Extremal equilibria for reaction diffusion equations in bounded domains and applications.*

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1 Introduction

In this paper we consider the following model problem

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(x,u) & \text{in} \quad \Omega \\ \mathcal{B}u &=& 0 & \text{on} \quad \partial\Omega \\ u(0) &=& u_0 & \end{array} \right. \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous in (x, u) and locally Lipschitz in u, uniformly in x (we may also consider more general cases for f including singular terms, see (2.7), (2.8) below). We denote by $\mathcal{B}$ the boundary conditions operator which is either of the form

$$\mathcal{B}u = u \quad \text{(Dirichlet boundary conditions)}$$

or,

$$\mathcal{B}u = \frac{\partial u}{\partial \vec{n}} + b(x)u \quad \text{(Robin boundary conditions)}$$

with a suitable smooth function b with no sign condition which includes the case $b(x) \equiv 0$, i.e., Neumann boundary conditions. We will also consider nonlinear boundary conditions of the form

$$\mathcal{B}u \equiv \frac{\partial u}{\partial \vec{n}} + b(x)u = g(x,u)$$

with $g : \Gamma \times \mathbb{R} \to \mathbb{R}$ a continuous function in (x, u) and locally Lipschitz uniformly in $x \in \Gamma$. In the special case in which $g(x, u) = g(x)$ we have inhomogeneous boundary conditions.

We pose the problem in a Banach space X of functions on Ω. Namely, either

$$u_0 \in X = C(\overline{\Omega}) \quad \text{or} \quad u_0 \in X = H^{2\alpha,q}_{\mathcal{B}}(\Omega)$$

where we denote by $H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ the Bessel potential spaces associated to the Laplacian with the boundary conditions given by $\mathcal{B}$, see [3].

We denote by $u(t, x; u_0)$ the solutions of problem (1.1). Under certain conditions those solutions are smooth enough and globally defined. Thus, we can define a nonlinear semigroup

$$S(t) : X \to X$$

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given by

$$S(t)u_0 = u(t, x; u_0).$$

Our aim is to prove that for a wide class of dissipative equations of the type (1.1) there exist two extremal equilibria (one maximal, one minimal). Also, the asymptotic dynamics of the solutions enters between these extremal equilibria, uniformly in space, for bounded sets of initial data. As a consequence, we will obtain a bound for the global attractor of problem (1.1). Namely, for a wide class of nonlinearities we will prove the following result (see Theorem 3.2 for a more precise statement)

Theorem 1.1 *There exist ordered extremal equilibria for problem (1.1), φ_m and φ_M, minimal and maximal, respectively, in the sense of any other equilibrium, ψ, satisfies $\varphi_m \le \psi \le \varphi_M$. Furthermore, the ordered set $\{v \in X : \ \varphi_m \le v \le \varphi_M\}$ uniformly attracts the dynamics of the systems, i.e,*

$$\varphi_m(x) \le \liminf_{t\to\infty} u(t, x; u_0) \le \limsup_{t\to\infty} u(t, x; u_0) \le \varphi_M(x) \tag{1.2}$$

uniformly in $x \in \Omega$ and for bounded sets of initial data $u_0 \in X$. Moreover, the minimal equilibrium is asymptotically stable from below and maximal one is asymptotically stable from above.

Finally, there exists a global attractor $\mathcal{A}$ for problem (1.1) which satisfies

$$\varphi_m \le \mathcal{A} \le \varphi_M$$

and $\varphi_m, \varphi_M \in \mathcal{A}$.

For this, the main property we will ask f to satisfy is the following structure condition

$$sf(x, s) \le C(x)s^2 + D(x)|s| \quad \text{for all } x \in \Omega, \quad s \in \mathbb{R}$$

with C and D in suitable spaces of functions in Ω together with the exponential decay of the semigroup generated by $\Delta + C$ with the boundary conditions given by $\mathcal{B}$.

In the applications, it is important to consider non–negative solutions when $f(x, 0) \ge 0$. In this framework, under suitable instability conditions of the state $u = 0$ for (1.1) we will prove the existence of a minimal positive equilibrium. Namely (see Theorem 4.2 for a more precise statement),

Theorem 1.2 *Suppose $f(x, 0) \ge 0$. For non–negative solutions of problem (1.1) we have*

- *i) If there exists a bounded nontrivial nonnegative solution then there exists a minimal* ***non–negative*** *equilibrium φ_m^+. Furthermore, either $\varphi_m^+ \equiv 0$ (if $f(x, 0) = 0$ for all $x \in \Omega$), or $\varphi_m^+(x) > 0$ for all $x \in \Omega$ (if $f(x_0, 0) > 0$ for some $x_0 \in \Omega$).*
- *ii) Furthermore, if $f(x, 0) \equiv 0$ and 0 is unstable in certain sense, then 0 is an isolated equilibrium. In such case, if there exists a bounded nontrivial nonnegative solution then there exists a minimal* ***positive*** *equilibrium.*

In all the cases above, if there exists the minimal positive equilibrium, it is asymptotically stable from below.

A straightforward corollary from the results above is the following: if there exist a unique equilibrium for (1.1), that is, $\varphi_m \equiv \varphi_M$, then it is globally asymptotically stable. In Section 4.2 we give conditions for the uniqueness of positive equilibria. Namely, if

$$\frac{f(x,s)}{s} \quad \text{is decreasing and strictly decreasing in a set of positive measure}$$

then there exists at most one positive equilibrium. In particular, we extend a known result by Brézis and Oswald (see [9]).

These arguments, based on dynamical arguments associated to the evolution equation (1.1), allow to recover, as a particular case and in relatively straightforward way, some well-known results about existence of positive solutions for elliptic problems by Amann [1], Berestycki [7], Lions [7, 18], Figueiredo [13], Hernández [17]. These results were obtained in the references above by specific elliptic techniques. Therefore, our arguments give dynamical information (e.g, stability, existence and properties of a global attractor) to these classical results. We will review these results in Section 5.

As a particular example of equations where our techniques can be applied, we will consider logistic equations, where the nonlinear term is of the form

$$f(x, s) = m(x)s - n(x)|s|^{\rho-1}s$$

with $m \in L^p(\Omega)$, for $p > N/2$, $n \geq 0$ is a continuous function and $\rho > 1$. For these problems we obtain conditions on the coefficients for the existence of (a necessarily unique) positive equilibria. For this we will need to distinguish between the case in which n vanishes slowly in a small subset of Ω or n vanishes fast. In the former case no further requirements are needed on m, while in the latter m must help to the dissipation near the set where n vanishes, since there the reaction is linear; see Section 6.

The paper is organized as follows. In Section 2 we review some results about existence of solutions of problems like (1.1) that we will use in the following. Then, in Section 3, we prove an abstract result about existence of extremal equilibria. As a consequence we obtain the existence of extremal equilibria for problems with Dirichlet, Neumann, Robin and nonlinear boundary conditions, i.e., we prove Theorem 1.1. The case of nonnegative solution is considered in Section 4. There we prove Theorem 1.2 and we also give conditions for the uniqueness of positive equilibria. In Section 5 we recover some known results about existence of positive solutions of elliptic problems mentioned above, by using the dynamical techniques developed before. Finally, in Section 6, we apply the results in the previous sections to the particular case of logistic equations.

2 Preliminaries

We start summarizing some existence and uniqueness result for nonlinear reaction–diffusion problems including (1.1). We consider the problem

$$\begin{cases} u_t + Au &= f(x,u) \quad \text{in} \quad \Omega \\ \mathcal{B}u &= 0 \quad \text{on} \quad \partial\Omega \\ u(0) &= u_0 \end{cases} \tag{2.1}$$

with a suitable nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

We will consider either operators in divergence form of the type

$$Au \equiv -\mathrm{div}(a(x)\nabla u) + c(x)u \tag{2.2}$$

with $a, c \in C^1(\Omega)$ such that $a(x) \geq a_0 > 0$ for $x \in \Omega$ or operators of the form

$$Au \equiv -\sum_{i,j=1}^{N} a_{ij}(x)\partial_i\partial_j u + \sum_{i=1}^{N} a_i(x)\partial_i u + a(x)u, \tag{2.3}$$

with suitable smooth coefficients.

The boundary operator $\mathcal{B}$ is either

$$\mathcal{B}u = u \quad \text{(Dirichlet boundary conditions)}$$

or,

$$\mathcal{B}u = \frac{\partial u}{\partial \vec{n}} + b(x)u \quad \text{(Robin boundary conditions)}$$

with no restriction on the sign of $b \in C^1(\partial\Omega)$.

In addition, for differential operators (2.2) we can consider nonlinear boundary conditions

$$\mathcal{B}u \equiv a(x)\frac{\partial u}{\partial n} + b(x)u = g(x, u)$$

for certain nonlinear function $g(x, u)$ on the boundary. Notice that in particular, if $g(x, u) = g(x)$ the problem has non-homogeneous Robin boundary conditions.

The nonlinear term $f(x, u)$ (and $g(x, u)$ in the case of nonlinear boundary conditions) will satisfy certain regularity properties guaranteeing the existence of solutions for (2.1). We remark that regularity conditions on the coefficients and nonlinear terms are close related to the class of initial data considered.

From the results in Mora [21], Daners and Koch–Medina [11], Lunardi [20], Amann [4], we can consider the problem (2.1) with initial data in $X = L^\infty(\Omega)$ or $X = C(\overline{\Omega})$.

For that, we will assume that coefficients in (2.3) are uniformly continuous and bounded in Ω. In the case of Robin boundary conditions, we will also assume that $b(x)$ is uniformly continuous and bounded, together with its derivative, Lunardi [20], p. 75. In order to apply the results in Daners and Koch–Medina [11], we will assume that $a_{ij}, a_i, a \in C^\mu(\overline{\Omega})$ and $b \in C^{1,\mu}(\Gamma)$, for some $0 < \mu \leq 1$; see Daners and Koch–Medina [11], p. 24. Notice that the case of operators in divergence form (2.2), with bounded uniformly continuous coefficients can be found in Lunardi [20], p. 119 and references therein. The case of C^1 coefficients in the framework of $L^q(\Omega)$, $1 < q < \infty$, is widely developed in Amann [3]. A work with a large amount of particular examples is Henry [16].

Under these conditions, references above show that this type of operators are sectorial in X. Therefore, they define analytic semigroups in $X = L^q(\Omega)$ with $1 < q \leq \infty$, $X = C(\overline{\Omega})$, or $X = C_D(\overline{\Omega})$. The latter space denote the subspace of continuous bounded functions vanishing on the boundary, in the case of Dirichlet boundary conditions. To unify notations, we denote by

$$C_{\mathcal{B}}(\overline{\Omega}) = \begin{cases} C_D(\overline{\Omega}) & \text{if } \mathcal{B}(u) = u \\ C(\overline{\Omega}) & \text{in any other case.} \end{cases}$$

Notice that the case $f \equiv 0$ in (2.1) can be written in an abstract form as

$$\begin{cases} u_t + Au &= 0 \\ u(0) &= u_0 \in X \end{cases}$$

for different spaces of initial data X as above.

Denoting by $S(t)$ the semigroup generated by $-A$, we have that if $u_0 \in \overline{D(A)}^X$ then $S(t)u_0$ is continuous for $t \geq 0$. In particular, if $D(A)$ is not dense in X and $u_0 \in X \setminus \overline{D(A)}^X$ then there exists a unique solution of the problem which is not continuous in $t = 0$ but it is so for $t > 0$.

Thus, on the one hand, the semigroup in $C_{\mathcal{B}}(\overline{\Omega})$ or in $L^q(\Omega)$, $1 < q < \infty$, is continuous at $t = 0$. On the other hand, it is not continuous at $t = 0$ in $L^\infty(\Omega)$, nor in $C(\overline{\Omega})$, in the case of Dirichlet boundary conditions.

In addition, in all the references above it is proved the existence of certain intermediate spaces between X and the domain of the operator $D(A)$. We denote these intermediate spaces by X^α, $0 \leq \alpha \leq 1$. Their main properties are that the embedings

$$X^\alpha \subset X^\beta$$

are compact if $\alpha > \beta$, and the semigroup has an smoothing effect in the sense that if $\alpha > \beta$,

$$\|e^{-tA}u_0\|_{X^\alpha} \leq M \frac{e^{\delta t}}{t^{\alpha-\beta}} \|e^{-tA}u_0\|_{X^\beta}, \quad \text{for } t > 0, \tag{2.4}$$

for certain $M > 0$, $\delta \in \mathbb{R}$. In particular, the semigroup $S(t) = e^{-tA}$ is compact X^β for all $0 \leq \beta \leq 1$.

These kind of inequalities combined with Sobolev embedings lead to estimates of the type

$$||e^{-tA}u_0||_{L^r(\Omega)} \leq M \frac{e^{\delta t}}{t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}} ||u_0||_{L^q(\Omega)}, \qquad t > 0 \tag{2.5}$$

or certain $M > 0$, $\delta \in \mathbb{R}$, and $1 \leq q \leq r \leq \infty$.

Finally, the maximum principle holds for these semigroups, e.g. Daners and Koch–Medina [11], p. 120.

As a consequence, we have one of the main tools for our work: the monotonicity methods. Namely, for problems like (2.1), once local existence of solutions is proved, see below, the following monotonicity properties with respect to the initial data and the nonlinear term hold:

(1) Given two ordered initial data, the corresponding solutions remain ordered as long as they exist.

(2) Given two functions, f y g, we denote by u_f and u_g the solutions of problem (1.1) with right hand side f and g, respectively. If $f(t, x, s) \leq g(t, x, s)$ for all $t \geq 0$, $s \in \mathbb{R}$ a.e. $x \in \Omega$ then $u_f(t, x; u_0) \leq u_g(t, x; u_0)$ a.e. $x \in \Omega$ as long as they exist. (the same applies to nonlinear boundary conditions with obvious modifications).

(3) Solutions of the parabolic problem starting at a subsolution (resp. supersolution) of the associated elliptic problem are monotonically increasing (resp. decreasing) (see Lemma 2.9).

We can now summarize the existence results for the nonlinear problem (2.1). First, assume $u_0 \in C_{\mathcal{B}}(\overline{\Omega})$. Then, if $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous in (x, u) and locally Lipschitz in u uniformly in x then it can be proved the existence and uniqueness of local solutions for (2.1). This can be done using a fixed point argument, see Theorems 7.3.1 and 7.3.2 in Lunardi [20], p. 276-277, or Mora [21].

Theorem 2.1 *Let Ω be a bounded domain. Suppose that f is a continuous function in (x, u) and locally Lipschitz in u. Then, for any $u_0 \in X = C_{\mathcal{B}}(\overline{\Omega})$ there exists a local solution of the problem (2.1) which satisfies $u \in C([0,T]; X) \cap C((0,T); D(A))$, for certain $T > 0$. This solution is given by the variation of constant formula*

$$u(t, x; u_0) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(x, u(s, x))\, ds, \quad 0 < t \leq T. \tag{2.6}$$

On the other hand, for applications it is very useful to find a similar result with certain variations. Namely, it is very convenient to be able to solve (2.1) for initial data in $C(\overline{\Omega})\setminus C_{\mathcal{B}}(\overline{\Omega})$ or even in $L^\infty(\Omega)\setminus C_{\mathcal{B}}(\overline{\Omega})$. In addition it will be useful to be able to include some singular term in the equations.

Following this idea, we will assume that f has a decomposition of the from

$$f(x,s) = g(x) + m(x)s + f_0(x,s) \tag{2.7}$$

with $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ a locally Lipschitz function in $s \in \mathbb{R}$ uniformly respect to $x \in \Omega$ and

$$f_0(x,0) = 0, \qquad \frac{\partial}{\partial s} f_0(x,0) = 0; \tag{2.8}$$

g is a suitable regular function (say bounded, in order to simplify the arguments); and $m \in L^p(\Omega)$ for certain $p > N/2$.

Under these assumptions the following result holds.

Theorem 2.2 *Let Ω be a bounded domain of $\mathbb{R}^N$. Suppose that f satisfies (2.7) and (2.8). Then, for any $u_0 \in L^\infty(\Omega)$ there exists a local solution of the problem $u \in C((0,T); C_{\mathcal{B}}(\overline{\Omega}))$, for certain $T > 0$. This solution is given by the variation of constants formula (2.6).*

To prove the result, the idea is the following: given any bounded initial data u_0, we truncate f_0 in a proper way such that it is bounded and globally Lipschitz. Now, from the existence and uniqueness results in $L^q(\Omega)$ above, we get the existence of a solution in a weaker sense. Finally, it can be shown that the solution constructed above is bounded in a certain time interval. This last property follows from (2.5) with $q = r = \infty$ and $q = p$ and $r = \infty$ since, in that case, from (2.6)

$$\|u(t;u_0)\|_{L^\infty(\Omega)} \le Me^{\delta t}\|u_0\|_{L^\infty(\Omega)} + \int_0^t Me^{\delta(t-s)}\big(\|g\|_{L^\infty(\Omega)} + K\big)\mathrm{d}s +$$
$$+ \int_0^t M\frac{e^{\delta(t-s)}}{(t-s)^{\frac{N}{2p}}}\|m\|_{L^p(\Omega)}\|u(s;u_0)\|_{L^\infty(\Omega)}\mathrm{d}s$$

where K represents the bound for the truncation of f_0.

