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Orderability in contact manifolds

Ordenabilidad en variedades de contacto

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PRESENTADA POR

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ORDERABILITY IN CONTACT MANIFOLDS
ORDENABILIDAD EN VARIEDADES DE CONTACTO

A thesis presented by
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supervised by
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"I've always believed in numbers, in equations, in logic and reason.But after a lifetime of such pursuits: I ask What truly is logic? Who decides reason? My quest has taken me to the physical, the metaphysical, the delusional, and back. I have made the most important discovery of my career - the most important discovery of my life. It is only in the mysterious equations of love that any logic or reasons can be found."

- Ron Howard, A Beautiful Mind.

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Agradecimientos

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Chapter 1

Introdución

"A person who cannot give up anything can change nothing."
- Hajime Isayama, Shingeki no kyojin.

Este documento es el fruto del trabajo de mis años de doctorado. Como consecuencia de dicho trabajo he publicado los dos artículos que explican los resultados más importantes que hemos logrado obtener durante este periodo de tiempo:

[**CPPP**] R. Casals, J.L. Pérez, A. del Pino, and F. Presas. *Existence h-principle for Engel structures.* Invent. Math. Volume **210**, Issue 2, (2017) 417-451.

[**PPP**] D. Pancholi, J.L. Pérez, and F. Presas. *A simple construction of positive loops of Legendrians.* Arkiv för Matematik, **56**, Number 2 (2018), 377 – 394.

Por orden cronológico, en primer lugar, tenemos un artículo cuya idea principal empezamos a desarrollar en 2015. Este artículo no se encontrará recogido en esta memoria porque hemos optado por escribir una monografía sobre un área de investigación especifica. Por eso hemos decidido centrar este documento en el segundo de los asuntos investigados en estos años. El área de investigación: Topología del moduli de estructuras Engel nos ha permitido publicar un resultado de gran impacto. Lo enunciamos a continuación.

Theorem. *Dada una variedad 4-dimensional M, una* ***estructura Engel*** *se define como una 2-distribución máximamente no integrable. La inclusión $i : \mathfrak{E}(M) \hookrightarrow \mathfrak{F}(M)$ del espacio de estructuras Engel $\mathfrak{E}(M)$ de una variedad de dimensión 4 dentro del espacio de full flags $\mathfrak{F}(M)$ del espacio tangente a M induce una sobreyectividad sobre todos los grupos de homotopia, es decir: $\pi_k(i) : \pi_k(\mathfrak{E}(M)) \to \pi_k(\mathfrak{F}(M))$ es sobreyectivo para todo $k \geq 0$. En particular, especificamos como construir estructuras Engel dado un full flag cualquiera.*

El segundo artículo anteriormente citado se corresponde al estudio y respuesta parcial de una pregunta que, finalmente, escogimos como objetivo principal para el desarrollo de esta tesis doctoral: construcciones simples de lazos de Legendrianas positivos. El contenido del mismo se expondrá con mayor detalle en las siguientes secciones.

1.1 Descripción del contenido

Fijemos una variedad de contacto (M^{2n+1}, ξ) de dimensión $2n+1$, es decir, una $(2n+1)$-variedad diferenciable M^{2n+1} junto con una una distribución de hiperplanos ξ máximamente no integrable. Para más detalles podemos consultar la sección 3.1 y, más concretamente, la definición 3.1.1.

Los objetos principales de estudio en esta tesis son los lazos Legendrianos. Una subvariedad $L \subset (M^{2n+1}, \xi)$ se dice Legendriana si su espacio tangente en cada punto es, a su vez, tangente a la distribución de contacto; es decir si:

$$T_pL \subset \xi_p, \text{ for all } p \in M.$$

A su vez, un lazo de Legendrianas no es más que una familia diferenciable uniparamétrica de Legendrianas $\{L_t\}_{t\in[0,1]}$ tal que sus extremos L_0, L_1 coinciden. En la sección 3.3 podemos consultar estos conceptos con mayor profundidad.

Dada la subvariedad Legendriana $L \subset (M^{2n+1}, \xi)$, denotemos por $\mathcal{L}eg(L)$ el espacio de todas las subvariedades Legendrianas que sean isótopas a L. Como explicaremos en la siguiente sección, Y. Eliashberg y L. Polterovich [**EP**] realizaron un estudio sobre la ordenabilidad de los grupos de contactomorfismos que permitió encontrar una relación entre la ordenabilidad de $\mathcal{L}eg(L)$ y la existencia de lazos Legendrianos positivos (definición 3.3.7). Dicha relación es extensible al recubridor universal $\mathcal{L}eg(L)$ de dicho grupo siempre que consideremos que los lazos anteriores sean además contráctiles (definición 3.3.8).

El objetivo principal de esta disertación es construir lazos positivos de subvariedades Legendrianas. Para ello hemos de imponer que se satisfagan ciertas hipótesis topológicas. En concreto, una elección habitual en esta memoria es asumir que la subvariedad Legendriana es loose.

La noción de subvariedad Legendriana loose requiere de la introducción de bastantes conceptos. Por ello, para su mayor comprensión nos referimos al capítulo 4 de la tesis y, más concretamente, a las definiciones 4.1.3 y 4.1.4. Por ahora, baste decir que una subvariedad Legendriana se dice loose si existe un sistema local de coordenadas especial dentro de una bola abierta que interseque a la subvariedad Legendriana de un modo específico.

En esta memoria ofrecemos una prueba alternativa más sencilla y concreta de un resultado enunciado recientemente por G. Liu [**Liu1**] que afirma que toda subvariedad Legendriana loose admite un lazo positivo. En nuestro caso, enunciamos lo siguiente:

Theorem 6.2.1. *Fijemos una subvariedad Legendriana cerrada loose L^n de la variedad de contacto (M^{2n+1}, ξ) con $n \geq 2$ y supongamos que el fibrado $T^*L \oplus \mathbb{R}$ tiene dos secciones linealmente independientes punto a punto. Entonces L admite un lazo positivo de Legendrianas.*

Es más, bajo ciertas condiciones extras añadidas al resultado anterior podemos conseguir que los lazos construidos a partir de él sean contráctiles:

Corollary 6.3.2. *Sea $n \geq 3$. Fijemos una subvariedad Legendriana cerrada loose L^n de la variedad de contacto (M^{2n+1}, ξ) y supongamos que el fibrado $T^*L \oplus \mathbb{R}$ tiene cuatro secciones linealmente independientes punto a punto. Entonces L admite un lazo positivo contrátil de Legendrianas.*

1.2 Antecedentes y motivación

Para intentar comprender la importancia de los lazos positivos hemos de remontarnos a los años 80, donde el estudio de la topología simpléctica iba a sufrir un significativo impulso. Concretamente, en 1985, cuando Mikhail Gromov [**Gr**] introdujo su conocido Teorema de *non-squeezing* que muestra la imposibilidad de embeber simplécticamente una bola en un cilindro infinito de menor radio.

Si analizamos un poco más en profundidad su resultado, el teorema de Gromov nos enuncia que ser una transformación simpléctica (es decir, una transformación que preserve la forma simpléctica) es una condición mucho más estricta y fundamentalmente diferente que simplemente ser una transformación que preserve el volumen. Éste fue un resultado crucial pues mostraba, por primera vez, que existía rigidez en la topología simpléctica.

Un segundo resultado que bebía del mismo espíritu que el resultado de Gromov (y que, de hecho, estaba íntimamente relacionado) fue el estudio de la ordenabilidad sobre el grupo de los Hamiltonianos (y su recubridor universal) de una variedad simpléctica dada, que dio origen al nacimiento de la métrica de Hofer [**Ho**] en 1990.

Llegados a este punto, una cuestión que surgía de forma natural era si se podían desarrollar ideas similares en el caso de contacto. Así, en el año 2000, Yakov Eliashberg y Leonid Polterovich [**EP**] introdujeron la noción de isotopía de contacto no negativa y probaron que dicha noción inducía una relación (también estudiada por Bhupal [**Bu**]) sobre la componente identidad del grupo de contactomorfismos $\mathcal{CD}iff_0(M, \xi)$ de una variedad de contacto dada (M, ξ). De igual forma, también se puede definir una relación sobre el recubridor universal de la componente identidad del grupo de contactomorfismos $\widetilde{\mathcal{CD}iff}_0(M, \xi)$ de una variedad de contacto dada (M, ξ).

Dicha relación induce un orden parcial no trivial sobre $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ si no existen lazos no negativos contráctiles de contactomorfismos. A su vez, se induce un orden sobre $\mathcal{CD}iff_0(M,\xi)$ si no existen lazos no negativos de contactomorfismos. La existencia de un orden parcial sobre $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ se puede interpretar como la versión algebraica del teorema de *non-squeezing* de Gromov para dominios de contacto [**EKP**]. Para más información, se puede consultar el capítulo 5.

Análogamente, existe una versión relativa de la construcción anterior adaptada al espacio $\mathcal{L}eg(L)$ y a su recubridor universal. Así surgió la noción de isotopía Legendriana no negativa que define una relación sobre dichos espacios. Y, por tanto, de forma semejante al caso anterior, induce un orden sobre ellos si no existen lazos positivos (contráctiles) de Legendrianas.

A raíz del trabajo realizado por Y. Eliashberg y L. Polterovich, el estudio de la ordenabilidad en variedades de contacto y en particular, en las variedades Legendrianas, se ha vuelto un tema de creciente interés. Así V. Colin, E. Ferrand and P. Pushkar [**CFP**] estudiaron la no existencia de lazos positivos de Legendrianas en ST^*M donde el recubridor universal de M es un espacio real n-dimensional. Además, V. Chernov y S. Nemirovsky [**CN1**, **CN2**, **CN3**] encontraron una aplicación de esta idea para estudiar la causalidad en espacio-tiempos globalmente hiperbólicos. Por otro lado, V. Colin y S. Sandon [**CS**] usaron la ordenabilidad del espacio de Legendrianas para construir métricas bi-invariantes sobre ellas. Por último, como dijimos previamente, G. Liu [**Liu1**, **Liu2**] probó la existencia de lazos positivos contráctiles para variedades Legendrianas loose.

1.3 Organización de la tesis

Como hemos afirmado previamente, el principal objetivo que nos marcamos al comenzar esta disertación era describir un resultado que nos permitiera encontrar lazos positivos de subvariedades Legendrianas siempre que éstas cumpliesen ciertas propiedades topológicas. Para alcanzar esta meta procedemos de la siguiente manera:

En primer lugar, en el capítulo 3, dotamos al lector del conocimiento básico necesario para poder comprender con detalle los resultados obtenidos. Para ello, en la sección 3.1 presentamos las definiciones y enunciados básicos relativos a la topología de contacto y detallamos su relación con la topología simpléctica. Las siguientes dos secciones (3.2 y 3.3) resultan esenciales para la comprensión de los resultados obtenidos pues en ellas introducimos los conceptos claves con los que vamos a trabajar: los lazos de contactomorfismos y las subvariedades Legendrianas. En la sección 3.4, introducimos el famoso teorema del entorno tubular de Weinstein el cuál usaremos más adelante. Por último, en la sección 3.5, analizamos la dicotomía existente en la topología de contacto: overtwisted vs tight. La mayor parte del material de este capítulo se puede encontrar descrito en [**Ge1**, **Ge2**, **McSal**].

Una condición indispensable que hemos de imponer a las variedades Legendrianas en nuestros resultados es que éstas han de ser subvariedades loose. Estos objetos nos van a dar la libertad suficiente como para poder realizar nuestras construcciones. Para ello usamos un resultado debido a E. Murphy [**Mu**]: Las subvariedades Legendrianas loose satisfacen un h-principio. En el capítulo 4 revisamos todos estos conceptos y damos una idea de la prueba del resultado de Murphy.

A continuación en el capítulo 5, revisamos toda la teoría de ordenabilidad en variedades de contacto introducidas por Y. Eliashberg y L. Polterovich [**EP**, **EKP**]. Definimos la relación de orden y los conceptos claves en las secciones 5.1 y 5.2, aportamos los principales ejemplos clásicos existentes en la sección 5.3 y, por último, en las secciones 5.4 y 5.5, detallamos la relación que se puede establecer entre la ordenabilidad en variedades de contacto y la métrica de Hofer y el teorema de *non-squeezing de Gromov* adaptados al caso de contacto.

Concluimos este trabajo con el capítulo 6 donde enumeramos y probamos los principales resultados obtenidos durante el desarrollo de este trabajo.

Chapter 2

Introduction

"I still choose to believe that all things do come at a price, that there's an ebb and a flow, a cycle, that the pain we went through did have a reward, and that anyone who's determined and perseveres will get something of value in return, even if it's not what they expected."

- Hiromu Arakawa, Hagane no Renkinjutsushi.

This document is the summary of the work I have been developing during my Ph. D. years. We have managed to write down and publish two articles summarizing the most important results we have achieved during this period:

[**CPPP**] R. Casals, J.L. Pérez, A. del Pino, and F. Presas. *Existence h-principle for Engel structures.* Invent. Math. Volume **210**, Issue 2, (2017) 417-451.

[**PPP**] D. Pancholi, J.L. Pérez, and F. Presas. *A simple construction of positive loops of Legendrians.* Arkiv för Matematik, **56**, Number 2 (2018), 377 – 394.

In chronological order, the first article is about a result we started to develop in 2015. In it, we prove the existence of a h-principle for Engel structures. This article is not included in this dissertation because we have chosen to write down a monograph focused on a self-contained and unitary description of a problem and its partial solution: the homotopy type of the space of Engel structures over a fix 4-fold. Let us state the main result of this part of my research

Given a 4-dimensional manifold M, an **Engel structure** is a maximally non-integrable 2-distribution. Denote by $\mathfrak{E}(M)$ the space of Engel structures of a given 4-manifold M and by $\mathfrak{F}(M)$ the space of full flags of the tangent space of a given 4-manifold M, then the inclusion map $i : \mathfrak{E}(M) \hookrightarrow \mathfrak{F}(M)$ induces surjections in all homotopy groups, that is, the map $\pi_k(i) : \pi_k(\mathfrak{E}(M)) \to \pi_k(\mathfrak{F}(M))$ is surjective for all $k \geq 0$. In particular, we explain how to construct Engel structures representing any given full flag.

The second article mentioned above summarizes the study and partial solution of the problem we decided to consider as the main topic for this thesis. In the following sections, we introduce the problem and we describe the content of the paper in more detail.

2.1 Review of the content

Consider a $(2n+1)$-dimensional contact manifold (M^{2n+1}, ξ), that is, a smooth $(2n+1)$-dimensional manifold M^{2n+1} endowed with a maximally non-integrable hyperplane distribution ξ. Confer section 3.1, and concretely Definition 3.1.1, for more details about this notion.

Legendrian loops are the main objects we are going to study in this thesis. On the one hand, given a contact manifold (M^{2n+1}, ξ), a submanifold $L \subset (M^{2n+1}, \xi)$ is called Legendrian if its tangent space at every point is also tangent to the contact distribution, meaning that:

$$T_pL \subset \xi_p, \text{ for all } p \in M.$$

On the other hand, a loop of Legendrians is a smooth 1-parametric family of Legendrians $\{L_t\}_{t\in[0,1]}$ satisfying that $L_0 = L_1$. For further information about these definitions, look section 3.3.

Given a Legendrian submanifold $L \subset (M^{2n+1}, \xi)$, let us denote by $\mathcal{L}eg(L)$ the space of all Legendrian submanifold isotopic to L. As we will see in the next section Y. Eliashberg and L. Polterovich [**EP**] made a research about orderability of the contactomorphism groups which allowed them to find a relation between the orderability of the space $\mathcal{L}eg(L)$ and the existence of positive loops of Legendrian submanifolds (see definition 3.3.7). This relation can be extended to the universal cover $\widetilde{\mathcal{L}eg}(L)$ of the space $\mathcal{L}eg(L)$ provided that those loops are also contractible (Definition 3.3.8).

The main goal of this dissertation is to describe a result which allows us to find positive loops of Legendrian submanifolds. In order to do that we have to impose some topological properties on them. Specifically, we need to assume that the Legendrian submanifold is also loose.

The notion of loose Legendrian submanifold requires several tools and definitions to be introduced. For this reason, and for a better understanding of this notion, we refer chapter 4. In particular, the reader should check definitions 4.1.3 and 4.1.4. For now, it is enough to say that a Legendrian submanifold will be loose if there exists a special local coordinates system on a open ball intersecting the Legendrian submanifold in a specific way.

Taking this goal into account, we aim to partially recover a recent result, due to G. Liu [**Liu1**], stating that any loose Legendrian submanifold admits a positive loop, under some mild topological assumptions on the Legendrian. Our version of the result which have a simpler and shorter proof, is the following:

Theorem 6.2.1. *Let $n \geq 2$. Fix a loose closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ). Assume that the bundle $T^*L \oplus \mathbb{R}$ has two pointwise linearly independent sections. Then L admits a positive loop of Legendrians.*

Moreover, under some extra topological assumptions, we are able to show contractibility of the constructed loops:

Corollary 6.3.2. *Let $n \geq 3$. Fix a loose closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ). Assume that the bundle $T^*L \oplus \mathbb{R}$ has four pointwise linearly independent sections. Then, L admits a contractible positive loop of Legendrians.*

2.2 Background and motivation

In order to understand the motivation to study the theory of positive loops we have to go back to the 80s where the field of symplectic topology was going to experience a significant boost. Concretely, in 1985, when Mikhail Gromov [**Gr**] stated his well-known Non-Squeezing theorem which affirms that one cannot embed a ball into a infinite cylinder via a symplectic map if the radius of the cylinder is less than the radius of the ball.

If we dig into Gromov's theorem, it shows us that being a symplectic transformation (that is, a transformation which preserves the symplectic form) is much more restrictive than just preserving volume. This result was crucial as it showed for the first time that there exists a rigidity phenomena in symplectic topology.

A second important result which shared the same spirit than Gromov's theorem (and that, in fact, was intimately related) was the study of orderability for the Hamiltonian group (and its universal cover) of a given symplectic manifold. This research gave birth to the Hofer metric [**Ho**] in 1990.

At this point, a question that arose naturally was if similar ideas could be developed in the contact case. In this way, in the year 2000, Yakov Eliashberg and Leonid Polterovich [**EP**] introduced the notion of non-negative contact isotopy and discovered that this concept induces a relation (also studied by Bhupal [**Bu**]) on the identity component of the contactomorphims group $\mathcal{CD}iff_0(M, \xi)$ of a given contact manifold (M, ξ). Equivalently, one can define a relation on the universal cover of the identity component of the contactomorphims group $\widetilde{\mathcal{CD}iff}_0(M, \xi)$ of a given contact manifold (M, ξ).

This relation induces a non-trivial partial order on $\widetilde{\mathcal{CD}iff}_0(M, \xi)$ if there is no contractible non-negative loops of contactomorphisms. Equally, a non-trivial partial order is induced on $\mathcal{CD}iff_0(M, \xi)$ if there is no non-negative loops of contactomorphisms. The existence of a partial order on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ can be understood as the algebraic version of the (non)-squeezing problem for contact domains [**EKP**]. One can check chapter 5 if more information is needed.

Analogously, there is a relative version of the above construction for the space of Legendrian isotopic submanifolds $\mathcal{L}eg(L)$ and its universal cover. In this way, the notion of non-negative Legendrian isotopy came up. Thus, this notion induces a relation on $\mathcal{L}eg(L)$ and $\widetilde{\mathcal{L}eg}(L)$ which will be a non-trivial partial order if there is no (contractible) non-negative loops of Legendrians.

As a result of the research done by Y. Eliashberg and L. Polterovich, the study of the orderability fora contact manifold or, in particular, for Legendrian submanifolds and of the existence of positive (contractible) loops has been an active research area in contact topology. Thus, for instance, V. Colin, E. Ferrand and P. Pushkar [**CFP**] studied the non–existence of positive loops of Legendrian submanifolds in ST^*M where the universal cover of M is the n–dimensional real space. In the field of Lorentzian geometry, V. Chernov and S. Nemirovsky [**CN1**, **CN2**, **CN3**] apply this topic to the study of causality in globally hyperbolic spacetimes. The orderability property of Legendrians allowed V. Colin and S. Sandon [**CS**] to introduce the existence of bi–invariant integer–valued metrics in the space of Legendrians [**CS**]. Lastly, as we have said before, G. Liu [**Liu1**, **Liu2**] announced, recently, the existence of (contractible) positive loops for loose Legendrian submanifolds.

2.3 Plan of the thesis

As we have stated previously, the main goal we proposed ourselves when we started writing this thesis is to find positive loops of Legendrian submanifolds knowing that we had to impose several topological assumptions on the Legendrians. In order to achieve this goal we proceed as follows:

First, in chapter 3, we provide the reader with the basic knowledge necessary to fully understand the results we are going to explain. In order to do that, in section 3.1. we introduce the basic statements and definitions concerning contact topology and we describe its relation with symplectic topology. The following two sections (3.2 and 3.3) are essential because they are devoted to introduce the key notions which we are going to work with: Legendrian submanifolds and loops of contactomorphisms (or Legendrians). In section 3.4, we present the well-known Weinstein's tubular neighborhood which we are going to use throughout this thesis. Lastly, in section 3.5 we study the existing dichotomy in contact topology: overtwisted vs tight. Most of the material in this chapter can be found in [**Ge1**, **Ge2**, **McSal**].

A required condition we have to impose to Legendrian submanifolds in our statements is that they must be loose. These objects are going to provide enough flexibility to make our constructions. We are going to use an h-principle result applied to Legendrian submanifolds due to E. Murphy [**Mu**]. In chapter 4 we review all these notions and we outline Murphy's theorem.

Next, in chapter 5 we go over the theory of orderability in contact topology introduced by Y. Eliashberg and L. Polterovich [**EP**, **EKP**]. We define the main notions and the order relation in section 5.1 and 5.2, we detail the main known examples in section 5.3 and , lastly, in section 5.4 and 5.5 we describe how the orderability properties can be used to develop an analogous Hofer's metric and Gromov's non-squeezing theorem for the contact case.

We conclude this thesis with chapter 6 where we state and prove the main results obtained during our research.

Chapter 3

Preliminary notions

"It's the job that's never started as takes longest to finish."
- J.R.R. Tolkien, The Lord of the Rings.

In this section we review some basic notions in symplectic and contact topology. We will pay particular attention to the theory of contact and symplectic Hamiltonians. We will also focus on the study of overtwisted contact manifolds and Legendrian submanifolds. More material on these subjects can be found for example in [**BEM**, **EL1**, **EL2**, **Ge1**, **Ge2**, **McSal**, **Sil**, **EKP**, **P2**].

3.1 Contact Structures

Given an n-dimensional smooth manifold M^n, a k-dimensional (tangent) *distribution* is just a smooth sub-bundle of dimension k of its tangent bundle T^nM. In other words, it is a smooth choice of a k-dimensional subspace of the tangent bundle at each point of M.

