UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

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(Un enfoque geométrico a los sistemas de Lie: formalismo de las deformaciones de álgebras de Poisson–Hopf)

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Eduardo Fernández Saiz

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PH. D. THESIS

A geometric approach to Lie systems: formalism of Poisson–Hopf algebra deformations

(Un enfoque geométrico a los sistemas de Lie: formalismo de las deformaciones de álgebras de Poisson–Hopf)

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By homely gifts and hindered Words
The human heart is told
Of Nothing -
"Nothing" is the force
That renovates the World -

Emily Dickinson

Abstract

The notion of quantum algebras is merged with that of Lie systems in order to establish a new formalism called Poisson–Hopf algebra deformations of Lie systems. The procedure can be naturally applied to Lie systems endowed with a symplectic structure, the so-called Lie–Hamilton systems. This is quite a general approach, as it can be applied to any quantum deformation and any underlying manifold. One of its main features is that, under quantum deformations, Lie systems are extended to generalized systems described by involutive distributions. As a consequence, a quantum deformed Lie system no longer has an underlying Vessiot–Guldberg Lie algebra or a quantum algebra one, but keeps a Poisson–Hopf algebra structure that enables us to obtain, in an explicit way, the *t*-independent constants of the motion from quantum deformed Casimir invariants, which are potentially useful in a further construction of the generalized notion of superposition rules. We illustrate this approach by considering the non-standard quantum deformation of $\mathfrak{sl}(2)$ applied to well-known Lie systems, such as the oscillator problem or Milne–Pinney equation, as well as several types of Riccati equations. In this way, we obtain their new generalized (deformed) counterparts that cover, in particular, a new oscillator system with a time-dependent frequency and a position-dependent mass. Based on a recently developed procedure to construct Poisson–Hopf deformations of Lie–Hamilton systems [13], a novel unified approach to nonequivalent deformations of Lie–Hamilton systems on the real plane with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2)$ is proposed. This, in particular, allows us to define a notion of Poisson–Hopf systems in dependence of a parameterized family of Poisson algebra representations [14]. Such an approach is explicitly illustrated by applying it to the three non-diffeomorphic classes of $\mathfrak{sl}(2)$ Lie–Hamilton systems. Furthermore *t*-independent constants of motion are given as well. Our methods can be employed to generate other Lie–Hamilton systems and their deformations for other Vessiot–Guldberg Lie algebras and their deformations. In addition, we study the deformed systems obtained from Lie–Hamilton systems associated to the oscillator algebra $\mathfrak{h}_4$, seen as a subalgebra of the 2-photon algebra $\mathfrak{h}_6$. As a particular application, we propose an epidemiological model of SISf type that uses the solvable Lie algebra $\mathfrak{b}_2$ as subalgebra of $\mathfrak{sl}(2)$, by restriction of the corresponding quantum deformed systems.

Keywords: Lie system, Poisson–Hopf algebra, Poisson coalgebra, quantum deformation.

Resumen

La noción de álgebras cuánticas se fusiona con la de sistemas de Lie para establecer un nuevo formalismo, las deformaciones del álgebra de Poisson–Hopf de los sistemas de Lie. El procedimiento puede aplicarse a sistemas de Lie dotados de una estructura simpléctica, los denominados sistemas de Lie–Hamilton. Este es un enfoque bastante general, ya que se puede aplicar a cualquier deformación cuántica y a cualquier variedad subyacente. Una de sus principales características es que, bajo deformaciones cuánticas, los sistemas de Lie se extienden a distribuciones involutivas generalizadas. Como consecuencia, un sistema de Lie deformado cuánticamente ya no tiene un álgebra de Vessiot–Guldberg Lie subyacente o un álgebra cuántica, sino que mantiene una estructura de álgebra de Poisson–Hopf que permite obtener, de manera explícita, las constantes del movimiento t-independientes a partir de los invariantes de Casimir deformados, que son potencialmente útiles en una construcción adicional de la noción generalizada de reglas de superposición. Ilustramos este enfoque considerando la deformación cuántica no estándar de $\mathfrak{sl}(2)$ aplicada a sistemas de Lie conocidos, como el problema del oscilador o la ecuación de Milne–Pinney, así como varios tipos de ecuaciones de Riccati. De esta manera, se obtienen sus análogos generalizados (deformados) que dan lugar, en particular, a un nuevo sistema de tipo oscilatorio con una frecuencia dependiente del tiempo y una masa dependiente de la posición. Basándonos en este procedimiento, desarrollado recientemente en [13], se presentan de modo unificado las deformaciones no equivalentes de los sistemas de Lie–Hamilton en el plano real con un álgebra de Lie de Vessiot–Guldberg isomorfa a $\mathfrak{sl}(2)$. Esto, en particular, nos permite definir una noción de sistemas de Poisson–Hopf en dependencia de una familia parametrizada de representaciones de álgebras de Poisson [14]. Este enfoque se ilustra explícitamente aplicándolo a las tres clases no difeomórficas de los sistemas $\mathfrak{sl}(2)$ Lie–Hamilton. Se dan las constantes de movimiento independientes de t. Nuestros métodos se pueden emplear para generar otros sistemas de Lie–Hamilton y sus deformaciones para otras álgebras de Lie de Vessiot–Guldberg y sus deformaciones. Análogamente, se estudian los sistemas deformados a partir de los sistemas de Lie–Hamilton basados en el álgebra del oscilador $\mathfrak{h}_4$, vista como subálgebra del álgebra de Lie $\mathfrak{h}_6$, la llamada 2-photon algebra. Como aplicación adicional, se estudian modelos epidemiológicos del tipi SISf obtenidos como deformación cuántica de sistemas de Lie–Hamilton basados en el álgebra resoluble en $\mathfrak{b}_2$, pero vista como subálgebra de $\mathfrak{sl}(2)$.

Palabras clave: sistema de Lie, álgebra de Poisson–Hopf, coálgebra de Poisson, deformación cuántica.

Breve extracto de la tesis

Introducción y objetivos

Desde su formulación original por Lie [100], los sistemas no autónomos de primer orden de ecuaciones diferenciales ordinarias que admiten una regla de superposición no lineal, los llamados sistemas de Lie, se han estudiado extensamente (ver [38; 43; 47; 154] y sus referencias). El teorema de Lie [100] establece que todo sistema de ecuaciones diferenciales de primer orden es un sistema de Lie si, y solo si, puede describirse como una curva en un álgebra de Lie de dimensión finita de campos vectoriales, la denominada álgebra de Lie de Vessiot–Guldberg. Aunque ser un sistema de Lie es más una excepción que una regla [41], se ha demostrado que los sistemas de Lie son de gran interés dentro de las aplicaciones físicas y matemáticas (ver [47] y sus referencias). Los sistemas de Lie que admiten un álgebra de Lie de Vessiot–Guldberg de campos vectoriales hamiltonianos, en relación con una estructura de Poisson, los sistemas de Lie–Hamilton, han encontrado incluso más aplicaciones que los sistemas de Lie estándar sin esta estructura geométrica asociada [9; 15; 49]. Los sistemas de Lie–Hamilton admiten un álgebra de Lie de dimensión finita adicional de funciones hamiltonianas, un álgebra de Lie–Hamilton, que permite la determinación algebraica de reglas de superposición y constantes del movimiento [29].

La mayoría de los enfoques de los sistemas de Lie se basan en la teoría de las álgebras de Lie y los grupos de Lie [119]. Sin embargo, el éxito de los grupos cuánticos [51; 105] y el formalismo de coálgebra dentro del análisis de sistemas superintegrables [8; 26], y el hecho de que las álgebras cuánticas aparezcan como deformaciones de las álgebras de Lie sugirieron la posibilidad de ampliar la noción y técnicas de los sistemas de Lie–Hamilton más allá de la teoría de Lie. En esta memoria se propone un enfoque en esta dirección (capítulo 3, [13]), donde se da un método para construir sistemas de Lie–Hamilton deformados cuánticamente (sistemas LH en resumen) mediante el formalismo de coálgebra y álgebras cuánticas. La idea subyacente es utilizar la teoría de los grupos cuánticos para deformar sistemas de Lie y sus estructuras geométricas asociadas. Más exáctamente, la deformación transforma un sistema LH con su álgebra de Lie de Vessiot–Guldberg en un sistema hamiltoniano cuya dinámica está determinada por un conjunto (finito) de generadores de una distribución generalizada de Stefan–Sussmann, sección 3.1.3. Mientras tanto, el álgebra inicial de Lie–Hamilton (álgebra LH en resumen) se identifica con un álgebra de Poisson–Hopf. Las estructuras deformadas permiten la construcción explícita de constantes de movimiento t-independientes mediante técnicas de álgebras cuánticas para el sistema deformado.

Este trabajo tiene como objetivo ilustrar el enfoque introducido en [13; 14] para construir deformaciones de los sistemas LH. Esto abarca, fundamentalmente, los siguientes objetivos: desarrollar el formalismo basado en deformaciones de estructuras de Poisson–Hopf, ya que estas estructuras permiten una sistematización adicional que abarca los sistemas LH no equivalentes entre sí, correspondientes a las álgebras isomórficas de LH, y ofrecer un procedimiento general para la obtención de nuevos sistemas LH.

Específicamente, se aporta un algoritmo para la consecución de deformaciones sistemas LH, lo cual pone de manifiesto la importancia del formalismo previamente desarrollado. Además, mostramos que las deformaciones de Poisson–Hopf de los sistemas LH basadas en un álgebra LH isomorfas a $\mathfrak{sl}(2)$ (teorema 4.1), se pueden describir genéricamente, facilitando así las funciones hamiltonianas deformadas y los campos vectoriales hamiltonianos deformados asociados; una vez estudiado el sistema no deformado.

Igualmente, se proporciona un nuevo método para construir sistemas LH con un álgebra LH isomorfa a un álgebra $\mathfrak{g}$ de Lie fija, sección 4.3. Nuestro enfoque se basa en el uso de la foliación simpléctica en $\mathfrak{g}$ inducida por el corchete de Kirillov–Konstant–Souriau, teorema 4.3. Como caso particular, se muestra explícitamente cómo nuestro procedimiento explica la existencia de tres tipos de sistemas LH en el plano relacionado con un álgebra LH isomorfa a $\mathfrak{sl}(2)$. Esto se debe al hecho de que cada uno de los tres tipos diferentes corresponde a uno de los tres tipos de hojas simplécticas en $\mathfrak{sl}(2)$. Análogamente, se puede generar el único tipo de sistemas LH en el plano admitiendo un álgebra de Lie de Vessiot–Guldberg isomorfa a $\mathfrak{so}(3)$.

Nuestra sistematización nos permite dar directamente el sistema deformado de Poisson–Hopf a partir de la clasificación de sistemas LH [9; 29], sugiriendo además una noción de sistemas Poisson–Hopf Lie basados en una familia z-parametrizada de morfismos del álgebra de Poisson (definición 4.2 en la sección 4.2).

Resultados más relevantes y estructura

Presentamos un procedimiento genérico que nos permite introducir la noción de deformación cuántica de sistemas LH, y basados en la noción de distribuciones involutivas en el sentido de Stefan–Sussman. La existencia de principios de superposición no lineal para sistemas no autónomos de primer orden de ecuaciones diferenciales ordinarias constituye una propiedad estructural que surge naturalmente del enfoque desde la teoría de grupos para las ecuaciones diferenciales iniciado por Lie, en el contexto del desarrollo del programa geométrico basado en grupos de transformación, así como de la clasificación analítica de ecuaciones diferenciales desarrollada por Painlevé y Gambier, entre otros, dando lugar a lo que hoy se conoce como la teoría de los sistemas de Lie [100; 149].

Recordamos que, más allá de los sistemas superintegrables [8; 26], las coálgebras se han aplicado recientemente a deformaciones bi-hamiltonianas integrables de sistemas de Lie–Poisson [24] y a deformaciones integrables de los sistemas de Rössler y Lorenz [12]. Este trabajo propone una generalización de los sistemas de Lie en esta línea. La idea es considerar sistemas de Lie deformados cuánticamente que poseen una estructura de álgebra de Poisson–Hopf que reemplaza el álgebra de Lie de Vessiot–Guldberg del sistema inicial, lo que nos permite una construcción de constantes del movimiento t-independientes expresadas en términos de Casimires invariantes deformados.

La estructura de la tesis es la siguiente. Los capítulos 1 y 2 están dedicados a la introducción de los principales aspectos de los sistemas LH y las álgebras de Poisson–Hopf, así como una síntesis de sus propiedades fundamentales. El formalismo general para construir deformaciones del tipo Poisson–Hopf de sistemas LH [13] se analiza con detalle en el capítulo 3. En él se presenta un procedimiento algorítmico esquematizado para determinar tanto las deformaciones como las constantes del movimiento.

En el capítulo 4 se presentan la deformaciones (no estándar) de sistemas basados en el álgebra de Lie simple $\mathfrak{sl}(2)$. Un enfoque unificado de las deformaciones de los sistemas Poisson–Hopf Lie con un álgebra LH isomorfa a $\mathfrak{g}$ se estudia en la sección 4.3.1. Dicho procedimiento se ilustra explícitamente, aplicándolo a las tres clases no difeomorfas de sistemas LH $\mathfrak{sl}(2)$ en el plano, obteniendo así su correspondiente deformación. En el capítulo 5 se estudian deformaciones de ecuaciones diferenciales relevantes. En primer lugar, se consideran las deformaciones de la ecuación de Milne–Pinney, de las cuales se derivan nuevos sistemas de tipo oscilatorio con una masa dependiente de la posición. Como segundo tipo relevante, se estudian las ecuaciones de Riccati, específicamente las ecuaciones compleja y acoplada. En el capítulo 6 se considera el problema de las deformaciones cuánticas de sistemas LH basados en el álgebra del oscilador $\mathfrak{h}_4$, con énfasis especial en su estructura como subálgebra de la llamada 2-photon álgebra $\mathfrak{h}_6$. Esto nos permite deducir deformaciones del oscilador armónico amortiguado (sección 6.3). El álgebra resoluble afín $\mathfrak{b}_2$, vista como subálgebra de $\mathfrak{sl}(2)$, se aplica en el capítulo 7 para proponer un nuevo modelo epidemiológico alternativo. Las técnicas desarrolladas con anterioridad se utilizan para construir y analizar un modelo SISf deformado, del que se obtienen las constantes del movimiento. Finalmente, en las conclusiones resumimos los resultados obtenidos y comentamos vías futuras de trabajo o actualmente en proceso de ejecución.

Conclusiones

En este trabajo hemos propuesto una noción de deformación de sistemas LH basada en las álgebras de Poisson–Hopf. Este enfoque difiere radicalmente de otros métodos empleados en la teoría de sistemas de Lie [15; 41; 47; 49; 154], dado que las deformaciones no corresponden formalmente a sistemas de Lie, sino a una noción extendida que precisa de una estructura de Hopf, de modo que el sistema sin deformar se obtiene por un paso al límite en el cual el parámetro de deformación desaparece. La introducción de una estructura de Poisson–Hopf permite una generalización de estos sistemas, en el sentido de que el álgebra finito-dimensional de Vessiot–Guldberg se reemplaza por una distribución involutiva en el sentido de Stefan–Sussman (capítulo 3).

En el capítulo 4 se estudia el análogo de las deformaciones cuánticas de $\mathfrak{sl}(2)$, estableciendo explícitamente las constantes del movimiento para los sistemas deformados cuánticamente. Los tres sistemas de Lie planos no equivalentes basados en el álgebra de Lie $\mathfrak{sl}(2)$ se describen de modo unificado, lo que proporciona una bella interpretación geométrica de estos sistemas y sus correspondientes deformaciones cuánticas. El capítulo 5 está dedicado al análisis de sistemas específicos de ecuaciones diferenciales y su contrapartida deformada. Consideramos en primer lugar la ecuación de Milne–Pinney, cuyas deformaciones nos proporcionan nuevos sistemas de tipo oscilatorio con la particularidad de que la masa de la partícula es dependiente de la posición, y donde se obtienen explícitamente las constantes del movimiento. Merece la pena observar que la deformación estándar o de Drinfel'd–Jimbo de $\mathfrak{sl}(2)$ no lleva a un oscilador del tipo mencionado, ya que, en este caso, la deformación viene descrita por $\operatorname{sinhc}(zqp)$ en lugar de $\operatorname{sinhc}(zq^2)$. Esto se deduce de la correspondiente realización simpléctica dada en [14]. Este hecho justifica que hayamos escogido la deformación no-estándar de $\mathfrak{sl}(2)$, para obtener aplicaciones físicas. A pesar de ello, la deformación de Drinfel'd–Jimbo podría proporcionar información suplementaria para la ecuación de Milne–Pinney, dando lugar a sistemas no equivalentes a los estudiados en la memoria. En cualquier caso, el método nos sugiere un enfoque alternativo de los sistemas con masa no constante, para los cuales los métodos clásicos son de limitado alcance. Un segundo tipo que se ha estudiado corresponde a las ecuaciones compleja y acoplada de Riccati, que se han analizado exhaustivamente en la literatura. Para ellos se obtienen la versión deformada y las constantes del movimiento. Los resultados principales de los capítulos 3-5 han sido publicados en [13] y [14]. En el capítulo 6 nos centramos en sistemas de tipo oscilatorio obtenidos a partir de los sistemas LH deformados basados en el álgebra de Lie $\mathfrak{h}_4$, vista como subálgebra de la llamada 2-photon algebra $\mathfrak{h}_6$. En particular, estas deformaciones se obtienen como restricción de los correspondientes sistemas deformados para $\mathfrak{h}_6$. Un ejemplo ilustrativo de este tipo de sistemas viene dado por el oscilador amortiguado deformado. Resta determinar un principio de superposición para tales sistemas, un problema actualmente en ejecución. En el capítulo 7 se usa el álgebra resoluble afín $\mathfrak{b}_2$, vista como subálgebra de $\mathfrak{sl}(2)$, para obtener sistemas deformados aplicables en el contexto de los modelos epidemiológicos. Esta idea constituye una novedad, dado que los métodos empleados habitualmente en este contexto son de naturaleza estocástica. Los resultados de este capítulo has sido enviados recientemente para su publicación.

Existe una plétora de posibilidades y aplicaciones que emergen del formalismo de deformaciones de Poisson–Hopf. Aunque los resultados han sido principalmente considerados en el plano, para el cual existe una clasificación explícita de sistemas LH [9; 15], el método es válido para variedades arbitrarias y álgebras de Vessiot–Guldberg de dimensiones mayores. Un estudio sistemático de estos sistemas sin duda dará lugar a nuevas propiedades de los sistemas deformados que merecen ser analizadas con detalle. En particular, las propiedades dinámicas de sistemas específicos de ecuaciones diferenciales pueden estudiarse mediante estas técnicas, donde se espera que nuevas propiedades sean descubiertas.

En relación con la actual pandemia COVID-19, podemos preguntarnos si existe una descripción en términos de los modelos SISf. Este modelo es una primera aproximación para procesos de infección primarios, en los cuales se consideran dos tipos de estados en la población: los infectados y los susceptibles de infección. El modelo no contempla la posibilidad de adquirir inmunidad. Parece que la COVID-19 está sujeta a ciertos tipos de inmunidad, aunque sólo hasta un treinta por ciento de la población. En este sentido, un modelo SIR que considera individuos inmunes no es un modelo apropiado para esta situación. Sería interesante tener un modelo que contemple individuos inmunes y no inmunes simultáneamente. Actualmente estamos buscando un modelo hamiltoniano estocástico que contemple estas variables.

Trabajos futuros

Una de las cuestiones principales a resolver es si el enfoque de Poisson–Hopf proporciona un método efectivo para deducir un análogo deformado de los principios de superposición para sistemas LH deformados. Asimismo, sería interesante saber si una tal descripción puede aplicarse simultáneamente a sistemas no equivalentes, como una extrapolación de la noción de contracción a los sistemas de Lie. Otros aspecto relevante es la posibilidad de obtener una descripción unificada de estos sistemas a partir de ciertos sistemas "canónicos" fijos, lo que implicaría una primera sistematización de los sistemas LH desde una perspectiva más amplia que la de las álgebras de Lie finito-dimensionales. Algunas vías de trabajo futuro pueden resumirse como sigue:

- En la clasificación de los sistemas LH en el plano juega un papel central la llamada 2-photon álgebra $\mathfrak{h}_6$, ya que es el álgebra de máxima dimensión que puede aparecer con las propiedades de un álgebra LH. El estudio de sus deformaciones cuánticas es, por tanto, una cuestión fundamental para completar el análisis de las deformaciones de los sistemas LH en el plano. Cabe señalar que existen esencialmente dos posibilidades diferentes para estas deformaciones, dependiendo de la estructura de dos subálgebras prominentes, el álgebra $\mathfrak{h}_4$ y $\mathfrak{sl}(2)$, que dan lugar a sistemas y deformaciones con diferentes propiedades. El primer caso, basado en el álgebra del oscilador $\mathfrak{h}_4$, ha sido parcialmente considerado en el capítulo 6. Sin embargo, resta aún obtener una regla de superposición efectiva, cuya implementación estamos analizando actualmente. El análisis debe completarse identificando clases particulares de sistemas de ecuaciones diferenciales que puedan deformarse mediante este procedimiento, y que puedan interpretarse como perturbaciones del sistema inicial. El segundo caso, basado en la extensión de los resultados obtenidos para $\mathfrak{sl}(2)$ al álgebra $\mathfrak{h}_6$, es estructuralmente muy distinto debido a la naturaleza de la deformación cuántica. Esperamos que nuevos sistemas con propiedades de interés surgan de este análisis. Desde el punto de vista de las aplicaciones, estos sistemas tienen muchas propiedades interesantes, tales como nuevos sistemas del tipo Lotka–Volterra o sistemas de tipo oscilatorio con masas y frecuencias dependientes de la posición y el tiempo, pero cuya dinámica pueda caracterizarse mediante la existencia de un procedimiento para la determinación exacta de las constantes del movimiento y las reglas de superposición. En análisis completo de los sistemas LH deformados basados en el álgebra $\mathfrak{h}_6$ está actualmente en proceso, para ser enviado próximamente para su publicación.
- Por otro lado, cabe observar que, actualmente, no existe una clasificación de los sistemas LH para las dimensiones $n \geq 3$. Un problema de interés que surge en este contexto es analizar la posibilidad de generar nuevos sistemas LH, tanto clásicos como deformados, mediante la extensión de los sistemas en el plano, en combinación con las proyecciones de las realizaciones de campos vectoriales. En este contexto, se sabe que las proyecciones de realizaciones del álgebra de Lie asociadas con una representación lineal dan lugar a realizaciones no lineales. Analizando la cuestión desde la perspectiva de las álgebras funcionales (hamiltonianas), es concebible la existencia de estructuras simplécticas compatibles que dan lugar a sistemas LH en dimensiones superiores, así como una dependencia de dichas formas simplécticas. Criterios de este tipo pueden combinarse con deformaciones cuánticas de álgebras de Lie conocidas, para obtener nuevas aplicaciones de estas en el contexto de ecuaciones diferenciales.
- Como complemento al modelo SISf basado en $\mathfrak{b}_2$ como subálgebra de $\mathfrak{sl}(2)$, es razonable desarrollar el modelo correspondiente a la misma álgebra, pero vista como una subálgebra de $\mathfrak{h}_4$. Nuevamente, el carácter esencialmente distinto de la deformación nos lleva a sistemas con propiedades muy diferentes, lo que sugiere comparar ambos modelos con detalle, analizando las soluciones numéricas deducidas de ambos enfoques. Un primer paso en esta dirección está actualmente en proceso.
- Desearíamos asimismo extender nuestro estudio a modelos epidemiológicos más complicados, aunque a primera vista no hayamos localizado nuevos sistemas de Lie, al menos en su forma usual. Sospechamos que la descripción hamiltoniana de los modelos compartimentales pueden asociarse a sistemas de Lie, como muestra el ejemplo desarrollado. En particular, debe analizarse con más detalle como las soluciones del modelo deformado (7.47) permiten recuperar las soluciones del modelo sin deformar, cuando el parámetro de deformación tiende a cero. Se precisa un análisis más detallado para determinar si tales modelos integrables modelizan procesos distintos a los infecciosos. En particular, nos interesaría saber si es posible

modelizar la dinámica subatómica mediante hamiltonianos deformados del tipo (7.46). Existe una teoría estocástica de los sistemas de Lie desarrollada en [95] que podría ser otro punto de partida para tratar estos sistemas. En el presente trabajo tuvimos la suerte de encontrar una teoría con fluctuaciones que coincidían con una expansión estocástica, pero esto es más una excepción que una regla. De hecho, parece que la forma más factible de proponer modelos estocásticos es utilizar la teoría estocástica de Lie en lugar de esperar un destello de suerte con las fluctuaciones. Como hemos dicho, encontrar soluciones particulares no es en absoluto trivial. La búsqueda analítica es una tarea muy atroz. Creemos que para ajustar soluciones particulares en el principio de superposición, es posible que sea necesario calcular estas soluciones particulares numéricamente. En [124] se pueden idear algunos métodos numéricos específicos para soluciones particulares de sistemas de Lie.

- Finalmente, a partir de la ecuación de Chebyshev, se ha demostrado que el punto de simetrías de Noether de esta ecuación se puede expresar para n arbitrario en términos de los polinomios de Chebyshev $T_n(x), U_n(x)$ de primer y segundo tipo, respectivamente. Además, se ha observado que la realización genérica del álgebra de simetría de puntos de Lie $\mathfrak{sl}(3,\mathbb{R})$ puede ampliarse a ecuaciones diferenciales ordinarias de segundo orden lineales más generales y las soluciones se pueden expresar en términos de funciones trigonométricas o hiperbólicas. En particular, los conmutadores de las simetrías de puntos genéricos muestran que varias de las relaciones algebraicas de las soluciones generales surgen realmente como consecuencia de la simetría. Las mismas conclusiones son válidas para la estructura de la subálgebra de cinco dimensiones de las simetrías de Noether. Se ha demostrado que la realización de los generadores de simetría sigue siendo válida para ecuaciones diferenciales de tipo hipergeométrico, lo que nos permite obtener realizaciones de $\mathfrak{sl}(3,\mathbb{R})$ en términos de funciones hipergeométricas en general y varios polinomios ortogonales en particular, como los polinomios de Chebyshev o Jacobi. Otro hecho notable que surge de este análisis es que los términos forzados son siempre independientes de las "velocidades" y_1', y_2'. Esto es nuevamente una consecuencia de la realización genérica elegida, y la cuestión de si otras realizaciones genéricas en términos de la solución general de la EDO (o sistema) permiten determinar términos forzosos que dependen explícitamente de las derivadas, e incluso conducen a ecuaciones diferenciales autónomas (sistemas), surge de manera natural. En este contexto, sería deseable obtener una realización de $\mathfrak{sl}(3,\mathbb{R})$ que no solo permita describir genéricamente las simetrías de punto y Noether de los polinomios de Jacobi, sino que también se aplique a las ecuaciones diferenciales asociadas a la familias restantes de polinomios ortogonales, específicamente los polinomios de Laguerre y Hermite. Esto permitiría construir más ecuaciones y sistemas no lineales que posean una subálgebra de simetrías de Noether, los generadores de las cuales se den en términos de estos polinomios ortogonales.

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M. de Cervantes, *El ingenioso hidalgo don Quijote de la Mancha*

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Edu.

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A mi madre y a mi hermano…

Introduction

A *Lie system* is a nonautonomous system of first-order ordinary differential equations whose general solution can be written as a function (a so-called *superposition rule*) depending on a certain number of particular solutions and some significant constants [58; 101; 149]. Superposition rules constitute a structural property that emerges naturally from the group-theoretical approach to differential equations initiated by Sophus Lie, Vessiot, and Guldberg, within the context of the development of the geometric program based on transformation groups, as well as from the analytic classification of differential equations developed by Painlevé and Gambier, among others. Indeed, Lie proved that every Lie system can be described by a finite-dimensional Lie algebra of vector fields, a *Vessiot–Guldberg Lie algebra* [101], and Vessiot used Lie groups to derive superposition rules [149].

In the frame of physical problems, it was not until the 80's when the power of superposition rules and Lie systems was fully recognized [154], motivating a systematic analysis of their applications in classical dynamics and their potential generalization to quantum systems (see [43; 44; 47; 154] and references therein).

Although Lie systems, as well as their refinements and generalizations, represent a valuable auxiliary tool in the integrability study of physical systems, it seems surprising that the methods employed have always remained within the limitations of Lie group and distribution theory, without considering other frameworks that have turned out to be a very successful approach to integrability, such as quantum groups and Poisson–Hopf algebras [2; 8; 26; 51; 105]. We recall that, beyond superintegrable systems [8; 26], Poisson coalgebras have been recently applied to integrable bi-Hamiltonian deformations of Lie–Poisson systems [24] and to integrable deformations of Rössler and Lorenz systems [12].

This work presents a novel generic procedure for the Poisson–Hopf algebra deformations of *Lie–Hamilton (LH) systems*, namely Lie systems endowed with a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields with respecto to a Poisson structure [49]. LH systems possess also a finite-dimensional Lie algebra of functions, a so-called *LH algebra*, governing their dynamics [49]. The proposed approach is based on the Poisson coalgebra formalism extensively used in the context of superintegrable systems, together with the notion of involutive distributions in the sense of Stefan–Sussman (see [120; 148; 152] for details). The main point will be to consider a Poisson–Hopf algebra structure that replaces the LH algebra of the non-deformed LH system, thus allowing us to obtain an explicit construction of t-independent constants of the motion, that will be expressed in terms of the deformed Casimir invariants. Moreover, the deformation will generally transform the Vessiot–Guldberg Lie algebra of the LH system into a set of vector fields generating an integrable distribution in the sense of Stefan–Sussman. Consequently, the deformed LH systems are not, in general, Lie systems anymore.

The novel approach is presented in the chapter 3, where the basic properties of LH systems and Poisson–Hopf algebras are reviewed (for details on the general theory of Lie and LH systems, chapters 1 and 2; the reader is referred to [9; 15; 29; 37; 38; 41; 42; 43; 44; 47; 49; 81; 85; 86; 101; 154]). To illustrate this construction, we consider the Poisson–Hopf algebra analogue of the so-called non-standard quantum deformation of $\mathfrak{sl}(2)$ [17; 116; 137] together with its deformed Casimir invariant, chapter 4.

Afterwards, relevant examples of deformed LH systems that can be extracted from this deformation are given. Firstly, the non-standard deformation of the Milney–Pinney equation is presented in part II, where this deformation is shown to give rise to a new oscillator system with a position-dependent mass and a time-dependent frequency (chapter 5), whose (time-independent) constants of the motion are also explicitly deduced. In secction 5.2 several deformed (complex and coupled) Riccati equations are obtained as a straightforward application of the formalism here presented. We would like to stress that, albeit these applications are carried out on the plane, thus provinid us a deeper insight in the proposed formalism, the method here presented is by no means constrained dimensionally, and its range of applicability goes far beyond the particular cases considered here.

Since its original formulation by Lie [101], Lie systems, have been studied extensively (see [38; 43; 47; 84; 120; 149; 154] and references therein). The Lie theorem [44; 101] states that every system of first-order differential equations is a Lie system if and only if it can be described as a curve in a finite-dimensional Lie algebra of vector fields, referred to as *Vessiot–Guldberg Lie algebra*.

Although being a Lie system is rather an exception than a rule [42], Lie systems have been shown to be of great interest within physical and mathematical applications (see [47] and references therein). Surprisingly, Lie systems admitting a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields relative to a Poisson structure, the *Lie–Hamilton systems*, have found even more applications than standard Lie systems with no associated geometric structure [9; 15; 29; 49]. Lie–Hamilton systems admit an additional finite-dimensional Lie algebra of Hamiltonian functions, a *Lie–Hamilton algebra*, that allows us to deduce an algebraic determination of superposition rules and constants of the motion of the system [29].

Apart from the theory of quasi-Lie systems [42] and superposition rules for nonlinear operators, most approaches to Lie systems rely strongly on the theory of Lie algebras and Lie groups. However, the success of quantum groups [51; 105] and the coalgebra formalism within the analysis of superintegrable systems [8; 26], and the fact that quantum algebras appear as deformations of Lie algebras, suggested the possibility of extending the notion and techniques of Lie–Hamilton systems beyond the range of application of the Lie theory. An approach in this direction was recently proposed in [14], where a method to construct quantum deformed Lie–Hamilton systems (LH systems in short) by means of the coalgebra formalism and quantum algebras was given.

The underlying idea is to use the theory of quantum groups to deform Lie systems and their associated structures. More exactly, the deformation transforms a LH system with its Vessiot–Guldberg Lie algebra into a Hamiltonian system whose dynamics is determined by a set of generators of a Steffan–Sussmann distribution. Meanwhile, the initial Lie–Hamilton algebra (LH algebra in short) is mapped into a Poisson–Hopf algebra. The deformed structures allow for the explicit construction of t-independent constants of the motion through quantum algebra techniques for the deformed system.

We illustrate how the approach introduced in [14] to construct deformations of LH systems via Poisson–Hopf structures allows a further systematization that encompasses the nonequivalent LH systems corresponding to isomorphic LH algebras. Specifically, we show that Poisson–Hopf deformations of LH systems based on a LH algebra isomorphic to $\mathfrak{sl}(2)$ can be described generically, hence providing the deformed Hamiltonian functions and the corresponding deformed Hamiltonian vector fields, once the corresponding counterpart of the non-deformed system is known. This provides a new method to construct LH systems with a LH algebra isomorphic to a fixed Lie algebra $\mathfrak{g}$. The approach is based on the symplectic foliation in $\mathfrak{g}^*$ induced by the Kirillov–Konstant–Souriou bracket on $\mathfrak{g}$. As a particular case, it is explicitly shown how our procedure explains the existence of three types of LH systems on the plane related to a LH algebra isomorphic to $\mathfrak{sl}(2)$. This is due to the fact that each one of the three different types corresponds to one of the three types of symplectic leaves in $\mathfrak{sl}^*(2)$. In analogy, one can generate the only LH system on the plane admitting a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{so}(3)$. This systematization enables us to give directly the Poisson–Hopf deformed system from the classification of LH systems [9; 29], further suggesting a notion of Poisson–Hopf Lie systems based on a z-parameterized family of Poisson algebra morphisms. Our methods seem to be extensible to study also LH systems and their deformations on other more general manifolds.

The structure of the thesis goes as follows. Chapters 1 and 2 are devoted to review the main aspects of LH systems and Poisson–Hopf algebras. The general procedure to construct Poisson–Hopf algebra deformations of LH systems [14], and other properties of the underlying formalism are given in Chapter 3. In 4, the (non-standard) Poisson–Hopf algebra deformation of LH systems based on the simple Lie algebra $\mathfrak{sl}(2)$ are analyzed in detail, while the unified approach to deformations of Poisson–Hopf Lie systems with a LH algebra isomorphic to a fixed Lie algebra $\mathfrak{g}$ is treated in Section 4.3.1. The procedure is explicitly illustrated, considering the three non-diffeomorphic classes of $\mathfrak{sl}(2)$-LH systems on the plane, from which the corresponding deformations are described in detail. Two general types of differential equations are considered in the context of their quantum deformations. First of all, the deformed Milne–Pinney equation is analyzed in detail, from which new oscillator systems with a position-dependent mass are derived. As a second relevant case, deformations of the Riccati equations are studied, specifically the deformed complex and deformed coupled Riccati equations. In Chapter 6 the problem of quantum deformations of LH systems based on the oscillator algebra $\mathfrak{h}_4$ is considered, with special emphasis of its structure as a subalgebra of the so-called 2-photon algebra $\mathfrak{h}_6$. This in particular leads to quantum deformations of the damped harmonic oscillator. In Chapter 7 we use the solvable Lie subalgebra $\mathfrak{b}_2$ of $\mathfrak{sl}(2)$ to propose a new and alternative epidemiological model. The techniques discussed in previous chapters are applied to analyze a defomed SISf model, which in particular is obtained by restriction of the corresponding $\mathfrak{sl}(2)$-deformed system, and for which the constants of the motion are explicitly constructed. Finally, in the Conclusions we summarize the results and outline some future work to be accomplished or already in progress.

Part I

Formalism of Poisson–Hopf Algebra Deformations

1 Lie Systems and Poisson–Hopf Algebras

1.1 Lie systems

Lie systems, besides their undeniable interest within Geometry, play a relevant role in many applications in Biology, Cosmology, Control Theory, Quantum Mechanics, among other disciplines. A specially interesting case is when a systems of first-order ordinary differential equations, which is the prototype of Lie system, can be endowed with a compatible symplectic structure, leading to the notion of Lie–Hamilton systems, of special interest within the frame of Classical Mechanics [58; 71]. Lie–Hamilton systems and their fundamental properties are conveniently described in terms of a t-dependent vector field that describes the dynamics. An illustrative example for this type of systems is given by the second-order Riccati equation.

1.1.1 Lie systems and superposition rules

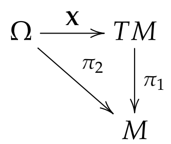
General properties of Lie systems, as well as additional geometrical applications, can be e.g. found in [47; 152].

Definition 1.1. *A superposition rule for a system* $\mathbf{X}$ *defined on an n-dimensional manifold M is a map*

$$\Phi : M^k \times M \longrightarrow M$$

such that $x(t) := \Phi(x_{(1)}(t), \ldots, x_{(k)}(t); \lambda)$ *is a general solution of the system* $\mathbf{X}$*, where* $x_i(t)$ *are particular solutions and* λ *is a point of the manifold M, corresponding to the initial condition of the Cauchy problem.*

Let M be an n-dimensional manifold and let π_i be the projections $\pi_1 : TM \longrightarrow M$ with $\pi_1(x, v) := x$ and $\pi_2 : \mathbb{R} \times M \longrightarrow M$ with $\pi_2(t, x) := x$. A smooth map $\mathbf{X} : \Omega \subseteq \mathbb{R} \times M \to TM$, where Ω is an open subset of $\mathbb{R} \times M$, is called a *t-dependent vector field* if the diagram



is commutative, i.e. $\pi_1 \circ \mathbf{X} = \pi_2$ in Ω. We observe that defining $\Omega_t := \{x \in M / (t, x) \in \Omega\}$ for each $t \in \mathbb{R}$, Ω_t is not empty and recovers the usual notion of vector field, denoted by $\mathbf{X}_t$. The notion of t-dependent vector field is very useful in the geometric theory of Lie systems and will play a relevant role in the formalism that will be developed.

Remark. It follows that if $\mathbf{X}$ is a t-dependent vector field, then it is equivalent to a linear morphism $\mathbf{X} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\mathbb{R} \times M)$ defined as $\mathbf{X}(f)(t, x) := (\mathbf{X}_t f)(x)$, for all $(t, x) \in \mathbb{R} \times \mathbb{M}$, such that it satisfies the Leibniz rule in $\mathcal{C}^\infty(M)$ at each point of $\mathbb{R} \times M$.

Let $\widetilde{\mathbf{X}}$ be a vector field over $\mathbb{R} \times M$ such that $\iota_{\widetilde{\mathbf{X}}} dt = 1$[1] and $(\widetilde{\mathbf{X}}\, \pi_2^* f)(t,x) = \mathbf{X}(f)(t,x)$. If the t-dependent vector field $\mathbf{X}$ is given by

$$\mathbf{X} = \sum_{j=1}^{n} \mathbf{X}_j(t,x) \frac{\partial}{\partial x_j}, \tag{1.1}$$

in local coordinates, then $\widetilde{\mathbf{X}}$ has the expression

$$\widetilde{\mathbf{X}} = \frac{\partial}{\partial t} + \sum_{j=1}^{n} \mathbf{X}_j(t,x) \frac{\partial}{\partial x_j},$$

called the *autonomization of* $\mathbf{X}$. An integral curve of the t-dependent vector field $\mathbf{X}$ is a map $\gamma : \mathbb{R} \to \mathbb{R} \times M$ such that $\pi_2 \circ \gamma$, where γ corresponds to an integral curve of $\widetilde{\mathbf{X}}$. It follows in particular that for any t with $\gamma(t) = (t, x(t))$ the components of $x(t)$ satisfy the following first order system:

$$\frac{dx_j}{dt} = \mathbf{X}_j(t,x), \quad i = 1, \ldots n. \tag{1.2}$$

This is the so-called associated system of $\mathbf{X}$. It is straightforward to verify that an arbitrary system of first-order ordinary differential equations of the type (1.2) determines a t-dependent vector field (1.1), the integral curves of which satisfy the system. This establishes a one-to-one correspondence between systems of type (1.2) and t-dependent vector fields.

If $\mathcal{A}$ is a family of vector fields on an n-dimensional manifold M, then $Lie(\mathcal{A})$ denotes the Lie algebra spanned by the vector fields and their successive commutators

$$\mathcal{A}, [\mathcal{A},\mathcal{A}], [\mathcal{A}[\mathcal{A},\mathcal{A}]], [\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{A}]]], \ldots$$

where $[\mathcal{A},\mathcal{B}]$ is a shorthand notation for the brackets $\{[\mathbf{X},\mathbf{Y}]/\mathbf{X} \in \mathcal{A} \;\; and \;\; \mathbf{Y} \in \mathcal{B}\}$. It follows from the construction that the Lie algebra $Lie(\mathcal{A})$ is the smallest Lie algebra of vector fields containing the set $\mathcal{A}$. The main result concerning the classical theory of Lie-systems is given by the Lie–Scheffers theorem (for more details see [101]).

Theorem 1.2 (Lie–Scheffers Theorem)**.** *A system of first-order ordinary differential equations* (1.2) *admits a superposition rule if and only if there exist smooth functions $\beta_j(t)$ such that associated t-dependent vector field has the form*

$$\mathbf{X}(t,x) = \sum_{j=1}^{\ell} \beta_j \mathbf{X}_j(x), \tag{1.3}$$

and such that the vector fields $\mathbf{X}_j$ (in a manifold M) span an ℓ-dimensional real Lie algebra $V^X := Lie(\{\mathbf{X}_j / j = 1, \ldots, \ell\})$.

The Lie algebra of vector fields V^X is usually called a *Vessiot–Guldberg Lie algebra*, where in addition the following numerical constraint must be satisfied:

$$dim(V^X) \leq m \cdot n = \dim(M). \tag{1.4}$$

The scalar m corresponds to the number of particular solutions of the system that are required for establishing a superposition rule. The relation (1.4) is also known as the *Lie condition*. It should be emphasized that a given system may admit different superposition rules, as a Vessiot–Guldberg algebra is not an invariant of the system, as happens e.g. with the Lie algebra of Lie-point symmetries. Thus, a specific system may admit nonisomorphic Vessiot–Guldberg algebras, and hence different superposition rules.

Definition 1.3. *A first-order system of differential equations that admits a superposition rule is a Lie system.*

[1]Let M be a smooth manifold, $\mathbf{X} \in \mathfrak{X}(M)$ and $\omega \in \Omega^p(M)$, then $\iota_{\mathbf{X}}\omega$ is the contraction of a differential form ω with respect to the vector field $\mathbf{X}$.

Example 1.4. Let $\mathbf{X}$ be a t-dependent vector field over $\mathbb{R}^2$ that expressed in local coordinates is given by

$$\mathbf{X} = -\frac{1}{t}X_1 + X_2 + \eta^2 X_3, \tag{1.5}$$

where

$$\mathbf{X}_1 = y\frac{\partial}{\partial y}, \quad \mathbf{X}_2 = y\frac{\partial}{\partial x}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial y}, \quad \mathbf{X}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.6}$$

These vector fields span a Vessiot–Guldberg Lie algebra V^X isomorphic to $gl(2)$, with domain $\mathbb{R}^2_{x\neq 0}$ (for more details see Appendix B) and the associated system takes the form

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -\frac{2}{t}y + \eta^2 x. \tag{1.7}$$

Hence, this system is a Lie system.

1.1.2 Lie–Hamilton systems

Definition 1.5. *A Lie system $\mathbf{X}$ is, furthermore, a Lie–Hamilton system [9; 15; 29; 47; 49; 81] if it admits a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields relative to a Poisson structure. This amounts to the existence, around each generic point of M, of a symplectic form, ω, such that:*

$$\mathcal{L}_{\mathbf{X}_i}\omega = 0 \tag{1.8}$$

for a basis $\mathbf{X}_1, \ldots, \mathbf{X}_\ell$ of V (cf. [9]). Then each vector field $\mathbf{X}_i$ admits a Hamiltonian function h_i given by the rule:

$$\iota_{\mathbf{X}_i}\omega = \mathrm{d}h_i, \tag{1.9}$$

where $\iota_{\mathbf{X}_i}\omega$ stands for the contraction of the vector field $\mathbf{X}_i$ with the symplectic form ω.

Since ω is non-degenerate, every function h induces a unique associated Hamiltonian vector field $\mathbf{X}_h$. This fact gives rise to a Poisson bracket on $\mathcal{C}^\infty(M)$ given by

$$\{\cdot,\cdot\}_\omega \; : \; \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \ni (f_1, f_2) \mapsto X_{f_2}f_1 \in \mathcal{C}^\infty(M), \tag{1.10}$$

turning $(\mathcal{C}^\infty(M), \{\cdot,\cdot\}_\omega)$ into a Lie algebra. The space $\mathrm{Ham}(\omega)$ of Hamiltonian vector fields on M relative to ω is also a Lie algebra relative to the commutator of vector fields. Moreover, we have the following exact sequence of Lie algebras morphisms (see [148])

$$0 \hookrightarrow \mathbb{R} \hookrightarrow (\mathcal{C}^\infty(M), \{\cdot,\cdot\}_\omega) \xrightarrow{\phi} (\mathrm{Ham}(\omega), [\cdot,\cdot]) \xrightarrow{\pi} 0, \tag{1.11}$$

where π is the projection onto 0 and ϕ maps each $f \in C^\infty(M)$ into the Hamiltonian vector field $\mathbf{X}_{-f}$. In view of the sequence (1.11), the Hamiltonian functions $h_1, \ldots, h_\ell$ and their successive Lie brackets with respect to (1.10) span a finite-dimensional Lie algebra of functions contained in $\phi^{-1}(V)$. This Lie algebra is called a *Lie–Hamilton (LH) algebra* $\mathcal{H}_\omega$ of $\mathbf{X}$ (see [49; 81] and references therein).

Let $\mathbf{X}$ be a system on an n-dimensional manifold M. A function $f \in \mathcal{C}^\infty(TM)$ is called a constant of the motion[2] of the system $\mathbf{X}$ if it is a first integral of the vector field $\widetilde{\mathbf{X}}$, i.e.,

$$\widetilde{\mathbf{X}}f = 0.$$

[2]The space of constant of the motion for a system form a $\mathbb{K}-$algebra.

Damped harmonic oscillator

Consider a t-dependent one-dimensional damped harmonic oscillator of the form

$$\frac{dx}{dt} = a(t)x + b(t)p + f(t), \qquad (x,p) \in T^*_x\mathbb{R}, \tag{1.12}$$
$$\frac{dp}{dt} = -a(t)p - c(t)x - d(t),$$

for arbitrary t-dependent functions $a(t), b(t), c(t), d(t), f(t)$. The system (1.12) is associated with the t-dependent vector field

$$\mathbf{X}_t = f(t)\mathbf{X}_1 - d(t)\mathbf{X}_2 + a(t)\mathbf{X}_3 + b(t)\mathbf{X}_4 - c(t)\mathbf{X}_5,$$

where

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial p}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad \mathbf{X}_4 = p\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = x\frac{\partial}{\partial p},$$

are such that $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle \simeq \mathbb{R}^2$ and $\langle \mathbf{X}_3, \mathbf{X}_4, \mathbf{X}_5 \rangle \simeq \mathfrak{sl}(2)$. Moreover $\langle \mathbf{X}_1, \ldots, \mathbf{X}_5 \rangle \simeq \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathbf{X}$ becomes a Lie system related to a Vessiot–Guldberg Lie algebra V_{do} isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$.

It can be easily shown that V_{do} consists of Hamiltonian vector fields with respect to the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}p$ on $T^*\mathbb{R}$. In fact, the Hamiltonian functions associated to the vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_5$ are given by

$$h_1 = p, \qquad h_2 = -x, \qquad h_3 = xp, \qquad h_4 = \frac{1}{2}p^2, \qquad h_5 = -\frac{1}{2}x^2,$$

respectively. The functions $h_1, \ldots, h_5$ along with h_0 span a Lie algebra $\mathcal{H}_\omega \simeq \mathfrak{h}_6$ with respect to the standard Poisson bracket on $T^*\mathbb{R}$.