Now, since $p > N/2$, by the singular Gronwall Lemma, see Henry [16] we obtain that $\|u(t;u_0)\|_{L^\infty(\Omega)}$ is bounded for bounded time intervals. Thus, in the estimate above we have that for T small and $0 \le t \le T$, the right hand side term is as close as we want to $M\|u_0\|_{L^\infty(\Omega)}$. Therefore, the f_0 truncation can be chosen such that $u(t;u_0)$ is a real solution of the original problem (without truncation) in this interval.

Finally notice that in any of the cases of the theorems above, bootstrap techniques allow to conclude that the solution is actually more regular. In fact, if $m \in L^p(\Omega)$ with $p > N$ it can be proved that $u(t) \in W^{2,p}(\Omega)$ and then $u(t) \in C^{1,\theta}(\overline{\Omega})$ for certain $0 < \theta < 1$. If $N/2 < p < N$, we can conclude that $u(t) \in C^\theta(\overline{\Omega})$ for certain $0 < \theta < 1$.

Besides the class of bounded initial data, it is usual to consider other spaces of initial data for problem (2.1). As we mentioned before, differential operators together with boundary conditions we consider, define analytic semigroups in $L^q(\Omega)$, $1 < q < \infty$. Hence, it is possible to use the intermediate spaces and the semigroup properties (2.4) and (2.5) to solve the problem in this framework. This is an standard approach, even more common than the one presented above for bounded data; see Henry [16], Pazy [22].

In particular, it is well-known that for $L^q(\Omega)$, $1 < q < \infty$ a suitable family of intermediate spaces are the Bessel potential spaces, that we denote by $X = H^{2\alpha,q}_{\mathcal{B}}(\Omega)$, see Henry [16], Amann [3]. Using the properties of these spaces and of the Nemitsky operator associated to the nonlinear term between spaces of that family we have the following results (see Arrieta et al. [6]).

Theorem 2.3 *Suppose that f satisfies (2.7) y (2.8). In addition, assume that f_0 satisfies*

$$|f_0(x,s) - f_0(x,r)| \leq c(1 + |s|^{\rho-1} + |r|^{\rho-1})|s-r| \tag{2.9}$$

for all $x \in \Omega$, $s, r \in \mathbb{R}$, with $\rho \geq 1$ such that

1. *if $2\alpha - \frac{N}{q} < 0$ then*
$$1 \leq \rho \leq \rho_C = 1 + \frac{2q}{N - 2\alpha q};$$
2. *if $2\alpha - \frac{N}{q} = 0$ then*
$$1 \leq \rho < \rho_C = \infty;$$
3. *if $2\alpha - \frac{N}{q} > 0$ then no growth restriction on f_0 is assumed.*

Then, for all $u_0 \in X = H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ there exists a unique local solution $u(t,x;u_0) \in C([0,\tau), X)$, $\tau > 0$, of the problem with initial data u_0 in the sense of it satisfies the variations of constant formula (2.6). This solution is a classical solution for $t > 0$.

We now consider the problem (1.1) with nonlinear boundary conditions as considered in Arrieta et al. [5]. Let $\Omega \subset \mathbb{R}^N$ be a regular bounded domain. We denote the boundary by $\Gamma = \partial\Omega$. Suppose that $\Gamma = \Gamma_0 \cup \Gamma_1$ is a regular partition of the boundary, i.e, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$.

$$\left\{ \begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& f(x,u) & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \Gamma_0 \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& g(x,u) & \text{on} \quad \Gamma_1 \\ u(0) &=& u_0 & \end{array} \right. \tag{2.10}$$

with $a, b, c \in C^1(\Omega)$ such that $a(x) \geq a_0 > 0$ for $x \in \Omega$, $f(x,\cdot), g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are continuous functions (x,u) and locally Lipschitz in u uniformly in $x \in \Omega$ and $x \in \Gamma$ respectively.

We pose the problem in some space in the class

$$\mathcal{E} = \{L^q(\Omega), W^{1,q}_{\Gamma_0}(\Omega), 1 < q < \infty\}$$

where we denote by $W^{1,q}_{\Gamma_0}(\Omega)$ the subspace of functions in $W^{1,q}(\Omega)$ vanishing on Γ_0.

Suppose that f and g satisfies the following growth condition

$(\mathbf{G_X})$: $f(x,\cdot), g(x,\cdot) : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz uniformly in $x \in \Omega$ and $x \in \Gamma$ respectively. In addition,

1. If $X = L^q(\Omega)$, we assume that f and g satisfy a relation of the form
$$|j(x,u) - j(x,v)| \leq c|u-v|(|u|^{\rho-1} + |v|^{\rho-1} + 1) \tag{2.11}$$
with ρ_f and ρ_g exponents, respectively, such that, for $N \geq 2$ (resp. $N = 1$)
$$\rho_f \leq \rho_\Omega := 1 + \frac{2q}{N}, \quad \rho_g \leq \rho_\Gamma := 1 + \frac{q}{N} \quad (\text{resp. } \rho_g < \rho_\Gamma := 1 + q)$$
2. If $X = W^{1,q}_{\Gamma_0}(\Omega)$, we assume that any of the following assumptions holds
   (a) $q > N$

(b) $q = N$ and f, g satisfy that for any $\eta > 0$ there exists $c_\eta > 0$ such that

$$|j(x,u) - j(x,v)| \le c_\eta(e^{\eta|u|^{\frac{N}{N-1}}} + e^{\eta|v|^{\frac{N}{N-1}}})|u - v| \tag{2.12}$$

(c) $1 < q < N$ and f, g satisfies (2.11) with exponents

$$\rho_f \le \rho_\Omega := 1 + \frac{2q}{N-q} \quad \rho_g \le \rho_\Gamma := 1 + \frac{q}{N-q}.$$

The local existence and uniqueness of solutions now follows from the next result (see Theorem 2.1 in Arrieta et al. [5]).

Theorem 2.4 *Let X an space in the class $\mathcal{E}$. Assume that f and g satisfy the growth condition $(\mathbf{G_X})$. Then, for any $u_0 \in X$ there exists a unique local solution $u(t, x; u_0) \in C([0, \tau), X)$, $\tau > 0$, of the problem with initial data u_0 in the sense of the variations of constants formula. This solution is classical for $t > 0$.*

Moreover, if $u_0 \in X$ then $u(t, x; u_0) \in Y$ for all $Y \in \mathcal{E}$ and $0 < t < \tau$.

We remark that the main difficult in the problem with nonlinear boundary conditions comes from the presence of two different nonlinear terms and the need of using scales of dual spaces. On the reason that for these kind of problems we restrict to operator in divergence form.

2.1 Global existence.

In this section we will prove the global existence of solutions of problem (2.1) provided certain structure condition for the nonlinear term holds (see (2.13) below). Then, we can define a nonlinear semigroup $S(t) : X \to X$ by $S(t)u_0 = u(t, x; u_0)$ and $t \ge 0$.

We now prove a result giving a sufficient condition for the global existence of problem (2.1).

Theorem 2.5 *Suppose that problem (2.1) has a local solution for a given initial data $u_0 \in X$ with X be either $C_{\mathcal{B}}(\Omega)$, $L^\infty(\Omega)$ or $H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ (see Theorems 2.1, 2.2 and 2.3). Assume in addition that there exist $C \in L^p(\Omega)$, $p > N/2$ and $0 \le D \in L^r(\Omega)$, $r > N/2$ such that*

$$f(x,s)s \le C(x)s^2 + D(x)|s| \quad \textit{for all } x \in \Omega, \quad s \in \mathbb{R}. \tag{2.13}$$

Then, the solution of the problem (2.1) is globally defined for all $t > 0$. Furthermore, given any bounded set $B \subset X$ we have that for all $0 < \varepsilon \le T$, $\cup_{t\in[\varepsilon,T]} S(t)B \subset C_{\mathcal{B}}(\overline{\Omega})$ is a compact set in $X \cap C_{\mathcal{B}}(\overline{\Omega})$.

Proof. Since the local solution exists, it is enough to show that u is bounded for t in compact sets bounded away from $t = 0$. Let v be the solution of problem

$$\begin{cases} v_t + Av &= C(x)v + D(x) \quad \text{in} \quad \Omega \\ \mathcal{B}v &= 0 \quad \text{on} \quad \partial\Omega \\ v(0) &= u_0. \end{cases}$$

Hypothesis (2.13) plus $D(x) \ge 0$ imply, by the comparison principle, that the solution of the nonlinear problem, that we denote by u, satisfies

$$|u(t, x; u_0)| \le v(t, x; |u_0|)$$

as long as both solutions exists.

We denote by $T(t)$ the linear semigroup generated by $-A + C(x)$ with boundary conditions given by $\mathcal{B}$. Then, v is given by the variation of constant formula

$$v(t; u_0) = T(t)u_0 + \int_0^t T(t-s)D\mathrm{d}s$$

Taking now $L^\infty(\Omega)$ norms we have that, using (2.5), for all $0 < \varepsilon \leq t \leq T$,

$$||v(t; u_0)||_{L^\infty(\Omega)} \leq Me^{\delta t}t^{-\alpha}||u_0||_X + \int_0^t Me^{\delta(t-s)}(t-s)^{-\frac{N}{2}\frac{1}{r}}||D||_{L^r(\Omega)}\,\mathrm{d}s$$

for certain α.

Now, the right hand side term is bounded for t in compact intervals bounded away from 0 (the integral term is convergent since $r > N/2$). From here, the $L^\infty(\Omega)$ bound in $[\varepsilon, T]$ follows.

Finally, we use the variation of constant formula (2.6) and the fact that $f(\cdot, u(\cdot))$ is bounded in $L^p(\Omega)$ with $p > N/2$. Then, from the classical results on parabolic regularity we get the compactness of $\cup_{t\in[\varepsilon,T]}S(t)B$ in $C_{\mathcal{B}}(\overline{\Omega})$ since it is bounded in $C^\theta(\overline{\Omega})$ for $0 < \theta \leq 1$. Now, (2.6) and (2.4) implies the compactness in X. ■

Notice that in the proof above, if $m \in L^p(\Omega)$, with $p > N$, in (2.7) then $\cup_{t\in[\varepsilon,T]}S(t)B$ is bounded in $C^{1,\theta}(\overline{\Omega})$.

In the case of operators in divergence from with nonlinear boundary conditions we have the following result about existence of global solutions (see Theorem 2.5 in Arrieta et al. [5]).

Theorem 2.6 *Suppose that there exists a local solution of problem (2.10) with initial data $u_0 \in X \in \mathcal{E}$. Assume in addition that there exist $C_0 \in L^p(\Omega)$, $p > N/2$, $0 \leq C_1 \in L^r(\Omega)$, $r > N/2$, $B_0 \in L^\sigma(\Gamma_1)$, $\sigma > N-1$ y $0 \leq B_1 \in L^\rho(\Gamma_1)$, $\rho > N-1$ such that*

$$f(x,s)s \leq -C_0(x)s^2 + C_1(x)|s| \quad \text{for all } x \in \Omega, \quad s \in \mathbb{R}; \tag{2.14}$$

$$g(x,s)s \leq -B_0(x)s^2 + B_1(x)|s| \quad \text{for all } x \in \Gamma_1, \quad s \in \mathbb{R}. \tag{2.15}$$

Then, the solution of problem (2.10) is globally defined for all $t > 0$. Moreover, for any bounded set $B \subset X$, we have that for all $0 < \varepsilon \leq T$, the set $\cup_{t\in[\varepsilon,T]}S(t)B \subset C_{\mathcal{B}}(\overline{\Omega})$ is a compact set in $X \cap C_{\mathcal{B}}(\overline{\Omega})$.

Remark 2.7 *Notice that from Theorems 2.5 and 2.6 solutions enters in a bounded set of $C_{\mathcal{B}}(\overline{\Omega})$. In particular, we can always work in $X = L^\infty(\Omega)$, $X = C(\overline{\Omega})$ or $X = C_{\mathcal{B}}(\overline{\Omega})$.*

2.2 A monotonicity lemma

As we said before we will use monotonicity of problem (1.1). In particular, we have that if $u_0 \leq v_0$ are two initial data, the corresponding solutions of (1.1) starting at them are ordered as long as they exist, that is, $u(t,x;u_0) \leq u(t,x;v_0)$. Also, if $f \leq g$ are two non-linear terms then $u_f(t,x;u_0) \leq u_g(t,x;u_0)$ as long as both solutions exists.

We prove now a lemma that will be useful in the following. For this we need the following results from Arrieta et al. [5] (see Theorem A.12 in [5])

Definition 2.8 *i) We say that $v \in C([0,\delta), X)$ is a* subsolution *for (1.1) if*

$$v(t) \leq S(t)v(s) + \int_s^t S(t-r)f(\cdot, v(r))\,dr \quad \text{for all} \quad 0 \leq s < t < \delta$$

where $S(t)$ is the linear semigroup generated by Δ. We say that v is a supersolution *if the reversed inequality holds.*
ii) We say that $v \in X$ is a subsolution *for the elliptic problem associated to (1.1) if*

$$v \leq \int_0^\infty S(t-r) f(\cdot, v)\, dr$$

where $S(t)$ is the linear semigroup generated by Δ. We say that v is a supersolution *if the reversed inequality holds.*

Then we have

Lemma 2.9 *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ locally Lipschitz. Let $v(t)$ a subsolution of (1.1). Then,*

$$v(t) \leq u(t; v(0))$$

while both solutions exist. The result for supersolutions is analogous with reverse inequality.

In particular, if $\underline{u} \in X$ a subsolution of the elliptic problem

$$\begin{cases} -\Delta u &= f(x,u) \quad in \quad \Omega \\ \mathcal{B}u &= 0 \qquad\quad on \quad \partial\Omega \end{cases} \tag{2.16}$$

then the solution starting at $\underline{u}$ is monotonically increasing.

If $\overline{u}$ is a supersolution of the elliptic problem, then the solution of the parabolic problem starting at $\overline{u}$ is monotonically decreasing.

Proof. We prove the result for subsolutions. The one for supersolutions is analogous.

The first part was proved in Theorem A.12 in [5].

Now, since $\underline{u}$ is a subsolution of the elliptic problem (2.16) we have that it is also so for the associated parabolic problem. Now, from the first part of the theorem we have

$$\underline{u} \leq u(s, x; \underline{u})$$

while solution u exists. Using now monotonicity we have

$$u(t, x; \underline{u}) \leq u(t+s, x; \underline{u})$$

for all $t, s \geq 0$, while both solutions exists. In particular, s is as small as we want.

So, $u(t, x; \underline{u})$ is decreasing in time. ■

Remark 2.10 *This result is an extension of Theorem 3.3 (for $C^{2,\alpha}(\Omega)$ solutions) and Theorem 3.4 (for $L^2(\Omega)$ solutions) in Sattinger [23] (p. 986) where the author proves the result in the classical framework which also assumes regularity of sub-supersolutions.*

3 Extremal equilibria

In this section we obtain the Theorem 1.1 for problem (1.1). First, we prove a general theorem for semigroups and then we apply it to the particular cases of the problems above.

3.1 An abstract result based on monotonicity and compactness.

We now show an abstract theorem giving sufficient conditions for the existence of extrema equilibria. The result will also give information about the dynamics of the problem. Fist, we prove a lemma that will be useful.

Lemma 3.1 *Let $S(t) : X \to X$ be a continuous semigroup for all $t > 0$. Assume that $u_0, v \in X$ are such that $S(t)u_0 \to v$ in X as $t \to \infty$. Then v is an equilibrium point for $S(t)$.*

Proof. From the assumptions $v = \lim_{t\to\infty} S(t)u_0$. Then, letting the system evolve and using the continuity of $S(t)$ for $t > 0$,

$$S(s)v = S(s) \lim_{t\to\infty} S(t)u_0 = \lim_{t\to\infty} S(s+t)u_0 = v.$$

That is, $S(s)v = v$ for all $s > 0$. Thus, v is an equilibrium point for the system. ∎

We now prove the main result.

Theorem 3.2 *Let $S(t)$ be an order-preserving continuous semigroup for $t > 0$ defined in a ordered complete metric space X. Assume that the order intervals in X are bounded in the norm of X. Suppose that either*

1. *any decreasing sequence order-bounded from below (resp. increasing order-bounded from above) is convergent in the norm of X;*
2. *or the semigroup $S(t)$ is asymptotically compact (see Hale [15]).*

Assume in addition that there exists an absorbing order interval, i.e, there exists two ordered elements $\eta_m, \eta_M \in X$ such that for all bounded set $B \subset X$ there exists $0 < T = T(B)$ such that for all $u_0 \in B$

$$\eta_m \leq S(t)u_0 \leq \eta_M$$

for all $t \geq T(B)$.

Then, there exist two ordered extremal equilibria $\varphi_m \leq \varphi_M$ such that any other equilibria ψ satisfies $\varphi_m \leq \psi \leq \varphi_M$. Furthermore, the set $\{v \in X : \ \varphi_m \leq v \leq \varphi_M\}$ uniformly attracts the dynamics of the system, i.e,

$$\varphi_m \leq \liminf_{t\to\infty} S(t)u_0 \leq \limsup_{t\to\infty} S(t)u_0 \leq \varphi_M \tag{3.1}$$

uniformly for bounded sets in X.

