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TESIS DOCTORAL

Stability and area law for radpidly mixing quantum dissipative systems
Estabilidad y ley de área para sistemas cuánticos disipativos con equilibración rápida

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

Ángelo Lucia

Directores

David Pérez García
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CHAPTER 1

Summary

Since its origins, the field of information theory has had strong ties to statistical mechanics: the terminology *entropy of information* was borrowed by Shannon from the thermodynamic entropy, as suggested by Von Neumann [5, 67, 82]. Traditionally information theory studies the storage of information (coding) and its transmission in noisy channels (communication capacity). By interpreting the physical interactions as communications channels, it has been possible to apply the same tools and ideas in order to understand how the collective behavior of a mechanical system composed of many (or infinite) parties emerges from the simple and limited interactions between its individual components. This has lead to understand the mechanism by which macroscopic properties emerge as effective behavior from microscopic interactions.

The same relationship has been developed recently between the corresponding *quantum generalizations* of both theories: quantum information (which is interested in the storage and manipulation of information in quantum mechanical systems) and many-body quantum physics. The ever-growing number of connections between the two fields goes in both directions, with tools and ideas from quantum information helping to solve long-standing problems in condensed matter physics, and new many-body models being developed for the storage and the transformation of quantum information. At the same time the spectacular improvements we have seen in the implementation and experimental control of small quantum systems is fueling the expectation that these experiments could be scaled up in size. Larger experiments means being closer to have practical applications, which has driven interest from top universities and research centers, national funding bodies such as EPSRC and NSF, to private companies with a strong focus on technological research as IBM, Microsoft and Google.

It is therefore of great importance to sharpen and deepen our understanding of these models, especially the ones that more likely resemble the physical world we live in. This means taking into account the fact that no experiment can be completely shielded from noise, nor it can be executed at zero temperature: we have then to consider open (as opposed to isolated) quantum systems, governed by a dissipative evolution. To make the problem tractable, a reasonable simplification is to assume a Markovian evolution, i.e. an

environment which holds no memory (or loses it fast enough that its effects on the system are negligible), and for which the future evolution only depends on the present state of the system and not the way in which it was obtained. These are evolutions controlled by a Lindblad equation, for which evolution at a fixed time slice is represented by a completely positive and trace preserving map, an object called *quantum channel* in the quantum information terminology.

Only recently has the quantum information community started to be interested in such models, and has started to consider these type of evolutions not only an issue to overcome but a resource which we need to learn how to exploit. This fundamental shift of perspective opened up a new area in the field, incorporating the idea of engineering artificial evolutions in which the dissipation works in our favor instead than against us [7, 43, 44, 55, 83], even by protecting the system from the effect of other, uncontrolled, noisy evolutions. Therefore a range of new interesting problems were faced: what is the computational power of these models, what conditions can guarantee the resilience against external noise, how long we have to wait for obtaining certain states, and so on.

This thesis is part of this effort to improve our knowledge of these models. We have focused on studying properties of quantum dissipative evolutions of spin systems on lattices in a particular regime that we denote **rapid mixing**. We consider dissipative evolutions with a unique fixed point, and which compress the whole space of input states into increasingly small neighborhoods of the fixed point. The time scale at which this compression takes place, or in other words the time we have to wait for any input state to become almost indistinguishable from the fixed point, is called the mixing time of the process. Rapid mixing is a condition on the scaling of this mixing time with the system size: if it is logarithmic, then we have rapid mixing.

The main contribution of this thesis is to show that rapid mixing has profound implications for the corresponding system: it is stable against external perturbations and its fixed point satisfies an area law for mutual information. The precise definitions of these properties will be given later.

It is somehow surprising that these properties can be derived just from an estimate on the time-scale of the convergence. It is a bit less surprising when we consider the other important ingredient in this setting, which are Lieb-Robinson bounds. Well known in the case of closed systems, but proven to hold also in the case of open systems, Lieb-Robinson bounds formalize the idea that information in a many-body model can only spread at a finite velocity, since its propagation has to be mediated by local interaction terms. Therefore, limiting the time-scale at which the evolution takes place implies limiting the scale of the distance at which it can create correlations. This simple observation lays at the heart of the technical part of this work.

CHAPTER 2

Resumen

Desde su origen, la teoría de la información ha tenido fuertes conexiones con la mecánica estadística: el mismo término *entropía de la información* fue elegido por Shannon a partir del término usado en termodinámica, bajo sugerencia de Von Neumann [5, 67, 82]. Tradicionalmente la teoría de la información estudia el almacenamiento (códigos) y la transmisión a través de canales con ruido (capacidad de comunicación). Al interpretar las interacciones físicas como canales de comunicación, ha sido posible aplicar las mismas técnicas e ideas para entender cómo un sistema mecánico compuesto de muchas (o infinitas) partes desarrolla un comportamiento colectivo a partir de las interacciones simples y limitadas entre sus componentes individuales. Esto ha permitido entender el mecanismo con el cual propiedades macroscópicas aparecen como efectos de interacciones microscópicas.

La misma relación se ha desarrollado recientemente entre las correspondientes *generalizaciones cuánticas* de ambas teorías: la información cuántica (que estudia el almacenamiento y la manipulación de la información en sistemas cuánticos) y la física de muchos cuerpos. Las conexiones entre los dos campos aumentan cada día y van en las dos direcciones: herramientas e ideas de la información cuántica ayudan a solucionar problemas abiertos en teoría de la materia condensada, y nuevos modelos de muchos cuerpos se desarrollan para aplicaciones de la información cuántica. Al mismo tiempo la implementación y el control experimental de pequeños sistemas cuánticos ha mejorado de forma espectacular, aumentando la posibilidad de que estos experimentos se puedan llevar a cabo a escala más grande. Experimentos más grandes significa estar cada vez más cerca de aplicaciones prácticas, lo cual ha orientado hacia el campo el interés de importantes universidades y centros de investigación, así como agencias nacionales de financiación como el EPSRC y la NSF, empresas privadas con fuerte inversión en la investigación y el desarrollo como IBM, Microsoft y Google.

Por lo tanto es muy importante afinar y mejorar nuestro conocimiento de estos modelos, y en particular los que describen de manera más fidedigna el mundo físico donde vivimos. Esto significa tener en cuenta que ningún experimento puede ser perfectamente aislado del ruido, ni puede ser efectuado a temperatura cero: tenemos que considerar sistemas cuánticos abiertos en vez de aislados, sujetos a una evolución disipativa. Para que el problema

sea abordable, una simplificación razonable es asumir que la evolución sea Markoviana, es decir que el sistema ambiente no tenga memoria de la evolución, o que la pierda lo suficientemente rápido para que su efecto en el sistema sea despreciable, y tal que la evolución futura sólo dependa del estado actual del sistema y no de su historia pasada. Este tipo de sistema está descrito por una ecuación de Lindblad, donde para cada instante temporal la evolución está dada por una aplicación completamente positiva y que preserva la traza, un objeto llamado *canal cuántico* en la terminología de la información cuántica.

Es reciente el interés de la comunidad de investigadores en información cuántica por estos modelos, y se ha empezado a ver este tipo de evoluciones no solamente como un problema que resolver, sino como un recurso que explotar. Este fundamental cambio de perspectiva ha abierto una nueva línea de investigación en el campo, incorporando ideas sobre cómo construir evoluciones artificiales en las cuales la disipación trabaja en nuestro favor en vez de en nuestra contra [7, 43, 44, 55, 83], incluso protegiendo el sistema de otros ruidos incontrolados. De esta manera se han planteado nuevos problemas interesantes: ¿cuál es el poder computacional de estos modelos? ¿Qué condiciones pueden garantizar la resistencia contra el ruido externo? ¿Cuánto tiempo hay que esperar para obtener de esta manera cierto tipo de estados?

Esta tesis es parte del esfuerzo para mejorar nuestro conocimiento de estos modelos. Nos hemos centrado en el estudio de las propiedades de evoluciones disipativas de sistemas cuánticos de espines en un retículo, bajo una hipótesis que llamamos **equilibración rápida**. Consideramos evoluciones disipativas con un único punto fijo, y que comprimen todo el espacios de estados iniciales en un entorno decreciente del punto fijo. La escala temporal a la cuál se tiene esta compresión, es decir el tiempo que tenemos que esperar para que cualquier estado inicial se encuentre arbitrariamente cercano al punto fijo, se llama tiempo de equilibración del proceso. La condición de equilibración rápida pide que este tiempo sea logarítmico en el tamaño del sistema.

La contribución principal de esta tesis es demostrar que la equilibración rápida tiene importantes consecuencias en las propiedades del sistema: este será estable bajo perturbaciones externas y sus puntos fijos satisfarán una ley de área para la información mutua. Las definiciones exactas serán dadas más adelante.

Es en cierta manera sorprendente que estas propiedades se puedan derivar de una cota en el tiempo de convergencia. Es menos sorprendente una vez que se considere el otro ingrediente fundamental en este contexto, las llamadas cotas de Lieb-Robinson. Muy conocidas en el caso de sistemas cerrados, pero igualmente válidas para sistemas abiertos, las cotas de Lieb-Robinson formalizan la idea que la información en un sistema de muchos cuerpos sólo puede moverse con una velocidad finita, dado que su propagación está mediada por las interacciones locales. Por lo tanto, controlar la escala temporal con la cuál la evolución converge permite a su vez controlar la escala espacial a la cuál se pueden crear correlaciones. Esta simple observación está en la base de la parte más técnica de este trabajo.

CHAPTER 3

Introduction

This dissertation is organized as follows. In section 3.1 we will define the main objects of interest, which are dynamical semigroups of quantum channels. We will recall the properties of their generators, which are denoted Lindblad generators. We will discuss why they are a sensible definition for modeling noisy quantum evolutions, and present the mixing time, a property which will be crucial in the main assumptions we make in order to prove our results. We will present the connection between the mixing time and other important properties of the semigroup, such as the spectral gap, log-Sobolev inequality, and hypercontractivity. These connections will show ways to prove rapid mixing. We will also introduce the notion of mutual information and we will discuss the area law property of states, together with its connections with efficiency of simulation and tensor network states. We will also discuss why stability is a crucial requirement for any mathematical model of a physical system. In section 3.2 we define the main assumptions and present a summary of the results obtained, together with a brief presentation of the technical tools that have been developed. Finally in section 3.3 we discuss future lines of research which are being developed at the time being.

The rest of the dissertation is composed of published papers which collect the results of the work done during the PhD. These chapters correspond to the following publications.

5. [S] T. S. Cubitt, A. Lucia, S. Michalakis, and D. Perez-Garcia. "Stability of Local Quantum Dissipative Systems". In: *Commun. Math. Phys.* 337.3 (Apr. 2015), pp. 1275–1315. DOI: 10.1007/s00220-015-2355-3

6. [R] A. Lucia, T. S. Cubitt, S. Michalakis, and D. Perez-Garcia. "Rapid mixing and stability of quantum dissipative systems". In: *Phys. Rev. A (Rapid Comm.)* 91.4 (Apr. 2015). DOI: 10.1103/physreva.91.040302

7. [A] F. G. S. L. Brandão, T. S. Cubitt, A. Lucia, S. Michalakis, and D. Perez-Garcia. "Area law for fixed points of rapidly mixing dissipative quantum systems". In: *J. Math. Phys.* 56.10 (Oct. 2015), p. 102202. DOI: 10.1063/1.4932612

The impact of these publications is reflected in the number of citations they have received, notwithstanding their young age: in particular [S] at the time of writing has already

been cited 18 times, while [R] received 3 and [A] received 1. Moreover, the results obtained have been presented as contributed talks in the most prestigious and relevant workshops of the field: the Quantum Information Processing and Communications 2013 (QIPC2013), the 17th Conference on Quantum Information Processing (QIP2014), and the Theory of Quantum Computation, Communication and Cryptography (TQC2015).

3.1 Background and current state of the topic

3.1.1 Notation

Let us fix the notation that we will use through this dissertation, although we will present in more detail some of the following objects later on.

Given a tensor product of two finite dimensional Hilbert spaces $\mathscr{H}_A \otimes \mathscr{H}_B$, the unique linear map $\mathrm{tr}_A : \mathscr{B}(\mathscr{H}_A \otimes \mathscr{H}_B) \to \mathscr{B}(\mathscr{H}_B)$ such that $\mathrm{tr}_A(a \otimes b) = b\,\mathrm{tr}(a)$ for all $a \in \mathscr{B}(\mathscr{H}_A)$ and all $b \in \mathscr{B}(\mathscr{H}_B)$ will be denoted the *partial trace* over A. A state over $\mathscr{B}(\mathscr{H})$ is given by a linear positive functional $\rho : \mathscr{B}(\mathscr{H}) \to \mathbb{R}$, normalized in such a way that $\rho(\mathbb{1}) = 1$. We will denote the trace of an operator X by $\mathrm{tr}\,X$, and we will denote by $\|\cdot\|_p$ the Schatten p-norm, i.e. $(\mathrm{tr}|X|^p)^{1/p}$. Where there is no risk of ambiguity, $\|\cdot\|$ will denote the usual operator norm (i.e. the Schatten ∞-norm).

We will consider a cubic lattice $\Gamma = \mathbb{Z}^D$, equipped with the graph distance metric, and a finite subset $\Lambda \subset \Gamma$. We will adhere to the physicists convention of calling any subset of Γ a lattice, even if it is not a lattice in the graph theoretic sense. The ball centered at $x \in \Lambda$ of radius r will be denoted by $b_x(r)$. At each site x of the lattice we will associate one elementary quantum system with a finite dimensional Hilbert space $\mathscr{H}_x$. We will use Dirac's notation for vectors: $|\varphi\rangle$ will denote a vector in $\mathscr{H}_x$, $\langle\varphi|$ its adjoint, and $\{|n\rangle\}_{n=0}^{\dim\mathscr{H}_x-1}$ the canonical basis for $\mathscr{H}_x$. Scalar product in $\mathscr{H}_x$ will be denoted by $\langle\varphi|\psi\rangle$, and rank-one linear maps by $|\varphi\rangle\langle\psi|$. For each finite subset $\Lambda \subseteq \Gamma$, we associate a Hilbert space given by

$$\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x,$$

and an algebra of observables defined by

$$\mathscr{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{B}(\mathscr{H}_x).$$

Since $\mathscr{H}_x$ is finite dimensional, $\mathscr{B}(\mathscr{H}_x)$ is a matrix algebra, and we will sometimes denote it by $\mathscr{M}_d$, with $d = \dim\mathscr{H}_x$.

If $\Lambda_1 \subset \Lambda_2$, there is a natural inclusion of $\mathscr{A}_{\Lambda_1}$ in $\mathscr{A}_{\Lambda_2}$ by identifying it with $\mathscr{A}_{\Lambda_1} \otimes \mathbb{1}$. The support of an observable $O \in \mathscr{A}_\Lambda$ is the minimal set Λ' such that $O = O' \otimes \mathbb{1}$, for some $O' \in \mathscr{A}_{\Lambda'}$, and will be denoted by $\mathrm{supp}\,O$.

A linear map $\mathscr{T} : \mathscr{A}_\Lambda \to \mathscr{A}_\Lambda$ will be called a *superoperator* to stress the distinction from operators in $\mathscr{A}_\Lambda$. The support of a superoperator $\mathscr{T}$ is the minimal set $\Lambda' \subseteq \Lambda$ such that $\mathscr{T} = \mathscr{T}' \otimes \mathbb{1}$, where $\mathscr{T}' \in \mathscr{B}(\mathscr{A}_{\Lambda'})$. A superoperator is said to be Hermiticity preserving if it maps Hermitian operators to Hermitian operators. It is said to be positive if it maps positive

operators (i.e. operators of the form O^*O) to positive operators. $\mathcal{T}$ is called *completely positive* if $\mathcal{T}\otimes\mathbb{1} : \mathcal{A}_\Lambda \otimes \mathcal{M}_n \to \mathcal{A}_\Lambda \otimes \mathcal{M}_n$ is positive for all $n \geq 1$. Finally, we say that $\mathcal{T}$ is trace preserving if $\mathrm{tr}\,\mathcal{T}(\rho) = \mathrm{tr}\,\rho$ for all $\rho \in \mathcal{A}_\Lambda$. The p-Schatten norms on $\mathcal{A}_\Lambda$ induce a corresponding family of norms on $\mathcal{B}(\mathcal{A}_\Lambda)$: we denote by $\|\cdot\|_{p\to q}$ the operator norm of a superoperator $\mathcal{T} : (\mathcal{A}_\Lambda, \|\cdot\|_p) \to (\mathcal{A}_\Lambda, \|\cdot\|_q)$, i.e. when we equip its domain with the p-Schatten norm and the codomain with the q-Schatten norm. We will sometimes need the following norm for superoperators, called diamond norm in the literature:

$$\|\mathcal{T}\|_\diamond = \sup_n \|\mathcal{T}\otimes\mathbb{1}_n\|_{1\to 1}.$$

3.1.2 Dynamical semigroups of quantum channels

3.1.2.1 Unitary evolutions and quantum channels

Quantum mechanics prescribes that a physical system is represented by Hilbert space $\mathcal{H}$. The measurable properties of the systems are encoded into a a state ρ, which is a positive operator having trace one. For simplicity, we will only consider the case in which $\mathcal{H}$ is a d-dimensional Hilbert space, and therefore $\mathcal{B}(\mathcal{H})$ is a matrix algebra $\mathcal{M}_d$. In the case of an isolated system, the physical evolution of the system is described by a unitary evolution of the state ρ, meaning that the state of the evolved system is given by $U\rho U^*$, with U a unitary operator over $\mathcal{H}$. It is immediate to see that this type of evolution is inherently reversible, meaning that its inverse $\rho \to U^*\rho U$ is also physically possible. Therefore, in order to consider dissipative quantum systems, by which we mean quantum systems which have a non-reversible evolution caused by the interaction of the system with an environment, we have to replace this unitary evolution with something more general.

Let us try to be as general as possible, and let us pin down the minimal assumptions a map $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ representing a physically realizable evolution should satisfy. Let ρ be the initial state, and $T(\rho)$ the evolved state. For sure, T should send states to states, and therefore it should be linear [1], positive and trace preserving. It is a surprising but important fact that positivity is not sufficient for such a map to have a consistent physical interpretation. In fact, imagine we extend our system with an auxiliary one, with its own Hilbert space $\mathcal{K}$ and its state τ. Then the state of the joint system is $\rho\otimes\tau \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Let us also assume that the evolution of τ is trivial. Is there a map $\tilde{T}$ that extends T to $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$, such that $\tilde{T}(\rho\otimes\tau) = T(\rho)\otimes\tau$ for every possible pair of states ρ and τ? Such a map exists and it is given by the tensor product T with the identity map, which is denoted by $T\otimes\mathbb{1}$.

If this were to be a physical evolution, it should again be positive. But surprisingly, $T\otimes\mathbb{1}$ does not need to be positive, if we only require positivity of T. We have to require the stronger property of *complete positivity*: we recall that we defined a map $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ to be completely positive if $T\otimes\mathbb{1}_n$ is a positive map for all $n \in \mathbb{N}$, where $\mathbb{1}_n$ is the identity of $\mathcal{M}_n$. A completely positive and trace preserving map is called a *quantum channel*, and they will be our main object of study.

[1] The operational interpretation of an *ensemble* of states $\{(p_i, |\varphi_i\rangle\langle\varphi_i|)\}$ is that of a probability distribution (p_i) over a set of possible states $\{|\varphi_i\rangle\}$. Therefore it is reasonable to expect that after the evolution the ensemble transformed to $\{(p_i, T(|\varphi_i\rangle\langle\varphi_i|))\}$, or in other words that $T(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|) = \sum_i p_i T(|\varphi_i\rangle\langle\varphi_i|)$.

Let us try to justify why quantum channels indeed represent physically motivated evolutions. Since we want to talk about non-isolated systems, let us assume we have an environment which is allowed to interact with our system. Denote the state of the environment by $|\varphi\rangle$ (which we can assume, without loss of generality, being pure). The pair system-environment is a closed system, so it evolves under a unitary U. At the end of the evolution we discard the environment and look only at the reduced density matrix of our original system, which we denote by ρ'. The sequence of operations we have described takes this form:

$$\rho \to \rho \otimes |\varphi\rangle\langle\varphi| \to U(\rho \otimes |\varphi\rangle\langle\varphi|)U^* \to \mathrm{tr}_E[U(\rho \otimes |\varphi\rangle\langle\varphi|)U^*] = \rho' \tag{3.1}$$

where tr_E is the partial trace over the environment. Note that every step of the process is a completely positive, trace preserving map: therefore the resulting evolution $\rho \to \rho'$ is a quantum channel.

Therefore, if we regard our system as being coupled to an environment, and that they jointly evolve as a closed system, we are led to the conclusion that the effective evolution of the system is described by a quantum channel. Interestingly, this is actually the only thing that can happen: every quantum channel can be interpreted in this way, as the restricted action of a unitary evolution acting on a larger system. This is the content of Stinespring's dilation theorem.

Stinespring's dilation theorem. *Let $T : \mathscr{M}_d \to \mathscr{M}_d$ be a completely positive and trace preserving map (a quantum channel). Then there exists a unitary $U \in \mathscr{M}_{d^2}$ and a normalized vector $\varphi \in \mathbb{C}^d$ such that*

$$T(\rho) = \mathrm{tr}_E[U(\rho \otimes |\varphi\rangle\langle\varphi|)U^*] \tag{3.2}$$

Therefore, evolutions represented by quantum channels are physically motivated as the restriction of a unitary evolution acting on a larger system: they are, in a sense, "effective" dynamics induced by unitary evolutions on sub-systems.

3.1.2.2 Weak coupling limit

Until now we have only considered a *single* application of a quantum channel, regarding it as a single time-step in a dynamical evolution, and we have seen that, due to Stinespring's theorem, that is equivalent to coupling the system with an environment and considering a unitary evolution of the pair. The clear advantage is, of course, that it is often easier to reason if we can ignore the internal evolution of the environment and only consider its effect on the system we are actually interested in.

But a dynamical system is more than just the application of a single time-step evolution: it is a composable sequence of such single steps, or a continuous-time description in which every fixed time slice $t \geq 0$ gives rise to a physically realizable evolution T_t. Mathematically we have then a problem: by tracing out the environment, we have destroyed any possible correlation that the unitary U might have created between the system and the environment, both in the form of quantum entanglement or in the form of classical correlations. We have only kept a shadow of them in the mixed state obtained, but the loss is not reversible. The

process described in eq. (3.1) is inherently not composable: the environment has changed because of its coupled evolution with our system.

While technically true, we might question whether this mathematical description is relevant for describing physical systems. It turns out that, in some cases, the effect of the system onto the environment is so negligible, it can be safely approximated to be irrelevant, and we can assume the environment to be not evolving. Imagine for example that the environment is a thermal bath at some temperature: for sure the interaction with the system will change the equilibrium and the temperature of the bath, but if the bath is much larger than the system it is a good approximation to consider a constant temperature, irrespective of what happens in the system.

Mathematically, this would mean that if $U\big(\rho\otimes|\varphi\rangle\langle\varphi|\big)U^*$ is not too distant from $\rho'\otimes|\varphi\rangle\langle\varphi|$, we can approximate the former with the latter: in the physics literature this is called the *weak coupling limit* or *Born approximation*[1, 13, 17, 70, 71]. At each "infinitesimal" time step, the environment is ideally thrown away, and is replaced by a fresh, identical one, which therefore does not hold any information about the previous evolution of the system. For this reason, this type of evolution is also called *Markovian*.

This naturally leads to consider the following dynamical system: the evolution of the system is described by a *semigroup*[2] of quantum channels $\{T_t:\mathscr{M}_d\to\mathscr{M}_d\}_{t\geq 0}$, such that T_0 is the identity map. The semigroup property $T_tT_s=T_{s+t}$ means that it is a homogeneous evolution, and that it is Markovian. As in the classical theory of dynamical semigroups, if T_t is strongly continuous in t (i.e. T_t is a C_0-semigroup), then it has an infinitesimal generator $\mathscr{L}$, satisfying the following relationships:

$$\mathscr{L}(x)=\frac{\mathrm{d}}{\mathrm{d}t}\,T_t(x)\bigg|_{t=0}=\lim_{t\downarrow 0}\frac{1}{t}(T_t-\mathbb{1})(x). \tag{3.3}$$

Note that for finite dimensional systems, strong continuity implies uniform continuity, and therefore we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t=\mathscr{L}T_t,\quad T_t=\exp(t\mathscr{L}).$$

A generalization of this approximation is to consider an environment evolving with time, but independently of the system itself (maybe because of its internal dynamics, or maybe because we have approximated the effect of the system on the environment in this way). This leads to consider dynamical cocycles instead of semigroups, i.e. families $\{T_{t,s}\}_{0\leq s\leq t}$ of quantum channels that satisfy the property $T_{t,s}T_{s,k}=T_{t,k}$ for all $k\leq s\leq t$. We can define (time-dependent) generators of cocycles in a similar way as we did for semigroups: we will not be interested in this non-homogeneous dynamics here, but we mention it for the sake of completeness.

3.1.2.3 Lindbladian generators

We have seen that the evolution of state ρ under a semigroup of quantum channels T_t is given by the solution of the differential equation $\dot\rho(t)=\mathscr{L}\rho(t)$, where $\rho(t)=T_t(\rho)$. The

[2] Strictly speaking, a representation of the semigroup $(\mathbb{R}_+,+)$

(super-)operator $\mathscr{L}$ is sometimes called *Liouvillian*, since this equation is a generalization of the Liouville-von Neumann equation. It cannot be an arbitrary operator, since we have imposed some restrictions on the semigroup it generates (it is a semigroup of quantum channels). Lindblad [48], Kossakowski, Gorini, and Sudarshan [27] proved that this imposes a particular form on the generator $\mathscr{L}$, which is called the Lindlbad-Kossakowski form, and $\mathscr{L}$ is usually called Lindbladian.

Theorem 1. *Let $\mathscr{L} : \mathscr{M}_d \to \mathscr{M}_d$. The following are equivalent*

1. *$\mathscr{L}$ generates a dynamical semigroup of quantum channels;*
2. *there exists a completely positive map $\varphi : \mathscr{M}_d \to \mathscr{M}_d$ and a matrix $\kappa \in \mathscr{M}_d$ such that*

$$\mathscr{L}(\rho) = \varphi(\rho) - \kappa\rho - \rho\kappa^*; \quad \varphi^*(\mathbb{1}) = \kappa + \kappa^*. \tag{3.4}$$

3. *there exists an Hermitian matrix H and a set of matrices $\{L_j \in \mathscr{M}_d\}_{j=0,\dots,d^2-1}$ such that*

$$\mathscr{L}(\rho) = -i\big[H,\rho\big] + \sum_{j=0}^{d^2-1} L_j \rho L_j^* - \frac{1}{2}\Big\{L_j^* L_j, \rho\Big\}; \tag{3.5}$$

where $\{a,b\} = ab + ba$ is the anticommutator.

H is (for obvious reasons) called the Hamiltonian, while the matrices L_j are called *Lindblad* or *jump operators*.

By Russo-Dye theorem [74], if $T : \mathscr{M}_d \to \mathscr{M}_d$ is positive and trace preserving, then it is a contraction under the trace norm: indeed we have that by duality with respect of the Hilbert-Schmidt product

$$\|T\|_{1\to 1} = \left\|T^*\right\|_{\infty\to\infty} = \left\|T^*(\mathbb{1})\right\|_\infty = \|\mathbb{1}\|_\infty = 1,$$

since the dual of a trace preserving map is a unital map. Therefore, its eigenvalues lie in the unit disk of the complex plane. Functional calculus shows that this implies that the spectrum of $\mathscr{L}$ is in the complex semiplane $\{z \in \mathbb{C} \mid \operatorname{Re} z \le 0\}$.

Eigenvalues of $\mathscr{L}$ lying on the purely imaginary axis correspond to eigenvalues of T_t lying on the boundary of the unit complex disk, and in that context they are also called *peripheral spectrum*. It can be shown that their associated Jordan blocks are one-dimensional, and they correspond to periodic states, with stationary states corresponding to eigenvalue 1[85].

For every other eigenvalue λ of $\mathscr{L}$, which has a strictly negative real part, the action of T_t on the corresponding generalized eigenspace is the one of a contraction exponential in time: the subspace is suppressed by a factor of $\exp(-t\operatorname{Re}\lambda)$. Therefore, it is the eigenvalue with the largest non-zero real part that determines the slowest rate of convergence of T_t to a map T_∞ that projects on the space of periodic points (and that acts unitarily on it). This is the reason for the following definition:

Definition 1 (Spectral gap)**.** We denote the *spectral* gap of a Lindbladian $\mathscr{L}$ the quantity

$$\operatorname{gap}\mathscr{L} = \min_{\lambda\in\sigma(\mathscr{L})\setminus\{0\}} |\operatorname{Re}\lambda|. \tag{3.6}$$

Let us assume for a moment that there are no periodic points, or in other words the peripheral spectrum is trivial: it is composed only of the eigenvalue 1. In this case T_∞ is actually a projection. Then from a simple Jordan decomposition, we can see that the spectral gap controls the scaling in time of the convergence to the space of fixed points, and we have that there exists a constant $c > 0$ such that

$$\left\| T_t(\rho) - T_\infty(\rho) \right\|_1 \leq c \exp\left(-t \operatorname{gap} \mathscr{L}\right), \tag{3.7}$$

for every initial state ρ.

We will go back to eq. (3.7) later on, when we talk about families of dynamical systems defined on an increasing sequence of lattice structures.

3.1.2.4 Local generators

Until now, we have only considered finite systems, which can be considered as one single big physical body subject to some dynamics. Most applications require instead a description of a *many-body* model: a system composed of many individual parts, which interact with each other in a defined and somehow regular way. If we only consider one single instance of a many-body model, then mathematically speaking there is no difference in considering the whole system as a single big body, with some internal degree of freedom evolving according to the mentioned interactions.

This point of view changes dramatically if we consider instead increasing sequences of many-body models defined on a graph or lattice structure. Let us recall the notation we have given for many-body systems on a lattice. Consider an infinite graph Γ (for example, $\mathbb{Z}^D$ for some integer D) equipped with the graph metric. Associate to every vertex x in the graph a finite Hilbert space $\mathscr{H}_x$, and let us assume for simplicity that they are all isomorphic (i.e., they have the same dimension d). Then for every finite $\Lambda \subset \Gamma$ we denote by $\mathscr{H}_\Lambda = \otimes_{x\in\Lambda}\mathscr{H}_x$, and $\mathscr{A}_\Lambda = \mathscr{B}(\mathscr{H}_\Lambda)$.

In this case, there is a well defined notion of locality: for every pair of finite subgraphs $\Lambda_1 \subset \Lambda_2 \subset \Gamma$, there is a natural embedding of $\mathscr{A}_{\Lambda_1}$ into $\mathscr{A}_{\Lambda_2}$, by identifying $X \in \mathscr{A}_{\Lambda_1}$ with $X \otimes \mathbb{1}_{\Lambda_2\setminus\Lambda_1} \in \mathscr{A}_{\Lambda_2}$. This allowed us to define the notion of *support*: given an operator $X \in \mathscr{A}_\Lambda$ we define the support of X, denoted by $\operatorname{supp} X$, as the minimal $\Lambda' \subset \Lambda$ such that there exists a $X' \in \mathscr{A}_{\Lambda'}$ that satisfy $X = X' \otimes \mathbb{1}$. In a sense, the support of X is independent of Λ, since $\operatorname{supp} X = \operatorname{supp} X \otimes \mathbb{1}$: therefore considering X to be acting on a larger system does not increase its support.

This is the first appearance of a simple but powerful idea that is behind most of this work: there exist some properties of the objects we are studying that do not depend on the size of the system, as long as this is large enough to contain them. If we take an increasing and absorbing sequence of finite $\Lambda_n \nearrow \Gamma$, then we can discuss about properties that are uniform in n.

Physical interactions usually become weaker when the distance between interacting bodies becomes larger. Therefore, if we can decompose the generator of the evolution $\mathscr{L}$ as a sum of local terms $\sum_{Z\subset\Lambda} \mathscr{L}_Z$, each of which is still of the Lindblad-Kossakowski form but it is only acting on the subsystem Z , it is reasonable to assume that their norm becomes

smaller as the diameter of their support Z becomes larger. In this case, we will say that $\mathscr{L}$ is a **local Lindbladian**.

If we do not specify at which rate $\|\mathscr{L}_Z\|_\diamond$ decays with $\operatorname{diam} Z$, this definition can be trivially satisfied by any Lindbladian. We will postpone for the moment to clarify this point, and we will do it in section 3.2.1.3.

3.1.2.5 Mixing time and spectral gap

Equation (3.7) captures the long-time properties of the dynamical system described by T_t: if t is larger than $\log(\epsilon/c)/\operatorname{gap}\mathscr{L}$ for some positive ϵ, then the set of possible input states will have been compressed inside a ϵ-neighborhood of the space of fixed points. The minimum time required for this to happen (which might be smaller than the time given by eq. (3.7)) is called the **mixing time** of the dynamical system. We give the formal definition only for systems without periodic points.

Definition 2 (mixing time)**.** We denote the *mixing time* of a dynamical system with no periodic points $T_t : \mathscr{M}_d \to \mathscr{M}_d$ the function

$$\tau(\epsilon) = \min\{t > 0 : \sup_\rho \left\| T_t(\rho) - T_\infty(\rho) \right\|_1 \leq \epsilon\},$$

where the supremum is taken over all states ρ.

Therefore, we can restate what we know about the spectral gap as follows:

$$\tau(\epsilon) \leq \frac{\log c - \log\epsilon}{\operatorname{gap}\mathscr{L}} \tag{3.8}$$

For any single finite-dimensional system, this analysis is usually satisfactory: more care should be taken if we consider families of dynamical systems defined on an increasing sequence of lattices Λ_n. In this case we want to control the scaling in n of $\tau(\epsilon)$. First of all, the quantity $\lambda_n = \operatorname{gap}\mathscr{L}_n$ can become smaller, as n increases, and therefore the bound on the mixing time will diverge. If instead the quantity $\lambda = \inf\lambda_n$ is bounded away from zero, we informally say that the system is gapped (meaning that it is gapped in the limit).

Nonetheless, there is another and deeper reason for which the bound eq. (3.8) will, in general, diverge with n, even with a strictly positive λ: the constant c will (in general) also depend on n. In fact, if we obtain this bound via the Jordan decomposition (which is not the optimal way to do it), it can grow faster than exponential in n. Some more careful analysis can be done to improve this dependence: in [85] it was shown that if $\mathscr{L}$ satisfies a condition called detailed balance, which will be presented in more detail in section 3.1.3.1, with respect to a full-rank state σ, then we can take c to be equal to $\left\|\sigma^{-1}\right\|_\infty^{1/2}$, which is equal to $\sigma_{\min}^{-1/2}$, the minimal eigenvalue of σ. This gives the following result, for which we will also present a different proof later:

Theorem 2. *If λ is the spectral gap of $\mathscr{L}$, and it satisfies detailed balance with respect to a full-rank state σ, then*

$$\tau(\epsilon) \leq \frac{\log\left(\sigma_{\min}^{-1/2}\right) - \log\epsilon}{\lambda} \tag{3.9}$$

Note that $\sigma_{\min}^{-1/2}$ has to scale at least exponentially in the system size, or worse - therefore, we obtain from eq. (3.9) a polynomial time mixing. If we assume that we know the whole spectrum of $\mathscr{L}$ but nothing else, we cannot improve too much on Theorem 2: as shown in [78], if we only know the spectrum of $\mathscr{L}$ and not some other property of it, the dependence of c in the system size cannot be improved to be slower than exponential.

For some applications, having a polynomial-time mixing is sufficient. In this work, we require a stronger condition, which because of [78] cannot be guaranteed only by some condition on the spectrum of $\mathscr{L}$: that $\tau(\epsilon)$ scales *logarithmically* in n (and in some cases we can have a relaxation to sub-linear scaling). It is the main contribution of this thesis to show that under this stronger assumption, some very interesting properties of the evolution and of its fixed point can be derived.

We will present such results in section 3.2: before that, let us present one important connection with logarithmic Sobolev inequalities.

3.1.3 Logarithmic Sobolev inequalities

The tools of hypercontractivity and logarithmic Sobolev inequalities (log-Sobolev for short) have been developed as part of Segal's program of giving mathematical rigor to Quantum Field Theory [75]. Log-Sobolev inequalities have been first introduced by Feissner (at the time a student of Leonard Gross, who had been in turn student of Segal) in his PhD thesis [24, 25] in order to generalize the classical Sobolev inequality to Gaussian measures in infinite dimensions. Then [29] used them to study ergodicity of Markov process in infinite dimensional spaces and were recognized afterwards to be an effective tool in analyzing finite dimensional systems too [21]. The application to classical spin systems has been introduced first by Holley, Stroock and Zegarlinski [35, 76, 77, 87, 88] and thereafter became a standard tool in statistical mechanics. They are intimately connected to hypercontractivity of semigroups, and have made an appearance in a wide range of different areas of mathematics.

For a modern review of the classical (commutative) theory of logarithmic Sobolev inequalities and its connections to Markov semigroups and concentration of measure, we refer the reader to [30]. Its quantum generalization has been developed in a series of papers [49–51, 66], and the connection between rapid mixing and log-Sobolev inequalities in the quantum setting is due to [39].

Hypercontractivity predates slightly the appearance of log-Sobolev inequality: the first examples can be traced back to a work of Nelson (also a student of Segal) [62, 63], when it was still not called in such a way. For a review of the subject we will refer to [18, 28]. It is finding its way into the quantum information community as a tool on its own [16, 40, 58, 81].

We will present in the rest of the section a simplified version of the quantum log-Sobolev theory, and its connection with hypercontractivity and rapid mixing.

In this dissertation we consider semigroups of trace preserving maps T_t, therefore describing the evolution of states, but an equivalent description could be done of the dual (under the Hilbert-Schmidt scalar product) evolution T_t^* where observables are evolving, and the limit state ρ_∞ is invariant in the sense that $\mathrm{tr}\big(\rho_\infty T_t^*(A)\big) = \mathrm{tr}\big(\rho_\infty A\big)$ for all operators A. In this case the semigroup is unital instead of trace preserving. This is the approach usually followed in the literature on logarithmic Sobolev inequalities. We will stick to our

notation, therefore denoting the evolution of observables as T_t^*, but the reader should be aware of this fact.

3.1.3.1 Spectral gap and detailed balance

Before presenting the definition of logarithmic Sobolev inequalities, or log-Sobolev inequalities for short, let us reformulate eq. (3.6) in a different but equivalent way, when we assume to have a full rank state $\sigma > 0$. For such state we can define a weighted scalar product on $\mathscr{B}(\mathscr{H})$ via

$$\langle A|B\rangle_\sigma = \mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} B\big] = \big\langle \sigma^{1/4} A \sigma^{1/4} \big| \sigma^{1/4} B \sigma^{1/4} \big\rangle_{HS},$$

and the corresponding induced norm $\|\cdot\|_\sigma = \langle\cdot|\cdot\rangle_\sigma^{1/2}$. It is easy to see that $\sigma_{\min}^{1/2}\|A\| \leq \|A\|_\sigma \leq \|A\|$, where $\sigma_{\min}$ is the minimal eigenvalue of σ. Moreover, we can define a generalization of the classical variance, as

$$\mathrm{Var}_\sigma(A) = \|A - \langle A|\mathbb{1}\rangle_\sigma\|_\sigma^2 = \mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} A\big] - \mathrm{tr}[\sigma A]^2.$$

Indeed, $\mathrm{Var}_\sigma(A)$ is positive and invariant under translations by multiples of the identity. In a similar way, given a Lindbladian $\mathscr{L}$, we can define a non-commutative generalization of the Dirichlet form:

$$\mathscr{E}(A,B) = \big\langle A \big| -\mathscr{L}^*(B)\big\rangle_\sigma = -\mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} \mathscr{L}^*(B)\big];$$

where $\mathscr{L}^*$ is the dual of $\mathscr{L}$ under the Hilbert-Schmidt scalar product, i.e. $\mathscr{E}(A,B) = -\mathrm{tr}\big[\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) B\big]$. We will write $\mathscr{E}(A) = \mathscr{E}(A,A)$. We say that $\mathscr{L}$ satisfies *detailed balance with respect to* σ if $\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) = \sigma^{1/2}\mathscr{L}^*(A)\sigma^{1/2}$ for all operators A, and therefore

$$\mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} \mathscr{L}^*(B)\big] = \mathrm{tr}\big[\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) B\big] = \mathrm{tr}\big[\sigma^{1/2}\mathscr{L}^*(A)\sigma^{1/2} B\big].$$

If so, $\mathscr{E}$ is a symmetric bilinear form, $\mathscr{L}^*$ is self-adjoint under $\langle\cdot|\cdot\rangle_\sigma$, and thus $\hat{\mathscr{L}}(\cdot) = \sigma^{1/4}\mathscr{L}^*(\sigma^{-1/4}\cdot\sigma^{-1/4})\sigma^{1/4}$ is self-adjoint under the Hilbert-Schmidt scalar product. Since $\mathscr{L}^*$ and $\hat{\mathscr{L}}$ are related by a similarity transformation, $\mathscr{L}^*$ has real spectrum, and contractivity of the generated semigroup implies that it is negative. Note that it also implies that σ is a steady state for $\mathscr{L}$, since for all A it holds

$$\mathrm{tr}[\mathscr{L}(\sigma) A] = \mathrm{tr}\big[\sigma^{1/2}\mathbb{1}\sigma^{1/2}\mathscr{L}^*(A)\big] = \mathrm{tr}\big[\sigma^{1/2}\mathscr{L}^*(\mathbb{1})\sigma^{1/2} A\big] = 0,$$

and thus $\mathscr{L}(\sigma) = 0$.

If the peripheral spectrum of T_t is trivial, then kernel of $\mathscr{L}$ is one dimensional, and by the Courant-Fischer-Weyl min-max principle, the second smaller eigenvalue of $\mathscr{L}$, which we have previously called the spectral gap, is given by:

$$\mathrm{gap}\,\mathscr{L} = \min_{A:\langle A|\mathbb{1}\rangle_\sigma = 0} \frac{-\langle\mathscr{L}^*(A)|A\rangle_\sigma}{\langle A|A\rangle_\sigma} = \min_{A:\mathrm{Var}_\sigma(A)\neq 0} \frac{\mathscr{E}(A)}{\mathrm{Var}_\sigma(A)}.$$

We have therefore re-expressed eq. (3.6) as a variational problem: $\mathrm{gap}\,\mathscr{L}$ is the maximal value that can take a constant c such that the quadratic functional $c\,\mathrm{Var}_\sigma(A)$ lower bounds the quadratic functional $\mathscr{E}(A)$.

$$c\,\mathrm{Var}_\sigma(A) \leq \mathscr{E}(A) \tag{3.10}$$

Consider now $A(t)$ the evolution on an observable A under $\mathscr{L}^*$, i.e. $A(t) = T_t^*(A)$. Since σ is invariant under T_t, it holds that $\langle A(t)|\mathbb{1}\rangle_\sigma = \langle A|\mathbb{1}\rangle_\sigma$, and therefore $\lim_{t\to\infty} A(t) = \langle A|\mathbb{1}\rangle_\sigma \mathbb{1}$. Therefore $\mathrm{Var}_\sigma(A(t))$ is equal to $\|A(t) - A(\infty)\|_\sigma^2$. We can consider the function $t \to \mathrm{Var}_\sigma(A(t))$: its derivative is given by $-2\mathscr{E}[A(t)]$. Therefore eq. (3.10) is really bounding the derivative of $\mathrm{Var}_\sigma(\cdot)$ with respect to the functional itself. This leads to the following bound:

$$\mathrm{Var}_\sigma(A(t)) \leq \mathrm{Var}_\sigma(A) e^{-2ct}.$$

Therefore, the spectral gap controls convergence when measured with $\mathrm{Var}_\sigma(\cdot)$. In turn, this implies that:

$$\|A(t) - A(\infty)\| \leq \sigma_{\min}^{-1/2} \|A(t) - A(\infty)\|_\sigma \leq \sigma_{\min}^{-1/2} \|A - A(\infty)\| e^{-ct}.$$

By duality this implies the following bound of the form eq. (3.7):

$$\sup_\rho \left\| T_t(\rho) - \sigma \right\|_1 \leq 2\sigma_{\min}^{-1/2} e^{-t\,\mathrm{gap}\,\mathscr{L}}.$$

or equivalently

Theorem 3. *If λ is the spectral gap of $\mathscr{L}$, then*

$$\tau(\epsilon) \leq \frac{\log\left(2\sigma_{\min}^{-1/2}\right) - \log\epsilon}{\lambda} \tag{3.11}$$

Notice that $\sigma_{\min}^{-1}$ scales **at least** exponentially in the system size (since it has to be at least smaller that $1/\dim \mathscr{H}_\Lambda$) - but in principle it could be even worse.

We could also have obtained the same bound, but without the multiplicative constant 2, by using the fact[39, 80]

$$\left\| \rho - \sigma \right\|_1 \leq \mathrm{Var}_\sigma^{1/2}(\sigma^{-1/2} \rho \sigma^{-1/2}),$$

and the fact that $\sup_\rho \mathrm{Var}_\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})$ is equal to $\left\|\sigma^{-1}\right\|_\infty = 1/\sigma_{\min}$ (where the sup is taken over states).

We have seen therefore that the detailed balance condition allows us to clearly express the relationship between the spectral gap and the mixing time, obtaining a pretty good prefactor for our bound (definitely better than what we could have obtained via the Jordan decomposition). The downside is that we need to assume that the fixed point is unique and full rank (a condition called *primitivity* of $\mathscr{L}$), and that $\sigma_{\min}$ can be controlled.

We will show next how in this setting it is natural to define other conditions on $\mathscr{L}$ that allow a better control on the mixing time than the one obtained via the spectral gap, which in turn will prove to be sufficient to prove our rapid mixing assumption.

3.1.3.2 Entropy and log-Sobolev inequalities

The idea of logarithmic Sobolev inequalities and other entropic inequalities is to generalize what has been done in the previous section with $\mathrm{Var}_\sigma(\cdot)$: find a positive functional $D(\cdot)$ that bounds the convergence of the semigroup T_t, then bound the derivative $\frac{\mathrm{d}}{\mathrm{d}t}D(T_t(\rho)-\sigma)$ in terms of the function itself, via a comparison with another functional defined in terms of $\mathscr{L}$.

Let us consider the following functional, denoted **relative entropy**

$$D(X\|Y)=\frac{1}{\mathrm{tr}\,X}\mathrm{tr}\big[X(\log X-\log Y)\big].$$

$D\big(\rho\|\sigma\big)$ is positive if ρ and σ are normalized states, and it is finite if the support of ρ is contained in the support of σ. It is also monotonically decreasing under the action of quantum channels [65]. Pinsker inequality [64] implies that $\big\|\rho-\sigma\big\|_1^2\leq 2D\big(\rho\|\sigma\big)$. Let us now differentiate and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}D\big(\rho(t)\|\sigma\big)=\mathrm{tr}\big[\mathscr{L}(\rho(t))(\log\rho(t)-\log\sigma)\big].$$

We can therefore denote by $\mathscr{K}(\rho)=-\frac{1}{\mathrm{tr}\,\rho}\mathrm{tr}\big[\mathscr{L}(\rho)(\log\rho-\log\sigma)\big]$. Compare this with the definition of $\mathscr{E}$. We can then define the following log-Sobolev type of inequality:

$$c[D\big(\rho\|\sigma\big)-\log\mathrm{tr}\,\rho]\leq\mathscr{K}(\rho) \tag{3.12}$$

where the optimal $c>0$ will be called log-Sobolev constant of $\mathscr{L}$, and denoted by α. As in the case of the spectral gap inequality, we can then conclude that

$$\sup_\rho\big\|\rho(t)-\sigma\big\|_1\leq\sup_\rho\sqrt{2D\big(\rho\|\sigma\big)}e^{-\alpha t}.$$

We then observe that $D\big(\rho\|\sigma\big)$ is upper bounded by $\big\|\log\sigma^{-1}\big\|_\infty=-\log(\sigma_{\min})$. Therefore, the convergence bound obtained via the log-Sobolev inequality is exponentially better than the one obtained through the spectral gap inequality (see eq. (3.11)):

Theorem 4. *Let α be the log-Sobolev constant of $\mathscr{L}$. Then*

$$\tau(\epsilon)\leq\frac{\log\big(\log\big(\sigma_{\min}^{-1/2}\big)\big)-2\log\epsilon}{2\alpha}. \tag{3.13}$$

If $\sigma_{\min}^{-1}$ is only exponential in the system size, then a system-size uniform log-Sobolev constant implies rapid mixing. Therefore, log-Sobolev bounds are a way of proving such an assumption for reversible, detailed balance generators.

This is the approach taken in [59, 60]. In [66] and [39] a bound equivalent to eq. (3.12) was denoted 1-log-Sobolev inequality, and is obtained by composing eq. (3.12) with the map $\rho\to\sigma^{-1/2}\rho\sigma^{-1/2}$. If we denote by $A=\sigma^{-1/2}\rho\sigma^{-1/2}$ and by $A(t)$ the evolution of A under $\mathscr{L}^*$, i.e. $A(t)=T_t^*(A)$, then detailed balance implies that

$$A(t)=T_t^*(\sigma^{-1/2}\rho\sigma^{-1/2})=\sigma^{-1/2}T_t(\rho)\sigma^{-1/2}=\sigma^{-1/2}\rho(t)\sigma^{-1/2}.$$

Thus eq. (3.12) can be restated as follows[3]:

$$c\,\mathrm{Ent}_1(A) \leq \mathscr{E}_1(A),$$

where $\mathrm{Ent}_1(A) = D\big(\rho\|\sigma\big) - \log\mathrm{tr}\,\rho$ and $\mathscr{E}_1(A) = \mathscr{K}(\rho)$. This version of the bound is clearly equivalent to eq. (3.12) if $\mathscr{L}$ satisfies detailed balance. The authors of [39] denote the optimal constant by α_1.

Unfortunately this is not what is denoted as log-Sobolev inequality in the classical literature (i.e. when we define all the above for a generator of a Markov chain over a probability space, which is the commutative equivalent of Lindbladian generators over quantum states). Instead, the classical log-Sobolev inequality is more similar to the following generalization:

$$c\,\mathrm{Ent}_2(A) \leq \mathscr{E}(A);$$

where $\mathscr{E}$ is the Dirichlet form defined earlier, $\mathrm{Ent}_2(A) = \mathrm{Ent}_1(I_{1,2}(A))$,

$$I_{1,2}(A) = \sigma^{-1/2}\big(\sigma^{1/4}A\sigma^{1/4}\big)^2\sigma^{-1/2} = \sigma^{-1/4}A\sigma^{1/2}A\sigma^{-1/4}$$

and therefore $\mathrm{Ent}_2(A) = D\Big(\big(\sigma^{-1/4}\rho\sigma^{-1/4}\big)^2\|\sigma\Big)$.

This bound is denoted 2-log-Sobolev inequality in [39, 59, 66] and the optimal constant α_2. Unfortunately we do not know if it is equivalent to eq. (3.12): under the additional hypothesis that $\mathscr{E}_1(I_{1,2}(A)) \geq \mathscr{E}(A)$ (denoted L_p*-regularity* in [66]), then one can at least show that $\alpha_2 \leq \alpha_1$, therefore recovering the classical result. Whether there exist Lindbladians which satisfy detailed balance but not L_p-regularity is still an open problem.

3.1.3.3 Hypercontractivity

As we have already seen, as a consequence of the Russo-Dye theorem we have that a positive and trace preserving map T is *contractive* with respect to the 1-norm, since $\|T\|_{1\to1} = 1$ - equivalently, a positive and unital map T^* satisfies $\|T^*\|_{\infty\to\infty} = 1$, i.e. is contractive with the ∞-norm. This applies in particular to quantum channels. Let us now introduce a non-commutative generalization of the L_p-norms[31, 49–51]: given a full rank state σ, for each $p \in [1,\infty)$ denote by

$$\|X\|_{p,\sigma}^p = \mathrm{tr}\big|\sigma^{1/2p}X\sigma^{1/2p}\big|^p = \big\|\sigma^{1/2p}X\sigma^{1/2p}\big\|_p^p.$$

It can be checked that $\|\cdot\|_{p,\sigma}$ is indeed a norm, and the usual properties that can be expected from L_p norms can be recovered, such as H"older inequality, duality, and interpolation theorems. In particular, they are increasing in p, thus for all $1 \leq p \leq q \leq \infty$ it holds $\|\cdot\|_{1,\sigma} \leq \|\cdot\|_{p,\sigma} \leq \|\cdot\|_{q,\sigma} \leq \|\cdot\|_\infty$. Moreover, $\lim_{p\to\infty}\|X\|_{p,\sigma} = \|X\|_\infty$ (the usual Schatten ∞-norm). Note that the norm defined in the previous section, which we denoted by $\|\cdot\|_\sigma$, corresponds to the $p = 2$ case. The space $\mathscr{B}(\mathscr{H})$ equipped with the $\|\cdot\|_{p,\sigma}$ norm will be denoted by $L_p(\sigma)$, and the operator norm of $T : L_p(\sigma) \to L_q(\sigma)$ will be denoted $\|T\|_{(p,\sigma)\to(q,\sigma)}$.

[3] we have removed a unimportant factor of 1/2 from the original definitions

Let us consider then a quantum channel T having σ as a fixed point, and consider its dual T^* w.r.t. the Hilbert-Schmidt scalar product. Let us assume again that T satisfies detailed balance w.r.t. σ. Then we know that T^* is unital, and thus $\|T^*\|_{\infty\to\infty} = 1$. Moreover, we have that for every operator A

$$\left\|T^*(A)\right\|_{1,\sigma} = \left\|\sigma^{1/2}T^*(A)\sigma^{1/2}\right\|_1 = \left\|T(\sigma^{1/2}A\sigma^{1/2})\right\|_1 \leq \left\|\sigma^{1/2}A\sigma^{1/2}\right\|_1 = \|A\|_{1,\sigma},$$

where we have used detailed balance and the fact that $\|T\|_{1\to 1} = 1$. This shows that $\|T^*\|_{(1,\sigma)\to(1,\sigma)} = 1$, and therefore by interpolation it holds that $\|T^*\|_{(p,\sigma)\to(p,\sigma)} = 1$ for all $p \in [1,\infty]$.

