

**UNIVERSIDAD COMPLUTENSE DE MADRID  
FACULTAD DE CIENCIAS MATEMÁTICAS**



**TESIS DOCTORAL**

**Global approximation theorems for partial differential  
equations and applications**

**Teoremas de aproximación global  
para ecuaciones en derivadas parciales  
y aplicaciones**

**MEMORIA PARA OPTAR AL GRADO DE DOCTOR**

**PRESENTADA POR**

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**Director**

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**Madrid  
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Memoria para optar al grado de doctor  
presentada por

**María Ángeles García Ferrero**

Bajo la dirección de

**Alberto Enciso Carrasco**

Dentro del Programa de Doctorado  
en Investigación Matemática

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# Abstract

Many open problems about the behavior of solutions to partial differential equations can be solved in a constructive way. For instance, if we are wondering whether intricate topological structures can emerge in the solutions to an equation, let us take a hypersurface with some prescribed topology and construct (not necessarily explicitly) a solution which exhibit this structure (up to small diffeomorphisms). Many of these questions arise also in physics and their answer provides a better understanding of the physical phenomena.

The strategy to deal with these problems was introduced by A. Enciso and D. Peralta-Salas in the context of vortex structures of stationary inviscid incompressible fluids and level sets of solutions to some elliptic equations. It is based on the construction of a suitable local solution to the equation and its subsequent promotion to a global solution up to controllable errors so that some properties are preserved. Nevertheless, this approach is limited by the existence of the approximation theorems that stand behind the step from local to global solutions, and almost the whole literature on this topic is devoted to (some classes of) elliptic operators.

The main aim of this thesis is the extension of this approximation theory to parabolic equations with some assumptions on their coefficients. Specifically, we prove that any solution in a spacetime region, under mild technical assumptions, can be approximated in a parabolic Hölder norm by a global solution to the same equation. If the parabolic operator satisfies further conditions (e.g. it looks like the heat equation outside a compact set), the global solution can be shown to fall off in space and time. Analog result holds for elliptic equations which behave like the Helmholtz equation outside a compact set.

Once we count on these theorems, we can focus on their applications. Regarding that, in this thesis we prove the existence of global solutions to parabolic equations with a local hot spot that moves in time along any prescribed curve, up to a small error. We also show that there are global solutions that exhibit isothermic hypersurfaces of prescribed topologies (possibly changing in time). Both results can be shown to hold for solutions of the heat equation on the flat torus for small space-time scales

Global approximation theorems are restricted to linear equations. However, with the previous strategy in mind, we can also use them to prescribe level sets of solutions to some nonlinear equations. In this context, this thesis accomplishes one result about the existence of minimal surfaces on the ball with a level set that looks like any given hypersurface satisfying some mild conditions.

In the previous setting, the existence and behaviour of the integral kernels which solve the equations are essential. The last result of this thesis is about

the structure of the kernel for the curl operator in a bounded domain. It plays a key role in fluid mechanics and electromagnetism and generalizes the well-known Biot-Savart operator, which recovers a vector field from its curl in  $\mathbb{R}^3$ . Namely, we show that the velocity field of an incompressible fluid with tangency boundary conditions on a bounded domain can be written in terms of its vorticity using an integral kernel that has an inverse-square singularity on the diagonal.

At the same time as these central results, other contributions have been obtained. They deal with questions in mathematical physics which are briefly expounded at the end of this monograph.

# Resumen

Muchos problemas sobre el comportamiento de soluciones de ecuaciones en derivadas parciales pueden resolverse de un modo constructivo. Consideremos por ejemplo que queremos saber si pueden aparecer estructuras topológicas complejas en las soluciones de una ecuación. Entonces podemos tomar una de esas hipersuperficies topológicas y construir una solución (no necesariamente explícita) que la contenga, salvo un pequeño difeomorfismo. Muchas de estas preguntas surgen también en física y responderlas nos permitirá entender mejor lo que subyace tras el fenómeno físico.

La estrategia para tratar este tipo de problemas se debe a A. Enciso y D. Peralta-Salas y fue inicialmente empleada en fluidos ideales con estructuras anudadas de vorticidad y para prescribir conjuntos de nivel de soluciones a ciertas ecuaciones elípticas. Se basa en la construcción de una solución local de la ecuación conveniente y su posterior extensión a una solución global salvo pequeños errores de modo que se preserven ciertas propiedades de la solución. Este enfoque está limitado por la existencia de los teoremas de aproximación que garantizan el paso de la solución local a la global con un error controlado. Prácticamente toda la literatura al respecto está dedicada a (ciertas clases de) operadores elípticos.

El principal objetivo de esta tesis es extender esta teoría de aproximación a ecuaciones parabólicas suponiendo cierta regularidad en sus coeficientes. En concreto, demostramos que cualquier solución en una región espacio-temporal con algunas condiciones técnicas puede aproximarse arbitrariamente por una solución global de la misma ecuación en una norma de Hölder parabólica. Si el operador parabólico satisface además otras condiciones (como que sea el del calor fuera de un conjunto compacto), se puede probar que la solución global decae tanto en espacio como en tiempo. Probamos también un resultado análogo para ecuaciones elípticas que se parecen a la de Helmholtz fuera de un compacto.

Una vez que disponemos de estos teoremas, podemos centrarnos en sus aplicaciones. En esta tesis probamos la existencia de soluciones globales a ecuaciones parabólicas con *hot spots* o puntos calientes (máximos en el espacio para cada tiempo) locales que se mueven siguiendo cualesquiera curvas dadas, salvo pequeños errores. También vemos que hay soluciones globales que poseen superficies isothermas (conjuntos de nivel para cada tiempo) de topologías prescritas, que incluso pueden cambiar con el tiempo. Ambos resultados se pueden observar también en soluciones de la ecuación del calor en el toro plano considerando escalas espacio-temporales pequeñas.

Aunque los teoremas de aproximación global se restringen a ecuaciones lineales, podemos emplearlos también para prescribir conjuntos de nivel de soluciones

a algunas ecuaciones no lineales. Esta tesis da un resultado sobre la existencia de superficies mínimas sobre la bola con un conjunto de nivel que se asemeja a cualquier hipersuperficie dada (verificando algunas condiciones sencillas).

En los teoremas de aproximación, la existencia y comportamiento de los núcleos integrales que resuelven las ecuaciones son esenciales. El último resultado de esta tesis versa sobre la estructura del núcleo integral para el rotacional en un dominio acotado. Este juega un papel clave en la mecánica de fluidos y el electromagnetismo y generaliza el famoso operador de Biot–Savart, que permite recuperar un campo vectorial de divergencia nula a partir de su rotacional en todo  $\mathbb{R}^3$ . En otras palabras, demostramos que la velocidad de un fluido incompresible tangente al contorno en un dominio acotado puede escribirse en función de su vorticidad usando un núcleo integral con una singularidad cuadrática en la diagonal.

Al mismo tiempo que estos, se han obtenido otros resultados sobre temas de física matemática, que se exponen brevemente al final de esta tesis.

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# Introduction

Many open problems in partial differential equations (PDEs) can be addressed through the construction of solutions with prescribed geometric or topological properties. The interest of these questions is not restricted to mathematics since many of them arise in the study of physical phenomena.

This thesis is devoted to the study of geometric questions in PDEs through this construction of solutions and the development of a fundamental tool for that: global approximation theorems.

## 1 Motivation

The level sets of solutions to PDEs are key objects in many fields, starting with the study of equipotential surfaces of electrostatic and gravitational potentials. The geometric and topological characterization concerns not only level sets but other invariant manifolds of solutions to PDEs. A natural question arises: whether there exist solutions to a PDE with level sets or other invariant manifolds with prescribed topological or geometric properties.

The theory to deal with this kind of questions was introduced by A. Enciso and D. Peralta-Salas in [28] and [29] to study the vortex lines of stationary solutions of inviscid incompressible fluids and the level sets of harmonic functions, respectively. There they presented a new strategy to address this kind of problems:

1. Construct a solution of the equation in a small set satisfying the desired geometric properties. This can be achieved thanks to the Cauchy-Kowaleskaya theorem or an appropriate boundary value problem.
2. Verify the robustness of the property under small perturbations of the solution. If for instance we are dealing with compact level sets, this can be accomplished by Thom's isotopy theorem.
3. Promote the local solution to a global solution up to controllable errors using a Runge-type global approximation theorem.

Therefore, provided that we can construct a (not necessarily explicit) suitable local solution, ensure the stability of the property and take a suitable approximation theorem for granted, we will be able to solve the problem. The flexibility of this strategy has allowed to use it in many contexts from the existence of solutions to general second-order elliptic PDEs with level sets of

arbitrary complicated topology [29] to the solution of the long-standing Lord Kelvin's conjecture on the existence of thin knotted vortex tubes in steady incompressible fluids [31].

Regarding the third step, a global approximation theorem for a differential operator  $P$  is, roughly speaking, a result ensuring that if a function  $v$  satisfies the equation  $Pv = 0$  in a set  $\Omega$ , under mild technical assumptions, we can conclude that there is a global solution  $u$  of the equation  $Pu = 0$  in  $\mathbb{R}^n$  which approximates  $v$ , meaning that the difference  $u - v$  can be made arbitrarily small in some suitable norm in  $\Omega$ . We say that this is a Runge-type theorem because it generalizes the classical theorem of Runge (1884) for the approximation of holomorphic functions in compact sets of the complex plane by polynomials. Seventy years later, Lax and Malgrange proved that the solutions of certain homogeneous elliptic PDEs in bounded subsets with connected complements are limits of solutions to the same equation in the whole space. Almost all of the subsequent results involve also elliptic equations. However, no obstruction exists in their extension to other equations under appropriate assumptions.

In the following three sections we informally present the main original results of this thesis about new global approximation theorems and their applications. Specific motivation for each topic is provided in the corresponding chapters. Section 5 contains a summary of the contributions of this thesis and their impact in the area, how they are organized in the following chapters and the corresponding

## 2 Global approximation theorems for PDEs

The first global approximation theorems date from the 1950s and are due to Lax and Malgrange. They proved that, for some classes of **elliptic** linear differential operators in  $\mathbb{R}^n$ , solutions in bounded sets  $\Omega$  can be uniformly approximated by solutions in  $\mathbb{R}^n$  provided that  $\Omega$  has a connected complement in  $\mathbb{R}^n$ . The classes of admissible elliptic operators, whose main example is the set of operators with analytic coefficients, were subsequently extended by Browder, including approximation in certain Hölder spaces. These results do not hold for uniform approximation in unbounded sets, nor do they provide any control on the growth of the solutions at infinity. This is fundamental for the construction of global solutions in the considered applications and has been achieved for certain equations. Next, we informally state an approximation theorem with decay at infinity, which generalizes the pointwise decay estimate for solutions to the Helmholtz equation in [31, 25]. We are always assuming that the operator  $P$  and its formal adjoint  $P^*$  are linear and has coefficients in some suitable Hölder space.

**Theorem 1** ([21]). *Suppose that  $P$  is an elliptic operator which behaves like the Helmholtz equation outside a compact set as specified in Definition 1.5. Let  $\Omega$  be a compact subset in  $\mathbb{R}^n$  with connected complement. If  $v$  is a solution to  $Pv = 0$  in a neighborhood of  $\Omega$ , then it can be approximated in a suitable Hölder*

norm by a solution  $u$  of  $Pu = 0$  in  $\mathbb{R}^n$  satisfying the decay condition

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x)|^2 dx < \infty.$$

If we now fix our attention to approximation theorems for **parabolic** equations, we just find some basic results ensuring uniform approximation in bounded sets for the heat equation  $-\partial_t u + \Delta u = 0$ . We extend these results to general linear parabolic equations and in possibly unbounded sets. Again, we will need some bound on the behavior of the global solution at infinity in many applications. With some restrictions on the parabolic operator, we show that the global solution can be assumed to decay both at spatial infinity and as  $t \rightarrow \infty$ .

Under the hypothesis that the coefficients of the linear parabolic operator  $L$  and its formal adjoint  $L^*$  lie in certain parabolic Hölder space, we can state our results, avoiding some minor details, as follows:

**Theorem 2** ([21]). *Let  $\Omega$  be a closed subset in  $\mathbb{R}^{n+1}$  such that for every hyperplane orthogonal to the time axis, its intersection with the complement of  $\Omega$  does not have any bounded connected components. Let us consider a function  $v$  satisfying the parabolic equation  $Lv = 0$  in a neighborhood of  $\Omega$ . Then there exists a solution  $u$  of  $Lu = 0$  in the whole space  $\mathbb{R}^{n+1}$  arbitrary close to  $v$  in  $\Omega$  in a suitable parabolic Hölder norm. If in addition the operator  $L$  has time-independent coefficients and behaves like the heat equation outside a compact set (possibly up to a nonpositive constant and suitably small perturbations) as specified in Definition 2.3 and  $\Omega$  is compact, then there exists a solution  $u$  of  $Lu = 0$  in  $\mathbb{R}^{n+1}$  that approximates  $v$  in the same way and satisfies the decay condition*

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x, t)|^2 dx \leq C e^{-t/C}$$

for some positive constant  $C$ .

A typical example of operator for which the decay condition holds is

$$Lu := -\frac{\partial u}{\partial t} + \Delta_g u - V(x)u,$$

where  $\Delta_g$  is the Laplacian operator associated with a Riemannian metric  $g$  that is Euclidean outside a compact set and  $V(x)$  is a nonnegative potential that is radial outside a compact set and tends to a constant fast enough at infinity. Furthermore, in the particular case that  $L$  is the heat operator up to a nonpositive constant, the decay becomes pointwise:

$$|u(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}.$$

As in the case of elliptic equations, we also prove a refinement of the previous theorem improving the uniform estimate in the case  $\Omega$  is a locally finite union of compact subsets.

Beyond elliptic and parabolic equations, new global approximation theorems for other equations can be expected, like for dispersive equations [24]. They can be obtained under suitable hypothesis both on the coefficients and the subset of approximation.

### 3 Solutions to PDEs with prescribed properties

The current applications of the strategy to construct solutions to PDEs with prescribed geometric or topological properties are many and assorted. All of them involve elliptic equations and have required the refinements of the classical global approximation theorem that we commented on earlier. They include the prescription of the level sets of solutions to the Laplace and Helmholtz equations [29, 15], the Allen–Cahn equation [32] and of eigenfunctions of the harmonic oscillator [25], and the proof of the existence and formation of knotted vortex structures in the 3D Euler and Navier–Stokes equations [28, 31, 27].

To the above contexts, we add an application to minimal surfaces. Making use of the new global approximation theorems, we also show the existence of solutions to parabolic equations with local hot spots moving along prescribed curves or with isothermic surfaces of prescribed topologies for all times.

#### Minimal surfaces with prescribed level sets

A clever construction of solutions to nonlinear equations with prescribed behavior in small sets yields the existence of minimal graphs on the  $n$ -dimensional unit ball  $\mathbb{B}^n$  whose transverse intersection with any horizontal hyperplane (equivalently, level set) has arbitrarily large  $(n - 1)$ -measure. Let us recall that if  $u$  is a solution of the equation

$$\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

in  $\mathbb{B}^n$ , the graph of  $u$ ,

$$\Sigma_u := \{(x, u(x)) : x \in \mathbb{B}^n\},$$

is a minimal hypersurface of  $\mathbb{R}^{n+1}$ .

The lack of a uniform bound for the  $(n - 1)$ -measure of (without loss of generality) the zero level set of a nontrivial minimal graph is a consequence of a more general result about the “freedom” in prescribing the geometry of the level set. Broadly speaking:

**Theorem 3** ([23]). *Given any hypersurface  $S$  of  $\mathbb{B}^n$  satisfying some mild conditions, there exists a minimal graph  $\Sigma_u$  on  $\mathbb{B}^n$  such that  $u^{-1}(0)$  has a connected component containing a small perturbation of  $S$ .*

The proof hinges on the construction of harmonic functions with the same property and small in a certain norm, which permits to promote them to solutions of the minimal surface equation through an iterative procedure that does not change much the shape of the level set. Taking a hypersurface with large enough  $(n - 1)$ -measure, the first claim is clearly recovered.

## Local hot spots and isothermic hypersurfaces

Combining a suitable construction of robust local solutions to parabolic equations with the previous global approximation theorems, we study the movement of local hot spots and isothermic hypersurfaces of solutions to parabolic equations. A (local) hot spot at time  $t$  of a solution  $u$  to the parabolic equation  $Lu = 0$  is a (local) maximum of  $u(\cdot, t)$ , whereas an isothermic hypersurface at time  $t$  is a connected component of a level set of  $u(\cdot, t)$ . For long time, these topics have attracted substantial attention, particularly concerning the large-time behavior of hot spots, the existence of stationary isothermic hypersurfaces and hot spots and the Matzoh ball soup problem [49, 10, 59, 61, 60].

Our first result claims that there are global solutions with local hot spots that move following, up to an arbitrarily small error, any prescribed curve in space for all times:

**Theorem 4** ([21]). *Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous curve in space (possibly periodic or with self-intersections) and take any continuous positive function on the line  $\delta(t)$ . If the coefficient of zeroth order of the parabolic operator  $L$  is nonpositive, then there is a solution to the parabolic equation  $Lu = 0$  in  $\mathbb{R}^{n+1}$  such that, at each time  $t \in \mathbb{R}$ ,  $u$  has a local hot spot  $X_t$  with*

$$|X_t - \gamma(t)| < \delta(t).$$

The second one asserts the existence of global solutions with an isothermic hypersurface of arbitrarily complicated compact topology (possibly changing in time):

**Theorem 5** ([21]). *Consider a possibly infinite subset  $\mathcal{I} \subset \mathbb{Z}$  and, for each index  $i \in \mathcal{I}$ , a compact orientable hypersurface without boundary  $\Sigma_i \subset \mathbb{R}^n$ . If  $i$  and  $i + 1$  belong to  $\mathcal{I}$ , we also assume that the domains bounded by  $\Sigma_i$  and  $\Sigma_{i+1}$  are disjoint. If the coefficient of zeroth order of the parabolic operator  $L$  is nonpositive, then there exists a solution of the equation  $Lu = 0$  in  $\mathbb{R}^{n+1}$  and constants  $\alpha_i > 0$  close to zero, such that  $\Sigma_i$  is a connected component of the isothermic hypersurface  $\{x \in \mathbb{R}^n : u(x, t) = \alpha_i\}$  up to a close to identity diffeomorphism whenever  $[t] = i$ .*

In both theorems, if the operator  $L$  belongs to the class which the decay estimate in the second part of Theorem 2 holds for and the time interval is finite, the global solution also exhibits this decay.

These results have some bearing on the possible behavior of solutions to the heat equation with periodic boundary conditions. Consider solutions to the



heat equation when the space variable takes values on the flat torus. For short times, there is no obstruction to the trajectories that local hot spots can follow on small space scales. Furthermore, the isothermic hypersurfaces can exhibit any prescribed, possibly rapidly changing topologies at small space-time scales.

## 4 The Biot-Savart operator of bounded domains

In the approximation theory for differential operators, the structure of the integral kernels of their inverses is essential. It is classical that a divergence-free vector field in the whole space can be recovered from its (well-behaved) curl by means of the Biot-Savart operator. This result applies to write the velocity field of an incompressible fluid in terms of its vorticity or the magnetic field in terms of the electric current.

The problem of mapping the vorticity contained in a bounded domain into its velocity field has been considered by several people, who have shown the existence of solutions for domains of different regularity and derived suitable estimates. But can these solutions be given by an integral formula? We answer positively to this question, overcoming the complications that the presence of tangency boundary conditions implies. The generalization of the classical Biot-Savart law to bounded domains can be stated as follows:

**Theorem 6** ([22]). *Consider the problem*

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

*Under necessary assumptions on  $\omega$ , the problem has a solution  $u$  which can be written as an integral of the form*

$$u(x) = \int_{\Omega} K_{\Omega}(x, y)w(y)dy,$$

*with  $K_{\Omega}(x, y)$  a matrix-valued integral kernel that is smooth outside the diagonal, where it has an inverse-square singularity.*

The proof combines the study of the usual Biot-Savart integral, refinements to consider the boundary terms and layer potentials.

## 5 Organization of the thesis and further remarks

This thesis goes beyond the state of the art in the context of global approximation theory for PDEs. Namely, we provide a general theory for parabolic equations including control of the growth for global solutions to certain class of

parabolic operators. An analogous approximation theorem with decay is also proved for elliptic equations.

Furthermore, we present some attractive applications of these theorems: the existence of global solutions to parabolic equations with local hot spots moving along prescribed curves or with isothermal hypersurfaces of prescribed topologies for all times, and the existence of nontrivial minimal graphs with arbitrarily large level sets.

Finally, we study the structure of the integral kernel related with the inverse of the curl operator for bounded domains. This generalizes the classical Biot–Savart operator and plays a role in the context of inviscid incompressible fluid confined in bounded regions.

The contents of this thesis are organized as follows. In Chapter 1 we introduce classic results in the theory of approximation for elliptic equations together with some improvements regarding unbounded sets and decay of the global solutions. In Chapter 2 we present a complete theory of approximation for parabolic operators, including approximation with decay. Their application to the movement of local hot spots and isothermic hypersurfaces is shown in Chapter 3. The prescription of level sets for solutions in the unit ball to the nonlinear equation for minimal surfaces is presented in Chapter 4. In Chapter 5 we study the Biot–Savart operator for bounded domains. To conclude, Appendix A contains a brief presentation of further results obtained during the Ph.D. studies that explore other subjects.

The central results of this thesis have been communicated in the following scientific publications:

- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, Approximation theorems for parabolic equations and movement of local hot spots, [arXiv: 1710.03782](#) (Chapters 2 and 3 and Section 1.3)
- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, Minimal graphs with micro-oscillations, *J. Differential Equations*, accepted, [arXiv: 1602.04978](#) (Chapter 4)
- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, The Biot-Savart operator of a bounded domain, *J. Math. Pures Appl.*, in press, [arXiv: 1702.04327](#) (Chapter 5)

Other less central results, which are explained in Appendix A, are collected in the following articles:

- M.A. García-Ferrero, D. Gómez-Ullate, Oscillation theorems for the Wronskian of an arbitrary sequence of eigenfunctions of Schrödinger’s equation, *Lett. Math. Phys.* 105 (2015) 551–573
- M.A. García-Ferrero, D. Gómez-Ullate, R. Milson, A Bochner type characterization for exceptional orthogonal polynomials, [arXiv: 1603.04358](#)

- M. Gadella, M.A. García-Ferrero, S. González-Martín, F.H. Maldonado-Villamizar, The Infinite Square Well with Point Interaction: A Discussion on the Different Parametrizations, *Int. J. Theor. Phys.* 53 (2014) 1614–1627
- P. Carrasco Millan, M.A. García-Ferrero, F.J. Llanes-Estrada, A. Porrás Riojano, E.M. Sanchez Garcia, Shannon entropy and particle decays, *Nucl. Phys. B* 930 (2018) 583–596

# *Introducción*

En el área de las ecuaciones en derivadas parciales (EDPs) existen problemas abiertos que pueden resolverse mediante la construcción de soluciones con propiedades geométricas o topológicas prescritas. El interés de estas preguntas no es exclusivamente matemático, puesto que muchas de ellas surgen también en el estudio de fenómenos físicos.

Esta tesis se centra en la construcción de soluciones a determinadas EDPs con ciertas características para dar respuesta a algunos de estos problemas y el desarrollo de una herramienta clave para ello: los teoremas de aproximación global.

## 1 Motivación

Los conjuntos de nivel de soluciones de EDPs son objetos de gran interés en muchos campos, comenzando por el estudio de las superficies equipotenciales de los potenciales electrostático y gravitatorio. Igualmente es importante caracterizar otras variedades invariantes de las soluciones de EDPs. Una pregunta que puede surgir de forma natural es si existen soluciones de una EDP que presenten conjuntos de nivel u otras de estas variedades invariantes con una topología o geometría dada.

La teoría para tratar con este tipo de preguntas fue introducida por A. Enciso y D. Peralta-Salas en [28] y [29] para estudiar las líneas de vorticidad de fluidos incompresibles no viscosos y los conjuntos de nivel de funciones armónicas, respectivamente. La estrategia que desarrollaron para tratar con estos problemas consta de los siguientes pasos:

1. Construir una solución a la ecuación que satisfaga las propiedades deseadas en un conjunto pequeño. En este paso pueden entrar en juego el teorema de Cauchy-Kowaleskaya o problemas con condiciones de frontera adecuadas.
2. Comprobar que la propiedad es robusta frente a pequeñas perturbaciones de la solución. Por ejemplo, si estamos tratando con conjuntos de nivel compactos, esto está garantizado por el teorema de isotopía de Thom.
3. Promocionar la solución local a una solución global salvo errores que podemos controlar usando un teorema de aproximación global de tipo Runge.

De este modo, siempre que podamos construir (aunque no sea explícitamente) una solución local adecuada, asegurar la robustez de la propiedad y garantizar el teorema de aproximación correspondiente, seremos capaces de dar solución al problema de partida. La flexibilidad de esta estrategia ha permitido su empleo en muchos contextos, desde la existencia de soluciones a EDPs elípticas con conjuntos de nivel de topología arbitrariamente compleja [29] hasta la solución de una conjetura de Lord Kelvin (1875) sobre la existencia de tubos de vorticidad anudados en fluidos incompresibles estacionarios [31].

Con respecto al tercer paso, diremos que un operador diferencial  $P$  verifica un teorema de aproximación global si, dada una función  $v$  que satisfaga la ecuación  $Pv = 0$  en un conjunto  $\Omega$  cumpliendo ciertas hipótesis técnicas, podemos concluir que existe una solución global  $u$  de  $Pu = 0$  en  $\mathbb{R}^n$  que aproxima  $v$ , en el sentido de que la diferencia  $u - v$  puede ser arbitrariamente pequeña en  $\Omega$  en una norma adecuada. Llamamos a estos teoremas «de tipo Runge» porque generalizan el clásico teorema de Runge (1884) sobre la aproximación de funciones holomorfas en conjuntos compactos del plano complejo con complemento conexo por polinomios. Setenta años después, Lax y Malgrange demostraron que la soluciones de ciertas ecuaciones elípticas homogéneas en subconjuntos acotados con complementos conexos son límites de soluciones de la misma ecuación en todo el espacio. Prácticamente todos los resultados posteriores son también para ecuaciones elípticas. Sin embargo, no existen restricciones que no permitan extender estos teoremas a otras ecuaciones con las hipótesis necesarias.

En las tres secciones siguientes se presentan sucintamente los principales resultados originales de esta tesis sobre la obtención de nuevos teoremas de aproximación global y sus aplicaciones. Cada uno de ellos tiene otras motivaciones específicas que se mencionan en los capítulos correspondientes. La Sección 5 contiene un resumen de las aportaciones de esta tesis y su impacto en el área, cómo están organizadas en los siguientes capítulos de esta tesis y los artículos en los que se han publicado.

## 2 Teoremas de aproximación global para EDPs

Los primeros teoremas de aproximación global datan de los años cincuenta. Sus autores, Lax y Malgrange, demostraron que, para ciertas clases de ecuaciones diferenciales lineales **elípticas** en  $\mathbb{R}^n$ , las soluciones en conjuntos acotados  $\Omega$  se pueden aproximar uniformemente por soluciones en  $\mathbb{R}^n$  siempre que el complemento de  $\Omega$  en  $\mathbb{R}^n$  sea conexo. Los operadores elípticos para los que probaron esta aproximación, cuyo principal ejemplo son los de coeficientes analíticos, fueron ampliados posteriormente por Browder, incluyendo resultados en ciertos espacios de Hölder. Pero en ningún caso son válidos para aproximación uniforme en regiones no acotadas ni dan ningún tipo de control sobre el crecimiento de las soluciones globales en el infinito. Estos aspectos son fundamentales para la construcción de soluciones en algunas aplicaciones y se han obtenido para determinadas ecuaciones. A continuación, se enuncia brevemente nuestro teorema de aproximación con decaimiento en infinito,

que generaliza el posible decaimiento puntual de las soluciones globales de la ecuación de Helmholtz en [31, 25]. A partir de ahora, vamos a asumir que el operador  $P$  y su adjunto formal  $P^*$  son lineales y sus coeficientes están en un espacio de Hölder adecuado.

**Teorema 1** ([21]). *Supongamos que  $P$  es un operador elíptico que se comporta como el de Helmholtz fuera de un conjunto compacto de acuerdo a la Definición 1.5 y que  $\Omega$  es un subconjunto compacto de  $\mathbb{R}^n$  con complemento conexo. Sea  $v$  una solución de  $Pv = 0$  en un entorno de  $\Omega$ . Entonces existe una solución  $u$  de  $Pu = 0$  en  $\mathbb{R}^n$  arbitrariamente próxima a  $v$  en una norma de Hölder adecuada y que verifica la condición de decaimiento*

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x)|^2 dx < \infty.$$

Si nos centramos ahora en los teoremas de aproximación para ecuaciones diferenciales **parabólicas**, solo encontraremos en la literatura algunos resultados básicos sobre aproximación uniforme en regiones acotadas para la ecuación del calor  $-\partial_t u + \Delta u = 0$ .

En esta tesis extendemos estos resultados a ecuaciones parabólicas generales y en regiones que pueden ser no acotadas. De nuevo, necesitaremos algún tipo de control sobre las soluciones globales en infinito para las aplicaciones. Con algunas restricciones en los operadores parabólicos, vemos que podemos tomar soluciones globales que decaigan en infinito, tanto en espacio como en tiempo.

Bajo la hipótesis de que los coeficientes del operador parabólico  $L$  y de su adjunto formal  $L^*$  pertenecen a cierto espacio de Hölder parabólico, podemos enunciar nuestros resultados, evitando detalles técnicos menores, del siguiente modo:

**Teorema 2** ([21]). *Sea  $\Omega$  un subconjunto cerrado de  $\mathbb{R}^{n+1}$  tal que para todo hiperplano ortogonal al eje temporal, su intersección con el complemento de  $\Omega$  no tiene ninguna componente conexa acotada. Sea  $v$  una solución de  $Lv = 0$  en un entorno de  $\Omega$ . Entonces existe un solución  $u$  de  $Lu = 0$  en todo  $\mathbb{R}^{n+1}$  arbitrariamente próxima a  $v$  en cierta norma de Hölder parabólica. Si además el operador  $L$  tiene coeficientes que no dependen del tiempo y se comporta como la ecuación del calor fuera de un conjunto compacto (salvo quizás una constante no positiva y pequeñas perturbaciones) según la Definición 2.3 y  $\Omega$  es compacto, entonces existe una solución  $u$  de  $Lu = 0$  en  $\mathbb{R}^{n+1}$  que aproxima  $v$  del mismo modo y además verifica la siguiente condición de decaimiento:*

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x, t)|^2 dx \leq C e^{-t/C}.$$

Un ejemplo típico de operador parabólico para el que la condición de decaimiento se cumple es

$$Lu := -\frac{\partial u}{\partial t} + \Delta_g u - V(x)u,$$

donde  $\Delta_g$  es el laplaciano asociado a una métrica riemanniana  $g$  que es euclídea fuera de un subconjunto compacto y  $V(x)$  es un potencial no negativo que es radial fuera de un conjunto compacto y tiende a una constante rápidamente en infinito. Además, en el caso concreto de que  $L$  se el operador del calor más una constante no positiva, el decaimiento se torna puntual:

$$|u(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}.$$

Como sucede para las ecuaciones elípticas, obtenemos también un refinamiento del teorema anterior mejorando la aproximación uniforme en el caso de que  $\Omega$  sea una unión localmente finita de subconjuntos compactos.

Más allá de las ecuaciones elípticas y parabólicas, cabe esperar nuevos teoremas de aproximación global para otras ecuaciones, como las dispersivas [24], bajo las las hipótesis adecuadas sobre los coeficientes y las regiones de aproximación.

### 3 Soluciones a EDPs con propiedades prescritas

Las aplicaciones existentes de la estrategia para construir soluciones a EDPs con propiedades geométricas o topológicas dadas son muchas y variadas. Todas ellas están asociadas a ecuaciones elípticas y precisan algunas mejoras de los teoremas clásicos de aproximación global que comentamos anteriormente. Estas incluyen la prescripción de conjuntos de nivel de soluciones de las ecuaciones de Laplace y Helmholtz [29, 15], de la ecuación de Allen–Cahn [32] y de autofunciones del oscilador armónico [25] y la demostración de la existencia y formación de estructuras de vorticidad anudadas en las ecuaciones de Euler y Navier–Stokes [28, 31, 27].

A las anteriores, nosotros añadimos una aplicación relativa a superficies mínimas. Haciendo uso de los nuevos teoremas de aproximación global para ecuaciones parabólicas, también probamos la existencia de soluciones globales a estas ecuaciones con *hot spots* o puntos calientes locales moviéndose según curvas dadas o con superficies isoterma de topología prescrita en todo tiempo.

#### Superficies mínimas con conjuntos de nivel prescritos

Una construcción audad de soluciones a la ecuación no lineal de las superficies mínimas con comportamiento prescrito en regiones pequeñas permite probar la existencia de grafos mínimos sobre la bola unidad  $n$ -dimensional  $\mathbb{B}^n$  cuya intersección transversa con un hiperplano horizontal (o conjunto de nivel) tiene medida  $((n - 1)$ -dimensional) arbitrariamente grande. Recordemos que si  $u$  es solución de la ecuación

$$\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

en  $\mathbb{B}^n$ , entonces el grafo de  $u$ ,

$$\Sigma_u := \{(x, u(x)) : x \in \mathbb{B}^n\},$$

es una hipersuperficie mínima en  $\mathbb{R}^{n+1}$ .

La falta de una cota uniforme para la medida  $(n - 1)$ -dimensional de los conjuntos de nivel de grafos mínimos no triviales es consecuencia de un resultado más general sobre la libertad para prescribir un conjunto de nivel. En términos generales:

**Teorema 3** ([23]). *Dada una hipersuperficie  $S$  de  $\mathbb{B}^n$  con frontera y sin auto-intersecciones, existe un grafo mínimo  $\Sigma_u$  en  $\mathbb{B}^n$  tal que  $u^{-1}(0)$  tiene una componente conexa que contiene una pequeña perturbación de  $S$ .*

La demostración se basa en la construcción de funciones armónicas con las mismas propiedades y pequeñas, lo que permite promocionarlas a soluciones de la ecuación de las superficies mínimas gracias a un proceso iterativo que no modifica mucho la apariencia del conjunto de nivel. Considerando una hipersuperficie de medida suficientemente grande, recuperamos el primer resultado.

## Puntos calientes locales e hipersuperficies isoterma

Si combinamos una construcción adecuada de soluciones locales robustas de ecuaciones parabólicas con los teoremas de aproximación global previos, podemos estudiar el movimiento de puntos calientes locales e hipersuperficies isoterma de soluciones a ecuaciones parabólicas. Un punto caliente o *hot spot* (local) para un tiempo  $t$  de una solución  $u$  de la ecuación parabólica  $Lu = 0$  es un máximo (local) de  $u(\cdot, t)$ , y una hipersuperficie isoterma es una componente conexa de un conjunto de nivel de  $u(\cdot, t)$ . Durante bastante tiempo, estos temas han recibido mucha atención, sobre todo el comportamiento de puntos calientes para tiempos grandes, la existencia de hipersuperficies isoterma y puntos calientes estacionarios y el problema de la *Matzoh ball soup* [49, 10, 59, 61, 60].

