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TESIS DOCTORAL

Cohomological characterization of universal bundles of the Grassmannian of lines

Caracterización cohomológica de fibrados universales de la Grassmaniana de rectas

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Memoria para optar al grado de doctor presentada por

ALICIA TOCINO SÁNCHEZ

Bajo la dirección de

ENRIQUE ARRONDO ESTEBAN

Madrid, 2015

To my sister

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Summary

Introduction

Probably one of the most important results about vector bundles over a projective space is the following:

Theorem 0.0.1. *(G. Horrock, [23]) A vector bundle F over $\mathbb{P}^n$ splits as a direct sum of line bundles if and only if it does not have intermediate cohomology (i.e. $H^j_*(F) = 0$ for $j = 1, 2, \ldots, n-1$).*

This theorem has been proved later using many different techniques, like restriction to hyperplanes, regularity theory or Beilinson theorem (in the general framework of derived categories). This kind of result is important, for example, since part of the celebrated Hartshorne's conjecture about complete intersections (see [21]) is equivalent to the splitting of vector bundles of low rank over the projective space. Horrocks theorem can be generalized from two different points of view:

The first point of view is to regard Horrocks theorem as a characterization of arithmetically Cohen-Macaulay (aCM for short) vector bundles (i.e. without intermediate cohomology) over the projective space. From this point of view, Horrocks theorem states that the only indecomposable aCM vector bundle over $\mathbb{P}^n$ is, up to a twist, the trivial line bundle. In this direction, Knörrer in [28] proved that line bundles and twists of the spinor bundles are the only indecomposable aCM vector bundles over quadrics. In a kind of converse result, Buchweitz, Greuel and Schreyer showed in [11] that the only smooth hypersurfaces in a projective space for which there exists, up to a twist, a finite number of aCM bundles are the hyperplanes and the quadrics. This explains why the characterization of aCM vector bundles over projective varieties is in general a difficult problem. There are for example results about low rank aCM vector bundles on different types of varieties, for example general hypersurfaces over $\mathbb{P}^5$ (see [12] and a series of works by C. Madonna), or different types of Fano threefolds (see [4] and [5]).

We will be more interested in the second point of view of Horrocks theorem, namely as a criterion to characterize vector bundles that split as a direct sum of line bundles. Such a characterization has been extended to other projective varieties. For example, G. Ottaviani characterized such vector bundles over Grassmannians and smooth quadrics (see [33] and [32]). Since we are going to use it, we state the particular case of $\mathbb{G}(1, n)$, the Grassmannian of lines in $\mathbb{P}^n$, where $\mathcal{Q}$ denotes the rank two universal quotient bundle (see Definition 1.1.9).

Theorem 0.0.2. *A vector bundle F over $\mathbb{G}(1,n)$ splits as a direct sum of line bundles if and only if the following conditions hold for $i = 0, 1, \ldots, n-2$:*

$$H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{with} \quad i \leq j < 2n-2-i \quad \textit{and} \quad j > 0.$$

An improvement of Ottaviani's criterion, which will be the starting point of this dissertation was given by E. Arrondo and F. Malaspina in [7]:

Theorem 0.0.3. *Let F be a vector bundles on the Grassmannian of lines $\mathbb{G}(1,n)$. Then F splits as a direct sum of line bundles if and only if:*

a. $H^j_*(F \otimes S^j\mathcal{Q}) = 0 \qquad j \in \{1, 2, \ldots, n-3, n-2\}$

b. $H^j_*(F \otimes S^i\mathcal{Q}) = 0 \qquad (i,j) \in \{(0, 2n-3), (1, 2n-4), \ldots, (n-3, n), (n-2, n-1)\}$

The idea behind the proof of this result (and similar ones in that paper) is quite simple: if a vector bundle F has a direct summand $\mathcal{O}(l)$, this is equivalent to have maps $\mathcal{O}(l) \to F$ and $F \to \mathcal{O}(l)$ whose composition is not zero. The tricky part is to relate the composition pairing $Hom(\mathcal{O}(l), F) \times Hom(F, \mathcal{O}(l)) \to Hom(\mathcal{O}(l), \mathcal{O}(l))$ with the perfect pairing given by Serre's duality.

As it is also done in [7], one could be interested in characterizing cohomologically other vector bundles, and not only line bundles. For instance, in 2005, L. Costa and R. M. Miró-Roig gave a characterization, for a given Schur functor, of the vector bundle $\mathbb{S}_\lambda\mathcal{Q}$, where $\mathcal{Q}$ is the universal quotient bundle of rank $k+1$ over the Grassmannian $\mathbb{G}(k,n)$ (see [13] and [14]). This characterization is done in terms of the other universal vector bundle (of rank $n-k$) and characterizes precisely $\oplus\mathbb{S}_\lambda\mathcal{Q}$ but not its twists. However we will be interested in using the same universal bundle $\mathcal{Q}$ for the characterization, and we also want the characterization to be up to a twist, as it happens for Horrocks theorem.

A result of the type we are looking for is the following by E. Arrondo and B. Graña (see [6]):

Theorem 0.0.4. *Let F be a vector bundle over $\mathbb{G}(1,4)$. Then F can be expressed as $\oplus\mathcal{O}(l_{i_0}) \bigoplus \oplus\mathcal{Q}(l_{i_1})$ if and only if:*

i. $H^1_*(F) = H^2_*(F) = H^3_*(F) = H^4_*(F) = H^5_*(F) = 0$ *(F does not have intermediate cohomology)*

ii. $H^1_*(F \otimes \mathcal{Q}) = H^2_*(F \otimes \mathcal{Q}) = H^3_*(F \otimes \mathcal{Q}) = H^4_*(F \otimes \mathcal{Q}) = H^5_*(F \otimes \mathcal{Q}) = 0$ *($F \otimes \mathcal{Q}$ does not have intermediate cohomology)*

The idea for obtaining such result was as follows. First, one looks at Theorem 0.0.2, which provides a characterization of the direct sums of line bundles. Since $\mathcal{Q}$ does not split, there must be some hypothesis of that theorem that it does not satisfy. There is precisely one of them, and one manage to prove (by techniques that are not relevant here) that the rest of the hypotheses (precisely the conditions of Theorem 0.0.4) work to characterize directs sums of twists of $\mathcal{O}$ and $\mathcal{Q}$. In fact,the authors continue to remove conditions until getting some description (very far from an actual classification) of aCM vector bundles over $\mathbb{G}(1,4)$.

From all these ideas in mind we can concentrate now on the main goals of this thesis, which we describe in the next item.