This leads to define a new property of a linear map $T : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$: we will say that T is **hypercontractive** if there exist some $p < q$ such that $\|T\|_{(p,\sigma)\to(q,\sigma)} \leq 1$. This in particular implies that T is contractive with respect to $\|\cdot\|_{(p,\sigma)}$.

If we have a dynamical semigroup of quantum channels T_t, then we can consider $\left\|T_t^*\right\|_{(p,\sigma)\to(q,\sigma)}$ for some $p < q$ as a measure of convergence of the semigroup: indeed for $t = 0$ we have that $T_0 = \mathbb{1}$ and $\|\mathbb{1}\|_{(p,\sigma)\to(q,\sigma)} = 1$ if and only if $p \geq q$. On the other hand, if σ is the unique fixed point of T_t, then $T_\infty^*(X) = \mathrm{tr}(\sigma X)\mathbb{1}$ and therefore $\left\|T_\infty^*(A)\right\|_\infty = \mathrm{tr}(\sigma X) \leq \|X\|_{(1,\sigma)}$, and $\left\|T_\infty^*\right\|_{(1,\sigma)\to\infty} = 1$.

Let $1 < p < \infty$, and q its H"older conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Because T_t^* is self-adjoint in $L_2(\sigma)$, then it holds that

$$\left\|T_t^*\right\|_{(p,\sigma)\to(2,\sigma)} = \left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)},$$

and therefore if $1 < p \leq 2$ then

$$\left\|T_{2t}^*\right\|_{(p,\sigma)\to(q,\sigma)} \leq \left\|T_t^*\right\|_{(p,\sigma)\to(2,\sigma)}\left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)} = \left\|T_t^*\right\|^2_{(2,\sigma)\to(q,\sigma)}.$$

In the light of the previous observation, let us focus (as done usually in the literature), on the behavior of $\left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)}$. The relationship between log-Sobolev inequality and hypercontractivity is contained in the following theorem

Theorem 5 ([66])**.** *Let $\mathscr{L}$ satisfy detailed balance and L_p-regularity. Then the following conditions are equivalent*

1. *For $q(t) = 1 + e^{2\alpha t}$,*
$$\left\|T_t^*\right\|_{(2,\sigma)\to(q(t),\sigma)} \leq 1.$$
2. *T_t^* satisfies a 2-log-Sobolev inequality with constant α.*

Note that point 1. of the previous theorem implies that, if $q(t) = 1 + e^{\alpha t}$, then $\left\|T_t^*\right\|_{(p(t),\sigma)\to(q(t),\sigma)} \leq 1$. For $t \to \infty$ we recover that $\left\|T_\infty^*\right\|_{(1,\sigma)\to(\infty,\sigma)} \leq 1$.

3.1.4 Area law

Another interesting problem in the study of dissipative semigroups is the description of the fixed point, or steady state, of the evolution. For some models of noise, the fixed point is the maximally mixed state, which is the state proportional to the identity. Such a state

represents the situation in which the noise has destroyed all the information on the physical system, and every measurement will give uniformly distributed random results. In other cases the noise model is different, and the steady state will be a thermal state corresponding to some Hamiltonian, meaning that it will be proportional to $e^{-\beta H}$ for some Hermitian H and some positive β representing the inverse temperature. Davies maps [19] are such an example. In other cases the evolution is engineered (or defined) to have a particular state as a steady state: one starts with a given state, which is interested in preparing, and from that derives a Lindbladian generator that produces that state as a fixed point. This is a common approach of classical Glauber dynamics and Metropolis sampling[53] and of Dissipative State Preparation [43, 83].

One would expect that, if the state satisfies some "good properties", then the resulting evolution would also have nice properties, as for example would converge quickly. This was proven rigorously in the case of classical spin systems and Glauber dynamics[53, 54], where the "good property" of the state ω is of this type: given two observables A and B, supported on separated regions that are d distant apart, then the value of $\omega(A\otimes B)$ becomes increasingly close to $\omega(A)\omega(B)$ as d increases. More specifically, it is required that the difference between the two goes to zero exponentially fast in d. This property is called *exponential decay of correlations*, since the quantity $\omega(A\otimes B)-\omega(A)\omega(B)$ measures how correlated the two region are. Under this assumption, for classical spin systems one can prove that the corresponding Glauber dynamics is rapid mixing (via proving a log-Sobolev inequality).

In this thesis we will be interested in the reserve problem: given a "well behaved" Lindbladian, what "good properties" of the fixed point can be assumed? We will start by presenting rigorous notions of correlations in many-body quantum systems.

3.1.4.1 Correlation measures

Consider a bipartite state $\rho_{AB} \in \mathscr{B}(\mathscr{H}_{AB})$. If ρ_{AB} is of the form $x\otimes y$ for some states x in $\mathscr{B}(\mathscr{H}_A)$ and y in $\mathscr{B}(\mathscr{H}_B)$, we say it is a product state. In this case measurement over the subsystem A are independent from measurements over the subsystem B, and vice versa: therefore the resulting statistics will be independent and there will be no correlations between the two subsystems. If ρ_{AB} is not product, then there are a number of different measures that quantify "how far" it is from being product.

The following notation is borrowed from [37]. We will denote by ρ_A (resp. ρ_B) the state $\mathrm{tr}_B\,\rho_{AB}$ (resp. $\mathrm{tr}_A\,\rho_{AB}$).

Definition 3 (Correlation measures)**.**

- *Covariance correlation*:

$$C(A:B) = \max_{\substack{M\in\mathscr{A}_A, N\in\mathscr{A}_B \\ \|M\|\le 1, \|N\|\le 1}} |\langle M\otimes N\rangle - \langle M\rangle\langle N\rangle| = \max_{\substack{M\in\mathscr{A}_A, N\in\mathscr{A}_B \\ \|M\|\le 1, \|N\|\le 1}} \left|\mathrm{tr}[M\otimes N(\rho_{AB}-\rho_A\otimes\rho_B)]\right|;$$

  where $\langle O\rangle = \mathrm{tr}(O\rho_{AB})$ is the expectation value of the observable O acting on ρ_{AB}.

- *Trace distance correlation*:

$$T(A:B) = \max_{\substack{F\in\mathscr{A}_{AB}\\ \|F\|\leq 1}} \left|\mathrm{tr}[F(\rho_{AB}-\rho_A\otimes\rho_B)]\right| = \left\|\rho_{AB}-\rho_A\otimes\rho_B\right\|_1.$$

- *Mutual information correlation*:

$$I(A:B) = S(\rho_A)+S(\rho_B)-S(\rho_{AB});$$

  where $S(\rho) = -\mathrm{tr}(\rho\log_2\rho)$ is the von Neumann entropy of the state ρ.

$C(A:B)$ is how correlations are usually measured in the condensed matter literature. It follows immediately from the definition that $C(A:B)$ is upper bounded by $T(A:B)$ (since $C(A:B)$ only depends on what can be measured with product observables, while $T(A:B)$ allows for general ones).

The relationship between trace distance and mutual information is given, in one direction by Pinsker's inequality [64], and in the other by an application of Alicki-Fannes-Audenaert inequalities [2, 4, 23]. We summarize it as follows:

Theorem 6.

$$\frac{1}{4}T(A:B)^2 \leq I(A:B) \leq 6T(A:B)\log_2 d_A + 4h_b(T(A:B)); \tag{3.14}$$

where $h_b(x) = -x\log_2 x-(1-x)\log_2(1-x)$ *denotes the binary entropy function, and* $d_A = \dim\mathscr{H}_A$.

3.1.4.2 Correlations in many-body systems

In our many-body scenario, we can consider the fixed point ρ_∞ of $\mathscr{L}$ over Λ, and for any region $A\subset\Lambda$ or any pair of regions $A,B\subset\Lambda$ we can consider the reduced density matrices $\rho_A = \mathrm{tr}_{\Lambda\setminus A}\rho_\infty$ and $\rho_{AB} = \mathrm{tr}_{\Lambda\setminus A\cup B}\rho_\infty$. We can then ask two types of questions (which we formulate for $I(A:B)$ but would be equally interesting for any other measure of correlations):

- Given $A,B\subset\Lambda$, how does $I(A:B)$ scale with $\mathrm{dist}(A:B)$?
- Given $A\subset\Lambda$, how does $I(A:A^c)$ scales with the size of A?

While of similar flavor, in the first case we are only considering finite regions, while in the second we are considering A^c, which is growing as Λ gets bigger. Therefore it should not be surprising that the conditions needed to give an answer to the first are less restrictive than the ones needed for the second. In the first case, we speak about **decay of correlations**: we expect that, if A and B are far apart in the lattice, they become more and more independent.

The second question is interesting for the following reason. For Haar random states, $I(A:A^c)$ is proportional to $|A|$. Instead, many physically motivated states show a different behavior, with $I(A:A^c)$ scaling as $|\partial A|$, where ∂A is defined as the subset of A of sites which directly interact with the complement of A. If the interactions are finite range and A is a ball, then $|A|$ is a polynomial of degree D while $|\partial A|$ has degree $D-1$. This situation is usually

called **area law** (the terminology originated in the study of black hole entropy, where the boundary is indeed a surface).

In the following we will be working with $T(A:B)$ and $I(A:B)$, with the reminder that because of Theorem 6 exponential decay in one of them implies exponential decay in the other.

3.1.4.3 Ground states of Hamiltonian

The problem of studying correlation decay, area laws and their relationship with dynamics has been extensively treated (although not completely solved) in the context of groundstates of Hamiltonians. A Hamiltonian is a Hermitian operator H acting on the Hilbert space $\mathscr{H}$ which represents the physical system. The action $\mathscr{L}(\rho) = -i[H,\rho]$ generates a group of automorphisms instead of simply a contraction semigroup, and it can be seen that it is a special case of eq. (3.5). Since any eigenvector of H is invariant under the action of $\mathscr{L}$, the evolution will have more than one fixed point: for physical reasons the one corresponding to the lowest eigenvalue of H plays a special role, and it is called the *groundstate* of H. It is a pure state. Given such a state $|\varphi\rangle_{AB}$, it holds that $I(A:B) = 2S(\varphi_A)$, where $\varphi_A = \mathrm{tr}_B |\varphi\rangle\langle\varphi|_{AB}$ is the reduced density matrix of $|\varphi\rangle\langle\varphi|_{AB}$ over A. Therefore, the mutual information reduces to (two times) the von Neumann entropy.

The crucial property in this setting is the so-called **spectral gap** of H: the difference between the two smallest eigenvalues of H. Using the standard convention, we will say that a family of Hamiltonians defined on an increasing and absorbing sequence $\Lambda_n \nearrow \Gamma$ is *gapped* if the gap is uniformly bounded away from zero - in other words, if the gap does not vanishes in the limit. Otherwise the Hamiltonians will be called *gapless* and one can be interested in specifying the speed at which the gap closes (whether polynomially or exponentially fast in n).

In the seminal work [34], Hastings and Koma proved that if a family of local Hamiltonians is gapped, then the ground state satisfies exponential decay of correlations uniformly in n. This result is interesting because it connects to the condensed matter theory of quantum phases: a quantum phase is an equivalence class of Hamiltonian systems, such that two Hamiltonians H_1 and H_2 are in the same equivalence class if they can be connected by a smooth path $H(t)$ of gapped Hamiltonians. Quantum phase transitions are therefore identified with points in the path where the gap closes. In that situation it is common for the correlation length to diverge, where by correlation length we mean a distance ξ such that $C(A:B) \leq e^{-\mathrm{dist}(A,B)/\xi}$.

Another property that is expected by condensed matter theorists is that groundstates of gapped local Hamiltonian satisfy an area law for the entanglement entropy. The intuitive argument (which unfortunately is not a formal proof) goes as follow: if we consider a finite region A, because of exponential decay of correlations, spins which are inside A and far away from its boundary are almost independent from the system outside A. Therefore, correlations and entropy can only come from spins which are closer to the boundary. Since any given d-dimensional spin can only contribute at most by $\log d$ to the total entropy, it follows that the total entropy is only scaling as the size of the boundary.

Whether this argument can be made rigorous is the content of the **area law conjecture** (that ground states of local gapped Hamiltonians satisfy an area law). It is considered a major open problem in condensed matter physics and has seen active development in recent years [22]. A solution in 1D was obtained by Hastings [32], and subsequently a different proof appeared in [10, 11], where they proved that in 1D exponential decay of correlations does actually imply an area law. This together with the result of [34] shows that a spectral gap, by implying exponential decay of correlations, also implies an area law in 1D.

In higher dimensions the problem is still open. Some advances have come from the computer science community [3], with a new proof of Hastings' and Koma's result, which allowed to greatly improve the dependence of the correlation length with respect to the spectral gap, making it fit better with the concrete cases in which we are able to estimate both, either analytically or numerically. The tools developed allowed for the construction of the first provable polynomial algorithm for approximating the groundstate of gapped Hamiltonians in 1D [46], as well as other combinatorial tools to study the structure of ground states. These advances, while very promising, have not led yet to a proof of an area law for groundstates in dimension larger or equal than 2.

3.1.4.4 Gibbs states and tensor network states

Gibbs states or thermal states are states proportional to $\exp(-\beta H)$, for some Hamiltonian H and a parameter β that represents the inverse temperature. They are of interest because they describe a system in equilibrium at finite temperature $1/\beta$, and therefore are naturally suited to fit in the open dynamics scenario: lots of the dissipative models we have mentioned are attempted descriptions of a thermalisation process that leads to a Gibbs state. So, even if they are not the only possible fixed point of dissipative maps, they are definitely an important class of them. Interestingly, they all satisfy an area law for the mutual information [86].

Another important class of states (this time pure states) which often satisfy an area law, are the so called tensor network states [14] - states whose amplitudes are given by the contraction of a given network of tensors. To be more specific, in the large family of tensor network states, the one dimensional Matrix Product States (MPS) and the two and higher dimensional Projected Entangled Pair States (PEPS) satisfy an area law by construction. The interest in these types of states is that they only require a polynomial (in the number of particles) number of parameters to be described, as opposed to the exponential dimension of the Hilbert space they live in. For this reason, they are used extensively in numerics, and they are believed to give good approximations of groundstates of local gapped Hamiltonian. While there are examples of states in 2D that satisfy an area law but are not approximable by a PEPS [26], it has been proved that under certain assumptions groundstates of 2D local Hamiltonians are well approximated by PEPS [33, 57].

In 1D, the situation is more clear: groundstates of local gapped Hamiltonian can be efficiently approximated by MPS. This is not only a theoretical result, but has also been important in understanding and developing algorithms that approximate 1D groundstates.

3.1.4.5 Area law and correlation length

We have mentioned an intuitive -but incomplete- argument that would connect a finite correlation length with an area law. Let us mention now an interesting formal connection presented in [86]. There the authors give a different definition of correlation length for the mutual information: given a finite region $A \subset \Lambda$, let $B_R = \{x \in \Lambda | \operatorname{dist}(x, A) > R\}$, define ξ_Λ as the minimal length R such that

$$I(A : B_R) < \frac{I(A : B_0)}{2}, \quad \forall A \subset \Lambda. \tag{3.15}$$

(Observe that $B_0 = \Lambda \setminus A$.) With this definition, then they can prove that $I(A : A^C) \leq 4|\partial A|\xi_\Lambda$, i.e. an area law.

While the result is sound, one should be careful in considering the relationship between eq. (3.15) and the usual decay of correlations, i.e. $I(A : B) \leq c \exp(-\operatorname{dist}(A, B)/k)$. It is tempting to argue that, if one has such decay of correlations, then because $\operatorname{dist}(A, B_R) = R$ one has that $I(A : B) \leq c \exp(-R/k)$, then it is sufficient to take ξ_Λ proportional to k to satisfy equation eq. (3.15). This argument breaks if the constant c in the decay of correlations bound is not independent of the size of the regions A and B, which is often the case as we will see later. If A is fixed as we change Λ, then the size of B_R is proportional to the total size of the lattice Λ, and therefore ξ_Λ has to grow with Λ. The resulting bound on $I(A : A^C)$ would still grow as a polynomial of lower degree than the geometrical dimension of the lattice, but now multiplied by a constant which is system-size dependent. This constant will in most cases make the bound trivial, since it will be larger than the general worst case bound, which is $\log \dim \mathscr{H}_A = |A| \log d$, where d is the dimension of the Hilbert space of a single site.

A similar problem was faced in [37], where they obtained under the assumption of a log-Sobolev inequality a bound of the form

$$I(A : A^c) \leq c \log\log \left\| \sigma^{-1} \right\| |\partial A|,$$

where σ is the fixed point of the evolution. Again, the right hand side of the bound would scale with the correct exponent to talk about an area law, but the multiplicative constant makes the bound trivial in most cases.

One of the main results of this thesis is to prove for the first time a fully satisfactory area law for fixed points of rapidly mixing evolutions (see section 3.2.3.4).

3.1.5 Stability of quantum systems

One of the properties of open quantum systems studied in this dissertation is stability. Before presenting the results obtained, it is worth explaining why it is so crucial. The mathematical structure we are considering is an attempt of describing a physical system composed of many particles. This might be either a naturally occurring system (for example, the original motivation of the Ising model was to study magnetization), or an artificially engineered system created to fulfill a task (computation, communication, memory, state preparation, etc.).

In the first case, the mathematical model will be of course an approximation to the real physics: it would be unreasonable to require that the quantities involved (coupling constants, energy levels, masses/charges/densities, etc.) can be measured with perfect precision. The only realistic hope is that they can be known with some level of precision. Once we plug this information in our mathematical model, we would like to have a tool that is capable of predicting the results of future experiments. If they change abruptly for even the smallest change in the parameters considered, the resulting predictions will rarely match reality and the model will be deemed to be useless, since it requires an impossible level of *fine-tuning* to work.

The situation is very similar in the case of artificial and engineered systems. In this case, the unreasonable assumption is that we have perfect control over the implementation of the artificial model, meaning that we can configure its parameters to arbitrary level of precision. No real system (not even macroscopic and classical) can be perfectly controlled in this way: the real implementation will be always at best a very good approximation of the mathematical model. If the resulting evolution depends heavily on these tiny differences, then we will end up implementing a different evolution than the one we thought of preparing, and the result will be different. The only practical models are the ones for which small errors in the implementation will give rise to small changes in the resulting system.

In both cases, the theoretical justification of a mathematical model relies on its **stability against perturbations**: we can of course talk about non-stable models, but one should be extremely careful in considering their physical implications and predictions, since in practice we will never be able to actually see them in reality. This argument is only made more stringent when we start considering, apart from experimental errors, physical sources of noises: no experiment will be ever completely isolated, no noise will ever be perfectly shielded.

We thus need tools that allow us to justify the soundness of physical models by proving that they are stable. In the case of local Hamiltonian, effort has focused in proving stability of the spectral gap, a parameter which has important consequences on the physical properties of the resulting models. Starting from the work of [12, 41] it culminated in [56] where it was proven that the spectral gap is stable (it does not close) under some physically reasonable conditions.

It should be stressed here that we are considering a special case of perturbations here: since we are considering many-body models, where every particle interacts only with its neighbors, it is natural to consider a perturbation/error that involves *every single interacting term*. Therefore, small perturbations will be microscopically or locally small, but since they will add up as we consider larger and larger systems, they are actually unbounded perturbations (but with a local structure). This is why we cannot simply apply standard perturbation theory but we need to develop specific tools for this type of perturbations.

Another main result of this thesis is to show that rapidly mixing systems are indeed stable against perturbations (see section 3.2.3.3).

3.2 Summary of the results

3.2.1 Assumptions

In this section, we present and discuss the main assumptions made in this work. The most characterizing one is for sure rapid mixing, a condition on the convergence time of the system to its unique fixed point.

We will be talking about families $\{\mathscr{L}^\Lambda\}_\Lambda$ of Lindbladian generators, where Λ runs over an infinite sequence of finite subsets of Γ. For each of them, we will denote by T_t^Λ the corresponding evolution, i.e. $T_t^\Lambda = \exp\big(t\mathscr{L}^\Lambda\big)$.

Definition 4 (Unique fixed point)**.** Let $\{\mathscr{L}^\Lambda\}_\Lambda$ be a family of Lindblad generators. We say it has a unique fixed point if, for every Λ, $\mathscr{L}^\Lambda$ has a unique fixed point and no periodic point (i.e. it has a trivial peripheral spectrum).

We will denote by T_∞^Λ the trace-preserving projector onto the fixed point of $T_t^\Lambda = \exp\big(t\mathscr{L}^\Lambda\big)$, given by $\lim_{t\to\infty} T_t^\Lambda$.

3.2.1.1 Rapid mixing

We have already argued why the spectral gap gives only partial information about the mixing time of a dissipative evolution, while log-Sobolev inequalities allow for a stronger control (but require some strong property of the fixed point). Our approach is more direct, and we will simply require that the mixing time scales logarithmically with the system size, leaving aside the question of how to prove such condition.

Definition 5 (Rapid mixing)**.** Let $\{T_t^\Lambda\}_\Lambda$ be a family of dissipative maps, where Λ runs over an infinite sequence of finite subsets of Γ. We say it satisfies rapid mixing if there exist $c, \gamma > 0$ and $\delta \geq 1$ such that

$$\sup_{\substack{\rho\geq 0\\ \operatorname{tr}\rho=1}} \left\| T_t^\Lambda(\rho) - T_\infty^\Lambda(\rho)\right\|_1 \leq c|\Lambda|^\delta\, e^{-t\gamma}. \tag{3.16}$$

As we already mentioned, in some cases it is possible to relax the rapid mixing assumption: this is covered partially by [S, sec. 4.5].

Just like proving the existence of a spectral gap for a Hamiltonian system, proving rapid mixing for a dissipative model is a hard task. Apart from "easy" examples, such as non-interacting models and dissipative state preparation for graph states [38], the other important class of models satisfying such assumption are the reversible Lindbladians satisfying a Log-Sobolev inequality [39], which includes classical models such as Glauber dynamics for the Ising model in the appropriate range of parameters [53].

3.2.1.2 Uniform families.

As we have explained in the previous sections, we are interested in studying the *scaling* of some properties of a family of Lindbladian generators $\mathscr{L}_n$, defined on an increasing and absorbing sequence of finite lattices Λ_n converging to an infinite graph Γ (in our case, Γ

will be $\mathbb{Z}^D$, but the same reasoning goes through if we consider any other graph in which balls grow polynomially in the diameter). But at the same time, since we are interested in describing physical models, we would like that different $\mathscr{L}_n$ represent the "same" physical system on a different scale, in such a way that the scaling actually tells us something about the physics we are trying to understand.

What does it mean for operators defined on different lattices to represent the "same physical system"? Of course the question is ill-defined, so no definitive answer can be given, but we can try to make some assumptions about a *rule* or *recipe* to obtain, from the same ingredients, all the $\mathscr{L}_n$ at different scales.

One possible way would be to assume that the local terms of each $\mathscr{L}_n$ are just the translation of a single local generator l_0: that is, there exists some finite $r > 0$ and a l_0 acting on $\mathscr{A}_{b_0(r)}$ such that for every n we have

$$\mathscr{L}_n = \sum_{x : b_x(r) \subset \Lambda_n} l_x$$

where l_x is the translated of l_0 by the vector x. This situation is usually referred to as **translational invariant**, since in the limit the interactions are invariant under translations of Γ (of course, it does not makes sense to talk about invariance under translation for finite lattices).

It should be noted that this is an excessively restrictive assumption: not only because we might want to study systems in which the interactions depend of the position in the lattice, but also because near the boundary of Λ_n the system becomes "under-determined": since there is no room to fit the support of l_0 there, there are fewer and fewer interactions involving sites near the boundary. Sometimes this case is denoted **open boundary conditions**. Since we are interested in systems with a unique fixed point, this assumption can be especially problematic, given that under-determination around the boundary might cause multiple fixed points to appear - therefore, we would be requiring two incompatible conditions on $\mathscr{L}_n$.

To overcome this limitations, we have proposed a definition of what we called **uniform families**, which we believe is a general enough way of describing "meaningful" sequences of Lindbladian generators. Let us denote by

$$\partial_d \Lambda = \{x \in \Lambda \mid \operatorname{dist}(x, \Gamma \setminus \Lambda) \leq d\}. \tag{3.17}$$

By convention, we will write $\partial\Lambda$ for $\partial_1\Lambda$.

Definition 6. Let $\Lambda \subset \Gamma$, a *boundary condition* for Λ is given by a Lindbladian $\mathscr{B}^{\partial\Lambda} = \sum_{d\geq 1} B_d^{\partial\Lambda}$, where $\operatorname{supp} B_d^{\partial\Lambda} \subset \partial_d\Lambda$.

The definition of boundary condition involves a different notion of locality than the one we used for defining local generators: the decay in norm is only required as interactions get *inside* the bulk of the system, but they are allowed to be strong between spins that are as distant as we want, as long as they have the same distance from the boundary. For example, if Λ is a square, this definition allows to couple opposite spins in the boundary: this situation is denoted *periodic boundary condition*, since one can imagine of wrapping up Λ on a torus,

and therefore making opposite spins in the boundary become nearest neighbors. This and other exotic ways of coupling spins in the boundary can be all described by the definition given above.

Definition 7. A *uniform family* of Lindbladians is given by the following:

(i) *bulk interaction*: a Lindbladian $\mathscr{M}_Z$ for every finite set $Z \subset \mathbb{Z}^D$;

(ii) *boundary conditions*: a family of *boundary conditions* $\{\mathscr{B}^{\partial\Lambda}\}_\Lambda$, for every finite $\Lambda \subset \mathbb{Z}^D$.

Given a uniform family of Lindbladians as we have just defined, for each finite $\Lambda \subset \Gamma$ we can define two Lindbladian generators acting on it:

$$\mathscr{L}^\Lambda = \sum_{Z \subset \Lambda} \mathscr{M}_Z \quad \text{open boundary evolution;} \tag{3.18}$$

$$\mathscr{L}^{\bar{\Lambda}} = \mathscr{L}^\Lambda + \mathscr{B}^{\partial\Lambda} \quad \text{closed boundary evolution.} \tag{3.19}$$

When speaking about $\mathscr{L}^{\bar{\Lambda}}$, we will refer to the terms $\mathscr{M}_Z$ as "bulk" interactions and to $\mathscr{B}_d^{\partial\Lambda}$ as "boundary" interactions.

Some comments on this definition are needed: the definition of a uniform does not involve a particular sequence of increasing lattice Λ_n, but instead allows to define one (actually two) Lindbladian for every finite Λ for which a boundary condition is given. If we take two finite $\Lambda_1 \subset \Lambda_2 \subset \Gamma$, and we look at the interactions involving the particles in the "bulk" of Λ_1, meaning the sites that are far away from $\Gamma \setminus \Lambda_1$, then it is easy to see that $\mathscr{L}^{\overline{\Lambda_1}}$ and $\mathscr{L}^{\overline{\Lambda_2}}$ have the same short-range interactions, and they only differ over the long-range ones: either because of the effect of the terms $\mathscr{M}_Z$ with Z extending outside Λ_1, or because of the difference between $\mathscr{B}^{\partial\Lambda_1}$ and $\mathscr{B}^{\partial\Lambda_2}$. Speaking informally, the microscopic details of the interactions are the same apart from some long-range terms. In the following section we will assume that the strength of the interactions, i.e. the norm of the corresponding operators, decays in their range: therefore, for uniform families, we have that the difference between the bulk interactions of $\mathscr{L}^{\overline{\Lambda_1}}$ and $\mathscr{L}^{\overline{\Lambda_2}}$ will be small. This is the fundamental property and defining characteristic of uniform families of Lindbladian: up to small errors, the microscopic details of the interactions in the bulk do no depend on how large the system is taken (as long as it is large enough).

3.2.1.3 Lieb-Robinson Assumptions

Up to now, our notion of local Lindbladian is incomplete: if we do not specify at what rate the norms of the interactions decay, then we can always trivially decompose a Lindbladian as a sum of local terms that are all zero but the one with support on the full space. If instead we impose that the norms decay as a function of the diameter of the support we obtain a highly non trivial condition. Since the decay rate is related to a property we will define later called Lieb-Robinson velocity, we denote these conditions **Lieb-Robinson assumptions**, and we give them only for the uniform families defined earlier.

Definition 8 (Lieb-Robinson Assumptions)**.** There exists an *increasing* function $\nu(r)$ satisfying $\nu(x+y) \leq \nu(x)\nu(y)$, and positive constants υ, b, and c, such that the following conditions hold:

$$\sup_{x\in\Gamma} \sum_{Z\ni x} \|\mathscr{M}_Z\|_\diamond |Z| \nu(\operatorname{diam} Z) \leq \upsilon < \infty, \tag{A-1}$$

$$\sup_{x\in\Gamma} \sup_r \nu(r) \sum_{d=r}^{N} \left\| B_d^{\partial b_x(N)} \right\|_\diamond \leq cN^b. \tag{A-2}$$

Note that if $\|\mathscr{M}_Z\|_\diamond$ decays exponentially in $\operatorname{diam} Z$ (or is zero for all Z with a large enough diameter, a situation denoted **finite range interactions**), then we can take $\nu(r) = \exp(\mu r)$ for some positive μ. If instead it decays polynomially, we are forced to consider slower functions, such as $\nu(r) = (1+r)^\mu$. In particular, if $\|\mathscr{M}_Z\|_\diamond \sim (\operatorname{diam} Z)^{-\alpha}$, then eq. (A-2) is satisfied for $\mu < \alpha - (2D+1)$, where D is the geometrical dimension of Γ (which means that α has to be larger than $2D+1$ for the condition to hold).

The motivation for such assumptions is that systems which satisfy them exhibit a finite speed of propagation: the support of a local observable spreads in time, usually linearly, up to an exponentially small tail. This implies that regions which are spatially separated, if they are uncorrelated at time zero, remain almost uncorrelated for a finite time, which depends (most of the time linearly) on their distance. More details will be given in section 3.2.2.1

3.2.1.4 Frustration freeness

Another condition we will need to impose in some cases is frustration freeness, inspired by a similar condition for Hamiltonians. It is worth noticing that in this case we do not require any specific behavior from the boundary terms, but only from the bulk ones.

Definition 9. We say that a uniform family $\mathscr{L} = \{\mathscr{M}, \mathscr{B}\}$ satisfies *frustration freeness* (or is *frustration free*) if for all Λ and all fixed points ρ_∞ of $\mathscr{L}^\Lambda$

$$\mathscr{M}_Z(\rho_\infty) = 0 \quad \forall Z \subset \Lambda. \tag{3.20}$$

Many interesting and natural examples of Lindbladians have this property: Davies generators and other types of Gibbs samplers for commuting Hamiltonians [36], as well as dissipative state engineering maps of PEPS.

3.2.2 Technical tools

Before presenting the main results, let us present some of the technical tools used to obtain them. They have been developed specifically in order to prove such results, but might have interest and applications in other contexts. We will start by presenting Lieb-Robinson bounds, a key tool in almost every many-body work, and then continue with some derived or inspired results.

In this section, we will assume that the generator $\mathscr{L}$ satisfies assumptions (A-1) and (A-2).

3.2.2.1 Lieb-Robinson bounds

In many-body systems interactions are assumed to be local or quasi-local: the spin present at each site of the lattice can only interact directly with its neighbors (finite range interactions), or if it can interact with distant spins the strength of the interactions has to decay quickly with the distance. Therefore, the interaction between distant spins is not direct, but it is mediated by the intermediate spins which have to "relay" between them. It is natural to expect then that interaction between distant spins is not instantaneous, but it will show a delay, which will get larger as more and more spins have to be involved in the relaying process, or in other words as the distance increases. This is not at all a relativistic effect, since there is no finite speed of light in our models: a better metaphor would be the speed of sound, since it is given by the medium in which the "information" propagates.

This intuitive argument is formalized by Lieb-Robinson bounds, and the resulting velocity of propagation is called Lieb-Robinson velocity. The first formal proof was obtained in the setting of Hamiltonian systems and groups of automorphisms [47, 72], and for this reason the propagation speed is also called group velocity. It was later on generalized to dissipative evolutions [61, 69].

A consequence of Lieb-Robinson bounds and the existence of such a finite velocity is that, if we consider a finite region A and we modify the generator of the evolution on sites that are distant from A, the resulting modified evolution will be almost indistinguishable from the original one on A, at least for short times: for times shorter than the one needed for information to propagate from the modified sites to A, spins in A have no way to "know" about the modification, and therefore will evolve in the same way as if the change had not been made. Since the velocity is given by assumption (A-1), it will be uniform for all system sizes. This fact allowed in [61] to prove existence of the thermodynamic limit of the finite-systems evolutions.

The effect of perturbing the dynamics in a distant region is given by the following lemma, which is derived from the usual Lieb-Robinson bounds.

Lemma 7 ([S, Lemma 5.4])**.** *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two local Lindbladians, and suppose $\mathscr{L}_2$ satisfies assumption* (A-1) *with parameters v and $\nu(r)$. Consider an operator O_X supported on $X \subset \Lambda$, and denote by $O_i(t)$ its evolution under $\mathscr{L}_i^*$, $i = 1,2$. Suppose that $\mathscr{L}_1 - \mathscr{L}_2 = \sum_{r\geq 0} M_r$, where M_r is a superoperator supported on Y_r which vanishes on $\mathbb{1}$, and* $\operatorname{dist}(X, Y_r) \geq r$*. Then the following holds:*

$$\|O_1(t) - O_2(t)\| \leq \|O_X\||X|\frac{e^{vt} - vt - 1}{v} \sum_{r=0}^{\infty} \|M_r\|_\diamond \nu^{-1}(r). \tag{3.21}$$

3.2.2.2 Open and closed evolution

The definition of uniform family allowed us to define two evolution families, one with boundary condition and one without. The definition of boundary condition we have given is justified by the following result: if assumptions eq. (A-1) and eq. (A-2) are satisfied, then the effect of the boundary conditions spreads from the boundary towards the bulk of the system with the same finite speed of propagation that characterizes Lieb-Robinson bounds.

Therefore, for short times and observables far away from the boundary, the two evolutions will be indistinguishable. This is proven by applying lemma 7.

Lemma 8 ([S, Lemma 5.6])**.** *Let O_A be an observable supported on $A \subset \Lambda$, and let $O_A(t)$ (resp. $\bar{O}_A(t)$) its evolution under $\mathscr{L}^{\Lambda *}$ (rep., $\mathscr{L}^{\bar{\Lambda} *}$). Let $r = \mathrm{dist}(A, \Gamma \setminus \Lambda)$. There exist positive constants c, v, and β such that:*

$$\left\| O_A(t) - \bar{O}_A(t) \right\| \leq c \|O_A\| |A| \frac{e^{vt} - 1 - vt}{v} v^{-\beta}(r). \tag{3.22}$$

3.2.2.3 Frustration-free localization

All the bounds presented in this section are valid for any observable localized in a certain region. The dual statement involving the evolution of states would then say something about the evolution of any state which at time zero decomposes as a product with respect to that given region. While working on the problem of determining an area law for the mutual information, we were faced with a similar but different problem: what happens for some specific states, which are not product but satisfy some other good property? More specifically, imagine that we prepare our system on $\Lambda_n = b_0(n)$ in the state ρ_n^∞ which is the fixed point of the (closed boundary) evolution on Λ_n, but then we extend the system with an arbitrary state τ in $\Lambda_{n+1} \setminus \Lambda_n$ and look at the evolution of the state $\rho_n^\infty \otimes \tau$ under the generator defined on Λ_{n+1}.

What can we say in this case? Standard Lieb-Robinson bounds will not give any useful insight, since the regions we are considering (Λ_n and $\Lambda_{n+1} \setminus \Lambda_n$) are at zero distance, and we cannot assume that ρ_n^∞ is product (or close to product) in any other bipartition of the system. On the other hand, if we assume frustration freeness, then since most of the terms in the generator of the evolution will be zero, the only non-zero ones corresponding to terms near the boundary of Λ_n. We will therefore expect that the evolution will be approximately trivial in the bulk of Λ_n, and it will begin invading it from the boundary at the Lieb-Robinson speed.

This intuitive idea is made formal in the following lemma. It should be noted that, as far as we know, it is not a direct consequence of the standard Lieb-Robinson bounds. Instead, in order to prove it, we replicated the ideas and techniques that are present in the proof of the Lieb-Robinson bound and have adapted them to this specific situation.

Lemma 9 ([A, Lemma 12])**.** *Let $\mathscr{L} = (\mathscr{M}, \mathscr{B})$ be an uniform family of Lindbladians, satisfying frustration freeness. Let $A \subset \Gamma$ be a finite region, and fix a positive natural number m. Let $B = A(m+1)$, $R = A(m+1) \setminus A(m)$ and ρ_∞^m a fixed point of $T_t^{\bar{A}(m)}$ and τ an arbitrary state on R.*

$$\left\| \left(T_t^{\bar{B}} - T_t^{\bar{B} \setminus A} \right) (\rho_\infty^m \otimes \tau) \right\|_1 \leq \mathrm{poly}(m) v^{-1}(m) \left[e^{vt} - 1 + t \right]; \tag{3.23}$$

where $T_t^{\bar{B} \setminus A}$ denotes the evolution generated by

$$\mathscr{L}^{\bar{B} \setminus A} = \sum_{Z \subset B \setminus A} \mathscr{M}_Z + \sum_{d \leq m+1} \mathscr{B}_d^{\partial B}.$$

3.2.3 Main results

Let us now present the main results obtained in [A, S, R].

3.2.3.1 Local rapid mixing

Definition 10 (Local rapid mixing)**.** Take $A \subset \Lambda$, and define the *contraction of T_t relative to A* as

$$\eta^A(T_t) := \sup_{\substack{\rho \geq 0 \\ \operatorname{tr}\rho = 1}} \left\| \operatorname{tr}_{A^c} \left[T_t(\rho) - T_\infty(\rho) \right] \right\|_1 = \sup_{\substack{O_A \in \mathscr{A}_A \\ \|O_A\| = 1}} \left\| T_t^*(O_A) - T_\infty^*(O_A) \right\|. \tag{3.24}$$

We say that $\mathscr{L}$ satisfies *local rapid mixing* if, for each $A \subset \Lambda$, we have that

$$\eta^A(T_t) \leq k(|A|) e^{-\gamma t}, \tag{3.25}$$

where $k(r)$ grows polynomially in r, $\gamma > 0$ and all the constants appearing above are independent of the system size.

We can also define a **local mixing time** as the inverse of η^A:

$$\tau^A(\epsilon) = \min\{t > 0 : \sup_\rho \left\| \operatorname{tr}_{A^c} [T_t(\rho) - T_\infty(\rho)] \right\| \leq \epsilon\}. \tag{3.26}$$

Then local rapid mixing implies that τ^A depends on $|A|$ and ϵ, but not on Λ: this implies that (up to small errors) local observables converge to their limit in a time scale which depends only on the observable support and **not** on the system size. Together with Lieb-Robinson bounds, it implies also that local observables are able to interact only with a finite region around them independent of the system size.

It seems surprising that just based on an estimate on the mixing time of a uniform family we could derive such stronger property, but this is what it is proved in [S, Proposition 6.6].

Theorem 10. *For uniform families satisfying Lieb-Robinson assumptions, rapid mixing implies local rapid mixing.*

3.2.3.2 Local indistinguishability

The implication of local rapid mixing for observables can be in a sense "dualized" to a corresponding property of the family of fixed points. The reasoning goes as follows: the limit $O(\infty) = \lim_{t\to\infty} O(t)$ of the evolution of an observable is its expectation value against the limit state, i.e. $O(\infty) = \operatorname{tr}(\rho_\infty O)\mathbb{1}$, since:

$$\operatorname{tr}\rho O(\infty) = \lim_{t\to\infty} \operatorname{tr}\rho T_t^*(O) = \lim_{t\to\infty} \operatorname{tr} T_t(\rho) O = \operatorname{tr}\rho_\infty O.$$

We have that on the one hand, Lieb-Robinson bounds imply that if O is a local observable, then for short times $O(t)$ does not depend on the interactions that are far away from its support; on the other hand, $O(t)$ converges to $O(\infty)$ independently of the system size. Therefore, $O(\infty)$ can in a sense only depend on interactions that are close to its support at time zero, and not on the ones that are too far away. Since $O(\infty)$ is equal to $\operatorname{tr}\rho_\infty O$, this

means that the observable O cannot distinguish between different fixed point of evolutions with increasing system size, as the only change between them is in the interactions far away from the support of O. This is formalized in the following lemma [S, Lemma 6.2]

Lemma 11. *Let $\mathscr{L} = \{\mathscr{M}, \mathscr{B}\}$ be a uniform family of dissipative evolutions that satisfies rapid mixing, and suppose each $T_t^{\bar{\Lambda}}$ has a unique fixed point and no other periodic points. Fix a Λ and let ρ_∞ be the unique fixed point of $T_t^{\bar{\Lambda}}$. Given $A \subset \Lambda$, for each $s \geq 0$ denote by ρ_∞^s the unique fixed point of $T_t^{\bar{A}(s)}$.*

Then we have:

$$\left\| \mathrm{tr}_{A^c}(\rho_\infty - \rho_\infty^s) \right\|_1 \leq |A|^\delta \Delta_0(s), \tag{3.27}$$

where $\Delta_0(s) = c(|A(s)|/|A|)^{\delta v/(v+\gamma)} v^{-\beta\gamma/(v+\gamma)}$, and c is a positive constant while β and v are given in lemma 8 and δ and γ in definition 5.

This property is coherent with the idea that uniform families represent some model in which the microscopic dynamic is well defined and independent of the system size.

3.2.3.3 Stability against perturbations

We have presented, in section 3.1.5, the importance of stability against perturbation for theoretically justifying the models we consider. Let us give a formal definition of stability. Since there are definitely different notions that might be useful in different context, we will take a very conservative approach, and impose as few assumptions as possible on the perturbations, while requiring the strongest possible notion of stability. There are of course a number of possible relaxations of this result.

Given a uniform family of Lindbladians $\mathscr{L}$, defined by its bulk terms $\mathscr{M}_Z$ and its boundary conditions $\mathscr{B}_d$, we will consider a perturbation of both the bulk terms $\mathscr{M}'_Z = \mathscr{M}_Z + E_Z$ and of the boundary conditions $\mathscr{B}'_d = \mathscr{B}_d + E_d$. The perturbation should be small compared to the norm of the original Lindbladians, so we will assume that for all E_Z and E_d it holds that $\|E_Z\|_\diamond \leq \epsilon \|M_Z\|_\diamond$ and $\|E_d\|_\diamond \leq \epsilon \|B_d\|_\diamond$, where $\epsilon > 0$.

Notice that such perturbation are small microscopically, but since they act on every local term, their sum $E^{\bar{\Lambda}} = \sum_{Z \subset \Lambda} E_Z + \sum_d E_d$ has norm which diverges with the system size, so that it is an unbounded perturbation once we forget about its local structure. This implies that we cannot simply apply standard perturbation theory results.

We still need some condition on the perturbation for it to be "physically realistic". Requiring $\mathscr{M}'_Z$ and $\mathscr{B}'_d$ to be Lindbladians would suffice, but we can relax this restriction and only impose the following conditions (which are always satisfied if the perturbed generators are Lindbladians):

- $E_Z^*(\mathbb{1}) = E_d^*(\mathbb{1}) = 0$;
- $S_t = \exp\left[t(\mathscr{L}^{\bar{\Lambda}} + E^{\bar{\Lambda}})\right]$ is a contraction for each $t \geq 0$.

With this perturbation model, under the rapid mixing assumption, we are able to prove the following stability result [S, Theorem 6.7].

Theorem 12. *Let $\mathscr{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying rapid mixing. Let S_t be defined as above. For an observable O_A supported on $A \subset \Lambda$, we have for all $t \geq 0$:*

$$\left\| T_t^*(O_A) - S_t^*(O_A) \right\| \leq c(|A|) \, \|O_A\| \left(\epsilon + |\Lambda| \nu^{-\eta}(d_A) \right), \tag{3.28}$$

where $d_A = \mathrm{dist}(A, \Lambda^c)$; η is positive and independent of Λ; $c(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Let us discuss the r.h.s. of eq. (3.28). The multiplicative term $\|O_A\|$ is an expected normalization constant, that takes into account the fact that the l.h.s. is 1-homogeneous. The constant $c(|A|)$ only depends on the support of A and not on Λ: therefore, if A is fixed once for all and does not scale with Λ, this is simply a constant prefactor. We will discuss later on what happens if this is not the case.

The local norm of the perturbation is ϵ, so it is expected that it would appear on the r.h.s. of the bound. More surprisingly it only appears as a linear factor. Let us stress again that if the perturbation is a sum of local terms acting on single sites, then ϵ is the (diamond) norm of one single term, not of the whole perturbation, and in particular is independent of the system size.

The other term $|\Lambda| \nu^{-\eta}(d_A)$ is a boundary correction term, which takes into account the effect of the perturbed boundary on the observable. As it is expected, it is decaying with the distance of A to the boundary, and it vanishes as Λ goes to the infinite lattice Γ. Therefore, for large enough Λ, is will be smaller than ϵ and can be considered negligible. In some intermediate cases it might not be optimal: for example, one would not expect such a term to appear in the case of translational invariant interactions and periodic boundary conditions, even if the observable is localized near the boundary. After all, in this setting, the location of the boundary is only a mathematical necessity, and it does not correspond to any difference in the physical interactions between the spins. In fact, we can "shift" the system to put the boundary as distant as possible from A, and therefore in such cases one should always consider d_A to be the half-diameter of the complement of A in Λ. With this observation, the decaying term $\nu^{-\eta}(d_A)$ will quickly cancel the contribution of the term $|\Lambda|$ and become negligible compared to ϵ.

It is worth noting that if we consider non-local observables (i.e. observables whose support grows with Λ), then the factor $c(|A|)$ cannot be assumed to be smaller than linear, and in particular this means that the bound cannot be improved to be non-divergent in Λ. In fact, it is easy to construct simple examples of non-interacting spins such that there are observables supported on the full lattice for which the l.h.s. of eq. (3.28) grows linearly with the system size. This means in particular that the factor $c(|A|)$ has to be at least linear. See [S, Example 4.8] for such a construction.

3.2.3.4 Area law with logarithmic correction

Regarding the problem of correlations decay for the fixed point of the evolution, we have the following result [A, Theorem 14]:

Theorem 13. *Let $\mathscr{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying rapid mixing. Then the fixed point of every $\mathscr{L}^{\bar{\Lambda}}$ satisfies:*

$$T(A:B) \leq 3(|A|+|B|)^{\delta} \Delta_0\left(\frac{d_{AB}}{2}\right), \tag{3.29}$$

where Δ_0 is the fast-decaying function given in lemma 11.

Because of Theorem 6, $I(A:B)$ will show a similar decay.

Let us now consider the question of whether ρ_∞ satisfies an area law or not. We have not been able to give a definite answer, but we have obtained a relaxed version: a scaling of $I(A:A^c)$ as $|\partial A|\log|A|$, which in terms of the radius of A goes as $r^{D-1}\log r$. In order to obtain such a result, we will need some extra assumptions [A, Proposition 16,Theorem 17].

Theorem 14. *Let $\mathscr{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying rapid mixing. Moreover, let us assume that $\mathscr{L}$ satisfies either of the following conditions:*

- *ρ_∞ is pure for every Λ;*
- *$\mathscr{L}$ satisfy frustration-freeness;*

Then the following holds for fixed point of every $\mathscr{L}^{\bar{\Lambda}}$, for some constant c independent of Λ:

$$I(A:A^c) \leq c|\partial A|\log|A| \tag{3.30}$$

Interestingly, the two alternative conditions required in Theorem 14 are independent from each other. Therefore, we suspect that their need is an artifact of the proof, and that the result holds without either assumption.

3.3 Outlook and future work

We have seen that a condition on the scaling of the mixing time of a quantum dynamical system can have deep impact on the properties it exhibits: both dynamical (like the stability property) and static (as the area law and the indistinguishability of its fixed points).

This brings us to reconsider the mixing time as a fundamental and characterizing property of these models, in the same spirit that the spectral gap plays in the context of Hamiltonian systems, as it was already proposed in [42]. In this dissertation we have identified a class of systems which, because of the mentioned properties, clearly plays a special role in a possible classification of dissipative evolutions. Whether this class is to be considered the "good" case or the "trivial" case is still to be determined: it will be important to see which properties can be recovered and which cannot hold when we lift the rapid mixing condition for something less restrictive, and how large and diverse is the class of rapid mixing models. This is an interesting line of future research, and the following observations can be considered as the first steps in its development.

3.3.1 Polynomial mixing time

If we consider Lindbladians as "dissipative machines" or preparation procedures to obtain useful quantum states, then from an algorithmic point of view polynomial time mixing is perfectly acceptable, and constitutes a more natural assumption than rapid mixing. In the same spirit, for computing purposes, it is sometimes useful to consider gapless Hamiltonians whose spectral gap only closes *polynomially* in the system size. In both cases, we have a situation that is clearly unfavorable in the thermodynamic limit, but that for finite systems can still be somehow treated in a reasonable amount of time and with a reasonable amount of resources.

In this situation, the connection with condensed matter theory (both formal results as wells as the "philosophy") becomes less useful: it would be akin to study the efficiency of different sorting algorithms by looking at how they perform on a infinite set of elements. Instead, a different set of tools would be needed

Describing what is the computational power and the properties of such class is an extremely interesting property and would be a way of clarifying whether dissipative state preparation can be scaled up in realistic experiments. In this setting stability, not necessarily in the same form as we have proven here, should be a crucial characteristic to consider.

3.3.2 Preparation of topological models

The area law result could be interpreted as a negative property of rapid mixing models: they are not able to generate states which violate area laws. On the other hand is it also known that some topological states cannot be obtained in sub-linear time [42], at least with realistic assumptions about the Lindbladian generators, even if they satisfy an area law. Since topological states are regarded as the key ingredient for a robust quantum memory, it would be interesting to study whether:

i. they can be prepared dissipatively in a stable way;
ii. there exist "good" dissipative processes that preserve the state, once the system has been prepared by other means.

The difference between the two lies in the fact that in the second case we are not asking for the process to robustly or rapidly prepare the topological model when started in any arbitrary state, but the only requirement is that these good properties hold when the process is started exactly in the state we want to preserve. If the system is robust, then any state which is close enough to the desired state would converge back to it, and any small perturbation of the generators will only slightly move this special fixed point. In principle, we would not be requiring any good properties outside these: there might be other fixed point, either stable or unstable.

In this line of research, we should mention the following works: the proposal of an "encoder" [20], which not only prepares a state in the 2D Toric Code groundstate, but it also allows to encode logical information, in the sense that a specific pair of qubits in the system will get mapped into the virtual qubits of the Toric Code. The process requires linear time,

and is highly not translation invariant. What is the effect of noise on this encoding procedure is still an open problem.

In [36], they study two different families of Lindbladian models arising from a commuting quantum Hamiltonian, both of which prepare the Gibbs state of the corresponding Hamiltonian. They show that for high enough temperature, both processes are gapped and therefore they have polynomial time mixing. In the corresponding classical case it is possible to show the stronger property of a log-Sobolev inequality and rapid mixing, so one could hope of strengthening the result. Notice that this would not contradict the result on the preparation of topological models: since here we are preparing the Gibbs state at some finite temperature, and we know that for the 2D Toric Code this shows no topological properties, there is in principle nothing forbidding a rapid mixing process to generate the Gibbs state of the 2D Toric Code.

It is known that if we take the temperature to zero, the Gibbs state converges to the ground state - under some assumption on the density of states at different energy levels, to get a good approximation it is sufficient that the temperature scales as the inverse of the logarithm of the number of particles [33]. Therefore, it would be interesting to study the behavior of the models considered in [36] as the temperature goes to zero, for specific commuting Hamiltonian such as the Toric Code. It is not clear whether the resulting map will have fixed points outside of the groundstate space or not, and whether the gap will close or stay open. In [84] it was considered a similar problem of preparing a Gibbs state for a Hamiltonian with a critical temperature, and it was shown that in the regime of phase coexistence there cannot be a unique fixed point. This does not solve the case of the Toric Code, since there is no critical temperature, but it can be an interesting comparison.

3.3.3 Proving rapid mixing

In this dissertation we have shown how rapid mixing implies properties of the fixed point of the evolution, and we have discussed log-Sobolev inequalities as a way of proving such condition in the case of detailed balance Lindbladians. It would be extremely useful to have tools and conditions that might allow to prove the rapid mixing hypothesis, either via log-Sobolev inequality or not. The work done in [36] can be considered as going in this direction: describing properties of the fixed point that imply conditions on the mixing time. This is inspired by the results obtained for classical models (see [53, 54] for a review), for which it was shown that a condition on the correlation decay of the fixed point of Glauber dynamics implies a log-Sobolev inequality and in turn rapid mixing. Moreover, this type of correlation decay is usually present for Gibbs states at high temperature.

In [36] a similar approach was taken, and a specific definition of correlation decay has been used to show that the corresponding dynamics has a spectral gap. It is also proven that this condition holds for high temperature, but also in the 1D case. It would be interesting to see if one could strengthen the result and prove a log-Sobolev inequality instead, which would be expected given the classical result.

The results presented in section 3.1.3.3 show that when we generalize log-Sobolev inequalities from the classical to the quantum setting, we do not obtain a single inequality but a family (indexed by $p \in [1,\infty)$) of inequalities. Classically they are all equivalent, but in the

quantum case this is not known, and therefore different authors have resorted to assume L_p-regularity: the assumption that the L_2 constant lower bounds all the others. To recover the connection with hypercontractivity one indeed needs the whole family of log-Sobolev inequalities to hold, but as we showed in section 3.1.3.2 one only needs the case L_1 in order to prove rapid mixing.

It is not known how generic the L_p-regularity condition is: the only general result being that some important class, such as Davies generators [39] and unital processes [66], satisfy it. In [39] it is conjectured that every detailed balanced Lindbladian satisfies L_p-regularity, and every primitive Lindbladian satisfies a weak version of it. It this conjecture fails to hold, then one might want to only look at strategies to prove the L_1 inequality, since that is the one which provides the rapid mixing bound. Showing which conditions have to be imposed on the fixed point in order to prove the log-Sobolev inequalities will help us understand whether the class of rapid mixing systems is "small" or "large".

