|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\|. \quad (83)$$

We denote by Φ^{KMS} the dual of Φ with respect to the σ_{tr} -KMS inner product (63). Then, applying duality (67) to (83), we get

$$\begin{aligned} \|\Phi : L_s^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_s^q(\mathcal{N} \subset \mathcal{M})\| &= \|\Phi^{\text{KMS}} : L_{s'}^q(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{s'}^p(\mathcal{N} \subset \mathcal{M})\| \\ &\leq \|\Phi^{\text{KMS}} : L_\infty^q(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^p(\mathcal{N} \subset \mathcal{M})\| \\ &= \|\Phi : L_1^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_1^q(\mathcal{N} \subset \mathcal{M})\|. \end{aligned}$$

Using duality one last time, we get

$$\begin{aligned}
\|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\| &= \|\Phi^{\text{KMS}} : L_1^{q'}(\mathcal{N} \subset \mathcal{M}) \rightarrow L_1^{p'}(\mathcal{N} \subset \mathcal{M})\| \\
&\leq \|\Phi^{\text{KMS}} : L_{s'}^{q'}(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{s'}^{p'}(\mathcal{N} \subset \mathcal{M})\| \\
&= \|\Phi : L_s^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_s^q(\mathcal{N} \subset \mathcal{M})\| \\
&\leq \|\Phi : L_\infty^p(\mathcal{N} \subset \mathcal{M}) \rightarrow L_\infty^q(\mathcal{N} \subset \mathcal{M})\|,
\end{aligned}$$

and hence all the norms coincide. This concludes the proof of (80). In the case $p = 2$, it is easy to see that all matrix norms are equal to each others and correspond to the spectral radius of Φ as a self-adjoint operator on L_2 , from which (81) follows. (see [38, Lemma 2.6] for independence of invariant state ρ). \square

We end this section with a Lemma that connects up the notion of cb-index to the operator space structure of amalgamated L_p norms (see [36, Theorem 3.9] for the tracial case).

Lemma 5.7. *Let $\mathcal{N} \subset \mathcal{M}$ be finite dimensional von Neumann subalgebras and $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be a conditional expectation. Then*

$$C_\tau(\mathcal{M} : \mathcal{N}) \geq \|\text{id} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\| \quad (84)$$

$$C_{\tau, \text{cb}}(\mathcal{M} : \mathcal{N}) \geq \|\text{id} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}}. \quad (85)$$

Proof. First by duality, we have

$$\|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\| = \|\text{id} : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|. \quad (86)$$

We show that it suffices to consider positive elements in the optimization of the norm on the left hand side. Denote

$$\|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|_+ = \sup_{\substack{x \geq 0 \\ \text{tr}(\sigma x) = 1}} \|x\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}.$$

For general $x \in \mathcal{M}$, we have

$$\|x\|_{L_1(\sigma_{\text{tr}})} = \left\| \sigma_{\text{tr}}^{1/2} x \sigma_{\text{tr}}^{1/2} \right\|_{L_1(\text{tr})} = \inf_{y' z' = \sigma_{\text{tr}}^{1/2} x \sigma_{\text{tr}}^{1/2}} \|y'\|_{L_2(\text{tr})} \|z'\|_{L_2(\text{tr})},$$

where $\|\cdot\|_{L_p(\text{tr})}$ denotes the usual p -Schatten norm. Take $y = \sigma_{\text{tr}}^{-1/2} y'$ and $z = z' \sigma_{\text{tr}}^{-1/2}$. We have

$$\begin{aligned}
\|x\|_{L_1(\sigma_{\text{tr}})} &= \inf_{yz=x} \left\| \sigma_{\text{tr}}^{1/2} y \right\|_{L_2(\text{tr})} \left\| z \sigma_{\text{tr}}^{1/2} \right\|_{L_2(\text{tr})} \\
&= \inf_{yz=x} \left\| \sigma_{\text{tr}}^{1/2} y y^\dagger \sigma_{\text{tr}}^{1/2} \right\|_{L_1(\text{tr})}^{1/2} \left\| \sigma_{\text{tr}}^{1/2} z^\dagger z \sigma_{\text{tr}}^{1/2} \right\|_{L_1(\text{tr})}^{1/2} \\
&= \inf_{yz=x} \left\| y y^\dagger \right\|_{L_1(\sigma_{\text{tr}})}^{1/2} \left\| z^\dagger z \right\|_{L_1(\sigma_{\text{tr}})}^{1/2}.
\end{aligned}$$