Objectives

The present dissertation has one main objective. We want to give a cohomological characterization for direct sums of twists of symmetric product of the quotient bundle $\mathcal{Q}$ in the Grassmannian of lines $\mathbb{G}(1, n)$. We will give such a classification for symmetric powers of order not bigger than $n-2$. The original reason was that a symmetric power $S^k\mathcal{Q}$ is aCM if and only if $k \leq n-2$, so that our result could be a help to understand aCM vector bundles over $\mathbb{G}(1, n)$. However such restriction is needed anyway, sine our proof has some obstruction when $k > n-2$ (see Remark 3.4.2).

As main ideas, we will use those from [6] and [7] that we already pointed out. Specifically, we will start from the splitting criterion of Theorem 0.0.3 (which is stronger than the one of Theorem 0.0.2, the starting point of Theorem 0.0.4). Contrary to what happened in the proof of Theorem 0.0.4, removing the only hypothesis of Theorem 0.0.3 will not be now sufficient to characterize direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$, so we will need to add more hypotheses for such characterization. As in [7], the idea to find direct summand of $\mathcal{Q}(l)$ will be to relate it with Serre's duality. With this ideas in mind, one can go on, removing in each step one condition and adding few ones, until arriving to the wanted classification.

On the other hand, as A. Kuznetsov kindly indicated to us, the techniques of derived categoris is also a natural way to get the kind of characterizations we just mentioned. Hence we decided to study those methods, in particular Beilinson's Theorem in order to compare the results one could obtain with our method and this other one.

Results

The main results of this thesis are in Chapters 3 and 4. Previously, in Chapter 1 we settle the main definitions and facts that will be used throughout this thesis. We start with some generalities on Grassmannians and the universal vector bundles. Then we recall the notion of Eagon-Northcott complex and apply it to the universal exact sequence of a Grassmannian of lines, which will become crucial in most of our proofs, since it will involve the symmetric powers of the universal bundle $\mathcal{Q}$. We finish that chapter recalling some general cohomologucal result and Serre's duality.

We devote Chapter 2 to Bott's theorem for Grassmannians. This is the main tool to compute the cohomology of many vector bundles obtained from the universal vector bundles, in particular the symmetric products of the universal bundle $\mathcal{Q}$. In order to do this we will introduce the notion of Young diagram and Schur functor. We will define concretely the algorithm that compute the cohomologies of some particular vector bundles expressed as a tensor product of Schur functor. Moreover, we have done, in collaboration with J. Caravantes, a program for SAGE that computes this algorithm for any Grassmannian of $k-$planes. We enclose the code of the program as an appendix at the end of the thesis.

In Chapter 3 we give the main result of this dissertation, namely Theorem 3.3.1, which gives a cohomological characterization in $\mathbb{G}(1, n)$ of direct sums of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$ with $k \leq n-2$. In order to make the cohomological condition more visible, we introduce, in a first section, a graphical representation of them. In a second section, we explain the use of Serre's duality and some cohomolgical conditions on a vector bundle F over $\mathbb{G}(1,n)$ to produce morphisms $S^k\mathcal{Q} \to F$ and $F \to \mathcal{Q}$ whose composition is a multiple of the identity. This will be one of the main ingredients to prove our main result in the third section of the chapter. Then there is a section of remarks about the sharpness of our result and how it compares with other known results. We conclude with a section explaining how to generalize our method to other varieties. As a concrete application, we introduce the main tool for the case of Grassmannian of $k-$planes.

Finally, in Chapter 4, after a first introductory section to derived categories, we pass to another section to introduce Beilinson's theorem, for projective spaces and for Grassmannians. In the last section, we use Beilinson's theorem to produce a splitting criterion for the Grassmannian of lines. We follow the steps given in [2] by V. Ancona and G. Ottaviani in the case of the projective space. We compare the criterion we got with the one obtained by G. Ottaviani, showing that ours is stronger, and with the one produced by E. Arrondo and F. Malaspina. In this second case we easily see that this new characterization made by using the Beilinson's theorem has much more conditions than the one we have taken as our starting point.

Conclusions

As a main conclusion, we can say that the use of the Serre's duality as in [7] produces much stronger cohomological characterizations of vector bundles than using the standard methods, for example Beilinson's theorem.

We could have applied this technique to characterize other types of vector bundles over $\mathbb{G}(1,n)$, or also for any arbitrary Grassmannian. The only difficulty would have been of clearness, since writing down the precise hypotheses in each step could become a mess.

However our main goal was to present this method and to show how it works, in the hope that it could be useful for other projective varieties. For example, a natural variety to try with would be a isotropic Grassmannian, for which A. Kuznetsov already studied in detail its derived category (see [29]).

Resumen

Introducción

Probablemente uno de los resultados más importantes sobre fibrados vectoriales sobre un espacio proyectivo es el siguiente:

Teorema 0.0.1. *(G. Horrock, [23]) Un fibrado vectorial F de $\mathbb{P}^n$ escinde como suma directa de fibrados lineales si y sólo si no tiene cohomología intermedia (i.e. $H^j_*(F) = 0$ for $j = 1, 2, \ldots, n-1$).*

Este teorema se ha probado después usando muchas técnicas diferentes, como restricciones a hiperplanos, teoremas de regularidad o el teorema de Beilinson (en el entramado general de categorías derivadas). Este tipo de resultado es importante, por ejemplo, ya que parte de la célebre conjetura de Hartshorne sobre intersecciones completas (véase [21]) es equivalente a la escisión de fibrados vectoriales de rango bajo sobre un espacio projectivo. El teorema de Horrocks puede generalizarse desde dos puntos de vista distintos:

El primer punto de vista es ver el teorema de Horrocks como una caracterización de fibrados vectoriales aritméticamente Cohen-Macaulay (aCM para acortar) sobre el espacio projectivo (i.e., sin cohomología intermedia). Desde este punto de vista, el teorema de Horrocks afirma que el único fibrado vectorial aCM indescomponible sobre $\mathbb{P}^n$ es, salvo twist, el fibrado lineal trivial. En esta dirección, Knörrer en [28] probó que los fibrados lineales y twists del fibrado spinor son los únicos fibrados vectoriales aCM indescomponibles sobre cuádricas. Para el resultado opuesto, Buchweitz, Greuel y Schreyer mostraron en [11], que las únicas hipersuperficies lisas es un espacio projectivo para las que existe, salvo twist, un número finito de fibrados aCM son los hiperplanos y las cuádricas. Ésto explica por qué la caracterización de fibrados vectoriales aCM sobre variedades projectivas es en general un problema difícil. Hay por ejemplo resultados sobre fibrados vectoriales aCM de rango bajo en distintos tipos de variedades, por ejemplo hipersuperficies generales sobre $\mathbb{P}^5$ (véase [12] y una serie de trabajos por C. Madonna), o distintos tipos de Fano threefolds (véase [4] y [5]).