Distributions are used, in natural language, to build up notions of integrability, and specifically of a foliation of a manifold. An *integral submanifold* for a distribution $\mathcal{D}$ is a submanifold $N \subset M$ such that its tangent bundle lies in the distribution; $TN \subseteq \mathcal{D}$. We say the distribution $\mathcal{D}$ is *integrable* if through each point of M there exists an integral submanifold for $\mathcal{D}$. Although it follows from the basic theory of ordinary differential equations that any 1-dimensional distribution is integrable, higher dimensional distributions need not to be integrable. If the sub-bundle has dimension greater than one, a condition known as the *Frobenius integrability condition* needs to be imposed. It states that the set of vector fields tangent to the distribution must be closed under Lie bracket.

The collection of integral submanifolds of a k-dimensional integrable distribution constitutes what is called a codimension k *foliation*.

Now let us focus on hyperplane distributions, that is, a smooth sub-bundle ξ of codimension 1. Locally, they can be written as the kernel of a non-vanishing differential 1-form α. Note that the 1-form can be globally defined if and only if the hyperplane distributions is co–orientable, meaning that TM/ξ is trivial. From now on, we shall always assume our hyperplane distributions to be coorientable. It turns out that the Frobenius integrability condition for a hyperplane distribution is equivalent to say that $\alpha \wedge d\alpha \equiv 0$. *Contact structures* are in a certain sense the opposite of integrable hyperplane distributions.

Definition 3.1.1. *Let M^{2n+1} be an odd-dimensional manifold. A (co-orientable)* ***contact structure*** *may be regarded as a maximally non-integrable hyperplane distribution $\xi = \ker \alpha \subset TM$, that is, α must satisfy that*

$$\alpha \wedge d\alpha^n \not\equiv 0.$$

The pair (M, ξ) is then called a ***contact manifold*** *and α is called a* ***contact form****.*

At this point we should observe, on the one hand, that the contact condition implies that M is orientable since $\alpha \wedge d\alpha^n$ is a volume form on M. On the other hand, any other contact form which determines the same contact structure, has to be of the form $\lambda\alpha$ for some smooth function $\lambda : M \to \mathbb{R} \setminus \{0\}$. Hence, the contact condition $\alpha \wedge d\alpha^n \not\equiv 0$ is independent of the specific choice of α since

$$(\lambda\alpha) \wedge (d(\lambda\alpha))^n = \lambda^{n+1}(\alpha \wedge d\alpha^n).$$

In addition, it is easy to see that if n is odd the sign of this volume form depends only on ξ, not the choice of α, hence the contact structure induces a natural orientation of M. This makes it possible, for n odd, given a manifold M equipped with a specific orientation, to speak of *positive* or *negative* contact structures.

Examples 3.1.2.

1. *On $\mathbb{R}^{2n+1}$ with coordinates $(z, x_1, y_1, \cdots, x_n, y_n)$ we can define the following contact structure*

$$\xi_{std} = \ker \alpha_{std} = \ker \left(dz + \sum_{j=1}^{2n} x_j dy_j \right).$$

   It is known as the ***standard contact structure*** *on $\mathbb{R}^{2n+1}$. In particular, on $\mathbb{R}^3$ we have $\alpha_{std} = dz + xdy$.*

2. On $\mathbb{R}^{2n+1}$ with coordinates $(z, r_1, \varphi_1, \cdots, r_n, \varphi_n)$ where (r_j, φ_j) are polar coordinates for the (x_j, y_j)-plane, the form

$$\widetilde{\alpha}_{std} = dz + \sum_{j=1}^{2n} r_j^2 d\varphi_j = dz + \frac{1}{2}\sum_{j=1}^{2n}(x_j dy_j - y_j dx_j)$$

is a contact form.

3. Consider the Cartesian coordinates on $\mathbb{R}^{2n+1}$ denoted by $(z; x_j, y_j)$. Then $\mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}$ is a contact manifold with contact structure given by

$$\xi_o = \ker(\alpha_o) = \frac{1}{2}\sum_{j=1}^{2n}(x_j dy_j - y_j dx_j).$$

4. Consider the cylindrical coordinates (r, φ, z) on $\mathbb{R}^3$, it easy to check that the form

$$\alpha_{ot} = \cos r dz + r \sin r d\varphi$$

is a contact form. It is called the standard ***overtwisted contact structure*** on $\mathbb{R}^3$. Later on, we will describe better this contact structure.

5. Consider the cotangent bundle T^*M of a smooth manifold M. For any point σ of T^*M define the ***Liouville form*** (also known as the ***canonical one-form*** λ_{can} by $\lambda_{can} = \sigma \circ (d\pi)_\sigma$ where π is the canonical fiber bundle projection of T^*M. In canonical coordinates (q^i, p_j), the tautological one-form is given by

$$\lambda_{can} = \sum_i p_i dq^i$$

The ***1-jet space*** $J^1(M) = \mathbb{R}^*M$ has a natural contact structure given as the kernel of $\alpha_{jet} = dz - \lambda_{can}$, where z is the coordinate on $\mathbb{R}$. Note that if $M = \mathbb{R}^{2n}$ we recover the first example.

6. Given any manifold M, let $\mathbb{P}T^*M = (T^*M \setminus \{0\text{-section}\})/\sim$ be the projectivization of the cotangent bundle and write π for the fiber bundle projection. For $u \in \mathbb{P}T_p^*M$, let $\xi_u^p \in T_u(\mathbb{P}^*TM)$ be the hyperplane such that $\pi_*\xi_u^p = \ker(u) \in T_{\pi(u)}M = T_pM$. Then ξ^p defines a contact structure on $\mathbb{P}T^*M$. Notice that the projectivization of the cotangent bundle can be naturally identified with the so-called space of contact elements. See [**Ge1**, **Ge2**] for more details.

7. Consider the oriented projectivization of the cotangent bundle

$$\mathbb{S}(T^*M) \xrightarrow{2:1} \mathbb{P}(T^*M)$$

It will be useful for our purposes to realize that $\mathbb{S}(T^*M)$ *can be regarded as a contact submanifold of* $J^1(M) = \mathbb{R} \times T^*M$ *by means of*

$$e : \mathbb{S}(T^*M) \hookrightarrow J^1(M),$$

where $e(\alpha) = \alpha \oplus \{0\}$ *and we have that* $e_*\xi^p = \xi_{jet}$.

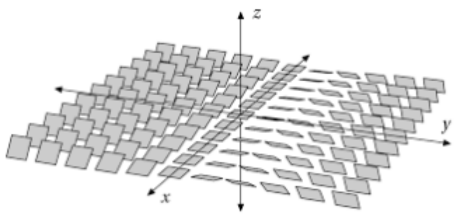

Figure 3.1: The standard contact structure on $\mathbb{R}^3$.

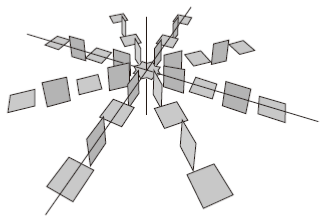

Figure 3.2: The OT contact structure on $\mathbb{R}^3$.

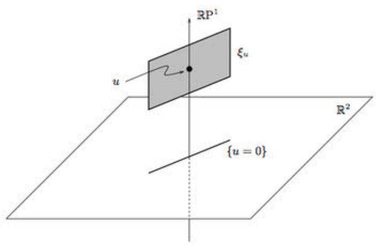

Figure 3.3: Example (6) for $M = \mathbb{R}^2$

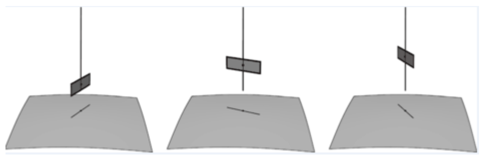

Figure 3.4: Animation of the contact structure in example (6).

Remark 3.1.3. *The name "contact structure" has its origins in the fact that one of the first historical sources of contact manifolds are, precisely, the spaces of contact elements.*

Definition 3.1.4. *A* ***symplectic form*** *on a even dimensional smooth manifold* M^{2n} *is a differential* 2*-form* ω *which satisfies that it is is closed* ($d\omega = 0$) *and non-degenerate (i.e.* ω^n *is a volume form on* M*). The manifold* M^{2n} *is then called* ***symplectic manifold****.*

Examples 3.1.5.

1. *On the Euclidean space* $\mathbb{R}^{2n}$ *with coordinates* $(x_1, y_1, \cdots, x_n, y_n)$ *we can define the following symplectic form*

$$\omega_{std} = \sum_{j=1}^{2n} dx_j \wedge dy_j.$$

It is known as the ***standard symplectic form****.*

2. *Given any manifold M, we can get a symplectic manifold by taking the cotangent bundle T^*M with symplectic form $\omega = -d\lambda_{can}$.*

Contact manifolds are intimately related to symplectic manifolds. In fact, observe that given a $(2n+1)$–dimensional manifold M, an equivalent condition for $\xi \subset TM$ to be a contact structure on M is that for any (local) 1–form α such that $\xi = \ker\alpha$ we have that $d\alpha^n|_\xi \neq 0$, that is, $(\xi_p, (d\alpha|_\xi)_p)$ is a symplectic vector bundle over M.

Moreover, a way to obtain a contact manifold from a symplectic manifold is the following: Let $\omega = d\lambda$ be an exact symplectic form on M^{2n} and consider the manifold $\mathbb{R} \times M^{2n}$. Then $\xi = \ker(dz - \lambda)$, where z is the coordinate in the $\mathbb{R}$-direction, is a contact structure on $\mathbb{R} \times M^{2n}$. Note that we can also make this construction by taking $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ instead of $\mathbb{R}$ since $dz - \lambda$ is $\mathbb{R}$-invariant. The manifold $QM = (\mathbb{R}^{2n}, \xi = \ker(dz - \lambda))$ is called the **contactization** or **prequantization** of (M^{2n}, ω). This notion can be generalised for symplectic manifolds whose symplectic form represents an integral cohomology class. See ([**A**], Appendix 4) for more details.

Examples 3.1.6.

1. *The contactization of the standard symplectic space $(\mathbb{R}^{2n}, \omega_{std})$ is the standard contact space $(\mathbb{R}^{2n+1}, \xi_{std})$.*
2. *The contactization of $(T^*M, -d\lambda_{can})$ is the 1-jet space $(\mathbb{R} \times T^*M, \xi_{jet}))$.*

Additionally, given a contact manifold $(M, \xi = \ker(\alpha))$, there is also a construction to create a symplectic manifold associated to it. Consider the manifold $SM := (\mathbb{R} \times M, d(e^t\alpha))$, where t denotes the coordinate on $\mathbb{R}$. The contact condition implies that $d(e^t\alpha)$ is a symplectic form on $\mathbb{R} \times M$. SM is called the **symplectization** of (M^{2n+1}, ξ).

Now notice that the vector field ∂_t satisfies that $\mathcal{L}_{\partial_t}\omega = \omega$, where by $\mathcal{L}_X$ we denote the Lie derivative along the flow of X. Vector fields with this property are known as Liouville vector fields:

Definition 3.1.7. *A* ***Liouville vector field*** *X on a symplectic manifold (M, ω) is a vector field satisfying the equation $\mathcal{L}_X\omega = \omega$. The triple (M, ω, L) of symplectic manifold with a fixed Liouville vector field L is known as a* ***Liouville manifold****.*

Example 3.1.8. *Consider the contact manifold $(\mathbb{S}^{2n-1}, \xi_o)$. We can identify $\mathbb{R} \times \mathbb{S}^{2n-1}$ with $\mathbb{R}^{2n} \setminus \{0\}$. Then*

$$S\mathbb{S}^{2n-1} \simeq (\mathbb{R}^{2n} \setminus \{0\},\ \omega_{std}).$$

Liouville vector fields have a special property: if X is a Liouville vector field, then the 1-form $\alpha = \iota_X\omega$ is a contact form on any hypersurface N transverse to X, where ι_X denotes the interior product of a vector field X. They are called ***hypersurface of contact type****.*

*Denote by $M_{*,N} = \bigcup_{t\in\mathbb{R}} \phi_t(N)$ where ϕ_t is the flow of a Liouville vector field X. It can be proven that $M_{*,N}$ is independent of N. Write M_* for $M_{*,N}$. Note that M_* is invariant under the $\mathbb{R}_+$-action given by $\lambda * p = \phi_{log\lambda} p$. Write N_∞ for $M_*/\mathbb{R}^*$. Note also that the contact plane on M_* descends to a contact plane field ξ_∞. We will call (N_∞, ξ_∞) the **ideal contact boundary** of the Liouville manifold. Note that, clearly, the symplectization SN_∞ of N_∞ is canonically identified with M_*.*

The next definition provides the natural equivalence between contact structures.

Definition 3.1.9. *Let $f : M_1 \to M_2$ be a diffeomorphism between two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) and denote by $f_* : TM_1 \to TM_2$ its differential. M_1 and M_2 are said to be **contactomorphic** if $f_*\xi_1 = \xi_2$. The diffeomorphism f is known as a **contactomorphism**.*

Remark 3.1.10. *Let $\xi_i = \ker \alpha_i, i = 1, 2$. In this case, the above definition is equivalent to the existence of a nowhere zero function $\lambda : M_1 \to \mathbb{R} \setminus \{0\}$ such that $f^*\alpha_2 = \lambda\alpha_1$. A contactomorphism f which preserves also a contact form, i.e. such that $f^*\alpha_2 = \alpha_1$, is sometimes called a **strict contactomorphism**.*

Note that $(\mathbb{R}^{2n+1}, \xi_{std})$ and $(\mathbb{S}^{2n+1} \setminus \{0\}, \xi_o)$ are contactomorphic, while $(\mathbb{R}^{2n+1}, \xi_{std})$ and $(\mathbb{R}^{2n+1}, \xi_{ot})$ are not. See [**Be**, **Ge**].

Definition 3.1.11. *A **contact vector field** X is a vector field whose (local) flow ,denoted by slight abuse of notation by ϕ_t, is a contactomorphism for all t.*

The condition for X to be a contact vector field on $(M, \xi = \ker \alpha)$ can be written as $\mathcal{L}_X\alpha = e^\lambda\alpha$ for some function $\lambda : M \to \mathbb{R}$, known as the **conformal factor** of X or its flow. Notice that this condition is independent of the choice of α. The local flow of X preserves α if and only if $\mathcal{L}_X\alpha = 0$. In this case, X is called a **strict contact vector field**.

Definition 3.1.12. *Associated with the contact form α there exists a unique contact vector field R_α defined by the equations:*

1. $\alpha(R_\alpha) = 1$
2. $d\alpha(R_\alpha, \cdot) = 0$.

*Such a vector field is called the **Reeb vector field** corresponding to the contact form α.*

Remark 3.1.13. *The Reeb vector field is a strict contact vector field since:*

$$\mathcal{L}_{R_\alpha}\alpha = 0.$$

Recall that a contact structure is a hyperplane distribution $\xi = \ker \alpha$ such that $d\alpha|_\xi$ is a symplectic form. Hence, it is a skew-symmetric form of maximal rank $2n$ and then, the form $d\alpha|_{T_pM}$ has a 1-dimensional kernel for each $p \in M^{2n+1}$. The contact condition $\alpha \wedge d\alpha \neq 0$ implies that a generator of this kernel is not annihilated by α. The Reeb vector field R_α is a generator of this kernel (2) normalized by α (1). Therefore, it is unique. The fact that R_α is a contact vector field follows from the Cartan formula.

Remark 3.1.14. *Once we choose a contact form α defining ξ, we can decompose the tangent bundle of M as the direct sum:*

$$TM = < R_\alpha > \oplus \xi$$

An important fact in contact geometry is that there are no local invariants apart from the dimension (this should be compared with Riemannian geometry, where the curvature is an obstruction for two metrics to be connjugated), meaning that every contact structure looks like the standard one near a point. More precisely,

Theorem 3.1.1 (Darboux's theorem for contact manifolds)**.** *Let $(M^{2n+1}, \xi = \ker\alpha)$ be a contact manifold and consider a point p on M. Then one can always find local chart $(U_p, \varphi = (z, x_1, y_1, \cdots, x_n, y_n))$ on a neighborhood U_p of p such that $p = (0, \cdots, 0)$ and*

$$\varphi^*(\alpha|_{\varphi(U)}) = dz + \sum_{j=1}^{2n} x_j dy_j.$$

The modern proof of Darboux's Theorem uses "*Moser's trick*", introduced by J. Moser [**Mo**] in the context of stability results for volume and symplectic forms. In fact, there exists an equivalent version of Darboux's Theorem for symplectic manifolds. The main idea of this method is to consider that the isotopy we are looking for is the flow of a time-dependent vector field X_t and translate the equation concerning the isotopy in terms of an equation for X_t. If that equation can be solved, then the isotopy can be found by integrating X_t. Remark that on a closed manifold the flow of X_t will be globally defined. There are several good references for Moser's method and its many corollaries [**McSal**, **Sil**, **Ge2**]. A result analogous to the classical Moser's stability theorem in symplectic geometry is that there are no non-trivial deformations of contact structures on closed manifolds. In other words,

Theorem 3.1.2 (Gray stability)**.** *Let $\xi_t, t \in [0,1]$ be a smooth family of contact structures on a closed manifold M. Then there exists an isotopy (i.e. a smooth family of diffeomorphisms) $(\psi_t)_{t\in[0,1]}$ of M such that*

$$(\psi_t)_*\xi_0 = \xi_t, \text{ for each } t \in [0,1].$$

Several remarks should be done about these theorems:

a) Contact forms do not satisfy stability, that is, in general one cannot find an isotopy ψ_t such that $\psi_t^*\alpha_t = \alpha_0$, considering α_t a smooth family of contact forms.

The reason why we can use the Moser trick to prove the Darboux theorems is that, locally, this method also yields solutions to the equation $\psi_t^* \alpha_t = \alpha_0$ by dropping the restriction $X_t \in \ker \alpha_t$.

b) In particular, on a compact manifold all deformations of contact structures come from diffeomorphisms of the underlying manifold. The theorem is not true if the contact structures do not agree off of a compact set. For example, Y. Eliashberg [**El1**] has shown that on the open manifold $\mathbb{R}^3$ there are likewise no non-trivial deformations of contact structures, but on $\mathbb{S}^1 \times \mathbb{R}^2$ there does exist a continuum of non-equivalent contact structures.

3.2 Contact Hamiltonians

Let (M,ω) be a symplectic manifold, a **symplectomorphism** of (M,ω) is a diffeomorphism $\psi \in \mathrm{Diff}(M)$ which preserves the symplectic form

$$\omega = \psi^* \omega.$$

Since ω is non-degenerate there is a one-to-one correspondence between vector fields and one-forms given by

$$\begin{array}{rcl} TM & \to & T^*M \\ X & \mapsto & \iota_X \omega. \end{array}$$

A vector field X is called **symplectic** if its correspondent one-form $\iota_X \omega$ is closed. If $\iota_X \omega$ is exact, then there exists a smooth function $H : M \to \mathbb{R}$ such that $\iota_X \omega = -dH$. Denote by X_H the associated symplectic vector field determined by H. It is called the **Hamiltonian vector field** associated to the **Hamiltonian function** H. Note that if M is closed, X_H generates a flow, known as the **Hamiltonian flow** associated to H.

Let us generalize this concept for time-dependent vector fields. Denote by $\mathrm{Symp}(M,\omega)$ the group of symplectomorphisms and by $\chi_{\mathrm{Symp}}(M,\omega)$ the space of symplectic vector fields then,

Proposition 3.2.1 ([**McSal**])**.** *Let M be a closed manifold and consider a smooth family of diffeomorphisms $\{\psi_t\}_{t\in[0,1]}$ generated by a unique time-dependent vector field $\{X_t\}_{t\in[0,1]}$ defined as*

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = Id,$$

then $X_t \in \chi_{\mathrm{Symp}}(M,\omega)$ for every $t \in [0,1]$ if and only if $\psi_t \in \mathrm{Symp}(M,\omega)$ for every $t \in [0,1]$.

A smooth family of symplectomorphisms $\{\psi_t\}_{t\in[0,1]}$ such that $\psi_0 = Id$ is called a **symplectic isotopy** of M.

As a consequence of the above proposition, a smooth family of functions defined by $H : M \times [0,1] \to \mathbb{R}$ on a closed symplectic manifold (M, ω) uniquely determines a time-dependent vector field X_{H_t}, called the **time-dependent Hamiltonian vector field** associated to the **time-dependent Hamiltonian function** $H(t,\cdot) := H_t(\cdot)$ by the relation

$$-dH_t = \iota(X_{H_t})\omega.$$

An immediate property of the time-dependent Hamiltonian function is that it is invariant, i.e. a first integral, under the flow $\{\psi_t\}$ generated by H_t, known as the **Hamiltonian flow**. Indeed,

$$X_{H_t} H_t = dH_t(X_{H_t}) = -\omega(X_{H_t}, X_{H_t}) = 0,$$

since ω is a symplectic form. Another important property is that the (local) flow of X_{H_t} preserves the symplectic form ω, i.e. that the Lie derivative of ω along X_{H_t}, denoted by $\mathcal{L}_{X_{H_t}}\omega$ vanishes. Actually, applying Cartan's formula, we get

$$\mathcal{L}_{X_{H_t}}\omega = d(\iota_{X_{H_t}}\omega) + \iota_{X_{H_t}} d\omega = d(dH_t) + 0 = 0.$$

Finally, a symplectomorphism $\psi \in \mathrm{Symp}(M, \omega)$ is called **Hamiltonian symplectomorphism** or **Hamiltonian diffeomorphism** if there exists a Hamiltonian isotopy ψ_t such that $\psi_0 = Id$ and $\psi_1 = \psi$. Notice that the Hamiltonian functions are determined up to time-dependent constant.

In the contact case the solution is even better: every contact vector field is Hamiltonian.

Theorem 3.2.2. *With a fixed choice of contact form α there is a one-to-one correspondence between contact vector fields X of $\xi = \ker(\alpha)$ and smooth functions $H : M \to \mathbb{R}$. This correspondence is given by:*

- $X \mapsto H_X := \alpha(X)$
- $H \mapsto X_H$ *defined uniquely by $i_{X_H}\alpha = H$ and $i_{X_H} d\alpha = dH(R_\alpha)\alpha - dH$, where by R_α we denote the Reeb vector field associated to α.*

H_X is called the ***associated contact Hamiltonian*** *to X, while X_H is known as the* ***contact Hamiltonian*** *vector field associated to H and its flow is known as the* ***Hamiltonian flow****.*

The above theorem gives us a bijection between the space of *contact vector fields* and the space of *smooth functions*. The fact that X_H is uniquely defined by the equations in the theorem follows from the fact that $d\alpha$ is non-degenerate on ξ and $R_\alpha \in \ker(dH(R_\alpha)\alpha - dH)$.

