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Inequivalent sets of commuting missing label operators for the nuclear surfon model

R. Campoamor-Stursberg†‡
† Dpto. Geometría y Topología
Universidad Complutense de Madrid
Plaza de Ciencias, 3
E-28040 Madrid, Spain
‡ Instituto de Matemática Interdisciplinar U.C.M.

E-mail: `rutwig@pdi.ucm.es`

Abstract. We exhibit a possible choice for the four available functionally independent labelling operators for the missing label problem associated to the reduction chain $\mathfrak{so}(5) \supset \mathfrak{so}(3)$.

1. Introduction

The labelling of the basis states of irreducible representations (irreps) of a semisimple Lie group G with respect to some subgroup H often leads to the so called missing label problem (MLP), i.e., to the problem of determining subgroup scalars[1] with respect to H that, added to the Casimir operators of G and H, enable to completely classify the states. Basis states for irreps of G are then specified by the common eigenstates of a complete set of labelling operators. The eight reduction chains with one missing label were solved in [1, 2]. Although some general results exist concerning arbitrary reduction chains, mainly about the properties of generating functions and the construction of integrity bases,[2] as well as the development of special techniques like the method of elementary permissible diagrams [1, 3, 4, 5, 6, 7], only for a small number of labelling problems with more than one labelling operator a general solution has been worked out.

In this paper we determine four functionally independent operators depending on a parameter that solve the missing label problem associated to the embedding $\mathfrak{so}(5) \supset \mathfrak{so}(3)$. This reduction chain appears in many applications involving the subalgebra of angular momentum, like the classification of states build up by quadrupole phonon states [8, 9]. The corresponding $n = 2$ missing label problem has been analyzed in [10, 11], where two commuting missing label operators of degrees four and six, respectively, were found. These two operators were shown to constitute, in some sense, the simplest possible choice to solve this labelling problem. A complete set of four independent labelling operators was however not given. In our analysis, it is verified that the pair of commuting operators found in [11] corresponds to the simplest possible solution, and the conjecture on the degree of these operators is confirmed. We moreover show the existence of another pair consisting of operators of degrees four and six that solve the labelling

[1] Polynomials in the enveloping algebra of G that commute with all generators of the subgroup H will be called subgroup scalars.
[2] An integrity basis is a set of elementary subgroup scalars in terms of which all can be expressed by polynomials.

problem. In particular, for special values of the parameter, seven special cases corresponding to the simplest possible solutions to the MLP are found.

2. Missing label operators

For semisimple Lie algebras $\mathfrak{s}$, the rank $l\mathcal{N}(\mathfrak{s}) = l$ specifies the maximal number of (functionally) independent polynomials in the generators such that they commute with the elements of $\mathfrak{s}$ [12]. The eigenvalues of these operators, when applied to representations, provide a first class of suitable labels to distinguish the multiplets and the states. For higher rank algebras, however, degeneracies are found that cannot be solved using the Casimir operators and/or the generators of the Cartan subalgebra. According to Racah, we have to find $f = \frac{1}{2}(\dim \mathfrak{s} - 3l)$ additional operators to distinguish the states. This gives a total number of labels equal to

$$i = \frac{1}{2}(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s})). \tag{1}$$

Reducing representations of a Lie algebra with respect to some (inner symmetry) subalgebra $\mathfrak{h}$ leads, in the generic case, to a similar situation, where the invariants of the chain do not provide enough labels to separate the states. The reduction chain provides $\frac{1}{2}(\dim \mathfrak{h} + \mathcal{N}(\mathfrak{h})) + l'$ labels, where l' denotes the number of common invariants of $\mathfrak{g}$ and $\mathfrak{h}$.[3] Using this fact, it is easy to see that in addition to the Casimir operators, exactly

$$n = \frac{1}{2}\left(\dim \mathfrak{s} - \mathcal{N}(\mathfrak{s}) - \dim \mathfrak{h} - \mathcal{N}(\mathfrak{h})\right) + l' \tag{2}$$

additional operators are necessary [13]. The scalar $m = 2n$ gives the total number of available operators.

We determine labelling operators by means of the analytical approach [13]. Given the Lie algebra $\mathfrak{s} = \left\{X_1, .., X_n \mid [X_i, X_j] = C_{ij}^k X_k\right\}$ in terms of generators and commutation relations, we realize the generators X_i in the space $C^\infty(\mathfrak{s}^*)$ by means of the differential operators

$$\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{3}$$

where $\{x_1, .., x_n\}$ are the coordinates on a dual basis. In this context, invariants of $\mathfrak{s}$ are the solutions of system of partial differential equations given by:

$$\widehat{X}_i F = 0, \quad 1 \leq i \leq n. \tag{4}$$

For polynomial solutions of (4), the symmetrization map defined by

$$Sym(x_{i_1}..x_{i_p}) = \frac{1}{p!} \sum_{\sigma \in S_p} x_{\sigma(i_1)}..x_{\sigma(i_p)} \tag{5}$$

recover the Casimir operators in their usual form, i.e., as a polynomial in the generators commuting with $\mathfrak{s}$. Observe that the preceding realization shows that the number $\mathcal{N}(\mathfrak{s})$ of independent solutions is given by:

$$\mathcal{N}(\mathfrak{s}) := \dim\, \mathfrak{s} - \operatorname{rank}\left(C_{ij}^k x_k\right), \tag{6}$$

[3] I.e., invariants of $\mathfrak{s}$ that depend only on variables of the subalgebra $\mathfrak{h}$.

Table 1. $\mathfrak{so}(5)$ brackets in a $\mathfrak{so}(3) = \{L_0, L_{\pm 1}\}$ basis.

[,]	Q_3	Q_2	Q_1	Q_0	Q_{-1}	Q_{-2}	Q_{-3}
L_0	$3Q_3$	$2Q_2$	Q_1	0	$-Q_{-1}$	$-2Q_{-2}$	$-3Q_{-3}$
L_1	0	$6Q_3$	Q_2	$2Q_1$	$6Q_0$	$10Q_{-1}$	Q_{-2}
L_{-1}	Q_2	$10Q_1$	$6Q_0$	$2Q_{-1}$	Q_{-2}	$6Q_{-3}$	0
Q_3	0	0	0	Q_3	Q_2	$10Q_1 + 15L_1$	$5Q_0 - 15L_0$
, Q_2		0	$-6Q_3$	$-Q_2$	$-15L_1$	$30Q_0 + 60L_0$	$10Q_{-1} - 15L_{-1}$
Q_1			0	$3L_1 - Q_1$	$-3L_0 - 3Q_0$	$15L_{-1}$	Q_{-2}
Q_0				0	$-Q_{-1} - 3L_{-1}$	$-Q_{-2}$	Q_{-3}
Q_{-1}					0	$-6Q_{-3}$	0
Q_{-2}						0	0

where $A(\mathfrak{s}) := \left(C_{ij}^k x_k\right)$ is the matrix associated to the commutator table of $\mathfrak{s}$ over the given basis.