Nosotros estamos más interesados en el segundo punto de vista del teorema de Horrocks, llamado como criterio que caracteriza los fibrados vectoriales que escinden como suma directa de fibrados lineales. Dicha caracterización ha sido generalizada a otras variedades projectivas. Por ejemplo, G. Ottaviani caracterizó dichos fibrados vectoriales sobre las Grassmannianas y sobre cuádricas lisas (véase [33] y [32]). Ya que vamos a usarlo, enunciamos a continuación el resultado para el caso particular de las Grassmannianas de rectas, $\mathbb{G}(1, n)$, donde $\mathcal{Q}$ denota el fibrado universal cociente de rango 2 (véase Definition 1.1.9).

Teorema 0.0.2. *Un fibrado vectorial F sobre $\mathbb{G}(1,n)$ escinde como suma directa de fibrados lineales si y sólo si se cumplen las siguientes condiciones para $i = 0, 1, \ldots, n-2$:*

$$H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{con} \quad i \leq j < 2n-2-i \quad \textit{y} \quad j > 0.$$

Una mejora del criterio de Ottaviani, que será el punto de inicio de esta tesis, lo dieron E. Arrondo y F. Malaspina en [7]:

Teorema 0.0.3 *Sea F un fibrado vectorial sobre las Grassmannianas de rectas $\mathbb{G}(1,n)$. Entonces F escinde como suma directa de fibrados lineales si y sólo si:*

a. $H^j_*(F \otimes S^j\mathcal{Q}) = 0 \qquad j \in \{1, 2, \ldots, n-3, n-2\}$

b. $H^j_*(F \otimes S^i\mathcal{Q}) = 0 \qquad (i,j) \in \{(0, 2n-3), (1, 2n-4), \ldots, (n-3, n), (n-2, n-1)\}$

La idea que hay detrás de la prueba de este resultado (y similares en el artículo) es bastante simple: si un fibrado vectorial F tiene un sumando directo $\mathcal{O}(l)$, ésto es equivalente a tener las aplicaciones $\mathcal{O}(l) \to F$ y $F \to \mathcal{O}(l)$ cuya composición es no nula. La parte difícil es relacionar la composición $Hom(\mathcal{O}(l), F) \times Hom(F, \mathcal{O}(l)) \to Hom(\mathcal{O}(l), \mathcal{O}(l))$ con un par perfecto dado por la dualidad de Serre.

Conforme está hecho también en [7], uno podría estar interesado en caracterizar cohomológicamente otros fibrados vectoriales, y no sólo los fibrados lineales. Por ejemplo, en 2005, L. Costa y R. M. Miró-Roig dieron una caracterización, para un functor de Schur dado, del fibrado vectorial $\mathbb{S}_\lambda\mathcal{Q}$, donde $\mathcal{Q}$ es el fibrado universal cociente de rango $k+1$ de la Grassmanniana $\mathbb{G}(k,n)$ (véase [13] y [14]). Esta caracterización está hecha en términos del otro fibrado universal (de rango $n-k$) y caracterizan precisamenete $\oplus\mathbb{S}_\lambda\mathcal{Q}$ pero no sus twists. Sin embargo nosotros estaeremos interesados en usar el mismo fibrado universal $\mathcal{Q}$ para la caracterización, y también queremos que la caracterización sea salvo twist, como ocurre con el teorema de Horrocks.

Un resultado del tipo que estamos buscando es el siguiente hecho por E. Arrondo y B. Graña (véase [6]):

Teorema 0.0.4. *Sea F un fibrado vectorial de $\mathbb{G}(1,4)$. Entonces F se puede expresar como $\oplus\mathcal{O}(l_{i_0}) \bigoplus \oplus\mathcal{Q}(l_{i_1})$ si y sólo si:*

i. $H^1_*(F) = H^2_*(F) = H^3_*(F) = H^4_*(F) = H^5_*(F) = 0$ *(i.e. F no tiene cohomología intermedia)*

ii. $H^1_*(F \otimes \mathcal{Q}) = H^2_*(F \otimes \mathcal{Q}) = H^3_*(F \otimes \mathcal{Q}) = H^4_*(F \otimes \mathcal{Q}) = H^5_*(F \otimes \mathcal{Q}) = 0$ *(i.e. $F \otimes \mathcal{Q}$ no tiene cohomología intermedia)*

La idea para obtener dicho resultado fue la siguiente. Primero, uno mira Teorema 0.0.2, que da una caracterización de sumas directas de fibrados lineales. Ya que $\mathcal{Q}$ no escinde, tiene que haber alguna hipótesis de ese teorema que no la satisfaga. Hay exactamente una de ellas, y uno es capaz de probar (mediante otras técnicas que aquí no son pertinentes) que el resto de hipótesis (precisamente las condiciones de Teorema 0.0.4) nos sirven para caracterizar sumas directas de twists de $\mathcal{O}$ y $\mathcal{Q}$.

De hecho, los autores continuan quitando condiciones hasta obtener alguna descripción (muy lejana de una clasificación actual) de fibrados vectoriales aCM sobre $\mathbb{G}(1,4)$.

Con todas estas ideas en la cabeza nos podemos concentrar ahora en nuestro objetivo principal de la tesis, que describimos en el siguiente apartado.

Objetivos

La presente tesis tiene un objetivo principal. Queremos dar una caracterización para sumas directas de twists de los productos simétricos del fibrado cociente $\mathcal{Q}$ en la Grassmanniana de rectas $\mathbb{G}(1,n)$. Daremos dicha clasificación para productos simétricos de orden no mayor de $n-2$. El motivo original fue que un producto simétrico $S^k\mathcal{Q}$ es aCM si y sólo si $k \leq n-2$, así que nuestro resultado podría ser una ayuda para entender los fibrados vectoriales aCM sobre $\mathbb{G}(1,n)$. Sin embargo dicha restricción es necesaria en cualquier caso, ya que nuestra prueba tiene alguna obstrucción cuando $k > n-2$ (véase Remark 3.4.2).

Como ideas principales, usaremas las de [6] y [7] que ya hemos señalado. Específicamente, empezaremos por el criterio de escisión de Teorema 0.0.3 (que es más fuerte que el que da Teorema 0.0.2, punto de partida de Teorema 0.0.4). Contrariamente a lo que ocurre en la prueba de Teorema 0.0.4, quitar la única hipótesis de Teorema 0.0.3 no será suficiente para caracterizar las sumas directas de twists de $\mathcal{O}$ y $\mathcal{Q}$, así que necesitaremos añadir más hipótesis para dicha caracterización. Como en [7], la idea de encontrar un sumando directo de $\mathcal{Q}(l)$ estará relacionada con la dualidad de Serre. Con todas estas ideas en la cabeza, uno puede ir quitando paso a paso condiciones y añadiendo algunas más, hasta llegar a la clasificación que queríamos.