Nuestro primer resultado establece que hay soluciones globales con puntos calientes locales que se mueven siguiendo cualquier curva en el espacio, salvo un error arbitrariamente pequeño, para todo tiempo:

**Teorema 4** ([21]). *Sea  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  una curva continua en el espacio (pudiendo ser periódica o con auto-intersecciones) y  $\delta(t)$ , una función continua positiva sobre  $\mathbb{R}$ . Si el coeficiente de orden cero del operador parabólico  $L$  no es positivo, entonces existe una solución de la ecuación  $Lu = 0$  en  $\mathbb{R}^{n+1}$  tal que, para todo tiempo  $t \in \mathbb{R}$ ,  $u$  tiene un punto caliente local  $X_t$  con*

$$|X_t - \gamma(t)| < \delta(t).$$

El segundo garantiza la existencia de soluciones globales con hipersuperficies isoterma de topología compacta arbitrariamente compleja (quizás variando con el tiempo):



**Teorema 5** ([21]). *Consideremos un subconjunto posiblemente infinito  $\mathcal{I} \subset \mathbb{Z}$  y, para cada índice  $i \in \mathcal{I}$ , una hipersuperficie compacta orientable sin frontera  $\Sigma_i \subset \mathbb{R}^n$ . Si  $i$  e  $i + 1$  pertenecen a  $\mathcal{I}$ , supongamos también que los dominios acotados por  $\Sigma_i$  y  $\Sigma_{i+1}$  son disjuntos. Si el coeficiente de orden cero de  $L$  no es positivo, entonces existe una solución de la ecuación  $Lu = 0$  en  $\mathbb{R}^{n+1}$  y constantes  $\alpha_i > 0$  próximas a cero, tales que  $\Sigma_i$  es una componente conexa de la hipersuperficie isoterma  $\{x \in \mathbb{R}^n : u(x, t) = \alpha_i\}$ , salvo un difeomorfismo próximo a la identidad, para  $[t] = i$ .*

En ambos teoremas, si el operador  $L$  pertenece a la clase para la cual tenemos aproximación con decaimiento de la segunda parte del Teorema 2 y el intervalo temporal ( $\mathcal{I}$  en Teorema 5) es finito, la solución global puede tomarse con dicho decaimiento.

Los resultados previos tienen cierta reproducibilidad en el comportamiento de soluciones de la ecuación del calor con condiciones periódicas. Consideremos soluciones de dicha ecuación con variables espaciales en el toro plano. En tiempos pequeños, no existen restricciones sobre las trayectorias que pueden seguir los puntos calientes locales a escalas espaciales pequeñas. Además, las hipersuperficies isotermas pueden presentar cualquier topología prescrita que cambie rápidamente a pequeñas escalas espacio-temporales.

## 4 El operador de Biot–Savart en dominios acotados

En la teoría de aproximación para operadores diferenciales, la estructura de los núcleos integrales de sus inversos resulta esencial. Es clásico que un campo de vectores de divergencia nula en todo el espacio puede escribirse en términos de su rotacional gracias al operador de Biot–Savart. Este resultado nos permite escribir el campo de velocidad de un fluido incompresible en función de su vorticidad o el campo magnético en términos de la corriente eléctrica que lo genera.

Si nos restringimos a dominios acotados e imponemos condiciones de tangencia en el contorno, el problema puede escribirse del siguiente modo:

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0 \quad \text{en } \Omega, \quad u \cdot \nu = 0 \quad \text{en } \partial\Omega,$$

donde  $\omega$  es de divergencia nula y satisface ciertas condiciones en la frontera de  $\Omega$ . Podemos encontrar resultados sobre la existencia de soluciones para dominios de diferente regularidad y la obtención de diferentes estimaciones. Pero ¿pueden estas soluciones escribirse como una fórmula integral? Nosotros respondemos afirmativamente a esta pregunta, resolviendo las complicaciones derivadas de la presencia de las condiciones de tangencia. La generalización de la ley de Biot–Savart a dominios acotados puede entonces expresarse del siguiente modo:

**Teorema 6** ([22]). *El problema anterior tiene una solución  $u$  que puede escribirse como una integral de la forma*

$$u(x) = \int_{\Omega} K_{\Omega}(x, y)w(y)dy,$$

con  $K_{\Omega}(x, y)$  un núcleo integral matricial que es suave fuera de la diagonal, donde tiene una singularidad cuadrática.

La demostración combina el estudio de la integral de Biot–Savart, refinamientos para tener en cuenta los términos de contorno y los potenciales de capa.

## 5 Organización de la tesis y comentarios adicionales

En esta tesis damos un paso más en el campo de la teoría de aproximación global para EDPs. Concretamente, desarrollamos una teoría general para ecuaciones parabólicas, incluyendo control del crecimiento de las soluciones globales para ciertos operadores parabólicos. Obtenemos también un teorema de aproximación con decaimiento análogo para ciertas ecuaciones elípticas.

Así mismo, presentamos algunas aplicaciones muy interesantes de estos teoremas: la existencia de soluciones globales de ecuaciones parabólicas con puntos calientes locales moviéndose según curvas dadas o con hipersuperficies isotermales de topologías prescritas para todos los tiempos, y la existencia de grafos mínimos con conjuntos de nivel arbitrariamente grandes.

Por último, estudiamos la estructura del núcleo integral asociado al inverso del operador rotacional en dominios acotados. Esto generaliza el clásico operador de Biot–Savart y juega un papel esencial en el contexto de los fluidos incompresibles no viscosos confinados en regiones acotadas.

Los contenidos de esta tesis están organizados del siguiente modo: En el Capítulo 1 introducimos los resultados clásicos sobre la aproximación de soluciones de ecuaciones elípticas junto con algunas mejoras para regiones no acotadas y decaimiento de las soluciones. En el Capítulo 2 presentamos una teoría completa de aproximación para operadores parabólicos, incluyendo aproximación con decaimiento. Su aplicación al movimiento de puntos calientes locales e hipersuperficies isotermales se explica en el Capítulo 3. La existencia de soluciones a la ecuación no lineal de las superficies mínimas con conjuntos de nivel prescritos se prueba en el Capítulo 4. En el Capítulo 5 estudiamos el operador de Biot–Savart para dominios acotados. Para finalizar, el Apéndice A contiene una breve reseña de otros resultados obtenidos durante el desarrollo de esta tesis sobre diferentes temas.

Los resultados centrales de esta tesis corresponden a las siguientes publicaciones:

- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, Approximation theorems for parabolic equations and movement of local hot spots, [arXiv: 1710.03782](#) (Capítulos 2 y 3 y Sección 1.3)
- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, Minimal graphs with micro-oscillations, *J. Differential Equations*, aceptado, [arXiv: 1602.04978](#) (Capítulo 4)
- A. Enciso, M.A. García-Ferrero, D. Peralta-Salas, The Biot-Savart operator of a bounded domain, *J. Math. Pures Appl.*, aceptado, [arXiv: 1702.04327](#) (Capítulo 5).

El resto de resultados, que se explican en el Apéndice A, están contenidos en los siguientes artículos:

- M.A. García-Ferrero, D. Gómez-Ullate, Oscillation theorems for the Wronskian of an arbitrary sequence of eigenfunctions of Schrödinger's equation, *Lett. Math. Phys.* 105 (2015) 551–573
- M.A. García-Ferrero, D. Gómez-Ullate, R. Milson, A Bochner type characterization for exceptional orthogonal polynomials, [arXiv: 1603.04358](#)
- M. Gadella, M.A. García-Ferrero, S. González-Martín, F.H. Maldonado-Villamizar, The Infinite Square Well with Point Interaction: A Discussion on the Different Parametrizations, *Int. J. Theor. Phys.* 53 (2014) 1614–1627
- P. Carrasco Millán, M.A. García-Ferrero, F.J. Llanes-Estrada, A. Porrás Riojano, E.M. Sánchez García, Shannon entropy and particle decays, *Nucl. Phys. B* 930 (2018) 583–596

## Chapter 1

# Global approximation theorems for elliptic equations

The precursor of the global approximation theory for solutions of partial differential equations is the uniform approximation of holomorphic functions by polynomials. The first result, due to Runge [70] in 1885, states that a holomorphic function on a compact subset  $\Omega$  of the complex plane  $\mathbb{C}$  having connected complement  $\mathbb{C} \setminus \Omega$  can be uniformly approximated in  $\Omega$  by polynomials. Subsequent results weakened the condition that the function is holomorphic on the boundary  $\partial\Omega$ , until the work of Mergelyan [65] in 1952 asserting that a function continuous on a compact set  $\Omega \subset \mathbb{C}$  and holomorphic in the interior of  $\Omega$  can be uniformly approximated with arbitrary accuracy on  $\Omega$  by polynomials if and only if  $\Omega$  has a connected complement. This approximation by polynomials is a particular case of the approximation by rational functions with poles in the bounded connected components of the complement of  $\Omega$  in  $\mathbb{C}$ . In addition, we can consider approximation by holomorphic functions in larger subsets  $\tilde{\Omega}$  containing  $\Omega$  providing that  $\tilde{\Omega} \setminus \Omega$  is connected.

Beyond complex analysis, in the 1950s Lax and Malgrange [56, 64] were pioneers in proving global approximation theorems for solutions of some classes of elliptic linear differential equations and compact subsets  $\Omega \subset \mathbb{R}^n$  with connected complement. This theory attained its peak with the results of Browder [11, 12] in the 1960s. Roughly speaking, these theorems assert that a function satisfying a homogeneous linear elliptic equation on a compact subset  $\Omega$  of  $\mathbb{R}^n$  can be uniformly approximated by a global solution to the same equation if the complement of  $\Omega$  in  $\mathbb{R}^n$  is connected. The admissible equations must satisfy that any solution in a subset is uniquely determined by its behaviour in a smaller region. A prime example of this approximation result is given by harmonic functions on subsets, which, by analyticity, can always be approximated by harmonic polynomials.

The flexibility granted by global approximation theorems, which permits to reproduce the behavior of a local solution by a global solution up to a small error, has recently found applications in a number of seemingly unrelated contexts such as the nodal sets of eigenfunctions of Schrödinger operators [25, 26], level sets of solutions to the Laplace and Helmholtz equations [29, 15], the Allen–Cahn equation [32], the minimal surface equation [23] (see Chapter 4) and, thanks to an extension of the theory to the Beltrami field equation,  $\operatorname{curl} u - u =$

0, in  $\mathbb{R}^3$ , vortex structures in the 3D Euler and Navier–Stokes equations [28, 31, 27].

Key to several of these results has been the extension of the previous global approximation theorems for some equations to approximation in unbounded domains [29], better-than-uniform approximation [28, 29] or approximation by global solutions which decay at infinity [31, 32, 25]. In the latter case we have proved this property for a class of elliptic operators which behave like the Helmholtz equation outside a compact set [21].

This chapter is organised as follows. In Section 1.1 we mention the classical approximation theory based on nonconstructive density results with concrete statements. In Section 1.2 we present global approximation theorems in unbounded domains. In the case of locally finite union of bounded subsets, the approximation can be better than uniform approximation. Finally, in Section 1.3, we develop global approximation theorems with decay at infinity for essentially flat elliptic operators.

## 1.1 Classical approximation theorems

The global approximation theory for elliptic equations started with the works of Lax [56] and Malgrange [64] depending on properties of the adjoint operator in spaces of compactly supported distributions. They were extended by Browder [11, 12] to a more general class of operators in a more sophisticated functional setting.

We state below a concrete result for operators with coefficients in suitable Hölder spaces. Throughout this chapter,  $C^r(\Omega)$  will denote the Hölder space of functions in  $\Omega$  with bounded continuous derivatives up to order  $[r]$ , the integral part of  $r$ , and such that the partial derivatives of order  $[r]$  are Hölder continuous with exponent  $r - [r] \in (0, 1)$ . Recall that the corresponding norm is

$$\|\varphi\|_{C^r(\Omega)} := \max_{|\alpha| \leq [r]} \sup_{x \in \Omega} |D^\alpha \varphi(x)| + \max_{\substack{|\alpha| = [r] \\ x, y \in \Omega \\ x \neq y}} \sup \frac{|D^\alpha \varphi(x) - D^\alpha \varphi(y)|}{|x - y|^{r - [r]}}.$$

Also,  $\Omega^c := \mathbb{R}^n \setminus \Omega$  will denote the complement of the subset  $\Omega$  in  $\mathbb{R}^n$  and we will say that a function satisfies an equation on a closed set  $\Omega$  if this is satisfied in an open set containing  $\Omega$ .

**Theorem 1.1** ([12]). *Let  $\Omega$  be a closed subset in  $\mathbb{R}^n$  such that  $\Omega^c$  does not have any bounded connected components. Assume that  $P$  is an elliptic operator of order  $m$  with coefficients in  $C_{loc}^r$  with  $r > 1$  a non-integer real number and its formal adjoint  $P^*$  has locally Hölder continuous coefficients and satisfies the unique continuation property. Then any function  $v$  that satisfies the equation  $Pv = 0$  on  $\Omega$  can be locally uniformly approximated in the  $C^{r+m}$ -norm by solutions of the equation in the whole space  $\mathbb{R}^n$ . That is, for any compact subset  $K \subset \Omega$  and  $\delta > 0$ , there exists a solution  $u$  of the equation  $Pu = 0$  in*

$\mathbb{R}^n$  that approximates  $v$  as

$$\|u - v\|_{C^{r+m}(K)} < \delta.$$

The *unique continuation property* for the adjoint operator  $P^*$  means that if  $u \in C^m(\Omega)$  for a connected open subset  $\Omega$  with  $P^*u = 0$  on  $\Omega$  and if  $u$  vanishes on a non-vacuous open subset of  $\Omega$ ,  $u$  must vanish identically on  $\Omega$ . This is in many cases equivalent to the uniqueness for the Cauchy problem, which is affirmatively answered for linear equations with real analytic coefficients by Holmgren's theorem. In the case of elliptic equations with nonanalytic coefficients, the first result is due to Carleman and his estimates with exponential weights. Since then, this area has recorded a lot of activity with significant contributions by Aronszajn, Calderón, Hörmander or Tataru among others.

In particular, for second-order uniformly elliptic operators, if the leading-order-term coefficients are Lipschitz continuous and the rest are locally bounded, the unique continuation property is satisfied [6] (actually a strong version of the unique continuation introduced here). Therefore, mixing these requirements for the coefficients of  $P$  and  $P^*$  with those in Theorem 1.1 we can state the following:

**Theorem 1.2.** *Let  $\Omega$  be a compact subset in  $\mathbb{R}^n$  with connected complement. Assume that  $P$  is a second-order elliptic operator and that the coefficients of  $P$  and  $P^*$  lie in  $C^r$  with  $r > 1$  a non-integer real number. Then any solution  $v$  of  $Pv = 0$  in  $\Omega$  can be uniformly approximated in  $C^{r+2}$ -norm by global solutions of the same equation in  $\mathbb{R}^n$ .*

In the general setting, other conditions on the operator  $P$  and its formal adjoint  $P^*$  in the considered spaces are required. In Theorem 1.1 they are implied by the regularity assumptions on the coefficients. In other cases, some regularity condition on the boundary of the subset  $\Omega$  is also required. This happens for instance in the case of  $C^0$ -uniform approximation in compact subsets for operators with  $C^1$  coefficients. The required condition, related with the Sobolev cone condition, is always satisfied by domains with smooth boundary.

Finally, two last comments regarding the analogy with the Runge theorems for holomorphic functions: If both  $P$  and  $P^*$  verify the unique continuation property, any solution can be approximated by a finite linear combination of the fundamental solution with poles outside  $\Omega$ . Similar results hold for approximation by solutions in a larger subset  $\tilde{\Omega} \supset \Omega$  provided that  $\Omega \setminus \tilde{\Omega}$  has not bounded connected components. If both  $\Omega$  and  $\tilde{\Omega}$  are bounded smooth sets, a quantitative version of the classical result for self-adjoint elliptic operators is provided in [69].

## 1.2 Approximation in unbounded domains

Theorem 1.2 leaves out the option of promoting local solutions in unbounded regions to global solutions up to a uniform error. For instance, in [29] harmonic

functions with level sets homeomorphic to some unbounded hypersurfaces are constructed. This requires the existence of global approximation theorems in unbounded domains, besides the robustness of the prescribed property (here, the level sets) in unbounded subsets.

In other constructions, better-than-uniform approximation theorems are essential. The subsets for which this approximation is achieved may be unbounded but they must be locally finite unions of compact subsets. Let us recall that  $\Omega := \bigcup_{i=1}^{\infty} \Omega_i \subset \mathbb{R}^n$  is the *locally finite* union of the compact subsets  $\Omega_i$  if, given any other compact set  $K \subset \mathbb{R}^n$ , the number of subsets  $\Omega_i$  that intersect  $K$  is always finite.

Approximation theorems in unbounded domains for harmonic functions are due to Gauthier, Goldstein and Ow [40] and Bagby [7]. In [29, Theorem 4.5] Enciso and Peralta-Salas generalize them for the operator  $\Delta - q$  with  $q$  a bounded, nonnegative potential in  $\mathbb{R}^n$  with some regularity, which satisfies a minimum principle but no longer a maximum principle. The proof relies on an iterative procedure that is built over an appropriate exhaustion by compact sets and combines the approximation theorem in bounded subsets, suitable fundamental solution estimates and a balayage-of-poles argument.

With straight forward modifications, we can extend this result to any second order elliptic operator  $P$  that admits a fundamental solution  $G(x, y)$  with suitable bounds like

$$|G(x, y)| \leq C|x - y|^{2-n} \quad (1.1)$$

and such that  $G(x, y) = G^*(y, x)$ , with  $G^*(x, y)$  the fundamental solution of  $P^*$ , provided that the approximation theorem in bounded subsets holds. With this hypothesis, we can state:

**Theorem 1.3.** *Let  $\Omega$  be a closed subset of  $\mathbb{R}^n$  whose complement does not have any bounded connected components. Suppose that  $P$  is a second-order elliptic operator which admits a fundamental solution  $G(x, y)$  satisfying the bound (1.1) and that the coefficients of  $P$  and  $P^*$  lie in  $C^r$ , with  $r > 1$  a non-integer real number. Then any function  $v$  that satisfies the equation  $Pv = 0$  in  $\Omega$  can be approximated in the  $C^{r+2}$  norm by a global solution to this equation in  $\mathbb{R}^n$ . That is to say, for any  $\delta > 0$  there exists a function  $u$  satisfying  $Pu = 0$  in  $\mathbb{R}^n$  with*

$$\|u - v\|_{C^{r+2}(\Omega)} < \delta.$$

The existence of the fundamental solution of  $P$  and the relation with the one of  $P^*$  is guaranteed [66] by the regularity of the coefficients asked in Theorem 1.3. However, in order to promote the bound (1.1) satisfied by the fundamental solutions in bounded subsets to the fundamental solution in  $\mathbb{R}^n$  we must require that the equation satisfies a minimum principle, what means that the zeroth order coefficient of  $P$  is nonpositive. In the case of elliptic operators in divergence form, existence and the Gaussian bound for the fundamental solutions are proved for less regular coefficients [45]. Unique continuation results are also available for less regular subleading-order coefficients (e.g. [53]).

A better-than-uniform approximation theorem firstly appeared for solutions to the Helmholtz equation in  $\mathbb{R}^3$  in [28, Lemma 7.1] and directly applied for solutions of the Beltrami equation  $\operatorname{curl} u = \lambda u$ . The proof is based on an iterative application of the Lax-Malgrange approximation theorem and a suitably chosen exhaustion of the whole space. Consequently, no obstructions exist to formulate it for a larger class of linear elliptic differential equations. Those with analytic coefficients are considered in [29, Lemma 7.2], but we can generalize the statement of the theorem as follows:

**Theorem 1.4.** *Let the closed set  $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$  be a locally finite union of pairwise disjoint compact sets of  $\mathbb{R}^n$  whose complements are connected and let  $P$  be a linear elliptic operator with coefficients of  $P$  and  $P^*$  in  $C^r$ , with  $r > 1$  a non-integer real number. If the function  $v$  is a solution of the equation  $Pv = 0$  on  $\Omega$ , then it can be approximated in  $\Omega$  by a global solution  $u$  of the aforementioned equation in the  $C^{r+2}$  better-than-uniform sense. That is, for any sequence of positive constants  $\{\delta_i\}_{i=1}^{\infty}$  there is a function  $u$  satisfying the equation  $Pu = 0$  in  $\mathbb{R}^n$  and such that*

$$\|u - v\|_{C^{r+2}(\Omega_i)} \leq \delta_i$$

for all  $i$ .

Equivalently, we can also say that for any continuous positive function  $\delta(x)$  on the closed subset  $\Omega$  and any integer  $k < r + 2$  there exist a solution  $u$  of  $Pu = 0$  in  $\mathbb{R}^n$  with the pointwise  $C^k$ -bound

$$\sum_{|\alpha| \leq k} |D^\alpha u(x) - D^\alpha v(x)| < \delta(x)$$

for all  $x \in \Omega$ .

## 1.3 Approximation theorems with decay

If the subset  $\Omega$  is compact with connected complement in  $\mathbb{R}^n$ , the global solution can exhibit a decay at infinity for certain elliptic equations. This was shown in [31, 32, 25] for the Helmholtz equation and inherited by Beltrami fields, solutions to the Allen-Cahn equation and eigenfunctions of the harmonic oscillator, respectively.

Namely, in these articles it is proved that the approximating global solution  $u$  of the equation  $\Delta u + \lambda^2 u = 0$  falls off at infinity as  $|D^\alpha u(x)| < C_\alpha (1 + |x|)^{\frac{1-n}{2}}$  for any multiindex  $\alpha$ . The proof goes as follows: We approximate  $v$  by a global solution of the Helmholtz equation  $w$  thanks to Theorem 1.2. Then we write  $w$  as an infinite series of spherical harmonics in a ball containing  $\Omega$ , whose radial parts are given by the corresponding Bessel functions. Finally, we approximate  $w$  in the ball by a suitable truncated series,  $u$ . Because of its construction,  $u$  satisfies the Helmholtz equation in  $\mathbb{R}^n$ , approximates  $v$  in  $\Omega$  and satisfies the above decay estimate.



Beyond the existing approximation theorem with pointwise decay condition for the Helmholtz equation, we can extend this result to a larger class of elliptic equations which behave like the Helmholtz equation outside a compact set. The decay condition corresponds to finite Agmon–Hörmander seminorm. In the case of the Helmholtz equation, we recover the above-mentioned pointwise estimate.

The requisites that the coefficients of the operator must satisfy at infinity are specified in the following definition:

**Definition 1.5.** An elliptic operator on  $\mathbb{R}^n$

$$Pu := \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

is *essentially flat* if there are constants  $R_0 > 0$ ,  $c_0 > 0$  and a function bounded as

$$|F(r)| \leq C(1+r)^{-1-\epsilon}$$

on  $(0, \infty)$ , with  $\epsilon > 0$ , such that

$$a_{ij}(x) = \delta_{ij}, \quad b_i(x) = 0, \quad c(x) = c_0 + F(|x|)$$

for all  $|x| > R_0$ .

The result can now be stated as follows:

**Theorem 1.6.** *Let  $\Omega$  be a compact subset in  $\mathbb{R}^n$  such that  $\Omega^c$  is connected and suppose that  $P$  is an essentially flat elliptic operator with coefficients, together with those of  $P^*$ , of class  $C^r$ , with  $r > 1$  a non-integer real number. Given a function  $v$  that satisfies the equation  $Pv = 0$  on  $\Omega$  and  $\delta > 0$ , there exists a solution  $u$  of the equation  $Pu = 0$  in the whole space  $\mathbb{R}^n$  that approximates  $v$  as*

$$\|u - v\|_{C^{r+2}(\Omega)} < \delta$$

and satisfies the decay condition

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x)|^2 dx < \infty. \quad (1.2)$$

Furthermore, if  $P$  is the Helmholtz operator ( $Pu := \Delta u + c_0 u$  with  $c_0 > 0$ ), then the function decays pointwise as

$$|u(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}}. \quad (1.3)$$

*Proof.* Let us take a domain  $\Omega'$  containing the closed set  $\Omega$ . Without any loss of generality one can assume that the domain  $\Omega'$  is contained in the ball  $B_{R_0}$ . Theorem 1.2 ensures that for any  $\delta' > 0$  there exists a solution  $w$  of the equation  $Pw = 0$  in the whole space  $\mathbb{R}^n$  such that

$$\|w - v\|_{C^{r+2}(\Omega')} < \delta'.$$

Let us now study the behavior of the functions  $w$  in a (one-sided) neighborhood of  $\partial B_{2R_0}$ . We start by expanding  $w$  in an orthonormal basis of spherical harmonics on the unit  $(n-1)$ -dimensional sphere, which we denote as  $\{Y_m(\omega)\}_{m=1}^\infty$  with  $\omega \in \mathbb{S}^{n-1}$ :

$$w(x) =: \sum_{m=1}^{\infty} w_m(r) Y_m(\omega). \quad (1.4)$$

Here  $r := |x|$ ,  $\omega := \frac{x}{|x|}$  and  $\Delta_{\mathbb{S}^{n-1}} Y_m = -\mu_m Y_m$ , where

$$\mu_m =: l_m(l_m + n - 2),$$

with  $l_m$  a certain nonnegative integer, is an eigenvalue of the Laplacian on the unit  $(n-1)$ -dimensional sphere. The coefficients in the expansion are computed as

$$w_m(r) := \int_{\mathbb{S}^{n-1}} w(r\omega) Y_m(\omega) d\omega.$$

The sum converges in  $H^{r+2}(B_{2R_0})$ , so one can take a large enough  $M$  to ensure that

$$\|w - \tilde{w}\|_{H^{r+2}(B_{2R_0})} < \delta', \quad (1.5)$$

where  $\delta'$  is a positive constant to be specified later and

$$\tilde{w} := \sum_{m=1}^M w_m(r) Y_m(\omega).$$

Since

$$Pw = \Delta w + [c_0 + F(|x|)] w$$

for  $|x| > R_0$  by the definition of an essentially flat operator, it automatically follows from the fact that  $Y_m(\omega)$  are linearly independent that the functions  $w_m(r)$  defined in (1.4) satisfy the ODE

$$w_m'' + \frac{n-1}{r} w_m' + \left( c_0 + F(r) - \frac{\mu_m}{r^2} \right) w_m = 0 \quad (1.6)$$

for  $R_0 < r < 2R_0$ . This is a linear ODE whose coefficients are bounded for  $r \geq R_0$ , so it is standard that  $w_m$  is indeed global, that is, it can be extended as a function in  $C_{\text{loc}}^2((0, \infty))$  that satisfies (1.6) for  $r > R_0$ .

Our goal now is to show that  $w_m$  falls off at infinity as

$$|w_m(r)| < C(1+r)^{-\frac{n-1}{2}}. \quad (1.7)$$

To see this, we rewrite equation (1.6) as

$$(r^{\frac{n-1}{2}} w_m)'' + \left( c_0 - \frac{\mu_m + \frac{1}{4}(n-1)(n-3)}{r^2} + F(r) \right) (r^{\frac{n-1}{2}} w_m) = 0. \quad (1.8)$$

As  $c_0$  is positive, the solutions of  $y'' + c_0 y = 0$  (which are sines and cosines)

are obviously bounded, so the Dini–Hukuwara theorem [8] ensures that the solutions of (1.8) are also bounded as

$$\|r^{\frac{n-1}{2}} w_m\|_{L^\infty((0,\infty))} < \infty$$

provided that

$$\int_{R_0}^{\infty} \left| \frac{\mu_m + \frac{1}{4}(n-1)(n-3)}{r^2} - F(r) \right| dr < \infty.$$

Since  $F(r) < C(1+r)^{-1-\epsilon}$  by the definition of an essentially flat elliptic operator, this condition is always satisfied, so we infer the bound (1.7). In particular, this shows that the function  $\tilde{w}$  is defined in all of  $\mathbb{R}^n$  and falls off at infinity as

$$|\tilde{w}(x)| < C(1 + |x|)^{-\frac{n-1}{2}}. \quad (1.9)$$

Let us now observe that, since

$$P\tilde{w} = 0$$

outside  $B_{R_0}$ , the function

$$f := P\tilde{w}$$

is supported in  $B_{R_0}$  and bounded as

$$\|f\|_{H^r(B_{R_0})} < C\delta$$

by the estimate (1.5).

Suitable resolvent estimates then show [47, Theorem 30.2.10] that the function

$$g := (P + i0)^{-1}f = \lim_{\epsilon \rightarrow 0^+} (P + i\epsilon)^{-1}f \quad (1.10)$$

satisfies the equation

$$Pg = f$$

on  $\mathbb{R}^n$  and the sharp decay condition of Agmon–Hörmander:

$$\left( \sup_{R>0} \frac{1}{R} \int_{B_R} g^2 dx \right)^{\frac{1}{2}} \leq C\delta'. \quad (1.11)$$

If we now define

$$u(x) := \tilde{w}(x) - g(x),$$

we infer that it satisfies the equation  $Pu = 0$  on  $\mathbb{R}^n$ , falls off at infinity as

$$\sup_{R>0} \frac{1}{R} \int_{B_R} u^2 dx < \infty$$

by the bounds (1.9) and (1.11), and is close to  $v$  in the sense that

$$\|u - v\|_{L^2(\Omega')} \leq \|u - \tilde{w}\|_{L^2(B_{R_0})} + \|\tilde{w} - w\|_{L^2(B_{R_0})} + \|w - v\|_{L^2(\Omega')} < C\delta'$$

by (1.11) and the definitions of  $\tilde{w}$  and  $w$ . Furthermore, since  $P(u - v) = 0$  in  $\Omega'$ , standard elliptic estimates yield the Hölder bound

$$\|u - v\|_{C^{r+2}(\Omega)} < C\delta'.$$

If we choose  $\delta'$  small enough, the first part of the theorem then follows.

Suppose now that

$$Pu := \Delta u + c_0 u$$

Then, by spherical symmetry,

$$f \equiv 0$$

(or, equivalently, the truncated function  $\tilde{w}$  also satisfies the equation  $P\tilde{w} = 0$  in  $\mathbb{R}^n$ ), which ensures that  $g \equiv 0$ . This implies the decay condition

$$|u(x)| < C(1 + |x|)^{-\frac{n-1}{2}},$$

so the theorem is then proved. □



## Chapter 2

# Global approximation theorems for parabolic equations

In contrast with the wide literature about approximation theorems for elliptic equations (see Chapter 1), the only theorems for parabolic equations are some basic results [50, 20, 42] ensuring uniform approximation on compact subsets for the heat equation. Paraphrasing Jones in [50]:

**Theorem 2.1.** *Let  $\Omega$  be an open subset in  $\mathbb{R}^{n+1}$ . The necessary and sufficient condition for any solution  $v$  of*

$$-\frac{\partial v}{\partial t} + \Delta v = 0.$$

*on  $\Omega$  to be locally uniformly approximated by solutions of the heat equation in  $\mathbb{R}^{n+1}$  is that the complement of every intersection of  $\Omega$  with slices orthogonal to the time axis does not have any compact components.*

In [20], Diaz generalises the previous result to approximation in  $\Omega$  by solutions in a larger subset  $\tilde{\Omega} \supset \Omega$  provided that every intersection of  $\tilde{\Omega} \setminus \Omega$  with slices orthogonal to the time axis does not have any compact components. Lastly, Gauthier and Tarkhanov in [42] correct the proof by Diaz and add the approximation by rational solutions, i.e. by finite linear combinations of the heat kernel with poles outside the closure of the subset  $\Omega$ . None of these results provide any control on the growth of the solutions at infinity (neither in space nor in time) and have not yet found any applications.

Our goal in this chapter is to prove flexible global approximation theorems for general linear parabolic equations. Namely, we prove that any solution in  $\Omega$  (possibly unbounded) can be approximated in a parabolic Hölder norm by a global solution of the equation provided that  $\Omega$  satisfies the topological condition of Theorem 2.1. If  $\Omega$  is bounded and the parabolic operator satisfies certain technical conditions (e.g., when it is the usual heat equation), the global solution can be shown to fall off in space and time. We also show that the former approximation can be better than uniform in the case that  $\Omega$  is a locally finite union of bounded subsets.

## 2.1 Main results

The following results and proofs remain valid on noncompact manifolds with minor modifications, but for concreteness we will restrict our attention to Euclidean spaces and consider parabolic operators of the form

$$Lu := -\frac{\partial u}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x,t) \frac{\partial u}{\partial x_i} + c(x,t)u \quad (2.1)$$

where  $t \in \mathbb{R}$  and the space variable  $x$  takes values in  $\mathbb{R}^n$  and the equation is uniformly parabolic in the sense that

$$\inf_{(x,t) \in \mathbb{R}^{n+1}} a_{ij}(x,t) \xi_i \xi_j \geq C_L |\xi|^2$$

for all  $\xi \in \mathbb{R}^n$ , with  $C_L$  a uniform constant. The coefficients of  $L$  and of its formal adjoint, which we denote by  $L^*$ , are assumed to be in a suitable parabolic Hölder space  $C^{r, \frac{r}{2}}(\mathbb{R}^{n+1})$ , with  $r > 2$  a real number that is not an integer. The definition of these spaces and the precise regularity assumptions on the coefficients are given in Section 2.2. The parabolic Hölder norm of a function  $f(x,t)$  on a spacetime domain  $\Omega \subset \mathbb{R}^{n+1}$  will be denoted by

$$\|f\|_{r,\Omega} := \|f\|_{C^{r, \frac{r}{2}}(\Omega)}.$$

The first result that we present is a global approximation theorem on closed sets for general second-order parabolic operators. Notice that, unlike Theorem 2.1, we will talk about solutions in closed subsets of  $\mathbb{R}^{n+1}$ , meaning that an equation is satisfied in a closed subset  $\Omega \subset \mathbb{R}^{n+1}$  if the equation holds in an open neighborhood of  $\Omega$ .

In order to state the result, if  $\Omega \subset \mathbb{R}^{n+1}$  is a subset in spacetime, it is convenient to denote by  $\Omega(t)$  its intersection with the time  $t$  slice, that is,

$$\Omega(t) := \{x \in \mathbb{R}^n : (x,t) \in \Omega\}.$$

Furthermore, the complement of a subset  $\Omega \subset \mathbb{R}^{n+1}$  will be denoted by  $\Omega^c := \mathbb{R}^{n+1} \setminus \Omega$ . Let us recall that the topological hypothesis on the subset  $\Omega$  that we require in this theorem is known to be also necessary for global approximation already in the case of the heat equation [50].

**Theorem 2.2.** *Let  $\Omega$  be a closed subset in  $\mathbb{R}^{n+1}$  such that  $\Omega^c(t)$  does not have any bounded connected components for all  $t \in \mathbb{R}$ . Let us consider a function  $v$  that satisfies the parabolic equation  $Lv = 0$  in  $\Omega$ . Then for any  $\delta > 0$  there exists a solution  $u$  of the equation  $Lu = 0$  in the whole space  $\mathbb{R}^{n+1}$  such that*

$$\|u - v\|_{r+2,\Omega} < \delta.$$

In many applications, a major drawback of all classical global approximation theorems that is also shared by Theorem 2.2 is that it does not provide any

bounds on the behavior of the global solution at infinity. Our second theorem ensures that, in certain cases, the global solution  $u$  approximating the local solution  $v$  can be assumed to decay both at spatial infinity and as  $t \rightarrow \infty$ . Roughly speaking, this result holds provided that the differential operator  $L$  behaves like the heat equation outside a compact set. The hypothesis on the coefficients of the equation are analog to those of Theorem 1.6, so with some abuse of notation, the associated parabolic operators will be also called *essentially flat*:

**Definition 2.3.** A parabolic operator on  $\mathbb{R}^n$  in divergence form

$$Lu := -\frac{\partial u}{\partial t} + \frac{1}{\rho(x)} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u, \quad (2.2)$$

with  $\rho(x)$  a positive function and  $A_{ij}$  symmetric, is *essentially flat* if:

- (i) There is some  $R_0 > 0$  and a function bounded as

$$|F(r)| \leq C(1+r)^{-1-\epsilon}$$

on  $(0, \infty)$ , with  $\epsilon > 0$ , such that

$$\rho(x) = 1, \quad A_{ij}(x) = \delta_{ij}, \quad c(x) = c_0 + F(|x|)$$

for all  $|x| > R_0$ .