Proof. Let X be a contact vector filed of ξ and define $H_X = \alpha(X)$. Recall that $\mathcal{L}_X\alpha = i_X d\alpha + d(i_X\alpha)$, then by hypothesis $\lambda\alpha = \mathcal{L}_X\alpha = i_X d\alpha + dH_X$. Applying this equation to R_α we have that $\lambda = dH_X(R_\alpha)$ and, consequently, $i_X d\alpha = dH_X(R_\alpha)\alpha - dH_X$. Hence, $X_{H_X} = X$.

Conversely, consider $H : M \to \mathbb{R}$ and let X_H be a vector field defined as in the theorem, then:

$$\mathcal{L}_{X_H}\alpha = i_{X_H} d\alpha + d(i_{X_H}\alpha) = dH(R_\alpha)\alpha.$$

Consequently, X_H is a contact vector field of ξ and $H_{X_H} = \alpha(X_H) = H$. □

Let us now consider a smooth family of functions $H_t : M \to \mathbb{R}, t \in [0,1]$ on a closed contact manifold $(M, \xi = \ker\alpha)$, known as a **time-dependent contact Hamiltonian**. Let X_{H_t} be the corresponding **time-dependent contact Hamiltonian vector fields** defined as above, then:

Corollary 3.2.3. *The globally defined flow ψ_t of the time-dependent contact Hamiltonian vector field X_{H_t} is a* ***contact isotopy*** *of (M, ξ), i.e. $\psi_t^*\alpha = e^{\lambda_t}\alpha$ for some smooth functions $\lambda_t : M \to \mathbb{R}$ with $\psi_0 = Id$ and for all $t \in [0,1]$. An isotopy is a loop of contactomorphisms if $\psi_0 = \psi_1 = Id$.*

A contactomorphism ψ can be then understood as the time-one map of a contact isotopy ψ_t.

Without loss of generality we will denote by X_t the time-dependent contact Hamiltonian vector field associated to H_t in case we do not need to specify the contact Hamiltonian function. Equally, depending on the situation, we will denote the time-dependent contact Hamiltonian associated to X_t by H_{X_t} or H_t. Finally if we want to specify the contact isotopy, we will denote H_{X_t} by H_{ψ_t} where ψ_t is the flow of X_t.

Definition 3.2.1. *An isotopy of contactomorphisms ψ_t is* ***non-negative*** *if its associated time-dependent contact Hamiltonian H_t is non-negative, i.e. $H_t(p) \geq 0$ for all p in M and for all t in $[0,1]$. If the inequality is strict, the isotopy is called* ***positive****. Analogously we can define* ***positive*** *and* ***non-negative loops of contactomorphisms****.*

This definition is independent of the choice of contact form α for the given co-orientation. Let us point out that when we have a loop of contactomorphisms we can choose the parameter to be defined as $t \in \mathbb{S}^1$. The above definitions can be adapted to this situation.

We can define three important operations on the space of loops of contactomorphisms: concatenation, composition and conjugation. Let $\{\psi_t^1, \cdots, \psi_t^k\}$ be k loops of contactomorphisms. The **concatenation** $\psi_t^1 \circledcirc \cdots \circledcirc \psi_t^k$ of the loops is defined as

$$\psi_t^1 \circledcirc \cdots \circledcirc \psi_t^k = \begin{cases} \psi_{kt}^1 & \text{if } t \in [0, \frac{1}{k}]; \\ \psi_{kt-1}^2 & \text{if } t \in [\frac{1}{k}, \frac{2}{k}]; \\ \vdots & \quad \vdots \\ \psi_{kt-k+2}^{k-1} & \text{if } t \in [1 - \frac{2}{k}, 1 - \frac{1}{k}] \\ \psi_{kt-k+1}^k & \text{if } t \in [1 - \frac{1}{k}, 1] \end{cases}.$$

Consider the Hamiltonians associated to the above loops denoted by $\{H_t^1, \cdots, H_t^k\}$, then the generating Hamiltonian of the concatenation is $H(\psi_t^1 \circledcirc \cdot \circledcirc \psi_t^k)$ is given by

$$H_{\psi_t^1 \circledcirc \cdots \circledcirc \psi_t^k}(p,t) = \begin{cases} kH^1(p, kt) & \text{if } t \in [0, \frac{1}{k}]; \\ kH^2(p, kt-1) & \text{if } t \in [\frac{1}{k}, \frac{2}{k}] \\ \quad \vdots & \quad \vdots \\ kH^{k-1}(p, kt-k+2) & \text{if } t \in [1 - \frac{2}{k}, 1 - \frac{1}{k}] \\ kH^k(p, kt-k+1) & \text{if } t \in [1 - \frac{1}{k}, 1] \end{cases}$$

Let ψ_t and ϕ_t be two loops of contactomorphisms generated by H_{ψ_t} and H_{ϕ_t}. The second operation is the **composition** $\psi_t \circ \phi_t$ of ψ_t and ϕ_t. In addition if ${\phi_t}^* \alpha = e^{f_t} \alpha$ then, as we will see in the next proposition, the associated Hamiltonian $H(\psi_t \circ \phi_t)$ for the composition is given by:

$$H_{\psi_t \circ \phi_t}(p,t) = H_{\psi_t}(p,t) + e^{f_t} H_{\phi_t}(\psi_t^{-1}(p), t).$$

Finally, the third operation is the **conjugation** $\Psi \circ \psi_t \circ \Psi^{-1}$ of the loop ψ_t by a contactomorphism Ψ. If Ψ satisfies that $\Psi^* \alpha = e^g \alpha$ then, as we will see in the next proposition, the associated Hamiltonian is defined as:

$$H(\Psi \circ \psi_t \circ \Psi^{-1})(p,t) = e^g H_{\psi_t}(\Psi^{-1}(p), t).$$

The next proposition contains some remarkable elementary formulas which will be used below on many occasions. Moreover, it also contains the details of the Hamiltonian's formulas explained above:

Proposition 3.2.4. *Let (M, ξ) be a contact manifold. Given two contact isotopies $\psi_t : (M, \xi) \to (M, \xi)$ and $\phi_t : (M, \xi) \to (M, \xi)$ such that $\phi_t^* \alpha = e^{f_t} \alpha$ for all $t \in [0,1]$ and a contactomorphism $\Psi : (M, \xi) \to (M, \xi)$ such that $\Psi^* \alpha = e^g \alpha$, the following properties holds:*

1. $H_{\phi_{t*} X_t} = e^{f_t} H_{X_t} \circ \phi_t^{-1}$
2. $H_{\psi_t \circ \phi_t} = H_{\psi_t} + e^{f_t} H_{\phi_t} \circ \psi_t^{-1}$

3. $H_{\phi_t^{-1}} = -e^{-f_t} H_{\phi_t} \circ \phi_t$

4. $H_{\Psi \circ \phi_t \circ \Psi^{-1}} = e^g H_{\phi_t} \circ \Psi^{-1}$

Proof.

1. By definition,

$$\begin{aligned} H_{\phi_{t*} X_t}(q) &= \alpha(\phi_{t*} X_t)(q) = (\phi_t^* \alpha)(X_t)(\phi_t^{-1}(q)) = \\ &= e^{f_t} \alpha(X_t)(\phi_t^{-1}(q)) = e^{f_t} H_{X_t}(\phi_t^{-1}(q)). \end{aligned}$$

2. Let $X_{\psi_t} \circ \psi_t = \frac{d}{dt}\psi_t$ and $X_{\phi_t} \circ \phi_t = \frac{d}{dt}\phi_t$ then, using the chain rule of partial derivatives,

$$X_{\psi_t \circ \phi_t} = \frac{d}{dt}(\psi_t \circ \phi_t) = X_{\psi_t} + \psi_{t*} X_{\phi_t}.$$

   Applying the first properties to the second summand of the right we get the result.

3. Recall that if $\phi_t^* \alpha = e^{f_t}\alpha$, then $(\phi_t^{-1})^* \alpha = e^{-f_t}\alpha$. Hence, applying the above formula

$$0 = H_{Id} = H_{\phi_t^{-1} \circ \ \phi_t} = H_{\phi_t^{-1}} + e^{-f_t} H_{\phi_t} \circ \phi_t.$$

4. It is easy to prove, just by differentiating $\Psi \circ \phi_t \circ \Psi$, that $X_{\Psi \circ \phi_t \circ \Psi} = \Psi_* X_{\phi_t}$. The result holds arguing as in the first property.

□

3.3 Legendrian submanifolds

A relevant class of submanifolds of a given contact manifold is the class of **Legendrian submanifolds**. The maximal non-integrability of the contact hyperplane field on a $(2n + 1)$-dimensional manifold implies that there are no m-dimensional submanifolds whose tangent bundle lies on the distribution, for $m > n$. However, it is in general possible to find submanifolds whose tangent spaces lie inside the contact field if $m \leq n$. They are called **isotropic submanifold**. More precisely:

Definition 3.3.1. *Let (M, ξ) be a contact manifold. A submanifold $L \subset M$ is called an* ***isotropic submanifold*** *if $T_p L \subset \xi_p$, for all $p \in L$.*

Observe that if $i : L \to M$ denotes the inclusion map and α is a contact form for ξ, then the condition for L to be an isotropic submanifold becomes $i^* \alpha \equiv 0$. It is easy to see that an isotropic submanifold of a $(2n+1)$-dimensional contact manifold have dimension less or equal to n.

Definition 3.3.2. *A* ***Legendrian submanifold*** $L \subset (M^{2n+1}, \xi)$ *is an isotropic submanifold of maximal dimension* n.

Example 3.3.3. *In particular, in a 3-dimensional contact manifold, we can find the well-known* ***Legendrian knots****, i.e. embeddedings of* $\mathbb{S}^1$ *that are always tangent to the distribution* ξ.

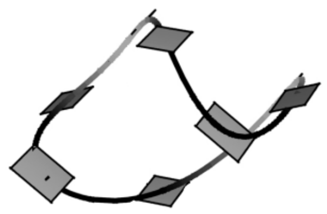

Figure 3.5: The Legendrian unknot for the standard contact structure on $\mathbb{R}^3$.

Example 3.3.4. *Given a smooth manifold* M*, consider the* 1*-jet space* $J^1(M)$ *with its canonical contact structure as we have seen in the fifth section of example 3.1.2. For every smooth function* $f : M \to \mathbb{R}$ *we can define the* ***1-jet of* f** *by* $(j_1 f)_p = (p, f_{*p}, f(p))$. *Define the subset*

$$M_f = \{(p, f_{*p}, f(p)) \text{ such that } p \in M\} \subset J^1(M),$$

then M_f *is a Legendrian submanifold.*

Definition 3.3.5. *Given a contact manifold* (M^{2n+1}, ξ)*, a* ***Legendrian embedding (immersion)*** *is an embedding (immersion)* $\phi : \Lambda \to M$ *such that its image is a Legendrian submanifold.*

Definition 3.3.6. *An* ***isotopy of Legendrian submanifolds*** *is a smooth* 1*–parametric family* $\phi_t : \Lambda \to M$ *of Legendrian embeddings with* $t \in I = [0,1]$*. That is, a smooth map* $\phi : \Lambda \times I \to M$ *such that* $\phi|_{\Lambda \times \{t\}}$ *is a Legendrian embedding for all* t*. By a* ***loop of Legendrians based at* L** *we mean an isotopy of Legendrians such that* $\phi_0(\Lambda) = \phi_1(\Lambda) = L$ *as submanifolds of* (M, ξ).

A basic fact about Legendrian submanifolds is that the isotopy extension theorem of differential topology – an isotopy of a closed submanifold extends to an isotopy of the ambient manifold – remains valid for them.

Theorem 3.3.1 (**Legendrian isotopy extension theorem** see, e.g., [**Ge2**]). *Let* ϕ_t *be a given isotopy of a closed Legendrian, then we can extend* ϕ_t *by an isotopy* ψ_t *of contactomorphisms satisfying* $\psi_t \circ \phi_0 = \phi_t$ *and* $\psi_0 = Id$.

We are now ready to give an equivalent definition to Def. 3.2.1 for Legendrian isotopies.

Definition 3.3.7. *Let α be a contact form for (M,ξ), an isotopy ϕ_t of Legendrians is called **non-negative** (resp. **positive**) if $\alpha\left(\frac{\partial\phi_t}{\partial t}(p)\right) \geq 0$ (resp. $\alpha\left(\frac{\partial\phi_t}{\partial t}(p)\right) > 0$) for all $p \in L$ and for all t.*

Clearly, again, this definition is independent of the chosen contact form compatible with the coorientation. Note that for a different parametrization $\widetilde{\phi} : \Lambda \times I \to M$, the difference of the vector fields $\frac{\partial\phi_t}{\partial t}$ and $\frac{\partial\widetilde{\phi}_t}{\partial t}$ lies in the tangent space to the Legendrian submanifold $\phi_t(\Lambda) := L_t$ at that point. Hence, this notion is also independent of the parametrization.

According to the above definition, a loop of Legendrians is called **non-negative** (resp. **positive**) provided the isotopy generating the loop is non-negative (resp. positive). Notice that to have a positive loop of Legendrians is much weaker than to have a positive loop of contactomorphisms. Any extension of a positive Legendrian loop needs be neither a loop of contactomorphisms nor is required to be positive. However, we can easily arrange the extension of a positive (resp. non-negative) loop of Legendrians to be positive (resp. non negative). This fact will be used afterwards.

Definition 3.3.8. *A loop of Legendrians ϕ_t is **contractible** if there exists a homotopy of loops of Legendrians $\phi_{t,s}$ such that $\phi_{t,1} = \phi_t$, $\phi_{t,0} = \phi_{0,1}$ and $\phi_{0,s} = \phi_{0,1} = \phi_{1,s}$.*

Denote by $\mathcal{L}eg(L)$ the space of all Legendrian submanifolds which are Legendrian isotopic to L and by $\mathcal{CD}iff(M,\xi)$ the contactomorphisms group of (M,ξ).

Remark 3.3.9. *The existence of a positive loop of a Legendrian L implies that the space $\mathcal{L}eg(L)$ is not orderable. Equivalently, the existence of a contractible positive loop of a Legendrian L implies that the space $\widetilde{\mathcal{L}eg}(L)$ is not orderable.*

Recall that, in the previous section, we define three operation on the space of loops of contactomorphisms. Analogously, we can define the same three operations on the space of loops of Legendrians:

1 <u>**Concatenation:**</u> Let $\{\phi_t^1, \cdots, \phi_t^k\}$ be k loops of Legendrians with fixed base point $L \subset M$. Note that the reparametrization of the loops given by

$$\begin{array}{rcl}
\widetilde{\phi}_t^1 &=& \phi_t^1, \\
\widetilde{\phi}_t^2 &=& \phi_t^2 \circ (\phi_0^2)^{-1} \circ \phi_1^1, \\
\widetilde{\phi}_t^3 &=& \phi_t^3 \circ (\phi_0^3)^{-1} \circ \phi_1^2 \circ (\phi_0^2)^{-1} \circ \phi_1^1, \\
\vdots & \vdots & \vdots \\
\widetilde{\phi}_t^k &=& \phi_t^k \circ (\phi_0^k)^{-1} \circ \phi_1^{k-1} \circ (\phi_0^{k-1})^{-1} \circ \cdots \circ \phi_1^1
\end{array}$$

satisfies $\widetilde{\phi}_1^j = \widetilde{\phi}_0^{j+1}$ for $1 \leq j \leq k-1$. Thus, we assume this property in the family of loops without loss of generality. The **concatenation** operation $\phi_t^1 \circledcirc \cdots \circledcirc \phi_t^k$ is defined in the loop space of $\mathcal{L}eg(L)$ as the usual concatenation of loops:

$$\phi_t^1 \circledcirc \cdots \circledcirc \phi_t^k = \begin{cases} \phi_{kt}^1 & \text{if } t \in [0, \frac{1}{k}]; \\ \phi_{kt-1}^2 & \text{if } t \in [\frac{1}{k}, \frac{2}{k}]; \\ \vdots & \vdots \\ \phi_{kt-k+2}^{k-1} & \text{if } t \in [1-\frac{2}{k}, 1-\frac{1}{k}] \\ \phi_{kt-k+1}^k & \text{if } t \in [1-\frac{1}{k}, 1] \end{cases} .$$

Fix extensions $\{\psi_t^1, \cdots, \psi_t^k\}$ and associated Hamiltonians $\{H_t^1, \cdots, H_t^k\}$, then the generating Hamiltonian of the concatenation is $H(\phi_t^1 \circledcirc \cdot \circledcirc \phi_t^k)$ is given by

$$H_{\phi_t^1 \circledcirc \cdots \circledcirc \phi_t^k}(p,t) = \begin{cases} kH^1(p, kt) & \text{if } t \in [0, \frac{1}{k}]; \\ kH^2(p, kt-1) & \text{if } t \in [\frac{1}{k}, \frac{2}{k}] \\ \vdots & \vdots \\ kH^{k-1}(p, kt-k+2) & \text{if } t \in [1-\frac{2}{k}, 1-\frac{1}{k}] \\ kH^k(p, kt-k+1) & \text{if } t \in [1-\frac{1}{k}, 1] \end{cases}$$

2 Composition:

Let ψ_t^1 and ψ_t^2 be two extensions of two loops of Legendrians with common base point L embedded in M and let H_t^1 and H_t^2 be their associated Hamiltonians. Realize that the composition of the loops $\psi_t^1 \circ \psi_t^2$ defines a loop of Legendrians given by $\phi_t = (\psi_t^1 \circ \psi_t^2)_{|\psi_0^1(L)}$. In addition, if ${\psi_t^1}^* \alpha = e^{f_t}\alpha$ then the associated Hamiltonian $H(\psi_t^1 \circ \psi_t^2)$ for the composition is given by

$$H_{\psi_t^1 \circ \psi_t^2}(p,t) = H_t^1(p,t) + e^{f_t} H_t^2((\psi_t^1)^{-1}(p), t).$$

Let us remark that this operation depends on the choice of extensions and is not canonically defined in the loop space of $\mathcal{L}eg(L)$.

3 Conjugation:

Finally, let ϕ_t be a loop of Legendrians based at L and let Ψ be a contactomorphism, then $\Psi \circ \phi_t$ is a loop of Legendrians of $\Psi(\phi_0(L))$. Now, consider the extension ψ_t of ϕ_t with the associated Hamiltonian H_t. If $\Psi^*\alpha = e^f\alpha$, then the contact isotopy $\Psi \circ \psi_t \circ \Psi^{-1}$ is an extension of the loop of Legendrians $\Psi \circ \phi_t$ and is generated by the Hamiltonian

$$H_{\Psi \circ \phi_t \circ \Psi^{-1}}(p,t) = e^{-f} H_t(\Psi^{-1}(p), t).$$

Let us remark that if Ψ preserves $\phi_0(L)$, then the conjugated loop is still a loop based at $\phi_0(L)$. Also, the conjugation of a positive (resp. non-negative) loop is positive (resp. non-negative).

Since isotopic Legendrians are isotopic through contactomorphisms, we have

Lemma 3.3.2. *If L admits a positive (contractible) loop of Legendrians through it, then any isotopic Legendrian also admits a positive (contractible) loop of Legendrians through it.*

Proof. Let $\varphi_0 : L \to (M, \xi)$ be a Legendrian embedding. Assume that there exists a positive (contractible) loop of Legendrians $\varphi_t : L \times \mathbb{S}^1 \to M$ through φ_0. Moreover, assume that there exists a Legendrian embedding $\phi_1 : L \to (M, \xi)$ isotopic to φ_0, i.e. there exists an isotopy of Legendrians $\tilde{\phi}_t : L \times [0,1] \to (M, \xi)$ such that $\tilde{\phi}_0 = \varphi_0$ and $\tilde{\phi}_1 = \phi_1$. By Theorem 3.3.1, there exists a contact isotopy $\psi_t : M \to M$ such that $\psi_t \circ \tilde{\phi}_0 = \tilde{\phi}_t$. Now, $\psi_1 \circ \varphi_t$ is a positive (contractible) loop of Legendrians embeddings based at $\psi_1 \circ \varphi_0 = \psi_1 \circ \tilde{\phi}_0 = \tilde{\phi}_1 = \phi_1$.

□

Before concluding this section, we need to introduce a result due to Y. Eliashberg and L. Polterovich [**EP**] adapted to the Legendrian case by V. Chernov and S. Nemirovski [**CN3**] which states that if a Legendrian isotopy class contains a non-constant non-negative loop of Legendrians, then it contains a positive loop. More precisely,

Lemma 3.3.3 (Prop. 4.5, [**CN3**])**.** *Let $\{\phi_t\}$ be a non-negative non-trivial Legendrian loop of closed Legendrians based at L. Then, there exists a positive loop of Legendrians $\{\phi'_t\}$ which satisfies that $\phi_0(L) = \phi'_0(L)$.*

If we assume that ϕ_t is contractible then ϕ'_t can be chosen to be contractible.

Proof. Given a smooth flow ψ_t in L, we lift it to a contact flow $\widetilde{\psi}_t$ in $\mathbb{R} \times T^*L$ which preserves the zero-section with associated Hamiltonians $\widetilde{H}_t$. Then, choosing an appropriate cut-off function, we construct a family of contactomorphisms $\widehat{\psi}_t$ with support arbitrary close to the zero-section. Moreover, $\widehat{\psi}_t$ coincides with ψ_t when restricted to L.

Now, let G_t be the associated Hamiltonian for an extension φ_t of the Legendrian loop ϕ_t. Recall that $G_t \geq 0$. We can assume that there exists a point p in the Legendrian and a time t_0 such that $G_{t_0}(p) > 0$. Hence, there exists a neighborhood U of $p \in L$ such that $G_{t_0}(q) > 0$, for all $q \in U$. As L is compact and the smooth flows of vector fields act transitively on L, there exists a finite set of flows f^i_t such that the open sets $U_1 = f^1_0(U), \cdots, U_n = f^n_0(U)$ cover L. Applying the construction above to $f^1_t, \cdots, f^n_t$, we get a family of contactomorphisms $\widehat{f}^1_t, \cdots, \widehat{f}^n_t$.