From this perspective, the missing label problem (short MLP) for a reduction chain $\mathfrak{s} \supset \mathfrak{s}'$ constitutes a special application of the analytical ansatz. More specifically, the labelling operators can be seen as those operators that commute with the generators of the subalgebra $\mathfrak{s}'$, thus correspond to a subsystem of (4). The latter has exactly $\mathcal{N}(f(\mathfrak{s}')) = \dim\mathfrak{s} - \dim\mathfrak{s}' - l'$ functionally independent solutions. Transforming this formula using (2), we arrive at the expression

$$\mathcal{N}(f(\mathfrak{s}')) = m + \mathcal{N}(\mathfrak{s}) + \mathcal{N}(\mathfrak{s}') - l'. \tag{7}$$

This shows that the subsystem corresponding to $\mathfrak{s}'$-generators has n more solutions as needed to solve the missing label problem [13]. The ansatz based on differential equations provides an alternative to the usually difficult task of finding integrity bases for the labelling operators [14]. Because of equation (7), if we find m solutions that are independent among each other and from the Casimir operators of the chain, a solution to the missing label problem is given by n polynomials of these solutions that commute mutually. This last step is in practice the most difficult one, and has been completely solved only for a small number of reduction chains.

To be used later, we introduce the counterpart of labelling operators in the space $C^\infty(\mathfrak{s})$, that will be useful to measure the non-commutativity of brackets. Let $P = a^{i_1\cdots i_p} X_{i_1}...X_{i_p}$ be a labelling operator. Replacing the generators X_{i_j} by the coordinate of the corresponding dual element, we obtain the homogeneous polynomial $P^* = a^{i_1\cdots i_p} x_{i_1}...x_{i_p}$. Observe that since the variables in $\mathfrak{s}^*$ commute, P^* will usually have less terms than P. It is clear that if $[O_1, O_2] \neq 0$, then the analytical counterpart of the commutator does not vanish, and the number of terms gives an approximate idea of how far the operators O_1, O_2 are from commuting.

3. The nuclear surfon model

For the non-canonical embedding $\mathfrak{so}(5) \supset \mathfrak{so}(3)$, the subalgebra $\mathfrak{so}(3)$ turns out to be the principal simple subalgebra of rank one [3]. We choose the basis of the orthogonal Lie algebra $\mathfrak{so}(5)$ to consist of generators $\{L_0, L_1, L_{-1}\}$ with brackets $[L_0, L_{\pm 1}] = \pm L_{\pm 1}$, $[L_1, L_{-1}] = 2L_0$ together with the irreducible tensor representation Q_μ ($\mu = -3..3$) of dimension seven. The brackets of $\mathfrak{so}(5)$ over this basis are given in Table 1.

In order to describe states when reducing representations of $\mathfrak{so}(5)$ with respect to this $\mathfrak{so}(3)$, we need, according to formula (2), two missing labels among the four available operators. In tems of the differential realization (3), these labelling operators correspond to solutions of the following system of equations:

$$\begin{aligned}
\widehat{L}_0F &:= l_1\frac{\partial F}{\partial l_1} - l_{-1}\frac{\partial F}{\partial l_{-.1}} + 3q_3\frac{\partial F}{\partial q_3} + 2q_2\frac{\partial F}{\partial q_2} + q_1\frac{\partial F}{\partial q_1} - q_{-1}\frac{\partial F}{\partial q_{-1}} - 2q_{-2}\frac{\partial F}{\partial q_{-2}} - 3q_{-3}\frac{\partial F}{\partial q_{-3}} = 0,\\
\widehat{L}_1F &:= -l_1\frac{\partial F}{\partial l_0} + 2l_0\frac{\partial F}{\partial l_{-1}} + 6q_3\frac{\partial F}{\partial q_2} + q_2\frac{\partial F}{\partial q_1} + 2q_1\frac{\partial F}{\partial q_0} + 6q_0\frac{\partial F}{\partial q_{-1}} + 10q_{-1}\frac{\partial F}{\partial q_{-2}} + q_{-2}\frac{\partial F}{\partial q_{-3}} = 0,\\
\widehat{L}_{-1}F &:= l_{-1}\frac{\partial F}{\partial l_0} - 2l_0\frac{\partial F}{\partial l_1} + q_2\frac{\partial F}{\partial q_3} + 10q_1\frac{\partial F}{\partial q_2} + 6q_0\frac{\partial F}{\partial q_1} + 2q_{-1}\frac{\partial F}{\partial q_0} + q_{-2}\frac{\partial F}{\partial q_{-1}} + 6q_3\frac{\partial F}{\partial q_{-2}} = 0.
\end{aligned} \tag{8}$$

For labelling purposes, we are only interested on polynomial solutions. These, after the corresponding symmetrization (5) of its terms, provide the classical subgroup scalars. The preceding system (8) has seven functionally independent solutions. It is obvious that the Casimir operators of $\mathfrak{so}(3)$ and $\mathfrak{so}(5)$ satisfy this system, thus can be interpreted as labelling operators of special kind. More specifically, they are composite functions of labelling operators of lower order.[4]

Because of the diagonal action of the generator L_0 of $\mathfrak{so}(3)$ on $L_{\pm 1}$ and the irreducible multiplet of dimension seven, polynomials $P = \alpha_{a_{-1}a_0a_1b_3\dots b_{-3}} l_0^{a_0} l_{-1}^{a_{-1}} l_1^{a_1} q_3^{b_3} \dots q_{-3}^{b_{-3}}$ satisfying the system (8) must also satisfy the linear condition

$$-a_{-1} + a_1 + 3b_3 + 2b_2 + b_1 - b_{-1} - 2b_{-2} - 3b_{-3} = 0. \tag{9}$$

Following the notation of [11], we denote by $[k, m]$ a homogeneous polynomial of degree $k + m$ such that its degree in the q_i-variables is k and its degree in $\{l_0, l_1, l_1\}$ is m. Observe that by (9), $k = b_3 + b_2 + b_1 + b_0 + b_{-1} + b_{-2} + b_{-3}$ and $m = a_{-1} + a_0 + a_1$. This notation enables us to rewrite P as a sum of homogeneous polynomials $[k_i, m_j]$. We will say that $[k_i, m_j]$ has bi-degree (k_i, m_i).