- (ii) One of the following conditions on  $c(x)$  and  $c_0$  holds

- (a)  $c_0 \geq 0$  and  $c(x)$  satisfies the mild nonpositivity condition that

$$\int_{\mathbb{R}^n} (C_L |\nabla \psi|^2 - c(x) \psi^2) \rho(x) dx > 0$$

for all nonzero  $\psi \in C_0^1(\mathbb{R}^n)$ , where  $C_L$  is the ellipticity constant,

- (b)  $c(x) \leq c_0 < 0$ .

The most important example of an essentially flat parabolic operator is, of course, the heat equation (possibly up to a nonpositive constant), that is,

$$Lu := -\frac{\partial u}{\partial t} + \Delta u + c_0 u,$$

with  $c_0 \leq 0$ . Another typical example of an essentially flat parabolic operator would be

$$Lu := -\frac{\partial u}{\partial t} + \Delta_g u - V(x)u,$$

where  $\Delta_g$  is the Laplacian operator associated with a Riemannian metric  $g$  that is Euclidean outside a compact set and the potential  $V(x)$  is the sum of a nonnegative radial potential tending to a constant fast enough at infinity and a nonnegative compactly supported perturbation.

To state our approximation theorem with decay, let us agree to denote by  $B_R$  the ball of radius  $R$  in  $\mathbb{R}^n$  centered at the origin:



**Theorem 2.4.** *Let  $\Omega$  be a compact subset in  $\mathbb{R}^{n+1}$  such  $\Omega^c(t)$  is connected for all  $t \in \mathbb{R}$  and suppose that  $L$  is an essentially flat parabolic operator. Given a function  $v$  satisfying the equation  $Lv = 0$  on  $\Omega$  and  $\delta > 0$ , there exists a solution  $u$  of the equation  $Lu = 0$  in the whole space  $\mathbb{R}^{n+1}$  that approximates  $v$  as*

$$\|u - v\|_{r+2, \Omega} < \delta$$

and satisfies the decay condition

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x, t)|^2 dx \leq C e^{-t/C}. \quad (2.3a)$$

for some positive constant  $C$ . Furthermore, if  $L$  is the heat equation up to a nonpositive constant, the function decays pointwise as

$$|u(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}. \quad (2.3b)$$

Finally we present a technical refinement of Theorem 2.2 which asserts that, when one has a solution of the equation  $Lv = 0$  defined on a locally finite union of compact subsets of  $\mathbb{R}^{n+1}$ , then the approximation by a global solution of the equation can be better than uniform. Specifically, the error in this approximation can be chosen to decrease as fast as one wishes as one goes to infinity. This refinement will be key crucially employed in some applications in Chapter 3.

Before stating the result, let us recall that  $\Omega := \bigcup_{i=1}^{\infty} \Omega_i \subset \mathbb{R}^{n+1}$  is the *locally finite* union of the compact subsets  $\Omega_i$  if, given any other compact set  $\mathcal{K} \subset \mathbb{R}^{n+1}$ , the number of subsets  $\Omega_i$  that intersect  $\mathcal{K}$  is always finite.

**Theorem 2.5.** *Let the closed set  $\Omega := \bigcup_{i=1}^{\infty} \Omega_i$  be a locally finite union of pairwise disjoint compact subsets  $\Omega_i$  of  $\mathbb{R}^{n+1}$  such that the complements of  $\Omega_i(t)$  are connected for all  $t \in \mathbb{R}$ , and let the function  $v$  satisfy the equation  $Lv = 0$  on  $\Omega$ . Then one can find a better-than-uniform approximation of  $v$  by a global solution of the equation, i.e., for any sequence of positive constants  $\{\delta_i\}_{i=1}^{\infty}$  there is a function  $u$  that satisfies the equation  $Lu = 0$  on  $\mathbb{R}^{n+1}$  and such that*

$$\|u - v\|_{r+2, \Omega_i} < \delta_i$$

for all  $i$ .

The rest of the chapter is organised as follows. In Section 2.2 we recall a few facts about fundamental solutions that will be needed in the rest of the chapter and set some notations. In Section 2.3 we state and prove several technical lemmas that are key in the proof of the global approximation results. The proofs of Theorems 2.2, 2.4 and 2.5 are respectively presented in Sections 2.4, 2.5 and 2.6.

## 2.2 Fundamental solutions for parabolic equations

In this section we state a few results on fundamental solutions for parabolic equations and establish some notation.

We start by recalling that the parabolic Hölder seminorm of a function  $f(x, t)$  defined on a spacetime region  $\Omega \subseteq \mathbb{R}^{n+1}$  is defined as

$$[f]_{\gamma, \frac{r}{2}, \Omega} := \sup_{\substack{(x,t), (y,s) \in \Omega \\ (x,t) \neq (y,s)}} \frac{|f(x, t) - f(y, s)|}{(|x - y| + |t - s|^{\frac{1}{2}})^\gamma},$$

where  $0 < \gamma < 1$ . The parabolic Hölder norm of order  $r$  (and we will hereafter assume that  $r > 0$  is a non-integer real number) can then be written as

$$\|f\|_{C^{r, \frac{r}{2}}(\Omega)} := \max_{2i+|\alpha| \leq [r]} \sup_{(x,t) \in \Omega} |D_x^\alpha \partial_t^i f(x, t)| + \max_{2i+|\alpha| = [r]} [D_x^\alpha \partial_t^i f]_{r-[r], \frac{r-[r]}{2}, \Omega},$$

where  $[\cdot]$  denotes the integral part.

We are now ready to state our regularity assumptions in terms of the parabolic Hölder norms of the coefficients of the operator  $L$  and of its formal adjoint, defined as

$$L^* u := \frac{\partial u}{\partial t} + \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^*(x, t) \frac{\partial u}{\partial x_i} + c^*(x, t) u \quad (2.4)$$

with

$$b_i^* := -b_i + 2 \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}, \quad c^* := c - \sum_{i=1}^n \frac{\partial b_i}{\partial x_i} + \sum_{i,j=1}^n \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j}.$$

Specifically, throughout we will assume that the coefficients of  $L$  and  $L^*$  are in a parabolic Hölder space  $C^{r, \frac{r}{2}}$  with a non-integer real  $r > 2$ :

$$a_{ij}, b_i, b_i^*, c, c^* \in C^{r, \frac{r}{2}}(\mathbb{R}^{n+1}). \quad (2.5)$$

When considering essentially flat operators, which we write as (2.2), we make the same regularity assumptions, which amount to saying that  $\rho(x)$ ,  $A_{ij}(x)$  and  $c(x)$  are in  $C^r(\mathbb{R}^n)$ .

Under the regularity assumption (2.5), it is standard (see [36, Chapter 1]) that the operators  $L$  and  $L^*$  admit fundamental solutions, which we denote by  $K(x, t; y, s)$  and  $K^*(x, t; y, s)$  respectively. Furthermore, the function  $K$  belongs to  $C^{r+2, \frac{r}{2}+1}$  outside the diagonal and is bounded as

$$|D_x^\alpha D_y^\beta \partial_t^k \partial_s^l K(x, t; y, s)| \leq C |t - s|^{-\frac{n+2k+2l+|\alpha|+|\beta|}{2}} e^{-\frac{|x-y|^2}{C|t-s|}} \quad (2.6)$$

for all multiindices with  $|\alpha| + |\beta| + 2k + 2l \leq [r] + 2$ . It is well known that the connection between  $K$  and  $K^*$  is

$$K^*(x, t; y, s) = K(y, s; x, t),$$

and that the solutions are causal in the sense that

$$\begin{aligned} K(x, t; y, s) &= 0 \quad \text{for } t < s, \\ K^*(x, t; y, s) &= 0 \quad \text{for } t > s. \end{aligned}$$

As  $K(x, t; y, s)$  is a fundamental solution, given any function  $\varphi$  in, say,  $C_c^\infty(\mathbb{R}^{n+1})$ , the function

$$w(x, t) := \int_{\mathbb{R}^{n+1}} K(x, t; y, s) \varphi(y, s) dy ds$$

satisfies the equation

$$Lw = \varphi.$$

In particular,

$$LK(\cdot, \cdot; y, s) = 0$$

for all  $(x, t) \neq (y, s)$ , and in fact,  $LK(x, t; y, s) = \delta(x - y) \delta(t - s)$  in the sense of distributions.

For future reference, let us record here that the parabolic operators  $L$  and  $L^*$  with  $C^{r, \frac{r}{2}}$  coefficients,  $r > 2$ , satisfy the unique continuation principle, which one can state as follows:

**Theorem 2.6** ([72]). *Let  $L$  be a uniformly parabolic operator in the previous sense whose leading-order coefficients lie in  $C^1$  (in  $x$  and  $t$ ) and the remaining ones are locally bounded. Consider an open subset in spacetime  $U \subset \mathbb{R}^{n+1}$  such that  $U(t)$  is connected for all  $t \in \mathbb{R}$ . If  $u$  satisfies  $Lu = 0$  or  $L^*u = 0$  in a connected subset  $\Omega \subset \mathbb{R}^{n+1}$  containing  $U$  and is identically zero in  $U$ , it then follows that  $u$  must vanish identically in the horizontal component of  $U$  in  $\Omega$ , that is,  $u(x, t) = 0$  for all  $(x, t) \in \Omega$  such that  $U(t) \neq \emptyset$ .*

## 2.3 Some technical lemmas on the sweeping of poles and discretization

In this section we state and prove some key technical lemmas that we will need for the proof of Theorem 2.2. The first lemma concerns the behavior of the fundamental solution  $K(x, t; y, s)$  when one perturbs a little the second pair of variables,  $(y, s)$ :

**Lemma 2.7.** *Let  $U$  be an open subset of  $\mathbb{R}^{n+1}$  and let  $(y, s)$  be a point in  $U$ . Then for any  $\epsilon > 0$  there is an open neighborhood  $B_{(y,s)}$  of  $(y, s)$  in  $U$  such that*

$$\|K(\cdot, \cdot; y, s) - K(\cdot, \cdot; y', s')\|_{r+2, U^c} < \epsilon \quad (2.7)$$

for all  $(y', s') \in B_{(y,s)}$ .

*Proof.* We can assume that  $U$  is bounded without loss of generality. Let us take a proper open subset  $\tilde{U} \subset U$  containing  $(y, s)$ . By the estimate (2.6) and the boundedness of  $\tilde{U}$ , for any  $\delta > 0$  we can take a compact subset  $\mathcal{K}_\delta \supset \tilde{U}$  of  $\mathbb{R}^{n+1}$  such that

$$\sup_{(y',s') \in \tilde{U}} \sup_{(x,t) \in \mathcal{K}_\delta^c} |K(x, t; y', s')| < \frac{\delta}{2}.$$

On the other hand, since  $K(x, t; y', s')$  depends continuously on  $(y', s') \in \mathbb{R}^{n+1} \setminus \{(x, t)\}$  and  $\mathcal{K}_\delta$  is compact, for each  $(y, s) \in \tilde{U}$  there is a small neighborhood  $B_{(y,s)} \subset \tilde{U}$  of  $(y, s)$  such that

$$\sup_{(y',s') \in B_{(y,s)}} \sup_{(x,t) \in \mathcal{K}_\delta \setminus \tilde{U}} |K(x, t; y, s) - K(x, t; y', s')| < \delta.$$

By the definition of the set  $\mathcal{K}_\delta$ ,

$$\|K(\cdot, \cdot; y, s) - K(\cdot, \cdot; y', s')\|_{0, \tilde{U}^c} < \delta \quad (2.8)$$

for all  $(y', s') \in B_{(y,s)}$ .

It is clear that  $K(x, t; y, s) - K(x, t; y', s')$ , as a function of  $(x, t)$ , solves the parabolic differential equation

$$L(K(x, t; y, s) - K(x, t; y', s')) = 0$$

in  $\tilde{U}^c$ . As the closure of the bounded set  $\tilde{U}$  is contained in  $U$  and the coefficients of  $L$  are in  $C^{r, \frac{r}{2}}(\mathbb{R}^{n+1})$ , standard interior Schauder estimates (applied to a uniform cover of  $\tilde{U}^c$  by bounded domains) then allow us to promote the uniform bound (2.8) to

$$\|K(\cdot, \cdot; y, s) - K(\cdot, \cdot; y', s')\|_{r+2, U^c} \leq C \|K(\cdot, \cdot; y, s) - K(\cdot, \cdot; y', s')\|_{0, \tilde{U}^c} < C\delta.$$

The lemma then follows.  $\square$

The second lemma shows that, outside a spacetime domain  $U$  satisfying a certain topological condition, one can approximate the fundamental solution  $K(\cdot, \cdot; y, s)$ , understood as a function of the first pair of variables with  $(y, s)$  fixed, by a linear combination of functions of the form  $K(\cdot, \cdot; y_j, s_j)$ , where the ‘‘poles’’  $(y_j, s_j)$  lie on a prescribed bounded domain contained in  $U$ :

**Lemma 2.8.** *Let  $U$  be a domain in  $\mathbb{R}^{n+1}$  such that  $U(t)$  is connected for all  $t \in \mathbb{R}$ . Consider a point  $(y, s) \in U$  and a bounded domain  $\mathcal{K} \subset U$  such that  $\mathcal{K}(s) \neq \emptyset$ . Then, for any  $\epsilon > 0$  there exists a finite set of points  $\{(y_j, s_j)\}_{j=1}^J$  in  $\mathcal{K}$  and real constants  $\{b_j\}_{j=1}^J \subset \mathbb{R}$  such that*

$$\left\| K(\cdot, \cdot; y, s) - \sum_{j=1}^J b_j K(\cdot, \cdot; y_j, s_j) \right\|_{r+2, U^c} < \epsilon. \quad (2.9)$$

*Proof.* We assume  $(y, s)$  does not belong to  $\mathcal{K}$ , as otherwise the statement is trivial. Let us take a proper bounded subdomain  $\tilde{U} \subset U$  containing  $(y, s)$  and  $\mathcal{K}$ . We can assume that  $\tilde{U}(t)$  is connected for all  $t$ .

Consider the space  $\mathcal{V}$  of all finite linear combinations of the fundamental solution with poles belonging to  $\mathcal{K}$ , i.e.

$$\mathcal{V} := \text{span}_{\mathbb{R}}\{K(x, t; z, \tau) : (z, \tau) \in \mathcal{K}\}.$$

Restricting these functions to the complement of  $\tilde{U}$ ,  $\mathcal{V}$  can be regarded as a subspace of the Banach space  $C_0(\tilde{U}^c)$  of bounded continuous functions on  $\tilde{U}^c$  that tend to zero at infinity.

By the Riesz–Markov theorem, the dual of  $C_0(\tilde{U}^c)$  is the space  $\mathcal{M}(\tilde{U}^c)$  of finite regular signed Borel measures on  $\mathbb{R}^{n+1}$  supported on  $\tilde{U}^c$ . Let us take  $\mu \in \mathcal{M}(\tilde{U}^c)$  orthogonal to all functions in  $\mathcal{V}$ , i.e.,

$$\int v d\mu = 0 \quad \text{for all } v \in \mathcal{V}.$$

Now we define a function  $F \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$  as

$$F(x, t) := \int K^*(x, t; z, \tau) d\mu(z, \tau) \tag{2.10}$$

where  $K^*(x, t; y, s)$  is the fundamental solution of the adjoint equation (2.4). Therefore,  $F$  satisfies the equation

$$L^*F = \mu$$

in the sense of distributions, which implies  $L^*F = 0$  in  $\tilde{U}$ . In addition,  $F$  is identically zero in  $\mathcal{K}$  because, for all  $(x, t) \in \mathcal{K}$ ,

$$F(x, t) = \int K^*(x, t; z, \tau) d\mu(z, \tau) = \int K(z, \tau; x, t) d\mu(z, \tau) = 0$$

by the definition of the measure  $\mu$ .

Hence, since  $\tilde{U}(t)$  is connected, the Unique Continuation Theorem 2.6 ensures that the function  $F$  vanishes on the horizontal components of  $\mathcal{K}$  in  $\tilde{U}$ . Particularly,  $F$  vanishes in  $\tilde{U}(s)$ . It then follows that

$$\int K(x, t; y, s) d\mu(x, t) = F(y, s) = 0.$$

This implies that  $K(x, t; y, s)$  can be uniformly approximated in  $C^0(\tilde{U}^c)$  by elements of the subspace  $\mathcal{V}$  as a consequence of the Hahn–Banach theorem.

Let us consider a function

$$v := \sum_{j=1}^J b_j K(\cdot, \cdot; y_j, s_j)$$

in  $\mathcal{V}$  such that

$$\|K(\cdot, \cdot; y, s) - v\|_{C^0(\tilde{U}^c)} \leq \delta$$

for sufficiently small  $\delta$ . Since

$$L(K(\cdot, \cdot; y, s) - v) = 0$$

in  $\tilde{U}^c$ , standard parabolic estimates allow us to promote the above uniform bound to

$$\|K(\cdot, \cdot; y, s) - v\|_{r+2, U^c} \leq C\|K(\cdot, \cdot; y, s) - v\|_{0, \tilde{U}^c} \leq C\delta.$$

We then conclude the desired result.  $\square$

The third lemma shows that the “convolution” of the fundamental solution and a compactly supported function can be approximated by a linear combination of fundamental solution  $K(x, t; y_j, s_j)$  with poles  $(y_j, s_j)$  lying on a certain spacetime region:

**Lemma 2.9.** *Let  $\varphi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  be an  $L^1$  function of compact essential support. For any neighborhood  $U$  of  $\text{supp } \varphi$  and  $\delta > 0$  there exists a finite set of points  $\{(y_j, s_j)\}_{j=1}^J$  in  $\text{supp } \varphi$  and constants  $\{c_j\}_{j=1}^J \subset \mathbb{R}$  such that*

$$\left\| \int_{\mathbb{R}^{n+1}} K(x, t; y, s) \varphi(y, s) dy ds - \sum_{j=1}^J c_j K(x, t; y_j, s_j) \right\|_{r+2, U^c} < \delta \quad (2.11)$$

*Proof.* Let  $U$  be a bounded open neighborhood of the support of  $\varphi$ . Since the support of  $\varphi$  is compact, we can take a finite collection of balls  $\{B_{(y_j, s_j)}\}_{j=1}^J$  as in Lemma 2.7, centered at points  $\{(y_j, s_j)\}_{j=1}^J$  in  $\text{supp } \varphi$  and with  $\epsilon < \frac{\delta}{\|\varphi\|_{L^1}}$ , such that

$$\text{supp } \varphi \subset \bigcup_{j=1}^J B_{(y_j, s_j)}.$$

Let  $(\chi_j)_{j=1}^J \subset C_c^\infty(\mathbb{R}^{n+1})$  be a partition of unity subordinated to  $\{B_{(y_j, s_j)}\}_{j=1}^J$ . Defining the function  $w$  as

$$w(x, t) := \sum_{j=1}^J K(x, t; y_j, s_j) \int \chi_j(y, s) \varphi(y, s) dy ds,$$

one can write

$$\begin{aligned} & \int K(x, t; y, s) \varphi(y, s) dy ds - w(x, t) \\ &= \sum_{j=1}^J \int \left( K(x, t; y, s) - K(x, t; y_j, s_j) \right) \chi_j(y, s) \varphi(y, s) dy ds \end{aligned}$$

for any  $(x, t) \in U^c$ . Therefore, by Lemma 2.7,

$$\begin{aligned} & \left\| \int K(\cdot, \cdot; y, s) \varphi(y, s) dy ds - w \right\|_{r+2, U^c} \\ & \leq \sum_{j=1}^J \int \|K(\cdot, \cdot; y, s) - K(\cdot, \cdot; y_j, s_j)\|_{r+2, U^c} |\varphi(y, s)| \chi_j(y, s) dy ds \\ & < \frac{\delta}{\|\varphi\|_{L^1}} \int |\varphi(y, s)| \sum_{j=1}^J \chi_j(y, s) dy ds = \delta. \end{aligned}$$

Estimate (2.11) then follows taking  $c_j := \int \chi_j(y, s) \varphi(y, s) dy ds$ .  $\square$

The next lemma in this section shows how the fundamental solution  $K(x, t; y, s)$  can be approximated in a certain spacetime region by a function satisfying  $Lw = 0$  in the whole  $\mathbb{R}^{n+1}$ :

**Lemma 2.10.** *Let  $D$  be an unbounded domain in  $\mathbb{R}^n$ . Consider the subset of  $\mathbb{R}^{n+1}$   $U = D \times (T_0, T_1)$  and a point  $(y, s) \in U$ . Then for any  $\delta > 0$  there exists a function  $w$  which satisfies  $Lw = 0$  in  $\mathbb{R}^{n+1}$  such that*

$$\|K(\cdot, \cdot; y, s) - w\|_{r+2, U^c} < \delta.$$

*Proof.* As  $D$  is unbounded, we can take a parametrized curve  $z : [0, \infty) \rightarrow D$  without self-intersections such that

$$z(0) = y, \quad \lim_{\ell \rightarrow +\infty} |z(\ell)| = +\infty.$$

For each nonnegative integer  $m$  we will denote by  $\mathcal{K}_m \subset D$  a bounded open neighborhood of the point  $z(m)$ , and  $W_m$  a bounded domain containing  $\mathcal{K}_m$  and  $\mathcal{K}_{m-1}$ .

Let us consider the following subsets of  $\mathbb{R}^{n+1}$

$$\mathcal{K}_m^T = \mathcal{K}_m \times (T_0, T_1), \quad W_m^T = W_m \times (T_0, T_1),$$

and denote by  $\mathcal{V}_m$  the space of finite linear combinations of the fundamental solution with poles in  $\mathcal{K}_m^T$ , i.e.

$$\mathcal{V}_m := \text{span}_{\mathbb{R}} \{K(x, t; z, \tau) : (z, \tau) \in \mathcal{K}_m^T\}.$$

As  $W_1^T(t)$  is connected for all  $t$  and  $\mathcal{K}_1^T(s)$  is nonempty, by Lemma 2.8 there exists  $v_1 \in \mathcal{V}_1$  such that

$$\|K(\cdot, \cdot; y, s) - v_1\|_{r+2, (W_1^T)^c} < \frac{\delta}{2}.$$

Since  $v_1$  is a finite linear combination of fundamental solutions with poles in  $\mathcal{K}_1^T$ , using Lemma 2.8 again it can be inductively shown that there are functions

$v_m \in \mathcal{V}_m$  such that

$$\|v_m - v_{m-1}\|_{r+2, (W_m^T)^c} < \frac{\delta}{2^m},$$

for all  $m \geq 2$ . Since the distance between the point  $y$  and  $W_m$  tends to infinity as  $m \rightarrow \infty$ , by a standard argument this implies that  $v_m$  converges  $C^{r+2, \frac{r}{2}+1}$ -uniformly on compact sets to a function  $w$  that solves

$$Lw = 0$$

in  $\mathbb{R}^{n+1}$ .

To finish the proof, it only remains to observe that

$$\begin{aligned} \|K(\cdot, \cdot; y, s) - w\|_{r+2, U^c} &\leq \|K(\cdot, \cdot; y, s) - v_1\|_{r+2, U^c} + \sum_{m=2}^{\infty} \|v_m - v_{m-1}\|_{r+2, U^c} \\ &< \sum_{m=1}^{\infty} \frac{\delta}{2^m} = \delta, \end{aligned}$$

where we have used the estimates derived above and the definition of  $w$ .  $\square$

Finally, the last lemma in this section is a preliminary global approximation result that we can readily prove using the previous results in this section and which will be instrumental in the proof of Theorem 2.2:

**Lemma 2.11.** *Let  $v$  be a function that satisfies the parabolic equation  $Lv = 0$  on a compact subset  $U \subset \mathbb{R}^{n+1}$  such that  $U^c(t)$  is connected for all  $t \in \mathbb{R}$ . Then for any  $\delta > 0$  there exists a function  $w$  satisfying  $Lw = 0$  in  $\mathbb{R}^{n+1}$  and such that it approximates  $v$  as*

$$\|v - w\|_{r+2, U} < \delta.$$

*Proof.* Let  $\tilde{U}$  be an open neighborhood of  $U$  where  $Lv = 0$ . Take a smooth function  $\chi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  equal to 1 in a closed set  $U' \subset \tilde{U}$  whose interior contains  $U$  and identically zero outside  $\tilde{U}$ . We define a smooth extension  $\tilde{v}$  of the function  $v$  to  $\mathbb{R}^{n+1}$  by  $\tilde{v} := \chi v$ . Notice that  $\tilde{v} = \tilde{v}' + \tilde{v}''$  where

$$\tilde{v}'(x, t) := \int K(x, t; y, s) L\tilde{v}(y, s) dy ds$$

and  $\tilde{v}'' := \tilde{v} - \tilde{v}'$  satisfies  $L\tilde{v}'' = 0$  in  $\mathbb{R}^{n+1}$ .

The support of  $L\tilde{v}$  is contained in  $\tilde{U} \setminus U'$ . Then Lemma 2.9 ensures we can approximate  $\tilde{v}'$  outside  $\tilde{U} \setminus U'$  by a finite combination of fundamental solutions with poles in  $\text{supp } L\tilde{v}$ . In particular, we choose  $\{(y_j, s_j)\}_{j=1}^J \subset \text{supp } L\tilde{v}$  and real constants  $\{c_j\}_{j=1}^J$  such that

$$\left\| \tilde{v}' - \sum_{j=1}^J c_j K(\cdot, \cdot; y_j, s_j) \right\|_{r+2, U} < \frac{\delta}{2}.$$



Since  $U^c(t)$  is connected for all  $t$ , for each index  $1 \leq j \leq J$  there exists an unbounded domain  $D_j \subset \mathbb{R}^n$  and a finite interval  $(T_{0j}, T_{1j})$  such that  $(y_j, s_j) \in D_j \times (T_{0j}, T_{1j})$  and  $D_j \times (T_{0j}, T_{1j}) \subset U^c$ . We can now apply Lemma 2.10 to each point  $(y_j, s_j)$ , thereby obtaining a function  $w'$  that satisfies  $Lw' = 0$  in  $\mathbb{R}^{n+1}$  and such that

$$\left\| \sum_{j=1}^J c_j K(\cdot, \cdot; y_j, s_j) - w' \right\|_{r+2, U} < \frac{\delta}{2}.$$

To conclude, consider the function  $w := w' + \tilde{v}''$ . By construction, it satisfies  $Lw = 0$  in  $\mathbb{R}^{n+1}$  and is bounded as

$$\|w - v\|_{r+2, U} = \|\tilde{v}'' - w'\|_{r+2, U} < \delta,$$

where we have used that  $v = \tilde{v}$  in  $U$  and the previous bounds. □

## 2.4 Proof of global approximation Theorem 2.2

Let  $\tilde{\Omega}$  be an open neighborhood of  $\Omega$  where  $Lv = 0$ . Let us take a smooth function  $\chi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  that is equal to 1 in a closed set  $\Omega' \subset \tilde{\Omega}$  whose interior contains  $\Omega$  and is identically zero outside  $\tilde{\Omega}$ . We define a smooth extension  $\tilde{v}$  of  $v$  to  $\mathbb{R}^{n+1}$  by  $\tilde{v} := \chi v$ .

Let  $U_1 \subset U_2 \subset \dots \subset \mathbb{R}^{n+1}$  be an exhaustion of  $\mathbb{R}^{n+1}$  by bounded spacetime domains. For concreteness, one can take  $U_m := B_m \times (-m, m)$ . Removing the first sets if necessary, there is no loss of generality in assuming that the intersection  $U_1 \cap (\mathbb{R}^{n+1} \setminus \Omega)$  is nonempty. Let  $\{\varphi_m : \mathbb{R}^{n+1} \rightarrow [0, 1]\}_{m=1}^\infty$  be a partition of unity subordinated to  $U_{m+1} \setminus U_{m-1}$ , where we have set  $U_0 := \emptyset$ . (That is, the support of  $\varphi_m$  is contained in  $U_{m+1} \setminus U_{m-1}$  and  $\sum_{m=1}^\infty \varphi_m(x, t) = 1$  for all  $(x, t) \in \mathbb{R}^{n+1}$ .)

Consider the functions

$$\tilde{v}_m(x, t) := \int_{\mathbb{R}^{n+1}} K(x, t; y, s) \varphi_m(y, s) L\tilde{v}(y, s) dy ds.$$

By the definition of the fundamental solution, it is clear that

$$L\left(\tilde{v} - \sum_{m=1}^{M+1} \tilde{v}_m\right) = 0 \text{ in a neighborhood of } U_M.$$

By Lemma 2.11 there exists a function  $w_1$  satisfying  $Lw_1 = 0$  on  $\mathbb{R}^{n+1}$  and such that

$$\|\tilde{v} - \tilde{v}_1 - \tilde{v}_2 - w_1\|_{r+2, U_1} < 1.$$

For a general  $M \geq 2$ , one can now inductively apply Lemma 2.11 to the function

$$\tilde{v} - \sum_{m=1}^{M+1} \tilde{v}_m - \sum_{m=1}^{M-1} w_m$$

to show that there is a function  $w_M$  satisfying  $Lw_M = 0$  on  $\mathbb{R}^{n+1}$  and such that

$$\left\| \tilde{v} - \sum_{m=1}^{M+1} \tilde{v}_m - \sum_{m=1}^M w_m \right\|_{r+2, U_M} < \frac{1}{M}. \quad (2.12)$$

Therefore the function  $\tilde{v}$  can be expressed in terms of the functions  $w_m$  as

$$\tilde{v} = \sum_{m=1}^{\infty} (\tilde{v}_m + w_m), \quad (2.13)$$

the convergence being uniform in  $C^{r+2, \frac{r}{2}+1}$  on compact subsets of  $\mathbb{R}^{n+1}$ :

Let us now approximate  $\tilde{v}_m$  by functions  $\hat{v}_m$  satisfying  $L\hat{v}_m = 0$  in  $\mathbb{R}^{n+1}$ . Since the support of  $\varphi_m L\tilde{v}$  is contained in

$$(\tilde{\Omega} \setminus \Omega') \cap (U_{m+1} \setminus U_{m-1}),$$

which is in turn a bounded subset whose complement contains  $\Omega \cup U_{m-1}$ , we can apply Lemma 2.9 to construct linear combinations of the form

$$\hat{v}_m(x, t) = \sum_{j=1}^{J_m} c_{m,j} K(x, t; y_{m,j}, s_{m,j})$$

that approximate  $\tilde{v}_m$  as

$$\|\hat{v}_m - \tilde{v}_m\|_{r+2, \Omega \cup U_{m-1}} \leq \frac{\delta}{2^{m+1}}. \quad (2.14)$$

Here  $\{(y_{m,j}, s_{m,j})\}_{j=1}^{J_m}$  is a finite set of points contained in the support of  $\varphi_m L\tilde{v}$  and  $c_{m,j}$  are real constants.

We will now apply Lemma 2.10 in order to sweep the poles of  $\hat{v}_m$  to infinity. For each  $m$ , let us consider an open set  $W_m \subset (U_{m-1} \cup \Omega)^c$  having  $J_m$  connected components of the form  $D_{m,j} \times (T_{0m,j}, T_{1m,j})$ , where  $D_{m,j}$  is an unbounded domain in  $\mathbb{R}^n$  and

$$D_{m,j} \times (T_{0m,j}, T_{1m,j}) \cap \bigcup_{j'=1}^{J_m} \{(y_{m,j'}, s_{m,j'})\} = \{(y_{m,j}, s_{m,j})\}.$$

Notice that it is possible to choose a set  $W_m$  as above because  $(y_{m,j'}, s_{m,j'}) \in U_{m-1}^c$  for all  $1 \leq j' \leq J_m$ . Lemma 2.10 then ensures the existence of functions  $\hat{w}_m$  satisfying  $L\hat{w}_m = 0$  in  $\mathbb{R}^{n+1}$  which approximate  $\hat{v}_m$  as

$$\|\hat{w}_m - \hat{v}_m\|_{r+2, W_m^c} \leq \frac{\delta}{2^{m+1}}. \quad (2.15)$$

Our goal now is to define a function  $u$  as

$$u := \sum_{m=1}^{\infty} (w_m + \widehat{w}_m). \quad (2.16)$$

To show that this makes sense, we will next prove that the sum on the right hand side converges  $C^{r+2, \frac{r}{2}+1}$ - uniformly on compact subsets of  $\mathbb{R}^{n+1}$ . In order to do this, given any compact spacetime domain  $\mathcal{K} \subset \mathbb{R}^{n+1}$  let us take an integer  $m_0$  such that  $\mathcal{K}$  is contained in  $U_{m_0-1}$  and the intersection  $W_m \cap \mathcal{K}$  is empty for all  $m > m_0$ . Then for any  $M > m_0$  we have the following estimate on the compact set  $\mathcal{K}$ :

$$\begin{aligned} \left\| \sum_{m=m_0}^M w_m + \sum_{m=m_0}^{M+1} \widehat{w}_m \right\|_{r+2, \mathcal{K}} &\leq \sum_{m=m_0}^{M+1} \|\widehat{w}_m - \widehat{v}_m\|_{r+2, \mathcal{K}} + \sum_{m=m_0}^{M+1} \|\widehat{v}_m - \widetilde{v}_m\|_{r+2, \mathcal{K}} \\ &\quad + \left\| \sum_{m=m_0}^{M+1} \widetilde{v}_m + \sum_{m=m_0}^M w_m \right\|_{r+2, \mathcal{K}} \\ &< \frac{\delta}{2} + \frac{\delta}{2} + \left\| \widetilde{v} - \sum_{m=1}^{M+1} \widetilde{v}_m - \sum_{m=1}^M w_m \right\|_{r+2, \mathcal{K}} + \|\widetilde{v}\|_{r+2, \mathcal{K}} \\ &\quad + \left\| \sum_{m=1}^{m_0-1} (\widetilde{v}_m + w_m) \right\|_{r+2, \mathcal{K}} \\ &\leq \delta + \frac{1}{M} + \|\widetilde{v}\|_{r+2, \mathcal{K}} + \left\| \sum_{m=1}^{m_0-1} (\widetilde{v}_m + w_m) \right\|_{r+2, \mathcal{K}}. \end{aligned}$$

Here we have used equations (2.13)–(2.15) to pass to the second line and equation (2.12) to pass to the third. The uniform convergence of the sum on compact sets then follows. Notice that, by uniform convergence, we immediately infer that  $Lu = 0$  on  $\mathbb{R}^{n+1}$ .