If, in addition, $S(t)$ is asymptotically compact then there exists the global attractor $\mathcal{A}$ for $S(t)$ which satisfies

$$\varphi_m \leq \mathcal{A} \leq \varphi_M.$$

Moreover, $\varphi_m, \varphi_M \in \mathcal{A}$.

Remark 3.3 *Requiring that order interval are bounded is equivalent to requiring that the positive cone is* normal *(see Theorem 1.5 in Amann [1], p. 627). We recall that a positive cone is* normal *if there exists a constant $\delta > 0$ such that if $0 \leq x \leq y$ then $||x|| \leq \delta ||y||$ for all $x, y \geq 0$ (i.e, the norm is* semimonotone*).*

For example, $L^p(\Omega)$ spaces, $1 \leq p \leq \infty$, have normal positive cone (with $\delta = 1$) and satisfies condition i) from the Theorem since due to Lebesgue's Theorem any decreasing sequence order-bounded from below (resp. increasing order-bounded from above) is convergent in the norm of X.

The space $C(\overline{\Omega})$ also has normal positive cone. However, not any decreasing sequence order–bounded from below is convergent. In that case, the asymptotic compactness of the semigroup is needed in order to apply the result.

Proof. Let $I = [\eta_m, \eta_M]$. Since I is an absorbing set there exists a time $T \geq 0$ such that

$$\eta_m \leq S(t+T)\eta_M \leq \eta_M \tag{3.2}$$

for all $t \geq 0$. In particular $S(t)\eta_M$ in bounded from below for all time from T on. Using now the order-preserving property of the semigroup and (3.2) we have

$$\eta_m \leq S(2T)\eta_M \leq S(T)\eta_M \leq \eta_M.$$

And by iterating the process

$$\eta_m \leq S(nT)\eta_M \leq S((n-1)T)\eta_M \leq \cdots \leq S(T)\eta_M \leq \eta_M \tag{3.3}$$

for all $n \in \mathbb{N}$. Thus, $\{S(nT)\eta_M\}_n$ is a monotonically decreasing sequence bounded from below (by η_m).

Thus, in any of the cases i) or ii) of the theorem there exists an increasing sequence $\{n_k\}$ going to ∞ such that $S(n_k T)\eta_M \to \varphi_M$ for certain element φ_M.

We show now that $\{S(nT)\eta_M\}_n$ actually converges to some element φ_M since the sequence is decreasing. Namely, suppose that there exists another subsequence $\{\tilde{n}_k\} \uparrow \infty$ such that $S(\tilde{n}_k T)\eta_M \to \psi$ for certain element $\psi \in X$. We can assume, without lack of generality $\tilde{n}_k \leq n_k \leq \tilde{n}_{k+1}$. By (3.3) we have

$$S(\tilde{n}_k T)\eta_M \leq S(n_k T)\eta_M \leq S(\tilde{n}_{k+1} T)\eta_M.$$

Then, taking limits as k to ∞ we have $\psi \leq \varphi_M \leq \psi$. Thus $\psi = \varphi_M$. Hence, we have proved that $S(nT)\eta_M$ converges to a unique element $\varphi_M \in X$.

Now we prove that, if fact, the whole solution $S(t)\eta_M$ converges to φ_M as $t \to \infty$. By (3.2) we have that, in particular,

$$S(T+t)\eta_M \leq \eta_M \tag{3.4}$$

for all $0 \leq t < T$. Let $\{t_n\}$ a time sequence tending to infinity. We can write $t_n = k_n T + \tau_n$ with $k_n \in \mathbb{N}$ and $0 \leq \tau_n < T$. We can assume that $\{k_n\}$ is a strictly increasing sequence. Then, on the one hand, taking $t = \tau_n$ in (3.4) we have

$$S(T+\tau_n)\eta_M \leq \eta_M.$$

And applying the semigroup at time $(k_n - 1)T$ on both sides we have,

$$S(t_n)\eta_M \leq S((k_n-1)T)\eta_M. \tag{3.5}$$

On the other hand, given any $0 \leq s < T$ we can take $t = T - s$ in (3.4) and let the semigroup act at time s to obtain

$$S(2T)\eta_M \leq S(s)\eta_M \quad \text{for all } 0 \leq s \leq T.$$

Now, letting the semigroup act at time $k_n T$ and taking $s = \tau_n$ we have,

$$S((k_n+2)T)\eta_M \leq S(t_n)\eta_M \tag{3.6}$$

Then, taking limits as n goes to infinity inequalities (3.5) and (3.6) we get

$$\lim_{n\to+\infty} S(t_n)\eta_M = \varphi_M.$$

Since the previous argument is valid for any time sequence $\{t_n\}$ we actually have

$$\lim_{t\to+\infty} S(t)\eta_M = \varphi_M.$$

From Lemma 3.1 we have than φ_M is an equilibrium point.

Moreover, we know that given any bounded set of initial data in X all the solutions starting at this set enter in *finite time* below η_M. We also know that the solution starting at η_M converges to φ_M. Then, property (3.1) holds uniformly for bounded sets in X.

Finally, let ψ another equilibrium. By (3.1), with $u_0 = \psi$ we get $\psi \leq \varphi_M$. Thus, φ_M is maximal in the set of equilibrium points, i.e, for any equilibrium, ψ, we have $\psi \leq \varphi_M$.

The results for φ_m can be obtained in an analogous way.

In the case that $S(t)$ is asymptotically compact, the existence of the attractor is obtained from Theorem 3.4.6 in Hale [15]. ■

Remark 3.4 *If we assume the existence of an absorbing order interval for points instead of bounded sets the results remain valid except the uniformity in bounded sets of X.*

Remark 3.5 *Once the result is established we can apply the results in Smith [24] to obtain that, under certain hypothesis, the generic convergence of solutions starting at any point in $[\eta_m, \eta_M]$ to a unique equilibrium point.*

3.2 Linear homogeneous boundary conditions

We now consider the problem (1.1) with linear homogeneous boundary conditions (Dirichlet, Neumann or Robin) under assumptions of Theorem 2.1, 2.2 or 2.3 so that the local existence for (1.1) holds.

We start with an uniform estimate on the asymptotic behaviour of the solutions of the nonlinear problem (1.1) in terms of the unique solution of a linear problem coming from (2.13). Notice that hypotheses in the next theorem are stronger than those in Theorem 2.5.

Theorem 3.6 *Suppose that f satisfies (2.7) and (2.8) with f_0 a continuous function in (x, u), locally Lipschitz in u. Assume that there exist $C \in L^p(\Omega)$, $p > N/2$, and $0 < D \in L^r(\Omega)$, $r > N/2$, such that*

$$sf(x, s) \leq C(x)s^2 + D(x)|s| \tag{3.7}$$

for all $s \in \mathbb{R}$ and $x \in \Omega$.

Let X denote either $C(\overline{\Omega})$, $L^\infty(\Omega)$ or $H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ and assume that the semigroup $S(t)$ generated by $\Delta + C(x)$ (with boundary conditions given by $\mathcal{B}$) in X has exponential decay.

Then, there exists a unique solution of the following problem

$$\left\{ \begin{array}{rcll} -\Delta\phi & = & C(x)\phi + D(x) & \text{in} \quad \Omega \\ \mathcal{B}\phi & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{3.8}$$

Furthermore, this solution is positive, belongs to $C_{\mathcal{B}}(\overline{\Omega})$ and solutions of the nonlinear problem (1.1) satisfies

$$\limsup_{t\to\infty} |u(t, x; u_0)| \leq \phi(x) \tag{3.9}$$

uniformly in $x \in \Omega$ for u_0 in bounded sets of X.

Proof. The proof follows the arguments of Theorem 5.1 in Arrieta et al. [6]. ■

In the case that f satisfies the growth condition (2.9), the hypothesis on D can be weakened in order to obtain the existence of the global solution. Following the argumentation of Theorem 5.2 in Arrieta et al. [6] can be obtained the following result

Notice that in Arrieta et al. [6] the setting was on unbounded domains. However, all the arguments carry out, with certain simplifications, for the case of bounded ones.

Theorem 3.7 *Suppose that f satisfies (2.7) and (2.8) with f_0 a continuous function in (x, u), locally Lipschitz in u and satisfies (2.9). Also assume that f satisfies (3.7) with $C \in L^p(\Omega)$, $p > N/2$ and*

$$D \in L^r(\Omega) \quad \text{with} \quad r > \left(1 - \frac{1}{\rho}\right)\frac{N}{2}.$$

Let X denote either $C(\overline{\Omega})$, $L^\infty(\Omega)$ or $H^{2\alpha,q}_{\mathcal{B}}(\Omega)$ and assume that the semigroup $S(t)$ generated by $\Delta + C(x)$ (with boundary conditions given by $\mathcal{B}$) in X has exponential decay.

Then, there exists a unique solution $0 \leq \phi(x) \in L^\infty(\Omega) \cap C_{\mathcal{B}}(\overline{\Omega})$ of (3.8). Moreover, solutions of (1.1) satisfy

$$\limsup_{t \to +\infty} |u(t, x; u_0)| \leq \phi(x) \tag{3.10}$$

uniformly in x for u_0 in bounded sets of initial data X.

Remark 3.8 *The exponential decay of the semigroup $S(t)$ is equivalent to the positivity for the first eigenvalue of $-\Delta - C$ with the corresponding boundary conditions given by $\mathcal{B}$.*

Remark 3.9 *Properties (3.9) and (3.10) give, in particular, an $L^\infty(\Omega)$ bound for solutions at large time. Namely, given a bounded set $B \subset X$ and any $\varepsilon > 0$ there exists a time $T = T(B) > 0$ such that for all $t \geq T$,*

$$-\phi(x) - \varepsilon \leq u(t, x; u_0) \leq \phi(x) + \varepsilon$$

for all $x \in \Omega$ and $u_0 \in B$. Therefore, we have the existence of a bounded absorbing set for the supremum norm.

As a consequence of Theorems 3.6 and 3.2 we have

Corollary 3.10 *Under the assumptions of Theorem 3.6 or 3.7, problem (1.1) has two extremal equilibria and the conclusions of Theorem 1.1 hold.*

Proof. Just notice that the semigroup is asymptotically compact (see Hale [15]) and from Theorem 3.6, taking $[\eta_m, \eta_M] = [-\phi - \delta, \phi + \delta]$ the assumptions in Theorem 3.2 hold. ■

Another straightforward consequence of Theorem 3.2 is the following.

Corollary 3.11 *Suppose that there exists a bounded absorbing set in $L^\infty(\Omega)$ for problem (1.1). Then, there exists two extremal equilibria and conclusions of Theorem 1.1 hold.*

Proof. Let $R > 0$ such that the absorbing set in the statement is contained in the ball of radius R centered at 0. Then, just taking $\eta_m = -R$, $\eta_M = R$ we are in the hypothesis of Theorem 3.2 with $X = C_{\mathcal{B}}(\overline{\Omega})$ or $X = L^\infty(\Omega)$. ■

Remark 3.12 *Notice that extremal equilibria can be sign-changing. As an example, we consider the problem (1.1) with Dirichlet boundary conditions.*

Assume that f satisfies (3.7) so that, from Theorem 3.6 and Corollary 3.10, there exist a maximal equilibrium φ_M and a minimal one φ_m.

If $f(x,0) \geq 0$ then the maximal equilibrium is non-negative. If $f(x,0) \leq 0$ then $\varphi_m \leq 0$. Thus, for the extremal equilibria to be sign-changing $f(x,0)$, must be sign-changing in Ω.

So, suppose $f(x,0) = 0$ and that φ_M is a positive equilibrium. Now, by a simple translation of the solutions of the former problem we can construct a sign-changing maximal equilibrium point. Namely, we take $h \in C_0^\infty(\Omega)$ such that $\varphi_M + h$ is sign-changing. Then, $v = u - h$ satisfies (1.1) with Dirichlet boundary conditions and nonlinear term $g(x,v) = f(x, v + h(x)) + \Delta h(x)$

Moreover, for this problem, the equilibria are of the form $\varphi - h$ where φ is an equilibrium of the original problem. Therefore, $\varphi_M - h$ is the maximal equilibrium for the new problem and it is sign-changing by construction. An analogous argument leads to the result for the minimal equilibrium.

It is easy to check that the new reaction term also satisfies (3.7).

Finally, notice that if both extremal equilibria are sign-changing any equilibria is so.

3.3 Convergence in more regular spaces

In the previous results we have obtained the uniform convergence from above to the maximal equilibria, i.e, in $C(\overline{\Omega})$. By the variation of constant formula and the smoothing effect of the semigroup convergence in more regular spaces can be obtained.

For this, we pose the problem in $L^q(\Omega)$. So, we have a scale of fractional power spaces X^α associated to $-\Delta$ with the corresponding boundary conditions given by $\mathcal{B}$ (see Henry [16] and Lunardi [20]). This scale satisfies $X^\alpha \subset W^{2\alpha,q}(\Omega)$, $\alpha < 1$, with continuous embedding. Thus, convergence in X^α implies convergence in $W^{2\alpha,q}(\Omega)$. But, by the Sobolev embedings we know that $W^{2\alpha,q}(\Omega) \subset C^{k,\theta}(\Omega)$, $k \in \mathbb{N}$, $0 \leq \theta \leq 1$, for $k + \theta < 2\alpha - N/q$.

So, if q is large enough we have $1 + \theta < 2 - \varepsilon - N/q$ for ε small enough and θ close to 1. Thus, X^α is continuously embedded in $C^{1,\theta}(\overline{\Omega})$ with θ as close to 1 as we want (taking ε smaller if needed). Therefore, convergence in X^α implies convergence in $C^{1,\theta}(\overline{\Omega})$. From the uniform convergence we will obtain, by mean of the variation of constants formula, the convergence in X^α and then in $C^{1,\theta}(\overline{\Omega})$.

To carry out the argument we will assume that $f(x,u)$ in a continuous function in (x,u), locally Lipschitz in the second argument. However, the same argument can be carried out for nonlinearities satisfying (2.7) and (2.8) with $m \in L^p(\Omega)$ for certain $p > N$. If $N/2 < p < N$ we can only conclude the $C^\theta(\overline{\Omega})$ convergence for certain $0 < \theta < 1$.

Theorem 3.13 *The maximal equilibrium point φ_M from Corollary 3.10 is globally asymptotically stable from above in $C^{1,\theta}(\overline{\Omega})$ norm for all $0 < \theta < 1$. An analogous result holds for the minimal equilibrium.*

Proof. It is clear that we just have to prove the convergence of the derivatives. For this, we can assume that $u_0 \geq \varphi_M$ is a regular initial data and that we have truncated f so that it is globally Lipschitz. Also, we can assume that we replace A and f by $A + \lambda I$ and $f + \lambda I$ with λ large enough so that $\delta < 0$ in (2.4).

Since φ_M is an equilibrium we have, from the variation of constants formula,

$$\varphi_M = \mathrm{e}^{A(t-\tau)}\varphi_M + \int_r^t \mathrm{e}^{A(t-s)} f(\varphi_M)\mathrm{d}s$$

for all $t \geq r$. Thus,

$$u(t) - \varphi_M = \mathrm{e}^{A(t-\tau)}(u(\tau) - \varphi_M) + \int_\tau^t \mathrm{e}^{A(t-s)}(f(u(s)) - f(\varphi_M))\mathrm{d}s.$$

Taking $X^\alpha \subset L^q(\Omega)$ norms,

$$||u(t) - \varphi_M||_\alpha \leq M\frac{\mathrm{e}^{\delta(t-\tau)}}{(t-r)^\alpha}||u(\tau) - \varphi_M||_\alpha + M\int_\tau^t \frac{\mathrm{e}^{\delta(t-s)}}{(t-s)^\alpha}||f(u(s)) - f(\varphi_M)||_0\mathrm{d}s.$$

Now, since f is Lipschitz and $u(t), \varphi_M \in L^\infty(\Omega)$,

$$||u(t) - \varphi_M||_\alpha \leq M\frac{\mathrm{e}^{\delta(t-\tau)}}{(t-r)^\alpha}||u(\tau) - \varphi_M||_{L^q(\Omega)} + ML\int_\tau^t \frac{\mathrm{e}^{\delta(t-s)}}{(t-s)^\alpha}||u(s) - \varphi_M||_{L^q(\Omega)}\mathrm{d}s.$$

So,

$$||u(t) - \varphi_M||_\alpha \leq \frac{M}{(t-\tau)^\alpha}||u(r) - \varphi_M||_0 + ML\int_\tau^t \frac{\mathrm{e}^{\delta(t-r)}}{(t-s)^\alpha}||u(s) - \varphi_M||_{L^q(\Omega)}\mathrm{d}s.$$

Now, given $\varepsilon > 0$ there exists a time $\tau > 0$ such that for all $s \geq \tau$

$$||u(s) - \varphi_M||_{L^q(\Omega)} < \varepsilon.$$

Hence, if $t \geq \tau + 1$ we have, since $\delta < 0$,

$$||u(t;u_0) - \varphi_M||_\alpha \leq C\sup_{s\geq\tau}||u(\tau;u_0) - \varphi_M||_{L^q(\Omega)} < C\varepsilon.$$

Therefore, we obtain X^α convergence, $\alpha < 1$. Since for q large enough, $X^\alpha \subset C^{1+\theta}(\overline{\Omega})$ the convergence of the derivatives holds. ■

3.4 Inhomogeneous and nonlinear boundary conditions.

We now consider the problem (2.10) with nonlinear boundary conditions posed in either $X = L^q(\Omega)$ or $X = W^{1,q}_{\Gamma_0}(\Omega)$ as in Section 2.1.