Then, we have by the factorization property (73) that

$$\begin{aligned}
\|x\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})} &\leq \inf_{x=yz} \left\| yy^\dagger \right\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}^{1/2} \left\| z^\dagger z \right\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})}^{1/2} \\
&\leq \|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|_+ \inf_{x=yz} \left\| yy^\dagger \right\|_{L_1(\sigma_{\text{tr}})}^{1/2} \left\| z^\dagger z \right\|_{L_1(\sigma_{\text{tr}})}^{1/2} \\
&= \|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|_+ \|x\|_{L_1(\sigma_{\text{tr}})}.
\end{aligned}$$

which proves

$$\|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|_+ = \|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|.$$

Therefore, by definition of the $L_1^\infty(\mathcal{N} \subset \mathcal{M})$ norm

$$\begin{aligned}
\|\text{id} : L_1(\sigma_{\text{tr}}) \rightarrow L_1^\infty(\mathcal{N} \subset \mathcal{M})\|_+ &= \sup_{\substack{x \geq 0 \\ \text{tr}(\sigma x) = 1}} \|x\|_{L_1^\infty(\mathcal{N} \subset \mathcal{M})} \\
&= \sup_{\substack{x \geq 0 \\ \text{tr}(\sigma x) = 1}} \inf_{\substack{a > 0, a \in \mathcal{N} \\ \text{tr}(\sigma_{\text{tr}} a) = 1}} \left\| a^{-1/2} x a^{-1/2} \right\|_\infty \\
&= \sup_{\substack{x \geq 0 \\ \text{tr}(\sigma x) = 1}} \inf_{\substack{a > 0, a \in \mathcal{N} \\ \text{tr}(\sigma_{\text{tr}} a) = 1}} \inf\{c > 0 \mid x \leq ca\} \\
&\stackrel{(1)}{=} \sup_{\substack{\rho \geq 0 \\ \text{tr}(\rho) = 1}} \inf_{\substack{\omega > 0, E_{\mathcal{N}^*}(\omega) = \omega \\ \text{tr}(\omega) = 1}} \inf\{c > 0 \mid \rho \leq c\omega\} \\
&\stackrel{(2)}{\leq} \sup_{\substack{\rho \geq 0 \\ \text{tr}(\rho) = 1}} \inf\{c > 0 \mid \rho \leq cE_{\mathcal{N}^*}(\rho)\} := C_\tau(\mathcal{M} : \mathcal{N}).
\end{aligned}$$

Here, the equality (1) uses the substitution $\rho = \sigma_{\text{tr}}^{\frac{1}{2}} x \sigma_{\text{tr}}^{\frac{1}{2}}$ and $\omega = \sigma_{\text{tr}}^{\frac{1}{2}} a \sigma_{\text{tr}}^{\frac{1}{2}}$. The inequality (2) follows by choosing $\omega = E_{\mathcal{N}^*}(\rho)$. This proves the inequality (84). Applying (84) to the inclusion $\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M})$ for all n yields the CB-version (85). \square

5.2 Proof of Lemma 5.1

The proof of i) is identical to that of [38, Lemma B.2] in the trace-symmetric case. ii) is a consequence of i) and Proposition 2.5 (see also [38, Lemma B.2] and [35, Lemma 3.14]). Recall that for a \mathcal{N} -bimodule map Φ , its module Choi operator $\chi_\Phi = \sum_{i,j} |i\rangle\langle j| \otimes \Phi(\xi_i^\dagger \xi_j) \in \mathcal{B}(l_2^n) \otimes \mathcal{M}$ where ξ_1, \dots, ξ_n is a module basis for the conditional expectation $E_{\mathcal{N}}$ such that

$$E_{\mathcal{N}}(\xi_i^\dagger \xi_j) = \delta_{ij} p_i$$

with some projections $p_i \in \mathcal{N}$. We first show the following inequality (which is actually an equality but we only need one direction here, for the other direction see [35])

$$\|\Phi : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}} \geq \|\chi_\Phi\|_{\mathcal{B}(l_2^n) \otimes \mathcal{M}}. \quad (87)$$