Por otro lado, como A. Kuznetsov amablemente nos indicó, las técnicas de categorías derivadas son también un modo natural para llegar al tipo de caracterizaciones que acabamos de mencionar. Por lo tanto decidimos estudiar esos métodos, en particular el teorema de Beilinson para así comparar los resultados que uno podría obtener con nuestro método y con este nuevo.

Resultados

Los resultados principales de esta tesis están en los capítulos 3 y 4. Previamente, en Capítulo 1 establecemos las definiciones y hechos principales que serán usados a lo largo de toda la tesis. Empezamos con algunas generalidades sobre Grassmannianas y sus fibrados vectoriales universales. Luego recordamos la noción de complejo de Eagon-Northcott y la aplicamos a la sucesión exacta universal de una Grassmanniana de rectas, que será crucial en la mayoría de nuestras demostraciones, ya que conllevará los productos simétricos del fibrado universal $\mathcal{Q}$. Terminaremos este capítulo recordando algunos resultados cohomológicos generales y la dualidad de Serre.

Dedicamos el Capítulo 2 al teorema de Bott para Grassmannianas. Esta es la herramienta principal para calcular la cohomología de muchos fibrados vectoriales obtenidos de los fibrados vectoriales universales, en particular el producto simétrico del fibrado universal $\mathcal{Q}$. Con el propósito de hacer esto introduciremos la noción de diagrama de Young y functor de Schur. Definiremos concretamente el algoritmo que calcula las cohomologías de algunos fibrados vectoriales particulares expresados como producto tensorial de functores de Schur. Además, hemos hecho, en colaboración con J. Caravantes, un programa para SAGE que computa este algoritmo para cualquier Grassmanniana de $k-$planos. Adjuntamos el código como apéndice al final de la tesis.

En el Capítulo 3 damos el resultado principal de esta disertación, nombrado Teorema 3.3.1, que da una caracterización cohomológica en $\mathbb{G}(1,n)$ de sumas directas de twists de $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$ with $k \leq n-2$. Para poder hacer las condiciones cohomológicas más visibles, introducimos, en la primera sección, una representación gráfica de ellas. En la segunda sección, explicamos el uso de la dualidad de Serre y algunas condiciones cohomológicas sobre un fibrado vectorial F en $\mathbb{G}(1,n)$ para producir morfismos $S^k\mathcal{Q} \to F$ y $F \to S^k\mathcal{Q}$ cuya composición es un mútiplo de la identidad. Este será uno de los principales ingredientes para probar nuestro resultado principal en la tercera sección del capítulo. Luego hay una sección de observaciones sobre la nitidez de nuestro resultado y cómo se compara con otros resultados conocidos. Concluimos con una sección donde se explica como generalizar nuestro mÃ©todo a otras variedades. Como una aplicación concreta, introducimos la herramienta principal para el caso de las Grassmannianas de $k-$planos.

Finalmente, en Capítulo 4, después de una sección introductoria de categorías derivadas, pasamos a otra sección para introducir el teorema de Beilinson, para el caso projectivo y para Grassmannianas. En la última sección, usamos el teorema de Beilinson para realizar un criterio de escisión para Grassmannianas de rectas. Seguimos los pasos dados en [2] por V. Ancona y G. Ottaviani en el caso del espacio projectivo. Comparamos el criterio que obtenemos con el obtenido por G. Ottaviani, mostrando que es más fuerte, y con el realizado por E. Arrondo and F. Malaspina. En este segundo caso podemos ver fácilmente que esta nueva caracterización hecha usando el teorema de Beilinson tiene muchas más condiciones que la que nosotros hemos tomado como nuestro punto de partida.

Conclusiones

Como una conclusión principal, podemos decir que el uso de la dualidad de Serre en [7] produce caracterizaciones cohomológicas mucho más fuertes que las obtenidas usando los métodos estándar, como por ejemplo el teorema de Beilinson.

Podríamos haber aplicado esta técnica para caracterizar otros tipos de fibrados vectoriales sobre $\mathbb{G}(1,n)$, o incluso para una Grassmanniana arbitraria. La única dificultad habría sido la claridad, ya que escribir las hipótesis precisas en cada paso podría convertirse en una confusión.

Sin embargo nuestro objetivo principal era presentar este método y mostrar cómo funciona, con la esperanza de que sea útil para otras variedades projectivas. Por ejemplo, sería natural tratar la variedad de la Grassmanniana isotrópica, para la que A. Kuznetsov ha estudiado ya en detalle su categoría derivada (véase [29]).

Chapter 1

Preliminaries

In this chapter we will show some notions that we will use throughout this work. First of all we introduce the concept of Grassmannian $\mathbb{G}(k,n)$ (sometimes we fix the notation $\mathbb{G} = \mathbb{G}(k,n)$) as a projective variety and its universal bundles $\mathcal{Q}$ and $\mathcal{S}$ (see [3]). Moreover we will discuss the Eagon-Northcott complexes (see [17]) and finally the Serre duality (see [20]) since we will use these techniques in the proof of the characterization we give in Section 3.3.

1.1 The Grassmannian

Let V be a $(n+1)-$dimensional vector space over an algebraically closed field $\mathbb{K}$ of characteristic zero and let $\mathbb{P}^n = \mathbb{P}(V)$ be projective space of all the hyperplanes in V, or equivalently we can consider $\mathbb{P}^n = \mathbb{P}(V)$ as the set of the lines in V^*.

Definition 1.1.1. We define the *Grassmann variety* or *Grassmannian*, $\mathbb{G}(k,n) = \mathbb{G}(k,\mathbb{P}^n)$, as the set consisting of all $k-$dimensional linear subspaces of $\mathbb{P}^n$. This variety is naturally identified with the set of $(k+1)-$dimensional linear subspaces of the dual V^* or even with the set of $(k+1)-$dimensional quotients of V.

By abuse of notation, we will identify a $k-$plane of $\mathbb{P}^n$ with the corresponding $(k+1)-$dimensional linear subspace of V^* (we will denote it by Λ).

Note that the projective space $\mathbb{P}^n$ appears as the particular case $k = 0$ ($\mathbb{G}(0,n) = \mathbb{P}^n$) and its dual, $\mathbb{P}^{n^*}$ appears for $k = n-1$ ($\mathbb{G}(n-1,n) = \mathbb{P}^{n^*}$).