3.3.4 Other work in different research lines

Another line of research which was developed during the PhD, other than the study of open dissipative dynamics, is the study of whether properties of the thermodynamic limit of a sequence of Hamiltonians can be inferred from the study of an increasing sequence of finite systems. More specifically, we were interested in the possible pitfalls of the common approach taken to study the large-system limit of Hamiltonian models, which consists in analyzing a sequence of finite cases, either experimentally or numerically, to then extrapolate some properties of the limit. In a number of important cases [6, 45, 52, 68, 79] this approach has been successful and has led to important insight on the properties of the physical models in the large-system limit. On the other hand, there is a general negative result: the problem of deciding whether a sequence of translation invariant local Hamiltonians is gapped or gapless in the limit is an undecidable problem [15]. This result shows that unpredictable behavior can be shown by this type of models, which lead us to be interested in exploring the possibilities of constructing such exotic examples.

The resulting work has been presented in a (yet to be published) paper [8], in which we present two families of models that show a surprising property: for any finite region smaller than a fixed threshold, the ground state and low-excited states are classical states (product states in the computational basis); above the threshold they show instead topological properties, which are characteristic of quantum models. By increasing the local dimension of the spins, the threshold can be made arbitrarily large, and already for local dimension $d = 10$ becomes bigger than the estimated number of particles in the universe. We denoted this phenomenon **size-driven phase transition**, as it can be seen as an abrupt change from a classical model to a quantum model driven by the change of the system size parameter.

The two constructions are based on different ideas, and have different thresholds scaling. They are both based on *tilings models*: a tiling is a covering of a region of the plane with unit squares with colored edges, such that colors on neighboring squares matches. It has been shown that the problem of deciding, given a finite set of tiles, whether they can tile the whole plane or not is undecidable [9, 73]. This result was a building block of the undecidability result of [15]. We modified their construction by using *plaquette* and *star*

interactions (instead of just having plaquettes), and with this we produced two families of models. The first one is based on the idea of constructing periodic patterns with very large periods (compared to the number of colors used in the tiling), in such a way that a specific pattern only appears once every period. By penalizing that pattern, we can induce a energy frustration for every lattice size larger than the period: this allows us to implement the transition between the classical and quantum models.

The other construction is based on the idea, already present in the results of undecidability, of encoding the history of a Turing Machine into the groundstate of the Hamiltonian. In this way, by giving an energy penalty if the machine halts, we have the same phenomenon as before of an energy frustration when the system size becomes large enough for the machine to halt. Since determining whether (and when) a Turing machine will halt is an undecidable problem, this was one of the key ingredient into showing undecidability of the spectral gap. We were able to highly optimize the cost of the encoding, in the sense of the minimal local Hilbert space dimension needed to write the history state of the Turing machine into the spin model. With this optimized encoding, we considered the so called *Busy Beavers* Turing machines: a machine with a very small dimension, but whose halting time is surprising large - it actually grows faster than any computable function. This gives us models which have a relatively small local dimension, but for which there is a frustration only for extremely large system sizes. Again, this frustration allows us to implement the phase transition.

CAPÍTULO 4

Introducción

Esta tesis está organizada como sigue. En el apartado 4.1 definimos los principales objetos de interés, que son los semigrupos dinámicos de canales cuánticos. Recordaremos las propiedades de sus generadores, llamados Lindbladianos. Se explicará por qué son un buen modelo para evoluciones cuánticas con ruido, y se definirá el tiempo de equilibración, una propiedad central en las hipótesis necesarias para probar los resultados principales. Se presentará la relación entre tiempo de equilibración y otras importantes propiedades del semigrupo, como el gap espectral, la desigualdad de log-Sobolev e hipercontractividad. Estas conexiones nos permitirán encontrar técnicas para probar la condición de equilibración rápida. Introduciremos también la noción de información mutua y se discutirá la ley de área para estados, así como sus conexiones con la dificultad de simulación y los estados de redes tensoriales. Finalmente, se discutirá por qué la estabilidad es una condición fundamental para cualquier modelo matemático de un sistema físico. En el apartado 4.2 se definirán las hipótesis principales y una síntesis de los resultados obtenidos, junto con una breve presentación de las herramientas técnicas desarrolladas para probarlos. En el apartado 4.3 se discutirán las lineas futuras de investigación en las cuales estamos trabajando actualmente.

El resto de la tesis está compuesta por las publicaciones que recogen los resultados obtenidos a lo largo del Doctorado. Los capítulos corresponden a las publicaciones siguientes.

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El impacto de estas publicaciones está reflejado en el número de citas recibidas, a pesar de su reciente publicación: en particular, [S] en el momento de la publicación de esta tesis ha recibido ya 18 citas, mientras que [R] ha recibido 3 y [A] una. Además, los resultados obtenidos han sido presentados como ponencia oral en las conferencias más prestigiosas del área: en Quantum Information Processing and Communications 2013 (QIPC2013), en el 17th Conference on Quantum Information Processing (QIP2014), y en Theory of Quantum Computation, Communication and Cryptography (TQC2015).

4.1 Objetos de estudio y resultados previos

4.1.1 Notación

Fijemos primero la notación que usaremos a lo largo de toda la tesis, aunque se presentarán estos objetos con más detalle más adelante.

Dado un producto tensorial de dos espacios de Hilbert de dimensión finita $\mathscr{H}_A \otimes \mathscr{H}_B$, la única función lineal $\mathrm{tr}_A : \mathscr{B}(\mathscr{H}_A \otimes \mathscr{H}_B) \to \mathscr{B}(\mathscr{H}_B)$ tal que $\mathrm{tr}_A(a \otimes b) = b\,\mathrm{tr}(a)$ para todo $a \in \mathscr{B}(\mathscr{H}_A)$ y todo $b \in \mathscr{B}(\mathscr{H}_B)$ se llamará la *traza parcial* sobre A. Un estado sobre $\mathscr{B}(\mathscr{H})$ está dado por un funcional lineal positivo $\rho : \mathscr{B}(\mathscr{H}) \to \mathbb{R}$, normalizado de manera que $\rho(\mathbb{1}) = 1$. Indicaremos la traza de un operador X con $\mathrm{tr}\,X$, y la norma p de Schatten como $\|\cdot\|_p$, es decir $(\mathrm{tr}|X|^p)^{1/p}$. Si no hay riesgo de ambigüedad, $\|\cdot\|$ denotará la norma de operador usual (la norma de Schatten ∞).

Consideraremos un retículo cúbico $\Gamma = \mathbb{Z}^D$, con la distancia de grafo, y un subconjunto finito $\Lambda \subset \Gamma$. Como es común en física, llamaremos a todo subconjunto de Γ un retículo, aunque no lo sea en el sentido de teoría de grafos. La bola centrada en $x \in \Lambda$ de radio r se denotará con $b_x(r)$. A cada vértice x del retículo le vamos a asociar un sistema cuántico elemental con un espacio de Hilbert de dimensión finita $\mathscr{H}_x$. Usaremos la notación de Dirac para vectores: $|\varphi\rangle$ será un vector en $\mathscr{H}_x$, $\langle\varphi|$ su adjunto, y $\{|n\rangle\}_{n=0}^{\dim\mathscr{H}_x-1}$ la base canónica de $\mathscr{H}_x$. El producto escalar en $\mathscr{H}_x$ se denotará como $\langle\varphi|\psi\rangle$, y las funciones de rango uno como $|\varphi\rangle\langle\psi|$. A cada subconjunto finito $\Lambda \subseteq \Gamma$ le asociamos un espacio de Hilbert dado por

$$\mathscr{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{H}_x,$$

y un álgebra de observables definida por

$$\mathscr{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathscr{B}(\mathscr{H}_x).$$

Dado que $\mathscr{H}_x$ tiene dimensión finita, $\mathscr{B}(\mathscr{H}_x)$ es una álgebra de matrices, y la indicaremos a veces como $\mathscr{M}_d$, donde $d = \dim\mathscr{H}_x$.

Si $\Lambda_1 \subset \Lambda_2$, hay una inclusión natural de $\mathscr{A}_{\Lambda_1}$ en $\mathscr{A}_{\Lambda_2}$, identificándola con $\mathscr{A}_{\Lambda_1} \otimes \mathbb{1}$. El soporte de un observable $O \in \mathscr{A}_\Lambda$ es el mínimo conjunto Λ' tal que $O = O' \otimes \mathbb{1}$, para algún $O' \in \mathscr{A}_{\Lambda'}$, y será indicado por $\mathrm{supp}\,O$.

Una función lineal $\mathscr{T} : \mathscr{A}_\Lambda \to \mathscr{A}_\Lambda$ se llamará *superoperador* para subrayar la distinción con los operadores en $\mathscr{A}_\Lambda$. El soporte de un superoperador $\mathscr{T}$ es el conjunto mínimo $\Lambda' \subseteq \Lambda$

tal que $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$, donde $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_{\Lambda'})$. Diremos que un superoperador preserva la hermiticidad si manda operadores hermíticos a operadores hermíticos. Diremos que es positivo si manda operadores positivos (es decir, operadores de la forma $O^* O$) a positivos. Diremos que $\mathcal{T}$ es *completamente positivo* si $\mathcal{T} \otimes \mathbb{1} : \mathcal{A}_\Lambda \otimes \mathcal{M}_n \to \mathcal{A}_\Lambda \otimes \mathcal{M}_n$ es positivo para todo $n \geq 1$. Diremos que $\mathcal{T}$ preserva la traza si $\operatorname{tr} \mathcal{T}(\rho) = \operatorname{tr} \rho$ para todo $\rho \in \mathcal{A}_\Lambda$. Las normas p de Schatten en $\mathcal{A}_\Lambda$ inducen una familia correspondiente de normas sobre $\mathcal{B}(\mathcal{A}_\Lambda)$: denotamos con $\|\cdot\|_{p\to q}$ la norma de operador de un superoperador $\mathcal{T} : (\mathcal{A}_\Lambda, \|\cdot\|_p) \to (\mathcal{A}_\Lambda, \|\cdot\|_q)$, es decir, cuando el dominio está equipado con la norma p y la imagen con la norma q. A veces nos hará falta la siguiente norma, llamada norma diamante:

$$\|\mathcal{T}\|_\diamond = \sup_n \|\mathcal{T} \otimes \mathbb{1}_n\|_{1\to 1}.$$

4.1.2 Semigrupos dinámicos de canales cuánticos

4.1.2.1 Evolución unitaria y canales cuánticos

La mecánica cuántica nos dice que un sistema físico viene representado por un espacio de Hilbert $\mathcal{H}$. Las propiedades medibles del sistema se codifican en un estado ρ, que es un operador positivo con traza uno. Para simplificar sólo consideraremos espacios de Hilbert de dimensión finita d. Por lo tanto $\mathcal{B}(\mathcal{H})$ es una álgebra de matrices $\mathcal{M}_d$. En el caso de sistemas aislados, la evolución física del sistema se describe con una evolución unitaria del estado ρ, donde el estado del sistema después de la evolución está dado por $U\rho U^*$, con U un operador unitario de $\mathcal{H}$. Es inmediato ver que este tipo de evolución es necesariamente reversible, dado que su inversa $\rho \to U^* \rho U$ es también físicamente posible. Por lo tanto, para incluir a sistemas disipativos, donde la evolución no es reversible a causa de una interacción con un espacio ambiente, tenemos que reemplazar la evolución unitaria por algo más general.

Vamos a intentar fijar unos requisitos lo más generales posibles que una aplicación $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ tiene que satisfacer para representar a una evolución físicamente realizable. Sea ρ el estado inicial, y $T(\rho)$ el estado evolucionado. T debe mandar estados a estados, y por lo tanto tiene que ser lineal [1], positivo y que preserve la traza. Es un hecho sorprendente pero muy importante que la positividad no sea suficiente para dar a T una interpretación física consistente. Imaginemos extender nuestro sistema con uno auxiliar, con su propio espacio de Hilbert $\mathcal{K}$ y estado τ. Entonces el estado del conjunto de los dos sistemas es $\rho \otimes \tau \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. Asumimos también que la evolución de τ sea trivial. ¿Existe una aplicación $\tilde{T}$ que extienda T en $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, de manera que $\tilde{T}(\rho \otimes \tau) = T(\rho) \otimes \tau$ para todos los estados ρ y τ? Esa función existe y está dada por el producto tensorial de T con la función identidad, y se denota por $T \otimes \mathbb{1}$.

Si esta fuese una evolución física, debería ser otra vez positiva. Pero sorprendentemente hay aplicaciones T positivas tales que $T \otimes \mathbb{1}$ no es positiva. Por esta razón necesitamos

[1] La interpretación de un *ensamble* de estados $\{(p_i, |\varphi_i\rangle\langle\varphi_i|)\}$ es la de una distribución de probabilidad (p_i) sobre un conjunto de estados posibles $\{|\varphi_i\rangle\}$. Por lo tanto es razonable esperar que después de la evolución el ensamble se haya transformado en $\{(p_i, T(|\varphi_i\rangle\langle\varphi_i|))\}$, o que en otras palabras $T(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|) = \sum_i p_i T(|\varphi_i\rangle\langle\varphi_i|)$.

pedir la condición más fuerte de que T sea completamente positiva: recordamos que una aplicación $T : \mathcal{M}_d \to \mathcal{M}_{d'}$ es completamente positiva si $T \otimes \mathbb{1}_n$ es positiva para todo $n \in \mathbb{N}$, donde $\mathbb{1}_n$ es la identidad de $\mathcal{M}_n$. Una aplicación completamente positiva y que preserva la traza se suele llamar un *canal cuántico*, y será nuestro objeto principal de estudio.

Vamos ahora a justificar por qué los canales cuánticos realmente representan una evolución motivada físicamente. Dado que nuestro interés está en sistemas no aislados, imaginemos tener un espacio ambiente que pueda interactuar con nuestro sistema original. Indicamos el estado del ambiente con $|\varphi\rangle$ (y podemos asumir, sin pérdida de generalidad, que sea un estado puro). El conjunto sistema-ambiente es un sistema aislado, y por lo tanto evoluciona con una unitaria U. Al final de la evolución, descartamos el ambiente y sólo miramos a la matriz de densidad reducida de nuestro sistema original, que llamaremos ρ'. La secuencia de operaciones que hemos descrito tiene la forma siguiente:

$$\rho \to \rho \otimes |\varphi\rangle\langle\varphi| \to U(\rho \otimes |\varphi\rangle\langle\varphi|)U^* \to \mathrm{tr}_E[U(\rho \otimes |\varphi\rangle\langle\varphi|)U^*] = \rho' \tag{4.1}$$

donde tr_E es la traza parcial sobre el ambiente. Nótese que cada paso es una aplicación completamente positiva y que preserva la traza : por lo tanto la evolución resultante $\rho \to \rho'$ es un canal cuántico.

Por lo tanto, si consideramos nuestro sistema acoplado con un espacio ambiente, y que los dos evolucionan juntos como un sistema aislado, llegamos a la conclusión de que la evolución del sistema original está dada por un canal cuántico. Esto es realmente lo único que puede pasar: todo canal cuántico se puede interpretar de esta manera, como la restricción a un subsistema de una evolución unitaria en un sistema más grande. Este es el contenido del teorema de dilatación de Stinespring.

Teorema de dilatación de Stinespring. *Sea $T : \mathcal{M}_d \to \mathcal{M}_d$ una aplicación completamente positiva y que preserva la traza (un canal cuántico). Entonces existen una unitaria $U \in \mathcal{M}_{d^2}$ y un vector normalizado $\varphi \in \mathbb{C}^d$ tales que*

$$T(\rho) = \mathrm{tr}_E[U\big(\rho \otimes |\varphi\rangle\langle\varphi|\big)U^*] \tag{4.2}$$

Por lo tanto, la evolución que representa un canal cuántico se justifica físicamente como la restricción de una evolución unitaria que actúa en un sistema más grande: eso son, en cierto sentido, las evoluciones efectivas inducidas en subsistemas por evoluciones unitarias.

4.1.2.2 Límite de acoplamiento débil

Hasta ahora hemos considerado *una sola* aplicación de un canal cuántico, interpretándola como un paso temporal individual de una evolución dinámica, y hemos visto que, por el teorema de Stinespring, es equivalente a acoplar el sistema con un ambiente y considerar una evolución unitaria conjunta. La ventaja es, claramente, que a menudo es más simple razonar ignorando la evolución interna del ambiente y considerar únicamente su efecto en el sistema que nos interesa.

Pero un sistema dinámico es más que una sola aplicación de un paso temporal: es una composición secuencial de tales pasos, o una descripción a tiempo continuo en la cual cada

instante temporal $t \geq 0$ produce una evolución físicamente realizable T_t. Matemáticamente tenemos un problema: al trazar el ambiente, hemos perdido toda correlación que la unitaria U podría haber creado entre el sistema y su ambiente, tanto en la forma de entrelazamiento cuántico como en la forma de correlaciones clásicas. Sólo hemos conservado una sombra de ellas en el estado mixto resultante, pero la pérdida es irreversible. El proceso descrito por la ecuación (4.1) no se puede componer de manera natural: el ambiente ha cambiado porque ha evolucionado junto con nuestro sistema.

Aunque técnicamente correcta, quizás nuestra descripción matemática no es del todo relevante para describir sistemas reales. Resulta que en algunos casos, el efecto del sistema en el ambiente es despreciable, y se puede aproximar asumiendo que el ambiente no evoluciona. Imaginemos por ejemplo que el ambiente sea un baño térmico de cierta temperatura: seguramente la interacción con el sistema cambiará el equilibrio y la temperatura del ambiente, pero si este es mucho más grande que el sistema no será una mala aproximación asumir que la temperatura es constante a lo largo de la evolución.

Matemáticamente, esto significa que si $U\big(\rho \otimes |\varphi\rangle\langle\varphi|\big)U^*$ no es demasiado distinto de $\rho' \otimes |\varphi\rangle\langle\varphi|$, podemos sustituir uno por el otro: en la literatura física esto se llama *límite de acoplamiento débil* o *aproximación de Born* [1, 13, 17, 70, 71]. En cada paso "infinitesimal", el ambiente se elimina y se reemplaza por uno nuevo e idéntico al original, que por lo tanto no contiene ninguna información sobre la evolución previa del sistema. Por esta razón, este tipo de evolución se llama también *Markoviana*.

Esto nos permite considerar el siguiente sistema dinámico: la evolución del sistema está descrita por un *semigrupo* [2] de canales cuánticos $\{T_t : \mathscr{M}_d \to \mathscr{M}_d\}_{t\geq 0}$, tales que T_0 es la aplicación identidad. La propiedad de semigrupo $T_t T_s = T_{s+t}$ implica que la evolución es homogénea y Markoviana. Como en la teoría clásica de semigrupos dinámicos, si T_t es fuertemente continuo en t (T_t es un C_0-semigrupo), entonces tiene un generador infinitesimal $\mathscr{L}$, que verifica las relaciones siguientes:

$$\mathscr{L}(x) = \frac{\mathrm{d}}{\mathrm{d}t}\, T_t(x)\bigg|_{t=0} = \lim_{t\downarrow 0} \frac{1}{t}(T_t - \mathbb{1})(x). \tag{4.3}$$

Nótese que para sistemas de dimensión finita, la continuidad fuerte implica la continuidad uniforme, y que por lo tanto podemos escribir

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = \mathscr{L} T_t, \quad T_t = \exp(t\mathscr{L}).$$

Una generalización de esta aproximación es considerar un ambiente que evoluciona en el tiempo, pero de manera independiente del sistema (por su propia dinámica interna, o quizás porque hemos aproximado el efecto del sistema sobre el ambiente de esta manera). Esto nos lleva a considerar co-ciclos en vez de semigrupos, es decir familias $\{T_{t,s}\}_{0\leq s\leq t}$ de canales cuánticos que satisfacen la propriedad $T_{t,s}T_{s,k} = T_{t,k}$ para todo $k \leq s \leq t$. Podemos definir generadores de co-ciclos (que serán dependientes del tiempo) de manera similar a como se hizo para semigrupos: dejaremos fuera de este trabajo estas evoluciones no homogéneas, pero las mencionamos por completitud.

[2] O de manera más exacta, de una representación del semigrupo $(\mathbb{R}_+, +)$.

4.1.2.3 Generadores Lindbladianos

Hemos visto que la evolución del estado ρ bajo un semigrupo de canales cuánticos T_t está dada por la solución de la ecuación diferencial $\dot{\rho}(t) = \mathscr{L}\rho(t)$, donde $\rho(t) = T_t(\rho)$. El superoperador $\mathscr{L}$ se llama también *Liouvilliano*, dado que esta ecuación es una generalización de la ecuación de Liouville-von Neumann. No puede ser cualquier operador, dado que hemos impuesto algunas restricciones sobre el semigrupo que genera (es un semigrupo de canales cuánticos). Lindblad [48], Kossakowski, Gorini, y Sudarshan [27] probaron que tales generadores tienen una forma particular, llamada forma de Lindblad-Kossakowski, y $\mathscr{L}$ se suele llamar Lindbladiano.

Teorema 15. *Sea $\mathscr{L} : \mathscr{M}_d \to \mathscr{M}_d$. Los siguientes hechos son equivalentes*

1. *$\mathscr{L}$ es generador de un semigrupo dinámico de canales cuánticos;*
2. *existen una aplicación completamente positiva $\varphi : \mathscr{M}_d \to \mathscr{M}_d$ y una matriz $\kappa \in \mathscr{M}_d$ tal que*
$$\mathscr{L}(\rho) = \varphi(\rho) - \kappa\rho - \rho\kappa^*; \quad \varphi^*(\mathbb{1}) = \kappa + \kappa^*. \tag{4.4}$$
3. *existen una matriz hermítica H y un conjunto de matrices $\{L_j \in \mathscr{M}_d\}_{j=0,\ldots,d^2-1}$ tales que*
$$\mathscr{L}(\rho) = -i\big[H,\rho\big] + \sum_{j=0}^{d^2-1} L_j\rho L_j^* - \frac{1}{2}\Big\{L_j^* L_j, \rho\Big\}; \tag{4.5}$$
donde $\{a,b\} = ab + ba$ es el anticonmutador.

A H se le llama (por razones obvias) el hamiltoniano, mientras que a las matrices L_j se les llama *operadores de salto* o *de Lindblad*.

Por el teorema de Russo-Dye [74], si $T : \mathscr{M}_d \to \mathscr{M}_d$ es una aplicación positiva y que preserva la traza, entonces es una contracción con respecto a la norma traza: de hecho, por dualidad con respecto al producto escalar de Hilbert-Schmidt,

$$\|T\|_{1\to 1} = \left\|T^*\right\|_{\infty\to\infty} = \left\|T^*(\mathbb{1})\right\|_\infty = \|\mathbb{1}\|_\infty = 1,$$

dado que el dual de una aplicación que preserva la traza es una aplicación que preserva la identidad. Por lo tanto, los autovalores de un canal cuántico están en el disco unidad complejo. Usando el cálculo funcional, se puede ver que esto implica que los autovalores de $\mathscr{L}$ están contenidos en el semiplano $\{z \in \mathbb{C} \mid \operatorname{Re} z \leq 0\}$.

Los autovalores de $\mathscr{L}$ que se quedan en el eje imaginario corresponden a autovalores de T_t en la frontera del disco unidad, y por lo tanto forman el llamado *espectro periférico*. Se puede probar que los bloques de Jordan asociados tienen dimensión 1, y corresponden a estados periódicos de la evolución, mientras que los estados estacionarios corresponden al autovalor 1.

Para cualquier otro autovalor λ de $\mathscr{L}$, al tener parte real estrictamente negativa, la acción de T_t en el autoespacio generalizado correspondiente es la de una contracción exponencial en el tiempo: el subespacio es aniquilado por un factor $\exp(-t\operatorname{Re}\lambda)$. Por lo tanto, es el

autovalor con la parte real no-nula más grande el que determina la razón de convergencia más lenta de T_t hacia una aplicación T_∞ que proyecta en el espacio generado por los estados periódicos (y que sobre ese espacio actúa como una unitaria). Esto justifica la siguiente definición:

Definición 11 (Gap espectral)**.** Definimos el *gap espectral* de un Lindbladiano $\mathscr{L}$ como

$$\operatorname{gap}\mathscr{L} = \min_{\lambda\in\sigma(\mathscr{L})\setminus\{0\}} |\operatorname{Re}\lambda|. \tag{4.6}$$

Asumimos por un momento que no haya estados periódicos, o dicho de otra manera que el espectro periférico esté trivialmente compuesto sólo del autovalor 1. En este caso T_∞ es realmente una proyección. Desde la descomposición de Jordan podemos ver que el gap espectral controla la razón de convergencia en el tiempo al espacio de puntos fijos, y que existe una constante $c > 0$ tal que

$$\left\| T_t(\rho) - T_\infty(\rho) \right\|_1 \leq c\exp\left(-t\operatorname{gap}\mathscr{L}\right), \tag{4.7}$$

para todo estado inicial ρ.

Volveremos a la ecuación (4.7) más adelante, cuando hablemos de familias de sistemas dinámicos definidos en una sucesión creciente de retículos.

4.1.2.4 Generadores locales

Hasta ahora, sólo hemos considerado sistemas finitos, que pueden ser considerados como un único cuerpo físico con su dinámica. La mayoría de aplicaciones requieren en su lugar una descripción de un modelo *a muchos cuerpos*: un sistema compuesto de muchas piezas individuales, que interactúan entre sí de una manera definida y con cierta regularidad. Si tenemos en cuenta una única instancia de un modelo de muchos cuerpos, matemáticamente hablando es lo mismo que considerar todo el sistema como un solo cuerpo grande, con unos cuantos grados de libertad interna evolucionando de acuerdo a las interacciones mencionadas.

Este punto de vista cambia si consideramos una sucesión de modelos de muchos cuerpos definidos en una estructura de grafo o de retículo. Recordamos la notación mencionada anteriormente. Γ será un grafo infinito con la métrica de grafo, por ejemplo $\mathbb{Z}^D$ para algún entero D. Asociamos a cada vértice x en el grafo un espacio de Hilbert complejo de dimensión finita $\mathscr{H}_x$, y asumimos por simplicidad que sean todos isomorfos (es decir, todos tienen la misma dimensión d). Para cada subconjunto finito $\Lambda \subset \Gamma$ indicamos con $\mathscr{H}_\Lambda = \otimes_{x\in\Lambda}\mathscr{H}_x$, y $\mathscr{A}_\Lambda = \mathscr{B}(\mathscr{H}_\Lambda)$.

En este caso hay una noción bien definida de localidad: para cada par de subgrafos finitos $\Lambda_1 \subset \Lambda_2 \subset \Gamma$, existe una inclusión natural de $\mathscr{A}_{\Lambda_1}$ en $\mathscr{A}_{\Lambda_2}$, dada por la identificación de $X \in \mathscr{A}_{\Lambda_1}$ con $X \otimes \mathbb{1}_{\Lambda_2\setminus\Lambda_1} \in \mathscr{A}_{\Lambda_2}$. Esto nos ha permitido definir la noción de *soporte*: dado un operador $X \in \mathscr{A}_\Lambda$ definimos el soporte de X, y lo indicamos como $\operatorname{supp} X$, como el mínimo $\Lambda' \subset \Lambda$ tal que existe un $X' \in \mathscr{A}_{\Lambda'}$ de manera que $X = X' \otimes \mathbb{1}$. En cierto sentido, el soporte de X es independiente de Λ, dado que $\operatorname{supp} X = \operatorname{supp} X \otimes \mathbb{1}$: considerar X actuando en un conjunto más grande no incrementa su soporte.

Esta es la primera aparición de una idea simple pero potente que está en la base de este trabajo: existen propiedades de los objetos que estudiamos que no dependen del tamaño del sistema, con ser este de tamaño suficiente para contenerlos. Si consideramos una sucesión creciente y absorbente de $\Lambda_n \nearrow \Gamma$ finitos, entonces podemos estudiar propiedades que son uniformes en n.

Las interacciones físicas suelen ser más débiles cuando la distancia entre los cuerpos que interactúan se hacen más largas. Por lo tanto si podemos descomponer el generador de la evolución $\mathscr{L}$ como una suma de términos locales $\sum_{Z\subset\Lambda}\mathscr{L}_Z$, cada uno de los cuales es nuevamente de la forma de Lindblad-Kossakowski pero solo actúa en un subsistema Z, es razonable pedir que sus normas sean mas pequeñas al crecer el diámetro de su soporte Z. En este caso diremos que $\mathscr{L}$ es un **Lindbladiano local**.

Si no especificamos a qué velocidad $\|\mathscr{L}_Z\|_\diamond$ decrece con respecto a diam Z, cualquier Lindbladiano satisface esta condición de manera trivial. Vamos a detallar más el decaimiento que necesitaremos en el apartado 4.2.1.3.

4.1.2.5 Tiempo de equilibración y gap espectral

La ecuación (4.7) describe las propiedades del sistema dinámico descrito por T_t para tiempos largos: si t es más grande que $\log(\epsilon/c)/\operatorname{gap}\mathscr{L}$ para algún ϵ positivo, entonces el conjunto de estados iniciales posibles ha sido comprimido en un ϵ-entorno del espacio de puntos fijos. El tiempo mínimo para que esto pase (que podría ser más pequeño que el tiempo dado por la ecuación (4.7)) se llamará **tiempo de equilibración** del sistema dinámico. Daremos la definición formal sólo para sistemas sin puntos periódicos.

Definición 12 (Tiempo de equilibración)**.** Llamamos *tiempo de equilibración* de un sistema dinámico $T_t : \mathscr{M}_d \to \mathscr{M}_d$ sin puntos periódicos a la función

$$\tau(\epsilon) = \min\{t > 0 : \sup_\rho \left\| T_t(\rho) - T_\infty(\rho)\right\|_1 \leq \epsilon\},$$

donde el supremo se toma sobre todos los estados ρ.

Por lo tanto, podemos reformular lo que sabemos sobre el gap espectral como sigue:

$$\tau(\epsilon) \leq \frac{\log c - \log\epsilon}{\operatorname{gap}\mathscr{L}} \tag{4.8}$$

Para todo sistema de dimensión finita, este análisis suele ser suficiente: más cuidado será necesario al considerar familias de sistemas dinámicos definidas en secuencias crecientes de retículos Λ_n. En este caso queremos controlar el crecimiento de $\tau(\epsilon)$ con respecto a n. Por emepzar, la cantidad $\lambda_n = \operatorname{gap}\mathscr{L}_n$ puede decrecer al crecer n, haciendo diverger la cota al tiempo de equilibración. Si por el contrario la cantidad $\lambda = \inf\lambda_n$ está acotada lejos de cero, diremos informalmente que el sistema tiene gap (en el sentido que tiene gap en el límite).

Sin embargo, puede haber una razón más profunda para que la cota de la ecuación (4.8) diverja en general en n, aunque se tenga un λ estrictamente positivo: la constante c dependerá en general de n también. De hecho si obtenemos esa cota a través de la descomposición de Jordan (aunque esta no sea necesariamente la manera óptima para obtenerla), puede

crecer mas rápido que una exponencial en n. Un análisis más cuidadoso puede mejorar esta dependencia: en [85] se demuestra que si $\mathscr{L}$ satisface una condición llamada reversibilidad con respecto a un estado de rango máximo σ, que presentaremos con más detalle en el apartado 4.1.3.1, entonces podemos elegir c igual a $\left\|\sigma^{-1}\right\|_\infty^{1/2}$, que a su vez es igual a $\sigma_{\text{mín}}^{-1/2}$, el mínimo autovalor de σ. Esto nos permite probar el siguiente resultado, del cual daremos después una demostración alternativa:

Teorema 16. *Si λ es el gap espectral de $\mathscr{L}$, que es reversible con respecto a un estado de rango máximo σ, entonces*

$$\tau(\epsilon) \leq \frac{\log\left(\sigma_{\text{mín}}^{-1/2}\right) - \log\epsilon}{\lambda} \tag{4.9}$$

Nótese que $\sigma_{\text{mín}}^{-1/2}$ tiene que escalar por lo menos de manera exponencial con el tamaño del sistema, o peor - por lo tanto de la ecuación (4.9) podemos obtener como mucho una cota polinomial al tiempo de equilibración. Si conocemos el espectro de $\mathscr{L}$ y nada más, no es posible mejorar mucho la cota del Teorema 16: como se muestra en [78], si no tenemos más información que el espectro de $\mathscr{L}$, la dependecia de c en el tamaño del sistema no puede ser mejorada en una cota más lenta que una exponencial.

Para algunas aplicaciones, tener un tiempo de equilibración polinomial es suficiente. En este trabajo, pediremos una condición más fuerte, que gracias a [78] sabemos que no puede ser garantizada solamente a partir de la información sobre el espectro de $\mathscr{L}$: que $\tau(\epsilon)$ tenga un crecimiento *logarítmico* en n (en algunos casos podemos relajar esta hipótesis a un crecimiento sub-lineal). Es la contribución principal de esta tesis probar que a partir de esta condición podemos probar algunas propiedades muy interesantes de la evolución y de su punto fijo.

Presentaremos tales resultados en el apartado 4.2: antes vamos a presentar una conexión importante con la desigualdad logarítmica de Sobolev.

4.1.3 Desigualdad logarítmica de Sobolev

Las herramientas de hipercontractividad y desigualdad logarítmica de Sobolev (en adelante, log-Sobolev) fueron introducidas como parte del programa de Segal de dar rigor matemático a la Teoría Cuántica de Campos [75]. La desigualdad de log-Sobolev fue introducida por primera vez por Feissner (en esa época, un estudiante de Leonard Gross, que a su vez había sido estudiante de Segal) en su Tesis Doctoral [24, 25] con el fin de generalizar la desigualdad clásica de Sobolev a medidas Gaussianas en dimensión infinita. Luego Gross [29] la usó para estudiar la ergodicidad de procesos de Markov en dimensión infinita, y finalmente fue reconocida como una herramienta útil para estudiar también procesos en dimensión finita [21]. La aplicación a sistemas de espín clásicos fue introducida por primera vez por Holley, Stroock, y Zegarlinski [35, 76, 77, 87, 88] y luego se convirtió en una herramienta estándar en mecánica estadística. Está fuertemente relacionada con la contractividad de semigrupos, y ha jugado un papel importante en muchas áreas de las matemáticas.

Para una reseña moderna de la teoría clásica (conmutativa) de la desigualdad de log-Sobolev y su conexión con semigrupos de Markov y concentración de la medida, véase

[30]. Su generalización cuántica ha sido desarrollada en una serie de artículos [49-51, 66], y la conexión entre equilibración rápida y desigualdad de log-Sobolev en el contexto de partículas cuánticas ha sido introducida en [39].

La hipercontractividad es ligeramente anterior a la desigualdad de log-Sobolev: el primer ejemplo de su uso se puede encontrar en un trabajo de Nelson (también estudiante de Segal)[62, 63], aunque todavía no había sido llamada así. Para una reseña del argumento véase [18, 28]. Está siendo reconocida en la comunidad de información cuántica como una herramienta muy potente [16, 40, 58, 81].

Presentaremos ahora una versión simplificada de la teoría de log-Sobolev, y su conexión con hipercontractividad y equilibración rapida.

En esta tesis consideramos semigrupos de aplicaciones que preservan la traza T_t, y que por lo tanto describen la evolución de estados, pero una descripción equivalente se obtiene de manera dual (con respecto al producto escalar de Hilbert-Schmidt) al considerar la aplicación T_t^*, que describe la evolución de los observables, y el estado límite ρ_∞ es invariante en el sentido que $\mathrm{tr}\big(\rho_\infty T_t^*(A)\big) = \mathrm{tr}\big(\rho_\infty A\big)$ para todo operador A. En este caso el semigrupo preserva la identidad en vez de la traza. Este enfoque es el que usualmente es elegido en la literatura sobre desigualdades de log-Sobolev. Mantendremos nuestra notación, e indicaremos la evolución de observables con T_t^*, pero el lector debería ser consciente de la diferencia.

4.1.3.1 Gap espectral y reversibilidad

Antes de presentar la definición de desigualdad de log-Sobolev, vamos a reformular la ecuación (4.6) de una manera distinta pero equivalente, en el caso de considerar un estado de rango máximo $\sigma > 0$. Dado un estado de este tipo, podemos definir un producto escalar pesado sobre $\mathscr{B}(\mathscr{H})$ como

$$\langle A|B\rangle_\sigma = \mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} B\big] = \big\langle \sigma^{1/4} A \sigma^{1/4} \big| \sigma^{1/4} B \sigma^{1/4} \big\rangle_{HS},$$

y la norma inducida correspondiente $\|\cdot\|_\sigma = \langle\cdot|\cdot\rangle_\sigma^{1/2}$. Es fácil ver que $\sigma_{\mathrm{mín}}^{1/2}\,\|A\| \le \|A\|_\sigma \le \|A\|$, donde $\sigma_{\mathrm{mín}}$ es el autovalor mínimo de σ. Además, podemos definir una generalización de la varianza clásica, como

$$\mathrm{Var}_\sigma(A) = \|A - \langle A|\mathbb{1}\rangle_\sigma\|_\sigma^2 = \mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} A\big] - \mathrm{tr}[\sigma A]^2.$$

En efecto, $\mathrm{Var}_\sigma(A)$ es positiva e invariante bajo traslaciones por múltiplos de la identidad. De manera similar, dado un Lindbladiano $\mathscr{L}$, podemos definir una generalización no-conmutativa de la forma de Dirichlet:

$$\mathscr{E}(A,B) = \big\langle A \big| -\mathscr{L}^*(B)\big\rangle_\sigma = -\,\mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} \mathscr{L}^*(B)\big];$$

donde $\mathscr{L}^*$ es el dual de $\mathscr{L}$ bajo el producto escalar de Hilbert-Schmidt, es decir $\mathscr{E}(A,B) = -\,\mathrm{tr}\big[\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) B\big]$. Escribiremos $\mathscr{E}(A) = \mathscr{E}(A,A)$. Diremos que $\mathscr{L}$ es *reversible con respecto a* σ si $\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) = \sigma^{1/2}\mathscr{L}^*(A)\sigma^{1/2}$ para todo operador A, y por lo tanto

$$\mathrm{tr}\big[\sigma^{1/2} A \sigma^{1/2} \mathscr{L}^*(B)\big] = \mathrm{tr}\big[\mathscr{L}(\sigma^{1/2} A \sigma^{1/2}) B\big] = \mathrm{tr}\big[\sigma^{1/2}\mathscr{L}^*(A)\sigma^{1/2} B\big].$$

En este caso, $\mathscr{E}$ es una forma bilinear simétrica, $\mathscr{L}^*$ es autoadjunto con respecto a $\langle\cdot|\cdot\rangle_\sigma$, y por lo tanto $\hat{\mathscr{L}}(\cdot) = \sigma^{1/4}\mathscr{L}^*(\sigma^{-1/4}\cdot\sigma^{-1/4})\sigma^{1/4}$ es autoadjunto con respecto al producto escalar de Hilbert-Schmidt. Dado que $\mathscr{L}^*$ y $\hat{\mathscr{L}}$ están relacionados por una semejanza, $\mathscr{L}^*$ tiene espectro real, y la contractividad del semigrupo generado implica que es negativo. Nótese que también implica que σ es un estado invariante por $\mathscr{L}$, dado que para todo A se cumple que

$$\operatorname{tr}[\mathscr{L}(\sigma)A] = \operatorname{tr}\big[\sigma^{1/2}\mathbb{1}\sigma^{1/2}\mathscr{L}^*(A)\big] = \operatorname{tr}\big[\sigma^{1/2}\mathscr{L}^*(\mathbb{1})\sigma^{1/2}A\big] = 0,$$

y por lo tanto $\mathscr{L}(\sigma) = 0$.

Si el espectro periférico de T_t es trivial, entonces el núcleo de $\mathscr{L}$ tiene dimensión uno, y por el principio min-max de Courant-Fischer-Weyl el segundo autovalor más pequeño de $\mathscr{L}$, que antes hemos llamado el gap espectral, está dado por

$$\operatorname{gap}\mathscr{L} = \min_{A:\langle A|\mathbb{1}\rangle_\sigma=0} \frac{-\langle\mathscr{L}^*(A)|A\rangle_\sigma}{\langle A|A\rangle_\sigma} = \min_{A:\operatorname{Var}_\sigma(A)\neq 0} \frac{\mathscr{E}(A)}{\operatorname{Var}_\sigma(A)}.$$

Hemos reexpresado entonces la ecuación (4.6) como un problema variacional: $\operatorname{gap}\mathscr{L}$ es el valor máximo que puede tomar una constante c tal que el funcional cuadrático $c\operatorname{Var}_\sigma(A)$ esté acotado superiormente por el funcional cuadrático $\mathscr{E}(A)$.

$$c\operatorname{Var}_\sigma(A) \leq \mathscr{E}(A) \tag{4.10}$$

Considérese ahora la evolución $A(t)$ de un observable A bajo $\mathscr{L}$, es decir $A(t) = T_t^*(A)$. Dado que σ es invariante por T_t, se tiene que $\langle A(t)|\mathbb{1}\rangle_\sigma = \langle A|\mathbb{1}\rangle_\sigma$, y por lo tanto $\lim_{t\to\infty} A(t) = \langle A|\mathbb{1}\rangle_\sigma\,\mathbb{1}$. Esto implica que $\operatorname{Var}_\sigma(A(t))$ es igual a $\|A(t) - A(\infty)\|_\sigma^2$. Consideremos la función $t \to \operatorname{Var}_\sigma(A(t))$: su derivada está dada por $-2\mathscr{E}[A(t)]$. Por lo tanto la ecuación (4.10) está realmente acotando la derivada de $\operatorname{Var}_\sigma(\cdot)$ por la función misma. Esto lleva a la cota siguiente:

$$\operatorname{Var}_\sigma(A(t)) \leq \operatorname{Var}_\sigma(A)e^{-2ct}.$$

Por lo tanto, el gap espectral controla la convergencia cuando esta es medida por $\operatorname{Var}_\sigma(\cdot)$. A su vez, esto implica que

$$\|A(t) - A(\infty)\| \leq \sigma_{\text{mín}}^{-1/2}\|A(t) - A(\infty)\|_\sigma \leq \sigma_{\text{mín}}^{-1/2}\|A - A(\infty)\|e^{-ct}.$$

Por dualidad esto implica la siguiente cota de la forma de la ecuación (4.7):

$$\sup_\rho \big\|T_t(\rho) - \sigma\big\|_1 \leq 2\sigma_{\text{mín}}^{-1/2}e^{-t\operatorname{gap}\mathscr{L}}.$$

o de manera equivalente

Teorema 17. *Si λ es el gap espectral de $\mathscr{L}$, entonces*

$$\tau(\epsilon) \leq \frac{\log\big(2\sigma_{\text{mín}}^{-1/2}\big) - \log\epsilon}{\lambda} \tag{4.11}$$

Nótese que $\sigma_{\text{mín}}^{-1}$ escala **por lo menos** exponencialmente con el tamaño del sistema (dado que tiene que ser por lo menos más pequeño que $1/\dim \mathscr{H}_\Lambda$), pero en principio podría ser peor.

Podríamos haber obtenido la misma cota, pero sin el factor multiplicativo de 2, usando el hecho que [39, 80]

$$\left\|\rho-\sigma\right\|_1 \leq \mathrm{Var}_\sigma^{1/2}(\sigma^{-1/2}\rho\sigma^{-1/2}),$$

y que $\sup_\rho \mathrm{Var}_\sigma(\sigma^{-1/2}\rho\sigma^{-1/2})$ es igual a $\left\|\sigma^{-1}\right\|_\infty = 1/\sigma_{\text{mín}}$ (donde el supremo está tomado sobre estados).

Hemos visto que la condición de reversibilidad nos permite expresar exactamente la relación entre el gap espectral y el tiempo de equilibración, obteniendo un prefactor bastante bueno a la cota (mucho mejor de lo que podríamos haber obtenido a través de la descomposición de Jordan). La desventaja es que hemos tenido que suponer que el punto fijo sea único y de rango máximo (en este caso decimos que $\mathscr{L}$ es *primitivo*), y que tengamos algún control sobre $\sigma_{\text{mín}}$.

En seguida vamos a mostrar cómo en este contexto es natural definir otras condiciones sobre $\mathscr{L}$ que nos permitirán tener un control mejor sobre el tiempo de equilibración que el que obtuvimos de la cota sobre el gap espectral. A su vez estas nuevas cotas serán suficientes para probar la condición de equilibración rápida.

4.1.3.2 Entropía y desigualdad de log-Sobolev

La idea de la desigualdad de log-Sobolev y de otras desigualdades entrópicas es de generalizar lo que hemos hecho en la sección anterior con $\mathrm{Var}_\sigma(\cdot)$: encontrar un funcional positivo $D(\cdot)$ que acote la convergencia de T_t, luego acotar la derivada $\frac{\mathrm{d}}{\mathrm{d}t}D(T_t(\rho)-\sigma)$ en términos de la función misma, comparándola con otro funcional definido en términos de $\mathscr{L}$.

Consideremos el funcional siguiente, que llamaremos **entropía relativa**

$$D(X\|Y) = \frac{1}{\operatorname{tr} X}\operatorname{tr}\big[X(\log X - \log Y)\big].$$

$D\big(\rho\|\sigma\big)$ es positivo si ρ y σ son estados normalizados, y es finito si el soporte de ρ está contenido en el soporte de σ. Decrece de manera monótona bajo la acción de canales cuánticos [65]. La desigualdad de Pisker [64] implica que $\left\|\rho-\sigma\right\|_1^2 \leq 2D\big(\rho\|\sigma\big)$. Al derivar obtenemos que

$$\frac{\mathrm{d}}{\mathrm{d}t}D\big(\rho(t)\|\sigma\big) = \operatorname{tr}\big[\mathscr{L}(\rho(t))(\log\rho(t)-\log\sigma)\big].$$

Podemos por lo tanto indicar con $\mathscr{K}(\rho) = -\frac{1}{\operatorname{tr}\rho}\operatorname{tr}\big[\mathscr{L}(\rho)(\log\rho-\log\sigma)\big]$. Compárese esta definición con la de $\mathscr{E}$. Podemos entones definir la siguiente desigualdad de tipo log-Sobolev:

$$c[D\big(\rho\|\sigma\big) - \log\operatorname{tr}\rho] \leq \mathscr{K}(\rho) \tag{4.12}$$

donde la constante óptima $c>0$ será llamada constante de log-Sobolev de $\mathscr{L}$, y la indicaremos con α. Como en el caso de la desigualdad del gap espectral, podemos concluir que

$$\sup_\rho \left\|\rho(t)-\sigma\right\|_1 \leq \sup_\rho \sqrt{2D\big(\rho\|\sigma\big)}e^{-\alpha t}.$$

Observamos que $D(\rho\|\sigma)$ está acotado por $\left\|\log\sigma^{-1}\right\|_\infty = -\log(\sigma_{\text{mín}})$. Por lo tanto, se obtiene una cota en la convergencia del semigrupo exponencialmente mejor que la que obtuvimos con el gap espectral (ver la ecuación (4.11)):

Teorema 18. *Sea α la constante de log-Sobolev de $\mathscr{L}$. Entonces*

$$\tau(\epsilon) \leq \frac{\log\left(\log\left(\sigma_{\text{mín}}^{-1/2}\right)\right) - 2\log\epsilon}{2\alpha}. \tag{4.13}$$

Si $\sigma_{\text{mín}}^{-1}$ es exponencial en el tamaño del sistema y la constante de log-Sobolev es uniforme en él, entonces el sistema tiene equilibración rápida. Por lo tanto, la desigualdad de log-Sobolev es una manera de probar esta hipótesis para sistemas reversibles.

Este es el enfoque elegido en [59, 60]. En [66] y [39] una cota equivalente a la ecuación (4.12) se denota como desigualdad 1-log-Sobolev, y se obtiene al componer la ecuación (4.12) con la aplicación $\rho \to \sigma^{-1/2}\rho\sigma^{-1/2}$. Si indicamos con $A = \sigma^{-1/2}\rho\sigma^{-1/2}$ y con $A(t)$ la evolución de A bajo $\mathscr{L}^*$, es decir $A(t) = T_t^*(A)$, entonces la condición de reversibilidad implica que

$$A(t) = T_t^*(\sigma^{-1/2}\rho\sigma^{-1/2}) = \sigma^{-1/2}T_t(\rho)\sigma^{-1/2} = \sigma^{-1/2}\rho(t)\sigma^{-1/2}.$$

Por lo tanto la ecuación (4.12) puede reexpresarse como sigue[3]

$$c\,\text{Ent}_1(A) \leq \mathscr{E}_1(A),$$

donde $\text{Ent}_1(A) = D(\rho\|\sigma) - \log\text{tr}\,\rho$ y $\mathscr{E}_1(A) = \mathscr{K}(\rho)$. Esta versión de la cota es claramente equivalente a la ecuación (4.12) si $\mathscr{L}$ es reversible. Los autores de [39] llaman a la constante óptima α_1.

Desafortunadamente esto no es lo que se suele llamar desigualdad de log-Sobolev en la literatura clásica (es decir, cuando todo lo anterior se define para generadores de cadenas de Markov sobre un espacio de probabilidad, que es el equivalente conmutativo de Lindbladianos sobre estados cuánticos). En lugar de ello, la desigualdad clásica es más bien parecida a la siguiente generalización:

$$c\,\text{Ent}_2(A) \leq \mathscr{E}(A);$$

donde $\mathscr{E}$ es la forma de Dirichlet definida anteriormente, $\text{Ent}_2(A) = \text{Ent}_1(I_{1,2}(A))$,

$$I_{1,2}(A) = \sigma^{-1/2}\left(\sigma^{1/4}A\sigma^{1/4}\right)^2\sigma^{-1/2} = \sigma^{-1/4}A\sigma^{1/2}A\sigma^{-1/4}$$

y por lo tanto $\text{Ent}_2(A) = D\left(\left(\sigma^{-1/4}\rho\sigma^{-1/4}\right)^2\|\sigma\right)$.

A esta cota se le llama desigualdad 2-log-Sobolev en [39, 59, 66] y a su constante óptima α_2. Lamentablemente no sabemos si es equivalente a la ecuación (4.12): bajo la hipótesis adicional de que $\mathscr{E}_1(I_{1,2}(A)) \geq \mathscr{E}(A)$ (llamada *L_p-regularidad* en [66]), al menos se puede probar que $\alpha_2 \leq \alpha_1$, recuperando el resultado clásico. El hecho de que existan Lindbladianos que sean reversibles pero que no sean L_p-regulares es todavía un problema abierto.

[3] Hemos eliminado un factor 1/2 de la definición original.

4.1.3.3 Hipercontractividad

Como hemos visto, a consecuencia del teorema de Russo-Dye, una aplicación positiva y que preserva la traza T es *contractiva* con respecto a la norma de la traza, dado que $\|T\|_{1\to 1} = 1$ - o de manera equivalente, una aplicación positiva y que preserva la unidad T^* verifica que $\|T^*\|_{\infty\to\infty}$=1, es decir es contractiva con respecto a la norma ∞. Esto se aplica en particular a los canales cuánticos. Vamos ahora a introducir una versión no conmutativa de las normas L_p [31, 49-51]: dado un estado de rango máximo σ, para cada $p \in [1,\infty)$ definimos

$$\|X\|_{p,\sigma}^p = \mathrm{tr}\left|\sigma^{1/2p} X \sigma^{1/2p}\right|^p = \left\|\sigma^{1/2p} X \sigma^{1/2p}\right\|_p^p.$$

Se puede comprobar que $\|\cdot\|_{p,\sigma}$ es de hecho una norma, y que se recuperan la propiedades usuales de los espacios L_p, como desigualdad de Hölder, dualidad, y teoremas de interpolación. En particular, estas normas son crecientes en p, y por lo tanto para todo $1 \le p \le q \le \infty$ se cumple que $\|\cdot\|_{1,\sigma} \le \|\cdot\|_{p,\sigma} \le \|\cdot\|_{q,\sigma} \le \|\cdot\|_\infty$. Además, $\lim_{p\to\infty} \|X\|_{p,\sigma} = \|X\|_\infty$ (la norma ∞ de Schatten usual). Nótese que la norma definida en la sección anterior, que denotamos por $\|\cdot\|_\sigma$, corresponde al caso $p = 2$. El espacio $\mathscr{B}(\mathscr{H})$ con la norma $\|\cdot\|_{p,\sigma}$ se denotará con $L_p(\sigma)$, y la norma de operador de una aplicación $T : L_p(\sigma) \to L_q(\sigma)$ se indicará con $\|T\|_{(p,\sigma)\to(q,\sigma)}$.

Consideremos entonces un canal cuántico T que tenga a σ como su punto fijo, y su dual T^* con respecto al producto escalar de Hilbert-Schmidt. Asumimos que T sea reversible con respecto a σ. Entonces sabemos que T^* preserva la unidad, y por lo tanto $\|T^*\|_{\infty\to\infty} = 1$. Además, tenemos que para cada operador A

$$\left\|T^*(A)\right\|_{1,\sigma} = \left\|\sigma^{1/2} T^*(A) \sigma^{1/2}\right\|_1 = \left\|T(\sigma^{1/2} A \sigma^{1/2})\right\|_1 \le \left\|\sigma^{1/2} A \sigma^{1/2}\right\|_1 = \|A\|_{1,\sigma},$$

donde hemos usado reversibilidad y el hecho de que $\|T\|_{1\to 1} = 1$. Podemos concluir entonces que $\|T^*\|_{(1,\sigma)\to(1,\sigma)} = 1$, y por interpolación que $\|T^*\|_{(p,\sigma)\to(p,\sigma)} = 1$ para todo $p \in [1,\infty]$.

Esto nos lleva a definir una nueva propiedad de una aplicación lineal $T : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$: diremos que T es **hipercontractiva** si existen $p < q$ tales que $\|T\|_{(p,\sigma)\to(q,\sigma)} \le 1$. En particular esto implica que T sea contractiva con respecto a la norma $\|\cdot\|_{(p,\sigma)}$.

Si consideramos un semigrupo dinámico de canales cuánticos T_t, entonces podemos considerar $\left\|T_t^*\right\|_{(p,\sigma)\to(q,\sigma)}$ para algun $p < q$ como una medida de convergencia del semigrupo: de hecho para $t = 0$ tenemos que $T_0 = \mathbb{1}$ y por lo tanto $\|\mathbb{1}\|_{(p,\sigma)\to(q,\sigma)} = 1$ si y sólo si $p \ge q$. Por otra parte, si σ es el único punto fijo de T_t, entonces $T_\infty^*(X) = \mathrm{tr}(\sigma X)\mathbb{1}$ y por lo tanto $\left\|T_\infty^*(A)\right\|_\infty = \mathrm{tr}(\sigma X) \le \|X\|_{(1,\sigma)}$, y $\left\|T_\infty^*\right\|_{(1,\sigma)\to\infty} = 1$.