To construct the labelling operators, we proceed basing on the degree of polynomial solutions. For any fixed $n \geq 2$, we determine the functionally independent operators $[k, m]$ such that $k + m = n$. Computing such solutions up to a sufficiently high n leads to a set of functions that, once symmetrized, constitute an integrity basis for the MLP. However, here we are only interested in finding two pairs of independent operators such that they solve the MLP for $\mathfrak{so}(5) \supset \mathfrak{so}(3)$ with the additional commutativity condition.

For $n = 2$, only two solutions to (8) exist:

$$[0, 2] = l_0^2 + l_1 l_{-1}, \quad [2, 0] = -\frac{2}{5}(q_3q_{-3} + q_1q_{-1}) + \frac{1}{15}q_2q_{-2} + q_0^2. \tag{10}$$

It follows at once that $[0, 2]$ is the Casimir operator of $\mathfrak{so}(3)$. In addition, the polynomial $C_2 = [0, 2] + [2, 0]$ corresponds to the quadratic Casimir operator of $\mathfrak{so}(5)$.[5]
A generic polynomial of degree 3 in the generators of $\mathfrak{so}(5)$ and satisfying constraint (9) has 26 terms. Due to the latter, it automatically gives a solution of the equation $\widehat{L_0}F = 0$. Inserting such a polynomial into the remaining equations of (8) and solving the corresponding system with respect to the coefficients shows that only the trivial solution is admissible, from which we conclude that system (8) has no polynomial solutions of order three. This provides additional information on the behavior of the quadratic solutions. It is well known that if O_1, O_2 are two

[4] This is more apparent when the Casimir operators are contracted with respect to the natural contraction determined by the embedding of the algebras [15].
[5] In the following, we will identify, for computational purposes, the polynomial function f that solves system (8) with the symmetrized polynomial $Sym(f)$. With an abuse of language, we will call f the labelling operator.

labelling operators, then their commutator $[O_1, O_2]$ also provides a labelling operator [1]. As a consequence of this fact, the two operators $[2,0]$ and $[0,2]$ commute after symmetrization, since their commutator would have terms of order three.

In order four, the three labelling operators $[0,2]^2$, $[0,2]\,[2,0]$ and $[2,0]^2$ are functionally dependent on those of order two, and therefore of no further use for solving the MLP. In the following we will only be interested on solutions that are not of this type. We call a polynomial $[k,m]$ indecomposable if it is not a function of polynomials of lower order. Among the seven linearly independent solutions of (8) of degree four, the only indecomposable ones are the following:

$$
\begin{aligned}
[1,3] = \;& \tfrac{1}{4}l_0l_1^2q_{-2} - \tfrac{3}{2}l_0l_1l_{-1}q_0 + \tfrac{1}{4}l_1^3q_{-3} - l_0^2l_{-1}q_1 + l_0^2l_1q_{-1} + l_0^3q_0 + \tfrac{1}{4}l_0l_{-1}^2q_2 - \tfrac{1}{4}l_{-1}^3q_3 \\
& -\tfrac{1}{4}l_1^2l_{-1}q_{-1} + \tfrac{1}{4}l_1l_{-1}^2q_1,
\end{aligned}
$$

$$
\begin{aligned}
[2,2] = \;& -\tfrac{1}{12}l_{-1}^2q_2q_0 + \tfrac{1}{12}l_0l_{-1}q_2q_{-1} + \tfrac{1}{6}l_0l_1q_0q_{-1} + \tfrac{1}{12}l_1^2q_1q_{-3} - \tfrac{1}{3}l_0^2q_1q_{-1} + \tfrac{7}{12}{l_0}^2q_0^2 \\
& +\tfrac{1}{60}l_1l_{-1}q_2q_{-2} + \tfrac{1}{12}l_{-1}^2q_3q_{-1} - \tfrac{1}{12}l_0l_{-1}q_3q_{-2} + \tfrac{1}{12}l_1l_{-1}q_0^2 - \tfrac{1}{12}l_1^2q_0q_{-2} + \tfrac{1}{12}l_1^2q_{-1}^2 \\
& -\tfrac{1}{6}l_0l_{-1}q_1q_0 - \tfrac{1}{12}l_1l_{-1}q_1q_{-1} + \tfrac{1}{12}l_{-1}^2q_1^2 + \tfrac{1}{15}l_0^2q_3q_{-3} + \tfrac{1}{60}l_0^2q_2q_{-2} - \tfrac{1}{12}l_0l_1q_1q_{-2} \\
& -\tfrac{11}{60}l_1l_{-1}q_3q_{-3} + \tfrac{1}{12}l_0l_1q_2q_{-3},
\end{aligned}
$$

$$
\begin{aligned}
[3,1] = \;& \tfrac{1}{4}l_1q_2q_0q_{-3} + \tfrac{1}{9}l_0q_2q_0q_{-2} - l_0q_0^3 - \tfrac{1}{2}l_0q_3q_0q_{-3} + \tfrac{17}{18}l_0q_1q_0q_{-1} + \tfrac{1}{36}l_{-1}q_2q_1q_{-2} \\
& -\tfrac{1}{4}l_{-1}q_3q_0q_{-2} - \tfrac{2}{9}l_1q_1^2q_{-3} + \tfrac{1}{36}l_1q_3q_{-2}^2 + \tfrac{1}{9}l_1q_1q_{-1}^2 + \tfrac{1}{18}l_0q_3q_{-1}q_{-2} - \tfrac{1}{9}l_{-1}q_1^2q_{-1} \\
& +\tfrac{1}{36}l_1q_1q_0q_{-2} - \tfrac{1}{6}l_0q_2q_{-1}^2 - \tfrac{1}{3}l_1q_3q_{-1}q_{-3} + \tfrac{2}{9}l_{-1}q_3q_{-1}^2 - \tfrac{1}{36}l_{-1}q_2^2q_{-3} - \tfrac{1}{6}l_0q_1^2q_{-2} \\
& -\tfrac{1}{6}l_1q_0^2q_{-1} + \tfrac{1}{6}l_{-1}q_1q_0^2 - \tfrac{1}{36}l_{-1}q_2q_0q_{-1} - \tfrac{1}{36}l_1q_2q_{-1}q_{-2} + \tfrac{1}{3}l_{-1}q_3q_1q_{-3} + \tfrac{1}{18}l_0q_2q_1q_{-3},
\end{aligned}
$$