Structure variety

Let us give now the structure of a variety to $\mathbb{G}(k,n)$. For this purpose we will cover it by affine charts and determine the patchings in the intersections. So let us fix a system of coordinates $x_0,\dots,x_n$ for $\mathbb{P}^n$ or equivalently a basis $\{w_0,\dots,w_n\}$ of V^*. Then we represent an element $\Lambda \in \mathbb{G}(k,n)$ by a

$(k+1)\times(n+1)$ matrix (that we will call *Plücker matrix*)

$$\Lambda = \begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix} \tag{1.1}$$

where the rows are the coordinates of a basis of Λ.

If we change the basis of Λ, the Plücker matrix changes by multiplying on the left by a non-degenerate square matrix of order $k+1$ corresponding to the change of basis in Λ.

If we assume that the minor corresponding to the first $k+1$ columns is not zero we can multiply by a suitable matrix thus Λ can be represented in a unique way by a matrix:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & b_{0k+1} & \dots & b_{0n} \\ 0 & 1 & \dots & 0 & b_{1k+1} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & b_{kk+1} & \dots & b_{kn} \end{pmatrix} \tag{1.2}$$

Hence, $\mathbb{G}(k,n)$ contains an open affine subset of dimension $(k+1)(n-k)$ of coordinates $b_{0k+1},\dots,b_{kn}$. This subset can be described as the set of $k-$planes that do not meet the $(n-k-1)-$plane of equations $x_0 = \dots = x_k = 0$. Since at least one of the minors of order $k+1$ of the Plücker matrix (1.1) is not zero, $\mathbb{G}(k,n)$ can be covered by $\binom{n+1}{k+1}$ affine pieces.

Notation 1.1.2. We will denote by $U_{i_0,\dots,i_k}$ the open affine subset of $\mathbb{G}(k,n)$ corresponding to subspaces that do not meet the $(n-k-1)-$plane of equations $x_{i_0} = \dots = x_{i_k} = 0$, or equivalently those subspaces such that the maximal minor of the Plücker matrix (1.1) obtained when considering the columns $i_0,\dots,i_k$ is not zero.

We can easily see that the change of coordinates from one piece to another is just given by quotients of polynomials on the coordinates. In conclusion we have that $\mathbb{G}(k,n)$ can be viewed as an abstract manifold of dimension $(k+1)(n-k)$.

1.1.1 The Plücker embedding

In order to view $\mathbb{G}(k,n)$ as a projective variety, one needs to consider the *Plücker embedding*

$$\begin{array}{rccc} \varphi_{k,n}: & \mathbb{G}(k,\mathbb{P}(V)) & \longrightarrow & \mathbb{P}(\bigwedge^{k+1} V^*) = \mathbb{P}^{\binom{n+1}{k+1}-1} \\ & L[v_0,\dots,v_k] & \longmapsto & [v_0\wedge\dots\wedge v_k] \end{array}$$

where $L[v_0,\dots,v_k]$ represents the linear span in $\mathbb{P}(V)$ of the points represented by independent vectors $v_0,\dots,v_k \in V^*$ and $[v_0\wedge\dots\wedge v_k]$ means the point of $\mathbb{P}(\bigwedge^{k+1} V^*)$ represented by $v_0\wedge\dots\wedge v_k$ (taking in $\bigwedge^{k+1} V^*$ the basis $\{\dots,w_{i_0}\wedge\dots\wedge w_{i_k},\dots\}$). We call $\mathbb{P}^{\binom{n+1}{k+1}-1} = \mathbb{P}^N$ Plücker space.

Let us fix a basis for V^* and the induced one for $\bigwedge^{k+1} V^*$, then $\varphi_{k,n}$ associates to the space generated by the rows $v_0,\dots,v_k$ of the Plücker matrix (1.1) the point in $\mathbb{P}(\bigwedge^{k+1} V)$ whose coordinates are just the maximal minors of the matrix. This means that, $\varphi_{k,n}$ associates to a space defined by (1.1) the point in $\mathbb{P}^{\binom{n+1}{k+1}-1}$ whose coordinates consist of all the minors of order $k+1$ of that matrix.

Moreover it is easy to check that $\varphi_{k,n}$ is well defined: If $v'_0, \dots, v'_k$ is another basis of the same space Λ and A is the matrix that changes from one basis into another, then $v'_0 \wedge \dots \wedge v'_k = (\det A)(v_0 \wedge \dots \wedge v_k)$, so that we obtain the same point in $\mathbb{P}(\bigwedge^{k+1} V)$.

Definition 1.1.3. The homogeneous coordinates in $\mathbb{P}(\bigwedge^{k+1} V)$ induced by a choice of coordinates in $\mathbb{P}(V)$ are called *Plücker coordinates* and they are denoted by $p_{i_0,\dots,i_k}$. Hence, $p_{i_0,\dots,i_k}$ are the coordinates corresponding to the determinant of the $(k+1) \times (k+1)$ matrix obtained by taking the columns $i_0, \dots, i_k$ of the Plücker matrix (1.1).

It is not hard to see that $\varphi_{k,n}$ provides an embedding of $\mathbb{G}(k,n)$ in $\mathbb{P}(\bigwedge^{k+1} V)$ as an algebraic variety. For each affine open set $V_{i_0,\dots,i_k} = \{p_{i_0,\dots,i_k} \neq 0\} \subseteq \mathbb{P}(\bigwedge^{k+1} V)$ we observe that

$$\varphi_{k,n}(\mathbb{G}(k,n)) \bigcap V_{i_0,\dots,i_k} = \varphi_{k,n}(U_{i_0,\dots,i_k})$$

so it is enough to prove that $\varphi_{k,n}|_{U_{i_0,\dots,i_k}}$ is an algebraic embedding in $V_{i_0,\dots,i_k}$. Let us work in $U_{0,\dots,k}$ and we can use (1.2) as a matrix representation. For this matrix we have $p_{0,\dots,k} = 1$. Consider the rest of Plücker coordinates as the coordinates of $V_{0,\dots,k}$. On the other hand, all the coordinates of $U_{0,\dots,k}$ (that are just the entries (b_{ij})) appear as coordinates of the map $\varphi_{k,n}$. More precisely we have that each b_{ij} appears up to a sign as the minor of the matrix (1.2) in which we take the columns $0, \dots, i-1, i+1, \dots, k, j$. This proves that $\varphi_{k,n}$ is an embedding. It is also algebraic because the rest of the Plücker coordinates are just polynomials in the previous coordinates. Hence,

Proposition 1.1.4. *There exists a canonical embedding*

$$\begin{array}{rccc} \varphi_{k,n}: & \mathbb{G}(k,\mathbb{P}(V)) & \longrightarrow & \mathbb{P}(\bigwedge^{k+1} V^*) = \mathbb{P}^{\binom{n+1}{k+1}-1} \\ & L[v_0,\dots,v_k] & \longmapsto & [v_0 \wedge \dots \wedge v_k] \end{array}$$

called the Plücker embedding, whose coordinates of the arrival space are called Plücker coordinates. Moreover, the image of $\mathbb{G}(k,n)$ through $\varphi_{k,n}$ is a projective variety in the considered ambient projective space.