The constants of the motion for the damped harmonic oscillator equations can be obtained by applying the coalgebra formalism introduced in [15]. Using the Casimir invariants of the underlying Lie algebra, it follows that these constants of the motion of the Lie system (1.12) are given by [29]

$$F^{(1)} = 0, \qquad F^{(2)} = 0,$$
$$F^{(3)} = \Big((x_{(2)} - x_{(3)})p_{(1)} + (x_{(3)} - x_{(1)})p_{(2)} + (x_{(1)} - x_{(2)})p_{(3)} \Big)^2.$$

By permutation of the indices corresponding to the variables of the non-trivial invariant $F^{(3)}$, we can find additional constants of the motion:

$$F^{(3)}_{14} = \Big((x_{(2)} - x_{(3)})p_{(4)} + (x_{(3)} - x_{(4)})p_{(2)} + (x_{(4)} - x_{(2)})p_{(3)} \Big)^2,$$
$$F^{(3)}_{24} = \Big((x_{(3)} - x_{(4)})p_{(1)} + (x_{(4)} - x_{(1)})p_{(3)} + (x_{(1)} - x_{(3)})p_{(4)} \Big)^2.$$

In order to derive a superposition rule, we just need to obtain the value of $p_{(1)}$ from the equation $k_1 = F^{(3)}$, where k_1 is a real constant; and then plug this value into the equation $k_2 = F^{(3)}_{24}$ to obtain [29]

$$x_{(1)} = x_{(3)} + \frac{(x_{(4)} - x_{(3)})\sqrt{k_1} + (x_{(2)} - x_{(3)})\sqrt{k_2}}{\sqrt{F^{(3)}_{14}}},$$
$$p_{(1)} = \frac{\sqrt{k_1}}{x_{(2)} - x_{(3)}} + p_{(3)} + \frac{\sqrt{k_2}(p_{(2)} - p_{(3)})}{\sqrt{F^{(3)}_{14}}} + \frac{\sqrt{k_1}(p_{(2)} - p_{(3)})(x_{(4)} - x_{(3)})}{\sqrt{F^{(3)}_{14}}(x_{(2)} - x_{(3)})}.$$

Clearly, the superposition rule obtained above is merely one among many possible superposition rules, that could be derived considering other choices of the constants of the motion.

A second-order Riccati equation in Hamiltonian form

Another relevant application of Lie–Hamilton systems is given by the class of Riccati equations, that have been analyzed in detail in [49]. The most general class of second-order Riccati equations is given by the family of second-order differential equations of the form

$$\frac{d^2x}{dt^2} + (f_0(t) + f_1(t)x)\frac{dx}{dt} + c_0(t) + c_1(t)x + c_2(t)x^2 + c_3(t)x^3 = 0, \tag{1.13}$$

with

$$f_1(t) = 3\sqrt{c_3(t)}, \qquad f_0(t) = \frac{c_2(t)}{\sqrt{c_3(t)}} - \frac{1}{2c_3(t)}\frac{dc_3}{dt}(t), \qquad c_3(t) \neq 0.$$

These equations arise by reducing third-order linear differential equations through a dilation symmetry and a time-reparametrization. Their interest is due to their use in the study of several physical and mathematical problems [49; 53].

It was recently discovered [50] that every second-order Riccati equation (1.13) admits a t-dependent non-natural regular Lagrangian of the form

$$L(t,x,v) = \frac{1}{v + U(t,x)},$$

with $U(t,x) = a_0(t) + a_1(t)x + a_2(t)x^2$ and $a_0(t), a_1(t), a_2(t)$ being certain functions related to the t-dependent coefficients of (1.13), see [53]. Therefore,

$$p = \frac{\partial L}{\partial v} = \frac{-1}{(v + U(t,x))^2}, \tag{1.14}$$

and the image of the Legendre transform $\mathbb{F}L : (t,x,v) \in \mathcal{W} \subset \mathbb{R} \times T\mathbb{R} \mapsto (t,x,p) \in \mathbb{R} \times T^*\mathbb{R}$, where $\mathcal{W} = \{(t,x,v) \in \mathbb{R} \times T\mathbb{R} \mid v + U(t,x) \neq 0\}$, is the open submanifold $\mathbb{R} \times \mathcal{O}$ where $\mathcal{O} = \{(x,p) \in T^*\mathbb{R} \mid p < 0\}$. The Legendre transform is not injective, as $(t,x,p) = \mathbb{F}L(t,x,v)$ for $v = \pm 1/\sqrt{-p} - U(t,x)$. Nevertheless, it can become so by restricting it to the open set $\mathcal{W}_+ = \{(t,x,v) \in \mathbb{R} \times T\mathbb{R} \mid v + U(t,x) > 0\}$. In such a case, $v = 1/\sqrt{-p} - U(t,x)$ and we can define over $\mathbb{R} \times \mathcal{O}$ the t-dependent Hamiltonian

$$h(t,x,p) = p\left(\frac{1}{\sqrt{-p}} - U(t,x)\right) - \sqrt{-p} = -2\sqrt{-p} - p\,U(t,x).$$

Its Hamilton equations read

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial h}{\partial p} = \dfrac{1}{\sqrt{-p}} - U(t,x) = \dfrac{1}{\sqrt{-p}} - a_0(t) - a_1(t)x - a_2(t)x^2, \\ \dfrac{dp}{dt} = -\dfrac{\partial h}{\partial x} = p\dfrac{\partial U}{\partial x}(t,x) = p(a_1(t) + 2a_2(t)x). \end{cases} \tag{1.15}$$

Since the general solution $x(t)$ of every second-order Riccati equation (1.14) can be recovered from the general solution $(x(t), p(t))$ of its corresponding system (1.15), the analysis of the latter provides information about general solutions of second-order Riccati equations.

The relevant point is to observe that the system (1.15) is actually a Lie system as shown in [48]. Indeed, consider the vector fields over $\mathcal{O}$ of the form

$$\mathbf{X}_1 = \frac{1}{\sqrt{-p}}\frac{\partial}{\partial x}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \qquad \mathbf{X}_4 = x^2\frac{\partial}{\partial x} - 2xp\frac{\partial}{\partial p},$$
$$\mathbf{X}_5 = \frac{x}{\sqrt{-p}}\frac{\partial}{\partial x} + 2\sqrt{-p}\frac{\partial}{\partial p}. \tag{1.16}$$

Their non-vanishing commutation relations read

$$\begin{gathered}[\mathbf{X}_1,\mathbf{X}_3]=\frac{1}{2}\mathbf{X}_1,\qquad [\mathbf{X}_1,\mathbf{X}_4]=\mathbf{X}_5,\qquad [\mathbf{X}_2,\mathbf{X}_3]=\mathbf{X}_2,\qquad [\mathbf{X}_2,\mathbf{X}_4]=2\mathbf{X}_3,\\ [\mathbf{X}_2,\mathbf{X}_5]=\mathbf{X}_1,\qquad [\mathbf{X}_3,\mathbf{X}_4]=\mathbf{X}_4,\qquad [\mathbf{X}_3,\mathbf{X}_5]=\frac{1}{2}\mathbf{X}_5,\end{gathered}\tag{1.17}$$

and therefore span a five-dimensional Lie algebra V of vector fields. The t-dependent vector field $\mathbf{X}_t$ associated to the system (1.15) is given by [48]

$$\mathbf{X}_t=\mathbf{X}_1-a_0(t)\mathbf{X}_2-a_1(t)\mathbf{X}_3-a_2(t)\mathbf{X}_4.\tag{1.18}$$

In view of expressions (1.17) and (1.18), the system (1.15) is a Lie system. Note also that a similar result would have been obtained by restricting the Legendre transform over the open set $\mathcal{W}_-=\{(t,x,v)\in\mathbb{R}\times\mathrm{T}\mathbb{R}\mid v+U(t,x)<0\}$.

Next, the vector fields (1.16) are additionally Hamiltonian vector fields relative to the Poisson bivector $\Lambda=\partial/\partial x\wedge\partial/\partial p$ on $\mathcal{O}$. Indeed, they admit the Hamiltonian functions

$$h_1=-2\sqrt{-p},\quad h_2=p,\quad h_3=xp,\quad h_4=x^2p,\quad h_5=-2x\sqrt{-p},\tag{1.19}$$

which span along with $h_6=1$ a six-dimensional real Lie algebra of functions with respect to the Poisson bracket induced by Λ (see [48] for details).

The above action enables us to write the general solution $\xi(t)$ of system (1.15) in the form $\xi(t)=\Phi(g(t),\xi_0)$, where $\xi_0\in\mathcal{O}$ and $g(t)$ is the solution of the equation

$$\frac{dg}{dt}=-\left(\mathbf{X}_1^R(g)-a_0(t)\mathbf{X}_2^R(g)-a_1(t)\mathbf{X}_3^R(g)-a_2(t)\mathbf{X}_4^R(g)\right),\qquad g(0)=e,\tag{1.20}$$

on G, with the $\mathbf{X}_\alpha^R$ being a family of right-invariant vector fields over G whose vectors $\mathbf{X}_\alpha^R(e)\in T_eG$ close on the same commutation relations as the $\mathbf{X}_\alpha$ (cf. [47]).

Let us now apply to Lie systems (1.20) the reduction theory for Lie systems. Since $T_eG\simeq\mathbb{R}^2\oplus_s\mathfrak{sl}(2,\mathbb{R})$, a particular solution of a Lie system of the form (1.20) but over $SL(2,\mathbb{R})$, which amounts to integrating (first-order) Riccati equations (cf. [47]), provides us with a transformation which maps system (1.20) into an easily integrable Lie system over $\mathbb{R}^2$. In short, the explicit determination of the general solution of a second-order Riccati equation reduces to solving Riccati equations.

In order to determine a superposition rule for the system (1.15), it suffices to consider two common functionally independent first-integrals for the diagonal prolongations $\widetilde{\mathbf{X}}_1,\widetilde{\mathbf{X}}_2,\widetilde{\mathbf{X}}_3,\widetilde{\mathbf{X}}_4,\widetilde{\mathbf{X}}_5$ to a certain $\mathrm{T}^*\mathbb{R}^{(m+1)}$, provided that these prolongations to $\mathrm{T}^*\mathbb{R}^m$ are linearly independent at a generic point. In the present case, it can be easily verified that $m=4$. The resulting first-integrals (see [29; 49; 133]) are explicitly given by

$$\begin{aligned}F_0&=(x_{(2)}-x_{(3)})\sqrt{p_{(2)}p_{(3)}}+(x_{(3)}-x_{(1)})\sqrt{p_{(3)}p_{(1)}}+(x_{(1)}-x_{(2)})\sqrt{p_{(1)}p_{(2)}},\\ F_1&=(x_{(1)}-x_{(2)})\sqrt{p_{(1)}p_{(2)}}+(x_{(2)}-x_{(0)})\sqrt{p_{(2)}p_{(0)}}+(x_{(0)}-x_{(1)})\sqrt{p_{(0)}p_{(1)}},\\ F_2&=(x_{(1)}-x_{(3)})\sqrt{p_{(1)}p_{(3)}}+(x_{(3)}-x_{(0)})\sqrt{p_{(3)}p_{(0)}}+(x_{(0)}-x_{(1)})\sqrt{p_{(0)}p_{(1)}}.\end{aligned}$$

Note that given a family of solutions $(x_{(i)}(t),p_{(i)}(t))$, with $i=0,\ldots,3$, of (1.15), then $d\bar{F}_j/dt=\widetilde{\mathbf{X}}_tF_j=0$ for $j=0,1,2$ and $\bar{F}_j=F_j(x_{(0)}(t),p_{(0)}(t),\ldots,x_{(3)}(t),p_{(3)}(t))$.

In order to derive a superposition rule, it remains to obtain the value of $p_{(0)}$ from the equation $k_1=F_1$, where k_1 is a real constant. Proceeding along these lines and from the results given in [29], we get

$$\sqrt{-p_{(0)}}=\frac{k_1+(x_{(2)}-x_{(1)})\sqrt{p_{(1)}p_{(2)}}}{(x_{(2)}-x_{(0)})\sqrt{-p_{(2)}}+(x_{(0)}-x_{(1)})\sqrt{-p_{(1)}}},$$

and then plug this value into the equation $k_2 = F_2$ to have

$$x_{(0)} = \frac{k_1\Gamma(x_{(1)}, p_{(1)}, x_{(3)}, p_{(3)}) + k_2\Gamma(x_{(2)}, p_{(2)}, x_{(1)}, p_{(1)}) - F_0 x_{(1)}\sqrt{-p_{(1)}}}{k_1(\sqrt{-p_{(1)}} - \sqrt{-p_{(3)}}) + k_2(\sqrt{-p_{(2)}} - \sqrt{-p_{(1)}}) - \sqrt{-p_{(1)}}F_0},$$
$$p_{(0)} = -\left[k_1/F_0(\sqrt{-p_{(3)}} - \sqrt{-p_{(1)}}) + k_2/F_0(\sqrt{-p_{(1)}} - \sqrt{-p_{(2)}}) + \sqrt{-p_{(1)}}\right]^2,$$

where $\Gamma(x_{(i)}, p_{(i)}, x_{(j)}, p_{(j)}) = \sqrt{-p_{(i)}}x_{(i)} - \sqrt{-p_{(j)}}x_{(j)}$. The above expressions give us a superposition rule $\Phi : (x_{(1)}, p_{(1)}, x_{(2)}, p_{(2)}, x_{(3)}, p_{(3)}; k_1, k_2) \in T^*\mathbb{R}^3 \times \mathbb{R}^2 \mapsto (x_{(0)}, p_{(0)}) \in T^*\mathbb{R}$ for the system (1.15). Finally, since every $x_{(i)}(t)$ is a particular solution for (1.13), the map $\Upsilon = \tau \circ \Phi$ furnishes the general solution of second-order Riccati equations in terms of three generic particular solutions $x_{(1)}(t), x_{(2)}(t), x_{(3)}(t)$ of (1.13), the corresponding $p_{(1)}(t), p_{(2)}(t), p_{(3)}(t)$ and two real constants k_1, k_2.

1.2 Poisson–Hopf algebras

The Hopf structure [2; 112; 144] is introduced through relevant examples, such as the universal enveloping algebra of a Lie algebra. The Hopf algebra structure originally appeared in the context of group cohomology, from which its use has been expanded to constitute nowadays an essential tool for the algebraic analysis. The next important definition will be the notion of Poisson structure and its compatibility with the Hopf algebras. In this section all vector spaces are defined over $\mathbb{C}$, where $V \otimes W$ denotes the tensor product of the two vector spaces V and W.

1.2.1 Hopf algebras

Definition 1.6. *An algebra is a pair (A, m), where A is a linear space and $m : A \otimes A \to A$ is a bilinear map. Furthermore, (A, m) is an algebra with unit if there is an element in A such that $m(1, a) = m(a, 1)$ for all $a \in A$. The algebra is associative if for arbitrary $x, y, z \in A$ the identity $m(m(x, y), z) = m(x, m(y, z))$ holds.*

If the map m is the inner product on the tensor product, it is possible define the unit map $\eta : \mathbb{C} \to A$ such that $\eta(1) = 1_A$. Then, the properties of algebra are

$$m \circ (\eta \circ id_A) = id_A = m \circ (id_A \circ \eta)$$
$$m \circ (m \otimes id_A) = id_A = m \circ (id_A \otimes m).$$

Definition 1.7. *A coassociative coalgebra with counit is a linear space A endowed with two linear maps: the coproduct $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to \mathbb{C}$ such that*

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta, \tag{1.21}$$
$$(id \otimes \epsilon) \circ \Delta = id = (\epsilon \otimes id) \circ \Delta, \tag{1.22}$$

moreover, the coalgebra is cocommutative if the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle id\otimes\Delta} \\ A \otimes A & \xrightarrow[\Delta\otimes id]{} & A \otimes A \otimes A. \end{array}$$

Coalgebras can be described essentially by a dualization process. More specifically, from the point of view of the commutative diagrams, they are obtained reversing the direction of the corresponding maps.

In view of this definitions, a *bialgebra* (A, m, Δ) is a linear space where (A, m) is an algebra and (A, Δ) is a coalgebra, such that it is verified

$$\begin{aligned} \Delta(xy) &= \Delta(x)\Delta(y), \\ \Delta(1) &= 1 \otimes 1, \\ \epsilon(xy) &= \epsilon(x)\epsilon(y) \end{aligned} \tag{1.23}$$

for all $x, y \in A$, i.e. the coproduct and the counit are algebra homomorphisms.

The existence of a compatible coproduct with the product in an algebra allows the representation of A over vector the space $V \otimes V'$, once the representations of A over V and V' are known. The representation theory can be framed in this context, the map $D : A \otimes V \to V$ is a representation of A on V if it satisfies

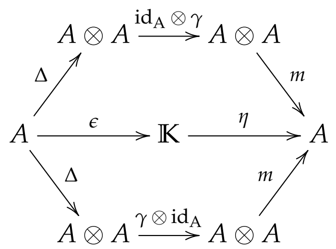
$$\begin{aligned} D \circ (id_A \otimes D) &= D \circ (m \otimes id_V), \\ D \circ (\eta \otimes id_V) &= id_V, \end{aligned}$$

i.e. D is a representation consistent with the inner product and unit.

Definition 1.8. *A bialgebra (A, m, Δ) is a Hopf algebra if there exist a linear map $\gamma : A \to A$, so-called antipode, such that*

$$m \circ (id_A \otimes \gamma) \circ \Delta = \eta \circ \epsilon = m \circ (\gamma \otimes id_A) \circ \Delta. \tag{1.24}$$

Let (A, m, Δ) be a bialgebra, if the antipode map exists, then it is unique. Hence, the Hopf algebra structure compatible with the bialgebra is unique. Furthermore, the antipode can be defined as the linear map such that the following diagram is commutative



(1.25)

where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. The following proposition puts together the most relevant properties of Hopf algebras.

Proposition 1.9. *Let (A, m, Δ) be a Hopf algebra. Then for all $x, y \in A$ it is verified*

$$\begin{aligned} \gamma(1) &= 1, \\ \epsilon(\gamma(x)) &= \epsilon(x), \\ \gamma(xy) &= \gamma(y)\gamma(x), \\ (\Delta \circ \gamma)(x) &= ((\gamma \otimes \gamma) \circ \sigma \circ \Delta)(x), \end{aligned}$$

where $\sigma(x \otimes y) = y \otimes x$ is a permutation. Hence, γ is an antihomomorphism and anticohomomorphism. Moreover, the antipode map can be non-invertible.

Example 1.10 (Lie algebra)**.** [88] Let $\mathfrak{g}$ be a Lie algebra, then the *universal enveloping algebra* of $\mathfrak{g}$,[3] denoted $\mathcal{U}(\mathfrak{g})$, is a general associative algebra that is obtained through the following quotient

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/\mathcal{I},$$

where $\mathcal{I}$ is an ideal spanned by $\{XY - YX - [X,Y]/X, Y \in \mathfrak{g}\}$ and the tensor algebra of $\mathfrak{g}$, $\mathcal{T}(\mathfrak{g})$ denotes the graded algebra $\bigoplus_k \mathcal{T}^k(\mathfrak{g})$. Given a generic element $X \in \mathfrak{g}$ such that

$$\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1, \quad \Delta(1) = 1 \otimes 1, \\
&\epsilon(X) = 0, \quad \epsilon(1) = 1, \\
&\gamma(X) = -X,
\end{aligned}$$

then we can extend the mappings to $\mathcal{U}(\mathfrak{g})$. This endowes the universal enveloping algebra with a Hopf algebra structure[4].

Remark (Friedrichs' theorem). Only the generators of a Lie algebra can be primitive elements of the universal enveloping algebra. For more details see [125].

Theorem 1.11 (Poincaré–Birkhoff–Witt)**.** *[92] Let $\mathfrak{g}$ be a Lie algebra and $\{X_1, \ldots, X_n\}$ a basis of $\mathfrak{g}$, then the set $\{X_1^{k_1} \cdots X_n^{k_n}/k_j \in \mathbb{N}\}$ is a basis of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.*

1.2.2 Poisson algebras

Definition 1.12. *A Poisson algebra is a vector space A over field $\mathbb{K}$ endowed with two bilinear maps: $m : A \otimes A \to A$, commutative, and the Poisson bracket[5] $\{\cdot,\cdot\} : A \otimes A \to A$, where (A, m) is an associative algebra and $(A, \{\cdot,\cdot\})$ is a Lie algebra. Then, the Poisson bracket obeys the Jacobi identity and it is antisymmetric.*

Let M be an n-dimensional manifold and consider its ring of smooth functions $\mathcal{C}^\infty(M)$. According to the last definition, a bivector $\Lambda \in \mathfrak{X}^2(M)$ induces a Poisson bracket on $\mathcal{C}^\infty(M)$ and it is an antisymmetric biderivation. If $(x_1, \ldots, x_n)$ are a local coordinates on M, a bivector Λ turns out to be

$$\Lambda = \lambda^{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad i < j, \quad \lambda^{ij} \in \mathcal{C}^\infty(M),$$

hence the bracket takes the form

$$\{f, g\} := \Lambda(f, g) = \lambda^{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \tag{1.26}$$

for all $f, g \in \mathcal{C}^\infty(M)$. Note also that, in a general context, the last condition is the vanishing of the Schouten—Nijenhuis bracket $[[\Lambda, \Lambda]] = 0$ [134]. Moreover, the tensor λ^{ij} can be degenerate, then this Poisson structure exists in odd-dimensional manifolds.

In this theoretical environment, a function $C \in \mathcal{C}^\infty(M)$ is a *Casimir element* of the Poisson algebra $(\mathcal{C}^\infty(M), \Lambda)$ if it satisfies

$$\{C, f\} = 0$$

for all $f \in \mathcal{C}^\infty(M)$, i.e. the Casimir element belongs to the center of the Poisson algebra.

[3] $\mathcal{U}(\mathfrak{g})$ is an algebra together with a morphism of Lie algebras $i : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ such that $F : \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras. Then, there is a single morphism $\widetilde{F} : \mathcal{U}(\mathfrak{g}) \to \mathfrak{g}'$ that verifies $\widetilde{F} \circ i = F$.

[4] The elements $X \in \mathfrak{g}$ are called primitive elements of the Hopf algebra if they can be written as $X \otimes 1 + 1 \otimes X$.

[5] The Poisson bracket induces a derivation in the product m.

Remark (Racah 1951). It was first shown by Racah that for semisimple Lie algebras $\mathfrak{s}$, the number of functionally independent Casimir invariants (operators) coincide with the the rank of $\mathfrak{s}$, i.e, the dimension of a Cartan subalgebra.

Example 1.13. [14] Consider the Lie algebra $\mathfrak{sl}_2$ of traceless real 2×2 matrices. This algebra admits the basis $\{e_1, e_2, e_3\}$ with commutation relations

$$[e_1, e_2] = e_1, \qquad [e_1, e_3] = 2e_2, \qquad [e_2, e_3] = e_3.$$

The elements of $\mathfrak{sl}_2$ can be considered as linear functions v_1, v_2, v_3 on $\mathfrak{sl}_2^*$, respectively. In this case, the corresponding Poisson bracket is given by

$$\{v_1, v_2\} = v_1, \qquad \{v_1, v_3\} = 2v_2, \qquad \{v_2, v_3\} = v_3.$$

This amounts to the Poisson bivector

$$\Lambda = v_1 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + 2v_2 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} + v_3 \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3}.$$

The Poisson structure on $\mathfrak{g}^*$ admits a Casimir given by the function

$$C = v_1 v_3 - v_2^2.$$

The surfaces S_k, where C takes a constant value k, are one-sided hyperboloids for $k > 0$, two-sided hyperboloids when $k > 0$, and cones for $k = 0$. The Poisson bivector on the neighbourhood of a generic point of such surfaces reads

$$\Lambda_{\mathfrak{g}} = v_1 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2},$$

in the coordinate system $\{v_1, v_2, C\}$. As the canonical Poisson bivector on $\mathbb{R}^2 \simeq T^*\mathbb{R}$ is given by

$$\Lambda_{\mathbb{R}^2} = \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

it turns out that Λ and $\Lambda_{\mathfrak{g}}$ locally describe the same Poisson bivector, but in different coordinates. Hence, there exits a Poisson algebra morphism $\phi : C^\infty(S_k) \to C^\infty(\mathbb{R}^2)$ given by

$$\phi(v_1) = x, \qquad \phi(v_2/v_1) = y, \qquad \phi(C_z) = k.$$

Therefore, $\phi(v_3) = k/x + y^2 x$, where k is an arbitrary constant.

Example 1.14. Consider the Lie–Hamilton algebra $\overline{\mathfrak{iso}(2)} = \langle e_0, e_1, e_2, e_3 \rangle$, where $\{e_0, e_1, e_2, e_3\}$ is a basis with commutation relations

$$[e_1, e_2] = e_0, \qquad [e_1, e_3] = e_2, \qquad [e_2, e_3] = -e_1$$

and e_4 spans the center of $\overline{\mathfrak{iso}(2)}$. Considering the elements of the above basis as linear functions $\{v_0, v_1, v_2, v_3\}$ on the dual to $\mathfrak{iso}(2)$, the corresponding Poisson bivector reads

$$\Lambda = v_0 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + v_2 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} - v_1 \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3}.$$

This Poisson algebra admits two Casimirs

$$C_1 := v_0, \qquad C_2 := 2v_0 v_3 - v_1^2 - v_2^2$$

which allow us to restrict the Poisson bracket to the surfaces S_{κ_1,κ_2} where the Casimirs take constant values κ_1, κ_2. These are symplectic manifolds where the Poisson bivector Λ can be mapped into a canonical form. In fact, the use of the coordinates v_1, v_2, C_1, C_2 leads to the Poisson bivector in the form

$$\Lambda = \frac{\partial}{\partial(v_1/v_0)} \wedge \frac{\partial}{\partial v_2}.$$

Hence, this leads to a representation of the Poisson bracket in terms of functions

$$\phi(v_1/v_0) = x, \qquad \phi(v_2) = y, \qquad \phi(C_1) = k_1, \qquad \phi(C_2) = k_2.$$

In consequence, a possible morphism of Poisson algebras is given by

$$\phi(v_0) = k_1, \qquad \phi(v_1) = k_1 x, \qquad \phi(v_2) = y, \qquad \phi(v_3) = \frac{k_2 + k_1^2 x^2 + y^2}{2k_1}.$$

Example 1.15. Consider now the Lie algebra $\mathfrak{so}(3) = \langle e_1, e_2, e_3 \rangle$, where the basis $\{e_1, e_2, e_3\}$ satisfies the commutation relations

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2.$$

Following the same methods sketched in the previous example, let us consider the Poisson algebra $(C^\infty(\mathfrak{so}(3)^*), \{\cdot, \cdot\})$. In this case, one obtains that the Poisson bivector associated with this Poisson manifold reads

$$\Lambda = v_3 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + v_1 \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3} + v_2 \frac{\partial}{\partial v_3} \wedge \frac{\partial}{\partial v_1}$$

and

$$C_1 = v_1^2 + v_2^2 + v_3^2$$

becomes a Casimir function for this Poisson algebra. Therefore, the symplectic foliation for the Lie algebra is given by surfaces parametrized by the value of C_1. Moreover, the Poisson bivector becomes a symplectic structure on each leaf admitting a canonical form. This canonical form can be obtained by writing Λ in the coordinate system v_1, v_2, C, then

$$\Lambda = \sqrt{k - v_1^2 - v_2^2} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2}.$$

To write the above in a canonical form, it is enough to introduce new variables $v_1 = r\cos\varphi$ $v_2 = r\sin\varphi$. This is natural as the symplectic leaves are spheres. Consequently,

$$\Lambda = \frac{\sqrt{k - r^2}}{r} \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \varphi} = \frac{\partial}{\partial(-\sqrt{k - r^2})} \wedge \frac{\partial}{\partial \varphi}.$$

Hence, it is enough to define the Poisson morphism as

$$\phi(-\sqrt{k - r^2}) = x, \qquad \phi(\varphi) = y.$$

Undoing the previous changes of coordinates, we obtain

$$h_1 = -\sqrt{k - x^2}\cos(y), \qquad h_2 = \sqrt{k^2 - x^2}\sin(y), \qquad h_3 = x,$$

where $h_i := \phi(v_i)$ for $i = 1, 2, 3$.

Remark (Schur's lemma). [138] Let $\mathfrak{g}$ be a Lie algebra and $\phi : \mathfrak{g} \to End(V)$ a representation of $\mathfrak{g}$. If this representation is irreducible, then ϕ can be extended to a representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. In view of this result, any x in the center of $\mathcal{U}(\mathfrak{g})$ has as its image $\phi(x)$ a multiple of the identity.

1.2.3 Poisson–Hopf algebras

Consider the Poisson algebras $(A, \{\cdot,\cdot\}_A)$ and $(B, \{\cdot,\cdot\}_B)$. A linear map $f : A \to B$ is a *homomorphism of Poisson algebras* if

$$\begin{aligned} f(xy) &= f(x)f(y), \\ f(\{x,y\}) &= \{f(x), f(y)\}, \end{aligned}$$

for all $x, y \in A$. In this context, we define the Poisson structure on the tensor product $A \otimes B$ as

$$\{x_1 \otimes y_1, x_2 \otimes y_2\}_{A\otimes B} := \{x_!, x_2\}_A \otimes y_1 y_2 + x_1 x_2 \otimes \{y_1, y_2\}_B.$$

Moreover, this Poisson bracket is the only Poisson structure such that, when it is projected onto either A or B, the original Poisson brackets are recovered [146].

Example 1.16 (Symmetric coalgebra)**.** The *symmetric algebra* $S(\mathfrak{g})$ of a (finite dimensional) Lie algebra $\mathfrak{g}$ is the smallest commutative algebra containing $\mathfrak{g}$. The second tensor power $\mathfrak{g} \otimes \mathfrak{g}$ of the Lie algebra is the space of real valued bilinear maps on the dual space. Recursively, the k^{th} tensor power $\mathfrak{g}^{\otimes k}$ is the space of real valued k-linear maps. Taking the direct sum of the tensor powers of all orders, we arrive at the tensor algebra $\mathcal{T}(\mathfrak{g})$ of $\mathfrak{g}$. Here, the multiplication is

$$\mathcal{T}(\mathfrak{g}) \times \mathcal{T}(\mathfrak{g}) \longrightarrow \mathcal{T}(\mathfrak{g}), \qquad (v, u) \mapsto v \otimes u. \tag{1.27}$$

We consider a basis $\{x_1, \ldots, x_r\}$ of the Lie algebra $\mathfrak{g}$. The space generated by the elements

$$x_i \otimes x_j - x_j \otimes x_i \tag{1.28}$$

is an ideal, denoted by $\mathcal{R}$. The quotient space $\mathcal{T}(\mathfrak{g})/\mathcal{R}$ is called a symmetric algebra and denoted by $S(\mathfrak{g})$. The elements of $S(\mathfrak{g})$ can be regarded as polynomial functions on $\mathfrak{g}^*$. Therefore, this space can be endowed with an appropriate Poisson bracket that makes $S(\mathfrak{g})$ into a Poisson algebra. It can be shown that a coalgebra structure can always be defined on $S(\mathfrak{g})$ introducing the comultiplication

$$\Delta : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g}), \qquad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \forall x \in \mathfrak{g} \subset S(\mathfrak{g}), \tag{1.29}$$

which is a Poisson algebra homomorphism. This makes $S(\mathfrak{g})$ into a Poisson-Hopf algebra. Furthermore, in the light of the coassociatity condition

$$\Delta^{(3)} := (\Delta \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \Delta) \circ \Delta, \tag{1.30}$$

we can define the third-order coproduct

$$\Delta^{(3)} : S(\mathfrak{g}) \to S(\mathfrak{g}) \otimes S(\mathfrak{g}) \otimes S(\mathfrak{g}), \qquad \Delta^{(3)}(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x \tag{1.31}$$

for all $x \in \mathfrak{g}$, where $\mathfrak{g}$ is understood as a subset of $S(\mathfrak{g})$. The mth-order coproduct map can be defined, recursively, as

$$\Delta^{(m)} : S(\mathfrak{g}) \to S^{(m)}(\mathfrak{g}), \qquad \Delta^{(m)} := (\overbrace{\mathrm{Id} \otimes \ldots \otimes \mathrm{Id}}^{(m-2)-\text{times}} \otimes \Delta^{(2)}) \circ \Delta^{(m-1)}, \qquad m \geq 3, \tag{1.32}$$

which, clearly, is also a Poisson algebra homomorphism.

Definition 1.17. *Let (A, m, Δ) be a Hopf algebra and $(A, \{\cdot,\cdot\})$ a Poisson structure for A, then the triple $(A, m, \Delta, \{\cdot,\cdot\})$ is a Poisson–Hopf algebra if the coproduct Δ is a homomorphism of Poisson algebras, i.e.*

$$\Delta(\{x,y\}_A) = \{\Delta(x), \Delta(y)\}_{A\otimes A},$$

for all $x, y \in A$. If the antipode map does not exist, the triple is a Poisson bialgebra [51; 60; 146].

2 Poisson Manifolds and Quantum Groups

2.1 Symplectic Geometry: Poisson Manifolds

2.1.1 Poisson bivectors and symplectic forms

If M is a smooth manifold such that its ring of functions is a Poisson algebra $(\mathcal{C}^\infty(M), \{\cdot,\cdot\})$, then M is called a *Poisson manifold* [148]. Clearly, if $\{\cdot,\cdot\}$ is a Poisson bracket, then $\{f,\cdot\}$ defines a vector field $\mathbf{X}_f \in \mathfrak{X}(M)$ for all $f \in \mathcal{C}^\infty(M)$. The vector field $\mathbf{X}_f$ is well-defined and satisfies the commutator

$$\{f,g\} = \mathbf{X}_f g = -\mathbf{X}_g f = dg(\mathbf{X}_f) = -df(\mathbf{X}_g).$$

Such a vector field $\mathbf{X}_f$ is called a *Hamiltonian vector field*. In these conditions, it can be easily verified that a Poisson bivector is a skew-bilinear form on $T^*(M)$.

In particular, if $\mathbf{X}_f, \mathbf{X}_g \in \mathfrak{X}(M)$ are Hamiltonian vector fields, they satisfy the identity

$$[\mathbf{X}_f, \mathbf{X}_g](h) = \{f,\{g,h\}\} - \{g,\{f,h\}\} = \{\{f,g\},h\} = \mathbf{X}_{\{f,g\}}(h).$$

In order words, for all functions $f,g \in \mathcal{C}^\infty(M)$, the Hamiltonian vector fields satisfy the commutator condition

$$[\mathbf{X}_f, \mathbf{X}_g] = \mathbf{X}_{\{f,g\}}. \tag{2.1}$$

Lemma 2.1. *If $(M,\{\cdot,\cdot\})$ is a Poisson manifold, then*

$$\mathcal{L}_{\mathbf{X}_f}\Lambda = 0, \tag{2.2}$$

for all $f \in \mathcal{C}^\infty(M)$.

Whenever (M,ω) is a symplectic manifold, the preceding equation (2.2) turns out to adopt the particular

$$\mathcal{L}_{\mathbf{X}_f}\omega = 0,$$

for all $f \in \mathcal{C}^\infty(M)$.

Remark. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇ and Λ be a bivector on M. If there exists a tensor field T^i_{jk} on M such that $T^i_{jk} = T^i_{kj}$ and

$$\nabla_k \Lambda^{ij} + T^i_{hk}\Lambda^{hj} + T^j_{hk}\Lambda^{ih} = 0,$$

then the bivector Λ is a Poisson bivector, i.e. Λ defines a Poisson structure on the Riemannian manifold.

If (M,ω) is a symplectic manifold, then ω is a closed non-degenerate 2-form. This determines on M a Poisson structure through the prescription

$$\{f,g\} := \omega(\mathbf{X}_f, \mathbf{X}_g),$$

where $\mathbf{X}_f$ and $\mathbf{X}_g$ are defined by (1.9).

2.1.2 General distributions and Poisson cohomology

Definition 2.2. *Let (M, Λ) be a Poisson manifold. If the set $\mathcal{D}_p(M)$ is defined as*

$$\mathcal{D}_p(M) := \{v \in T_p(M) / \exists f \in \mathcal{C}^\infty(M), \mathbf{X}_f(p) = v\},$$

then the set of linear subspaces $\mathcal{D}(M) = \{\mathcal{D}_p(M)\}$ is called a general distribution. Moreover, it is called the characteristic distribution of the Poisson structure whenever for all $p \in M$ there are vector fields $\mathbf{X}_{f_1}, \ldots, \mathbf{X}_{f_k} \in \mathcal{D}(M)$ such that for each point in M they span the subspace $\mathcal{D}_p(M)$.

Example 2.3. Let $\{f_1, f_2, f_3\}$ be the set of Hamiltonian functions

$$f_1 = x, \quad f_2 = x^2 + zy, \quad f_3 = 1,$$

and $\omega = dx \wedge dy$ the canonical symplectic form in the plane. Then the Hamiltonian vector fields are given by

$$\mathbf{X}_{f_1} = -\frac{\partial}{\partial y}, \quad \mathbf{X}_{f_2} = -2x\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}, \quad \mathbf{X}_{f_3} = 0.$$

In this case, the distribution $\mathcal{D} =< \mathbf{X}_{f_1}, \mathbf{X}_{f_2}, \mathbf{X}_{f_3} >$ has rank 2 if and only if $z \neq 0$, as

$$\det \begin{pmatrix} 0 & -1 \\ z & -2x \end{pmatrix} = z.$$

A point $(x, y, z) \in M$ such that $z \neq 0$ is called *regular*, otherwise it is called *singular*.

In this context, it is possible to establish a simple criterion that ensures the complete integrability of a distribution (the proof can be found e.g. in [28; 148]).

Definition 2.4. *Let $\mathcal{D}(M)$ be a general distribution on a Poisson manifold, M. The distribution is invariant if there are vector fields on M such that for all $p \in M$, these vector fields span $\mathcal{D}_p$ at p, and such that for all $t \in \mathbb{R}$ and $p \in M$ the exponential map*

$$(\exp t\mathbf{X})_*(\mathcal{D}_p(M)) = \mathcal{D}_{\exp t\mathbf{X}(p)}(M).$$

is defined.

Theorem 2.5 (Stefan–Sussmann Frobenius theorem)**.** *A general distribution $\mathcal{D}(M)$ is completely integrable if and only if it is invariant.*

If the manifold M is additionally a Poisson manifold, the leaves of $\mathcal{D}(M)$ are called *symplectic leaves*, with $\mathcal{D}(M)$ being referred to as a *symplectic foliation* on M.

Theorem 2.6. *Let (M, Λ) be a Poisson manifold and $\mathcal{D}(M)$ its characteristic distribution. Then $\mathcal{D}(M)$ is completely integrable and the Poisson structure defines a symplectic structure on each leaf of the characteristic distribution.*

For completeness, we briefly recall some elementary notions about cohomology groups.

Definition 2.7. *Let (M, Λ) be a Poisson manifold, and δ_Λ the coboundary operator such that $\delta_\Lambda(X) = [\Lambda, X] \in \mathfrak{X}^{k+1}(M)$ for all $X \in \mathfrak{X}^k(M)$. Then, the cohomology groups are defined as*

$$H^k_\Lambda(M) := \frac{\ker(\delta_\Lambda : \mathfrak{X}^k(M)) \to \mathfrak{X}^{k+1}(M)}{\operatorname{im}(\delta_\Lambda : \mathfrak{X}^{k-1}(M)) \to \mathfrak{X}^k(M))}. \tag{2.3}$$

Remark. For low values of k, the cohomology groups have a concise geometrical interpretation:

k=0: If $\mathbf{X}_f \in \mathfrak{X}(M)$ is a Hamiltonian vector field, then $\delta_\Lambda(f) = \mathbf{X}_f$. Hence, $H^0_\Lambda(M)$ is the set of functions $f \in \mathcal{C}^\infty(M)$ such that $\{f, \cdot\} = 0$, i.e., the Casimir functions of the Poisson structure. Hence, $H^1_\Lambda(M)$ coincides with the center of $\mathcal{C}^\infty(M)$.

k=1: Let $\mathbf{X} \in \mathfrak{X}(M)$ be a vector field. If $\mathbf{X}$ is a infinitesimal automorphism of the Poisson structure, then $\mathcal{L}_\mathbf{X}\Lambda = \delta_\Lambda\mathbf{X} = 0$. Therefore, $H^1_\Lambda(M)$ is the quotient of the space of infinitesimal automorphisms of the Poisson manifold by the space of the Hamiltonian vector fields.

k=2: This group has the class of the element Λ defined by $[\Lambda, \Lambda] = 0$. If this class is 0, there is a vector field $\mathbf{X}$ such that $\delta_\Lambda(\mathbf{X}) = \mathcal{L}_\mathbf{X}\Lambda$. Poisson manifolds with this property are called exact Poisson manifolds.

If ω is a $k-$form on M, then the identity $\delta_\Lambda(\sharp(\omega)) = \sharp(d\omega)$ is satisfied[1]. It follows that the map $\sharp$ induces a homomorphism of $H^k_{dR}(M) \to H^k_\Lambda(M)$, where $H^k_{dR}(M)$ denotes the de Rham cohomology. If the bivector is non-degenerate, then the map is an isomorphism.

2.2 Poisson–Lie groups and Lie bialgebras

Definition 2.8. *Let G be a Lie group. If $(\mathcal{C}^\infty(G), \Delta, \{\cdot, \cdot\})$ is a Poisson–Hopf algebra, then G is a Poisson–Lie group.*

Let G be a Poisson–Lie group with its Poisson structure defined by the bivector Λ. For all $f \in \mathcal{C}^\infty(G)$ we have a vector field locally determined by

$$\mathbf{X}_f := \{f, \cdot\} = X^{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}.$$

As the right-invariant vector fields $\{\mathbf{R}_i\}$ are a basis for $T_g(G)$ for arbitrary elements $g \in G$, we conclude that there exist functions α^i such that $\mathbf{X}_f = \alpha^i\mathbf{R}_i$. Hence, all Poisson structures of a Poisson–Lie group can be written in local coordinates as

$$\Lambda = \lambda^{ij}\mathbf{R}_i \wedge \mathbf{R}_j,$$

where $\lambda^{ij} \in \mathcal{C}^\infty(G)$ and $i < j$.

Let $\mathfrak{g}$ be a Lie algebra of a Poisson–Lie group G. Then, the Lie algebra has a Lie bracket associated with the group structure. Moreover, there is a Lie structure in $\mathfrak{g}^*$ due to the linearization of the Poisson structure in G. The Lie structure in $\mathfrak{g}^*$ is well-defined as

$$[v_1, v_2] := (d\{f_1, f_2\})|_e \tag{2.4}$$

for all $f_1, f_2 \in \mathcal{C}^\infty(G)$, where $\{\cdot, \cdot\}$ is the Poisson bracket and $v_i := df_i|_e$. Hence, the cocommutator is defined by means of the following relation

$$\delta([X, Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \tag{2.5}$$

for all $X, Y \in \mathfrak{g}$, such that if $\Lambda_R : G \to \mathfrak{g} \otimes \mathfrak{g}$ is the right-translation of the Poisson bivector on G, then $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is the tangent map $\delta := T_e\Lambda_R$. Thus, δ^* defines a Lie bracket on $\mathfrak{g}^*$, such that $\delta^*(v \otimes v') = [v, v']$, and $v, v' \in \mathfrak{g}^*$. In this context, the Lie algebra $(\mathfrak{g}, \delta)$ is called a *Lie bialgebra*.

If there is an element $\rho \in \mathfrak{g} \otimes \mathfrak{g}$, such that $\delta(\rho) = [1 \otimes X + X \otimes 1, \rho]$, then the Lie bialgebra $(\mathfrak{g}, \delta, \rho)$ is called a *coboundary Lie bialgebra*. The element ρ is the *classical r-matrix* of $\mathfrak{g}$.

[1] The map $\sharp : T^*(M) \to T(M)$ is a homomorphism, such that for all $\beta, \alpha \in T^*(M)$ is defined by $\beta(\sharp(\alpha)) = \Lambda(\alpha, \beta)$. If the bivector Λ is non-degenerate then the map $\sharp$ is an isomorphism.

Let $(\mathfrak{g}, \delta, \rho)$ be a coboundary Lie bialgebra. If $\{X_1, \ldots, X_n\}$ is a basis of $\mathfrak{g}$ then the classical r-matrix turns out to be $\rho = \rho^{ij} X_i \otimes X_j$. By virtue of the elements

$$\rho_+ := \frac{1}{2}(\rho^{i,j} + \rho^{j,i}) X_i \otimes X_j, \;\; \rho_{12} := \rho^{ij} X_i \otimes X_j \otimes 1, \tag{2.6}$$

$$\rho_{13} := \rho^{ij} X_i \otimes 1 \otimes X_j, \;\; \rho_{23} := \rho^{ij} 1 \otimes X_i \otimes X_j, \tag{2.7}$$

$(\mathfrak{g}, \delta, \rho)$ is a Lie bialgebra if, and only if, the following identities are satisfied:

$$(ad_{\mathfrak{g}} \otimes ad_{\mathfrak{g}})(\rho_+) = 0, \tag{2.8}$$

and *the modified classical Yang-Baxter equation* holds

$$(ad_{\mathfrak{g}} \otimes ad_{\mathfrak{g}} \otimes ad_{\mathfrak{g}})[[\rho, \rho]] = 0, \tag{2.9}$$

where $[[\cdot, \cdot]]$ is the Schouten-–Nijenhuis bracket

$$[[\rho, \rho]] := [\rho_{12}, \rho_{13}] + [\rho_{12}, \rho_{23}] + [\rho_{13}, \rho_{23}].$$

The first condition guarantees the antisymmetry of δ^*, while the last equation ensures that the Jacobi identity in $\mathfrak{g}^*$ is satisfied.

Definition 2.9. *Two Lie bialgebras, $(\mathfrak{g}, \delta_1)$ and $(\mathfrak{g}, \delta_2)$ are equivalent if there is a Lie automorphism Θ of $\mathfrak{g}$ such that*

$$\delta_2 = (\Theta^{-1} \otimes \Theta^{-1}) \circ \delta_1 \circ \Theta,$$

that is, the next diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\delta_1} & \mathfrak{g} \otimes \mathfrak{g} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta\otimes\Theta} \\ \mathfrak{g} & \xrightarrow[\delta_2]{} & \mathfrak{g} \otimes \mathfrak{g} \end{array}$$

is commutative.

Remark. (see [146])

- If $\mathfrak{g}$ is a semisimple Lie algebra, then all its Lie bialgebras are coboundary Lie bialgebras.
- If $\mathfrak{g}$ is an abelian Lie algebra, then there is a non-coboundary Lie bialgebra $(\mathfrak{g}, \delta)$, and if the Lie algebra $\mathfrak{g}$ is not abelian, then there is a non-trivial coboundary Lie bialgebra $(\mathfrak{g}, \delta)$.
- In general, the Lie bialgebra $(\mathfrak{g}, \delta)$ can be a coboundary Lie bialgebra, while its dual bialgebra, $(\mathfrak{g}^*, \delta^*)$ is not a coboundary Lie bialgebra.
- Let $\mathfrak{g}$ be a Lie algebra, the commutation relations

$$[X_i, X_j] = c_{ij}^k X_k, \quad [v_i, v_j] = \lambda_{ij}^k v_k, \quad [v_i, X_j] = c_{jk}^i v_k - \lambda_{ik}^j X_k,$$

  span a Lie algebra over $\mathfrak{g} \otimes \mathfrak{g}^*$ called a *Manin–Lie algebra*, $\mathfrak{g} \bowtie \mathfrak{g}^*$.

2.2.1 Quantum deformations and quantum algebras

In this section, we shall consider the algebra $A = C^\infty(M)$ with M being a Poisson manifold. Albeit quantum deformations can be defined in a rather general context, we restrict ourselves to the syudy on Poisson manifolds. For the general case, see e.g. [7; 51; 60; 148].