$$
\begin{aligned}
[4,0] = \;& -\tfrac{1}{9}q_1^3q_{-3} - \tfrac{3}{5}q_3q_0^2q_{-3} - \tfrac{1}{36}q_2^2q_{-1}q_{-3} + \tfrac{1}{675}q_2^2q_{-2}^2 + \tfrac{1}{100}q_3q_2q_{-2}q_{-3} - \tfrac{1}{9}q_3q_{-1}^3 \\
& -\tfrac{1}{15}q_2q_0^2q_{-2} - \tfrac{5}{108}q_1^2q_{-1}^2 - \tfrac{1}{540}q_2q_1q_{-1}q_{-2} + \tfrac{1}{18}q_1^2q_0q_{-2} + \tfrac{7}{30}q_3q_1q_{-1}q_{-3} \\
& +\tfrac{1}{18}q_2q_0q_{-1}^2 - \tfrac{3}{100}q_3^2q_{-3}^2 - \tfrac{1}{36}q_3q_1q_{-2}^2 + \tfrac{1}{6}q_2q_1q_0q_{-3} + \tfrac{1}{6}q_3q_0q_{-1}q_{-2}.
\end{aligned}
$$

The fourth order Casimir operator of $\mathfrak{so}(5)$ can be recovered from a linear combination of these operators by simply considering $C_4 = [4,0] + [3,1] + [2,2] + [1,3]$.[6] Now, among the operators $[2,0]], C_2, C_4, [1,3], [2,2], [3,1], [4,0]$, at most five are functionally independent. This is easily verified checking a Jacobian. Therefore we can extract two independent linear combinations of the operators $[1,3], [2,2], [3,1], [4,0]$. as labelling operators. However, it can be verified that no such pair of labelling operators commutes, unless one of them coincides with C_4. This follows from the following table, specifying the number of terms in the analytical counterpart of the commutator of the elementary subgroups scalars $[1,3], [2,2], [3,1], [4,0]$:

$[\ ,\ ]$	$[4,0]$	$[3,1]$	$[2,2]$	$[1,3]$
$[4,0]$	–	152	282	130
$[3,1]$		–	370	218
$[2,2]$			–	88

This means that subgroup scalars of higher order have to be considered. Following a reasoning similar to that applied for $n = 3$, it can be verified that (8) does not admit polynomial solutions of order $n = 5$. This further implies that any quadratic labelling operators automatically commutes with the preceding fourth order operators. Observe further that since the latter do not commute, solutions of degree seven must exist. In order six, only five order indecomposable operators $[2,4]$, $[3,3]$ $[4,2]$ $[5,1]$ and $[6,0]$ exist, given in Table 2.

[6] Using the contraction procedure of [16], it follows that the operators of the previous lemma are the components of the decomposition of the Casimir operator C_4 [15].

Table 2. Indecomposable sixth order polynomial solutions to system (8)

$$
\begin{aligned}
[2,4] = {} & (q_{-3}q_3 - \tfrac{1}{4}q_0^2)l_0^4 + \tfrac{1}{2}(q_0(q_1l_{-1} - q_{-1}l_1) + l_1q_2q_{-3} - l_{-1}q_3q_{-2})l_0^3 + l_0^2(l_1l_{-1}q_0^2 - \tfrac{1}{8}(l_{-1}^2q_2 + l_1^2q_{-2})q_0 - \tfrac{1}{4}l_1l_{-1}(-2q_1q_{-1} + q_{-2}q_2 - 2q_{-3}q_3) \\
& + \tfrac{1}{4}\left((5q_3q_{-1} - q_1^2)l_{-1}^2 + (+5q_1q_{-3} - q_{-1}^2)l_1^2)\right) + (\tfrac{3}{16}q_{-1}q_{-3} - \tfrac{1}{64}q_{-2}^2)l_1^4 + \tfrac{1}{32}l_{-1}^2l_1^2(26q_1q_{-1} - q_{-2}q_2 - 50q_0^2 + 2q_{-3}q_3) + (\tfrac{3}{16}q_3q_1 - \tfrac{1}{64}q_2^2)l_{-1}^4 \\
& + \tfrac{1}{8}l_1^3((9q_0q_{-3} - 1q_{-1}q_{-2})l_0 + \tfrac{1}{16}(6q_{-1}^2 + 2q_1q_{-3} - 5q_0q_{-2})l_{-1}) + \tfrac{1}{16}l_{-1}^3((6q_1^2 + 2q_3q_{-1} - 5q_2q_0)l_1 + \tfrac{1}{8}(q_2q_1 - 9q_3q_0)l_0) \\
& + \tfrac{1}{8}l_{-1}^2l_1l_0(4q_2q_{-1} - q_3q_{-2} - 11q_1q_0) + \tfrac{1}{8}l_0l_1^2l_{-1}(+q_2q_{-3} - 4q_1q_{-2} + 11q_0q_{-1}).
\end{aligned}
$$