We also consider $i_0 \leq \dots \leq i_k$ since $p_{i_0 \dots i_k} = sg(\sigma) p_{\sigma(i_0)\dots\sigma(i_k)}$ for any permutation σ of the elements $\{i_0, \dots, i_k\}$. In fact $i_0 < \dots < i_k$ since $p_{i_0,\dots,i_k}$ will be zero if there exists any of those equalities.

Let us give an example that will help us to understand the definitions given before.

Example 1.1.5. Let us consider the variety $\mathbb{G}(1,3) \subseteq \mathbb{P}^5$. Each line in $\mathbb{P}^3$ can be represented by two independent points spanning it and so the matrix associated will be:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

where a_i and b_j are the coordinates of the two points. In this case, the six Plücker coordinates will be given by:

$$p_{0,1} = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \qquad p_{0,2} = \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} \qquad p_{0,3} = \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix}$$

$$p_{1,2} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \qquad p_{1,3} = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \qquad p_{2,3} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

The vanishing of the determinant of the matrix:

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

allows us to write the equation that defines the Grassmannian, seen as embedded in $\mathbb{P}^5$, which is:

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

This is the only equation of $\mathbb{G}(1,3)$ since it defines a $4-$dimensional irreducible variety that contains $\mathbb{G}(1,3)$ (that is also $4-$dimensional).

1.1.2 The universal bundles

First of all, let us recall the definition of vector bundle over an algebraic variety and its main properties.

Definition 1.1.6. A *vector bundle of rank r* (or line bundle if $r = 1$) over an algebraic variety X is an algebraic variety F equipped with a morphism $\pi : F \longrightarrow X$ such that there exists a covering $X = \bigcup_{i\in I} U_i$ by (Zariski) open subsets such that:

(i) For each $i \in I$ there is an isomorphism $\psi_i : \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{K}^r$ satisfying that the composition $\pi \circ \psi^{-1} : U_i \times \mathbb{K}^r \longrightarrow U_i$ is the first projection.

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\pi"] & U_i \times \mathbb{K}^r \arrow[dl] \\ U_i & \end{tikzcd}$$

(ii) For each $i, j \in I$ there is an $(r \times r)-$matrix A_{ij} (called transition matrix, or transition function if $r = 1$) whose entries are regular functions in $U_i \bigcap U_j$ satisfying that the composition

$$\varphi_{ij} = \psi_j \circ \psi^{-1}_{i|U_i \bigcap U_j} : (U_i \bigcap U_j) \times \mathbb{K}^r \longrightarrow \pi^{-1}(U_i \bigcap U_j) \longrightarrow (U_i \bigcap U_j) \times \mathbb{K}^r$$

takes the form

$$\varphi_{ij}(x, v) = (x, A_{ij}(x)v)$$

Hence,

$$\begin{array}{lcl} (U_i \cap U_j) \times \mathbb{K}^r & \longrightarrow \pi^{-1}((U_i \cap U_j) \longrightarrow & (U_i \cap U_j) \times \mathbb{K}^r \\ (x, v) & \longmapsto & (x, A_{ij}(x) \cdot v) \end{array}$$

It is clear that any subpartition of the covering still satisfies conditions (i) and (ii). In the case that only one open set is needed, i.e. $F = X \times \mathbb{K}^r$ and π is the first projection, we say that F is a *trivial bundle* of rank r.

Remark 1.1.7. Condition (i) is saying that, for any $x \in X$ the set $\pi^{-1}(x)$ (called the *fiber of the vector bundle at the point* x, F_x), is bijective to $\mathbb{K}^r$, and that locally the fibers are glued to produce a trivial bundle.

From this, it is clear that, in condition (ii), the first coordinate of $\varphi_{ij}(x, v)$ must be x. Thus, condition (ii) is just saying that the fibers of F glue together in different trivial representations in a linear way. In other words, the fibers of the vector bundle have to be regarded as vector spaces.

Now consider an element $\Lambda \in \mathbb{G}(k, n)$ as a $k-$dimensional subspace of the vector space V^*.

Notation 1.1.8. We will use * for denoting the dual of a vector space and $^\vee$ for denoting the dual of a vector bundle.

Depending on which point of view we take there are two incidence diagrams in which the maps are natural projections:

$$\begin{tikzcd} & I = \{(p, \Lambda) \in \mathbb{P}(V) \times \mathbb{G} \,|\, p \in \Lambda\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}(V) & & \mathbb{G} \end{tikzcd} \tag{1.3}$$

$$\begin{tikzcd} & \mathcal{Q}^\vee = \{(v, \Lambda) \in V^* \times \mathbb{G} \,|\, v \in \Lambda\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ V^* & & \mathbb{G} \end{tikzcd} \tag{1.4}$$

Definition 1.1.9. We can define the universal bundles by considering the maps p and q of the first incidence diagram (1.3). Consider also in $\mathbb{P}^n = \mathbb{P}(V)$ the Euler exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}(1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

and pull it back to I via p and them push it forward to $\mathbb{G}(k, n)$. Then we get the *universal exact sequence* of $\mathbb{G}(k, n)$

$$0 \longrightarrow \mathcal{S}^\vee \xrightarrow{\varphi} V \otimes \mathcal{O} \xrightarrow{\phi} \mathcal{Q} \longrightarrow 0 \tag{1.5}$$

where the bundles

$$\mathcal{S}^\vee = q_* p^*(\Omega_{\mathbb{P}^n}(1)) \qquad \mathcal{Q} = q_* p^*(\mathcal{O}_{\mathbb{P}^n}(1)) \tag{1.6}$$

are called *universal subbundle* of rank $n-k$ and *universal quotient bundle* of rank $k+1$ respectively.

Remark 1.1.10. We can associate to each universal bundle its corresponding locally free sheaf but we will not distinguish between a vector bundle and its locally free sheaf of sections.