Definition 2.10. *The algebra A_z is a z-parametric deformation of the algebra A, if A_z is a formal series algebra $A[[z]]$ such that the quotient A/kA_z is isomorphic to A.*

Furthermore, in terms of the product, A_z is a *quantization* of the Poisson algebra $(A, m, \{\cdot,\cdot\})$ if there is an associative $*_z$-product such that it is a deformation of m given by

$$f *_z g := fg + \frac{1}{z}\{f,g\} + o(z^2),$$

$f *_z a = a *_z f$ for all $a \in \mathbb{C}$ and $f, g \in A$. If there is a homomorphism of Poisson algebras Φ such that

$$\Phi(f) *_z \Phi(g) = \Phi(f *_z g),$$

then the $*_z$-product is invariant. It follows that the limit

$$\{f,g\} = \lim_{z\to 0} \frac{1}{z}(f *_z g - g *_z f)$$

recovers the classic bracket, implying the relation

$$[f,g] := f *_z g - g *_z f = z\{f,g\} + o(z^2). \tag{2.10}$$

Let (A, m, Δ) be a Hopf algebra and $(A, \{\cdot,\cdot\})$ be a Poisson structure for A. Then $(A_z \Delta_z)$ is a quantum deformation of $(A, m, \Delta, \{\cdot,\cdot\})$ if there is a coproduct Δ_z such that

$$\Delta_z(f *_z g) = \Delta_z(f) *_z \Delta_z(g), \tag{2.11}$$

for all $f, g \in A$.

Remark. Let ω be a bilinear form on the vector space $\mathcal{C}^\infty(M)$ of a Poisson Manifold M and let z be a parameter. Hence, a deformation of ω is a formal power series. If ω determines a Poisson bracket, a deformation of the Poisson–Lie bracket will be denoted as $\omega_z(f,g) = \{f,g\}_z$.

Definition 2.11. *A Hopf algebra (A_z, m_z, Δ_z) is a quantization of a co-Poisson–Hopf algebra (A, m, Δ, δ) if A_z is a deformation of a Hopf algebra (A, m, Δ) and there is a $*_z$-co-product defined by*

$$\Delta_z(X) = \Delta(X) + \frac{z}{2}\delta(X) + o(z^2) \tag{2.12}$$

compatible with a product m_z.

Remark. If $\mathfrak{g}$ is a Lie bialgebra over $\mathbb{R}$ or $\mathbb{C}$, then admits a quantization.

Within the Hopf algebras, quantum groups represent a specially interesting class that have shown to be of capital importance in Geometry [51; 105]. For example, as algebraic groups are well described by its Hopf algebra of functions, the deformed version of the latter Hopf algebra describes a quantized version of the algebraic group, which generally does not correspond anymore to an algebraic group.

Definition 2.12. *Let $\mathfrak{g}$ be a Lie algebra, $(\mathcal{U}_z(\mathfrak{g}), m_z, \Delta_z)$ is a quantum algebra if it is a quantization of a co-Poisson–Hopf algebra $(\mathcal{U}(\mathfrak{g}), m, \Delta_0, \delta)$.*

Example 2.13. Let us consider $\mathfrak{sl}(2)$ with the standard basis $\{J_3, J_+, J_-\}$ satisfying the commutation relations

$$[J_3, J_\pm] = \pm 2 J_\pm, \qquad [J_+, J_-] = J_3.$$

In this basis, the Casimir operator reads

$$C = \frac{1}{2} J_3^2 + (J_+ J_- + J_- J_+). \tag{2.13}$$

Considering the non-standard (triangular or Jordanian) quantum deformation $\mathcal{U}_z(\mathfrak{sl}(2))$ of $\mathfrak{sl}(2)$ [116] (see also [17] and references therein), we are led to the following deformed coproduct

$$\begin{aligned} \Delta_z(J_+) &= J_+ \otimes 1 + 1 \otimes J_+, \\ \Delta_z(J_l) &= J_l \otimes \mathrm{e}^{2zJ_+} + \mathrm{e}^{-2zJ_+} \otimes J_l, \qquad l \in \{-, 3\} \end{aligned}$$

and the commutation rules

$$\begin{aligned} [J_3, J_+]_z &= 2\,\mathrm{shc}(2zJ_+) J_+, \qquad [J_+, J_-]_z = J_3, \\ [J_3, J_-]_z &= -J_-\,\mathrm{ch}(2zJ_+) - \mathrm{ch}(2zJ_+) J_-. \end{aligned}$$

Here shc denotes the cardinal hyperbolic sinus function defined by

$$\mathrm{sinhc}(\xi) := \begin{cases} \dfrac{\sinh(\xi)}{\xi}, & \text{for } \xi \neq 0, \\ 1, & \text{for } \xi = 0. \end{cases}$$

It is known that every quantum algebra $\mathcal{U}_z(\mathfrak{g})$ related to a semi-simple Lie algebra $\mathfrak{g}$ admits an isomorphism of algebras $\mathcal{U}_z(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ (see [51, Theorem 6.1.8]). This allows us to obtain a Casimir operator of $U_z(\mathfrak{sl}(2))$ from C in (2.13) as (see [51] for details)

$$C_z = \frac{1}{2} J_3^2 + \mathrm{shc}(2zJ_+) J_+ J_- + J_- J_+ \,\mathrm{shc}(2zJ_+) + \frac{1}{2}\,\mathrm{ch}^2(2zJ_+),$$

As expected, this coincides with the expression formerly given in [17].

If G is a Lie group with Lie algebra $\mathfrak{g}$, then we know that there is a duality between the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and $\mathcal{C}^\infty(G)$ [146]. The function $\rho : \mathfrak{g} \to End\mathcal{C}^\infty(G)$ defined by

$$(\rho(X)\phi)(g) := \frac{d}{dt}\phi(exp^{tX} g) \mid_{t=0},$$

where $X \in \mathfrak{g}$, $\phi \in \mathcal{C}^\infty(G)$ and $g \in G$, can be extended to a homomorphism of $\mathcal{U}(\mathfrak{g})$ into $End_{\mathbb{C}}\mathcal{C}^\infty(G)$. There is an right invariant action $R : G \to \mathcal{C}^\infty(G)$ such that $R_g(\phi)(h) = \phi(gh)$, and such that the Leibniz rule is fulfilled. Then, it is possible to define a bilinear map $\langle \cdot, \cdot \rangle$ of $\mathcal{C}^\infty(G) \otimes \mathcal{U}(\mathfrak{g})$ onto $\mathbb{C}$

$$\langle \phi, x \rangle = (p(x)\phi)(e), \tag{2.14}$$

where e is the neutral element of G. Hence, the map defined by

$$\mathcal{C}^\infty(G) \longrightarrow \mathcal{U}(\mathfrak{g})^* \tag{2.15}$$

$$\phi \longmapsto \langle \phi, \cdot \rangle \tag{2.16}$$

is an immersion, according to the Gelfand–-Naimark theorem [69]. We conclude that the ring of functions $\mathcal{C}^\infty(G)$ can be considered as a dual of $\mathcal{U}(\mathfrak{g})$ [2].

[2] It is not true that both Hopf algebras, $\mathcal{C}^\infty(G)$ and $\mathcal{U}(\mathfrak{g})$, are duals of each other, due to some problems that arise in infinite dimensional Lie algebras [2; 51].

Remark (Quantum group). It ought to be observed that currently there is no universal and unified definition of quantum groups [7; 51], albeit all definitions explicitly refer to the Hopf algebra structure. Alternative approaches are e.g. given by the following:

- Approach in Noncommutative Geometry: as a deformation of algebraic groups. Here matrix groups are subjected to satisfy certain algebraic identities.
- The Faddeev theory [65]: it uses solutions of the quantum Yang–Baxter equation (YBE). This approach is the preferred one in Quantum Field Theory.

In the following we will use the notion of quantum group as introduced by Drinfel'd in [60]: A quantum group is a non-commutative Hopf algebra (deformation of the universal enveloping algebra of a Lie algebra) that gives rise to a Poisson–Hopf algebra $(\mathcal{C}^\infty(G), \{\cdot,\cdot\}, \Delta)$.

3 Poisson–Hopf Algebra Deformations of Lie Systems

3.1 Formalism

For the sake of simplicity we consider explicit computations merely on $\mathbb{R}^2$, but we stress that this approach can be applied, mutatis mutandis, to construct Poisson–Hopf algebra deformations of Lie–Hamilton systems defined on any manifold.

3.1.1 Lie–Hamilton systems

Let us consider the local coordinates $\{x_1, \ldots, x_n\}$ on an n-dimensional manifold M. Geometrically, every non-autonomous system of first-order differential equations on M of the form

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(t, x_1, \ldots, x_n), \qquad i = 1, \ldots, n, \tag{3.1}$$

where $f_i : \mathbb{R}^{n+1} \to \mathbb{R}$ are arbitrary functions, amounts to a t-dependent vector field $\mathbf{X} : \mathbb{R} \times M \to \mathrm{T}M$ given by

$$\mathbf{X}_t : \mathbb{R} \times M \ni (t, x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} f_i(t, x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \in \mathrm{T}M. \tag{3.2}$$

This justifies to represent (3.2) and its related system of differential equations (3.1) by $\mathbf{X}_t$ (cf. [47]).

According to the *Lie–Scheffers Theorem* [43; 44; 101], see the theorem 1.2, a system $\mathbf{X}$ is a Lie system if, and only if,

$$\mathbf{X}_t(x_1, \ldots, x_n) := \mathbf{X}(t, x_1, \ldots, x_n) = \sum_{i=1}^{\ell} b_i(t) \mathbf{X}_i(t, x_1, \ldots, x_n), \tag{3.3}$$

for some t-dependent functions $b_1(t), \ldots, b_\ell(t)$ and vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_\ell$ on M that span an ℓ-dimensional real Lie algebra V of vector fields, i.e. the Vessiot–Guldberg Lie algebra of $\mathbf{X}$.

A Lie system $\mathbf{X}$ is, furthermore, a LH one [9; 15; 29; 47; 49; 81] if it admits a Vessiot–Guldberg Lie algebra V of Hamiltonian vector fields relative to a Poisson structure. This amounts to the existence, around each generic point of M, of a symplectic form, ω, such that:

$$\mathcal{L}_{\mathbf{X}_i} \omega = 0, \tag{3.4}$$

for a basis $\mathbf{X}_1, \ldots, \mathbf{X}_\ell$ of V (cf. Lemma 4.1 in [9]). To avoid minor technical details and to highlight our main ideas, hereafter it will be assumed, unless otherwise stated, that the symplectic form and remaining structures are defined globally. More accurately, a local description around a generic point in M could easily be carried out.

Each vector field $\mathbf{X}_i$ admits a Hamiltonian function h_i given by the rule:

$$\iota_{\mathbf{X}_i} \omega = \mathrm{d}h_i, \tag{3.5}$$

where $\iota_{\mathbf{X}_i} \omega$ stands for the contraction of the vector field $\mathbf{X}_i$ with the symplectic form ω. Since ω is non-degenerate, every function h induces a unique associated Hamiltonian vector field $\mathbf{X}_h$ (Chapter 1).

3.1.2 Poisson–Hopf algebras

The core in what follows is the fact that the space $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ can be endowed with a *Poisson–Hopf algebra* structure. We recall that an associative algebra A with a *product* m and a *unit* η is said to be a *Hopf algebra* over $\mathbb{R}$ [2; 51; 105] if there exist two homomorphisms called *coproduct* $(\Delta : A \longrightarrow A \otimes A)$ and *counit* $(\epsilon : A \longrightarrow \mathbb{R})$, along with an antihomomorphism, the *antipode* $\gamma : A \longrightarrow A$, such that the following diagram (1.25) is commutative, section 1.2.

If A is a commutative Poisson algebra and Δ is a Poisson algebra morphism, then $(A, m, \eta, \Delta, \epsilon, \gamma)$ is a *Poisson–Hopf algebra* over $\mathbb{R}$. We recall that the Poisson bracket on $A \otimes A$ reads

$$\{a \otimes b, c \otimes d\}_{A\otimes A} = \{a, c\} \otimes bd + ac \otimes \{b, d\}, \qquad \forall a, b, c, d \in A.$$

In our particular case, $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ becomes a Hopf algebra relative to its natural associative algebra with unit provided that

$$\Delta(f)(x_1, x_2) := f(x_1 + x_2), \qquad m(h \otimes g)(x) := h(x)g(x),$$
$$\epsilon(f) := f(0), \qquad \eta(1)(x) := 1, \qquad \gamma(f)(x) := f(-x),$$

for every $x, x_1, x_2 \in \mathcal{H}_\omega$ and $f, g, h \in \mathcal{C}^\infty(\mathcal{H}_\omega^*)$. Therefore, the space $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ becomes a Poisson–Hopf algebra by endowing it with the Poisson structure defined by the Kirillov–Kostant–Souriau bracket related to a Lie algebra structure on $\mathcal{H}_\omega$.

3.1.3 Deformations of Lie–Hamilton systems and generalized distributions

In this section we propose a systematic procedure to obtain deformations of LH systems by using LH algebras and deformed Poisson–Hopf algebras that lead to appropriate extensions of the theory of LH systems. Explicitly, the construction is based upon the following four steps:

1. Consider a LH system $\mathbf{X}$ (3.3) on $\mathbb{R}^{2n}$ with respect to a symplectic form ω and admitting a LH algebra $\mathcal{H}_\omega$ spanned by a basis of functions $h_1, \ldots, h_\ell \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ with structure constants c_{ij}^k, i.e.
$$\{h_i, h_j\}_\omega = \sum_{k=1}^{\ell} c_{ij}^k h_k, \qquad i, j = 1, \ldots, \ell.$$

2. Introduce a Poisson–Hopf algebra deformation of $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ with deformation parameter $z \in \mathbb{R}$ (in a quantum group setting we would have $q := \mathrm{e}^z$) as the space of smooth functions $F(h_{z,1}, \ldots, h_{z,\ell})$ with fundamental Poisson bracket given by
$$\{h_{z,i}, h_{z,j}\}_\omega = F_{z,ij}(h_{z,1}, \ldots, h_{z,\ell}), \tag{3.6}$$
where $F_{z,ij}$ are certain smooth functions also depending smoothly on the deformation parameter z and such that
$$\lim_{z\to 0} h_{z,i} = h_i, \qquad \lim_{z\to 0} \nabla h_{z,i} = \nabla h_i, \qquad \lim_{z\to 0} F_{z,ij}(h_{z,1}, \ldots, h_{z,\ell}) = \sum_{k=1}^{\ell} c_{ij}^k h_k, \tag{3.7}$$
where ∇ stands for the gradient relative to the Euclidean metric on $\mathbb{R}^{2n}$. Hence,
$$\lim_{z\to 0} \{h_{z,i}, h_{z,j}\}_\omega = \{h_i, h_j\}_\omega. \tag{3.8}$$

3. Define the deformed vector fields $\mathbf{X}_{z,i}$ by the rule
$$\iota_{\mathbf{X}_{z,i}}\omega := \mathrm{d}h_{z,i}, \tag{3.9}$$
so that
$$\lim_{z\to 0} \mathbf{X}_{z,i} = \mathbf{X}_i. \tag{3.10}$$

4. Define the deformed LH system of the initial system $\mathbf{X}$ (3.3) by

$$\mathbf{X}_z := \sum_{i=1}^{\ell} b_i(t)\mathbf{X}_{z,i}. \tag{3.11}$$

Now some remarks are in order. First, note that for a given LH algebra $\mathcal{H}_\omega$ there exist as many Poisson–Hopf algebra deformations as non-equivalent Lie bialgebra structures δ on $\mathcal{H}_\omega$ [51], where the 1-cocycle δ essentially provides the first-order deformation in z of the coproduct map Δ. For three-dimensional real Lie algebras the full classification of Lie bialgebra structures is known, and some classification results are also known for certain higher-dimensional Lie algebras (see [25; 11] and references therein). Once a specific Lie bialgebra $(\mathcal{H}_\omega, \delta)$ is chosen, the full Poisson–Hopf algebra deformation can be systematically obtained by making use of the Poisson version of the quantum duality principle for Hopf algebras, as we will explicitly see in the next section for an $(\mathfrak{sl}(2), \delta)$ Lie bialgebra.

Second, the deformed vector fields $\mathbf{X}_{z,i}$ (3.9) will not, in general, span a finite-dimensional Lie algebra, which implies that (3.11) is not a Lie system. In fact, the sequence of Lie algebra morphisms (1.11) and the properties of Hamiltonian vector fields [148] lead to

$$[\mathbf{X}_{z,i}, \mathbf{X}_{z,j}] = [\varphi(h_{z,i}), \varphi(h_{z,j})] = \varphi(\{h_{z,i}, h_{z,j}\}_\omega) = \varphi(F_{z,ij}(h_{z,1}, \dots, h_{z,l})) = -\sum_{k=1}^{\ell} \frac{\partial F_{z,ij}}{\partial h_{z,k}} \mathbf{X}_{z,k}.$$

In other words,

$$\left[\mathbf{X}_{z,i}, \mathbf{X}_{z,j}\right] = \sum_{k=1}^{\ell} G^k_{z,ij}(x,y)\mathbf{X}_{z,k}, \tag{3.12}$$

where the $G^k_{z,ij}(x,y)$ are smooth functions relative to the coordinates x, y and the deformation parameter z. Despite this, the relations (3.8) and the continuity of φ imply that

$$[\mathbf{X}_i, \mathbf{X}_j] = \varphi(\{h_i, h_j\})_\omega = \varphi\left(\lim_{z\to 0}\{h_{z,i}, h_{z,j}\}_\omega\right) = \lim_{z\to 0} \varphi\{h_{z,i}, h_{z,j}\}_\omega = \lim_{z\to 0}[\mathbf{X}_{z,i}, \mathbf{X}_{z,j}].$$

Hence

$$\lim_{z\to 0} G^k_{z,ij}(x,y) = \text{constant}$$

holds for all indices. Geometrically, the conditions (3.12) establish that the vector fields $\mathbf{X}_{z,i}$ span an involutive smooth generalized distribution $\mathcal{D}_z$. In particular, the distribution $\mathcal{D}_0$ is spanned by the Vessiot–Guldberg Lie algebra $\langle \mathbf{X}_1, \dots, \mathbf{X}_l \rangle$. This causes $\mathcal{D}_0$ to be integrable on the whole $\mathbb{R}^{2n}$ in the sense of Stefan–Sussman [148; 120; 152]. The integrability of $\mathcal{D}_z$, for $z \neq 0$, can only be ensured on open connected subsets of $\mathbb{R}^{2n}$ where $\mathcal{D}_z$ has constant rank [148].

Third, although the vector fields $\mathbf{X}_{z,i}$ depend smoothly on z, the distribution $\mathcal{D}_z$ may change abruptly. For instance, consider the case given by the LH system $\mathbf{X} = \partial_x + ty\partial_x$ relative to the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and admitting a LH algebra $\mathcal{H}_\omega = \langle h_1 := y,\ h_2 := y^2/2 \rangle$. Let us define $h_{z,1} := y$ and $h_{z,2} := y^2/2 + zx$. Then $\mathbf{X}_z = \partial_x + t(y\partial_x - z\partial_y)$ and $\dim \mathcal{D}_0(x,y) = 1$, but $\dim \mathcal{D}_z(x,y) = 2$ for $z \neq 0$. Hence, the deformation of LH systems may change in an abrupt way the dynamical and geometrical properties of the systems $\mathbf{X}_z$ (cycles, periodic solutions, etc).

Fourth, the deformation parameter z provides an additional degree of freedom that enables the control or modification of the deformed system $\mathbf{X}_z$. In fact, as z can be taken small, perturbations of the initial Lie system $\mathbf{X}$ can be obtained from the deformed one $\mathbf{X}_z$ in a natural way.

And, finally, we stress that, by construction, the very same procedure can be applied to other $2n-$dimensional manifolds different to $\mathbb{R}^{2n}$, to higher dimensions as well as to multiparameter Poisson–Hopf algebra deformations of Lie algebras endowed with two or more deformation parameters.

Remark. The coalgebra method employed in [15] to obtain superposition rules and constants of motion for LH systems on a manifold M relies almost uniquely in the Poisson–Hopf algebra structure related to $\mathcal{C}^\infty(\mathfrak{g}^*)$ and a Poisson map

$$D : \mathcal{C}^\infty(\mathfrak{g}^*) \to \mathcal{C}^\infty(M),$$

where we recall that $\mathfrak{g}$ is a Lie algebra isomorphic to a LH algebra, $\mathcal{H}_\omega$, of the LH system.

Relevantly, quantum deformations allow us to repeat this scheme by substituting the Poisson algebra $\mathcal{C}^\infty(\mathfrak{g}^*)$ with a quantum deformation $\mathcal{C}^\infty(\mathfrak{g}_z^*)$, where $z \in \mathbb{R}$, and obtaining an adequate Poisson map

$$D_z : \mathcal{C}^\infty(\mathfrak{g}_z^*) \to \mathcal{C}^\infty(M).$$

The above procedure enables us to deform the LH system into a z-parametric family of Hamiltonian systems whose dynamics is determined by a Steffan–Sussmann distribution and a family of Poisson algebras. If z tends to zero, then the properties of the (classical) LH system are recovered by a limiting process, hence enabling to construct new deformations exhibiting physically relevant properties.

3.1.4 Constants of the motion

The fact that we are handling Poisson–Hopf algebra allows us to apply the coalgebra formalism established in [15] in order to obtain t-independent constants of the motion for $\mathbf{X}_z$.

Let $S(\mathcal{H}_\omega)$ be the *symmetric algebra* of $\mathcal{H}_\omega$, i.e. the associative unital algebra of polynomials on the elements of $\mathcal{H}_\omega$. The Lie algebra structure on $\mathcal{H}_\omega$ can be extended to a Poisson algebra structure in $S(\mathcal{H}_\omega)$ by requiring $[v, \cdot]$ to be a derivation on the second entry for every $v \in \mathcal{H}_\omega$. Then, $S(\mathcal{H}_\omega)$ can be endowed with a Hopf algebra structure with a non-deformed (trivial) coproduct map Δ defined by

$$\Delta : S(\mathcal{H}_\omega) \to S(\mathcal{H}_\omega) \otimes S(\mathcal{H}_\omega), \qquad \Delta(v_i) := v_i \otimes 1 + 1 \otimes v_i, \qquad i = 1, \ldots, \ell, \tag{3.13}$$

which is a Poisson algebra homomorphism relative to the Poisson structure on $S(\mathcal{H}_\omega)$ and the one induced in $S(\mathcal{H}_\omega) \otimes S(\mathcal{H}_\omega)$. Recall that every element of $S(\mathcal{H}_\omega)$ can be understood as a function on $\mathcal{H}_\omega^*$. Moreover, as $S(\mathcal{H}_\omega)$ is dense in the space $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ of smooth functions on the dual $\mathcal{H}_\omega^*$ of the LH algebra $\mathcal{H}_\omega$, the coproduct in $S(\mathcal{H}_\omega)$ can be extended in a unique way to

$$\Delta : \mathcal{C}^\infty(\mathcal{H}_\omega^*) \to \mathcal{C}^\infty(\mathcal{H}_\omega^*) \otimes \mathcal{C}^\infty(\mathcal{H}_\omega^*).$$

Similarly, all structures on $S(\mathcal{H}_\omega)$ can be extended turning $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ into a Poisson–Hopf algebra. Indeed, the resulting structure is the natural one in $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ given in section 2.2.

Let us assume now that $\mathcal{C}^\infty(\mathcal{H}_\omega^*)$ has a Casimir invariant

$$C = C(v_1, \ldots, v_\ell),$$

where $v_1, \ldots, v_\ell$ is a basis for $\mathcal{H}_\omega$. The initial LH system allows us to define a Lie algebra morphism $\phi : \mathcal{H}_\omega \to \mathcal{C}^\infty(M)$, where M is a submanifold of $\mathbb{R}^{2n}$ where all functions $h_i := \phi(v_i)$, for $i = 1, \ldots, \ell$, are well defined. Then, the Poisson algebra morphisms

$$D : \mathcal{C}^\infty(\mathcal{H}_\omega^*) \to \mathcal{C}^\infty(M), \qquad D^{(2)} : \mathcal{C}^\infty(\mathcal{H}_\omega^*) \otimes \mathcal{C}^\infty(\mathcal{H}_\omega^*) \to \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M), \tag{3.14}$$

defined respectively by

$$D(v_i) := h_i(x_1, y_1), \qquad D^{(2)}(\Delta(v_i)) := h_i(x_1, y_1) + h_i(x_2, y_2), \qquad i = 1, \ldots, \ell, \tag{3.15}$$

lead to the t-independent constants of motion $F^{(1)} := F$ and $F^{(2)}$ for the Lie system $\mathbf{X}$ given in (3.3) where

$$F := D(C), \qquad F^{(2)} := D^{(2)}(\Delta(C)). \tag{3.16}$$

The very same procedure can also be applied to any Poisson–Hopf algebra with deformed coproduct Δ_z and Casimir invariant $C_z = C_z(v_1, \ldots, v_\ell)$, where $\{v_1, \ldots, v_\ell\}$ fulfill the same Poisson brackets (3.6), and such that

$$\lim_{z\to 0} \Delta_z = \Delta, \qquad \lim_{z\to 0} C_z = C.$$

Following [15], the element C_z turns out to be the cornerstone in the construction of the deformed constants of the motion for the 'generalized' LH system $\mathbf{X}_z$.

Part II

Applications to the Theory of Quantum Poisson–Hopf Algebras

4 Poisson–Hopf Algebra Deformations of $\mathfrak{sl}(2)$-related Systems

4.1 Poisson–Hopf algebra deformations of $\mathfrak{sl}(2)$

Once the general description of our approach has been introduced, we present in this section the general properties of the Poisson analogue of the so-called non-standard quantum deformation of the simple real Lie algebra $\mathfrak{sl}(2)$. This deformation will be applied in the sequel to get deformations of the Milne–Pinney equation or Ermakov system and of some Riccati equations, since all these systems are known to be endowed with a LH algebra $\mathcal{H}_\omega$ isomorphic to $\mathfrak{sl}(2)$ [9; 15; 29].

Let us consider the basis $\{J_3, J_+, J_-\}$ for $\mathfrak{sl}(2)$ with Lie brackets and Casimir operator given by

$$[J_3, J_\pm] = \pm 2J_\pm, \qquad [J_+, J_-] = J_3, \qquad C = \tfrac{1}{2}J_3^2 + (J_+J_- + J_-J_+). \tag{4.1}$$

Amongst the three possible quantum deformations of $\mathfrak{sl}(2)$, we shall hereafter consider the non-standard (triangular or Jordanian) quantum deformation, $U_z(\mathfrak{sl}(2))$ (see [17; 116; 137] for further details). The Hopf algebra structure of $U_z(\mathfrak{sl}(2))$ has the following deformed coproduct and compatible deformed commutation rules

$$\Delta_z(J_+) = J_+ \otimes 1 + 1 \otimes J_+, \qquad \Delta_z(J_j) = J_j \otimes \mathrm{e}^{2zJ_+} + \mathrm{e}^{-2zJ_+} \otimes J_j, \qquad j \in \{-, 3\},$$

$$[J_3, J_+]_z = \frac{\sinh(2zJ_+)}{z}, \qquad [J_3, J_-]_z = -J_- \cosh(2zJ_+) - \cosh(2zJ_+)J_-, \qquad [J_+, J_-]_z = J_3.$$

The counit and antipode can be explicitly found in [17; 116], and the deformed Casimir reads [17]

$$C_z = \frac{1}{2}J_3^2 + \frac{\sinh(2zJ_+)}{2z}J_- + J_- \frac{\sinh(2zJ_+)}{2z} + \frac{1}{2}\cosh^2(2zJ_+).$$

Let $\mathfrak{g}$ be the Lie algebra of G. It is well known (see [51; 105]) that quantum algebras $\mathcal{U}_z(\mathfrak{g})$ are Hopf algebra duals of quantum groups G_z. On the other hand, quantum groups G_z are just quantizations of Poisson–Lie groups, which are Lie groups endowed with a multiplicative Poisson structure, i.e. a Poisson structure for which the Lie group multiplication is a Poisson map. In the case of $\mathcal{U}_z(\mathfrak{sl}(2))$, such Poisson structure on $SL(2)$ is explicitly given by the Sklyanin bracket coming from the classical r-matrix

$$r = zJ_3 \wedge J_+, \tag{4.2}$$

which is a solution of the (constant) classical Yang–Baxter equation.

Moreover, the 'quantum duality principle' [60; 136] states that quantum algebras can be thought of as 'quantum dual groups' G_z^*, which means that any quantum algebra can be obtained as the Hopf algebra quantization of the dual Poisson–Lie group G^*. The usefulness of this approach to construct explicitly the Poisson analogue of quantum algebras was developed in [25].

In the case of $\mathcal{U}_z(\mathfrak{sl}(2))$, the Lie algebra $\mathfrak{g}^*$ of the dual Lie group G^* is given by the dual of the cocommutator map δ that is obtained from the classical r-matrix as

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \qquad \forall x \in \mathfrak{g}. \tag{4.3}$$

In our case, from (4.1) and (4.2) we explicitly obtain

$$\delta(J_3) = 2z\, J_3 \wedge J_+, \qquad \delta(J_+) = 0, \qquad \delta(J_-) = 2z\, J_- \wedge J_+,$$

and the dual Lie algebra $\mathfrak{g}^*$ reads

$$[j^+, j^3] = -2z\, j^3, \qquad [j^+, j^-] = -2z\, j^-, \qquad [j^3, j^-] = 0, \tag{4.4}$$

where $\{j^3, j^+, j^-\}$ is the basis of $\mathfrak{g}^*$, and $\{J_3, J_+, J_-\}$ can now be interpreted as local coordinates on the dual Lie group G^*. The dual Lie algebra (4.4) is the so-called 'book' Lie algebra, and the complete set of its Poisson–Lie structures was explicitly obtained in [11] (see also [10], where book Poisson–Hopf algebras were used to construct integrable deformations of Lotka–Volterra systems). In particular, if we consider the coordinates on G^* given by

$$v_1 = J_+, \qquad v_2 = \tfrac{1}{2} J_3, \qquad v_3 = -J_-,$$

the Poisson–Lie structure on the book group whose Hopf algebra quantization gives rise to the quantum algebra $\mathcal{U}_z(\mathfrak{sl}(2))$ is given by the fundamental Poisson brackets [11]

$$\{v_1, v_2\}_z = -\operatorname{sinhc}(2zv_1) v_1, \qquad \{v_1, v_3\}_z = -2v_2, \qquad \{v_2, v_3\}_z = -\cosh(2zv_1) v_3, \tag{4.5}$$

together with the coproduct map

$$\Delta_z(v_1) = v_1 \otimes 1 + 1 \otimes v_1, \qquad \Delta_z(v_k) = v_k \otimes \mathrm{e}^{2zv_1} + \mathrm{e}^{-2zv_1} \otimes v_k, \qquad k = 2, 3, \tag{4.6}$$

which is nothing but the group law for the book Lie group G^* in the chosen coordinates (see [10; 11; 25] for a detailed explanation). Therefore, (4.5) and (4.6) define a Poisson–Hopf algebra structure on $C^\infty(G^*)$, which can be thought of as a Poisson–Hopf algebra deformation of the Poisson algebra $C^\infty(\mathfrak{sl}(2)^*)$, since we have identified the local coordinates on $C^\infty(G^*)$ with the generators of the Lie–Poisson algebra $\mathfrak{sl}(2)^*$.

Notice that we have introduced in (4.5) the hereafter called *cardinal hyperbolic sinus function* (see Appendix B) defined by[1]

$$\operatorname{sinhc}(x) := \frac{\sinh(x)}{x}. \tag{4.7}$$

Summarizing, the Poisson–Hopf algebra given by (4.5) and (4.6), together with its Casimir function

$$C_z = \operatorname{sinhc}(2zv_1)\, v_1 v_3 - v_2^2, \tag{4.8}$$

will be the deformed Poisson–Hopf algebra that we will use in the sequel in order to construct deformations of LH systems based on $\mathfrak{sl}(2)$. Note that the usual Poisson–Hopf algebra $C^\infty(\mathfrak{sl}(2)^*)$ is smoothly recovered under the $z \to 0$ limit leading to the non-deformed Lie–Poisson coalgebra

$$\{v_1, v_2\} = -v_1, \qquad \{v_1, v_3\} = -2v_2, \qquad \{v_2, v_3\} = -v_3, \tag{4.9}$$

with undeformed coproduct (3.13) and Casimir

$$C = v_1 v_3 - v_2^2. \tag{4.10}$$

We stress that this application of the 'quantum duality principle' would allow one to obtain the Poisson analogue of any quantum algebra $\mathcal{U}_z(\mathfrak{g})$, which by following the method here presented could be further applied in order to construct the corresponding deformation of the LH systems associated to the Lie–Poisson algebra $\mathfrak{g}$. In particular, the Poisson versions of the other quantum algebra deformations of $\mathfrak{sl}(2)$ can be obtained in the same manner with no technical obstructions (for instance, see [25] for the explicit construction of the 'standard' or Drinfel'd–Jimbo deformation).

[1] Some properties of this function along with its relationship with Lie systems are given in the Appendix A

Remark. Since $mathcal U_z(\mathfrak{sl}(2))$ is a Poisson algebra, one can define a Lie algebra representation $v \in \mathfrak{sl}(2) \mapsto [v, \cdot]_z \in \mathrm{End}(mathcal U_z(\mathfrak{sl}(2)))$, which makes $mathcal U_z(\mathfrak{sl}(2))$ into a $\mathfrak{sl}(2)$-space. Similarly, $mathcal C^\infty_z(\mathfrak{sl}(2)^*)$ is also a $\mathfrak{sl}(2)$-space relative to the Lie algebra representation induced by the Poisson structure on $mathcal C^\infty_z(\mathfrak{sl}(2)^*)$, i.e.

$$\rho_2 : v \in \mathfrak{sl}(2) \mapsto \{v, \cdot\}_{\mathfrak{sl}(2),z} \in \mathrm{End}(mathcal C^\infty_z(\mathfrak{sl}(2)^*)).$$

There exists a z-parametrized family of linear morphisms of the form

$$\phi_z : P \in mathcal U_z(\mathfrak{sl}(2)) \mapsto f_P \in mathcal C^\infty_z(\mathfrak{sl}(2)^*), \qquad \forall z \in \mathbb{R},$$

satisfying that $\phi_z([v, \cdot]) = \{v, \phi_z(\cdot)\}_{\mathfrak{sl}(2)^*,z}$ for every $v \in \mathfrak{sl}(2)$, i.e. ϕ_z is a morphism of $\mathfrak{sl}(2)$-spaces.

There is a canonical way of constructing ϕ_0 by setting $\phi_0([P])$ to be the unique symmetric polynomial in the equivalence class $[P]$. This construction is no longer available for $mathcal U_z(\mathfrak{sl}(2))$. To define ϕ_z is enough to use that every class of equivalence $[P]$ in $mathcal U_z(\mathfrak{sl}(2))$ has a unique decomposition as a linear combination of the elements of every *Poincaré–Birkhoff–Witt* basis for $mathcal U_z(\mathfrak{sl}(2))$. Then, ϕ_z is the linear morphism on $mathcal U_z(\mathfrak{sl}(2))$ that acts as the identity on the elements of the chosen Poincaré–Birkhoff–Witt basis.

To illustrate the above point, let us recall that $mathcal U_z(\mathfrak{sl}(2))$ can be defined as the algebra generated by the operators

$$J_-, \qquad J_3, \qquad K := e^{2zJ_+}, \qquad K^{-1} := e^{-2zJ_+}.$$

Then, a Poincaré–Birkhoff–Witt basis is given by the polynomials $J_-^m K^p J_3^l$, where $m, l \geq 0$ and $p \in \mathbb{Z}$. In other words, every element in $mathcal U_z(\mathfrak{sl}(2))$ admits a unique representation as a polynomial in this basis. This allows us to define a morphism of $\mathfrak{sl}(2)$-spaces:

$$\phi_z : P(J_-, K, J_3) \in mathcal U_z(\mathfrak{sl}(2)) \mapsto f_P(J_-, K, J_3) \in mathcal C^\infty_z(\mathfrak{sl}(2)^*), \qquad \forall z \in \mathbb{R}.$$

Hence, any Casimir element of the Poisson–Hopf algebra $mathcal U_z(\mathfrak{sl}(2))$ gives rise to a Casimir of $mathcal C^\infty_z(\mathfrak{sl}(2)^*)$. For instance, C_z (4.8) is the Poisson analog of $-\frac{1}{2}C_z$ (4.2).

4.2 Poisson–Hopf deformations of $\mathfrak{sl}(2)$ Lie–Hamilton systems

This section concerns the analysis of Poisson–Hopf deformations of LH systems on a manifold M with a Vessiot–Guldberg algebra isomorphic to $\mathfrak{sl}(2)$. Our geometric analysis will allow us to introduce the notion of a Poisson–Hopf Lie system that, roughly speaking, is a family of non-autonomous Hamiltonian systems of first-order differential equations constructed as a deformation of a LH system by means of the representation of the deformation of a Poisson–Hopf algebra in a Poisson manifold.

Let us endow a manifold M with a symplectic structure ω and consider a Hamiltonian Lie group action $\Phi : SL(2, \mathbb{R}) \times M \to M$. A basis of fundamental vector fields of Φ, let us say $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$, enable us to define a Lie system

$$\mathbf{X}_t = \sum_{i=1}^{3} b_i(t)\mathbf{X}_i,$$

for arbitrary t-dependent functions $b_1(t), b_2(t), b_3(t)$, and $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ spanning a Lie algebra isomorphic to $\mathfrak{sl}(2)$. As is well known, there are only three non-diffeomorphic classes of Lie algebras of Hamiltonian vector fields isomorphic to $\mathfrak{sl}(2)$ on the plane [9]. Since $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ admit Hamiltonian functions h_1, h_2, h_3, the t-dependent vector field $\mathbf{X}$ admits a t-dependent Hamiltonian function

$$h = \sum_{i=1}^{3} b_i(t)h_i.$$

Due to the cohomological properties of $\mathfrak{sl}(2)$ (see e.g. [148]), the Hamiltonian functions h_1, h_2, h_3 can always be chosen so that the space $\langle h_1, h_2, h_3\rangle$ spans a Lie algebra isomorphic to $\mathfrak{sl}(2)$ with respect to $\{\cdot,\cdot\}_\omega$.

Let $\{v_1, v_2, v_3\}$ be the basis for $\mathfrak{sl}(2)$ given in (4.4) and let M be a manifold where the functions h_1, h_2, h_3 are smooth. Further, the Poisson–Hopf algebra structure of $\mathcal{C}^\infty(\mathfrak{sl}^*(2))$ is given by (4.9). In these conditions, there exists a Poisson algebra morphism $D : \mathcal{C}^\infty(\mathfrak{sl}^*(2)) \to \mathcal{C}^\infty(M)$ satisfying

$$D(f(v_1, v_2, v_3)) = f(h_1, h_2, h_3), \quad \forall f \in \mathcal{C}^\infty(\mathfrak{sl}^*(2)).$$

Recall that the deformation $\mathcal{C}^\infty(\mathfrak{sl}^*_z(2))$ of $\mathcal{C}^\infty(\mathfrak{sl}^*(2))$ is a Poisson–Hopf algebra with the new Poisson structure induced by the relations (4.5). Let us define the submanifold $\mathcal{O} =: \{\theta \in \mathfrak{sl}^*(2) : v_1(\theta) \neq 0\}$ of $\mathfrak{sl}^*(2)$. Then, the Poisson structure on $\mathfrak{sl}^*(2)$ can be restricted to the space $\mathcal{C}^\infty(\mathcal{O})$. In turn, this enables us to expand the Poisson–Hopf algebra structure in $\mathcal{C}^\infty(\mathfrak{sl}^*(2))$ to $\mathcal{C}^\infty(\mathcal{O})$. Within the latter space, the elements

$$\begin{aligned} v_{z,1} &:= v_1, \qquad v_{z,2} := (2zv_1)v_2, \\ v_{z,3} &:= (2zv_1)\frac{v_2^2}{v_1} + \frac{c}{4(2zv_1)v_1}, \end{aligned} \tag{4.11}$$

are easily verified to satisfy the same commutation relations with respect to $\{\cdot,\cdot\}$ as the elements v_1, v_2, v_3 in $\mathcal{C}^\infty(\mathfrak{sl}^*_z(2))$ with respect to $\{\cdot,\cdot\}_z$ (4.5), i.e.

$$\begin{aligned} &\{v_{z,1}, v_{z,2}\} = -(2zv_{z,1})v_{z,1}, \qquad \{v_{z,1}, v_{z,3}\} = -2v_{z,2}, \\ &\{v_{z,2}, v_{z,3}\} = -\operatorname{ch}(2zv_{z,1})v_{z,3}. \end{aligned} \tag{4.12}$$

In particular, from (4.11) with $z = 0$ we find that

$$\{v_{0,1}, v_{0,2}\} = -v_{0,1}, \qquad \{v_{0,1}, v_{0,3}\} = \frac{2\{v_{0,1}, v_{0,2}\}v_{0,2}}{v_{0,1}} = -2v_{0,2},$$

$$\{v_{0,2}, v_{0,3}\} = -\frac{v_{0,2}^2}{v_{0,1}^2}\{v_{0,2}, v_{0,1}\} - \frac{c}{4v_{0,1}^2}\{v_{0,2}, v_{0,1}\} = -\frac{v_{0,2}^2}{v_{0,1}} - \frac{c}{4v_{0,1}} = -v_{0,3}.$$

The functions $v_{z,1}, v_{z,2}, v_{z,3}$ are not functionally independent, as they satisfy the constraint

$$(2zv_{1,z})v_{1,z}v_{3,z} - v_{2,z}^2 = c/4. \tag{4.13}$$

The existence of the functions $v_{z,1}, v_{z,2}, v_{z,3}$ and the relation (4.13) with the Casimir of the deformed Poisson–Hopf algebra is by no means casual. Let us explain why $v_{z,1}, v_{z,2}, v_{z,3}$ exist and how to obtain them easily.

Around a generic point $p \in \mathfrak{sl}^*(2)$, there always exists an open U_p containing p where both Poisson structures give a symplectic foliation by surfaces. Examples of symplectic leaves for $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_z$ are displayed in Fig. 4.1.

The splitting theorem on Poisson manifolds [148] ensure that if U_p is small enough, then there exist two different coordinate systems $\{x, y, C\}$ and $\{x_z, y_z, C_z\}$ where the Poisson bivectors related to $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_z$ read $\Lambda = \partial_x \wedge \partial_y$ and $\Lambda_z = \partial_{x_z} \wedge \partial_{y_z}$. Hence, C_z and C are Casimir functions for Λ_z and Λ, respectively. Moreover, $x_z = x_z(x, y, C), y_z = y_z(x, y, C), C_z = C_z(x, y, C)$. It follows from this that

$$\Phi : f(x_z, y_z, C_z) \in C_z^\infty(U_p) \mapsto f(x, y, C) \in C^\infty(U_p)$$

is a Poisson algebra morphism.

If $\{v_1, v_2, v_3\}$ are the standard coordinates on $\mathfrak{sl}^*(2)$ and the relations (4.5) are satisfied, then $v_i = \xi_i(x_z, y_z, C_z)$ holds for certain functions $\xi_1, \xi_2, \xi_3 : \mathbb{R}^3 \to \mathbb{R}$. Hence, the $\hat{v}_{z,i} = \xi_i(x, y, C)$ close the same commutation relations relative to $\{\cdot,\cdot\}$ as the v_i do with respect to $\{\cdot,\cdot\}_z$. As C is a Casimir invariant, the functions $v_{z,i} := \xi_i(x, y, c)$, with a constant value of c, still close the same commutation

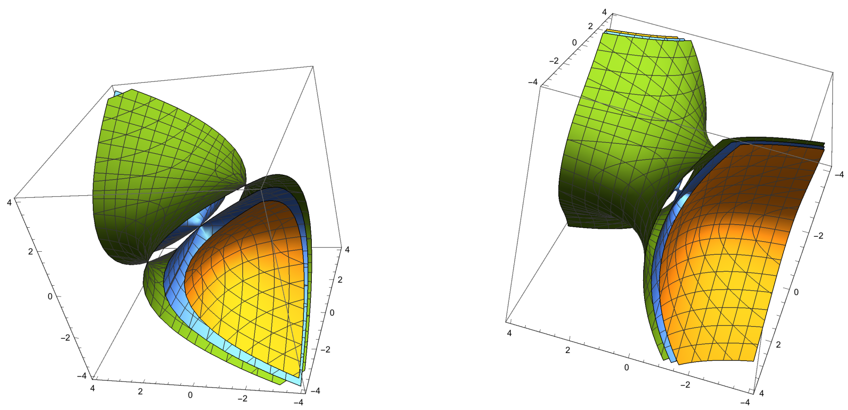


FIGURE 4.1: Representatives of the submanifolds in $\mathfrak{sl}^*(2)$ given by the surfaces with constant value of the Casimir for the Poisson structure in $\mathfrak{sl}^*(2)$ (left) and its deformation (right). Such submanifolds are symplectic submanifolds where the Poisson bivectors Λ and Λ_z admit a canonical form.

relations among themselves as the v_i. Moreover, the functions $v_{z,i}$ become functionally dependent. Indeed,

$$C_z = C_z(v_1, v_2, v_3) = C_z(\xi_1(x_z, y_z, C_z), \xi_2(x_z, y_z, C_z), \xi_3(x_z, y_z, C_z)).$$

Hence, $c = C_z(\xi_1(x_z, y_z, c), \xi_2(x_z, y_z, c), \xi_3(x_z, y_z, c))$ and we conclude that $c = C_z(v_{z,1}, v_{z,2}, v_{z,3})$.

The previous argument allows us to recover the functions (4.11) in an algorithmic way. Actually, the functions x_z, y_z, C_z and x, y, C can be easily chosen to be

$$x_z := v_1, \qquad y_z := -\frac{v_2}{(2zv_1)v_1}, \qquad C_z := (2zv_1)v_1v_3 - v_2^2,$$

as well as

$$x = v_1, \qquad y = -v_2/v_1, \qquad C = v_1v_3 - v_2^2.$$

Therefore,

$$\xi_1(x_z, y_z, C_z) = x_z, \qquad \xi_2(x_z, y_z, C_z) = -y_z(2zx_z)x_z,$$
$$\xi_3(x_z, y_z, C_z) = \frac{C_z + x_z^2 y_z^{22}(2zx_z)}{(2zx_z)x_z}.$$

Assuming that $C_z = c/4$, replacing x_z, y_z by $x = v_1, y = -v_2/v_1$, respectively, and taking into account that $v_{z,i} := \xi_i(x, y, c)$, one retrieves (4.11).

It is worth mentioning that due to the simple form of the Poisson bivectors in splitting form for three-dimensional Lie algebras, this method can be easily applied to such a type of Lie algebras.

Next, the above relations enable us to construct a Poisson algebra morphism

$$D_z : f(v_1, v_2, v_3) \in C^\infty(\mathfrak{sl}_z^*(2)) \mapsto D(f(v_{1,z}, v_{2,z}, v_{3,z})) \in C^\infty(M)$$

for every value of z allowing us to pass the structure of the Poisson–Hopf algebra $C^\infty(\mathfrak{sl}_z^*(2))$ to $C^\infty(M)$. As a consequence, $D_z(C_z)$ satisfies the relations

$$\{D_z(C_z), h_{z,i}\}_\omega = 0, \qquad i = 1, 2, 3.$$

Using the symplectic structure on M and the functions $h_{z,i}$ written in terms of $\{h_1, h_2, h_3\}$, one can easily obtain the deformed vector fields $\mathbf{X}_{z,i}$ in terms of the vector fields $\mathbf{X}_i$. Finally, as $\mathbf{X}_{z,t} = \sum_{i=1}^3 b_i(t)\mathbf{X}_{z,i}$ holds, it is straightforward to verify that the brackets

$$\mathbf{X}_{z,i} D(C_z) = \{D(C_z), h_{z,i}\} = 0,$$

imply that the function $D(C_z)$ is a t-independent constant of the motion for each of the deformed LH system $\mathbf{X}_{z,t}$.

Consequently, deformations of LH-systems based on $\mathfrak{sl}(2)$ can be treated simultaneously, starting from their classical LH counterpart. The final result is summarized in the following statement.

Theorem 4.1. *If $\phi : \mathfrak{sl}(2) \to C^\infty(M)$ is a morphism of Lie algebras with respect to the Lie bracket in $\mathfrak{sl}(2)$ and a Poisson bracket in $C^\infty(M)$, then for each $z \in \mathbb{R}$ there exists a Poisson algebra morphism $D_z : C^\infty(\mathfrak{sl}_z^*(2)) \to C^\infty(M)$ such that for a basis $\{v_1, v_2, v_3\}$ satisfying the commutation relations (4.9) is given by*

$$\begin{aligned}&D_z(f(v_1,v_2,v_3))\\&= f\left(\phi(v_1), (2z\phi(v_1))\phi(v_2), (2z\phi(v_1))\frac{\phi^2(v_2)}{\phi(v_1)} + \frac{c}{4(2z\phi(v_1))\phi(v_1)}\right).\end{aligned}$$

Provided that $h_i := \phi(v_i)$, the deformed Hamiltonian functions $h_{z,i} := D_z(v_i)$ adopt the form

$$h_{z,1} = h_1, \qquad h_{z,2} = (2zh_1)h_2,$$
$$h_{z,3} = (2zh_1)\frac{h_2^2}{h_1} + \frac{c}{4(2zh_1)h_1},$$

which satisfy the commutation relations (4.12).