$$
\begin{aligned}
[3,3] = {} & (q_2q_0q_{-2} - \tfrac{4}{3}\left(q_1^2q_{-2} - q_2q_{-1}^2\right) + 8q_1q_0q_{-1} - 9q_0^3)l_0^3 + \tfrac{1}{12}(6q_0q_{-1}q_{-2} + 12q_1q_{-1}q_{-3} - 27q_0^2q_{-3} - 4q_{-1}^3 - q_1q_{-2}^2)l_1^3 + \tfrac{4}{3}\left(q_1q_{-1}^2 - q_2q_{-1}q_{-2}\right)l_1l_0^2 \\
& + \tfrac{1}{12}(q_2^2q_{-1} - 12q_3q_1q_{-1} + 4q_1^3 + 27q_3q_0^2 - 6q_2q_1q_0)l_{-1}^3 + (3\left(q_2q_0q_{-3} - q_0^2q_{-1}\right) + q_1q_0q_{-2} - 4q_1^2q_{-3})l_1l_0^2 + 12l_0l_1^2\left(q_2q_{-1}q_{-3} - q_0q_{-1}^2\right) \\
& + \tfrac{1}{3}l_{-1}(12q_3q_{-1}^2 + 9\left(q_1q_0^2 - q_3q_0q_{-2}\right) + q_2q_1q_{-2} - 3q_2q_0q_{-1} - 4q_1^2q_{-1})l_0^2 + \tfrac{1}{12}l_0(9q_0^2q_{-2} - 36q_1q_0q_{-3} + 4q_1q_{-1}q_{-2} - q_2q_{-2}^2)l_1^2 \\
& + \tfrac{1}{2}l_0l_1l_{-1}(10q_1q_0q_{-1} - 9q_0^3 - 2q_2q_{-1}^2 + 2q_2q_1q_{-3} + 2q_3q_{-1}q_{-2} - 2q_1^2q_{-2} - 18q_3q_0q_{-3} + q_2q_0q_{-2}) + \tfrac{1}{3}\left(q_2q_1q_{-1} + q_3q_1q_{-2} - q_1^2q_0\right)l_{-1}^2l_0 \\
& + \tfrac{1}{12}(9q_2q_0^2 - 36q_3q_0q_{-1} - q_2^2q_{-2})l_{-1}^2l_0 - \tfrac{1}{12}l_1l_{-1}(((8q_1^2q_{-1} - 9q_1q_0^2 - 36q_3q_1q_{-3} + 18q_3q_0q_{-2} - 2q_2q_1q_{-2} - 12q_3q_{-1}^2 + 3q_2^2q_{-3})l_{-1}) \\
& + ((36q_3q_{-1}q_{-3} + 9q_0^2q_{-1} + 2q_2q_{-1}q_{-2} - 3q_3q_{-2}^2 - 8q_1q_{-1}^2 - 18q_2q_0q_{-3} + 12q_1^2q_{-3})l_1))
\end{aligned}
$$

$$
\begin{aligned}
[4,2] = {} & \tfrac{1}{2}(27l_1l_{-1} - 135l_0^2)q_0^4 + (27\,(l_0l_{-1}q_1 - l_0l_1q_{-1}) - \tfrac{9}{2}\left(l_1^2q_{-2} + l_{-1}^2q_2\right))q_0^3 + q_0^2(l_0(\tfrac{3}{2}(l_1q_1q_{-2} - l_{-1}q_2q_{-1}) + \tfrac{27}{2}(l_{-1}q_3q_{-2} - l_1q_2q_{-3})) \\
& + (27q_3q_{-3} + 81q_1q_{-1} + 9q_2q_{-2})l_0^2 + \tfrac{27}{2}l_{-1}(5q_3q_{-3} - q_1q_{-1})l_1 + \tfrac{1}{2}(3q_1^2 + 9q_3q_{-1})l_{-1}^2 + \tfrac{1}{2}(3q_{-1}^2 + 9q_1q_{-3})l_1^2) + 16l_0^2\left(q_1^3q_{-3} + q_3q_{-1}^3\right) \\
& - \tfrac{1}{6}l_0^2(108(q_3q_0q_{-1}q_{-2} + q_2q_1q_0q_{-3}) - 6(q_3q_1q_{-2}^2 + q_2^2q_{-1}q_{-3}) + 84(q_1^2q_0q_{-2} + q_2q_0q_{-1}^2) - 20q_2q_1q_{-1}q_{-2} + q_2^2q_{-2}^2 + 64q_1^2q_{-1}^2) + 9l_1^2q_3q_{-3}q_{-1}^2 \\
& - \tfrac{1}{2}l_1^2q_2q_{-2}q_{-1}^2 - l_1^2\left((q_{-1}q_{-3} + \tfrac{3}{4}q_{-2}^2)q_1^2 + (4q_1q_{-1}q_{-2} - 3q_2q_{-1}q_{-3} - 9q_3q_{-2}q_{-3} + \tfrac{1}{2}q_2q_{-2}^2)q_0 - q_1q_{-1}^3 + 9q_3q_1q_{-3}^2 + q_2q_1q_{-2}q_{-3} - \tfrac{3}{4}q_2^2q_{-3}^2\right) \\
& + l_{-1}^2((-9q_3q_2q_{-3} - 3q_3q_1q_{-2} + \tfrac{1}{2}q_2^2q_{-2} + 4q_2q_1q_{-1})q_0 + (9q_3q_{-3} - \tfrac{1}{2}q_2q_{-2})q_1^2 - (q_3q_1 + \tfrac{3}{4}q_2^2)q_{-1}^2 + 9q_3^2q_{-1}q_{-3} - q_1^3q_{-1} + q_3q_2q_{-1}q_{-2} - \tfrac{3}{4}q_3^2q_{-2}^2) \\
& + (-6l_0l_1q_2 + 4l_1l_{-1}q_3)q_{-1}^3 + (6l_0l_{-1}q_{-2} + 4l_1l_{-1}q_{-3})q_1^3 + q_1^2(l_0((-28q_{-1}l_{-1} + 12l_1q_{-3})q_0 - (6l_{-1}q_2q_{-3} + 8l_1q_{-1}q_{-2})) + \tfrac{1}{3}l_{-1}l_1(3q_0q_{-2} + 4q_{-1}^2)) \\
& + q_{-1}^2((-12l_0l_{-1}q_3 + l_1l_{-1}q_2 + 28l_0l_1q_1)q_0 + (8l_{-1}q_2q_1 + 6q_{-2}q_3l_1)l_0) + l_1l_{-1}(\tfrac{5}{2}q_3q_1q_{-2}^2 - 9q_3q_0q_{-1}q_{-2} + \tfrac{1}{3}q_2q_1q_{-1}q_{-2} + \tfrac{5}{2}q_2^2q_{-1}q_{-3} - \tfrac{1}{6}q_2^2q_{-2}^2 \\
& - 36q_3q_1q_{-1}q_{-3} - 9q_2q_1q_0q_{-3}) + l_0l_1((6l_{-1}q_2^2q_{-3} + 5l_1q_2q_{-1}q_{-2} - 6l_1q_3q_{-2}^2 - 5l_{-1}q_2q_1q_{-2})q_0 + (6q_3q_1q_{-2}q_{-3} + \tfrac{1}{2}\left(q_2q_1q_{-2}^2 - q_2^2q_{-2}q_{-3}\right)) \\
& + 2l_0l_1q_2q_1q_{-1}q_{-3} + l_0l_{-1}(\tfrac{1}{2}\left(q_3q_2q_{-2}^2 - q_2^2q_{-1}q_{-2}\right) - 6q_3q_2q_{-1}q_{-3} - 2q_3q_1q_{-1}q_{-2}).
\end{aligned}
$$