Now consider the second incidence diagram (1.4). We can also view the vector bundle $\mathcal{Q}$ as the dual of the bundle $V^* \times \mathbb{G}$. Therefore we can also consider the quotient vector bundle,

$$V^* \times \mathbb{G}/\mathcal{Q}^\vee \simeq \mathcal{S}$$

Remark 1.1.11. The notation we use is consistent if we look the Grassmannian space as the quotient space of V. It may not be standard notation but we prefer $\mathcal{S}$ and $\mathcal{Q}$ to have the same behavior when dualizing since $\mathbb{G}(k, \mathbb{P}^n) \simeq \mathbb{G}(n-k-1, \mathbb{P}^{n^*})$. Hence, the universal exact sequence for $\mathbb{G}(n-k-1, \mathbb{P}^{n^*})$ is the dual of the universal exact sequence (1.5). In this way, we can interchange the bundles $\mathcal{S}$ and $\mathcal{Q}$:

- the universal quotient bundle can be interpreted as the dual of the subbundle $\mathcal{Q}^\vee$ of the trivial vector bundle $V^* \times \mathbb{G}$,

$$\mathcal{Q}^\vee = \{(v, l) \text{ such that } v \text{ is a vector of the linear space } \vec{l} \text{ that defines } L\}$$

- the universal vector bundle $\mathcal{S} := V^* \times \mathbb{G}/\mathcal{Q}^\vee$ can be interpreted as the dual of the subbundle of $V^* \times \mathbb{G}$ that consists in

$$\mathcal{S} = \{(h, l) \text{ such that } H \text{ vanishes in } L\}$$

From identities (1.6) we obtain the following properties.

Proposition 1.1.12. *There exist natural identifications:*

$$H^0(\mathbb{G}(k,n), \mathcal{Q}) = V \qquad H^0(\mathbb{G}(k,n), \mathcal{S}) = V^*$$

Although $H^j(\mathbb{G}(k,n), \mathcal{Q}) = 0$ and $H^j(\mathbb{G}(k,n), \mathcal{S}) = 0$ for all $j > 0$.

Under these identifications s independent sections of $\mathcal{Q}$ correspond to a linear subspace $A \subset \mathbb{P}^n$ of codimension s.

If $s \leq k+1$, the dependency locus of these sections is just the set of $k-$planes meeting A in dimension at least $k-s+1$.

Analogously, $r+1$ independent sections of $\mathcal{S}$ correspond to a linear subspace $B \subset \mathbb{P}^n$ of dimension r.

If $r+1 \leq n-k$, the dependency locus of this sections is the set of $k-$planes meeting B.

If we take the particular cases $s = k+1$ and $r+1 = n-k$ in Proposition 1.1.12, then we get the same dependency locus. Concretely we get the following corollary:

Corollary 1.1.13. *The set of $k-$planes that meet a fixed $(n-k-1)-$plane is the dependency locus of $k+1$ sections of $\mathcal{Q}$, or $n-k$ sections of $\mathcal{S}$. In particular we have the identification of invertible bundles*

$$\bigwedge^{k+1} \mathcal{Q} \simeq \bigwedge^{n-k} \mathcal{S}$$

(that can be deduced from the universal exact sequence (1.5)) and also isomorphisms to $\mathcal{O}(1)$.

Remark 1.1.14. Since the cotangent bundle over $\mathbb{G}(k,n)$ is $\Omega_{\mathbb{G}(k,n)} = \mathcal{S}^\vee \otimes \mathcal{Q}^\vee$ by taking the top exterior power (i.e. the determinant) it follows that the canonical bundle over $\mathbb{G}(k,n)$ is just $\omega_{\mathbb{G}(k,n)} \simeq \mathcal{O}(-n-1)$.

Remark 1.1.15. Let us write in this remark more natural isomorphisms that we know between the universal bundles over the Grassmannians $\mathbb{G}(k,n)$:

- $\mathcal{Q}^\vee \simeq \bigwedge^k \mathcal{Q}(-1)$. For the particular case $k=1$ we get $\mathcal{Q}^\vee \simeq \mathcal{Q}(-1)$ and if we apply the $j-$th symmetric power results $S^j\mathcal{Q}^\vee \simeq (S^j\mathcal{Q})(-j)$
- $\bigwedge^j \mathcal{S}^\vee \simeq \bigwedge^{n-k-j} \mathcal{S}(-1)$ (where $\bigwedge^j$ denotes the $j-$th wedge power)

1.2 The Eagon-Northcott complex

The complexes we are describing in this section will be used many times in the proof of the previous lemmas to the main theorem. For more details see [17].

Theorem 1.2.1. *Given an exact sequence of vector bundles:*

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with $rank(F_1) = r_1$, $rank(F_2) = r_2$ and $rank(F) = r$ then there exists the following natural maps:

$$\bigwedge^{i} F^\vee \otimes S^{j-i}F_2^\vee \longrightarrow \bigwedge^{i} F^\vee \otimes F_2^\vee \otimes S^{j-i-1}F_2^\vee \longrightarrow \bigwedge^{i} F^\vee \otimes F^\vee \otimes S^{j-i-1}F_2^\vee \longrightarrow \bigwedge^{i+1} F^\vee \otimes S^{j-i-1}F_2^\vee$$

From these maps arise the following long exact sequence for each $k \leq r_1$:

$$0 \longrightarrow S^kF_2^\vee \longrightarrow F^\vee \otimes S^{k-1}F_2^\vee \longrightarrow \bigwedge^{2} F^\vee \otimes S^{k-2}F_2^\vee \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{i} F^\vee \otimes S^{j-i}F_2^\vee \longrightarrow \bigwedge^{i+1} F^\vee \otimes S^{j-i-1}F_2^\vee \longrightarrow \dots$$

$$\dots \longrightarrow F_2^\vee \otimes \bigwedge^{k-1} F^\vee \longrightarrow \bigwedge^{k} F^\vee \longrightarrow \bigwedge^{k} F_1^\vee \longrightarrow 0$$

Example 1.2.2. In the case of $\mathbb{G}(k,n)$, if we apply Theorem 1.2.1 and Remark 1.1.15 to the universal exact sequence (1.5) we obtain the following Eagon-Northcott complex:

$$0 \longrightarrow S^j\mathcal{Q}(-j) \longrightarrow V^* \otimes S^{j-1}\mathcal{Q}(-j+1) \longrightarrow \dots \longrightarrow \bigwedge^{j-1} V^* \otimes \mathcal{Q}(-1) \longrightarrow \bigwedge^j V^* \otimes \mathcal{O} \longrightarrow \bigwedge^j \mathcal{S} \longrightarrow 0 \tag{$R_j^\vee$}$$

By dualizing ($R_j^\vee$) we give its dual complex (R_j):

$$0 \longrightarrow \bigwedge^j \mathcal{S}^\vee \longrightarrow \bigwedge^j V \otimes \mathcal{O} \longrightarrow \bigwedge^{j-1} V \otimes \mathcal{Q} \longrightarrow \dots \longrightarrow \bigwedge^2 V \otimes S^{j-2}\mathcal{Q} \longrightarrow V \otimes S^{j-1}\mathcal{Q} \longrightarrow S^j\mathcal{Q} \longrightarrow 0 \tag{R_j}$$