The Hamiltonian vector fields $\mathbf{X}_{z,i}$ associated with $h_{z,i}$ through (3.9) turn out to be

$$\mathbf{X}_{z,1} = \mathbf{X}_1, \qquad \mathbf{X}_{z,2} = \frac{h_2}{h_1}(\cosh(2zh_1) - (2zh_1))\mathbf{X}_1 + (2zh_1)\mathbf{X}_2,$$
$$\begin{aligned}\mathbf{X}_{z,3} = &\left[\frac{h_2^2}{h_1^2}(\mathrm{ch}(2zh_1) - 2\,\mathrm{shc}(2zh_1)) - \frac{c\,\mathrm{ch}(2zh_1)}{4h_1^2\mathrm{shc}^2(2zh_1)}\right]\mathbf{X}_1\\&+2\,\frac{h_2}{h_1}\,(2zh_1)\mathbf{X}_2,\end{aligned}$$

and satisfy the following commutation relations coming from

$$[\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = \mathrm{ch}(2zh_{z,1})\mathbf{X}_{z,1}, \qquad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = 2\,\mathbf{X}_{z,2},$$
$$[\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = \mathrm{ch}(2zh_{z,1})\mathbf{X}_{z,3} + 4z^2\mathrm{shc}^2(2zh_{z,1})h_{z,1}h_{z,3}\,\mathbf{X}_{z,1}.$$

As a consequence, the deformed Poisson–Hopf system can be generically described in terms of the Vessiot–Guldberg Lie algebra corresponding to the non-deformed LH system as follows:

$$\begin{aligned}\mathbf{X}_{z,t} = \sum_{i=1}^{3} b_i(t)\mathbf{X}_{z,i} = &\left[b_1(t) + b_2(t)\,\frac{h_2}{h_1}(\cosh(2zh_1) - (2zh_1))\right]\mathbf{X}_1\\&+b_3(t)\left[\frac{h_2^2}{h_1^2}(\mathrm{ch}(2zh_1) - 2\,\mathrm{shc}(2zh_1)) - \frac{c\,\mathrm{ch}(2zh_1)}{4h_1^2\mathrm{shc}^2(2zh_1)}\right]\mathbf{X}_1\\&+\mathrm{shc}(2zh_1)\left(b_2(t) + 2b_3(t)\frac{h_2}{h_1}\right)\mathbf{X}_2.\end{aligned}$$

This unified approach to nonequivalent deformations of LH systems possessing a common underlying Lie algebra suggests the following definition.

Definition 4.2. *Let $(C^\infty(M), \{\cdot,\cdot\})$ be a Poisson algebra. A Poisson–Hopf Lie system is pair consisting of a Poisson–Hopf algebra $C^\infty(\mathfrak{g}_z^*)$ and a z-parametrized family of Poisson algebra representations $D_z : C^\infty(\mathfrak{g}_z^*) \to C^\infty(M)$ with $z \in \mathbb{R}$.*

Next, constants of the motion for $\mathbf{X}_{z,t}$ can be deduced by applying the coalgebra approach introduced in [15] in the way briefly described in Section 3. In the deformed case, we consider the Poisson algebra morphisms

$$D_z : \mathcal{C}^\infty\left(\mathfrak{sl}_z^*(2)\right) \to \mathcal{C}^\infty(M),$$
$$D_z^{(2)} : \mathcal{C}^\infty\left(\mathfrak{sl}_z^*(2)\right) \otimes \mathcal{C}^\infty\left(\mathfrak{sl}_z^*(2)\right) \to \mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M),$$

which by taking into account the coproduct (4.6) are defined by

$$D_z(v_i) := h_{z,i}(\mathbf{x}_1) \equiv h_{z,i}^{(1)}, \quad i = 1,2,3,$$
$$D_z^{(2)}\left(\Delta_z(v_1)\right) = h_{z,1}(\mathbf{x}_1) + h_{z,1}(\mathbf{x}_2) \equiv h_{z,1}^{(2)},$$
$$D_z^{(2)}\left(\Delta_z(v_k)\right) = h_{z,k}(\mathbf{x}_1)\mathrm{e}^{2zh_{z,1}(\mathbf{x}_2)} + \mathrm{e}^{-2zh_{z,1}(\mathbf{x}_1)}h_{z,k}(\mathbf{x}_2) \equiv h_{z,k}^{(2)}, \quad k = 2,3,$$

where $\mathbf{x}_s$ $(s = 1,2)$ are global coordinates in M. We remark that, by construction, the functions $h_{z,i}^{(2)}$ satisfy the same Poisson brackets (4.12). Then t-independent constants of motion are given by (see (3.16))

$$F_z \equiv F_z^{(1)} := D_z(C_z), \qquad F_z^{(2)} := D_z^{(2)}\left(\Delta_z(C_z)\right),$$

where C_z is the deformed Casimir (4.8). Explicilty, they read

$$F_z = \operatorname{sinhc}\left(2zh_{z,1}^{(1)}\right) h_{z,1}^{(1)}h_{z,3}^{(1)} - \left(h_{z,2}^{(1)}\right)^2 = \frac{c}{4},$$
$$F_z^{(2)} = \operatorname{sinhc}\left(2zh_{z,1}^{(2)}\right) h_{z,1}^{(2)}h_{z,3}^{(2)} - \left(h_{z,2}^{(2)}\right)^2.$$

4.3 Deformations of $\mathfrak{sl}(2)$ Lie–Hamilton systems in $\mathbb{R}^2$

We now apply last theorem to the three classes of LH systems in the plane with a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2)$ according to the local classification performed in [9], which was based on the previous results [73]. Consider the manifold $M = \mathbb{R}^2$ and the coordinates $\mathbf{x} = (x, y)$. According to [9; 29], there are only thee classes of LH systems, denoted by P_2, I_4 and I_5 and they correspond to a positive, negative and zero value of the Casimir constant c, respectively. Recall that these are non-diffeomorphic, so that there does not exist any local t-independent change of variables mapping one into another.

Table 4.1 summarizes the three cases, covering vector fields, Hamiltonian functions, symplectic structure and t-independent constants of motion. The particular LH systems which are diffeormorphic within each class are also mentioned [29]. Notice that for all of them it is satisfy the following commutation relations for the vector fields and Hamiltonian functions (the latter with respect to corresponding ω):

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \qquad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3,$$
$$\{h_1, h_2\}_\omega = -h_1, \qquad \{h_1, h_3\}_\omega = -2h_2, \qquad \{h_2, h_3\}_\omega = -h_3.$$

By applying the theorem 4.1 with the results of Table 4.1 we obtain the corresponding deformations which are displayed in Table 4.2. It is straightforward to verify that the classical limit $z \to 0$ in Table 4.2 recovers the corresponding starting LH systems and related structures of Table 4.1, in agreement with the relations (3.7) and (3.10).

TABLE 4.1: The three classes of LH systems on the plane with underlying Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2)$. For each class, it is displayed, in this order, a basis of vector fields $\mathbf{X}_i$, Hamiltonian functions h_i, symplectic form ω, the constants of motion F and $F^{(2)}$ as well as the corresponding specific LH systems.

• Class P_2 with $c = 4 > 0$

$$\mathbf{X}_1 = \frac{\partial}{\partial x} \qquad \mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \mathbf{X}_3 = (x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}$$

$$h_1 = -\frac{1}{y} \qquad h_2 = -\frac{x}{y} \qquad h_3 = -\frac{x^2+y^2}{y} \qquad \omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}$$

$$F = 1 \qquad F^{(2)} = \frac{(x_1 - x_2)^2 + (y_1 + y_2)^2}{y_1 y_2}$$

– Complex Riccati equation
– Ermakov system, Milne–Pinney and Kummer–Schwarz equations with $c > 0$

• Class I_4 with $c = -1 < 0$

$$\mathbf{X}_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \qquad \mathbf{X}_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \qquad \mathbf{X}_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$$

$$h_1 = \frac{1}{x-y} \qquad h_2 = \frac{x+y}{2(x-y)} \qquad h_3 = \frac{xy}{x-y} \qquad \omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{(x-y)^2}$$

$$F = -\frac{1}{4} \qquad F^{(2)} = -\frac{(x_2 - y_1)(x_1 - y_2)}{(x_1 - y_1)(x_2 - y_2)}$$

– Split-complex Riccati equation
– Ermakov system, Milne–Pinney and Kummer–Schwarz equations with $c < 0$
– Coupled Riccati equations

• Class I_5 with $c = 0$

$$\mathbf{X}_1 = \frac{\partial}{\partial x} \qquad \mathbf{X}_2 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} \qquad \mathbf{X}_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$$

$$h_1 = -\frac{1}{2y^2} \qquad h_2 = -\frac{x}{2y^2} \qquad h_3 = -\frac{x^2}{2y^2} \qquad \omega = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^3}$$

$$F = 0 \qquad F^{(2)} = \frac{(x_1 - x_2)^2}{4y_1^2 y_2^2}$$

– Dual-Study Riccati equation
– Ermakov system, Milne–Pinney and Kummer–Schwarz equations with $c = 0$
– Harmonic oscillator
– Planar diffusion Riccati system

4.3.1 A method to construct Lie–Hamilton systems

Last chapter showed that deformations of a LH system with a fixed LH algebra $\mathcal{H}_\omega \simeq \mathfrak{g}$ can be obtained through a Poisson algebra $C^\infty(\mathfrak{g}^*)$, a given deformation and a certain Poisson morphism $D : C^\infty(\mathfrak{g}^*) \to C^\infty(M)$. This section presents a simple method to obtain D from an arbitrary $\mathfrak{g}^*$ onto a symplectic manifold $\mathbb{R}^{2n}$.

Theorem 4.3. *Let $\mathfrak{g}$ be a Lie algebra whose Kostant–Kirillov–Souriau Poisson bracket admits a symplectic foliation in $\mathfrak{g}^*$ with a $2n$-dimensional $\mathcal{S} \subset \mathfrak{g}^*$. Then, there exists a LH algebra on the plane given by*

$$\Phi : \mathfrak{g} \to C^\infty(\mathbb{R}^{2n})$$

relative to the canonical Poisson bracket on the plane.

Proof. The Lie algebra $\mathfrak{g}$ gives rise to a Poisson structure on $\mathfrak{g}^*$ through the Kostant–Kirillov–Souriau bracket $\{\cdot,\cdot\}$. This induces a symplectic foliation on $\mathfrak{g}^*$, whose leaves are symplectic manifolds relative to the restriction of the Poisson bracket. Such leaves are characterized by means of the Casimir functions of the Poisson bracket. By assumption, one of these leaves is $2n$-dimensional. In such a case, the Darboux Theorem warrants that the Poisson bracket on each leave is locally symplectomorphic to the Poisson bracket of the canonical symplectic form on $\mathbb{R}^{2n} \simeq T^*\mathbb{R}^n$. In particular, there exist some Darboux coordinates mapping the Poisson bracket on such a leaf into the canonical symplectic bracket on $T^*\mathbb{R}^n$. The corresponding change of variables into the canonical form in Darboux coordinates can be understood as a local diffeomorphism $h : \mathcal{S}_k \to \mathbb{R}^{2n}$ mapping the Poisson bracket Λ_k on the leaf $\mathcal{S}_k$ into the canonical Poisson bracket on $T^*\mathbb{R}^n$. Hence, h gives rise to a canonical Poisson algebra morphism $\phi_h : C^\infty(\mathcal{S}_k) \to C^\infty(T^*\mathbb{R}^n)$.

As usual, a basis $\{v_1, \ldots, v_r\}$ of $\mathfrak{g}$ can be considered as a coordinate system on $\mathfrak{g}^*$. In view of the definition of the Kostant–Kirillov–Souriau bracket, they span an r-dimensional Lie algebra. In fact, if $[v_i, v_j] = \sum_{k=1}^r c_{ij}^k v_k$ for certain constants c_{ij}^k, then $\{v_i, v_j\} = \sum_{k=1}^r c_{ij}^k v_k$. Since $\mathcal{S}_k$ is a symplectic submanifold, there is a local immersion $\iota : \mathcal{S}_k \hookrightarrow \mathfrak{g}^*$ which is a Poisson manifold morphism. In consequence,

$$\{\iota^* v_i, \iota^* v_j\} = \sum_{k=1}^r c_{ij}^k \iota^* v_k.$$

Hence, the functions $\iota^* v_i$ span a finite-dimensional Lie algebra of functions on $\mathcal{S}$. Since $\mathcal{S}$ is $2n$-dimensional, there exists a local diffeomorphism $\phi : \mathcal{S} \to \mathbb{R}^{2n}$ and

$$\Phi : v \in \mathfrak{g} \mapsto \phi \circ \iota^* v \in C^\infty\left(\mathbb{R}^{2n}\right)$$

is a Lie algebra morphism. □

Let us apply the above mechanism to explain the existence of three types of LH systems on the plane. We already know that the Lie algebra $\mathfrak{sl}(2)$ gives rise to a Poisson algebra in $C^\infty(\mathfrak{g}^*)$. In the standard basis v_1, v_2, v_3 with commutation relations (4.9), the Casimir is (4.10). It turns out that the symplectic leaves of this Casimir are of three types:

- A one-sheeted hyperboloid when $v_1 v_3 - v_2^2 = k < 0$.
- A conical hyperboloid surface when $v_1 v_3 - v_2^2 = 0$.
- A two-sheeted hyperboloid when $v_1 v_3 - v_2^2 = k > 0$.

In each of the three cases we have the Poisson bivector

$$\Lambda = -v_1 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} - 2v_2 \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_3} - v_3 \frac{\partial}{\partial v_2} \wedge \frac{\partial}{\partial v_3}.$$

Then, we have a changes of variables passing from the above form into Darboux coordinates

$$\bar{v}_1 = v_1, \qquad \bar{v}_2 = -v_2/v_1, \qquad C = v_1 v_3 - v_2^2.$$

Then,

$$v_1 = v_1, \qquad v_2 = -\bar{v}_1 \bar{v}_2, \qquad v_3 = (C + \bar{v}_1^2 \bar{v}_2^2)/\bar{v}_1.$$

On a symplectic leaf, the value of C is constant, say $C = c/4$, and the restrictions of the previous functions to the leaf read

$$\iota^* v_1 = v_1, \qquad \iota^* v_2 = -\bar{v}_1 \bar{v}_2, \qquad \iota^* v_3 = c/(4\bar{v}_1) + \bar{v}_1 \bar{v}_2^2.$$

This can be viewed as a mapping $\Phi : \mathfrak{sl}(2) \to C^\infty(\mathbb{R}^2)$ such that

$$\phi(v_1) = x, \qquad \phi(v_2) = -xy, \qquad \phi(v_3) = c/(4x) + xy^2,$$

which is obviously a Lie algebra morphism relative to the standard Poisson bracket in the plane. It is simple to proof that when c is positive, negative or zero, one obtains three different types of Lie algebras of functions and their associated vector fields span the Lie algebras P_2, I_4 and I_5 as enunciated in [9]. Observe that since $\phi(v_1)\phi(v_3) - \phi(v_2)^2 = c/4$, there exists no change of variables on $\mathbb{R}^2$ mapping one set of variables into another for different values of c. Hence, Theorem 4.3 ultimately explains the real origin of all the $\mathfrak{sl}(2)$-LH systems on the plane.

It is known that $\mathfrak{su}(2)$ admits a unique Casimir, up to a proportional constant, and the symplectic leaves induced in $\mathfrak{su}^*(2)$ are spheres. The application of the previous method originates a unique Lie algebra representation, which gives rise to the unique LH system on the plane related to $\mathfrak{so}(3)$. All the remaining LH systems on the plane can be generated in a similar fashion. The deformations of such Lie algebras will generate all the possible deformations of LH systems on the plane.

TABLE 4.2: Poisson–Hopf deformations of the three classes of $\mathfrak{sl}(2)$-LH systems written in Table 1. The symplectic form ω is the same given in Table 1 and $F \equiv F_z$.

• Class P_2 with $c = 4 > 0$

$$\mathbf{X}_{z,1} = \frac{\partial}{\partial x} \qquad \mathbf{X}_{z,2} = x\,\mathrm{ch}(2z/y)\frac{\partial}{\partial x} + y\,\mathrm{shc}(2z/y)\frac{\partial}{\partial y}$$

$$\mathbf{X}_{z,3} = \left(x^2 - \frac{y^2}{\mathrm{shc}^2(2z/y)}\right)\mathrm{ch}(2z/y)\frac{\partial}{\partial x} + 2xy\,\mathrm{shc}(2z/y)\frac{\partial}{\partial y}$$

$$\mathbf{h}_{z,1} = -\frac{1}{y} \qquad \mathbf{h}_{z,2} = -\frac{x}{y}\,\mathrm{shc}(2z/y) \qquad \mathbf{h}_{z,3} = -\frac{x^2\,\mathrm{shc}^2(2z/y) + y^2}{y\,\mathrm{shc}(2z/y)}$$

$$F_z^{(2)} = \frac{(x_1-x_2)^2}{y_1y_2}\,\mathrm{sinhc}(2z/y_1)\,\mathrm{sinhc}(2z/y_2)\,\mathrm{e}^{2z/y_1}\mathrm{e}^{-2z/y_2} + \frac{(y_1+y_2)^2}{y_1y_2}\,\frac{\mathrm{sinhc}^2(2z/y_1+2z/y_2)}{\mathrm{sinhc}(2z/y_1)\,\mathrm{sinhc}(2z/y_2)}\,\mathrm{e}^{2z/y_1}\mathrm{e}^{-2z/y_2}$$

• Class I_4 with $c = -1 < 0$

$$\mathbf{X}_{z,1} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

$$\mathbf{X}_{z,2} = \frac{x+y}{2}\,\mathrm{ch}\left(\frac{2z}{x-y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + \frac{x-y}{2}\,\mathrm{shc}\left(\frac{2z}{x-y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\mathbf{X}_{z,3} = \frac{1}{4}\,\mathrm{ch}\left(\frac{2z}{x-y}\right)\left[(x+y)^2 + (x-y)^2\mathrm{shc}^{-2}\left(\frac{2z}{x-y}\right)\right]\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) + \frac{1}{2}\,(x-y)^2\,\mathrm{shc}\left(\frac{2z}{x-y}\right)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\mathbf{h}_{z,1} = \frac{1}{x-y} \quad \mathbf{h}_{z,2} = \frac{(x+y)\,\mathrm{shc}\left(\frac{2z}{x-y}\right)}{2(x-y)} \quad \mathbf{h}_{z,3} = \frac{(x+y)^2\mathrm{shc}^2\left(\frac{2z}{x-y}\right) - (x-y)^2}{4(x-y)\,\mathrm{shc}\left(\frac{2z}{x-y}\right)}$$

$$F_z^{(2)} = \frac{(x_1-x_2+y_1-y_2)^2}{4(x_1-y_1)(x_2-y_2)}\,\mathrm{sinhc}\left(\frac{2z}{x_1-y_1}\right)\,\mathrm{sinhc}\left(\frac{2z}{x_2-y_2}\right)\,\mathrm{e}^{-\frac{2z}{x_1-y_1}}\,\mathrm{e}^{\frac{2z}{x_2-y_2}} - \frac{(x_1+x_2-y_1-y_2)\,\mathrm{sinhc}\left(\frac{2z}{x_1-y_1}+\frac{2z}{x_2-y_2}\right)}{4(x_1-y_1)(x_2-y_2)}\left[\frac{\mathrm{e}^{\frac{2z}{x_2-y_2}}\,(x_1-y_1)}{\mathrm{sinhc}\left(\frac{2z}{x_1-y_1}\right)} + \frac{\mathrm{e}^{-\frac{2z}{x_1-y_1}}\,(x_2-y_2)}{\mathrm{sinhc}\left(\frac{2z}{x_2-y_2}\right)}\right]$$

• Class I_5 with $c = 0$

$$\mathbf{X}_{z,1} = \frac{\partial}{\partial x} \qquad \mathbf{X}_{z,2} = x\,\mathrm{ch}\left(z/y^2\right)\frac{\partial}{\partial x} + \frac{y}{2}\,\mathrm{shc}\left(z/y^2\right)\frac{\partial}{\partial y}$$

$$\mathbf{X}_{z,3} = x^2\,\mathrm{ch}\left(z/y^2\right)\frac{\partial}{\partial x} + xy\,\mathrm{shc}\left(z/y^2\right)\frac{\partial}{\partial y}$$

$$\mathbf{h}_{z,1} = -\frac{1}{2y^2} \qquad \mathbf{h}_{z,2} = -\frac{x}{2y^2}\,\mathrm{shc}\left(z/y^2\right) \qquad \mathbf{h}_{z,3} = -\frac{x^2}{2y^2}\,\mathrm{shc}\left(z/y^2\right)$$

$$F_z^{(2)} = \frac{(x_1-x_2)^2}{4y_1^2y_2^2}\,\mathrm{sinhc}\left(z/y_1^2\right)\,\mathrm{sinhc}\left(z/y_2^2\right)\,\mathrm{e}^{z/y_1^2}\mathrm{e}^{-z/y_2^2}$$

5 Deformation of ODEs

5.1 Deformed Milne–Pinney equation and oscillator systems

In this section we construct the non-standard deformation of the well-known Milne–Pinney (MP) equation [111; 123], which is known to be a LH system [9; 29]. Recall that the MP equation corresponds to the equation of motion of the isotropic oscillator with a time-dependent frequency and a 'centrifugal' term. As we will show in the sequel, the main feature of this deformation is that the new oscillator system has both a position-dependent mass and a time-dependent frequency.

5.1.1 Non-deformed system

The MP equation [111; 123] has the following expression

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\Omega^2(t)x + \frac{c}{x^3}, \tag{5.1}$$

where $\Omega(t)$ is any t-dependent function and $c \in \mathbb{R}$. By introducing a new variable $y := \mathrm{d}x/\mathrm{d}t$, the system (5.1) becomes a first-order system of differential equations on $T\mathbb{R}_0$, where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\Omega^2(t)x + \frac{c}{x^3}. \tag{5.2}$$

This system is indeed part of the one-dimensional Ermakov system [47; 62; 96; 98] and diffeomorphic to the one-dimensional t-dependent frequency counterpart [9; 15; 29] of the Smorodinsky–Winternitz oscillator [68].

The system (5.2) determines a Lie system with associated t-dependent vector field [29]

$$\mathbf{X} = \mathbf{X}_3 + \Omega^2(t)\mathbf{X}_1, \tag{5.3}$$

where

$$\mathbf{X}_1 := -x\frac{\partial}{\partial y}, \qquad \mathbf{X}_2 := \frac{1}{2}\left(y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x}\right), \qquad \mathbf{X}_3 := y\frac{\partial}{\partial x} + \frac{c}{x^3}\frac{\partial}{\partial y}, \tag{5.4}$$

span a Vessiot–Guldberg Lie algebra V^{MP} of vector fields isomorphic to $\mathfrak{sl}(2)$ (for any value of c) with commutation relations given by

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_1, \qquad [\mathbf{X}_1, \mathbf{X}_3] = 2\mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = \mathbf{X}_3. \tag{5.5}$$

The vector fields of V^{MP} are defined on $\mathbb{R}^2_{x\neq 0}$, where they span a regular distribution of order two.

Furthermore, $\mathbf{X}$ is a LH system with respect to the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ and the vector fields (5.4) admit Hamiltonian functions given by

$$h_1 = \frac{1}{2}x^2, \qquad h_2 = -\frac{1}{2}xy, \qquad h_3 = \frac{1}{2}\left(y^2 + \frac{c}{x^2}\right), \tag{5.6}$$

that fulfill the following commutation relations with respect to the Poisson bracket induced by ω:

$$\{h_1, h_2\}_\omega = -h_1, \qquad \{h_1, h_3\}_\omega = -2h_2, \qquad \{h_2, h_3\}_\omega = -h_3. \tag{5.7}$$

Then, the functions h_1, h_2, h_3 span a LH algebra $\mathcal{H}_\omega^{\rm MP} \simeq \mathfrak{sl}(2)$ of functions on $\mathbb{R}^2_{x\neq 0}$; the t-dependent Hamiltonian associated with the t-dependent vector field (5.3) reads

$$h = h_3 + \Omega^2(t)h_1. \tag{5.8}$$

We recall that this Hamiltonian is a natural one, that is, it can be written in terms of a kinetic energy T and potential U by identifying the variable y as the conjugate momentum p of the coordinate x:

$$h = T + U = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2(t)x^2 + \frac{c}{2x^2}. \tag{5.9}$$

Hence h determines the composition of a one-dimensional oscillator with a time-dependent frequency $\Omega(t)$ and unit mass with a Rosochatius or Winternitz potential; the latter is just a centrifugal barrier whenever $c > 0$ (see [21] and references therein). The LH system (5.2) thus comes from the Hamilton equations of h and, obviously, when c vanishes, these reduce to the equations of motion of a harmonic oscillator with a time-dependent frequency.

We stress that it has been already proved in [9; 29] that the MP equations (5.2) comprise the *three* different types of possible $\mathfrak{sl}(2)$-LH systems according to the value of the constant c: class P_2 for $c > 0$; class I_4 for $c < 0$; and class I_5 for $c = 0$. This means that any other LH system related to a Vessiot–Guldberg Lie algebra of Hamiltonian vector fields isomorphic to $\mathfrak{sl}(2)$ must be, up to a t-independent change of variables, of the form (5.2) for a positive, zero or negative value of c.

This implies that the second-order Kummer–Schwarz equations [42; 102] and several types of Riccati equations [45; 61; 64; 106; 142; 143; 153] are comprised within $\mathcal{H}_\omega^{\rm MP}$ (depending on the sign of c). The relationships amongst all of these systems are ensured by construction and these can be explicitly obtained through either diffeomorphisms or changes of variables (see [9; 29] for details).

The constants of motion for the MP equations can be obtained by applying the coalgebra formalism introduced in [15] and briefly summarized in section 2.4. Explicitly, let us consider the Poisson–Hopf algebra $C^\infty(\mathcal{H}_\omega^{\rm MP*})$ with basis $\{v_1, v_2, v_3\}$, coproduct (3.13), fundamental Poisson brackets (4.9) and Casimir (4.10). The Poisson algebra morphisms (3.14)

$$D : C^\infty(\mathcal{H}_\omega^{\rm MP*}) \to C^\infty(\mathbb{R}^2_{x\neq 0}), \quad D^{(2)} : C^\infty(\mathcal{H}_\omega^{\rm MP*}) \otimes C^\infty(\mathcal{H}_\omega^{\rm MP*}) \to C^\infty(\mathbb{R}^2_{x\neq 0}) \otimes C^\infty(\mathbb{R}^2_{x\neq 0}),$$

defined by (3.15), where h_i are the Hamiltonian functions (5.6), lead to the t-independent constants of the motion $F^{(1)} := F$ and $F^{(2)}$ given by (3.16), through the Casimir (4.10), for the Lie system $\mathbf{X}$ (5.2); namely [15]

$$\begin{aligned} F &= h_1(x_1,y_1)h_3(x_1,y_1) - h_2^2(x_1,y_1) = \frac{c}{4}, \\ F^{(2)} &= \big([h_1(x_1,y_1) + h_1(x_2,y_2)]\,[h_3(x_1,y_1) + h_3(x_2,y_2)]\big) - \big(h_2(x_1,y_1) + h_2(x_2,y_2)\big)^2 \\ &= \frac{1}{4}(x_1y_2 - x_2y_1)^2 + \frac{c}{4}\frac{(x_1^2 + x_2^2)^2}{x_1^2x_2^2}. \end{aligned} \tag{5.10}$$

We observe that $F^{(2)}$ is just a Ray–Reid invariant for generalized Ermakov systems [96; 129] and that it is related to the one obtained in [8; 20] from a coalgebra approach applied to superintegrable systems.

By permutation of the indices corresponding to the variables of the non-trivial invariant $F^{(2)}$, we find two other constants of the motion:

$$F_{13}^{(2)} = S_{13}(F^{(2)}), \qquad F_{23}^{(2)} = S_{23}(F^{(2)}), \tag{5.11}$$

where S_{ij} is the permutation of variables $(x_i, y_i) \leftrightarrow (x_j, y_j)$. Since $\partial(F^{(2)}, F_{23}^{(2)})/\partial(x_1, y_1) \neq 0$, both constants of motion are functionally independent (note that the pair $(F^{(2)}, F_{13}^{(2)})$ is functionally independent as well). From these two invariants, the corresponding superposition rule can be derived in a straightforward manner. Its explicit expression can be found in [15].

5.1.2 Deformed Milne–Pinney equation

In order to apply the non-standard deformation of $\mathfrak{sl}(2)$ described in chapter 4 to the MP equation, we need to find the deformed counterpart $h_{z,i}$ $(i = 1, 2, 3)$ of the Hamiltonian functions h_i (5.6), so fulfilling the Poisson brackets (4.5), by keeping the canonical symplectic form ω.

This problem can be rephrased as the one consistent in finding symplectic realizations of a given Poisson algebra, which can be solved once a particular symplectic leave is fixed as a level set for the Casimir functions of the algebra, where the generators of the algebra can be expressed in terms of the corresponding Darboux coordinates. In the particular case of the $\mathcal{U}_z(\mathfrak{sl}(2))$ algebra, the explicit solution (modulo canonical transformations) was obtained in [18] where the algebra (4.5) was found to be generated by the functions

$$\begin{aligned} v_1(q,p) &= \frac{1}{2}\, q^2, \\ v_2(q,p) &= -\frac{1}{2}\frac{\sinh zq^2}{zq^2}\, qp, \\ v_3(q,p) &= \frac{1}{2}\frac{\sinh zq^2}{zq^2}\, p^2 + \frac{1}{2}\frac{zc}{\sinh zq^2}, \end{aligned}$$

where $\omega = \mathrm{d}q \wedge \mathrm{d}p$, and the Casimir function (4.8) reads $C_z = c/4$. In practical terms, such a solution can easily be found by solving firstly the non-deformed case $z \to 0$ and, afterwards, by deforming the $v_i(q,p)$ functions under the constraint that the Casimir C_z has to take a constant value. With this result at hand, the corresponding deformed vector fields $\mathbf{X}_{z,i}$ can be computed by imposing the relationship (3.9) and the final result is summarized in the following statement.

Proposition 5.1. *(i) The Hamiltonian functions defined by*

$$h_{z,1} := \frac{1}{2}x^2, \qquad h_{z,2} := -\frac{1}{2}\operatorname{sinhc}(zx^2)\, xy, \qquad h_{z,3} := \frac{1}{2}\left(\operatorname{sinhc}(zx^2)\, y^2 + \frac{1}{\operatorname{sinhc}(zx^2)}\frac{c}{x^2}\right), \tag{5.12}$$

close the Poisson brackets (4.5) with respect to the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ *on* $\mathbb{R}^2_{x\neq 0}$, *namely*

$$\begin{gathered} \{h_{z,1}, h_{z,2}\}_\omega = -\operatorname{sinhc}(2zh_{z,1})h_{z,1}, \qquad \{h_{z,1}, h_{z,3}\}_\omega = -2h_{z,2}, \\ \{h_{z,2}, h_{z,3}\}_\omega = -\cosh(2zh_{z,1})h_{z,3}, \end{gathered} \tag{5.13}$$

where $\operatorname{sinhc}(x)$ *is defined in (4.7). Relations (5.13) define the deformed Poisson algebra* $C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})$.

(ii) The vector fields $\mathbf{X}_{z,i}$ *corresponding to* $h_{z,i}$ *read*

$$\begin{aligned} \mathbf{X}_{z,1} &= -x\frac{\partial}{\partial y}, \qquad \mathbf{X}_{z,2} = \left(\cosh(zx^2) - \frac{1}{2}\operatorname{sinhc}(zx^2)\right) y\frac{\partial}{\partial y} - \frac{1}{2}\operatorname{sinhc}(zx^2)\, x\frac{\partial}{\partial x}, \\ \mathbf{X}_{z,3} &= \operatorname{sinhc}(zx^2)\, y\frac{\partial}{\partial x} + \left[\frac{c}{x^3}\frac{\cosh(zx^2)}{\operatorname{sinhc}^2(zx^2)} + \frac{\operatorname{sinhc}(zx^2) - \cosh(zx^2)}{x}\, y^2\right]\frac{\partial}{\partial y}, \end{aligned}$$

which satisfy

$$\begin{gathered} [\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = \cosh(zx^2)\, \mathbf{X}_{z,1}, \qquad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = 2\mathbf{X}_{z,2}, \\ [\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = \cosh(zx^2)\, \mathbf{X}_{z,3} + z^2\left(c + x^2y^2\, \operatorname{sinhc}^2(zx^2)\right) \mathbf{X}_{z,1}. \end{gathered} \tag{5.14}$$

Since $\lim_{z\to 0}\operatorname{sinhc}(zx^2)=1$ and $\lim_{z\to 0}\cosh(zx^2)=1$, it can directly be checked that all the classical limits (3.7), (3.8) and (3.10) are fulfilled. As expected, the Lie derivative of ω with respect to each $\mathbf{X}_{z,i}$ vanishes.

At this stage, it is important to realize that, albeit (5.13) are genuine Poisson brackets defining the Poisson algebra $C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})$, the commutators (5.14) show that $\mathbf{X}_{z,i}$ do not span a new Vessiot–Guldberg Lie algebra; in fact, the commutators give rise to linear combinations of the vector fields $\mathbf{X}_{z,i}$ with coefficients that are functions depending on the coordinates and the deformation parameter.

Consequently, proposition 5.1 leads to a deformation of the initial Lie system (5.3) and of the LH one (5.8) defined by

$$\mathbf{X}_z := \mathbf{X}_{z,3}+\Omega^2(t)\mathbf{X}_{z,1}, \qquad h_z := h_{z,3}+\Omega^2(t)h_{z,1}. \tag{5.15}$$

Thus we obtain the following z-parametric system of differential equations that generalizes (5.2):

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \operatorname{sinhc}(zx^2)\,y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\Omega^2(t)x+\frac{c}{x^3}\frac{\cosh(zx^2)}{\operatorname{sinhc}^2(zx^2)}+\frac{\operatorname{sinhc}(zx^2)-\cosh(zx^2)}{x}\,y^2.\end{aligned} \tag{5.16}$$

From the first equation, we can write

$$y=\frac{1}{\operatorname{sinhc}(zx^2)}\frac{\mathrm{d}x}{\mathrm{d}t},$$

and by substituting this expression into the second equation in (5.16), we obtain a deformation of the MP equation (5.1) in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\left(\frac{1}{x}-\frac{zx}{\tanh(zx^2)}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=-\Omega^2(t)\,x\operatorname{sinhc}(zx^2)+\frac{c\,z}{x\tanh(zx^2)}.$$

Note that this really is a deformation of the MP equation in the sense that the limit $z\to 0$ recovers the standard one (5.1).

5.1.3 Constants of motion for the deformed Milne–Pinney system

An essential feature of the formalism here presented is the fact that t-independent constants of motion for the deformed system $\mathbf{X}_z$ (5.15) can be deduced by using the coalgebra structure of $C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})$. Thus we start with the Poisson–Hopf algebra $C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})$ with deformed coproduct Δ_z given by (4.6) and, following section 2.4 [15], we consider the Poisson algebra morphisms

$$D_z : C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})\to C^\infty(\mathbb{R}^2_{x\neq 0}), \quad D_z^{(2)} : C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})\otimes C^\infty(\mathcal{H}^{\mathrm{MP}*}_{z,\omega})\to C^\infty(\mathbb{R}^2_{x\neq 0})\otimes C^\infty(\mathbb{R}^2_{x\neq 0}),$$

which are defined by

$$\begin{aligned}&D_z(v_i)=h_{z,i}(x_1,y_1):=h^{(1)}_{z,i}, \quad i=1,2,3,\\ &D_z^{(2)}\left(\Delta_z(v_1)\right)=h_{z,1}(x_1,y_1)+h_{z,1}(x_2,y_2):=h^{(2)}_{z,1},\\ &D_z^{(2)}\left(\Delta_z(v_j)\right)=h_{z,j}(x_1,y_1)\mathrm{e}^{2zh_{z,1}(x_2,y_2)}+\mathrm{e}^{-2zh_{z,1}(x_1,y_1)}h_{z,j}(x_2,y_2):=h^{(2)}_{z,j}, \quad j=2,3,\end{aligned}$$

where $h_{z,i}$ are the Hamiltonian functions (5.12), so fulfilling (5.13). Hence (see [18])

$$\begin{aligned}h^{(2)}_{z,1} &= \frac{1}{2}(x_1^2+x_2^2),\\ h^{(2)}_{z,2} &= -\frac{1}{2}\left(\operatorname{sinhc}(zx_1^2)\,x_1y_1\mathrm{e}^{zx_2^2}+\mathrm{e}^{-zx_1^2}\operatorname{sinhc}(zx_2^2)\,x_2y_2\right),\\ h^{(2)}_{z,3} &= \frac{1}{2}\left(\operatorname{sinhc}(zx_1^2)\,y_1^2+\frac{c}{x_1^2\operatorname{sinhc}(zx_1^2)}\right)\mathrm{e}^{zx_2^2}+\frac{1}{2}\,\mathrm{e}^{-zx_1^2}\left(\operatorname{sinhc}(zx_2^2)\,y_2^2+\frac{c}{x_2^2\operatorname{sinhc}(zx_2^2)}\right).\end{aligned}$$

Recall that, by construction, the functions $h_{z,i}^{(2)}$ fulfill the Poisson brackets (5.13). The t-independent constants of motion are then obtained through

$$F_z = D_z(C_z), \qquad F_z^{(2)} = D_z^{(2)}\left(\Delta_z(C_z)\right),$$

where C_z is the Casimir (4.8); these are

$$\begin{aligned} F_z &= \operatorname{sinhc}\left(2zh_{z,1}^{(1)}\right)h_{z,1}^{(1)}h_{z,3}^{(1)} - \left(h_{z,2}^{(1)}\right)^2 = \frac{c}{4}, \\ F_z^{(2)} &= \operatorname{sinhc}\left(2zh_{z,1}^{(2)}\right)h_{z,1}^{(2)}h_{z,3}^{(2)} - \left(h_{z,2}^{(2)}\right)^2 \\ &= \frac{1}{4}\left[\operatorname{sinhc}(zx_1^2)\operatorname{sinhc}(zx_2^2)\,(x_1y_2 - x_2y_1)^2 + c\,\frac{\operatorname{sinhc}^2\left(z(x_1^2+x_2^2)\right)}{\operatorname{sinhc}(zx_1^2)\operatorname{sinhc}(zx_2^2)}\,\frac{(x_1^2+x_2^2)^2}{x_1^2x_2^2}\right]\mathrm{e}^{-zx_1^2}\mathrm{e}^{zx_2^2}, \end{aligned} \tag{5.17}$$

so providing the corresponding deformed Ray–Reid invariant, being (5.10) its non-deformed counterpart with $z = 0$. Notice that this invariant is related to the so-called 'universal constant of the motion' coming from $\mathcal{U}_z(\mathfrak{sl}(2))$ and given in [20]. As in (5.11), other equivalent constants of motion can be deduced from $F_z^{(2)}$ by permutation of the variables.

5.1.4 A new oscillator system with position-dependent mass

If we set $p := y$, the t-dependent Hamiltonian h_z in (5.15) can be written, through (5.12), as:

$$h_z = T_z + U_z = \frac{1}{2}\operatorname{sinhc}(zx^2)\,p^2 + \frac{1}{2}\Omega^2(t)x^2 + \frac{c}{2x^2\operatorname{sinhc}(zx^2)},$$

so deforming h given in (5.9). The corresponding Hamilton equations are just (5.16).

It is worth mentioning that h_z can be interpreted naturally within the framework of position-dependent mass oscillators (see [16; 55; 56; 70; 113; 126; 127; 128] and references therein). The above Hamiltonian naturally suggests the definition of a position-dependent mass function in the form

$$m_z(x) := \frac{1}{\operatorname{sinhc}(zx^2)} = \frac{zx^2}{\sinh(zx^2)}, \qquad \lim_{z\to 0} m_z(x) = 1, \qquad \lim_{x\to\pm\infty} m_z(x) = 0. \tag{5.18}$$

Then h_z can be rewritten as

$$h_z = \frac{p^2}{2m_z(x)} + \frac{1}{2}m_z(x)\Omega^2(t)\left[x^2\operatorname{sinhc}(zx^2)\right] + \frac{c}{2m_z(x)}\left[\frac{1}{x^2\operatorname{sinhc}^2(zx^2)}\right].$$

Thus the Hamiltonian h_z can be regarded as a system corresponding to a particle with position-dependent mass $m_z(x)$ under a deformed oscillator potential $U_{z,\mathrm{osc}}(x)$ with time-dependent frequency $\Omega(t)$ and a deformed potential $U_{z,\mathrm{RW}}(x)$ given by

$$\begin{aligned} U_{z,\mathrm{osc}}(x) &:= x^2\operatorname{sinhc}(zx^2) = \frac{\sinh(zx^2)}{z}, \\ U_{z,\mathrm{RW}}(x) &:= \frac{1}{x^2\operatorname{sinhc}^2(zx^2)} = \left(\frac{zx}{\sinh(zx^2)}\right)^2, \end{aligned} \tag{5.19}$$

such that

$$\begin{aligned} &\lim_{z\to 0} U_{z,\mathrm{osc}}(x) = x^2, \qquad \lim_{x\to\pm\infty} U_{z,\mathrm{osc}}(x) = +\infty, \\ &\lim_{z\to 0} U_{z,\mathrm{RW}}(x) = \frac{1}{x^2}, \qquad \lim_{x\to\pm\infty} U_{z,\mathrm{RW}}(x) = 0. \end{aligned}$$

The deformed mass and the oscillator potential functions are represented in figures 5.1 and 5.2

The Hamilton equations (5.16) can easily be expressed in terms of $m_z(x)$ as

$$\dot{x} = \frac{\partial h_z^{\mathrm{MP}}}{\partial p} = \frac{p}{m_z(x)},$$
$$\dot{p} = -\frac{\partial h_z^{\mathrm{MP}}}{\partial x} = -m_z(x)\Omega^2(t)\, x \,\mathrm{sinhc}(zx^2) + \frac{c}{m_z(x)} \frac{\cosh(zx^2)}{x^3\,\mathrm{sinhc}^3(zx^2)} + p^2 \frac{m_z'(x)}{2m_z^2(x)},$$

and the constant of the motion (5.17) turns out to be

$$F_z^{(2)} = \frac{1}{4}\left[\frac{(x_1p_2 - x_2p_1)^2}{m_z(x_1)m_z(x_2)} + c\, m_z(x_1)m_z(x_2)\,\mathrm{sinhc}^2\big(z(x_1^2+x_2^2)\big)\,\frac{(x_1^2+x_2^2)^2}{x_1^2x_2^2}\right]\mathrm{e}^{-zx_1^2}\mathrm{e}^{zx_2^2}.$$

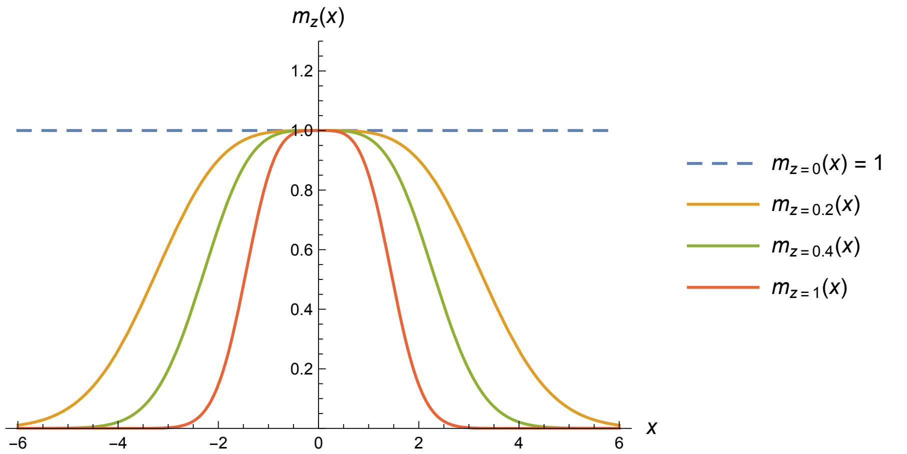


FIGURE 5.1: The position-dependent mass (5.18) for different values of the deformation parameter z.

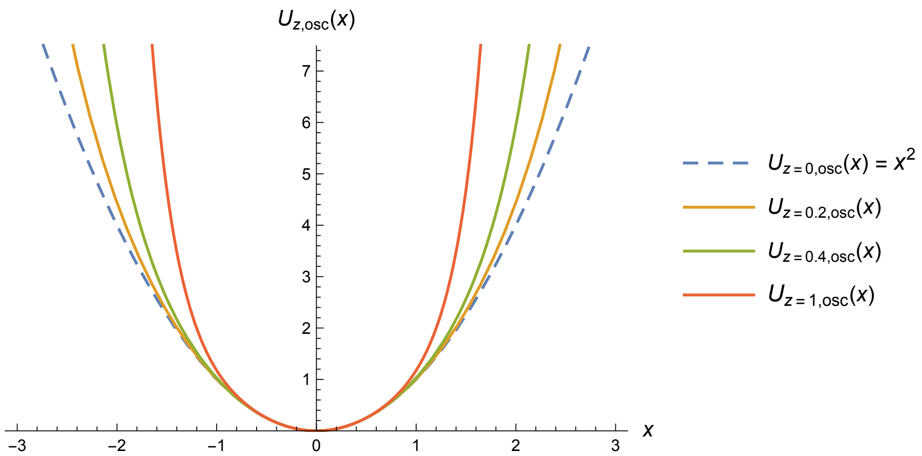


FIGURE 5.2: The deformed oscillator potential (5.19) for different values of the deformation parameter z.

5.2 Deformed Riccati equation

5.2.1 Deformed complex Riccati equation

In this section we consider the complex Riccati equation given by

$$\frac{\mathrm{d}z}{\mathrm{d}t} = b_1(t) + b_2(t)z + b_3(t)z^2, \qquad z \in \mathbb{C}, \tag{5.1}$$

where $b_i(t)$ are arbitrary t-dependent real coefficients. We recall that (5.1) is related to certain planar Riccati equations [61; 153] and that several mathematical and physical applications can be found in [118; 135].

By writing $z = u + iv$, we find that (5.1) gives rise to a system of the type (3.1), namely

$$\frac{\mathrm{d}u}{\mathrm{d}t} = b_1(t) + b_2(t)u + b_3(t)(u^2 - v^2), \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = b_2(t)v + 2b_3(t)uv. \tag{5.2}$$

Thus the associated t-dependent vector field reads

$$\mathbf{X} = b_1(t)\mathbf{X}_1 + b_2(t)\mathbf{X}_2 + b_3(t)\mathbf{X}_3, \tag{5.3}$$

where

$$\mathbf{X}_1 = \frac{\partial}{\partial u}, \qquad \mathbf{X}_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad \mathbf{X}_3 = (u^2 - v^2)\frac{\partial}{\partial u} + 2uv\frac{\partial}{\partial v}, \tag{5.4}$$

span a Vessiot–Guldberg Lie algebra $V^{\mathrm{CR}} \simeq \mathfrak{sl}(2)$ with the same commutation relations (5.5). It has already be proven that the system $\mathbf{X}$ is a LH one belonging to the class P_2 [9; 29] and that their vector fields span a regular distribution on $\mathbb{R}^2_{v\neq 0}$. The symplectic form, coming from (3.4), and the corresponding Hamiltonian functions (3.5) turn out to be

$$\omega = \frac{\mathrm{d}u \wedge \mathrm{d}v}{v^2}, \qquad h_1 = -\frac{1}{v}, \qquad h_2 = -\frac{u}{v}, \qquad h_3 = -\frac{u^2 + v^2}{v}, \tag{5.5}$$

which fulfill the commutation rules (5.7) so defining a LH algebra $\mathcal{H}^{\mathrm{CR}}_{\omega}$. A t-dependent Hamiltonian associated with $\mathbf{X}$ reads

$$h = b_1(t)h_1 + b_2(t)h_2 + b_3(t)h_3. \tag{5.6}$$

In this case, the constants of the motion (3.16) are found to be $F = 1$ and [29]

$$F^{(2)} = \frac{(u_1 - u_2)^2 + (v_1 + v_2)^2}{v_1 v_2}. \tag{5.7}$$

As commented above, the Riccati system (5.2) is locally diffeomorphic to the MP equations (5.2) with $c > 0$, both belonging to the same class P_2 [9]. Explicitly, the change of variables

$$x = \pm\frac{c^{1/4}}{\sqrt{|v|}}, \qquad y = \mp\frac{c^{1/4}\,u}{\sqrt{|v|}}, \qquad u = -\frac{y}{x}, \qquad |v| = \frac{c^{1/2}}{x^2}, \qquad c > 0, \tag{5.8}$$

map, in this order, the vector fields (5.4) on $\mathbb{R}^2_{x\neq 0}$, the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$, Hamiltonian functions (5.6) and the constant of motion (5.10) onto the vector fields (5.4) on $\mathbb{R}^2_{v\neq 0}$, (5.5) and (5.7) (up to a multiplicative constant $\pm\frac{1}{2}c^{1/2}$).