Suppose that the assumptions in Theorem 2.6 hold. Also assume that the following dissipativity condition holds

(**D**) : The first eigenvalue, λ_1, of the next problem is positive

$$\begin{cases} -\mathrm{div}(a(x)\nabla u) + (c(x) + C_0(x))u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_0 \\ a(x)\frac{\partial u}{\partial \vec{n}} + (b(x) + B_0(x))u = 0 & \text{on } \Gamma_1 \end{cases} \qquad (3.1)$$

Remark 3.14 *Property (D) is nothing else but the exponential decay of the linear semigroup generated by $\Delta + C$ with the corresponding boundary conditions.*

Notice that require exponential decay for the linear semigroup impose conditions not only C_0 but also on B_0.

We have the following result for (2.10), analogous to Theorem 3.6 for linear boundary conditions (see Proposition 2.5 in Arrieta et al. [5]).

Theorem 3.15 *Under the assumptions above, there exists a unique solution ϕ of problem*

$$\left\{\begin{array}{rcll} -\mathrm{div}(a(x)\nabla\phi)+(c(x)+C_0(x))\phi &=& C_1(x) & in \quad \Omega \\ \phi &=& 0 & on \quad \Gamma_0 \\ a(x)\frac{\partial\phi}{\partial\vec{n}}+(b(x)+B_0(x))\phi &=& B_1(x) & on \quad \Gamma_1 \end{array}\right. \tag{3.2}$$

Furthermore, $0\le\phi\in L^\infty(\Omega)\cap C(\overline{\Omega})$ and, if we denote by $u(t,x;u_0)$ the solution of (2.10),

$$\limsup_{t\to\infty}|u(t,x;u_0)|\le\phi(x) \quad \textit{uniformly in } x\in\overline{\Omega} \tag{3.3}$$

where the limit above is uniform for bounded sets of initial data $u_0\in X$.

In particular, if $g(x,u)\equiv g(x)$ we have inhomogeneous Robin boundary conditions. Then, we can take $B_0(x)\equiv 0$ and $B_1(x)=|g(x)|$.

Once we have the estimate (3.3) since $\phi\in C(\overline{\Omega})$ we can truncate f and g and assume that f and g in (2.10) are globally Lipschitz. Also, we can assume that $X=L^q(\Omega)$ and (3.3) remains valid. So, even if $\Gamma_0\ne\emptyset$ we can assume that $\phi+\delta\in X$ and the order interval $[-\phi-\delta,\phi+\delta]\cap X$ is an absorbing set.

Because of the smoothing property of the semigroup, orbits of bounded sets are relatively compact in X (see Hale [15]). Then, we can apply Theorem 3.2 to get that conclusions in Theorem 1.1 are valid in this case. Thus, we have

Theorem 3.16 *Under the assumptions above the results of Theorem 1.1 hold for problem (2.10) in $X\in\mathcal{E}$.*

Then, similar to Theorem 3.13, convergence in $W^{1,q}_{\Gamma_0}(\Omega)$ is obtained.

4 Non-negative solutions

In this section we consider only non-negative solutions of (1.1). We begin with some direct consequence of the results in the previous sections. For this, we consider problem (1.1) with homogeneous linear boundary conditions.

We assume that f satisfies (2.7) and (2.8). We also assume that $f(x,0)\ge 0$ so that the nonlinear semigroup generated by the solutions of (1.1) preserves the positivity, i.e, for any $u_0\ge 0$, $u(t,x;u_0)\ge 0$ for all $t\ge 0$.

An interesting question is when problem (1.1) has a positive equilibrium even in the case $f(x,0)=0$. The next result gives a sufficient condition for the existence of positive equilibrium in that case. Essentially, we need the global boundness of solutions starting near of 0 as well as the instability of 0 to avoid the case in which the maximal equilibrium was 0.

Proposition 4.1 *Suppose f as in Theorem 3.6 or 3.7. Also assume that $f(\cdot,0)\equiv 0$.*

Assume in addition that 0 *is unstable for problem (1.1) and the semigroup generated by $\Delta+C$ has exponential decay.*

Then there exists a maximal equilibrium for (1.1) which is positive. In that case, the minimal non-negative equilibrium is 0*.*

Proof. From Corollary 3.10 we know that there exists a maximal equilibrium φ_M which is asymptotically stable from above. We recall that we obtain the existence of φ_M as the limit of $u(t,x;\phi)$ (see (3.8)). Since $f(\cdot,0)\equiv 0$ wee have that φ_M is non-negative, by the comparison principle. Now, using that 0 is unstable we have that φ_M is positive since otherwise $\varphi_M\equiv 0$

(by the maximum principle φ_M is either 0, or positive). Then, 0 would be stable which is a contradiction. ■

In the case of nonlinear boundary condition the result is also valid provided $f(\cdot,0) \equiv g(\cdot,0) \equiv 0$.

4.1 Existence of a minimal positive equilibrium.

In this section, we study the existence of minimal positive equilibrium. In particular, we will prove Theorem 1.2. We will prove that in case $u = 0$ is an equilibrium for (1.1), i.e, $f(x,0) \equiv 0$, the existence of a bounded solution plus certain instability property of 0 (see (4.1) in Theorem 4.2 below) implies the existence of a minimal positive equilibrium.

Note that part ii) improves somehow the conclusions of Proposition 4.1.

Theorem 4.2 *Consider problem (1.1) with homogeneous linear boundary conditions. Suppose that f satisfies (2.7) and (2.8), with f_0 a continuous function in (x,u), locally Lipschitz in u and $f(x,0) \geq 0$. Then,*

i) If there exists a bounded nontrivial nonnegative solution then there exists a minimal ***non-negative*** *equilibria φ_m^+. Even more, $\varphi_m^+ \equiv 0$ (if $f(x,0) = 0$ a.e. $x \in \Omega$), or $\varphi_m^+(x) > 0$ for all $x \in \Omega$ (if $f(x_0,0) > 0$ for some $x_0 \in \Omega$).*

ii) Furthermore, assume $f(x,0) \equiv 0$ and there exists $M \in L^p(\Omega)$ with $p > N/2$ such that

$$f(x,s) \geq M(x)s \quad a.e.\ x \in \Omega, \quad 0 \leq s \leq s_0. \tag{4.1}$$

Also assume that $M(x)$ is such that 0 is unstable for problem

$$\begin{cases} v_t - \Delta v = M(x)v & in \quad \Omega \\ \mathcal{B}v = 0 & on \quad \partial\Omega \end{cases} \tag{4.2}$$

which is equivalent to $\lambda_1(-\Delta - M(x)) < 0$ where we denote by $\lambda_1(-\Delta - M(x))$ the first eigenvalue of $-\Delta + M(x)$ with the corresponding boundary conditions given by $\mathcal{B}$.

Then, 0 is an isolated equilibrium point. If there exists a bounded nontrivial nonnegative solution then there exists a minimal ***positive*** *equilibrium φ_m^+.*

In any of the previous cases, if there exists the minimal positive equilibrium, it is asymptotically stable from below.

As a consequence of Proposition 4.1 and the theorem above we have

Corollary 4.3 *Suppose the assumptions of point i) with $f(x,0) \not\equiv 0$ or point ii) in the previous theorem hold. Also assume that f satisfies the assumption in Theorem 3.6 or 3.7 In addition, assume that the semigroup generated by $\Delta + C(x)$ has exponential decay.*

Then, there exists two ordered extremal positive equilibria $0 < \varphi_m^+ \leq \varphi_M$ (which may coincide). Moreover, φ_m^+ is asymptotically stable from below and φ_M it is so from above.

Furthermore, there exists an attractor for positive solutions $\mathcal{A}_+$ which satisfies

$$\mathcal{A}_+ \subset [\varphi_m^+, \varphi_M]$$

and $\varphi_m^+, \varphi_M \in \mathcal{A}_+$.

For nonlinear boundary conditions, the result in the case of point i) is valid for operators in divergence form provided that the nonlinearity on the boundary satisfies $g(\cdot, 0) \geq 0$ with either $f(x, 0)$ or $g(x, 0)$ not identically zero and condition (D) from Section 3.4 holds.

Notice that if 0 is an equilibrium, i.e, $f(\cdot, 0) \equiv 0$, then 0 is the minimal non-negative one. In the case of non-linear boundary conditions 0 is the minimal equilibrium provided $f(\cdot, 0) \equiv g(\cdot, 0) \equiv 0$.

Proof of Theorem 4.2. i) If $f(x, 0) \equiv 0$ then 0 is an equilibrium and that is the minimal non-negative solution. Thus, suppose that $f(x, 0) \not\equiv 0$. Then, $u(t, x; 0)$, is increasing since 0 is an strict subsolution of the associated elliptic problem (see Lemma 2.9).

Since $u(t, x; 0)$ is increasing and bounded, there exists the pointwise limit

$$\lim_{t \to \infty} u(t, x; 0) = \varphi_m^+(x).$$

Now, from Ascoli–Arzelá Theorem the convergence is uniform in x. Thus, $0 \leq \varphi_m^+$ is a positive equilibrium for (1.1). Moreover, this equilibrium is minimal since any other equilibria for (1.1), ψ, satisfies $0 < \psi$, and by the comparison principle, we must have $u(t, x; 0) \leq \psi(x)$. So, taking limits as $t \to \infty$ we have $\varphi_m^+ \leq \psi$.

The asymptotic stability from below of φ_m^+ follows from the fact that given any initial data $u_0 \in X$ such that $0 \leq u_0 \leq \varphi_m^+$ we have, by comparison, $u(t, x; 0) \leq u(t, x; u_0) \leq \varphi_m^+$ and $u(t, x; 0) \to \varphi_m^+$ as $t \to \infty$.

ii) Suppose that $f(x, 0) \equiv 0$ and satisfies

$$f(x, s) \geq M(x)s \quad \text{for all } x \in \Omega, \quad 0 \leq s \leq s_0. \tag{4.3}$$

Also assume that there exists bounded global solution apart from the trivial one starting at the non-negative initial data $v_0 \in C_0^1(\overline{\Omega})$.

In the following it is enough to consider positive initial data with negative normal derivative at the boundary. Otherwise, we let the system evolve and the solution at a small time t satisfies these conditions by the maximum principle. Then, we take as initial data this solution at time t.

Let ϕ the first eigenfunction (which we assume positive) for the problem

$$\left\{ \begin{array}{rcll} -\Delta v - M(x)v & = & \lambda v & \text{in} \quad \Omega \\ \mathcal{B}v & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{4.4}$$

with norm $||\phi||_{L^\infty(\Omega)} = 1$.

We prove the result in three steps.

Step A: Convergence of the solutions $u(t, x; \gamma\phi)$, with $0 < \gamma < \gamma_0$ to equilibria. Given $s_0 > \gamma > 0$, $\gamma\phi$ is a subsolution for the elliptic problem associated to (1.1) since

$$-\Delta(\gamma\phi) = M(x)\gamma\phi + \lambda_1\gamma\phi \leq M(x)\gamma\phi \leq f(x, \gamma\phi)$$

where $\lambda_1 = \lambda_1(-\Delta - M) < 0$. Therefore, $u(t, x; \gamma\phi)$ is monotonic increasing in time (see Lemma 2.9). Since we are assuming that there exists a bounded solution (starting at $v_0 \in C_0^1(\Omega)$) we have that there exists γ_0 small enough such that $\gamma\phi \leq v_0$ for all $0 < \gamma \leq \gamma_0 \leq s_0$. Hence, $u(t, x; \gamma\phi) \to \varphi_\gamma$ uniformly in $x \in \Omega$ as $t \to \infty$ to certain function φ_γ (by Ascoli-Arzelá Theorem). Furthermore, from Lemma 3.1, φ_γ is an equilibrium.

Thus, given $\gamma < \tilde{\gamma} \leq \gamma_0$ we have that $u(t, x; \gamma\phi) \to \varphi_\gamma$ and $u(t, x; \tilde{\gamma}\phi) \to \varphi_{\tilde{\gamma}}$. But $\gamma\phi \leq \tilde{\gamma}\phi$. So, by comparison, $u(t, x; \gamma\phi) \leq u(t, x; \tilde{\gamma}\phi)$ and then $\varphi_\gamma \leq \varphi_{\tilde{\gamma}}$. Thus, $\{\varphi_\gamma\}_{0<\gamma\leq\gamma_0}$ is an ordered set.

Step B: Equilibria are uniformly bounded away from zero. More precisely, we prove that given $u_0 \in C(\overline{\Omega})$, $u_0 \geq 0$, there exists a time $t_0 = t_0(u_0)$ such that for all $t \geq t_0$

$$u_f(t; u_0) \geq s_0 \phi,$$

where we denote by u_f the solution of problem (1.1). We define g by

$$g(x,s) = \begin{cases} m(x)s, & 0 \leq s \leq s_0 \\ \tilde{g}(x,s), & s_0 < s \end{cases}$$

such that $g(x,s) \leq f(x,s)$ for all $x \in \Omega$, $s \geq 0$. We denote by u_g the solution of problem (1.1) with nonlinear term g. Then, by the comparison principle, for all $0 \leq u_0 \in C(\overline{\Omega})$,

$$u_g(t,x;u_0) \leq u_f(t,x;u_0), \quad \text{for all} \quad t \geq 0.$$

Again, by comparison, for $u_0 = \gamma\phi$ with $0 < \gamma < \gamma_0$,

$$\mathrm{e}^{\varepsilon t}\gamma\phi = u_g(t,x;\gamma\phi) \leq u_f(t,x;\gamma\phi)$$

as long as $u_g(t,x;\gamma\phi)$ is below s_0. This happens for all $t \geq 0$ such that $\mathrm{e}^{\varepsilon t}\gamma \leq s_0$, i.e, for all $0 \leq t \leq t_0$ with $t_0 = \varepsilon^{-1}\ln(s_0/\gamma)$.

Moreover, notice that for all $t \geq 0$, solution $u_g(t;\gamma\phi)$ is increasing since it is so for $0 \leq t \leq t_0$.

Now, fixed $0 \leq u_0 \in C^1(\overline{\Omega})$ with negative normal derivative at the boundary, there exists $0 < \gamma < \gamma_0$ such that $0 < \gamma\phi < u_0$. Thus, if $t > t_0$,

$$u_f(t;u_0) \geq u_g(t;u_0) \geq u_g(t;\gamma\phi) \geq u_g(t_0;\gamma\phi) = s_0\phi.$$

Step C: Equilibria of the form φ_γ converge to a positive equilibrium as $\gamma \to 0$. Since φ_γ is an equilibrium, it satisfies

$$\begin{cases} -\Delta\varphi_\gamma &= f(x,\varphi_\gamma) & \text{in} & \Omega \\ \varphi_\gamma &= 0 & \text{on} & \partial\Omega \end{cases} \tag{4.5}$$

Now $||\varphi_\gamma||_{L^\infty(\Omega)} \leq C$ for certain constant C independent of γ. So, for another constant that we still denote C,

$$||f(\cdot,\varphi_\gamma)||_{L^p(\Omega)} \leq C$$

with $p > N/2$. Thus, by elliptic regularity

$$||\varphi_\gamma||_{C^\theta(\Omega)} \leq C, \quad 0 < \theta < 1.$$

In particular, the set $\{\varphi_\gamma\}_{0<\gamma<\gamma_0}$ is equicontinuous. Furthermore, it is an ordered set. So, we have $\varphi_\gamma \to \varphi^* \in C(\overline{\Omega})$ as $\gamma \to 0$ for some φ^*. Moreover, we can pass to the limit in (4.5) to obtain that φ^* is an equilibrium. We also have that

$$\varphi^* \geq s_0\phi.$$

since, from Step B, for all $0 < \gamma < \gamma_0$, $\varphi_\gamma > s_0\phi$. So φ^* is positive.

Finally, it is clear that given any equilibrium ψ there exists a positive constant $\gamma > 0$ such that $\gamma\phi \leq \psi$. Thus, by comparison, $\varphi_\gamma \leq \psi$. Now, $\varphi^* \leq \varphi_\gamma$. And then $\varphi^* \leq \psi$. Therefore, φ^* is minimal.

So, we have that φ^* is the minimal positive equilibrium. To obtain the asymptotic stability from below of φ^* just take into account that if $0 < \gamma < \gamma_0$ then $u(t;\gamma\phi) \to \varphi^*$ as $t \to \infty$.