The loop $\phi^j_t = \widehat{f}^j_1 \circ \phi_t$ with extension $\varphi^j_t = \widehat{f}^j_1 \circ \phi_t \circ (\widehat{f}^j_1)^{-1}$ is positive in U_j at t_0. Therefore $\Phi_t = (\varphi^1_t \circ \cdots \circ \varphi^n_t)$ is an extension of a non–negative loop of Legendrians based at $\phi_0(L)$

that is strictly positive for $t = t_0$. Now, fix k big enough such that $H(\Phi_t)$ is positive for $t \in [t_0, t_0 + \frac{2}{2k}]$. Consider a finite open covering $(t_0, t_0 + \frac{2}{2k}), (t_0 + \frac{1}{2k}, t_0 + \frac{3}{2k}), \cdots, (t_0 - \frac{1}{2k}, t_0 + \frac{1}{2k})$ of $\mathbb{S}^1$. Then the conjugated loop $(\Phi_{-s})^{-1} \circ (\Phi_{t-s})_{|\Phi_0(L)}$ with extension $(\Phi_{-s})^{-1} \circ \Phi_{t-s} \circ \Phi_{-s}$ is positive in the interval $(t_0 + s, t_0 + s + \frac{2}{2k})$ and is based at L. Hence, the composition of this loop for $s = 0, \frac{1}{2k}, \frac{2}{2k}, \cdots, \frac{2k-1}{2k}$ is a positive loop based at L.

The proof follows with no changes in the contractible case. □

3.4 Tubular Neighborhoods

Throughout the thesis, we are going to use several types of neighborhoods theorems which we are going to explain in this section. Consider a smooth submanifold $X \subset M$ and denote by $\mathcal{N}X = \left\{(p, v) \,|\, p \in X,\, v \in \frac{T_pM}{T_pX}\right\}$, its normal bundle. Let $i : X \hookrightarrow M$ be the inclusion map, $i_0 : X \hookrightarrow \mathcal{N}X$ be the zero-section of $\mathcal{N}X$ and $\mathcal{U}_0$ a neighborhood of the zero-section in $\mathcal{N}X$ then:

Theorem 3.4.1. *There exists a neighborhood $\mathcal{U}$ of X in M and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that the following diagram:*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\psi"] & & \mathcal{U} \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd},$$

*commutes. This neighborhood is known as a **tubular neighborhood** of X.*

Notice that $\mathcal{N}X$ is also a tubular neighborhood of X. For a proof of this theorem, see Lecture 6 of [**Sil**].

Definition 3.4.1. *Let (M, ω) be a symplectic manifold, a submanifold $L \subset (M, \omega)$ is called **Lagrangian** if $\omega|_L = 0$.*

Example 3.4.2. *Consider the symplectic manifold $(T^*M, d\lambda_{can})$, then the zero-section $M \subset (T^*M, d\lambda_{can})$ is a Lagrangian submanifold.*

These submanifolds are the analogous in symplectic geometry of the Legendrian submanifolds for contact manifolds. Lagrangian submanifolds have the following well-known property:

Proposition 3.4.2. *Let $L \subset (M, \omega)$ be a Lagrangian submanifold, then the vector bundles $\mathcal{N}L$ and T^*L are canonically identified.*

Proof. Define the following nondegenerate bilinear pairing $\omega' : \mathcal{N}L = \frac{TM}{TL} \times TL$ as $\omega'([x], y) = \omega(x, y)$. Notice that it is well-defined because L is Lagrangian. The proof follows by considering the next isomorphism:

$$\begin{array}{rccc} \Omega: & \dfrac{TM}{TL} & \to & T^*L \\ & [x] & \mapsto & \omega'([x], y) \end{array}$$

□

Remark 3.4.3. *There exists analogous versions of the above Theorem for Legendrian and Lagrangian submanifolds. If L is a Lagrangian submanifold of M then T^*L is a tubular neighborhood of L. Equivalently if L is a Legendrian submanifold of $(M, \ker\alpha)$, we can always get a Lagrangian submanifold by symplectization: $\mathbb{R} \times L$ is a Lagrangian submanifold of the symplectic manifold SM. Hence, $R \times T^*L$ is a tubular neighborhood of L.*

Now, we can extend these results for the symplectic and the contact case. Concretely,

Theorem 3.4.4 (Weinstein tubular neighborhood)**.** *Let (M, ω) be a symplectic manifold and $L \subset M$ be a compact Lagrangian submanifold. Consider $i_0 : L \hookrightarrow (T^*L, d\lambda_{can})$ the Lagrangian embedding as the zero-section and $i : L \hookrightarrow M$ the Lagrangian embedding given by inclusion. Then there exists neighborhoods $\mathcal{U}$ of L in M, $\mathcal{U}_0$ of L in T^*L and a diffeomorphism $\psi : \mathcal{U}_0 \to \mathcal{U}$ such that the following diagram:*

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\psi"] & & \mathcal{U} \\ & L \arrow[ul, "i_0"] \arrow[ur, "i"'] & \end{tikzcd},$$

commutes and $\psi^*\omega = d\lambda_{can}$.

Proof. The proof follows from Theorem 3.4.1 and from Weinstein Lagrangian neighborhood theorem (See Theorem 8.4 from [**Sil**] for more details). □

Again, notice that we can get an analogous result for Legendrian submanifolds just by symplectization.

Corollary 3.4.5 (Weinstein tubular neighborhood for Legendrian)**.** *Let $(M, \xi = \ker(\alpha))$ be a contact manifold and $L \subset M$ be a compact Legendrian submanifold of (M, ξ). Then there exists a neighborhood $\mathcal{U}$ of L contactomorphic to $(J^1(L), \xi_{jet}) = (T^* \times L, \ker(dz - \lambda_{can}))$.*

3.5 Overtwisted contact manifolds

In example 3.1.2, we defined the overtwisted contact structure $\alpha_{ot} = \cos r dz + r \sin r d\varphi$ in cylindrical coordinates (r, φ, z) on $\mathbb{R}^3$. We mentioned that this contact structure was not contactomorphic to the standard one. At this point, interesting questions concerning a classification of contact structures can be posed. It turns out that there exists a well-known dichotomy of contact structures. They break into one of two types: *tight* or *overtwisted*. Overtwisted contact structures obey an h-principle and are in general easy to understand. Tight contact structures have a more subtle, geometric nature. It was first studied by Eliashberg [**El1**] for 3-dimensional manifolds and then by Borman, Eliashberg and Murphy [**BEM**] for any dimension.

Definition 3.5.1. *A contact structure ξ_{ot} in a 3-manifold M^3 is called **overtwisted** if there exists an embedded 2–disk $\mathbb{D}^2 \subset M$, known as an **overtwisted disk**, whose boundary is Legendrian and whose center is also tangent to the distribution while the rest of the disk is transverse to ξ. If ξ is not overtwisted, it is called **tight**.*

In the example 3.1.2, consider the disk $\mathbb{D}_\pi = \{z = 0, r \le \pi\} \subset \mathbb{R}^3$. Notice that $\mathbb{D}_\pi$ is tangent to $\xi_{ot} = \ker \alpha_{ot}$ along its boundary and its characteristic foliation (i.e. the intersection of its tangent space with the distribution) consists of all radial lines with singular points at the center and at the boundary. If the interior of $\mathbb{D}^2$ is pushed up slightly, the singular points at the boundary can be made to disappear. Only the singular point at the centre remains, and the characteristic foliation now looks as in Figure 3.6, where the boundary is a closed leaf of the foliation. This disk is known as the **standard overtwisted disk**.

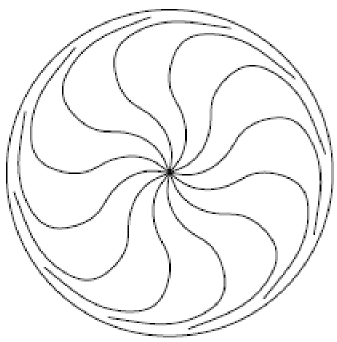

Figure 3.6: Overtwisted disk.

For $\delta > 0$ small enough, the contact domain $(U_{ot}, \alpha_{ot}) := (\mathbb{D}^2_{\pi+\delta} \times [-\delta, \delta], \alpha_{ot})$ can always be embedded in the neighborhood of an overtwisted disk. It will be called an **3-dimensional overtwisted contact germ**.

The above construction give us a notion of an overtwisted contact structure in dimension 3. Let us generalize this concept.

There exists a sequence of positive constants $R(n)$ in $\mathbb{R}^+$, whose value is computed in [**CMP**], that provides the following

Definition 3.5.2. *Let (M,ξ) be a contact manifold of dimension $2n+1>3$. (M,ξ) is called overtwisted if there exists a contact embedding $\phi_{ot}:(U_{ot}\times\mathbb{D}^{2n-2}_{R(n)},\ker(\alpha_{ot}+\lambda_{std}))\hookrightarrow(M,\xi)$, where $\lambda_{std}=\sum x_i dy_i - y_i dx_i$ is the standard Liouville form on the closed ball $\mathbb{D}^{2n-2}_{R(n)}$ and (U_{ot},α_{ot}) is a 3-dimensional overtwisted contact germ. The domain will be known as an **overtwisted contact germ** in dimension $2n+1$.*

Let $\mathcal{C}ont(M)$ be the **space of co-oriented contact structures** on M and consider the set $\mathcal{DC}ont(M)=\{(\xi,\alpha,J:\xi\to\xi)\}$ where $\xi\in\mathcal{C}ont(M)$, α is an associated contact form and J is an almost-complex structure compatible with $(\xi,d\alpha)$. This set is known as the **space of decorated contact structures**. Finally, we define the **space of formal contact structures** of M as the set of pairs $\mathcal{FC}ont(M)=\{(\xi,J)\}$, where ξ is a co-oriented distribution of rank $2n$ on M and $J:\xi\to\xi$ is an almost-complex structure. Two contact structures ξ_1 and ξ_2 are **formally equivalent** if there exists a family of formal contact structures $\{(\xi_t,J_t)\}$ that connects them. We say that a formal contact structure is overtwisted if it is genuine in some open set B and is overtwisted in B.

Fix a closed set $A\subset M$ and a contact structure ξ_A on a germ of neighborhood of A. Denote by $\mathcal{C}ont_{ot}(M,A,\xi_A)$ the space of contact structures that are overtwisted in $M\setminus A$ and coincide with ξ_A on an arbitrarily small neighborhood U_A of A. Equivalently, define $\mathcal{FC}ont_{ot}(M,A,\xi_A)$ to be the space of overtwisted formal contact structures that agree with ξ_A on U_A. Finally, denote by j the inclusion map $j:\mathcal{C}ont_{ot}(M,A,\xi_A)\to\mathcal{FC}ont_{ot}(M,A,\xi_A)$.

Theorem 3.5.1 ([**BEM**]). *If $M\setminus A$ is connected, then the inclusion map j induces an isomorphism*

$$j_0:\pi_0(\mathcal{C}ont_{ot}(M,A,\xi_A))\to\pi_0(\mathcal{FC}ont_{ot}(M,A,\xi_A)).$$

In particular, on any closed manifold M, any formal contact structure is homotopic to an overtwisted contact structure which is unique up to isotopy.

Y. Eliashberg [**El2**] also gave a complete classification of contact structures on $\mathbb{R}^3$. It turns out that there are three types of contact structures: tight, overtwisted at infinity and tight at infinity. We have already defined overtwisted and tight contact structures. Let us define the other ones.

Definition 3.5.3. *The contact manifold (M,ξ) is called **overtwisted at infinity** if for any compact subset $K\subset M$, each noncompact connected component of the contact manifold $(M\setminus K,\xi)$ is overtwisted. Otherwise, it is called **tight at infinity**.*

Y. Eliashberg [**El2**] proved that any two contact structures on $\mathbb{R}^3$ overtwisted at infinity are contactomorphic. This result can be extended, without changes in the argument, to general open manifolds of arbitrary dimension. Concretely,

Lemma 3.5.2. *Let M be an open manifold and let (M, ξ_0) and (M, ξ_1) be two contact structures overtwisted at infinity such that ξ_0 and ξ_1 are formally equivalent. Then, there exists a diffeomorphism $\Psi : M \to M$ such that $\Psi_* \xi_0 = \xi_1$.*

The proof follows, verbatim, [**El2**].

The next result will be very useful for our purposes:

Proposition 3.5.3. *Let $(M, \ker(\alpha))$ be an overtwisted contact manifold. Then $(M \times \mathbb{R}^2, \ker(\alpha + r^2 d\theta))$ is overtwisted at infinity.*

Notice that the proposition does not hold if the dimension of M is 1 since there is no notion of overtwistedness in this case. We also need the following two elementaries lemmas in order to prove the above statement:

Lemma 3.5.4. *Let $(M, \xi = \ker(\alpha))$ be a contact manifold satisfying that the Reeb vector field R_α is complete. Denote the associated flow ϕ_t^R. Choose $f : \mathbb{D}_r^2 \to \mathbb{R}$ a smooth function and $\lambda \in \Omega^1(\mathbb{R}^2)$ a primitive for $\omega_0 = dx \wedge dy$. Define on $M \times \mathbb{D}_r^2$ the contact forms $\alpha_0 = \alpha + \lambda$ and $\alpha_1 = \alpha + \lambda + df$. Then the diffeomorphism*

$$\begin{aligned} \Psi : M \times \mathbb{D}_r^2 &\to M \times \mathbb{D}_r^2 \\ (p, x, y) &\mapsto (\phi^R_{f(x,y)}(p), x, y) \end{aligned}$$

satisfies $\Psi^* \alpha_0 = \alpha_1$.

The proof follows from straightforward computations.

Lemma 3.5.5. *Given a contact manifold (M, ξ) there always exists a contact form defining ξ such that the Reeb vector field R_α associated to it is complete.*

Proof. Consider an arbitrary contact form β such that $\ker(\beta) = \xi$ and denote by R_β the Reeb vector field corresponding to β. Take also an arbitrary proper Morse function $f : M \to \mathbb{R}$. Note that as the contact manifold is open we can always find a proper smooth Morse function on it.

Now, define the sets:

$$\begin{aligned} A_n &= [2n, 2n+1] \\ B_n &= [2n+1, 2n+2] \end{aligned}$$

Notice that $\bigcup_n (A_n \cup B_n)$ is a cover for $\mathbb{R}$ whose elements are all compact. Hence, as f is a proper map, $\bigcup_n (f^{-1}(A_n) \cup f^{-1}(B_n))$ is a cover for M whose elements are also compact.

By compactness, as f is proper, there exist a minimum finite amount of time $t = \lambda_n \in \mathbb{R}^+$ such that any flow trajectory of R_β spends time higher or equal than λ_n to go from any point on $f^{-1}(2n)$ to an arbitrary point on $f^{-1}(2n+1)$. Note that we can make the same argument for the sets B_n.

Consider the contact form
$$\widetilde{\alpha} = \lambda_n \beta.$$

Then the flow of the associated Reeb vector field $R_{\widetilde{\alpha}} = \dfrac{1}{\lambda_n} R_\beta$ takes $t \geq 1$ to go from $f^{-1}(2n)$ to $f^{-1}(2n+1)$.

Now, define the map $g : M \to \mathbb{R}$ such that $g(p) = \lambda_n$ in $f^{-1}(A_n)$ for all $n \in \mathbb{Z}$ where, as above, λ_n is the minimum amount of time that it takes to go through A_n and extends it smoothly with a choice of a strictly positive smooth function, for the sets $f^{-1}(B_n)$. Suppose that the Reeb vector field R_α corresponding to the contact form $\alpha = g\beta$ is not complete, that is, its flow ϕ_t^R satisfies that $\phi_t^R(p) \xrightarrow{t \to t_0} \infty$ for some $t_0 \in \mathbb{R}^+$. As f is proper, this means that $f(\phi_t^R(p)) \xrightarrow{t \to t_0} +\infty$.

By definition of g, as the interval $[f(\phi_t^R(p)), +\infty)$ contains an infinite number of set of the type A_n, it takes an infinity amount of time to go from $f(\phi_t^R(p))$ to $+\infty$. But t_0 is finite. This contradicts the above supposition, so we are forced to conclude that R_α is complete.

□

Proof of Proposition 3.5.3. Observe that, by Lemma 3.5.5, there exists a smooth function $g : M \to \mathbb{R}$ such that $\tilde{\alpha} = e^g \alpha$ satisfies that $R_{\tilde{\alpha}}$ is complete. In fact, we have the following diffeomorphism

$$\begin{array}{rcl} \psi : M \times \mathbb{R}^2 & \to & M \times \mathbb{R}^2 \\ (p, r, \theta) & \mapsto & (p, e^{g/2} r, \theta), \end{array}$$

that clearly satisfies $\psi^*(\tilde{\alpha} + r^2 d\theta) = e^g(\alpha + r^2 d\theta)$. So $(M \times \mathbb{R}^2, \alpha + r^2 d\theta)$ is contactomorphic to $(M \times \mathbb{R}^2, \tilde{\alpha} + r^2 d\theta)$. Therefore, we can assume without loss of generality that the Reeb vector field associated to α is complete to begin with.

It is sufficient to show that for any $K > 0$, the manifold $W = (M \times \mathbb{R}^2) \setminus (M \times \mathbb{D}^2_K(0,0))$ is overtwisted. Let us prove it.

First, we realize that, since $(M, \ker(\alpha))$ is overtwisted, there exists a positive constant $R = R(n)$ such that $(M \times \mathbb{D}^2_R((0,0)), \ker(\alpha + \lambda_{std}))$ is overtwisted [**CMP**]. Now, let us consider the manifold $(M \times \mathbb{D}^2_R((0, K + 3R)), \ker(\alpha + \lambda_{std}))$ embedded in W. We apply Lemma 3.5.4 with $f(x,y) = -(k+3R)x$ to show that $(M \times \mathbb{D}^2_R, \ker(\alpha + \lambda_{std})))$ is contactomorphic to $(M \times \mathbb{D}^2_R((0, K+3R)), \ker(\alpha + \lambda_{std}))$. Hence, $(M \times \mathbb{D}^2_R((0, K+3R)), \ker(\alpha + \lambda_{std}))$ is overtwisted and thus, $M \times \mathbb{R}^2$ is overtwisted at infinity.

□

We claim

Lemma 3.5.6. *The overtwisted contact germ $(U_{ot} \times \mathbb{D}^{2n-2}_{R(n)}, \alpha_{ot} + \lambda_{std})$ contains an open ball $\mathbb{B}_{ot}$ overtwisted at infinity.*

Proof. In dimension 3, the ball $\mathbb{B}_{ot}$ can be chosen to be the interior of the whole domain U_{ot}. The reason is that the contact flow ∂_z pushes the overtwisted disk $\mathbb{D}^2_\pi \times \{0\} \subset U_{ot}$ arbitrarily close to the boundary.

In higher dimension, consider the open manifold $C_{ot} = (\mathbb{R}^3 \times \mathbb{R}^{2n-2}, \alpha_{ot} + \lambda_{std})$. Notice that it contains the contact germ $(U_{ot} \times \mathbb{D}^{2n-2}_{R(n)}, \alpha_{ot} + \lambda_{std})$. Moreover, C_{ot} admits a formal contact embedding into $(U_{ot} \times \mathbb{D}^{2n-2}_{R(n)}, \alpha_{ot} + \lambda_{std})$ since both domains are contractible and there is no topological obstruction to upgrade a smooth embedding, that of course exists, into a contact formal one. Then Corollary 1.4 of [**BEM**], changes the formal embedding into a contact one.

Recall example 3.1.2. There we review that $(\mathbb{R}^3, \ker(dz + r^2 d\theta))$ and $(\mathbb{R}^3, \ker(dz + xdy - ydx)$ are contactomorphic. In particular, $(M \times \mathbb{R}^2, \ker(\alpha + r^2 d\theta))$ and $(M \times \mathbb{R}^2, \ker(\alpha + \lambda_{std})$ are contactomorphic. Then, applying proposition 3.5.3, C_{ot} is overtwisted at infinity by induction on n.

Hence we conclude that, indeed, the overtwisted contact germ contains an open ball overtwisted at infinity.

□

Now, recall the definition of the space of decorated contact structures. Notice that the forgetful map $f : \mathcal{DC}ont(M) \to \mathcal{C}ont(M)$ has contractible fibers. Therefore it induces a homotopy equivalence and thus it has a homotopy inverse $\iota : \pi_*(\mathcal{C}ont(M)) \to \pi_*(\mathcal{DC}ont(M))$. Composing ι with the forgetful map $\pi : \pi_*(\mathcal{DC}ont) \to \pi_*(\mathcal{FC}ont)$, we get a natural map

$$j : \pi_*(\mathcal{C}ont(M)) \hookrightarrow \pi_*(\mathcal{FC}ont(M)).$$

There is a natural inclusion i given by:

$$\begin{aligned} i : \mathcal{FC}ont(M) &\hookrightarrow \mathcal{FC}ont(M \times \mathbb{R}^2) \\ (\xi, J) &\mapsto \left(\xi \oplus \mathbb{R}^2, \left(\begin{array}{c|c} J & 0 \\ \hline 0 & i \end{array} \right) \right). \end{aligned} \tag{3.1}$$

Lemma 3.5.7. *The inclusion map (3.1) induces an isomorphism*

$$i_0 : \pi_0(\mathcal{FC}ont(M)) \to \pi_0(\mathcal{FC}ont(M \times \mathbb{R}^2)),$$

if M is an open manifold.

Proof. Assume $\dim M = 2n - 1$. Notice that a formal contact structure on M is a reduction of the structure group to $1 \times U(n-1)$. Hence, considering a formal contact structure on M is equivalent to having a section of the associated bundle $SO(2n-1)/U(n-1)$. Analogously, having a formal contact structure on $M \times \mathbb{R}^2$ is equivalent to choosing a section of the associated $(SO(2n+1)/U(n))$–bundle.

By Lemma 8.1.2 from [**Ge**], the spaces $SO(2n-1)/U(n-1)$ and $SO(2n)/U(n)$ (respectively; $SO(2n+1)/U(n)$ and $SO(2n+2)/U(n+1)$) are diffeomorphic. We claim that the homotopy groups π_k of the spaces $SO(2n-1)/U(n-1)$ and $SO(2n+1)/U(n)$ are isomorphic whenever $k < 2n-1$. To check it, we write the long homotopy sequence associated to the fibration $U(n) \to SO(2n) \to SO(2n)/U(n)$. Now consider the following commutative diagram in which the vertical arrows are the morphisms associated to the natural inclusions $U(n) \to U(n+1)$ and $SO(2n) \to SO(2n+2)$ which induce isomorphisms in the π_k homotopy groups for $k < 2n-1$:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(U(n)) \arrow[r] \arrow[d] & \pi_k(SO(2n)) \arrow[r] \arrow[d] & \pi_k(SO(2n)/U(n)) \arrow[r] \arrow[d, "h"] & \pi_{k-1}(U(n)) \arrow[r] \arrow[d] & \pi_{k-1}(SO(2n)) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & \pi_k(U(n+1)) \arrow[r] & \pi_k(SO(2n+2)) \arrow[r] & \pi_k(SO(2n+2)/U(n+1)) \arrow[r] & \pi_{k-1}(U(n+1)) \arrow[r] & \pi_{k-1}(SO(2n)+2) \arrow[r] & \cdots \end{tikzcd}$$

Then, applying the *Five Lemma*, we conclude that the vertical arrow h is an isomorphism for all $k < 2n-1$.