$$
\begin{aligned}
[5,1] = {} & -8l_0q_0^5 - \tfrac{4}{3}(l_1q_{-1} - l_{-1}q_1)q_0^4 + \tfrac{2}{9}q_0^3((2q_2q_{-2} - 9q_3q_{-3} + 58q_1q_{-1})l_0 + (6q_2q_{-3} + q_1q_{-2})l_1 - (q_2q_{-1} + 6q_3q_{-2})l_{-1}) + q_2^2q_{-3}l_1(\tfrac{1}{12}q_3q_{-3} - \tfrac{1}{81}q_2q_{-2}) \\
& + \tfrac{43}{81}(l_{-1}q_1^3q_{-1}^2 - l_1q_1^2q_{-1}^3) - \tfrac{1}{108}(l_0q_3^2q_{-2}^3 + l_0q_2^3q_{-3}^2) + q_0^2(\tfrac{5}{27}\left(l_{-1}q_2q_1q_{-2} - l_1q_2q_{-1}q_{-2}\right) + \tfrac{1}{3}\left(l_1q_3q_{-2}^2 - l_{-1}q_2^2q_{-3} + l_0q_2q_1q_{-3} + l_0q_3q_{-1}q_{-2}\right)) \\
& + \left(\tfrac{11}{9}\left(l_{-1}q_3q_{-1}^2 - l_1q_1^2q_{-3}\right) + \tfrac{5}{3}\left(l_{-1}q_3q_1q_{-3} - l_1q_3q_{-1}q_{-3}\right) + \tfrac{16}{9}\left(l_1q_1q_{-1}^2 - l_{-1}q_1^2q_{-1}\right) - \tfrac{38}{27}\left(l_0q_2q_{-1}^2 + l_0q_1^2q_{-2}\right)\right)q_0^2 + \tfrac{1}{36}(l_1q_1^3q_{-2}^2 - l_{-1}q_2^2q_{-1}^3) \\
& - \tfrac{1}{18}(l_1q_3q_2q_{-1}q_{-2}q_{-3} - l_{-1}q_3q_2q_1q_{-2}q_{-3}) + \tfrac{4}{27}(l_1q_3q_{-1}^4 - l_{-1}q_1^4q_{-3}) + l_0q_0(\tfrac{4}{27}\left(q_1^3q_{-3} + q_3q_{-1}^3\right) + \tfrac{1}{18}\left(q_3q_1q_{-2}^2 + q_2^2q_{-1}q_{-3}\right) + q_3^2q_{-3}^2(l_1q_{-1} - l_{-1}q_1) \\
& + \tfrac{10}{81}q_1q_{-1}(l_{-1}q_2^2q_{-3} - l_1q_3q_{-2}^2) + \tfrac{1}{18}l_0q_0(20q_1q_{-1} - 11q_2q_{-2} + 18q_3q_{-3})q_3q_{-3} + \tfrac{4}{81}q_2^2q_{-2}^2l_0q_0 - \left(\tfrac{403}{81}q_1^2q_{-1}^2 + \tfrac{5}{54}q_2q_1q_{-1}q_{-2}\right)l_0q_0 + \tfrac{68}{81}q_1^3q_{-3}q_{-1}l_1 \\
& + (\tfrac{8}{9}q_1q_{-1} - \tfrac{1}{12}q_{-2}q_2 + \tfrac{8}{27}q_3q_{-3})(q_{-1}^2q_2 + q_1^2q_{-2})l_0 + (\tfrac{7}{162}q_{-2}q_2 - \tfrac{32}{81}q_1q_{-1})(q_3q_{-1}q_{-2} + q_1q_{-3}q_2)l_0 + \tfrac{19}{162}q_2q_{-2}(l_{-1}q_3q_{-1}^2 - l_1q_{-3}q_1^2) \\
& + q_0(\tfrac{41}{162}q_1q_{-1} + \tfrac{7}{18}q_3q_{-3} + \tfrac{2}{81}q_{-2}q_2)(q_2l_{-1}q_{-1} - l_1q_{-2}q_1) + q_0(\tfrac{4}{27}q_{-2}q_2 - \tfrac{47}{54}q_1q_{-1} - \tfrac{5}{6}q_3q_{-3})(q_{-3}q_2l_1 - l_{-1}q_3q_{-2}) - \tfrac{2}{27}l_0l_1q_{-1}^2(q_{-1}q_2 + 4q_{-2}q_3) \\
& + \tfrac{2}{27}q_0q_1^2l_{-1}(q_1q_{-2} + 4q_{-3}q_2) - (\tfrac{38}{27}q_{-3}q_3q_{-1}q_1 + \tfrac{17}{81}q_1q_{-1}q_{-2}q_2 - \tfrac{1}{81}q_2^2q_{-2}^2)(q_1l_{-1} - l_1q_{-1}) + l_1q_1q_{-3}^2(\tfrac{8}{9}q_3q_1 - \tfrac{1}{108}q_2^2) + l_1q_{-3}q_{-1}(\tfrac{68}{81}q_1^3 - \tfrac{5}{108}q_2^2q_{-1}) \\
& - \tfrac{5}{108}l_1q_{-3}q_{-1}^2q_2^2 + (\tfrac{1}{81}q_2q_{-2} - \tfrac{1}{12}q_3q_{-3})q_3q_{-2}^2l_1 + l_{-1}q_3^2q_{-1}(\tfrac{1}{108}q_{-2}^2 - \tfrac{8}{9}q_{-1}q_{-3}) + l_{-1}q_1q_3(\tfrac{5}{108}q_1q_{-2}^2 - \tfrac{68}{81}q_{-1}^3)
\end{aligned}
$$