Then, taking a regular initial data $0 \le u_0 \le \varphi^*$ we have that there exists $0 < \gamma < \gamma_0$ such that $0 < \gamma\phi \le u_0 \le \varphi^*$. By comparison

$$0 \le u(t; \gamma\phi) \le u(t; u_0) \le \varphi^*.$$

So, taking limits as $t \to \infty$ we obtain the asymptotic stability from below of φ^*. ■

Remark 4.4 *In the case ii) of the theorem, we proved indeed that the order interval $(0, \varphi_m^+)$ is contained in $W^s(\varphi_m^+)$, the stable manifold of φ_m^+.*

Notice that only the linear instability of 0 is not enough to obtain the previous result. It is necessary that 0 is an isolated equilibrium to avoid $\varphi^* \equiv 0$ (see Step B in the Proof). In our case, this property is implied by condition (4.1). However, if

$$\lim_{s \to 0} \frac{f(x,s)}{s} = m(x)$$

uniformly in x, with $m \in L^p(\Omega)$, $p > N/2$, then given any $\varepsilon > 0$ there exists $s_0 > 0$ such that for all $0 \le s \le s_0$

$$f(x,s) \ge (m(x) - \varepsilon)s \quad \text{a.e.} \quad x \in \Omega.$$

If 0 is a linearly unstable equilibrium for (1.1) then $\lambda_1(-\Delta - m) < 0$ since $m(x) = \partial_s f(x,0)$. Taking now ε small enough we have $\lambda_1(-\Delta - (m - \varepsilon)) < 0$ by the continuity of the first eigenvalue respect to the potential. Thus, there exists a potential $M(x) = m(x) - \varepsilon$ in the hypotheses of the theorem.

Notice that the uniformity of the limit above is nothing but a condition on way the derivative of f at $u = 0$ is approached. For example, for logistic nonlinear terms (see Section 6) we will consider $f(x,s) = m(x)s - n(x)s^\rho$, with $\rho > 1$, $n \ge 0$ bounded and $m \in L^p(\Omega)$, $p > N/2$. Thus, the limit above is satisfied.

Analogously, for problems with nonlinear boundary conditions we have

Theorem 4.5 *Suppose that f and g are continuous functions in (x,u), locally Lipschitz in u and $f(x,0), g(x,0) \ge 0$. Then,*

i) If there exists a bounded nontrivial nonnegative solution then there exists a ***non-negative*** *minimal equilibrium φ_m^+. Even more, $\varphi_m^+ \equiv 0$ (if $f(x,0) = g(x,0) = 0$ for all $x \in \Omega$), or $\varphi_m^+(x) > 0$ for all $x \in \Omega$ (if $f(x_0,0) > 0$ or $g(x_0,0) > 0$ at some $x_0 \in \Omega$ or $x_0 \in \Gamma$ resp.).*

ii) Suppose $f(x,0) \equiv g(x,0) \equiv 0$. Assume that there exist $s_0 > 0$, $M(x) \in L^p(\Omega)$, $p > N/2$, y $N(x) \in L^q(\Gamma_1)$, $q > N - 1$, such that for all $0 \le s \le s_0$,

$$f(x,s) \ge M(x)s \quad \textit{for all} \quad x \in \Omega, \qquad g(x,s) \ge N(x)s \quad \textit{for all} \quad x \in \Gamma_1$$

and 0 is unstable for the following problem

$$\begin{cases} u_t - \operatorname{div}(a(x)\nabla u) + c(x)u &= M(x)u \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \Gamma_0 \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &= N(x)u \quad \text{on} \quad \Gamma_1 \\ u(0) &= u_0 \end{cases}$$

Then, 0 is an isolated equilibrium point. If there exists a bounded nontrivial nonnegative solution then there exists a minimal ***positive*** *equilibrium φ_m^+.*

In any of the previous cases, if there exists the minimal positive equilibrium, it is asymptotically stable from below.

Proof. We just point out the differences with the proof of Theorem 4.2. The proof of part i) is analogous to that of Theorem 4.2 except that now 0 is subsolution if $f(x,0)$ or $g(x,0)$ are not identically zero. Otherwise, zero is an equilibrium.

Part ii) is also analogous to that of Theorem 4.2. Now, for Step A, we take ϕ the first eigenfunction (which we assume positive and normalized as $||\phi||_{L^\infty(\Omega)} = 1$) of problem

$$\left\{ \begin{array}{rcll} -\mathrm{div}(a(x)\nabla v) + (c(x) - M(x))v &=& \lambda v & \text{in} \quad \Omega \\ v &=& 0 & \text{on} \quad \Gamma_0 \\ \frac{\partial v}{\partial \vec{n}} + (b(x) - N(x))v &=& 0 & \text{on} \quad \Gamma_1 \end{array} \right. \tag{4.6}$$

Now, for all $0 < \gamma < s_0$, $\gamma\phi$ is subsolution of (2.10) since

$$-\mathrm{div}(a(x)\gamma\phi) + c(x)\gamma\phi = M(x)\gamma\phi + \lambda_1\gamma\phi \leq M(x)\gamma\phi \leq f(x,\gamma\phi)$$

where $\lambda_1 < 0$ is the first eigenvalue of (4.6) and

$$\frac{\partial\gamma\phi}{\partial\vec{n}} + b(x)\gamma\phi = N(x)\gamma\phi \leq g(x,\gamma\phi).$$

Again, for $0 < \gamma < \gamma_0$, $u(t,x;\gamma\phi)$ is increasing has to converge to an equilibrium.

For Step B, we consider the auxiliary functions

$$F(x,s) = \left\{ \begin{array}{ll} M(x)s, & 0 \leq s \leq s_0 \\ \tilde{g}(x,s), & s_0 < s \end{array} \right. \quad \text{and} \quad G(x,s) = \left\{ \begin{array}{ll} N(x)s, & 0 \leq s \leq s_0 \\ \tilde{g}(x,s), & s_0 < s \end{array} \right.$$

such that $f(x,s) \geq F(x,s)$ and $g(x,s) \geq G(x,s)$ for all $s \geq 0$ and $x \in \Omega$ or $x \in \Gamma_1$ respectively. Following the argument in the proof of Step B of Theorem 4.2 we have that 0 is an isolated equilibrium.

Step C follows as above. ■

Following the idea of Theorem II.1 in Berestycki [7] (see Section 5.2.2) we now state an interesting consequence of the two previous results for problems with Dirichlet boundary conditions.

Corollary 4.6 *Let Ω_1 be a connected subdomain of Ω. Suppose $f(x,0) \equiv 0$. Also assume that f satisfies*

$$f(x,s) \geq M(x)s \quad \text{in } \Omega_1 \quad 0 \leq s \leq s_0 \tag{4.7}$$

for some $M \in L^r(\Omega)$, $r > N/2$ such that 0 is unstable for

$$\left\{ \begin{array}{rcll} v_t - \Delta v &=& M(x)v & \text{in} \quad \Omega_1 \\ v &=& 0 & \text{on} \quad \partial\Omega_1. \end{array} \right.$$

Finally assume that there exists a bounded nontrivial nonnegative solution of (1.1) in Ω.

Then there exists a minimal positive equilibrium of (1.1) in Ω with Dirichlet boundary conditions. Moreover, the minimal equilibrium is asymptotically stable from below.

Proof. We denote by u_{Ω_1} the solution of (1.1) in Ω_1 with Dirichlet boundary conditions and by u_Ω the solution of the problem in Ω.

Notice that, by hypothesis, there exists a bounded nontrivial nonnegative solution $u_\Omega(t,x;v_0)$ of the problem in Ω (for certain initial data v_0 which we can assume in $C_0^1(\overline{\Omega})$) . Then, given $w_0 \in C_0^1(\overline{\Omega}_1)$ such that $0 \le w_0 \le v_0$ we have, by comparison

$$0 \le u_{\Omega_1}(t,x;w_0) \le u_\Omega(t,x;v_0) \quad \text{for} \quad x \in \Omega_1. \tag{4.8}$$

Thus, there exists a bounded solution for the problem in Ω_1. Therefore, Theorem 4.2 applies to the problem in Ω_1. So, there exists a minimal positive equilibrium, which is asymptotically stable from below, $\varphi_m^{\Omega_1}$ for the Dirichlet problem in Ω_1. Then, given an initial data u_0 in Ω_1 with $0 \le u_0 \le \varphi_m^{\Omega_1}$ in Ω_1 we have $u_{\Omega_1}(t,x;u_0) \to \varphi_m^{\Omega_1}$ as $t \to \infty$.

Now, we extend $\varphi_m^{\Omega_1}$ by zero to Ω (which we denote the same). Then it becomes a subsolution (in a weak sense) of the elliptic problem in Ω

$$\begin{cases} -\Delta w &= f(x,u) \quad \text{in} \quad \Omega \\ w &= 0 \quad\quad\quad \text{on} \quad \partial\Omega \end{cases}$$

since

$$\begin{cases} -\Delta \varphi_m^{\Omega_1} &= f(x,\varphi_m^{\Omega_1}) \quad \text{in} \quad \Omega_1 \\ \varphi_m^{\Omega_1} &= 0 \quad\quad\quad \text{on} \quad \partial\Omega_1. \end{cases}$$

Then, taking $0 \le \xi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \nabla\varphi_m^{\Omega_1}\nabla\xi = \int_{\Omega_1} \nabla\varphi_m^{\Omega_1}\nabla\xi = \int_{\Omega_1} -\Delta\varphi_m^{\Omega_1}\xi + \int_{\partial\Omega_1} \frac{\partial\varphi_m^{\Omega_1}}{\partial n}\xi \le \int_{\Omega_1} f(x,\varphi_m^{\Omega_1})\xi = \int_\Omega f(x,\varphi_m^{\Omega_1})\xi$$

where we have used that $\frac{\partial\varphi_m^{\Omega_1}}{\partial n} \le 0$ in $\partial\Omega_1$, $\xi \ge 0$ in $\partial\Omega_1$, $f(x,0) = 0$ and $\varphi_m^{\Omega_1} = 0$ out of Ω_1. So, $\varphi_m^{\Omega_1}$ is a subsolution of the problem in Ω.

Hence, from Lemma 2.9 we have that $u_\Omega(t,x;\varphi_m^{\Omega_1})$ is monotonically increasing. Moreover, we now show it is bounded. Indeed, from (4.8), we get that $\varphi_m^{\Omega_1} \le v_1$, in Ω, where v_1 belongs to the omega limit set of v_0, which is nonempty. Thus, $u_\Omega(t,x;\varphi_m^{\Omega_1}) \le u_\Omega(t,x;v_1)$ and is bounded.

Thus, $u_\Omega(t,x;\varphi_m^{\Omega_1})$ must converge to certain equilibrium that we denote by φ_m^Ω. Let us see that φ_m^Ω is the minimal positive equilibrium for the problem in Ω. Let $0 \le u_0 \in C(\overline{\Omega})$ (which we can assume is positive with negative normal derivative at the boundary). Let $w_0 \in C_0(\overline{\Omega}_1)$ such that $u_0 \ge w_0$ in Ω_1. Then, by comparison, we have, in Ω_1,

$$u_\Omega(t,x;u_0) \ge u_\Omega(t,x;w_0) \ge u_{\Omega_1}(t,x;w_0), \quad x \in \Omega_1.$$

In particular, taking w_0 small enough so that $w_0 \le \varphi_m^{\Omega_1}$ and taking limits as $t \to \infty$,

$$\liminf_{t\to\infty} u_\Omega(t,x;u_0) \ge \varphi_m^{\Omega_1}(x), \quad \text{for} \quad x \in \Omega_1.$$

Now, since $u_\Omega(t,x;u_0) \ge 0$ for all $t > 0$, $x \in \Omega$ and extending $\varphi_m^{\Omega_1}$ by zero to Ω, we have

$$\liminf_{t\to\infty} u_\Omega(t,x;u_0) \ge \varphi_m^{\Omega_1}(x), \quad x \in \Omega.$$

In particular, taking $u_0 = \psi$ a positive equilibrium in Ω, we have

$$\psi(x) = \liminf_{t\to\infty} u_\Omega(t,x;\psi) \ge \varphi_m^{\Omega_1}(x), \quad x \in \Omega.$$

Letting the semigroup of solutions in Ω act in both sides of the inequality, we have

$$\psi(x) = u(t,x;\psi) \ge u(t,x;\varphi_m^{\Omega_1}) \quad x \in \Omega.$$

Taking limits as $t \to \infty$

$$\psi(x) \geq \varphi_m^\Omega(x), \quad x \in \Omega.$$

Thus, φ_m^Ω is the minimal equilibrium of the problem in Ω.

For the asymptotic stability from below for the minimal equilibrium, let ϕ the first eigenfunction of $-\Delta - M$ in Ω_1 (which we assume positive with $||\phi||_{L^\infty(\Omega)} = 1$). Notice that $\gamma\phi$, for $0 < \gamma$ small enough, is a subsolution of the stationary problem in Ω_1. Its extension by zero to Ω is a subsolution of the stationary problem in Ω. Thus, by Lemma 2.9, $u_\Omega(t, x; \gamma\phi)$ is increasing in time. Moreover, following the arguments in Step A of the proof of Theorem 4.2, converges to an equilibrium ψ.

Now, given $u_0 \leq \varphi_m^\Omega$, there exists $1 > \gamma > 0$ small enough such that $\gamma\phi \leq u_0$ in Ω. By monotonicity,

$$\varphi_m^\Omega(x) \geq u_\Omega(t, x; u_0) \geq u_\Omega(t, x; \gamma\phi), \quad x \in \Omega.$$

Taking limit as t goes to ∞ we have

$$\varphi_m^\Omega(x) \geq \limsup_{t\to\infty} u_\Omega(t, x; u_0) \geq \liminf_{t\to\infty} u_\Omega(t, x; u_0) \geq \psi(x), \quad x \in \Omega$$

for certain equilibrium ψ of the problem in Ω. But, since φ_m^Ω is minimal, we must have $\psi = \varphi_m^\Omega$. So,

$$\lim_{t\to\infty} u_\Omega(t, x; u_0) = \varphi_m^\Omega(x), \quad x \in \Omega.$$

Thus, φ_m^Ω is asymptotically stable from below. ■

4.2 On the stability and uniqueness of positive equilibria

In Theorem 4.2, we proved that if there is a bounded nontrivial nonnegative solution then there exists a minimal positive equilibrium, φ_m^+, which is asymptotically stable from below. However, we can obtain more information about the stability of the minimal equilibrium.

Let consider the linearization around φ_m^+ of the elliptic problem associated to (1.1)

$$\begin{cases} -\Delta u - \partial_s f(x, \varphi_m^+)u &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

This problem has a first eigenvalue λ_1 and a first eigenfunction $\phi_1(x)$ which we may assume is positive and with L^∞-norm equal to 1. Moreover, we must have $\lambda_1 \geq 0$. Otherwise, if $\lambda_1 < 0$, the equilibrium φ_m^+ is unstable in the direction of ϕ_1. Thus, the solution of evolution problem (1.1) starting at $\varphi_m^+ - \varepsilon\phi_1$ for $\varepsilon > 0$ small enough must go away from φ_m^+. On the other hand, we know that given any initial data $0 < u_0 \leq \varphi_m^+$, we have $u(t, x; u_0) \to \varphi_m^+(x)$ as $t \to \infty$. Now, taking $0 < u_0 = \varphi_m^+ - \varepsilon\phi_1 \leq \varphi_m$ (u_0 is positive for ε small enough since the normal derivative of φ_m^+ at the boundary is negative) and using comparison we have

$$u(t, x; \varphi_m^+ - \varepsilon\phi_1) \leq \varphi_m^+(x).$$

Thus, taking limits as $t \to \infty$,

$$\varphi_m^+(x) = \lim_{t\to\infty} u(t, x; \varphi_m^+ - \varepsilon\phi_1) = \varphi_m^+(x),$$

that is,

$$u(t, x; \varphi_m^+ - \varepsilon\phi_1) \to \varphi_m^+(x) \quad \text{as} \quad t \to \infty$$

which get in contradiction with the instability of φ_m^+. So, we must have $\lambda_1 \geq 0$.

Now, if $\lambda_1 > 0$ then φ_m^+ is linearly asymptotically stable and hence locally asymptotically stable. Otherwise, if $\lambda_1 = 0$ we cannot ensure, in general, the stability of φ_m^+ (we have a one-dimensional center manifold). However, under certain convexity hypothesis on f the stability can be obtained. We have the following result

Proposition 4.7 *Let φ be a positive equilibrium which is asymptotically stable from below (resp. from above) of problem (1.1). Then, the first eigenvalue λ_1 of the linearized problem around φ*

$$\begin{cases} -\Delta v &= \partial_s f(x,\varphi)v + \lambda_1 v \quad \text{in} \quad \Omega \\ \quad v &= 0 \qquad\qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases} \tag{4.9}$$

is non-negative. Moreover, if λ_1 is positive then φ is also locally asymptotically stable from above (resp. from below) and thus locally asymptotically stable. Furthermore, assume that f satisfies

$$f(x,s) \leq \partial_s f(x,s)s \quad \text{or} \quad f(x,s) \geq \partial_s f(x,s)s \tag{4.10}$$

in Ω for $0 < s \leq ||\varphi||_{L^\infty(\Omega)}$ with strict inequality in a set of positive measure, for any fixed s. Then, $\lambda_1 > 0$ and therefore φ is stable.

Proof. Arguing as above, we must have $\lambda_1 \geq 0$. Suppose $\lambda_1 = 0$. Then, there exists a unique positive solution (up to multiples) of problem (4.9) which we denote by $\phi(x)$.