To obtain the corresponding (non-standard) deformation of the complex Riccati system (5.2), the very same change of variables (5.8) can be considered since, in our approach, the symplectic form (5.5) is kept non-deformed. Thus, by starting from proposition 5.1 and applying (5.8) (with $c = 4$ for simplicity), we get the following result.

Proposition 5.2. *(i) The Hamiltonian functions given by*

$$h_{z,1} = -\frac{1}{v}, \qquad h_{z,2} = -\operatorname{sinhc}(2z/v)\,\frac{u}{v}, \qquad h_{z,3} = -\frac{\operatorname{sinhc}^2(2z/v)\,u^2 + v^2}{\operatorname{sinhc}(2z/v)\,v},$$

fulfill the commutation rules (5.13) with respect to the Poisson bracket induced by the symplectic form ω (5.5) defining the deformed Poisson algebra $C^\infty(\mathcal{H}^{\mathrm{CR}*}_{z,\omega})$.

(ii) The corresponding vector fields $\mathbf{X}_{z,i}$ read

$$\mathbf{X}_{z,1} = \frac{\partial}{\partial u}, \qquad \mathbf{X}_{z,2} = u\cosh(2z/v)\frac{\partial}{\partial u} + v\operatorname{sinhc}(2z/v)\frac{\partial}{\partial v},$$

$$\mathbf{X}_{z,3} = \left(u^2 - \frac{v^2}{\operatorname{sinhc}^2(2z/v)}\right)\cosh(2z/v)\frac{\partial}{\partial u} + 2uv\operatorname{sinhc}(2z/v)\frac{\partial}{\partial v},$$

which satisfy

$$[\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = \cosh(2z/v)\,\mathbf{X}_{z,1}, \qquad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = 2\mathbf{X}_{z,2},$$
$$[\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = \cosh(2z/v)\,\mathbf{X}_{z,3} + 4z^2\left(1 + \frac{u^2}{v^2}\,\mathrm{sinhc}^2(2z/v)\right)\mathbf{X}_{z,1}.$$

Next the deformed counterpart of the Riccati Lie system (5.3) and of the LH one (5.6) is defined by

$$\mathbf{X}_z := b_1(t)\mathbf{X}_{z,1} + b_2(t)\mathbf{X}_{z,2} + b_3(t)\mathbf{X}_{z,3}, \qquad h_z := b_1(t)h_{z,1} + b_2(t)h_{z,2} + b_3(t)h_{z,3}. \tag{5.9}$$

And the t-independent constants of motion turn out to be $F_z = 1$ and

$$F_z^{(2)} = \left(\mathrm{sinhc}(2z/v_1)\,\mathrm{sinhc}(2z/v_2)\frac{(u_1-u_2)^2}{v_1v_2} + \frac{\mathrm{sinhc}^2(2z/v_1+2z/v_2)}{\mathrm{sinhc}(2z/v_1)\,\mathrm{sinhc}(2z/v_2)}\frac{(v_1+v_2)^2}{v_1v_2}\right)\mathrm{e}^{2z/v_1}\mathrm{e}^{-2z/v_2}.$$

Therefore the deformation of the system (5.2), defined by $\mathbf{X}_z$ (5.9), reads

$$\frac{\mathrm{d}u}{\mathrm{d}t} = b_1(t) + b_2(t)u\cosh(2z/v) + b_3(t)\left(u^2 - \frac{v^2}{\mathrm{sinhc}^2(2z/v)}\right)\cosh(2z/v),$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = b_2(t)v\,\mathrm{sinhc}(2z/v) + 2b_3(t)uv\,\mathrm{sinhc}(2z/v).$$

5.2.2 Deformed coupled Riccati equations

As a last application, let us consider two coupled Riccati equations given by [106]

$$\frac{\mathrm{d}u}{\mathrm{d}t} = a_0(t) + a_1(t)u + a_2(t)u^2, \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = a_0(t) + a_1(t)v + a_2(t)v^2, \tag{5.10}$$

constituting a particular case of the systems of Riccati equations studied in [15; 45].

Clearly, the system (5.10) is a Lie system associated with a t-dependent vector field

$$\mathbf{X} = a_0(t)\mathbf{X}_1 + a_1(t)\mathbf{X}_2 + a_2(t)\mathbf{X}_3, \tag{5.11}$$

where

$$\mathbf{X}_1 = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \qquad \mathbf{X}_2 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad \mathbf{X}_3 = u^2\frac{\partial}{\partial u} + v^2\frac{\partial}{\partial v}, \tag{5.12}$$

close on the commutation rules (5.5), so spanning a Vessiot–Guldberg Lie algebra $V^{2\mathrm{R}} \simeq \mathfrak{sl}(2)$. Furthermore, $\mathbf{X}$ is a LH system which belongs to the class I_4 [9; 29] restricted to $\mathbb{R}^2_{u\neq v}$. The symplectic form and Hamiltonian functions for $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ read

$$\omega = \frac{\mathrm{d}u\wedge\mathrm{d}v}{(u-v)^2}, \qquad h_1 = \frac{1}{u-v}, \qquad h_2 = \frac{1}{2}\left(\frac{u+v}{u-v}\right), \qquad h_3 = \frac{uv}{u-v}. \tag{5.13}$$

The functions h_1, h_2, h_3 satisfy the commutation rules (5.7), thus spanning a LH algebra $\mathcal{H}_\omega^{2\mathrm{R}}$. Hence, the t-dependent Hamiltonian associated with $\mathbf{X}$ is given by

$$h = a_0(t)h_1 + a_1(t)h_2 + a_2(t)h_3. \tag{5.14}$$

The constants of the motion (3.16) are now $F = -1/4$ and [29]

$$F^{(2)} = -\frac{(u_2-v_1)(u_1-v_2)}{(u_1-v_1)(u_2-v_2)}. \tag{5.15}$$

The LH system (5.10) is locally diffeomorphic to the MP equations (5.2) but now with $c < 0$ [9]. Such a diffeomorphism is achieved through the change of variables given by

$$x = \pm \frac{(4|c|)^{1/4}}{\sqrt{|u-v|}}, \qquad y = \mp \frac{(4|c|)^{1/4}(u+v)}{2\sqrt{|u-v|}}, \qquad c<0,$$
$$u = \pm \frac{|c|^{1/2}}{x^2} - \frac{y}{x}, \qquad v = \mp \frac{|c|^{1/2}}{x^2} - \frac{y}{x}, \tag{5.16}$$

which map the MP vector fields (5.4) with domain $\mathbb{R}^2_{x\neq 0}$, symplectic form $\omega = dx \wedge dy$, Hamiltonian functions (5.6) and constant of motion (5.10) onto (5.12) with domain $\mathbb{R}^2_{u\neq v}$, (5.13) and (5.15) (up to a multiplicative constant $\pm|c|^{1/2}$), respectively.

As in the previous section, the (non-standard) deformation of the coupled Riccati system (5.10) is obtained by starting again from proposition 5.1 and now applying the change of variables (5.16) with $c = -1$ (without loss of generality) finding the following result.

Proposition 5.3. *(i) The Hamiltonian functions given by*

$$h_{z,1} = \frac{1}{u-v}, \qquad h_{z,2} = \frac{1}{2}\operatorname{sinhc}\left(\tfrac{2z}{u-v}\right)\left(\frac{u+v}{u-v}\right), \tag{5.17}$$
$$h_{z,3} = \frac{\operatorname{sinhc}^2\left(\tfrac{2z}{u-v}\right)(u+v)^2 - (u-v)^2}{4\operatorname{sinhc}\left(\tfrac{2z}{u-v}\right)(u-v)},$$

satisfy the commutation relations (5.13) with respect to the symplectic form ω (5.13) and define the deformed Poisson algebra $C^\infty(\mathcal{H}^{2\mathrm{R}*}_{z,\omega})$.
(ii) Their corresponding deformed vector fields turn out to be

$$\mathbf{X}_{z,1} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v},$$
$$\mathbf{X}_{z,2} = \frac{1}{2}(u+v)\cosh\left(\tfrac{2z}{u-v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) + \frac{1}{2}(u-v)\operatorname{sinhc}\left(\tfrac{2z}{u-v}\right)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right),$$
$$\mathbf{X}_{z,3} = \frac{1}{4}\left[(u+v)^2 + \frac{(u-v)^2}{\operatorname{sinhc}^2\left(\tfrac{2z}{u-v}\right)}\right]\cosh\left(\tfrac{2z}{u-v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) + \frac{1}{2}(u^2-v^2)\operatorname{sinhc}\left(\tfrac{2z}{u-v}\right)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right),$$

which fulfill

$$[\mathbf{X}_{z,1}, \mathbf{X}_{z,2}] = \cosh\left(\tfrac{2z}{u-v}\right)\mathbf{X}_{z,1}, \qquad [\mathbf{X}_{z,1}, \mathbf{X}_{z,3}] = 2\mathbf{X}_{z,2},$$
$$[\mathbf{X}_{z,2}, \mathbf{X}_{z,3}] = \cosh\left(\tfrac{2z}{u-v}\right)\mathbf{X}_{z,3} - z^2\left[1 - \left(\frac{u+v}{u-v}\right)^2 \operatorname{sinhc}^2\left(\tfrac{2z}{u-v}\right)\right]\mathbf{X}_{z,1}.$$

The deformed counterpart of the coupled Ricatti Lie system (5.11) and of the LH one (5.14) is defined by

$$\mathbf{X}_z := a_0(t)\mathbf{X}_{z,1} + a_1(t)\mathbf{X}_{z,2} + a_2(t)\mathbf{X}_{z,3}, \qquad h_z := a_0(t)h_{z,1} + a_1(t)h_{z,2} + a_2(t)h_{z,3}. \tag{5.18}$$

And the t-independent constants of motion are $F_z = -1/4$ and

$$F_z^{(2)} = \frac{e^{-\frac{2z}{u_1-v_1}}\, e^{\frac{2z}{u_2-v_2}}}{4(u_1-v_1)(u_2-v_2)}\left[\operatorname{sinhc}\left(\tfrac{2z}{u_1-v_1}\right)\operatorname{sinhc}\left(\tfrac{2z}{u_2-v_2}\right)(u_1-u_2+v_1-v_2)^2\right.$$
$$\left. - \left(\frac{e^{\frac{2z}{u_1-v_1}}(u_1-v_1)}{\operatorname{sinhc}\left(\tfrac{2z}{u_1-v_1}\right)} + \frac{e^{-\frac{2z}{u_2-v_2}}(u_2-v_2)}{\operatorname{sinhc}\left(\tfrac{2z}{u_2-v_2}\right)}\right)\operatorname{sinhc}\left(\tfrac{2z}{u_1-v_1} + \tfrac{2z}{u_2-v_2}\right)(u_1+u_2-v_1-v_2)\right].$$

Therefore, the deformation of the system (5.10) is determined by $\mathbf{X}_z$ (5.18). Note that the resulting system presents a strong interaction amongst the variables (u, v) through z, which goes far beyond the initial (naive) coupling corresponding to set the same t-dependent parameters $a_i(t)$ in both one-dimensional Riccati equations; namely

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}t} &= a_0(t) + \frac{a_1(t)}{2}\left[(u+v)\cosh\left(\tfrac{2z}{u-v}\right) + (u-v)\,\mathrm{sinhc}\left(\tfrac{2z}{u-v}\right)\right] \\
&\quad + \frac{a_2(t)}{4}\left[\left((u+v)^2 + \frac{(u-v)^2}{\mathrm{sinhc}^2\left(\tfrac{2z}{u-v}\right)}\right)\cosh\left(\tfrac{2z}{u-v}\right) + 2(u^2-v^2)\,\mathrm{sinhc}\left(\tfrac{2z}{u-v}\right)\right], \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= a_0(t) + \frac{a_1(t)}{2}\left[(u+v)\cosh\left(\tfrac{2z}{u-v}\right) - (u-v)\,\mathrm{sinhc}\left(\tfrac{2z}{u-v}\right)\right] \\
&\quad + \frac{a_2(t)}{4}\left[\left((u+v)^2 + \frac{(u-v)^2}{\mathrm{sinhc}^2\left(\tfrac{2z}{u-v}\right)}\right)\cosh\left(\tfrac{2z}{u-v}\right) - 2(u^2-v^2)\,\mathrm{sinhc}\left(\tfrac{2z}{u-v}\right)\right].
\end{aligned}$$

6 Oscillator Systems from $\mathfrak{h}_4$

6.1 Deformed oscillators from the oscillator algebra $\mathfrak{h}_4$

In this paragraph we analyze a quite different type of deformation based on the Poincaré algebra $\mathfrak{h}_4$, which is the second of the relevant subalgebras of the two-photon Lie algebra $\mathfrak{h}_6$, corresponding to the highest dimensional Lie algebra of vector fields on the real plane that appears in the context of planar Lie–Hamilton systems admits two non-equivalent quantum deformations leading to correspondingly non-equivalent Lie–Poisson systems. The Lie algebra $\mathfrak{h}_4$ is amidst the notable non-semisimple Lie algebras used in physics, where it provides a unified description of coherent, squeezed and intelligent states of light ([156] and references therein), also having applications in the theory of integrable systems [19] and the description of damped harmonic oscillators [29], among others.

The two-photon Lie algebra $\mathfrak{h}_6$, as considered in [156], is spanned by the six operators

$$N = a_+a_-, \quad A_+ = a_+, \quad A_i = a_i, \quad B_+ = a_+^2, \quad B_i = a_i^2, \quad M = \mathrm{I}, \tag{6.1}$$

where a_+ and a_- are the generators of the boson algebra $[a_-, a_+] = \mathrm{I}$. Over this basis, the commutation relations are given by (see [23]):

$$\begin{aligned} &[A_\pm, B_\pm] = 0, \quad [A_-, A_+] = M, \quad [M, \cdot] = 0, \quad [B_-, B_+] = 4N + 2M, \\ &[A_\pm, B_\mp] = \mp 2A_\mp, \quad [N, A_\pm] = \pm A_\pm, \quad [N, B_\pm] = \pm 2B_\pm. \end{aligned} \tag{6.2}$$

It follows at once from these relations that the operators $\{A_+, A_-, M\}$ span a subalgebra isomorphic to the Heisenberg-Weyl algebra $\mathfrak{h}_3$, which can be extended to the oscillator algebra $\mathfrak{h}_4$ spanned by these operators along with the number operator N.

The two-photon Lie algebra admits two independent Casimir operators, one corresponding to the centre generator M and a second one having degree four in the generators and given by

$$C_4 = \left(MB_+ - A_+^2\right)\left(MB_- - A_-^2\right) - \left(MN - A_-A_+ + \frac{1}{2}M^2\right)^2. \tag{6.3}$$

For computational convenience, let us consider the change of basis $D = -(N + \frac{1}{2}M)$. Now, if we understand the generators $\left\{M, A_-, -A_+, D, \frac{B_-}{2}, -\frac{B_+}{2}\right\}$ of the two-photon algebra as functions on $\mathfrak{h}_6^*$ in the natural way [19], we can consider the coordinates $\{v_0, \cdots, v_5\}$ on $\mathfrak{h}_6$. Taking into account the induced Kirillov–Konstant–Souriau Poisson structure on $\mathcal{C}^\infty\left(\mathfrak{h}_6^*\right)$ with respect to the canonical symplectic 2-form in the plane $\omega = \mathrm{d}x \wedge \mathrm{d}y$, we are led to the brackets

$$\begin{aligned} &\{v_1, v_2\} = v_0, \quad \{v_1, v_3\} = -v_1, \quad \{v_1, v_4\} = 0, \quad \{v_1, v_5\} = -v_2, \quad \{v_2, v_3\} = v_2, \\ &\{v_2, v_4\} = -v_1, \quad \{v_2, v_5\} = 0, \quad \{v_3, v_4\} = 2v_4, \quad \{v_3, v_5\} = -2v_5, \quad \{v_4, v_5\} = v_3. \end{aligned} \tag{6.4}$$

The Casimir functions are found to v_0 and

$$C_3 = 2\left(v_1^2 v_5 - v_2^2 v_4 - v_1 v_2 v_3\right) - v_0\left(v_3^2 + 4v_4 v_5\right). \tag{6.5}$$

We observe that the cubic invariant is a consequence of the Lie algebra $\mathfrak{h}_6$ having the structure of a semi-direct product of $\mathfrak{su}(1,1)$ and a Heisenberg algebra [34]. Defining

$$V_3 = v_0 v_3 + v_1 v_2, \quad V_2 = v_0 v_4 - \frac{1}{2} v_1^2, \quad V_3 = v_0 v_5 + \frac{1}{2} v_2^2 \tag{6.6}$$

it follows at once that $v_0 C_3 = V_3^2 + 4 V_1 V_2$, showing its relation to the fourth-order Casimir operator (6.3).

Using now the identity $\iota_{\mathbf{X}_i}\omega = \mathrm{d}v_i$, the corresponding Hamiltonian vector fields are given by

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial y}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \mathbf{X}_4 = y\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = x\frac{\partial}{\partial y}, \tag{6.7}$$

with the nontrivial commutation relations

$$\begin{aligned} &[\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_1, \quad [\mathbf{X}_1, \mathbf{X}_5] = \mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_2, \quad [\mathbf{X}_2, \mathbf{X}_4] = \mathbf{X}_1, \\ &[\mathbf{X}_3, \mathbf{X}_4] = -2\mathbf{X}_4, \quad [\mathbf{X}_3, \mathbf{X}_5] = 2\mathbf{X}_5, \quad [\mathbf{X}_4, \mathbf{X}_5] = -\mathbf{X}_3. \end{aligned} \tag{6.8}$$

6.2 Poisson–Hopf deformation of $\mathfrak{h}_4$ Lie–Hamilton systems

As observed earlier, the subalgebra of $\mathfrak{h}_6$ generated by $\{N, M, A_+, A_-\}$ is isomorphic to the four-dimensional Poincaré algebra $\mathfrak{h}_4$. This Lie algebra possesses two Casimir operators, given respectively by M and $C_2 = 2MN - A_+A_- - A_-A_+$. The quantum deformation of $\mathfrak{h}_4$ (see [22]) is determined by the coassociative coproduct

$$\begin{aligned} &\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1, \\ &\Delta(N) = 1 \otimes N + N \otimes \mathrm{e}^{zA_+}, \quad \Delta(A_-) = 1 \otimes A_- + A_- \otimes \mathrm{e}^{zA_+} + zN \otimes \mathrm{e}^{zA_+} M, \end{aligned} \tag{6.9}$$

from which the deformed commutation relations result as:

$$[N, A_+] = \frac{\mathrm{e}^{zA_+} - 1}{z}, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = M\mathrm{e}^{zA_+}. \tag{6.10}$$

Al already used, we consider the change of basis $D = -N - \frac{1}{2}M$. By (6.10), the deformed Hamiltonian functions for $\mathfrak{h}_4$ are given by

$$v_{0,z} = v_0, \quad v_{1,z} = \mathrm{e}^{zv_2} v_1,\ v_{2,z} = v_2, \quad v_{3,z} = \frac{1 - \mathrm{e}^{zv_2}}{z} v_1, \tag{6.11}$$

corresponding to the classical Hamiltonian functions v_i in the limit $z \to 0$:

$$\lim_{z\to 0} v_{j,z} = v_j, \quad 0 \le j \le 3 \tag{6.12}$$

and Poisson brackets

$$\{v_{1,z}, v_{2,z}\}_z = v_{0,z}\mathrm{e}^{zv_{2,z}}, \quad \{v_{1,z}, v_{3,z}\}_z = -v_{1,z}, \quad \{v_2, v_3\}_z = \frac{\mathrm{e}^{zv_{2,z}} - 1}{z}. \tag{6.13}$$

With respect to the canonical 2-form $\omega = \mathrm{d}x \wedge \mathrm{d}y$, the Poisson structure of $\mathcal{U}(\mathfrak{h}_4)$ is given by the Hamiltonian functions v_j $(0 \le j \le 3)$ with Poisson brackets (6.13) and Casimir invariants v_0 and $C_2 = v_0 v_3 - v_1 v_2$.

The t-dependent vector field $\mathbf{X}_t = \sum_{j=1}^{3} f_i(t)\mathbf{X}_{j,z}$ leads to the Poisson–Hopf system

$$\begin{aligned} \frac{dx}{dt} &= \mathrm{e}^{-xz} f_1(t) + \frac{1 - \mathrm{e}^{-xz}}{z} f_3(t), \\ \frac{dy}{dt} &= yz\mathrm{e}^{-xz} f_1(t) + f_2(t) - y\mathrm{e}^{-xz} f_3(t), \end{aligned} \tag{6.14}$$

for arbitrary functions $f_j(t)$ $(1 \leq j \leq 3)$. For the choice $f_3(t) = 0$, the preceding system corresponds to the quantum deformation of the Lie–Hamilton system based on the Heisenberg Lie algebra $\mathfrak{h}_3$. We further observe that, under this latter assumption, the classical and deformed systems essentially have the same form, as the system consists in this case of a separable equation and a linear first-order inhomogeneous equation. In this sense, the Poisson–Hopf system associated to $\mathfrak{h}_3$ does not lead to new types of systems.

For the generic deformation based on the Poincaré algebra, the deformed noncentral invariant is given by

$$C_{2,z} = v_{0,z}v_{3,z} - v_{1,z}\frac{e^{-z_{2,z}} - 1}{z}.$$

The constants of the motion of the deformed system (6.14), computed by means of the coalgebra, are easily verified to take the form

$$\begin{aligned} F_z(C_{2,z}) &= \frac{e^{-zx}}{z}\left(zx + e^{-zx} - 1\right)y, \\ F_z^{(2)}(C_{2,z}) &= \frac{2(y_1 + y_2) - e^{-z(x_1+x_2)}\left(2 + z(x_1 + x_2)\right)\left(y_1 e^{zx_2} + y_2 e^{zx_1}\right)}{z} \end{aligned} \tag{6.15}$$

with the expected classical limits (see Table 2 in [29])

$$\lim_{z\to 0} F_z(C_{2,z}) = 0, \quad \lim_{z\to 0} F_z^{(2)}(C_{2,z}) = (x_1 - x_2)(y_1 - y_2). \tag{6.16}$$

6.3 Poisson–Hopf deformation of $\mathfrak{h}_6$ Lie–Hamilton systems from $\mathfrak{h}_4$

The non-standard (also called Jordanian) quantum deformation of the two photon algebra $\mathfrak{h}_6$ is a natural extension of the preceding quantum deformation of the Poincaré algebra $\mathfrak{h}_4$ seen previously. It has been extensively studied, for which reason we omit the details (these can be found e.g. in [23; 19]) and merely indicate the coproduct and commutation relations. The coassociative coproduct for the quantum two-photon algebra $\mathcal{U}_z(\mathfrak{h}_6)$ is given by:

$$\begin{aligned} &\Delta(A_+) = 1 \otimes A_+ + A_+ \otimes 1, \quad \Delta(M) = 1 \otimes M + M \otimes 1, \\ &\Delta(N) = 1 \otimes N + N \otimes e^{zA_+}, \quad \Delta(B_+) = 1 \otimes B_+ + B_+ \otimes e^{-2zA_+}, \\ &\Delta(A_-) = 1 \otimes A_- + A_- \otimes e^{zA_+} + zN \otimes e^{zA_+}M, \\ &\Delta(B_-) = 1 \otimes B_- + B_- \otimes e^{2zA_+} + zN \otimes e^{zA_+}(A_- - zMN) - zA_- \otimes e^{zA_+}N. \end{aligned} \tag{6.17}$$

The corresponding compatible deformed commutation relations are thus given by:

$$\begin{aligned} &[N, A_+] = \frac{e^{zA_+} - 1}{z}, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = Me^{zA_+}, \\ &[N, B_+] = 2B_+, \quad [N, B_-] = -2B_- - zA_-N, \quad [A_+, B_+] = 0, \\ &[A_-, B_+] = 2\frac{1 - e^{-zA_+}}{z}, \quad [A_-, B_-] = -zA_-^2, \quad [\cdot, M] = 0, \\ &[A_+, B_-] = -\left(1 + e^{zA_+}\right)A_- + ze^{zA_+}MN, \\ &[B_-, B_+] = 2\left(1 + e^{-zA_+}\right)N + 2M - 2zA_-B_+. \end{aligned} \tag{6.18}$$

It follows from (6.18) that the deformed Hamiltonian functions adopt the form have the form

$$v_{0,z} = v_0, \quad v_{1,z} = e^{zv_2} v_1, \quad v_{2,z} = v_2, \quad v_{3,z} = \frac{1-e^{zv_2}}{z} v_1,$$
$$v_{4,z} = \frac{1}{2} e^{zv_2} v_1^2, \quad v_{5,z} = -\frac{1}{2}\left(\frac{e^{-zv_2}-1}{z}\right)^2, \tag{6.19}$$

which faithfully reproduce the Hamiltonian functions v_i in the limit $z \to 0$:

$$\lim_{z\to 0} v_{j,z} = v_j, \quad 0 \leq j \leq 5. \tag{6.20}$$

The explicit Poisson brackets for the deformation are thus

$$\{v_{1,z}, v_{2,z}\}_z = v_{0,z} e^{zv_{2,z}}, \quad \{v_{1,z}, v_{3,z}\}_z = -v_{1,z}, \quad \{v_{1,z}, v_{4,z}\}_z = -\frac{zv_{1,z}^2}{2},$$
$$\{v_{1,z}, v_{5,z}\}_z = \frac{e^{-zv_{2,z}}-1}{z}, \quad \{v_2, v_3\}_z = \frac{e^{zv_{2,z}}-1}{z},$$
$$\{v_{2,z}, v_{4,z}\}_z = -\frac{v_{1,z}}{2}(1+e^{zv_{2,z}}) - \frac{ze^{zv_{2,z}} v_{0,z}}{4}(2v_{3,z}+v_{0,z}-1),$$
$$\{v_{2,z}, v_{5,z}\}_z = 0, \quad \{v_{3,z}, v_{4,z}\}_z = 2v_{4,z} - \frac{zv_{1,z}}{4}(2v_{3,z}+v_{0,z}-1), \quad \{v_{3,z}, v_{5,z}\}_z = -2v_{5,z},$$
$$\{v_{4,z}, v_{5,z}\}_z = v_3 \frac{1+e^{-zv_{2,z}}}{2} + \left(\frac{v_{0,z}}{4} - \frac{1}{4}\right)(e^{-zv_{2,z}}-1) - zv_1 v_5, \quad \{v_{0,z}, \cdot\} = 0. \tag{6.21}$$

As follows from inspection of the coassociative coproduct (6.17) of the quantum two-photon algebra, the deformed generators of $\mathfrak{h}_4$ give rise to a subalgebra of the quantum deformation of $\mathfrak{h}_6$, hence the deformed Hamiltonian functions can be identified with the $v_{j,z}$ in equation (6.19) for $0 \leq j \leq 3$, while the corresponding Hamiltonian vector fields can be identified with the fields $\mathbf{X}_{j,z}$ given in (6.24) for these indices. We observe that, in this case, the Hamiltonian functions $\{v_{3,z}, v_{4,z}, v_{5,z}\}$ do not close as a subalgebra, as they involve the remaining generators, indicating that we are dealing with a different quantum deformation of $\mathfrak{h}_6$ as that considered earlier extending the deformation on the subalgebra $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$. A cumbersome but routine computation shows that the Casimir invariants for (6.21) are $v_{0,z}$ for being a central element and the function

$$C_{3,z} = zv_{0,z}v_{1,z}v_{5,z}(2v_{3,z}+v_{0,z}-1) - \frac{2v_{4,z}(e^{zv_{2,z}}-1)^2}{z^2} e^{-2zv_{2,z}} + 2v_{1,z}^2 v_{5,z} - v_{0,z}v_{3,z}^2$$
$$-4v_{0,z}v_{4,z}v_{5,z} - \frac{(e^{zv_{2,z}}-1)(2v_{3,z}(e^{zv_{2,z}}+1)+(v_{0,z}-1)(1-e^{zv_{2,z}}))}{ze^{2zv_{2,z}}}. \tag{6.22}$$

It is straightforward to verify that in the limit we obtain the cubic invariant (6.5)

$$\lim_{z\to 0} C_{3,z} = C_3. \tag{6.23}$$

The Hamiltonian vector fields $\mathbf{X}_{j,z}$ associated to the functions $v_{j,z}$ are given by

$$\mathbf{X}_{0,z} = \mathbf{X}_0, \quad \mathbf{X}_{1,z} = e^{z\,v_{2,z}}\mathbf{X}_1 + zv_{1,z}\mathbf{X}_2, \quad \mathbf{X}_{2,z} = \mathbf{X}_2, \quad \mathbf{X}_{3,z} = \frac{1-e^{z\,v_{2,z}}}{z}\mathbf{X}_1 - v_{1,z}\mathbf{X}_2$$
$$\mathbf{X}_{4,z} = v_{1,z}\mathbf{X}_1 + zv_{4,z}\mathbf{X}_2, \quad \mathbf{X}_{5,z} = e^{-z\,v_{2,z}}\frac{e^{-z\,v_{2,z}}-1}{z}\mathbf{X}_2. \tag{6.24}$$

with non-vanishing commutators

$$[\mathbf{X}_{1,z}, \mathbf{X}_{2,z}] = -e^{z\,v_{2,z}}\mathbf{X}_{2,z}, \quad [\mathbf{X}_{1,z}, \mathbf{X}_{3,z}] = \mathbf{X}_{1,z}, \quad [\mathbf{X}_{1,z}, \mathbf{X}_{4,z}] = zv_{1,z}\mathbf{X}_{1,z},$$
$$[\mathbf{X}_{1,z}, \mathbf{X}_{5,z}] = e^{-z\,v_{2,z}}\mathbf{X}_{2,z}, \quad [\mathbf{X}_{2,z}, \mathbf{X}_{3,z}] = -e^{z\,v_{2,z}}\mathbf{X}_{2,z}, \quad [\mathbf{X}_{2,z}, \mathbf{X}_{4,z}] = \mathbf{X}_{1,z},$$
$$[\mathbf{X}_{3,z}, \mathbf{X}_{4,z}] = -(1+e^{z\,v_{2,z}})\mathbf{X}_{4,z} - zv_{4,z}e^{z\,v_{2,z}}\mathbf{X}_{2,z}, \quad [\mathbf{X}_{3,z}, \mathbf{X}_{5,z}] = 2\mathbf{X}_{5,z},$$
$$[\mathbf{X}_{4,z}, \mathbf{X}_{5,z}] = e^{-z\,v_{2,z}}v_{1,z}(e^{-z\,v_{2,z}}-1)\mathbf{X}_{2,z} - e^{-z\,v_{2,z}}\mathbf{X}_{3,z}. \tag{6.25}$$

Hence, for both the vector fields and the commutators the limit is found to recover the classical Lie-Hamilton system:

$$\lim_{z\to 0} \mathbf{X}_{j,z} = \mathbf{X}_j, \quad \lim_{z\to 0} \left[\mathbf{X}_{j,z}, \mathbf{X}_{k,z}\right] = \left[\mathbf{X}_j, \mathbf{X}_k\right], \quad 0 \le j \le 5. \tag{6.26}$$

For generic functions $f_i(t)$, the deformed Poisson–Hopf system determined by the t-dependent vector field $\mathbf{X}_t = \sum_{j=1}^{5} f_i(t)\mathbf{X}_{i,z}$ is explicitly given by

$$\begin{aligned}\frac{dx}{dt} &= \mathrm{e}^{-xz} f_1(t) + \frac{1-\mathrm{e}^{-xz}}{z} f_3(t) + y\mathrm{e}^{-xz} f_4(t), \\ \frac{dy}{dt} &= yz\mathrm{e}^{-xz} f_1(t) + f_2(t) - y\mathrm{e}^{-xz} f_3(t) + \frac{y^2 z}{2}\mathrm{e}^{-xz} f_4(t) + \frac{\mathrm{e}^{xz}\left(\mathrm{e}^{xz}-1\right)}{z} f_5(t).\end{aligned} \tag{6.27}$$

The t-independent constants of the motion of this system are obtained by application of the coalgebra formalism (see [26]), from which the following expressions are obtained:

$$\begin{aligned}&F(C_{3,z}) = 0, \qquad F^{(2)}(C_{3,z}) = 0, \\ &F^{(3)}(C_{3,z}) = \frac{\mathrm{e}^{-z(x_1+x_2)}}{2z}\left(4\mathrm{e}^{zx_1} - 2\mathrm{e}^{2zx_1} + 4\mathrm{e}^{zx_2} - 2\mathrm{e}^{-2zx_2} + \mathrm{e}^{2z(x_1+x)} - 2\mathrm{e}^{z(x_1+x_2)} - 3\right) \times \\ &\qquad (y_1\mathrm{e}^{zx_2} + y_2\mathrm{e}^{zx_1}) - \frac{y_1^2}{z^2}\mathrm{e}^{-2zx_1}\left(\mathrm{e}^{2zx_1}-1\right)\left(\mathrm{e}^{zx_2}-1\right) - \frac{y_2^2}{z^2}\left(\mathrm{e}^{2zx_2}-1\right) \times \\ &\qquad \left(\mathrm{e}^{zx_1}-1\right)\mathrm{e}^{-2zx_2} + \frac{y_1y_2}{z^2}\mathrm{e}^{-z(x_1+x_2)}\left(\mathrm{e}^{z(2x_1+3x_2)} + \mathrm{e}^{z(3x_1+2x_2)} - 2\mathrm{e}^{z(x_1+2x_2)}\right. \\ &\qquad -2\mathrm{e}^{z(2x_1+x_2)} - 2\mathrm{e}^{2z(x_1+x_2)} + 4\mathrm{e}^{z(x_1+x_2)} - 2\mathrm{e}^{3zx_2} + 6\mathrm{e}^{2zx_2} - 5\mathrm{e}^{zx_2} - 2\mathrm{e}^{3zx_1} \\ &\qquad \left. +6\mathrm{e}^{2zx_1} - 5\mathrm{e}^{zx_1} + 2\right).\end{aligned} \tag{6.28}$$

It can be routinely verified that $\lim_{z\to 0} F^{(2)}(C_{3,z}) = 0$, which is in agreement with the well known fact that $F^{(2)}(C_3) = 0$ holds for the classical counterpart. Applying again the same procedure, another deformed invariant $F^{(3)}(C_{3,z})$ can be constructed so that in the limit we recover the first nonvanishing t-independent constant of the motion

$$F^{(3)}(C_3) = \left(x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)\right)^2$$

of the classical Lie–Hamilton system [29].

6.3.1 Deformation of the damped oscillator

As a physically interesting application of the preceding deformation, let us consider a one-dimensional damped oscillator of the form

$$\begin{aligned}\frac{dx}{dt} &= a(t)x + b(t)y + f(t), \\ \frac{dp}{dt} &= -c(t)x - a(t)p - g(t),\end{aligned} \tag{6.29}$$

where $a(t), b(t), c(t), f(t)$ and $g(t)$ are arbitrary functions. The system (6.29) is actually a Lie–Hamilton system associated to the t-dependent vector field

$$\mathbf{X}_t = f(t)\mathbf{X}_1 - g(t)\mathbf{X}_2 + a(t)\mathbf{X}_3 + b(t)\mathbf{X}_4 - c(t)\mathbf{X}_5,$$

where

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \quad \mathbf{X}_2 = \frac{\partial}{\partial p}, \quad \mathbf{X}_3 = x\frac{\partial}{\partial x} - p\frac{\partial}{\partial p}, \quad \mathbf{X}_4 = p\frac{\partial}{\partial x}, \quad \mathbf{X}_5 = x\frac{\partial}{\partial p} \tag{6.30}$$

are Hamiltonian vector fields with respect to the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}p$. The Hamiltonian functions h_i associated to the vector fields (6.30) are respectively

$$h_1 = p, \quad h_2 = -x, \quad h_3 = xp, \quad h_4 = \frac{1}{2}p^2, \quad h_5 = -\frac{1}{2}x^2, \tag{6.31}$$

showing that the resulting Vessiot–Guldberg Lie algebra of the Lie–Hamilton system is isomorphic to the two-photon Lie algebra $\mathfrak{h}_6$ as given in Equation (6.7).[1] The quantum deformation of (6.29) therefore corresponds to a system of type (6.27), for the choice of functions $f_1(t) = f(t), f_2(t) = -g(t), f_3(t) = a(t), f_4(t) = b(t)$ and $f_5(t) = c(t)$. As follows from (6.28), the deformed system possesses a non-vanishing z-dependent constant of the motion $F^{(2)}(C_{3,z})$.

We observe that the operators $\left\{B_+, B_-, N + \frac{1}{2}M\right\}$ span a simple non-compact Lie algebra isomorphic to $\mathfrak{su}(1,1) \simeq \mathfrak{sl}(2,\mathbb{R})$. As both of these subalgebras admit quantum deformations, it is natural to ask whether these can be extended to the whole two-photon algebra in a consistent way, such that all Poisson–Hopf deformations of Lie–Hamilton systems in the plane can be described uniformly in terms of subalgebras of the two photon algebra. In this sense, an extension of the $\mathfrak{sl}(2,\mathbb{R})$-related deformed systems remain to be described. Various preliminary results in this direction have been obtained, with a complete description currently in progress.

[1]The invariants and nonlinear superposition rule for this type of Lie–Hamilton system have been analyzed in detail in [29], for which reason we skip their detailed expressions.

7 Applications to Biology

7.1 An epidemic SIS model

If one wants to study the evolution of a SIS epidemic exposed to a constant heat source, like centrally heated buildings; one can make use of quantum stochastic differential equations. It was shown that the stochastic SIS-epidemic model can be interpreted as a Hamiltonian system. Lie–Hamiltonian systems admit a quantum deformation, so does the stochastic SIS-epidemic model, because it is a Lie–Hamilton system.

This chapter [63] proposes a quantum version of a stochastic SIS-epidemic model without using stochastic calculus, but using the proper Hamiltonian approximation for the mean and the variance.

Epidemic models try to predict the spread of an infectious disease afflicting a specific population, see [31; 109]. These models are rooted in the works of Bernoulli in the 18th century, when he proposed a mathematical model to defend the practice of inoculating against smallpox [82]. This was the start of germ theory.

At the beginning of the 20th century, the emergence of compartmental models was starting to develop. Compartmental models are deterministic models in which the population is divided into compartments, each representing a specific stage of the epidemic. For example, S represents the susceptible individuals to the disease, I designates the infected individuals, whilst R stands for the recovered ones. The evolution of these variables in time is represented by a system of ordinary differential equations whose independent variable, the time, is denoted by t. Some of these first models are the Kermack–McKendrick [90] and the Reed–Frost [1] epidemic models, both describing the dynamics of healthy and infected individuals among other possibilities. There are several types of compartmental models [110], as it can be the SIS model, in which after the infection the individuals do not acquire immunity, the SIR model, in which after the infection the individuals acquire immunity, the SIRS model, for which immunity only lasts for a short period of time, the MSIR model, in which infants are born with immunity, etc. In this present work our focus is on the SIS model.

7.2 The SIS model

The susceptible-infectious-susceptible (SIS) epidemic model assumes a population of size N and one single disease disseminating. The infectious period extends throughout the whole course of the disease until the recovery of the patient with two possible states, either infected or susceptible. This implies that there is no immunization in this model. In this approach the only relevant variable is the instantaneous density of infected individuals $\rho = \rho(\tau)$ depending on the time parameter τ, and taking values in $[0, 1]$. The density of infected individuals decreases with rate $\gamma\rho$, where γ is the recovery rate, and the rate of growth of new infections is proportional to $\alpha\rho(1-\rho)$, where the intensity of contagion is given by the transmission rate α. These two processes are modelled through the compartmental equation

$$\frac{d\rho}{d\tau} = \alpha\rho(1-\rho) - \gamma\rho. \tag{7.1}$$

One can redefine the timescale as $t := \alpha\tau$ and introduce $\rho_0 := 1 - \gamma/\alpha$, so we can rewrite (7.1) as

$$\frac{d\rho}{dt} = \rho(\rho_0 - \rho). \tag{7.2}$$

Clearly, the equilibrium density is reached if $\rho = 0$ or $\rho = \rho_0$. Although compartmental equations have proven their efficiency for centuries, they are still based on strong hypotheses. For example, the SIS model works more efficiently under the random mixing and large population assumptions. The first assumption is asking homogeneous mixing of the population, that is, individuals contact with each other randomly and do not gather in smaller groups, as abstaining themselves from certain communities. This assumption is nevertheless rarely justified. The second assumption is the rectangular and stationary age distribution, which means that everyone in the population lives to an age L, and for each age up to L, which is the oldest age, there is the same number of people in each subpopulation. This assumption seems feasible in developed countries where there exists very low infant mortality, for example, and a long live expectancy. Nonetheless, it looks reasonable to implement probability at some point to permit random variation in one or more inputs over time. Some recent experiments provide evidence that temporal fluctuations can drastically alter the prevalence of pathogens and spatial heterogeneity also introduces an extra layer of complexity as it can delay the pathogen transmission.

7.2.1 The SIS model with fluctuations

It is needless to point out that fluctuations should be considered in order to capture the spread of infectious diseases more closely. Nonetheless, the introduction of these fluctuations is not trivial [63]. One way to account for fluctuations is to consider stochastic variables. On the other hand, it seems that in the case of SIS models there exist improved differential equations for the mean and variance of infected individuals.

Recently, in [114], the model assumes the spreading of the disease as a Markov chain in discrete time in which at most one single recovery or transmission occurs in the duration of this infinitesimal interval.

As a result [91], the first two equations for instantaneous mean density of infected people $\langle\rho\rangle$ and the variance $\sigma^2 = \langle\rho^2\rangle - \langle\rho\rangle^2$ are

$$\begin{aligned} \frac{d\langle\rho\rangle}{dt} &= \langle\rho\rangle\left(\rho_0 - \langle\rho\rangle\right) - \sigma^2, \\ \frac{d\sigma^2}{dt} &= 2\sigma^2\left(\rho_0 - \langle\rho\rangle\right) - \Delta_3 - \frac{1}{N}\langle\rho(1-\rho)\rangle + \frac{\gamma}{N\alpha}\langle\rho\rangle, \end{aligned} \tag{7.3}$$

where $\Delta_3 = \langle\rho^3\rangle - \langle\rho\rangle^3$. This system finds excellent agreement with empirical data [114]. Equations (7.2) and (7.3) are equivalent when σ becomes irrelevant compared to $\langle\rho\rangle$. Therefore, a generalization of compartmental equations only requires mean and variance, neglecting higher statistical moments. The skewness coefficient vanishes as a direct consequence of this assumption, so that $\Delta_3 := 3\sigma^2\langle\rho\rangle$. For a big number of individuals $(N \gg 1)$, the resulting equations are

$$\begin{aligned} \frac{d\ln\langle\rho\rangle}{dt} &= \rho_0 - \langle\rho\rangle - \frac{\sigma^2}{\langle\rho\rangle}, \\ \frac{1}{2}\frac{d\ln\sigma^2}{dt} &= \rho_0 - 2\langle\rho\rangle. \end{aligned} \tag{7.4}$$

The system right above can be obtained from a stochastic expansion as it is given in [150], as well.

7.2.2 Hamiltonian character of the model

The investigation of the geometric and the algebraic foundations of a system permits to employ several powerful techniques of geometry and algebra while performing the qualitative analysis of the system. This even results in an analytical/general solution of the system in our case. For example [97] and [115] for Lie symmetry approach to solve the classical SIS model. The Hamiltonian analysis of a system plays an important role in the geometrical analysis of a given system.

In [114], the SIS system (7.4) involving fluctuations has been recasted in Hamiltonian form in the following way: the dependent variables are the mean $\langle \rho \rangle$ and the variance σ^2, and they both depend on time. Then, we define the dynamical variables $q = \langle \rho \rangle$ and $p = 1/\sigma$, so the system (7.4) turns out to be

$$\begin{aligned} \frac{dq}{dt} &= q\rho_0 - q^2 - \frac{1}{p^2}, \\ \frac{dp}{dt} &= -p\rho_0 + 2pq. \end{aligned} \tag{7.5}$$

We employ the abbreviation SISf for system (7.5) to differentiate it from the classical SIS model in (7.2). The letter “f" accounts for “fluctuations".

We have computed the general solution to this system, finding a more general solution than the one provided by Nakamura and Martínez in [114]. Indeed, we have found obstructions in their model solution. We shall comment this in the last section gathering all our new results.

The general solution for this system reads:

$$\begin{aligned} q(t) &= \frac{\rho e^{\rho t}(C_1\rho^2 - 4)e^{\rho t} + 2C_1C_2\rho^2}{(C_1^2\rho^2 - 4)e^{2\rho t} + 4C_2\rho^2(C_1e^{\rho t} + C_2)}, \\ p(t) &= C_1 + \frac{C_1^2\rho^2 - 4}{4\rho^2 - C_2} + C_2e^{-\rho t}. \end{aligned} \tag{7.6}$$

In order to develop a geometric theory for this system of differential equations, we need to choose certain particular solutions that we shall make use of. Here we present three different choices and their corresponding graphs according to the change of variables $q = \langle \rho \rangle$ and $p = 1/\sigma$.