Sea $1 < p < \infty$, y q su conjugado de Hölder, es decir $\frac{1}{p} + \frac{1}{q} = 1$. Dado que T_t^* es autoadjunto en $L_2(\sigma)$, se cumple que

$$\left\|T_t^*\right\|_{(p,\sigma)\to(2,\sigma)} = \left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)},$$

y por lo tanto si $1 < p \le 2$ entonces

$$\left\|T_{2t}^*\right\|_{(p,\sigma)\to(q,\sigma)} \le \left\|T_t^*\right\|_{(p,\sigma)\to(2,\sigma)} \left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)} = \left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)}^2.$$

Dada la observación anterior, nos centramos (como es común en la literatura) en el comportamiento de $\left\|T_t^*\right\|_{(2,\sigma)\to(q,\sigma)}$. La relación entre desigualdad de log-Sobolev e hipercontractividad está contenida en el teorema siguiente:

Teorema 19 ([66])**.** *Sea $\mathscr{L}$ un Lindbladiano reversible y que sea L_p-regular. Entonces las condiciones siguientes son equivalentes*

1. *Para $q(t) = 1 + e^{2\alpha t}$,*
$$\left\| T_t^* \right\|_{(2,\sigma)\to(q(t),\sigma)} \leq 1.$$
2. *T_t^* satisface una desigualdad 2-log-Sobolev con constante óptima α.*

Obsérvese que el punto 1. del teorema anterior implica que, si $q(t) = 1 + e^{\alpha t}$, entonces $\left\| T_t^* \right\|_{(p(t),\sigma)\to(q(t),\sigma)} \leq 1$. Pasando al límite para $t \to \infty$ recuperamos que $\left\| T_\infty^* \right\|_{(1,\sigma)\to(\infty,\sigma)} \leq 1$.

4.1.4 Ley de área

Otro problema interesante en el estudio de semigrupos disipativos es la descripción del punto fijo, o estado invariante, de la evolución. Para algunos modelos de ruido el punto fijo es el estado máximamente mixto, proporcional a la identidad. Este estado representa la situación en la cual el ruido ha destruido toda información sobre el sistema físico, y cualquier medida producirá resultados uniformemente distribuidos. En otros casos el modelo de ruido es distinto, y el estado invariante será un estado térmico correspondiente a algún hamiltoniano, proporcional a $e^{-\beta H}$ para algún operador hermítico H y un β positivo que representa el inverso de la temperatura. Este es el caso por ejemplo de las aplicaciones de Davies [19]. En otros casos todavía la evolución es artificial y construida para tener un estado particular como punto fijo: se considera con un estado que queremos preparar, y de ahí derivamos un generador Lindbladiano que produce ese estado como punto fijo. Este es el enfoque que tienen la dinámica de Glauber clásica y el muestreo de Metropolis [53] y la Preparación Disipativa de Estados [43, 83].

Uno se esperaría que, si el estado satisficiera alguna propiedad "buena", la evolución resultante también tendría algunas propiedades buenas, por ejemplo de convergencia rápida. Esto ha sido probado de manera rigurosa en el caso de espines clásicos y dinámica de Glauber [53, 54], donde la propiedad "buena" del estado ω es de este tipo: dados dos observables A y B, soportados en regiones que distan d, el valor de $\omega(A \otimes B)$ se acerca al de $\omega(A)\omega(B)$ al crecer d. Mas precisamente, se requiere que la diferencia entre los dos decaiga a cero exponencialmente rápido en d. Esta propiedad se suele llamar *decaimiento exponencial de correlaciones*, dado que la cantidad $\omega(A \otimes B) - \omega(A)\omega(B)$ mide cómo de correladas son las dos regiones. Bajo esta hipótesis, para sistemas de espines clásicos se puede probar que la dinámica de Glauber correspondiente tiene equilibración rápida (a través de una desigualdad de log-Sobolev).

En esta tesis hemos afrontado el problema inverso: dado un Lindbladiano con "buenas" propiedades, ¿qué propiedades del punto fijo podemos derivar? Para empezar, presentaremos de manera rigurosa la noción de correlaciones en sistemas de muchos cuerpos.

4.1.4.1 Medidas de correlaciones

Considérese un estado bipartito $\rho_{AB} \in \mathscr{B}(\mathscr{H}_{AB})$. Si ρ_{AB} es de la forma $x \otimes y$ para algún estado x en $\mathscr{B}(\mathscr{H}_A)$ y un estado y en $\mathscr{B}(\mathscr{H}_B)$, diremos que es un estado producto. En este caso

cada medida sobre el subsistema A será independiente de las medidas en el subsistema B, y viceversa: por lo tanto las estadísticas obtenidas serán independientes y no habrá correlaciones entre los dos subsistemas. Si ρ_{AB} no es un producto, hay varias maneras distintas de cuantificar "cuánto de lejos" está de ser un producto.

La notación siguiente está tomada de [37]. Denotaremos con ρ_A (resp. ρ_B) el estado $\mathrm{tr}_B \rho_{AB}$ (resp. $\mathrm{tr}_A \rho_{AB}$).

Definición 13 (Medidas de correlaciones)**.**

- *Correlaciones en covarianza*:

$$C(A:B) = \max_{\substack{M\in\mathscr{A}_A, N\in\mathscr{A}_B \\ \|M\|\leq 1, \|N\|\leq 1}} |\langle M\otimes N\rangle - \langle M\rangle\langle N\rangle| = \max_{\substack{M\in\mathscr{A}_A, N\in\mathscr{A}_B \\ \|M\|\leq 1, \|N\|\leq 1}} \left|\mathrm{tr}[M\otimes N(\rho_{AB}-\rho_A\otimes\rho_B)]\right|;$$

  donde $\langle O\rangle = \mathrm{tr}(O\rho_{AB})$ es el valor esperado del observable O con respecto a ρ_{AB}.

- *Correlaciones en traza*:

$$T(A:B) = \max_{\substack{F\in\mathscr{A}_{AB} \\ \|F\|\leq 1}} \left|\mathrm{tr}[F(\rho_{AB}-\rho_A\otimes\rho_B)]\right| = \left\|\rho_{AB}-\rho_A\otimes\rho_B\right\|_1.$$

- *Correlaciones en información mutua*:

$$I(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB});$$

  donde $S(\rho) = -\mathrm{tr}(\rho\log_2\rho)$ es la entropía de von Neumann del estado ρ.

En la teoría de la materia condensada, las correlaciones suelen medirse con $C(A:B)$. De la definición se sigue de manera inmediata que $C(A:B)$ está acotada superiormente por $T(A:B)$ (dado que $C(A:B)$ sólo depende de medidas con observables productos, mientras que $T(A:B)$ permite operadores más generales).

La relación entre la distancia en traza y la información mutua está dada, en una dirección, por la desigualdad de Pinsker [64], y en la otra por una aplicación de las desigualdades de Alicki-Fannes-Audenaert [2, 4, 23]. Sintetizamos las dos como sigue:

Teorema 20.

$$\frac{1}{4}T(A:B)^2 \leq I(A:B) \leq 6T(A:B)\log_2 d_A + 4h_b(T(A:B)); \tag{4.14}$$

donde $h_b(x) = -x\log_2 x - (1-x)\log_2(1-x)$ *es la función de entropía binaria, y* $d_A = \dim\mathscr{H}_A$.

4.1.4.2 Correlaciones en sistemas de muchos cuerpos

En el caso de sistemas de muchos cuerpos, consideremos el punto fijo ρ_∞ de $\mathscr{L}$ en Λ, y para cada región $A\subset\Lambda$ o cada par de regiones $A,B\subset\Lambda$ consideremos la matriz reducida $\rho_A = \mathrm{tr}_{\Lambda\setminus A}\rho_\infty$ y $\rho_{AB} = \mathrm{tr}_{\Lambda\setminus A\cup B}\rho_\infty$. Podemos entonces preguntar dos tipos de cuestiones (donde usaremos $I(A:B)$ pero hubiesen sido igualmente interesantes para cualquier otra medida de correlaciones):

- Dados $A, B \subset \Lambda$, ¿cómo escala $I(A:B)$ con respecto a dist$(A:B)$?
- Dado $A \subset \Lambda$, ¿cómo escala $I(A:A^c)$ con respecto al tamaño de A?

Aunque sean preguntas similares, en el primer caso sólo consideramos regiones finitas, mientras que en el segundo consideramos A^c, que crece al crecer Λ. Entonces no debería sorprender que la condiciones necesarias para dar una respuesta a la primera pregunta sean menos restrictivas que por la segunda. En el primer caso hablamos de **decaimiento de correlaciones**: esperamos que al considerar regiones A y B más alejadas, estas se vuelvan más independientes.

La segunda pregunta es interesante por lo siguiente. Para estados aleatorio elegidos con la medida de Haar, $I(A:A^c)$ es proporcional a $|A|$. Por el otro lado, muchos estados de interés físico tienen un comportamiento muy distinto, y $I(A:A^c)$ escala como $|\partial A|$, donde ∂A se define como el subconjunto de A de vértices que interactúan directamente con el complementario de A. Si las interacciones son finitas y A es una bola, entonces $|A|$ es un polinomio de grado D mientras que $|\partial A|$ es de grado $D-1$. Esta situación se denomina **ley de área** (con una terminología tomada del estudio de la entropía de agujeros negros, donde la frontera es efectivamente una superficie).

En lo que sigue trabajaremos con $T(A:B)$ y con $I(A:B)$, pero recordamos que a causa del Teorema 20 el decaimiento exponencial de una cantidad implica el decaimiento exponencial de la otra.

4.1.4.3 Estados fundamentales de hamiltonianos

El problema de estudiar decaimientos de correlaciones, leyes de área y sus relaciones con la dinámica ha sido afrontado de manera extensa en el contexto de estados fundamentales de hamiltonianos, aunque estas relaciones no han sido totalmente determinadas aún. Un hamiltoniano es un operador hermítico H sobre un espacio de Hilbert $\mathscr{H}$ que representa un sistema físico. El generador $\mathscr{L}(\rho) = -i[H,\rho]$ genera un grupo de automorfismos en vez de simplemente un semigrupo de contracciones, y se puede ver como caso especial de la ecuación (4.5). Dado que todo autovector de H es invariante bajo la acción de $\mathscr{L}$ la evolución tendrá más de un punto fijo: pero hay motivaciones físicas que justifican que los que corresponden al autovalor mínimo de H tengan un papel especial, y son llamados **estados fundamentales** de H. Son estados puros. Dado un estado puro $|\varphi\rangle_{AB}$, se verifica que $I(A:B) = 2S(\varphi_A)$, donde $\varphi_A = \mathrm{tr}_B |\varphi\rangle\langle\varphi|_{AB}$ es la matriz de densidad reducida de $|\varphi\rangle\langle\varphi|_{AB}$ sobre A. Por lo tanto, la información mutua se reduce a (dos veces) la entropía de von Neumann.

La propiedad crucial en este contexto es el llamado **gap espectral** de H: la diferencia entre los dos autovalores más pequeños de H. Según convención, diremos que una familia de hamiltonianos definidos en una secuencia creciente y absorbente $\Lambda_n \nearrow \Gamma$ tiene un **gap** si el gap es uniformemente acotado por encima de cero - en otras palabras, si el gap no se cierra en el límite. En caso contrario se dirá que el hamiltoniano no tiene gap, y se puede estudiar con que velocidad el gap se cierra (si polinomialmente o exponencialmente rápido en n).

En su artículo fundacional [34], Hastings y Koma probaron que si una familia de hamiltonianos locales tiene gap, entonces el estado fundamental verifica un decaimiento exponencial de correlaciones uniforme en n. Este resultado es interesante porque conecta con la teoría de fases de la materia condensada: una fase cuántica es una clase de equivalencia de hamiltonianos, tal que dos hamiltonianos H_1 y H_2 son equivalentes si se pueden conectar con un camino regular $H(t)$ de hamiltonianos con gap uniforme en t. Las transiciones de fases corresponden por lo tanto a los puntos del camino donde el gap se cierra. En esa situación es común que la longitud de correlación diverja, donde por longitud de correlación intendemos la distancia ξ tal que $C(A:B) \leq e^{-\operatorname{dist}(A,B)/\xi}$.

Otra propiedad prevista por la teoría de materia condensada es que el estado fundamental de hamiltonianos con gap verifique una ley de área para la entropía de entrelazamiento. El argumento intuitivo (pero que no constituye una prueba rigurosa), es el siguiente: si consideramos una región finita A, por el decaimiento exponencial de correlaciones los espines que están dentro de A y lejos de la frontera serán casi independientes de los que están fuera de A. Por lo tanto, las correlaciones y la entropía sólo pueden ser dadas por los espines que están cerca de la frontera. Dado que cada espín de dimensión d sólo puede contribuir con un factor $\log d$ a la entropía total, se sigue que esta escalará con el tamaño de la frontera.

Que este argumento se pueda hacer riguroso es el contenido de la **conjetura de la ley de área** (que los estados fundamentales de hamiltonianos con gap verifiquen una ley de área). Es un importante problema abierto en la teoría de materia condensada y en los últimos años ha visto importantes avances [22]. Una solución fue obtenida para el caso de dimensión 1 por Hastings [32], y luego una demostración alternativa fue presentada en [10, 11], donde se probó que en 1D el decaimiento exponencial de correlaciones implica una ley de área. Junto con el resultado de Hastings y Koma [34], esto implica que un gap espectral, al implicar un decaimiento de correlaciones, también implica una ley de área en 1D.

En dimensión más alta el problema sigue abierto. Algunos avances han llegado de la comunidad de ciencias de la computación [3], con una nueva prueba del resultado de Hastings y Koma, que ha permitido mejorar la dependencia de la longitud de correlación con el gap espectral, produciendo unas cotas más fieles en los casos concretos para los cuales somos capaces de calcular (analíticamente o numéricamente) las dos cantidades. Las herramientas desarrolladas han permitido también la construcción del primer algoritmo para aproximar estados fundamentales de hamiltonianos con gap en 1D para el que es posible probar una complejidad polinomial [46], así como otras herramientas combinatorias para estudiar la estructura de los estados fundamentales. Estos avances, aunque muy prometedores, todavía no han llevado a una demostración de una ley de área para estados en dimensión mayor o igual a 2.

4.1.4.4 Estados de Gibbs y estados de redes de tensoriales

Los estados de Gibbs o estados térmicos son estados proporcionales a $\exp(-\beta H)$, por algún hamiltoniano H y un parámetro β que representa al inverso de la temperatura. Son interesantes porque describen un sistema en equilibrio a temperatura finita $1/\beta$, y porque encajan de manera natural en el caso de dinámica abierta: muchos de los modelos disipativos que

hemos mencionado son intentos de describir un proceso de termalización que lleva a un estado de Gibbs. Por lo tanto, aunque no son los únicos puntos fijos posibles de los sistemas disipativos, constituyen ciertamente una clase importante. Cabe destacar que verifican una ley de área para la información mutua [86].

Otra clase importante de estados (esta vez puros) que a menudo satisfacen una ley de área son los llamados estados de redes tensoriales [14] - estados cuyas amplitudes vienen dadas por la contracción de una red de tensores. Para ser más específicos, en la gran familia de estados de redes tensoriales, en 1D los **Matrix Product States (MPS)** y en dimensión mayor o igual a 2 los **Projected Entangled Pair States (PEPS)** satisfacen una ley de área por construcción. El interés en ese tipo de estados es que solamente requieren una cantidad polinomial de parámetros (en el número de partículas) para describirlos, al contrario de la dimensión exponencial del espacio de Hilbert en el cual viven. Por esta razón, son ampliamente usados en el calculo numérico, y se cree que son buenas aproximaciones de los estados fundamentales de hamiltonianos con gap. Aunque haya ejemplos de estados en 2D que verifiquen una ley de área pero que no sean aproximables por un PEPS [26], se probó que bajo ciertas hipótesis en 2D el estado fundamental de un hamiltoniano local se puede aproximar por un PEPS [33, 57].

En 1D, la situación es bastante más clara: los estados fundamentales de hamiltonianos locales con gap se pueden aproximar de manera eficiente con MPS. Esto no solo es un resultado teórico, sino que ha sido muy importante a la hora de entender y desarrollar algoritmos que aproximen estados fundamentales en 1D.

4.1.4.5 Ley de área y longitud de correlación

Hemos mencionado una demostración intuitiva -pero incompleta- que conectaría una longitud de correlación finita con una ley de área. Hay que mencionar al respecto un resultado riguroso presentado en [86]. En ese trabajo los autores dan una noción distinta de longitud de correlación para la información mutua: dada una región finita $A \subset \Lambda$, sea $B_R = \{x \in \Lambda | \operatorname{dist}(x, A) > R\}$, y definimos ξ_Λ como la longitud mínima R tal que

$$I(A : B_R) < \frac{I(A : B_0)}{2}, \quad \forall A \subset \Lambda. \tag{4.15}$$

(Nótese que $B_0 = \Lambda \setminus A$.) Con esta definición, pueden probar que $I(A : A^C) \leq 4|\partial A|\xi_\Lambda$, es decir una ley de área.

Aunque el resultado es correcto, hay que tener cuidado en considerar la relación entre la ecuación (4.15) y el decaimiento de correlaciones usual, es decir $I(A : B) \leq c\exp(-\operatorname{dist}(A, B)/k)$. Se podría razonar que, en el caso de decaimiento exponencial de correlaciones, al cumplirse que si $\operatorname{dist}(A, B_R) = R$ entonces $I(A : B) \leq c\exp(-R/k)$, entonces es suficiente elegir ξ_Λ proporcional a k para verificar la ecuación (4.15). Este argumento no funciona de la forma esperada si la constante c que aparece en el decaimiento de correlaciones no es independiente del tamaño de las regiones A y B, lo que suele ser el caso como veremos más adelante. Si A está fijado y hacemos crecer Λ, entonces el tamaño de B_R es proporcional al tamaño total del sistema, con lo cual también ξ_Λ crecerá con Λ. La cota obtenida de esta manera será todavía polinómica de grado menor que la dimensión

geométrica del retículo, pero será multiplicada por una constante que depende del tamaño del sistema. Esta constante hará que en muchos casos la cota sea trivial, dado que será más grande que la cota general dada por $\log\dim\mathscr{H}_A = |A|\log d$, donde d es la dimensión del espacio de Hilbert de una única partícula.

Un problema similar se encontró en [37], donde se demostró bajo la hipótesis de una desigualdad de log-Sobolev una cota de la forma

$$I(A:A^c) \leq c\log\log\left\|\sigma^{-1}\right\||\partial A|,$$

donde σ es el punto fijo de la evolución. Otra vez, el término de la derecha escala con el exponente adecuado a una ley de área, pero la constante multiplicativa hace que esta cota sea peor que la trivial en la mayor parte de los casos.

Uno de los resultados principales de esta tesis es demostrar por primera vez una ley de área satisfactoria para puntos fijos de evoluciones con equilibración rapida (véase el apartado 4.2.3.4).

4.1.5 Estabilidad de sistemas cuánticos

Una de las propiedades de los sistemas cuánticos abiertos estudiada en esta tesis es la estabilidad. Antes de presentar el resultado obtenido, merece la pena explicar por qué es tan crucial. La estructura matemática que estamos considerando es un intento de describir un sistema físico compuesto de muchas partículas. Este podría ser o bien un sistema que ocurre en la naturaleza (por ejemplo, la motivación original del modelo de Ising fue la de estudiar la magnetización), o bien un sistema artificial creado para cumplir con un trabajo (computación, comunicación, memorias, preparación de estados, etc.)

En el primer caso, el modelo matemático será obviamente una aproximación de la física real: sería poco razonable pedir que las cantidades involucradas (constantes de interacción, niveles de energías, masas/cargas/densidades, etc.) se puedan medir con precisión infinita. La única expectativa realista es que se puedan medir con cierto nivel de precisión. Una vez que introduzcamos esta información en nuestro modelo matemático, nos gustaría tener una herramienta que sea capaz de predecir los resultados de nuestros experimentos. Si estas predicciones cambian de manera brusca al más mínimo cambio en los parámetros considerados, entonces las predicciones que nos proporciona raramente tendrán algo que ver con la realidad y el modelo resultará ser bastante inútil, dado que requeriría un *afinamiento* imposible para funcionar.

La situación es muy similar en el caso de sistemas artificiales. En este caso, la hipótesis irrazonable es que se pueda tener control perfecto sobre la implementación del modelo artificial, en el sentido de que podemos configurar sus parámetros a cualquier nivel de precisión. Ningún sistema real (ni siquiera macroscópico y clásico) puede ser controlado de manera perfecta: la implementación en realidad siempre será como mucho una aproximación fiel del modelo matemático. Si las evoluciones resultantes dependen de manera crucial de estas pequeñas diferencias, entonces acabaremos implementando una evolución muy distinta de la que pensábamos, y con resultados diferentes. Por lo tanto los únicos modelos prácticos son aquellos para los que pequeños errores en la implementación darán a lugar a pequeños cambios en el sistema final.

En los dos casos, la justificación teórica de un modelo matemático requiere que éste sea **estable contra perturbaciones**: podemos obviamente hablar de modelos que no sean estables, pero hay que tener mucho cuidado al considerar su implementación física y sus predicciones, dado que en la práctica no seremos nunca capaces de verlas en la realidad. Este razonamiento sólo se hace más fuerte al considerar, además de errores experimentales, fuentes de ruidos físico: ningún experimento estará perfectamente aislado, ningún ruido será completamente eliminado.

Por lo tanto necesitamos herramientas que justifiquen la robustez de los modelos físicos gracias a su estabilidad. En el caso de hamiltonianos locales, el enfoque ha sido probar estabilidad del gap espectral, un parámetro que tiene importantes consecuencias en las propiedades físicas de los modelos correspondientes. Al partir del trabajo de [12, 41] se llegó al de [56], en el cual se probó que el gap espectral es estable (es decir que no se cierra) bajo algunas condiciones físicamente razonables.

Hay que subrayar que estamos considerando un tipo especial de perturbaciones: dado que trabajamos con modelos de muchos cuerpos, donde cada partícula solo interactúa con sus vecinas, es natural considerar que las perturbaciones y los errores involucren a *cada término de interacción local*. Por lo tanto, las perturbaciones serán pequeñas a nivel microscópico y localmente despreciables, pero se sumarán al considerar sistemas más grandes, y serán realmente perturbaciones no acotadas (pero con mucha estructura local). Por esta razón no podemos simplemente aplicar la teoría de perturbaciones estándar, sino que tenemos que desarrollar técnicas específicas para este tipo de perturbaciones.

Otro resultado importante de esta tesis es probar que los sistemas con equilibración rápida son estables bajo perturbaciones (véase el apartado 4.2.3.3).

4.2 Resumen de los resultados

4.2.1 Hipótesis

En este apartado presentaremos y discutiremos las hipótesis principales hechas en este trabajo. La más importante de ellas es la de equilibración rápida, una condición sobre el tiempo de convergencia del sistema hacia su punto fijo.

Hablaremos de familias de generadores Lindbladianos $\{\mathscr{L}^\Lambda\}_\Lambda$ donde Λ es una sucesión creciente de subconjuntos finitos de Γ. Para cada uno de ellos, denotaremos con T_t^Λ la evolución correspondiente, es decir $T_t^\Lambda = \exp\left(t\mathscr{L}^\Lambda\right)$.

Definición 14 (Punto fijo único)**.** Sea $\{\mathscr{L}^\Lambda\}_\Lambda$ una familia de generadores Lindbladianos. Diremos que tiene un único punto fijo si, para todo Λ, $\mathscr{L}^\Lambda$ tiene un único punto fijo y ningún punto periódico (es decir, tiene un espectro periférico trivial).

Indicaremos con T_∞^Λ el proyector sobre el punto fijo de $T_t^\Lambda = \exp\left(t\mathscr{L}^\Lambda\right)$ que preserva la traza, dado por $\lim_{t\to\infty} T_t^\Lambda$.

4.2.1.1 Equilibración rápida

Ya hemos descrito por qué el gap espectral sólo nos proporciona información parcial sobre el tiempo de equilibración de un sistema disipativo, mientras que la desigualdad de log-Sobolev permite tener un control mayor (pero también requiere alguna propiedad más fuerte del punto fijo). Nuestro enfoque será más directo, y pediremos simplemente que el tiempo de equilibración escale de manera logarítmica con el tamaño del sistema, dejando al lado el problema de cómo probar tal condición.

Definición 15 (Equilibración rápida)**.** Sea $\{T_t^\Lambda\}_\Lambda$ una familia de aplicaciónes disipativas, donde Λ varía en una secuencia infinita de subconjuntos de Γ. Diremos que tiene equilibración rápida si existen $c, \gamma > 0$ y $\delta \geq 1$ tales que

$$\sup_{\substack{\rho \geq 0 \\ \operatorname{tr}\rho = 1}} \left\| T_t^\Lambda(\rho) - T_\infty^\Lambda(\rho) \right\|_1 \leq c|\Lambda|^\delta \, e^{-t\gamma}. \tag{4.16}$$

Como hemos mencionado anteriormente, en algunos casos es posible relajar la hipótesis de equilibración rápida: este caso está tratado en parte en [S, sec. 4.5].

Probar que un modelo disipativo tiene equilibración rápida es difícil, de la misma manera que lo es probar la existencia del gap espectral para un sistema hamiltoniano. Aparte de casos "simples", como modelos sin interacciones y preparación de *graph states* [38], la otra clase importante de modelos que verifican esta propiedad son Lindbladianos reversibles que satisfacen una desigualdad de log-Sobolev [39], lo que incluye modelos clásicos como la dinámica de Glauber para el modelo de Ising en el rango apropiado de parámetros [53].

4.2.1.2 Familias uniformes

Como ya hemos explicado, estamos interesados en estudiar el comportamiento de algunas propiedades de los generadores Lindbladianos $\mathscr{L}_n$, definidos en una sucesión creciente y absorbente de retículos finitos Λ_n que convergen a un grafo infinito Γ (en nuestro caso, Γ será $\mathbb{Z}^D$, pero el mismo razonamiento hubiese funcionado con cualquier otro grafo en el cual las bolas crecen de manera polinómica con su diámetro). Pero al mismo tiempo, dado que estamos interesados en modelos físicos, queremos que distintos $\mathscr{L}_n$ representen "el mismo" sistema físico en escalas distintas, de manera que estudiar esta sucesión realmente nos diga algo sobre la física que estamos modelizando.

¿Qué significa que operadores definidos en retículos distintos representen "el mismo" sistema físico? Obviamente la pregunta tiene muchas respuestas posibles, pero intentaremos hacer algunas hipótesis sobre una *regla* o *receta* para obtener, a partir de los mismos ingredientes, todos los $\mathscr{L}_n$ en las distintas escalas.

Una hipótesis posible podría ser que todos los términos locales de cada $\mathscr{L}_n$ sean simplemente el trasladado de un único generador local l_0: es decir, existe un $r > 0$ finito y un l_0 que actua en $\mathscr{A}_{b_0(r)}$ de manera que para todo n se cumpla

$$\mathscr{L}_n = \sum_{x : b_x(r) \subset \Lambda_n} l_x$$

donde l_x es el trasladado de l_0 por el vector x. A esta situación se le suele denominar **invarianza translacional**, dado que en el límite las interacciones son invariantes bajo translaciones de Γ (no tiene sentido obviamente hablar de invarianza bajo traslaciones para retículos finitos).

Se debería notar que esta es una restricción excesiva: no solamente porque nos gustaría poder estudiar sistemas donde las interacciones dependen de la posición en el retículo, sino porque también cerca de la frontera de Λ_n el sistema deviene "indeterminado": dado que no hay espacio para contener el soporte de l_0 ahí, habrá cada vez menos interacciones que involucren los vértices cercanos a la frontera. A veces a esta situación se le llama **condiciones de contorno abiertas**. Dado que estamos interesados en sistemas con un único punto fijo, esta hipótesis puede ser especialmente problemática, puesto que la indeterminación cerca de la frontera puede crear múltiples puntos fijos - y estaremos pidiendo dos condiciones incompatibles entre sí.

Para superar esta limitación, hemos propuesto una definición que hemos llamado **familias uniformes**, y creemos que puede ser una definición lo suficientemente general para describir sucesiones "interesantes" de generadores Lindbladianos. Denotaremos con

$$\partial_d \Lambda = \{x \in \Lambda \mid \operatorname{dist}(x, \Gamma \setminus \Lambda) \leq d\}. \tag{4.17}$$

Por convención, escribiremos $\partial\Lambda$ en lugar $\partial_1\Lambda$.

Definición 16. Sea $\Lambda \subset \Gamma$. Una *condición de contorno* para Λ viene dada por un Lindbladiano $\mathscr{B}^{\partial\Lambda} = \sum_{d\geq 1} B_d^{\partial\Lambda}$, donde $\operatorname{supp} B_d^{\partial\Lambda} \subset \partial_d\Lambda$.

En la definición de condición de contorno se usa una noción de localidad distinta de la que hemos usado para definir los generadores locales: el decaimiento en la norma es sólo necesario cuando las interacciones entran *dentro* del centro del sistema, mientras que se permite que sean fuertes entre espines muy distantes siempre que estén a la misma distancia de la frontera. Por ejemplo, si Λ es un cuadrado, esta condición permite acoplar espines opuestos en la frontera, una condición conocida como *condiciones de contorno periódicas*, dado que podemos imaginar haber envuelto Λ en un toro, de manera que vértices opuestos en la frontera se vuelvan vecinos. Estas y otras condiciones más exóticas pueden ser descritas por la definición dada arriba.

Definición 17. Una *familia uniforme* de Lindbladianos está dada por:

(i) *interacciones centrales*: un Lindbladiano $\mathscr{M}_Z$ para todo $Z \subset \mathbb{Z}^D$ finito;

(ii) *condiciones de contorno*: una familia de *condiciones de contorno* $\{\mathscr{B}^{\partial\Lambda}\}_\Lambda$, para todo $\Lambda \subset \mathbb{Z}^D$ finito.

Dada una familia uniforme de Lindbladianos así definida, para cada $\Lambda \subset \Gamma$ finito podemos definir dos generadores que actúan en el:

$$\mathscr{L}^\Lambda = \sum_{Z\subset\Lambda} \mathscr{M}_Z \quad \text{condiciones de contorno abiertas;} \tag{4.18}$$

$$\mathscr{L}^{\bar{\Lambda}} = \mathscr{L}^\Lambda + \mathscr{B}^{\partial\Lambda} \quad \text{condiciones de contorno cerradas.} \tag{4.19}$$

Al hablar de $\mathscr{L}^{\bar{\Lambda}}$, llamaremos a los términos $\mathscr{M}_Z$ interacciones centrales y a los $\mathscr{B}_d^{\partial\Lambda}$ interacciones de contorno.

Son necesarios unos comentarios sobre esta definición: no estamos considerando una secuencia concreta de retículos crecientes Λ_n, sino que permitimos definir un Lindbladiano (y de hecho dos) para todo Λ finito por el cual está dada una condición de contorno.

Si elegimos dos $\Lambda_1 \subset \Lambda_2 \subset \Gamma$ finitos, y miramos a las interacciones que involucran las partículas en el centro de Λ_1, entendiendo por esto los vértices que están lejos de $\Gamma \setminus \Lambda_1$, entonces es facil ver que $\mathscr{L}^{\overline{\Lambda}_1}$ y $\mathscr{L}^{\overline{\Lambda}_2}$ tienen las mismas interacciones de corto alcance, y que sólo difieren en los términos de largo alcance: o bien por el efecto de los términos de $\mathscr{M}_Z$ con Z que se extiende fuera de Λ_1, o bien por las diferencias entre $\mathscr{B}^{\partial\Lambda_1}$ y $\mathscr{B}^{\partial\Lambda_2}$. Hablando informalmente, podemos decir que los detalles microscópicos de las interacciones son los mismos excepto por algún término de largo alcance. En la sección siguiente asumiremos que la fuerza de las interacciones (la norma de los operadores correspondientes), decrece en su alcance: por lo tanto, para familias uniformes, la diferencia entre las interacciones centrales de $\mathscr{L}^{\overline{\Lambda}_1}$ y $\mathscr{L}^{\overline{\Lambda}_2}$ será pequeña. Esta es la propiedad fundamental y caracterizante de las familias uniformes de Lindbladianos: a excepción de pequeños errores, los detalles microscópicos de las interacciones centrales no dependen de cómo de grande se ha elegido el sistema (con ser este lo suficientemente grande para contenerlas).

4.2.1.3 Hipótesis de Lieb-Robinson

Hasta ahora, nuestra definición de Lindbladiano local es incompleta: si no especificamos a qué ritmo las normas de las interacciones decaen, siempre podemos descomponer un Lindbladiano en una suma de términos locales todos nulos menos el último con soporte en todo el espacio. Si en vez de esto imponemos que las normas decrezcan en función del diámetro del soporte obtenemos una condición altamente no trivial. Dado que el ritmo de decaimiento está relacionado con una propiedad que presentaremos en seguida, llamada velocidad de Lieb-Robinson, llamaremos estas condiciones **hipótesis de Lieb-Robinson**, y las daremos sólo para las familias uniformes definidas anteriormente.

Definición 18 (Hipótesis de Lieb-Robinson)**.** Existe una función *creciente* $\nu(r)$ que verifica $\nu(x+y) \leq \nu(x)\nu(y)$, y constantes positivas υ, b, y c, tales que:

$$\sup_{x\in\Gamma} \sum_{Z\ni x} \|\mathscr{M}_Z\|_\diamond |Z| \nu(\operatorname{diam} Z) \leq \upsilon < \infty, \tag{A-1}$$

$$\sup_{x\in\Gamma} \sup_r \nu(r) \sum_{d=r}^{N} \left\| B_d^{\partial b_x(N)} \right\|_\diamond \leq cN^b. \tag{A-2}$$

Nótese que si $\|\mathscr{M}_Z\|_\diamond$ decae exponencialmente en $\operatorname{diam} Z$ (o si es cero para todo Z de diámetro más grande que una cierta constante, una situación llamada **interacciones de alcance finito**) entonces podemos elegir $\nu(r) = \exp(\mu r)$ para algún μ positivo. Si por el contrario decae polinomialmente, tenemos que considerar funciones más lentas, como $\nu(r) = (1+r)^\mu$. En particular, si $\|\mathscr{M}_Z\|_\diamond \sim (\operatorname{diam} Z)^{-\alpha}$, entonces la ecuación (A-2) se cumple si $\mu < \alpha - (2D+1)$, donde D es la dimensión geométrica de Γ (lo que significa que α tendrá que ser mayor que $2D+1$ para que la condición se cumpla).

La motivación tras estas hipótesis es que los sistemas que verifican estas condiciones muestran una velocidad finita de propagación: el soporte de un observable local crece en el tiempo, a menudo de manera lineal, salvo por una cola exponencialmente pequeña. Esto implica que las regiones que están separadas en el espacio, si están descorreladas a tiempo cero, quedarán casi descorreladas por un tiempo finito, que depende (a menudo de manera lineal) de su distancia. Daremos más detalles sobre el tema en el apartado 4.2.2.1

4.2.1.4 Sistemas libres de frustración

Otra condición que necesitaremos imponer en ciertos casos es la llamada **ausencia de frustración**, parecida a una propiedad de hamiltonianos con el mismo nombre. Merece la pena notar que es solamente una condición sobre las interacciones centrales, y no sobre las de contorno.

Definición 19. Diremos que una familia uniforme $\mathscr{L} = \{\mathscr{M}, \mathscr{B}\}$ es *libre de frustración* si para todo Λ y todo punto fijo ρ_∞ de $\mathscr{L}^{\bar{\Lambda}}$ se cumple

$$\mathscr{M}_Z(\rho_\infty) = 0 \quad \forall Z \subset \Lambda. \tag{4.20}$$

Hay muchos ejemplos interesantes y naturales de Lindbladianos que verifican esta propiedad: entre ellos, los generadores de Davies y otros tipos de muestreo de Gibbs para Hamiltoninanos conmutativos [36], así como la preparación disipativa de estados para PEPS.

4.2.2 Herramientas

Antes de presentar los resultados principales, haremos un resumen de algunas herramientas usada en su demostración. Han sido desarrolladas para este objetivo, pero pueden resultar igualmente interesantes en otros contextos. Empezaremos con las cotas de Lieb-Robinson, una herramienta estándar en los problemas de muchos cuerpos, y continuaremos con unos resultados derivados de ellas, o simplemente inspirados.

En esta sección, daremos por hecho que el generador $\mathscr{L}$ verifica las hipótesis (A-1) y (A-2).

4.2.2.1 Cotas de Lieb-Robinson

En los sistemas de muchos cuerpos se asume que las interacciones son locales o quasi-locales: el espín en cada vértice del retículo sólo puede interactuar directamente con sus vecinos (interacciones de alcance finito), o si puede interactuar con espines mas alejados la fuerza de esa interacción tiene que decaer rápidamente con la distancia. Por lo tanto, la interacción entre espines distantes no es directa, sino mediada por los espines intermedios que tienen que "transmitir" la información. Es de esperar que dicha interacción no sea por lo tanto inmediata, sino que tendrá un retraso, y más grande si la distancia crece dado que más espines intermedios tendrán que verse involucrados. Esto no es un efecto relativístico, dado que no hay velocidad finita de la luz en nuestros modelos: una metáfora más exacta

sería la de la velocidad del sonido, que está dada por el medio en el cual la información se propaga.

Esta visión intuitiva está formalizada por las cotas de Lieb-Robinson, y la velocidad de propagación resultante se llama velocidad de Lieb-Robinson. La primera demostración formal fue obtenida para sistemas hamiltonianos y grupos de automorfismos, [47, 72], y por esta razón también se le llama velocidad de grupo. Mas tarde fue generalizada a evoluciones disipativas [61, 69].

Una consecuencia de las cotas de Lieb-Robinson y de la existencia de tal velocidad es que, si consideramos una región finita A y modificamos el generador de la evolución en unos vértices que están lejos de A, la evolución modificada será casi indistinguible de la original en A, por lo menos por un corto tiempo: antes de que la información haya tenido tiempo de viajar desde los puntos donde se han hecho las modificaciones hasta A, los espines de A no "saben" que hubo modificación alguna, y por lo tanto evolucionarán como si no hubiese habido ninguna. Dado que la velocidad está dada por la hipótesis (A-1), será uniforme en el tamaño del sistema. Esto ha permitido en [61] demostrar la existencia del límite termodinámico.

El efecto de perturbar la dinámica en una región lejana viene dado por el siguiente lema, que se deriva de la cota usual de Lieb-Robinson.

Lema 21 ([S, Lemma 5.4])**.** *Sean $\mathscr{L}_1$ y $\mathscr{L}_2$ dos Lindbladianos locales, y suponemos que $\mathscr{L}_2$ verifica la hipótesis* (A-1) *con parámetros v y $\nu(r)$. Consideramos un operador O_X soportado en $X \subset \Lambda$, e indicamos con $O_i(t)$ su evolución bajo $\mathscr{L}_i^*$, $i = 1,2$. Suponemos que $\mathscr{L}_1 - \mathscr{L}_2 = \sum_{r\geq 0} M_r$, donde M_r es un superoperador soportado en Y_r y que se anula en $\mathbb{1}$, con* $\operatorname{dist}(X, Y_r) \geq r$*. Entonces se verifica lo siguiente:*

$$\|O_1(t) - O_2(t)\| \leq \|O_X\||X|\frac{e^{vt} - vt - 1}{v} \sum_{r=0}^{\infty} \|M_r\|_\diamond \nu^{-1}(r). \tag{4.21}$$

4.2.2.2 Evoluciones abiertas y cerradas

La definición de familia uniforme nos ha permitido definir dos evoluciones, una con condiciones de contorno y una sin. La definición de condiciones de contorno que hemos dado está justificada por el siguiente resultado: si se verifican las hipótesis de la ecuación (A-1) y la ecuación (A-2), entonces el efecto de la condición de contorno se difunde desde la frontera hacia el centro del sistema con la misma velocidad finita de propagación de las cotas de Lieb-Robinson. Por lo tanto, para tiempos cortos y observables lejanos de la frontera, las dos evoluciones serán indistinguibles. Esto ha sido probado usando el lema 21.

Lema 22 ([S, Lemma 5.6])**.** *Sea O_A un observable soportado en $A \subset \Lambda$, y sea $O_A(t)$ (resp. $\bar{O}_A(t)$) su evolución bajo $\mathscr{L}^{\Lambda *}$ (rep., $\mathscr{L}^{\bar{\Lambda} *}$). Sea $r = \operatorname{dist}(A, \Gamma \setminus \Lambda)$. Entonces existen constantes positivas c, v, y β tales que:*

$$\left\| O_A(t) - \bar{O}_A(t) \right\| \leq c\|O_A\||A|\frac{e^{vt} - 1 - vt}{v} \nu^{-\beta}(r). \tag{4.22}$$

4.2.2.3 Localización libre de frustración

Las cotas presentadas en esta sección son válidas para todo observable localizado. El enunciado dual sobre la evolución de estados diría algo sobre la evolución de cualquier estado que a tiempo cero se descompone como un producto con respecto a esa región. Trabajando en el problema de determinar una ley de área para la información mutua, hemos tenido que afrontar un problema parecido pero distinto: ¿qué pasa para otros estados que, aunque no tengan esta estructura de producto, satisfacen alguna otra propiedad "buena"? Mas exactamente, imaginemos preparar nuestro sistema en $\Lambda_n = b_0(n)$ en el estado ρ_n^∞ que es el punto fijo de la evolución (con condiciones de contorno cerradas) en Λ_n, pero luego extendemos el sistema con un estado arbitrario τ en $\Lambda_{n+1} \setminus \Lambda_n$ y miramos la evolución del estado $\rho_n^\infty \otimes \tau$ bajo el generador definido en Λ_{n+1}.

¿Qué podemos decir en este caso? Las cotas de Lieb-Robinson estándar no nos dan ninguna información, dado que las regiones que estamos considerando (Λ_n y $\Lambda_{n+1} \setminus \Lambda_n$) están a distancia cero, y no podemos asumir que ρ_n^∞ sea producto (o casi producto) en ninguna otra bipartición del sistema. Por otra parte, si asumimos que no haya frustración, la mayor parte de los términos del generador serán cero sobre el estado considerado, y los únicos que no son ceros son los que están cerca de la frontera de Λ_n. Entonces esperamos que la evolución sea aproximadamente trivial en el centro de Λ_n, y que la parte no trivial se difunda de la frontera hacia el centro a la velocidad de Lieb-Robinson.

Esta idea intuitiva se formaliza rigurosa en el lema siguiente. Se debe notar que, por lo que sabemos, esto no es una consecuencia directa de la cota estándar de Lieb-Robinson. Para demostrarla hemos tenido que reproducir las ideas y las técnicas de la demostración de las misma cotas de Lieb-Robinson y adaptarlas a esta situación específica.

Lema 23 ([A, Lemma 12])**.** *Sea $\mathscr{L} = (\mathscr{M}, \mathscr{B})$ una familia de Lindbladianos uniforme y libre de frustración. Sea $A \subset \Gamma$ una región finita. Fíjese un natural positivo m. Sea $B = A(m+1)$, $R = A(m+1) \setminus A(m)$ y ρ_∞^m el punto fijo de $T_t^{\bar{A}(m)}$ y τ un estado cualquiera en R. Entonces*

$$\left\| \left(T_t^{\bar{B}} - T_t^{\bar{B}\setminus A} \right) (\rho_\infty^m \otimes \tau) \right\|_1 \leq \mathrm{poly}(m) \nu^{-1}(m) \left[e^{\nu t} - 1 + t \right]; \tag{4.23}$$

donde $T_t^{\bar{B}\setminus A}$ es la evolución generada por

$$\mathscr{L}^{\bar{B}\setminus A} = \sum_{Z \subset B \setminus A} \mathscr{M}_Z + \sum_{d \leq m+1} \mathscr{B}_d^{\partial B}.$$

4.2.3 Resultados principales

Podemos ahora presentar los resultados principales obtenidos en [A, S, R].

4.2.3.1 Equilibración rápida local

Definición 20 (Equilibración rápida local)**.** Para $A \subset \Lambda$, definimos la *contracción de T_t con respecto a A* como

$$\eta^A(T_t) := \sup_{\substack{\rho \geq 0 \\ \mathrm{tr}\,\rho = 1}} \left\| \mathrm{tr}_{A^c} \left[T_t(\rho) - T_\infty(\rho) \right] \right\|_1 = \sup_{\substack{O_A \in \mathscr{A}_A \\ \|O_A\| = 1}} \left\| T_t^*(O_A) - T_\infty^*(O_A) \right\|. \tag{4.24}$$

Diremos que $\mathscr{L}$ verifica *equilibración rápida local* si, para todo $A \subset \Lambda$, se cumple que

$$\eta^A(T_t) \leq k(|A|)e^{-\gamma t}, \tag{4.25}$$

donde $k(r)$ crece polinomialmente en r, $\gamma > 0$ y todas las contantes que aparecen arriba son independientes del tamaño del sistema.

Podemos igualmente definir un **tiempo de equilibración local** como la inversa de η^A:

$$\tau^A(\epsilon) = \text{mín}\{t > 0 : \sup_{\rho} \left\| \text{tr}_{A^c}[T_t(\rho) - T_\infty(\rho)] \right\| \leq \epsilon\}. \tag{4.26}$$

La condición de equilibración rápida local implica que τ^A dependa de $|A|$ y ϵ, pero no de Λ: esto implica que, salvo por pequeños errores, los observables locales convergen a su límite en una escala temporal que depende solamente del soporte del observable y **no** del tamaño del sistema. Junto con la cota de Lieb-Robinson, esto implica que los observables locales sólo pueden interactuar con una región finita a su alrededor e independiente del tamaño del sistema.

Parece sorprendente que solamente basándonos en una cota sobre el tiempo de equilibración de una familia uniforme hayamos podido derivar una propiedad tan fuerte, tal y como probamos en [S, Proposition 6.6].

Teorema 24. *Para familias uniformes que satisfacen las hipótesis de Lieb-Robinson, equilibración rápida implica equilibración rápida local.*

4.2.3.2 Indistinguibilidad local

El resultado sobre equilibración local para observables puede en cierto sentido "dualizarse" en una propiedad de la familia de los puntos fijos. El razonamiento es el siguiente: el límite $O(\infty) = \lim_{t\to\infty} O(t)$ de la evolución de un observable es el valor esperado con respecto al estado límite, es decir $O(\infty) = \text{tr}(\rho_\infty O)\mathbb{1}$, dado que:

$$\text{tr}\,\rho O(\infty) = \lim_{t\to\infty} \text{tr}\,\rho T_t^*(O) = \lim_{t\to\infty} \text{tr}\,T_t(\rho)O = \text{tr}\,\rho_\infty O.$$

Por una parte las cotas de Lieb-Robinson implican que si O es un observable local, entonces para tiempos cortos $O(t)$ no depende de las interacciones que están lejos de su soporte; por otra parte, $O(t)$ converge a $O(\infty)$ en un tiempo independiente del tamaño del sistema. Por lo tanto, $O(\infty)$ sólo depende de las interacciones que están cerca de su soporte a tiempo cero, y no de las que están alejadas. Dado que $O(\infty)$ es igual a $\text{tr}\,\rho_\infty O$, esto implica que el observable O no puede distinguir entre distintos puntos fijos de evoluciones definidas en sistemas más grandes, dado que la única diferencia en las interacciones está alejada del soporte de O. El lema siguiente formaliza este argumento [S, Lemma 6.2]

Lema 25. *Sea $\mathscr{L} = \{\mathscr{M}, \mathscr{B}\}$ una familia uniforme de evoluciones disipativas con equilibración rápida, y asumamos que cada $T_t^{\bar{\Lambda}}$ tenga un punto fijo único y ningún otro punto periódico. Fíjese un Λ y sea ρ_∞ el único punto fijo de $T_t^{\bar{\Lambda}}$. Dado $A \subset \Lambda$, para todo $s \geq 0$ indicamos por ρ_∞^s el único punto fijo de $T_t^{\bar{A}(s)}$.*

Entonces se cumple que:

$$\left\| \mathrm{tr}_{A^c}(\rho_\infty - \rho_\infty^s) \right\|_1 \leq |A|^\delta \Delta_0(s), \tag{4.27}$$

donde $\Delta_0(s) = c(|A(s)|/|A|)^{\delta\nu/(\nu+\gamma)} v^{-\beta\gamma/(\nu+\gamma)}$*, y* c *es una constante positiva mientras que* β *y* ν *vienen dadas por el lema 22 y* δ *y* γ *por la Definición 15.*

Esta propriedad es coherente con la idea que las familias uniformes representen modelos en los que la dinámica microscópica está bien definida y es independiente del tamaño del sistema.

4.2.3.3 Estabilidad bajo perturbaciones

En el apartado 4.1.5 hemos presentado la importancia de la estabilidad bajo perturbaciones con el fin de justificar a nivel teórico los modelos que consideramos. Vamos ahora a dar una definición más formal de estabilidad. Dado que hay muchas maneras distintas de definirla que pueden tener sentido en contextos distintos, haremos una elección muy conservadora e impondremos las mínimas hipótesis posibles en la perturbación, mientras que pediremos la noción más fuerte de estabilidad. Hay naturalmente muchas maneras de relajar este resultado.

Dada una familia uniforme de Lindbladianos $\mathscr{L}$, definida por sus interacciones centrales $\mathscr{M}_Z$ y por sus condiciones de contorno $\mathscr{B}_d$, consideraremos una perturbación al mismo tiempo de los términos centrales $\mathscr{M}'_Z = \mathscr{M}_Z + E_Z$ y de las condiciones de contorno $\mathscr{B}'_d = \mathscr{B}_d + E_d$. La perturbación debería ser pequeña comparada con la norma del los Lindbladianos originales, así que vamos a asumir que para todo E_Z y E_d se cumpla $\|E_Z\|_\diamond \leq \epsilon \|M_Z\|_\diamond$ y $\|E_d\|_\diamond \leq \epsilon \|B_d\|_\diamond$, por algún $\epsilon > 0$.

Nótese que una perturbación de este tipo es pequeña a nivel microscópico, pero dado que actúa en cada término local, la suma $E^\Lambda = \sum_{Z\subset\Lambda} E_Z + \sum_d E_d$ tiene una norma que es divergente con el tamaño del sistema, y por lo tanto es una perturbación no acotada una vez que olvidamos la estructura local. Esto implica que no podemos simplemente aplicar la teoría de perturbaciones estándar.

Todavía necesitamos alguna condición sobre la perturbación para que sea "físicamente realista". Sería suficiente pedir que $\mathscr{M}'_Z$ y $\mathscr{B}'_d$ sean Lindbladianos, pero podemos relajar esta condición y solamente pedir las condiciones siguientes (que se cumplen de manera automática si los generadores perturbados son Lindbladianos):

- $E_Z^*(\mathbb{1}) = E_d^*(\mathbb{1}) = 0$;
- $S_t = \exp\left[t(\mathscr{L}^\Lambda + E^\Lambda) \right]$ is a contraction for each $t \geq 0$.

Con este modelo de perturbación, y bajo la hipótesis de equilibración rápida, hemos podido probar el siguiente resultado de estabilidad [S, Theorem 6.7].

Teorema 26. *Sea $\mathscr{L}$ una familia uniforme de Lindbladianos locales con un único punto fijo y con equilibración rápida. Sea S_t definido como arriba. Para un observable O_A soportado en $A \subset \Lambda$, se cumple que para todo $t \geq 0$:*

$$\left\| T_t^*(O_A) - S_t^*(O_A) \right\| \leq c(|A|) \, \|O_A\| \left(\epsilon + |\Lambda| \nu^{-\eta}(d_A) \right), \tag{4.28}$$

donde $d_A = \mathrm{dist}(A, \Lambda^c)$; η es positivo e independiente de Λ; $c(|A|)$ es independiente de Λ y t, y está acotado por un polinomio en $|A|$.

Comentamos el término de la derecha de la ecuación (4.28). El factor multiplicativo $\|O_A\|$ es una normalización esperada, dado que el término de la izquierda es 1-homogéneo. La constante $c(|A|)$ sólo depende del soporte de A y no de Λ: por lo tanto, si fijamos A y hacemos crecer Λ, esta será simplemente una constante fija. Discutiremos más adelante qué pasa cuando no es este el caso.

La norma local de la perturbación es ϵ, así que no es inesperado que aparezca en el lado derecho de la cota. Más sorprendente es que sólo aparezca como un factor lineal. Recordamos nuevamente que aunque la perturbación sea una suma de términos locales actuando en cada vértice, ϵ es la norma (diamante) de uno sólo de ellos y no de toda la perturbación, y que por lo tanto es independiente del tamaño del sistema.

El otro término $|\Lambda|\nu^{-\eta}(d_A)$ es un factor de corrección de la frontera, que tiene en consideración el efecto de la condición de contorno perturbada sobre el observable. Como es de esperar, decae con la distancia entre A y la frontera, y tiende a cero con Λ que tiende al retículo infinito Γ. Por lo tanto, para Λ suficientemente grande, será más pequeño que ϵ, y por lo tanto despreciable. En algún caso intermedio este factor puede parecer lejos de ser óptimo: por ejemplo, no se esperaría un factor de este tipo en el caso de interacciones invariantes por translaciones, aunque los observables estén localizados cerca de la frontera. Al fin y al cabo, en este caso, la frontera es sólo una necesidad matemática, y no corresponde a ninguna diferencia en las interacciones físicas entre los espines. De hecho, podemos trasladar el sistema para mover la frontera lo más lejos posible de A, y por lo tanto en esos casos se debería considerar d_A como la mitad del diámetro del complementario de A en Λ. Con esta observación, el decaimiento del termino $\nu^{-\eta}(d_A)$ cancelará rápidamente la contribución de $|\Lambda|$ y el resultado será despreciable con respecto a ϵ.

Hay que observar que si consideramos observables no-locales (es decir, observables cuyo soporte crece con Λ), entonces el factor $c(|A|)$ no puede ser más pequeño que lineal, y en particular esto implica que no se puede mejorar la cota para que no sea divergente en Λ. De hecho, es sencillo construir ejemplos simples de espines sin interacciones tales que existen observables soportados en todo en retículo y tales que el término de la izquierda de la ecuación (4.28) crece de manera lineal en el tamaño del sistema. Esto implica en particular que el factor $c(|A|)$ tiene que ser por lo menos lineal. Véase [S, Example 4.8] para dicha construcción.