From the equation for φ we have

$$-\Delta\varphi = f(x,\varphi),$$

and by (4.10) (we assume the first case for definiteness; the other one is analogous)

$$-\Delta\varphi \leq \partial_s f(x,\varphi)\varphi.$$

Multiplying this inequality by $\phi(x) > 0$ we get

$$-\phi\Delta\varphi \leq \partial_s f(x,\varphi)\varphi\phi = -\varphi\Delta\phi$$

where we have used the equation for ϕ in the last equality. Integrating in Ω and using the fact that (4.10) is strict in a set of positive measure we have

$$\int_\Omega \nabla\varphi\nabla\phi - \int_{\partial\Omega} \frac{\partial\varphi}{\partial n}\phi = \int_\Omega -\phi\Delta\varphi < \int_\Omega -\varphi\Delta\phi = \int_\Omega \nabla\phi\nabla\varphi - \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\varphi$$

which is a contradiction since the integrals on $\partial\Omega$ are zero because of the boundary conditions of φ and ϕ. Thus, $\lambda_1 > 0$. ■

In the following it will be useful to use the concept of μ-increasing function we define below.

Definition 4.8 *We say that $h(x,s)$ is μ-*increasing *(resp. μ-decreasing) if given $s_1 < s_2$ we have $h(x,s_1) \leq h(x,s_2)$ (resp. $h(x,s_1) \geq h(x,s_2)$) with strict inequality in a set of positive measure.*

Remark 4.9 *Under regularity assumptions on f, property (4.10) is equivalent to*

$$\frac{f(x,s)}{s} \quad \mu\text{-increasing} \quad \text{or} \quad \frac{f(x.s)}{s} \quad \mu\text{-decreasing}$$

respectively.

Proposition 4.7 can be also obtained as a consequence of a more general argument that we now estate

Proposition 4.10 *Suppose that there exists a positive equilibrium for (1.1). Then,*
i) If $f(x,s) \leq \partial_s f(x,s)s$ *for* $0 \leq s \leq ||\varphi||_{L^\infty(\Omega)}$ *then* φ *is asymptotically stable.*
ii) If $f(x,s) \geq \partial_s f(x,s)s$ *for* $0 \leq s \leq ||\varphi||_{L^\infty(\Omega)}$ *then* φ *is linearly unstable.*

Proof. Let

$$m_0(x) = f(x,\varphi(x))/\varphi(x) \quad \text{and} \quad m_1(x) = \partial_s f(x,\varphi(x)).$$

Notice that φ is a positive eigenfunction of the problem

$$\begin{cases} -\Delta w &= \dfrac{f(x,\varphi)}{\varphi} w = m_0(x)w \quad \text{in} \quad \Omega \\ \mathcal{B}w &= 0 \qquad\qquad\qquad\qquad \text{on} \quad \partial\Omega \end{cases}$$

Thus, zero is the first eigenvalue of $-\Delta - m_0(x)$ with boundary conditions given by $\mathcal{B}$. Moreover,

$$-\Delta - m_1(x) = -\Delta - m_0(x) + (m_0(x) - m_1(x)).$$

Now, by the assumptions in i) or ii), either $m_0(x) \leq m_1(x)$, or $m_0(x) \geq m_1(x)$ respectively, with strict inequalities in a set of positive measure. Thus, zero cannot be the first eigenvalue of the problem with potential m_1, that is, the linearisation in φ. So, φ is either asymptotically stable in case i) or linearly unstable in case ii). ■

Remark 4.11 *Notice that Proposition 4.10 is also true for general second order elliptic operators, not necessarily in divergence form, see (2.3). In such case we need to use the result on monotonicity of the first eigenvalue respect to the potential m in Berestycki et al. [8].*

Under certain hypothesis on the nonlinear term closely related with (4.10) (see Remark 4.13 below) the uniqueness of positive solution for the elliptic problem associated to (1.1) can be obtained. This together with the instability of 0 will imply, in particular, the existence of a unique positive equilibrium which is globally asymptotically stable for positive solutions.

As an example, we state a result for problem with Dirichlet boundary conditions. Uniqueness is a direct consequence of the following results. The proof is and adaptation to the space-dependent nonlinear terms of Lemma 10.3.3, p. 160, in Cazenave and Haraux [10] and therefore, we omit it.

Proposition 4.12 *Suppose that* $f(x,0) \equiv 0$ *and* $f(x,\cdot)$ *is concave in* $[0,\infty)$. *Let* $\varphi > 0$ *a solution of*

$$\begin{cases} -\Delta\varphi &= f(x,\varphi) \quad \text{in} \quad \Omega \\ \varphi &= 0 \qquad\quad\ \ \text{on} \quad \partial\Omega \end{cases} \tag{4.11}$$

and $\psi \geq 0$ *solving*

$$\begin{cases} -\Delta\psi &\geq f(x,\psi) \quad \text{in} \quad \Omega \\ \psi &= 0 \qquad\quad\ \ \text{on} \quad \partial\Omega \end{cases} \tag{4.12}$$

that is, ψ *is a supersolution of (4.11).*

Then, either $\psi \equiv 0$ *or* $\psi \geq \varphi$. *In particular, there exists at most one positive solution of (4.11).*

Remark 4.13 *Suppose that f is concave, smooth and $f(x,0) \equiv 0$. By the mean value theorem, for certain $0 < \xi_x < s$*

$$\frac{f(x,s)}{s} = \frac{f(x,s) - f(x,0)}{s} = \partial_s f(x,\xi_x) \geq \partial_s f(x,s)$$

where we have used the concavity of f for the last inequality. Thus, f satisfies the second property in (4.10) and it is clear that this is equivalent to the fact that $f(x,s)/s$ is decreasing in s.

A related results can be found in Brézis and Oswald [9]. There, the following results about existence and uniqueness of positive solution for (4.11) are proved.

Theorem 4.14 (Brézis y Oswald) *Suppose that f is a continuous function in the second argument and satisfies the following property*

$$\frac{f(x,s)}{s} \quad \text{is decreasing in } s. \tag{4.13}$$

Then, there exists at most one positive solution of (4.11).

Also in that work it is proved the following existence result.

Theorem 4.15 (Brézis y Oswald) *Suppose that $f(x,s)$ is a continuous function in the second argument. Assume that for all $\delta > 0$ there exists a constant $C_\delta \geq 0$ such that $f(x,s) \geq -C_\delta s$ for all $0 \leq s \leq \delta$, a.e. $x \in \Omega$. Also assume that for any $u \geq 0$, $f(\cdot,s) \in L^\infty(\Omega)$ and there exists a constant $C > 0$ such that $f(x,s) \leq C(s+1)$. Let*

$$a_0(x) = \lim_{s\to 0^+} \frac{f(x,s)}{s} \qquad \text{and} \qquad a_\infty(x) = \lim_{s\to +\infty} \frac{f(x,s)}{s}.$$

Assume

$$\lambda_1(-\Delta - a_0) < 0 < \lambda_1(-\Delta - a_\infty).$$

Then, there exists at least a positive solution of (4.11).

We now prove a related result showing uniqueness for (4.11) when $f(x,s)/s$ is μ-increasing or μ-decreasing without assuming $f(\cdot,u) \in L^\infty(\Omega)$. Namely, we have

Theorem 4.16 *Suppose that there exists the maximal positive solution for (4.11). Assume in addition that, either*

$$\frac{f(x,s)}{s} \quad \text{is } \mu\text{-decreasing in } s;$$

or

$$\frac{f(x,s)}{s} \quad \text{is } \mu\text{-increasing in } s.$$

Then, there exists a unique positive solution of (4.11).

Proof. Let φ be the maximal positive solution of (4.11) and $\psi \leq \varphi$ any other solution. Then,

$$-\Delta\varphi = f(x,\varphi) \qquad -\Delta\psi = f(x,\psi).$$

Multiplying the first equation by ψ, the second one by φ, subtracting and integrating in Ω, we have

$$0 = \int_\Omega \frac{f(x,\varphi)}{\varphi}\varphi\psi - \int_\Omega \frac{f(x,\psi)}{\psi}\varphi\psi = \int_\Omega \left(\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi}\right)\varphi\psi.$$

Now, since $\psi \leq \varphi$ using the condition for $f(x,s)/s$ we have

$$\frac{f(x,\varphi)}{\varphi} - \frac{f(x,\psi)}{\psi} \neq 0$$

in a set of positive measure, and it does not change its sign. So, we must have $\psi \equiv 0$. ■

Notice that Brézis and Oswald results (see Theorems 4.14 and 4.15 above) require f to be bounded. However, under our assumptions, we only require f to satisfy (2.7) and (2.8). See Section 6 for examples in which these conditions are satisfied and f is not bounded.

Remark 4.17 *We now show another proof of Theorem 4.16 which is valid for more general operators and boundary conditions. In particular it is valid for (1.1). With the notations in Theorem 4.16, let*

$$m_0(x) = \frac{f(x,\varphi(x))}{\varphi(x)} \quad \text{and} \quad m_1(x) = \frac{f(x,\psi(x))}{\psi(x)}.$$

Then, φ is a positive solution for

$$\begin{cases} -\Delta\varphi = m_0(x)\varphi & \text{in } \Omega \\ \mathcal{B}\varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, the first eigenvalue of $-\Delta - m_0(x)$ with the boundary conditions given by $\mathcal{B}$ is zero. Analogously, the first eigenvalue of $-\Delta - m_1(x)$ with the boundary conditions given by $\mathcal{B}$ is also zero. Now,

$$-\Delta - m_1(x) = -\Delta - m_0(x) + (m_1(x) - m_0(x)).$$

So, the first eigenvalue of $-\Delta - m_1(x)$ is non-zero since either $m_0(x) \leq m_1(x)$, or $m_0(x) \geq m_1(x)$, with strict inequality in a non-zero measure set. This is a contradiction, so $\varphi = \psi$.

From the previous theorem we have the following Corollary.

Corollary 4.18 *If $f(x,s)/s$ is μ-increasing or μ-decreasing and there exist two positive equilibria for (1.1) then they are not ordered.*

Concerning stability, we have

Corollary 4.19 *Is $f(x,s)/s$ is μ-decreasing and there exists the maximal positive equilibrium φ_M of (1.1) then φ_M is the unique positive equilibrium for (1.1). Furthermore, it is globally asymptotically stable for positive solutions.*

Corollary 4.20 *Suppose f as in Theorems 3.6 or 3.7. Also assume that the linear semigroup generated by $\Delta + C$, $S_{\Delta+C}$, has exponential decay. Finally, assume that f is such that $f(x,s)/s$ is μ-decreasing and 0 is an unstable equilibrium.*

Then, there exist a unique positive equilibrium which is globally asymptotically stable.

Proof. Since f satisfies (3.7) we know that there exists a maximal non-negative equilibrium which is stable from above (see Corollary 3.10). Now, such equilibrium is either 0 or positive (by the maximum principle). But it cannot be 0 since we are assuming that 0 is unstable (in particular, it is unstable from above). Thus, the maximal equilibrium must be positive. The uniqueness follows from Theorem 4.16. Stability from below (and therefore the global asymptotic stability of the unique positive equilibrium) follows from the instability of 0. ■

5 Some known elliptic results revisited.

In this section we review some other results concerning the existence of positive equilibria that were obtained using elliptic methods as they are the topological degree and (elliptic) sub-supersolutions. Here, these results are recovered by mean of the dynamical techniques developed above.

5.1 The principal eigenvalue

We begin with a brief summary about *principal eigenvalue* and its relation with the first eigenvalue of second order elliptic operators. This notion appears quite often in the literature for the study of the existence of equilibria and its stability.

Let $\Omega \subset \mathbb{R}^N$ bounded domain and $m \in L^p(\Omega)$ with $p > N/2$.

Definition 5.1 *We say that $\lambda \in \mathbb{R}$ is a* principal eigenvalue *for $m(x)$, and we denote it by $\lambda_0(m)$, if there exists a positive solution of*

$$\left\{ \begin{array}{rcll} -\Delta u &=& \lambda m(x) u & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{5.1}$$

We now consider the eigenvalue problem

$$\left\{ \begin{array}{rcll} -\Delta u - m(x)u &=& \mu u & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{5.2}$$

and denote by $\mu_1(m)$ the first eigenvalue which always has an associated positive eigenfunction.

First, notice that $-\lambda_0(m)$ is a principal eigenvalue for $-m$. Moreover, taking $m(x) \equiv 1$ we have that $\lambda_0(1)$ is the first eigenvalue of

$$\left\{ \begin{array}{rcll} -\Delta u &=& \lambda u & \text{in} \quad \Omega \\ u &=& 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{5.3}$$

that is, $\lambda_0(1) = \mu_1(0)$.

Moreover, given $m \in L^p(\Omega)$, $p > N/2$, $\lambda_0(m) = 1$ if and only if $\mu_1(m) = 0$. If $m(x) \geq 0$ for a.e. $x \in \Omega$ then $\text{sign}(\lambda_0(m) - 1) = \text{sign}(\mu_1(m))$. Thus, the semigroup generated by $\Delta + \lambda m(x)$ has exponential decay if and only if $\lambda < \lambda_0(m)$. If $m(x) \leq 0$ for a.e. $x \in \Omega$ then $\text{sign}(\lambda_0(m) - 1) \neq \text{sign}(\mu_1(m))$ and the semigroup generated by $\Delta + \lambda m(x)$ has exponential decay if and only if $\lambda > \lambda_0(m)$.

It is known that is $m(x)$ does not change its sign in Ω then the principal eigenvalue is unique (see Figueiredo [13] or López–Gómez [19]).

In the following we will consider potential functions $m(x)$ that might be sign-changing in Ω. Let consider operators of the form

$$Lu = -\sum_{i^1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + a_0(x) u$$

strongly elliptic, with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ and $a_0 \in L^{N/2}(\Omega)$, $a_0(x) \geq 0$ in Ω. For these operators we have that for potentials $m \in L^r(\Omega)$, $r > N/2$, the principal eigenvalue $\lambda_0(m)$ is strictly increasing in m and continuous respect to m in $L^{N/2}(\Omega)$ norm (see Proposition 1.12A and 1.12B in Figueiredo [13]).

We now define the function $\mu(\lambda)$ mapping $\lambda \in \mathbb{R}$ to the first eigenvalue of

$$\left\{ \begin{array}{rcll} -\Delta u - \lambda m u & = & \mu(\lambda) u & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right.$$

Then, λ is a principal eigenvalue for (5.1) if and only if $\mu(\lambda) = 0$. By studying the zeros of $\mu(\lambda)$ we can determine the number of principal eigenvalues of problem (5.1). In this direction, the next lemma estates an interesting property of $\mu(\lambda)$.

Lemma 5.2 (Lemma 6.1 in López–Gómez [19], p. 283) *The function $\mu(\lambda)$ is analytic in λ and concave. Moreover, if $m \in C(\overline{\Omega})$ is positive in some point of Ω then*

$$\lim_{\lambda \to \infty} \mu(\lambda) = -\infty;$$

if $m(x)$ is negative in some point of Ω then

$$\lim_{\lambda \to -\infty} \mu(\lambda) = -\infty.$$

Moreover, we have

Lemma 5.3 *Suppose that $m \in C(\overline{\Omega})$ is sign-changing in Ω. Also assume that there exists λ^* such that $\mu(\lambda^*) > 0$. Then, there exists two principal eigenvalues, $\lambda_m < \lambda^* < \lambda_M$. Moreover, the trivial solution of*

$$\left\{ \begin{array}{rcll} -\Delta u - \lambda m u & = & 0 & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. \tag{5.4}$$

is stable for all $\lambda_m < \lambda < \lambda_M$ and unstable for all $\lambda > \lambda_m$ and $\lambda > \lambda_M$.

5.2 Existence of positive equilibria

We now review some known results concerning existence of positive equilibria.

Note that we not only recover several classical results below, but our result gives the existence of a maximal, φ_M, and minimal, φ_m^+, positive equilibria, Theorems 4.2 or 4.5. Moreover they are stable from above and from below, respectively, and and the order interval $[\varphi_m^+, \varphi_M]$ attracts bounded sets of non-negative initial data uniformly.

5.2.1 Amann [1] results revisited

In Amann [1] the author prove some results about existence of postive solutions of ellipitic problems by mean topological techniques: fixed points theorems, Leray-Schauder degree, index, , etc. Most of these elliptic results can be obtained by means of the asymptotic behaviour of the associated parabolic problems as we now show. This proof, in particular, provides dynamical information.

The problem considered in [1] is

$$\left\{ \begin{array}{rcll} Lu & = & f(x,u) & \text{in} \quad \Omega \\ \mathcal{B}u & = & g(x) & \text{on} \quad \partial\Omega \end{array} \right. \tag{5.5}$$

where Ω is a regular bounded domain in $\mathbb{R}^N$ and L is a strongly elliptic operator of the form

$$Lu = -\sum_{i,j=1}^{N} a_{ij}(x) D_i D_k u + \sum_{i=1}^{N} a_i(x) D_i u + a(x) u$$

with $a_{ij}, a_i, a \in C^\mu(\overline{\Omega})$, (a_{ij}) symmetric and $a \geq 0$.

Boundary conditions are of the form

$$\mathcal{B}u = b(x)u + \delta \frac{\partial u}{\partial \nu}$$

where $b \in C^{1+\mu}(\partial\Omega)$ and either,

1. $\delta = 0$ y $b = 1$;

2. or, $\delta = 1$ y $b \geq 0$ (if $b \equiv 0$ then $a > 0$ is required).

Let $\hat{\mu} = \mu$ if $N \geq 2$ and $\hat{\mu} = 0$ if $N = 1$.