Finally, from this one can easily show that  $u$  approximates  $v$  in  $\Omega$  in the  $C^{r+2, \frac{r}{2}+1}(\Omega)$  norm:

$$\begin{aligned} \|u - v\|_{r+2, \Omega} &= \|u - \widetilde{v}\|_{r+2, \Omega} \\ &\leq \sum_{m=1}^{\infty} \|\widehat{w}_m - \widetilde{v}_m\|_{r+2, \Omega} \\ &\leq \sum_{m=1}^{\infty} \left( \|\widehat{w}_m - \widehat{v}_m\|_{r+2, \Omega} + \|\widehat{v}_m - \widetilde{v}_m\|_{r+2, \Omega} \right) \\ &< 2 \sum_{m=1}^{\infty} \frac{\delta}{2^{m+1}} = \delta. \end{aligned}$$

The theorem then follows.

*Remark 2.12.* It is classical [50] that the topological condition that  $\Omega^c(t)$  has no bounded components for all  $t \in \mathbb{R}$  (which, when  $\Omega$  is compact, is equivalent

to saying that  $\Omega^c(t)$  is connected) is necessary, as shown through the analysis of the heat equation.

## 2.5 Approximation with decay: Proof of Theorem 2.4

In this section we prove Theorem 2.4.

Without any loss of generality one can assume that  $Lv = 0$  in a domain  $\Omega'$  containing  $\Omega$  and which is contained in turn in  $B_{R_0} \times (-T, T)$ , where we recall that  $B_R$  denotes the ball in  $\mathbb{R}^n$  of radius  $R$ . Theorem 2.2 ensures that for any  $\delta' > 0$  there exists a solution  $w$  of the equation  $Lw = 0$  in the whole space  $\mathbb{R}^{n+1}$  such that

$$\|w - v\|_{r+2, \Omega'} < \delta'. \quad (2.17)$$

Let us denote by  $\{\psi_k(x)\}_{k=1}^\infty$  a basis of Dirichlet eigenfunctions of the (formally self-adjoint) elliptic operator

$$Pu := \frac{1}{\rho(x)} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( A_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u$$

on the ball  $B_{2R_0}$ , which satisfy

$$P\psi_k = -\lambda_k \psi_k \quad \text{in } B_{2R_0}, \quad \psi_k|_{\partial B_{2R_0}} = 0. \quad (2.18)$$

With the suitable normalization, these eigenfunctions, which are of class  $C^{r+2}$ , can be assumed to define an orthonormal basis of  $L^2(B_{2R_0}, \rho dx)$ . Here

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

are the eigenvalues of the operator  $P$ , and  $\lambda_1$  is strictly positive because, integrating by parts the eigenvalue equation and using the definition of an asymptotically flat operator,

$$\begin{aligned} \lambda_1 &= \lambda_1 \int_{B_{2R_0}} \psi_1^2 \rho(x) dx \\ &= \int_{B_{2R_0}} \left( \sum_{1 \leq i, j \leq n} A_{ij}(x) \partial_i \psi_1 \partial_j \psi_1 dx - c(x) \rho(x) \psi_1^2 \right) dx \\ &\geq \int_{B_{2R_0}} (C_L |\nabla \psi_1|^2 - c(x) \psi_1^2) \rho(x) dx > 0. \end{aligned} \quad (2.19)$$

For  $(x, t) \in B_{2R_0} \times \mathbb{R}$ , one can expand  $w$  as

$$w(x, t) = \sum_{k=1}^{\infty} w_k(t) \psi_k(x),$$

with

$$w_k(t) := \int_{B_{2R_0}} w(x, t) \psi_k(x) \rho(x) dx,$$

where the sum converges e.g. in  $C^0((-2T, 2T), L^2(B_{2R_0}))$ . In particular, the coefficients  $w_k(t)$  satisfy the equation

$$\frac{dw_k}{dt} + \lambda_k w_k = 0,$$

so they are given by

$$w_k(t) = W_k e^{-\lambda_k t}, \quad W_k := e^{-2\lambda_k T} \int_{B_{2R_0}} w(x, -2T) \psi_k(x) \rho(x) dx.$$

If we now set

$$w^K(x, t) := \sum_{k=1}^K w_k(t) \psi_k(x),$$

the convergence of the series implies that for any  $\delta' > 0$  one can choose a large  $K$  so that

$$\sup_{-2T < t < 2T} \|w^K(\cdot, t) - w(\cdot, t)\|_{L^2(B_{2R_0})} < \delta', \quad (2.20)$$

where  $\delta'$  is the same unspecified small positive constant as above.

Let us now study the behavior of the eigenfunctions  $\psi_k$  in a (one-sided) neighborhood of  $\partial B_{2R_0}$ . Our goal is to approximate  $\psi_k$  by global solutions to the corresponding eigenvalue equation which fall off at infinity. We start by expanding  $\psi_k$  in a basis of spherical harmonics on the unit  $(n-1)$ -dimensional sphere, which we denote as  $\{Y_m(\omega)\}_{m=1}^{\infty}$  with  $\omega \in \mathbb{S}^{n-1}$  and assume to be normalized so that they are an orthonormal basis of  $L^2(\mathbb{S}^{n-1})$ :

$$\psi_k(x) =: \sum_{m=1}^{\infty} \varphi_{km}(r) Y_m(\omega). \quad (2.21)$$

Here  $r := |x|$ ,  $\omega := \frac{x}{|x|}$  and  $\Delta_{\mathbb{S}^{n-1}} Y_m = -\mu_m Y_m$ , where

$$\mu_m =: l_m(l_m + n - 2),$$

with  $l_m$  a certain nonnegative integer, is an eigenvalue of the Laplacian on the unit  $(n-1)$ -dimensional sphere. Notice that the coefficient  $\varphi_{km}(r)$  is precisely

$$\varphi_{km}(r) = \int_{\mathbb{S}^{n-1}} \psi_k(r\omega) Y_m(\omega) d\sigma(\omega),$$

where  $d\sigma$  is the standard measure on the unit sphere, so  $\varphi_{km}(r)$  is a function of class  $C^{r+2}$ .

Since

$$P\psi_k = \Delta\psi_k + [c_0 + F(|x|)]\psi_k$$

for  $|x| > R_0$  by the definition of an essentially flat operator, it automatically follows from the fact that  $Y_m(\omega)$  are linearly independent that the functions  $\varphi_{km}(r)$  defined in (2.21) satisfy the equation

$$\varphi_{km}'' + \frac{n-1}{r}\varphi_{km}' + \left(\lambda_k + c_0 + F(r) - \frac{\mu_m}{r^2}\right)\varphi_{km} = 0 \quad (2.22)$$

for  $R_0 < r < 2R_0$ . This is a linear ordinary differential equation whose coefficients are bounded for  $r \geq R_0$ , so it is standard that  $\varphi_{km}$  is indeed global, that is, it can be extended as a function in  $C_{\text{loc}}^2((0, \infty))$  that satisfies (2.22). In addition, it decays at infinity as follows:

**Lemma 2.13.** *Let  $\varphi_{km}$  satisfy the equation (2.22). Then it falls off at infinity as*

$$|\varphi_{km}(r)| < C(1+r)^{-\frac{n-1}{2}}. \quad (2.23)$$

*Proof.* We begin by showing that

$$\Lambda_k := \lambda_k + c_0 > 0. \quad (2.24)$$

If condition (a) of the definition of an essentially flat operator holds, this is verified since  $c_0 \geq 0$  and  $\lambda_k > 0$  by (2.19). Otherwise, it suffices to multiply the eigenvalue equation by  $\psi_k$  and integrate by parts to get

$$\begin{aligned} \lambda_k &= \lambda_k \int_{B_{2R_0}} \psi_k^2 \rho \, dx \\ &= \sum_{1 \leq i, j \leq n} \int_{B_{2R_0}} A_{ij}(x) \partial_i \psi_k \partial_j \psi_k \, dx - \int_{B_{2R_0}} c(x) \psi_k^2 \rho \, dx \\ &\geq C_L \left( \inf_{\mathbb{R}^n} \rho \right) \int_{B_{2R_0}} |\nabla \psi_k|^2 \, dx - c_0 \\ &\geq C_L \left( \inf_{\mathbb{R}^n} \rho \right) \lambda_1(B_{2R_0}) \int_{B_{2R_0}} \psi_k^2 \, dx - c_0, \end{aligned} \quad (2.25)$$

where  $\lambda_1(B_{2R_0}) \geq C'/R_0^2$  is the first Dirichlet eigenvalue of the Euclidean ball of radius  $2R_0$  and we have used that  $c(x) \leq c_0$  and the uniform ellipticity of the operator.

Let us now rewrite equation (2.22) as

$$\left(r^{\frac{n-1}{2}} \varphi_{km}\right)'' + \left(\Lambda_k - \frac{\mu_m + \frac{1}{4}(n-1)(n-3)}{r^2} + F(r)\right) \left(r^{\frac{n-1}{2}} \varphi_{km}\right) = 0 \quad (2.26)$$

As  $\Lambda_k$  is positive by (2.24), the solutions of  $y'' + \Lambda_k y = 0$  (which are sines and cosines) are bounded, so the Dini–Hukuwara theorem (see e.g. [8]) ensures that the solutions of (2.26) are also bounded as

$$\left\| r^{\frac{n-1}{2}} \varphi_{km} \right\|_{L^\infty((0, \infty))} < \infty$$

provided that

$$\int_{R_0}^{\infty} \left| \frac{\mu_m + \frac{1}{4}(n-1)(n-3)}{r^2} - F(r) \right| dr < \infty.$$

Since  $F(r) < C(1+r)^{-1-\epsilon}$  by the definition of an essentially flat operator, this condition is always satisfied, so we infer the bound (2.23).  $\square$

We now have the tools to show that for each  $k$ , the function  $\psi_k$  can be approximated by a global solution to the same eigenvalue equation and such that satisfies a suitable decay condition:

**Lemma 2.14.** *Let  $\psi_k$  satisfy (2.18). Then there exists a solution  $\widehat{\psi}_k$  to the equation  $P\widehat{\psi}_k + \lambda_k\widehat{\psi}_k = 0$  in  $\mathbb{R}^n$  which approximates  $\psi_k$  in the sense that for any  $\delta_k > 0$*

$$\|\psi_k - \widehat{\psi}_k\|_{L^2(B_{R_0})} < \delta_k, \quad (2.27)$$

and such that falls off at infinity as

$$\sup_{R>0} \frac{1}{R} \int_{B_R} \widehat{\psi}_k^2 dx < \infty. \quad (2.28)$$

*Proof.* The expansion (2.21) of  $\psi_k$  converges in  $H^{r+2}(B_{2R_0})$ , so one can take a large enough  $M_k$  to ensure that

$$\|\psi_k - \widetilde{\psi}_k\|_{H^{r+2}(B_{2R_0})} < \delta'_k, \quad (2.29)$$

where  $\delta'_k$  is a positive constant to be specified later and

$$\widetilde{\psi}_k(r\omega) := \sum_{m=1}^{M_k} \varphi_{km}(r) Y_m(\omega).$$

By Lemma 2.13 the function  $\widetilde{\psi}_k$  is defined in all of  $\mathbb{R}^n$  and falls off at infinity as

$$|\widetilde{\psi}_k(x)| < C(1+|x|)^{-\frac{n-1}{2}}. \quad (2.30)$$

Let us now observe that, since

$$P\widetilde{\psi}_k + \lambda_k\widetilde{\psi}_k = 0$$

outside  $B_{R_0}$ , the function

$$f_k := P\widetilde{\psi}_k + \lambda_k\widetilde{\psi}_k$$

is supported in  $B_{R_0}$  and, by the estimate (2.29), bounded as

$$\|f_k\|_{H^r(B_{R_0})} < C\delta'_k,$$

with  $C$  a constant independent of  $\delta_k$ . The hypotheses on  $L$  imply that

$$c(x) + \lambda_k = \Lambda_k + O(|x|^{-1-\epsilon}), \quad |x| \rightarrow \infty$$

with  $\Lambda_k > 0$  by equation (2.24). Consider the function

$$g_k := (P + \lambda_k + i0)^{-1} f_k, \quad (2.31)$$

which must be understood as the limit of  $(P + \lambda_k + i\epsilon)^{-1} f_k$  when  $\epsilon \rightarrow 0^+$ . Suitable resolvent estimates then show [47, Theorem 30.2.10] that (2.31) satisfies the equation

$$Pg_k + \lambda_k g_k = f_k$$

on  $\mathbb{R}^n$  and the sharp decay condition of Agmon–Hörmander:

$$\left( \sup_{R>0} \frac{1}{R} \int_{B_R} g_k^2 dx \right)^{\frac{1}{2}} \leq C \|f_k\|_{L^2(B_{R_0})} \leq C \delta'_k. \quad (2.32)$$

If we now define

$$\widehat{\psi}_k(x) := \widetilde{\psi}_k(x) - g_k(x),$$

we get a global solution of  $P\widehat{\psi}_k + \lambda_k \widehat{\psi}_k = 0$ . By (2.29) and (2.32) and taking  $\delta'_k$  small enough, we can obtain estimate (2.27):

$$\|\psi_k - \widehat{\psi}_k\|_{L^2(B_{R_0})} < \|\psi_k - \widetilde{\psi}_k\|_{L^2(B_{R_0})} + \|\widetilde{\psi}_k - \widehat{\psi}_k\|_{L^2(B_{R_0})} < \delta'_k + CR_0^{\frac{1}{2}} \delta'_k < \delta_k. \quad (2.33)$$

In view of the bounds (2.30) and (2.32),  $\widehat{\psi}_k$  falls off at infinity as (2.28). □

Finally, for each  $k \in 1, \dots, K$  let us choose the constant  $\delta_k$  of Lemma 2.14 so that

$$\|w_k(\cdot)\|_{L^\infty((-T, T))} \delta_k = e^{\lambda_k T} W_k \delta_k < 2^{-k} \delta'.$$

We then infer from (2.17), (2.20) and (2.27) that the finite sum

$$u(x, t) := \sum_{k=1}^K w_k(t) \widehat{\psi}_k(x)$$

is defined on all  $\mathbb{R}^{n+1}$ , decays as

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x, t)|^2 dx < C e^{-\lambda_1 t},$$

satisfies the equation

$$Lu = 0$$

and approximates  $v$  on  $\Omega'$  as

$$\|u - v\|_{L^2(\Omega')} \leq \|v - w\|_{L^2(\Omega')} + \|w - w^K\|_{L^2(\Omega')} + \sum_{k=1}^K \|w_k(\psi_k - \widehat{\psi}_k)\|_{L^2(\Omega')} < C \delta'.$$



Since  $P(u - v) = 0$  in  $\Omega'$ , this  $L^2$  estimate can be readily promoted to the pointwise estimate using parabolic estimates as we did before to get

$$\|u - v\|_{r+2, \Omega} < C\delta' < \delta.$$

where  $\delta'$  has been chosen small enough. The result for a general essentially flat operator then follows.

It only remains to show that when  $L$  is the heat operator up to a nonpositive constant (that is,  $Lu = -\frac{\partial u}{\partial t} + \Delta u + c_0 u$ ,  $c_0 \leq 0$ ), the decay is indeed pointwise. For this, observe that in Lemma 2.14, by spherical symmetry, the truncated function  $\tilde{\psi}_k$  also satisfies the equation

$$P\tilde{\psi}_k + \lambda_k \tilde{\psi}_k = 0$$

on  $\mathbb{R}^n$ , so  $f_k \equiv 0$  and therefore  $g_k \equiv 0$ . This immediately gives the decay condition

$$|u(x, t)| < C(1 + |x|)^{-\frac{n-1}{2}} e^{-\lambda_1 t},$$

so the theorem is then proved.

*Remark 2.15.* It follows from the proof that the nonpositivity assumption (ii)-(a) in the definition of an essentially flat operator can be replaced by the slightly weaker and more natural condition that

$$\int_{\mathbb{R}^n} \left( \sum_{1 \leq i, j \leq n} A_{ij}(x) \partial_i \psi \partial_j \psi - \rho(x) c(x) \psi^2 \right) dx > 0$$

for all nonzero  $\psi \in C_0^1(\mathbb{R}^n)$ .

## 2.6 Better-than-uniform approximation: Proof of Theorem 2.5

Since the set  $\Omega$  is the locally finite union of  $\Omega_i$ , it is clear that there exists an exhaustion  $\emptyset =: \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \dots$  by compact sets of  $\mathbb{R}^{n+1}$  such that:

- (i) The union of the interiors of the sets  $\mathcal{K}_j$  is  $\mathbb{R}^{n+1}$ .
- (ii) For each  $j$ , the complements of the sets  $\mathcal{K}_j(t)$  and of  $(\Omega \cup \mathcal{K}_j)(t)$  are connected for all  $t \in \mathbb{R}$ .
- (iii) If the set  $\mathcal{K}_j$  meets a component  $\Omega_i$  of  $\Omega$ , then  $\Omega_i$  is contained in the interior of  $\mathcal{K}_{j+1}$ .

We can safely assume that  $\Omega \cap (\mathcal{K}_j \setminus \mathcal{K}_{j-1}) \neq \emptyset$  for all  $j \geq 1$ .

The proof relies on an induction argument that is conveniently presented in terms of a sequence of positive numbers  $\{\epsilon_k\}_{k=1}^\infty$  related to the approximation

parameters  $\{\delta_i\}_{i=1}^\infty$  and to the above exhaustion by compact sets  $\mathcal{K}_k$  through the conditions

$$\epsilon_k < \frac{1}{6} \min\{\delta_i : i \text{ for which } \Omega_i \cap \mathcal{K}_{k+1} \neq \emptyset\} \quad \text{and} \quad \sum_{j=k+1}^\infty \epsilon_j < \epsilon_k, \quad (2.34)$$

which are required to hold for all  $k \geq 1$ . For later convenience, we set  $\epsilon_0 := 0$ .

The induction hypothesis is that there are functions  $v_j : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  satisfying the equation  $Lv_j = 0$  in  $\mathbb{R}^{n+1}$  and such that, for any integer  $J \geq 1$ , the following estimates hold:

$$\left\| v - \sum_{j=1}^J v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)} < \epsilon_J, \quad (2.35a)$$

$$\left\| v - \sum_{j=1}^J v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_J \setminus \mathcal{K}_{J-1})} < \epsilon_J + 2\epsilon_{J-1}, \quad (2.35b)$$

$$\|v_J\|_{r+2, \mathcal{K}_{J-1}} < \epsilon_J + \epsilon_{J-1}. \quad (2.35c)$$

Let us start by noticing that, by Lemma 2.11, there exists a function  $v_1$  satisfying the equation  $Lv_1 = 0$  in  $\mathbb{R}^{n+1}$  and the estimate

$$\|v - v_1\|_{r+2, \Omega \cap \mathcal{K}_2} < \epsilon_1.$$

Since we chose  $\mathcal{K}_0 = \emptyset$  and  $\epsilon_0 = 0$ , it is a trivial matter that the induction hypotheses (2.35) hold for  $J = 1$ . We shall hence assume that the induction hypotheses hold for all  $1 \leq J \leq J'$  and use this assumption to prove that they also hold for  $J = J' + 1$ .

To this end, let us construct a function  $w_{J'}$  on the set  $\Omega \cup \mathcal{K}_{J'}$  by setting  $w_{J'} := 0$  in the set  $\mathcal{K}_{J'}$  and defining  $w_{J'}$  on each component  $\Omega_i$  of the set  $\Omega$  as

$$w_{J'}|_{\Omega_i} := \begin{cases} v - \sum_{j=1}^{J'} v_j & \text{if } \Omega_i \cap (\mathcal{K}_{J'+2} \setminus \overset{\circ}{\mathcal{K}}_{J'+1}) \neq \emptyset, \\ 0 & \text{if } \Omega_i \cap (\mathcal{K}_{J'+2} \setminus \overset{\circ}{\mathcal{K}}_{J'+1}) = \emptyset. \end{cases}$$

Here  $\overset{\circ}{\mathcal{K}}_j$  stands for the interior of the set  $\mathcal{K}_j$ . The definition of the exhaustion and the first induction hypothesis (2.35a) guarantee that the function  $w_{J'}$  satisfies the equation  $Lw_{J'} = 0$  in its domain and that one has the estimate

$$\|w_{J'}\|_{r+2, \mathcal{K}_{J'} \cup (\Omega \cap \mathcal{K}_{J'+1})} \leq \left\| v - \sum_{j=1}^{J'} v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} < \epsilon_{J'}. \quad (2.36)$$

A further application of Lemma 2.11 allows us to take a function  $v_{J'+1}$  which satisfies the equation  $Lv_{J'+1} = 0$  on  $\mathbb{R}^{n+1}$  and which is close to the above function  $w_{J'}$  in the sense that

$$\|w_{J'} - v_{J'+1}\|_{r+2, \mathcal{K}_{J'+2} \cap (\Omega \cup \mathcal{K}_{J'})} < \epsilon_{J'+1}. \quad (2.37)$$

Equation (2.37) and the way we have defined the function  $w_{J'}$  in the set  $\mathcal{K}_{J'+2} \setminus \mathcal{K}_{J'+1}$  ensure that the first induction hypothesis (2.35a) also holds for  $J = J' + 1$ . Moreover, from the relations (2.35a), (2.36) and (2.37) one finds that

$$\begin{aligned} \left\| v - \sum_{j=1}^{J'+1} v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} &\leq \left\| v - \sum_{j=1}^{J'} v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} \\ &\quad + \|v_{J'+1}\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} \\ &< \epsilon_{J'} + \|w_{J'} - v_{J'+1}\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} + \|w_{J'}\|_{r+2, \Omega \cap (\mathcal{K}_{J'+1} \setminus \mathcal{K}_{J'})} \\ &< \epsilon_{J'+1} + 2\epsilon_{J'}. \end{aligned}$$

This proves the second induction hypothesis (2.35b) for  $J = J' + 1$ . Furthermore,

$$\|v_{J'+1}\|_{r+2, \mathcal{K}_{J'}} \leq \|w_{J'} - v_{J'+1}\|_{r+2, \mathcal{K}_{J'}} + \|w_{J'}\|_{r+2, \mathcal{K}_{J'}} < \epsilon_{J'+1} + \epsilon_{J'}$$

by the relations (2.36) and (2.37), so the third induction hypothesis (2.35c) also holds for  $J = J' + 1$ . This completes the induction argument.

The desired global solution  $u$  can now be defined as

$$u := \sum_{j=1}^{\infty} v_j,$$

with this sum converging uniformly in  $C^{r+2, \frac{r}{2}+1}$  by a standard argument thanks to the definition of the constants  $\epsilon_j$  (see conditions (2.34)) and the third induction hypothesis (2.35c). As the functions  $v_j$  verify the equation, it is easily checked that the function  $u$  also satisfies the equation  $Lu = 0$  in  $\mathbb{R}^{n+1}$ . In addition to this, from the definition of the constants (2.34) and the induction hypotheses (2.35) it follows that, for any integer  $J$ , in the set  $\Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)$  we have the estimate

$$\begin{aligned} \|u - v\|_{r+2, \Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)} &\leq \left\| v - \sum_{j=1}^{J+1} v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)} \\ &\quad + \|v_{J+2}\|_{r+2, \Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)} + \left\| \sum_{j=J+3}^{\infty} v_j \right\|_{r+2, \Omega \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J)} \\ &< (\epsilon_{J+1} + 2\epsilon_J) + (\epsilon_{J+2} + \epsilon_{J+1}) + \sum_{j=J+3}^{\infty} (\epsilon_j + \epsilon_{j-1}) \\ &< 6\epsilon_J \\ &< \min\{\delta_i : \Omega_i \cap (\mathcal{K}_{J+1} \setminus \mathcal{K}_J) \neq \emptyset\}. \end{aligned}$$

The better-than-uniform approximation lemma then follows.

## Chapter 3

# Movement of local hot spots and topology of isothermic hypersurfaces

Our goal in this chapter is to present some applications of the approximation theorems for parabolic operators developed in Chapter 2 to the study of the movement of local hot spots and isothermic hypersurfaces. Let us recall that a point  $X \in \mathbb{R}^n$  is a *(local) hot spot* at time  $t$  of a solution  $u(x, t)$  to the parabolic equation  $Lu = 0$  if it is a (local) maximum of  $u(\cdot, t)$ . An *isothermic hypersurface* at time  $t$  is a connected component of a level set of  $u(\cdot, t)$ .

These topics have attracted considerable attention for decades, particularly concerning the large-time behavior of hot spots [17, 48, 49, 10], and the existence of stationary hot spots [62, 63] and isothermic hypersurfaces and the Matzoh ball soup problem [59, 61, 60, 71]. A related problem, which concerns the behavior of the second Neumann eigenfunction of a bounded domain, is the celebrated hot spots conjecture of Rauch, the first counterexample to which was found in [13].

Even for the heat equation in the whole Euclidean space  $\mathbb{R}^n$ , despite the large literature on the subject, not much is known about how local hot spots (or any other critical points of  $u(\cdot, t)$ ) can move other than the fact that [17] if the initial condition is nonnegative and compactly supported, the global hot spots must converge to a point (specifically, the center of mass of the initial datum). If the initial condition is convex, the convexity is preserved in the evolution, so there is a unique local (and global) hot spot at each time. Generalizations of this fact can be found in [57] and references therein. Our results show that the movement of local hot spots and the topology of isothermic hypersurfaces of solutions to general parabolic equations on  $\mathbb{R}^{n+1}$  can be remarkably rich.

In the case of compact Riemannian manifolds, it is well known that the large-time behavior of solutions of the heat equation  $-\frac{\partial u}{\partial t} + \Delta u = 0$  is much more rigid than in the whole Euclidean space. In particular, for large times one has that

$$u(x, t) = c_0 + c_1 e^{-\lambda_1 t} \psi_1(x) + O(e^{-\lambda_2 t}),$$

where  $0 < \lambda_1 \leq \lambda_2 \leq \dots$  are the eigenvalues of the Laplacian on the manifold  $M$ ,  $\psi_j(x)$  are the corresponding eigenfunctions and

$$c_0 := \frac{1}{|M|} \int_M u(x, 0) dx, \quad c_1 := \frac{\int_M u(x, 0) \psi_1(x) dx}{\int_M \psi_1(x)^2 dx}$$

are constants. In particular, it is apparent from this formula that all the local hot spots of  $u(x, t)$  converge to those of the first nontrivial eigenfunction  $\psi_1(x)$  as  $t \rightarrow \infty$  for generic metrics on  $M$  and generic initial data, and this observation can be easily refined. We show that the movement of local hot spots and topology of isothermic hypersurfaces found in solutions of the heat equation in the flat torus can be just as complex as those on  $\mathbb{R}^{n+1}$  at small scales.

### 3.1 Main results

The next two theorems show that the class of movements of local hot spots and isothermic hypersurfaces of solutions to a general parabolic equation on  $\mathbb{R}^{n+1}$  is very wide. Let us recall that we are considering uniformly parabolic operators

$$Lu := -\frac{\partial u}{\partial t} + \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u \quad (3.1)$$

whose coefficients, together with those of its formal adjoint, are in a suitable parabolic Hölder space  $C^{r, \frac{r}{2}}(\mathbb{R}^{n+1})$ , with  $r > 2$  a non-integer real number (Section 2.2). The existence of solutions with decay are shown in the case of essentially flat parabolic operators (Definition 2.3).

More precisely, the first result asserts that there are global solutions having local hot spots that move along any prescribed spatial curve for all times, up to a small error:

**Theorem 3.1.** *Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous curve in space (possibly periodic or with self-intersections) and take any continuous positive function on the line  $\delta(t)$ . If the coefficient  $c(x, t)$  is nonpositive, then there is a solution to the parabolic equation  $Lu = 0$  on  $\mathbb{R}^{n+1}$  such that, at each time  $t \in \mathbb{R}$ ,  $u$  has a local hot spot  $X_t$  with*

$$|X_t - \gamma(t)| < \delta(t).$$

*Furthermore, if  $L$  is also essentially flat and  $[T_1, T_2]$  is any finite interval, there is a solution to the parabolic equation  $L\tilde{u} = 0$  on  $\mathbb{R}^{n+1}$  satisfying the decay conditions (2.3) and such that, at each time  $t \in [T_1, T_2]$ ,  $\tilde{u}$  has a local hot spot  $\tilde{X}_t$  with*

$$|\tilde{X}_t - \gamma(t)| < \delta(t).$$

Notice that we are not claiming that, given times  $t < t'$ , the local hot spot  $X_{t'}$  is in any way related to the evolution of the local hot spot  $X_t$ . In particular, even

if we choose the curve  $\gamma$  to be constant (say  $\gamma(t) = x_0$  for all  $t$ ), the local hot spot we construct does not need to be “almost stationary”.

Since there are continuous curves  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  whose image is dense in  $\mathbb{R}^n$  and one can take an error function  $\delta(t)$  that tends to zero as  $|t| \rightarrow \infty$ , an immediate corollary is the following:

**Corollary 3.2.** *If the coefficient  $c(x, t)$  is nonpositive, there is a solution to the parabolic equation  $Lu = 0$  on  $\mathbb{R}^{n+1}$  having a local hot spot  $X_t$  at each time  $t$  such that  $\{X_t : t \in \mathbb{R}\}$  is dense in  $\mathbb{R}^n$ , that is,*

$$\inf_{t \in \mathbb{R}} |X_t - x| = 0$$

for all  $x \in \mathbb{R}^n$ .

The next result, which complements the previous theorem, shows that there are global solutions that exhibit an isothermic hypersurface of arbitrarily complicated compact topology (which can change in time). In order to state this result, let us denote by

$$\Gamma_{u,\alpha}(t) := \{x \in \mathbb{R}^n : u(x, t) = \alpha\}$$

the level set  $u = \alpha$  at time  $t$ . All the hypersurfaces that we consider in this chapter are of class  $C^2$ .

**Theorem 3.3.** *Take a possibly infinite subset  $\mathcal{I} \subset \mathbb{Z}$  and, for each index  $i \in \mathcal{I}$ , a compact orientable hypersurface without boundary  $\Sigma_i \subset \mathbb{R}^n$ . If  $i$  and  $i+1$  belong to  $\mathcal{I}$ , we also assume that the domains bounded by  $\Sigma_i$  and  $\Sigma_{i+1}$  are disjoint. If the coefficient  $c(x, t)$  is nonpositive, then there exist a solution of the equation  $Lu = 0$  on  $\mathbb{R}^{n+1}$ , constants  $\alpha_i > 0$  close to zero, and diffeomorphisms  $\Phi_t$  of  $\mathbb{R}^n$  arbitrarily close to the identity in the  $C^{r+2}$  norm such that  $\Phi_t(\Sigma_i)$  is a connected component of the isothermic hypersurface  $\Gamma_{u,\alpha_i}(t)$  whenever  $\lfloor t \rfloor = i$ . Furthermore:*

- (i) *If the coefficients of the operator  $L$  depend analytically on  $x$ , one can take  $\alpha_i = 0$  for all  $i \in \mathcal{I}$ .*
- (ii) *If the set  $\mathcal{I}$  is finite, one can take some  $\alpha > 0$  such that  $\alpha_i = \alpha$  for all  $i \in \mathcal{I}$ .*
- (iii) *If  $L$  is essentially flat (or the heat equation up to a nonpositive constant) and the set  $\mathcal{I}$  is finite, then the function  $u$  satisfies the decay conditions (2.3).*

*Remark 3.4.* It is worth mentioning that the behavior described in Theorems 3.1 and 3.3 is structurally stable, meaning that a suitably small perturbation of the function  $u(x, t)$  (e.g., small in  $C^1(\mathbb{R}^{n+1})$ ) still presents the same prescribed collection of hot spots or isothermic hypersurfaces up to a diffeomorphism close to the identity.

Our last objective in this chapter is to show that, although global approximation theorems do not apply in compact manifolds, previous results have some bearing on the possible behavior of solutions to the heat equation on the flat torus  $\mathbb{T}^n$ , with  $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ . Specifically, we will next state two theorems, related to Theorems 3.1 and 3.3 above, about solutions to the heat equation when the space variable takes values on the torus. The first of them shows that, for suitably short times, there is no obstruction to the (possibly knotted or self-intersecting) trajectories that local hot spots can follow on suitably small scales:

**Theorem 3.5.** *Let  $\gamma : [T_1, T_2] \rightarrow B_1$  be a (possibly self-intersecting) continuous curve contained in the unit  $n$ -dimensional ball. Given  $\delta > 0$ , there exists an arbitrarily small  $\epsilon > 0$  and a solution of the heat equation  $-\frac{\partial u}{\partial t} + \Delta u = 0$  on  $\mathbb{T}^n \times \mathbb{R}$  having a local hot spot  $X_t$  contained in the ball of radius  $\sqrt{\epsilon}$  for all times in  $[\epsilon T_1, \epsilon T_2]$  with the property that*

$$\left| \frac{1}{\sqrt{\epsilon}} X_{\epsilon\tau} - \gamma(\tau) \right| < \delta$$

for all  $\tau \in [T_1, T_2]$ .

Intuitively speaking, the reason for which one considers times of order  $\epsilon$  and balls of radius  $O(\sqrt{\epsilon})$  in space is that it is at this scale that the behavior of solutions to the heat equation on the torus can be shown to resemble those of the heat equation on the whole Euclidean space, and viceversa.

Our second theorem in this direction is an analog of Theorem 3.3 at these spacetime scales that shows that there are global solutions of the heat equation on the torus that feature isothermic hypersurfaces of prescribed, possibly rapidly changing topologies:

**Theorem 3.6.** *For each index  $0 \leq i \leq N$ , take a compact orientable hypersurface without boundary  $\Sigma_i$  contained in the unit ball and assume that the domains bounded by  $\Sigma_i$  and  $\Sigma_{i+1}$  are disjoint. Given  $\delta > 0$  and an integer  $k$ , there exists an arbitrarily small  $\epsilon > 0$  and a solution of the heat equation  $-\frac{\partial u}{\partial t} + \Delta u = 0$  on  $\mathbb{T}^n \times \mathbb{R}$  with the following property: for each time  $t \in [0, \epsilon N]$  there is a connected component  $S_t$  of the isothermic hypersurface  $\Gamma_{u,0}(t)$  contained in the ball of radius  $\sqrt{\epsilon}$  and such that*

$$\frac{1}{\sqrt{\epsilon}} S_{\epsilon\tau} = \Phi_\tau(\Sigma_i)$$

for all  $\tau \in [0, N]$  with  $\lfloor \tau \rfloor = i$ , where  $\Phi_\tau$  is a diffeomorphism of the unit ball with  $\|\Phi_\tau - \text{id}\|_{C^k(B_1)} < \delta$ .

The rest of the chapter is organized as follows. The results for local hot spots and isothermic hypersurfaces in  $\mathbb{R}^{n+1}$  (Theorems 3.1 and 3.3) are established in Sections 3.2 and 3.3, respectively. The results for the heat equation on the torus (Theorems 3.5 and 3.6) hinge on the previous ones and a lemma which is stated and shown in Section 3.4.

## 3.2 Movement of local hot spots: Proof of Theorem 3.1

Without loss of generality we can assume that the curve  $\gamma(t)$  is smooth, since otherwise in the argument below one can replace  $\gamma(t)$  by any function  $\tilde{\gamma}(t)$  in  $C^\infty(\mathbb{R})$  such that

$$|\gamma(t) - \tilde{\gamma}(t)| < \frac{\delta(t)}{4}.$$

For each integer  $k$ , let us consider the bounded spacetime domain

$$\Omega_k := \left\{ (x, t) \in \mathbb{R}^{n+1} : k - \frac{1}{5} < t < k + \frac{6}{5}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| < \frac{\delta_k}{4} \right\},$$

where  $e \in \mathbb{R}^n$  is a fixed unit vector and  $\delta_k$  is a small constant that we will define in terms of the function  $\delta(t)$  later on. Notice that

$$|x - \gamma(t)| < \delta_k$$

for all  $(x, t) \in \Omega_k$ .