4.2.3.4 Ley de área con corrección logarítmica

Sobre el problema de decaimiento de correlaciones para el punto fijo de la evolución, hemos obtenido el siguiente resultado [A, Theorem 14]:

Teorema 27. *Sea $\mathscr{L}$ una familia uniforme de Lindbladianos locales con único punto fijo y equilibración local. Entonces el punto fijo de cada $\mathscr{L}^{\bar{\Lambda}}$ verifica:*

$$T(A:B) \leq 3(|A|+|B|)^{\delta}\Delta_0\left(\frac{d_{AB}}{2}\right), \tag{4.29}$$

donde Δ_0 es la función de decaimiento rápido dada en el lema 25.

A causa del Teorema 20, $I(A:B)$ tendrá el mismo decaimiento.

Consideremos ahora la cuestión de si ρ_∞ verifica una ley de área. No hemos podido dar una respuesta definitiva, pero hemos obtenido un resultado ligeramente menos fuerte: $I(A:A^c)$ escala como $|\partial A|\log|A|$, que en términos del radio de A va como $r^{D-1}\log r$. Para obtener tal resultado, tuvimos que hacer alguna hipótesis más [A, Proposition 16, Theorem 17].

Teorema 28. *Sea $\mathscr{L}$ una familia uniforme de Lindbladianos locales con único punto fijo y equilibración rápida. Además, asumimos que $\mathscr{L}$ verifique una de las condiciones siguientes:*

- *ρ_∞ es un estado puro para todo Λ;*
- *$\mathscr{L}$ es libre de frustración;*

Entonces se cumple lo siguiente para todo punto fijo de $\mathscr{L}^{\bar{\Lambda}}$, para alguna constante c independiente de Λ:

$$I(A:A^c) \leq c|\partial A|\log|A| \tag{4.30}$$

Es interesante notar que las dos condiciones alternativas en el Teorema 28 son independientes entre sí. Por lo tanto, sospechamos que su necesidad sea un efecto de la demostración, y que se pueda probar el resultado sin ninguna de ellas.

4.3 Perspectivas y trabajos futuros

Hemos visto que una condición sobre el crecimiento del tiempo de equilibración de un sistema dinámico cuántico puede tener un gran impacto en las propiedades que este presenta: sean estas dinámicas (como la estabilidad) o estáticas (como la ley de área y la indistinguibilidad de los puntos fijos).

Esto nos lleva a reconsiderar el tiempo de equilibración como una propiedad fundamental y característica de estos modelos, de la misma manera en la cual lo es el gap espectral para los sistemas hamiltonianos. Esta idea ya había sido propuesta en [42]. En esta tesis hemos identificado una clase de sistemas que, dadas las propiedades mencionadas arriba, claramente tienen un papel especial en una posible clasificación de evoluciones disipativas. Todavía no nos resulta claro si esta clase de modelos surge de considerar el caso "bueno" o más bien el caso "trivial": será importante ver qué propiedades se pueden recuperar y cuáles no al sustituir la condición de equilibración rápida por algo menos restrictivo, y cómo de amplia y variada es la clase de modelos que sí satisfacen equilibración rápida. Es una línea de investigación interesante, y las observaciones siguientes son los primeros pasos en su desarrollo.

4.3.1 Tiempo de equilibración polinomial

Si consideramos a los Lindbladianos como "máquinas disipativas" o procedimientos de preparación de estados cuánticos con cierta utilidad, entonces desde un punto de vista algorítmico un tiempo de equilibración polinomial es perfectamente aceptable, y constituye una hipótesis más natural que la de equilibración rápida. En el mismo espíritu, para efectuar computaciones, a veces es útil considerar hamiltonianos cuyo gap se cierra pero solamente de manera *polinómica* en el tamaño del sistema. En los dos casos, tenemos una situación que es claramente una desventaja en el límite termodinámico, pero que para sistemas finitos todavía puede ser manejada en un tiempo razonable y con una cantidad razonable de recursos.

En esta situación, la conexión con la teoría de la materia condensada (con sus resultados matemáticos pero al mismo tiempo con su "filosofía") se vuelve menos útil: sería parecido a estudiar la eficiencia de un algoritmo de ordenamiento mirando cómo se comporta en un conjunto infinito de elementos. Necesitamos herramientas distintas.

Describir cuál es el poder computacional y las propiedades de esta clase de evoluciones es un problema muy interesante y sería una manera de clarificar si las propuestas de preparación disipativa de estados puedan ser implementadas en experimentos a grandes escalas. En este contexto la estabilidad, no necesariamente del mismo tipo que hemos considerado en este trabajo, sería la propiedad fundamental a considerar.

4.3.2 Preparación de modelos topológicos

El resultado sobre la ley de área puede también verse como una propiedad negativa de los modelos con equilibración rápida: no pueden generar estados que no tengan una ley de área. Por otra parte sabemos que algunos estados topológicos no pueden ser obtenidos en tiempo sub-lineal [42], al menos con hipótesis realistas sobre los generadores, aunque sí cumplan con una ley de área. Dado que los estados topológicos se ven como ingredientes cruciales para una memoria cuántica robusta, sería interesante estudiar si:

i. pueden ser preparados con un proceso disipativo de manera que este sea estable;
ii. existen "buenos" procesos disipativos que mantengan el estado, una vez que este haya sido preparado de otra forma.

La diferencia entre los dos casos es que en el segundo no estamos pidiendo que el proceso prepare el estado topológico de manera robusta y rápida al partir de cualquier estado inicial, sino que solamente pedimos que estas propiedades se cumplan cuando el proceso arranca justamente en el estado que queremos preservar. Si el sistema es robusto, entonces todo estado que esté lo suficientemente cerca del estado topológico convergerá hacia él, y toda pequeña perturbación de los generadores solamente cambiará poco este punto fijo. En principio, no haría falta pedir otras propiedades fuera de estas: podría haber otros punto fijos, estables o también inestables.

En esta línea de investigación, debemos mencionar los trabajos siguientes: la propuesta de un proceso de "codificación" [20], que no solamente prepara un estado fundamental del Toric Code en 2D, sino que también permite codificar información lógica, en el sentido de

que un par específico de qubits del sistema será copiado en los qubits virtuales del código. El proceso requiere tiempo lineal, y no es invariante por traslaciones. Cuál es el efecto del ruido en este modelo es todavía un problema abierto.

En [36] se estudian dos familias de modelos Lindbladianos derivados de hamiltonianos conmutativos, de lo cuales preparan el estado de Gibbs. Se demuestra que para temperaturas suficientemente altas, ambos procesos tienen un gap espectral y que por lo tanto tienen un tiempo de equilibración polinomial. En el caso clásico correspondiente es posible demostrar que además cumplen la condición más fuerte de la desigualdad de log-Sobolev y equilibración rápida, por lo tanto es posible que también el resultado cuántico se pueda mejorar. Nótese que esto no contradiría el resultado sobre la preparación de modelos topológicos mencionado antes: dado que estamos considerando el estado de Gibbs a temperatura finita, y sabemos que para el Toric Code en 2D no hay propiedades topológicas en este caso, no hay nada en principio en contra de que puedan existir procesos disipativos con equilibración rápida que preparen el estado de Gibbs del Toric Code en 2D.

Sabemos que al mandar la temperatura a cero, el estado de Gibbs converge al estado fundamental, y bajo alguna hipótesis sobre la densidad de estados en los distintos niveles de energía, para tener una aproximación buena es suficiente que la temperatura escale como el inverso del logaritmo del número de partículas [33]. Por lo tanto sería interesante estudiar el comportamiento de los modelos considerados en [36] cuando la temperatura tiende a cero, para el caso específico del hamiltoniano del Toric Code. No está claro si la aplicación resultante tendrá algún punto fijo distinto del subespacio fundamental, ni si tendrá un gap espectral. En [84] se considera un problema similar de preparar un estado de Gibbs para un hamiltoniano con temperatura crítica, y se demuestra que en el régimen de coexistencia de fases no puede haber un único punto fijo. Esto no resuelve el problema del Toric Code, dado que este último no tiene temperatura crítica, pero es ciertamente una comparación interesante.

4.3.3 Demostrar equilibración rápida

En esta tesis hemos demostrado cómo la equilibración rápida implica propiedades del punto fijo de la evolución, y hemos presentados la desigualdad de log-Sobolev como una manera de probar esta condición en el caso de Lindbladianos reversibles. Sería extremadamente útil tener herramientas y condiciones que nos permitan probar la condición de equilibración rápida, a través de la desigualdad de log-Sobolev o no. El trabajo hecho en [36] puede también verse como ir en esta dirección: determinar condiciones sobre el punto fijo que implican cotas en el tiempo de equilibración. La inspiración viene de los resultados obtenidos para modelos clásicos (véase [53, 54] para una reseña), para los cuales se ha probado que una condición sobre el decaimiento de correlaciones del punto fijo de una dinámica de Glauber implica una desigualdad de log-Sobolev y a su vez esta implica equilibración rápida. Además, este tipo de decaimiento de correlaciones suele estar presente en estados de Gibbs a temperatura alta.

En [36] los autores elegieron un enfoque parecido, y una definición especifica de decaimiento de correlaciones fue usada para probar que la dinámica correspondiente tiene un gap espectral. Igualmente demostraron que esta condición se cumple para temperaturas

altas, y además en el caso en 1D. Sería muy interesante ver si se puede mejorar el resultado y probar una desigualdad de log-Sobolev, lo que correspondería al resultado clásico.

Los resultados presentado en el apartado 4.1.3.3 muestran que al generalizar la desigualdad de log-Sobolev del caso clásico al caso cuántico, no obtenemos una única desigualdad, sino una familia (indexada por $p \in [1,\infty)$) de desigualdades. Clásicamente son todas equivalentes, pero en el caso cuántico no se sabe, y por lo tanto varios autores han tenido que asumir la regularidad L_p: la hipótesis que la constante de L_2 sea una cota inferior de todas las otras. Para recuperar la conexión con la hipercontractividad se necesita efectivamente la familia entera de desigualdades, pero como hemos mostrado en el apartado 4.1.3.2 es sólo el caso L_1 el que interesa para demostrar equilibración rápida.

No sabemos cómo de genérica es la condición de regularidad L_p: el único resultado general que tenemos es que clases importantes de Lindbladianos, como los generadores de Davies [39] y los procesos que preservan la unidad [66] sí la verifican. En [39] se ha conjeturado que cada Lindbladiano reversible sea L_p regular, y que cada Lindbladiano primitivo verifique una versión débil de dicha condición. Si la conjetura es falsa, entonces uno podría restringir su estudio a la desigualdad de L_1, dado que es la que nos provee la cota en el tiempo de equilibración. Mostrar qué condiciones tienen que imponerse en el punto fijo para probar esa desigualdad de log-Sobolev nos ayudará a entender si la clase de sistemas con equilibración rápida es "pequeña" o "grande".

4.3.4 Otros trabajos en distintas lineas de investigación

Otra línea de investigación que ha sido desarrollada a lo largo del Doctorado, aparte del estudio de las dinámicas disipativas abiertas, ha sido el estudio de qué propiedades del límite termodinámico de una sucesión de hamiltonianos se pueden inferir del estudio de algunos casos finitos. En particular, nos hemos interesado en posibles problemas que puedan surgir al estudiar, como es costumbre, el límite a través de una secuencia de casos finitos, de manera numérica o experimental, para luego extrapolar información sobre el límite. En un número de casos muy relevantes esta manera de afrontar el problema ha tenido éxito [6, 45, 52, 68, 79] y ha proporcionado importante información sobre propiedades del modelo físico en la escala de sistemas grandes. Por otra parte, se ha probado un resultado general negativo: el problema de decidir si una sucesión de hamiltonianos locales invariantes por traslaciones en 2D tiene un gap espectral en el límite es un problema indecidible [15]. Esto implica que estos modelos pueden exhibir comportamientos impredecibles, y nos ha llevado a explorar las posibilidades de construir ejemplos exóticos.

El resultado de esta investigación ha sido presentado en un artículo (todavía no publicado) [8], en el cual presentamos dos familias de modelos que presentan una propiedad sorprendente: para toda región finita más pequeña que un umbral dado, el estado fundamental y los estados de bajas energías son estados clásicos (estados productos en la base computacional); por encima del umbral estos muestran por el contrario propiedades topológicas, que son características de ciertos modelos cuánticos. Si consideramos espines con dimensión local más grande, podemos hacer crecer el umbral de manera espectacular, y ya sólo con considerar dimensión local $d = 10$ este se vuelve más grande que el número estimado de partículas en el universo. Hemos llamado a este fenómeno **transición de fase**

inducida por el tamaño, dado que puede considerarse como un cambio brusco entre un modelo clásico y uno cuántico al incrementar el parámetro del tamaño del sistema.

Las dos construcciones están basadas en ideas distintas, y tienen umbrales distintos. Ambas se basan en **problemas de teselación**: una teselación es un recubrimiento de una región del plano con cuadrados de lado uno y bordes coloreados, de manera que los colores de los bordes de cuadrados adyacentes se correspondan. Se ha demostrado que el problema de decidir si, dado un conjunto finito de teselas, es posible teselar el plano entero es un problema indecidible [9, 73]. Este resultado está en la base del resultado de indecidiblidad del gap de [15]. Hemos modificado su construcción con el uso de interacciones de plaquetas y estrellas (en vez de usar simplemente plaquetas como en la construcción original), y hemos construido dos familias de modelos. La primera se basa en la idea de construir patrones periódicos de periodo muy grande (comparado con el número de colores usados en la teselación), de manera que un patrón específico sólo ocurra una vez cada periodo. Al penalizar tal patrón podemos inducir una frustración de la energía para todo retículo más grande que el periodo: esto nos permite implementar la transición entre el modelo clásico y el cuántico.

La otra construcción se basa en la idea, ya presente en los resultados previos de indecidibilidad, de codificar la historia de una máquina de Turing en el estado fundamental del hamiltoniano. De esta manera podemos dar una penalización energética si la máquina termina su computación, y de la misma manera que en el caso anterior introducir una frustración de la energía si el sistema es suficientemente grande para que la máquina termine. Dado que el problema de determinar si (y cuándo) una máquina de Turing termina es un problema indecidible, este ha sido el ingrediente fundamental para probar la indecidibilidad del gap espectral. Hemos sido capaces de optimizar mucho el coste de la codificación, en el sentido de necesitar un espacio de Hilbert local de dimensión mucho más pequeña para inscribir la historia de una máquina de Turing en el modelo de espines. Con este código optimizado, hemos considerados las llamadas máquinas *Busy Beavers*: una máquina de dimensión muy pequeña, pero cuyo tiempo de terminación es increíblemente largo - de hecho crece más rápidamente que cualquier función computable. De esta manera hemos obtenido modelos que tienen una dimensión local relativamente pequeña, pero para los cuales hay una frustración en la energía sólo para sistemas extremamente grandes. Nuevamente, esta frustración nos permite implementar la transición de fase.

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CHAPTER 5

Stability of Local Quantum Dissipative Systems

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Stability of Local Quantum Dissipative Systems

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Abstract: Open quantum systems weakly coupled to the environment are modeled by completely positive, trace preserving semigroups of linear maps. The generators of such evolutions are called Lindbladians. In the setting of quantum many-body systems on a lattice it is natural to consider Lindbladians that decompose into a sum of local interactions with decreasing strength with respect to the size of their support. For both practical and theoretical reasons, it is crucial to estimate the impact that perturbations in the generating Lindbladian, arising as noise or errors, can have on the evolution. These local perturbations are potentially unbounded, but constrained to respect the underlying lattice structure. We show that even for polynomially decaying errors in the Lindbladian, local observables and correlation functions are stable if the unperturbed Lindbladian has a unique fixed point and a mixing time that scales logarithmically with the system size. The proof relies on Lieb–Robinson bounds, which describe a finite group velocity for propagation of information in local systems. As a main example, we prove that classical Glauber dynamics is stable under local perturbations, including perturbations in the transition rates, which may not preserve *detailed balance*.

Contents

1. Background and Previous Work

The physical properties of a closed many-body quantum system are encoded in its Hamiltonian. Theoretical models of such systems typically assume some form of local structure, whereby the Hamiltonian decomposes into a sum over interactions between subsets of nearby particles. Similarly, the behavior of an open many-body quantum system is encoded in its Liouvillian. Again, this is typically assumed to have a local structure, decomposing into a sum over local Liouvillians acting on subsets of nearby particles.

Crucial to justifying such theoretical models is the question of whether their physical properties are stable under small perturbations to the local interactions. If the physical properties of a many-body Hamiltonian or Liouvillian depend sensitively on the precise mathematical form of those local terms, then it is difficult to conclude anything about physical systems, whose interactions will always deviate somewhat from theory.

Quantum information theory has motivated another perspective on many-body Hamiltonians. Rather than studying models of naturally occurring systems, it studies how many-body systems can be engineered to produce desirable behavior, such as long-term storage of information in quantum memories [12,19,20,48], processing of quantum information for quantum computing [11,13,14,34,51], or simulation of other quantum systems that are computationally intractable by classical means [3,7,8,28,30]. Again, stability of these systems under local perturbations is crucial, otherwise even tiny imperfections may destroy the desired properties. Stability in this context has been studied for self-correcting topological quantum memories, where one in addition requires robustness against local sources of *dissipative* noise, and the relevant quantity is the minimum time needed to introduce logical errors in the system. It has been known since [1,12] that a self-correcting quantum memory with local interactions is possible in four spatial dimensions. With the breakthrough of the Haah code [19], it seems it may be possible to engineer such self-correcting quantum memories in three dimensions.

Recently, and partially motivated by the dissipative nature of noise, this "engineering" approach has been extended to open quantum systems and many-body Liouvillians. First theoretical [38,61], and then experimental [5,39] work has shown that creating many-body quantum states as fixed points of engineered, dissipative Markovian evolutions can be more robust against undesirable errors and maintain coherence of quantum information for longer times. Intuitively, there is an inherent robustness in such models: the target state is independent of the initial state. If the dissipation is engineered perfectly, the system will always be driven back towards the desired state. This idea can be used to engineer dissipative systems both for carrying out computation via dissipative dynamics [61] and for storing quantum information [53]. However, it does not guarantee stability against errors *in the engineered Liouvillian itself*. Once again, stability against local perturbations—this time for many-body Liouvillians rather than Hamiltonians—is of crucial importance. Indeed, in [53] the authors consider depolarizing noise acting on top of a dissipative interaction engineered for protecting a logical qubit encoded in the system, and provide positive numerical evidence for stability of a specific 4D model.

In the case of closed systems governed by Hamiltonians, recent breakthroughs have given rigorous mathematical justification to our intuition that the physical properties of many-body Hamiltonians are stable. Starting with [10,35], it culminated in the work of [49] which showed that, under a set of mathematically well-defined and physically reasonable conditions, gapped many-body Hamiltonians are stable under perturbations

to the local interactions.[1] More precisely, in the presence of *frustration-freeness*, *local topological quantum order* (LTQO), and *local gap*, the spectral gap of a Hamiltonian with local (or quasi-local) interactions is stable against small (quasi-) local perturbations (see [49] for a formal definition of these conditions). The bound on the amount of imperfection tolerated by the system depends on the decay of the local gaps, the decay of the local topological order, and the strength (and decay rate) of the interactions. Furthermore, except for frustration freeness which is a technical condition required in the proof, these conditions are in a sense tight. There exist simple counterexamples to stability if any one of the conditions is lifted.

2. Stability of Open Quantum Systems

In this work, we study stability of many-body Liouvillians. We consider dynamics generated by rapidly decaying interactions, where the notion of rapid decay is made precise in Sect. 3. Moreover, in order to have a well-defined notion of scaling with system size, we restrict to Liouvillians whose local terms depend only on the subsystem on which they act, and thus are not redefined as we consider larger systems. We call such families of Liouvillians *uniform*.

Our main result shows that, under the above assumptions on the structure of the Liouvillian, logarithmic mixing time implies the desired stability in the dissipative setting.

However, although the result is analogous to [49], the proof and even the definition of stability in the case of Liouvillians necessarily differ substantially from the Hamiltonian case. For Hamiltonians, the relevant issue is stability of the spectral gap. Via the quasi-adiabatic technique [21,25], this in turn implies a smooth transition between the initial and perturbed ground states, showing that both are within the same phase. Note that the existence of a smooth transition (no closing of the spectral gap in the thermodynamic limit) does not imply that both ground states are close in norm, as the simple example $H = \sum_{i=1}^{N} |0\rangle\langle 0|_i$ vs. $H(\varepsilon) = \sum_{i=1}^{N} (|0\rangle + \varepsilon\,|1\rangle)(\langle 0| + \varepsilon\,\langle 1|)_i/(1+\varepsilon^2)$ shows.[2] It does however imply a well-behaved perturbation in the expectation value of local observables—such as order parameters—and correlation functions, which in most experimental situations are the only measurable quantities.

For Liouvillians, we are interested in a definition of stability more related to the evolution itself, which accounts at the same time for both the speed of convergence and the properties of the fixed point. Here, we consider the strongest definition of stability: we want our systems (initial and perturbed) to evolve similarly for all times and all possible initial states. Thus, not only should the speed of convergence to the fixed points be similar, the fixed points themselves should be close and so should the approach to the fixed points. Such "dynamical" definition of stability is radically different from other notions of stability previously considered in the literature, such as local thermodynamical stability [2,56]. While there are cases in which both concepts are of interest, they are generally independent concepts.

[1] Note that, in stark contrast to traditional perturbation theory, the perturbations considered here simultaneously change *all* the local interactions by a small amount. The strength of the total perturbation therefore scales with system size and standard perturbation theory does not apply. It is the structure of local ground states of the Hamiltonian that ensures stability.

[2] Note that each Hamiltonian is the sum of non-interacting projections for any $\varepsilon \in \mathbb{R}$. In particular, for each ε, there is a unitary $U(\varepsilon)$ acting on a single site, such that $H(\varepsilon) = U(\varepsilon)^{\otimes N} H U^{\dagger}(\varepsilon)^{\otimes N}$.

Requiring stability of the evolution is significantly stronger than stability of the spectral gap alone,[3] and is more directly relevant to the applications discussed above. As in the Hamiltonian case, the analogous simple example shows that one cannot expect to attain such stability if we consider global measurements on the system. Therefore, in analogy with the Hamiltonian case, restrict our attention to local observables and few-body correlation functions. Since there are technical subtleties involved in extending this stronger definition of stability to dynamics with multiple fixed points, we defer consideration of multiple fixed points to a future paper, and restrict our attention here to dissipative dynamics with unique fixed points. It is important to note, however, that we do not make *any* assumption on the form of the unique fixed point. In particular, we do not assume that it is full-rank (primitivity); our results apply equally well to Liouvillians with pure fixed points. (Pure-state fixed points are particularly relevant to quantum information applications, such as dissipative state engineering and computation.)

A key technical ingredient in the stability proof for Hamiltonians is the quasi-adiabatic evolution technique [21,25], which directly uses the fact that Hamiltonian evolution is reversible. This is of course no longer true for Liouvillians, so we must use a different proof approach. We make use of the fact that evolution under a Liouvillian converges to a steady-state, together with dissipative generalizations [50] of the Lieb–Robinson bounds that are the other crucial ingredient in [49].

Among systems which satisfy our assumptions, one finds classical Glauber dynamics [46]. This immediately shows that Glauber dynamics is stable against errors. To the best of our knowledge, this is new even to the classical literature (related results, but with different assumptions, were given in [27]). Given the importance of Glauber dynamics to sampling from the thermal distributions of classical spin systems [41,46], we expect our results to have applications also to classical statistical mechanics.

The paper is structured as follows: after setting up notation and basic definitions in the next section, we state our main stability result in Sect. 4 and discuss the assumptions it requires. In Sect. 5 we prove various technical results used in the main proof, which is given in Sect. 6. We apply these results in Sect. 7 to the important example of classical Glauber dynamics, before concluding with a discussion of the results and related open questions in Sect. 8.

3. Setup and Notation

We will consider a cubic lattice[4] $\Gamma = \mathbb{Z}^D$. The ball centered at $x \in \Lambda$ of radius r will be denoted by $b_x(r)$. At each site x of the lattice we will associate one elementary quantum system with a finite dimensional Hilbert space $\mathcal{H}_x$. We will use the Dirac notation for vectors: $|\phi\rangle$ will denote a vector in $\mathcal{H}_x$, $\langle\phi|$ its adjoint, and $\{|n\rangle\}_{n=0}^{\dim \mathcal{H}_x}$ the canonical basis for $\mathcal{H}_x$. Scalar product in $\mathcal{H}_x$ will be denoted by $\langle\phi|\psi\rangle$, and rank-one linear maps by $|\phi\rangle\langle\psi|$. For each finite subset $\Lambda \subseteq \Gamma$, the associated Hilbert space is given by

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x, \tag{1}$$

[3] Due to the recent work in [57], it is not clear whether the spectral gap in Liouvillians is the relevant quantity for convergence questions.

[4] We restrict to cubic lattices for the sake of exposition. The results can be extended to more general settings, replacing the lattice $\mathbb{Z}^D$ with a graph with polynomial growth.

and the algebra of observables supported on Λ is defined by

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x).$$

If $\Lambda_1 \subset \Lambda_2$, there is a natural inclusion of $\mathcal{A}_{\Lambda_1}$ in $\mathcal{A}_{\Lambda_2}$ by identifying it with $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}$. The support of an observable $O \in \mathcal{A}_\Lambda$ is the minimal set Λ' such that $O = O' \otimes \mathbb{1}$, for some $O' \in \mathcal{A}_{\Lambda'}$, and will be denoted by $\operatorname{supp} O$. We will denote by $\|\cdot\|_p$ the Schatten p-norm over $\mathcal{A}_\Lambda$. Where there is no risk of ambiguity, $\|\cdot\|$ will denote the usual operator norm (i.e. the Schatten ∞-norm).

A linear map $\mathcal{T}\colon \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ will be called a *superoperator* to distinguish it from operators acting on states. The support of a superoperator $\mathcal{T}$ is the minimal set $\Lambda' \subseteq \Lambda$ such that $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$, where $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_{\Lambda'})$. A superoperator is said to be Hermiticity preserving if it maps Hermitian operators to Hermitian operators. It is said to be positive if it maps positive operators (i.e. operators of the form O^*O) to positive operators. $\mathcal{T}$ is called *completely positive* if $\mathcal{T} \otimes \mathbb{1}\colon \mathcal{A}_\Lambda \otimes M_n \to \mathcal{A}_\Lambda \otimes M_n$ is positive for all $n \geqslant 1$. Finally, we say that $\mathcal{T}$ is trace preserving if $\operatorname{tr} \mathcal{T}(\rho) = \operatorname{tr} \rho$ for all $\rho \in A_\Lambda$. For a general review on superoperators, see [62].

The dynamics of the system is generated by a superoperator $\mathcal{L}$, which plays a similar role to the Hamiltonian in the non-dissipative case. The evolution will be given by the one parameter semigroup $T_t = e^{t\mathcal{L}}$. The natural assumptions to make about T_t are that it is a continuous semigroup of completely positive and trace preserving maps (CPTP, sometimes also called *quantum channels*). Such maps are always contractive, meaning that $\|T_t\|_{1\to 1,cb} \leqslant 1$, where the completely-bounded norm is defined as:

$$\|T\|_{1\to 1,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{1\to 1} = \sup_n \sup_{\substack{X\in\mathcal{A}_\Lambda\otimes M_n \\ X\neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_1}{\|X\|_1}. \tag{2}$$

We will also be interested in the $\|\cdot\|_{\infty\to\infty,cb}$ completely-bounded norm of superoperators, which is defined as follows:

$$\|T\|_{\infty\to\infty,cb} = \sup_n \|T \otimes \mathbb{1}_n\|_{\infty\to\infty} = \sup_n \sup_{\substack{X\in\mathcal{A}_\Lambda\otimes M_n \\ X\neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_\infty}{\|X\|_\infty}.$$

The relationship between $\|\cdot\|_{1\to 1,cb}$ and $\|\cdot\|_{\infty\to\infty,cb}$ is the following:

$$\|T\|_{1\to 1,cb} = \left\|T^*\right\|_{\infty\to\infty,cb},$$

where T^* is the dual of T, satisfying $\operatorname{tr} A\, T(B) = \operatorname{tr} T^*(A)\, B$. We will denote $\|\cdot\|_{\infty\to\infty,cb}$ simply by $\|\cdot\|_{cb}$ when there is no risk of confusing different completely-bounded norms.

Remark 1. As shown in [29], the supremum in Eq. (2) is reached when n is equal to the dimension of the space on which T is acting: if $T\colon \mathcal{M}_n \to \mathcal{M}_n$, then $\|T \otimes \mathbb{1}_n\|_{1\to 1} = \|T\|_{1\to 1,cb}$.

The generator $\mathcal{L}$ of the semigroup $T_t = e^{t\mathcal{L}}$, is called a *Liouvillian*. All such generators can be written in the following general form, often called the *Lindblad form* [15,42] (see [62]):

Proposition 1. *$\mathcal{L}$ generates a continuous semigroup of CPTP maps if and only if it can be written in the form*:

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^* - \frac{1}{2}\sum_j \{L_j^* L_j, \rho\}, \tag{3}$$

where H is a Hermitian matrix, $\{L_j\}_j$ a set of matrices called the Lindblad operators, $[\cdot,\cdot]$ denotes the commutator and $\{\cdot,\cdot\}$ the anticommutator.

We will use the term *Lindbladian* and Liouvillian interchangeably. Since we consider Lindbladians $\mathcal{L}$ corresponding to local dissipative dynamics, we assume that $\mathcal{L}$ is a *local Lindbladian* of the form:

$$\mathcal{L} = \sum_{u\in\Lambda}\sum_{r\geqslant 0} \mathcal{L}_{u,r}, \quad \operatorname{supp}\mathcal{L}_{u,r} = b_u(r), \tag{4}$$

where each term in the sum above can be written in the form given by Eq. (3).

Such a decomposition is obviously always trivially possible. We are interested in the cases in which the norms of $\mathcal{L}_{u,r}$ decay with r. Concretely, let us define the *strength of interaction* for a Lindbladian as the pair (J, f) given by:

$$J = \sup_{u,r} \left\|\mathcal{L}_{u,r}\right\|_{1\to 1,cb}, \quad f(r) = \sup_u \frac{\left\|\mathcal{L}_{u,r}\right\|_{1\to 1,cb}}{J}. \tag{5}$$

The behavior of $f(r)$ as r goes to infinity corresponds to various interaction regimes, listed in order of decreasing decay rate:

- finite range interaction: $f(r)$ is compactly supported;
- exponentially decaying: $f(r) \leqslant e^{-\mu r}$, for some $\mu > 0$;
- quasi-local interaction: $f(r)$ decays faster than any polynomial;
- power-law decay: $f(r) \leqslant (1+r)^{-\alpha}$, for some positive $\alpha > 0$.

As we will see later, our result will apply whenever $\mathcal{L}$ has finite range, exponentially decaying, or quasi-local interactions. It will also hold in the power-law decay regime, but we will require a lower bound on the decay exponent α, depending on the dimension of the underlying lattice. Not to overload the exposition, we will assume that $\mathcal{L}$ has finite range or exponentially decaying interactions, unless otherwise specified. The modifications needed to work with quasi-local interactions and power-law decay are presented in Sect. 6.4. Also, we will say that functions we construct along the way are *fast-decaying*, if their decay rate is within the same decay class of $f(r)$ we are considering (or faster).

As shown in [63], from the spectral decomposition of $\mathcal{L}$ (and T_t) one can define two new CPTP maps which represent the infinite-time limit of the semigroup T_t. We will denote by T_∞ the projector onto the subspace of stationary states (fixed points), and by T_ϕ the projector onto the subspace of periodic states. They correspond, respectively, to the kernel of $\mathcal{L}$ and to the eigenspace of purely imaginary eigenvalues of $\mathcal{L}$, which we denote $\mathcal{F}_\mathcal{L}$ and $\mathcal{X}_\mathcal{L}$, respectively. Both subspaces are invariant under T_t: in particular, T_t acts as the identity over $\mathcal{F}_\mathcal{L}$, while it is a unitary operator over $\mathcal{X}_\mathcal{L}$. Note, also, that both subspaces are spanned by positive operators (i.e. density matrices) [62, Props. 6.8, 6.12]. We will denote by $T_{\phi,t}$ the composition $T_t \circ T_\phi$.

Since we plan to exploit the local structure of $\mathcal{L}$, we will often make use of the restriction of $\mathcal{L}$ to a subset of the lattice. Given $A \subset \Lambda$, we define the *truncated*, or *localized*, generator:

$$\mathcal{L}_A = \sum_{b_u(r)\subseteq A} \mathcal{L}_{u,r}. \tag{6}$$

3.1. Uniform families. We are interested in how properties of dissipative dynamics scale with the size of the system. Hence, we are concerned with sequences of Lindbladians defined on lattices of increasing size, where all the Lindbladians in the sequence are from the same "family". To make this precise, we need to pin down how Lindbladians from the same family, but on different size lattices, are related to one-another. Our results will apply to very general sequences of Lindbladians, which we call *uniform families*. Before giving the precise definition, it is helpful to consider some special cases.

For local Hamiltonians on a lattice, one often considers translationally-invariant systems with various types of boundary conditions (e.g. open or periodic boundaries). There is then a natural definition of what it means to consider the same translationally-invariant Hamiltonian on different lattice sizes. Translationally-invariant Lindbladians are an important special case of a uniform family. In this special case, all the local terms in the Lindbladian that act in the "bulk" of the lattice are the same. Another way of thinking about this is to formally consider the translationally-invariant Lindbladian $\mathcal{M}$ defined on the infinite lattice $\Gamma = \mathbb{Z}^d$, and then consider each member of the family to be a restriction of this infinite Lindbladian to a finite sub-lattice $\Lambda \subset \Gamma$ of some particular size:

$$\mathcal{L} = \mathcal{M}_\Lambda.$$

This gives us translationally-invariant Lindbladians with *open boundary condition*. But of course, this is only one particular choice of boundary terms (in this case, no boundary terms at all). We are also interested in more general boundary conditions, such as periodic boundaries. So, in addition to the "bulk" interactions coming from $\mathcal{M}$, we allow additional terms that play the role of *boundary conditions*:

$$\mathcal{L} = \mathcal{M}_\Lambda + \mathcal{L}^{\partial\Lambda}.$$

We allow greater freedom in the boundary terms $\mathcal{L}^{\partial\Lambda}$. For one thing, they are allowed to depend on the size of the lattice Λ. But more importantly, we allow strong interactions that *cross the boundary* of Λ, coupling sites that would otherwise be far apart. For example, the case of *periodic boundary conditions* corresponds to adding interaction terms that connect opposite boundaries of Λ, as if on a torus (see Fig. 1).

Now that we have given an intuition of what a uniform family is, it is time to present the formal definition. This includes all the special cases discussed so far, but also captures much more general families of Lindbladians that are not necessarily translationally-invariant, and many other types of boundary conditions (e.g. cylindrical boundaries, or

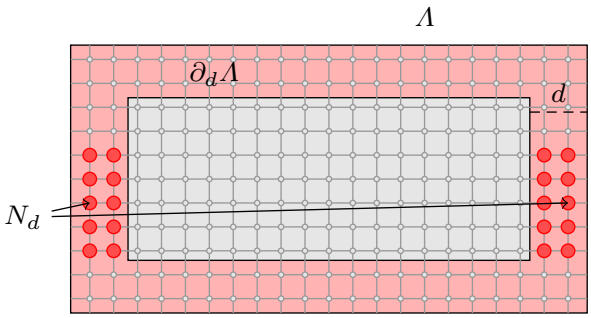

Fig. 1. Partition of the lattice Λ into the bulk and the boundary of thickness d, $\partial_d\Lambda$ (see Definition 2). The *dark red* regions on the boundary correspond to the interaction term N_d coupling distant regions in Λ (color figure online)

boundary terms that give the sphere topology, or terms that force fixed states on the boundary[5]).

Definition 2. Given $\Lambda \subset \Gamma$, a *boundary condition* with strength (J, f) for Λ is a Lindbladian $\mathcal{L}^{\partial\Lambda} = \sum_{d\geqslant 1} N_d$, where

$$\|N_d\|_{1\to 1,c.b.} \leqslant J\,|\partial_d\Lambda|\,f(d)$$

with

$$\partial_d\Lambda := \{x \in \Lambda |\, \mathrm{dist}(x, \Lambda^c) \leqslant d\}, \quad \mathrm{supp}\, N_d \subset \partial_d\Lambda.$$

Definition 3. A *uniform family* of Lindbladians $\mathcal{L}$ with strength (J, f) is given by the following:

(i) *infinite Lindbladian*: a Lindbladian $\mathcal{M}$ on all of $\mathbb{Z}^D$ with strength (J, f);
(ii) *boundary conditions*: a set of *boundary conditions* $\mathcal{L}^{\partial\Lambda}$, with strength (J, f) and $\Lambda = b_u(L)$, for each $u \in \mathbb{Z}^D$ and $L \geqslant 0$.

We say that the family is *translationally invariant* if $\mathcal{M}$ is translationally invariant and $\mathcal{L}^{\partial b_u(L)}$ is independent of u.

Given a uniform family $\mathcal{L}$, we fix the following notation for evolutions defined on a subset Λ:

$$\mathcal{L}^{\Lambda} = \mathcal{M}_{\Lambda} \quad \text{“open boundary” evolution;} \tag{7}$$

$$\mathcal{L}^{\overline{\Lambda}} = \mathcal{M}_{\Lambda} + \mathcal{L}^{\partial\Lambda} \quad \text{“closed boundary” evolution,} \tag{8}$$

with the respective evolutions $T_t^{\Lambda} = \exp(t\mathcal{L}^{\Lambda})$ and $T_t^{\overline{\Lambda}} = \exp(t\mathcal{L}^{\overline{\Lambda}})$.

Remark 2. In the rest of the paper, we will make use of the following notation:

$$A(s) = \{x \in \Lambda |\, \mathrm{dist}(x, A) \leqslant s\}.$$

Since we are interested in observables whose support is not connected, we want to consider more general regions than balls: in particular, we are interested in disjoint unions of convex regions (for example, to calculate two-point correlation functions). Consider what happens to such a region $A = A_0 \sqcup A_1$ when we grow it by taking $A(s)$. When s becomes sufficiently large, $A_0(s)$ will merge with $A_1(s)$. At this point, $A(s)$ will not be a disjoint union of balls anymore. To avoid such complications, for s large enough that disjoint balls merge, we will replace $A(s)$ by the smallest ball containing it. This will not hurt us, as $|A(s)|$ will still grow asymptotically at the same rate, which will be sufficient for our purposes (see Fig. 2).

Definition 4. We say that $\mathcal{L}$ has a unique fixed point if, for all $\Lambda = b_u(L)$, $\mathcal{X}_{\mathcal{L}^{\overline{\Lambda}}} = \mathcal{F}_{\mathcal{L}^{\overline{\Lambda}}} = \{\rho_{\infty}^{\overline{\Lambda}}\}$. In other words, $T_{\phi}^{\overline{\Lambda}}(\rho) = T_{\infty}^{\overline{\Lambda}}(\rho) = \rho_{\infty}^{\overline{\Lambda}}$, for all density matrices ρ.

Note that if for all pure states ρ, we have $T_t^{\overline{\Lambda}}(\rho) > 0$ (positive definite), for $t > 0$, then the evolution has a unique fixed point $\rho_{\infty} > 0$ (see [62, Thm. 6.7]).

We will drop the superscript from $T_t^{\overline{\Lambda}}$, and simply write T_t, when we consider some fixed $\Lambda \subset \Gamma$. In that case, we will refer to the number of lattice sites in Λ as the *system size*.

[5] Or even Möbius strips, Klein bottles, and other exotic topologies.

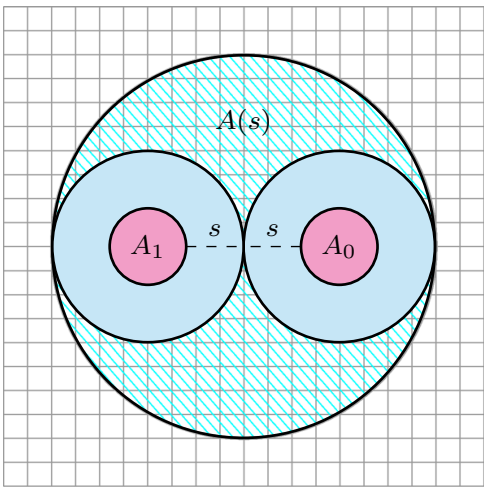

Fig. 2. The convention on how to grow a region $A = A_0 \sqcup A_1$

4. Main Result

4.1. Assumptions for stability. In Hamiltonian systems, the spectral gap (the difference between the two lowest energy levels) plays a crucial role in a number of settings, from defining quantum phases and phase transitions [55] to understanding the entanglement and correlations present in the system [22–24] and analyzing its stability to perturbations [10,49]. On the other hand, it is known that for Lindbladians, the spectral gap (in this setting, the least negative real part of the non-zero eigenvalues) alone is not sufficient to fully characterize the convergence properties of the dissipative evolution [31,57]. Therefore, we will instead impose a more general requirement on the convergence of the dynamics. (The dependence of this requirement on spectral properties of $\mathcal{L}$, i.e. properties depending on the eigenvalues—like the gap—and eigenvectors—like the condition number, is an active area of research.)

Definition 5 (*Rapid mixing*). Given a one-parameter semigroup of CPTP maps T_t, define the *contraction* of T_t as the following quantity:

$$\eta(T_t) = \frac{1}{2} \sup_{\substack{\rho \geqslant 0 \\ \operatorname{tr} \rho = 1}} \left\| T_t(\rho) - T_{\phi,t}(\rho) \right\|_1 . \tag{9}$$

Given a family of such semigroups $\{T_t^\alpha\}_\alpha$, each of which is acting on $\mathcal{B}(H_\alpha)$ for some Hilbert space H_α of finite dimension d_α, we say that it satisfies *rapid mixing* if there exist constants $c, \gamma, \delta > 0$, such that for each α:

$$\eta(T_t^\alpha) \leqslant c \log^\delta(d_\alpha)\, e^{-t\gamma} . \tag{10}$$

We will write RM(γ, δ) for short.

If each H_α has a tensor product structure of the type defined in Eq. (1), then the rapid mixing assumption can be restated as a logarithmic scaling with system size of the mixing time. Since the dimension of $\mathcal{H}_\Lambda$ is $(\dim \mathcal{H}_x)^{|\Lambda|}$, for uniform families condition (10) is equivalent to:

$$\eta(T_t^{\overline{\Lambda}}) \leqslant c\, |\Lambda|^\delta\, e^{-t\gamma} \quad \forall \Lambda . \tag{11}$$

Let us recall a result from [31].

Theorem 6 (Contraction for commuting Lindbladians). *Let $\{\mathcal{L}_j\}_{j=0}^n$ be a set of commuting Lindbladians. Define $\mathcal{L} = \sum_j \mathcal{L}_j$ and the corresponding evolutions $T_t^j = e^{t\mathcal{L}_j}$ and $T_t = e^{t\mathcal{L}}$. Then:*

$$\eta(T_t) \leqslant \sum_j \eta(T_t^j). \tag{12}$$

In particular, consider the definition of $T_t^{\overline{\Lambda}}$ given in Remark 2 for $\Lambda \subset \Gamma$ being a disjoint union of balls. Then the previous theorem implies that, if $\mathcal{L}$ is translationally-invariant and it satisfies Eq. (10) for each of the connected components of Λ, then it also satisfies the same equation (up to constants) for Λ.

Finally, for translationally-invariant uniform families of Lindbladians, it is sufficient to satisfy Eq. (10) for lattices centered at the origin: $\Lambda = b_0(L)$, $L \geqslant 1$.

4.2. Stability. With the required assumptions laid out, we can now state our main result.

Theorem 7. *Let $\mathcal{L}$ be a uniform family of local Lindbladians with a unique fixed point, satisfying rapid mixing* [*Eq.* (11)]*, and consider a perturbation of the form:*

$$E^{\overline{\Lambda}} = \sum_{u\in\Lambda}\sum_{r\geqslant 0} E_{u,r} + \sum_{d\geqslant 1} E_d,$$

where $E_{u,r}$ is supported on $b_u(r)$ and each E_d is supported on $\partial_d\Lambda$ (see Definition 2) and

$$\left\| E_{u,r} \right\|_{1\to 1,cb} \leqslant \varepsilon\, e(r), \quad \|E_d\|_{1\to 1,cb} \leqslant \varepsilon\, |\partial_d\Lambda|\, e(d),$$

where $\varepsilon > 0$ is a constant (the strength of the perturbation) and $e(r)$ is a fast-decaying function. Consider the perturbed evolution

$$S_t = \exp t(\mathcal{L}^{\overline{\Lambda}} + E^{\overline{\Lambda}})$$

and suppose that the following assumptions hold:

(i) $E_{u,r}^*(\mathbb{1}) = E_d^*(\mathbb{1}) = 0$ [*or, equivalently:* $\operatorname{tr} E_{u,r}(O_A) = \operatorname{tr} E_d(O_A) = 0$*, for all operators O_A*].
(ii) *S_t is a contraction for each $t \geqslant 0$.*

For an observable O_A supported on $A \subset \Lambda$, we have for all $t \geqslant 0$:

$$\left\| T_t^*(O_A) - S_t^*(O_A) \right\| \leqslant c(|A|)\, \|O_A\| \left(\varepsilon + |\Lambda|\, \nu_\eta^{-1}(d_A)\right), \tag{13}$$

where $d_A = \operatorname{dist}(A, \Lambda^c)$; η is positive and independent of Λ; $\nu_\eta^{-1}(d) \leqslant (1+d)^{-D-1}$; $c(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Remark 3. Note that, for a fixed A, if we let Λ grow then d_A will increase with the linear size of Λ and consequently $|\Lambda|\, \nu_\eta^{-1}(d_A)$ will vanish in the limit.

Remark 4. The assumptions (i)–(ii) on the perturbation E are satisfied whenever $\mathcal{M}_{u,r} + E_{u,r}$ and $N_d + E_d$ (as in Definition 3) are Lindbladians, but the theorem also covers more general perturbations.

Remark 5. Since we are free to choose an O_A with support on two non connected regions, we can apply Theorem 20 to two-point correlation functions (or more generally k-point correlation functions, for fixed k) and still obtain that the error introduced by the perturbation depends linearly on the strength of the perturbation (and not on its global norm).

A set of tools already applied in the setting of classical Markov chains [16–18,46], and recently generalized to quantum dissipative systems [9,32,52], are the so-called *Logarithmic Sobolev inequalities* (in short, log-Sobolev inequalities). Introduced in a different setting to study hypercontractivity of semigroups [33], they provide the right asymptotic regime needed to satisfy the rapid mixing condition: in fact, the existence of a system size independent log-Sobolev constant implies a logarithmic scaling of the mixing time, which is exactly what is required in Definition 5. Without going into the technical details of log-Sobolev inequalities (which can be found in [32]), we summarize this fact in the following corollary:

Corollary 8. *Let $\mathcal{L}$ belong to a uniform family of translationally-invariant Lindbladians with a unique fixed point for each system size. If $\mathcal{L}$ satisfies the log-Sobolev inequality with a system-size independent constant, then the dissipative dynamics are stable, in the sense of Theorem 7.*

In particular, in [60] it was shown that product evolutions, i.e. Lindbladians that can be decomposed as a sum of independent terms $\mathcal{L}_k$ acting on a single subsystem, satisfy a log-Sobolev inequality with a log-Sobolev constant lower bounded by the minimum of the spectral gaps of $\mathcal{L}_k$ (times a factor depending on the maximum dimension of the subsystems). Moreover, the authors of [60] were able to show that Davies maps associated to a graph state Hamiltonian [26] (which are not in a product form, but can be analyzed in a similar way) and the ones associated to free-fermionic Hamiltonians have a system-size independent log-Sobolev constant.

In all such cases, Corollary 8 implies that the evolution of local observables is stable.

4.3. Local observables vs. global observables. The bound in Eq. (13) scales with the size of the support of the observable O_A. Although the dependence is polynomial, for observables with large support the result is not useful. Still, in most realistic experiments, we are interested in the behavior of observables with fixed support and low-degree correlation functions, making the above result widely applicable. Nonetheless, one might ask more generally for a system-size independent bound on:

$$\sup_{\rho} \|T_\infty(\rho) - S_\infty(\rho)\|_1, \tag{14}$$

where S_∞ is the fixed-point projector for the evolution of the perturbed Lindbladian. However, this is not possible; the limitation to local observables is in some sense strict. There is no hope of finding such a bound for global observables, as the following simple example shows.[6]

Example 1. Consider N independent amplitude damping processes, with uniform rate γ (which we can suppose w.l.o.g. equal to 1). This Lindbladian can be written as

$$\mathcal{L}_N = \sum_{k=1}^{N} \mathbb{1}_{1,\ldots,k-1} \otimes \mathcal{L}_1 \otimes \mathbb{1}_{k+1,\ldots,N},$$

[6] Indeed, all global stability results for quantum Linbladians we are aware of have a dependency on the total Hilbert space dimension [58].

where

$$\mathcal{L}_1(\rho) = |0\rangle\langle 1|\,\rho\,|1\rangle\langle 0| - \frac{1}{2}\{\rho, |1\rangle\langle 1|\}$$

is an amplitude damping process on a single qubit, describing the decay of the state $|1\rangle$ into $|0\rangle$ at a constant rate $\gamma = 1$. This Lindabladian has gap $1/2$ and $e^{t\mathcal{L}_N} = (e^{t\mathcal{L}_1})^{\otimes N}$ has mixing time of order $O(\log N)$ [31, Sec. V. C.]. The fixed point is the pure state $|0\ldots0\rangle\langle 0\ldots0|$.

Now consider $\mathcal{L}_1^\varepsilon$, a rotation of $\mathcal{L}_1$, which fixes $|\alpha_0\rangle = \sqrt{1-\varepsilon^2}\,|0\rangle + \varepsilon\,|1\rangle$. We have $\left\|\mathcal{L}_1 - \mathcal{L}_1^\varepsilon\right\|_{1\to 1} = O(\varepsilon)$, but the new fixed point $|\alpha_0\rangle\langle\alpha_0|^{\otimes N}$ is almost orthogonal to the original one, since the overlap between the two is

$$\langle 0\ldots0|\alpha_0\ldots\alpha_0\rangle = \langle 0|\alpha_0\rangle^N = (1-\varepsilon^2)^{N/2} \sim e^{-N\varepsilon^2/2} \to 0 \quad \text{as } N \to \infty.$$

This shows that, in general, there is no good bound on (14) (note that we have $\| \,|0\ldots0\rangle\langle 0\ldots0| - |a_0\ldots\alpha_0\rangle\langle a_0\ldots\alpha_0|\,\|_1 \geqslant 1 - |\,\langle 0\ldots0|\alpha_0\ldots\alpha_0\rangle\,|^2$) and that the dependence on the support of the observable in Eq. (13) cannot be improved: to see this consider the observable $O_r = |0\ldots0\rangle\langle 0\ldots0|_{1\ldots r}$ acting on $r \leqslant N$ spins. O_r has norm one, and

$$O_\infty := \lim_{t\to\infty} T_t^*(O_r) = \mathbb{1}, \quad O_\infty^\varepsilon := \lim_{t\to\infty} T_t^{\varepsilon\,*}(O_r) = \langle 0|\alpha_0\rangle^{2r}\,\mathbb{1} = (1-\varepsilon^2)^r\,\mathbb{1}.$$

Consequently, we have:

$$\left\|O_\infty - O_\infty^\varepsilon\right\| = 1 - (1-\varepsilon^2)^r = r\varepsilon^2 + o(\varepsilon^2).$$

This implies that any upper bound to $\left\|O_\infty - O_\infty^\varepsilon\right\|$ has to be at least linear in r, which is the size of the support of O_r.

4.4. Do we need all the assumptions?. It is reasonable to ask if the assumptions of Theorem 7 are all necessary. We have just shown that we must necessarily consider local observables if we are to have meaningful bounds, but what about the other conditions? We will now present three examples, each consisting of a family of Lindbladiands with periodic boundary conditions, such that, in order:

- The family is uniform and translationally invariant, satisfies rapid mixing, but does not have a unique fixed point.
- The family has a unique fixed point, but is not uniform and fails to satisfy rapid mixing.
- The family (presented in Appendix 8) has a unique fixed point, satisfies rapid mixing, but is not uniform.

All these systems will be shown to be unstable.

Example 2. Consider a 1D chain composed of N 4-level systems, with an independent Lindbladian acting on each site, having the following Lindblad operators

$$L_1 = |0\rangle\langle 1|\,, \quad L_2 = |0\rangle\langle 3|\,, \quad L_3 = |2\rangle\langle 1|\,, \quad L_4 = |2\rangle\langle 3|\,,$$

and denote by

$$\mathcal{L}_0(\rho) = \sum_{i=1}^{4} L_i \rho L_i^* - \frac{1}{2}\{\rho, L_i^* L_i\}.$$

The global Lindbladian $\mathcal{L}_N$ is given by applying $\mathcal{L}_0$ independently on each site $k = 1, \ldots, N$:

$$\mathcal{L}_N = \sum_{k=1}^{N} \mathbb{1}_{1,\ldots,k-1} \otimes \mathcal{L}_0 \otimes \mathbb{1}_{k+1,\ldots,N}.$$

Then we have that

$$\mathcal{L}_0(|i\rangle\langle j|) = \begin{cases} 0 & \text{if } i = j \in \{0, 2\} \\ |0\rangle\langle 0| + |2\rangle\langle 2| - 2\,|i\rangle\langle j| & \text{if } i = j \in \{1, 3\} \\ -[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)]\,|i\rangle\langle j| & \text{if } i \neq j. \end{cases}$$

Diagonal states of the form $|i\rangle\langle i|$ evolve according to the classical Markov process embedded in the Lindbladian, while off-diagonal elements $|i\rangle\langle j|$ evolve as

$$T_t(|i\rangle\langle j|) = \exp(-t[\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j)])\,|i\rangle\langle j|\,;$$

where $\chi_{\{1,3\}}$ denotes the indicator function of the set $\{1, 3\}$. This implies that the space of fixed points $\mathcal{F}_{\mathcal{L}_0}$ is given by $\text{span}\{|0\rangle\langle 0|\,, |2\rangle\langle 2|\,, |0\rangle\langle 2|\,, |2\rangle\langle 0|\}$. Since $\mathcal{L}_0$ has gap equal to 1, Theorem 6 implies that $\mathcal{L}_N$ satisfies rapid mixing. $\mathcal{L}_N$ forms a uniform family, but it does not satisfy the unique fixed point condition.