Now obstruction theory shows that i_0 is an isomorphism if M is open, because M retracts to a $2n-2$ dimensional skeleton. $\square$

Let us remark that if M is closed, the same argument only provides the surjectivity of i_0.

Chapter 4

Loose

"It's the questions we can't answer that teach us the most. They teach us how to think. If you give a man an answer, all he gains is a little fact. But give him a question and he'll look for his own answers."

- P. Rothfuss, The Wise Man's Fear.

Just as overtwisted contact structures abide by an h-principle, there also exists a subclass of Legendrian embeddings, referred to as **loose**, which satisfy an h–principle type resolution [**Mu**]. This result, see Theorem 6.2.1 below, will be essential for our statements. In fact, loose Legendrians submanifolds are the main objects on which our constructions are based.

This chapter is then consecrated to introduce these objects and to outline Murphy's h-principle for loose Legendrian submanifolds .

4.1 Loose Legendrians submanifolds

Loose Legendrian h-principle was born in order to give an answer to the following question: When two Legendrian embeddings are isotopic through Legendrian embeddings? Just as we did for the overtwisted case, let us first define the notion of a formal Legendrian submanifold:

Definition 4.1.1. *Given a smooth n-dimensional submanifold $L \subset (M^{2n+1}, \xi)$ on a $(2n+1)$-contact manifold (M^{2n+1}, ξ), a **formal Legendrian embedding** (ϕ, Φ_s) is an embedding $\phi : L \to M$ together with a family of maps $\Phi_s : TL \to \phi^* TM$ covering ϕ such that:*

1. *Φ_s is a monomorphism for all $s \in [0,1]$ satisfying that $\Phi_0 = d\phi$ and $\Phi_1(TL) \subset \phi^* \xi$.*

2. *Furthermore $\Phi_1(TL)$ is Lagrangian with respect to the linear conformal symplectic structure on ξ.*

In this case, we will say that L or $\phi(L)$ is a ***formal Legendrian submanifold.***

Note that a Legendrian enbedding can be thought of as a formal Legendrian embedding by letting $\Phi_s = d\phi$ for all s. In particular, two Legendrian embeddings ϕ_0 and ϕ_1 are formally isotopic if there exists a smooth isotopy ϕ_t between them and a homotopy of monomorphisms $\Phi_{t,s} : TL \to \phi_t^* TM$ such that $\Phi_{t,0} = d\phi_t$, $\Phi_{0,s} = d\phi_0$, $\Phi_{1,s} = d\phi_1$ and $\Phi_{t,1}(TL) \subset \phi_t^* \xi$ is Lagrangian.

E. . Murphy [**Mu**] presented the notion of **loose Legendrian submanifolds**. Before introducing them it, we need to recall the following definition:

Definition 4.1.2. *Let $\gamma(s) = (x(s), y(s), z(s))$ be a parametrized curve in $(\mathbb{R}^3, \xi_{std})$. Then:*

1. *The* ***front projection*** *of $\gamma(s)$ is given by:*

$$\gamma_F(s) = (x(s), z(s)).$$

2. *The* ***Lagrangian projection*** *of $\gamma(s)$ is given by:*

$$\gamma_L(s) = (x(s), y(s)).$$

Remark that $(\mathbb{R}^3, \xi_{std} = \ker dz - ydx)$ can be understood as the contact manifold given by $(J^1(\mathbb{R}), \ker dz - \lambda_{can})$.

Hence, we can generalise the above projections by considering the first jet space $J^1(M)$ of a given manifold M with the usual contact structure $\xi_{jet} = \ker(dz - \lambda_{can}))$. Then:

1. The **front projection** π_F on $J^1(M)$ as the natural projection

$$\pi_F : J^1(M) \to \mathbb{R} \times M.$$

2. The **Lagrangian projection** π_L on $J^1(M)$ as the natural projection

$$\pi_F : J^1(M) \to T^* M.$$

the **front projection** π_F on $J^1(M)$ as the natural projection $\pi_F : J^1(M) \to \mathbb{R} \times L$.

Loose Legendrian submanifolds are characterized by the following local model:

Consider an open ball $\mathbb{D}^3$ of radius 1 around the origin in $(\mathbb{R}^3, \xi_{std})$ and let $L_0 \subset \mathbb{D}^3$ be a a properly embedded Legendrian curve whose front projection is like in Figure 4.1 and which is equal to $\{y = z = 0\}$ near the boundary. Consider an open polydisc $U_\Gamma(\rho) = \{|q| < \rho,\ |p| < \rho\} \subset T^*\mathbb{R}^{n-1}$ of radius ρ where $(q = (q^i)_i, p = (p_i)_i)$ are the canonical coordiantes for $T^*\mathbb{R}^{n-1}$. We will denote it by U_Γ when there is no need to specify the radius. Write $\Gamma(\rho) \subset T^*\mathbb{R}^{n-1}$ for the intersection of $U_\Gamma(\rho)$ with the zero-section $\Gamma = \{p = 0\} \subset T^*\mathbb{R}^{n-1}$ of $T^*\mathbb{R}^{n-1}$. Then, $(L_0 \times \Gamma(\rho)) \subset (\mathbb{D}^3 \times U_\Gamma(\rho), \ker(\alpha_{std} + \lambda_{std}))$ is a Legendrian submanifold. Notice that $(\mathbb{R}^3 \times T^*\mathbb{R}^{n-1}, \ker(\alpha_{std} + \lambda_{std})) = (\mathbb{R}^{2n+1}, \xi_{std})$.

img/Defn_loose.pdf

Figure 4.1: The front projection of a stabilized Legendrian arc.

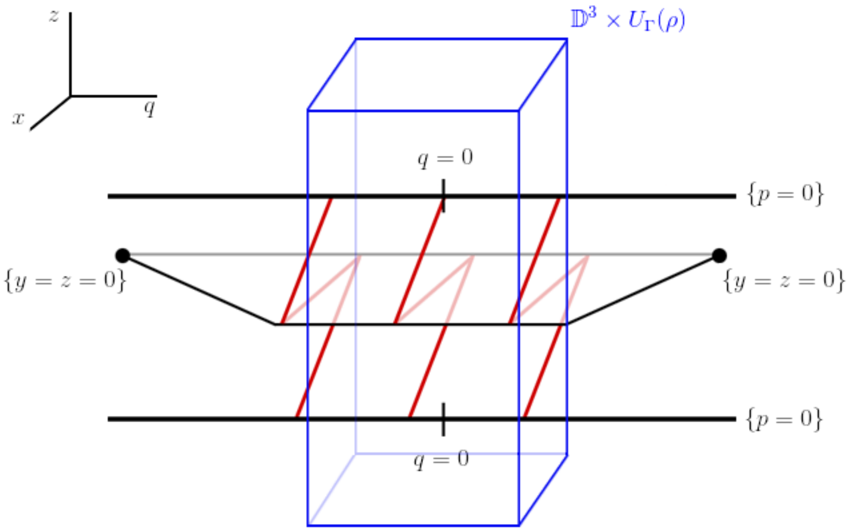

Figure 4.2: Loose chart.

Definition 4.1.3. *The pair $(L_0 \times \Gamma(\rho), \mathbb{D} \times U_\Gamma(\rho))$ together with the contact structure* $\ker(\alpha_{std} + \lambda_{std})$ *is known as a* ***loose chart*** *for $\rho > 1$.*

Definition 4.1.4. *A Legendrian submanifold $L^n \subset (M^{2n+1}, \xi)$ with $n \geq 2$ is called* ***loose*** *if there exists an open set $U \subset M$ such that $((U \cap L, U), \xi)$ is contactomorphic to a loose chart.*

Remark 4.1.5. *We will say that a* ***Legendrian embedding is loose*** *if its image is loose.*

Remark 4.1.6 (Prop. 4.4., [**Mu**])**.** *Any loose chart contains two disjointly embedded* *loose charts, and therefore a loose chart contains infinitely many disjointly embedded loose charts.*

Proof. On the one hand, for any given $\delta > 0$, denote by $m_\delta : \mathbb{R} \to (0, \infty)$ to be a smoothing of the function $x \mapsto min(x, \delta)$ such that it satisfies:

1. $m_\delta(x) = x$ for all $x > 2\delta$
2. $m_\delta(x) = \delta$ for all $x \le \frac{1}{2}\delta$
3. $\delta \le m_\delta(x) \le x$ for all $x \in (\frac{1}{2}\delta,\, 2\delta)$
4. $m_\delta'(x) \le 1$ for all $x \in \mathbb{R}$.

On the other hand, denote by hL_0 the Legendrian curve obtained by scaling L_0 using the contactomorphism on $(\mathbb{R}^3, \xi_{std})$ defined by $(x, y, z) \mapsto (hx, hy, h^2 z)$. Notice that L_0 and hL_0 are equals near the boundary, and that they are Legendrian isotopic rel. boundary.

Now, we construct a new Legendrian L inside $\mathbb{D}^3 \times U_\Gamma(\rho)$ such that L and $L_0 \times \Gamma(\rho)$ are Legendrian isotopic rel. boundary.

$$\{(q, x, z) \text{ such that } (x, z) \in m_\delta(|q| - 1 - \rho)L_0\}\,.$$

Notice that $|p| = m_\delta'(|q| - 1 - \rho) \le 1$, therefore $L \subset \mathbb{D}^3 \times U_\Gamma(\rho)$ and that L and $L_0 \times \Gamma(\rho)$ are equals near the boundary and their front are smoothly ambient isotopic rel. boundary. Hence, they are Legendrian isotopic rel. boundary.

Thus, there is a compactly supported contact isotopy taking $L_0 \times \gamma(\rho)$ to L, and therefore it suffices to find two disjoint loose charts in $(\mathbb{D}^3 \times U_\Gamma(\rho), L)$. See figure 4.3.

Next, define $\widetilde{\rho} = \rho - 1 - \frac{1}{2}\delta$ for a small $\delta > 0$. On $\mathbb{D}^3 \times U_\Gamma(\widetilde{\rho})$, we have that $L = \delta L_0 \times \Gamma(\widetilde{\rho})$. Notice that L_0 and δL_0 are equals near the boundary, and that they are Legendrian isotopic rel. boundary. Scaling using the contactomorphism $(q, p, x, y, z) \mapsto (\frac{1}{\delta}q, \frac{1}{\delta}p, \frac{1}{\delta}x, \frac{1}{\delta}y, \frac{1}{\delta^2}z)$ we see that $(L, \mathbb{D}^{2n+1}_{\widetilde{\rho}})$ and $(L_0 \times \Gamma(\frac{\widetilde{\rho}}{\delta}), \mathbb{D} \times U_\Gamma(\frac{\widetilde{\rho}}{\delta}))$ are contactomorphic. Note that choosing δ to be small enough we can make $\frac{\widetilde{\rho}}{\delta} > 1$.

□

The corresponding h–principle can be stated as follows.

Theorem 4.1.1 ([**Mu**]). *Suppose $n > 1$ and let $\phi : L \to (M^{2n+1}, \xi)$ be a formal Legendrian embedding. Then there exists a loose Legendrian embedding $\widetilde{\phi} : L \to (M^{2n+1}, \xi)$ such that they are formally isotopic. Moreover, given two formally isotopic loose Legendrians embeddings $\phi_0, \phi_1 : L \to (M^{2n+1}, \xi)$, they are isotopic through loose Legendrian embeddings.*

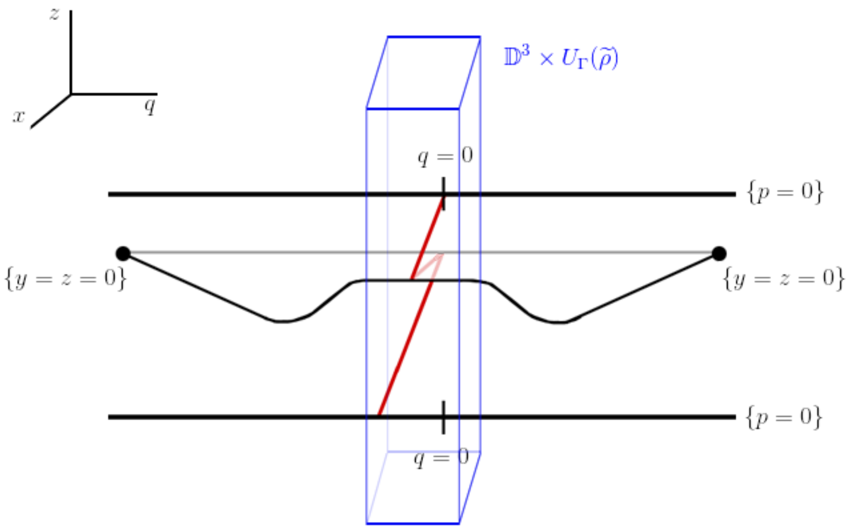

Figure 4.3: New loose chart constructed in remark 4.1.6 for the new Legendrian L.

This statement give us a complete classification of loose Legendrian embeddings up to isotopy. However, Murphy also states a more general theorem which gives us an understanding of parametric families of loose Legendrians embeddings.

For a contact manifold (M^{2n+1}, ξ), fix a smooth manifold $L^n \subset M$, an open disk $\mathbb{D}^n \subseteq L$, an open set $U \subset M$, a contactomorphism between U and $\mathbb{B}^{2n+1}_{std}$ and a embedding $\varphi : \mathbb{D}^n \to \mathbb{B}^{2n+1}_{std}$, denote by $\mathcal{L}eg^{form}_l(L, U)$ the space of all formal Legendrians embeddings (f, F_s) such that $f^{-1}(U) = \mathbb{D}^n$, (f, F_s) is a genuine Legendrian embedding on $\mathbb{D}^n$ and $f_{|\mathbb{D}^n} = \varphi$ with respect to the chosen contactomorphism, then

Theorem 4.1.2 ([**Mu**])**.** *Suppose $n > 1$ and $k > 0$ fixed. Let $(\phi_t, \Phi_{s,t})$ be a smooth family in $\mathcal{L}eg^{form}_l(L, U)$ for all $t \in \mathbb{D}^k$ such that it is genuine for all $t \in \delta\mathbb{D}^k$. Then the family $(\phi_t, \Phi_{s,t})$ is isotopic thought formal Legendrian embeddings rel $\delta\mathbb{D}^k$ to a family of genuine Legendrian embeddings.*

Note that, on the one hand, the second part of Theorem 4.1.1 is a corollary of Theorem 4.1.2 when $k = 1$. There is one difficulty which is that in Theorem 4.1.1 there's no reason why both loose Legendrian embeddings have to share the same loose chart and, therefore, we cannot directly apply Theorem 4.1.2. What we have to do is, first, use Darboux Theorem to find a contact isotopy between both charts and use it to construct a new isotopy ψ_t relative to $\partial\mathbb{D}^1$ such that, now, $\psi_0 = \phi_0$ and $\psi_1 = \phi_1$ have contactomorphic loose charts. Then, since the space of formal Legendrian embeddings is a Serre fibration over the set of smooth embeddings, for each $t \in \mathbb{D}^1$, ψ_t can be realized as a formal Legendrian embedding $(\psi_t, \Psi_{s,t})$. Notice that $(\psi_t, \Psi_{s,t})$ is a formal Legendrian isotopy between ϕ_0 and ϕ_1 and it satisfies the hypothesis of Theorem 4.1.2.

On the other hand, the first part (surjectivity) of Theorem 4.1.1 is an analog statement for $k = 0$. The proof presented here is a bit more sophisticated. It will fairly follow the proof of Theorem 4.1.2 which we are going to explain in the next sections. Concretely, first, notice we can assume there exists an open set U such that ϕ is Legendrian on $\phi^{-1}(U)$ and $(\phi(L) \cap U, U)$ is a loose chart. Then, we will replace ϕ by a wrinkled Legendrian embedding (i.e. a Legendrian embedding with singularities, [see definition 4.2.1 below for a specific description]) and use the loose chart to approximate it to a Legendrian embedding as we will explain in Section 4.3.

Henceforth, the rest of the chapter is devoted to prove Theorem 4.1.2. Its proof is composed by three steps.

1. First of all, we use a result due to Eliashberg and Cieliebak to find a contactomorphism between the neighborhood of any formal Legendrian embedding and an open set in $J^1(L)$, after a possible formal Legendrian isotopy. This will define a global model in $J^1(L)$ for any formal Legendrian embedding to work with.

   Proposition 4.1.3 (Proposition 2.1 [**Mu**]). *Consider a formal Legendrian embedding $f : L \to (M, \xi)$, covered by maps $F_s : TL \to TM$. Then, after a smooth isotopy from f to $\widetilde{f}$ there exist an open set $U \subseteq M$ containing $\widetilde{f}(L)$ and a map $\varphi : U \to J^1(L)$ which is a contactomorphism onto its image, such that $\pi_F \circ \varphi \circ \widetilde{f} = Id_{|L}$, where by π_F we denote the front projection.*

   Moreover, we have the same result parametrically. Consider the family $(f_t, F_{s,t})$ of formal Legendrian embeddings for $t \in \mathbb{D}^k$. Then there exist an isotopic family of maps $\widetilde{f}_t$, an open set $U \subseteq M \times \mathbb{D}^k$ and a smooth family of maps $\varphi_t : U \cap (M \times \{t\}) \to J^1(L)$ such that each of them is a contactomorphism onto its image and $\pi_F \circ \varphi_t \circ \widetilde{f}_t$ is the identity for all t. Additionally, we can make this construction relative to a closed set $A \subseteq L \times \mathbb{D}^k$ where f_t is Legendrian: we consider $\widetilde{f}$ such that $\widetilde{f}_t = f_t$ on A and φ_t maps $\widetilde{f}_t(A \cap \{t\})$ to the zero section in $J^1(L)$.

   A proof of it can be found in [**Mu**].

2. In Section 4.2 we introduce the concept of wrinkled embeddings. We will, then, use the above result to approximate any family of formal Legendrian embeddings to a family of wrinkled Legendrians embeddings.

3. Wrinkled Legendrians embeddings are Legendrian embeddings with singularities.

Therefore, we just need to resolve these singularities in order to end the proof. Here, we use the loose charts to approximate the wrinkled singularities. This is the content of Section 4.3.

4.2 Wrinkles

As we have said above, in this section we are going to define wrinkled embeddings which will serve us as a tool in order to approximate our formal Legendrian embeddings to genuine ones except in a finite collection of singularities. For that purpose, let us start by describing the models singularities which are going to appear.

Consider the contact manifold $(\mathbb{R}^3, \xi_{std} = \ker dz - ydx)$. Define the Legendrian curve $\psi : \mathbb{R} \to \mathbb{R}^3$ given by:

$$\psi(u) = (\psi^x(u), \psi^y(u), \psi^z(u)) = \left(u^3 - u, \frac{15}{4}\left(u^2 - \frac{1}{3}\right), \frac{9}{4}u^5 - \frac{5}{2}u^3 + \frac{5}{4}u\right).$$

By computing its derivative:

$$\psi'(u) = \left(\psi^{x'}(u), \psi^{y'}(u), \psi^{z'}(u)\right) = \left(3\left(u^2 - \frac{1}{3}\right), \frac{15}{2}u, \frac{45}{4}\left(u^2 - \frac{1}{3}\right)^2\right),$$

it is easy to check that this curve satisfies the Legendrian condition:

$$\psi^{z'}(u) - \psi^y(u)\psi^{x'}(u) = 0.$$

Consider the front projection of ψ:

$$\psi^F(u) = (\psi^x(u), \psi^z(u)).$$

It is singular at $u = \pm\frac{1}{\sqrt{3}}$. See Figure 4.1 for its graph.

Consider now the rescaling ψ^F_δ of ψ^F given by

$$\psi^F_\delta(u) = (\psi^x_\delta(u), \psi^z_\delta(u)) = \left(\delta^{\frac{3}{2}}\psi^x\left(\frac{u}{\sqrt{\delta}}\right), \delta^{\frac{5}{2}}\psi^z\left(\frac{u}{\sqrt{\delta}}\right)\right) = \left(u^3 - \delta u, \frac{9}{4}u^5 - \frac{5\delta}{2}u^3 + \frac{5\delta^2}{4}u\right).$$

Now the singular points are $u = \pm\frac{\sqrt{3\delta}}{3}$ for each δ. Therefore, ψ^F_δ is a smooth graphical curve when $\delta < 0$. Notice also that, by using a cut-off function, we can assume ψ^F_δ to be compactly supported.

Finally, consider the map $\mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by

$$(x,u) \mapsto (x, \psi^F_{x_{n-1}}(u)), \quad \text{where } x = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}. \tag{4.1}$$

The singular points for this map are defined by the set $\{3u^2 = x_{n-1}\}$. We find two types of singularities in this set:

1. When $x_{n-1} > 0$, we have a **cusp singularity**. This singularity is diffeomorphic to the existing singularity for ψ^F times $\mathbb{R}^{n-1}$.
2. The second type of singularity is a codimension 2 singularity which occurs when $x_{n-1} = u = 0$. It is called an **unfurled swallowtail**.

Figure 4.4 shows the image of the above map and, therefore, both singularities in the case $n = 2$.

img/SingWrinkle.JPG

Figure 4.4: Image of a cusp singularity and unfurled swallowtail singularity.

Now, we are in position to define the wrinkle map. A **wrinkle map** is a map $w : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $w(x,u) = (x, \psi^F_{1-\|x\|^2}(u))$, where $x \in \mathbb{R}^{n-1}$. The wrinkle map is singular on the sphere $\{\|x\|^2 + 3u^2 = 1\}$. Again, it has two kinds of singularities:

1. Unfurled swallowtails singularities along the equator $\{u = 0\}$.
2. Cusp singularities along the complement, that is on the upper $\{u > 0\}$ and lower $\{u < 0\}$ hemispheres.

Without loss of generality we will also refer to these spheres and their images as **wrinkles**. For parametric families of wrinkles there exists also another type of singularities. Define the family of wrinkle maps given by $w_t(x,u) = (x, \psi^F_{t-\|x\|^2}(u))$. An **embryo** singularity occurs when $x = u = t = 0$. It is a codimension 1 singularity in time which is isolated in space and allows wrinkles to appear or disappear in a homotopy of wrinkles.

This was the last model singularity we needed to describe. Thus, now, we are able to give the main notion of this section:

Definition 4.2.1. *A smooth map $f : M^n \to N^{n+1}$ is a **wrinkled embedding** if it is a topological smooth embedding except on some finite collection of codimension 1 spheres $\mathbb{S}^{n-1}_j \subseteq M$ which bound n-dimensional disks $\mathbb{D}^n_j \subseteq M$ and, near each of them, the map f is required to be modeled by a wrinkle map.*

We should remark that as we said above, in parametric families, wrinkled embeddings are allows to have embryo singularities, but no singularities of higher codimension.