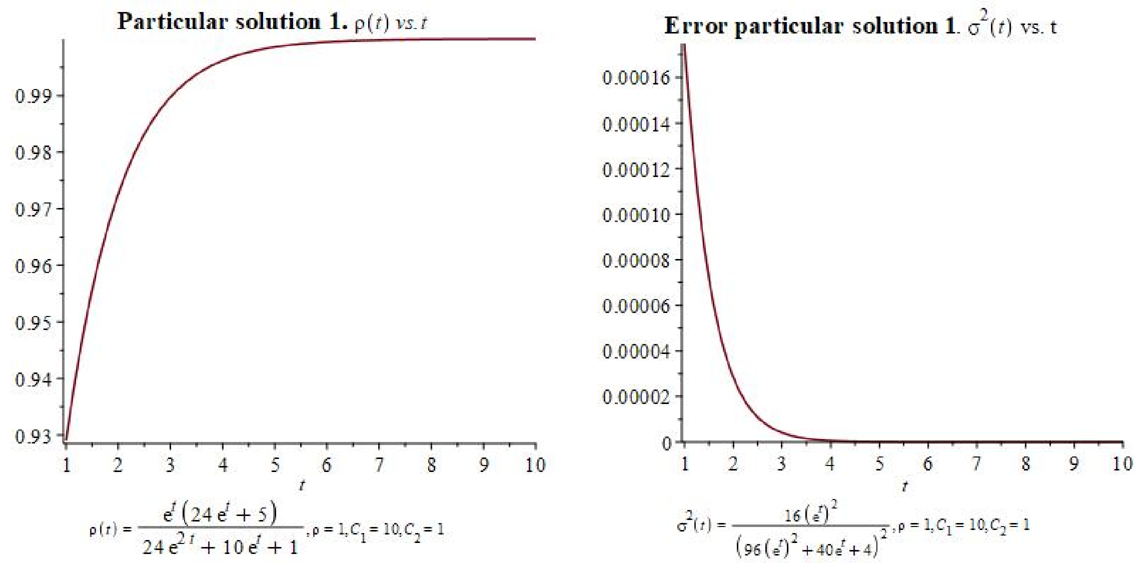


FIGURE 7.1: The first particular solution

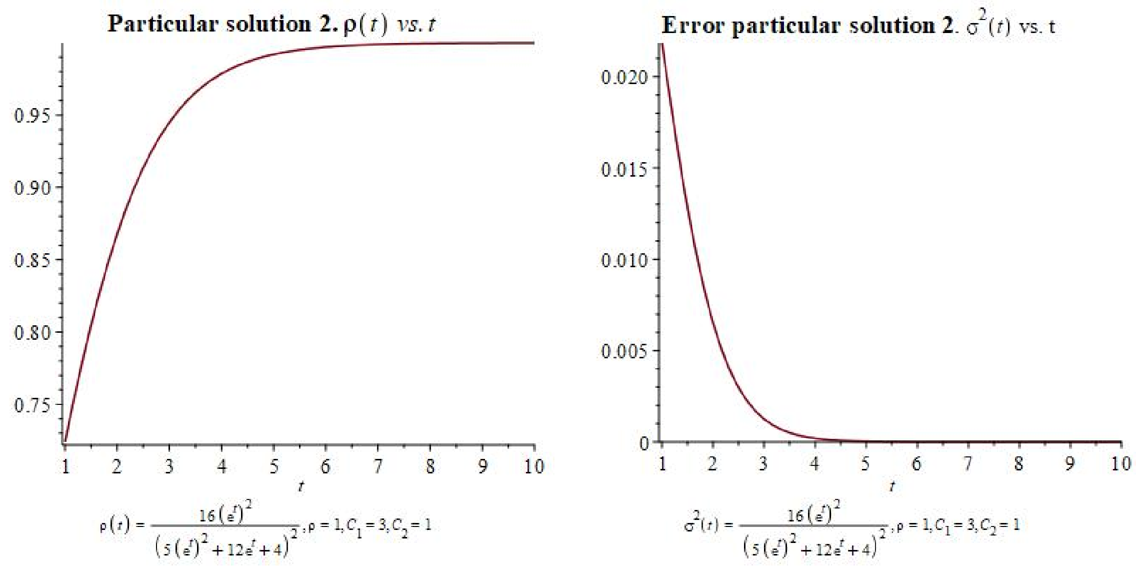


FIGURE 7.2: The second particular solution

Let us turn now to interpret these equations geometrically on a symplectic manifold. The symplectic two-form $\omega = dq \wedge dp$ is a canonical skew-symmetric tensorial object in two-dimensions. For a chosen (real-valued) Hamiltonian function $h = h(q, p)$, the dynamics is governed by a Hamiltonian vector field X_h defined through the Hamilton equation

$$\iota_{X_h}\omega = dh. \tag{7.7}$$

In terms of the coordinates (q, p), the Hamilton equations (7.7) become

$$\frac{dq}{dt} = \frac{\partial h}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial h}{\partial q}. \tag{7.8}$$

It is possible to realize that the SISf system (7.5) is a Hamiltonian system since it fulfills the Hamilton equations (7.7). To see this, consider the Hamiltonian function

$$h = qp\,(\rho_0 - q) + \frac{1}{p}. \tag{7.9}$$

and substitute it into (7.8). A direct calculation will lead us to (7.5). The skew-symmetry of the symplectic two-form implies that the Hamiltonian function is constant all along the motion. In holonomic Classical Mechanics, where the Hamiltonian is taken to be the total energy, this corresponds to the conservation of energy.

7.2.3 Lie Analysis of the SISf model

The model (7.5) can be generalized to a model represented by a time-dependent vector field

$$X_t = \rho_0(t)X_1 + X_2, \tag{7.10}$$

where the constitutive vector fields are computed to be

$$X_1 = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}, \quad X_2 = \left(-q^2 - \frac{1}{p^2}\right)\frac{\partial}{\partial q} + 2qp\frac{\partial}{\partial p}. \tag{7.11}$$

The generalization comes from the fact that $\rho_0(t)$ is no longer a constant, but it can evolve in time.

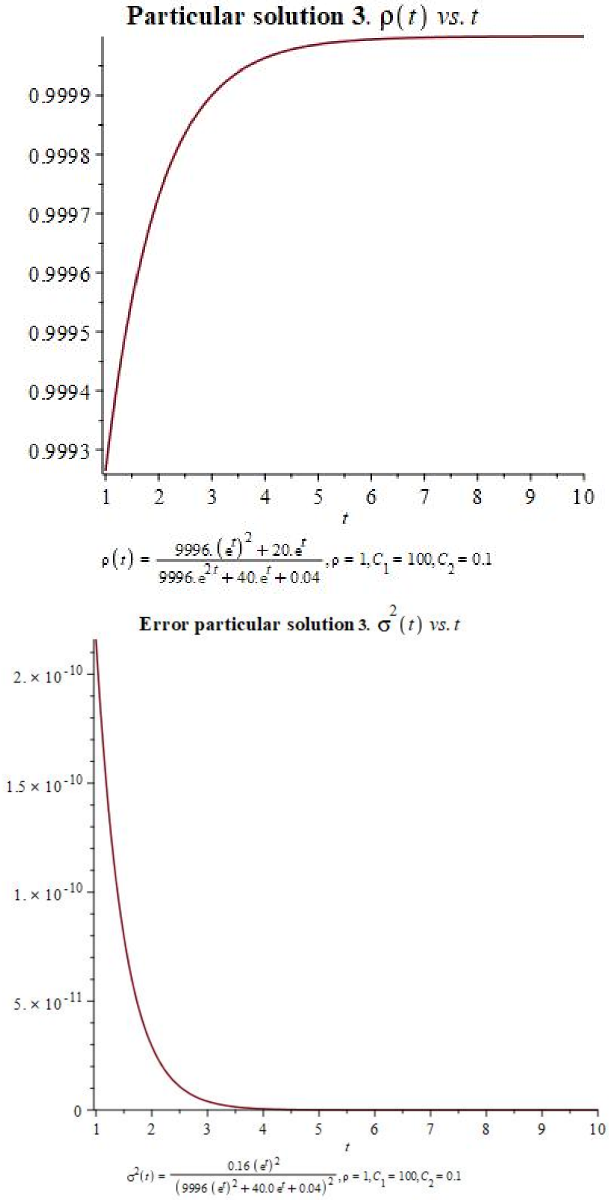


FIGURE 7.3: The second particular solution

First, for the vector fields in (7.11), a direct calculation shows that the Lie bracket

$$[X_1, X_2] = X_2 \tag{7.12}$$

is closed within the Lie algebra. This implies that the SISf model (7.5) is a Lie system. The Vessiot-Guldberg algebra spanned by X_1, X_2 is an imprimitive Lie algebra of type I_{14} according to the classification presented in [12].

If we copy the configuration space twice, we will have four degrees of freedom (q_1, p_1, q_2, p_2) and we will archieve precisely two first-integrals as a consequence of the Fröbenius theorem. A first-integral for X_t has to be a first-integral for X_1 and X_2 simultaneously. We define the diagonal prolongation $\widetilde{X}_1$ of the vector field X_1 in the decomposition (7.12). Then we look for a first integral F_1 such that $\widetilde{X}_1[F_1]$ vanishes identically. Notice that if F_1 is a first-integral of the vector field $\widetilde{X}_1$ then it is a first integral of $\widetilde{X}_2$ due to the commutation relation. For this reason, we start by integrating the prolonged vector field

$$\widetilde{X}_1 = q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2}, \tag{7.13}$$

through the following characteristic system

$$\frac{dq_1}{q_1} = \frac{dq_2}{q_2} = \frac{dp_1}{-p_1} = \frac{dp_2}{-p_2}. \tag{7.14}$$

Fix the dependent variable q_1 and obtain a new set of dependent variables, say (K_1, K_2, K_3), which are computed to be

$$K_1 = \frac{q_1}{q_2}, \qquad K_2 = q_1 p_1, \qquad K_3 = q_1 p_2. \tag{7.15}$$

Now, this induces the following basis in the tangent space

$$\frac{\partial}{\partial K_1} = q_2 \frac{\partial}{\partial q_1} - \frac{q_2 p_1}{q_1}\frac{\partial}{\partial p_1} - \frac{q_2 p_2}{q_1}\frac{\partial}{\partial p_1}, \qquad \frac{\partial}{\partial K_2} = \frac{1}{q_1}\frac{\partial}{\partial p_1}, \qquad \frac{\partial}{\partial K_3} = \frac{1}{q_1}\frac{\partial}{\partial p_2}. \tag{7.16}$$

provided that q_1 is not zero. Introducing the coordinate changes exhibited in (7.15) into the diagonal projection $\widetilde{X}_2$ of the vector field X_2, we arrive at the following expression

$$\widetilde{X}_2 = \left(2K_1 - \left(1 + \frac{1}{K_1^2}\right)\right)\frac{\partial}{\partial K_1} + \left(\left(\frac{1}{K_2^2} + \frac{1}{K_3^2}\right)K_2^2 - \left(1 + \frac{1}{K_1^2}\right)K_2\right)\frac{\partial}{\partial K_2}$$
$$+ \left(2\frac{K_3}{K_2} - \left(1 + \frac{1}{K_1^2}\right)K_3\right)\frac{\partial}{\partial K_3}.$$

To integrate the system once more, we use the method of characteristics again and obtain

$$\frac{d\ln|K_1|}{1 - \frac{1}{K_1^2}} = \frac{d\ln|K_2|}{\frac{1}{K_2} + \frac{K_2}{K_3} - 1 - \frac{1}{K_1^2}} = \frac{d\ln|K_3|}{\frac{2}{K_2} - \left(1 + \frac{1}{K_1^2}\right)}. \tag{7.17}$$

We obtain two first integrals by integrating in pairs (K_1, K_2) and (K_1, K_3), where K_1 is fixed. After some cumbersome calculations we obtain

$$K_2 = \frac{K_1\left(4k_2^2K_1^2 + 4k_1k_2K_1 + k_1^2 - 4\right)}{2(K_1+1)(K_1-1)k_2(2k_2K_1 + k_1)}, \qquad K_3 = \frac{K_1\left(k_2K_1^2 + k_1K_1 + \frac{k_1^2-4}{4k_2}\right)}{(K_1+1)(K_1-1)}. \tag{7.18}$$

By substituting back the coordinate transformation (7.15) into the solution (7.18), we arrive at the following implicit equations

$$\begin{aligned} q_1 &= -\frac{q_2\left(k_1k_2 \pm \sqrt{4k_2^2p_2^2q_2^2 + k_1^2k_2p_2q_2 - 4k_2^3p_2q_2 - 4k_2p_2q_2 + 4k_2^2}\right)}{2k_2(-p_2q_2 + k_2)}, \\ p_1 &= \frac{4q_1^2k_2^2 + 4q_1q_2k_1k_2 + q_2^2k_1^2 - 4q_2^2}{2k_2(2q_1^3k_2 + q_1^2k_1q_2 - 2q_1k_2q_2^2 - k_1q_2^3)}. \end{aligned} \tag{7.19}$$

Let us notice that the equations (7.19) depend on a particular solution (q_2, p_2) and two constants of integration (k_1, k_2) which are related to initial conditions.

Let us show now the graphs and values of the initial conditions for which the solution reminds us of sigmoid behavior, which is the expected growth of $\rho(t)$. As particular solution for (q_2, p_2), we have made use of particular solution 2 given in Figure 7.2 through its corresponding values of q, p through the change of variables $q = \langle\rho\rangle$ and $p = 1/\sigma$.

Since the solution (7.19) is quite complicated, one may look for a solution of a linearized model. We first employ the following change of coordinates

$$\{u = \ln|K_1|, v = \ln|K_2|, w = \ln|K_3|\}. \tag{7.20}$$

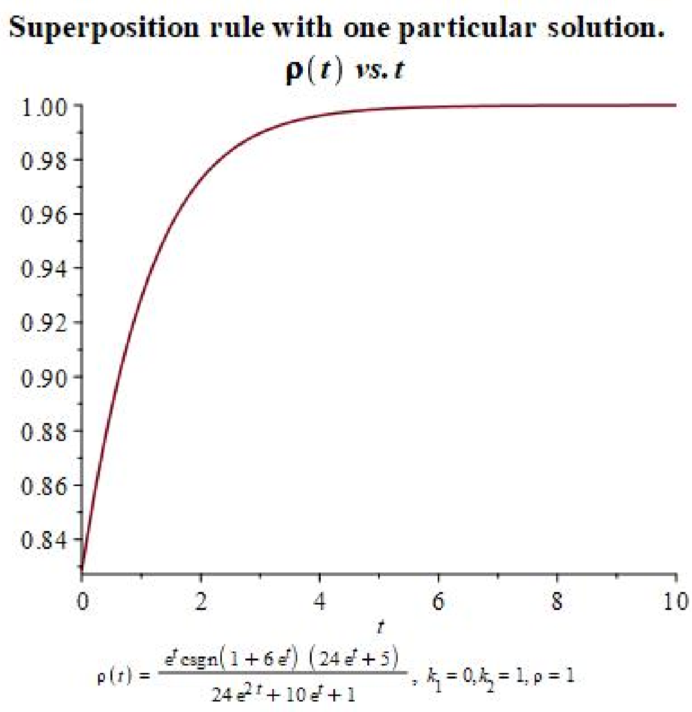


FIGURE 7.4: Superposition rule for exact solution

In terms of these new variables, the system (7.17) reads

$$\frac{du}{1-e^{-2u}} = \frac{dv}{e^{-v}+e^{v-w}-1-e^{-2u}} = \frac{dw}{2e^{-v}-(1+e^{-2u})}. \tag{7.21}$$

One can solve the system above by introducing a linear approximation

$$\begin{aligned} 1-e^{-2u} &\simeq 2u, \\ e^{-v}+e^{v-w}-1-e^{-2u} &\simeq 2u-w, \\ 2e^{-v}-(1+e^{-2u}) &\simeq 2u-2v, \end{aligned} \tag{7.22}$$

after which (7.21) reads

$$\frac{du}{2u} = \frac{dv}{2u-w} = \frac{dw}{2u-2v}. \tag{7.23}$$

We can solve now v and w in terms of u and obtain

$$v(u) = k_1 u^{-\sqrt{2}/2} + k_2 u^{\sqrt{2}/2} + u, \qquad w(u) = \sqrt{2}\left(k_1 u^{-\sqrt{2}/2} - k_2 u^{\sqrt{2}/2}\right). \tag{7.24}$$

We need to isolate the constants of integration k_1 and k_2. Hence, the two first integrals read now

$$k_1 = u^{\frac{\sqrt{2}}{2}}(\sqrt{2}v - \sqrt{2}u + w)/2\sqrt{2}, \qquad k_2 = u^{-\frac{\sqrt{2}}{2}}(\sqrt{2}v - \sqrt{2}u - w)/2\sqrt{2}. \tag{7.25}$$

Now, if we substitute the coordinate changes in (7.20) and in (7.15), we arrive at the following general solution

$$q_1 = q_2 \exp\left(-\frac{\ln(q_2 p_2)}{1+k_1+k_2}\right), \qquad p_1 = \frac{1}{q_2}\exp\left(\frac{\sqrt{2}}{2}\frac{(k_1-k_2)\ln(q_2 p_2)}{1+k_1+k_2}\right). \tag{7.26}$$

which can be written as

$$q_1 = q_2 \left(q_2 p_2\right)^{\frac{-1}{1+k_1+k_2}}, \qquad p_1 = \frac{1}{q_2}\left(q_2 p_2\right)^{\frac{\sqrt{2}}{2}\frac{k_1-k_2}{1+k_1+k_2}}. \tag{7.27}$$

Notice that the solution depends on a particular solution (q_2, p_2) and two constants of integration (k_1, k_2), as in (7.19).

Let us show now the graphs and values of the initial conditions for which the solution reminds us of sigmoid behavior, which is the expected growth of $\rho(t)$. As particular solution for (q_2, p_2), we have made use of particular solution 2 given in Figure 7.2 through its corresponding values of q, p through the change of variables $q =< \rho >$ and $p = 1/\sigma$.

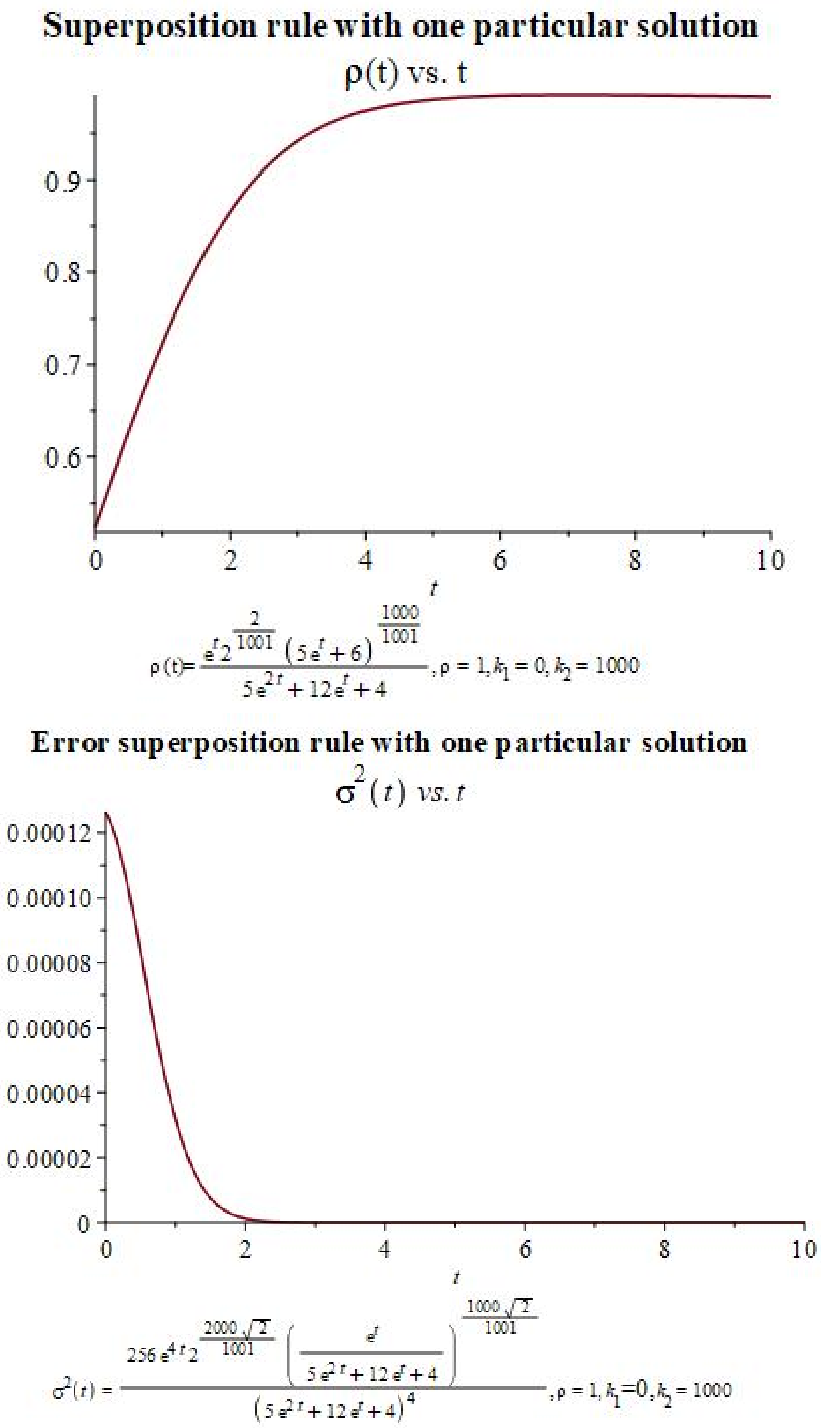


FIGURE 7.5: Superposition rule for linear approximation

7.3 Lie–Hamilton analysis of the SISf model

In this section, we shall show that the SISf model (7.5) is a Lie-Hamilton system [101]. Among the developed methods for Lie–Hamilton systems, we consider a very important recent method for the obtainance of solutions as superposition principles through the Poisson coalgebra method [12].

We have already proven in (7.12) that (7.5) defines a Lie system. In order to see if it is a Lie–Hamilton system, we first need to check whether the vector fields in (7.12) are Hamiltonian vector fields. Consider now the canonical symplectic form $\omega = dq \wedge dp$. It is easy to check that the vector fields X_1 and X_2 in (7.12) are Hamiltonian with respect to the Hamiltonian functions

$$h_1 = -qp, \qquad h_2 = -q^2 p + \frac{1}{p}, \tag{7.28}$$

respectively. It is easy to see that the Poisson bracket of these two functions reads $\{h_1, h_2\} = h_2$. It means that the Hamiltonian functions form a finite dimensional Lie algebra, denoted in the literature as $I_{14A}^{r=1} \simeq \mathbb{R} \ltimes \mathbb{R}$, and it is isomorphic to the one defined by vector fields X_1, X_2. The Hamiltonian function for the total system is

$$h = \rho_0(t) h_1 + h_2 = -q^2 p + \frac{1}{p} - \rho_0(t) qp \tag{7.29}$$

and it is exactly the Hamiltonian function (7.9) proposed in [114].

Lie-Hamilton systems can also be integrated in terms of a superposition rule. We need to find a Casimir function for the Poisson algebra, but unfortunately, there exists no nontrivial Casimir in this particular case. It is interesting to see how a symmetry of the Lie algebra $\{X_1, X_2\}$ commutes with the Lie bracket, i.e. the vector field

$$Z = -\frac{1}{2}\frac{p(C_2 p^2 q^2 + 4C_1 pq + C_2)}{(pq-1)(pq+1)}\frac{\partial}{\partial p} + \frac{C_1 p^4 q^4 + C_2 p^3 q^3 - C_2 pq - C_1}{p(pq-1)^2(pq+1)^2}\frac{\partial}{\partial q} \tag{7.30}$$

fulfills $[X_1, Z] = 0, \quad [X_2, Z] = 0$. Notice too that Z is a conformal vector field, that is,

$$\mathcal{L}_Z \omega = -(C_2/2)\omega. \tag{7.31}$$

Since it is a Hamiltonian system, one would expect that a first integral for Z, let us say f, would Poisson commute with the Poisson algebra $\{h_1, h_2\}$, since $Z = -\hat{\Lambda}(df)$. Nonetheless, this is not the case unless $f =$ constant. This implies that the Casimir is a constant, hence trivial and the coalgebra method can not be directly applied. However, there is a way in which we can circumvent this problem by considering an inclusion of the algebra $I_{14A}^{r=1}$ as a Lie subalgebra of a Lie algebra to another class admitting a Lie–Hamiltonian algebra with a non-trivial Casimir. In this case, we will consider the algebra, denoted by $I_8 \simeq \mathfrak{iso}(1,1)$, due to the simple form of its Casimir. If we obtain the superposition rule for I_8, we simultaneously obtain the superposition for $I_{14A}^{r=1}$ as a byproduct.

7.3.1 Superposition rules for the SISf model: $\mathfrak{iso}(1,1)$

The Lie–Hamilton algebra $\mathfrak{iso}(1,1)$ has the commutation relations

$$\{h_1, h_2\} = h_0, \quad \{h_1, h_3\} = -h_1, \quad \{h_2, h_3\} = h_2, \quad \{h_0, \cdot\} = 0, \tag{7.32}$$

with respect to $\omega = dx \wedge dy$ in the basis $\{h_1 = y, h_2 = -x, h_3 = xy, h_0 = 1\}$. The Casimir associated to this Lie–Hamilton algebra is $C = h_1 h_2 + h_3 h_0$. Let us apply the coalgebra method to this case. Mapping the representation without coproduct, the first iteration is trivial, i.e., $F = 0$. We could use the second-order coproduct and third-order coproduct $\Delta^{(2)}$ and $\Delta^{(3)}$, or the second-order coproduct $\Delta^{(2)}$ together with the permuting sub-indices property. We need three constants of motion, this would be equivalent to integrating the diagonal prolongation $\widetilde{X}$ on $(\mathbb{R}^2)^3$. Using the coalgebra method and sub-index permutation, one obtains

$$\begin{aligned} F^{(2)} &= (x_1 - x_2)(y_1 - y_2) = k_1, \\ F^{(2)}_{23} &= (x_1 - x_3)(y_1 - y_3) = k_2, \\ F^{(2)}_{13} &= (x_3 - x_2)(y_3 - y_2) = k_3. \end{aligned} \tag{7.33}$$

From them, we can choose two functionally independent constants of motion. Our choice is $F^{(2)} = k_1, F^{(2)}_{23} = k_2$. The introduction of k_3 simplifies the final result, with expression

$$\begin{aligned} x_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(x_2 + x_3) + \frac{k_2 - k_1 \pm B}{2(y_2 - y_3)}, \\ y_1(x_2, y_2, x_3, y_3, k_1, k_2, k_3) &= \frac{1}{2}(y_2 + y_3) + \frac{k_2 - k_1 \mp B}{2(x_2 - x_3)}, \end{aligned} \tag{7.34}$$

where

$$B = \sqrt{k_1^2 + k_2^2 + k_3^2 - 2(k_1k_2 + k_1k_3 + k_2k_3)}. \tag{7.35}$$

In the case that matters to us, $I^{r=1}_{14A}$, the third constant k_3 is a function $k_3 = k_3(x_2, y_2, x_3, y_3)$ and $B \geq 0$. Notice though that this superposition rule is expressed in the basis (7.32), therefore, we need the change of coordinates between the present $\mathfrak{iso}(1,1)$ and our problem (7.28). See that the commutation relation $\{h_1, h_3\} = -h_1$ in (7.32) coincides with the commutation relation $\{h_1, h_2\} = h_2$ of our pandemic system (7.28). So, by comparison, we see there is a change of coordinates

$$x = -qp, \qquad y = q - \frac{1}{qp^2}. \tag{7.36}$$

This way, introducing this change (7.36) in (7.34), the superposition principle for our Hamiltonian pandemic system reads

$$\begin{aligned} q &= A \times C^{-1} \\ p &= -C \times D^{-1} \end{aligned} \tag{7.37}$$

where

$$A := \left(\frac{\frac{1}{2}q_2 + \frac{1}{2}q_3 + (k_2 - k_1 \pm B)}{(2p_2 - 2p_3)} \right)^2 \left(\frac{1}{2}p_2 + \frac{1}{2}p_3 + \frac{(k_2 - k_1 \mp B)}{(2q_2 - 2q_3)} \right),$$

$$C := \left(\frac{\frac{1}{2}q_2 + \frac{1}{2}q_3 + (k_2 - k_1 \pm B)}{(2p_2 - 2p_3)} \right)^2 - 1$$

and

$$D := \frac{1}{2}q_2 + \frac{1}{2}q_3 + \frac{(k_2 - k_1 \pm B)}{(2p_2 - 2p_3)} \left(\frac{1}{2}p_2 + \frac{1}{2}p_3 + \frac{(k_2 - k_1 \mp B)}{(2q_2 - 2q_3)} \right).$$

Here, (q_2, p_2) and (q_3, p_3) are two particular solutions and k_1, k_2, k_3 are constants of integration.

Now, we show the graphics for $< \rho >= q(t)$ and $\sigma^2 = 1/p^2$ using the two particular solutions in Figure 7.2 and Figure 7.3 provided in the introduction. Notice that we have renamed $c = (k_2 - k_1 \pm B)$ and $k = (k_2 - k_1 \mp B)$.

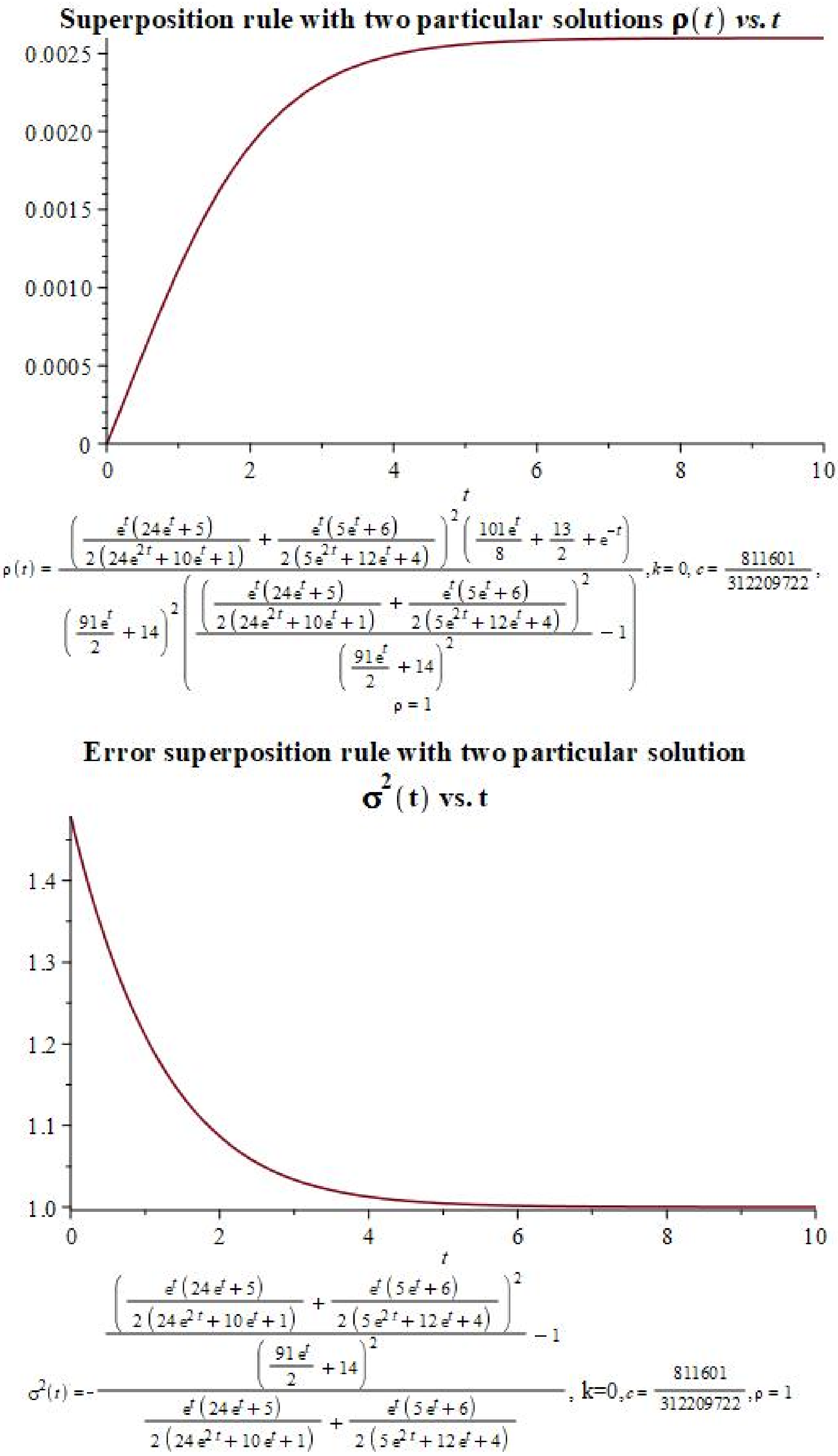


FIGURE 7.6: Superposition rule with two particular solutions

7.4 A deformed SISf model

For the SISf model (7.5), we start with the Vessiot–Guldberg algebra (7.12) labelled as $I_{14A}^{r=1}$. To obtain a deformation of a Lie algebra $I_{14A}^{r=1}$, we need to rely on a bigger Lie algebra, in this case, we make use of $\mathfrak{sl}(2)$. To this end, consider the vector fields X_1 and X_2 in (7.11), and let X_3 be a vector field given by

$$X_3 := \frac{p^2q^2(-2p^2q^2+c+6)+c}{2(p^2q^2-1)^2}\frac{\partial}{\partial q} - \frac{p^3q(c+2)}{(p^2q^2-1)^2}\frac{\partial}{\partial p}, \tag{7.38}$$

where $c \in \mathbb{R}$. Then, $\{X_1, X_2, X_3\}$ span a Vessiot–Lie algebra V isomorphic to $\mathfrak{sl}(2)$ that satisfies the following commutation relations

$$[X_1, X_2] = X_2, \qquad [X_1, X_3] = -X_3, \qquad [X_2, X_3] = 2X_1. \tag{7.39}$$

This vector field X_3 admits a Hamiltonian function, say h_3, with respect to the canonical symplectic form on $\mathbb{R}^2$, so that we have the family

$$h_1 = -qp, \qquad h_2 = \frac{1}{p} - q^2p, \qquad h_3 = \frac{2p^3q^2+c}{2-2p^2q^2}. \tag{7.40}$$

Hence, $\{h_1, h_2, h_3\}$ span a Lie–Hamilton algebra $\mathcal{H}_\omega$; isomorphic to $\mathfrak{sl}(2)$ where the commutation relations with respect to the Poisson bracket induced by the canonical symplectic form ω on $\mathbb{R}^2$ are given by

$$\{h_1, h_2\}_\omega = h_2, \qquad \{h_1, h_3\}_\omega = -h_3, \qquad \{h_2, h_3\}_\omega = 2h_1. \tag{7.41}$$

Step 1. Applying the non-standard deformation of $\mathfrak{sl}(2)$ in [13] we arrive at the Hamiltonian functions

$$h_{z;1} = -shc(2zh_{z;2})qp, \qquad h_{z;2} = \frac{1}{p} - q^2p, \qquad h_{z;3} = -\frac{p\left(shc(2zh_{z;2})2q^2p^2+c\right)}{2shc(2zh_{z;2})(q^2p^2-1)}, \tag{7.42}$$

Accordingly, the Poisson brackets are computed to be

$$\begin{gathered}\{h_{z;1}, h_{z;2}\}_\omega = shc(2zh_{z;2})h_{z;2}, \qquad \{h_{z;2}, h_{z;3}\}_\omega = 2h_{z;1}, \\ \{h_{z;1}, h_{z;3}\}_\omega = -cosh(2zh_{z;2})h_{z;3},\end{gathered} \tag{7.43}$$

Step 2. The vector fields $X_{z;1}$ and $X_{z;2}$ associated to the Hamiltonian functions $h_{z;1}$ and $h_{z;2}$ exhibited in (7.42) are

$$\begin{aligned} X_{z,1} &= \frac{cosh\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)^2}\left[(1-p^4q^4)\frac{\partial}{\partial q} + (2p^5q^4-p^3q^2)\frac{\partial}{\partial p}\right] \\ &\qquad + \frac{shc\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)}\left[q\frac{\partial}{\partial q} - p(p^2q^2+1)\frac{\partial}{\partial p}\right], \\ X_{z,2} &= \left(-q^2-\frac{1}{p^2}\right)\frac{\partial}{\partial q} + 2qp\frac{\partial}{\partial p}. \end{aligned} \tag{7.44}$$

We do not write explicitly the expression of the vector field $X_{z;3}$ because it does not play a relevant role in our system. The deformed vector fields keep the commutation relations

$$[X_{z;1}, X_{z;2}] = cosh\left(2z\left(\frac{1}{p}-q^2p\right)\right)X_{z;2}. \tag{7.45}$$

Step 3. The total Hamiltonian function for the deformed model is

$$h_z = \rho(t)h_{z;1} + h_{z;2} = -\rho(t)shc(2zh_{z;2})qp + \frac{1}{p} - q^2p. \tag{7.46}$$

so that the deformed dynamics is computed to be

$$\begin{aligned}\frac{dq}{dt} &= \left(\frac{cosh\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)^2}(1-p^4q^4) + \frac{shc\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)}\, q \right)\rho_0(t) - q^2 - \frac{1}{p^2},\\ \frac{dp}{dt} &= \left(\frac{cosh\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)^2}(2p^5q^4-p^3q^2) - p\frac{shc\left(2z(\frac{1}{p}-q^2p)\right)}{(p^2q^2-1)}(p^2q^2+1) \right)\rho_0(t) - 2qp.\end{aligned} \tag{7.47}$$

This system describes a family of z-parametric differential equations that generalizes the SISf model (7.5), where the demographic interaction and both rates allow a more realistic representation of the epidemic evolution. According to the kind of deformation, this may be called a quantum family SISf model. Note that the SISf model can be recovered in the limit when z tends to zero.

7.4.1 Constants of motion

For the present case, the constants of motion are computed to be

$$F^{(1)} = \frac{c}{4}, \qquad F^{(2)} = \left(h_2^{(1)} + h_2^{(2)}\right)\left(h_3^{(1)} + h_3^{(2)}\right) - \left(h_1^{(1)} + h_1^{(2)}\right)^2, \tag{7.48}$$

after the quantization, the latter one becomes

$$F_z^{(2)} = shc\left(2zh_{z;2}^{(2)}\right) h_{z;2}^{(2)} h_{z;3}^{(2)} - \left(h_{z;1}^{(2)}\right)^2, \tag{7.49}$$

where $h_{z;j}^{(2)} := D_z^{(2)}(\Delta_z(v_j))$. This coproduct Δ_z can be described as a follows

$$\Delta_z(v_2) = v_2 \otimes 1 + 1 \otimes v_2, \qquad \Delta_z(v_j) = v_j \otimes e^{2zv_2} + e^{-2zv_2} \otimes v_j, \qquad j = 1, 3.$$

More explictly, using the expressions given in (5.12), we have

$$\begin{aligned} h_{z;j}^{(2)} =& h_{z;j}(q_1,p_1)e^{2zh_{z;2}(q_2,p_2)} + h_{z;j}(q_2,p_2)e^{-2zh_{z;2}(q_1,p_1)}, \qquad j = 1,3\\ h_{z;2}^{(2)} =& h_{z;2}(q_1,p_1) + h_{z;2}(q_2,p_2).\end{aligned} \tag{7.50}$$

So, to retrieve another constant of motion we can apply the trick of permuting indices. Then, here we have a second constant of motion, writing it implicitly,

$$F_{z,(23)}^{(2)} = shc\left(2zh_{z;2(23)}^{(2)}\right) h_{z;2(23)}^{(2)} h_{z;3(23)}^{(2)} - \left(h_{z;1(23)}^{(2)}\right)^2, \tag{7.51}$$

where the sub-index (23) means that the variables (q_2, p_2) are interchanged with (q_3, p_3) when they appear in the deformed Hamiltonian functions $h_{z;j}$ and

$$\begin{aligned} h_{z;j(23)}^{(2)} =& h_{z;j}(q_1,p_1)e^{2zh_{z;2}(q_3,p_3)} + h_{z;j}(q_3,p_3)e^{-2zh_{z;2}(q_1,p_1)}, \qquad j = 1,3\\ h_{z;2(23)}^{(2)} =& h_{z;2}(q_1,p_1) + h_{z;2}(q_3,p_3).\end{aligned}$$

In (7.49), we have

$$\begin{aligned} h_{z;2}(q_1,p_1) &= -shc(2zh_{z;2})q_1p_1, \quad h_{z;2} = \frac{1}{p_1} - q_1^2p_1, \quad h_{z;3} = -\frac{p_1\left(shc(2zh_{z;2})2q_1^2p_1^2 + c\right)}{2shc(2zh_{z;2})(q_1^2p_1^2-1)},\\ h_{z;2}(q_2,p_2) &= -shc(2zh_{z;2})q_2p_2, \quad h_{z;2} = \frac{1}{p_2} - q_2^2p_2, \quad h_{z;3} = -\frac{p_2\left(shc(2zh_{z;2})2q_2^2p_2^2 + c\right)}{2shc(2zh_{z;2})(q_2^2p_2^2-1)},\end{aligned} \tag{7.52}$$

whilst in (7.51)

$$h_{z;2}(q_1,p_1) = -shc(2zh_{z;2})q_1p_1, \quad h_{z;2} = \frac{1}{p_1} - q_1^2p_1, \quad h_{z;3} = -\frac{p_1\left(shc(2zh_{z;2})2q_1^2p_1^2+c\right)}{2shc(2zh_{z;2})(q_1^2p_1^2-1)},$$
$$4pt]h_{z;2}(q_3,p_3) = -shc(2zh_{z;2})q_3p_3, \quad h_{z;2} = \frac{1}{p_3} - q_3^2p_3, \quad h_{z;3} = -\frac{p_3\left(shc(2zh_{z;2})2q_3^2p_3^2+c\right)}{2shc(2zh_{z;2})(q_3^2p_3^2-1)}. \tag{7.53}$$

If we set these two first integrals equal to a constant, $F^{(2)}_{z,(23)} = k_{23}$ and $F^{(2)}_{z} = k_{12}$, with $k_{23}, k_{12} \in \mathbb{R}$, one is able to retrieve a superposition principle for $q_1 = q_1(q_2, q_3, p_2, p_3, k_{12}, k_{23})$ and $p_1 = p_1(q_2, q_3, p_2, p_3, k_{12}, k_{23})$. Notice that here (q_2, p_2) and (q_3, p_3) are two pairs of particular solutions and k_{12}, k_{23} are two constants over the plane to be related to initial conditions [63].

8 Conclusions

In this work, the notion of Poisson–Hopf deformation of LH systems has been proposed. This framework differs radically from other approaches to the LH systems theory [15; 41; 47; 49; 154], as our resulting deformations do not formally correspond to LH systems, but to an extended notion of them that requires a (non-trivial) Hopf structure and is related with the non-deformed LH system by means of a limiting process in which the deformation parameter z vanishes. Moreover, the introduction of Poisson–Hopf structures allows for the generalization of the type of systems under inspection, since the finite-dimensional Vessiot–Guldberg Lie algebra is replaced by an involutive distribution in the Stefan–Sussman sense (Chapter 3).

In Chapter 4, the Poisson analogue of the non-standard quantum deformation of $\mathfrak{sl}(2)$ has been studied, establishing explicitly the constants of the motion for the quantum deformed systems. The three non-equivalent LH systems in the plane based on the Lie algebra $\mathfrak{sl}(2)$ have been described in unified form, which provides a nice geometrical interpretation of both these systems and their corresponding quantum deformations. Chapter 5 is devoted to the analysis of specific systems of differential equations and their deformed counterpart. We first consider the Milne–Pinney equation, the deformations of which provide us with new oscillator systems with the particularity that the mass of the particle is dependent on the position, and where the constants of the motion are explicitly obtained. In particular, the Schrödinger problem for position-dependent mass Hamiltonians is directly connected with the quantum dynamics of charge carriers in semiconductor heterostructures and nanostructures (see, for instance, [27; 79; 151]). In this context, it is worthy to be observed that the standard or Drinfel'd–Jimbo deformation of $\mathfrak{sl}(2)$ would not lead to an oscillator with a position-dependent mass as, in that case, the deformation function would be $\operatorname{sinhc}(zqp)$ instead of $\operatorname{sinhc}(zq^2)$; this can clearly be seen in the corresponding symplectic realization given in [14]. This fact explains that we have chosen the non-standard deformation of $\mathfrak{sl}(2)$ due to its physical applications. In spite of this, the Drinfel'd–Jimbo deformation could provide additional deformation for the Milne–Pinney equation, leading to systems that are non-equivalent to those studied here. In any case, these examples suggest an alternative approach to dynamical systems with a nonconstant mass, for which the classical tools are of limited applicability. The second type of LH systems that has been studied corresponds to the complex and coupled Riccati equations, which have been extensively studied in the literature. For them, the deformed versions of the corresponding LH systems and their constants of the motion have been obtained. The main results of Chapters 3-5 have been published in [13] and [14]. In Chapter 6 we focus on oscillator systems obtained as a deformation of LH systems based on the oscillator algebra $\mathfrak{h}_4$, seen as a subalgebra of the 2-photon algebra $\mathfrak{h}_6$. In particular, these deformations can be obtained as the restriction of the corresponding deformed $\mathfrak{h}_6$ Poisson–Hopf systems. An illustrative example of this type of deformations is given by a generalization of the damped oscillator. It remains to explicitly determine a superposition rule for such systems, a problem that it is currently in progress. In Chapter 7 the affine Lie algebra $\mathfrak{b}_2$, seen as a subalgebra of $\mathfrak{sl}(2)$, is used to obtain quantum deformed systems applicable in the context of epidemiological models. This approach constitutes a novelty, as the techniques usually employed for this type of models are essentially of stochastic nature. The results of this chapter have recently been submitted for publication.

There is a plethora of problems and applications that emerge from the Poisson–Hopf algebra deformation formalism. Although the results have been principally focused on the two-dimensional case, for which an explicit classification of LH algebras exists [9; 15], the results are valid for arbitrary manifolds and higher-dimensional Vessiot–Guldberg Lie algebras. A systematic analysis of the known systems would certainly lead to a richer spectrum of properties for the deformed systems that deserve further investigation. In particular, the dynamical properties of specific systems of differential equations can be studied with these techniques, and it is expected that some new and intriguing features will emerge from this analysis.

As a byproduct, and related to the current COVID-19 pandemic, one may wonder whether a description of the pandemic could be related to a SISf-pandemic model. The SISf model is a very first approximation for a trivial infection process, in which there are only two possible states for an individual in the population: they are either infected or susceptible to the infection. Hence, this model does not provide the possibility of acquiring immunity at any point. It seems that COVID-19 provides some certain types of immunity, but only a thirty percent of the infected individuals, hence, a SIR model that considers "R" for recovered individuals (not susceptible anymore, i.e., immune) is not a proper model for the current situation. It would be interesting to have a model contemplating immune and nonimmunized individuals simultaneously. Currently we are still in search of a stochastic Hamiltonian model that includes potential immunity and nonimmunity.

8.1 Future work

One of the most important questions to be addressed is whether the Poisson–Hopf algebra approach can provide an effective procedure to derive a deformed analogue of superposition principles for deformed LH systems. It would also be interesting to know whether such a description is simultaneously applicable to the various non-equivalent deformations, like an extrapolation of the notion of Lie algebra contraction to Lie systems. Another open problem worthy to be considered is the possibility of getting a unified description of such systems in terms of a certain amount of fixed 'elementary' systems, thus implying a first rough systematization of LH-related systems from a more general perspective than that of finite-dimensional Lie algebras. Some possible future work in this direction can be summarized as follows:

- In the classification of LH systems on the plane, the so-called 2-photon algebra plays a central role, as it is the highest dimensional algebra that can appear with the properties of a LH algebra. The study of their quantum deformations is, therefore, a fundamental question to complete the analysis of the deformations of the LH systems on the plane. It should be noted that there are essentially two different possibilities for these deformations, depending on the structure of two prominent subalgebras, the algebra h_4 and $\mathfrak{sl}(2)$, which gives rise to systems and deformations with different properties. The first case, based on the oscillator algebra $\mathfrak{h}_4$, has partially been considered in Chapter 6. However, there still remains the problem of obtaining an effective superposition rule, the implementation of which is currently being studied. The analysis must be completed identifying particular classes of systems of differential equations that can be deformed by this procedure, and that can be interpreted as perturbations of the initial system. The second case, based on the extension of the results for $\mathfrak{sl}(2)$ to the 2-photon algebra, is structurally quite different due to the corresponding quantum deformation. It is expected that new systems with diverging properties will emerge from this analysis. From the point of view of applications, these systems have a multitude of interesting properties, such as new systems of the Lotka–Volterra type or oscillator systems with variable frequencies or masses dependent on one or more parameters, but whose dynamics can be characterized by the existence of a procedure for the systematic and explicit construction of the constants of motion and the superposition rules. The complete analysis of the Poisson–Hopf deformations of the LH systems based on the 2-photon algebra is currently in progress, to be sent soon for publication.

- On the other hand, it should be noted that, currently, there is no classification of LH systems for dimension $n \geq 3$. A problem of interest that arises in this context is to analyze the possibility of generating new LH systems, both classical and deformed, by means of the extension of the systems on the plane, in combination with the projections of the realizations of vector fields. In this context, it is known that projections of Lie algebra realizations associated with a linear representation give rise to non-linear realizations. Analyzing the question from the perspective of functional algebras (Hamiltonians), it is conceivable that there exist compatible symplectic structures that give rise to LH systems in higher dimensions, as well as a dependence of the symplectic forms of the representation. Criteria of this type can be combined with quantum deformations of known Lie algebras, in order to obtain new applications of these in the context of differential equations.

- As a complement to the SISf models based on $\mathfrak{b}_2$ as a subalgebra of $\mathfrak{sl}(2)$, it is natural to construct the corresponding model but considering $\mathfrak{b}_2$ as a subalgebra of the oscillator algebra $\mathfrak{h}_4$. Again, the quite distinct quantum deformation leads to systems with different properties, and both approaches should be compared in detail, analyzing the numerical solutions deduced from both approaches. The first steps in this direction are also currently under scrutiny.