Consider now the following additional Lindbladian

$$\mathcal{E}_0(\rho) = \frac{2}{N}\left[|0\rangle\langle 2|\,\rho\,|2\rangle\langle 0| - \frac{1}{2}\{\rho, |2\rangle\langle 2|\}\right].$$

Then, we have:

$$(\mathcal{L}_0 + \mathcal{E}_0)(|i\rangle\langle j|) = \begin{cases} 0 & \text{if } i = j = 0 \\ |0\rangle\langle 0| + |2\rangle\langle 2| - 2\,|i\rangle\langle j| & \text{if } i = j = 1, 3 \\ \frac{2}{N}(|0\rangle\langle 0| - |i\rangle\langle j|) & \text{if } i = j = 2 \\ -\left(\chi_{\{1,3\}}(i) + \chi_{\{1,3\}}(j) + \frac{\chi_{\{i,j\}}(2)}{N}\right)|i\rangle\langle j| & \text{if } i \neq j. \end{cases}$$

Again, this implies that $\mathcal{F}_{\mathcal{L}_0+\mathcal{E}_0} = \{|0\rangle\langle 0|\}$. Consequently $\mathcal{L}_N + \mathcal{E}_N$ has a unique fixed point. It is not a uniform family, and it does not satisfy rapid mixing, as it is not even globally gapped. To see this, note that for $\sigma = |200\ldots0\rangle\langle 200\ldots0| - |020\ldots0\rangle\langle 020\ldots0|$:

$$(\mathcal{L}_N + \mathcal{E}_N)(\sigma) = -\frac{2}{N}\,\sigma.$$

Analogously, $\mathcal{L}_N + \mathcal{E}_N^*$ satisfies the same conditions as $\mathcal{L}_N + \mathcal{E}_N$, but the unique fixed point is now $|2\ldots2\rangle\langle 2\ldots2|$.

All three systems described above are unstable, since we can transform one into the other by applying a perturbation of order $O(1/N)$, yet the fixed points of $\mathcal{L}_N + \mathcal{E}_N$ and $\mathcal{L}_N + \mathcal{E}_N^*$ are locally orthogonal (while $\mathcal{L}_N$ has both of them as fixed points).

4.5. Relaxations of the rapid mixing condition. In the case of finite range or exponentially decaying interactions, the proof of Theorem 7 still holds if we relax Eq. (11) by requiring only a polynomial decay in time, i.e. a bound of the form

$$\eta(T_t^{\overline{\Lambda}}) \leqslant c\,|\Lambda|^{\delta}\,\gamma(t), \tag{15}$$

if $\gamma(t)$ is a fast enough decaying function, where the threshold decay rate is determined by system-size independent constants (such as the Lieb–Robinson bound constants and the geometrical dimension of the underlying lattice structure).

Determining the precise value of such threshold requires an argument similar to the one given for the case of power-law decaying interactions in Sect. 6.4, and is presented in Sect. 6.5.

5. Toolbox for the Proof

Before presenting the proof of Theorem 7, we need to introduce some useful tools. We present them in full generality, including the case of power-law decay of interactions, without restricting here to exponentially decaying interactions.

5.1. Lieb–Robinson bounds for Lindbladian evolution. We first recall a generalization of Lieb–Robinson bounds to non-Hamiltonian evolution, due to [50,54], which we use to derive a number of useful tools that allow us to approximate the support of an evolving observable with a finite set which grows linearly in time. The following condition is sufficient for the bounds to hold.

Assumption 1 (*Lieb–Robinson condition*). Let $\mathcal{L} = \sum_{u,r} \mathcal{L}_{u,r}$ be a local Lindbladian. There exist positive constants μ and v, such that:

$$\sup_{x\in\Lambda} \sum_{u\in\Lambda} \sum_{r\geqslant \mathrm{dist}(u,x)} \left\|\mathcal{L}_{u,r}\right\|_{1\to 1,cb} |b_u(r)|\, \nu_\mu(r) \leqslant \frac{v}{2} < \infty; \tag{16}$$

where $\nu_\mu(\cdot)$ is one of the following:

$$\nu_\mu(r) = e^{\mu r}, \tag{LR-1}$$

$$\nu_\mu(r) = (1+r)^\mu. \tag{LR-2}$$

Note that both functions are submultiplicative, in the sense that $\nu_\mu(r+s) \leqslant \nu_\mu(r)\nu_\mu(s)$. Moreover, $\nu_a(r)^b = \nu_{ab}(r)$.

The constant v is called the *Lieb–Robinson velocity* of $\mathcal{L}$, while the reciprocal function $\nu_\mu^{-1}(r) = 1/\nu_\mu(r)$ is called the *Lieb–Robinson decay* of $\mathcal{L}$.

Note that if $\mathcal{L}$ has interaction strength (J, f), then condition (16) can be replaced by:

$$J \sup_{x,y\in\Lambda} \sum_{n\geqslant 0} |b_x(r)\backslash b_x(r-1)| \sum_{r\geqslant n} f(r) \left|b_y(r)\right| \nu_\mu(r) \leqslant \frac{v}{2} < \infty. \tag{17}$$

Since our systems are embedded in the lattice $\mathbb{Z}^D$, we have that $v < \infty$, as long as:

$$\sum_{n\geqslant 0} n^{D-1} F_\mu(n) < \infty, \quad F_\mu(n) := \sum_{r\geqslant n} r^D\, f(r)\, \nu_\mu(r). \tag{18}$$

Remark 6. Condition (LR-1) is satisfied when $\mathcal{L}$ has finite-range or exponentially decaying interactions, while condition (LR-2) is satisfied when $\mathcal{L}$ has quasi-local interactions. If $\mathcal{L}$ has interactions decaying as a power-law with exponent α, then condition (LR-2) is satisfied whenever $\alpha > 2D+1$ [by choosing $\mu < \alpha - (2D+1)$].

Theorem 9 (Lieb–Robinson bound). *Suppose $\mathcal{L}$ is a local Lindbladian satisfying Assumption 1. Let O_X be an observable supported on $X \subset \Lambda$, and denote by $O_X(t) = T_t^*(O_X)$ its evolution under $\mathcal{L}$. Let $K : \mathcal{A}_Y \to \mathcal{A}_Y$ be a super-operator supported on $Y \subset \Lambda$ which vanishes on $\mathbb{1}$. Then, the following bound holds* [50,54]*:*

$$\|K(O(t))\| \leqslant \|K\|_{\infty\to\infty,cb}\, \|O_X\|\, C(X,Y) \frac{\left(e^{vt}-1\right)}{v_\mu(\mathrm{dist}(X,Y))}, \tag{19}$$

where $C(X,Y) = \min(|X|, |Y|)$.

From now on, we will only consider Lindbladians which satisfy Eq. (17) with either of the two possible assumptions on $v_\mu(\cdot)$.

Lemma 10 (Comparing different dynamics). *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two local Lindbladians, and suppose $\mathcal{L}_2$ has Lieb–Robinson speed and decay bounded by v and v_μ^{-1}. Consider an operator O_X supported on $X \subset \Lambda$, and denote by $O_i(t)$ its evolution under $\mathcal{L}_i$, $i = 1, 2$. Suppose that $\mathcal{L}_1 - \mathcal{L}_2 = \sum_{r\geqslant 0} M_r$, where M_r is a superoperator supported on Y_r which vanishes on $\mathbb{1}$, and $\mathrm{dist}(X, Y_r) \geqslant r$. Then the following holds:*

$$\|O_1(t) - O_2(t)\| \leqslant \|O_X\|\, |X|\, \frac{e^{vt} - vt - 1}{v} \sum_{r=0}^{\infty} \|M_r\|_{1\to 1,cb}\, v_\mu^{-1}(r). \tag{20}$$

Proof. Let $h(t) = O_1(t) - O_2(t)$. Calculating its derivative, we obtain

$$h'(t) = \mathcal{L}_1^* O_1(t) - \mathcal{L}_2^* O_2(t) = \mathcal{L}_1^* h(t) + (\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(t).$$

Since $h(0) = 0$, this differential equation for $h(t)$ has solution

$$\begin{aligned} h(t) = O_1(t) - O_2(t) &= \int_0^t e^{(t-s)\mathcal{L}_1^*} (\mathcal{L}_1^* - \mathcal{L}_2^*) O_2(s)\, \mathrm{d}s \\ &= \sum_{r\geqslant 0} \int_0^t e^{(t-s)\mathcal{L}_1^*} M_r^* O_2(s)\, \mathrm{d}s, \end{aligned} \tag{21}$$

giving us a useful integral representation for $O_1(t) - O_2(t)$. From this, we obtain the estimate

$$\|O_1(t) - O_2(t)\| \leqslant \sum_{r\geqslant 0} \int_0^t \left\| M_r^* O_2(s) \right\| \mathrm{d}s,$$

where we have used the fact that $e^{t\mathcal{L}_1^*}$ is a contraction with respect to $\|\cdot\|_\infty$ for each $t \geqslant 0$.

We can now apply the Lieb–Robinson bound [Eq. (19)] to each of the terms in the sum in the previous estimate, to obtain:

$$\|O_1(t) - O_2(t)\| \leqslant \sum_{r\geqslant 0} \|M_r\|_{1\to 1,cb}\, \|O_X\|\, C(X,Y_r) v_\mu^{-1}(\mathrm{dist}(X,Y_r)) \int_0^t (e^{vs} - 1)\, \mathrm{d}s,$$

which implies the claimed bound. □

A particular application of the previous lemma is when $\mathcal{L}_2$ is a restriction of $\mathcal{L}_1$ onto a smaller region. Since this case occurs frequently, and is of particular interest, we state it as a separate lemma. Similar results were presented in [6,50].

Lemma 11 (Localizing the evolution). *Let O_A be an observable supported on a finite $A \subset \Lambda$. Denote by $O_A(t) = T_t^*(O_A)$ its evolution under a local Lindbladian $\mathcal{L}$ with strength (J, f). Given $r > 0$, denote by $O_A(r;t)$ its evolution under the localized Lindbladian $\mathcal{L}_{A(r)}$.*

Then, the following bound holds:

$$\|O_A(t) - O_A(r;t)\| \leqslant \|O_A\|\,|A|\,J\frac{e^{vt} - 1 - vt}{v}\nu_\beta^{-1}(r), \tag{22}$$

where $\nu_\beta^{-1}(r)$ decays exponentially if $\mathcal{L}$ satisfies condition (LR-1), while decays as $(1+r)^{-\beta}$ if $\mathcal{L}$ satisfies condition (LR-2). In this case, if we denote by α the decay rate of $\mathcal{L}$, then β is given by:

$$\beta = \begin{cases} \alpha - 3D & \text{if } \alpha \geqslant 5D-1; \\ \frac{1}{2}(\alpha - D - 1) & \text{if } \alpha \leqslant 5D-1. \end{cases}$$

Proof. First, let us decompose $\mathcal{L} - \mathcal{L}_{A(r)}$ as a telescoping sum

$$\mathcal{L} - \mathcal{L}_{A(r)} = \sum_{l \geqslant r} \mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)}.$$

Since each element in the sum is the difference between restrictions on different subsets of the same global Lindbladian, it is easy to explicitly write their difference

$$\mathcal{L}_{A(l+1)} - \mathcal{L}_{A(l)} = \sum_{\delta=0}^{l+1} \sum_{\operatorname{dist}(u,A)=\delta} \mathcal{L}_u(l+1-\delta).$$

We group the terms in the sum by their distance from A: let

$$d = \operatorname{dist}(A, b_u(l+1-\delta)) = \max\{0, 2\delta - l - 1\}$$

and

$$M_0 = \sum_{l \geqslant r} \sum_{\delta=0}^{\frac{l+1}{2}} \sum_{\operatorname{dist}(u,A)=\delta} \mathcal{L}_u(l+1-\delta); \tag{23}$$

$$M_d = \sum_{l \geqslant r} \sum_{\substack{\operatorname{dist}(u,A)=\delta \\ \delta = \frac{l+1+d}{2}}} \mathcal{L}_u(l+1-\delta). \tag{24}$$

Then, we can write:

$$\sum_{d \geqslant 0} M_d = \mathcal{L} - \mathcal{L}_{A(r)}; \quad \operatorname{dist}(A, \operatorname{supp} M_d) = d.$$

Applying Lemma 10, we obtain:

$$\|O_A(t) - O_A(r;t)\| \leqslant \|O_A\|\,|A|\,J\frac{e^{vt}-1-vt}{v}\zeta(r);$$

where, by denoting $q(l) = |A(l)\backslash A(l-1)|$, $\zeta(r)$ is the following:

$$\begin{aligned}\zeta(r) &= \frac{1}{J}\sum_{d\geqslant 0}\|M_d\|_{1\to 1,cb}\,\nu_\mu^{-1}(d)\\ &\leqslant \sum_{l\geqslant r}\sum_{\delta=0}^{\frac{l+1}{2}} q(\delta)f(l+1-\delta) + \sum_{\delta=\frac{l+1}{2}}^{l+1} q(\delta)f(l+1-\delta)\nu_\mu^{-1}(2\delta-l-1). \quad (25)\end{aligned}$$

If $\delta \geqslant (l+1)/2$, since $\nu_\mu(\cdot)$ is submultiplicative, we have:

$$\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)\nu_\mu(2\delta-l-1).$$

Otherwise, since $\nu_\mu(\cdot)$ is increasing, we have that $\nu_\mu(\delta) \leqslant \nu_\mu(l+1-\delta)$. Plugging these inequalities in the above sum, we get:

$$\zeta(r) \leqslant \sum_{l\geqslant r}\sum_{\delta=0}^{l+1}\left[q(\delta)\nu_\mu^{-1}(\delta)\right]\left[f(l+1-\delta)\nu_\mu(l+1-\delta)\right].$$

Since f satisfies Eq. (17), which in particular implies

$$\sum_{\delta\geqslant 0} f(\delta)\nu_\mu(\delta)\,|b_0(\delta)| < \infty,$$

then the sequence $f(\delta)\nu_\mu(\delta)$ is decreasing. We distinguish two cases: If ν_μ is of the type (LR-1), then the decay of $f(\delta)\nu_\mu(\delta)$ is exponential. Since $q(\delta)$ grows polynomially, $q(\delta)\nu_\mu^{-1}(\delta)$ is exponentially decaying. Then, the convolution of the two sequences, which is exactly:

$$\sum_{\delta=0}^{l+1}\left[q(\delta)\nu_\mu^{-1}(\delta)\right]\left[f(l+1-\delta)\nu_\mu(l+1-\delta)\right]$$

is exponentially decaying too, which implies an exponential decay rate for $\zeta(r)$. Thus, there exists some $\beta > 0$ such that $\zeta(r) \leqslant \nu_\beta^{-1}(r)$, and this concludes the proof for the case of exponential decay. Let us suppose now that ν_μ is of type (LR-2). Then, $f(\delta)\nu_\mu(\delta)$ decays as $(1+\delta)^{\mu-\alpha}$, while $q(\delta)\nu_\mu^{-1}(\delta)$ decays as $(1+\delta)^{D-1-\mu}$. This implies[7] that their convolution decays as $(1+l)^{-\min(\alpha-\mu,\mu-D+1)}$ and thus

$$\zeta(r) \leqslant c(1+r)^{-\min(\alpha-\mu-1,\mu-D)} = \nu_\beta^{-1}(r).$$

Recalling that condition (LR-2) requires $\mu < \alpha - (2D+1)$, a simple calculation shows that the above decay rate is maximized for

$$\mu < \min\left(\alpha - 2D - 1, \frac{\alpha + D - 1}{2}\right),$$

which gives the claimed formula for β. □

[7] Consider two positive decreasing sequences (x_n) and (y_n). Since $0 < p < 1$ implies that $(x+y)^p \leqslant x^p + y^p$, it holds that $(x * y)_n^p \leqslant \sum_k x_k^p y_{n-k}^p = (x^p * y^p)_n$.

Another specialization of Lemma 10, similar in spirit to the one just presented, is when we compare the evolution of local observables under $\mathcal{L}^{A(r)}$ and $\mathcal{L}^{\overline{A(r)}}$, as defined in Definition 3.

Lemma 12. *Let O_A be an observable supported on $A \subset \Lambda$. Given $r > 0$, it holds that*

$$\left\| T_t^{*\overline{A(r)}}(O_A) - T_t^{*A(r)}(O_A) \right\| \leqslant \|O_A\| \, |A| \; J \, \frac{e^{vt} - 1 - vt}{v} \nu_\beta^{-1}(r). \tag{26}$$

Proof. Without loss of generality, we consider the case of $A(r)$ being a convex set. By construction, $\mathcal{L}^{\overline{A(r)}} - \mathcal{L}^{A(r)} = \mathcal{L}^{\partial A(r)}$, and $\mathcal{L}^{\partial A(r)} = \sum_{d \geqslant 1} N_d$, where each N_d acts on sites that are closer than d to the border of $A(r)$. We group these terms by their distance from A. Let $k = \frac{1}{2} \operatorname{diam} A$ and set:

$$M_0 = \sum_{i=0}^{k} N_{r+1+i},$$
$$M_j = N_{r+1-j}, \quad j = 1, \ldots, r.$$

It is easy to see that $\operatorname{dist}(A, \operatorname{supp} M_j) = j$. By applying Lemma 10, we have that:

$$\left\| T_t^{*\overline{A(r)}}(O_A) - T_t^{*A(r)}(O_A) \right\| \leqslant \|O_A\| \, |A| \, \frac{e^{vt} - 1 - vt}{v} \sum_{j=0}^{r} \left\| M_j \right\|_{1 \to 1, c.b.} \nu_\mu^{-1}(j).$$

We are left to prove that the sum appearing on the r.h.s. is fast-decaying in r. From Definition 3 it follows that for $j > 0$:

$$\left\| M_j \right\|_{1 \to 1, c.b.} \leqslant J \left| \partial_{r-j} A(r) \right| f(r + 1 - j) = J \, |A(r) \backslash A(j)| \, f(r + 1 - j),$$

while for $j = 0$:

$$\|M_0\|_{1 \to 1, c.b.} \leqslant \sum_{i=0}^{k} J \, |\partial_{r+i} A(r)| \, f(r + 1 + i).$$

Setting $h_{m,n} = |b_0(m) \backslash b_0(n)|$, we have that:

$$\sum_{j=0}^{r} \left\| M_j \right\|_{1 \to 1, c.b.} \nu_\mu^{-1}(j) \leqslant J \zeta(r), \tag{27}$$

where

$$\zeta(r) := \sum_{i=0}^{k} h_{r+k,k-i} \, f(r + 1 + i) + \sum_{j=1}^{r} h_{r+k,k+j} \, f(r + 1 - j) \nu_\mu^{-1}(j).$$

An argument similar to the one in the proof of Lemma 11 shows that $\zeta(r)$ is fast-decaying. Indeed, $h_{r+k,k-i} \, f(r + 1 + i)$ scales asymptotically as $r^D f(r)$, while $h_{r+k,k+j} \, f(r + 1 - j)$ scales as $(r - j)^D f(r + 1 - j)$. If $\mathcal{L}$ satisfies (LR-1), then $\zeta(r)$ will be exponentially decaying, with rate $\min(\alpha, \mu) - 1 = \mu - 1$.

If otherwise $\mathcal{L}$ satisfies (LR-2), then $\zeta(r)$ has a polynomial decay, with rate $\min(\alpha - D, \mu) - 1 = \mu - 1$. In both cases, then:

$$\zeta(r) \leqslant \nu_{\mu-1}^{-1}(r).$$

Notice that the constant β defined in Lemma 11 is smaller than $\mu - 1$. □

5.2. Local rapid mixing. The rapid mixing condition implies a local version of mixing that will be a useful tool for the proof of Theorem 7. We state its definition here.

Definition 13 (*Local rapid mixing*). Take $A \subset \Lambda$, and define the *contraction of* T_t *relative to* A as

$$\begin{aligned}\eta^A(T_t) &:= \sup_{\substack{\rho\geqslant 0\\ \operatorname{tr}\rho=1}} \left\|\operatorname{tr}_{A^c}\left[T_t(\rho)-T_{\phi,t}(\rho)\right]\right\|_1\\ &= \sup_{\substack{\rho\geqslant 0\\ \operatorname{tr}\rho=1}}\ \sup_{\substack{O_A\in\mathcal{A}_A\\ \|O_A\|=1}} \operatorname{tr}\left(O_A\left[T_t(\rho)-T_{\phi,t}(\rho)\right]\right)\\ &= \sup_{\substack{\rho\geqslant 0\\ \operatorname{tr}\rho=1}}\ \sup_{\substack{O_A\in\mathcal{A}_A\\ \|O_A\|=1}} \operatorname{tr}\left(\rho\left[T_t^*(O_A)-T_{\phi,t}^*(O_A)\right]\right). \end{aligned} \tag{28}$$

We say that $\mathcal{L}$ satisfies *local rapid mixing* if, for each $A \subset \Lambda$, we have that

$$\eta^A(T_t) \leqslant k(|A|)e^{-\gamma t}, \tag{29}$$

where $k(r)$ grows polynomially in r, $\gamma > 0$ and all the constants appearing above are independent of the system size.

Remark 7. It follows from the definition that $\eta^A(T_t) \leqslant \eta^B(T_t)$ whenever $A \subset B$. In particular, $\eta^A(T_t) \leqslant \eta(T_t)$.

Note that, in contrast with Definition 5, the quantity $\eta^A(T_t)$ depends on the evolution *on the whole system* Λ, and not just on the subset A. Thus local rapid mixing is a very strong condition: the term $k(r)$ appearing in Eq. (29) only depends on the support of A, so the local mixing time (i.e. the time it takes for the reduced density matrix on the subset A to converge) is required to be independent of system size.

Example 3. A simple dissipative system satisfying Definition 13 is the tensor product of amplitude damping channels acting (with the same rate) on different qubits. Note that, though it might seem a trivial example, there are interesting dissipative systems of this form: among others, dissipative preparation of graph states [31] can be brought into this form by a non-local unitary rotation (which of course does not change the convergence rates).

6. Proof of Main Result

We are now ready to prove our main resul, Theorem 7. The proof proceeds in three steps. First, we show that the assumptions of Theorem 7 imply that the fixed points of $\mathcal{L}^{\overline{A}}$ for different Λ are locally indistinguishable. Then, we prove that rapid mixing implies local rapid mixing. Finally, we show how local rapid mixing and the uniqueness of the initial fixed point imply the desired stability result.

6.1. Step 1: closeness of the fixed points. Topological quantum order (TQO), namely the property of certain orthogonal quantum states to be locally indistinguishable from each other, is a widely studied property of ground state subspaces in the Hamiltonian setting. In the dissipative setting on the other hand, where the concept of ground states is no longer applicable, one may define the analogous concept for periodic states of

Lindbladians. Below we describe the concept of LTQO [49], which extends the concept of TQO to the invariant subspace (periodic states) of local restrictions of the global Lindbladian.

We note that, in contrast to the Hamiltonian case, in order to prove the desired stability result we do not require extra assumptions like LTQO, or frustration-freeness. Indeed, we show in this section that rapid mixing *implies* LTQO and a property similar to frustration-freeness. These properties will play a role in the proof of stability, via Lemma 17.

Definition 14 (*LTQO*). Consider a Lindbladian $\mathcal{L}$. Take a convex set $A \subset \Lambda$ and let $A(\ell) = \{x \in \Lambda | \operatorname{dist}(x, A) \leqslant \ell\}$. Given two states $\rho_i \in \mathcal{X}_{\mathcal{L}_{A(\ell)}}$, $i = 1, 2$, consider their reduced density matrices on A:

$$\rho_i^A = \operatorname{tr}_{A(\ell)\setminus A} \rho_i, \quad i = 1, 2.$$

We say that $\mathcal{L}$ has *LTQO* if for each $\ell \geqslant 0$:

$$\left\| \rho_1^A - \rho_2^A \right\|_1 \leqslant p(|A|)\, \Delta_0(\ell), \tag{30}$$

where $\Delta_0(\ell)$ is a fast-decaying function, and $p(\cdot)$ is a polynomial.

As a first step in the proof, we will show that the conditions of Theorem 7 imply that the fixed point of T_t, the fixed point of $T_t^{\overline{A}}$ and the periodic points of T_t^A are difficult to distinguish locally, in the same spirit as the LTQO condition.

Lemma 15. *Let $\mathcal{L}$ be a uniform family satisfying condition (11), and suppose each $\mathcal{L}^{\overline{A}}$ has a unique fixed point and no other periodic points. Let O_A be an observable supported on $A \subset \Lambda$, ρ a periodic point of $T_t^{A(s)}$ and ρ_∞^s the unique fixed point of $T_t^{\overline{A(s)}}$. Then, we have*

$$\left|\operatorname{tr} O_A(\rho - \rho_\infty^s)\right| \leqslant \|O_A\| \left(\frac{J}{v} |A| + c\, |A|^\delta \right) \Delta_0(s), \tag{31}$$

where

$$\Delta_0(s) = (|A(s)| / |A|)^{\delta v/(v+\gamma)} \nu_{\beta'}^{-1}(s), \quad \beta' = \beta\gamma/(v+\gamma),$$

with c, γ, δ the constants defined in the rapid mixing condition $RM(\gamma, \delta)$, β the rate defined in Lemma 12 and v the Lieb–Robinson velocity.

Proof. Fix a $t := t(s) \geqslant 0$, to be determined later. Since $T_t^{A(s)}$ acts on its space of periodic points as a unitary evolution, there exists a periodic point of $\mathcal{L}^{A(s)}$, ρ', such that $\rho = T_t^{A(s)}(\rho')$. Then, by the triangle inequality, we have:

$$\left|\operatorname{tr} O_A(\rho - \rho_\infty^s)\right| \leqslant \left|\operatorname{tr} O_A[T_t^{A(s)} - T_t^{\overline{A(s)}}](\rho'))\right| + \left|\operatorname{tr} O_A(T_t^{\overline{A(s)}}(\rho') - \rho_\infty^s)\right|. \tag{32}$$

The first term is bounded by Lemma 12, since

$$\operatorname{tr} O_A(T_t^{A(s)}(\rho') - T_t^{\overline{A(s)}}(\rho')) = \operatorname{tr} \rho'(T_t^{*A(s)}(O_A) - T_t^{*\overline{A(s)}}(O_A))$$

and

$$\left|\operatorname{tr} \rho'(T_t^{*A(s)}(O_A) - T_t^{*\overline{A(s)}}(O_A))\right| \leqslant \|\rho'\|_1 \|T_t^{*A(s)}(O_A) - T_t^{*\overline{A(s)}}(O_A)\|_\infty.$$

The second term is bounded using the rapid mixing condition on $T_t^{\overline{A(s)}}$. By putting the two bounds together, we obtain

$$\left|\operatorname{tr} O_A(\rho - \rho_\infty^s)\right| \leqslant \|O_A\|\,|A|\,\frac{J}{v}e^{vt}\nu_\beta^{-1}(s) + \|O_A\|\,c\,|A(s)|^\delta\,e^{-\gamma t}.$$

Setting $p(s) = (|A(s)|\,/\,|A|)^\delta$ and choosing $t(s)$ such that

$$e^{vt(s)}\nu_\beta^{-1}(s) = p(s)e^{-\gamma t(s)},$$

we have that $t(s) = \ln(\nu_\beta(s)\cdot p(s))^{1/(v+\gamma)}$. Under such choice, it holds that

$$e^{-\gamma t(s)} = (\nu_\beta(s)p(s))^{-\gamma/(v+\gamma)} = \nu_{\beta'}^{-1}(s)p(s)^{-\gamma/(v+\gamma)},$$

where $\beta' = \beta\gamma/(v+\gamma)$. Defining

$$\Delta_0(s) := (|A(s)|\,/\,|A|)^{\delta v/(v+\gamma)}\nu_{\beta'}^{-1}(s),$$

concludes the proof. □

Corollary 16 (LTQO). *Under the assumptions of Lemma 15, the Lindbladian $\mathcal{L}^{\overline{\Lambda}}$ satisfies LTQO (Definition 14) for all Λ.*

Proof. Take $A \subset \Lambda$, and $s \geqslant 0$. Let ρ_1 and ρ_2 be two periodic points of $T_t^{A(s)}$. Then, by the triangle inequality, we have that:

$$\begin{aligned}|\operatorname{tr} O_A(\rho_1 - \rho_2)| &\leqslant \left|\operatorname{tr} O_A(\rho_1 - \rho_\infty^s)\right| + \left|\operatorname{tr} O_A(\rho_\infty^s - \rho_2)\right| \\ &\leqslant 2\,\|O_A\|\left(\frac{J}{v}\,|A| + c\,|A|^\delta\right)\Delta_0(s).\end{aligned}$$

Since $\|\rho_1^A - \rho_2^A\|_1 = \sup_{\|O_A\|=1}|\operatorname{tr} O_A(\rho_1 - \rho_2)|$, the result follows immediately. □

Lemma 17. *Under the same notation and assumptions of Lemma 15, we have the following bound for ρ_∞ the unique fixed point of T_t:*

$$\sup_{\|O_A\|=1}\left|\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s)\right| \leqslant \|O_A\|\left(\frac{J}{v}\,|A| + c\,|A|^\delta\right)\Delta_0(s). \tag{33}$$

Proof. By the triangle inequality:

$$\left|\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s))\right| \leqslant \left|\operatorname{tr} O_A(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty)\right| + \left|\operatorname{tr} O_A(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s)\right|.$$

The first term on the right can be bounded using Lemmas 11 and 12 along with $T_t(\rho_\infty) = \rho_\infty$:

$$\begin{aligned}&\left|\operatorname{tr} O_A\left(T_t(\rho_\infty) - T_t^{\overline{A(s)}}(\rho_\infty)\right)\right| \\ &= \left|\operatorname{tr}\rho_\infty(T_t^*(O_A) - T_t^{*\overline{A(s)}}(O_A))\right| \\ &\leqslant \|\rho_\infty\|_1\left(\|T_t^*(O_A) - T_t^{*A(s)}(O_A)\|_\infty + \|T_t^{*A(s)}(O_A) - T_t^{*\overline{A(s)}}(O_A)\|_\infty\right) \\ &\leqslant \|O_A\|\,|A|\,\frac{J}{v}e^{vt}\nu_\beta^{-1}(s).\end{aligned}$$

The second term is bounded using the rapid mixing condition:

$$\left|\operatorname{tr} O_A\left(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s\right)\right| \leqslant \|O_A\|\, c\,|A|^\delta\, p(s) e^{-\gamma t}.$$

By making the same choice of $t = t(s)$ as in Lemma 15, we get the desired bound. □

Corollary 18 (Approximate frustration-freeness). *Under the same notation and assumptions of Lemma 15, denote by ρ_∞ the unique fixed point of T_t, and by ρ a periodic point of $T_t^{A(s)}$. Then, we have the following bound*:

$$\sup_{\|O_A\|=1} |\operatorname{tr} O_A(\rho_\infty - \rho)| \leqslant 2\,\|O_A\| \left(\frac{J}{v}\,|A| + c\,|A|^\delta\right) \Delta_0(s). \tag{34}$$

Proof. By the triangle inequality and Lemmas 15 and 17, we have:

$$\begin{aligned} |\operatorname{tr} O_A(\rho_\infty - \rho))| &\leqslant \left|\operatorname{tr} O_A(\rho_\infty - \rho_\infty^s)\right| + \left|\operatorname{tr} O_A(\rho_\infty^s - \rho)\right| \\ &\leqslant 2\,\|O_A\| \left(\frac{J}{v}\,|A| + c\,|A|^\delta\right) \Delta_0(s). \end{aligned}$$

□

6.2. Step 2: from global to local rapid mixing. As a second step in the proof, we show that the assumptions on $\mathcal{L}$ imply local rapid mixing.

Proposition 19 (From global to local rapid mixing). *Let $\mathcal{L}$ be a uniform family of Lindbladians with unique fixed point. Then, if condition (11) is satisfied, $\mathcal{L}$ satisfies local rapid mixing.*

Proof. Let O_A be an observable supported on A with $\|O_A\| = 1$. Denote by s_0 the minimum $s \geqslant 0$ such that $A(s) = \Lambda$. Fix $0 \leqslant s \leqslant s_0$, and let $B = A(s)$. Then, by the triangle inequality, we can bound the norm of $(T_t^* - T_\infty^*)$ as follows:

$$\left\|(T_t^* - T_\infty^*)O_A\right\| \leqslant \left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| + \left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\|. \tag{35}$$

We bound the first term on the right using Lemmas 11 and 12:

$$\left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| \leqslant |A|\,\frac{J}{v}(e^{vt} - 1 - vt)e^{-\beta s}. \tag{36}$$

The second term is bounded by the rapid mixing condition (11), setting $p(s) = (|A(s)|\,/\,|A|)^\delta$:

$$\left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| \leqslant \eta(T_t^{\overline{B}}) \leqslant c\,|A|^\delta\, p(s) e^{-\gamma t}. \tag{37}$$

Finally, the third term is bounded by using Lemma 17:

$$\left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\| = \left|\operatorname{tr} O_A(\rho_\infty^s - \rho_\infty)\right| \leqslant \left(\frac{J}{v}\,|A| + c\,|A|^\delta\right) \Delta_0(s). \tag{38}$$

Substituting bounds (36), (37) and (38) into Eq. (35), we obtain, for $0 \leqslant s \leqslant s_0$ and for all $t \geqslant 0$:

$$\eta^A(T_t) \leqslant \frac{J}{v}\,|A|\,e^{vt}e^{-\beta s} + c\,|A|^{\delta}\,p(s)e^{-\gamma t} + \left(\frac{J}{v}\,|A| + c\,|A|^{\delta}\right)\Delta_0(s).$$

We want to show that we can choose $s = s(t) \in [0, s_0]$ in such a way that both $e^{vt}e^{-\beta s}$ and $e^{-t\gamma}\,p(s)$ are exponentially decaying in t. Choose $s := s(t) = t(v+\gamma)/\beta$. Since $\Delta_0(s) = (|A(s)|\,/\,|A|)^{\delta v/(v+\gamma)}\nu_{\beta'}^{-1}(s)$, denoting

$$\bar{p}(t) = p \circ s(t) = p(t(v+\gamma)/\beta),$$

we have that

$$\Delta_0(s(t)) = \bar{p}(t)^{v/(v+\gamma)}e^{-\gamma t}.$$

Therefore, since $\bar{p}(t) \geqslant 1$,

$$\begin{aligned}\eta^A(T_t) &\leqslant \frac{J}{v}\,|A|\,e^{-\gamma t} + c\,|A|^{\delta}\,\bar{p}(t)e^{-\gamma t} + \left(\frac{J}{v}\,|A| + c\,|A|^{\delta}\right)\bar{p}(t)^{v/(v+\gamma)}e^{-\gamma t}\\ &\leqslant 2\left(\frac{J}{v}\,|A| + c\,|A|^{\delta}\right)\bar{p}(t)e^{-\gamma t}, \quad \forall t \leqslant \frac{\beta}{v+\gamma}s_0.\end{aligned}$$

When $t \geqslant \beta/(v+\gamma)s_0$, we can simply bound $\eta^A(T_t)$ by $\eta(T_t)$ (see Remark 7), obtaining:

$$\eta^A(T_t) \leqslant c\,|A|^{\delta}\,p(s_0)e^{-\gamma t} \leqslant c\,|A|^{\delta}\,\bar{p}(t)e^{-\gamma t}, \quad \forall t \geqslant \frac{\beta}{v+\gamma}s_0.$$

This completes the proof. □

6.3. Step 3: from local rapid mixing to stability. We now prove that local rapid mixing alone implies stability. This is the last step in the proof of Theorem 7, as we already proved in the previous sections that the condition of Theorem 7 imply local rapid mixing. However, the following result also stands independently: if a system can be shown to satisfy local rapid mixing by other means, it will also be stable. Moreover, the same proof holds if we relax the assumption on prefactor $k(|A|)$ in Eq. (29): a similar (but weaker) stability result will hold true as long as $|A|$ is independent of system size.

Theorem 20. *Let $\mathcal{L}$ be a local Lindbladian satisfying local rapid mixing, and having a unique fixed point ρ_∞ such that*

$$T_\phi^*(O_A) = T_\infty^*(O_A) = \mathrm{tr}(O_A\rho_\infty)\mathbb{1}.$$

Then, using the notation of Theorem 7, for all observables O_A supported on $A \subset \Lambda$ we have that

$$\left\|T_t^*(O_A) - S_t^*(O_A)\right\| \leqslant c(|A|)\;\|O_A\|\left(\varepsilon + |\Lambda|\,\nu_\eta^{-1}(d_A)\right), \tag{39}$$

where $d_A = \mathrm{dist}(A, \Lambda^c)$; η is positive and independent of Λ; $\nu_\eta^{-1}(d) \leqslant (1+d)^{-D-1}$; $c(|A|)$ is independent of Λ and t, and is bounded by a polynomial in $|A|$.

Proof. Let $O_0(t) = T_t^*(O_A)$ and $O_1(t) = S_t^*(O_A)$ and write the difference $O_0 - O_1$ using the integral representation from Eq. (21):

$$O_0(t) - O_1(t) = \int_0^t S_{t-s}^* E^* T_s^*(O_A)\,\mathrm{d}s.$$

The triangle inequality implies:

$$\|O_0(t) - O_1(t)\| \leqslant \sum_u \sum_r \int_0^t \left\|E_{u,r}^* O_0(s)\right\| \mathrm{d}s + \sum_d \int_0^t \left\|E_d^* O_0(s)\right\| \mathrm{d}s,$$

where we used the fact that S_t is a contraction.

Fix a $K \in \{E_{u,r}\}_{u,r} \cup \{E_d\}_d$, and let $\delta = \mathrm{dist}(A, \mathrm{supp}\, K)$. We can split the integral at a time t_0 (to be fixed later, depending on δ). We bound the first part of the integral with Lieb–Robinson bounds:

$$\int_0^{t_0} \left\|K^* O_0(s)\right\| \mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\, \|O_A\|\, |A|\, \frac{e^{vt_0} - vt_0 - 1}{v\nu_\mu(\delta)}.$$

Now pick $t_0 = t_0(\delta)$ such that

$$\nu_\mu^{-1}(\delta)\frac{e^{vt_0} - vt_0 - 1}{v} \leqslant \nu_{\mu/2}^{-1}(\delta).$$

We can choose $t_0(\delta) = \frac{\mu}{2}\frac{\log v}{v}\delta = O(\delta)$, for exponentially decaying (or faster) $\nu_\mu^{-1}(\delta)$.

If $t \leqslant t_0(\delta)$, then we have bounded the entire integral, and we are done. Otherwise, we treat the second part of the integral as follows:

$$\begin{aligned}
&\int_{t_0(\delta)}^t \left\|K^* O_0(s)\right\| \mathrm{d}s = \int_{t_0(\delta)}^t \left\|K^*(O_0(s) - T_\infty^*(O_A))\right\| \mathrm{d}s \\
&\quad\leqslant \|K\|_{1\to 1,cb}\, \|O_A\| \int_{t_0(\delta)}^\infty \eta^A(T_s)\,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\, \|O_A\|\, q(|A|) \int_{t_0(\delta)}^\infty e^{-\gamma s}\,\mathrm{d}s \\
&\quad= \|K\|_{1\to 1,cb}\, \|O_A\|\, k(|A|)\frac{1}{\gamma} e^{-\gamma t_0(\delta)}
\end{aligned}$$

where we used $K^* T_\infty^*(O_A) = K^*(\mathrm{tr}(\rho_\infty O_A)\mathbb{1}) = \mathrm{tr}(\rho_\infty O_A) K^*(\mathbb{1}) = 0$, together with the local rapid mixing condition.

Since $t_0(\delta)$ is linear in δ, we have that:

$$h(\delta) := e^{-\frac{\mu\delta}{2}} + \frac{1}{\gamma} e^{-\gamma t_0(\delta)}$$

is exponentially decaying in δ.

Putting the different bounds together, we obtain:

$$\int_0^t \left\|K^* O_0(s)\right\| \mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\, \|O_A\|\, k_1(|A|) h(\delta),$$

where $k_1(|A|) = \max(k(|A|), |A|)$.

Returning to the sum, we have proven that:

$$\|O_0(t) - O_1(t)\| \leqslant \varepsilon\, k_1(|A|)\, \|O_A\| \Big[\underbrace{\sum_u \sum_r e(r) h(\mathrm{dist}(A, b_r(u)))}_{I_1(A;\, e,h)} + \underbrace{\sum_d |\partial_d \Lambda|\, e(d) h(\mathrm{dist}(A, \partial_d \Lambda))}_{I_2(A;\, e,h)} \Big]. \tag{40}$$

It suffices to show that I_1 and I_2 are finite (and independent of system size), and that I_2 decays exponentially in $\mathrm{dist}(A, \Lambda^c)$. Let us decompose the I_1 as follows

$$\begin{aligned} I_1(A;\, e, h) &= \sum_u \sum_r e(r) h(\mathrm{dist}(A, b_r(u))) \\ &= \sum_{\mathrm{dist}(u,A)=0} \sum_r e(r) h(0) + \sum_{d>0} \sum_{\mathrm{dist}(u,A)=d} \left(\sum_{r=0}^{d} e(r) h(d-r) + \sum_{r=d+1}^{\infty} e(r) h(0) \right) \\ &= h(0)\, |A| \sum_r e(r) + \sum_{d>0} q(d) \left(\sum_{r=0}^{d} e(r) h(d-r) + h(0) \sum_{r=d+1}^{\infty} e(r) \right), \end{aligned}$$

where $q(d) = |\{u : \mathrm{dist}(u, A) = d\}|$ grows polynomially in d.

The first term is clearly bounded, since $e(r)$ is summable. Since e and h are both exponentially decaying functions, their discrete convolution $e \star h(d) = \sum_{r=0}^{d} e(r) h(d-r)$ is also exponentially decaying, and consequently summable against any polynomial. The same holds for $\sum_{r>d} e(r)$. This proves that the second term is also bounded.

On the other hand, we have that

$$I_2(A;\, e, h) = \sum_d |\partial_d \Lambda|\, e(d) h(\mathrm{dist}(A, \partial_d \Lambda)) \leqslant |\Lambda|\, (e \star h(d_A) + \sum_{d \geqslant d_A} e(d)),$$

where $d_A = \mathrm{dist}(A, \Lambda^c)$. We have just proven that $e \star h(d_A)$ and $\sum_{d \geqslant d_A} e(d)$ are exponentially decaying. This implies that there exists a positive η such that $\nu_\eta^{-1}(d_A)$ upper bounds both. Denoting $c(|A|) = k_1(|A|) I_1(A;\, e, h)$, we have the desired bound. □

6.4. Power-law decay. As we stated before, the results and proofs presented above still hold when $\mathcal{L}$ has quasi-local or power-law interactions. In the latter case, this is only true when certain conditions are met on the decay of $\mathcal{L}$. In what follows, we highlight the changes one needs to make in the case of power-law decay, in order for the main stability results to hold.

Definition 21 (*Compatibility condition*). Let $\mathcal{L}$ be a local Lindbladian, and suppose it satisfies (LR-2) and rapid mixing RM(γ, δ). Let μ and v be the Lieb–Robinson constants for $\mathcal{L}$ defined in Assumption 1 and β the constant defined in Lemma 11. Then we say that $\mathcal{L}$ satisfies the weak compatibility condition for stability, if the following inequality is satisfied.

$$\beta\gamma - \delta D v > 0; \tag{CC-1}$$

we say that $\mathcal{L}$ satisfies the strong compatibility condition for stability if

$$\mu \frac{\beta\gamma - \delta D v}{\beta(\gamma + v)} > D + 2. \tag{CC-2}$$

Moreover, if the perturbation E, defined in Theorem 7, is decaying polynomially and not exponentially, it must satisfy

$$\sum_n n^D \sum_{r>n} e(r) < \infty \tag{CC-e}$$

for the theorem to hold.

Remark 8. Clearly, the strong version of the compatibility condition implies the weak one. If $\mathcal{L}$ has quasi-local interactions, then the (polynomial) decay rate α of the interactions can be chosen to be larger than any fixed value. Consequently, since β and μ can be taken to be linear in α, quasi-local Lindbladians $\mathcal{L}$ satisfy the strong compatibility condition (CC-2).

Under the weak compatibility condition, all the results presented in Sects. 6.1 and 6.2 still hold true, while under the strong compatibility condition also the results presented in Sect. 6.3 are still valid, and in particular our main result, Theorem 20.

We will now show this in the cases in which we made explicit use of condition (LR-1), and give the needed modifications to the proofs of Lemma 15, Proposition 19 and Theorem 20 in order to make them valid for power-law decaying interactions.

From now on, we proceed under the working hypothesis that $\mathcal{L}$ satisfies (LR-2) and that the above compatibility conditions are satisfied.

Proof (*Modifications in the proof of Lemma* 15). The argument below follows closely the proof of the original lemma, but now one must check that $\Delta_0(s)$ is still decaying. Recall the definition of $\Delta_0(s)$ from the original proof of Lemma 15:

$$\Delta_0(s) = (|A(s)| \,/\, |A|)^{\delta v/(v+\gamma)} \nu_{\beta'}^{-1}(s), \quad \beta' = \beta\gamma/(v+\gamma).$$

Since $(|A(s)| \,/\, |A|)^{\delta v/(v+\gamma)}$ grows as $(1+s)^{\delta D v/(v+\gamma)}$, we have:

$$\Delta_0(s) \sim (1+s)^{-\gamma'},$$

where $\gamma' = \frac{\beta\gamma - \delta D v}{v+\gamma}$ is positive because of (CC-1). □

Proof (*Modifications in the proof of Proposition* 19). Keeping the notation introduced in the original proof of this proposition, we have already shown that, for each $0 \leqslant s \leqslant s_0$:

$$\eta^A(T_t) \leqslant \frac{J}{v} |A| \, e^{vt} \nu_\beta^{-1}(s) + c \, |A|^\delta \, p(s) e^{-\gamma t} + \left(\frac{J}{v} |A| + c \, |A|^\delta \right) \Delta_0(s).$$

At this point, we can no longer choose $s = s(t)$ to scale linearly in t, since the decay $\nu_\beta^{-1}(s)$ is polynomial in s and the prefactor e^{vt} would render the bound trivial. Still, we may choose $s = s(t) \in [0, s_0]$ in such a way that the r.h.s. above is exponentially decaying in t.

Fix $k > 0$ (to be determined later), and consider:

$$s(t) = e^{kt} - 1,$$

in such a way that for $t \leqslant \log(1+s_0)/k$, we have:

$$\bar{p}(t) = p \circ s(t) = \left(\left| A(e^{kt} - 1) \right| / |A| \right)^{\delta} \sim e^{kD\delta t}.$$

Then, the r.h.s. of the desired bound for $\eta^A(T_t)$ contains the following exponentials:

$$e^{vt} \nu_\beta^{-1}(s) = e^{-(\beta k - v)t}; \quad p(s) e^{-\gamma t} \sim e^{-(\gamma - kD\delta)t},$$

and

$$\Delta_0(s) \sim (1+s)^{-\gamma'} = e^{-k\gamma' t},$$

where

$$\gamma' = \frac{\beta\gamma - \delta D v}{v + \gamma}$$

is defined in the modified proof of Lemma 15. We want to show that we can choose k in such a way that all the exponential functions appearing above are decaying, i.e. each exponent is negative for $t > 0$. (CC-1) implies that $\Delta_0(s)$ is decaying for all $k > 0$. Let

$$k' = \frac{v + \gamma}{\beta + \delta D},$$

such that $\beta k' - v = \gamma - k' D\delta = k'\gamma'$, making all of the above exponents equal to $-(\beta\gamma - \delta D v)/(\beta + \delta D)$ and negative [due to (CC-1)], as desired.

When $t \geqslant \log(1+s_0)/k'$, as in the proof for exponentially decaying interactions, we bound $\eta^A(T_t)$ by $\eta(T_t)$ (see Remark 7), thus obtaining:

$$\eta^A(T_t) \leqslant c\,|A|^{\delta}\, p(s_0) e^{-\gamma t} \leqslant c\,|A|^{\delta}\, \bar{p}(t) e^{-\gamma t} \sim c\,|A|^{\delta}\, e^{-k'\gamma' t}.$$

□

Proof (*Modifications in the proof of Theorem 20*). Following the same steps as in the original proof, but now using the constants for the local rapid mixing obtained in the modified proof of Proposition 19, we have that, for each $0 \leqslant t_0 \leqslant t$:

$$\int_0^t \left\| K^* O_0(s) \right\| \mathrm{d}s \;\leqslant\; \|K\|_{1\to 1,cb}\, \|O_A\| \left(|A|\, \frac{1}{v} e^{vt_0} \nu_\mu^{-1}(d) + k(|A|) e^{-t_0 \frac{\beta\gamma - \delta D v}{\beta + \delta D}} \right),$$

where $d = \mathrm{dist}(A, \mathrm{supp}\, K)$.

Let us define $t_0(d) = k \log(1+d)$ for some positive k (to be determined later), and denote $h(d) = \nu_{vk-\mu}(d) + \nu_{-k\frac{\beta\gamma - \delta D v}{\beta + \delta D}}(d)$, such that

$$\int_0^t \|K\, O_0(s)\| \,\mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\, \|O_A\|\, k_1(|A|) h(d),$$

where $k_1(|A|) = \max(k(|A|), |A|/v)$. Then we have that h has a maximum decay rate of

$$\mu' = \sup_{k \geqslant 0} \min \left(\mu - vk, k \frac{\beta\gamma - \delta D v}{\beta + \delta D} \right).$$

The optimal choice of k is $k = \frac{\mu}{\beta}\frac{\beta+\delta D}{v+\gamma}$, in such a way that $\mu' = \frac{\mu}{\beta}\frac{\beta\gamma-\delta D v}{v+\gamma}$. μ' is positive because of condition (CC-1).

Recalling the following definitions from the original proof of Theorem 20:

$$q(d) = |\{u : \mathrm{dist}(u, A) = d\}|, \quad l(d) = |\partial_d \Lambda|\, e(d),$$
$$x \star y(d) = \sum_{r=0}^{d} x(r) y(d-r), \qquad d_A = \mathrm{dist}(A, \Lambda^c),$$

we need to show that

$$I_1(A;\, e, h) = h(0)\,|A| \sum_r e(r) + \sum_{d>0} q(d) \left(e \star h(d) + h(0) \sum_{r>d} e(r) \right) e \star h(d)$$

is finite, and that

$$I_2(A;\, e, h) = l \star h(d_A) + \sum_{d \geqslant d_A} l(d) \leqslant \nu_\eta^{-1}(d_A)$$

for some positive η. Notice that

$$I_1(A;\, e, h) \leqslant (1 + |A|) h(0) \sum_d q(d) \sum_{r>d} e(r) + \sum_d q(d)\, e \star h(d).$$

Since $q(d)$ grows as $(1+d)^D$, $\sum_d q(d) \sum_{r \geqslant d} e(r)$ is finite if $\sum_n n^D \sum_{r>n} e(r) < \infty$, which is condition (CC-e).

On the other hand, $e \star h$ decays as the slowest of the two functions. Since we have already assumed that $\sum_d q(d) \sum_{r>d} e(r)$ is finite, we only need to satisfy that $\sum_d q(d) \sum_{r>d} h(r)$ is finite. For this to happen, it suffices that $\mu' > D + 2$, which is condition (CC-2).

In order to bound $I_2(A; e, h)$, note that $l(d) \leqslant |\Lambda|\, e(d)$, and therefore

$$I_2(A; e, h) \leqslant |\Lambda| \, (e \star h(d_A) + \sum_{d \geqslant d_A} e(d)).$$

We have already proven that conditions (CC-2) and (CC-e) imply that the r.h.s. of the latter bound is decaying polynomially in d_A at least as fast as $\nu_{D+1}^{-1}(d_A)$.

This concludes the proof. □

6.5. Relaxing rapid mixing. In this section, we will show that, in the case of exponentially decaying interactions (LR-1), the proof of Theorem 7 still holds if

$$\sum_n n^D \sum_{d>n} \int_d^\infty \left(1 + \frac{v}{\beta} s - \frac{1}{\beta} \log \gamma(s)\right)^{\delta D} \gamma(s)\, \mathrm{d}s < \infty. \tag{41}$$

We will directly prove Proposition 19, without the intermediate step of a Lemma 15. Nonetheless, results of that kind can be proven using exactly the same arguments that we will use in the following proof.

Proof (*Proof of Proposition* 19). Using the same notation as in the original proof, we have that for $0 \leqslant s \leqslant s_0$:

$$\left\|(T_t^* - T_\infty^*)O_A\right\| \leqslant \left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \left\|(T_t^{\overline{B}*} - T_\infty^{\overline{B}*})O_A\right\| + \left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\|.$$

We will bound the first two terms as in the original proof [using Lemma 11, Lemma 12 and Eq. (15)] while we rewrite the third term as in the proof of Lemma 15:

$$\begin{aligned}
\left\|(T_\infty^{\overline{B}*} - T_\infty^*)O_A\right\| &= \left|\mathrm{tr}\, O_A(\rho_\infty^s - \rho_\infty)\right| \\
&\leqslant \left|\mathrm{tr}\, O_A(\rho_\infty - T_t^{\overline{A(s)}}(\rho_\infty))\right| + \left|\mathrm{tr}\, O_A(T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s)\right| \\
&= \left|\mathrm{tr}\, O_A(T_t(\rho_\infty) - T_t^{\overline{A(s)}}(\rho_\infty))\right| + \|O_A\| \left\|T_t^{\overline{A(s)}}(\rho_\infty) - \rho_\infty^s\right\|_1 \\
&\leqslant \left\|(T_t^* - T_t^{\overline{B}*})O_A\right\| + \eta(T_t^{\overline{B}}).
\end{aligned}$$

Thus we have that

$$\left\|(T_t^* - T_\infty^*)O_A\right\| \leqslant 2\frac{J}{v}\,|A|\, e^{vt} v_\beta^{-1}(s) + 2c\,|A|^\delta\, p(s)\gamma(t),$$

where $p(s) = (|(A(s)|\,/\,|A|)^\delta \sim (1+s)^{\delta D}$.