Notice also that, as f_* is smooth at the cusps and at the unfurled swallowtails, given a wrinkled embedding $f : M^n \to N^{n+1}$ the map $Gf_* : M \to Gr_{n+1,n}$ taking a point in M to a tangent n-plane $f_*(T_pM)$ in N is smooth and well defined everywhere despite the singularities. Now, we are in condition to state the following:

Theorem 4.2.1 ([**EM2**]). *Let $f_t : M^n \to N^{n+1}$ be a parametric family of smooth embeddings with $t \in \mathbb{D}^k$. Consider a smooth homotopy $G_t^s : M \to Gr_n(N)$ with $s \in [0,1]$ of maps covering f_t such that $G_t^0 = Gf_{t*}$. Then there is a family of wrinkled embeddings $F_t^s : M \to N$ so that:*

1. *$F_t^0 = f_0$,*
2. *F_t^s is C^0-close to f_t for all $s \in [0,1]$,*
3. *$GF_{t\,*}^s$ is C^0-close to G_t^s (with respect to a given local trivialization of $Gr_n(N)$).*

Moreover, this statement also works relative to a close set $A \subseteq M \times \mathbb{D}^k$ where the homotopy G_t^s is constant in s.

This result will serve us to approximate any family of formal Legendrian embeddings to a family of wrinkled Legendrians embeddings. first, we need to introduce this concept.

Given a contact manifold $\mathbb{R} \times L$, the space of non-vertical planes is equivalent to the 1-jet space $J^1(L)$ and, moreover, considering the canonical contact structure on $J^1(L)$, the subset $\{(p, \sigma_p, h_p)\} \subset J^1(L)$ for some sections $\sigma : L \to T^*L$ and $h : L \to \mathbb{R}$, then Λ is Legendrian if and only if $\sigma = G((id \times h)_*)$ for every $p \in L$.

Now, recall the map $\phi^F : \mathbb{R} \to \mathbb{R}^2$. We can recover the value of $\psi^y(u)$ by applying the Legendrian condition: $\psi^y(u) = dz/dx$. This computation also works in the cusps singularities and we can repeat it for any plane-curve considering it as the front projection of a curve in $\mathbb{R}^3$. Then, the lifted curve ψ from ψ^F is in fact an embedding.

This situation is not valid if we are in a unfurled swallowtail or in an embryo singularity. It can be easily check by computing the lift from formula 4.1 over the set $\{x_{n-1} = u = 0\}$. Because they are built on the same singularity, this holds for wrinkles as well as embryo singularities.

Because of this, we cannot apply theorem 4.2.1 to prove an approximation result about Legendrians, as not all wrinkles embedding arises as the front projection of any smooth Legendrian.

Definition 4.2.2. *Consider the smooth manifolds L and a contact manifold (M, ξ). A topological embedding $f : L \to (M, \xi)$ is called a **wrinkled Legendrian embedding** if it satisfies the following properties:*

1. $\mathrm{Im}(f_*) \subseteq \xi$

2. $\mathrm{Rank}(f_{*|f(L)\setminus A}) = n$ *for any codimension* 2 *subset* $A \subset f(L)$ *diffeomorphic to a disjoint union of spheres* $\{\mathbb{S}^{n-2}_j\}$ *called* ***Legendrian wrinkles****.*

3. *For each sphere* $\mathbb{S}^{n-2}_j$*, there exists a Darboux chart* U_j *containing it such that* $L \cap \mathcal{U}_j$ *is diffeomorphic to* $\mathbb{R}^n$ *and the front projection* $\pi_j \circ f : L \cap U_j \to \mathbb{R}^{n+1}$ *of* f *is a wrinkled embedding (smooth outside of a compact set).*

Remark 4.2.3. *In summary, a wrinkled Legendrian embedding is a smooth Legendrian embedding outside a codimension* 2 *set. This singular set is nothing more than a collection of spheres such that each of them comes from an unfurled swallowtail singular set of the front projection of a wrinkle. Moreover, we also require a global trivialization of these spheres given by non-necessary disjoint Darboux charts* U_j *in a way that different choices of* U_j *are considered to be different as wrinkled Legendrian embeddings.*

For parametric families of wrinkled Legendrians we also allow ***Legendrian embryos****; which are Legendrian lifts from the front projection of embryo singularities.*

Hence, in order to give an approximation theorem for wrinkles Legendrian we need to define a map from wrinkled Legendrian embeddings to formal Legendrian embeddings. As wrinkled Legendrian embeddings are already Legendrian outside the singular set, we only need to give a correspondence between them on the Darboux charts.

Given a Darboux chart U_j and considering our map $f : \mathbb{R}^n \to U_j$, perturb its front projection f^F to a smooth embedding $\widetilde{f^F} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, just rounding out all the singularities. Then, lifting it by fixing the y-coordinates, we get a C^1-closed smooth embedding $\widetilde{f}$ which is not Legendrian but such that its differential f_* is homotopic to a Legendrian bundle map, see [**Mu**] for more details.

This construction defines a map from wrinkled Legendrian embeddings to formal Legendrian embeddings as we wanted. While the explicit map depends on many choices, the map is canonical up to homotopy, and in particular can be made continuous in any k–parametric family.

Finally, we are able to states the next result:

Proposition 4.2.2 ([**MU**])**.** *Every parametric family of formal Legendrian embeddings* $(f_t : L \to (M,\xi), F_{s,t} : TL \to TM)$ *with* $t \in \mathbb{D}^k$ *is homotopic through formal Legendrian embeddings to a family* $\overline{f}_t$ *of wrinkled Legendrian embeddings. If the family of formal Legendrian embeddings is already a wrinkled Legendrian embedding on a closed subset* $A \subseteq L \times \mathbb{D}^k$*, we can make this construction relative to the subset A, jut by taking* $\overline{f}_t = f_t$ *on this subset.*

The proof follows from Theorem 4.2.1 together with Proposition 4.1.3.

4.3 Surgery of Singularities

Proposition 4.2.2 give us a statement pretty similar to Theorem 6.2.1. To prove it, we only need to know how to get rid of the singularities of the wrinkled Legendrians embeddings. To do that, we are going to use loose Legendrians to perform a kind of surgery on the singularities.

Definition 4.3.1. *Consider a wrinkled Legendrian embedding $f : L \to (M, \xi)$. A compact codimension 1 embedded submanifold $\Phi \subseteq L$ is a* ***marking for* f** *if:*

1. *Its boundary is a disjoint union of spheres which are mapped via f to a subset of the Legendrian wrinkles.*
2. *$\Phi = \{u = 0, \|x\| \geq 1\}$ in the Darboux neighborhoods modelled by a wrinkle in the front projection.*
3. *The interior of Φ is disjoint from the singular set of f.*

In parametric families $f_T : L \to (M, \xi)$ we need to add two new requirements:

1. *The family $\Phi_t \subseteq L$ has to vary smoothly in t whenever we are disjoint from the set of $t \in \mathbb{D}^k$ corresponding the embryo singularities.*
2. *$\Phi_s = \left\{u = 0, \|x\|^2 \geq s\right\}$ at a Darboux neighborhood modelling an embryo singularity in the front projection where $s \in (-\varepsilon, \varepsilon)$ is a local coordinate in $\mathbb{D}^k$ transverse to the embryo singular set.*

Intuitively, the idea of a marking is giving us a place where we can desingularize wrinkles. Remember the unfurled swallowtail described above (recall Figure 4.4). The singular set of f can be thought of as the singularity that occurs when the zigzags in the front projection are pulled tight, into a smooth graphical curve. If we just put some "tiny zigzags" on the marking Φ in a consistent way, the resulting front will only have cusp singularities and therefore its lift will be smooth everywhere on a neighborhood of Φ and will be equal to f outside of that neighborhood. For a precise description of this process you can see [**Mu**]. We can also follow the same reasoning parametrically since, by definition, Φ_t is required to behave as our model embryo at embryo singularities. Hence, summarizing, we have proven the following

Lemma 4.3.1. *Given any family $f_t : L \to (M, \xi)$ of wrinkled Legendrian embeddings with $t \in \mathbb{D}^k$ and a family of markings Φ_t for them, the family f_t is C^0-close to a family of wrinkle Legendrian embeddings $\widetilde{f_t} : L \to (M, \xi)$ such that $\widetilde{f_t}$ is smooth on a neighborhood of Φ_t and equal to f_t outside of that neighborhood.*

Remark 4.3.2. *The above construction motivates our definition of loose Legendrians: intuitively they are smooth Legendrians which look like resolutions of some wrinkled Legendrian embeddings along some marking.*

Proof of Theorem 4.1.2. As we have said before, our first step must be to apply Theorem 4.2.2 to approximate our family of formal Legendrian embeddings by a family of wrinkled Legendrian embeddings, which is equal to the original genuine Legendrian family on $(L \times \partial\mathbb{D}^k) \cup (U \times \mathbb{D}^k)$ where U will be the open set defining the fixed loose chart. By slight abuse of notation let us, again, denote this family by $f_t : L \to M$ and denote by K the number of smoothly embedded codimension 1 submanifolds of $\mathbb{D}^k$ where embryo singularities appear.

Recall that, by assumption:

1. $f_t^{-1}(U) = \mathbb{D}^n$ is a fixed disk,
2. $f_{t|\mathbb{D}^n}$ is constant in t
3. $(U, f_t(\mathbb{D}^n))$ is a fixed loose chart.

Then, by Remark 4.1.6, we can choose disjoint open sets U_i with $i = 1, \cdots, K$ such that each $f_t(L) \cap U_i$ is a loose chart.

Now define the **inside-out wrinkle** by the map $\overline{w} : \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by $\overline{w}(x, u) = (x, \psi_{\|x\|^2-1}(u))$.

Analogously to the wrinkle map case, the inside-out wrinkle map is singular on the hyperbola $\{\|x\|^2 - 3u^2 = 1\}$ and has two kinds of singularities:

1. Unfurled swallowtails singularities along the subset $\{u = 0, \|x\|^2 = 1\}$.
2. Cusp singularities along the complement.

Let $f^w : \mathbb{R}^n \to J^1(\mathbb{R}^n)$ be a wrinkled Legendrian embedding whose front projection is $\overline{w}$, then $\Phi = \{u = 0, \|x\| \le 1\}$ is a marking for it. Moreover, let $\widetilde{f^w}$ be the resolution of f^w along Φ, then $\widetilde{f^w}(\mathbb{R}^n) \cap \mathbb{B}^{2n+1}(\rho)$ is contactomorphic to a loose chart for any $\rho > 1$.

Now, let us define a family of wrinkled Legendrian embeddings by replacing the loose chart on $f_t^{-1}(U_i)$ with an inside-out wrinkle with the same boundary conditions and constant in t. Denote this family by g_t. For each i, define the markings Φ_t^i of g_t satisfying the following assumptions:

1. Φ_t^i is disjoint from all other components of the singular set.
2. Φ_t^i is either diffeomorphic to cylinder $\mathbb{S}^{n-1} \times [0,1]$ or a disk $\mathbb{D}^{n-1}$ for all $t \in \mathbb{D}^k$.
3. Φ_t^i is a disk modelled as above contained in $g_t^{-1}(U_i)$ near $\partial\mathbb{D}^k$.
4. for the ith connected component of the embryo set, these embryos are all contained in Φ_t^i
5. The boundary of Φ_t^i is exactly the sphere of Legendrian wrinkles created by this embryo set other than the component in U_i.

Applying Lemma 4.3.1 for each Φ_t^i one at a time, we get a family $\widetilde{g_t}$ of smooth Legendrians. Moreover, it is isotopic to f_t via an isotopy supported in the union of the U_i with

$t \in \partial\mathbb{D}^k$. Thus we have a collar isotopy constant in t between $\{f_t\}_{t\in\partial\mathbb{D}^k}$ and $\{\widetilde{g_t}\}_{t\in\mathbb{D}^k}$. Hence, considering the union of this collar with $\{\widetilde{g_t}\}_{t\in\mathbb{D}^k}$, we get a family of genuine Legendrians extending the family $\{f_t\}_{t\in\partial\mathbb{D}^k}$ over the disk.

To conclude the proof, we just need to o show that this extension is formally Legendrian isotopic through $\mathbb{D}^k$ families relative to $\partial\mathbb{D}^k$ to the original f_t.

Suppose that Φ_t^i is diffeomorphic to a cylinder for a fixed t and consider a neighborhood in M which is the union of three open sets: the neighborhood of Φ_t in the proof of Lemma 4.3.1, the loose chart U_i containing an inside-out wrinkle and the Darboux chart containing the wrinkle on the other boundary component of Φ_t^i.

Notice that the front projection of g_t in this neighborhood has the model singularities at the wrinkle and the inside-out wrinkle, but it is smooth elsewhere, since g_t is equal to the zero section near the interior of Φ_t^i. Hence, if we resolve g_t along the small disk marking of U_i, we get the original wrinkled Legendrian embedding f_t. In our front projection, we have a $\mathbb{D}^{n-1}$ family of zig-zags in the loose chart, and a disjoint wrinkle. This is formally Legendrian isotopic to a loose chart, since by definition a wrinkle is formally Legendrian isotopic to the zero section. On the other hand if we resolve g_t along Φ_t^i we obtain $\widetilde{g_t}$. In our front projection here, we still just have a $\mathbb{D}^{n-1}$ family of zig-zags which is and a smooth front everywhere else; therefore this is formally Legendrian isotopic to a loose chart as well. Therefore f_t is formally Legendrian isotopic to $\widetilde{g_t}$.

Since this formal Legendrian isotopy is canonical up to contractible choices, it is continuous in t. We also observe that resolving g_t along Φ_t either when Φ_t^i contains an embryo or when Φ_t^i is a disk containing no singularities besides the inside-out wrinkle, the result is the same: a front which is smooth except for a disk of zig-zags. $\square$

Chapter 5

Orderability

"You can be anything you want to be, just turn yourself into anything you think that you could ever be."
- Freddie Mercury, Innuendo.

In [**EP**] a natural partial order on the universal cover of the groups of contactomorphisms was introduced which leads to a new viewpoint on geometry and dynamics of contactomorphisms. Concretely, the authors studied them as an analog of the Hofer metric (a detailed description of this metric can be found in [**P2**]) for the group of Hamiltonian symplectomorphisms on closed symplectic manifolds. Moreover, the existence of a partial order on the universal cover of the groups of contactomorphisms can be viewed as the algebraic counterpart of the (non)-squeezing problem for contact domains (see [**EKP**]) and gives rise to new numerical invariants of contactomorphisms.

5.1 The normal cone

The main results and concepts of this chapter drinks from different notions from the theory of partially ordered groups to the universal cover of the groups of contactomorphisms. So we start with the following basic definitions and constructions.

Definition 5.1.1. *Let $\mathcal{G}$ be a group, a* ***normal cone*** *is a subset $\mathcal{C} \subset \mathcal{G}$ such that:*

1. $f, g \in \mathcal{C} \Rightarrow fg \in \mathcal{C}$
2. $f \in \mathcal{C}, h \in \mathcal{D} \Rightarrow hfh^{-1} \in \mathcal{C}$
3. $1 \in \mathcal{C}$.

Given a normal cone $\mathcal{C} \subset \mathcal{G}$, one can define the relation $\geq$ on $\mathcal{D}$ by $f \geq g$ if and only if $fg^{-1} \in \mathcal{C}$. It is not hard to check that this relation is *reflexive* and *transitive* but it does not have to be *anti-symmetric*. If it is also *anti-symmetric* then we have a partial order on $\mathcal{D}$. In fact, it is known as a **bi-invariant partial order** induced by $\mathcal{C}$. Bi-invariant partial orders have many numerical invariants. Let us define some of them.

Remark 5.1.2. *Notice that the normality of the cone $\mathcal{C}$ implies that for every $f, g, d, e \in \mathcal{D}$,*

$$\text{If } f \geq g \text{ and } d \geq e \text{ then } fd \geq ge.$$

Definition 5.1.3. *Consider a bi-invariant partial order on $\mathcal{D}$, an element $f \in \mathcal{C}\backslash\{1\}$ is called **dominant** if for every $g \in \mathcal{D}$ there exists a number $p \in \mathbb{N}$ such that $f^p \geq g$.*

Definition 5.1.4. *For a dominant f and any $g \in \mathcal{D}$, **the relative growth of f with respect to** g is*

$$\gamma(f, g) = \lim_{k \to \infty} \frac{\gamma_k(f, g)}{k},$$

where

$$\gamma_k(f, g) = \mathit{Inf}\left\{p \in \mathbb{Z} \mid f^p \geq g^k\right\}, \text{ with } k \in \mathbb{N}.$$

Remark 5.1.5. *The number $\gamma_k = \gamma_k(f, g)$ is always finite and the above limit always exists.*

Proof. On the one hand, given $k \in \mathbb{N}$, let us suppose that $f^p \geq g^k$ and choose $q \in \mathbb{N}$ such that $f^q \geq g^{-1}$. Then, $f^{-p} \leq g^{-k}$ and so, as $f^q \geq g^{-1}$, we have that $f^{kq} \geq g^{-k} \geq f^{-p}$ and $p \geq -kq$. Hence γ_k is finite as, by definition, $\gamma_k \geq -kq$.

On the other hand, by definition, $f^{\gamma_n} \geq g^n$, $f^{\gamma_m} \geq g^m$ and, therefore, $f^{\gamma_n + \gamma_m} \geq g^{n+m}$. Hence, we conclude that the sequence γ_k is subadditive. Consequently, the sequence $u_k = \gamma_k + kq$ is also subadditive. In addition it is non-negative as we have shown above. Then, by Fekete's subadditive lemma, the limit $\lim_{k \to \infty} \frac{u_k}{k}$ exists and then the limit $\gamma(f, g)$. $\square$

Notice that $\gamma(f, g)$ can be positive, negative or equal to 0. Finally, it is also easy to verify (check [**EP**] for a proof) that

Proposition 5.1.1. *If f and g are both dominants then $\gamma(g, f)$ is also defined and*

$$\gamma(f, g)\gamma(g, f) \geq 1.$$

5.2 Orderability on the group of contactomorphisms

Fix a contact form α for ξ and let (M, ξ) be a co-orientable closed connected contact manifold. Denote by $\mathcal{CD}iff(M, \xi)$ the **group of contactomorphisms of** (M, ξ) and by $\mathcal{CD}iff_0(M,\xi)$ the identity component of the previous group, i.e. **the group of contactomorphisms isotopic to the identity contactomorphism Id**.

Let $\mathcal{P}Cont_0(M,\xi)$ be the set of all smoothly parametrized paths $\{\phi_t\}_{t\in[0,1]}$ with $\phi_0 = Id$. The **universal cover** $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ of $\mathcal{CD}iff_0(M,\xi)$ is then $\mathcal{P}Cont_0(M,\xi)/\sim$ where $\sim$ denotes the equivalence relation of being homotopic with fixed endpoints, i.e. $\phi_t \sim \psi_t$ if and only if $\phi_1 = \psi_1$ and we can connect ϕ_t and ψ_t via a smooth family $\phi^s = \{\phi_t^s\}_{t,s\in[0,1]}$ such that $\phi_t^0 = \phi_t$, $\phi_t^1 = \psi_t$ and ϕ_1^s is independent of s.

As we have seen before, a contactomorphism ψ can be understood as the time-one map of a contact isotopy ψ_t. By slight abuse of notation, let us denote by $\widetilde{\psi_t}$ the class $[\psi_t]$ of all paths in $\mathcal{P}Cont_0(M,\xi)$ homotopic with fixed endpoints to $\psi_t \in \mathcal{P}Cont_0(M,\xi)$, then we can define the covering map Θ by:

$$\Theta : \widetilde{\mathcal{CD}iff}_0(M,\xi) \to \mathcal{CD}iff_0(M,\xi)$$
$$\widetilde{\psi_t} \mapsto \psi$$

Remark 5.2.1. *In effect, the sets $\mathcal{CD}iff_0(M,\xi)$ and $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ are groups with respect to the composition. Moreover, the covering map Θ is a group homomorphism.*

Consequently, we can define a normal cone on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$.

Proposition 5.2.1. *Let $\widetilde{\mathcal{C}}(M,\xi)$ be the set of $\widetilde{\psi_t} \in \widetilde{\mathcal{CD}iff}_0(M,\xi)$ which can be represented by a non-negative path joining Id with $\Theta(\widetilde{\psi_t}) = \psi$, then $\widetilde{\mathcal{C}}(M,\xi)$ is a normal cone in $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ known as the* ***non-negative normal cone*** *of $\widetilde{\mathcal{CD}iff}_0(M,\xi)$.*

Proof. We just need to verify the characteristics which define a normal cone:

1. Let ψ, ϕ be the time-one map from the contact isotopies ψ_t, $\phi_t \in \mathcal{P}Cont_0(M,\xi)$, and suppose that $\widetilde{\psi_t}, \widetilde{\phi_t} \in \widetilde{\mathcal{C}}(M,\xi)$. As the covering map is a group homomorphism $\Theta(\widetilde{\psi_t} \circ \widetilde{\phi_t}) = \psi \circ \phi \in \mathcal{CD}iff_0(M,\xi)$. Let $\psi_t \circ \phi_t$ be a path joining Id with $\psi \circ \phi$. Applying the second property of Proposition 3.2.4 we get that $H_{\psi_t \circ \phi_t} \geq 0$ and, thus, $\widetilde{\psi_t} \circ \widetilde{\phi_t} \in \widetilde{\mathcal{C}}(M,\xi)$.
2. Analogously, but applying the fourth property of Proposition 3.2.4 instead of the second one, we get the result.
3. This is trivial.

□

As we have seen before, the non-negative normal cone $\widetilde{\mathcal{C}}(M,\xi)$ induces a relation $\geq$ on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ defined by $\widetilde{\psi_t} \geq \widetilde{\phi_t}$ if and only if $\widetilde{\psi_t} \circ \widetilde{\phi_t}^{-1} \in \widetilde{\mathcal{C}}(\mathcal{M},\xi)$. This relation is analogous to the following one:

$\widetilde{\psi}_t \geq \widetilde{\phi}_t$ if there exist a non-negative contact isotopy joining Id with $\psi_1 \circ \phi_1^{-1}$ homotopic to $\psi_t \circ \phi_t^{-1}$.

At this stage, we would like to know when this relation defines a non-trivial partial order on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$. Before that, we need to introduce a Lemma which is the one which Lemma 3.3.3 is based on. Its proof is analogous to the proof of Lemma 3.3.3.

Lemma 5.2.2 (Proposition 2.1.B, [**EP**]). *If a closed contact manifold (M,ξ) admits a non-constant contractible non-negative loop of contactomorphisms, then it admits a contractible strictly positive loop of contactomorphisms.*

The next Theorem gives us a necessary and sufficient condition for orderability of a closed contact manifold.