$$
\begin{aligned}
[6,0] = {} & -729q_0^6 - 54q_1^4q_{-2}^2 + 54q_3q_{-3}\left(9q_2q_0^2q_{-2} + 162q_1q_0^2q_{-1} - 32q_1^2q_{-1}^2 + 6q_2q_1q_{-1}q_{-2}\right) + 6q_2q_{-2}\left(6q_3q_{-1}^3 - 10q_1^2q_{-1}^2 + 6q_{-3}q_1^3 - 63q_1q_0^2q_{-1}\right) \\
& - 162q_0^2\left(q_{-2}^2q_3q_1 + q_2^2q_{-3}q_{-1}\right) + 54\left(q_0^2\left(27q_3^2q_{-3}^2 - 8q_{-3}q_1^3 - 8q_3q_{-1}^3 - 13q_1^2q_{-1}^2\right) - q_3^2\left(-q_0q_{-2}^3 + q_{-1}^2q_{-2}^2\right)\right) - 64q_1^3q_{-1}^3 + q_2^3q_0q_{-3}^2 + q_2^3q_{-2}^3 \\
& - 54\left(q_1^2q_{-3}^2 + q_{-1}^4\right)q_2^2 - 3q_2^2q_{-2}^2\left(4q_1q_{-1} + 9q_0^2\right) - 324q_0^3\left(q_1^2q_{-2} + q_2q_{-1}^2\right) - 18q_{-2}q_2\left(q_{-2}^2q_3q_1 + q_2^2q_{-3}q_{-1}\right) - 756q_0q_1q_{-1}\left(q_3q_{-1}q_{-2} + q_2q_1q_{-3}\right) \\
& + 972\left(q_0^3\left(q_3q_{-1}q_{-2} + q_2q_1q_{-3}\right) - \left(q_3^2q_{-1}q_{-2}q_{-3} + \left(q_2q_1q_{-3}^2 + q_1^2q_{-2}q_{-3}\right)q_3\right)q_0\right) + 243\left(6q_1q_{-1} - 30q_3q_{-3} + q_2q_{-2}\right)q_0^4 + 288q_{-1}q_1\left(q_{-3}q_1^3 + q_3q_{-1}^3\right) \\
& + 864q_{-3}^2q_3q_1^3 + 972q_2q_{-1}^2q_{-3}q_3q_0 + 90q_{-2}q_2\left(q_1^2q_{-2} + q_2q_{-1}^2\right)q_0 + 396q_{-1}q_0q_1\left(q_1^2q_{-2} + q_2q_{-1}^2\right) + 180q_1q_{-1}\left(q_{-2}^2q_3q_1 + q_2^2q_{-3}q_{-1}\right) + 864q_{-3}q_3^2q_{-1}^3
\end{aligned}
$$

There is a first interesting fact concerning these labelling operators. Although their symmetrized expressions belong to an integrity basis for this MLP [14], they cannot be deduced neither from the reduction chain $\mathfrak{so}(5) \supset \mathfrak{so}(3)$ itself nor from the associated inhomogeneous contraction. This is due essentially to the fact that the Lie $\mathfrak{so}(5)$ has no primitive Casimir operator of degree six. According to [15], these labelling operators are purely formal, and do not inherit an obvious physical meaning. A routine computation shows that these elementary scalars do not commute mutually, as shown in the following table:[7]

$[\ ,\ ]$	$[2,4]$	$[3,3]$	$[4,2]$	$[5,1]$	$[6,0]$
$[2,4]$	$-$	1070	2112	2980	2112
$[3,3]$		$-$	2862	3562	2490
$[4,2]$			$-$	3840	2602
$[5,1]$				$-$	2154

Clearly all these operators commute with $[0,2]$ and the quadratic Casimir operator C_2 of $\mathfrak{so}(5)$, thus with $[2,0]$. The next case to be analyzed corresponds to the commutator of an operator of degree four with another of degree six. Here, in accordance with [11], we find the first nontrivial pairs of commuting labelling operators. To this extent, we analyze the commutator of an arbitrary linear combination of the scalars $[4,0]$, $[3,1]$, $[2,2]$ and $[1,3]$ with an operator of degree six formed by $[2,4]$, $[3,3]$, $[4,2]$, $[5,1]$ and $[6,0]$ and products of lower order operators.[8] Proceeding in this way, we find the two following pairs of linearly independent operators $\left\{X_1^1, X_2^1\right\}$ and $\left\{X_1^2, X_2^2\right\}$, where

$$\begin{array}{rl} X_1^1 = & [4,0] + [3,1] + (4-3\alpha)\,[2,2] + \alpha\,[1,3]\,, \quad \alpha \neq 1, \\ X_2^1 = & -\frac{27}{5}\,[6,0] - 162\,[5,1] + [4,2] - 216\,[3,3] - [2,0]\left(5310\,[2,2] + \frac{2025}{2}\,[4,0]\right) \\ & + [0,2]\,(2124\,[3,1] + 528\,[1,3]) + 768\,[2,4] \end{array} \tag{11}$$

and

$$\begin{array}{rl} X_1^2 = & \left(\frac{4}{3} - \beta\right)[4,0] + \frac{3}{2}\,(1-\beta)\,[3,1] + 3\beta\,[2,2] + [1,3]\,, \quad \beta \neq \frac{1}{3} \\ X_2^2 = & -\frac{12}{5}\,[6,0] + 108\,[5,1] + [4,2] + 324\,[3,3] - [2,0]\,(180\,[2,2] - 1035\,[3,1]) \\ & -[0,2]\,(17172\,[2,2] + 1728\,[2,4] + 3998\,[1,3])\,. \end{array} \tag{12}$$

The corresponding symmetrized operators $O_i^j = Sym(X_i^j)$ in the enveloping algebra of $\mathfrak{so}(5)$ satisfy the requirements

$$\begin{array}{ll} \left[O_1^1, O_2^1\right] = 0, & \left[O_1^1, O_2^2\right] \neq 0, \\ \left[O_1^2, O_2^1\right] \neq 0, & \left[O_1^2, O_2^2\right] = 0. \end{array} \tag{13}$$

Observe that for the excluded values of the parameters α and β, the fourth order labelling operator is reduced to the Casimir operator C_4 of $\mathfrak{so}(5)$. Equation (13) further confirms that the sets $\mathcal{F}_{1,\alpha} = \left\{O_1^1, O_2^1\right\}$ and $\mathcal{F}_{2,\beta} = \left\{O_1^2, O_2^2\right\}$ are inequivalent sets, since no element of $\mathcal{F}_{1,\alpha}$ commutes with an element of $\mathcal{F}_{2,\beta}$.

We claim that the two preceding pairs of labelling operators can be taken as a possible choice for the four available operators that solve this MLP. This is equivalent to show that the seven operators $[0,2]\,, [2,0]\,, C_4, X_1^1, X_1^2, X_2^1, X_2^2$ are functionally independent. To prove this assertion, it suffices to find a set of seven independent variables such that the Jacobian

[7] As before, the table specifies the number of terms for the analytical counterpart of the commutator.
[8] Observe that for the four dimensional operator, the products and powers of quadratic labelling operators need not to be considered, since they commute with the sixth order solutions.

of these operators with respect to these variables does not vanish. We take the variables $\{l_0, l_{-1}, q_0, q_{-1}, q_{-2}, q_{-3}, q_3\}$ and consider

$$\frac{\partial\left([0,2],[2,0],C_4,X_1^1,X_1^2,X_2^1,X_2^2\right)}{\partial\left(l_0,l_{-1},q_0,q_{-1},q_{-2},q_{-3},q_3\right)}\neq 0. \tag{14}$$

Since the operators are independent, they can be taken as a fundamental set of solutions to system (8). We remark that this can also be proved using an indirect argument based on equation (13).