- We would also like to extend our study to more complicated compartmental models, although at a first glance we have not been able to identify more Lie systems, at least in their current form. We suspect that the Hamiltonian description of these compartmental models could nonetheless behave as a Lie system, as it has happened in our presented case. This shall be part of our future endeavors. Moreover, one could inspect in more meticulous detail how the solutions of the quantum-deformed system (7.47) recover the nondeformed solutions when the introduced parameter tends to zero. We need to further study how this precisely models a heat bath, and if this new integrable system could correspond to other models apart from infectious models. We would like to figure out whether it is possible to modelize subatomic dynamics with the resulting deformed Hamiltonian (7.46). There exists a stochastic theory of Lie systems developed in [95] that could be another starting point to deal with compartmental systems. In the present work we were lucky to find a theory with fluctuations that happened to match a stochastic expansion, but this is rather more of an exception than a rule. Indeed, it seems that the most feasible way to propose stochastic models is using the stochastic Lie theory instead of expecting a glimpse of luck with fluctuations. As stated, finding particular solutions is by no means trivial. The analytic search is a very intricate task. We think that in order to fit particular solutions in the superposition principle, one may need to compute these particular solutions numerically. Some specific numerical methods for particular solutions of Lie systems can be devised in [124].

- Finally, starting from the Chebyshev equation, it has been shown that the point of Noether symmetries of this equation can be expressed for arbitrary n in terms of the Chebyshev polynomials $T_n(x), U_n(x)$ of first and second kind, respectively. Moreover, it has been observed that the generic realization of the Lie point symmetry algebra $\mathfrak{sl}(3,\mathbb{R})$ can be enlarged to more general linear homogeneous second-order ODEs, the solutions of which are expressible in terms of trigonometric or hyperbolic functions. In particular, the commutators of the generic point symmetries show that various of the algebraic relations of the general solutions actually arise as a consequence of the symmetry. The same conclusions hold for the structure of the five-dimensional subalgebra of Noether symmetries. The realization of the symmetry generators has been shown to remain valid for differential equations of hypergeometric type, enabling us to obtain realizations of $\mathfrak{sl}(3,\mathbb{R})$ in terms of hypergeometric functions in general and various orthogonal polynomials in particular, such as the Chebyshev or Jacobi polynomials. Another remarkable fact emerges from this analysis; namely, that the forcing terms are always independent on the "velocities" y_1', y_2'. This is again a consequence of the chosen generic realization, and the question whether other generic realizations in terms of the general solution of the ODE (or system) enable to determine forcing terms that explicitly depend on the derivatives, and even lead to autonomous differential equations (systems), arises naturally. In this context, it would be desirable to obtain a realization of $\mathfrak{sl}(3,\mathbb{R})$ that not only enables to describe generically the point and Noether symmetries of the Jacobi polynomials, but also applies to the differential equations associated to the remaining families of orthogonal polynomials, specifically the Laguerre and Hermite polynomials. This would allow to construct further non-linear equations and systems possessing a subalgebra of Noether symmetries, the generators of which are given in terms of these orthogonal polynomials.

Part III

Appendix

A Lie algebras: Elementary properties

A.1 Lie algebras

In this Appendix, we recall the main structural properties of Lie algebras used in this work. Details can be found in [141; 155].

Given a Lie algebra $\mathfrak{g}$ of dimension n over a basis $\{X_1, \cdots, X_n\}$ with commutators

$$[X_i, X_j] = C_{ij}^k X_k, \tag{A.1}$$

the structure constants $\left\{C_{ij}^k\right\}$ correspond to the coefficients of a skew-symmetric 2-covariant 1-contravariant tensor μ defined on the linear space underlying $\mathfrak{g}$ and satisfying the Jacobi identity. If $\{\omega^1, \cdots, \omega^n\}$ denotes the dual basis of 1-forms to $\{X_1, \cdots, X_n\}$, the Maurer-Cartan equations of $\mathfrak{g}$ are defined as

$$d\omega^k = -\frac{1}{2} C_{ij}^k \omega^i \wedge \omega^j, \ 1 \leq i, j, k \leq n. \tag{A.2}$$

In particular, the Jacobi identity is satisfied if and only of the 2-forms $d\omega^k$ are closed [132].

Definition A.1. *The adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is given by*

$$ad(X)(Y) = [X, Y], \ X, Y \in \mathfrak{g}. \tag{A.3}$$

Definition A.2. *The Killing form κ of $\mathfrak{g}$ is the bilinear symmetric form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by*

$$\kappa(X, Y) = \mathrm{Tr}(ad(X) \cdot ad(Y)), \ X, Y \in \mathfrak{g}. \tag{A.4}$$

In particular, it follows that a Lie algebra $\mathfrak{g}$ is semisimple if and only if κ is non-degenerate, i.e., $\det() \neq 0$. For real Lie algebras, the signature of the Killing form further determines the isomorphism class [155].

Proposition A.3 (Levi decomposition). *Any Lie algebra $\mathfrak{g}$ admits a decomposition*

$$\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus} \mathfrak{r}, \tag{A.5}$$

where $\mathfrak{r}$ is a maximal solvable ideal of $\mathfrak{g}$ and $\mathfrak{s} \simeq \mathfrak{g}/\mathfrak{r}$ is the maximal semisimple subalgebra.

The semisimple algebra $\mathfrak{s}$ of (A.5) is usually called the Levi subalgebra of $\mathfrak{g}$.

The Lie algebras $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{sl}(3, \mathbb{R})$ are semisimple by the preceding criterion. Moreover, they are simple Lie algebras [139].

B The hyperbolic sinc function

B.1 The hyperbolic sinc function

The hyperbolic counterpart of the well-known sinc function is defined by

$$\operatorname{sinhc}(x) := \frac{\sinh(x)}{x} = \begin{cases} \frac{\sinh(x)}{x}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$$

The power series around $x = 0$ reads

$$\operatorname{sinhc}(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n+1)!}\,.$$

And its derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sinhc}(x) = \frac{\cosh(x)}{x} - \frac{\sinh(x)}{x^2} = \frac{\cosh(x) - \operatorname{sinhc}(x)}{x}\,.$$

Hence the behaviour of $\operatorname{sinhc}(x)$ and its derivative remind that of the hyperbolic cosine and sine functions, respectively. We represent them in figure B.1.

A novel relationship of the sinhc function (and also of the sinc one) with Lie systems can be established by considering the following second-order ordinary differential equation

$$t\,\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\,\frac{\mathrm{d}x}{\mathrm{d}t} - \eta^2 t\, x = 0, \tag{B.1}$$

where η is a non-zero real parameter. Its general solution can be written as

$$x(t) = A \operatorname{sinhc}(\eta t) + B\,\frac{\cosh(\eta t)}{t}\,, \qquad A, B \in \mathbb{R}.$$

Notice that if we set $\eta = i\lambda$ with $\lambda \in \mathbb{R}^*$ we recover the known result for the sinc function:

$$t\,\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\,\frac{\mathrm{d}x}{\mathrm{d}t} + \lambda^2 t\, x = 0, \qquad x(t) = A \operatorname{sinc}(\lambda t) + B\,\frac{\cos(\lambda t)}{t}\,. \tag{B.2}$$

Next the differential equation (B.1) can be written as a system of two first-order differential equations by setting $y = \mathrm{d}x/\mathrm{d}t$, namely

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{2}{t}\,y + \eta^2 x.$$

These equations determine a Lie system with associated t-dependent vector field

$$\mathbf{X} = -\frac{2}{t}\,\mathbf{X}_1 + \mathbf{X}_2 + \eta^2 \mathbf{X}_3, \tag{B.3}$$

where

$$\mathbf{X}_1 = y\frac{\partial}{\partial y}, \qquad \mathbf{X}_2 = y\frac{\partial}{\partial x}, \qquad \mathbf{X}_3 = x\frac{\partial}{\partial y}, \qquad \mathbf{X}_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

fulfill the commutation relations

$$[\mathbf{X}_1, \mathbf{X}_2] = \mathbf{X}_2, \qquad [\mathbf{X}_1, \mathbf{X}_3] = -\mathbf{X}_3, \qquad [\mathbf{X}_2, \mathbf{X}_3] = 2\mathbf{X}_1 - \mathbf{X}_4, \qquad [\mathbf{X}_4, \cdot\,] = 0.$$

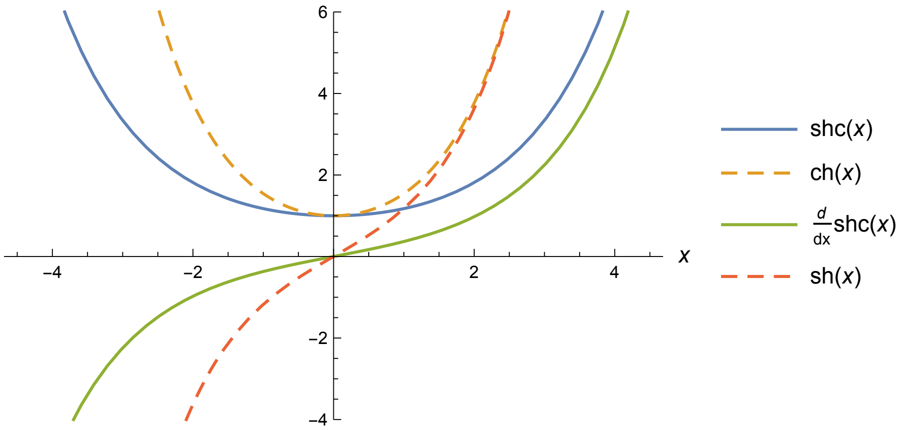


FIGURE B.1: The hyperbolic sinc function versus the hyperbolic cosine function and the derivative of the former versus the hyperbolic sine function.

Hence, these vector fields span a Vessiot–Guldberg Lie algebra V isomorphic to $\mathfrak{gl}(2)$ with domain $\mathbb{R}^2_{x\neq 0}$. In fact, V is diffeomorphic to the class $\mathrm{I}_7 \simeq \mathfrak{gl}(2)$ of the classification given in [9]. The diffemorphism can be explictly performed by means of the change of variables $u = y/x$ and $v = 1/x$, leading to the vector fields of class I_7 with domain $\mathbb{R}^2_{v\neq 0}$ given in [9]

$$\mathbf{X}_1 = u\frac{\partial}{\partial u}, \qquad \mathbf{X}_2 = -u^2\frac{\partial}{\partial u} - uv\frac{\partial}{\partial v}, \qquad \mathbf{X}_3 = \frac{\partial}{\partial u}, \qquad \mathbf{X}_4 = -v\frac{\partial}{\partial v}.$$

Therefore $\mathbf{X}$ (B.3) is a Lie system but not a LH one since there does not exist any compatible symplectic form satisfying (3.4) for class I_7 as shown in [9].

Finally, we point out that the very same result follows by starting from the differential equation (B.2) associated with the sinc function.

C Orthogonal systems and symmetries of ODEs

C.1 Point symmetries of ordinary differential equations

Symmetries of differential equations can be formulated by various different approaches, from the classical one by means of vector fields and their k^{th}-order prolongations to their reformulation in terms of differential forms (see e.g. [30; 83; 84; 100; 152] and references therein). From the computational point of view, a rather convenient approach to determine symmetries of differential equations is based on the reformulation of the symmetry condition in terms of differential operators [84]. As in the following we adopt this method for the computation of symmetries, we briefly review the main facts of the procedure (see e.g. [141] for details). It is well known that a scalar second-order ordinary differential equation

$$y'' = \omega\left(x, y, y'\right) \tag{C.1}$$

can be formulated in equivalent form in terms of the partial differential equation (PDE)

$$\mathbf{A}f = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \omega\left(x, y, y'\right)\frac{\partial}{\partial y'}\right)f = 0. \tag{C.2}$$

A vector field $X = \xi\left(x, y\right)\frac{\partial}{\partial x} + \eta\left(x, y\right)\frac{\partial}{\partial y} \in \mathfrak{X}\left(\mathbb{R}^2\right)$ is called a (Lie) point symmetry generator of the equation (C.1) if the prolonged vector field

$$\dot{X} = X + \left(\frac{d\eta}{dx} - \frac{d\xi}{dx}y'\right)\frac{\partial}{\partial y'} \tag{C.3}$$

satisfies the commutator

$$\left[\dot{X}, \mathbf{A}\right] = -\frac{d\xi}{dx}\mathbf{A}. \tag{C.4}$$

We observe in particular that the condition on the prolongation of the symmetry generator X is automatically given by the commutator. If we expand the latter, it follows that the only non-vanishing component is that related to the basic vector field $\frac{\partial}{\partial y'}$. From this we extract the equation defining the symmetry condition

$$\begin{aligned}&\left(y'\right)^3\frac{\partial^2\xi}{\partial y^2} + \left(y'\right)^2\left(2\frac{\partial^2\xi}{\partial x\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\omega}{\partial y'} - \frac{\partial^2\eta}{\partial y^2}\right) + \xi\frac{\partial\omega}{dx} + \eta\frac{\partial\omega}{\partial y} + \frac{\partial\eta}{\partial x}\frac{\partial\omega}{\partial y'} - \frac{\partial^2\eta}{\partial x^2}\\&+\omega\left(2\frac{\partial\xi}{\partial x} - \frac{\partial\eta}{\partial y}\right) + y'\left(3\omega\frac{\partial\xi}{\partial x} + \frac{\partial\omega}{\partial y'}\left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right) + \frac{\partial^2\xi}{\partial x^2} - 2\frac{\partial^2\eta}{\partial x\partial y}\right) = 0.\end{aligned} \tag{C.5}$$

As the components ξ and η of the symmetry generator X do not depend on y', the latter equation can be separated into a system of partial differential equations. In particular, for a second-order linear homogeneous differential equation

$$y'' + g_1\left(x\right)y' + g_2\left(x\right)y = 0, \tag{C.6}$$

we have $\omega = -g_1(x)\,y' - g_2 y(x)$, and the preceding symmetry condition separates into the following four partial differential equations (PDEs in short):

$$\begin{aligned}&\frac{\partial^2\xi}{\partial y^2}=0;\ \left(2\frac{\partial^2\xi}{\partial x\partial y}-2g_1(x)\frac{\partial\xi}{\partial y}-\frac{\partial^2\eta}{\partial x\partial y}\right)=0;\\&\frac{\partial^2\xi}{\partial x^2}-g_1(x)\frac{\partial\xi}{\partial x}-3y\,g_2(x)\frac{\partial\xi}{\partial y}-2\frac{\partial^2\eta}{\partial x\partial y}+\frac{dg_1(x)}{dx}\xi=0;\\&g_2(x)\left(\frac{\partial\eta}{\partial y}-2\frac{\partial\xi}{\partial x}-\eta\right)-g_1(x)\frac{\partial\eta}{\partial x}-\frac{\partial^2\eta}{\partial x^2}-y\frac{dg_2(x)}{dx}\xi=0.\end{aligned}\tag{C.7}$$

As the equation (C.6) is linear and homogeneous, it is well known that the symmetry algebra is maximal and isomorphic to the rank two simple Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ [30; 84; 100]. For linear homogeneous ordinary differential equations, the vector field $X_1 = y\frac{\partial}{\partial y}$ is always a point symmetry. Moreover, if

$$y(x) = \lambda_1 T(x) + \lambda_2 U(x);\ \lambda_1, \lambda_2 \in \mathbb{R} \tag{C.8}$$

denotes the general solution of (C.6), two additional independent symmetries of the equation can be chosen as

$$X_2 = T(x)\frac{\partial}{\partial y},\ X_3 = U(x)\frac{\partial}{\partial y}. \tag{C.9}$$

These three symmetries satisfy the commutators

$$[X_1, X_i] = -X_i,\ i = 2,3;\ [X_2, X_3] = 0, \tag{C.10}$$

therefore span a solvable Lie algebra of type $A_{3,3}$ [104; 139]. It is well known that a scalar second-order ODE admits a Lie algebra of point symmetries of dimension $n = 0,1,2,3,8$ [100; 103]. From the various studies concerning the structure of linearizable differential equations (see e.g. [6; 40; 103; 104] and references therein), it follows that invariance with respect to the solvable algebra $A_{3,3}$ implies that the symmetry algebra of the ODE (C.6) is maximal, hence isomorphic to $\mathfrak{sl}(3,\mathbb{R})$. By means of a local transformation, the ODE can be reduced to the free particle equation $z'' = 0$.

For later use, for the general solution (C.8) of equation (C.6) we denote the Wronskian $W\{T(x), U(x)\}$ as

$$\mathbf{W} = \det\begin{pmatrix} T(x) & U(x) \\ \frac{d}{dt}T(x) & \frac{d}{dt}U(x)\end{pmatrix}. \tag{C.11}$$

C.1.1 Symmetries of the Chebyshev equation

Chebyshev polynomials possibly constitute the simplest case of orthogonal polynomials, and possess various interesting structural properties that have found extensive application in numerical analysis [130]. In contrast to the other classical orthogonal polynomials, the Laguerre, Legendre and Hermite polynomials, which appear in a wide variety of physical problems and therefore are of considerable importance in the description of natural phenomena [89], Chebyshev polynomials appear only marginally (e.g. in connection with the Lissajous figures [71], although they have found extensive application in approximation theory and numerical methods [130].

The Chebyshev polynomials of first and second kind are defined by

$$T_n(x) = \cos(n \arccos x) = \frac{1}{2}\left[\left(x + i\sqrt{1-x^2}\right)^n + \left(x - i\sqrt{1-x^2}\right)^n\right], \tag{C.12}$$

$$U_n(x) = \sin(n \arccos x) = \frac{1}{2i}\left[\left(x + i\sqrt{1-x^2}\right)^n - \left(x - i\sqrt{1-x^2}\right)^n\right]. \tag{C.13}$$

An alternative formulation of Chebyshev polynomials is given by means of the functions

$$V_n(x) = \cos(n \arcsin x),\ W_n(x) = \sin(n \arcsin x) \tag{C.14}$$

satisfying the following identities:

$$\begin{array}{ll} V_{2m+1}(x) = (-1)^m U_{2m+1}(x), & V_{2m}(x) = (-1)^m T_{2m}(x), \\ W_{2m+1}(x) = (-1)^m T_{2m+1}(x), & W_{2m}(x) = (-1)^m U_{2m}(x). \end{array} \tag{C.15}$$

For the weight function $\varphi(x) = \left(\sqrt{1-x^2}\right)^{-1}$ and the closed interval $[-1,1]$, the Chebyshev polynomials satisfy the orthogonality relations

$$\int_{-1}^{1} T_n(x)\, T_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0 \end{cases} \tag{C.16}$$

and

$$\int_{-1}^{1} U_n(x)\, U_m(x) \frac{dx}{\sqrt{1-x^2}} = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}. \tag{C.17}$$

The Chebyshev polynomials can also be constructed by means of an iterative process using the Rodrigues formula (see e.g. [52; 145]). For any $n \geq 0$, $T_n(x)$ and $U_n(x)$ are respectively given by a n^{th}-order differential operator

$$T_n(x) = (-1)^n 2^n \frac{n!}{(2n)!} \sqrt{1-x^2} \frac{\mathrm{d}^n}{\mathrm{d}\,x^n} \left[1-x^2\right]^{n-\frac{1}{2}}, \tag{C.18}$$

$$U_n(x) = (-1)^{n-1} 2^n \frac{n!\,n}{(2n)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\,x^{n-1}} \left[1-x^2\right]^{n-\frac{1}{2}}. \tag{C.19}$$

From the latter identities we can easily deduce the relations

$$\frac{d}{dx} T_n(x) = \frac{n}{\sqrt{1-x^2}} U_n(x); \quad \frac{d}{dx} U_n(x) = -\frac{n}{\sqrt{1-x^2}} T_n(x), \tag{C.20}$$

from which we get

$$\frac{d}{dx} T_n(x) \frac{d}{dx} U_n(x) + n \frac{T_n(x)\, U_n(x)}{1-x^2} = 0 \tag{C.21}$$

for any n. Using these relations, it is straightforward to verify that the polynomials $T_n(x)$ and $U_n(x)$ are independent solutions of the linear homogeneous second-order ODE

$$\left(1-x^2\right) y'' - x\, y' + n^2 y = 0. \tag{C.22}$$

As observed in the previously, three of the symmetry generators are immediate:

$$X_1 = y\frac{\partial}{\partial y}, \; X_2 = T_n(x) \frac{\partial}{\partial y}, \; X_3 = U_n(x) \frac{\partial}{\partial y}. \tag{C.23}$$

We conclude that the Chebyshev equation (C.22) has maximal symmetry $\mathcal{L} \simeq \mathfrak{sl}(3,\mathbb{R})$, and hence constitutes a linearizable equation. Discarding the case $n = 0$ for being reducible, if we compute the point symmetries for equation (C.22) and $n = 1$, we find that the symmetries (C.23), together with the following vector fields, form a basis of $\mathcal{L}$:[1]

$$\begin{array}{l} X_4 = \sqrt{1-x^2} y \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right); \; X_5 = \left(x^2-1\right) y \frac{\partial}{\partial x} + y^2 x \frac{\partial}{\partial y}; \; X_6 = x\sqrt{1-x^2} \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right); \\ X_7 = x\left(x^2-1\right)\frac{\partial}{\partial x} + \left(x^2+1\right) y \frac{\partial}{\partial y}; \; X_8 = \sqrt{1-x^2}\left(\left(x^2-1\right)\frac{\partial}{\partial x} + yx\frac{\partial}{\partial y}\right). \end{array} \tag{C.24}$$

As $T_1(x) = x$ and $U_1(x) = \sqrt{1-x^2}$, it follows at once that the symmetry generators (C.24) can all be expressed in terms of the Chebyshev polynomials for $n = 1$, using appropriately the relations (C.18)-(C.19) and those derived from them. It is therefore natural to ask whether this realization for the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ can be modified in order to describe the point symmetries of the Chebyshev equation for arbitrary n. The answer, which is in the affirmative, will be proven to remain valid for differential equations having solutions of trigonometric and hyperbolic types.

[1] This specific realization differs from that considered in [67].

C.2 Functional realization of $\mathfrak{sl}(3,\mathbb{R})$

As follows from equation (C.12), the Chebyshev polynomials constitute a particular case of trigonometric functions of the type

$$y(x) = \lambda_1 \sin H(x) + \lambda_2 \cos H(x), \tag{C.25}$$

with $H(x)$ being an arbitrary differentiable function and $\lambda_1, \lambda_2 \in \mathbb{R}$. Functions of the form (C.25) can be shown to be solutions to the linear second-order homogeneous equation

$$\frac{dy^2}{dx^2} - \left(\frac{\frac{d^2H}{dx^2}}{\frac{dH}{dx}}\right)\frac{dy}{dx} + \left(\frac{dH}{dx}\right)^2 y = 0. \tag{C.26}$$

In analogy with the previous example, it is reasonable to ask whether for this equation, that also exhibits maximal $\mathfrak{sl}(3,\mathbb{R})$-symmetry, the symmetry generators can be described generically in terms of the fundamental solutions $T(x) = \sin H(x)$ and $U(x) = \cos H(x)$. Making the substitution $g(x) = \left(\frac{dH}{dx}\right)^2$, the ODE (C.26) transforms onto

$$\frac{dy^2}{dx^2} - \frac{g'(x)}{2g(x)}\frac{dy}{dx} + g(x)\, y = 0. \tag{C.27}$$

Skipping the assumption that $g(x)$ is obtained from the derivative of $H(x)$, we can formulate the symmetry problem for the more general ODE (C.27). Without loss of generality, we can suppose that the general solution of this equation is given by $y(x) = \lambda_1 T(x) + \lambda_2 U(x)$, where $T(x)$ and $U(x)$ are two independent solutions.

Proposition C.1. *For arbitrary functions $g(x) \neq 0$, the vector fields*

$$X_4 = -\frac{T'(x)}{g(x)} y \frac{\partial}{\partial x} + T(x)\, y^2 \frac{\partial}{\partial y}; \quad X_5 = -\frac{U'(x)}{g(x)} y \frac{\partial}{\partial x} + U(x)\, y^2 \frac{\partial}{\partial y} \tag{C.28}$$

are point symmetries of (C.27).

Proof. Let $T(x)$ be a solution of the ODE (C.27). Denoting $\frac{dT}{dx} = T'(x)$, the prolongation $\dot{X}_4$ of the vector field X_4 is explicitly given by

$$\dot{X}_4 = -\frac{T'(x)}{g(x)} y \frac{\partial}{\partial x} + T(x)\, y^2 \frac{\partial}{\partial y} + \left(y^2 T'(x) + yy' \frac{T(x)\, g(x) - T'(x)\, g'(x)}{g^2(x)} + y'^2 \frac{T'(x)}{g(x)}\right)\frac{\partial}{\partial y'}. \tag{C.29}$$

We further define the quantity $R = \left(\frac{d^2T}{dx^2} - \frac{g'(x)}{2\,g(x)}\frac{dT}{dx} + g(x)\, T(x)\right)$, which reduces to zero as $T(x)$ solves the equation. If we now evaluate the commutator (C.4), after some simplification we obtain the following expression for the symmetry condition:

$$\left(\frac{-2y'^2}{g(x)} + \frac{yy'g'(x)}{g^2(x)} + y^2\right) R - \frac{yy'}{g(x)}\frac{dR}{dx} = 0, \tag{C.30}$$

showing that X_4 is a point symmetry. Permuting $T(x)$ and $U(x)$, the same argument shows that X_5 is also a symmetry of the ODE. □

Clearly the vector fields $\{X_1, \cdots, X_5\}$ are independent, as their component in $\frac{\partial}{\partial y}$ depends on different powers of y and $U(x), T(x)$ are independent solutions of the ODE. In order to complete a basis of symmetries, we use the fact that the commutator of two point symmetries is a point symmetry [30; 59]. To this extent, we consider the additional vector fields

$$\begin{aligned} X_6=[X_2,X_4] &= -\tfrac{T(x)U'(x)}{g(x)}\tfrac{\partial}{\partial x}+\tfrac{2T(x)U(x)g(x)+T'(x)U'(x)}{g(x)}y\tfrac{\partial}{\partial y},\\ X_7=[X_2,X_5] &= -\tfrac{T(x)T'(x)}{g(x)}\tfrac{\partial}{\partial x}+\tfrac{2T^2(x)g(x)+(T'(x))^2}{g(x)}y\tfrac{\partial}{\partial y},\\ X_8=[X_3,X_5] &= -\tfrac{T'(x)U(x)}{g(x)}\tfrac{\partial}{\partial x}+\tfrac{2T(x)U(x)g(x)+T'(x)U'(x)}{g(x)}y\tfrac{\partial}{\partial y} \end{aligned} \tag{C.31}$$

We observe that, according to the ODE (C.27), the solutions $T(x)$ and $U(x)$ are functionally related through $g(x)$, i.e.,

$$g(x)=\frac{T'(x)}{C_1-T^2(x)}=\frac{U'(x)}{C_2-U^2(x)} \tag{C.32}$$

for some constants C_1, C_2. As $T(x)$ and $U(x)$ do not themselves reduce to constants, a routine but cumbersome computation shows that $\{X_6, X_7, X_8\}$ are linearly independent vector fields. As a consequence, $\{X_1,\cdots,X_8\}$ are linearly independent and can be taken as a basis of the symmetry algebra $\mathcal{L}$ of the differential equation (C.27). We will see that this realization describes the symmetries of the ODE for arbitrary choices of $g(x)$, always yielding the same commutators.

We now proceed to compute the structure constants of $\mathfrak{sl}(3,\mathbb{R})$ over the preceding basis. Up to now, the only known commutators are the following:

$$\begin{array}{llll} [X_1,X_2]=-X_2, & [X_1,X_3]=-X_3, & [X_1,X_4]=X_4, & [X_1,X_5]=X_5,\\ [X_1,X_6]=0, & [X_1,X_7]=0, & [X_1,X_8]=0, & [X_2,X_3]=0,\\ [X_2,X_4]=X_6, & [X_2,X_5]=X_7, & [X_3,X_5]=X_8. & \end{array} \tag{C.33}$$

We observe in particular that X_1 acts diagonally on the remaining generators, hence it must belong to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(3,\mathbb{R})$.[2] Further, as $\mathfrak{h}$ is an Abelian subalgebra of $\mathfrak{sl}(3,\mathbb{R})$, the brackets (C.33) imply that a second generator of $\mathfrak{h}$ must be a linear combination of X_6, X_7 and X_8. However, as the functions $g(x)$, $U(x)$ and $T(x)$ are unknown, we ignore the coefficients of the remaining commutators. One possibility to circumvent this difficulty is to forget provisionally that $\{X_1,\cdots,X_8\}$ arise as symmetries of an ODE and focus only on the algebraic problem. We suppose that $\{X_1,\cdots,X_8\}$ are independent generators of an eight-dimensional Lie algebra, and that the commutators (C.33) hold. Using the Jacobi condition, we can derive a parameterized expression for the remaining brackets, or alternatively we can compute the Maurer–Cartan equations associated to this basis [132]. Proceeding like this, it follows that $\{X_1,\cdots,X_8\}$ define a Lie algebra if the following conditions hold:

$$\begin{array}{lll} [X_2,X_6]=2a_1a_2X_2, & [X_2,X_7]=2a_1X_2, & [X_2,X_8]=a_1a_2X_2+a_1X_3,\\ [X_3,X_4]=\alpha X_1+a_2X_6+a_3X_7+a_2X_8, & & [X_3,X_6]=a_1\left(a_2X_3-a_3X_2\right),\\ [X_3,X_7]=a_1\left(a_2X_2+X_3\right), & & [X_3,X_8]=2a_1a_2X_3,\\ [X_4,X_5]=0, & [X_4,X_6]=-2a_1a_2X_4, & [X_4,X_7]=-a_1\left(X_4+a_2X_5\right),\\ [X_4,X_8]=a_1\left(a_3X_5-a_2X_4\right), & & [X_5,X_6]=-a_1\left(X_4+a_2X_5\right),\\ [X_5,X_7]=-2X_5, & [X_5,X_8]=-2a_1a_2X_5, & [X_6,X_7]=a_1\left(X_6-a_2X_7\right),\\ [X_6,X_8]=a_1\left(\alpha X_1+a_2X_6+2a_3X_7+a_2X_8\right), & & [X_7,X_8]=a_1\left(X_8-a_2X_7\right), \end{array} \tag{C.34}$$

where $a_1,a_2,a_3\in\mathbb{R}$ are arbitrary constants and $\alpha=-3a_1\left(a_3+a_2^2\right)$. As the Lie algebra must be isomorphic to $\mathfrak{sl}(3,\mathbb{R})$, its Killing form κ must be non-degenerate [132]. A routine computation shows that κ has the following determinant:

$$\det\kappa=-559872\,a_1\left(a_3+a_2^2\right)^4. \tag{C.35}$$

Thus, if κ is non-degenerate, then we must always have $a_1\neq 0$ and $a_3\neq -a_2^2$. As $a_1\neq 0$, a change of scale always allows us to suppose that $a_1=1$.

[2] For the basic definitions on Lie algebras, see [139; 155].

We now return to the interpretation of $\{X_1, \cdots, X_8\}$ as point symmetries of (C.27) in the realization (C.28)-(C.31). In order to satisfy the commutators (C.34), the functions $T(x), U(x)$ and their derivatives $T'(x), U'(x)$ will have to satisfy certain supplementary constraints that depend on the specific values of a_2 and a_3. One one hand, such constraints will enable us to deduce relations between $T(x)$, $U(x)$ and further $g(x)$, in order to appear as the solutions of the differential equation (C.27). On the other hand, we will derive the admissible values of a_2 and a_3 for which the commutators are compatible with the generators being realized as vector fields.[3]

Developing formally the commutator of X_2 and X_6, imposition of the identity $[X_2, X_6] + 2a_2X_2 = 0$ forces the functional relation

$$a_2T'(x) + T^2(x)U'(x) - U'(x) - T(x)U(x)T'(x) = 0. \tag{C.36}$$

It is not difficult to see that this equation admits an integrating factor, enabling us to rewrite (C.36) as

$$\frac{d}{d\,x}\left((U(x) - a_2T(x))\left(T^2(x) - 1\right)^{-\frac{1}{2}} + \beta\right) = 0 \tag{C.37}$$

for some constant β. As a consequence, we obtain that

$$U(x) = a_2T(x) - \beta\left(T^2(x) - 1\right)^{\frac{1}{2}}, \tag{C.38}$$

where $\beta \neq 0$ as $\mathbf{W} \neq 0$. Now the commutator $[X_2, X_7] + X_2 = 0$ is satisfied if

$$(T'(x))^2 + (T^2(x) - 1)g(x) = 0 \tag{C.39}$$

holds. Analyzing now the bracket $[X_3, X_4]$ we obtain, after some simplification, the numerical relation

$$\beta^2 - a_3 - a_2^2 = 0. \tag{C.40}$$

With these conditions, the only remaining commutator that still imposes a constraint is $[X_6, X_8]$. Developing the latter leads to the condition

$$a_2\left(T(x)^2 - 1\right)\left(a_2T(x) - \sqrt{a_3 + a_2^2}\sqrt{T^2(x) - 1}\right) = 0. \tag{C.41}$$

As the solution $T(x) = \text{cons.} \in \mathbb{R}$ is excluded by the previous conditions (as otherwise the Wronskian is $\mathbf{W} = 0$), we necessarily have that $a_2 = 0$. In particular, we have that the admissible functions for which the realization (C.28)-(C.31) are point symmetries of the ODE (C.27) are given by

$$g(x) = \left(T'(x)\right)^2\left(1 - T^2(x)\right)^{-1}, \tag{C.42}$$

$$U(x) = \sqrt{a_3}\sqrt{T^2 - 1}. \tag{C.43}$$

As a consequence, the squares of the functions $U(x)$ and $T(x)$ are related by

$$U^2(x) - a_3T^2(x) + a_3 = 0. \tag{C.44}$$

We observe in particular that, in addition, the identity

$$g(x)T(x)U(x) + T'(x)U'(x) = 0 \tag{C.45}$$

linking $g(x)$ with the solution of the ODE is satisfied.

In view of this result, we will essentially obtain two types of functions (trigonometric and hyperbolic) for which the symmetries are given by (C.28)-(C.31), depending on the sign of the parameter a_3. From (C.34), we obtain that the commutator table for the point symmetry algebra $\mathcal{L}$ and the given realization is the following:[4]

[3] In other words, the values for which the vector fields define a realization of the Lie algebra $\mathfrak{sl}(3,\mathbb{R})$. Details can be found in [73].

[4] As the Lie bracket is skew-symmetric, we only display the commutators $[X_i, X_j]$ for $i < j$.

$[\cdot,\cdot]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_2$	$-X_3$	X_4	X_5	0	0	0
X_2		0	0	X_6	X_7	0	$2X_2$	X_3
X_3			0	$3a_3X_1 + a_3X_7$	X_8	$-a_3X_2$	X_3	0
X_4				0	0	0	$-X_4$	a_3X_5
X_5					0	$-X_4$	$-2X_5$	0
X_6						0	X_6	$-3a_3X_1 + 2a_3X_7$
X_7							0	X_8
X_8								0

(C.46)

C.2.1 Noether symmetries

The differential equation (C.27) can be seen as the equation of motion of a particle in one dimension. As such systems are always integrable and conservative (see e.g. [80; 121; 131]), it follows that there exists a Lagrangian $L(x,y,y')$ such that (C.27) arises as the Lagrange equation of second kind

$$\frac{d}{d\,x}\left(\frac{\partial\,L}{\partial\,y'}\right) - \frac{\partial\,L}{\partial\,y} = 0. \tag{C.47}$$

In particular, as the ODE (C.27) can be reduced to the free particle equation $z''(s) = 0$ by means of a local transformation [30; 119], it follows that the symmetry algebra $\mathcal{L}$ must contain a five-dimensional subalgebra $\mathcal{L}_{NS}$ corresponding to Noether symmetries [140].

Recall that a point symmetry $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ is called a Noether symmetry if there exists a function $V(x,y)$ such that the condition

$$\dot{X}(L) + A(\xi)\,L - A(V) = 0 \tag{C.48}$$

is satisfied. As a consequence, the function

$$\psi = \xi(x,y)\left[y'\frac{\partial L}{\partial y'} - L\right] - \eta(x,y)\frac{\partial L}{\partial y'} + V(x,y) \tag{C.49}$$

will be a constant of the motion of the system [71; 131].

For the Lagrangian defined as

$$L\left(x,y,y'\right) = \frac{1}{2\sqrt{g(x)}}\left((y')^2 - g(x)y^2\right), \tag{C.50}$$

the equation of motion is equivalent to the differential equation (C.27), so that, without loss of generality, we can suppose that L is the Lagrangian of the system.

Evaluating the symmetry condition (C.48) for L and separating the resulting expression into powers of y', we obtain the system of PDEs for the components of a Noether symmetry:

$$\frac{\partial\xi}{\partial y} = 0, \quad \xi\,(x,y)\,g'\,(x) - 4g\,(x)\,\frac{\partial\eta}{\partial y} + 2g\,(x)\,\frac{\partial\xi}{\partial x} = 0, \tag{C.51}$$

$$2g^2\,(x)\,\frac{\partial\xi}{\partial y} + -4g\,(x)\,\frac{\partial\eta}{\partial x} + 4g\,(x)^{\frac{3}{2}}\,\frac{\partial V}{\partial y} = 0, \tag{C.52}$$

$$\xi\,(x,y)\,g\,(x)\,g'\,(x)\,y^2 + 4g^2\,(x)\,y\,\eta\,(x,y) + 2g^2\,(x)\,y^2\frac{\partial\xi}{\partial x} + 4g\,(x)^{\frac{3}{2}}\,\frac{\partial V}{\partial x} = 0. \tag{C.53}$$

The first condition implies that $\xi(x,y) = \varphi(x)$. Inserting this into the second equation further shows that $\eta(x,y)$ satisfies the equation

$$\frac{\partial \eta}{\partial y} = \frac{1}{4}\left(\frac{\varphi'(x)\, g'(x)}{g(x)} + 2\varphi'(x)\right) \tag{C.54}$$

with solution $\eta(x,y) = \frac{1}{4}\left(\frac{\varphi'(x)g'(x)}{g(x)} + 2\varphi'(x)\right) y + \theta(x)$. Therefore, the generic form of a Noether symmetry is given by

$$X = \varphi(x)\frac{\partial}{\partial x} + \left(\frac{1}{4}\left(\frac{\varphi(x)\, g'(x)}{g(x)} + 2\varphi'(x)\right) y + \theta(x)\right)\frac{\partial}{\partial y}. \tag{C.55}$$

Reordering the terms and simplifying, the third equation can be brought to the form

$$\begin{aligned}&4g(x)^{\frac{5}{2}}\frac{\partial V}{\partial y} + y\left[\varphi(x)\left(g'(x)^2 - g(x)\, g''(x)\right) - g(x)\left(2g(x)\,\varphi''(x) + g'(x)\,\varphi'(x)\right)\right]\\ &- 4g^2(x)\,\theta'(x) = 0,\end{aligned} \tag{C.56}$$

from which the expression for $V(x,y)$ is obtained as

$$\begin{aligned}V(x,y) &= -\frac{1}{8}\frac{\left[\varphi(x)\left(g'(x)^2 - g(x)\, g''(x)\right) - g(x)\left(2g(x)\,\varphi''(x) + g'(x)\,\varphi'(x)\right)\right]}{g(x)^{\frac{5}{2}}} y^2 + h(x)\\ &+ \frac{g^2(x)\,\theta'(x)}{g(x)^{\frac{5}{2}}} y.\end{aligned} \tag{C.57}$$

Inserting $\xi(x,y)$, $\eta(x,y)$ and $V(x,y)$ into the equation (C.53) and simplifying the resulting expression, we finally obtain the conditions to be satisfied by $\varphi(x)$ and $\theta(x)$ in order to define a Noether symmetry:

$$\theta(x)\, g(x) - \frac{1}{2}\frac{g'(x)}{g(x)}\theta'(x) + \theta''(x) = 0, \tag{C.58}$$

$$\begin{aligned}&4g(x)^2\,\varphi'''(x) + \varphi'(x)\left(4g(x)^2 g''(x) + 16g(x)^4 - 5g(x)\, g'(x)^2\right) +\\ &\varphi(x)\left(2g(x)^2 g'''(x) + 5g'(x)^3 - 7g(x)\, g'(x)\, g''(x) + 8g(x)^3 g'(x)\right) = 0.\end{aligned} \tag{C.59}$$

Now, as any Noether symmetry of (C.27) must be a linear combination of $\{X_1, \cdots, X_8\}$, we conclude from (C.28) and (C.31) that

$$X = k_1X_1 + k_2X_2 + k_3X_3 + k_6X_6 + k_7X_7 + k_8X_8, \tag{C.60}$$

as only these symmetries are at most linear in the variable y, being thus compatible with the form (C.55). Now X_2 and X_3 are clearly Noether symmetries if we take either $\theta(x) = T(x)$ or $\theta(x) = U(x)$. As a consequence, the three remaining Noether symmetries must be a linear combination of X_1, X_6, X_7 and X_8. Instead of inspecting the preceding equation (C.59) for any arbitrary linear combinations, we check whether X_6 or X_8 as given in (C.31) satisfy the condition. Take for instance X_6.[5]

Using the constraint (C.45) and the fact that $T(x), U(x)$ are independent solutions of the differential equation (C.27), the expansion of condition (C.48) applied to X_6 reduces to

$$\begin{aligned}\dot{X}_6(L) + A(\xi)\, L - A(V) &= y'\left(\frac{U(x)\, T'(x) + T(x)\, U'(x)}{\sqrt{g(x)}} y - \frac{\partial V}{\partial y}\right)\\ &- \frac{y^2}{2\sqrt{g(x)}}\left(T'(x)\, U'(x) - 5g(x)\, T(x)\, U(x)\right) - \frac{\partial V}{\partial x}.\end{aligned} \tag{C.61}$$

[5]Again, permuting $T(x)$ and $U(x)$, the analysis is extensible to the vector field X_8.

From the term in y' we obtain the auxiliary function

$$V(x,y) = \frac{U(x)\,T'(x) + T(x)\,U'(x)}{2\sqrt{g(x)}}y^2 + h(x). \quad \text{(C.62)}$$

Inserting this into the last term of (C.61) and manipulating algebraically the expression we obtain that

$$\frac{y^2\left(5g(x)\,T(x)\,U(x) - T'(x)\,U'(x)\right)}{2\sqrt{g(x)}} - \frac{\partial V}{\partial x} = -\frac{y^2\left[U(x)\,\Lambda_1 + T(x)\,\Lambda_2 + 3\,\Lambda_3\right]}{2\sqrt{g(x)}} + h'(x), \quad \text{(C.63)}$$

where the Λ_i $(1 \leq i \leq 3)$ are defined as

$$\begin{aligned}\Lambda_1 &= \left(T''(x) - \frac{g'(x)}{2g(x)}T'(x) + g(x)\,T(x)\right),\\ \Lambda_2 &= \left(U''(x) - \frac{g'(x)}{2g(x)}U'(x) + g(x)\,U(x)\right),\\ \Lambda_3 &= \left(T(x)\,U(x)\,g(x) + T'(x)\,U'(x)\right).\end{aligned}$$

As $T(x)$ and $U(x)$ are solutions of (C.27), it is immediate that $\Lambda_1 = \Lambda_2 = 0$, while $\Lambda_3 = 0$ follows from the constraint (C.45). We conclude that for $h(x) = \alpha \in \mathbb{R}$, the point symmetry X_6 is also a Noether symmetry. Changing $T(x)$ for $U(x)$ further shows that X_8 also constitutes a Noether symmetry of the equation. Using that Noether symmetries are preserved by commutators [140], it follows that $[X_6, X_8] = -3a_3X_1 + 2a_3X_7$ is also a Noether symmetry.

Proposition C.2. *Any Noether symmetry X of the ODE (C.27) has the form*

$$X = \lambda_1 X_2 + \lambda_2 X_3 + \lambda_3 X_6 + \lambda_4 X_8 + \lambda_5 [X_6, X_8] \quad \text{(C.64)}$$

for some scalars $\lambda_1, \cdots, \lambda_5 \in \mathbb{R}$. In particular, the Lie algebra $\mathcal{L}_{NS}$ of Noether symmetries admits the following Levi decomposition

$$\mathcal{L}_{NS} = \mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{V_2} \mathbb{R}^2, \quad \text{(C.65)}$$

where the Levi subalgebra $\mathfrak{s} = \mathfrak{sl}(2,\mathbb{R})$ is generated by X_6, X_8 and $Y = [X_6, X_8]$. The symmetries X_2, X_3 transform according to the 2-dimensional irreducible representation V_2 of $\mathfrak{s}$.

The first part follows from the previous computations. Now, using Table (C.46), the commutators of the Noether symmetries are the following:

$[\cdot,\cdot]$	Y	X_6	X_8	X_2	X_3
Y	0	$-2a_3X_6$	$2a_3X_8$	$-a_3X_2$	a_3X_3
X_6		0	Y	0	a_3X_2
X_8			0	$-X_3$	0
X_2				0	0
X_3					0

(C.66)

showing that the Levi subalgebra is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and generated by X_6, X_8 and Y. The symmetries X_2 and X_3 are easily seen to form a maximal solvable ideal of $\mathcal{L}_{NS}$, hence the Levi decomposition of the Lie algebra is given by (C.65).

From (C.49) it is immediate to verify that the constants of the motion associated to the symmetries X_2 and X_3 are

$$\psi_1 = \frac{y'T(x) - y\,T'(x)}{\sqrt{g(x)}}, \quad \psi_2 = \frac{y'U(x) - y\,U'(x)}{\sqrt{g(x)}}. \quad \text{(C.67)}$$

For the symmetries in the Levi subalgebra $\mathfrak{sl}(2,\mathbb{R})$, the constants of the motion are quadratic in y' and easily seen to be functionally dependent on ψ_1 and ψ_2, just as it is expected from the free particle equation $z'' = 0$ [80; 71].

C.3 Orthogonal functions as solutions to the ODE (C.27)

Among the many interesting questions arising in the theory of special functions, considerable attention has been devoted to the problem of obtaining and characterizing orthogonal polynomials by means of differential equations [4; 32; 66; 78; 87; 99]. In this context, it is well known that the so-called classical orthogonal polynomials constitute essentially the only class to be determined by a second-order differential equation of Sturm-Liouville type [4; 78].
An important structural result in the theory of classical orthogonal polynomials states that the Rodrigues formula

$$F_n(x) = \frac{1}{p(x)} D^n \left[p(x) Q(x)^n \right], \tag{C.68}$$

for quadratic polynomials $Q(x) = (b-x)(x-a)$, where $p(x)$ is a weight function in the finite interval (a,b), provides a polynomial $F_n(x)$ of degree n in x for any $n \geq 0$ only if the weight function has the form

$$p(x) = (b-x)^{\alpha}(x-a)^{\beta}; \; \alpha > -1, \; \beta > -1. \tag{C.69}$$

The $F_n(x)$ correspond to the class of Jacobi polynomials. For polynomials $Q(x)$ of degrees $d \leq 1$ two other cases are given, corresponding to the Laguerre and Hermite polynomials (see e.g. [147]). The possible orders of linear differential equations satisfied by non-classical or generalized orthogonal polynomials have been analyzed by various authors, albeit no generically valid analogue of the Rodrigues formula has been found for these generalizations [78; 93]. The formula (C.68) is interesting in its own right, as it allows to derive the second-order linear ordinary differential equation satisfied by the polynomial $F_n(x)$ for any $n \geq 0$. For the case of a weight function (C.69), the ODE has the following form:[6]

$$\begin{aligned} &(x-a)(b-x) F_n''(x) + (a(1+\alpha) + b(1+\beta) - (2+\alpha+\beta)x) F_n'(x) \\ &+ n(n+1+\alpha+\beta) F_n(x) = 0. \end{aligned} \tag{C.70}$$

As seen before, the Chebychev polynomials $T_n(x)$ and $U_n(x)$ arise from (C.68) for $a = -1, b = 1$ and $\alpha = \beta = -\frac{1}{2}$.[7] In this context, the question arises whether, besides the Chebyshev case previously considered, there are other possible values of α, β, a and b such that the ODE (C.70) has the form (C.27). Starting from the function

$$g(x) = \frac{n(n+1+\alpha+\beta)}{(x-a)(b-x)},$$

the constraint

$$-\frac{g'(x)}{2g(x)} = \frac{(a(1+\alpha) + b(1+\beta) - (2+\alpha+\beta)x)}{(x-a)(b-x)} \tag{C.71}$$

leads, after comparison of the differential equations (C.27) and (C.70), to the equation

$$a + b - 2x = 2(a(1+\alpha) + b(1+\beta) - (2+\alpha+\beta)x). \tag{C.72}$$

It follows at once that this identity holds only if $\alpha = \beta = -\frac{1}{2}$, without any conditions on a and b. As a consequence, the orthogonal polynomials of the form

$$P_n(x) = (b-x)^{\frac{1}{2}}(x-a)^{\frac{1}{2}} D^n \left[(b-x)^{n-\frac{1}{2}}(x-a)^{n-\frac{1}{2}} \right], \; n \geq 0 \tag{C.73}$$

are always solutions of the ODE

$$(x-a)(b-x) F_n''(x) + \left(\frac{a}{2} + \frac{b}{2} - x \right) F_n'(x) + n^2 F_n(x) = 0 \tag{C.74}$$

of type (C.27), with $g(x)$ given by

$$g(x) = \frac{n^2}{(x-a)(b-x)}. \tag{C.75}$$

The ODE (C.74) is clearly of hypergeometric type (see [3; 52; 89]), hence the orthogonal functions obtained will always be expressible in terms of hypergeometric functions.