Define the map

$$\Psi : \mathcal{T}_1(l_2^n) \rightarrow \mathcal{M}, \Psi(|i\rangle\langle j|) = \xi_i^\dagger \xi_j.$$

Its (standard) Choi matrix is $J_\Psi = \sum_{i,j} |i\rangle\langle j| \otimes \xi_i^\dagger \xi_j \in \mathcal{B}(l_2^n) \otimes \mathcal{M}$. Then, by Proposition 2.5 and the duality $\mathcal{T}_1(l_2^n)^* = \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned} \|\Psi : \mathcal{T}_1(l_2^n) \rightarrow L_\infty^1(\mathcal{N} \subset \mathcal{M})\|_{\text{cb}} &= \|J_\Psi\|_{\mathcal{B}(l_2^n) \otimes_{\min} L_\infty^1(\mathcal{N} \subset \mathcal{M})} \\ &\stackrel{(1)}{=} \|J_\Psi\|_{\mathbb{M}_n(L_\infty^1(\mathcal{N} \subset \mathcal{M}))} \\ &\stackrel{(2)}{=} \|J_\Psi\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))} \end{aligned} \quad (88)$$

where (1) is a consequence of the second part of Proposition 2.5 and (2) comes from the operator space structure of L_∞^1 in Proposition 5.5. Because J_Ψ is a positive element in $\mathbb{M}_n(\mathcal{M})$, we have by (69):

$$\begin{aligned} \|J_\Psi\|_{L_\infty^1(\mathbb{M}_n(\mathcal{N}) \subset \mathbb{M}_n(\mathcal{M}))} &= \sup_{\substack{a \in \mathbb{M}_n(\mathcal{N}), a > 0 \\ \|a\|_{L_1(\sigma_{\text{tr}}^{(n)})} = 1}} \left\| a^{1/2} J_\Psi a^{1/2} \right\|_{L_1(\sigma_{\text{tr}}^{(n)})} \\ &\stackrel{(1)}{=} \sup_{\substack{a \in \mathbb{M}_n(\mathcal{N}), a > 0 \\ \|a\|_{L_1(\sigma_{\text{tr}}^{(n)})} = 1}} \langle a, J_\Psi \rangle_{\sigma_{\text{tr}}^{(n)}} \\ &\stackrel{(2)}{=} \sup_{\substack{a \in \mathbb{M}_n(\mathcal{N}), a > 0 \\ \|a\|_{L_1(\sigma_{\text{tr}}^{(n)})} = 1}} \langle a, (\text{id}_n \otimes E_{\mathcal{N}})(J_\Psi) \rangle_{\sigma_{\text{tr}}^{(n)}} \\ &\stackrel{(3)}{=} \|(\text{id}_n \otimes E_{\mathcal{N}})(J_\Psi)\|_{\mathbb{M}_n(\mathcal{N})} \\ &= \left\| \sum_i |i\rangle\langle i| \otimes p_i \right\|_\infty \\ &\leq 1, \end{aligned}$$

where in (1) we use that $[\sigma_{\text{tr}}^{(n)}, \mathbb{M}_n(\mathcal{N})] = 0$, whereas (2) follows by $(\text{id}_n \otimes E_{\mathcal{N}})(a) = a$ and self-adjointness of $\text{id}_n \otimes E_{\mathcal{N}}$ with respect to $\langle \cdot, \cdot \rangle_{\sigma_{\text{tr}}^{(n)}}$. Moreover, (3) follows by duality. Combining this bound with (88), we obtain

$$\|\Psi : \mathcal{T}_1(l_2^n) \rightarrow L_\infty^1(\mathcal{N} \subset \mathcal{M})\|_{\text{cb}} \leq 1. \quad (89)$$