In addition, it is assumed that $g \in C^{2-\delta+\hat{\mu}}(\partial\Omega), f \in C^{\hat{\mu}}(\overline{\Omega} \times I)$ with $0 < \hat{\mu} < 1$ and there exists $\omega \geq 0$ such that

$$f(x,\xi) - f(x,\eta) > -\omega(\xi - \eta)$$

for all $x \in \overline{\Omega}$, $\xi, \eta \in I$, $\xi > \eta$, where I is a closed interval of $\mathbb{R}$. This property holds in particular if $f \in C^1$ and $\partial_s f(x,s) > -\omega$.

Solutions are looked for in

$$D_I = \{v \in C(\overline{\Omega}) : v(x) \in I \text{ for all } x \in \overline{\Omega}\}.$$

If $I = \mathbb{R}^+$ this is equivalent to work with non-negative solutions.

Theorem 5.4 (Theorem 9.6 in Amann [1], p. 649) *Suppose $g \geq 0$ and $f(\cdot, 0) \geq 0$. Let $\hat{f} \in C(\overline{\Omega})$. Assume $m \in C^{\hat{\mu}}(\overline{\Omega})$ with $m(x) > 0$ for almost every $x \in \Omega$. Also assume that $(x,\xi) \in \overline{\Omega} \times \mathbb{R}_+$*

$$f(x,\xi) \leq \hat{f}(x) + \lambda m(x)\xi \tag{5.6}$$

*Then, problem (5.5) has a **minimal non-negative** solution provided $\lambda < \lambda_0(m)$, where $\lambda_0(m)$ is the principal eigenvalue associated to m.*

In our context, this result in the case of Dirichlet, Neumann or Robin homogeneous boundary conditions is a straightforward consequence of Corollary 4.3 and remarks below. Indeed, notice that (5.6) implies property (3.7) holds with $C(x) = \lambda m(x)$ and $D(x) = \hat{f}(x)$, and $\lambda < \lambda_0(m)$ implies that the semigroup generated by $\Delta + C(x)$ has exponential decay.

For the case of an operator in divergence form with either Neumann or Robin inhomogeneous, or non-linear boundary conditions, the result follows from Proposition 4.5.

In fact, our results in Sections 3 and 4 applies to more general problems since we do not require $0 \leq m \in C^{\hat{\mu}}(\overline{\Omega})$ but $m \in L^p(\Omega)$ with $p > N/2$ and no restrictions on the sign of $m(x)$. Notice that in the case of Robin or non-linear boundary conditions we do not impose any restriction on the sign of $b(x)$.

The following result is proved in [1] by mean of fixed point techniques studying the spectral radius of the right derivative of the integral operator whose fixed points are solutions of (5.5).

Theorem 5.5 (Theorem 9.8 in [1], p. 650) *Let $I = \mathbb{R}_+$, $g = 0$ and $f(\cdot, 0) = 0$. Suppose that there exists $\partial_u f(\cdot, 0) \in C^{\hat{\mu}}(\overline{\Omega})$ and is continuous in a neighborhood of* 0, *and $\partial_u f(x, 0) > 0$ for almost every $x \in \Omega$.*

If there exists a supersolution of the problem (5.5), $\psi > 0$, then there exists a maximal positive solution of (5.5) in the order interval $[0, \psi]$ provided $\lambda_0(\partial_u f(\cdot, 0)) < 1$.

As before, in our context, this result is a straightforward consequence of the results in Sections 3 and 4. Namely, since $\lambda_0(\partial_u f(x,0)) < 1$, then $u \equiv 0$ is an unstable equilibrium. Moreover, since f is sufficiently smooth, we are under assumptions of Theorem 4.2 (see comments after Remark 4.4).

Notice that to obtain the result in our setting we do not need as much regularity in the coefficients. We do not need positivity of $\partial_u f(\cdot,0)$ either.

The next result, state conditions ensuring the existence of a positive supersolution which make possible to apply the previous theorem. The proof in Amann [1] is obtained combining Theorems 5.4 and 5.5 and a fixed point theorem for compact increasing maps whose right derivative at 0 and the spectral radius of certain majorant at infinity satisfy certain hypothesis.

Theorem 5.6 (Theorem 9.9 in [1], p. 650) *Under the assumption of Theorem 5.4 assume that there exist $\hat{f} \in C(\overline{\Omega})$ and $m \in C^{\hat{\mu}}(\overline{\Omega})$, with $m(x) > 0$ for almost every $x \in \Omega$, such that, for some $\lambda \geq 0$,*

$$f(x,\xi) \leq \hat{f}(x) + \lambda m(x)\xi$$

for all $(x,\xi) \in \overline{\Omega} \times \mathbb{R}_+$.

Then, there exists at least a positive solution of problem (5.5) provided $\lambda_0(\partial_u f(\cdot,0)) < 1$ and $\lambda < \lambda_0(m)$.

With our arguments, this theorem is a straightforward consequence of Proposition 4.1.

Also, notice that we are under the assumptions of Theorem 4.2 or 4.5 depending on the boundary conditions. In particular, the regularity of f together with the assumption on the linearized problem at 0, i.e., $\lambda_0(\partial_u f(\cdot,0)) < 1$, give the existence and stability from below of a minimal positive equilibrium (see Theorem 4.2 or 4.5). On the other hand, the condition $\lambda < \lambda_0(m)$ gives the existence and asymptotic stability from above of a maximal positive equilibrium (see Theorem 1.1). Notice that f satisfies property (3.7) with $C(x) = \lambda m(x)$ y $D(x) = \hat{f}(x)$ and the semigroup generated by $\Delta + C$ has exponential decay (see Section 5.1).

Notice that we do not require m to satisfy $m(x) \geq 0$ nor the above regularity of $\hat{f}(x)$ or $m(x)$.

In Amann [1] the author prove the following consequence of the previous theorem giving conditions on m and $\partial_u f(x,0)$ to be under the assumptions of the theorem above.

Corollary 5.7 (Corollary 9.10 in [1], p. 651) *Under the assumptions of the previous theorem, if*

$$\min_{x\in\overline{\Omega}} \partial_u f(x,0) > \lambda_0(1) \quad \textit{and} \quad \max_{x\in\overline{\Omega}} m(x) < \lambda_0(1)$$

then there exists at least a positive solution of problem (5.5).

Again, our arguments prove the result. Namely, the hypothesis on $\partial_u f(x,0)$ implies that 0 is an unstable equilibrium for the parabolic problem associated to the elliptic problem (5.5). On the other hand, the hypothesis on $m(x)$ allow us to apply Proposition 4.1 with $C = \beta = \max_{x\in\overline{\Omega}} m(x)$. Notice that the semigroup generated by $-L - \beta$ has exponential decay since $\beta < \lambda_0(1)$ (see Section 5.1).

Furthermore, our result gives the existence of a maximal equilibrium and from Theorems 4.2 or 4.5 we have the existence of a minimal positive equilibrium. In particular, there exists a positive solution of (5.5). Indeed, we obtain the existence of a minimal and a maximal positive solution. Moreover, the minimal solution is asymptotically stable from below and the maximal one is so from above, and the order interval $[\varphi_m^+, \varphi_M]$ attracts bounded sets of non-negative initial data uniformly.

5.2.2 H. Berestycki and P. L. Lions [7] revisited

In Berestycki and Lions [7] the authors prove the next result for a problem of the form

$$\begin{cases} -\Delta u &= f(x,u) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{5.7}$$

Theorem 5.8 (Theorem II.1, p. 18 in Berestycki and Lions [7]) *Let Ω_1 be a connected subdomain of Ω. Suppose $f(x,0) \equiv 0$ and*

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s} > \mu_1 \quad \textit{uniformly in } x \in \overline{\Omega}_1 \tag{5.8}$$

where μ_1 is the first eigenvalue of $-\Delta$ in Ω_1 with Dirichlet boundary conditions; and

$$\limsup_{s\to +\infty} \frac{f(x,s)}{s} < \lambda_1 \quad \textit{uniformly in } x \in \overline{\Omega} \tag{5.9}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions.

Then, there exists a positive solution of problem (5.7) in $W^{2,p}(\Omega)$ (for all $p < \infty$).

In Remark II.1 in Berestycki and Lions [7] the authors notice that the theorem remains valid for more general operators. Namely, strongly elliptic operators of the form

$$L = -\sum_{i,j} D_j(a_{ij}(x)D_i) + \sum_i b_i(x)D_i + c(x)$$

with $a_{ij} = a_{ji} \in C(\overline{\Omega})$, $b_i \in L^\infty(\Omega)$ and $0 \le c \in L^\infty(\Omega)$.

As a consequence of the results, taking $\Omega_1 = \Omega$, they obtain

Corollary 5.9 (Corollary II.1, p. 19 in Berestycki and Lions [7]) *Suppose that $f(x,0) \equiv 0$ and*

$$\liminf_{s\to 0^+} \frac{f(x,s)}{s} \;>\; \lambda_1 \quad \textit{uniformly in } x \in \overline{\Omega} \tag{5.10}$$

$$\limsup_{s\to +\infty} \frac{f(x,s)}{s} \;<\; \lambda_1 \quad \textit{uniformly in } x \in \overline{\Omega} \tag{5.11}$$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω with Dirichlet boundary conditions.

Then, there exists a positive solution of problem (5.7) in $W^{2,p}(\Omega)$ (for all $p < \infty$).

These results are proved in Berestycki and Lions [7] using the sub-supersolutions method for elliptic problems: first, they build a subsolution from the first eigenfunction for the problem in Ω_1; then, they build a sequence of elliptic problems approaching (5.7) such that the non-linear term in these problems is bounded; from here, any solution of the original problem is bounded; finally, they build an iterative algorithm which gives a sequence converging to the solution of (5.7).

In our context, Theorem 5.8 can be obtained as a consequence of Proposition 4.1 and Corollary 4.6: on the one hand, property (5.9) implies the existence of two constants $C < \lambda_1$ and $D > 0$ such that, for all $s \ge 0$

$$f(x,s) \le Cs + D, \quad x \in \Omega,$$

i.e., (3.7); on the other hand, property (5.8) implies the linear instability of the zero solution.

Notice that from Proposition 4.1 we have the existence of a maximal positive equilibrium which is globally asymptotically stable from above. Moreover, we are under assumption of Corollary 4.6. Thus, there exists a minimal equilibrium which is globally asymptotically stable from below.

In the case of Corollary 5.9, Proposition 4.1 and Theorem 4.2 give the results.

5.2.3 P. L. Lions [18] revisited

In Lions [18] the author study problem

$$\begin{cases} -\Delta u &= f(u) & \text{in} & \Omega \\ u &= 0 & \text{on} & \partial\Omega \end{cases} \tag{5.12}$$

with Ω a bounded domain in $\mathbb{R}^N$ and f a locally Lipschitz function which satisfies

$$\limsup_{s\to+\infty} \frac{f(s)}{s} < \lambda_1 \tag{5.13}$$

where λ_1 is the first eigenvalue of $-\Delta$.

First, the following results is stated, which gives sufficient conditions for the existence of a maximal positive solution.

Theorem 5.10 (Theorem 1.3 in [18], p. 447) *Assume f satisfies (5.13), $f(0) = 0$ and*

$$\liminf_{s\to 0} \frac{f(s)}{s} > \lambda_1. \tag{5.14}$$

Then, there exists a maximal positive solution of (5.12). If f vanishes at some $\beta > 0$, the maximal positive solution is so among those below β.

To prove the theorem, the author refers to Amann [1], Amann [2] and Berestycki and Lions [7] (see Sections 5.2.1 and 5.2.2 above).

In our context, the result is a consequence of Proposition 4.1 and Theorem 4.2. If f vanishes at $\beta > 0$ the result follows from the proof of Proposition 4.1 and Theorem 4.2 considering now the supersolution β instead of ϕ.

As before, from Proposition 4.1 we have the existence of a maximal positive equilibrium which is globally asymptotically stable from above. Moreover, we are under assumption of Theorem 4.2. Thus, there exists a minimal equilibrium which is globally asymptotically stable from below.

Also in Lions [18] the author proves, by mean of sub-supersolutions techniques, the following result

Theorem 5.11 (Theorem 2.3 in [18], p. 453) *Suppose f is a locally Lipschitz function such that $f(0) > 0$. Also assume that f satisfies that either there exists $\beta > 0$ such that $f(\beta) = 0$, or $f(s) > 0$ for all $s > 0$ and*

$$\lim_{s\to\infty} \frac{f(s)}{s} = K$$

with $K < \lambda_1$. Then, for all $0 < \lambda < \lambda_1/K$, there exists the minimal positive solution of

$$\begin{cases} -\Delta u &= \lambda f(u) & \textit{in} & \Omega \\ u &= 0 & \textit{on} & \partial\Omega \end{cases} \tag{5.15}$$

Remark 5.12 *If $K = 0$ in the theorem above then the results is valid for all $\lambda > 0$.*

With our techniques, the result is a consequence of Corollary 4.3. But, we also obtain the asymptotic stability from below of the minimal positive equilibrium.

Namely, first assume that $\lambda = 1$. Since 0 is a subsolution of the elliptic problem (5.15), the solution of the parabolic problem starting at 0 is monotonically increasing. If we could find a supersolution of the elliptic problem, then we will have the existence of a bounded solution (the one starting at that supersolution) and then we will have that $u(t, x; 0)$ converges to a minimal equilibrium which is positive from the maximum principle (see Theorem 4.2 i)). The convergence is uniform by the Ascoli-Arzelá Theorem.

Now, the hypotheses above implies the existence of a supersolution. Indeed, if there exists $\beta > 0$ such that $f(\beta) = 0$ then β is a supersolution of the elliptic problem. On the other hand, if $f(s) > 0$ for all $s \geq 0$ and

$$\lim_{s \to \infty} \frac{f(s)}{s} = K \tag{5.16}$$

with $K < \lambda_1$, then f satisfies

$$f(s)s \leq Cs^2 + D|s|$$

for some $C < \lambda_1$ and $D > 0$ (see (3.7)). Thus, from Corollary 4.3 we obtain the existence of a maximal equilibrium and so a supersolution of the elliptic problem. In fact, it is enough to notice that ϕ (the solution of (3.8)) is a supersolution of (5.12).

Asymptotic stability from below of the minimal equilibrium follows from Corollary 4.3 or Theorem 4.2.

The result for all $0 < \lambda < \lambda_1/K$ is a straightforward consequence of the following remark: if we denote by $g(u) = \lambda f(u)$ then the problem with g is under assumption of the theorem with $\lambda = 1$ since

$$\lim_{s \to \infty} \frac{g(s)}{s} = \lim_{s \to \infty} \frac{\lambda f(s)}{s} < \lambda K < \lambda_1.$$

In the last case, we also obtain the existence of a maximal equilibrium. With this, we have a bound for the asymptotic dynamics of the non-negative solutions. Furthermore, the region between the two extremal equilibria attracts the asymptotic dynamics of non-negative solutions uniformly.

5.2.4 Figueiredo [13] revisited

In Figueiredo [13] the author considers a problem similar to the one studied in Amann [1] (see Section 5.2.1) with Dirichlet boundary conditions and L is an strongly elliptic operator in divergence form

$$Lu = -\sum_{i^1}^{N} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + a_0(x)u$$

with $a_{ij} \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ and $a_0 \in L^{N/2}(\Omega)$, $a_0(x) \geq 0$ in Ω. Moreover, $f \in C^\alpha(\overline{\Omega} \times \mathbb{R}^+)$ and satisfies:

(c_0) There exists a continuous function, $f_0(x) \geq 0$, and $s_0 > 0$ such that

$$f(x, s) \geq f_0(x)s \quad \text{for all } 0 < s < s_0, \ x \in \overline{\Omega}.$$

(c_∞) There exist two continuous functions $f_\infty(x), c(x)$ with $c(x) \geq 0$ such that

$$f(x, s) \leq f_\infty(x)s + c(x) \quad \text{for all } s \geq 0, \ x \in \overline{\Omega}.$$

(nd) For all $M > 0$ there exists $k \geq 0$ such that $s \to f(x, s) + ks$ is not increasing for $|s| \leq M$.

Then it is proved

Theorem 5.13 (Theorem 2.2 in [13], p. 53) *Assume f satisfies (c_0), (c_∞) and (nd). Also assume that*

$$\lambda_0(f_0) < 1 \tag{5.17}$$

$$\lambda_0(f_\infty) > 1 \tag{5.18}$$

(the last condition on f_∞ is needed if $f_\infty(x_0) > 0$ for some $x_0 \in \Omega$). Then, there exists a positive solution of the Dirichlet problem

$$\begin{cases} -\Delta u &= f(x,u) \quad in \quad \Omega \\ \quad u &= 0 \qquad\quad on \quad \partial\Omega \end{cases} \tag{5.19}$$

Also in Figueiredo [13] the author proves

Lemma 5.14 (Lemma 2.3 in [13], p. 56) *Under the assumptions of the theorem above, problem (5.19) has a maximal positive solution.*

The author obtains these results by mean of the sub-supersolution method: hypothesis (c_∞) and (5.18) imply the existence of a positive supersolution of the elliptic problem and hypothesis (c_0) and (5.17) imply the existence of a subsolution below the supersolution. From here a solution is constructed as in Amann [1].

With our techniques we can recover these results and obtain stronger conclusions (also notice that our assumptions are weaker). Namely, we obtained the existence of two extremal positive solutions (which may coincide) and asymptotic stability properties. Indeed, property (5.17) implies the instability of $u \equiv 0$ while property (5.18) implies the exponential decay of the semigroup generated by $-L + f_\infty$. Now, by (c_∞) we have the dissipativity property (3.7). The result follows from Corollary 4.3 and Theorem 4.2.