We add an alternating displacement of the spatial centers of the domains  $\Omega_k$  through  $(-1)^k$  so that their closures are pairwise disjoint and we can finally apply the better-than-uniform approximation Theorem 2.5.

### Step 1: Construction of local solutions in $\Omega_k$

Let  $\phi$  be a nonnegative function on  $\mathbb{R}^n$ , not identically zero, that is supported in the ball of radius  $\frac{1}{4}$ . In each domain  $\Omega_k$  we shall construct a function  $v_k$  such that

$$Lv_k = 0 \quad \text{in } \Omega_k,$$

$v_k(x, t) = 0$  if  $x \in \partial_L \Omega_k$  and

$$v\left(x, k - \frac{1}{5}\right) = \phi\left(\frac{x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e}{\delta_k}\right).$$

Here

$$\partial_L \Omega_k := \left\{ (x, t) \in \mathbb{R}^{n+1} : k - \frac{1}{5} < t < k + \frac{6}{5}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| = \frac{\delta_k}{4} \right\}$$

is the lateral boundary of  $\Omega_k$ .

To this end, let us introduce the new coordinates

$$\xi := x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e, \quad \tau := t.$$

This change of coordinates maps the domain  $\Omega_k$  onto  $B_{\frac{\delta_k}{4}} \times (k - \frac{1}{5}, k + \frac{6}{5})$ . With some abuse of notation, we will still denote by  $v_k(\xi, \tau)$  the expression of the



function  $v_k(x, t)$  in the new coordinates. The equation for  $v_k$  then reads

$$-\frac{\partial v_k}{\partial \tau} + \sum_{i,j=1}^n a_{ij} \left( \xi + \gamma(\tau) - (-1)^k \frac{\delta_k}{2} e, \tau \right) \frac{\partial^2 v_k}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^n \left[ \dot{\gamma}_i(\tau) + b_i \left( \xi + \gamma(\tau) - (-1)^k \frac{\delta_k}{2} e, \tau \right) \right] \frac{\partial v_k}{\partial \xi_i} + c \left( \xi + \gamma(\tau) - (-1)^k \frac{\delta_k}{2} e, \tau \right) v_k = 0,$$

in the domain  $|\xi| < \frac{\delta_k}{4}$ ,  $k - \frac{1}{5} < \tau < k + \frac{6}{5}$ , and we have the initial and boundary conditions

$$v_k(\xi, \tau) = 0 \quad \text{if } |\xi| = \delta_k/4, \quad v_k \left( \xi, k - \frac{1}{5} \right) = \phi(\xi/\delta_k).$$

It is standard that there is a unique solution to this initial-boundary value problem. As  $c \leq 0$  by assumption, the maximum principle then implies that

$$v_k(x, t) > 0$$

for all  $(x, t) \in \Omega_k$ . In particular, for each  $t \in (k - \frac{1}{5}, k + \frac{6}{5})$  one can take a point  $\widehat{X}_t^k$  (a global maximum at frozen time  $t$ ) satisfying

$$v_k(\widehat{X}_t^k, t) = \sup_{|x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e| < \frac{\delta_k}{4}} v_k(x, t) > 0.$$

Notice that this point is not necessarily unique, so if there is more than one global maximum we will choose  $\widehat{X}_t^k$  so that the distance of this maximum to the set

$$S_t^k := \left\{ x \in \mathbb{R}^n : \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| = \frac{\delta_k}{4} \right\}$$

is as small as possible. If there are more than one maxima that minimize this distance, any of them will do.

## Step 2: Construction of the global solution

Let us start defining some quantities to prove the existence of a global solution with suitably located maxima.

Since the points  $\widehat{X}_t^k$  are in the interior of the spacetime domain  $\Omega_k$ , the quantity

$$\epsilon_k := \inf \left\{ |\widehat{X}_t^k - x| : k - \frac{1}{5} < t < k + \frac{6}{5}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| = \frac{\delta_k}{4} \right\}$$

is strictly positive (although possibly not uniformly), and not larger than  $\delta_k/4$ . We will set

$$\eta_k := \inf \left\{ v_k(\widehat{X}_t^k, t) - v_k(x, t) : k - \frac{1}{5} < t < k + \frac{6}{5}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| > \frac{\delta_k}{4} - \frac{\epsilon_k}{2} \right\}.$$

Notice that this quantity is positive because we have chosen the maximum  $\widehat{X}_t^k$  so as to minimize the distance to the set  $S_t^k$ .

Let us now define the closed sets

$$\widetilde{\Omega}_k := \left\{ (x, t) : k - \frac{1}{8} \leq t \leq k + \frac{9}{8}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| \leq \frac{\delta_k}{4} - \frac{\epsilon_k}{2} \right\}$$

and

$$\Omega := \bigcup_{k=-\infty}^{\infty} \widetilde{\Omega}_k. \quad (3.2)$$

Notice that, by the definition of  $\epsilon_k$  and  $\eta_k$ ,

$$\inf \left\{ v(\widehat{X}_t^k, t) - v(x, t) : k - \frac{1}{8} \leq t \leq k + \frac{9}{8}, \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| = \frac{\delta_k}{4} - \frac{\epsilon_k}{2} \right\} \geq \eta_k.$$

The function  $v$  defined by setting

$$v(x, t) := v_k(x, t) \quad \text{if } (x, t) \in \widetilde{\Omega}_k \text{ for some } k$$

then satisfies the equation  $Lv = 0$  in an open neighborhood of the closed set  $\Omega$ . Since  $\Omega$  is the locally finite union of compact sets such that the complement of  $\Omega(t)$  is connected for all  $t \in \mathbb{R}$ , Theorem 2.5 ensures that there is a function  $u$  satisfying the equation

$$Lu = 0$$

in  $\mathbb{R}^{n+1}$  and such that

$$\|u - v\|_{r+2, \widetilde{\Omega}_k} < \frac{\eta_k}{3}$$

for all integers  $k$ .

It then follows that for all  $k - \frac{1}{8} < t < k + \frac{9}{8}$  and all  $x$  with

$$\left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| = \frac{\delta_k}{4} - \frac{\epsilon_k}{2}$$

we have

$$\begin{aligned} u(\widehat{X}_t^k, t) - u(x, t) &> \left[ v_k(\widehat{X}_t^k, t) - \frac{\eta_k}{3} \right] - \left[ v_k(x, t) + \frac{\eta_k}{3} \right] \\ &= \left[ v_k(\widehat{X}_t^k, t) - v_k(x, t) \right] - \frac{2\eta_k}{3} \\ &\geq \frac{\eta_k}{3} > 0, \end{aligned}$$

where we have used the definitions of the various sets and quantities. It then follows that, for each  $t \in (k - \frac{1}{8}, k + \frac{9}{8})$ , the function  $u(\cdot, t)$  must have a local maximum  $\widehat{X}_t^k$  in the region

$$\left\{ x \in \mathbb{R}^n : \left| x - \gamma(t) + (-1)^k \frac{\delta_k}{2} e \right| < \frac{\delta_k}{4} - \frac{\epsilon_k}{2} \right\}.$$

The first assertion of the theorem then follows if one now chooses the constants  $\delta_k$  so that

$$\delta_k < \inf_{|t| < |k|+2} \delta(t),$$

where  $\delta(t)$  is the function appearing in the statement of the theorem. Notice that this local maximum is not necessarily unique and that, in general,  $X_t^k \neq \widehat{X}_t^k$ .

## Decaying global solution for essentially flat operators

Let us now pass to second part of the theorem, which asserts that if  $L$  is essentially flat and one only controls the existence of a local hot spot in a finite time interval  $[T_1, T_2]$ , then the solution falls off in space and time. The result follows from the above construction, but instead of defining  $\Omega$  as in (3.2) one sets

$$\Omega := \bigcup_{k=K_1}^{K_2} \widetilde{\Omega}_k,$$

where  $K_1$  (respectively,  $K_2$ ) is any integer smaller than  $T_1$  (respectively, larger than  $T_2$ ). The construction is then exactly as above but instead of using Theorem 2.5 to obtain a better-than-uniform approximation of the function  $v$ , one directly uses Theorem 2.4 to approximate  $v$  on the set  $\Omega$  (which is now compact) by a solution of the equation  $L\tilde{u} = 0$  on  $\mathbb{R}^{n+1}$ , with an error

$$\|\tilde{u} - v\|_{r+2, \Omega} < \min_{K_1 \leq k \leq K_2} \frac{\eta_k}{3},$$

and the decay condition

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |\tilde{u}(x, t)|^2 dx \leq C e^{-t/C}.$$

As before, in the case of the heat equation one can replace this by the pointwise decay estimate

$$|\tilde{u}(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}.$$

The theorem then follows.

## Movement of multiple local hot spots

A minor modification of the proof permits to control more than one hot spot (even countably many) simultaneously. One only needs to consider the analog of the domain  $\Omega$  defined for each curve, and it is not hard to see that intersections or even accumulation points of the set of curves do not introduce any serious complications. With a little work, one then finds the following generalization of Theorem 3.1:

**Theorem 3.7.** *Let  $\gamma_i : \mathbb{R} \rightarrow \mathbb{R}^n$  be a family of (possibly intersecting) parametrized curves in space labeled by the integers  $i$  in a possibly infinite subset  $\mathcal{I} \subseteq \mathbb{Z}$ . Take*

any continuous positive functions on the line  $\delta_i(t)$ . If the coefficient  $c(x, t)$  is nonpositive, then there is a solution to the parabolic equation  $Lu = 0$  on  $\mathbb{R}^{n+1}$  such that, for each time  $t \in \mathbb{R}$  and  $i \in \mathcal{I}$ ,  $u$  has a local hot spot  $X_t^i$  with

$$|X_t^i - \gamma_i(t)| < \delta_i(t).$$

Furthermore, if  $L$  is also essentially flat (or the heat equation up to a nonpositive constant),  $\mathcal{I}$  is finite and  $[T_1, T_2]$  is any finite interval, there is a solution to the parabolic equation  $L\tilde{u} = 0$  on  $\mathbb{R}^{n+1}$  satisfying the decay conditions (2.3) and such that, for each time  $t \in [T_1, T_2]$  and  $i \in \mathcal{I}$ ,  $\tilde{u}$  has a local hot spot  $\tilde{X}_t^i$  with

$$|\tilde{X}_t^i - \gamma_i(t)| < \delta_i(t).$$

### 3.3 Isothermic hypersurfaces: Proof of Theorem 3.3

Let  $D_i$  denote the bounded domain of  $\mathbb{R}^n$  whose boundary is  $\Sigma_i$ .

#### Step 1: Construction of local solutions

We start by constructing, for each integer  $i \in \mathcal{I}$ , a function  $v_i(x, t)$  on  $D_i \times (i - \frac{1}{5}, i + \frac{6}{5})$  that satisfies the equation

$$Lv_i = 0$$

with initial and boundary conditions

$$v_i\left(i - \frac{1}{5}, x\right) = \phi_i(x), \quad v_i(t, \cdot)|_{\Sigma_i} = 0.$$

Here  $\phi_i$  is any smooth, nonnegative function supported on  $D_i$  that is not identically zero. Since the coefficient  $c(x, t)$  is nonpositive, the maximum principle ensures that

$$v_i(x, t) > 0$$

for all  $(x, t) \in D_i \times (i - \frac{1}{5}, i + \frac{6}{5})$ , and Hopf's boundary point lemma shows that the normal derivative of  $v_i$  is negative:

$$\frac{\partial v_i}{\partial \nu} < 0.$$

Let us now fix some positive constants  $\epsilon_i$ , with  $i \in \mathcal{I}$ . Fixing  $t$  and regarding it simply as a parameter, Thom's isotopy theorem (see e.g. [1, Theorem 20.2] or [29, Theorem 3.1 and Remark 3.2]) then ensures that there is some  $\alpha_i > 0$  such that, for any  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$ , given a positive constant  $\alpha \leq \alpha_i$  there is a diffeomorphism  $\Phi_t^i$  of  $\mathbb{R}^n$  with

$$\|\Phi_t^i - \text{id}\|_{C^{r+2}} < \epsilon_i$$

and such that

$$\Gamma_{i,\alpha}(t) = \Phi_t^i(\Sigma_i), \quad (3.3)$$

where

$$\Gamma_{i,\alpha}(t) = \{x \in D_i : v_i(x, t) = \alpha\}.$$

It is worth mentioning that Thom's isotopy theorem does not grant exactly (3.3), but only the existence of a connected component of  $\Gamma_{i,\alpha}(t)$  of the form  $\Phi_t^i(\Sigma_i)$ . However, the maximum principle ensures that, for  $t \geq i - \frac{1}{8}$ , if  $\alpha$  is small enough (say smaller than  $\alpha_i$ , without any loss of generality), the set  $\Gamma_{i,\alpha}(t)$  is connected. Indeed,  $v_i(\cdot, t)$  is strictly positive on  $D_i$  for  $t > i - \frac{1}{5}$ , so for very small  $\alpha$  the level set  $\Gamma_{i,\alpha}(t)$  must be contained in a small (uniformly in  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$ ) neighborhood of  $\Sigma_i$ . But Thom's isotopy theorem (which is basically an application of the implicit function theorem) guarantees that the only connected component in a small neighborhood of  $\Sigma_i$  is  $\Phi_t^i(\Sigma_i)$ , which implies (3.3).

Observe that, for small enough  $\alpha_i$  and  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$ ,  $\Gamma_{i,\alpha}(t)$  is the boundary of the set

$$D_{i,\alpha,t} := \{x \in D_i : v_i(x, t) > \alpha\}.$$

Moreover, if  $\alpha_i$  is small enough,  $\nabla v_i$  does not vanish on  $\Gamma_{i,\alpha}(t)$ , so, again by Thom's isotopy theorem, there is some  $\eta_i > 0$  such that, if  $w$  is a function with

$$\max_{t \in [i - \frac{1}{8}, i + \frac{9}{8}]} \|v_i(t, \cdot) - w(t, \cdot)\|_{C^{r+2}(D_{i, \frac{\alpha_i}{2}, t})} < \eta_i,$$

then we have

$$\{x \in D_{i, \frac{\alpha_i}{2}, t} : w(x, t) = \alpha\} = \tilde{\Phi}_t^i(\Sigma_i) \quad (3.4)$$

for all  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$ , where  $\tilde{\Phi}_t^i$  is a diffeomorphism of  $\mathbb{R}^n$  with

$$\|\tilde{\Phi}_t^i - \text{id}\|_{C^{r+2}} < 2\epsilon_i.$$

Notice that, if  $i$  and  $i + 1$  are in  $\mathcal{I}$ , as the intersection  $D_i \cap D_{i+1}$  is empty, so is  $D_{i,\alpha,t} \cap D_{i+1,\alpha',t}$ .

## Step 2: Construction of the global solution

We are now ready to complete the proof of Theorem 3.3. Define the closed set

$$\Omega := \bigcup_{i \in \mathcal{I}} \Omega_i,$$

with

$$\Omega_i := \left\{ (x, t) \in \mathbb{R}^n \times \left[ i - \frac{1}{8}, i + \frac{9}{8} \right] : x \in \overline{D_{i, \frac{\alpha_i}{2}, t}} \right\}$$

and observe that the previous discussion ensures that the closed sets  $\Omega_i$  are pairwise disjoint. If we define a function  $v$  on  $\Omega$  by setting

$$v(x, t) := v_i(x, t)$$

if  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$  and  $x \in \overline{D_{i, \frac{\alpha_i}{2}, t}}$  for some  $i \in \mathcal{I}$ , it is then clear that  $Lv = 0$  in  $\Omega$ .

As  $\Omega$  is a locally finite union of pairwise disjoint sets and the complement of  $\Omega(t)$  in  $\mathbb{R}^n$  is connected for all  $t$ , Theorem 2.5 then ensures that there is a function satisfying the equation

$$Lu = 0$$

in  $\mathbb{R}^{n+1}$  such that

$$\|u - v\|_{r+2, \Omega_i} < \eta_i.$$

Equation (3.4) then implies that, if  $t \in [i - \frac{1}{8}, i + \frac{9}{8}]$  for some  $i \in \mathcal{I}$ , we have that there is a connected component of the set

$$\{x \in \mathbb{R}^n : u(x, t) = \alpha_i\} \quad (3.5)$$

given by

$$\Phi_t^i(\Sigma_i),$$

where  $\Phi_t^i$  is a diffeomorphism of  $\mathbb{R}^n$  with  $\|\Phi_t^i - \text{id}\|_{C^{r+2}} < 2\epsilon_i$ . This is the only connected component of the set (3.5) that intersects with (and is in fact contained in)  $D_{i, \frac{\alpha_i}{2}, t}$ . This proves the first part of the theorem.

## Special cases

Let us now prove the three final assertions of the theorem. Firstly, notice that if  $\mathcal{I}$  is finite, one can obviously take the same  $\alpha$  (e.g.,  $\alpha := \min_{i \in \mathcal{I}} \alpha_i$ ) for all  $i$ . Concerning the decay, notice that if  $\mathcal{I}$  is finite, the set  $\Omega$  is compact, so when  $L$  is essentially flat one can use Theorem 2.4 to show that there is a function, which we still call  $u$ , satisfying

$$Lu = 0$$

in  $\mathbb{R}^{n+1}$ , the decay condition

$$\sup_{R>0} \frac{1}{R} \int_{B_R} |u(x, t)|^2 dx \leq C e^{-t/C}, \quad (3.6)$$

and such that

$$\|u - v\|_{r+2, \Omega} < \min_{i \in \mathcal{I}} \eta_i.$$

Moreover, if  $Lu = -\frac{\partial u}{\partial t} + \Delta u + c_0 u$ , (3.6) can be replaced by the pointwise decay condition

$$|u(x, t)| \leq C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}.$$

Finally, suppose that the coefficients of  $L$  are analytic in  $x$ . By the density of analytic functions in the space of  $C^2$  functions, without loss of generality one can henceforth assume that the hypersurfaces  $\Sigma_i$  are analytic too. It is then standard that  $v_i(x, t)$  is analytic in  $x$  up to the boundary, so in particular there is a slightly larger domain  $D'_i \supset \overline{D_i}$  such that the equation  $Lv_i = 0$  holds in

$D'_i \times (i - \frac{1}{5}, i + \frac{6}{5})$ . Thom's isotopy theorem then ensures that one can take  $\alpha_i = 0$ , no matter whether the set  $\mathcal{I}$  is finite or not.

### 3.4 The heat equation on the flat torus

In this section we shall denote by

$$L_0 u := -\frac{\partial u}{\partial t} + \Delta u$$

the heat operator. In view of Theorems 3.1 and 3.3 about local hot spots and isothermic hypersurfaces of solutions to (in particular) the equation  $L_0 v = 0$  on  $\mathbb{R}^{n+1}$  and Remark 3.4, which ensures the structure stability of these objects, the results presented in Theorems 3.5 and 3.6 about local hot spots and isothermic hypersurfaces of solutions to the equation  $L_0 u = 0$  on  $\mathbb{T}^n \times \mathbb{R}$  will stem from the following lemma. Informally speaking, this lemma ensures that, given a solution of  $L_0 v = 0$  on a certain compact spacetime region  $\Omega \subset \mathbb{R}^{n+1}$ , there is a solution of  $L_0 u = 0$  on  $\mathbb{T}^n \times \mathbb{R}$  that approximates  $v$  on  $\Omega$  after a suitable parabolic rescaling of variables in  $\mathbb{T}^n \times \mathbb{R}$ . In this section, as the coefficients of the heat equation are constant, for simplicity we will use regular Hölder norms instead of the parabolic ones.

**Lemma 3.8.** *Let  $\Omega$  be a compact subset of  $\mathbb{R}^{n+1}$  such  $\Omega^c(t)$  is connected for all  $t \in \mathbb{R}$ . Given a function  $v$  that satisfies the equation  $L_0 v = 0$  in  $\Omega$ , a positive real  $k$  and  $\delta > 0$ , there exists an arbitrarily small  $\epsilon > 0$  and a solution  $u(x, t)$  of the equation  $L_0 u = 0$  in  $\mathbb{T}^n \times \mathbb{R}$  that approximates  $v$  modulo a rescaling as*

$$\|u(\sqrt{\epsilon} \cdot, \epsilon \cdot) - v\|_{C^k(\Omega)} < \delta.$$

*Proof.* Theorem 2.4 ensures that there is a solution of the equation  $L_0 w = 0$  in  $\mathbb{R}^{n+1}$  which approximates  $v$  as

$$\|v - w\|_{C^k(\Omega)} < \delta' \tag{3.7}$$

for any  $\delta' > 0$  and decays in space and time as

$$|w(x, t)| < C(1 + |x|)^{-\frac{n-1}{2}} e^{-t/C}.$$

Let us consider some time  $T$  such that the compact region  $\Omega$  is contained in the half-space  $t > T$  and set

$$f(x) := w(x, T).$$

The Fourier transform of  $f$ , which we denote by  $\widehat{f}(\xi)$ , is well defined as a complex-valued tempered distribution on  $\mathbb{R}^n$ . In terms of  $\widehat{f}$ , for all times  $t > T$  one can write  $w(x, t)$  as

$$w(x, t) = \int_{\mathbb{R}^n} e^{ix \cdot \xi - (t-T)|\xi|^2} \widehat{f}(\xi) d\xi,$$

where the integral is to be understood in the sense of distributions.

Let  $\Omega'$  be a bounded neighborhood of  $\Omega$  in  $\mathbb{R}^{n+1}$  that is contained in the half-space  $t > T$ . Since smooth compactly supported functions are dense in the space of tempered distributions and  $w$  is a smooth function on  $\Omega'$ , it is standard that for any  $\delta' > 0$  there is a complex-valued function  $F \in C_c^\infty(\mathbb{R}^n)$  with

$$\|w - w_1\|_{L^\infty(\Omega')} < \delta', \quad (3.8)$$

where

$$w_1(x, t) := \int_{\mathbb{R}^n} e^{ix \cdot \xi - (t-T)|\xi|^2} F(\xi) d\xi.$$

Let us denote by  $S$  the support of the function  $F$ . Since  $w$  is a real-valued function, we can safely assume that  $F(-\xi) = \overline{F(\xi)}$ , so that  $w_1$  is also real-valued.

It is well known that any finite regular complex-valued Borel measure supported in  $S$ , such as

$$F(\xi) d\xi,$$

can be approximated in the weak topology by a purely discrete measure, that is, a finite linear combination of the form

$$\sum_{j=-J}^J a_j \delta(\xi - \xi_j)$$

with  $\delta(\xi - \xi_j)$  the Dirac measure supported on a point  $\xi_j \in S$  and  $a_j \in \mathbb{C}$ . The points  $\xi_j$  can be assumed to lie on the support of the measure (in this case, the support of the function  $F$ ) and, since the set of rational points  $\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$ , in fact the points  $\xi_j$  can be taken in  $S \cap \mathbb{Q}^n$ . Moreover, we can also take

$$\xi_{-j} = -\xi_j \quad \text{and} \quad a_{-j} = \overline{a_j}.$$

As the set  $\overline{\Omega'}$  is compact, it follows that

$$\|w_1 - w_2\|_{L^\infty(\Omega')} < \delta' \quad (3.9)$$

where

$$w_2(x, t) := \sum_{j=-J}^J a_j e^{ix \cdot \xi_j - (t-T)|\xi_j|^2}$$

with  $a_j \in \mathbb{C}$ ,  $\xi_j \in S \cap \mathbb{Q}^n$  as above. Notice that  $w_2$  is a real-valued function by the choice of the constants  $a_j$  and the points  $\xi_j$ . Combining the inequalities (3.7)–(3.9) one gets

$$\|v - w_2\|_{L^\infty(\Omega')} < 3\delta',$$

and the fact that  $L_0 v = L_0 w_2 = 0$  allows us to use parabolic estimates to promote the above uniform bound to a  $C^k$  estimate of the form

$$\|v - w_2\|_{C^k(\Omega)} < C\delta'. \quad (3.10)$$



Let  $M$  denote the least common multiple of the denominators of the rational points  $\xi_j$ , so that  $M\xi_j \in \mathbb{Z}^n$  for all  $1 \leq j \leq J$ , and let be  $N$  any positive integer. It then follows that the function

$$u(x, t) := \sum_{j=-J}^J a_j e^{iMNx \cdot \xi_j - (M^2N^2t - T)|\xi_j|^2}$$

is well defined on  $\mathbb{T}^n \times \mathbb{R}$ , as the periodicity in  $x$  follows from the fact that  $NM\xi_j \in \mathbb{Z}^n$ . An elementary computation also shows that

$$L_0 u = 0.$$

Moreover, setting

$$\epsilon := \frac{1}{M^2N^2}$$

it follows directly from (3.10) and the definition of  $u$  that

$$\|u(\sqrt{\epsilon} \cdot, \epsilon \cdot) - v\|_{C^k(\Omega)} < C\delta'$$

The claim then follows by choosing  $\delta'$  small enough. □

## Chapter 4

# Minimal graphs with micro-oscillations

This chapter is devoted to the construction of minimal surfaces with prescribed level sets. Beyond the application of the classical approximation Theorem 1.1 for elliptic equations in bounded sets, the trick here is how to construct the solution to the non-linear minimal graph equation.

Namely, we consider minimal graphs on the unit ball  $\mathbb{B}^n$  of  $\mathbb{R}^n$ . More precisely, let  $\mathbb{B}^n$  denote the unit ball of  $\mathbb{R}^n$  and let  $u$  be a function satisfying the equation

$$\nabla \cdot \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad (4.1)$$

in  $\mathbb{B}^n$ . This is equivalent to saying that the graph of  $u$ ,

$$\Sigma_u := \{(x, u(x)) : x \in \mathbb{B}^n\}$$

is a minimal hypersurface of  $\mathbb{R}^{n+1}$ .

Area bounds for minimal graphs play a key role in the theory of minimal surfaces. Since  $\Sigma_u$  is a global minimizer of the area functional among the surfaces with fixed boundary, if  $u$  is bounded on  $\mathbb{B}^n$  it is clear that the  $n$ -dimensional Hausdorff measure  $\mathcal{H}^n(\Sigma_u)$  is bounded by a non-uniform constant that depends on the oscillation of  $u$ ,  $\text{osc } u := \max_{\mathbb{B}^n} u - \min_{\mathbb{B}^n} u$ . On the contrary, if we just specify the value of the oscillation of  $u$ , we can consider a hyperplane of slope  $\frac{\text{osc } u}{2}$ , which is a minimal hypersurface whose area  $\mathcal{H}^n(\Sigma_u)$  is bounded from below by a constant increasing as  $\text{osc } u$ . Therefore, there is not a uniform estimate holding for the area of minimal graphs on  $\mathbb{B}^n$ .

Our objective is to explore the non existence of a higher-codimension analog of the uniform estimate. We will be interested in bounds for the  $(n - 1)$ -dimensional Hausdorff measure of the hypersurface, or rather of its transverse intersection with a hyperplane. To this end, let us denote by  $\Pi$  the portion of the horizontal hyperplane that is contained in the unit ball of  $\mathbb{R}^{n+1}$ :

$$\Pi := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0, |x| < 1\}.$$

Actually, our goal in this chapter is to show a more general result claiming that we can prescribe the geometry of the transverse intersection of a minimal graph on  $\mathbb{B}^n$  with  $\Pi$ , the zero nodal set, up to a small deformation:

**Theorem 4.1.** *Let  $S$  be a compact, connected, properly embedded, orientable hypersurface of  $\mathbb{B}^n$  with nonempty boundary. Then, for any integer  $k$  and  $\epsilon > 0$ , there is a minimal graph over the unit  $n$ -ball and an open subset  $\Pi' \subset \Pi$  such that the intersection  $\Sigma_u \cap \Pi'$  is given by  $\Phi(S)$ , where  $\Phi : \Pi \rightarrow \Pi$  is a diffeomorphism with  $\|\Phi - \text{id}\|_{C^k} < \epsilon$ .*

If one chooses  $S$  to be a compact hypersurface of  $\mathbb{B}^n$  with area  $\mathcal{H}^{n-1}(S) > c$  and  $\epsilon$  is small enough, the immediate corollary is a codimension 1 analog of the existence of minimal graphs with arbitrarily large area:

**Corollary 4.2.** *The  $(n - 1)$ -dimensional measure of the transverse intersection of a minimal graph over the unit  $n$ -ball with a hyperplane is not uniformly bounded. Specifically, given any constant  $c$ , there is some  $u$  satisfying equation (4.1) for which  $\Sigma_u$  and  $\Pi$  intersect transversally but*

$$\mathcal{H}^{n-1}(\Sigma_u \cap \Pi) > c.$$

Of course, the reason for which transverse intersections are considered is that for  $u := 0$  (that is,  $\Sigma_u = \Pi$ ) one trivially has  $\mathcal{H}^{n-1}(\Sigma_u \cap \Pi) = \infty$ . This fact strongly suggests that Corollary 4.2 should hold for graphs that are a small perturbation of the hyperplane, but the passage from this heuristic argument to an actual proof is nontrivial.

These minimal graphs with micro-oscillations play the opposite role with respect to area bounds that hyperplanes. Even hyperplanes with arbitrarily large  $n$ -measure do not have larger  $(n - 1)$ -measure of its intersection with  $\Pi$  than the diameter of the ball. Our construction of minimal graphs leads to arbitrarily large  $(n - 1)$ -measure of the transverse intersection but with  $n$ -measure less than twice the area of the ball.

The key point of Theorem 4.1 is that it can be analyzed in the linear regime of the minimal surface equation. In fact, the strategy that we have used to prove it (see Section 4.1) is to construct harmonic functions  $v$  on the ball that are small in a  $C^k$  norm and whose zero set  $v^{-1}(0)$  contains the hypersurface  $S$  up to a small diffeomorphism. The smallness assumption then permits to promote them to solutions of the minimal surface equation through an iterative procedure that does not change much the geometry of the zero set. Hence, in Corollary 4.2, the large  $(n - 1)$ -measure of the intersection of the minimal graph  $\Sigma_u$  with the hyperplane  $\Pi$  comes from micro-oscillations that do not significantly contribute to the curvature of the minimal hypersurface (which is almost flat).

To conclude, it is worth mentioning that Theorem 4.1 remains valid when we consider the intersection of a minimal hypersurface with the portion  $\Pi'$  of any (non-vertical) hyperplane contained in  $\mathbb{B}^n \times \mathbb{R}$  (in this case, the minimal hypersurface can be constructed as a graph over  $\Pi'$ ).

## 4.1 Proof of the main theorem

In this section we prove Theorem 4.1. For this, a well known result of Whitney ensures that, by perturbing  $S$  a little if necessary, one can assume that  $S$  is analytic. Now let us consider an open extension  $S'$  of  $S$  (that is, an open, connected, analytic hypersurface  $S'$  of  $\mathbb{B}^n$  containing  $S$ ) and let us denote by  $\Omega$  a small neighborhood of the hypersurface  $S$  whose closure is contained in  $\mathbb{B}^n$  and such that  $\mathbb{R}^n \setminus \Omega$  is connected. Such a choice of  $S'$  and  $\Omega$  is always possible because  $S$  is connected and its boundary is nonempty.

An important ingredient in the proof of the main theorem is the construction of a harmonic function on  $\mathbb{R}^n$  for which a small deformation of  $S'$  is a structurally stable (portion of a) connected component of its zero set (similar to the construction in [30]):

**Lemma 4.3.** *For any  $\epsilon > 0$  there is a harmonic function  $v$  on  $\mathbb{R}^n$  and some  $\delta > 0$  such that the zero set  $u^{-1}(0)$  of any function  $u$  with  $\|u - v\|_{C^k(\Omega)} < \delta$  satisfies*

$$u^{-1}(0) \cap \Omega' = \Psi(S'),$$

where  $\Omega'$  is an open subset of  $\Omega$  and  $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a diffeomorphism with  $\|\Psi - \text{id}\|_{C^k(\mathbb{R}^n)} < \epsilon$ .

*Proof.* Let us choose an orientation of  $S'$  and denote by  $\nu$  the corresponding unit normal vector. A natural way to define a harmonic function associated with  $S'$  and with some control on its zero set and on its gradient is via the following Cauchy problem:

$$\Delta \tilde{v} = 0, \quad \tilde{v}|_{S'} = 0, \quad \left. \frac{\partial \tilde{v}}{\partial \nu} \right|_{S'} = 1. \quad (4.2)$$

The Cauchy–Kowaleskaya theorem ensures the existence of a solution  $\tilde{v}$  of the above problem in a small neighborhood of  $S'$ , which can be taken to be  $\Omega$  without any loss of generality. Since  $\mathbb{R}^n \setminus \Omega$  is connected, the global approximation Theorem 1.2 ensures the existence of a harmonic function  $v : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\|v - \tilde{v}\|_{C^k(\Omega)} < \delta,$$

where  $\delta$  is a small quantity to be specified later.

Now let  $u$  be close to  $v$  in the sense that  $\|u - v\|_{C^k(\Omega)} < \delta$ . Then

$$\|u - \tilde{v}\|_{C^k(\Omega)} < 2\delta,$$

so, since  $S$  is a component of the nodal set of  $\tilde{v}$  and the gradient of  $\tilde{v}$  does not vanish by (4.2), Thom's isotopy theorem [1, Theorem 20.2] implies that, for small enough  $\delta$ , there is an open subset  $\Omega'$  of  $\mathbb{B}^n$  and a diffeomorphism  $\Psi$  of  $\mathbb{R}^n$  with  $\|\Psi - \text{id}\|_{C^k(\mathbb{R}^n)} < \epsilon$  such that  $\Psi(S') = u^{-1}(0) \cap \Omega'$ . The lemma then follows.  $\square$

The observation now is that one can construct a solution to the minimal graph equation on the ball whose zero set is a small perturbation of that of the harmonic function constructed in the previous lemma. More precisely, we have the following:

**Lemma 4.4.** *Given any  $\delta > 0$ , there is a function  $u$  satisfying the minimal surface equation (4.1) in  $\mathbb{B}^n$  and a positive constant  $\lambda$  such that  $\|\lambda u - v\|_{C^k(\mathbb{B}^n)} < \delta$ .*

*Proof.* Assuming that  $k \geq 2$  without loss of generality and taking any  $\alpha \in (0, 1)$ , let us define a function  $F : C^{k,\alpha}(\mathbb{B}^n) \rightarrow C^{k-2,\alpha}(\mathbb{B}^n)$  as

$$F(u) := \frac{1}{2} \nabla u \cdot \nabla \log(1 + |\nabla u|^2).$$

Equation (4.1) is then expressible as

$$\Delta u - F(u) = 0. \tag{4.3}$$

Let  $v$  be the harmonic function on  $\mathbb{R}^n$  that we constructed in Lemma 4.3. Take a small positive constant  $\epsilon$  that will be fixed later and consider the iterative scheme

$$\begin{aligned} u_0 &:= \gamma v \\ u_{j+1} &:= \gamma v + w_j \end{aligned} \tag{4.4}$$

where

$$\gamma := \frac{\epsilon}{2\|v\|_{C^{k,\alpha}(\mathbb{B}^n)}}$$

and the function  $w_j$  is the unique solution to the boundary value problem

$$\Delta w_j = F(u_j) \quad \text{in } \mathbb{B}^n, \quad w_j = 0 \quad \text{on } \partial\mathbb{B}^n. \tag{4.5}$$

Our goal is to show that, for small enough  $\epsilon$ ,  $u_j$  converges in  $C^{k,\alpha}(\mathbb{B}^n)$  to a function  $u$  that satisfies the minimal graph equation (4.3) in  $\mathbb{B}^n$  and is close to  $\gamma v$  in a suitable sense. To this end, let us start by noticing that, as an application of the maximum principle to the boundary problem (4.5), the functions  $w_j$  must satisfy

$$\|w_j\|_{C^0(\mathbb{B}^n)} \leq C\|F(u_j)\|_{C^0(\mathbb{B}^n)}.$$

Standard elliptic estimates then yield

$$\begin{aligned} \|w_j\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq C(\|w_j\|_{C^0(\mathbb{B}^n)} + \|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}) \\ &\leq C(\|F(u_j)\|_{C^0(\mathbb{B}^n)} + \|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}) \\ &\leq C\|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)}. \end{aligned}$$

On the other hand, if we assume that  $\|u_j\|_{C^{k,\alpha}} < \epsilon$ , one can exploit the above estimate to infer in equation (4.4) that

$$\begin{aligned} \|u_{j+1}\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq \gamma\|v\|_{C^{k,\alpha}(\mathbb{B}^n)} + \|w_j\|_{C^{k,\alpha}(\mathbb{B}^n)} \\ &\leq \frac{\epsilon}{2} + C\|F(u_j)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ &\leq \frac{\epsilon}{2} + C\|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)}^3 \\ &\leq \frac{\epsilon}{2} + C\epsilon^3 < \epsilon, \end{aligned} \tag{4.6}$$

so the norm of  $u_{j+1}$  is less than  $\epsilon$  too. Here we have used that

$$\|F(w)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \leq C\|w\|_{C^{k,\alpha}(\mathbb{B}^n)}^3$$

and the fact that  $\gamma\|v\|_{C^{k,\alpha}(\mathbb{B}^n)}$  and  $C\epsilon^3$  are bounded above by  $\epsilon/2$ . Notice, in particular, that since the first function  $u_0$  of the iteration satisfies

$$\|u_0\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq \frac{\epsilon}{2},$$

the induction argument (4.6) then implies that

$$\|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)} < \epsilon \tag{4.7}$$

for all  $j$ .