Note that the standard Choi matrix of $\Phi \circ \Psi$ is the module Choi matrix of Φ , i.e. $J_{\Phi \circ \Psi} = \chi_\Phi$. Then, by Proposition 2.5 again

$$\|\chi_\Phi\|_{\mathcal{B}(l_2^n) \otimes \mathcal{M}} = \|J_{\Phi \circ \Psi}\|_{\mathcal{B}(l_2^n) \otimes \mathcal{M}} = \|\Phi \circ \Psi : \mathcal{T}_1(l_2^n) \rightarrow \mathcal{M}\|_{\text{cb}} \leq \|\Phi : L_\infty^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M}\|_{\text{cb}},$$

where the last inequality uses (89). This proves (87). Now consider Φ to be a unital \mathcal{N} -bimodule map that is self-adjoint w.r.t. the KMS inner product $\langle \cdot, \cdot \rangle_{\sigma_{\text{tr}}}$. It follows that $\Phi \circ E_{\mathcal{N}} = E_{\mathcal{N}} \circ \Phi = E_{\mathcal{N}}$. Indeed, for any $x \in \mathcal{M}, y \in \mathcal{N}$:

$$\Phi(y) = y, \quad \Phi(\mathbf{1}) = \mathbf{1}, \quad \langle y, E_{\mathcal{N}} \circ \Phi(x) \rangle_{\sigma_{\text{tr}}} = \langle \Phi \circ E_{\mathcal{N}}(y), x \rangle_{\sigma_{\text{tr}}} = \langle E_{\mathcal{N}}(y), x \rangle_{\sigma_{\text{tr}}} = \langle y, E_{\mathcal{N}}(x) \rangle_{\sigma_{\text{tr}}}.$$

Thus, $(\Phi - E_{\mathcal{N}})^k = \Phi^k - E_{\mathcal{N}}$ and we control the norm of the difference between the module Choi operators of Φ^k and $E_{\mathcal{N}}$ as follows:

$$\begin{aligned}
& \|\chi_{\Phi^k} - \chi_{E_{\mathcal{N}}}\|_{\mathcal{B}(l_2^2) \otimes \mathcal{M}} \\
& \stackrel{(1)}{\leq} \left\| \Phi^k - E_{\mathcal{N}} : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}} \\
& = \left\| (\Phi - E_{\mathcal{N}})^k : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}} \\
& = \left\| \text{id} : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \right\|_{\text{cb}} \cdot \left\| (\Phi - E_{\mathcal{N}})^k : L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \right\|_{\text{cb}} \\
& \quad \cdot \left\| \text{id} : L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}} \\
& \stackrel{(2)}{=} \left\| \text{id} : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}} \lambda^k.
\end{aligned}$$

Here, (1) is a consequence of the inequality (87) proved above. (2) follows the duality

$$\begin{aligned}
\left\| \text{id} : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \right\|_{\text{cb}} &= \left\| \text{id} : L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}} \\
&= \left\| \text{id} : L_{\infty}^1(\mathcal{N} \subset \mathcal{M}) \rightarrow \mathcal{M} \right\|_{\text{cb}}^{1/2}
\end{aligned}$$

and by Equation (80), we have for the $L_{\infty}^2 \rightarrow L_{\infty}^2$ norm

$$\begin{aligned}
& \left\| (\Phi - E_{\mathcal{N}})^k : L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \rightarrow L_{\infty}^2(\mathcal{N} \subset \mathcal{M}) \right\|_{\text{cb}} \\
& = \sup_n \left\| \text{id}_{M_n} \otimes (\Phi - E_{\mathcal{N}})^k : L_{\infty}^2(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \rightarrow L_{\infty}^2(M_n(\mathcal{N}) \subset M_n(\mathcal{M})) \right\| \\
& = \sup_n \left\| \text{id}_{M_n} \otimes (\Phi - E_{\mathcal{N}})^k : L_2(M_n(\mathcal{M})) \rightarrow L_2(M_n(M_n(\mathcal{M}))) \right\| \\
& = \left\| (\Phi - E_{\mathcal{N}})^k : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M}) \right\| \leq \lambda^k
\end{aligned}$$

Therefore, the assertion ii) follows from i) and Lemma 5.7. □

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