5.2.5 A result about logistic equations

In Hernández [17] the existence of positive solution for elliptic logistic equations with unbounded potential. Namely, the author studies the equation

$$\begin{cases} -\Delta u &= \lambda m(x) u - u^\rho \quad \text{in} \quad \Omega \\ \quad u &= 0 \qquad\qquad\quad \text{on} \quad \partial\Omega \end{cases} \tag{5.20}$$

with $\rho > 1$, $\lambda \in \mathbb{R}$ y $m \in L^p(\Omega)$, $p > N/2$, with $m(x) \geq m_0 > 0$ in Ω, for certain positive constant m_0. The author proves the following theorem

Theorem 5.15 *Let m as above. Suppose $\rho > 1$ and*

$$\frac{\rho N}{2(\rho - 1)} < p. \tag{5.21}$$

Then, for all $\lambda > \lambda_1$, the principal eigenvalue of m, there exists a unique positive solution $u > 0$ of (5.20) with $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ for some $q > N/2$. Therefore, $u \in C(\overline{\Omega})$. If, in addition,

$$\frac{\rho N}{\rho - 1} < p \tag{5.22}$$

then $u \in C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$.

The proof in Hernández [17] is based again in the sub-supersolutions method to prove the existence of a positive solution. Namely, the supersolution is obtained by computing the maximum in u of the nonlinear term f. The subsolution is built as a small multiple of the first eigenfunction of the principal eigenvalue for m. The uniqueness follows from a similar argument of our Theorem 4.16. Finally, the regularity is obtained by elliptic regularity.

Equations like (5.20) are considered in the next section (see Proposition 6.1 below). The existence result is a consequence of Proposition 4.1. Uniqueness follows from Theorem 4.16. Moreover, results from Section 4.2 implies the global asymptotic stability for positive solutions of the unique equilibrium for problem (5.20). Regularity is a consequence of elliptic regularity.

Finally, notice that our results are valid for more general operators than Laplacian (including non-variational ones). Moreover, we do not require $m(x)$ to be positive in Ω. Also, we do not require any relation between the growth of the non-linearity, ρ, and the regularity of m, i.e. p, as in (5.21) and (5.22). In fact, we get $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Thus, $u \in C(\overline{\Omega})$ just requiring $p > N/2$. If $p > N$ we have $u \in C^{1,\beta}(\overline{\Omega})$ for some $0 < \beta < 1$ (see comment below the proof of Theorem 2.2 and the last part of Theorem 2.5).

6 A model example: logistic equations

Our aim in this section is to apply the previous results to a model equation: a logistic autonomous equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u &=& f(x,u), & \text{in} \quad \Omega \\ u &=& 0, & \text{on} \quad \partial\Omega \\ u(0) &=& u_0 & \end{array} \right. \tag{6.1}$$

where

$$f(x,s) = m(x)s - n(x)|s|^{\rho-1}s, \quad \rho > 1,$$

with

$$m \in L^p(\Omega) \quad \text{for some} \quad p > N/2$$

and

$$n(x) \geq 0 \quad \text{in} \quad \Omega \quad \text{is a continuous function not identically zero.}$$

Therefore, f is in the hypotheses of Theorem 2.2 and so there exists a local solution for initial data in $X = L^\infty(\Omega)$.

First, notice that

$$f(x,s)s = m(x)s^2 - n(x)|s|^{\rho+1}. \tag{6.2}$$

Thus, if $\lambda_1(\Delta + m) > 0$ then

$$f(x,s)s = m(x)s^2 - n(x)|s|^{\rho+1} \leq m(x)s^2.$$

Thus, we can take $C = m$ and $D = 0$ and get the existence of a unique equilibrium $\varphi_m = \varphi_M = 0$ which is globally asymptotically stable.

On the contrary, if $\lambda_1(\Delta + m) < 0$ then suppose that we decompose m in the form $m(x) = m_1(x) + m_2(x)$, $x \in \Omega$, with $m_2 \geq 0$. Then, at least formally, by Young inequality

$$f(x,s)s \leq m_1(x)s^2 + \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)} \right]^{\rho'} |s| \tag{6.3}$$

for certain positive constant $\beta > 0$. Thus, f satisfies the structure condition (3.7) with

$$C(x) = m_1(x) \quad \text{y} \quad D(x) = \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'}.$$

Notice that f satisfies the growth restriction (2.9) with growth rate ρ. Thus, the existence of two extremal equilibria which are globally stable from above and from below respectively will follow from the results in Section 3. More precisely, to apply Theorem 3.7 and Corollary 3.10 we need to choose $m_1 \in L^p(\Omega)$, for some $p > N/2$, such that the linear semigroup generated by $\Delta + m_1$ has exponential decay and m_2 such that $D(x)$ belongs to some $L^r(\Omega)$ with $r > \frac{N}{2}(1-\frac{1}{\rho})$.

In such a case, from Theorem 4.2 we obtain that the maximal equilibrium is positive. Analogously, the minimal one is negative since $f(\cdot, 0) \equiv 0$ and 0 is unstable.

Moreover, the uniqueness of positive equilibria will follow from Theorem 4.16 since

$$\frac{f(x,s)}{s} = m(x) - n(x)|s|^{\rho-1}$$

is μ-decreasing (we recall that $n(x) \geq 0$ is not identically null). Analogously, we obtain uniqueness of negative solutions. Therefore, the minimal and maximal solutions are, respectively, globally asymptotically stable for nonnegative or non-positive solutions of (6.1). Notice that since $m \in L^p(\Omega)$ then f do not satisfies the hypotheses of Brézis and Oswald Theorem (see Theorem 4.15).

As we will see below, to construct m_1 and m_2 as above we will distinguish the case in which $n > 0$ in Ω or vanishes slowly in a small region, from the case in which it vanishes very fast or in a large set. In the first case, we will show that we can always choose m_1 and m_2 such that C and D satisfy conditions above.

In the second case, we will see that, to apply the results of the previous sections, we need that m contributes to the dissipation near of the set where n vanishes, since there the reaction is linear. More precisely we have, in the former case,

Proposition 6.1 *Suppose that either $n(x) \geq \gamma > 0$ in $\overline{\Omega}$ or $1/n \in L^s(\Omega)$, $s > N/2\rho$. Then, for all $m \in L^p(\Omega)$ with $p > N/2$ there exist C and D satisfying (3.7) and such that the semigroup generated by $\Delta + C$ has exponential decay and $D \in L^r(\Omega)$ for some $r > \frac{N}{2}\left(1 - \frac{1}{\rho}\right)$.*

Proof. Suppose that there exists $\gamma > 0$ such that $n(x) \geq \gamma$ in $\overline{\Omega}$ or $1/n \in L^s(\Omega)$, $s > N/2\rho$. Then, we take

$$C(x) = m_1(x) = m(x) - \lambda \quad \text{y} \quad m_2(x) = \lambda$$

with λ large enough such that the semigroup generated by $\Delta + m_1$ has exponential decay. In such case, in (6.3) we have

$$0 \leq D(x) = \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)}\right]^{\rho'} \leq \beta \frac{\lambda^{\rho'}}{n^{\rho'/\rho}(x)} \in L^r(\Omega) \quad \text{with} \quad r > \frac{N}{2}\left(1 - \frac{1}{\rho}\right).$$

■

We now consider the case in which $n(x)$ vanishes fast or in a large subset of Ω. In this case, we have

Proposition 6.2 *Let $\Omega_0 = \{x \in \Omega : \ n(x) = 0\}$ and Ω_δ be a neighborhood of Ω_0 such that $n(x) \geq \delta > 0$ for all $x \in \Omega \setminus \overline{\Omega}_\delta$. Suppose that the first eigenvalue of $-\Delta - m$ with Dirichlet boundary conditions in Ω_δ, $\lambda_1^{\Omega_\delta}(-\Delta - m)$, is positive.*

Then, there exist C and D satisfying (3.7). Also they are such that the semigroup generated by $\Delta + C$ has exponential decay and $D \in L^r(\Omega)$ for some $r > \frac{N}{2}(1 - \frac{1}{\rho})$.

Proof. By (6.2), if $x \in \Omega_\delta$ then

$$f(x,s) \le m(x)s^2$$

and we take $C(x) = m(x)$ and $D(x) = 0$ if $x \in \Omega_\delta$.

On the other hand, if $x \in \Omega \setminus \overline{\Omega}_\delta$ then, for A large enough, we write

$$m_1(x) = \big(-m^-(x) - A \big), \quad m_2(x) = \big(m^+(x) + A \big) \ge 0, \quad x \in \Omega \setminus \overline{\Omega}_\delta.$$

Then, we set $C(x) = m_1(x) = -m^-(x) - A$ for $x \in \Omega \setminus \overline{\Omega}_\delta$.

From Lemma 6.3 below and the sign of $-m^-(x)$ in $\Omega \setminus \overline{\Omega}_\delta$ we have that the linear semigroup generated by $\Delta + C(x)$ with Dirichlet boundary conditions in Ω has exponential decay.

Arguing as in (6.3) we have, for $x \in \Omega \setminus \overline{\Omega}_\delta$,

$$D(x) = \beta \left[\frac{m_2(x)}{n^{1/\rho}(x)} \right]^{\rho'} \le \beta \delta^{-\rho'/\rho} m_2^{\rho'}(x) \in L^r(\Omega \setminus \overline{\Omega}_\delta), \quad r > \frac{N}{2}(1 - \frac{1}{\rho})$$

where we have used that $m^+ \in L^p$, $p > N/2$. Therefore, $D \in L^r(\Omega)$, with $r > \frac{N}{2}(1 - \frac{1}{\rho})$ and we can apply Theorem 3.7 and Corollary 3.10. ∎

A similar result, assuming $C^{3+\alpha}$ regularity on the set Ω_0 can be found in Fraile et al. [14].

Notice that if Ω_0 is a regular set and $\lambda_1^{\Omega_0}(-\Delta - m) > 0$ then for some neighborhood Ω_δ close enough to Ω_0 we have

$$\lambda_1^{\Omega_\delta}(-\Delta - m) > 0$$

and $n(x) \ge \delta > 0$ for all $x \in \Omega \setminus \overline{\Omega}_\delta$.

We now prove the lemma used in the proof Proposition 6.2.

Lemma 6.3 *With the previous notation, suppose $\lambda_1^{\Omega_\delta}(-\Delta - m) > 0$.*

Then, for A large enough, taking

$$C_0(x) = \begin{cases} m(x) & \text{if } x \in \Omega_\delta \\ -A & \text{if } x \in \Omega \setminus \overline{\Omega_\delta} \end{cases}$$

we have that the first eigenvalue of $-\Delta - C_0$ with Dirichlet boundary condition in Ω is positive.

Proof. Notice that

$$\lambda_1(-\Delta - C_0) = \inf_{\varphi \in H_0^1(\Omega)} \frac{J(\varphi)}{\int_\Omega \varphi^2}$$

where

$$J(\varphi) = \int_\Omega |\nabla \varphi|^2 - \int_\Omega C_0 \varphi^2.$$

Now, given $\varphi \in H_0^1(\Omega)$ we set $P(\varphi) = \xi \in H_0^1(\Omega_\delta)$ where ξ solves

$$\left\{ \begin{array}{rcll} -\Delta\xi - m\xi & = & 0 & \text{in } \Omega_\delta \\ \xi & = & \varphi & \text{on } \partial\Omega_\delta. \end{array} \right. \tag{6.4}$$

Notice that $\varphi_{|\partial\Omega_\delta} \in H^{1/2}(\partial\Omega_\delta)$ and $\lambda_1^{\Omega_\delta}(-\Delta - m) > 0$. Thus, (6.4) is well-posed.

Now, let $\eta = \varphi - \xi \in H_0^1(\Omega_\delta)$ which solves

$$\left\{ \begin{array}{rcll} -\Delta\eta - m\eta & = & -\Delta\varphi - m\varphi & \text{in } \Omega_\delta \\ \eta & = & 0 & \text{on } \partial\Omega_\delta. \end{array} \right.$$

Then, we extend η to Ω by zero and we still denote it by η. Thus, on the one hand $\varphi = \eta + (\varphi - \eta)$ and

$$\int_\Omega C_0\varphi^2 = \int_\Omega C_0[\eta^2 + (\varphi-\eta)^2 + 2\eta(\varphi-\eta)].$$

But, since $\eta = 0$ out of Ω_δ we have

$$\int_\Omega C_0\varphi^2 = \int_{\Omega_\delta} m\eta^2 + 2\int_{\Omega_\delta} m\eta\xi + \int_{\Omega_\delta} m\xi^2 - \int_{\Omega\setminus\overline{\Omega}_\delta} A\varphi^2$$

where we have used that $\varphi - \eta = \xi$ in Ω_δ.

On the other hand, since $\varphi = \eta + (\varphi - \eta)$ we have

$$\int_\Omega |\nabla\varphi|^2 = \int_\Omega |\nabla\eta|^2 + 2\int_\Omega \nabla\eta\nabla(\varphi-\eta) + \int_\Omega |\nabla(\varphi-\eta)|^2.$$

But, η vanishes out of Ω_δ. So,

$$\int_\Omega |\nabla\varphi|^2 = \int_{\Omega_\delta} |\nabla\eta|^2 + 2\int_{\Omega_\delta} \nabla\eta\nabla\xi + \int_{\Omega_\delta} |\nabla\xi|^2 + \int_{\Omega\setminus\overline{\Omega}_\delta} |\nabla\varphi|^2$$

where we have used that $\varphi - \eta = \xi$ in Ω_δ.

Thus,

$$\begin{aligned} J(\varphi) = \int_\Omega |\nabla\varphi|^2 - \int_\Omega C_0\varphi^2 \;\; &= \int_{\Omega_\delta} \left[|\nabla\eta|^2 - m\eta^2\right] + \int_{\Omega\setminus\overline{\Omega}_\delta} \left[|\nabla\varphi|^2 + A\varphi^2\right] \\ &\quad + 2\int_{\Omega_\delta} [\nabla\eta\nabla\xi - m\eta\xi] + \int_{\Omega_\delta} \left[|\nabla\xi|^2 - m\xi^2\right]. \end{aligned} \tag{6.5}$$

But, since $\eta \in H_0^1(\Omega_\delta)$,

$$\int_{\Omega_\delta} \left[|\nabla\eta|^2 - m\eta^2\right] \geq \lambda_1^{\Omega_\delta}(-\Delta - m)\int_\Omega \eta^2.$$

Now, multiplying (6.4) by ξ and integrating by parts

$$\int_{\Omega_\delta} \left[|\nabla\xi|^2 - m\xi^2\right] = \int_{\partial\Omega_\delta} \frac{\partial\xi}{\partial n}\varphi.$$

Using the relation between φ and ξ we have

$$\left|\int_{\partial\Omega_\delta} \frac{\partial\xi}{\partial n}\varphi\right| \leq C||\varphi||^2_{H^{1/2}(\partial\Omega_\delta)}. \tag{6.6}$$

Thus, given $\varepsilon > 0$

$$\int_{\partial\Omega_\delta} \frac{\partial\xi}{\partial n}\varphi \geq -\varepsilon\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla\varphi|^2 - C_\varepsilon\int_{\Omega\setminus\overline{\Omega}_\delta} |\varphi|^2.$$

Then,

$$\int_{\Omega\setminus\overline{\Omega}_\delta} \left[|\nabla\varphi|^2 + A\varphi^2\right] + \int_{\partial\Omega_\delta} \frac{\partial\xi}{\partial n}\varphi \geq (1-\varepsilon)\int_{\Omega\setminus\overline{\Omega}_\delta} |\nabla\varphi|^2 + (A - C_\varepsilon)\int_{\Omega\setminus\overline{\Omega}_\delta} \varphi^2.$$

Taking $\varepsilon = 1/2$ we have, for A large enough,

$$\int_{\Omega\setminus\overline{\Omega}_\delta} \left[|\nabla\varphi|^2 + A\varphi^2\right] + \int_{\Omega_\delta} \left[|\nabla\xi|^2 - m\xi^2\right] \geq \gamma||\varphi||^2_{H^1(\Omega\setminus\overline{\Omega}_\delta)}$$

for certain positive constant $\gamma > 0$.

Finally, since $\eta \in H_0^1(\Omega_\delta)$ and ξ solves (6.4) we have

$$\int_{\Omega_\delta} [\nabla\eta\nabla\xi - m\eta\xi] = 0.$$

Thus, in (6.5) we get

$$J(\varphi) \geq \lambda_1^{\Omega_\delta} \int_{\Omega_\delta} \eta^2 + \gamma ||\varphi||^2_{H^1(\Omega\setminus\overline{\Omega}_\delta)} \geq 0$$

Notice that from (6.6),

$$||\varphi||^2_{H^1(\Omega\setminus\overline{\Omega}_\delta)} \geq a(\int_{\Omega\setminus\overline{\Omega}_\delta} |\varphi|^2 + ||\xi||^2_{H^1(\Omega_\delta)})$$

for certain positive constant a. Thus,

$$J(\varphi) \geq \beta \int_\Omega |\varphi|^2$$

for some $\beta > 0$, and then $\lambda_1(-\Delta - C_0) > \beta > 0$. ■

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