To estimate the difference  $u_{j+1} - u_j$ , let us use the bound (4.7) to write

$$\begin{aligned} \|F(u_j) - F(u_{j-1})\|_{C^{k-2,\alpha}(\mathbb{B}^n)} &\leq C(\|u_j\|_{C^{k,\alpha}(\mathbb{B}^n)}^2 + \|u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}^2)\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)} \\ &\leq C\epsilon^2\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}. \end{aligned}$$

Since

$$\Delta(u_{j+1} - u_j) = F(u_j) - F(u_{j-1}) \quad \text{in } \mathbb{B}^n, \quad u_{j+1} - u_j = 0 \quad \text{on } \partial\mathbb{B}^n,$$

standard elliptic estimates then yield

$$\begin{aligned} \|u_{j+1} - u_j\|_{C^{k,\alpha}(\mathbb{B}^n)} &\leq C\|F(u_j) - F(u_{j-1})\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ &< C\epsilon^2\|u_j - u_{j-1}\|_{C^{k,\alpha}(\mathbb{B}^n)}. \end{aligned} \tag{4.8}$$

Taking  $\epsilon$  small enough for  $C\epsilon^2 < 1$ , we infer from (4.7) and (4.8) that, as  $j \rightarrow \infty$ ,  $u_j$  converges in  $C^{k,\alpha}(\mathbb{B}^n)$  to some function  $u$  with

$$\|u\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq \epsilon. \tag{4.9}$$

Since the sequence  $w_j$  converges to  $w$  in  $C^{k,\alpha}(\mathbb{B}^n)$ , the function  $u$  satisfies the equation

$$u = \gamma v + w, \tag{4.10}$$

where  $w$  is the unique solution to the problem

$$\Delta w = F(u) \quad \text{in } \mathbb{B}^n, \quad w = 0 \quad \text{on } \partial\mathbb{B}^n.$$

As  $v$  is a harmonic function, one can then take the Laplacian of (4.10) to show that  $u$  is a solution of the minimal graph equation (4.1) with boundary conditions  $u = \gamma v$  on  $\partial\mathbb{B}^n$ .

Taking  $\lambda := 1/\gamma$ , one can now use the bound (4.9), the relation (4.10) and the definition of  $\gamma$  to check that

$$\begin{aligned} \|\lambda u - v\|_{C^{k,\alpha}(\mathbb{B}^n)} &= \lambda \|u - \gamma v\|_{C^{k,\alpha}(\mathbb{B}^n)} \leq C\lambda \|F(u)\|_{C^{k-2,\alpha}(\mathbb{B}^n)} \\ &\leq C\lambda \|u\|_{C^{k,\alpha}(\mathbb{B}^n)}^3 \leq C\epsilon^2 < \delta \end{aligned}$$

provided that  $\epsilon$  is sufficiently small. □

Theorem 4.1 readily follows from Lemmas 4.3 and 4.4.

## Chapter 5

# The Biot–Savart operator of a bounded domain

The structure of the integral kernels of the inverses of operators is key in the development of global approximation theorems. In this chapter we show the existence and behaviour of an integral kernel for the inverse of the curl operator in bounded domains.

The Biot–Savart operator,

$$\text{BS}(\omega)(x) := \int_{\mathbb{R}^3} \frac{\omega(y) \times (x - y)}{4\pi|x - y|^3} dy,$$

plays a key role in fluid mechanics and electromagnetism as an inverse of the curl operator. More precisely, if  $\omega$  is a well-behaved divergence-free vector field on  $\mathbb{R}^3$ , then  $u := \text{BS}(\omega)$  is the only solution to the equation

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0$$

that falls off at infinity. Consequently, in fluid mechanics the Biot–Savart operator maps the vorticity  $\omega$  of a fluid into its associated velocity field  $u$ . If we talked about electromagnetism,  $\omega$  would play the role of a stationary electric current flow and  $u$ , the generated magnetic field.

Our concern is the problem of mapping the vorticity of a fluid contained in a bounded domain  $\Omega$  of  $\mathbb{R}^3$  into its velocity field. To put it differently, given a divergence-free vector field  $\omega$  we want to solve the problem

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u \cdot \nu|_{\partial\Omega} = 0, \quad (5.1)$$

where the tangency condition  $u \cdot \nu|_{\partial\Omega} = 0$  means that the fluid stays inside the domain.

The problem (5.1) has been considered by a number of people, who have shown the existence of solutions for domains of different regularity and derived estimates in  $L^p$  or Hölder spaces. Up to date accounts of the problem can be found e.g. in [3, 4, 18] and references therein. However, the question of whether the solution is given by an integral formula generalizing the classical Biot–Savart law remains wide open.



We fill this gap by showing that one can construct a solution to the problem (5.1) through a generalized Biot–Savart operator  $BS_\Omega$  with a reasonably well-behaved integral kernel  $K_\Omega(x, y)$  which has an inverse-square singularity on the diagonal

We will only state our result for the flat-space problem, but it will be apparent from the construction that the result holds true (*mutatis mutandis*) for bounded domains in any Riemannian 3-manifold. The existence of a Biot–Savart operator on a compact Riemannian 3-manifold without boundary can be obtained from the Green’s function of the Hodge Laplacian, computed in [19], as explicitly discussed in [75]. It should be stressed that this connection between the Green’s function of the Laplacian in a domain (with certain boundary conditions) and the Biot–Savart operator is no longer true in the presence of a boundary condition. Indeed, the combination of the boundary condition with the fact that the field must be divergence-free makes the structure of the generalized Biot–Savart kernel rather involved.

Beyond our interest on the structure of integral kernels in the context of the approximation theory, we have other motivations to construct the Biot–Savart operator in bounded domains. On the one hand, the integral kernel of the Biot–Savart operator on a compact 3-manifold without boundary has been recently employed [33] to show that the helicity is the only regular integral invariant of volume-preserving transformations. Proving a similar result for the case of manifolds with boundary presents additional difficulties, but in any case it is important to have a thorough understanding of the associated Biot–Savart operator. A particular case has been recently established in [55]. Applications to electrodynamics of the Biot–Savart operator for domains in the 3-sphere (whose existence as an integral operator is not discussed, however) can be found in [67]. On the other hand, the celebrated connection between the helicity of a field and its asymptotic linking number, unveiled by Arnold [5], needs that the inverse of the curl operator be expressed as a generalized Biot–Savart integral.

## 5.1 Main result

Before stating the existence of a Biot–Savart integral operator on  $\Omega$ , let us see that, for problem (5.1) to admit a solution, the field  $\omega$  must satisfy several hypotheses. Firstly, since the divergence of a curl is zero, it is obvious that  $\omega$  must be divergence-free. Secondly, it is easy to see that if  $\Gamma_1, \dots, \Gamma_m$  denote the connected components of  $\partial\Omega$ , then one must have

$$\int_{\Gamma_j} \omega \cdot \nu \, d\sigma = 0 \quad \text{for all } 1 \leq j \leq m. \quad (5.2)$$

To see why this is true, it is enough to take the harmonic functions  $\psi_i$  defined by the boundary value problems

$$\Delta\psi_i = 0 \quad \text{in } \Omega, \quad \psi_i|_{\Gamma_j} = \delta_{ij}$$

and observe that

$$\int_{\Gamma_j} \omega \cdot \nu \, d\sigma = \int_{\Omega} \nabla \cdot (\psi_j \omega) \, dx = \int_{\Omega} \nabla \psi_j \cdot \nabla \times u \, dx = \int_{\partial\Omega} u \cdot (\nabla \psi_j \times \nu) \, d\sigma = 0.$$

In addition to using that  $\omega$  is divergence-free and integrating by parts, we have exploited that the gradient  $\nabla \psi_j$  is proportional to  $\nu$  at the boundary.

In order to state our result, let us define the function

$$\ell(y) := \log \left( 2 + \frac{1}{\text{dist}(y, \partial\Omega)} \right). \quad (5.3)$$

Let us also recall that a vector field  $h$  on  $\Omega$  is said to be harmonic and tangent to the boundary if

$$\nabla \times h = 0, \quad \nabla \cdot h = 0, \quad h \cdot \nu|_{\partial\Omega} = 0.$$

By Hodge theory (see e.g. [14]), the dimension of the linear space of tangent harmonic fields is the genus of  $\partial\Omega$  (if  $\partial\Omega$  is disconnected, this is defined as the sum of the genus of the connected components of the boundary). The main result of the chapter can then be presented as follows:

**Theorem 5.1.** *Let  $\omega \in W^{k,p}(\Omega)$  be a divergence-free vector field satisfying the hypothesis (5.2), with  $k \geq 0$  and  $1 < p < \infty$ . Then the boundary-value problem (5.1) has a solution  $u \in W^{k+1,p}(\Omega)$  that satisfies the estimate*

$$\|u\|_{W^{k+1,p}(\Omega)} \leq C \|\omega\|_{W^{k,p}(\Omega)} \quad (5.4)$$

and which can be represented as an integral of the form

$$u(x) = \text{BS}_{\Omega}(\omega)(x) := \int_{\Omega} K_{\Omega}(x, y) \omega(y) \, dy, \quad (5.5)$$

where  $K_{\Omega}(x, y)$  is a matrix-valued integral kernel that is smooth outside the diagonal and satisfies the pointwise bound

$$|K_{\Omega}(x, y)| \leq C \frac{\ell(y)}{|x - y|^2}.$$

Furthermore, the solution is unique modulo the addition of a harmonic field tangent to the boundary.

Let us emphasize that the core of the result is not the estimate (5.4), which is not new, but the existence of an integral kernel  $K_{\Omega}(x, y)$  with the above properties. It is worth mentioning that the proof of Theorem 5.1 grants the existence of the integral kernel  $K_{\Omega}(x, y)$  for domains of class  $C^2$ , but for simplicity we will assume throughout that the boundary of this domain is smooth.

Although one could have expected that the estimate (5.4) might imply the existence of an integral kernel using an argument based on the Lax-Milgram theorem (see e.g. [46], also [58]), the argument is not conclusive as one needs

to control the tangential component of  $u$  on the boundary. This difficulty is not only technical, and it is related to the fact that the kernel  $K_\Omega(x, y)$  (which does not satisfy the equation  $\nabla \times K_\Omega(x, y) = 0$  outside the diagonal) is strongly non-unique.

This non-uniqueness is a serious difficulty that arises in the analysis of the integral kernel of the Biot–Savart operator on a domain. It means that many other integral kernels give the solution of the problem (5.1) through integration. Specifically, if  $A_i(x, y)$  is any function satisfying the boundary condition  $A_i(x, \cdot)|_{\partial\Omega} = 0$  and we define a matrix-valued kernel as

$$K'_{ij}(x, y) = \partial_{y_j} A_i(x, y),$$

it is easy to see that

$$\int_{\Omega} K'(x, y) \omega(y) dy = 0$$

for any vector field with  $\nabla \cdot \omega = 0$ . Hence  $K_\Omega(x, y) + K'(x, y)$  is also an admissible kernel for the Biot–Savart operator of the domain (in fact, its action on divergence-free fields, after integrating in the variable  $y$ , coincides with that of  $K_\Omega(x, y)$ ).

Let us briefly discuss the strategy of the proof. The first step is to make use of the properties of the Biot–Savart operator and of layer potentials to show that one can construct a solution to the problem (5.1) as

$$u = \text{BS}(\omega) - \text{BS}(\nabla \mathcal{S}f) - \nabla \mathcal{S}g, \quad (5.6)$$

where

$$\mathcal{S}f(x) := - \int_{\partial\Omega} \frac{f(y)}{4\pi|x-y|} d\sigma(y)$$

denotes the single layer operator and  $f$  and  $g$  are suitably chosen scalar functions defined on  $\partial\Omega$ . The hard part of the proof, of course, is to show that, at the end of the day, the function  $u$  defined by (5.6) can be written in terms of an integral kernel  $K_\Omega(x, y)$  as in (5.5). This is done through a carefully crafted analysis of the singularities and boundary terms appearing in the formula (5.6). In particular, in order to deal with the boundary terms we define an extension operator that allows us to write the single layer potential (and, crucially for our purposes, less explicit generalizations thereof) generated by the normal component of a vector field as the normal component of another vector field whose divergence on  $\Omega$  is given in terms of certain integral kernel.

The above strategy is implemented as follows. Firstly, as it plays a key role in the proof, in Section 5.2 we present some estimates and identities for the usual Biot–Savart integral. In Section 5.3 we construct the aforementioned extension operator that we need to deal with boundary terms. In Section 5.4 we employ layer potentials to construct a solution  $u$  of the form (5.6). This explicit formula is carefully analyzed in Section 5.5 to establish the existence of the desired integral kernel. The result and the proof hold true without any major modifications on any Riemannian 3-manifold, using the layer potentials and the Biot–Savart operator associated with the metric. To conclude, for completeness

we discuss in Section 5.6 the uniqueness of the solutions (possibly with nonzero but prescribed divergence and normal component on the boundary).

## 5.2 Estimates and identities for the Biot–Savart integral

This section is a brief but reasonably self-contained presentation of several results and identities for the Biot–Savart integral that will be used later. These results are essentially standard.

Given a vector field  $F \in C^1(\overline{\Omega})$ , we will state the results in terms of the field

$$w(x) := \int_{\Omega} \frac{F(y) \times (x - y)}{4\pi|x - y|^3} dy.$$

In what follows,  $\epsilon_{ijk}$  will denote the Levi-Civita's permutation symbol and  $B_r(x)$  (resp.  $B_r$ ) will denote the three-dimensional ball of radius  $r$  centered at the point  $x$  (resp. at the origin).

**Lemma 5.2.** *The derivative of the field  $w$  at any point  $x \in \Omega$  is given by*

$$\begin{aligned} \partial_j w_k(x) = \epsilon_{klm} \int_{\Omega} [F_l(y) - F_l(x)] \frac{|x - y|^2 \delta_{jm} - 3(x_j - y_j)(x_m - y_m)}{4\pi|x - y|^5} dy \\ - \epsilon_{klm} F_l(x) A_m^j(x), \end{aligned} \quad (5.7)$$

where  $A_m^j$  stands for the  $m^{\text{th}}$  component of a certain continuous vector field  $A^j \in L^\infty(\Omega)$ .

*Proof.* Let us take a smooth function  $\eta(r)$  which vanishes for  $r < \frac{1}{2}$  and is equal to 1 for  $r > 1$  and set  $\eta_\delta(r) := \eta(r/\delta)$ , with  $\delta$  a small positive constant. Then let us define the vector field

$$w^\delta(x) := \int_{\Omega} \eta_\delta(|x - y|) \frac{F(y) \times (x - y)}{4\pi|x - y|^3} dy.$$

It is apparent that  $w^\delta$  is a smooth vector field on  $\mathbb{R}^3$ . Moreover, it is not hard to see that  $w^\delta$  converges to  $w$  uniformly, since for any  $x \in \Omega$  one has

$$\begin{aligned} |w(x) - w^\delta(x)| &= \left| \int_{B_\delta(x)} (1 - \eta_\delta(|x - y|)) \frac{F(y) \times (x - y)}{4\pi|x - y|^3} dy \right| \\ &\leq (1 + \|\eta\|_{L^\infty}) \|F\|_{L^\infty(\Omega)} \int_{B_\delta} \frac{dz}{4\pi|z|^2} \\ &\leq C\delta. \end{aligned}$$

The derivative of the  $k^{\text{th}}$  component of  $w^\delta$  can be readily computed as

$$(5.8)$$

$$\begin{aligned}
\partial_j w_k^\delta(x) &= \epsilon_{klm} \int_{\Omega} F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\
&= \epsilon_{klm} \int_{\Omega} [F_l(y) - F_l(x)] \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\
&\quad - \epsilon_{klm} F_l(x) \int_{\Omega} \partial_{y_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\
&= \epsilon_{klm} \int_{\Omega} [F_l(y) - F_l(x)] \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\
&\quad - \epsilon_{klm} F_l(x) A_m^{j,\delta}(x), \quad (5.9)
\end{aligned}$$

where for each  $x \in \Omega$  we have set

$$A_m^{j,\delta}(x) := \int_{\partial\Omega} \nu_j(y) \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} d\sigma(y).$$

As  $\delta \rightarrow 0$ , the first term converges to the first line of (5.7), since the difference

$$\begin{aligned}
M &:= \int_{\Omega} [F_l(y) - F_l(x)] \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\
&\quad - \int_{\Omega} [F_l(y) - F_l(x)] \frac{|x-y|^2 \delta_{jm} - 3(x_j - y_j)(x_m - y_m)}{4\pi|x-y|^5} dy
\end{aligned}$$

can be estimated using the fact that  $\eta_\delta(r) = 1$  for  $r > \delta$  and the mean value theorem as

$$\begin{aligned}
|M| &\leq \int_{B_\delta(x)} |F_l(y) - F_l(x)| \left| \partial_{x_j} \left( (1 - \eta_\delta(|x-y|)) \frac{x_m - y_m}{4\pi|x-y|^3} \right) \right| dy \\
&\leq C \int_{B_\delta(x)} \frac{|F_l(y) - F_l(x)|}{|x-y|^3} dy \\
&\leq C \|\nabla F_l\|_{L^\infty(\Omega)} \int_{B_\delta(x)} \frac{1}{|x-y|^2} dy \\
&\leq C\delta. \quad (5.10)
\end{aligned}$$

Since  $A^{j,\delta}(x)$  obviously converges to the field

$$A^j(x) := \int_{\partial\Omega} \nu_j(y) \frac{x - y}{4\pi|x-y|^3} d\sigma(y) \quad (5.11)$$

for all  $x \in \Omega$ , equations (5.9) and (5.10) show that the derivative  $\partial_j w_k$  is indeed given by the formula (5.7) of the statement. As  $A^j$  is obviously smooth in  $\Omega$ , it only remains to show that  $A^j(x)$  is bounded when  $x$  approaches  $\partial\Omega$ .

For this, let us take an arbitrary point of the boundary, which we can assume to be the origin. Rotating the coordinate axes if necessary, one can parametrize  $\partial\Omega$  in a neighborhood  $U_\rho$  of the origin as the graph

$$Y \in D_\rho \mapsto (Y, h(Y)) \in \mathbb{R}^3, \quad (5.12)$$

where  $D_\rho := \{Y \in \mathbb{R}^2 : |Y| < \rho\}$  is the two-dimensional disk of a small radius  $\rho$  and the function  $h$  satisfies

$$h(0) = 0, \quad \nabla h(0) = 0.$$

Let us now analyze the behavior of  $A^j(x)$  when  $x = (0, 0, -t)$  and  $t \rightarrow 0^+$ . Since the unit normal and the surface measure can be written in terms of  $h$  as

$$\nu = \frac{(-\nabla h, 1)}{\sqrt{1 + |\nabla h|^2}}, \quad d\sigma = \sqrt{1 + |\nabla h|^2} dY$$

and  $x \in U_\rho$  for small enough  $t$ , it follows that

$$\begin{aligned} |A_j(x)| &\leq \left| \int_{\partial\Omega \cap U_\rho} \nu_j(y) \frac{x - y}{4\pi|x - y|^3} d\sigma(y) \right| + \left| \int_{\partial\Omega \setminus U_\rho} \nu_j(y) \frac{x - y}{4\pi|x - y|^3} d\sigma(y) \right| \\ &\leq \frac{1}{4\pi} \left| \int_{D_\rho} \bar{\nu}_j(Y) \frac{(Y, t + h(Y))}{(|Y|^2 + (t + h(Y))^2)^{3/2}} dY \right| + C, \end{aligned} \quad (5.13)$$

where we have used that the second integral is bounded by a constant independent of  $t$  and

$$\bar{\nu}_j := \sqrt{1 + |\nabla h|^2} \nu_j$$

It is clear then that

$$\begin{aligned} \left| \int_{D_\rho} \bar{\nu}_j(Y) \frac{Y}{(|Y|^2 + (t + h(Y))^2)^{3/2}} dY \right| &= \left| \int_{D_\rho} \frac{\bar{\nu}_j(0)Y + O(|Y|^2) + tO(Y)}{(|Y|^2 + t^2)^{3/2}} dY \right| \\ &\leq \left| \bar{\nu}_j(0) \int_{D_\rho} \frac{Y}{(|Y|^2 + t^2)^{3/2}} dY \right| + C \int_{D_\rho} \frac{dY}{|Y|} \\ &\leq C\rho, \end{aligned}$$

where we have used that the first integral in the second line vanishes by parity. Likewise,

$$\begin{aligned} \left| \int_{D_\rho} \bar{\nu}_j(Y) \frac{t + h(Y)}{(|Y|^2 + (t + h(Y))^2)^{3/2}} dY \right| &= \left| \int_{D_\rho} \frac{\bar{\nu}_j(0)t + O(|Y|^2) + tO(Y)}{(|Y|^2 + t^2)^{3/2}} dY \right| \\ &\leq 2\pi t |\bar{\nu}_j(0)| \int_0^\rho \frac{r dr}{(r^2 + t^2)^{3/2}} + C \int_{D_\rho} \frac{dY}{|Y|} \\ &\leq 2\pi |\bar{\nu}_j(0)| \int_0^\infty \frac{r dr}{(r^2 + 1)^{3/2}} + C\rho \\ &\leq C, \end{aligned}$$

Hence we infer from (5.13) that  $\|A^j\|_{L^\infty(\Omega)} \leq C$  and the lemma follows.  $\square$

*Remark 5.3.* By the regularity of  $F_l$  and the definition of the principal value, equation (5.7) can be written as

$$\partial_j w_k(x) = \epsilon_{klm} \text{PV} \int_{\Omega} F_l(y) \frac{|x - y|^2 \delta_{jm} - 3(x_j - y_j)(x_m - y_m)}{4\pi|x - y|^5} dy,$$

where we have taken into account that

$$\int_{\partial B_r(x)} \nu_j(y) \frac{x-y}{4\pi|x-y|^3} d\sigma(y) = 0.$$

Lemma 5.2 immediately yields the following characterization of the divergence and curl of  $w$ :

**Proposition 5.4.** *The vector field  $w$  is divergence-free in  $\mathbb{R}^3$  and its curl at a point  $x \in \Omega$  is given by*

$$\nabla \times w(x) = F(x) + \nabla \int_{\Omega} \frac{\nabla \cdot F(y)}{4\pi|x-y|} dy - \nabla \int_{\partial\Omega} \frac{F(y) \cdot \nu(y)}{4\pi|x-y|} d\sigma(y). \quad (5.14)$$

*Proof.* It follows from the proof of Lemma 5.2 that  $\nabla \cdot w = \lim_{\delta \rightarrow 0} \nabla \cdot w^\delta$  in  $\Omega$ , with  $w^\delta$  defined as before. For simplicity, let us write

$$E := \frac{1}{4\pi|x-y|}.$$

Since first line of equation (5.9) shows that

$$\begin{aligned} \nabla \cdot w^\delta(x) &= \epsilon_{klm} \int_{\Omega} F_l(y) \partial_{x_k} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\ &= \epsilon_{klm} \int_{\Omega} F_l(y) \left( \eta'_\delta(|x-y|) \frac{(x_k - y_k)(x_m - y_m)}{4\pi|x-y|^4} - \eta_\delta(|x-y|) \partial_{x_k} \partial_{x_m} E \right) dy \\ &= 0, \end{aligned}$$

and  $w$  is clearly divergence-free outside  $\Omega$ , it follows that  $\nabla \cdot w = 0$  everywhere.

To compute  $\nabla \times w$ , let us begin by computing the  $i^{\text{th}}$  component of  $\nabla \times w^\delta$  using again equation (5.9) and the fact that  $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ :

$$\begin{aligned} \epsilon_{ijk} \partial_j w_k^\delta(x) &= \int_{\Omega} F_i(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} \right) dy \\ &\quad + \int_{\Omega} F_j(y) \partial_{x_j} (\eta_\delta(|x-y|) \partial_{x_i} E) dy \\ &= \partial_{x_j} \int_{\Omega} F_i(y) \eta_\delta(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} dy \\ &\quad - \int_{\Omega} F_j(y) \partial_{y_j} (\eta_\delta(|x-y|) \partial_{x_i} E) dy \\ &= \partial_{x_j} \int_{\Omega} F_i(y) \eta_\delta(|x-y|) \frac{x_j - y_j}{4\pi|x-y|^3} dy \\ &\quad - \int_{\partial\Omega} F \cdot \nu(y) \eta_\delta(|x-y|) \partial_{x_i} E d\sigma(y) + \int_{\Omega} \nabla \cdot F(y) \eta_\delta(|x-y|) \partial_{x_i} E dy. \end{aligned}$$

Taking the limit  $\delta \rightarrow 0$  and using that  $E$  is a fundamental solution of the Laplacian one then obtains

$$\begin{aligned} (\nabla \times w)_i(x) &= \lim_{\delta \rightarrow 0} \epsilon_{ijk} \partial_j w_k^\delta(x) \\ &= \partial_{x_j} \int_{\Omega} F_i(y) \frac{x_j - y_j}{4\pi|x - y|^3} dy - \int_{\partial\Omega} F \cdot \nu(y) \partial_{x_i} E d\sigma(y) \\ &\quad + \int_{\Omega} \nabla \cdot F(y) \partial_{x_i} E dy \\ &= F_i(x) - \partial_{x_i} \int_{\partial\Omega} F \cdot \nu(y) E d\sigma(y) + \partial_{x_i} \int_{\Omega} \nabla \cdot F(y) E dy. \end{aligned}$$

The proposition then follows.  $\square$

We are now ready to state the basic  $L^p$  estimate for  $w$ :

**Proposition 5.5.** *For any nonnegative integer  $n$  and  $1 < p < \infty$ , if  $F \in W^{n,p}(\Omega)$ , then the field  $w$  can be estimated as*

$$\|w\|_{W^{n+1,p}(\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)}.$$

*Proof.* Notice that the integral appearing in the first term of the formula (5.7) is the action of a Calderón–Zygmund operator in  $\mathbb{R}^3$  on  $F_l 1_\Omega$ , where  $1_\Omega$  denotes the indicator function of the domain. As the field  $A^j$  is bounded, it is then standard that

$$\|w\|_{W^{1,p}(\Omega)} \leq C \|F\|_{L^p(\Omega)}. \quad (5.15)$$

This is the bound of the statement with  $n = 0$ .

We claim that, if  $Z^1, \dots, Z^n$  are smooth vector fields in  $\overline{\Omega}$  which are tangent to  $\partial\Omega$ , then

$$\|Z^1 \cdots Z^n w\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)}, \quad (5.16)$$

where, as it is customary (see e.g. [51]), we are regarding the vector field  $Z^j$  as a first-order differential operator (namely,  $Z^j = Z_i^j(x) \partial_{x_i}$ ) and the function  $F$  can be taken smooth. It is easy to see that the proposition readily follows from this estimate. Indeed, Proposition 5.4 ensures that  $w$  is divergence-free, so we can take the curl of (5.14) to find that

$$\Delta w = -\nabla \times F$$

in  $\Omega$ . Since (5.16) means that  $\|w\|_{W^{n+1-\frac{1}{p},p}(\partial\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)}$ , standard elliptic estimates then yield

$$\begin{aligned} \|w\|_{W^{n+1,p}(\Omega)} &\leq C (\|\nabla \times F\|_{W^{n-1,p}(\Omega)} + \|w\|_{W^{1,p}(\Omega)} + \|w\|_{W^{n+1-\frac{1}{p},p}(\partial\Omega)}) \\ &\leq C \|F\|_{W^{n,p}(\Omega)}, \end{aligned}$$

as claimed.



Hence it only remains to prove (5.16). In view of the trace inequality

$$\|f\|_{W^{s,q}(\partial\Omega)} \leq C \|f\|_{W^{s+\frac{1}{q},q}(\Omega)},$$

it suffices to show that for any  $j$  and  $n$  one can write

$$\partial_j(Z^1 \cdots Z^n w_k) = \tilde{\mathcal{I}}_n + \tilde{\mathcal{J}}_n,$$

where the terms  $\tilde{\mathcal{I}}_n$  and  $\tilde{\mathcal{J}}_n$  (which depend on  $j$ ,  $k$  and  $n$ ) are respectively bounded as

$$\|\tilde{\mathcal{I}}_n\|_{W^{s,q}(\partial\Omega)} \leq C \|F\|_{W^{s+n-1,q}(\partial\Omega)}, \quad \|\tilde{\mathcal{J}}_n\|_{L^p(\Omega)} \leq C \|F\|_{W^{n,p}(\Omega)}$$

for all real  $s$  and all  $1 < q < \infty$ . In fact, it is slightly more convenient to prove an analogous estimate for the quantity

$$Z^1 \cdots Z^n \partial_j w_k = \mathcal{I}_n + \mathcal{J}_n;$$

this clearly suffices for our purposes as the commutator term

$$\partial_j(Z^1 \cdots Z^n w_k) - Z^1 \cdots Z^n \partial_j w_k$$

only involves  $n^{\text{th}}$  order derivatives of  $w$ , of which at least  $n - 1$  are taken along tangent directions on the boundary. The ideas of the proof are mostly standard, but we provide a sketch of the proof as we have not found a suitable reference in the literature.

Let us start with the case  $n = 1$ . We can differentiate the formula (5.9) to obtain, for any tangent vector field  $Z$ ,

$$\begin{aligned} Z \partial_j w_k^\delta(x) &= \epsilon_{klm} Z_i(x) \int_{\Omega} F_l(y) \partial_{x_i} \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\ &= -\epsilon_{klm} Z_i(x) \int_{\Omega} F_l(y) \partial_{y_i} \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\ &=: \mathcal{I}_1 + \mathcal{J}_1, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_1 &:= -\epsilon_{klm} \int_{\partial\Omega} Z(x) \cdot \nu(y) F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y), \\ \mathcal{J}_1 &:= \epsilon_{klm} Z_i(x) \int_{\Omega} \partial_i F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy. \end{aligned}$$

Writing the volume integral as

$$\begin{aligned} \mathcal{J}_1 &= \epsilon_{klm} Z_i(x) \int_{\Omega} [\partial_i F_l(y) - \partial_i F_l(x)] \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy \\ &\quad - Z F_l(x) \epsilon_{klm} A_m^{j,\delta}(x) \end{aligned}$$

to obtain a principal-value-type formula, it is clear in view of the boundedness

of  $A_m^j$  that the  $L^p$  norm of  $\mathcal{J}_1$  is controlled in terms of  $F$  with a  $\delta$ -independent constant:

$$\|\mathcal{J}_1\|_{L^p(\Omega)} \leq C\|F\|_{W^{1,p}(\Omega)}.$$

To analyze the boundary term,  $\mathcal{I}_1$ , we next restrict our attention to points  $x$  lying on  $\partial\Omega$ . We can perform the analysis locally, parametrizing a portion of the boundary as a graph using the notation (5.12), which amounts to writing  $x = (X, h(X))$ , possibly after a rotation of the coordinate axes. A basis for the space of tangent vectors at the point  $x$  is

$$T_1(X) := (1, 0, \partial_1 h(X)), \quad T_2(X) := (0, 1, \partial_2 h(X)).$$

By abuse of notation, let us denote by  $f(X)$  the value of a function  $f(x)$  at the point  $x = (X, h(X)) \in \partial\Omega$ , one can therefore write

$$Z(X) = a_1(X) T_1(X) + a_2(X) T_2(X)$$

in terms of two smooth functions  $a_j(X)$ . This implies that

$$Z(X) \cdot \nu(Y) = \frac{a_1(X) [\partial_1 h(X) - \partial_1 h(Y)] + a_2(X) [\partial_2 h(X) - \partial_2 h(Y)]}{\sqrt{1 + |\nabla h(Y)|^2}}.$$

Using the cancellation that stems from this formula it is not hard to see that, as the singular part of  $\mathcal{I}_1$  is

$$\int_{D_\rho} g(Y) F_l(Y) \left[ Z(X) \cdot \nu(Y) \eta_\delta(|X-Y|) \frac{|X-Y|^2 \delta_{jm} - 3(X_j - Y_j)(X_m - Y_m)}{|X-Y|^5} + \mathcal{O}\left(\frac{1}{|X-Y|}\right) \right] dY$$

with  $g(Y)$  a smooth function,  $\mathcal{I}_1$  defines a singular integral operator on the boundary, so we have

$$\|\mathcal{I}_1\|_{L^q(\partial\Omega)} \leq C\|F\|_{L^q(\partial\Omega)}$$

for all  $1 < q < \infty$ . Furthermore, the coefficients are smooth and the derivatives have the right singularities, so a straightforward computation shows that  $\mathcal{I}_1$  behaves like a zeroth-order pseudodifferential operator on the boundary, leading to the estimate

$$\|\mathcal{I}_1\|_{W^{s,q}(\partial\Omega)} \leq C\|F\|_{W^{s,q}(\partial\Omega)}.$$

The case  $n = 1$  then follows.