Theorem 5.2.3. *The relation $\geq$ on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ is a partial order if and only there are no contractible loops of contactomorphisms of (M,ξ) generated by a strictly positive time-periodic contact Hamiltonian. We will say that the contact manifold (M,ξ) is **orderable** if the relation $\geq$ defines a partial order on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$.*

The proof follows immediately from the following Proposition and the above Lemma.

Proposition 5.2.4 (Proposition 2.1.A, [**EP**]). *The relation $\geq$ on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ is a non-trivial partial order if and only if every non-negative contractible loop of contactomorphisms is the constant loop.*

Proof. Suppose that every non-negative contractible loop of contactomorphisms is constant. Let $\widetilde{\psi}_t \in \widetilde{\mathcal{CD}iff}_0(M,\xi)$ such that $\widetilde{\psi}_t \geq Id$ and $\widetilde{\psi}_t \leq Id$ then we have to prove that $\widetilde{\psi}_t = Id$.

As $\widetilde{\psi}_t \geq Id$ there exist a non-negative contact isotopy $\{\psi_t^1\}$ such that $\psi_0^1 = Id$ and $\psi_1^1 = \psi$. Analogously, as $\widetilde{\psi}_t \leq Id$ there exist a non-positive contact isotopy $\{\psi_t^2\}$ such that $\psi_0^2 = Id$ and $\psi_1^2 = \psi$. Moreover, we can assume that both path are homotopic with fix end points.

Without loss of generality one can assume that $\psi_t^1 = \psi_t^2 = 1$ near $t = 0$ and that $\psi_t^1 = \psi_t^2 = \psi$ near $t = 1$. Consider the concatenation given by $\psi_t^1 \circledcirc -\psi_t^2$, where by $-\psi_t^2$ we denote the isotopy ψ_t^2 with the opposite orientation. Hence, we get a non-negative contractible loop of contactomorphisms which, by our assumption, must be constant. Therefore both isotopies ψ_t^1 and ψ_t^2 are constant and then, $\widetilde{\psi}_t = Id$.

The proof of the converse statement is analogous. □

Analogously, we can define a normal cone $\mathcal{C}(M,\xi)$ on $\mathcal{CD}iff_0(M,\xi)$ as follows:

$\psi \in \mathcal{C}(M,\xi)$ if and only if there exist a non-negative contact isotopy such that $\psi_1 = \psi$.

It is called the **non-negative normal cone** of $\mathcal{CD}iff_0(M,\xi)$. This cone defines a relation $\geq$ on $\mathcal{CD}iff_0(M,\xi)$ given by

$\psi_t \geq \phi_t$ if there exist a non-negative contact isotopy joining ϕ_1 with ψ_1.

Therefore we have an equivalent result to Proposition 5.2.5 and Theorem 5.2.6 for $\mathcal{CD}iff_0(M,\xi)$.

Proposition 5.2.5. *The relation $\geq$ on $\mathcal{CD}iff_0(M,\xi)$ is a partial order if and only if every non-negative loop of contactomorphisms is the constant loop.*

Theorem 5.2.6. *The relation $\geq$ on $\mathcal{CD}iff_0(M,\xi)$ is a partial order if and only there are no loops of contactomorphisms of (M,ξ) generated by a strictly positive time-periodic contact Hamiltonian. In this case, we will say that the contact manifold (M,ξ) is* ***strongly orderable****.*

To prove both results we only have to follow the same argument than above but taking into account that the loop you produce in the proof of Proposition 5.2.5 does not have to be contractible.

5.3 Simplest examples

The simplest case where we can find an orderable contact manifold is provided by the contact manifold $(\mathbb{S}^1 = \mathbb{R}/\mathbb{Z},\ \xi)$ where ξ is composed by the 0-dimensional tangent subspaces and whose co-orientation is determined by the orientation of $\mathbb{S}^1$. Let $\mathrm{Diff}^+(\mathbb{S}^1)$ be the group of orientation preserving diffeomorphisms of $\mathbb{S}^1$ then $\mathcal{CD}iff_0(M,\xi) = \mathrm{Diff}^+(\mathbb{S}^1)$. Therefore its universal cover $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ is conformed by the orientation preserving diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x) + 1$. Remember that the non-negative normal cone are the elements of $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ that can be represented non-negative path starting at the identity, hence $\mathcal{C}(M,\xi)$ is formed by those diffeomorphisms $f \in \widetilde{\mathcal{CD}iff}_0(M,\xi)$ such that $f(x) \geq Id(x) = x$ for all $x \in \mathbb{R}$. It turns out that the induced partial order on $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ is provided by $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in \mathbb{R}$. In fact, notice that there are no non-negative contractible loops of diffeomorphisms .

The first non-trivial contact manifold which admits a partial order is the space of co-oriented contact elements, i.e. the positive projectivization of a cotangent bundle $\mathbb{P}^*_+TM$ which carries a canonical contact structure as we have seen in Example 3.1.2 (5). This result is due to Eliashberg and Polterovich [**EP**] and states the following:

Theorem 5.3.1 ([**EP**]). *If a closed manifold M admits a non-degenerate closed 1-form then the non-negative normal cone $\widetilde{\mathcal{C}}(\mathbb{P}^*_+TM,\xi)$ induces the non-trivial partial order on $\widetilde{\mathcal{CD}iff}_0(\mathbb{P}^*_+TM,\xi)$.*

Now, consider a symplectic manifold (M,ω) such that it has a closed Lagrangian submanifold $L \subset (M,\omega)$ and let QM be its contactization of (M,ω).

Theorem 5.3.2 ([**EP**]). *Suppose that the above Lagrangian L satisfies that:*

1. *The connection on QM is flat over L.*
2. *The relative homotopy group $\pi_2(M,L)$ vanishes.*

Then QM is orderable.

Remark 5.3.3. *Such a Lagrangian exists for example when (M,ω) is the standard symplectic torus $(\mathbb{R}^{2n}/\mathbb{Z}^{2n}, dp \wedge dq)$.*

Consider the standard contact sphere $\mathbb{S}^{2n-1}$ and the standard contact projective space $\mathbb{PR}^{2n-1}$. It is well known [**W**, **Wo**, **Ki**] that both contact manifolds are contactizations of the symplectic manifold $(\mathbb{C}P^n,\omega)$ where ω is the Fubini-Study symplectic form normalized to be integral (see [**Sil**] for more details). Let us notice that neither of them satisfies the hypothesis of the above Theorem.

Nevertheless on the one hand, the standard contact projective space $\mathbb{PR}^{2n-1}$ is orderable. The orderability follows from the theory of the nonlinear Maslov index introduce by Givental see [**Giv**]. On the other hand, we have the following result:

Theorem 5.3.4 ([**EKP**]). *Let $n \geq 2$, then there exists a positive contractible loop of contactomorphisms of the standard contact sphere $\mathbb{S}^{2n-1}$. In particular, it is not orderable.*

5.4 The Hofer's metric

Any Hamiltonian function H on a symplectic manifold (M^{2n},ω) can be used to define a Hamiltonian system. In classical mechanics, the Hamiltonian function is known as the energy function and the symplectic manifold is then called the phase space. As we have seen before the Hamiltonian function induces a Hamiltonian symplectomorphism defined as the flow of the Hamiltonian vector field associated to H. In particular, it preserves, then, the Hamiltonian system.

In 1990 the study of the minimal amount of energy needed in order to generate a Hamiltonian system defined by a given Hamiltonian symplectomorphism led Hofer [**Ho**] to a remarkable discovery: a bi–invariant metric on (the universal cover of) the group of Hamiltonian diffeomorphisms $\mathcal{H}am(M^{2n},\omega)$ of a compact symplectic manifold (M^{2n},ω).

The Hofer metric is defined as follows. First of all, consider the length of the Hamiltonian isotopy $\{\psi_{H_t}\}_{t\in[0,1]}$ generated by a Hamiltonian function H_t as

$$l(\psi_{H_t}) = \int_0^1 \|H_t\|_\infty = \int_0^1 \max_{p\in M} H_t(p) - \min_{p\in M} H_t(p)dt.$$

Then, as usual, we define the distance between two Hamiltonian symplectomorphisms ψ and ϕ by

$$d(\psi,\phi) := \mathrm{Inf}\{l(\gamma_t)\},$$

where the infimum is taken over all Hamiltonian paths γ_t connecting ψ and ϕ. It is easy to verify that d is a pseudo-distance function. Finally, we define the pseudo-norm of a Hamiltonian symplectomorphisms to be the infimum of the lengths of all Hamiltonian isotopies from the identity to ϕ, i.e.:

$$\|\psi\| = d(Id,\psi).$$

The above pseudo-norm, which we will denote by $\|\cdot\|_{HO}$, is known as the Hofer pseudo-norm, and the induced pseudo-metric, denoted by d_{HO},, is called the Hofer pseudo-metric.

Proposition 5.4.1. *The above definition gives rise to a bi-invariant pseudo-metric on $\mathcal{H}am(M,\omega)$, i.e. d_{HO} satisfies that:*

$$d_{HO}(\psi,\phi) = d_{HO}(\varphi\circ\psi,\varphi\circ\phi) = d_{HO}(\psi\circ\varphi,\phi\circ\varphi),$$

for all $\psi,\phi,\varphi\in\mathcal{H}am(M,\omega)$.

Similarly, we can define the Hofer pseudo-metric for elements in $\widetilde{\mathcal{H}am(M^{2n},\omega)}$ to be the infimum of the length of all isotopies representing the element and, analogously, this metric gives rise to a bi-invariant pseudo-metric on $\widetilde{\mathcal{H}am(M,\omega)}$.

It is highly non-trivial to check whether such a pseudo-metric is non-degenerate, that is, if it is actually a metric. It was first discovered and proved by Hofer in [**Ho**] for the $2n$-dimensional symplectic euclidean space. This was also proven by Viterbo [**Vi**] in 1992. It was, then, generalized by Polterovich [**P1**] to some larger class of symplectic manifolds, and finally proved in the full generality by Lalonde and McDuff in [**LMc**].

Now, a natural question arises: What can we say about contact Hamiltonians? In contrast to the symplectic case, no interesting bi-invariant metrics on the group contact Hamiltonians were known. As an attempt to answer this question. Eliashberg and Polterovich [**EP**] noticed that for certain contact manifold $\widetilde{\mathcal{CD}iff}_0(M,\xi)$ carries a bi-invariant partial order from which we get a natural metric space associated to $\mathcal{CD}iff_0(M,\xi)$. Let us describe their construction.

First, we need to discuss the notion of dominants in the context of contact Hamiltonians. Let us denote by $\mathcal{C}^+(M,\xi) \subset \mathcal{C}(M,\xi) \subset \widetilde{\mathcal{CD}iff}_0(M,\xi)$ the set of $\widetilde{\psi}_t \in \widetilde{\mathcal{CD}iff}_0(M,\xi)$ which can be represented by a strictly positive contact isotopy, then

Proposition 5.4.2 ([**EP**])**.** *All the elements of $\mathcal{C}^+(M,\xi)$ are dominants.*

Following Definition 5.1.4, the above result allows us to define the relative growth $\gamma(\widetilde{\psi},\widetilde{\phi})$ for every $\widetilde{\psi} \in \mathcal{C}^+(M,\xi)$ and every $\widetilde{\phi} \in \widetilde{\mathcal{CD}iff}_0(M,\xi)$. The calculation of the relative growth is not trivial. However Eliashberg and Polterovich did calculate the relative growth for some special cases.

Examples 5.4.1.

1. *Let $M = \mathbb{S}^1$. Recall that $\widetilde{\mathcal{CD}iff}_0(\mathbb{S}^1,\xi)$ is conformed by the orientation preserving diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x)+1$. Then we have that*
$$\gamma(f,g) = \frac{\mathrm{Rot}(g)}{\mathrm{Rot}\, f}.$$

2. *Let (QM,ξ) be a contactization space. All prequantization space admits admits a contact form α which produces a 1-periodic Reeb flow ψ_t and consider the lift to the universal cover $\widetilde{\psi}_t$. Then if (QM,ξ) is orderable, we know that*
$$\gamma(\widetilde{\psi}_1,\widetilde{\psi}_t) = t.$$

Remark 5.4.2. *For every $\widetilde{\psi},\widetilde{\phi},\widetilde{\varphi} \in \mathcal{C}^+(M,\xi)$ we have that*
$$\gamma(\widetilde{\psi},\widetilde{\varphi}) \geq \gamma(\widetilde{\psi},\widetilde{\phi})\gamma(\widetilde{\phi},\widetilde{\varphi})$$

Definition 5.4.3. *Define the function $\kappa : \mathcal{C}^+(M,\xi) \times \mathcal{C}^+(M,\xi) \to [0, +\infty)$ by*
$$\kappa(f,g) = \max log\gamma(\widetilde{\psi},\widetilde{\phi}), log\gamma(\widetilde{\phi},\widetilde{\psi}).$$

Remark 5.4.4. *It follows from Proposition 5.1.1 and Remark 5.4.2 that κ is a pseudo-metric.*

Now, consider the following relation: $\widetilde{\psi} \sim \widetilde{\phi}$ if and only if $\kappa(\widetilde{\psi},\widetilde{\phi}) = 0$ and define $Z = \mathcal{C}^+(M,\xi)/\sim$. Then κ projects to a metric d_{EP} on Z. Finally, define the following partial order $\geq$ on Z:
$$[\widetilde{\psi}] \geq [\widetilde{\phi}] \text{ if and only if } \gamma(\widetilde{\psi},\widetilde{\phi}) \leq 1.$$

Therefore $(Z,d,\geq)$ is a partially ordered metric space; i.e. a metric space endowed with a partial order such that if $\widetilde{\psi} \geq \widetilde{\phi} \geq \widetilde{\varphi}$ for every $\widetilde{\psi},\widetilde{\phi},\widetilde{\varphi} \in Z$ then $d_{EP}(\widetilde{\psi},\widetilde{\varphi}) \geq d_{EP}(\widetilde{\phi},\widetilde{\varphi})$.

5.5 Orderability vs Non-squeezing

Gromov's non-squeezing Theorem [**Gr1**] states that the standard symplectic ball cannot be symplectically embedded into any cylinder of smaller radius. That is

Theorem 5.5.1 ([**Gr1**]). *Let* $(\mathbb{R}^{2n}, \omega_{std})$ *the standard symplectic euclidean space and consider the domains*

$$\mathbb{B}^{2n}_R = \left\{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} \mid \pi \sum_{i=1}^{n} x_i^2 + y_i^2 < R\right\},$$
$$\mathbb{C}^{2n}_R = \mathbb{B}^2_R \times \mathbb{R}^{2n-2} = \left\{(x_1, y_1, \cdots, x_n, y_n) \in \mathbb{R}^{2n} \mid \pi(x_1^2 + y_1^2) < R\right\}.$$

If $\mathbb{R}_1 > \mathbb{R}_2$ *then the ball* $\mathbb{B}^{2n}_{R_1}$ *cannot be symplectically embedded into the cylinder* $\mathbb{C}^{2n}_{R_2}$

Note that the cylinder has infinite volume, hence it should be possible to find a volume–preserving embedding of the ball (for any radius) into the cylinder. Therefore the above result is telling us that being a symplectic transformation (i.e. preserving ω) is a much stricter than just being a volume-preserving embedding (i.e. preserving ω^n).

By contrast, the analogous statement in contact geometry is trivially false. In fact, consider the contactomorphism $(x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$ with $\lambda \in \mathbb{R}^+$. Then, we can squeeze any arbitrary Darboux balls into an arbitrarily small neighbourhood of a point.

However, Eliashberg, Kim and Polterovich [**EKP**] found an important non-squeezing phenomenon concerning prequantized balls in the contact manifold $\mathbb{R}^{2n} \times \mathbb{S}^1$ and a more restrictive notion of contact squeezing:

Definition 5.5.1. *Given two open subsets* U *and* V *of a contact manifold* (M, ξ) *, we say that* **U** ***can be squeezed into*** **V** *if there exists a contact isotopy* $\psi_t : \overline{U} \to M$, $t \in [0, 1]$ *such that*

$$\psi_0 = Id$$
$$\psi_1(\overline{U}) \subset V.$$

The isotopy is, then, known as a ***contact squeezing*** *of* U *into* V*. Let* W *be another open subset of* M*, we say that* **U** ***can be contactly squeezed into*** **V** ***inside*** **W** *if* $\overline{V} \subset W$ *and* $\psi_t(\overline{U}) \subset W$ *for all* t*.*

Note that, by the isotopy extension theorem, if $\overline{U}$ is compact then any contact squeezing of U into V inside W extends to a contactomorphism of V supported in W.

Let $(M, \xi) = (\mathbb{R}^{2n} \times \mathbb{S}^1, \ker(dz - \lambda_{can}))$ the prequantization space of $(\mathbb{R}^{2n}, d\lambda_{can})$ and denote by $\widehat{U} = U \times \mathbb{S}^1 \subset \mathbb{R}^{2n} \times \mathbb{S}^1$, the prequantization of an arbitrary domain $U \subset \mathbb{R}^{2n}$.

Then, Eliashberg, Kim and Polterovich showed that

Theorem 5.5.2 ([**EKP**]). *(Non-squeezing) Assume $R_2 \le k \le R_1$ for some positive integer k, then $\widehat{\mathbb{B}}^{2n}_{R_1}$ cannot be squeezed into $\widehat{\mathbb{C}}^{2n}_{R_2}$.*

Theorem 5.5.3 ([**EKP**]). *(Squeezing) Let $n > 1$ and $R_1, R_2 < 1$, then $\widehat{\mathbb{B}}^{2n}_{R_1}$ can be squeezed into $\widehat{\mathbb{B}}^{2n}_{R_2}$.*

The above results on contact (non)-squeezing are closely related to the concept of orderability of the group of contactomorphisms of the standard contact sphere.

To review it, let us first remark that given a contact manifold (M, ξ), every contactomorphism ψ of it uniquely lifts to a symplectomorphism $S\psi$ of its symplectization SM:

$$S\psi(p,t) = (\psi(p), t - f(p)),$$

for all $(p,t) \in SM$, where f is the function defined by the conformal factor of ψ. Moreover, there is a one-to-one correspondence between contact isotopies and Hamiltonian functions on SM.

Let $\Delta = \psi_{t,s}$ with $t \in \mathbb{S}^1, s \in [0,1]$ be a homotopy of $\psi_{t,1}$, a positive contractible loop of contactomorphisms in $\mathbb{S}^{2n-1}$, to the constant loop $f_{t,0} = Id$ and assume $f_{0,s} = Id$ for all s. For each $s \in [0,1]$, let $SH_{\phi_{t,s}}$ denote the Hamiltonian function on $(\mathbb{R}^{2n} \smallsetminus \{0\}) \times \mathbb{S}^1$ generating the loop $\phi_{t,s}$, $t \in \mathbb{S}^1$. Define:

$$\mu(\Delta) = -\min_{p,s,t} \frac{SH_{\phi_{t,s}}(p)}{\pi |z|^2}.$$

Then Theorem 5.5.3 implies the following result:

Theorem 5.5.4 ([**EKP**]). *Let $n > 1$, then $\mu(\Delta) \ge 1$ for every homotopy Δ of a positive contractible loop of contactomorphisms of the sphere $\mathbb{S}^{2n-1}$ to the constant loop.*

Moreover, Theorem 5.5.2 is a consequence of the following statement:

Theorem 5.5.5 ([**EKP**]). *Let $n > 1$, then $Inf_\Delta \mu(\Delta) = 1$*

Let us, now, give a link between non-squeezing in prequantization spaces of Liouville manifolds and orderability of its ideal contact boundary.

Let (M, ω, L) be a Liouville manifold with the ideal contact boundary (N, ξ) and consider the contactization $QM = (M \times \mathbb{S}^1, \xi = \ker(dz - \lambda)$ of M. Let us suppose that N is not orderable, then there exists a positive contractible loop of contactomorphisms ψ_t. Consider the corresponding symplectomorphism $S\psi_t$ of SN and define the homotopy $S\phi_{t,s}$ between it (at time $s = 1$) and the constant loop (at $s = 0$). Assume $S\psi_{0,s} = Id$

and denote by $SH_{\psi_{t,s}}$ the correspondent Hamiltonian on QM generating the loop $\psi_{t,s}$, for each $t \in \mathbb{S}^1$. Then:

Theorem 5.5.6 ([**EKP**]). *Consider a positive Hamiltonian $G : M \to \mathbb{R}$ and define the domain*

$$A_R = \{G < R\} \times \mathbb{S}^1 \subset QM, \quad \textit{for all } R > 0.$$

Suppose that there exists a constant $\mu > 0$ such that

$$SH_{\psi_{t,s}}(p) > -\mu G(p), \quad \textit{for all } p \in SN,\ t \in \mathbb{S}^1,\ s \in [0,1].$$

Then

1. *For all $R < \mu^{-1}$ there exists $\gamma > 0$ such that A_R can be contactly squeezed into $A_{\frac{R}{1+\gamma R}}$.*
2. *Moreover, given $\rho > \dfrac{1}{R^{-1} - \mu}$, then A_R can be contactly squeezed into itself inside A_ρ.*

Finally, another way of using a positive loop of contactomorphisms of $\mathbb{S}^{2n-1}$ for producing a contact embedding of domains in $\mathbb{R}^{2n} \times \mathbb{S}^1$ (see Section 2.2. of [**EKP**]) gives us the following result:

Proposition 5.5.7 ([**EKP**]). *For all $R_1, R_2 > 0$, there exists a contact embedding of $\widehat{\mathbb{B}}^{2n}_{R_1}$ into $\widehat{\mathbb{B}}^{2n}_{R_2}$ For $n > 1$, this embedding can be chosen isotopic to the natural inclusion through smooth embeddings.*

In fact as a consequence of the above Proposition and Darboux Theorem, we have

Corollary 5.5.8. *For every $R > 0$ there exists a contact embedding of $\widehat{\mathbb{B}}^{2n}_R$ into an arbitrarily small neighbourhood of a point in any contact manifold.*

Chapter 6

A simple construction of positive loops of Legendrians

"If you have never wept bitter tears because a wonderful story has come to an end and you must take your leave of the characters with whom you have shared so many adventures, whom you have loved and admired, for whom you have hoped and feared, and without whose company life seems empty and meaningless. If such a thing have not been part of your own experience, you probably won't understand what Bastian did next."

- Michael Ende, The Neverending Story.

Now we are ready to state the main results of this thesis. In substance, we will construct positive loops Legendrian submanifolds in several instances. In section 6.1, we will introduce a key remark which will allow us to make the required constructions. In particular, we will partially recover G. Liu's result [**Liu1**] stating that any loose Legendrian admits a positive loop, under some extra mild topological assumptions on the Legendrian. As we have seen before, this will mean that the space of Legendrian isotopic loose Legendrian submanifolds is not orderable. This will be the content of Section 6.2 Moreover, in Section 6.3, we will show contractibility of the constructed loops under an extra topological assumption.