We have claimed that the result only holds in the case of exponentially decaying or faster decay of interaction. Suppose $v_\beta(s) = (1+s)^\beta$ [i.e., if $\mathcal{L}$ satisfies (LR-2)]. Defining $s = s(t)$ as

$$s(t) = e^{\frac{v}{\beta+\delta D}t}\gamma(t)^{-\frac{1}{\beta+\delta D}} - 1,$$

then it holds that

$$\frac{e^{vt}}{\gamma(t)} = p(s(t))v_\beta(s(t)) \quad \forall t \leqslant t_0,$$

where t_0 is such that $s(t_0) = s_0$. Thus

$$\delta_0(t) := e^{vt} v_\beta^{-1}(s) = e^{\frac{\delta D}{\beta+\delta D}vt}\gamma(t)^{\frac{\beta}{\beta+\delta D}}.$$

We have that this last function is decaying in t if

$$\gamma(t) < e^{-(v\delta D/\beta)t}.$$

This forces $\gamma(t)$ to be exponentially decaying, and thus there is no possible relaxation of the rapid mixing condition.

On the other hand, if $v_\beta(s) = e^{\beta s}$ [i.e., if $\mathcal{L}$ satisfies (LR-1)], we define

$$s(t) = \frac{v}{\beta}t - \frac{1}{\beta}\log\gamma(t),$$

such that $e^{vt}v_\beta^{-1}(s) = \gamma(t)$ and

$$\bar{p}(t) = p \circ s(t) \sim \left(1 + \frac{v}{\beta}t - \frac{1}{\beta}\log\gamma(t)\right)^{\delta D}$$

grows polynomially.

In this case, we have proved that

$$\left\|(T_t^* - T_\infty^*)O_A\right\| \leqslant 2\left(\frac{J}{v}|A| + c\,|A|^\delta\right)\bar{p}(t)\gamma(t),$$

and this concludes the proof since Eq. (41) implies that $\bar{p}(t)\gamma(t)$ is decaying in t. □

Proof (*Proof of Theorem* 20) Following the same steps as in the original proof, we have that for any $K \in \{E_{u,r}\}_{u,r} \cup \{E_d\}_d$, and let $\delta = \mathrm{dist}(A, \mathrm{supp}\, K)$

$$\int_0^t \left\|K^* O_0(s)\right\| \mathrm{d}s \leqslant \|K\|_{1\to 1,cb}\, \|O_A\|\, k_1(|A|)h(\delta).$$

where $h(\delta)$ is now

$$h(\delta) = e^{-\mu\delta/2} + \int_{\frac{\mu}{2}\frac{\log v}{v}\delta}^{\infty} \bar{p}(s)\gamma(s)\,\mathrm{d}s.$$

We want to show that $h(\delta)$ is decaying fast enough for the r.h.s. of Eq. (40) to be summable. This is the case (see the proof of Theorem 20 in Sect. 6.4) if Eq. (41) holds. □

7. Glauber Dynamics

7.1. Quantum embedding of Glauber dynamics. As an example of a non-trivial dynamics for which we can now prove stability using our results, we turn to one of the most studied dynamics in classical statistical mechanics: Glauber dynamics, a Markov process that samples thermal states of local (classical) Hamiltonians on lattices. Apart from being an interesting model in itself, it has important applications in Monte-Carlo Markov chain algorithms for numerical many-body physics [41]. Determining whether Glauber dynamics is stable against noise or errors is therefore an important question and, as far as we are aware, still open (with partial results obtained under the assumption of attractiveness [27]).

In this section, we present a natural embedding of Glauber dynamics into the Linbdlabian setting, showing how this embedded dynamics inherits properties from the classical Markov chain.[8] We will then apply the results of Sect. 4 to prove, in the appropriate regime, stability of Glauber dynamics.

We will consider a lattice spin system over $\Gamma = \mathbb{Z}^D$ or $\Gamma = (\mathbb{Z}/L\mathbb{Z})^D$, with (classical) configuration space of a single spin a finite set S. For simplicity, we will consider the case $S = \{+1, -1\}$. For each $\Lambda \subset \Gamma$, we will denote by Ω_Λ the space of configurations over Λ, namely S^Λ. Λ^c will denote the complementary of Λ in Γ, namely $\Gamma\backslash\Lambda$.

Definition 22. A finite range, translationally-invariant potential $\{J_A\}_{A\subset\Gamma}$ is a family of real functions indexed by the non empty finite subsets of Γ satisfying the following properties:

1. $J_A : \Omega_A \to \mathbb{R}$.

[8] A similar construction was proposed in [4].

2. For all $A \subset \Gamma$ and all $x \in \Gamma$:

$$J_A(\sigma) = J_{A+x}(\eta) \quad \text{if } \sigma(y+x) = \eta(y) \quad \forall y \in A.$$

3. There exists a positive $r > 0$ such that $J_A = 0$ if $\operatorname{diam} A > r$, called the range of interaction.

Given a finite-range, translationally-invariant potential, we can define a Hamiltonian for each finite lattice $\Lambda \subset \Gamma$ and each *boundary condition* $\tau \in \Omega_{\Lambda^c}$ by

$$H_\Lambda^\tau(\sigma) = -\sum_{A \cap \Lambda \neq 0} J_A(\sigma \times \tau) \quad \forall \sigma \in \Omega_\Lambda,$$

where $\sigma \times \tau$ is the configuration that agrees with σ over Λ and with τ over Λ^c. For each such Hamiltonian, we define the Gibbs state state as

$$\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp(-H_\Lambda^\tau(\sigma)),$$

where Z_Λ^τ is a normalizing constant.[9] The convex hull of the set of Gibbs states over Λ will be denoted by $\mathcal{G}(\Lambda)$:

$$\mathcal{G}(\Lambda) = \operatorname{conv}\{\mu_\Lambda^\tau | \tau \in \Omega_{\Lambda^c}\}.$$

Definition 23. The Glauber dynamics for a potential J is the Markov process on Ω_Λ with the following generator:

$$(Q_\Lambda f)(\sigma) = \sum_{x \in \Lambda} c_J(x, \sigma) \nabla_x f(\sigma),$$

where $\nabla_x f(\sigma)$ if defined as $f(\sigma^x) - f(\sigma)$, and σ^x is the configuration obtained by flipping the spin at position x:

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } x \neq y \\ -\sigma(x) & \text{if } x = y. \end{cases}$$

The numbers $c_J(x, \sigma)$ are called transition rates and must satisfy the following assumptions:

1. Positivity and boundedness: there exist positive constants c_m and c_M such that:

$$0 < c_m \leqslant c_J(x, \sigma) \leqslant c_M < \infty \quad \forall x, \sigma.$$

2. Finite range: $c_J(x, \cdot)$ depends only on spin values in $b_r(x)$.
3. Translational invariance: for all $k \in \Gamma$,

$$c_J(x, \sigma') = c_J(x + k, \sigma) \quad \text{if } \sigma'(y) = \sigma(y + k) \quad \forall y.$$

4. Detailed balance: for all $x \in \Gamma$ and all σ

$$\exp\left(-\sum_{A \ni x} J_A(\sigma)\right) c_J(x, \sigma) = c_J(x, \sigma^x) \exp\left(-\sum_{A \ni x} J_A(\sigma^x)\right).$$

[9] Following [46], in our notation we have incorporated the usual inverse temperature parameter β directly into the potential J.

These assumptions are sufficient to ensure that Q_Λ generates a Markov process which has the Gibbs states over Λ as stationary points.

Definition 24. A quantum embedding of the classical Glauber dynamics for a potential J is generated by the following Lindblad operators

$$L_{x,\eta} = \sqrt{c_J(x,\eta)}\,|\eta^x\rangle\langle\eta| \otimes \mathbb{1}, \quad \forall x \in \Lambda, \forall \eta \in \Omega_{b_x(r)}; \tag{42}$$

$$\mathcal{L}_{x,\eta}(\rho) = L_{x,\eta}\rho L_{x,\eta}^* - \frac{1}{2}\left\{\rho, c_J(x,\eta)\,|\eta\rangle\langle\eta|\right\};$$

$$\mathcal{L}_\Lambda(\rho) = \sum_{x\in\Lambda}\sum_{\eta} L_{x,\eta}\rho L_{x,\eta}^* - \frac{1}{2}\{\rho, K\}, \quad K = \sum_\sigma \left(\sum_x c_J(x,\sigma)\right)|\sigma\rangle\langle\sigma|\,; \tag{43}$$

plus a dephasing channel acting independently and uniformly on all sites $x \in \Lambda$:

$$D_{x,0} = \sqrt{\gamma}\,|0\rangle\langle 0|\,, \quad D_{x,1} = \sqrt{\gamma}\,|1\rangle\langle 1|\,, \quad \mathcal{D}(\rho) = \sum_{x\in\Lambda}\sum_{i=0,1} D_{x,i}\rho D_{x,i}^* - |\Lambda|\,\gamma\rho. \tag{44}$$

$\mathcal{L}_\Lambda$ satisfies translational invariance because the transition rates c_J do, and it easy to see that this family of Lindbladians is uniform.

Remark 9. Take $|\alpha\rangle\langle\beta|$ an element of the computational basis, and let $d(\alpha,\beta)$ be the Hamming distance between α and β. Then it holds that

$$\mathcal{D}(|\alpha\rangle\langle\beta|) = -\gamma d(\alpha,\beta)\,|\alpha\rangle\langle\beta|\,.$$

In other words, $\mathcal{D}$ is a Schur multiplier in the computational basis, represented by $(-\gamma d(\alpha,\beta))_{\alpha,\beta}$.

On the other hand, we have that for all x:

$$\sum_{\eta\in\Omega_{b_x(r)}} \mathcal{L}_{x,\eta}(|\alpha\rangle\langle\beta|) = \begin{cases} c_J(x,\alpha)\left(|\alpha^x\rangle\langle\beta^x| - |\alpha\rangle\langle\beta|\right) & \text{if } \alpha|_{b_x(r)} = \beta|_{b_x(r)}, \\ -\frac{1}{2}\left(c_J(x,\alpha) + c_J(x,\beta)\right)|\alpha\rangle\langle\beta| & \text{otherwise.} \end{cases} \tag{45}$$

Since $d(\alpha^x,\beta^x) = d(\alpha,\beta)$, $[\mathcal{D}, \sum_\eta \mathcal{L}_{x,\eta}] = 0$ for all $x \in \Lambda$, and in particular $\mathcal{D}$ and $\mathcal{L}_\Lambda$ commute.

This quantum dissipative system inherits various properties from its classical counterpart.

Definition 25. Let μ be a full-rank positive state. Denote by

$$\Gamma_\mu(\rho) = \mu^{\frac{1}{2}}\rho\mu^{\frac{1}{2}}.$$

We say that $\mathcal{L}$ is in detailed balance [37,44,45,59] with respect to μ if $\Gamma_\mu \circ \mathcal{L} = \mathcal{L}^* \circ \Gamma_\mu$.

Proposition 26. *Let μ_Λ^τ be a Gibbs state over Λ. Then $\mathcal{L}_\Lambda$ and $\mathcal{D}$ are in detailed balance with respect to μ_Λ^τ.*

Proof. Note that $\Gamma_{\mu_\Lambda^\tau}$ is a Schur multiplier in the computational basis:

$$\Gamma_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) = \mu_\Lambda^\tau(\eta_1)^{\frac{1}{2}}\mu_\Lambda^\tau(\eta_2)^{\frac{1}{2}}\,|\eta_1\rangle\langle\eta_2|\,.$$

From the detailed balance condition for the transition rates $c_J(x,\sigma)$, it follows that for all $x \in \Lambda$, denoting $\mathcal{L}_x = \sum_{\eta\in\Omega_{b_x(r)}} \mathcal{L}_{x,\eta}$,

$$\begin{aligned}
&\Gamma_{\mu_\Lambda^\tau} \circ \mathcal{L}_x \circ \Gamma^{-1}_{\mu_\Lambda^\tau}(|\eta_1\rangle\langle\eta_2|) \\
&= \delta^x_{\eta_1,\eta_2}\left(c_J(x,\eta_1)\frac{\mu_\Lambda^\tau(\eta_1^x)}{\mu_\Lambda^\tau(\eta_1)}\right)|\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1)+c_J(x,\eta_2)}{2}\,|\eta_1\rangle\langle\eta_2| \\
&= \delta^x_{\eta_1,\eta_2} c_J(x,\eta_1^x)\,|\eta_1^x\rangle\langle\eta_2^x| - \frac{c_J(x,\eta_1)+c_J(x,\eta_2)}{2}\,|\eta_1\rangle\langle\eta_2| \\
&= \mathcal{L}_x^*(|\eta_1\rangle\langle\eta_2|),
\end{aligned}$$

where

$$\delta^x_{\eta_1,\eta_2} = \begin{cases} 1 & \text{if } \eta_1|_{b_x(r)} = \eta_2|_{b_x(r)} \\ 0 & \text{otherwise.} \end{cases}$$

To prove detailed balance for $\mathcal{D}$, note that Schur multipliers commute, thus $[\mathcal{D}, \Gamma_\mu] = 0$. This, together with the fact that $\mathcal{D}^* = \mathcal{D}$, implies that $\mathcal{D}$ is in detailed balance w.r.t. μ_Λ^τ. □

The above proposition implies that Gibbs states are stationary states for the quantum Glauber dynamics. Let us prove that there are no other fixed points apart from the classical ones (i.e. states that are diagonal in the computational basis). Clearly, $\mathcal{D}$ has all classical states as stationary points. We just have to check $\mathcal{L}_\Lambda$.

Proposition 27. *The set of fixed points of $\mathcal{L}_\Lambda$ is equal to $\mathcal{G}(\Lambda)$, the set of Gibbs states over Λ.*

Proof. Let ρ be a fixed point of $\mathcal{L}_\Lambda$. We want to prove that ρ is diagonal, i.e. that it is of the form

$$\rho = \sum_\sigma p_\sigma\,|\sigma\rangle\langle\sigma|\,.$$

Consider a non-diagonal element $|\alpha\rangle\langle\beta|$, and suppose $\alpha(x) \neq \beta(x)$ for some $x \in \Lambda$. Then, from Eq. (45), we have that for all $y \in b_x(r)$,

$$\mathcal{L}_y(|\alpha\rangle\langle\beta|) = -\frac{1}{2}(c_J(y,\alpha)+c_J(y,\beta))\,|\alpha\rangle\langle\beta|\,.$$

For $y \notin b_x(r)$, $\mathcal{L}_y$ is not supported on x, and thus cannot change the configuration there. This implies that the evolution cannot change the configurations over the set $\Delta(r)$, where $\Delta = \{x \in \Lambda | \alpha(x) \neq \beta(x)\}$. In turn, this implies that $\mathcal{L}_\Delta$ commutes with $\mathcal{L} - \mathcal{L}_\Delta$ [since it acts as a Schur multiplier whose entries depend only on the sites in $\Delta(r)$]. Finally, this means that

$$\begin{aligned}
\left\|e^{t\mathcal{L}_\Lambda}(|\alpha\rangle\langle\beta|)\right\|_1 \leqslant \left\|e^{t\mathcal{L}_\Delta}(|\alpha\rangle\langle\beta|)\right\|_1 &= \exp\left(-t\frac{1}{2}\left(\sum_{x\in\Delta} c_J(x,\alpha)+c_J(x,\beta)\right)\right) \\
&\leqslant \exp\left(-t\frac{1}{2}c_m d(\alpha,\beta)\right) \to 0.
\end{aligned}$$

Since the off-diagonal elements are killed, ρ must be of the form $\sum_\sigma p_\sigma |\sigma\rangle\langle\sigma|$. Writing the equation $\mathcal{L}_\Lambda(\rho) = 0$ we obtain

$$\sum_\sigma \sum_x c_J(x,\sigma) p_\sigma |\sigma^x\rangle\langle\sigma^x| - \sum_\sigma \sum_x c_J(x,\sigma) p_\sigma |\sigma\rangle\langle\sigma| = 0,$$

which implies

$$\sum_x c_J(x,\sigma^x) p_{\sigma^x} = \sum_x p_\sigma c_J(x,\sigma).$$

The last equation is simply a rewriting of the fact that (p_σ) is a stationary distribution for Q_Λ, that is, it is exactly a Gibbs state on Λ. □

Since $\mathcal{L}_\Lambda$ and $\mathcal{L}_\Lambda + \mathcal{D}$ have the same stationary distributions, even locally, all properties that depend just on the structure of the fixed-point sets will be shared by both: this is the case, for example, of frustration freeness (which we will prove next) and LTQO (which will be proved later).

Proposition 28. *$\mathcal{L}_\Lambda$ (and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) is frustration free.*

Proof. By the previous proposition, we have that $\mathcal{X}_{\mathcal{L}_\Lambda} = \mathcal{G}(\Lambda)$. We know [41] that for Gibbs states it holds that

$$\Delta \subset \Lambda \Rightarrow \mathcal{G}(\Lambda) \subset \mathcal{G}(\Delta),$$

but this is exactly the frustration-freeness condition for $\mathcal{L}_\Lambda$. □

7.2. Stability of Glauber dynamics. We want to show that the contraction of the semigroup generated by $\mathcal{L}_\Lambda + \mathcal{D}$ can be controlled by the contraction of the classical Glauber dynamics. To fix notation, denote by $\mathcal{C}\colon \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ the projector on the diagonal subspace with respect to the computational basis. $\mathcal{C}$ is a completely positive, trace preserving map, and it also satisfies $\mathcal{C} = \lim_{t\to\infty} \exp(t\mathcal{D})$. Since $\mathcal{L}_\Lambda$ commutes with $\mathcal{D}$, it also commutes with $\mathcal{C}$. Then we can prove the following:

Lemma 29. *If $T_t = \exp(t(\mathcal{L}_\Lambda + \mathcal{D}))$, then*

$$\eta(T_t) \leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})). \tag{46}$$

Proof. Fix an initial state ρ. Then we can write

$$\begin{aligned} \|T_t(\rho) - T_\infty(\rho)\|_1 &\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty(\rho)\|_1 + \|T_t \circ (1-\mathcal{C})(\rho)\|_1 \\ &\leqslant \|T_t \circ \mathcal{C}(\rho) - T_\infty \circ \mathcal{C}(\rho)\|_1 + \|\exp(t\mathcal{D}) \circ (1-\mathcal{C})(\rho)\|_1 \\ &\leqslant \eta(T_t \circ \mathcal{C}) + \eta(\exp(t\mathcal{D})), \end{aligned}$$

where we have used the fact that $\mathcal{L}_\Lambda$ and $\mathcal{D}$ commute, and that the fixed points of $\mathcal{L}_\Lambda$ are invariant under $\mathcal{C}$. □

We know, because of Theorem 6, that

$$\eta(\exp(t\mathcal{D})) \leqslant |\Lambda| e^{-\frac{\gamma}{2}t}, \tag{47}$$

and this implies the following result.

Corollary 30. *If the classical Glauber dynamics satisfies rapid mixing, then also the quantum embedded Glauber dynamics generated by $\mathcal{L}_\Lambda + \mathcal{D}$ does.*

Remark 10. Convergence rates of classical Glauber dynamics are a well studied subject. It is known that, in some regimes, classical Glauber dynamics satisfies a Log Sobolev inequality with system-size independent Log Sobolev constant (for a review on the subject see [46]). In such situations the classical chain has a logarithmic mixing time, and thus satisfies rapid mixing.

For this class of classical dynamical systems it is possible to apply our main result Theorem 7. In particular, we can arbitrary perturb the transition rates $c_J(x,\sigma)$ by some $e(x,\sigma)$, not necessary preserving detailed balance. If we denote by $\mathcal{E}$ the maximum of $|e(x,\sigma)|$, the difference between the perturbed and the original evolution of local observables can be bounded by $\mathcal{E}$ times a factor depending on the size of the support of the observables taken into account.

Theorem 31. *Let Q_Λ the generator of a classical Glauber dynamics, having a unique fixed point and satisfying a Log Sobolev inequality with constant independent of system size. Let E be the generator of another classical Markov process of the form*

$$(Ef)(\sigma) = \sum_{x\in\Lambda} e(x,\sigma)\nabla_x f(\sigma).$$

Suppose that $\mathcal{E} = \sup_{x,\sigma} |e(x,\sigma)| < \infty$ and that $e(x,\cdot)$ has support bounded uniformly in x. Denote by T_t the evolution generated by Q_Λ and by S_t the evolution generated by $Q_\Lambda + E$. Then, for each function f supported on $A \subset \Lambda$, it holds that

$$\|T_t(f) - S_t(f)\|_\infty \leqslant c(|A|)\,\|f\|_\infty\,\mathcal{E},$$

for some $c(\cdot)$ independent of system size and polynomially growing.

Remark 11. It is known [43,47] that the Ising model on $\mathbb{Z}^2$ or $(\mathbb{Z}/n\mathbb{Z})^2$ has a system size independent Log Sobolev constant for high temperatures (when the inverse temperature β is lower than the critical value β_c), or at any temperature in presence of an external magnetic field. In this regime the Glauber dynamics sampling the Ising model is stable (in the sense of Theorem 7).

7.3. Weak mixing and LTQO. As a nice observation, though not necessary to prove Theorem 31, we want show that *weak mixing*, a condition on Gibbs states defined in [46], is equivalent to the LTQO condition given in Sect. 6. The weak mixing conditions for two-dimensional systems has been shown [47] to imply L_2 convergence of the corresponding Glauber dynamics.

Definition 32. We say that the Gibbs measures in $\mathcal{G}(\Lambda)$ satisfy the *weak mixing* condition in $V \subset \Lambda$ if there exist constants C and m such that, for every subset $\Delta \subset V$, the following holds:

$$\sup_{\tau,\tau'\in\Omega_{V^c}} \left\| \mu^\tau_{V,\Delta} - \mu^{\tau'}_{V,\Delta} \right\|_1 \leqslant C \sum_{\substack{x\in\Delta,\\ y\in\partial_r^+ V}} e^{-m\,\mathrm{dist}(x,y)}, \tag{48}$$

where $\partial_r^+ V = \{x \in V^c \,|\, \mathrm{dist}(x,V) \leqslant r\}$ and $\mu^\tau_{V,\Delta} = \mathrm{tr}_{V\setminus\Delta}\,\mu^\tau_V$.

Proposition 33. *If $\mathcal{G}(\Lambda)$ satisfies the weak mixing condition for each $V \subset \Lambda$, then $\mathcal{L}_\Lambda$ (and consequently $\mathcal{L}_\Lambda + \mathcal{D}$) satisfies LTQO.*

Proof. Take $A \subset \Lambda$, $\ell \geqslant 0$, and let V be $A(\ell)$. The weak mixing condition for V implies that there exist constants C and m such that

$$\sup_{\tau,\tau' \in \Omega_{V^c}} \left\| \mu^{\tau}_{V,A} - \mu^{\tau'}_{V,A} \right\|_1 \leqslant C \sum_{\substack{x \in A, \\ y \in \partial_r^+ V}} e^{-m \operatorname{dist}(x,y)} \leqslant C e^{-m\ell} |A| \left| \partial_r^+ A(\ell) \right| .$$

This is the LTQO condition with $\Delta_0(\ell) = C e^{-m\ell} |A| \left| \partial_r^+ A(\ell) \right|$. The bound, proven for states of the form μ^{τ}_V, can be extended by convexity to all $\mathcal{G}(V)$. Let $\eta_0, \eta_1 \in \mathcal{G}(V)$. By definition, η_0 and η_1 are convex combination of states of the form μ^{τ}_V, thus we can write

$$\eta_0 = \sum_i p_i \mu^{\tau_i}_V, \quad \eta_1 = \sum_j q_j \mu^{\sigma_j}_V, \quad \sum_i p_i = \sum_j q_j = 1; \quad p_i, q_j \geqslant 0.$$

Then we have

$$\begin{aligned} \left\| \eta_{0,A} - \eta_{1,A} \right\|_1 &= \left\| \sum_i p_i \mu^{\tau_i}_{V,A} - \sum_j q_j \mu^{\sigma_j}_{V,A} \right\|_1 \\ &= \left\| \sum_i p_i \Big(\sum_j q_j \mu^{\tau_i}_{V,A} \Big) - \sum_j q_j \Big(\sum_i p_i \mu^{\sigma_j}_{V,A} \Big) \right\|_1 \\ &\leqslant \sum_{i,j} p_i q_j \left\| \mu^{\tau_i}_{V,A} - \mu^{\sigma_j}_{V,A} \right\|_1 \leqslant \sup_{\tau,\sigma} \left\| \mu^{\tau}_{V,A} - \mu^{\sigma}_{V,A} \right\|_1 . \end{aligned}$$

□

8. Conclusions and Open Questions

In the context of local perturbations of local Hamiltonians, changes in the ground state can be detected by the lack of smoothness of the expectation value of local observables. Via the quasi-adiabatic technique [25], the regularity of such expectation values can be related to the study of the effect that the perturbation has on the spectral gap of the Hamiltonian. In [49], the stability of the spectral gap was shown under the assumptions of frustration-freeness and local indistinguishability between ground states of local patches of the original Hamiltonian.

In this paper we have studied a class of open quantum systems described by local Lindbladian evolutions with unique fixed points, focusing on the problem of the smoothness of evolution of local observables in the presence of local perturbations. Given any initial configuration, the system will converge toward the fixed point with a certain rate. The slowest rate over all possible initial configurations defines a mixing property of the Lindbladian, and we consider how this scales with the system size. In the case of power-law decay of interactions, we show that a logarithmic scaling is sufficient for the stability of the evolution of local observables, while for exponentially decaying and finite range interactions a scaling at least as fast as a certain polynomial, determined by Eq. (41), is also sufficient. Moreover, the same assumptions imply certain properties of

the fixed point, such as LTQO. It should be emphasized that Log Sobolev inequalities provide strong enough convergence-time estimates to satisfy our assumptions, but that our results also apply more generally.

The most important open question involves state engineering of degenerate topologically ordered states, such as topologically protected quantum codes. For such states, all known preparation maps have a convergence time that is slower than required for our result to apply [36]. It is an interesting question whether it is possible to exploit the very weak requirements in terms of locality of the boundary condition in our definition of uniform families (see Definition 2) to construct faster mixing maps for which one could prove stability, since logical observables partially supported on such boundaries are not necessarily localizable in the sense of [36].

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Appendix A. The Non-Stable Example

The following example will satisfy all the conditions of Theorem 7, except forming an uniform family, and will be shown to be unstable. Interestingly, the system *is* rapid mixing, showing that without the correct structure with respect to system size scaling, rapid mixing alone is not sufficient to imply stability of local observables. This example is the generalization to dissipative systems of the globally gapped but not locally gapped example in [49]. We will show that the characteristics of the dynamics are essentially determined by a classical Markov chain embedded into the Lindbladian. For a general review on convergence of Markov chains, see [40].

Example 4. Consider a chain of $2N$ classical spins, with values in $\{0, 1\}$. Let us define a generator Q^{2N} of a classical Markov chain over the configuration space $\{0, 1\}^{2N}$. We will define Q^{2N} in a translationally-invariant way as follows:

$$Q_c = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -\frac{2}{3N} & 0 & 0 & \frac{2}{3N} \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad Q_r = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$Q_l = \begin{array}{c} \\ |10\rangle \\ |00\rangle \\ |11\rangle \\ |01\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |10\rangle & |00\rangle & |11\rangle & |01\rangle \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \delta_0 = |0\rangle\langle 0| \,, \quad \delta_1 = |1\rangle\langle 1| \,.$$

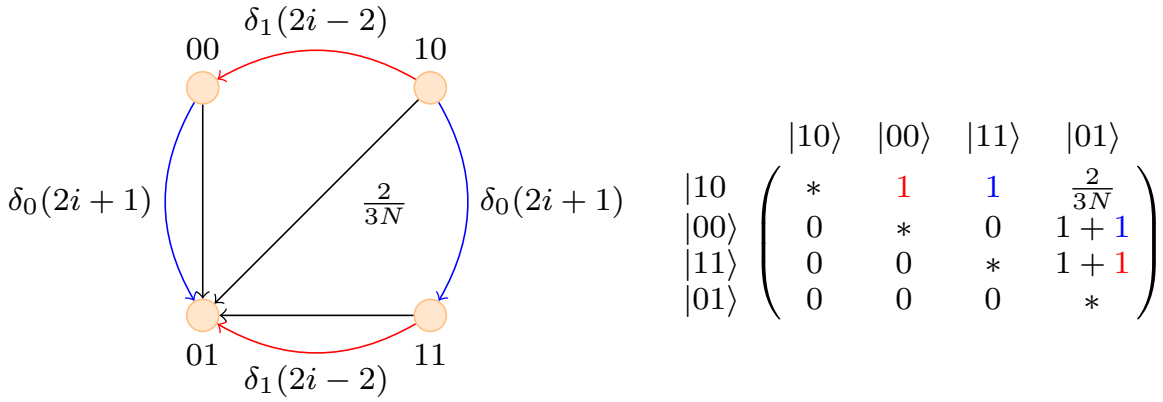

Fig. 3. The transition matrix for Q_i on the spins $(2i-1, 2i)$. The *blue* and the *red* transitions are present depending on the nearby sites: the *blue ones* if there is a 0 on the *right*, the *red ones* if there is a 1 on the *left*. *Asterisks in the diagonal* are such that the sum of each *row* is zero (color figure online)

We then define for each $i = 1, \ldots, N$, a generator matrix Q_i acting on spins $(2i-2, \ldots, 2i+1)$ by

$$Q_i = \mathbb{1} \otimes Q_c \otimes \mathbb{1} + \mathbb{1} \otimes Q_r \otimes \delta_0 + \delta_1 \otimes Q_l \otimes \mathbb{1};$$

and $Q^{2N} = \sum_{i=1}^{N} Q_i$.

The matrix Q_i can only change spins $(2i-1, 2i)$: its transition graph restricted to such spins is presented in Fig. 3.

By construction, Q^{2N} is upper triangular. Thus the elements on the diagonal are the eigenvalues. The unique steady state is then $|0101\ldots01\rangle$, and the smallest non-zero eigenvalue, corresponding to the state $|1010\ldots10\rangle$, is $\frac{2}{3}$. Furthermore, it is easy to see that the diameter of the graph of the transitions of Q^{2N} is N, and in turn this implies that the mixing time for Q^{2N} is of order $O(\log N)$.[10]

Let us now embed this classical Markov chain into a Lindbad operator, in a similar fashion as we have done in Sect. 7 with Glauber dynamics. We will consider then a chain of $2N$ qubits, and define the following Lindblad operators: if k is odd, then

$$\begin{aligned} L_{k,1} &= \sigma_x^{k+1} |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}\,, \\ L_{k,2} &= \sigma_x^{k} \quad |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}\,, \\ L_{k,3} &= \sqrt{\frac{2}{3N}} \sigma_x^k \otimes \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |0\rangle\langle 0|_{k+1}\,; \end{aligned}$$

if k is even, then

$$\begin{aligned} L_{k,1} &= \sigma_x^{k} \quad |0\rangle\langle 0|_k \otimes |0\rangle\langle 0|_{k+1}\,, \\ L_{k,2} &= \sigma_x^{k+1} |1\rangle\langle 1|_k \otimes |1\rangle\langle 1|_{k+1}\,, \\ L_{k,3} &= 0. \end{aligned}$$

The Lindbladian is then defined translationally-invariantly as

$$\mathcal{L}^{2N} = \sum_{k=1}^{2N} \sum_{i=1}^{3} \mathcal{L}_{k,i} + \mathcal{D}_k;$$

[10] This can be seen from the upper triangular form of Q^{2N}, noticing that the polynomials appearing in $e^{tQ^{2N}}$ have degree of at most the diameter of the transition graph.

where $\mathcal{D}_k$ is a dephasing channel acting on site k, as in Eq. (44). Since $L_{k,3}$ depends on N, the family we have defined is not a uniform family.

It is easy to see that the action of $\mathcal{L}^{2N}$ on diagonal states of the form $|\alpha\rangle\langle\alpha|$, with $\alpha \in \{0, 1\}^{2N}$, is equal to that of Q^{2N} acting on α: this is indeed an embedding of Q^{2N}.

Then, by a similar argument as in Sect. 7, we can prove that the fixed points of $\mathcal{L}^{2N}$ are exactly the same as those of Q^{2N} (namely, the unique state $|0101\ldots01\rangle\langle0101\ldots01|$), and that the mixing time of $\mathcal{L}^{2N}$ is bounded by the sum of the mixing times of Q^{2N} and of $\mathcal{D}$. Since both of them are mixing in time $O(\log N)$, we see that $\mathcal{L}^{2N}$ satisfies rapid mixing.

But the system is unstable: if we perturb $\mathcal{L}^{2N}$ by removing the terms generated by $L_{k,3}$ (which is a perturbation of order $O(\frac{1}{N})$), the diagonal state $|1010\ldots10\rangle\langle1010\ldots10|$ becomes a stationary state, and it is clearly locally ortogonal from the original one $|0101\ldots01\rangle\langle0101\ldots01|$.

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CHAPTER 6

Rapid mixing and stability of quantum dissipative systems

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Rapid mixing and stability of quantum dissipative systems

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The physics of many materials is modeled by quantum many-body systems with local interactions. If the model of the system is sensitive to noise from the environment, or small perturbations to the original interactions, it will not properly model the robustness of the real physical system it aims to describe, or be useful when engineering novel systems for quantum information processing. We show that local observables and correlation functions of local Liouvillians are stable to local perturbations if the dynamics is rapidly mixing and has a unique fixed point. No other condition is required.

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Traditionally, the study of quantum many-body systems has focused on constructing simplified models that capture the underlying physics of real materials in order to explain their physical properties and behavior. More recently, quantum information theory has added a complementary perspective, by asking how quantum many-body systems can be artificially engineered to produce useful behavior, such as long-term storage of information [1–4], or processing of information in a quantum computer [5–9]. This has come full circle, with one of the most important applications of quantum information processing being the simulation of other quantum systems which are computationally intractable by classical means [10–13]. Whether studying theoretical models of many-body physics, or artificially engineering their dynamics for information processing purposes, it is crucial that the properties of the model are stable under perturbations *to the model itself*. If the physical predictions of a model undergo dramatic changes when the local interactions are modified by a small amount, it is difficult to argue that the idealized model captures the correct physics of the real physical system. Similarly, if the correct behavior of an engineered quantum system relies on infinitely precise control of all the local interactions, the proposal will not be of much practical use.

In the case of closed systems modeled by Hamiltonian dynamics, recent breakthroughs have given rigorous mathematical justification for our intuition that the physical properties of many-body Hamiltonians are stable to small perturbations. Starting with [14,15], it culminated in the work of [16] which showed that, under a set of mathematically well-defined and physically reasonable conditions, the properties of gapped many-body Hamiltonians are stable under perturbations to the local interactions.

However, even the most carefully isolated physical systems are susceptible to external noise and dissipation. Broadly, many-body theory has traditionally viewed dissipation as a source of errors to be modeled theoretically and minimized experimentally. Recently, the quantum information "engineering" approach has been extended to dissipative quantum systems, with the aim of *exploiting* dissipation. Both theoretical [17,18] and experimental [19–23] work has shown that creating many-body quantum states as fixed points of engineered, dissipative Markovian dynamics can be more robust against undesirable noise, both in maintaining coherence of quantum information for longer times [17,18,24], and in carrying out universal quantum computation via dissipative dynamics [18].

Intuitively, there is an inherent robustness in such proposals: since a dissipative system converges to its steady state eventually, regardless of the state in which it was initialized, the long-term behavior of the system is insensitive to the system's current state. Indeed, this remains the case even if some external process completely changes the state of the system partway through the evolution; if the dissipation is engineered perfectly, the system will inexorably be driven back towards the desired steady state. However, once again, this robustness relies on the hitherto unproven assumption that the physical behavior of the system is insensitive to small implementation errors in engineering the local interactions of the system itself. Therefore, both in justifying theoretical models of real, noisy physical many-body systems, and in the new proposals for exploiting dissipation to carry out quantum information processing tasks, it is crucial to go beyond stability of closed systems, and derive stability results for open, dissipative systems. While earlier works, such as [24], have produced numerical evidence for stability of particular models, we are interested in producing general analytical results.

In this Rapid Communication, we prove that *rapid mixing implies stability against local perturbations*. Our result shows that rapidly mixing systems with unique fixed point are stable in the strongest possible sense: all local observables and correlation functions are stable against local perturbations, independent of the system size. This is true not only in the infinite time limit (i.e., for the steady state), but also for all intermediate times. In other words, we prove that local observables of the perturbed system are good approximations to the unperturbed observables *throughout the entire evolution*. We prove our result for the more general and difficult case of quantum dissipative Markovian dynamics.

Single site noise processes, and all "noninteracting" dissipative processes trivially satisfy our rapid mixing condition. For interacting models proving estimates on the mixing times is generally a hard task; nonetheless, it is known that dissipative state preparation for graph states (a resource for some error-correcting codes and some quantum computation models)

is rapidly mixing [18,25]. Moreover, as classical Markovian dissipative dynamics is a special case of quantum dissipative dynamics, our results also apply to the classical setting; indeed, our results imply stability of classical systems even to quantum perturbations. As an example, we apply our result to prove stability of the important and widely studied classical Glauber dynamics.

For the sake of simplicity of the exposition, we restrict our attention to translationally invariant, nearest-neighbor, dissipative interactions on spins arranged on a D-dimensional square lattice with periodic boundary conditions. The proof in the general case follows the same ideas, but becomes notationally involved. It is available in [26].

a. Terminology. Rapid mixing corresponds to the assumption that the convergence of the density matrix $\rho(t)$ of the system to its steady state ρ_∞, as a function of time t, is of the form $\|\rho(t) - \rho_\infty\|_1 \leqslant c\,\mathrm{poly}(L)e^{-\gamma t}$ for some constants c, γ independent of system size, where L is the linear size of the system. Since we are considering finite dimensional systems, the exponential convergence with respect to time is a general property; the nontrivial content of the rapid mixing condition is how γ and the multiplicative prefactor depend on L.

Local perturbation means that the local interactions of the system can be modified *everywhere*. Indeed, our result applies more generally to arbitrary perturbations composed of a sum of local terms, not only to modifications of the strength of the original local interactions of the system. This is the natural (and standard) model of perturbations in physical systems with local interactions. Note that the total perturbation is a sum of all the local terms, and therefore may diverge with system size regardless of how weak the local perturbations are. Standard perturbation theory breaks down completely in this setting, as the overall perturbation is usually unbounded. It is, instead, the *local* structure of the perturbation that permits stability in our setting. Moreover, recall that a linear map from operators to operators is called a *superoperator*. The *support* of a superoperator is defined to be the smallest set $\Gamma \subset \mathbb{Z}_L^D$, such that the operator acts trivially outside of Γ.

The restriction to local observables and correlation functions, apart from being justified by practical considerations of what can be measured in experiments, also has a fundamental theoretical justification: global observables on the full system cannot be stable to local perturbations. (This is equally true for Hamiltonian systems.) It is easy to construct simple examples that demonstrate this [27]. But it is also intuitively obvious from the above discussion: global observables can "see" the effect of the local perturbations integrated over the entire system, and this effect diverges with system size.

While our result is motivated by the work of [16] for Hamiltonian systems, both the result itself and several of the concepts and techniques required for the proof are different in the dissipative case. In the Hamiltonian case, stability is proven under the assumption that the system is frustration-free, has local topological quantum order (LTQO), and is locally gapped. In the dissipative case, our result derives stability for all rapidly mixing systems (which can be viewed as the dissipative analog of the local-gap condition for Hamiltonians), without any need for frustration-freeness (i.e., detailed balance), or LTQO for the steady state. We are able to derive the necessary properties of the steady state from the rapid mixing condition alone. Moreover, the technical proof in the Hamiltonian setting relies on the fact that Hamiltonian dynamics is reversible. This is by definition false for dissipative systems, necessitating a different mathematical approach.

b. Main result. Let $\Lambda \simeq \mathbb{Z}_L^D$ denote the D-dimensional square lattice. The dynamics is then generated by a local Liouvillian $\mathcal{L} = \sum_{u\in\Lambda} L_u$ (the dissipative analog of a local Hamiltonian), where each L_u has the well-known Lindblad form (the most general form that preserves complete positivity of the density matrix): $L_u(A) = i[H_u, A] + \sum_j [K_{u,j}^\dagger A K_{u,j} - \frac{1}{2}(K_{u,j}^\dagger K_{u,j} A + A K_{u,j}^\dagger K_{u,j})]$, where $K_{u,j}$ are arbitrary operators and H_u is Hermitian. The L_u terms are related by translation, with each term acting only on u and its neighbors. The evolution of an observable A in the Heisenberg picture is then given by $A(t) = e^{t\mathcal{L}}(A)$, which is the solution to the differential Liouville master equation $\dot{A}(t) = \mathcal{L}A(t)$. We can assume without loss of generality that the strength of the local interactions L_u is bounded as follows (in the completely bounded norm): $\sup_u \|L_u\|_{cb} := \sup_u \sup_n \|L_u \otimes \mathbb{1}_n\| \leqslant 1$. We will also assume that $\mathcal{L}$ has a unique fixed point, i.e., in the Heisenberg picture $A(\infty) := \lim_{t\to\infty} A(t) = \mathrm{Tr}(A\rho_\infty)\mathbb{1}$ for any observable A [28].

In the Heisenberg picture, the rapid mixing condition states that, for any observable A, $A(t)$ converges fast to $A(\infty)$. More precisely, there exist positive constants c, δ, and γ independent of system size, such that

$$\|A(t) - A(\infty)\| \leqslant cL^\delta e^{-\gamma t}. \tag{1}$$

The perturbed evolution is given by a different local Liouvillian, $\tilde{\mathcal{L}}$, such that $\tilde{\mathcal{L}} = \mathcal{L} + \sum_u E_u$, where the perturbation terms E_u are local, and their strength is bounded by ε (i.e., $\sup_u \|E_u\|_{cb} \leqslant \varepsilon$).

Theorem 1. For an observable A supported on $X \subset \Lambda$, let $A(t) = e^{t\mathcal{L}}(A)$ and $\tilde{A}(t) = e^{t\tilde{\mathcal{L}}}(A)$ be the time evolution of the observables in the Heisenberg picture, under the original and perturbed Liouvillians, respectively.

Then for all $t \geqslant 0$,

$$\|A(t) - \tilde{A}(t)\| \leqslant C_X \|A\| \varepsilon, \tag{2}$$

for some $C_X > 0$ not depending on the system size and independent of t.

Note that we do not require the support X of the observable A to be connected. Our result therefore immediately applies to two-point (or, more generally, k-point) correlation functions.

In fact, the same result applies to systems with quasilocal interactions, where the interactions L_u and perturbations E_u act on arbitrarily distant spins, but the interaction strength decays exponentially with distance. The results also generalize to interactions with polynomially decaying strength, and to the nontranslationally invariant case with arbitrary boundary conditions (under natural uniformity conditions that make the concept of scaling with system size meaningful). Moreover, the rapid mixing condition given in Eq. (1) can be weakened to slower-than-exponential decay: $\|A(t) - A(\infty)\| \leqslant c\,L^\delta \gamma(t)$, where $\gamma(t)$ is decaying at least as $(1+t)^{-(D+2+\delta+\eta)}$, for some arbitrarily small $\eta > 0$ [26].

c. Sketch of the proof. The main technical tool we need is the Lieb-Robinson bound [29,30]. In many-body quantum systems, where the evolution is generated by local interactions,

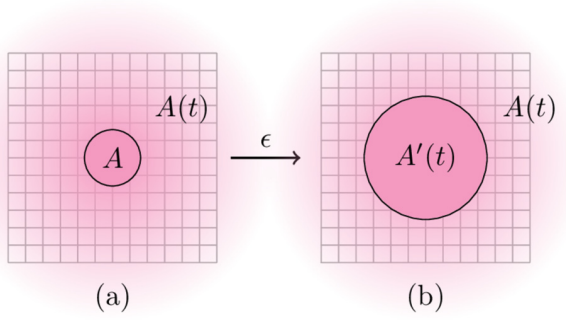

FIG. 1. (Color online) (a) The support of a local observable A spreads linearly in time at the Lieb-Robinson velocity, up to an exponentially small error. (b) The time-evolved observable $A(t)$ can be approximated to small error ε by a local observable $A'(t)$.

there exists an effective "light cone" outside of which the amount of information that can escape is negligible. The effective velocity that limits the light cone is called the Lieb-Robinson velocity, and is in general many orders of magnitude smaller than the actual speed of light [31].

The existence of such light cones implies that localized observables spread linearly in time, up to negligible tails outside the cones (Fig. 1). Since the system is rapidly mixing by assumption, by the time the system has relaxed and reached its steady state, any finite region of the lattice has only had time to interact with a bounded region around it, namely, a region of size proportional to the mixing time. There is effectively no further evolution after that time scale. This implies that a local observable feels the effects of only part of the total perturbation: the local perturbations acting near the support of the observable. One might then be tempted to consider just this effective perturbation and obtain a bound for the evolution of the observable under examination. However, this is not yet sufficient for our purposes, as this reduced perturbation still scales (sublinearly) with the system size, so diverges for large system sizes.

We improve on this idea by showing that, under the same conditions, evolution of a local observable can be approximated in a finite region around its support, with a localized evolution that only takes a *finite* time to reach its steady state. Since we are working with a translation invariant model with periodic boundary conditions, the localized evolution we choose is the one given by the global Liouvillian, but defined on a smaller lattice size. After proving this stronger property, it is then straightforward to apply the original approach of restricting the perturbation to a finite region, leading to the proof of the main result.

d. Proof of main result. To fix notation, we will consider a normalized observable A supported on a region X, and will denote by $X(s)$ the region X "grown" by s, i.e., $X(s) = \{u \in \Lambda : \mathrm{dist}(u,X) \leqslant s\}$. Without loss of generality, we can assume that $X(s)$ is always a disjoint union of convex regions [32]. We will consider the Liouvillian $\mathcal{L}_s$ acting on $X(s)$, defined by translational invariance and periodic boundary conditions. The evolution of A under this new Liouvillian will be denoted by $A_s(t)$. Now, since $\mathcal{L}_s$ is none other than the same local Liouvillian on a smaller lattice, the rapid mixing condition of (1) applies, immediately giving

$$\|A_s(t) - A_s(\infty)\| \leqslant c_X(1+s)^{\delta}\gamma(t), \tag{3}$$

for some appropriate constant c_X, recalling that the linear size of $X(s)$ is bounded by $\mathrm{diam}(X) + 2s$.

Consider a superoperator $\mathcal{T}$ supported on a region Y, such that $d = \mathrm{dist}(X,Y) > 0$, and assume that $\mathcal{T}(\mathbb{1}) = 0$. The dissipative version of the Lieb-Robinson bound states that there exist some positive constants k_X, v, and μ, independent of system size, such that for all $t \geqslant 0$: $\|\mathcal{T}[A(t)]\| \leqslant k_X \|\mathcal{T}\|_{cb}(e^{vt} - 1)e^{-\mu d}$. A known consequence of Lieb-Robinson bounds is that we can approximate the evolution of a local observable by a *localized* evolution, i.e., by a time-evolved observable whose support only grows linearly with time. Since Lieb-Robinson bounds depend only on the microscopic structure of the evolution, the presence of a boundary condition has a negligible effect on the localized evolution of local observables. Therefore, one may add periodic boundary conditions to the localized evolution coming from the standard Lieb-Robinson bounds, while still obtaining a good approximation for the original evolution of the local observables. More formally, we obtain the following bound, valid for all $s \geqslant 0$:

$$\|A(t) - A_s(t)\| \leqslant k_X(e^{vt} - 1)e^{-\mu s}. \tag{4}$$

A number of properties of the system can be derived from Eqs. (3) and (4). By the definition of the fixed point, we have that $A(\infty) = \mathrm{Tr}(A\rho_\infty)\mathbb{1} = \mathrm{Tr}[A(t)\rho_\infty]\mathbb{1}$. Then by the triangle inequality:

$$\begin{aligned}
&\|A(\infty) - A_s(\infty)\| \\
&\quad = |\,\mathrm{Tr}[A\rho_\infty] - \mathrm{Tr}[A\rho_\infty^s]| \\
&\quad \leqslant |\,\mathrm{Tr}\{[A(t) - A_s(t)]\,\rho_\infty\}\,| + |\,\mathrm{Tr}\{[A_s(t) - A_s(\infty)]\,\rho_\infty\}\,| \\
&\quad \leqslant \|A(t) - A_s(t)\| + \|A_s(t) - A_s(\infty)\|.
\end{aligned}$$

Together with Eqs. (4) and (3) and choosing t linear in s, it implies that $\|A(\infty) - A_s(\infty)\|$ decays with s. This in turn implies a stronger convergence bound for $A(t)$, since

$$\begin{aligned}
\|A(t) - A(\infty)\| &\leqslant \|A(t) - A_s(\infty)\| + \|A_s(\infty) - A(\infty)\| \\
&\leqslant 2\,\|A(t) - A_s(t)\| + 2\,\|A_s(t) - A_s(\infty)\|\,.
\end{aligned} \tag{5}$$

Again by applying Eqs. (4) and (3), the right-hand side is bounded by a decaying function $\Delta(t)$, if s is chosen to scale linearly in t. The big difference with respect to the rapid mixing condition is that we have managed to remove the dependence on the system size from the pre-factor of the right-hand side, since the bounds in Eqs. (4) and (3) are system size independent. Of course, this was possible because A is a local observable. Note that the assumption made on $\gamma(t)$ implies that $\Delta(t)$ goes to zero at least as $(1+t)^{-(D+2+\eta)}$.

Once we have established such size-independent bounds, we can directly show—by one last application of Lieb-Robinson bounds—stability of the evolution of $A(t)$. Let us decompose the quantity we want to bound as follows: $A(t) - \tilde{A}(t) = \sum_u \int_0^t e^{(t-s)\tilde{\mathcal{L}}} E_u A(t)\,ds$. Let us take norms and use the fact that $e^{(t-s)\tilde{\mathcal{L}}}$ is norm contractive:

$$\|A(t) - \tilde{A}(t)\| \leqslant \sum_u \int_0^t \|E_u A(t)\|\,ds. \tag{6}$$

For each $u \in \Lambda$, call $d = \mathrm{dist}(X,u)$ and fix a time scale $t_0 = t_0(d)$ to be determined later. For short times, i.e., for times $t \leqslant t_0$, we can apply the standard Lieb-Robinson bounds and thus $\int_0^{t_0} \|E_u A(t)\|\, ds \leqslant k_X \varepsilon\, e^{v t_0 - \mu d}$. For long times, i.e., $t \geqslant t_0$, we bound the integral by using Eq. (5) and the fact that $E_u(\mathbb{1}) = 0$ [33]: $\int_{t_0}^{\infty} \|E_u A(t)\|\, ds \leqslant \varepsilon \int_0^{t_0} \Delta(s) ds$. We can now choose $t_0(d) = \frac{\mu}{2v} d$, such that the integral is entirely bounded by a function decaying in d. By putting this back into Eq. (6), we can sum over all terms u and obtain the claimed result:

$$\|A(t) - \tilde{A}(t)\| \leqslant \varepsilon \sum_u \left[k_X e^{-(\mu/2)d} + \int_0^{\mu d/2v} \Delta(s) ds \right] \leqslant C_X\, \varepsilon.$$

The sum is convergent because $\int_0^d \Delta(s) ds$ decays to zero at least as fast as $(1+t)^{-(D+1+\eta)}$.

e. Glauber dynamics. One of the systems which satisfies the conditions of our theorem is classical Glauber dynamics [34] (the continuous-time version of the Metropolis algorithm), in the regime in which it has a system-size-independent Log-Sobolev constant. By embedding this dynamics into a quantum Liouvillian in a careful way, our result immediately implies that Glauber dynamics is stable against local perturbations (even those that do not preserve detailed balance). (Related results, but with different assumptions, were given in [35].) Given the importance of Glauber dynamics to sampling from the thermal distributions of classical spin systems [34,36], we expect our results to have applications also to classical statistical mechanics.

f. Conclusions. We have considered the influence that a small but extensive perturbation to the generators of a dissipative quantum many-body master equation can have on the evolution of local observables. We have shown that, if the system relaxes to its unique fixed point sufficiently fast, the observables are stable to such local perturbations throughout the entire evolution: the effect of the observables depends linearly on the microscopic strength of the perturbation, independently of the system size, even though the magnitude of the overall perturbation diverges with system size. Stability is therefore a result of the local structure of the perturbations.

While the requirement of rapid mixing does not cover all possible interesting quantum systems, the result already has important applications in well-studied models: it applies to dissipative state preparation of graph states [25], a resource for universal quantum computation; to classical Glauber dynamics, one of the most important models in statistical mechanics; and to the modeling of local noise—e.g., the physically important case of independent local depolarizing noise—as well as any other noise model which acts independently on every particle in the system. The latter case justifies the choice of a particular type of noise in a theoretical model without requiring perfect knowledge of the form of physical noise (which is essentially unknowable by definition).

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[27] Consider as an example, two dissipative processes acting on a qubit and having as a fixed point two pure states with an overlap of $1-\varepsilon<1$. Then, by taking the tensor product of N of each process, we note that the fixed points of the composed processes become orthogonal as $N\to\infty$, for any fixed $\varepsilon>0$.

[28] It is easy to see that in a system with degenerate fixed points space, a small local perturbation could destroy the degeneracy by selecting some of the fixed points over the others. Therefore it is not possible to generalize our result to such scenario without either requiring extra assumptions or weakening the notion of stability.

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[31] Indeed, the models we consider are nonrelativistic, so the existence of such a limit speed is not at all trivial. It could more accurately be described as a speed of sound, as it depends on the microscopic structure of the "medium," i.e., the local interaction terms.

[32] When s is sufficiently large for disconnected regions in X to touch each other, we simply merge them by considering their convex hull.

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CHAPTER 7

Area law for fixed points of rapidly mixing dissipative quantum systems

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Area law for fixed points of rapidly mixing dissipative quantum systems

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We prove an area law with a logarithmic correction for the mutual information for fixed points of local dissipative quantum system satisfying a rapid mixing condition, under either of the following assumptions: the fixed point is pure or the system is frustration free. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4932612]

I. INTRODUCTION

One of the problems common to quantum information and condensed matter physics is understanding correlations and entanglement in many-body quantum states. This is motivated by the observation that many interesting states behave very differently from random states: while in the latter, the entanglement entropy of a sub-region scales as the volume of the region, and in the states that occur in quantum many-body systems, this entanglement entropy is often seen to scale only with the boundary of the region. This surprising behaviour has been called the *area law* (the terminology comes from 3D systems), and it is a well-studied conjecture that ground states of (gapped) local Hamiltonians should satisfy an area law. Even in the case of critical systems and gapless Hamiltonians, evidence suggests that the ground state still has a sub-volume growth of the entanglement, with a rate proportional to the boundary of the region times a logarithmic correction.[1–3] While it has been formally proven only for 1D systems,[4–8] area laws have also been proven in specific cases in higher dimensions (harmonic lattice systems,[9] models satisfying local topological quantum order,[10] perturbations of gapped Hamiltonians satisfying an area law,[11,12] and with a logarithmic correction for fermionic systems[13,14]) and are the subject of active research.