In the case of  $n \geq 2$ , the proof goes along the same lines but there is another kind of term that one needs to consider. To see this, we next consider the case  $n = 2$ , which illustrates all the difficulties that appear in the general case. For this we take another vector field  $Z'$  that is tangent to the boundary and differentiate the formula for  $Z\partial_j w_k^\delta$  along this field, thereby obtaining

$$\begin{aligned}
Z' Z \partial_j w_k^\delta(x) = & \\
& - \epsilon_{klm} Z'_n(x) \partial_{x_n} \int_{\partial\Omega} Z(x) \cdot \nu(y) F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y) \\
& + \epsilon_{klm} Z'_n(x) \partial_{x_n} \int_{\Omega} Z_i(x) \partial_i F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) dy.
\end{aligned}$$

The volume integral can be dealt with just as in the case of  $n = 1$ . Indeed, the action of  $\partial_{x_n}$  on  $Z_i(x)$  is harmless, while when  $\partial_{x_n}$  acts on the singular term, one just replaces the derivative  $\partial_{x_n}$  by  $-\partial_{y_n}$  and integrates by parts. This yields another volume integral that can be related to a principal value as above, leading to  $W^{2,p}(\Omega) \rightarrow L^p(\Omega)$  bounds, and a boundary integral with the same structure as above (with  $DF$  playing the role of  $F$ ), which leads to  $W^{s+1,q}(\partial\Omega) \rightarrow W^{s,q}(\partial\Omega)$  bounds.

The estimates for the surface integral are also as above when the derivative  $\partial_{x_n}$  acts on the field  $Z(x)$ . When it acts on the singular term, however, one needs to refine the argument a little bit. Again we start by replacing the  $\partial_{x_n}$  by  $-\partial_{y_n}$ , so we have to control the integral

$$\mathcal{I} := \int_{\partial\Omega} Z(x) \cdot \nu(y) F_l(y) \partial_{y_n} \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y).$$

The point is that, as one can decompose the  $n^{\text{th}}$  unit vector as

$$e_n = T(y) + b(y) \nu(y),$$

where  $T$  is a tangent vector and  $\nu$  is the unit normal, one can also write

$$\partial_{y_n} = T(y) + b(y) \nu(y) \cdot \nabla_y,$$

where the vector field  $T$  is here interpreted as a differential operator as before.

We can now integrate the tangent field by parts to arrive at

$$\begin{aligned}
\mathcal{I} = & - \int_{\partial\Omega} [Z(x) \cdot \nu(y)] T F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y) \\
& + \int_{\partial\Omega} \mathcal{O}(1) F_l(y) \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y) \\
& + \int_{\partial\Omega} [Z(x) \cdot \nu(y)] F_l(y) b(y) [\nu(y) \cdot \nabla_y] \partial_{x_j} \left( \eta_\delta(|x-y|) \frac{x_m - y_m}{4\pi|x-y|^3} \right) d\sigma(y).
\end{aligned}$$

The first term admits bounds  $W^{s+1,q}(\partial\Omega) \rightarrow W^{s,q}(\partial\Omega)$  just as before, the second term (which we get from tangential derivatives that act on the unit normal and from the divergence of  $T$  as a vector field on  $\partial\Omega$ ) is clearly bounded  $W^{s,q}(\partial\Omega) \rightarrow W^{s,q}(\partial\Omega)$ , and we just have to control the last integral. For this we need another cancellation, which hinges on the well-known fact that

$$\nu(y) \cdot (x - y)$$

is of order  $|x - y|^2$  when  $x, y \in \partial\Omega$ . This appears here because the singular term in the last integral is

$$\begin{aligned} & [Z(x) \cdot \nu(y)][\nu(y) \cdot \nabla_y] \partial_{x_j} \left( \eta_\delta(|x - y|) \frac{x_m - y_m}{|x - y|^3} \right) = \\ & 3\eta_\delta(|x - y|) [Z(x) \cdot \nu(y)] \left[ \nu(y) \cdot (x - y) \frac{|x - y|^2 \delta_{jm} - 5(x_j - y_j)(x_m - y_m)}{|x - y|^7} \right. \\ & \quad \left. + \frac{\nu_j(y)(x_m - y_m) + \nu_m(y)(x_j - y_j)}{|x - y|^5} \right] + \mathcal{O}\left(\frac{1}{|x - y|}\right). \end{aligned}$$

This readily yields the estimate  $W^{s+1,q}(\partial\Omega) \rightarrow W^{s,q}(\partial\Omega)$ . The general case is handled by repeatedly applying these ideas.  $\square$

### 5.3 The extension operator $\mathcal{E}_T$

In this section we construct an extension operator that will be of use in the construction of the kernel  $K_\Omega(x, y)$ . For this, let us denote by  $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}$  the signed distance to the set  $\partial\Omega$ , which is smooth in the set  $\tilde{\mathcal{U}} := \rho^{-1}((-\rho_0, \rho_0))$  provided that  $\rho_0$  is small enough. Notice that  $\tilde{\mathcal{U}}$  is a tubular neighborhood of the boundary  $\partial\Omega$ , so one can then identify  $\tilde{\mathcal{U}}$  with  $\partial\Omega \times (-\rho_0, \rho_0)$  via a diffeomorphism

$$x \in \tilde{\mathcal{U}} \mapsto (x', \rho) \in \partial\Omega \times (-\rho_0, \rho_0).$$

We will often write  $\rho_x \equiv \rho(x)$ .

Taking local normal coordinates  $X \equiv (X_1, X_2)$  on  $\partial\Omega$ , the Euclidean metric reads as

$$ds^2|_{\tilde{\mathcal{U}}} = G_\rho + d\rho^2, \quad (5.17)$$

where

$$G_\rho := h_{ij}(X, \rho) dX_i dX_j$$

defines a  $\rho$ -dependent metric on  $\partial\Omega$  that coincides with the induced surface metric on  $\partial\Omega$  at  $\rho = 0$ . Hence the volume reads as

$$dx = d\sigma_\rho(x') d\rho,$$

where  $d\sigma_\rho$  is a  $\rho$ -dependent metric on  $\partial\Omega$  that can be written in local coordinates as

$$d\sigma_\rho = \sqrt{\det(h_{ij}(X, \rho))} dX_1 dX_2.$$

Obviously the connection with the surface measure on  $\partial\Omega$  is

$$d\sigma_\rho = (1 + O(\rho)) d\sigma.$$

Consider the portion of the tubular neighborhood  $\tilde{\mathcal{U}}$  contained in  $\bar{\Omega}$ ,

$$\mathcal{U} := \tilde{\mathcal{U}} \cap \bar{\Omega} = \rho^{-1}([0, \rho_0]).$$

Let us denote by  $\omega^\perp$  the  $\rho$ -component of a vector field  $\omega$ , so that one can decompose  $\omega$  in  $\mathcal{U}$  as

$$\omega = \omega^\parallel + \omega^\perp \partial_\rho,$$

where  $\omega^\parallel$  is orthogonal to  $\partial_\rho$ . To put it differently,

$$\omega^\perp := \omega \cdot \nabla \rho, \quad \omega^\parallel := \omega - \omega^\perp \nabla \rho.$$

Consider the operator  $T_0$  defined, for  $x \in \partial\Omega$ , by the integral

$$T_0 f(x) := \int_{\partial\Omega} \frac{(x-y) \cdot \nu(x)}{4\pi|x-y|^3} f(y) d\sigma(y). \quad (5.18)$$

It is well known (see e.g. [73, Section 7.11]) that, under the assumption that the boundary is smooth,  $T_0$  is a pseudodifferential operator on  $\partial\Omega$  of order  $-1$ . (More generally, notice that for a domain with  $C^2$  boundary, the kernel of the operator is bounded by  $C/|x-y|$ .) In this chapter we will employ operators on  $\partial\Omega$  of the form

$$Tf(x) = \int_{\partial\Omega} K_T(x, y) f(y) d\sigma(y) \quad (5.19)$$

with

$$K_T(x, y) := \frac{(x-y) \cdot \nu(x)}{4\pi|x-y|^3} + K'_T(x, y) \quad (5.20)$$

and  $K'_T$  bounded on  $\partial\Omega \times \partial\Omega$ . We will also assume that the derivative of  $K'_T(x, y)$  is bounded by  $\frac{C}{|x-y|}$ .

If  $\chi(t)$  is a smooth cut-off function that is equal to 1 for  $t < \rho_0/2$  and that vanishes for  $t > \rho_0$ , one can exploit the above identification of  $\mathcal{U}$  with  $\partial\Omega \times [0, \rho_0]$  to define a vector field  $\mathcal{E}_T \omega : \bar{\Omega} \rightarrow \mathbb{R}^3$  as

$$\mathcal{E}_T \omega(x) := \left[ \chi(\rho_x) \int_{\partial\Omega} \left( \frac{(x'-y) \cdot \nu(x')}{4\pi|x-y|^3} + K'_T(x', y) \right) \omega^\perp(y, \rho_x) d\sigma_{\rho_x}(y) \right] \partial_\rho, \quad (5.21)$$

where  $K'_T$  is the kernel defined in (5.20). Notice that  $\mathcal{E}_T \omega$  is supported in  $\mathcal{U}$ . With some abuse of notation, we will also denote by  $K_T(x, y)$  the extension of the kernel (5.20) to  $\bar{\Omega} \times \partial\Omega$  as it appears in the integral of (5.21).

The basic properties of the extension operator that we will need later are the following:

**Proposition 5.6.** *Let  $T$  be an operator of the form (5.19). For any divergence-free vector field  $\omega \in C^1(\bar{\Omega})$  one has*

$$T(\omega \cdot \nu) = (\mathcal{E}_T \omega) \cdot \nu$$

on  $\partial\Omega$ , where  $\mathcal{E}_T \omega$  is defined by (5.21), and for each  $x \in \Omega$  the divergence of  $\mathcal{E}_T \omega$  can be written as

$$\nabla \cdot \mathcal{E}_T \omega(x) = \int_{\partial\Omega} K_{T, \text{div}}(x, y) \omega(y, \rho_x) d\sigma_{\rho_x}(y)$$

with a kernel of the form

$$\begin{aligned} K_{T,\text{div}}(x, y) \omega(y, \rho_x) &:= \chi(\rho_x) \left[ \frac{3(x' - y) \cdot \nu(x') (y - x) \cdot \nu(x')}{4\pi|x - y|^5} \omega^\perp(y, \rho_x) \right. \\ &\quad \left. + \frac{3(x' - y) \cdot \nu(x') (x - y) - |x - y|^2 \nu(x')}{4\pi|x - y|^5} \cdot \omega^\parallel(y, \rho_x) \right] + \tilde{K}_T(x, y) \omega(y, \rho_x). \end{aligned} \quad (5.22)$$

where  $|\tilde{K}_T(x, y)| \leq \frac{C}{|x-y|}$  and is supported in  $\mathcal{U} \times \partial\Omega$ .

*Proof.* Since  $x \notin \partial\Omega$ , one can easily compute the divergence of  $\tilde{\omega} := \mathcal{E}_T \omega$  as

$$\begin{aligned} \nabla \cdot \tilde{\omega}(x) &= \partial_{\rho_x} \left[ \chi(\rho_x) \int_{\partial\Omega} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} + K'_T(x', y) \right) \omega^\perp(y, \rho_x) d\sigma_{\rho_x}(y) \right] \\ &= \chi(\rho_x) J_1 + J_2, \end{aligned}$$

where

$$J_1 := \int_{\partial\Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \omega^\perp(y, \rho_x) \right) d\sigma_{\rho_x}(y)$$

and  $J_2$  denotes a term of the form

$$J_2 = \int_{\partial\Omega} \tilde{K}_T(x, y) \omega(y, \rho_x) d\sigma(y)$$

with a kernel  $\tilde{K}_T$  as in the statement.

To simplify the expression of  $J_1$ , we shall use that, by (5.17), one can write the divergence of  $\omega$  as

$$0 = \nabla \cdot \omega(y, \rho_x) = \partial_{\rho_x} \omega^\perp(y, \rho_x) + \nabla^\parallel \cdot \omega^\parallel(y, \rho_x),$$

where  $\nabla^\parallel \cdot \omega^\parallel(y, \rho_x)$  is the divergence of the field  $\omega^\parallel$  (understood as a tangent vector field on  $\partial\Omega$ ) with respect to the divergence operator on  $\partial\Omega$  associated with the measure  $d\sigma_{\rho_x}$ . This allows us to write

$$\begin{aligned} J_1 &= \int_{\partial\Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \right) \omega^\perp(y, \rho_x) d\sigma_{\rho_x}(y) \\ &\quad + \int_{\partial\Omega} \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \partial_{\rho_x} \omega^\perp(y, \rho_x) d\sigma_{\rho_x}(y) \\ &= \int_{\partial\Omega} \partial_{\rho_x} \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \right) \omega^\perp(y, \rho_x) d\sigma_{\rho_x}(y) \\ &\quad - \int_{\partial\Omega} \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \nabla^\parallel \cdot \omega^\parallel(y, \rho_x) d\sigma_{\rho_x}(y) \\ &= \int_{\partial\Omega} (\omega^\perp(y, \rho_x) \partial_{\rho_x} + \omega^\parallel(y, \rho_x) \cdot \nabla) \left( \frac{(x' - y) \cdot \nu(x')}{4\pi|x - y|^3} \right) d\sigma_{\rho_x}(y) \end{aligned}$$

$$= \int_{\partial\Omega} \left[ \frac{3(x' - y) \cdot \nu(x') (y - x) \cdot \nu(x')}{4\pi|x - y|^5} \omega^\perp(y, \rho_x) + \frac{3(x' - y) \cdot \nu(x') (x - y) - |x - y|^2 \nu(x')}{4\pi|x - y|^5} \cdot \omega^\parallel(y, \rho_x) \right] d\sigma_{\rho_x}(y),$$

where we have integrated by parts the tangential divergence  $\nabla^\parallel \cdot \omega^\parallel$ . The proposition then follows.  $\square$

## 5.4 Construction of the solution

Given the divergence-free field  $\omega$  in  $W^{k,p}(\Omega)$  (which, by density, we can and will assume to be in fact of class  $C^\infty(\overline{\Omega})$ ), our objective in this section is to construct a vector field  $u$  that satisfies the equations

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0$$

in  $\Omega$  and is tangent to the boundary. To this end, let us start by considering the vector field  $v$  on  $\Omega$  defined as

$$v(x) := v_1(x) + v_2(x),$$

$$v_1(x) := \int_{\Omega} \frac{\omega(y) \times (x - y)}{4\pi|x - y|^3} dy, \quad (5.23)$$

$$v_2(x) := - \int_{\Omega} \frac{\nabla \mathcal{S}f(y) \times (x - y)}{4\pi|x - y|^3} dy, \quad (5.24)$$

where for  $x \in \Omega$  we define the scalar function  $\mathcal{S}f$  through the single layer potential

$$\mathcal{S}f(x) := - \int_{\partial\Omega} \frac{f(y)}{4\pi|x - y|} d\sigma(y),$$

with  $f$  a function on  $\partial\Omega$  to be determined. The basic properties of the single layer potential can be found e.g. in [35, Chapter 3].

It follows from Proposition 5.4 that  $v$  is a divergence-free field whose curl is given by

$$\begin{aligned} \nabla \times v(x) &= \omega(x) - \nabla \mathcal{S}f(x) + \nabla \int_{\partial\Omega} \frac{\partial_\nu \mathcal{S}f(y) - \omega \cdot \nu(y)}{4\pi|x - y|} d\sigma(y) \\ &= \omega(x) + \nabla \int_{\partial\Omega} \frac{f(y) + \partial_\nu \mathcal{S}f(y) - \omega \cdot \nu(y)}{4\pi|x - y|} d\sigma(y), \end{aligned} \quad (5.25)$$

where we have used that  $\Delta(\mathcal{S}f) = 0$  in  $\Omega$  for any function  $f$ . As a side remark, observe that since we have defined  $\mathcal{S}f$  only on  $\Omega$ , the fact that the derivative of the extension of  $\mathcal{S}f$  to the whole space  $\mathbb{R}^3$  is discontinuous across  $\partial\Omega$  does not play a role here. Consequently,  $\partial_\nu \mathcal{S}f(x)$  will necessarily stand for the interior derivative of  $\mathcal{S}f$  at the boundary point  $x$  in the direction of the outer normal,

that is,

$$\partial_\nu \mathcal{S}f(x) := \lim_{y \in \Omega, y \rightarrow x} \nu(x) \cdot \nabla \mathcal{S}f(y),$$

where for all  $y \in \Omega$  one has

$$\nabla \mathcal{S}f(y) = \int_{\partial\Omega} f(z) \frac{y - z}{4\pi|y - z|^3} d\sigma(z).$$

Consider the operator  $T_0$  defined in (5.18), which appears in the analysis of the normal derivative of  $\mathcal{S}f$  at the boundary through the formula

$$\partial_\nu \mathcal{S}f = (T_0 - \frac{1}{2}I)f.$$

In view of equation (5.25), our goal now is to choose the function  $f$  so that

$$f + \partial_\nu \mathcal{S}f - \omega \cdot \nu = 0,$$

or equivalently

$$(\frac{1}{2}I + T_0)f = \omega \cdot \nu. \quad (5.26)$$

Since  $\frac{1}{2}I + T_0$  is precisely the operator that one needs to invert in order to solve the exterior Neumann boundary value problem for the Laplacian, it is well known (see e.g. [35, Section 3.E]) that there is a function  $f$  satisfying (5.26) if and only if

$$\int_{\partial\Omega_j} \omega \cdot \nu d\sigma = 0, \quad 1 \leq j \leq m - 1. \quad (5.27)$$

Here  $\Omega_1, \dots, \Omega_{m-1}$  are the bounded connected components of  $\mathbb{R}^3 \setminus \bar{\Omega}$ . Since each  $\partial\Omega_j$  is a connected component of  $\partial\Omega$ , the hypothesis (5.2) ensures that the condition (5.27) holds, so one can find a function  $f$  on  $\partial\Omega$  satisfying (5.26). Notice, moreover, that (5.26) implies that  $f$  can be written as

$$f = (2I - 4T)(\omega \cdot \nu), \quad (5.28)$$

with  $T$  an operator of the form (5.19).

Since  $T$  is a pseudodifferential operator on  $\partial\Omega$  of order  $-1$ , equation (5.28) yields the estimate

$$\|f\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} \leq C\|\omega \cdot \nu\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)},$$

and therefore, by the properties of the single layer potential,

$$\|\mathcal{S}f\|_{W^{k+1,p}(\Omega)} \leq C\|f\|_{W^{k-\frac{1}{p},p}(\partial\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}.$$

With this function  $f$ , by construction one then has that

$$\nabla \times v = \omega, \quad \nabla \cdot v = 0,$$



while Proposition 5.5 ensures that  $v$  is bounded as

$$\|v\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)} + C\|\nabla \mathcal{S}f\|_{W^{k,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}.$$

Hence one can now find a solution to the problem (5.1) by setting

$$u := v - \nabla \varphi, \quad (5.29)$$

where  $\varphi$  is a solution to the Neumann boundary value problem

$$\Delta \varphi = 0 \quad \text{in } \Omega, \quad \partial_\nu \varphi|_{\partial\Omega} = v \cdot \nu.$$

It is well-known that the solution exists and is unique up to an additive constant because

$$\int_{\partial\Omega} v \cdot \nu \, d\sigma = \int_{\Omega} \nabla \cdot v \, dx = 0.$$

Moreover, the function  $\varphi$  can be written as a single layer potential

$$\varphi = \mathcal{S}g,$$

where the function  $g$  and  $v \cdot \nu$  are related through the operator  $T_0$  as

$$\left(\frac{1}{2}I - T_0\right)g = -v \cdot \nu.$$

Therefore,

$$g = -(2I + 4\tilde{T})(v \cdot \nu) \quad (5.30)$$

with  $\tilde{T}$  a pseudodifferential operator on  $\partial\Omega$  of the form (5.19), so

$$\|g\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|v \cdot \nu\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \leq C\|v\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}$$

By the properties of single layer potentials it then follows that

$$\|\nabla \varphi\|_{W^{k+1,p}(\Omega)} \leq \|\mathcal{S}g\|_{W^{k+2,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}.$$

Hence in this section we have proved the following:

**Theorem 5.7.** *The field  $u$  given by (5.29) solves the problem (5.1) and satisfies the estimate (5.4).*

## 5.5 Existence and bounds for the integral kernel

In this section we show that the solution  $u$  is actually obtained as the integral of the product of a matrix-valued integral kernel and the field  $\omega$ . Specifically, in terms of the function  $\ell$  defined in (5.3), we aim to prove the following:

**Theorem 5.8.** *The solution  $u$  constructed in Theorem 5.7 (see equation (5.29)) is of the form*

$$u(x) = \int_{\Omega} K_{\Omega}(x, y) \omega(y) dy$$

for some matrix-valued kernel satisfying  $|K_{\Omega}(x, y)| \leq C \frac{\ell(y)}{|x-y|^2}$ .

Without loss of generality, we can assume that  $\omega \in C^1(\overline{\Omega})$ . We will construct the kernel by treating separately the three summands appearing in the decomposition

$$u = v_1 + v_2 - \nabla\varphi$$

presented in equations (5.23), (5.24) and (5.29). Since

$$v_1(x) = \int_{\Omega} K_{\Omega}^1(x, y) \omega(y) dy$$

with

$$K_{\Omega}^1(x, y) \omega(y) := \frac{\omega(y) \times (x - y)}{4\pi|x - y|^3},$$

we will only need to show that  $v_2$  and  $\nabla\varphi$  can also be written in a similar fashion.

### Step 1: The first term of the kernel of $v_2$

Let us start with  $v_2$ , which by equations (5.24) and (5.28) can be written as the limit as  $\delta \rightarrow 0$  of the functions

$$\begin{aligned} v_2^{\delta}(x) &:= \frac{1}{8\pi^2} \int_{\Omega} \int_{\partial\Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} [\omega \cdot \nu(z) - 2T(\omega \cdot \nu)(z)] d\sigma(z) dy \\ &=: \frac{V_1(x) - 2V_2(x)}{8\pi^2}, \end{aligned} \quad (5.31)$$

with  $T$  a pseudodifferential operator on  $\partial\Omega$  of order  $-1$  and of the form (5.19).

In this subsection we work out the details for  $V_1(x)$ :

$$V_1(x) := \int_{\Omega} \int_{\partial\Omega \setminus B_{\delta}(y)} \frac{z - y}{|z - y|^3} \times \frac{x - y}{|x - y|^3} \omega \cdot \nu(z) d\sigma(z) dy. \quad (5.32)$$

Integrating by parts and using the fact that the divergence of  $\omega$  is zero, one readily obtains that

$$V_1(x) = I_1 + I_2 \quad (5.33)$$

with

$$\begin{aligned}
I_1 &:= \int_{\Omega} \int_{\Omega \setminus B_{\delta}(y)} \partial_{z_j} \left( \frac{z-y}{|z-y|^3} \right) \times \frac{x-y}{|x-y|^3} \omega_j(z) dz dy \\
&= - \int_{\Omega} \int_{\Omega \setminus B_{\delta}(y)} \frac{x-y}{|x-y|^3} \times \left( \frac{|z-y|^2 I - 3(z-y) \otimes (z-y)}{|z-y|^5} \cdot \omega(z) \right) dz dy,
\end{aligned} \tag{5.34}$$

$$I_2 := \int_{\Omega} \int_{\Omega \cap \partial B_{\delta}(y)} \frac{z-y}{|z-y|^3} \times \frac{x-y}{|x-y|^3} \omega \cdot \nu(z) d\sigma(z) dy.$$

The dot in the second line of  $I_1$  denotes the multiplication of the vector field  $\omega(z)$  by a symmetric matrix and  $\nu$  in  $I_2$  denotes the outward normal of  $\partial B_{\delta}(y)$ . The next two lemmas describe the behaviour of the integrals  $I_1$  and  $I_2$ .

**Lemma 5.9.**

$$\lim_{\delta \rightarrow 0} I_2 = 0.$$

*Proof.* For this, observe that for  $\omega \in C^1(\overline{\Omega})$  one has

$$\int_{\Omega \cap \partial B_{\delta}(y)} \omega(z) \cdot \nu(z) \frac{z-y}{|z-y|^3} d\sigma(z) = \omega(y) \cdot \int_{\Omega \cap \partial B_{\delta}(y)} \frac{\nu(z) \otimes (z-y)}{|z-y|^3} d\sigma(z) + O(\delta).$$

To analyze the remaining integral, we shall begin by noticing that, in terms of the variable  $\Theta := \frac{1}{\delta}(z-y)$ , the set  $\Omega \cap \partial B_{\delta}(y)$  can be described as

$$\Omega \cap \partial B_{\delta}(y) = \{\Theta \in S_{y,\delta}\},$$

where  $S_{y,\delta}$  is a subset of the unit sphere  $\mathbb{S}^2 \subset \mathbb{R}^3$  with

$$S_{y,\delta} = \mathbb{S}^2 \quad \text{if } \text{dist}(y, \partial\Omega) > \delta.$$

Denoting by  $d\Theta$  the canonical area form on the unit sphere, it is clear that

$$\int_{\Omega \cap \partial B_{\delta}(y)} \frac{\nu(z) \otimes (z-y)}{|z-y|^3} d\sigma(z) = \int_{S_{y,\delta}} (\Theta \otimes \Theta) d\Theta.$$

Since the norm of the latter integral is obviously bounded by a constant that does not depend on  $y$  or  $\delta$  and the integral is zero when  $S_{y,\delta} = \mathbb{S}^2$ , it then follows that

$$|I_2| \leq C \int_{\Omega_{\delta}} \frac{|\omega(y)|}{|x-y|^2} dy + C\delta \leq C\delta,$$

with  $\Omega_{\delta} := \{y \in \Omega : \text{dist}(y, \partial\Omega) \leq \delta\}$ , thereby proving that the boundary term vanishes in the limit  $\delta \rightarrow 0$ .  $\square$

**Lemma 5.10.**  $I_1$  can be written as

$$I_1 = \int_{\Omega} K_{\delta}(x, z) \omega(z) dz \tag{5.35}$$

with  $K_\delta$  a matrix kernel satisfying

$$|K_\delta(x, z)| \leq \frac{C \ell(z)}{|x - z|^2}. \quad (5.36)$$

*Proof.* If we write (5.34) as (5.35), then

$$K_\delta(x, z) := - \int_{\Omega \setminus B_\delta(z)} \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 I - 3(z - y) \otimes (z - y)}{|z - y|^5} dy,$$

where the cross product of a vector field with a matrix has the obvious meaning. In order to prove estimate (5.36), let us take some small but fixed number  $a > 0$ . Clearly,

$$\left| \int_{\Omega \setminus B_a(z)} \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 I - 3(z - y) \otimes (z - y)}{|z - y|^5} dy \right| \leq C, \quad (5.37)$$

so it suffices to study the behavior of the integral

$$I_3 := \int_{\Omega \cap (B_a(z) \setminus B_\delta(z))} \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 I - 3(z - y) \otimes (z - y)}{|z - y|^5} dy. \quad (5.38)$$

Let  $\rho_z$  be the distance between the point  $z$  and the boundary. We study  $I_3$  separately depending on whether  $\rho_z$  is less or greater than  $a$ .

*First case:*  $\rho_z \geq a$ . Setting

$$e := \frac{x - z}{R} \quad \text{and} \quad R := |x - z| \quad (5.39)$$

and defining

$$w := \frac{y - z}{R},$$

one has

$$\begin{aligned} I_3 &= \int_{B_a(z) \setminus B_\delta(z)} \frac{x - y}{|x - y|^3} \times \frac{|z - y|^2 I - 3(z - y) \otimes (z - y)}{|z - y|^5} dy \\ &= \frac{1}{R^2} \int_{B_{a/R} \setminus B_{\delta/R}} \frac{e - w}{|e - w|^3} \times \frac{|w|^2 I - 3w \otimes w}{|w|^5} dw. \end{aligned}$$

This integral is uniformly convergent as  $R \rightarrow 0$  because the integrand is bound by  $C|w|^{-5}$  for large  $w$ . For small  $R_0$  and  $\delta < R R_0$  one has the asymptotic behavior

$$\begin{aligned} I_3 &\leq \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} (e + O(w)) \times \frac{|w|^2 I - 3w \otimes w}{|w|^5} dw \\ &= e \times \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} \frac{|w|^2 I - 3w \otimes w}{|w|^5} dw + \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} O(|w|^{-2}) dw \\ &= \frac{1}{R^2} \int_{B_{R_0} \setminus B_{\delta/R}} O(|w|^{-2}) dw < \frac{C}{R^2} \end{aligned}$$

with  $C$  a constant independent of  $\delta$ . Here we have used that the first integral in the second line vanishes because the average of the matrix  $w \otimes w$  over any sphere is  $-\frac{1}{3}I$ . This shows that

$$|I_3| \leq \frac{C}{|x - z|^2} \quad (5.40)$$

whenever the distance between  $z$  and  $\partial\Omega$  is at least  $a$ .

*Second case:  $\rho_z < a$ .* A similar argument yields the same bound but with a constant that can grow as  $\ell(z)$ . In this case, one begins by straightening out the boundary, so we locally identify the boundary with an horizontal plane. This amounts to saying that there is a diffeomorphism  $\Phi_z$  of the ball  $B_a$  such that

$$\Omega \cap (B_a(z) \setminus B_\delta(z)) = \{z + \Phi_z(w) : w \in \mathcal{B}_{a,\delta,\rho_z}\},$$

with  $\mathcal{B}_{a,\delta,r}$  being the intersection of the annulus of outer radius  $a$  and inner radius  $\delta$  with the half-space of points whose third coordinate is at most  $r$ :

$$\mathcal{B}_{a,\delta,r} := \{w = (w_1, w_2, w_3) \in \mathbb{R}^3 : \delta < |w| < a, w_3 < r\} \quad (5.41)$$

Furthermore, upon choosing a suitable orientation of the axes one can take

$$\|\Phi_z - I\|_{C^k(B_a)} < Ca$$

and one can assume without loss of generality that

$$\Phi_z(0) = 0.$$

In this case one can write

$$\begin{aligned} I_3 &= \int_{\mathcal{B}_{a,\delta,\rho_z}} \frac{x - z - \Phi_z(w)}{|x - z - \Phi_z(w)|^3} \times \frac{|\Phi_z(w)|^2 I - 3\Phi_z(w) \otimes \Phi_z(w)}{|\Phi_z(w)|^5} |\det D\Phi_z(w)| dw \\ &= \int_{\mathcal{B}_{a,\delta,\rho_z}} \frac{x - z - w}{|x - z - w|^3} \times \frac{|w|^2 I - 3w \otimes w}{|w|^5} dw + \int_{\mathcal{B}_{a,\delta,\rho_z}} \frac{x - z - w}{|x - z - w|^3} \times \frac{O(1)}{|w|^2} dw \\ &=: I_{31} + I_{32}. \end{aligned}$$

One can introduce the variable  $q := \frac{w}{R}$  and define  $e$  and  $R$  as in equation (5.39). The second integral  $I_{32}$  can be readily bounded by  $C/R^2$ , while one can argue as before to obtain

$$\begin{aligned} I_{31} &= \frac{1}{R^2} \int_{\mathcal{B}_{a/R,\delta/R,\rho_z/R}} \frac{e - q}{|e - q|^3} \times \frac{|q|^2 I - 3q \otimes q}{|q|^5} dq \\ &= e \times \frac{1}{R^2} \int_{\mathcal{B}_{a/R,\delta/R,\rho_z/R}} \frac{|q|^2 I - 3q \otimes q}{|q|^5} dq. \end{aligned}$$

The point now is that

$$I_4 := \int_{\mathcal{B}_{a/R,\delta/R,\rho_z/R}} \frac{|q|^2 I - 3q \otimes q}{|q|^5} dq \quad (5.42)$$

is bounded by  $C \log(2 + 1/\rho_z)$ , with  $C$  a constant that does not depend on  $\delta$ , which proves the  $\delta$ -independent bound

$$|I_{31}| < \frac{C \ell(z)}{|x - y|^2}. \quad (5.43)$$

In order to see this, let us take spherical coordinates  $(r, \theta, \phi)$ :

$$q =: \frac{r}{R} \Theta(\theta, \phi), \quad \Theta(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

and use again the shorthand notation

$$d\Theta := \sin \theta \, d\theta \, d\phi$$

for the surface measure on the unit sphere. With  $\theta_z \in (0, \frac{\pi}{2})$  defined by

$$\cos \theta_z = \frac{\rho_z}{a},$$

the integral (5.42) can be written as

$$\begin{aligned} I_4 &= \int_0^{2\pi} \int_0^{\theta_z} \int_{\delta}^{\frac{\rho_z}{\cos \theta}} \frac{I - 3\Theta \otimes \Theta}{r} \, dr \, \sin \theta \, d\theta \, d\phi \\ &\quad + \int_0^{2\pi} \int_{\theta_z}^{\pi} \int_{\delta}^a \frac{I - 3\Theta \otimes \Theta}{r} \, dr \, \sin \theta \, d\theta \, d\phi \end{aligned}$$

Since

$$\int_{\delta}^{\frac{\rho_z}{\cos \theta}} \frac{dr}{r} = \log \frac{a}{\delta} + \log \frac{\rho_z}{a \cos \theta}, \quad \int_{\delta}^a \frac{dr}{r} = \log \frac{a}{\delta},$$

one obtains

$$\begin{aligned} |I_4| &= \left| \log \frac{a}{\delta} \int_{\mathbb{S}^2} (I - 3\Theta \otimes \Theta) \, d\Theta + \int_0^{2\pi} \int_0^{\theta_z} (I - 3\Theta \otimes \Theta) \log \frac{\rho_z}{a \cos \theta} \sin \theta \, d\theta \, d\phi \right| \\ &= \left| \int_0^{2\pi} \int_0^{\theta_z} (I - 3\Theta \otimes \Theta) \log \frac{\rho_z}{a \cos \theta} \sin \theta \, d\theta \, d\phi \right| \\ &\leq C(1 + |\log \rho_z|) \leq C\ell(z), \end{aligned}$$

where  $C$  is independent of  $\delta$  and to pass to the second line we have used again that the integral of  $I - 3\Theta \otimes \Theta$  over the sphere vanishes. This completes the proof of (5.43).

The bounds for  $I_{31}$  and  $I_{32}$  then imply that

$$|I_3| \leq \frac{C\ell(z)}{|x - z|^2} \quad (5.44)$$

if the distance between  $z$  and  $\partial\Omega$  is less than  $a$ . Hence the kernel  $K_\delta$  satisfies the estimate (5.36) and the lemma then follows.  $\square$

By estimate (5.36) the kernel

$$K_{\Omega}^{21}(x, z) := \frac{1}{8\pi^2} \lim_{\delta \rightarrow 0} K_{\delta}(x, z),$$

which is formally given by

$$K_{\Omega}^{21}(x, z) = -\frac{1}{8\pi^2} \text{PV} \int_{\Omega} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3(z-y) \otimes (z-y)}{|z-y|^5} dy,$$

is a well-defined kernel bounded as

$$|K_{\Omega}^{21}(x, z)| \leq \frac{C\ell(z)}{|x-z|^2}.$$

Moreover, it then follows from Lemmas 5.9 and 5.10 that  $K_{\Omega}^{21}(x, z)$  satisfies

$$\lim_{\delta \rightarrow 0} V_1(x) = \int_{\Omega} K_{\delta}^{21}(x, z) w(z) dz. \quad (5.45)$$

## Step 2: The second term of the kernel of $v_2$

To complete our analysis of  $v_2$ , it remains to consider the term in equation (5.31) having the quantity  $T(\omega \cdot \nu)$ , which we have called  $V_2(x)$ . Our goal is to show that

$$\lim_{\delta \rightarrow 0} V_2(x) = \int_{\Omega} K_{\Omega}^{22}(x, z) \omega(z) dz$$

for some kernel that we will bound as  $|K_{\Omega}^{22}(x, z)| \leq C \frac{\ell(z)}{|x-z|}$ .