6.1 A key remark: Non-orderability on the product

Consider a $(2n+1)$-dimensional manifold M^{2n+1} endowed with a co-oriented contact structure $\xi = \ker\alpha$ and let $L^n \subset (M^{2n+1}, \xi)$ be a Legendrian submanifold. Denote by $\mathbb{D}^n_\varepsilon$ the closed Euclidean ball of radius ε in $\mathbb{R}^n$; and by $\mathring{\mathbb{D}}^n_\varepsilon$ the open ball.

Theorem 6.1.1. *Fix an $\varepsilon > 0$ positive constant and consider the contact manifold $(M \times \mathring{\mathbb{D}}^2_\varepsilon(r,\theta), \ker(\alpha + r^2 d\theta))$. Any closed Legendrian submanifold in $M \times \mathring{\mathbb{D}}^2_\varepsilon$ admits a positive loop of Legendrians.*

This statement is is the pillar which all the results in this dissertation are based on. It gives us a way of finding a positive loop of Legendrian for any closed Legendrian submanifold in the product contact manifold $M \times \mathring{\mathbb{D}}^2_\varepsilon$. Its proof is, essentially, a consequence of Lemma 5.2.2.

Proof of Theorem 6.1.1. Consider the contact manifold $(M \times \mathring{\mathbb{D}}^2_\varepsilon, \ker\beta = \ker(\alpha + r^2 d\theta))$ where $\varepsilon > 0$ is fixed and $(M, \ker\alpha)$ is a contact manifold. The contact vector field $X = \frac{\partial}{\partial\theta}$ generates a non–negative loop of contactomorphisms on it. Moreover, it is positive away from $M \times \{0\}$ as $\beta(\partial\theta) = r^2$.

Let L be a Legendrian submanifold in $M \times \mathbb{D}^2$. For dimensional reasons there exists a point of L which is not in the contact submanifold $M \times \{0\}$. Hence the loop restricted to the Legendrian is a non–negative non–trivial loop of Legendrians. Now we can apply Lemma 5.2.2 to complete the proof. □

Corollary 6.1.2. *Any Legendrian submanifold in $\mathbb{R}^{2n+1}$ admits a positive loop of Legendrians.*

Proof. The standard contact manifold $\mathbb{R}^{2n+1}$ is nothing but $\mathbb{R}^{2n-1} \times \mathbb{R}^2$ with the contact structure given by $\alpha_{std} + r^2 d\theta$ where α_{std} is the standard contact form on $\mathbb{R}^{2n-1}$. Let L be a closed Legendrian in $\mathbb{R}^{2n+1}$, by compactness $L \subset \mathbb{R}^{2n-1} \times \mathring{\mathbb{D}}^2_\varepsilon$, for $\varepsilon > 0$ large enough. The corollary follows from Theorem 6.1.1 applied to $\mathbb{R}^{2n-1} \times \mathring{\mathbb{D}}^2_\varepsilon$. □

Remark 6.1.3. *Actually, it can be shown that $\mathbb{R}^{2n+1}$ admits a positive loop of contactomorphisms. This is even true for $M \times \mathbb{R}^2$ just by checking that the proof of Lemma 5.2.2 works also for open manifolds. The only delicate issue is that the contact vector fields defined in that proof should be complete.*

6.2 The main result

As a consequence of Theorem 6.1.1, assuming a non very restrictive topological hypothesis, we are able to construct positive loops of loose Legendrian submanifolds. This is the content of the following Theorem.

Theorem 6.2.1. *Let $n \geq 2$. Fix a loose closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ). Assume that the bundle $T^*L \oplus \mathbb{R}$ has two pointwise linearly independent sections. Then L admits a positive loop of Legendrians.*

Recall, by the Weinstein neighborhood Theorem (Thm. 3.4.4), that $T^*L \oplus \mathbb{R} = J^1(L)$ is the normal bundle of the Legendrian submanifold and determines the contact structure on a small neighborhood of the submanifold.

The dimension restriction comes from the definition of loose Legendrian submanifolds. In the proof we will use, as a basic step, Murphy's h–principle for loose Legendrians submanifolds [**Mu**] which we have seen in Chapter 4.

In 3–dimensional contact topology, there is an analogous older notion [**EF**] for loose Legendrian submanifolds. A Legendrian knot in a contact 3–fold whose complement is overtwisted is called **loose**. They also satisfy an h–principle.

If $2n+1 \geq 5$, any Legendrian submanifold whose complement is overtwisted is loose. This is a consequence of the parametric and relative nature of the h–principle for overtwisted contact structures (see [**BEM**]).

For didactical reasons, we will first prove the following particular case of Theorem 6.2.1.

Theorem 6.2.2. *Let $n \geq 1$. Assume that a closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ) satisfies that the bundle $T^*L \oplus \mathbb{R}$ has two pointwise linearly independent sections. If $M \backslash L$ is overtwisted, then L admits a positive loop of Legendrians.*

The main idea of the proof is to construct a neighborhood U_L of L contactomorphic to $N \times \mathbb{R}^2$, for some contact manifold (N, ξ), with the contact form defined as in Theorem 6.1.1. In order to do that, we need to use Lemma 3.5.2 together with Proposition 3.5.3. Lemma 3.5.2 allows us to find a contactomorphisms between two overtwisted at infinity contact structures which are formally equivalent. Proposition 3.5.3 gives us a condition for $N \times \mathbb{R}^2$ to be overtwisted at infinity.

Hence, we need U_L to satisfy the hypothesis of Lemma 3.5.2. This is the content of Lemma 6.2.3. The proof of Theorem 6.2.2 will then follow from Theorem 6.1.1.

Lemma 6.2.3. *For any Legendrian submanifold $L \subset (M, \xi)$ satisfying the hypothesis of Theorem 6.2.2, there exists a neighborhood U_L of L diffeomorphic to $N \times \mathbb{R}^2$ such that (U_L, ξ) is overtwisted at infinity and N is an open manifold if $n \geq 2$.*

Proof. By the first hypothesis and the Legendrian neighborhood theorem [**Ge**], a small tubular neighborhood V_L of L is diffeomorphic to $N \times \mathbb{R}^2$. By the second hypothesis, there exists an overtwisted disk contact germ which does not intersect L. The germ contains an open ball overtwisted at infinity $\mathbb{B}_{ot}$ by Lemma 3.5.6. V_L is disjoint from the overtwisted ball $\mathbb{B}_{ot}$. Define U_L to be the embedded connected sum of V_L with $\mathbb{B}_{ot}$ along a tubular neighborhood of a path connecting their boundaries (see Figure 6.1). U_L is overtwisted at infinity by construction and is diffeomorphic to $N \times \mathbb{R}^2$.

□

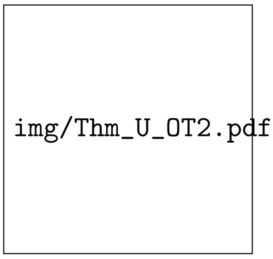

Figure 6.1: Construction of U_L.

Proof. Recall that Proposition 3.5.3 does not hold if N is a 1-dimensional manifold. Therefore, we have to distinguish two cases:

Proof of Theorem 6.2.2 for $n > 1$.

It follows from Lemma 6.2.3 that there exists a diffeomorphism $\Phi : U_L \to N \times \mathbb{R}^2$. In addition, (U_L, ξ) is overtwisted at infinity. In order to apply Lemma 3.5.2 we need to find a contact structure on $N \times \mathbb{R}^2$ overtwisted at infinity and formally homotopic to ξ.

By Lemma 3.5.7 the submanifold $N \times \{0\}$ can be equipped with a formal contact structure (ξ_N, J_N) such that $(\xi_N \oplus \mathbb{R}^2, J_N \oplus i)$ represents the same formal contact class as ξ. Applying Theorem 3.5.1, there exists an overtwisted contact structure $\xi_{ot} = \ker(\alpha_{ot})$ on N, formally homotopic to ξ_N. Therefore, the contact structure $\xi' = \ker(\alpha_{ot} + r^2 d\theta)$ in $N \times \mathbb{R}^2$ is overtwisted at infinity by Proposition 3.5.3 and formally homotopic to ξ.

By Lemma 3.5.2, there is a diffeomorphism $F : U_L \to N \times \mathbb{R}^2$ taking ξ to ξ' and preserving co–orientations. By the compactness of L, we have that $F(L) \subset N \times \mathbb{D}^2_\varepsilon$, for $\varepsilon > 0$ large enough.

By Theorem 6.1.1, $F(L)$ admits a positive loop of Legendrians ϕ_t. Thus, the family $\phi_t \circ F^{-1}$ is a positive loop of Legendrians for L.

Proof of Theorem 6.2.2 for $n = 1$.

In this case, we cannot apply Proposition 3.5.3 to find an overtwisted at infinity contact structure on $N \times \mathbb{R}^2$. Hence, we need to argue as follows:

Let $L \hookrightarrow (M, \xi)$ denote the Legendrian embedding. A tubular neighborhood U_L can be identified with $L \times \mathring{\mathbb{D}}^2_\varepsilon \subset (M, \xi)$. By Lemma 6.2.3, U_L is overtwisted at infinity and diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$.

Consider now the contact manifold $(\mathbb{S}^1(z) \times \mathring{\mathbb{D}}^2_\varepsilon(r, \theta), \eta = \ker(dz + r^2 d\theta))$. Here, integrating ∂_z gives a positive loop of contactomorphism with Hamiltonian $H = 1$, in particular it is autonomous. Fix a sequence of transverse knots $\gamma_k = (z, \varepsilon(1 - 1/k), 0)$, with $k \in \mathbb{Z}_{>0}$. Then, the contact manifold obtained as a sequence of half Lutz twists (see 4.3 [**Ge**]) along each of them is overtwisted at infinity. It admits a positive loop of contactomorphisms by [**CP**]. Denote it by $(\mathbb{S}^1 \times \mathring{\mathbb{D}}^2_\varepsilon, \eta^\gamma)$.

Finally, ξ and η^γ are formally equivalent because there exists only one class of formal contact structures on $\mathbb{S}^1 \times \mathbb{D}^2_\varepsilon$. Again, the claim follows by using Lemma 3.5.2.

□

We remark that this result covers the 3–dimensional situation that is not included in Theorem 6.2.1.

Observe that the hypothesis of $T^*L \oplus \mathbb{R}$ having two independent sections is pretty mild. If L is orientable, then some sufficient conditions for this hypothesis to be satisfied are:

- $\chi(L) = 0$. This, in particular, covers odd dimensional Legendrians.
- $w_n(L) = 0$. This implies that $w_n(T^*L \oplus \mathbb{R}) = 0$ and by the definition of this obstruction class in the even dimensional case, the vanishing of the class implies the existence of two independent sections. In particular, this covers even dimensional Legendrians with even Euler characteristic.
- Any Legendrian submanifold whose tangent bundle is trivialized by direct sum with $\mathbb{R}$. This covers all the spheres.

There are simple examples of manifolds not satisfying that property. For instance, $L = \mathbb{CP}^2$ is a manifold whose 1–jet bundle $T^*\mathbb{CP}^2 \oplus \mathbb{R}$ does not admit two independent sections.

Now, we are ready to prove Theorem 6.2.1. Again, we will make use of Theorem 6.1.1. Hence we need to construct a neighborhood of L contactomorphic to $(N \times \mathring{\mathbb{D}}^2_\varepsilon, \ker(\alpha_N + r^2 d\theta))$ for some contact manifold (N, α_N).

We will first prove a simple case.

Proposition 6.2.4. *Let $n \geq 2$. Fix a loose closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ). Assume that $\chi(L) = 0$,then L admits a positive loop of Legendrians.*

Proof. Assume that T^*L has a never–vanishing section. Using Weinstein's tubular neighborhood theorem, we find a neighborhood U_L of $(L, \ker(\alpha))$ contactomorphic to $(R(z) \times T^*L, \ker(dz - \lambda_{std}))$. As $(T^*L \setminus \{0\}, d\lambda_{std})$ and $(R \times \mathbb{S}(T^*L), d(e^t \lambda_{std}))$ admit a diffeomorphism preserving the Liouville forms, the natural inclusion $\mathbb{S}(T^*L) \hookrightarrow R \times T^*L$ is a contact embedding. By the tubular neighborhood theorem for contact submanifolds ([**Ge**]), there exists a neighborhood V of $\mathbb{S}(T^*L)$ contactomorphic to $\mathbb{S}(T^*L) \times \mathring{\mathbb{D}}^2_\varepsilon$.

The never–vanishing section of T^*L provides an embedding $\sigma : L \to \mathbb{S}(T^*L) \subset R \times T^*L$. Thus, we obtain a family of embeddings $\sigma_t : L \to R \times T^*L$ defined as $\sigma_t = t\sigma$. Since σ_0 is a Legendrian embedding, the whole family σ_t can be lifted into a family (σ_t, Φ_t) of formal Legendrian embeddings.

Apply Theorem 4.1.1 to (σ_1, Φ_1) as a formal Legendrian embedding into the manifold to create a family (σ_t, Φ_t) with $t \in [1,2]$ of formal Legendrian embeddings in V satisfying that σ_2 is a loose Legendrian embedding. The family (σ_t, Φ_t) with $t \in [0,2]$ satisfies the hypothesis of the second part of Theorem 4.1.1 and so, it can be deformed relative to $t = 0, 2$ into a Legendrian isotopy inside M. We are, because of Lemma 3.3.2, reduced to find a positive loop for the loose Legendrian σ_2. But this is true by Theorem 6.1.1 applied to V. □

Let us now prove the general statement.

Proof of Theorem 6.2.1. By hypothesis, we have that a neighborhood U_L of L is diffeomorphic to $N \times \mathbb{R}^2$, for an open manifold N. By Lemma 3.5.7, we assume that there is a formal contact structure (ξ_N, J_N) on N such that $(\xi_N \oplus \mathbb{R}^2, J_N \oplus i)$ is the formal contact class of ξ. By Theorem [**EM**], the formal contact structure $\xi_N = \ker \alpha_N$ can be assumed to be contact.

We are in the hypothesis of [**EM**]. Therefore, the formal contact embedding $e_0 : N \hookrightarrow N \times \{0\} \subset N \times \mathbb{R}^2 \simeq U_L$ admits an isotopy of formal contact embeddings $e_t : N \to U_L$ satisfying that e_1 is a contact embedding. By the contact neighborhood theorem ([**Ge**], Theorem 2.5.15), there exists $\phi_1 : N \times \mathring{\mathbb{D}}^2_\varepsilon \hookrightarrow U_L$, for sufficiently small $\varepsilon > 0$, such that

1. $(\phi_1)_{|N\times 0} = e_1$.
2. Fix the contact form $\alpha = \alpha_N + r^2 d\theta$ in the manifold $N \times \mathring{\mathbb{D}}^2_\varepsilon$. The map ϕ_1 is a contact embedding.

By construction we have $L \subset N$. Define the family of embeddings $\varphi_t : L \to U_L$, $t \in [0,1]$ as $\varphi_t = (e_t)_{|L}$.

Promote the family φ_t into a family of formal Legendrian embeddings (φ_t, Φ_t), $t \in [0,1]$. Apply Theorem 4.1.1 to (φ_1, Φ_1) as formal Legendrian embedding of the manifold $\phi_1(N \times \mathring{\mathbb{D}}^2_\varepsilon)$, to create a family of formal Legendrians embeddings (φ_t, Φ_t) $t \in [1,2]$ such that (φ_2, Φ_2) is a loose Legendrian embedding into $\phi_1(N \times \mathring{\mathbb{D}}^2_\varepsilon)$. Since, by hypothesis φ_0 is loose, we can apply the second part of Theorem 4.1.1 to show that φ_0 and φ_2 are Legendrian isotopic in M.

But the image of φ_2 lies in $\phi_1(N \times \mathring{\mathbb{D}}^2_\varepsilon)$. Thus, $(\phi_1)^{-1} \circ \varphi_2$ is a Legendrian embedding into $(N \times \mathring{\mathbb{D}}^2_\varepsilon, \ker(\alpha_N + r^2 d\theta))$. Theorem 6.1.1 concludes that φ_2 possesses a positive loop. Lemma 3.3.2 provides one for the original Legendrian embedding φ_0. □

6.3 Contractible positive loops

Let us move to the study of positive contractible loops. Equivalently to the non-necessary contractible case, we will start by stating a result which will play the role of Theorem 6.1.1 for this case.

Theorem 6.3.1. *Let $(M, \xi = \ker(\alpha))$ be a contact manifold and on the product $M \times \mathring{\mathbb{D}}^4_\varepsilon(r_1, \theta_1, r_2, \theta_2)$ define the contact form $\widetilde{\alpha} = \alpha + r_1^2 d\theta_1 + r_2^2 d\theta_2$. Define the domain*

$$M^+ = \{(p, r_1, \theta_1, r_2, \theta_2) \in M \times \mathring{\mathbb{D}}^4_\varepsilon \text{ such that } 0 < r_1 < r_2\}.$$

Any Legendrian embedding $L \hookrightarrow M^+ \subset M \times \mathring{\mathbb{D}}^4_\varepsilon$ admits a contractible positive loop of Legendrians on $M \times \mathring{\mathbb{D}}^4_\varepsilon$.

Proof. Notice that U(2) acts by contactomorphisms on $M \times \mathring{\mathbb{D}}^4_\varepsilon$.

Now consider the contact vector fields $X_1 = \partial_{\theta_1}$ and $X_2 = \partial_{\theta_2}$ with associated Hamiltonians $H_1 = r_1^2$ and $H_2 = r_2^2$, respectively. The contact vector field $X = X_2 - X_1 = \partial_{\theta_2} - \partial_{\theta_1}$, whose associated Hamiltonian is $H = r_2^2 - r_1^2$, generates a loop that preserves M^+ and is positive on this domain. Denote by A_t the unitary matrix

$$\begin{pmatrix} e^{2\pi i t} & 0 \\ 0 & e^{-2\pi i t} \end{pmatrix},$$

then the flow associated to X reads as $\phi_t(p, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \left(p, A_t \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

Realize that A_t is contractible in U(2) since $det(A_t) = 1$ and SU(2) is simply connected. Therefore, there exists a family of loops $\widetilde{A}_{t,s} \in \mathrm{U}(2)$ with $s \in [0,1]$ such that

$$\widetilde{A}_{t,0} = Id,$$
$$\widetilde{A}_{t,1} = A_t.$$

Hence, $\phi_{t,s}(p, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \left(p, \widetilde{A}_{t,s} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$ is the contraction of the positive loop.

□

The main consequence of the above Theorem is the following corollary which can be understood as the analogous version (for contractible loops) to Theorem 6.2.1.

Corollary 6.3.2. *Let $n \geq 3$. Fix a loose closed Legendrian submanifold L^n in a contact manifold (M^{2n+1}, ξ). Assume that the bundle $T^*L \oplus \mathbb{R}$ has four pointwise linearly independent sections. Then, L admits a contractible positive loop of Legendrians.*

Proof. We mimic the proof of Theorem 6.2.1. A neighborhood U_L of L is diffeomorphic to $N \times \mathbb{R}^4$. By an application of classical h–principles, we can find an isotopy $\phi_t : N \times \mathring{\mathbb{D}}^4_\varepsilon \to U_L$ such that is the identity for $t = 0$ and is a contact embedding for $t = 1$.

Denote by $\varphi_0 : L \to U_L$ the given Legendrian embedding. We create a path of formal Legendrian embeddings (φ_t, Φ_t) starting at φ_0 an such that $\varphi_1(L) \subset \phi_1(N^+) \subset \phi_1(N \times$

$\mathring{\mathbb{D}}^4_\varepsilon$). Finally, applying twice Theorem 4.1.1 and Theorem 6.3.1, we conclude the result. □

Again, the hypotheses can be easily checked. They are satisfied, for instance, by Legendrian spheres of dimension $n \geq 3$. Let us consider two more corollaries from Theorem 6.3.1.

Corollary 6.3.3. *If $L \subset (\mathbb{R}^{2n+1}, \xi_{std})$ with $n \geq 2$, then L admits a contractible positive loop of Legendrians.*

Proof of Corollary 6.3.3. $(\mathbb{R}^{2n+1}, \xi_{std})$ is contactomorphic to $(\mathbb{R}^{2n-3} \times \mathbb{R}^4, \ker(\alpha_{std} + r_1^2 d\theta_1 + r_2^2 d\theta_2))$. By compactness of the Legendrian submanifold, we can assume that $L \subset \mathbb{R}^{2n-3} \times \mathbb{D}^2_R \times \mathbb{D}^2_R$, for some $R > 0$.

Applying Lemma 3.5.4 to $N = \mathbb{R}^{2n-3} \times \mathbb{D}^2_R(0,0)$, the domains $\mathbb{R}^{2n-3} \times \mathbb{D}^2_R(0,0) \times \mathbb{D}^2_R(0,0)$ and $\mathbb{R}^{2n-3} \times \mathbb{D}^2_R(0,0) \times \mathbb{D}^2_R(10R,0)$ are contact isotopic. Therefore, we can assume that the Legendrian embedding can be pushed into $\mathbb{R}^{2n-3} \times \mathbb{D}^2_R(0,0) \times \mathbb{D}^2_R(10R,0) \subset (\mathbb{R}^{2n-3})^+$. We apply Theorem 6.3.1 to conclude the result. □

Observe that this statement can be proven using the fact that $\mathbb{S}^{2n+1}$ admits a contractible positive loop (as we have seen in Theorem 5.3.4), placing $\mathbb{R}^{2n+1} \subset \mathbb{S}^{2n+1}$ and making sure that the restrictions of the contact isotopies to the Legendrian submanifold do not cross $\infty \in \mathbb{S}^{2n+1}$. This can be done by a genericity argument whenever $n \geq 2$. However, the proof presented above is more elementary.

Corollary 6.3.4. *Let $\mathbb{R}^{2n+1}$ be the Euclidean space equipped with the overtwisted at infinity contact structure ξ. If $L \subset (\mathbb{R}^{2n+1}, \xi)$ and $n > 2$ then L admits a contractible positive loop of Legendrians.*

Proof. Consider $(\mathbb{R}^{2n-3}, \widetilde{\xi_{ot}} = \ker(\widetilde{\alpha_{ot}}))$ with $\widetilde{\xi_{ot}}$ any overtwisted contact structure on $\mathbb{R}^{2n-3}$. $(\mathbb{R}^{2n-3} \times \mathbb{R}^4, \ker(\widetilde{\alpha_{ot}} + r_1^2 d\theta_1 + r_2^2 d\theta_2))$ is the overtwisted at infinity contact structure on $\mathbb{R}^{2n+1}$. The complementary of L is overtwisted, thus L is loose. The result follows immediately from Corollary 6.3.2.

□

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