It is clear that any linear combination $X_1^1+\mu C_4$ or $X_2^1+\mu C_4$ is also a possible fourth order labelling operator that commutes with X_1^2 and X_2^2, respectively. In view of these possibilities, it is natural to look for commuting pairs of operators with the lowest possible number of components. Since the found operators of degree six cannot be modified, this leads to look for operators of degree four having four or less components. Having in mind that $\alpha\neq 1$ and $\beta\neq\frac{1}{3}$, it can be shown that, up to scalars, only seven non-equivalent operators with less of four components exist:

(i) $X_1^1(0)=[4,0]+[3,1]+4\,[2,2]$,
(ii) $X_1^1(\frac{4}{3})=[4,0]+[3,1]+\frac{4}{3}\,[1,3]$,
(iii) $X_1^1-C_4=3\,[2,2]-1\,[1,3]$,
(iv) $X_2^1(\frac{4}{3})=-\frac{1}{2}\,[3,1]+4\,[2,2]+[1,3]$,
(v) $X_2^1(1)=[4,0]+9\,[2,2]+[1,3]$,
(vi) $X_2^1(0)=\frac{4}{3}\,[4,0]+\frac{3}{2}\,[3,1]+[1,3]$,
(vii) $X_2^1-C_4=[4,0]+\frac{3}{2}\,[3,1]-3\,[2,2]$.

The two component solution found in [11] is equivalent to the symmetrized operators obtained from $\left\{X_1^1-C_4,X_1^2\right\}$. The discrepancy in the coefficients of the scalars $[2,2]$ and $[1,3]$ is due to a different normalization factor to that used in [11]. Moreover, it follows from the previous list that this is the only solution with two terms. In this sense, the pair proposed in [11] is actually the simplest possible choice for solving the missing label problem. The pair $\left\{X_2^1,X_2^2\right\}$ constitutes a new solution and has no analogue in the previous analysis. The non-equivalence of these sets of labelling operators refers to their independence and to the fact they do not mutually commute. Thus the class of labelling operators is divided into two incompatible sets with respect to the commutativity requirement. Starting from either the operators of $\mathcal{F}_{1,\alpha}$ or $\mathcal{F}_{2,\beta}$, labelling operators of higher even order may be constructed. In particular, for arbitrary constants a,b, the pairs $\left\{(a\,[2,0]+b\,[0,2])\,X_1^1,X_1^2\right\}$ and $\left\{(a\,[2,0]+b\,[0,2])\,X_2^1,X_2^2\right\}$ are solutions to the MLP consisting of two operators of degree six, although one of them is decomposable. It was further verified that there do not exist two independent of operators of the form $O=a_1\,[6,0]+a_2\,[5,1]+a_3\,[4,2]+a_4\,[3,3]+a_5\,[2,4]$ that commute.

Looking for higher order solutions to system (8), we found that it has only three solutions of degree seven, which correspond to operators of bi-degree $[5,2]$, $[4,3]$ and $[3,4]$ respectively.[9] These operators can be shown to appear in the commutator of fourth order elementary scalars:

$$[[4,0],[3,1]]=[5,2]\,,\ [[4,0],[1,3]]=[4,3]\,,\ [[2,2],[3,1]]=[3,4]\,. \tag{15}$$

As a consequence of the non-existence of elementary scalars of degree three and five, it follows that no linear combination of $[5,2]$, $[4,3]$ and $[3,4]$ commutes with a fourth order labelling

[9] Their symmetrized expressions coincide with those found by other authors for the integrity basis. The computation of indecomposable solutions up to order nine also agreed completely with the results of [14].

operator. In agreement with the results of [14], we found also seven indecomposable scalars of degree nine among the 13 solutions of (8) having this degree. However, no lineal combination of the latter commutes with an operator of degree four. Due to the computational complications for these higher order operators, we did not further extended the search of commuting pairs supplementary to those already found. A question left open in this analysis concerns the existence of a pair of commuting operators consisting of an operator of degree four and an operator of odd degree.

Final remarks

We have obtained one possible choice for the four functionally independent label operators available among the 28 subgroups scalars computed in [14] for the reduction chain $\mathfrak{so}(5) \supset \mathfrak{so}(3)$. Two inequivalent sets of solutions have been found, each consisting of operators of degree four and six, respectively. While the fourth operator can be interpreted as a linear combination of "broken" Casimir operators with respect to the contraction defined by the embedding of $\mathfrak{so}(3)$ into $\mathfrak{so}(5)$ [15, 16], the operators of order six have no obvious interpretation in terms of the initial data. As observed in [16], it is unlikely that such operators, which have to be determined by pure formal means, have really a significant physical meaning other than that conferred by the labelling of basis states. As shown in [17, 18], for degenerate representations the number of missing label operators needed is lower than that given by the generic formula (2). This actually happens for the applications of this chain to the description of quadrupole vibrations of the nuclear surface about a spherical equilibrium shape and their coupling to giant dipole resonances [19, 20], where the totally symmetric representations are considered. In these situations, linear combinations of the operators X_1^1, X_1^2 constitute the natural choice, and allow to determine the most general form of a fourth order labelling operator. The numerical part, left untouched here, concerns the determination of the complete eigenvalue spectrum for these missing label operators.

Finally, one striking point about the missing label problem $\mathfrak{so}(5) \supset \mathfrak{so}(3)$ is its difficulty when compared to similar problems with more than one labelling operator, like the Wigner's scheme $\mathfrak{su}(4) \supset \mathfrak{su}(2) \times \mathfrak{su}(2)$ [21]. Although the decomposition of the fourth order Casimir operator provides a sufficient number of labelling operators, the commutativity requirement is not fulfilled, regardless of the linear combinations considered. This forces to determine higher order operators with no apparent link to the information provided by the embedding. One may wonder whether this difficulty is related to the fact that the subalgebra $\mathfrak{so}(5)$ is principal, and therefore possesses special characteristics not shared by other types of subalgebras. Another obstruction to find suitable pairs of commuting operators seems to be the non-existence of solutions of orders three and five, and the fact that the seventh order labelling operators arise as commutators of operators of order four.

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