Let us begin by using Proposition 5.6 to write

$$T(\omega \cdot \nu) = \tilde{\omega} \cdot \nu$$

in  $\partial\Omega$ , where

$$\tilde{\omega} := \mathcal{E}_T \omega$$

is the extension of  $T(\omega \cdot \nu)$  associated to the operator  $T$  defined by the integral (5.21). To analyze the term  $V_2(x)$  in equation (5.31), let us write

$$V_2(x) = \int_{\Omega} \int_{\partial\Omega \setminus B_{\delta}(y)} \frac{z-y}{|z-y|^3} \times \frac{x-y}{|x-y|^3} (\tilde{\omega} \cdot \nu)(z) d\sigma(z) dy,$$

where of course the integration variables  $(y, z)$  range over  $\Omega \times \partial\Omega$ . Integrating by parts one then finds

$$V_2(x) = I_5 + I_6 - I_7,$$

where

$$I_5 := \int_{\Omega} \int_{\Omega \setminus B_{\delta}(y)} \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 I - 3(z-y) \otimes (z-y)}{|z-y|^5} \cdot \tilde{\omega}(z) dy dz,$$

$$\begin{aligned}
I_6 &:= \int_{\Omega} \int_{\Omega \setminus B_{\delta}(y)} \frac{z-y}{|z-y|^3} \times \frac{x-y}{|x-y|^3} \nabla \cdot \tilde{\omega}(z) dy dz, \\
I_7 &:= \int_{\Omega} \int_{\Omega \cap \partial B_{\delta}(y)} \frac{z-y}{|z-y|^3} \times \frac{x-y}{|x-y|^3} (\tilde{\omega} \cdot \nu)(z) d\sigma(z) dy.
\end{aligned} \tag{5.46}$$

Let us discuss the structure of these integrals in the following lemmas.

**Lemma 5.11.**

$$\lim_{\delta \rightarrow 0} I_7 = 0.$$

*Proof.* Arguing as in the case of  $I_2$  in Lemma 5.9 the result follows.  $\square$

**Lemma 5.12.**  $I_5$  can be written as

$$I_5 = \int_{\Omega} \mathcal{K}_{\delta}(x, \bar{w}) \omega(z) dz \tag{5.47}$$

with  $\mathcal{K}_{\delta}$  a matrix kernel satisfying

$$|\mathcal{K}_{\delta}(x, \bar{w})| \leq \frac{C \ell(\bar{w})}{|x - \bar{w}|}. \tag{5.48}$$

*Proof.* Notice that, identifying  $\mathcal{U}$  with  $\partial\Omega \times [0, \rho_0)$  via the coordinates

$$z \mapsto (z', \rho_z)$$

as in Section 5.3, we can write (5.21) as

$$\tilde{\omega}(z) = \nabla \rho(z) \int_{\partial\Omega} K_T(z, w) \omega^{\perp}(w, \rho_z) d\sigma(w)$$

with  $K_T(z, w)$  the kernel which extends (5.20) to  $\bar{\Omega} \times \partial\Omega$ . Then we can write  $I_5$  as (5.47) with

$$\begin{aligned}
\mathcal{K}_{\delta}(x, \bar{w}) &:= \int_{\delta} K_T(z, w) \left( \frac{x-y}{|x-y|^3} \times \frac{|z-y|^2 \nabla \rho(z) - 3(z-y) \cdot \nabla \rho(z) (z-y)}{|z-y|^5} \right) \\
&\quad \otimes \nabla \rho(\bar{w}) dy d\sigma(z') \tag{5.49}
\end{aligned}$$

provided that the latter exists. We are denoting by  $\bar{w}$  the point in  $\mathcal{U}$  of coordinates  $(w, \rho_z)$ , and the subscript  $\delta$  in the integral will henceforth mean that we integrate over points  $(y, z')$  such that

$$|y - z| > \delta \quad \text{and} \quad |w - z'| > \delta.$$

Of course,  $(y, z') \in \Omega \times \partial\Omega$ , but in what follows we shall not explicitly write the domain of integration (which will be apparent) to keep the notation simple.

To analyze the behavior of this integral, it suffices to analyze the behavior of the integrand at the points where it can be not uniformly in  $L^1_{\text{loc}}$  (that is, in the region where  $|x - y|$  is small). To this end, in addition to exploiting the



diffeomorphism  $\mathcal{U} \rightarrow \partial\Omega \times [0, \rho_0)$  we will take local normal coordinates on  $\partial\Omega$ , thereby identifying a point  $x$  in the interior of  $\mathcal{U}$  (with  $x'$  in certain open subset of  $\partial\Omega$ ) with the triple  $(X, \rho_x)$ , with  $X \equiv (X_1, X_2)$ . Since we are using normal coordinates on  $\partial\Omega$ , it is standard that, given two points  $x, y$  of coordinates  $(X, \rho_x), (Y, \rho_y)$ , one has

$$|x - y|^2 = (\rho_x - \rho_y)^2 + |X - Y|^2 + \text{h.o.t.}, \quad (5.50)$$

where in what follows we write h.o.t. to denote higher order terms. Hence by the extension of equation (5.20) the kernel  $K_T(z, w)$  can be written in these coordinates as

$$K_T(z, w) = \frac{1}{4\pi} \frac{q(\zeta)}{(\rho_z^2 + |\zeta|^2)^{3/2}} + \text{h.o.t.}, \quad (5.51)$$

where  $\zeta := Z - W$  and  $q(\zeta) := q_{ij}\zeta_i\zeta_j$  is a quadratic form (here we have used that  $(z' - w) \cdot \nu(z')$  vanishes to second order). Let us set  $\tilde{y} := y - z$  and write

$$|x - y| = |x - z - \tilde{y}| = |x - \bar{w} - \zeta - \tilde{y}| + \text{h.o.t.},$$

where we have identified  $\zeta$  with the point of coordinates  $(\zeta, 0)$  and made use of (5.50). This permits us to write

$$\mathcal{K}_\delta(x, \bar{w}) = \int_\delta \frac{q(\zeta)}{(\rho_z^2 + |\zeta|^2)^{3/2}} \frac{x - \bar{w} - \zeta - \tilde{y}}{|x - \bar{w} - \zeta - \tilde{y}|^3} \times \frac{|\tilde{y}|^2 e_3 - 3\tilde{y}\tilde{y}_3}{4\pi|\tilde{y}|^5} \otimes e_3 d\tilde{y} d\zeta + \text{h.o.t.},$$

where the subscript  $\delta$  now refers to the region

$$|\tilde{y}| > \delta, \quad |x - \bar{w} - \zeta - \tilde{y}| > \delta.$$

Let us now write

$$e := \frac{x - \bar{w}}{R}, \quad R := |x - \bar{w}|$$

and observe that, for small  $R$ , one can rescale the variables as

$$\tilde{y}' := \frac{\tilde{y}}{R}, \quad \zeta' := \frac{\zeta}{R}, \quad \rho'_z := \frac{\rho_z}{R}$$

and write the above integral as

$$\mathcal{K}_\delta(x, \bar{w}) = \frac{1}{R} \int_{\delta/R} \frac{q(\zeta')}{(\rho_z'^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}'}{|e - \zeta' - \tilde{y}'|^3} \times \frac{|\tilde{y}'|^2 e_3 - 3\tilde{y}'\tilde{y}'_3}{4\pi|\tilde{y}'|^5} \otimes e_3 d\tilde{y}' d\zeta' + \text{h.o.t.},$$

where of course the rescaled domains of integration become unbounded in the limit  $R \rightarrow 0$ .

Our goal now is to show that

$$I_8 := \int_{\delta/R} \frac{q(\zeta')}{(\rho_z'^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}'}{|e - \zeta' - \tilde{y}'|^3} \times \frac{|\tilde{y}'|^2 e_3 - 3\tilde{y}'\tilde{y}'_3}{|\tilde{y}'|^5} d\tilde{y}' d\zeta' \quad (5.52)$$

is bounded as

$$|I_8| \leq C \ell(\bar{w})$$

uniformly as  $\delta \rightarrow 0$ . Observe that this will show (5.48).

To prove the bound for  $I_8$  it suffices to analyze the behavior of the integrand at the points where it can be not uniformly in  $L_{\text{loc}}^1$  (that is, in a neighborhood of the regions  $e - \zeta' - \tilde{y}' = 0$  and  $\tilde{y} = 0$ ). Let us start with this first case. Around  $e - \zeta' - \tilde{y}' = 0$  one can write

$$e - \tilde{y}' = (B, b),$$

with  $B \equiv (B_1, B_2) \in \mathbb{R}^2$ . It is clear from that it is enough to consider values of  $\tilde{y}$  that are close to zero, say with  $|\tilde{y}| < \frac{1}{4}$ . It then follows that

$$|B|^2 + b^2 \geq \frac{1}{2}, \quad (5.53)$$

so that the integral in  $\zeta'$  can be written in terms of the variable

$$E := \zeta' - B$$

as

$$\begin{aligned} I_9 &:= \int_{\delta/R} \frac{q(\zeta')}{(\rho_z^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}'}{|e - \zeta' - \tilde{y}'|^3} d\zeta' \\ &= \int_{D_{a/R}(B) \setminus D_{\delta/R}(B)} \frac{q(\zeta')}{(\rho_z^2 + |\zeta'|^2)^{3/2}} \frac{e - \zeta' - \tilde{y}'}{|e - \zeta' - \tilde{y}'|^3} d\zeta' + O(1) \\ &= \int_{D_{a/R} \setminus D_{\delta/R}} \frac{q(E+B)}{(\rho_z^2 + |E+B|^2)^{3/2}} \frac{(E, b)}{(b^2 + |E|^2)^{3/2}} dE + O(1). \end{aligned}$$

The possible problem can arise as  $b \rightarrow 0$ . Since in this case  $B$  is bounded away from zero by (5.53), however, one can write  $I_9$  as

$$\begin{aligned} I_9 &= \frac{q(B)}{(\rho_z^2 + |B|^2)^{3/2}} \int_{D_{a/R} \setminus D_{\delta/R}} \frac{(E, b)}{(b^2 + |E|^2)^{3/2}} dE + O(1) \\ &= \frac{q(B)}{(\rho_z^2 + |B|^2)^{3/2}} \left( \int_{D_{a/R} \setminus D_{\delta/R}} \frac{E dE}{(b^2 + |E|^2)^{3/2}}, \int_{D_{a/R} \setminus D_{\delta/R}} \frac{b dE}{(b^2 + |E|^2)^{3/2}} \right) + O(1). \end{aligned}$$

The first integral vanishes by parity, so one can estimate  $I_9$  in terms of the second integral as

$$|I_9| \leq C \int_{\mathbb{R}^2} \frac{b dE}{(b^2 + |E|^2)^{3/2}} + O(1) = C \int_{\mathbb{R}^2} \frac{d\bar{E}}{(1 + |\bar{E}|^2)^{3/2}} + O(1) \leq C,$$

where we have defined the new  $\mathbb{R}^2$ -valued variable  $\bar{E} := E/b$  and used that  $dE = b^2 d\bar{E}$ . It remains now to consider the behavior of the integral with respect to  $\tilde{y}$  around  $\tilde{y} = 0$ . This can be handled exactly as in the case of  $I_4$ , which yields to a bound of the form  $C \log(2 + 1/\rho_z)$ . Combining both results

one immediately obtains that

$$|I_8| \leq C \ell(\bar{w}),$$

thereby establishing (5.48).  $\square$

**Lemma 5.13.**  $I_6$  can be written as

$$I_6 = \int_{\Omega} \mathcal{K}'_{\delta}(x, \bar{w}) \omega(z) dz \quad (5.54)$$

with  $\mathcal{K}'_{\delta}$  a matrix kernel satisfying

$$|\mathcal{K}'_{\delta}(x, \bar{w})| \leq \frac{C \ell(\bar{w})}{|x - \bar{w}|}. \quad (5.55)$$

*Proof.* The proof follows by repeatedly applying the same ideas that we have used in Lemma 5.12. In view of the formula for the divergence of  $\tilde{\omega}$  given in Proposition 5.6, the integral kernel of (5.54) can be written as

$$\mathcal{K}'_{\delta}(x, \bar{w}) := \int_{\delta} \left( \frac{x - y}{|x - y|^3} \times \frac{z - y}{|z - y|^3} \right) \otimes K_{T, \text{div}}(z, w) dy d\sigma(z'),$$

where as before the subscript  $\delta$  means that one only integrates over  $(y, z')$  with

$$|y - z| > \delta, \quad |z' - w| > \delta.$$

Note that the kernel  $K_{T, \text{div}}$ , which was introduced in (5.22), diverges as the inverse square of the distance. Setting, in local coordinates  $(Z, \rho_z)$  as above,

$$\zeta := Z - W, \quad \tilde{y} := y - z, \quad e := \frac{x - \bar{w}}{R}$$

with  $R := |x - \bar{w}|$ , and rescaling the integral as before, one finds that

$$\mathcal{K}'_{\delta}(x, \bar{w}) = \frac{1}{R} I_{10} + \text{h.o.t.},$$

where  $I_{10}$  is a certain integral with respect to rescaled variables  $(\tilde{y}', \zeta')$  that is shown to be bounded by  $C \ell(\bar{w})$  by tediously repeating the steps taken before, with only minor modifications.  $\square$

Estimates (5.48) and (5.55) ensure the existence of the kernel

$$K_{\Omega}^{22}(x, \bar{w}) := \lim_{\delta \rightarrow 0} (\mathcal{K}_{\delta}(x, \bar{w}) + \mathcal{K}'_{\delta}(x, \bar{w}))$$

bounded as

$$|K_{\Omega}^{22}(x, \bar{w})| \leq \frac{C \ell(\bar{w})}{|x - \bar{w}|}.$$

Lemmas 5.11, 5.12 and 5.13 imply that  $K_{\Omega}^{22}(x, \bar{w})$  satisfies

$$\lim_{\delta \rightarrow 0} V_2(x) = \int_{\Omega} K_{\Omega}^{22}(x, \bar{w}) \omega(\bar{w}) d\bar{w},$$

what completes our treatment of  $v_2$ .

### Step 3: The kernel of $\nabla\varphi$

The analysis of  $\nabla\varphi$  is similar and we just sketch the remaining steps. Let us recall

$$\nabla\varphi(x) = - \int_{\partial\Omega} \frac{x-y}{4\pi|x-y|^3} g(y) d\sigma(y)$$

with  $g$  given in terms of  $v \cdot \nu$  by (5.30). In order to prove that one can write

$$\nabla\varphi(x) = \int_{\Omega} K_{\Omega}^3(x, z) \omega(z) dz$$

with a kernel bounded as

$$|K_{\Omega}^3(x, z)| \leq \frac{C \ell(z)}{|x-z|^2}$$

one starts off by writing

$$\begin{aligned} \nabla\varphi(x) &= - \int_{\partial\Omega} \frac{x-y}{2\pi|x-y|^3} v \cdot \nu(y) d\sigma(y) - \int_{\partial\Omega} \frac{x-y}{\pi|x-y|^3} \tilde{T}(v \cdot \nu)(y) d\sigma(y) \\ &=: - \lim_{\delta \rightarrow 0} \frac{J_1 + 2J_2}{2\pi}, \end{aligned}$$

where  $\tilde{T}$  is an operator of the form (5.19). Integrating by parts and arguing as before, one can readily infer that

$$\begin{aligned} J_1 &= - \int_{\Omega \setminus B_{\delta}(x)} v(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dy \\ &= - \frac{1}{8\pi^2} \left( \int_{\Omega \setminus B_{\delta}(x)} V_1(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dy \right. \\ &\quad \left. - 2 \int_{\Omega \setminus B_{\delta}(x)} V_2(y) \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dy \right) \\ &=: - \frac{1}{8\pi^2} (J_{11} - 2J_{12}). \end{aligned}$$

Given the expression of  $V_1$ , it turns out that one can integrate by parts to write

$$\begin{aligned} J_{11} &= \int_{\delta} \omega(z) \frac{|z-w|^2 I - 3(z-w) \otimes (z-w)}{|z-w|^5} \times \frac{y-w}{|y-w|^3} \\ &\quad \cdot \frac{|x-y|^2 I - 3(x-y) \otimes (x-y)}{|x-y|^5} dw dy dz, \end{aligned}$$

where the subscript  $\delta$  means that

$$|x - y| > \delta, \quad |z - w| > \delta.$$

Hence essentially the same reasoning that we used with  $I_3$  in the proof of Lemma 5.10 allows us to show that

$$\lim_{\delta \rightarrow 0} J_{11} = \int K_{\Omega}^{31}(x, z) \omega(z) dz$$

with a kernel satisfying the bound

$$|K_{\Omega}^{31}(x, z)| \leq \frac{C \ell(z)}{|x - z|^2}$$

that is given by

$$K_{\Omega}^{31}(x, z) = \lim_{\delta \rightarrow 0} \int_{\Omega \setminus B_{\delta}(x)} \int_{\Omega \setminus B_{\delta}(z)} \frac{|z - w|^2 I - 3(z - w) \otimes (z - w)}{|z - w|^5} \times \frac{y - w}{|y - w|^3} \cdot \frac{|x - y|^2 I - 3(x - y) \otimes (x - y)}{|x - y|^5} dw dy.$$

Likewise, arguing exactly as in the analysis of  $V_2$  one can show that one has

$$\begin{aligned} J_{12} &= \int_{\delta} \frac{z - w}{|z - w|^3} \times \frac{y - w}{|y - w|^3} \cdot \frac{|x - y|^2 I - 3(x - y) \otimes (x - y)}{|x - y|^5} \tilde{\omega} \cdot \nu(z) d\sigma(z) dw dy \\ &= \int_{\delta} \tilde{\omega}(z) \cdot \frac{|z - w|^2 I - 3(z - w) \otimes (z - w)}{|z - w|^5} \times \frac{y - w}{|y - w|^3} \\ &\quad \cdot \frac{|x - y|^2 I - 3(x - y) \otimes (x - y)}{|x - y|^5} dz dw dy \\ &\quad - \int_{\delta} \frac{z - w}{|z - w|^3} \times \frac{y - w}{|y - w|^3} \cdot \frac{|x - y|^2 I - 3(x - y) \otimes (x - y)}{|x - y|^5} \nabla \cdot \tilde{\omega}(z) dz dw dy. \end{aligned}$$

Writing  $\tilde{\omega}$  and  $\nabla \cdot \tilde{\omega}$  in terms of  $\omega$  using Proposition 5.6 as above one finds that

$$\lim_{\delta \rightarrow 0} J_{12} = \int K_{\Omega}^{32}(x, \bar{w}) \omega(\bar{w}) d\bar{w},$$

where the kernel  $K_{\Omega}^{32}(x, \bar{w})$  is bounded as

$$|K_{\Omega}^{32}(x, \bar{w})| \leq \frac{C \ell(\bar{w})}{|x - \bar{w}|}.$$

The treatment of  $J_2$  can be accomplished using a completely analogous reasoning, yielding

$$J_2 = \int K_{\Omega}^{33}(x, z) \omega(z) dz$$

with

$$|K_{\Omega}^{33}(x, z)| \leq \frac{C \ell(z)}{|x - z|}.$$

The details, which are tedious but now straightforward, are omitted. This completes the proof of Theorem 5.8.

## 5.6 Uniqueness and an application to the full div-curl system

As a last step in the proof of Theorem 5.1, in this easy section we consider the uniqueness of the solution. For completeness, we do so in the context of the general div-curl system

$$\nabla \times v = \omega, \quad \nabla \cdot v = f \quad \text{in } \Omega, \quad v \cdot \nu|_{\partial\Omega} = g. \quad (5.56)$$

Here  $\omega$  is a divergence-free field satisfying (5.2) and the functions  $f$  and  $g$  satisfy the well known compatibility condition

$$\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, d\sigma. \quad (5.57)$$

**Proposition 5.14.** *The system (5.56), with  $\omega$ ,  $f$  and  $g$  as above, admits a solution  $v$ , which is bounded as*

$$\|v\|_{W^{k+1,p}(\Omega)} \leq C(\|\omega\|_{W^{k,p}(\Omega)} + \|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)}).$$

*Furthermore, the solution is unique modulo the addition of a harmonic field tangent to the boundary, and the dimension of this linear space equals the genus of  $\partial\Omega$ .*

*Proof.* Uniqueness is immediate: if  $v$  and  $v'$  are two solutions to the problem, their difference  $w := v - v'$  satisfies

$$\nabla \times w = 0, \quad \nabla \cdot w = 0 \quad \text{in } \Omega, \quad w \cdot \nu|_{\partial\Omega} = 0, \quad (5.58)$$

so it is a harmonic field on  $\Omega$  tangent to the boundary. The dimension of the linear space of solutions to (5.58) is known to be given by the genus of  $\partial\Omega$  by Hodge theory [14].

To prove the existence of a solution, let  $\phi$  be the only solution to the problem

$$\Delta\phi = f \quad \text{in } \Omega, \quad \partial_{\nu}\phi|_{\partial\Omega} = g, \quad \int_{\Omega} \phi \, dx = 0,$$

which is granted to exist by the hypothesis (5.57) and satisfies

$$\|\phi\|_{W^{k+2,p}(\Omega)} \leq C(\|f\|_{W^{k,p}(\Omega)} + \|g\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)}).$$

The field  $u := v - \nabla\phi$  then satisfies

$$\nabla \times u = \omega, \quad \nabla \cdot u = 0 \quad \text{in } \Omega, \quad u \cdot \nu|_{\partial\Omega} = 0,$$

so Theorem 5.7 ensures that there is a solution satisfying

$$\|u\|_{W^{k+1,p}(\Omega)} \leq C\|\omega\|_{W^{k,p}(\Omega)}.$$

The statement then follows. □

## Appendix A

### Other results

This appendix briefly presents a series of less central results in the area of mathematical physics, which have also been obtained by the candidate during her Ph.D. studies. Section A.1 contains two results [38, 39] related with orthogonal polynomial systems and eigenfunctions of a Schrödinger problem. In Section A.2 we discuss the article [37] about self-adjoint extensions of the Hamiltonian operator on the infinite square well potential. Finally, in Section A.3 we introduce the Shannon entropy in the context of channels of particle decays as it is studied in [16].

#### A.1 Orthogonal polynomial systems

An orthogonal polynomial system is an infinite sequence of univariate polynomials  $\{P_n(x)\}_n$  that are orthogonal with respect to a positive measure on an open interval  $I$  of the real line, that is,

$$\int_I P_n(x)P_m(x)W(x)dx = c_n\delta_{mn},$$

where  $W(x)$  is the orthogonal weight function and  $\delta_{mn}$  is the Kronecker delta. Orthogonal polynomials play a key role in the solutions of notable mathematical and physical problems, such as the quantum harmonic oscillator.

On the other hand, a Sturm–Liouville problem consists in a real second-order linear differential equation of the form

$$-\frac{d}{dx}\left(P(x)\frac{dy}{dx}\right) + R(x)y = \lambda W(x)y \quad (\text{A.1})$$

in some interval  $I \subset \mathbb{R}$  together with suitable boundary conditions.  $\lambda$  is a spectral parameter and functions  $P(x)$  and  $W(x)$  are positive in  $I$ . After dividing by  $W(x)$ , we can rewrite equation A.1 as

$$-T[y] = \lambda y$$

with

$$T[y] = p\frac{d^2y}{dx^2} + q\frac{dy}{dx} + ry. \quad (\text{A.2})$$



The eigenfunctions of a Sturm–Liouville problem corresponding to different eigenvalues are orthogonal with respect to  $W(x)$ . If these eigenfunctions are polynomials, then they will form a orthogonal polynomial system.

Bochner [9] proved that if an operator of the form (A.2) admits eigenfunctions which are polynomials of every degree, then its coefficients  $p$ ,  $q$  and  $r$  are polynomials. If in addition the polynomial eigenfunctions satisfy a Sturm–Liouville problem, they must be, up to affine transformations, the classical polynomials of Hermite, Laguerre and Jacobi.

If we relax the condition of Bochner’s result about the existence of eigenpolynomials for every degree, we can also get complete systems of orthogonal polynomials satisfying a Sturm–Liouville problem. These systems, discovered by Gómez-Ullate, Kamran and Milton [43], are called exceptional orthogonal polynomials. Section A.1.2 contains a more complete introduction to them and a result [39] that paves the road to their complete classification.

Another feature of the classical orthogonal polynomials is that they appeared in bound states of the Schrödinger equation for certain common potentials. Up to a change of variable and multiplication by a nonzero pre-factor,

$$\varphi_n(x) = \mu(x)P_n(z(x)),$$

they are essentially the eigenfunctions of the operator

$$H[\varphi] = -\frac{d^2\varphi}{dx^2} + V(x)\varphi, \quad x \in I, \quad (\text{A.3})$$

for particular potentials. This allows us to apply certain results concerning eigenfunctions of a Schrödinger problem to classical orthogonal polynomials. In particular, in A.1.1 we comment the result Section [38] about the number of zeros of Wronskian determinants of arbitrary sequences of classical orthogonal polynomials.

### A.1.1 Counting zeros of Wronskians with classical orthogonal polynomials entries

Karlin and Szegö [52] studied Wronskian determinants whose entries belong to an orthogonal polynomial system, working out the number of real zeros in the interval of orthogonality if the polynomials have consecutive degrees. We extend this result in [38] by giving explicit formulas for the number of real zeros of the Wronskian determinant of arbitrary sequences of classical orthogonal polynomials, under very mild conditions satisfied for generic values of the parameters.

Actually, the formulas for counting zeros apply to Wronskian determinants whose entries are general eigenfunctions of a Schrödinger problem (A.3) with an arbitrary regular potential. Namely, let  $\varphi_k$  be a solution of  $H[\varphi_k] = \lambda_k\varphi_k$ , with  $\lambda_0 < \lambda_1 < \dots$  the corresponding eigenvalues. Then,

**Theorem A.1.** *Under some non-degeneracy conditions on the set of eigenfunctions  $\{\varphi_n\}_{n=0}^\infty$ , the Wronskian  $\text{Wr}[\varphi_{k_1}, \varphi_{k_2}, \dots, \varphi_{k_\ell}]$ , with  $k_1 < k_2 < \dots < k_\ell$ , has*

$$\sum_{j=1}^{\ell} (-1)^{\ell-j} (k_j - j + 1) \quad (\text{A.4})$$

*simple real zeros in  $I$ .*

The non-degeneracy conditions which we assume are, roughly speaking, that any two Wronskians which differ just in one entry do not have any common root in the interval  $I$ . We also consider that this may fail at  $x = 0$ , what happens when the potential  $V$  is even. In this case, all odd eigenfunctions vanish at  $x = 0$  and the multiplicity of the root at this point can be larger than one. Moreover the formula (A.4) suffers a small correction.

This result generalizes the work of Adler and Krein [2, 54], who gave necessary and sufficient conditions characterising the sequences of eigenfunctions such that the Wronskian has constant sign: those corresponding to a sequence of  $k$ 's of the form  $0, 1, 2, \dots, M_0; M_1, M_1 + 1; \dots; M_s, M_s + 1$ . This turns out to be crucial to characterise the regularity of the transformed potential in (A.3) after applying a sequence of Darboux transformations.

### A.1.2 Exceptional orthogonal polynomials: towards their complete classification

Exceptional orthogonal polynomials are complete systems of orthogonal polynomials that satisfy a Sturm-Liouville problem but they differ from the classical families of Hermite, Laguerre and Jacobi in that there are a finite number of exceptional degrees for which no polynomial eigenfunction exists. As opposed to their classical counterparts, the differential equation A.2 contains rational instead of polynomial coefficients, yet the eigenvalue problem has an infinite number of polynomial eigenfunctions that form the basis of a weighted Hilbert space.

The recent development of exceptional polynomial systems has received contributions both from the mathematics community working on orthogonal polynomials and special functions, and from mathematical physicists. Among the physical applications, exceptional polynomial systems appear mostly as solutions to exactly solvable quantum mechanical problems, describing both bound states and scattering amplitudes.

In the mathematical literature, two main questions have centered the research activity in relation to exceptional polynomial systems: describing their mathematical properties and achieving a complete classification.

Among the mathematical properties, the study of their zeros deserve particular attention. Their interlacing, asymptotic behaviour and monotonicity as a function of parameters have been widely investigated. The results commented

in Section A.1.1 have some bearing in this context, since the Wronskian determinants of certain sequences of classical orthogonal polynomials define a family of exceptional orthogonal polynomials.

The quest for a complete classification of exceptional polynomials has been a fundamental problem that with our result in [39] is now close to being solved. The first attempts to classify exceptional polynomial systems proceeded by increasing the number of gaps in the degree sequence of the exceptional family [43, 44]. But this approach proved to be unfeasible for the purpose of achieving a complete classification. However, a fundamental idea towards the full classification was also launched in [44], namely that every exceptional polynomial system might be obtained from a classical system by applying a finite number of Darboux transformations. The main result in [39] is the following:

**Theorem A.2.** *Every exceptional orthogonal polynomial system can be obtained by applying a finite sequence of Darboux transformations to a classical orthogonal polynomial system.*

The essential consequence of this result is that the program to classify exceptional polynomial systems becomes constructive: start from the three classical systems of Hermite, Laguerre and Jacobi and apply all possible Darboux transformations to describe the entire exceptional class.

## A.2 Self-adjoint extensions of the Hamiltonian for a particle in a box

Let us consider the standard problem of a free particle in a one dimensional infinitely deep square potential well of width  $2c$ :

$$V(x) = \begin{cases} 0 & \text{if } x \leq c \\ \infty & \text{if } |x| > c \end{cases}.$$

Stationary states are then obtained through the Schrödinger problem

$$H[\varphi] = E\varphi, \quad \varphi(\pm c) = 0, \quad (\text{A.5})$$

with  $H = -\frac{d^2}{dx^2}$  the kinetic operator, on the Hilbert space  $L^2([-c, c])$ . This means that the action of the Hamiltonian operator for a free particle confine to  $[-c, c]$  is described by  $H$  and its definition domain

$$\mathcal{D}_0(H) = \{\phi \in L^2([-c, c]) : H[\phi] \in L^2([-c, c]), \phi(\pm c) = 0\}. \quad (\text{A.6})$$

The operator  $(H, \mathcal{D}_0(H))$  is symmetric since for all  $\phi, \psi \in \mathcal{D}_0(H)$  we have

$$(H\phi, \psi) = (\phi, H\psi),$$

or equivalently,  $H = H^*$  with  $H^*$  its formal adjoint. However,  $(H, \mathcal{D}_0(H))$  is not a self-adjoint operator because  $\mathcal{D}_0(H) \subsetneq \mathcal{D}(H^*)$ . The search of self-adjoint

extensions then consists in the search of definition domains for  $H$  which coincide with the definition domains of  $H^*$ , besides the symmetry requirement.

The Hamiltonian operator on the infinite square well has an infinite number of self-adjoint determinations, each one characterized by its own domain and corresponding to one distinct quantum observable. These self-adjoint extensions are parametrized by four real parameters, which relate the boundary values of the wave functions and of their first derivatives at both walls of the well  $(\varphi(\pm c), \varphi'(\pm c))$ .

Our interest in [37] lie in those self-adjoint extensions which can be identified with the presence of Dirac delta potentials in the centre of the well. We choose a point perturbation of the type  $a\delta(x) + b\delta'(x)$ , where  $\delta(x)$  is the Dirac delta and  $\delta'(x)$  its derivative in the distributional sense, on the infinite square well. In this case, the formal Hamiltonian takes the form:

$$-\frac{1}{2m} \frac{d^2}{dx^2} + V(x) + a\delta(x) + b\delta'(x). \quad (\text{A.7})$$

This perturbation is obtained by choosing a suitable self-adjoint extension of  $H$ . However, this self-adjoint extension is determined by some matching conditions imposed to the wave functions at the origin. Then, the question that we investigate in [37] is how we can characterize this self-adjoint extension (parametrized by  $a$  and  $b$ ) by the four parameters relating the boundary values of  $\varphi$  and  $\varphi'$  at the walls that we mentioned earlier.

The conclusion is quite surprising. One would have expected that any point perturbation characterized by  $a$  and  $b$  gives unique values for the four parameters which characterize the self-adjoint extension of  $H$  in the infinite square well. However, this relation is not one to one: for any value of the infinitely many energy levels, we have a different relation.

### A.3 Shannon entropy and particle decays

Shannon entropy was introduced in 1948 in the context of information theory. It measures the uncertainty associated to a random variable, or when ignoring the value taken by that variable, the average missing information content.

Let  $X$  be a random variable taking values in a finite set  $\mathcal{X}$  whose probability distribution is  $p(x_j) = P(X = x_j)$ ,  $x_j \in \mathcal{X}$ . Shannon entropy provides an idea of the information needed, in average, to determine the value that the variable takes and it is defined as

$$S(X) = - \sum_{j=1}^{|\mathcal{X}|} p(x_j) \log p(x_j).$$

Fixed  $|\mathcal{X}|$ , the minimum  $S(X) = 0$  is achieved when the random variable takes always the same value. On the contrary,  $S(X)$  is maximum and equal to  $\log(|\mathcal{X}|)$  when the probability distributions is uniform ( $p(x_j) = \frac{1}{|\mathcal{X}|}$ ,  $j = 1, \dots, |\mathcal{X}|$ ).

The range of application of Shannon entropy is very wide. In [16], we study it in the context of particle decays.

A particle can decay into different combinations of other particles over the time. Every of these different transformations is called decay channel and it is characterized by a partial width  $\Gamma_j$  related with the Feynman amplitude of the process. The total decay width of the original particle is just the sum of the partial widths for each of its possible decay channels,  $\Gamma = \sum_j \Gamma_j$ , and is equal to the inverse of the average lifetime.

We can also characterize the decay by the branching ratios,  $BR_j = \frac{\Gamma_j}{\Gamma}$ . Their sum is unity  $\sum_j BR_j = 1$  so they provide a probability distribution for the decay channels. Hence, we can consider the set of all decay channels of a particle with the corresponding branching ratios and apply to them the Shannon entropy function:

$$S = - \sum_j BR_j \log BR_j. \quad (\text{A.8})$$

This quantity must be interpreted as an average of the necessary information to determine which channel the particle decay proceeds through. In [16] we evaluate Shannon entropy with actual data from meson and gauge boson decays taken from [68] and explore several new features of this magnitude.

Experimentally, the total decay width of a particle is known, but not all the partial widths are, not even all the channels. Therefore we can just calculate a lower bound by considering one channel with bind the remaining probability together:  $1 - \sum_{j=1}^N BR_j$ , with  $N$  the number of known channels. Hence, the entropy function A.8 obeys the following formula

$$S(N) = - \sum_{j=1}^N BR_j \log BR_j - \left(1 - \sum_{j=1}^N BR_j\right) \log \left(1 - \sum_{j=1}^N BR_j\right). \quad (\text{A.9})$$

We use A.9 to quantify how important a given new reported decay channel is, from the point of view of the information that it adds to the already known ones. If we order the decay channels in decreasing branching ratios, we see that the generic behaviour of the entropy against the number of considered (known) channels increases linearly for the first few (more probable ones) and then saturates towards the upper bound  $\log(N + 1)$ . We also see an anticorrelation between the entropy and the maximum branching fraction of any decay channel. Simple derived functions that help quantify the amount of entropy that a given decay channel adds to the distribution after its discovery are discussed too. We finally discuss the use of the total number of accounted channels in the base of the logarithm as a normalisation of the entropy to compare different particles.

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