[6] For the general ODE for the classical orthogonal polynomials, see e.g. [147].
[7] This case is generally seen as a special case of ultraspherical functions. See [3] for details.

C.3.1 Solutions of trigonometric type

If we consider the values $b = -a$, the ODE (C.74) admits the general solution of trigonometric type

$$F_n(x) = C_1 \cos\left(n \arctan \frac{x}{\sqrt{a^2 - x^2}}\right) + C_2 \sin\left(n \arctan \frac{x}{\sqrt{a^2 - x^2}}\right). \quad \text{(C.76)}$$

With $H(x) = n \arctan\left(x\,(a^2 - x^2)^{-\frac{1}{2}}\right)$, taking $T(x) = \sin H(x)$ and $U(x) = \cos H(x)$, the constraints (C.43) and (C.43) are satisfied for $a_3 = -1$, while the relations (C.44) and (C.45) follow at once. In this case, the symmetry generators of the differential equation are given by

$$X_1 = y\frac{\partial}{\partial y};\; X_2 = \sin H(x)\frac{\partial}{\partial y};\; X_3 = \cos H(x)\frac{\partial}{\partial y};\; X_4 = \frac{\sin H(x)}{H'(x)} y\frac{\partial}{\partial x} + y^2 \cos H(x)\frac{\partial}{\partial y};$$
$$X_5 = -\frac{\cos H(x)}{H'(x)} y\frac{\partial}{\partial x} + y^2 \sin H(x)\frac{\partial}{\partial y};\; X_6 = \frac{\sin^2 H(x)}{H'(x)}\frac{\partial}{\partial x} + \frac{y}{2}\sin(2H(x))\frac{\partial}{\partial y};$$
$$X_7 = -\frac{\sin(2H(x))}{2H'(x)}\frac{\partial}{\partial x} - y\left(1 + \sin^2 H(x)\right)\frac{\partial}{\partial y};\; X_8 = -\frac{\cos^2 H(x)}{H'(x)} y\frac{\partial}{\partial x} + \frac{y}{2}\sin(2H(x))\frac{\partial}{\partial y}.$$

For this basis of generators, the commutator table of $\mathfrak{sl}(3,\mathbb{R})$ is explicitly given by

$[\cdot,\cdot]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	$-X_2$	$-X_3$	X_4	X_5	0	0	0
X_2		0	0	X_6	X_7	0	$2X_2$	X_3
X_3			0	$3X_1 - X_7$	X_8	X_2	X_3	0
X_4				0	0	0	$-X_4$	$-X_5$
X_5					0	$-X_4$	$-2X_5$	0
X_6						0	X_6	$3X_1 - 2X_7$
X_7							0	X_8
X_8								0

(C.77)

We observe that for $a = 1$, the preceding solution (C.76) can be simplified by means of the trigonometric identity

$$\arcsin x = \arctan \frac{x}{\sqrt{1 - x^2}},$$

and thus we recover the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$. The basis of symmetries of (C.22) is explicitly given by

$$X_1 = y\frac{\partial}{\partial y};\; X_2 = T_n(x)\frac{\partial}{\partial y};\; X_3 = U_n(x)\frac{\partial}{\partial y};\; X_4 = \frac{\sqrt{1-x^2}T_n(x)}{n} y\frac{\partial}{\partial x} + y^2 U_n(x)\frac{\partial}{\partial y};$$
$$X_5 = -\frac{\sqrt{1-x^2}U_n'(x)}{n} y\frac{\partial}{\partial x} + y^2 T_n(x)\frac{\partial}{\partial y};\; X_6 = \frac{\sqrt{1-x^2}T_n^2(x)}{n}\frac{\partial}{\partial x} + yT_n(x)U_n(x)\frac{\partial}{\partial y};$$
$$X_7 = -\frac{\sqrt{1-x^2}T_n(x)U_n(x)}{n}\frac{\partial}{\partial x} + y\left(1 + T_n^2(x)\right)\frac{\partial}{\partial y};$$
$$X_8 = -\frac{\sqrt{1-x^2}U_n^2(x)}{n}\frac{\partial}{\partial x} + yT_n(x)U_n(x)\frac{\partial}{\partial y}. \quad \text{(C.78)}$$

For $n = 1$, we recover exactly the vector fields in (C.24), showing that the generic realization describes naturally the basis of symmetries of the Chebyshev equation (C.22) for arbitrary values of n.

For $a \neq 1$, the orthogonal polynomials deduced from the Rodrigues formula are still deeply related to the Chebyshev case, and by means of a new scaled variable $z = x\,a^{-1}$, it can be shown that the functions

$$T_n\left(\frac{x}{a}\right),\; U_n\left(\frac{x}{a}\right),\; n \geq 0 \quad \text{(C.79)}$$

solve the differential equation (C.74). This case hence does not add essentially new variants.

For $a \neq 0$ and $b + a \neq 0$, the general solution of (C.74) can be written as

$$F_n(x) = C_1 \cos\left(n \arctan \frac{a+b+2x}{2\sqrt{(a-x)(x-b)}}\right) + C_2 \sin\left(n \arctan \frac{a+b+2x}{2\sqrt{(a-x)(x-b)}}\right). \quad (C.80)$$

In this case, the polynomials $P_n(0)$ have nonzero terms in any even and odd order, hence they will be expressible in terms of linear combinations of Jacobi polynomials [3; 145]. As an example, we enumerate the first five polynomials that arise from this choice of the parameters:

n	$P_n(x)$
0	1
1	$\frac{1}{2}((a+b)-2x)$
2	$\frac{3}{4}\left((a^2+6ab+b^2)-8(a+b)x+8x^2\right)$
3	$\frac{15}{8}\left((a^3+15a^2b+15ab^2+b^3)-6(a+3b)(3a+b)x+48(a+b)x^2-32x^3\right)$
4	$\frac{105}{16}\left((a^4+28a^3b+70a^2b^2+28ab^3+b^4)-32(a+b)\left(a^2+6ab+b^2\right)x\right)+\frac{105}{16}\left(32\left(5a^2+14ab+5b^2\right)x^2-256(a+b)x^3+128x^4\right)$
5	$\frac{945}{32}(a+b)\left(a^4+44a^3b+166a^2b^2+44ab^3+b^4\right)-\frac{4725}{16}\left(5a^2+10ab+5b^2\right)\times\left(a^2+10ab+5b^2\right)x+\frac{4725}{2}(a+b)\left(5a^2+22ab+5b^2\right)x^2-4725\left(7a^2+18ab+7b^2\right)x^3+37800(a+b)x^4-15120x^5$

The orthogonality relation for these polynomials is given by the formula

$$\int_a^b P_n(x)P_m(x)(b-x)^{-\frac{1}{2}}(x-a)^{-\frac{1}{2}}\,dx = \delta_n^m \frac{\prod_{l=0}^{n-1}(2l+1)}{2^{2n+1}}. \quad (C.81)$$

Let us finally observe that for the case $a=0$, $b=1$, the differential equation

$$x(1-x)F_n''(x) + \left(\frac{1}{2}-x\right)F_n'(x) + n^2F_n(x) = 0 \quad (C.82)$$

is of the classical Jacobi type [89]. Its general solution can be conveniently expressed as[8]

$$F_n(x) = C_1\cos\left(2n \arctan\frac{\sqrt{x}}{\sqrt{b-x}}\right) + C_2\sin\left(2n\arctan\frac{\sqrt{x}}{\sqrt{b-x}}\right). \quad (C.83)$$

The Jacobi polynomials $\mathfrak{F}_n\left(0,\frac{1}{2},x\right)$ are obviously a solution to this equation, hence considering the transformation $x = 1-2z$, the orthogonal polynomials obtained from the Rodrigues formula can be easily related to the Chebyshev polynomials $T_n(1-2z)$.

C.3.2 Solutions of hyperbolic type

Functions of hyperbolic type can be obtained as solutions of the ODE (C.27) choosing the auxiliary function $g(x) = -\left(\frac{dH}{dx}\right)^2$. In this case, we write the general solution as

$$y(x) = \lambda_1 \sinh H(x) + \lambda_2 \cosh H(x), \quad (C.84)$$

[8] The general solution can also be expressed in terms of hyperbolic functions.

and the relations (C.44)-(C.45) are satisfied for $a_3 = 1$ taking $U(x) = \cosh H(x)$, $T(x) = \sinh H(x)$.[9] The basis of point symmetries is explicitly given by

$$X_1 = y\frac{\partial}{\partial y};\; X_2 = \cosh H(x)\frac{\partial}{\partial y};\; X_3 = \sinh H(x)\frac{\partial}{\partial y};\; X_4 = \frac{\cosh H(x)}{H'(x)}y\frac{\partial}{\partial x} + y^2\sinh H(x)\frac{\partial}{\partial y};$$
$$X_5 = \frac{\sinh H(x)}{H'(x)}y\frac{\partial}{\partial x} + y^2\cosh H(x)\frac{\partial}{\partial y};\; X_6 = \frac{\cosh^2 H(x)}{H'(x)}\frac{\partial}{\partial x} + \frac{y}{2}\sinh 2H(x)\frac{\partial}{\partial y};$$
$$X_7 = \frac{\sinh(2H(x))}{2H'(x)}\frac{\partial}{\partial x} + y\left(1+\cosh^2 H(x)\right)\frac{\partial}{\partial y};\; X_8 = \frac{\sinh^2 H(x)}{H'(x)}y\frac{\partial}{\partial x} + \frac{y}{2}\sinh(2H(x))\frac{\partial}{\partial y}.$$

We will see that for suitable choices of $H(x)$, the hyperbolic functions of (C.84) define orthonormal systems of functions in the interval $[-1,1]$.

We start for example from the function

$$g(x) = n^2\left(1-x^2\right) \tag{C.85}$$

From (C.27) we have the differential equation

$$y'' + \frac{x}{1-x^2}y' + n^2\left(1-x^2\right)y = 0. \tag{C.86}$$

It is not difficult to justify that no n^{th}-order polynomials satisfy this equation for $n > 0$. The general solution can be written in terms of hyperbolic functions as

$$y(x) = C_1\sinh\left[F_n(x)\right] + C_2\cosh\left[F_n(x)\right], \tag{C.87}$$

where $F_n(x)$ is defined as

$$F_n(x) = \frac{n}{2}\left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right). \tag{C.88}$$

Making the substitution $u = \left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)$, it is straightforward to verify that

$$\int_{-1}^{1}\cosh\left(F_n(x)\right)\cosh\left(F_m(x)\right)\sqrt{1-x^2}dx = \frac{\mathrm{i}}{2}\int_{-\mathrm{i}\pi}^{0}\cosh\left(\frac{n}{2}v\right)\cosh\left(\frac{m}{2}v\right)dv \tag{C.89}$$

holds. The latter integral can be easily solved, and for $n \neq m$ we obtain that

$$\begin{aligned}\frac{\mathrm{i}}{2}\int_{-\mathrm{i}\pi}^{0}\cosh\left(\frac{n}{2}v\right)\cosh\left(\frac{m}{2}v\right)dv &= \frac{\mathrm{i}}{2}\left[\frac{\sinh\left(\frac{m-n}{2}v\right)}{m-n} + \frac{\sinh\left(\frac{m+n}{2}v\right)}{m+n}\right]_{-\mathrm{i}\pi}^{0}. \\ &= \frac{1}{2}\left[\frac{\sin\left(\frac{m-n}{2}\pi\right)}{m-n} + \frac{\sin\left(\frac{m+n}{2}\pi\right)}{m+n}\right].\end{aligned} \tag{C.90}$$

Now observe that if n and m have different parity, i.e., $n = 2p$ and $m = 2q+1$, then

$$\frac{1}{2}\left[\frac{\sin\left(\frac{\pi}{2}\right)}{2q+1-2p} + \frac{\sin\left(\frac{\pi}{2}\right)}{2q+2p+1}\right] = -\frac{2q+1}{(-2q-1+2p)(2q+2p+1)} \neq 0, \tag{C.91}$$

whereas for the case of n, m having the same parity, the integral (C.89) vanishes identically. An analogous result is obtained if we compute the integrals for the hyperbolic sine functions.

As a consequence, two possibilities are given to construct an orthogonal system of functions with $g(x)$ of type (C.85):

[9] Observe that interchanging the role of $T(x)$ and $U(x)$ always changes the sign of a_3.

1. For $g(x) = 4n^2(1-x^2)$, the fundamental solutions

$$P_n(x) = \cosh\left[n\left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)\right], \tag{C.92}$$

$$Q_n(x) = \sinh\left[n\left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)\right] \tag{C.93}$$

of the ODE

$$y'' + \frac{x}{1-x^2}y' + 4n^2\left(1-x^2\right)y = 0$$

define orthogonal systems of functions for the weight function $w(x) = \sqrt{1-x^2}$ in the interval $[-1,1]$.

2. For $g(x) = (2n+1)^2(1-x^2)$, the fundamental solutions

$$P_n(x) = \cosh\left[\frac{2n+1}{2}\left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)\right], \tag{C.94}$$

$$Q_n(x) = \sinh\left[\frac{2n+1}{2}\left(x\sqrt{x^2-1} - \ln\left(x+\sqrt{x^2-1}\right)\right)\right] \tag{C.95}$$

of the ODE

$$y'' + \frac{x}{1-x^2}y' + (2n+1)^2\left(1-x^2\right)y = 0$$

define orthogonal systems of functions for the weight function $w(x) = \sqrt{1-x^2}$ in the interval $[-1,1]$.

Considering different choices of the function $g(x)$, other orthogonal systems of functions can formally be obtained as solutions of the ODE (C.27).

C.4 Non-linear deformations

We have seen previously that the linear homogeneous ODE (C.27) admits five Noether symmetries if the equation is derived, via the Helmholtz condition, from the Lagrangian (C.50). The corresponding constants of the motion are obtained from (C.49), where the two constants linear in the velocity y' generate the remaining invariants. This fact, to a certain extent, is a direct consequence of the maximal symmetry of the equation, implying in particular that it is linearizable [121]. In this situation, we can ask how to modify the ODE (C.27) by addition of a "forcing" term such that the maximal symmetry is broken, but such that the resulting equation preserves a given subalgebra of Noether symmetries. In order to avoid those variational symmetries with a constant of the motion linear in y', this subalgebra should be chosen as the Levi subalgebra $\mathcal{L}_{NS}$. For our specific purposes, the problem can be formulated in the following terms: Does there exist a (non-linear) ODE and a Lagrangian L' such that the vector fields X_6 and X_8 of (C.31) are Noether symmetries?

Consider an arbitrary function $G(x,y,y')$ and define the extended Lagrangian

$$L_0 = \frac{1}{2\sqrt{g(x)}}\left((y')^2 - g(x)y^2\right) - G(x,y,y'). \tag{C.96}$$

The equation of motion associated to (C.96) has the form

$$y'' - \frac{g'(x)}{2g(x)}y' + g(x)y - \sqrt{g(x)}\left(\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) - \frac{\partial G}{\partial y}\right) = 0, \tag{C.97}$$

which can be interpreted as a "deformation" of the ODE (C.27) by the forcing term $G(x,y,y')$. We now require that X_6 and X_8 from (C.31) are Noether symmetries of L_0, imposing additionally that the symmetry condition (C.48) is satisfied for the same function $V(x,y)$ valid for the Lagrangian

(C.50). This will imply in particular that both differential equations share exactly the same symmetry generators, hence the addition of the forcing term can actually be seen as a symmetry breaking. On the other hand, by the properties of commutators, the deformed equation will have al least a $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries.

We observe again that, as the vector fields X_6 and X_8 are obtained by interchanging the role of $T(x)$ and $U(x)$, it suffices to compute the symmetry condition for only one of these symmetries. We make the computations for X_6. The Noether symmetry condition

$$\dot{X}_6(L_0) + A(\xi) L_0 - A(V) \tag{C.98}$$

with $V(x,y)$ as given in (C.62) leads, after some simplification using the constraint (C.45), to the following partial differential equation for $G(x,y,y')$:

$$\begin{aligned}&\left(2T(x)U(x) + \frac{T(x)U'(x)g(x)}{2}\right)G(x,y,y') + yT(x)U(x)\frac{\partial G}{\partial y} - \frac{T(x)U'(x)}{g(x)}\frac{\partial G}{\partial x}\\ &+\left(y\left(T(x)U'(x)+T'(x)U(x)\right) - y'T(x)\left(U(x)+\frac{g'(x)U'(x)}{g(x)^2}\right)\right)\frac{\partial G}{\partial y'} = 0.\end{aligned} \tag{C.99}$$

Albeit complicated in form, this PDE can be solved. As we are interested in those solutions being valid for both the symmetries X_6 and X_8, suppose that the Noether condition (C.48) is also satisfied for X_8. The corresponding PDE for $G(x,y,y')$ is obtained from (C.99) replacing $T(x)$ by $U(x)$. Taking the difference of these two equations leads to the auxiliary equation[10]

$$-\frac{\mathbf{W}}{2g(x)^2}\left(g'(x)\,y'\frac{\partial G}{\partial y'} + 2g(x)\frac{\partial G}{\partial x} - g'(x)\,G(x,y,y')\right) = 0. \tag{C.100}$$

The general solution is easily found to be

$$G(x,y,y') = \varphi\left(y, \frac{y'}{\sqrt{g(x)}}\right)\sqrt{g(x)}, \tag{C.101}$$

with φ an arbitrary function of its arguments. Inserting the latter into (C.99) and solving for φ shows that the only functions $G(x,y,y')$ for which X_6 and X_8 are Noether symmetries of the Lagrangian L_0 in (C.96) are

$$G(x,y,y') = \frac{\alpha\sqrt{g(x)}}{y^2},\ \alpha\in\mathbb{R}. \tag{C.102}$$

The perturbed equation of motion preserving the $\mathfrak{sl}(2,\mathbb{R})$-subalgebra of Noether symmetries is therefore

$$y'' - \frac{g'(x)}{2g(x)}y' + g(x)\,y - 2\alpha\frac{g(x)}{y^3} = 0. \tag{C.103}$$

We observe that if we only require invariance by either X_6 or X_8, the possibilities are wider. However, in these cases the forcing term will always contain explicitly the function $T(x)$ and $U(x)$, in addition to $g(x)$. For this reason we leave this case aside.

Proposition C.3. *For arbitrary functions $g(x)\neq 0$, the nonlinear ODE (C.103) possesses a Lie algebra of point symmetries isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. Moreover, any point symmetry is a Noether symmetry for the Lagrangian*

$$L_0 = \frac{1}{2\sqrt{g(x)}}\left((y')^2 - g(x)\,y^2\right) - \frac{\alpha\sqrt{g(x)}}{y^2}. \tag{C.104}$$

[10] Recall that $\mathbf{W}$ is the Wronskian of (C.27).

Proof. For the equation of motion associated to the Lagrangian L_0 we have the auxiliary function

$$\omega\left(x,y,y'\right) = \frac{g'\left(x\right)}{2g\left(x\right)}y' - g\left(x\right)y + 2\alpha\frac{g\left(x\right)}{y^3}. \tag{C.105}$$

The symmetry condition for point symmetries obtained from (C.5) is given by the system

$$\frac{\partial^2\xi}{\partial y^2} = 0; \quad \frac{g'\left(x\right)}{g\left(x\right)}\frac{\partial\xi}{\partial y} + 2\frac{\partial^2\xi}{\partial x\partial y} - \frac{\partial^2\eta}{\partial y^2} = 0; \tag{C.106}$$

$$\frac{\partial^2\xi}{\partial x^2} + \frac{g'\left(x\right)}{2g\left(x\right)}\frac{\partial\xi}{\partial x} + 3g\left(x\right)\left(\frac{2\alpha}{y^3} - y\right)\frac{\partial\xi}{\partial y} + \frac{g'\left(x\right)}{2g\left(x\right)}\left(1 - \frac{g'\left(x\right)}{g\left(x\right)}\right)\xi\left(x,y\right) - 2\frac{\partial^2\eta}{\partial x\partial y} = 0; \tag{C.107}$$

$$g\left(x\right)\left(y - \frac{2\alpha}{y^3}\right)\left(\frac{\partial\eta}{\partial y} - 2\frac{\partial\xi}{\partial x} - g'\left(x\right)\xi\left(x,y\right)\right) - \left(1 + \frac{6\alpha}{y^4}\right)g\left(x\right)\eta\left(x,y\right) - \frac{\partial^2\eta}{\partial x^2}$$
$$+ \frac{g'\left(x\right)}{2g\left(x\right)}\frac{\partial\eta}{\partial x} = 0. \tag{C.108}$$

From the first two equations we immediately obtain that

$$\xi\left(x,y\right) = F_{11}\left(x\right)y + F_{12}\left(x\right);\ \eta\left(x,y\right) = \frac{y^2}{2}\left(\frac{g'\left(x\right)F_{11}\left(x\right)}{g\left(x\right)} + 2F_{11}'\left(x\right)\right) + F_{21}\left(x\right)y + F_{22}\left(x\right). \tag{C.109}$$

Inserting these functions into the equations (C.107) and (C.108) and separating with respect to the powers of y, the free terms provide us with the two constraints $F_{11}\left(x\right) = F_{22}\left(x\right) = 0$. This simplification further leads to the relation

$$F_{12}\left(x\right)g'\left(x\right) + 2F_{12}'\left(x\right)g\left(x\right) - 4g\left(x\right)F_{21}\left(x\right) = 0, \tag{C.110}$$

enabling us to write the components ξ and η of a point symmetry X as

$$\xi\left(x,y\right) = F_{12}\left(x\right), \tag{C.111}$$

$$\eta\left(x,y\right) = \frac{y}{4}\left(2F_{12}'\left(x\right) + \frac{g'\left(x\right)}{g\left(x\right)}F_{12}\left(x\right)\right). \tag{C.112}$$

In order to satisfy the equation (C.108), the function $F_{12}\left(x\right)$ must be a solution to the differential equation

$$\frac{d^3F_{12}}{dx^3} + \frac{4g''\left(x\right)g\left(x\right) + 16g\left(x\right)^3 - 5\left(g'\left(x\right)\right)^2}{4g\left(x\right)^2}\frac{dF_{12}}{dx} + \frac{g'''\left(x\right)}{2g\left(x\right)} - \frac{7g'\left(x\right)g''\left(x\right)}{4g\left(x\right)^3}$$
$$\frac{+8g'\left(x\right)g\left(x\right)^3 + 5\left(g'\left(x\right)\right)^3}{4g\left(x\right)^3} = 0. \tag{C.113}$$

This shows that the dimension of the symmetry algebra is at most 3. Now, since the Noether symmetries X_6, X_8 and $[X_6, X_8]$ are also point symmetries of the equation [141], they automatically satisfy the ODE (C.113). From the commutator of the Noether symmetries it follows at once that $\mathcal{L} \simeq \mathcal{L}_{NS} \simeq \mathfrak{sl}\left(2,\mathbb{R}\right)$. □

This result implies in particular that the perturbed or deformed differential equation (C.103) is not linearizable, and hence genuinely non-linear. It is worthy to be mentioned that differential equations invariant under $\mathfrak{sl}(2,\mathbb{R})$ have been analyzed by various authors in connection with non-linear equations, as well as in the context of systems admitting constants of the motion of a specific type [33; 57; 74; 77; 122].

We now determine the constants of the motion of the Lagrangian (C.104) using formula (C.49): for the symmetry X_6, the appropriate function $V\left(x,y\right)$ is given by (C.62) for $h\left(x\right) = 0$, resulting in the expression

$$\psi_1 = -\frac{(U'(x)\,T'(x) + 2U(x)\,T(x)\,g(x))}{g(x)\,\sqrt{g(x)}}\,yy' + \frac{U(x)\,T'(x)}{2\sqrt{g(x)}}y^2 + \frac{\alpha T(x)\,U'(x)}{\sqrt{g(x)}y^2}$$
$$-\frac{T(x)\,U'(x)}{g(x)^{\frac{3}{2}}}\,(y')^2. \tag{C.114}$$

For X_8, the function $V(x,y)$ is the same, and the constant of the motion ψ_2 is obtained by simply permuting $U(x)$ and $T(x)$. In particular, taking the difference $\psi_2 - \psi_1$ we obtain the simplified invariant

$$\psi_2 - \psi_1 = \frac{\mathbf{W}}{\sqrt{g(x)}}\left(\frac{(y')^2}{g(x)} + \frac{y^2}{2} - \frac{\alpha}{y^2}\right). \tag{C.115}$$

Now, for $[X_6, X_8] = -a_3(3X_1 - 2X_7)$ the function $V(x,y)$ satisfying condition (C.48) is $V(x,y) = \frac{2y^2T(x)T'(x)}{\sqrt{g(x)}}$ and the constant of the motion

$$\psi_3 = -\frac{T(x)\,T'(x)\left((y')^2 + g(x)\left(y^2 + \frac{2\alpha}{y^2}\right)\right) + y\,y'\left(4g(x)\,T^2(x) - 3g(x) + 2\left(T'(x)^2\right)\right)}{g(x)\,\sqrt{g(x)}}. \tag{C.116}$$

Clearly the constants $\psi_0 = \psi_2 - \psi_1$ and ψ_3 are independent, and both quadratic in y'. In contrast to the non-deformed ODE (C.27), for $\alpha \neq 0$ these constants of the motion are not obtainable from linear invariants of the equation. Technically, we could use these invariants to reduce the order of the equation [5; 54; 119]. However, for generic functions $g(x)$ the reduced equation has an even more intricate form, and does not greatly simplify the integration of the non-linear equation. With the exception of $g(x) = 1$, that is easily seen to lead to the classical Pinney equation [123], for arbitrary non-constant $g(x)$ it is more practical to use numerical methods for the resolution.

As an example to illustrate this fact, we consider $g(x) = x$ and the deformed equation

$$y'' - \frac{y'}{2x} + x\,y - \frac{2\alpha\,x}{y^3} = 0. \tag{C.117}$$

For $\alpha = 0$ we recover the non-deformed ODE with general solution

$$y(x) = C_1 \sin\left(\frac{2}{3}x\sqrt{x}\right) + C_2 \cos\left(\frac{2}{3}x\sqrt{x}\right)$$

and Wronskian $\mathbf{W} = -\sqrt{x}$. For $\alpha \neq 0$ the constants of the motion of (C.103) are

$$\psi_0 = -\frac{1}{\sqrt{x}}\left(\frac{(y')^2}{x} + \frac{y^2}{2} - \frac{\alpha}{y^2}\right),$$
$$\psi_3 = \frac{-\sqrt{x}\left((y')^2 y^2 - xy^4 - 2\alpha x\right)\sin\left(\frac{4}{3}x\sqrt{x}\right) + xy^3y'\cos\left(\frac{4}{3}x\sqrt{x}\right)}{2x\sqrt{x}^2}.$$

In spite of their apparent simplicity, the fact that both constants of the motion depend explicitly on the independent variable x makes their use rather complicated, so that we skip this step.
In the following Figure we compare the solutions of (C.117) for the values $\alpha = 1$ and $\alpha = 0$ with the same initial conditions (the dashed line corresponds to the solution with $\alpha = 0$):

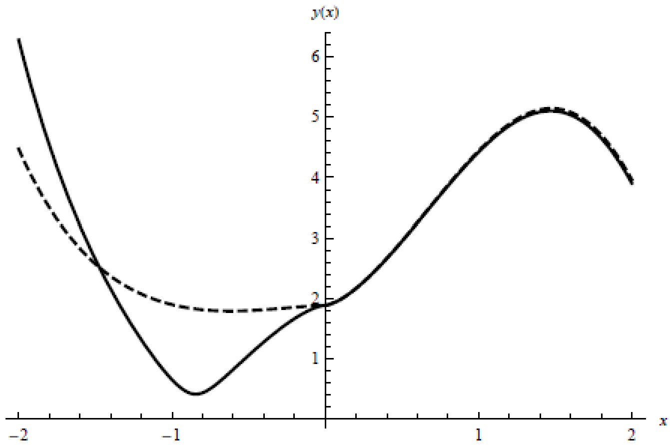


FIGURE C.1: $y(x)$ for the non-linear ODE (C.117) and the linearizable ODE.

C.4.1 Non-linear systems in $N = 2$ dimensions

Just as the scalar ODE (C.27) has been perturbed using a subalgebra of Noether symmetries, we can consider the problem of deforming systems of differential equations along the same lines. We recall that, in different contexts, variations of this ansatz have already been considered in the literature (see [76] and references therein), although usuallly related to the time-dependent oscillator equations.

We start from the decoupled system in $N = 2$ dimensions given by:

$$y_1'' - \frac{g'(x)}{2g(x)} y_1' + g(x) y_1 = 0,$$
$$y_2'' - \frac{g'(x)}{2g(x)} y_2' + g(x) y_2 = 0. \qquad \text{(C.118)}$$

Clearly this system is linearizable and reducible to the free particle system $\{z_1'' = 0, z_2'' = 0\}$, hence the Lie algebra of point symmetries is isomorphic to the rank three simple Lie algebra $\mathfrak{sl}(4, \mathbb{R})$ of dimension 15 [5; 141]. The system (C.118) also arises as the equations of motion associated to the Lagrangian

$$L = \frac{1}{2\sqrt{g(x)}} \left((y_1')^2 + (y_2')^2 - g(x) \left(y_1^2 + y_2^2 \right) \right). \qquad \text{(C.119)}$$

As for the scalar case, a point symmetry

$$X = \xi(x, y_1, y_2) \frac{\partial}{\partial x} + \eta_1(x, y_1, y_2) \frac{\partial}{\partial y_1} + \eta_2(x, y_1, y_2) \frac{\partial}{\partial y_2}$$

is a Noether symmetry of (C.118) if the constraint (C.48) is satisfied for some function $V(x, y_1, y_2)$. The constant of the motion associated to X is then given by

$$\psi = \xi \left[y_1' \frac{\partial L}{\partial y_1'} + y_2' \frac{\partial L}{\partial y_2'} - L \right] - \eta_1 \frac{\partial L}{\partial y_1'} + \eta_2 \frac{\partial L}{\partial y_2'} + V(x, y_1, y_2). \qquad \text{(C.120)}$$

As (C.118) has maximal symmetry, we know that the subalgebra of Noether symmetries must have dimension 8 [72]. Now, instead of solving the symmetry condition (C.48), we use the results obtained for the scalar case (C.27) to determine the Noether symmetries. We first consider the case of Noether symmetries of the form

$$Y = \eta_1(x, y_1, y_2) \frac{\partial}{\partial y_1} + \eta_2(x, y_1, y_2) \frac{\partial}{\partial y_2}. \qquad \text{(C.121)}$$

For this choice, the symmetry condition equals

$$\sqrt{g(x)}\left((y_1')^2\frac{\partial\eta_1}{\partial y_1}+(y_2')^2\frac{\partial\eta_2}{\partial y_2}\right)+\frac{y_1'y_2'}{\sqrt{g(x)}}\left(\frac{\partial\eta_1}{\partial y_2}+\frac{\partial\eta_2}{\partial y_1}\right)+\frac{1}{\sqrt{g(x)}}\left(\frac{\partial\eta_1}{\partial x}+\frac{\partial\eta_2}{\partial x}\right)$$
$$-\left(\frac{\partial V}{\partial y_1}+\frac{\partial V}{\partial y_2}\right)-\left(\frac{\partial V}{\partial x}+\sqrt{g(x)}\,(y_1\eta_1+y_2\eta_2)\right)=0. \tag{C.122}$$

From the quadratic powers in y_1' and y_2' it follows at once that

$$\eta_1(x,y_1,y_2)=k\,y_2+f_{12}(x),\;\eta_2(x,y_1,y_2)=-k\,y_1+f_{13}(x). \tag{C.123}$$

Inserting these expressions into (C.122) we further obtain $V(x,y_1,y_2)=(y_1f_{12}(x)+y_2f_{13}(x))$, and the symmetry condition reduces to

$$\frac{y_1\left(f_{12}''(x)-\frac{g'(x)}{2g(x)}f_{12}'(x)+g(x)f_{12}(x)\right)+y_2\left(f_{13}''(x)-\frac{g'(x)}{2g(x)}f_{13}'(x)+g(x)f_{13}(x)\right)}{\sqrt{g(x)}}=0. \tag{C.124}$$

It follows at once that the functions $f_{12}(x)$ and $f_{13}(x)$ must be solutions to the ODE (C.27). This proves that, for the special type (C.121) of vector fields we obtain five independent Noether symmetries

$$Y_1=U(x)\frac{\partial}{\partial y_1},\;Y_2=U(x)\frac{\partial}{\partial y_2},\;Y_3=T(x)\frac{\partial}{\partial y_1},\;Y_4=T(x)\frac{\partial}{\partial y_2},\;Y_5=y_2\frac{\partial}{\partial y_1}-y_1\frac{\partial}{\partial y_2}. \tag{C.125}$$

In order to obtain the three remaining symmetries, we apply the results of the preceding sections. A routine computation shows that the vector fields

$$Z_1=-\frac{T(x)U'(x)}{g(x)}\frac{\partial}{\partial x}+\frac{(T'(x)U'(x)+2g(x)T(x)U(x))}{g(x)}\left(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}\right), \tag{C.126}$$
$$Z_2=-\frac{U(x)T'(x)}{g(x)}\frac{\partial}{\partial x}+\frac{(T'(x)U'(x)+2g(x)T(x)U(x))}{g(x)}\left(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}\right) \tag{C.127}$$

are Noether symmetries of (C.118) for the function

$$V(x,y_1,y_2)=\frac{U(x)T'(x)+T(x)U'(x)}{2\sqrt{g(x)}}\left(y_1^2+y_2^2\right). \tag{C.128}$$

The vector fields Z_1,Z_2 and $Z_3=[Z_1,Z_2]$ are independent, and thus $\{Z_1,Z_2,Z_3,Y_1,\cdots,Y_5\}$ form a basis of the Lie algebra $\mathcal{L}_{NS}$ of Noether symmetries of the system (C.118). In particular, $\{Z_1,Z_2,Z_3\}$ generate a copy of $\mathfrak{sl}(2,\mathbb{R})$ isomorphic to the Levi subalgebra of $\mathcal{L}_{NS}$. We further observe that Y_5 corresponds to the infinitesimal generator of a rotation in the plane [73].

Now we analyze the existence of Lagrangians

$$L_0=\frac{1}{2\sqrt{g(x)}}\left((y_1')^2+(y_2')^2-g(x)\left(y_1^2+y_2^2\right)-G(x,y_1,y_2,y_1',y_1')\right) \tag{C.129}$$

such that the vector fields Z_1 and Z_2 are Noether symmetries. The procedure is completely analogous to that of scalar ODEs previously considered, for which reason we omit the detailed computations. Imposing that (C.48) is satisfied for Z_1 leads, after some heavy algebraic manipulation,[11] to the PDE:

$$2T(x)\left(U'(x)g'(x)+2U(x)g(x)^2\right)G(x,y_1,y_2,y_1',y_1')-2T(x)U(x)g(x)U'(x)\frac{\partial G}{\partial x}$$
$$+\left(2g(x)^2y_1(T(x)U'(x)+T'(x)U(x))-2T(x)\left(U'(x)g'(x)+2g(x)^2U(x)\right)y_1'\right)\frac{\partial G}{\partial y_1}$$
$$+\left(2g(x)^2y_2(T(x)U'(x)+T'(x)U(x))-2T(x)\left(U'(x)g'(x)+2g(x)^2U(x)\right)y_2'\right)\frac{\partial G}{\partial y_1}$$
$$2T(x)U(x)g(x)^2\left(y_1\frac{\partial G}{\partial y_1}+y_2\frac{\partial G}{\partial y_2}-\frac{U'(x)}{g(x)}\frac{\partial G}{\partial x}\right)=0. \tag{C.130}$$

[11] Using that $T(x)$ and $U(x)$ are solutions to the ODE (C.27), as well as the constraint (C.45).

For the vector field Z_2, the corresponding PDE satisfied by G is obtained from (C.130) permuting $T(x)$ and $U(x)$. In this form, however, the equations are of little use, as all intervening functions are unknown. We can transform the PDEs using the constraints satisfied by $T(t), U(t)$ and $g(t)$ in order to obtain an equivalent pair of differential equations. Skipping the routine computations, such a set is given by the equations

$$2g(x)\frac{\partial G}{\partial t}+y_1'g'(x)\frac{\partial G}{\partial y_1'}+y_2'g'(x)\frac{\partial G}{\partial y_2'}-2g'(x)G(x,y_1,y_2,y_1',y_1')=0, \tag{C.131}$$

$$\begin{aligned}-2T(x)U(x)\left(y_1\frac{\partial G}{\partial y_1}+y_2\frac{\partial G}{\partial y_2}-G\right)+\left(T(x)U(x)y_1'-y_1A_0\right)\frac{\partial G}{\partial y_1'}\\ +\left(T(x)U(x)y_2'-y_2A_0\right)\frac{\partial G}{\partial y_2'}=0,\end{aligned} \tag{C.132}$$

where $A_0=T(x)U'(x)+T'(x)U(x)$. The first of these equations has the general solution

$$G(x,y_1,y_2,y_1',y_1')=g(x)\,\Phi\left(y_1,y_2,\frac{y_1'}{\sqrt{g(x)}},\frac{y_2'}{\sqrt{g(x)}}\right). \tag{C.133}$$

Inserting the latter into equation (C.132) and analyzing the terms depending on y_1',y_2', it is not difficult to verify that the condition

$$\frac{\partial\Phi}{\partial y_1'}=\frac{\partial\Phi}{\partial y_2'}=0 \tag{C.134}$$

must be satisfied. Therefore the integrability condition reduces to the linear first-order PDE

$$y_1\frac{\partial\Phi}{\partial y_1}+y_2\frac{\partial\Phi}{\partial y_2}-\Phi(y_1,y_2)=0 \tag{C.135}$$

with solution $\Phi(y_1,y_2)=\varphi\left(y_2y_1^{-1}\right)y_1^{-2}$. This shows that the most general function G satisfying the system (C.131)-(C.132) is given by

$$G(x,y_1,y_2,y_1',y_1')=\frac{g(x)}{y_1^2}\,\varphi\left(\frac{y_2}{y_1}\right). \tag{C.136}$$

Proposition C.4. *For any non-zero function* $\varphi\left(\frac{y_2}{y_1}\right)$*, the non-linear system*

$$y_1''-\frac{g'(x)}{2g(x)}y_1'+g(x)y_1+\frac{g(x)}{y_1^3}\varphi\left(\frac{y_2}{y_1}\right)+\frac{g(x)y_2}{2y_1^4}\varphi'\left(\frac{y_2}{y_1}\right)=0, \tag{C.137}$$

$$y_2''-\frac{g'(x)}{2g(x)}y_2'+g(x)y_2-\frac{g(x)}{2y_1^3}\varphi'\left(\frac{y_2}{y_1}\right)=0 \tag{C.138}$$

possesses exactly three Noether symmetries.

The proof is essentially the same as that of Proposition 3 for the scalar case. The system corresponds to the equations of motion associated to the Lagrangian

$$L_0=\frac{1}{2\sqrt{g(x)}}\left(\left(y_1'\right)^2+\left(y_2'\right)^2-g(x)\left(y_1^2+y_2^2\right)-\frac{g(x)}{y_1^2}\varphi\left(\frac{y_2}{y_1}\right)\right) \tag{C.139}$$

Successive reduction of the Noether symmetry condition (C.48) shows that any such symmetry has the components

$$\begin{aligned}\xi(x,y_1,y_2)&=\phi(x),\\ \eta_1(x,y_1,y_2)&=\left(\frac{\phi(x)g'(x)}{4g(x)}+\frac{\phi'(x)}{2}\right)y_1,\\ \eta_2(x,y_1,y_2)&=\left(\frac{\phi(x)g'(x)}{4g(x)}+\frac{\phi'(x)}{2}\right)y_2,\end{aligned}$$

where the function $\phi(x)$ satisfies the third-order equation (C.113). As the solutions to the latter correspond exactly to the symmetries Z_1, Z_2 and Z_3 preserved by the deformation, we conclude that the dimension of the Noether symmetry algebra is three, hence it must be isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. As a consequence, the system (C.137)-(C.138) cannot be linearizable.

As happened in the scalar case, the integration of the deformed system is quite cumbersome in spite of the two constants of the motion, as these contain explicitly the independent variable x and their expression differs from being an easy one.

As an example to illustrate one of these deformed systems, we consider the auxiliary functions $g(x) = \frac{49}{1-x^2}$ and $\varphi(\frac{y_2}{y_1}) = \alpha\frac{y_2^2}{y_1^2}$. We are thus deforming the uncoupled system consisting of two Chebyshev equations. For the chosen forcing term, the deformed system equals

$$y_1'' - \frac{x}{(1-x^2)}y_1' + \frac{49}{(1-x^2)}y_1 - \frac{14\alpha y_2^2}{\sqrt{x^2-1}y_1^5} = 0, \tag{C.140}$$

$$y_2'' - \frac{x}{(1-x^2)}y_2' + \frac{49}{(1-x^2)}y_2 + \frac{7\alpha y_2}{\sqrt{x^2-1}y_1^4} = 0. \tag{C.141}$$

Solving numerically the system for $\alpha = 0.21$ and the initial conditions $y_1(0) = -1$, $y_1'(0) = 0$, $y_2(0) = 0$, $y_2'(0) = -7$,[12] the solutions $y_1(x)$ and $y_2(x)$ give rise to the following graphical representation:

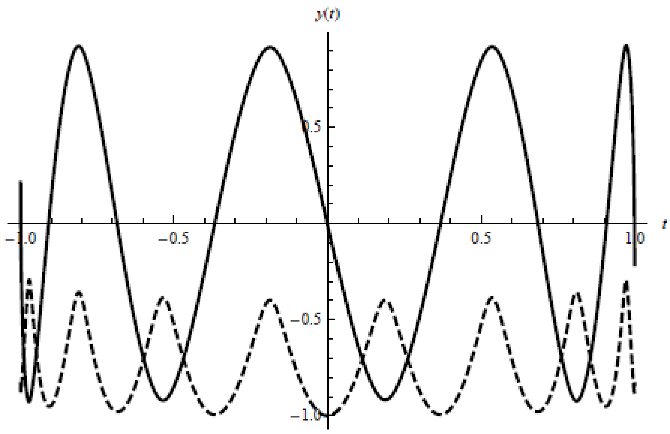


FIGURE C.2: Solutions $y_1(x)$ (dashed) and $y_2(x)$ for the system (C.140)-(C.141).

Both solutions are approximatively oscillations, however with varying frequency. The resulting trajectory in the plane $\{y_1, y_2\}$ has a relatively complicated structure, as shows the following plot.

[12]For $\alpha = 0$ the solutions are clearly the Chebyshev polynomials $y_1(t) = U_7(t)$, $y_2 = T_7(t)$.

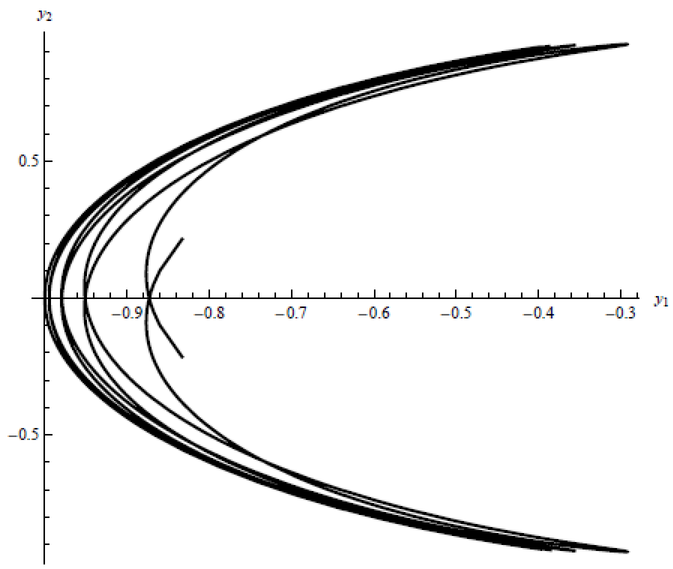


FIGURE C.3: Plane trajectory of the solutions $(y_1(x), y_2(x))$ of system (C.140)-(C.141).

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CV abreviado

A continuación presentamos una relación de los trabajos publicados, enviados a publicación y en proceso de ejecución relacionados con la temática de esta memoria. Asimismo se indican las conferencias y comunicaciones presentadas con relación a esta temática.

Publicaciones

Campoamor-Stursberg, R. and Fernández-Saiz, E. (2015). Realizations of sl(3;R) in Terms of Chebyshev Polynomials and Orthogonal Systems of Functions. Symmetry Breaking and Variational Symmetries. *In book: Ordinary and Partial Differential Equations*. Chapter: 3. Nova Science Publishers.

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2018). Poisson–Hopf algebra deformations of Lie–Hamilton systems. *Journal of Physics A: Mathematical and Theoretical*, **51**(6), 065202.

Ballesteros, A.; Campoamor-Stursberg, R.; Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2018). A unified approach to Poisson-Hopf deformations of Lie-Hamilton systems based on sl(2). *Dobrev, V. (ed.). Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics* Volume 1, 2018. pp. 347–366.

Esen, O.; Fernández-Saiz, E.; Sardón, C. and Zając, M. (2020). Geometry and solutions of an epidemic SIS model permitting fluctuations and quantization, submitted *arXiv preprint arXiv:2008.02484 [q-bio.PE]*.

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. A refinement of Poisson–Hopf deformations of Lie–Hamilton systems: The setup of explicit deformed superposition rules and applications to the oscillator algebra. *In progress*.

Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and Sardon, C. An alternative epidemic SIS model based on the Poincaré algebra $\mathfrak{h}_4$. *In progress*.

Conferencias y comunicaciones

Fernández–Saiz, E. (2016). Realizations of $\mathfrak{sl}(3;\mathbb{R})$ in terms of Chebyshev polynomials. Poster session in *10th International Summer School on Geometry, Mechanics, and Control*, in Madrid.

Fernández–Saiz, E. (2016). Lie symmetries and differential equations: Harrison-Estabrook method. Seminar in Burgos

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2017). Quantum algebras in Lie–Hamilton systems. Poster session in *11th International Young Researcher Workshop on Geometry, Mechanics, and Control*, in La Laguna.

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2017). Quantum algebras in Lie–Hamilton systems: oscilator system. Poster session in *V Iberoamerican Meeting on Geometry, Mechanics and Control*, in La Laguna (Tenerife).

Campoamor-Stursberg, R. and Fernández–Saiz, E. (2017). Realizations of sl(3,R): Chebyshev polynomials. Poster session in *A Workshop to honour Prof. Orlando Ragnisco in his 70th anniversary*, in Burgos.

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2017). Poisson–Hopf algebra deformation of Lie systems. Talk in *X. International Symposium*

Quantum Theory and Symmetries 12-th edition of the International Workshop "Lie Theory and Its Applications in Physics", in Varna.

Fernández–Saiz, E. (2018). Poisson-Hopf algebra deformations of Lie systems. Scientific Divulgation Seminar in Madrid

Ballesteros, A.; Campoamor-Stursberg, R.: Fernández-Saiz, E.; Herranz, F. J. and de Lucas, J. (2018). Poisson–Hopf algebra deformation of Lie systems. Talk in *13th Young Researchers Workshop*, in Madrid.

Fernández–Saiz, E. (2018). Sistemas de Lie y sus aplicaciones. Seminar (Red de Doctorandos en Matemáticas UCM), in Madrid.

Fernández–Saiz, E. (2018). Estructuras de Poisson: un billete de ida y vuelta a la mecánica cuántica. Scientific Divulgation Seminar (*3ª Jornada PhDay Complutense - Ciencias Matemáticas*), in Madrid.

Participación en organización de congresos

Fernández–Saiz, E. (2016) in the Organizing Committee of *X Workshop of Young Researchers in Mathematics*, in Madrid.

Fernández–Saiz, E. (2017) in the Organizing Committee of *Nonlinear Integrable Systems*, in Burgos.