Recently, a different class of states arising in quantum many-body systems has been attracting attention in the quantum information literature: fixed points of (local) dissipative processes. More precisely, fixed points of semigroups of trace preserving completely positive linear maps. The motivation is two-fold: on the one hand, such processes model most of the different types of noise that can be found in nature and therefore provide a more realistic model for physical systems, since in practice no system will be completely isolated. On the other hand, proposals have been made to artificially engineer such dissipative interactions in order to have a determined quantum state as a fixed point, effectively making them "dissipative machines" for producing useful/interesting quantum states.[15,16] This dissipative state engineering has been experimentally shown to be a robust mechanism to maintain coherence.[17,18]

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A natural question then arises: is there an area law (either strict or with a logarithmic correction) in this context? Note that since fixed points of dissipative evolutions are generically not pure, we must find another measure of entanglement or correlations, since local entropy is no longer a useful measure for mixed states (as the trivial example of the maximally mixed state shows).[19] proposed instead using the *mutual information*, a measure of correlations between two parts of a quantum state. It has the advantage of coinciding with entanglement entropy for pure states, and it upper bounds operational measures of entanglement in the mixed-state case, such as the distillable entanglement.[20] In Ref. 19, it was also shown that thermal states of local Hamiltonians satisfy an area law for the mutual information. However, thermal states do not cover all the possible fixed points of dissipative systems. Pinning down under which conditions an area law for the mutual information holds for general dissipative quantum many-body systems is thus an interesting open problem.

For Hamiltonian systems, the main assumption that is usually made is the presence of a spectral gap: a non-vanishing separation between the two lowest energy levels of the Hamiltonian. In the dissipative setting, instead of spectral assumptions, it is more natural to make assumptions on the speed of convergence of the dissipation towards its fixed point (a quantity that is not controlled by the spectrum alone[21]) or equivalently on the so-called *mixing time*. In this work, we restrict to systems for which the mixing time scales logarithmically with the system size. In a previous paper,[22] some of the authors showed that such systems — which we called *rapid mixing* — are stable under local perturbations. Similarly to the gap of closed quantum systems, proving rapid mixing for dissipative systems is a daunting task. There are however some interesting key examples: state preparation of graph states,[23] classical Glauber dynamics for the Ising model in 2D (in some range of parameters),[24,25] among others.

Rapid mixing is implied in many cases by a well studied property of dissipative evolutions, namely, the existence of a system-size independent Log-Sobolev constant for primitive reversible Liouvillians.[26–30] Under such assumptions, in Ref. 31, it was proven a bound on the mutual information of the form:

$$I(A : A^c) \leq c \log\log \left\|\rho^{-1}\right\| |\partial A|. \tag{1}$$

In the latter bound, ρ is the fixed point of the evolution, and therefore, $\left\|\rho^{-1}\right\|$ will usually depend at least exponentially on the total system size, and sometimes even worse. As recognized by the authors of Ref. 31, this poses a serious problem in considering Eq. (1) a satisfactory area law. It indicates however that rapid mixing seems to be the required condition to have an area law in the dissipative setting. This is exactly what we prove in the present paper, with the following results:

1. if the system satisfies rapid mixing and the fixed point is pure, then it satisfies an area law with a logarithmic correction for the entanglement entropy;
2. if the system satisfies rapid mixing and is frustration free, meaning that the local terms of the Liouvillian share a common steady state, then such fixed point satisfies an area law with a logarithmic correction for the mutual information.

Compared with (1), the bounds we obtain do not have any dependence on the total system size. Moreover, we do not require primitivity or reversibility of the generators of the evolution, and we only require rapid mixing instead of a system-size independent Log-Sobolev constant (a strictly weaker assumption, since the Log-Sobolev constant is undefined for non-primitive Liouvillians).

It is known that there is a connection between area laws and decay of correlations.[7,19] Therefore, it does not come as a surprise that with the tools we have developed for proving the area law, we can also prove a decay of correlations measured with the mutual information. It is worth noting that with the results available in the literature and to the best of our knowledge, it is not possible to derive the area law from the type of decay of correlations we will show here.

As we have mentioned earlier, for ground states of closed systems, a logarithmic correction is usually considered a signature of a gapless Hamiltonian. For open systems and mixed states, the situation is less clear: already in Ref. 19, it was shown that thermal states of local Hamiltonian satisfy an area law without a logarithmic correction irrespective of the gap of the Hamiltonian. For these states, the bound we obtained is therefore not optimal. We do not know whether there exist

systems which saturate our bound or if instead the correction is only an artefact of the proof. If there exist systems that saturate our bound, it would then imply that their fixed point have a very interesting property: while still satisfying an exponential decay of correlations, they do not satisfy an area law without a logarithmic correction. Having an example of such state which can also be efficiently prepared with a dissipative process would be interesting on its own, as it could lead to new insight into the relationship between area laws and decay of correlations.

We conjecture that rapid mixing alone, without any additional assumptions, should imply an area law for mutual information, but we do not have a formal proof. The fact that we have two different proofs of an area law, requiring different extra assumptions (pure fixed point on the one hand, frustration freeness on the other) is strong evidence for this conjecture.

The paper is organized as follows. In Section II, we set up the problem and introduce the necessary notation and definitions. In Section III, we prove two lemmas regarding localization properties of the fixed point of the dissipative maps: Lemma 11 is based only on the rapid mixing assumption and will be used in Section IV to prove decay of correlations and in Section V to prove the area law for pure fixed points; Lemma 12 instead requires the extra assumption of frustration freeness and will be used in Section V to prove the area law for the mutual information in the case of mixed fixed points.

II. SETUP AND NOTATION

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a finite dimensional complex Hilbert space, representing a bipartite quantum system. We denote by $\mathcal{B}(\mathcal{H}_{AB})$ the space of bounded linear operators on $\mathcal{H}_{AB}$. A *state* is given by a positive semi-definite operator $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$, normalized to have trace equal to one. The *reduced density matrix* of the subsystem A (respectively, B) will be denoted by ρ_A (respectively, ρ_B), and it is given by $\rho_A = \mathrm{Tr}_A \rho_{AB}$ (respectively, $\rho_B = \mathrm{Tr}_B \rho_{AB}$), where the partial trace Tr_A is defined to be the unique linear operator $\mathrm{Tr}_A : \mathcal{B}(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_B)$ such that $\mathrm{Tr}_A(x \otimes y) = y\mathrm{Tr}(x)$ for all x in $\mathcal{B}(\mathcal{H}_A)$ and all y in $\mathcal{B}(\mathcal{H}_B)$ (Tr_B is similarly defined).

We will use the standard Dirac notation for Hilbert spaces, denoting vectors as $|\phi\rangle$, adjoint vectors as $\langle\phi|$, $\langle\phi|\psi\rangle$ for the scalar product, and $|\phi\rangle\langle\psi|$ for rank one linear maps. The canonical basis will be indexed by natural numbers starting from zero: $|0\rangle, |1\rangle, \ldots, |n\rangle$.

We will denote by $\mathrm{poly}(x)$ any polynomial in the variable x with real coefficients and arbitrary degree.

A. Measures of correlations

Given a state $\rho_{AB} \in \mathcal{B}(\mathcal{H}_{AB})$ of a bipartite system, there are a number of possible measures of how "distant" the state ρ_{AB} is from being a product state, i.e., of the form $x \otimes y$ for some x in $\mathcal{B}(\mathcal{H}_A)$ and y in $\mathcal{B}(\mathcal{H}_B)$. Since a product state represents a system in which measurements over the subsystem A are independent of measurements over the subsystem B, we will talk of *correlation measures* between subsystems A and B. We will need to define and use three of such measures. We will follow the same terminology of Ref. 31.

Definition 1 (Correlation measures).

- Covariance correlation:

$$C(A:B) = \max_{\substack{M \in \mathcal{B}(\mathcal{H}_A), N \in \mathcal{B}(\mathcal{H}_B) \\ \|M\| \leq 1, \|N\| \leq 1}} |\langle M \otimes N\rangle - \langle M\rangle\langle N\rangle|$$
$$= \max_{\substack{M \in \mathcal{B}(\mathcal{H}_A), N \in \mathcal{B}(\mathcal{H}_B) \\ \|M\| \leq 1, \|N\| \leq 1}} |\mathrm{Tr}\left[M \otimes N(\rho_{AB} - \rho_A \otimes \rho_B)\right]|,$$

where $\langle O\rangle = \mathrm{Tr}(O\rho_{AB})$ *is the expectation value of the observable* O *acting on* ρ_{AB}.

- Trace distance correlation*:*

$$T(A:B) = \max_{\substack{F\in\mathcal{B}(\mathcal{H}_{AB})\\ \|F\|\leq 1}} |\mathrm{Tr}\left[F(\rho_{AB}-\rho_A\otimes\rho_B)\right]| = \|\rho_{AB}-\rho_A\otimes\rho_B\|_1.$$

- Mutual information correlation*:*

$$I(A:B) = S(\rho_A)+S(\rho_B)-S(\rho_{AB}),$$

where $S(\rho) = -\mathrm{Tr}(\rho\log_2\rho)$ *is the von Neumann entropy of the state* ρ*.*

When it is not clear from context which state ρ_{AB} we are considering, we indicate it in a subscript to avoid ambiguity and write $C(A:B)_\rho$, $T(A:B)_\rho$, and $I(A:B)_\rho$.

As it should be clear from the definition, $C(A:B)$ is always upper bounded by $T(A:B)$. Moreover, by Pinsker's inequality,[32]

$$C(A:B)\leq T(A:B)\leq 2\sqrt{I(A:B)}.$$

Therefore, mutual information is the strongest correlation measure. It is also a well known consequence of the Alicki-Fannes-Audenaert inequalities[33–35] that there is a non-linear inverse relationship between trace distance and mutual information: there is a differentiable function $f(x)$ vanishing at zero such that $I(A:B)\leq f(T(A:B))$. Such non-linear equivalence between the two measures will allow us to take bounds on $T(A:B)$ (which in the context of our assumptions will be easier to deduce) and obtain information on the behavior of $I(A:B)$.

We state this result in a form that will be more convenient for us and, for the sake of completeness, present a short proof of it.

For $x\in[0,1]$, $h_b(x) = -x\log_2 x-(1-x)\log_2(1-x)$ denotes the binary entropy function.

Theorem 2. *The following inequalities hold:*

Fannes-Audenaert:[33,34]

Let $\rho,\sigma\in\mathcal{B}(\mathbb{C}^d)$*, and let* $\delta = \|\rho-\sigma\|_1 < 1$*. Then,*

$$|S(\rho)-S(\sigma)|\leq 2\delta\log_2(d-1)+2h_b(\delta). \tag{2}$$

Alicki-Fannes:[35]

Let $\rho^{AB},\sigma^{AB}\in\mathcal{B}(\mathbb{C}^{d_A}\otimes\mathbb{C}^{d_B})$*, and let* $\delta = \left\|\rho^{AB}-\sigma^{AB}\right\|_1 < 1$*. Then,*

$$|S(\rho^{AB}|\rho^B)-S(\sigma^{AB}|\sigma^B)|\leq 4\delta\log_2 d_A+2h_b(\delta), \tag{3}$$

where the conditional von Neumann entropy $S(\rho^{AB}|\rho^B)$ *is defined as* $S(\rho^{AB}|\rho^B) = S(\rho^{AB})-S(\rho^B)$*.*

Combining the two previous inequalities, we will obtain the desired non-linear bound on $I(A:B)$.

Corollary 3. Let $\delta = \left\|\rho^{AB}-\sigma^{AB}\right\|_1 < 1$*. Then,*

$$|I(A:B)_\rho - I(A:B)_\sigma|\leq 6\delta\log_2 d_A+4h_b(\delta). \tag{4}$$

In particular, if we take $\sigma^{AB} = \rho^A\otimes\rho^B$*, we have that* $I(A:B)_\sigma = 0$*,* $\delta = T(A:B)_\rho$*, and thus,*

$$I(A:B)_\rho\leq 6T(A:B)_\rho\log_2 d_A+4h_b(T(A:B)_\rho). \tag{5}$$

Proof. Note that $\left\|\rho^{AB}-\sigma^{AB}\right\|_1\leq\delta\Rightarrow\left\|\rho^A-\sigma^A\right\|_1\leq\delta$. Applying triangle inequality and Equations (2) and (3) gives

$$|I(A:B)_\rho - I(A:B)_\sigma|\leq|S(\rho^A)-S(\sigma^A)|+|S(\rho^{AB}|\rho^B)-S(\sigma^{AB}|\sigma^B)|.$$

□

B. Many-body quantum systems

Let us now recall the standard definitions and the common notation for many-body quantum systems.

We will consider a quantum system defined on the square lattice $\Gamma = \mathbb{Z}^D$ equipped with the graph metric, where at each site $x \in \Gamma$, we associate a finite-dimensional complex Hilbert space $\mathcal{H}_x$. We choose to work with a square lattice for simplicity of exposition, but the results presented can be generalized straightforwardly to graphs with polynomial growth (i.e., with balls size growing polynomially with the diameter). The ball centered at x of radius r will be denoted by $b_x(r)$. We will use the following convention: given a subset $A \subset \Lambda$, we will denote by $A(s)$ the smallest disjoint union of balls containing $\{x \in \Lambda | \text{dist}(x, A) \leq s\}$. For each finite subset $\Lambda \subset \Gamma$ of the lattice, we will associate a Hilbert space $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$ and an algebra of observables $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$. We will equip $\mathcal{A}_\Lambda$ with the Hilbert-Schmidt scalar product $\langle A, B\rangle = \text{Tr}(A^*B)$.

A linear map $\mathcal{T} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ will be called a *superoperator*[36] to stress the fact that it is an operator acting on operators. Its *support* is defined to be the minimal set $\Lambda' \subseteq \Lambda$ such that $\mathcal{T} = \mathcal{T}' \otimes \mathbb{1}$, where $\mathcal{T}' \in \mathcal{B}(\mathcal{A}_{\Lambda'})$. Positivity is defined as usual for linear maps: $\mathcal{T}$ is said to be positive if it maps positive operators to positive operators. $\mathcal{T}$ is called *completely positive* if $\mathcal{T} \otimes \mathbb{1} : \mathcal{A}_\Lambda \otimes M_n \to \mathcal{A}_\Lambda \otimes M_n$ is positive for all $n \geq 1$. Finally, we say that $\mathcal{T}$ is trace preserving if $\text{Tr}\mathcal{T}(\rho) = \text{Tr}\rho$ for all $\rho \in A_\Lambda$.

A *dissipative evolution* for a quantum system is given by a one-parameter continuous semigroup of completely positive and trace preserving superoperators $\{T_t : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda\}_t$. If $\rho \in \mathcal{B}(\mathcal{H}_\Lambda)$ is the state of the system at time zero, then the evolution of ρ at time $t \geq 0$ is given by $\rho(t) = T_t(\rho)$. The assumptions on T_t guarantee that $\rho(t)$ is again a state, i.e., a positive and trace one operator. This is usually called the Schrödinger picture.

We will make use of the following norm for superoperators:

$$\|T\|_\diamond = \sup_n \|T \otimes \mathbb{1}_n\|_{1\to 1} = \sup_n \sup_{\substack{X\in\mathcal{A}_\Lambda\otimes M_n \\ X\neq 0}} \frac{\|T \otimes \mathbb{1}_n(X)\|_1}{\|X\|_1}. \tag{6}$$

Dissipative maps are contractive with respect to such norm, in the sense that $\|\mathcal{T}\|_\diamond \leq 1$.

Given a semigroup of dissipative maps $\{T_t\}$, it has a generator $\mathcal{L} : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$ which satisfies $\frac{\mathrm{d}}{\mathrm{d}t}T_t(\rho) = \mathcal{L}T_t(\rho)$. Such superoperator is called a *Lindbladian* or *Liouvillian* (we will use the former). The assumptions made on T_t force a particular structure on $\mathcal{L}$, which is called the *Lindblad form*.[37,38] A superoperator $\mathcal{L}$ is said to be in the Lindblad form if it can be written as

$$\mathcal{L}(\rho) = i[\rho, H] + \sum_j L_j \rho L_j^* - \frac{1}{2}\{L_j^* L_j, \rho\}, \tag{7}$$

where H is a Hermitian operator, $(L_j)_j$ are arbitrary operators (called the Lindblad operators), $[\cdot,\cdot]$ denotes the commutator, and $\{\cdot,\cdot\}$ denotes the anti-commutator. We refer to Refs. 36 and 39 for details on the theory of Lindblad operators.

As shown in Ref. 36, given a semigroup of dissipative maps $\{T_t\}_t$, we can define a new map T_∞ representing the "infinite time limit" of the evolution, or in other words, the projector onto the space of fixed points of the evolution. T_∞ is again a completely positive, trace preserving superoperator, since it can be obtained as $\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N} T_n$.

C. Uniform families

Given a generator $\mathcal{L}$, we can decompose it as a sum of *local terms*, i.e., terms which are still of the Lindblad form but with controlled support,

$$\mathcal{L} = \sum_{Z\subset\Lambda} \mathcal{L}_Z, \quad \text{supp}\mathcal{L}_Z = Z, \quad \mathcal{L}_Z \text{ is Lindbladian.}$$

When $\|\mathcal{L}_Z\|_\diamond$ is decaying with $\text{diam}\, Z$, we will generically say that the evolution is *local*. More stringent assumptions on the decay rate of the norms of the local generators will be required and are formalized in Assumptions (A1) and (A2) in this section.

Since we are interested in dissipative evolutions defined on increasing sequences of lattices and how their properties depend on the lattice size (often referred to as the *system size*), we need to define a meaningful way of growing the evolutions with the lattice size, adding, and modifying the necessary generator terms appropriately. As presented in Ref. 22, the following definition of *uniform families* of dissipative evolutions is one solution to this, which is general enough to cover a wide range of models and situations.

Definition 4. Given $\Lambda \subset \Gamma$, *a* boundary condition *for* Λ *is given by a Lindbladian* $\mathcal{B}^{\partial\Lambda} = \sum_{d\geq 1} \mathcal{B}_d^{\partial\Lambda}$, *where* $\mathrm{supp} B_d^{\partial\Lambda} \subset \partial_d\Lambda := \{x \in \Lambda \mid \mathrm{dist}(x, \Lambda^c) \leq d\}$.

Definition 5. A uniform family *of Lindbladians is given by the following:*

(i) infinite Lindbladian: a local Lindbladian $\mathcal{M}$ *defined all of* $\mathbb{Z}^D$*:* $\mathcal{M} = \sum_{Z\subset\mathbb{Z}^D} \mathcal{M}_Z$;
(ii) boundary conditions: a family of boundary conditions $\{\mathcal{B}^{\partial\Lambda}\}_\Lambda$, *where* $\Lambda = b_u(L)$, *for each* $u \in \mathbb{Z}^D$ *and* $L \geq 0$.

Definition 6. A local Lindbladian $\mathcal{L} = \sum_{Z\subset\mathbb{Z}^D} \mathcal{L}_Z$ *is said to be* translationally invariant *if* $\mathcal{L}_{Z+u} = \mathcal{L}_Z$, $\forall u \in \mathbb{Z}^D$.

We say that a uniform family $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ *is* translationally invariant *if* $\mathcal{M}$ *is translationally invariant, and moreover,* $\mathcal{B}^{\partial b_u(L)}$ *is independent of u.*

Given a uniform family $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$, we fix the following notation for evolutions defined on $\Lambda = b_u(L) \subset \Gamma$:

$$\mathcal{L}^\Lambda = \sum_{Z\subset\Lambda} \mathcal{M}_Z \qquad \text{“open boundary” evolution,} \tag{8}$$

$$\mathcal{L}^{\overline{\Lambda}} = \mathcal{L}^\Lambda + \mathcal{L}^{\partial\Lambda} \qquad \text{“closed boundary” evolution,} \tag{9}$$

with the respective evolutions $T_t^\Lambda = \exp(t\mathcal{L}^\Lambda)$ and $T_t^{\overline{\Lambda}} = \exp(t\mathcal{L}^{\overline{\Lambda}})$.

Until now, we have made no specific assumption on the decay rate of the norms of the local generators. As mentioned above, in order to meaningfully talk about locality of the evolution, we need to impose that $\|\mathcal{L}_Z\|_\diamond$ is decaying with diam Z. The rate at which such function decays classifies the system into one of the specific cases more usually considered in the literature: compactly supported (usually called finite-range interactions), exponentially decaying, super-polynomially decaying, power-law decaying, etc. We will take a more general approach and will simply assume from now on that our family of Liouvillians satisfies the following assumptions.

Definition 7 (Lieb-Robinson assumptions). There exists an increasing *function* $\nu(r)$ *satisfying* $\nu(x + y) \leq \nu(x)\nu(y)$, *such that the following conditions hold:*

$$\sup_{x\in\Gamma} \sum_{Z\ni x} \|\mathcal{M}_Z\|_\diamond |Z| \nu(\mathrm{diam}\, Z) \leq v < \infty, \tag{A1}$$

$$\sup_{x\in\Gamma} \sup_r \nu(r) \sum_{d=r}^{N} \left\| \mathcal{B}_d^{\partial B(x,N)} \right\|_\diamond \leq \mathrm{poly}(N). \tag{A2}$$

If $\|\mathcal{M}_Z\|_\diamond$ is exponentially decaying or is compactly supported, then one can take $\nu(r) = \exp(\mu r)$ for some positive μ. On the other hand, if $\|\mathcal{M}_Z\|_\diamond$ decays only polynomially, then we must take $\nu(r) = (1 + r)^\mu$. In the latter case, the Lieb-Robinson bounds only hold if μ is bigger than a constant depending on the geometrical dimension of the lattice Γ, i.e., D if $\Gamma = \mathbb{Z}^D$. The details of when we can apply Lieb-Robinson bounds in this case can be found in our previous work.[22] From now on, we will simply assume that $\nu(\cdot)$ decays sufficiently fast for the Lieb-Robinson bounds to apply.

D. Frustration freeness

The following definition is inspired by the analogous concept defined for closed systems and Hamiltonian dynamics. It captures the idea that a fixed point of a local evolution might or might not be *locally steady*. The local dissipative terms could in general have a non-trivial action on the fixed point, so that it is only the sum of such local effects that adds up to zero and leaves the state invariant. Assuming that the system is frustration free means excluding such cases.

Definition 8. We say that a uniform family $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ *satisfies* frustration freeness *(or is* frustration free*) if for all* Λ *and all fixed points* ρ_∞ *of* $T_t^{\tilde{\Lambda}}$,

$$\mathcal{M}_Z(\rho_\infty) = 0 \quad \forall Z \subset \Lambda. \tag{10}$$

Remark 9. If $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ is frustration free, then each fixed point of $T_t^{\tilde{\Lambda}}$ is also a fixed point of T_t^{Λ}.

While not true in general, the frustration freeness condition is satisfied by a large class of interesting dissipative systems, as the following examples show:

1. dissipative state engineering procedures defined in Refs. 15 and 16;
2. locally reversible classical Markov chains;
3. locally detailed balanced quantum Markov processes and in particular, Gibbs samplers for commuting Hamiltonians.[40]

III. LOCALIZATION RESULTS

A well known property of many-body systems with local interactions, either dissipative or Hamiltonian, is the existence of a finite speed of propagation. This describes how the support of a localized observable spreads in time during the evolution: up to an exponentially small correction, the support spreads linearly with time. The finite velocity at which such linear growth occurs is often called a Lieb-Robinson velocity or sometimes a group velocity. (It is a property of the model and not a consequence of some relativistic effect — we are considering only non-relativistic models here.) While the original work focused on Hamiltonian systems and groups of automorphisms,[41,42] the existence of such finite speed of propagation in the lattice has been generalized to dissipative evolutions,[43,44] and in Ref. 22, we showed that the definition of boundary condition we have given allows us to recover the same type of localization properties of the evolution.

Nonetheless, all the Lieb-Robinson localization bounds have a time-dependency, becoming worse as time increases, until the bound they provide becomes trivial and does not give any information at all about the properties of the fixed point. In Secs. III A and III B, we want to produce results which might be interpreted as "infinite time" versions of Lieb-Robinson bounds. To do so, we will need to make an extra assumption on the evolution: we will assume that the convergence to the fixed point is fast, in the sense that scales logarithmically with the system size. This is formalized in the definition of *rapid mixing* below. Such a definition can be in some cases relaxed to allow convergence which is only scaling sub-linearly with respect to system size. (We will not pursue such generalizations here and instead refer to Ref. 22 for guidance on which changes are necessary to the results below.)

The localization lemma that we prove, Lemma 11, will be only sufficient to prove an area law for the pure fixed point case not for the mutual information. Therefore, we also prove a stronger result, Lemma 12, for which we will need to add the extra hypothesis of frustration freeness.

A. Rapid mixing

In this section, we want to briefly recall a result proven in Ref. 22. We start by recalling the definition of *rapid mixing*.

Definition 10 (Rapid mixing). Let $\{T_t^\Lambda\}_\Lambda$ *be a family of dissipative maps, we say it satisfies rapid mixing if there exist* $c, \gamma, \delta > 0$ *such that*

$$\sup_{\substack{\rho \geq 0 \\ \mathrm{Tr}\,\rho = 1}} \left\| T_t^\Lambda(\rho) - T_\infty^\Lambda(\rho) \right\|_1 \leq c|\Lambda|^\delta e^{-t\gamma}. \tag{11}$$

In analogy to the spectral gap for Hamiltonians, proving that a family of Lindbladians is rapid mixing is not an easy task. Nonetheless, there exists a large class of interesting systems for which we already have mixing time estimates that imply rapid mixing.

1. (Trivially) non-interacting particle systems.
2. Dissipative state engineering for graph states.[23]
3. Quantum and classical Markov processes satisfying a Log-Sobolev inequality.[29] This includes in particular Glauber dynamics for the Ising model in 2D, either above the critical temperature or with non-zero magnetic field.[24]

Lemma 11. Let $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ *be a uniform family of dissipative evolutions that satisfies rapid mixing, and suppose each* $T_t^{\bar{\Lambda}}$ *has a unique fixed point and no other periodic points. Fix a* Λ *and let* ρ_∞ *be the unique fixed point of* $T_t^{\bar{\Lambda}}$. *Given* $A \subset \Lambda$, *for each* $s \geq 0$ *denote by* ρ_∞^s *the unique fixed point of* $T_t^{\bar{A}(s)}$.

Then, we have

$$\left\| \mathrm{Tr}_{A^c}(\rho_\infty - \rho_\infty^s) \right\|_1 \leq |A|^\delta \Delta_0(s), \tag{12}$$

for some fast-decaying function $\Delta_0(s)$ *and some positive constant* δ.

The decay rate of $\Delta_0(s)$ is in the same class as $\nu^{-1}(s)$, where $\nu(s)$ is defined by Assumptions (A1) and (A2): it is exponential if $\nu^{-1}(s)$ is exponential and polynomial if $\nu^{-1}(s)$ is polynomial. In the latter case, the degree of the polynomial controlling the decay is smaller than that of $\nu^{-1}(s)$, but the loss is independent of the system size — again, this corresponds to requiring a sufficiently fast-decaying $\nu^{-1}(s)$.

B. Localizing with frustration freeness

In Sec. III B, we want to show a property of the fixed points of a uniform family of Lindbladians verifying frustration freeness. We want to study the behavior of a system when it is prepared and started in a state, which is the fixed point of the same family but of a slightly smaller region. A reasonable guess would be that frustration freeness implies that the evolution should be localized "around the boundary," and that for short times, nothing at all would happen in the "bulk" (where the state is left invariant by the local interaction terms because of frustration freeness). This intuition is formalized in the following lemma.

Lemma 12. Let $\mathcal{L} = (\mathcal{M}, \mathcal{B})$ *be a uniform family of Lindbladians, satisfying frustration freeness. Let* $A \subset \Gamma$ *be a finite region and fix a positive natural number* m. *Let* $B = A(m+1)$, $R = A(m+1) \setminus A(m)$, *and* ρ_∞^m *a fixed point of* $T_t^{\bar{A}(m)}$ *and* τ *an arbitrary state on* R *(see Figure 1),*

$$\left\| \left(T_t^{\bar{B}} - T_t^{\bar{B}\setminus A} \right) (\rho_\infty^m \otimes \tau) \right\|_1 \leq \mathrm{poly}(m) \nu^{-1}(m) \left[e^{vt} - 1 + t \right], \tag{13}$$

where $T_t^{\bar{B}\setminus A}$ *denotes the evolution generated by* $\mathcal{L}^{\bar{B}\setminus A} = \sum_{Z \subset B\setminus A} \mathcal{M}_Z + \sum_{d \leq m+1} \mathcal{B}_d^{\partial B}$.

In order to prove such result, we will first prove a Lieb-Robinson-type of lemma. Denote by $\rho(t) = T_t^{\bar{B}}(\rho_\infty^m \otimes \tau)$. For each $X \subset B$, denote $\mathbb{L}_X$ the algebra generated by $\{\mathcal{M}_Z \mid Z \subset X\}$, which is the set of interaction terms of $\mathcal{L}^B$ whose support is contained in X.

Lemma 13. Under the same assumptions of Lemma 12, for each $K \in \mathbb{L}_X$, *the following "Lieb-Robinson-like" bound holds for some positive* v,

$$\|K(\rho(t))\|_1 \leq \mathrm{poly}(m)|X|\|K\|_\diamond (e^{vt} + t - 1)\nu^{-1}(\mathrm{dist}(X, R)). \tag{14}$$

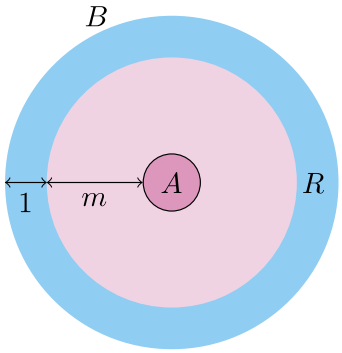

FIG. 1. The construction of the sets B and R.

Proof. Denote $C(Z,t) = \sup_{T\in\mathbb{L}_Z} \frac{\|T(\rho(t))\|_1}{\|T\|_\diamond}$. Frustration-freeness implies that $C(Z,0)$ is 0 if $Z \cap R = \emptyset$ (since $\mathrm{Tr}_R\rho(0) = \rho_\infty^m$), while it is bounded by 1 otherwise. Moreover, let $\Delta(r) = \sum_{d\geq r} \|\mathcal{B}_d^{\partial B}\|_\diamond$ and for each $Z \subset B$, let $\delta(Z) = \Delta(\mathrm{dist}(Z,R))$. Assumption (A2) implies that $\sup_r \nu(r)\Delta(r) \leq \mathrm{poly}(m)$.

We are now going to replicate the proof technique of Lieb-Robinson bounds: denote $\tilde{\mathcal{L}}^X = \mathcal{L}^{\bar{B}} - \mathcal{L}^{\bar{B}\setminus X}$, and notice that, since they have disjoint support, $[K, \mathcal{L}^{\bar{B}\setminus X}] = 0$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}K(\rho(t)) = K\mathcal{L}^{\bar{B}}\rho(t) = \mathcal{L}^{\bar{B}\setminus X}K(\rho(t)) + K\tilde{\mathcal{L}}^X(\rho(t)),$$

and consequently,

$$K(\rho(t)) = e^{t\mathcal{L}^{\bar{B}\setminus X}}K(\rho(0)) + \int_0^t e^{(t-s)\mathcal{L}^{\bar{B}\setminus X}}K\tilde{\mathcal{L}}^X\rho(s)\mathrm{d}s.$$

By taking norms

$$\|K(\rho(t))\|_1 \leq \|K(\rho(0))\|_1 + \|K\|_\diamond \int_0^t \|\tilde{\mathcal{L}}^X\rho(s)\|_1\mathrm{d}s,$$

and thus,

$$\begin{aligned} C(X,t) &\leq C(X,0) + \sum_{Z\cap X\neq\emptyset}\int_0^t \|\mathcal{M}_Z\|_\diamond C(Z,s)\mathrm{d}s + \sum_{d>m+1}\int_0^t \|\mathcal{B}_d^{\partial B}\|_\diamond \mathrm{d}s \\ &\leq C(X,0) + \delta(X)t + \sum_{Z\cap X\neq\emptyset}\|\mathcal{M}_Z\|_\diamond \int_0^t C(Z,s)\mathrm{d}s. \end{aligned} \quad (15)$$

By recursively applying Equation (15), we obtain that

$$C(X,t) \leq \sum_{n=0}^{\infty}[a_n + b_n]\frac{t^n}{n!},$$

where $a_0 = C(X,0)$,

$$a_n = \sum_{Z_1\cap X\neq 0}\cdots\sum_{Z_n\cap Z_{n-1}\neq 0}\|\mathcal{M}_{Z_1}\|\ldots\|\mathcal{M}_{Z_n}\|\,C(Z_n,0),$$

$b_0 = 0$, $b_1 = \delta(X)$, and

$$b_{n+1} = \sum_{Z_1\cap X\neq 0}\cdots\sum_{Z_n\cap Z_{n-1}\neq 0}\|\mathcal{M}_{Z_1}\|\ldots\|\mathcal{M}_{Z_n}\|\,\delta(Z_n).$$

Let us bound the two coefficients independently. The coefficients a_n are treated in the same way as is done in the standard proof of Lieb-Robinson bounds:[45] recalling that $C(Z,0)$ is zero unless $Z \cap R \neq \emptyset$, we have that

$$a_n = \sum_{Z_1\cap X\neq 0}\cdots\sum_{\substack{Z_n\cap Z_{n-1}\neq 0\\ Z_n\cap R\neq 0}}\|\mathcal{M}_{Z_1}\|\ldots\|\mathcal{M}_{Z_n}\|.$$

We have that a_1 is bounded by using Assumption (A1),

$$a_1 \leq \sum_{i\in X}\sum_{Z_1\ni i}\sum_{Z_1\cap R\neq 0}\|\mathcal{M}_{Z_1}\| \leq \sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R))\sum_{\substack{Z_1\ni i\\ Z_1\cap R\neq 0}}\|\mathcal{M}_{Z_1}\|\,\nu(\mathrm{diam}(Z_1)) \leq v\sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R)).$$

Similarly, we bound a_2 as follows:

$$a_2 \leq \sum_{i\in X}\sum_{Z_1\ni i}\sum_{j\in Z_2}\sum_{\substack{Z_2\ni j\\ Z\cap Y\neq 0}}\|\mathcal{M}_{Z_1}\|\,\|\mathcal{M}_{Z_2}\|.$$

We now use the fact that $\nu(\mathrm{dist}(i,j))\nu(\mathrm{dist}(j,R)) \geq \nu(\mathrm{dist}(i,R))$, and thus,

$$\begin{aligned}
a_2 &\leq \sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R))\sum_{Z_1\ni i}\|\mathcal{M}_{Z_1}\|\sum_{j\in Z_2}\nu(\mathrm{dist}(i,j))\sum_{\substack{Z_2\ni j\\ Z\cap Y\neq 0}}\nu(\mathrm{dist}(j,Y))\,\|\mathcal{M}_{Z_2}\|\\
&\leq \sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R))\sum_{Z_1\ni i}\|\mathcal{M}_{Z_1}\|\,\nu(\mathrm{diam}(Z_1))\sum_{j\in Z_1}\sum_{\substack{Z_2\ni j\\ Z\cap Y\neq 0}}\nu(\mathrm{diam}(Z_2))\,\|\mathcal{M}_{Z_2}\|\\
&\leq v\sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R))\sum_{Z_1\ni i}\|\mathcal{M}_{Z_1}\|\,\nu(\mathrm{diam}(Z_1))|Z_1| \leq v^2\sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R)).
\end{aligned}$$

Proceeding in a similar way, we can bound a_n by $v^n\sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R))$. Let us now turn our attention to b_n. Let $Z_1\cap X\neq\emptyset$. Then, for all u in Z_1, it holds that $\mathrm{dist}(X,R)\leq \mathrm{dist}(u,X)+\mathrm{dist}(u,R)\leq \mathrm{diam}\,Z_1+\mathrm{dist}(u,R)$. In particular, this holds for $y_{Z_1}\in Z_1$ such that $\mathrm{dist}(y_{Z_1},R)=\mathrm{dist}(Z_1,R)$. Therefore, we have that $1\leq \nu(\mathrm{diam}\,Z_1)\nu(\mathrm{dist}(Z_1,R))\nu^{-1}(\mathrm{dist}(X,R))$. We can use the previous inequality to bound b_2 as follow:

$$\begin{aligned}
b_2 = \sum_{Z_1\cap X\neq\emptyset}\|M_{Z_1}\|\,\delta(Z_1) &\leq \nu^{-1}(\mathrm{dist}(X,R))\sum_{x\in X}\sum_{Z_1\ni x}\|M_{Z_1}\|\,\nu(\mathrm{diam}\,Z_1)\Delta(\mathrm{dist}(Z_1,R))\nu(\mathrm{dist}(Z_1,R))\\
&\leq v\,\mathrm{poly}(m)|X|\nu^{-1}(\mathrm{dist}(X,R)).
\end{aligned}$$

For b_3, we reason similarly as follows:

$$\begin{aligned}
b_3 &= \sum_{Z_1\cap X\neq\emptyset}\sum_{Z_2\cap Z_1\neq\emptyset}\|M_{Z_1}\|\,\|M_{Z_2}\|\,\delta(Z_2)\\
&\leq \sum_{Z_1\cap X\neq\emptyset}\frac{\|M_{Z_1}\|}{\nu(\mathrm{dist}(Z_1,R))}\sum_{z\in Z_1}\sum_{Z_2\ni z}\|M_{Z_2}\|\,\nu(\mathrm{diam}\,Z_2)\Delta(\mathrm{dist}(Z_2,R))\nu(\mathrm{dist}(Z_2,R))\\
&\leq v\mathrm{poly}(m)\sum_{Z_1\cap X\neq\emptyset}\|M_{Z_1}\|\,\nu^{-1}(\mathrm{dist}(Z_1,R))|Z_1| \leq v\mathrm{poly}(m)\nu^{-1}(\mathrm{dist}(X,R))\sum_{x\in X}\sum_{Z_1\ni x}\|M_{Z_1}\|\,|Z_1|\nu(\mathrm{diam}\,Z_1)\\
&\leq v^2\,\mathrm{poly}(m)\nu^{-1}(\mathrm{dist}(X,R))|X|.
\end{aligned}$$

Following the same argument, we can thus bound the general term b_{n+1} by

$$v^n\,\mathrm{poly}(m)|X|\nu^{-1}(\mathrm{dist}(X,R)).$$

We can then bound

$$\sum_n a_n\frac{t^n}{n!} \leq (e^{vt}-1)\sum_{i\in X}\nu^{-1}(\mathrm{dist}(i,R)) \leq (e^{vt}-1)|X|\nu^{-1}(\mathrm{dist}(X,R))$$

and

$$\sum_n b_n\frac{t^n}{n!} \leq \mathrm{poly}(m)v^{-1}(e^{vt}-1)|X|\nu^{-1}(\mathrm{dist}(X,R)) + t[\delta(X)-\mathrm{poly}(m)|X|\nu^{-1}(\mathrm{dist}(X,R))].$$

Note that, because of Assumption (A2), the last term in the r.h.s. can be bounded as

$$\delta(X)-\mathrm{poly}(m)|X|\nu^{-1}(\mathrm{dist}(X,R)) \leq \mathrm{poly}(m)|X|\nu^{-1}(\mathrm{dist}(X,R)).$$

This concludes the proof. □

We can now prove Equation (13).

Proof. Applying Duhamel's formula,[46] we have that

$$\left(T_t^{\bar{B}} - T_t^{\bar{B}\setminus A}\right)(\rho_\infty^m \otimes \tau) = \int_0^t T_{t-s}^{\bar{B}\setminus A} \tilde{\mathcal{L}}^A \rho(s) \mathrm{d}s,$$

and therefore,

$$\left\|\left(T_t^{\bar{B}} - T_t^{\bar{B}\setminus A}\right)(\rho_\infty^m \otimes \tau)\right\|_1 \leq \sum_{Z\cap A\neq\emptyset} \int_0^t \|\mathcal{M}_Z \rho(s)\|_1 \mathrm{d}s + \sum_{d>m+1} \|B_d^{\partial B}\|_\diamond t.$$

The second term on the r.h.s. is bounded by $t\,\mathrm{poly}(m)\nu^{-1}(m)$. Let us focus on the first term on the r.h.s. If $Z \subset A(m)$, we can bound the r.h.s with Equation (14). In particular, we have the following:

$$\sum_{\substack{Z\cap A\neq\emptyset \\ Z\subset A(m)}} \|\mathcal{M}_Z \rho(s)\|_1 \leq \mathrm{poly}(m)(e^{vs} + t - 1) \sum_{\substack{Z\cap A\neq\emptyset \\ Z\subset A(m)}} \|\mathcal{M}_Z\|_\diamond |Z| \nu^{-1}(\mathrm{dist}(Z,R)).$$

Observe that since $Z \cap A \neq \emptyset$, it holds that $\mathrm{dist}(Z,R) + \mathrm{diam}\, Z \geq m$, and therefore,

$$\sum_{\substack{Z\cap A\neq\emptyset \\ Z\subset A(m)}} \|\mathcal{M}_Z\|_\diamond |Z| \nu^{-1}(\mathrm{dist}(Z,R)) \leq \nu^{-1}(m) \sum_{\substack{Z\cap A\neq\emptyset \\ Z\subset A(m)}} \|\mathcal{M}_Z\|_\diamond |Z| \nu(\mathrm{diam}\, Z) \leq \nu^{-1}(m) v,$$

where we have used the Lieb-Robinson assumption.

If $Z \not\subseteq A(m)$, then it must hold that $Z \cap R \neq \emptyset$. Then, we showed in the previous lemma that

$$\sum_{\substack{Z\cap A\neq\emptyset \\ Z\cap R\neq\emptyset}} \|\mathcal{M}_Z\|_1 \leq v|A|\nu^{-1}(\mathrm{dist}(A,R)) = v|A|\nu^{-1}(m).$$

Putting it all together, we have that

$$\left\|\left(T_t^{\bar{B}} - T_t^{\bar{B}\setminus A}\right)(\rho_\infty^m \otimes \tau)\right\|_1 \leq \mathrm{poly}(m)\nu^{-1}(m)\left[e^{vt} - 1 + t\right].$$

□

IV. DECAY OF CORRELATIONS

In this section, we show that as a straightforward consequence of Lemma 11, the hypotheses on $\mathcal{L}$ imply that its fixed points have a particular character: they have fast decay of correlations, meaning that the correlations between two spatially separated regions are fast-decaying in distance. How fast this decay is given by the decaying function Δ_0 defined in Lemma 11.

Theorem 14. *Under the same assumption as in Lemma 11, fix two regions A and $B \subset \Lambda$, let $d_{AB} > 0$ be the distance between them. Then, we have that*

$$T(A:B) \leq 3(|A| + |B|)^\delta \Delta_0\left(\frac{d_{AB}}{2}\right), \tag{16}$$

where the correlations are calculated with respect to ρ_∞, and δ and Δ_0 are defined in Lemma 11.

Proof. Let $C = A \cup B$, and denote by ρ_{AB} the reduced density matrix of ρ_∞ over C, and by ρ_A and ρ_B the reduced state on A and B, respectively.

Consider ρ_∞^s the unique fix point of $T_t^{\overline{C(s)}}$ and denote by ρ_A^s and ρ_B^s its reduced density matrices over A and B, respectively. If $s \leq \frac{d_{AB}}{2}$, then $C(s)$ has two disjoint components corresponding to $A(s)$ and $B(s)$, and thus, ρ_∞^s decomposes as a tensor product over such bipartition, and its reduced density matrix over C is given by $\rho_A^s \otimes \rho_B^s$.

By Lemma 11, we have that for any observable O_C with operator norm equal to 1 and supported on C,

$$|\mathrm{Tr} O_C(\rho_\infty - \rho_\infty^s)| \leq |C|^\delta \Delta_0(s).$$

This implies that

$$\left\|\rho_{AB} - \rho_A^s \otimes \rho_B^s\right\|_1 \leq |C|^\delta \Delta_0(s).$$

Since the trace norm does not increases under the partial trace, then

$$\left\|\rho_A - \rho_A^s\right\|_1 \leq |C|^\delta \Delta_0(s),$$

and the same holds for B. This in turn implies that

$$\left\|\rho_A \otimes \rho_B - \rho_A^s \otimes \rho_B^s\right\|_1 \leq 2|C|^\delta \Delta_0(s),$$

and by applying the triangle inequality, we obtain the desired result. □

Remark 15. By Equation (4), we have that the mutual information $I(A:B)$ decays with $\mathrm{dist}(A,B)$ at essentially the same rate as $T(A:B)$.

Note that the dependence on $|A|$ and $|B|$ of the bound in Equation (16), which is harmless when A and B are finite regions — as in the case of two-point correlation functions — becomes significant when one of the two regions is proportional to the system size. In Ref. 19, the authors defined a correlation length ξ for the mutual information, as the minimal length such that for all $L \geq \xi$, it holds that

$$I(b_x(R-L) : b_x(R)^c) \leq \frac{1}{2} I(b_x(R) : b_x(R)^c), \quad \forall x, \forall R \geq 0.$$

From this property, they are able to derive an area law for the mutual information of the type

$$I(A : A^c) \leq 4|\partial A|\xi.$$

If we were to use (16) to determine a correlation length ξ, we would obtain a value for ξ which depends on the system size and thus obtain a bound comparable to Equation (1) obtained in Ref. 31. In Sec. V, we want to show that it is possible to greatly improve this bound at the cost of adding some extra hypotheses on the evolution.

V. AREA LAW FOR MUTUAL INFORMATION

Until now, we have avoided as much as possible to put restrictions on the function $\nu(r)$ appearing in Assumptions (A1) and (A2), and we have simply required it to be fast enough for Lieb-Robinson bounds to hold. Indeed, in most of the bounds obtained in the previous results, it appears $\nu(r)$ (or $\Delta_0(r)$, which depends on $\nu(r)$), so that weaker assumptions will simply lead to weaker bounds.

This gets more complicated when it gets to prove the area law bounds, since we are actually interested in pinning down the case in which the bound on mutual information takes the form of $|\partial A| \log |A|$, an area law with a logarithmic correction. As will be clear in the proof, the logarithmic correction depends on $\nu(r)$ (and consequently $\Delta_0(r)$) to be exponential. Slower rates will still lead to a bound on the mutual information, but where the logarithmic correction is replaced by a super-logarithmic correction. The resulting bound can hardly be called an area law, a sub-volume law would be more correct. We will avoid such generalization, as they make the proof unnecessarily complicated and focus on the more interesting case of exponentially decaying interactions, for which we can prove a proper area law with logarithmic correction.

A. The pure fixed point case

In this section, let us suppose that each $T_t^{\bar{\Lambda}}$ has a unique fixed point, and moreover, that this fixed point is pure. This setting is of particular interest in view of dissipative state engineering,[15,16] since ideally one would like to be able to create pure states (at least before noise and errors are taken into account).

Let us denote by $|\phi_\Lambda\rangle$ the pure fixed point of $T_t^{\bar{\Lambda}}$.

Proposition 16. Let $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ be a uniform family of dissipative evolutions, satisfying rapid mixing and having a unique pure fixed point $|\phi_\Lambda\rangle$ for each Λ. Fix $A \subset \Lambda$, and let ρ_A denote the reduced density matrix of $|\phi_\Lambda\rangle$ on A. Then, it holds that

$$S(\rho_A) \leq c \log |A| \cdot |\partial A|,$$

for some constant $c > 0$.

Proof. Let $\ell \geq 0$, to be determined later, and denote by σ the reduced density matrix of $|\phi_{A(\ell)}\rangle$ on A. Then, we have trivially that

$$S(\sigma) \leq \log_2 \dim \mathcal{H}_{A(\ell)\setminus A} \leq c_0 \ell\, |\partial A|,$$

for some positive constant c_0. On the other hand, by Lemma 11, we have that $d := \|\rho_A - \sigma\|_1 \leq |A|^\delta \Delta_0(\ell)$, and thus by Equation (2),

$$S(\rho_A) \leq S(\sigma) + |S(\sigma) - S(\rho_A)| \leq c_0 \ell\, |\partial A| + 2d|A| + 2h_b(d) \leq c_0 \ell\, |\partial A| + 2|A|^{\delta+1}\Delta_0(\ell) + 2h_b(d).$$

Fix an ε such that $0 < \varepsilon < 1/2$ and let us choose ℓ such that $2|A|^{\delta+1}\Delta_0(\ell) \leq \varepsilon$. This implies that ℓ scales as $\log |A|$, and thus, $S(\rho_A) \leq c_1 \log |A| \cdot |\partial A| + \varepsilon + 2h_b(\varepsilon/2|A|)$, for some positive c_1. By taking $c \geq c_1$, we can absorb the terms depending on ε in the other one and obtain the claimed estimate $S(\rho_A) \leq c \log |A| \cdot |\partial A|$. □

While of interest, the case of a pure fixed point is a very specific one. Therefore, we want to give results applicable in the generic case of a mixed fixed point. To obtain such results, we will need to make an additional assumption, namely, that the system is frustration free.

B. The frustration-free case

Theorem 17 (Area law for mutual information). *Let $\mathcal{L} = \{\mathcal{M}, \mathcal{B}\}$ be a uniform family of dissipative evolutions, satisfying rapid mixing frustration freeness and having a unique pure fixed point. Let ρ_∞ be the fixed point of $\mathcal{L}^\Lambda$ for some $\Lambda = b_u(L) \subset \Gamma$. Then, we have that*

$$I(A : A^c)_{\rho_\infty} \leq c|\partial A| \log |A|, \tag{17}$$

for some positive c independent of the system-size.

Proof. For each $n \geq 0$, let ρ_∞^n be the fixed point of $T_t^{\bar{A}(n)}$. Fix a positive n_0 to be determined later. Then, it holds that

$$I(A : A^c)_{\rho_\infty} = I(A : A^c)_{\rho_\infty^{n_0}} + \sum_{n=n_0}^{L-1} \left[I(A : A^c)_{\rho_\infty^{n+1}} - I(A : A^c)_{\rho_\infty^n} \right]. \tag{18}$$

We want to show that it is possible to choose n_0 in such a way that $I(A : A^c)_{\rho_\infty^{n_0}} \leq c|\partial A| \log |A|$ and the sum in the r.h.s. is arbitrarily small.

For each $n \geq 0$, we have that by rapid mixing (11),

$$\left\| \rho_\infty^{n+1} - T_t^{\bar{A}(n+1)}(\rho_\infty^n \otimes \tau) \right\|_1 \leq |A| \phi_1(n) e^{-\gamma t},$$

where $\phi_1(n)$ is a polynomial in n. On the other hand, Equation (13) implies that

$$\left\| T_t^{\bar{A}(n+1)}(\rho_\infty^n \otimes \tau) - T_t^{\bar{A}(n+1)\setminus A}(\rho_\infty^n \otimes \tau) \right\|_1 \leq |A| \phi_2(n) \nu^{-1}(n) [e^{vt} + t - 1],$$

and $\phi_2(n)$ is polynomial in n. Let us choose t_n such that

$$\varepsilon_n := \phi_1(n) e^{-\gamma t_n} + \phi_2(n) \nu^{-1}(n) [e^{v t_n} + t_n - 1]$$

is exponentially decaying in n. This can be done by taking t_n which scales proportionally to $\frac{1}{v+\gamma} \log \left(\nu(n) \frac{\phi_1(n)}{\phi_2(n)} \right)$, which is essentially linear if $\nu(n)$ grows exponentially. We can put the two bounds together using the triangle inequality in such a way that

$$\left\| \rho_\infty^{n+1} - T_{t_n}^{\bar{A}(n+1)\setminus A}(\rho_\infty^n \otimes \tau) \right\|_1 \leq |A| \varepsilon_n.$$

Observe that since $T_t^{\bar{A}(n+1)\setminus A}$ does not act on A, $I(A:A^c)_{T_t^{\bar{A}(n+1)\setminus A}(\rho_\infty^n\otimes\tau)} \leq I(A:A^c)_{\rho_\infty^n}$.

Let us assume that n_0 is big enough so that $\varepsilon_n|A|^2 \leq 2^{-n}$ for all $n \geq n_0$, we can then apply inequality (4) and obtain

$$I(A:A^c)_{\rho_\infty^{n+1}} - I(A:A^c)_{\rho_\infty^n} \leq I(A:A^c)_{\rho_\infty^{n+1}} - I(A:A^c)_{T_t^{\bar{A}(n+1)\setminus A}(\rho_\infty^n\otimes\tau)} \leq 6\varepsilon_n|A|^2 + 4h_b(\varepsilon_n|A|).$$

Then,

$$I(A:A^c)_{\rho_\infty} \leq I(A:A^c)_{\rho_\infty^{n_0}} + 6\sum_{n=n_0}^{L-1} 2^{-n} + 4\sum_{n=n_0}^{L-1} h_b(\varepsilon_n|A|).$$

Observe that if $0 \leq x \leq 1/e$, then $(x-1)\log_2(1-x) \leq -x\log_2 x$, and the latter is an increasing function in that interval. Therefore,

$$h_b(\varepsilon_n|A|) \leq -2\varepsilon_n|A|\log_2(\varepsilon_n|A|) \leq \frac{2^{-n+1}}{|A|}\left(n + \log_2|A|\right).$$

Therefore, $\sum_{n=n_0}^{L-1} h_b(\varepsilon_n|A|)$ is the tail of a series which is converging geometrically, and therefore, it is exponentially decaying as n_0 increases. The same is true for $\sum_{n=n_0}^{L-1}\lambda^n$, so that both of them can be made smaller than $I(A:A^c)_{\rho_\infty^{n_0}}$.

By taking n_0 proportional to $\log|A|$, we can bound $I(A:A^c)_{\rho_\infty^{n_0}}$ by the logarithm of $\dim\mathcal{H}_{A^c}$, which is proportional to $|A(n_0)\setminus A|$, and therefore,

$$I(A:A^c)_{\rho_\infty^{n_0}} \leq c\,|\partial A|\log|A|.$$

In conclusion, we have bounded the r.h.s. of (18) by $c|\partial A|\log|A|$, and this concludes the proof. □

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