Remark 1.2.3. The universal exact sequence (1.5) is the analogue in $\mathbb{G}(k,n)$ of the Euler sequence in the projective space. The long Koszul exact sequence in the projective space comes by taking the top exterior product in the left map of the Euler sequence, while taking the smaller exterior products produces the Koszul exact sequence truncated at the left. In the case of Grassmannaians of lines we construct these complexes by making the $j-$th wedge power of φ in the universal exact sequence

$$\ldots \longrightarrow \bigwedge^j V^* \otimes \mathcal{O} \overset{\bigwedge^j \varphi^*}{\longrightarrow} \bigwedge^j \mathcal{S} \longrightarrow 0 \qquad (R_j^\vee)$$

$$0 \longrightarrow \bigwedge^j \mathcal{S}^\vee \overset{\bigwedge^j \varphi}{\longrightarrow} \bigwedge^j V \otimes \mathcal{O} \longrightarrow \ldots \qquad (R_j)$$

or, on the other hand, by making the $j-$th symmetric power of ϕ in the universal exact sequence.

$$\ldots \longrightarrow S^j V \otimes \mathcal{O} \overset{S^j\phi}{\longrightarrow} S^j\mathcal{Q} \longrightarrow 0$$

$$0 \longrightarrow S^j\mathcal{Q}^\vee \overset{S^j\phi^*}{\longrightarrow} S^j V^* \otimes \mathcal{O} \longrightarrow \ldots$$

Remark 1.2.4. *In Proposition 2.3.1, with the language of Young diagrams and Schur functors, we will give a generalization of the complex (R_{n-1}) for a Grassmannian of $k-$planes in $\mathbb{P}^n$, $\mathbb{G}(k,n)$:*

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \bigwedge^{n-1} V \otimes \mathcal{O} \longrightarrow \bigwedge^{n-2} V \otimes \mathcal{Q} \longrightarrow \ldots \longrightarrow \bigwedge^{2} V \otimes S^{n-3}\mathcal{Q} \longrightarrow V \otimes S^{n-2}\mathcal{Q} \longrightarrow S^{n-1}\mathcal{Q} \longrightarrow 0 \qquad (R_{n-1})$$

1.3 Cohomology and Serre duality

For main definitions, facts and properties about cohomology and Serre duality see [20].

Definitions and notations about cohomology

Notation 1.3.1. We denote with small letter h just the dimension of the corresponding cohomology

$$h^p(F) = \dim(H^p(F)).$$

Definition 1.3.2. Suppose F a coherent sheaf of a smooth projective variety X with dimension n. We say that F has *no intermediate cohomology* if

$$h^j(F(l)) = 0 \text{ for all } j = 1, 2 \ldots, n-1 \text{ and for all } l \in \mathbb{Z}.$$

Notation 1.3.3. We denote $H^p_*(F) := \bigoplus_{l\in\mathbb{Z}} H^p(F(l))$.

Properties and facts about Serre duality

We recall Serre duality (Theorem 7.6 from [20]) that we state for the case of locally free sheaves (vector bundles) since it is the one we are going to use.

Theorem 1.3.4. *Let $\mathcal{F}$ be a vector bundle over X with dim$(X) = N$, then:*

(a) $H^N(X, \omega_X) \cong \mathbb{K}$. *Fix one such isomorphism;*

(b) *the natural pairing*

$$Ext^j(\mathcal{F}, \omega_X) \times H^{N-j}(X, \mathcal{F}) \longrightarrow H^N(X, \omega_X) \cong \mathbb{K} \tag{1.7}$$

is a perfect pairing of finite-dimensional vector spaces over $\mathbb{K}$;

(c) *for every $j \geq 0$ there is a natural functorial isomorphism*

$$Ext^j(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{N-j}(X, \mathcal{F})^*$$

where * denotes the dual vector space, which for $j = 0$ is the one induced by the pairing of (b).

Remark 1.3.5. Suppose now $X = \mathbb{G}(1, n)$ and recall from Remark 1.1.14 that $\omega_X = \mathcal{O}(-n-1)$. Since (c) can be expressed as:

$$H^{N-j}(X, \mathcal{F}^\vee \otimes \omega_X) \cong H^j(X, \mathcal{F})^*. \tag{1.8}$$

We can thus write:

$$H^{2n-2-j}(\mathbb{G}(1, n), \mathcal{F}^\vee \otimes \mathcal{O}(-n-1)) \cong H^j(\mathbb{G}(1, n), \mathcal{F})^*.$$

Remark 1.3.6. One of the ideas that are used in the proof of Theorem 0.0.3 and our main theorem is the interpretation of the Serre duality as extensions. Observe, for the case of Grassmannian of lines $X = \mathbb{G}(1, n)$, that $H^{2n-2}(\omega_X) = H^{2n-2}(\mathcal{O}(-n-1)) = Ext^{2n-2}(\mathcal{O}, \mathcal{O}(-n-1))$ and the element that generates it is precisely,

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow \bigwedge^{n-1} V \otimes \mathcal{O}(-n) \longrightarrow \bigwedge^{n-2} V \otimes \mathcal{Q}(-n) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{2} V \otimes S^{n-3}\mathcal{Q}(-n) \longrightarrow V \otimes S^{n-2}\mathcal{Q}(-n) \longrightarrow V^* \otimes S^{n-2}\mathcal{Q}(-n+1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-2} V^* \otimes \mathcal{Q}(-2) \longrightarrow \bigwedge^{n-1} V^* \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

that comes from the Eagon-Northcott complexes of Example 1.2.2 ($(R_{n-1}^\vee \otimes \mathcal{O}(-1))$ and $(R_{n-1}) \otimes \mathcal{O}(-n)$, see Remark 1.2.4).

Remark 1.3.7. For any coherent sheaf $\mathcal{F}$ on X where $dim(X) = N$ by Theorem 5.2 (b) of [20] we have that:

$$h^j(\mathcal{F}(l)) = 0 \qquad l >> 0 \qquad j > 0.$$

And if $\mathcal{F}$ is also locally free, then:

$$h^j(\mathcal{F}(l)) = h^{N-j}(\mathcal{F}^\vee(-l) \otimes \omega_X) = 0 \qquad N > j \qquad l << 0.$$

From these equalities we get that if $0 < j < N$:

$$\sum_l h^j(\mathcal{F}(l)) < \infty.$$

Chapter 2

Cohomological tools

In the first section of this chapter we will introduce the notion of Young diagram, its corresponding irreducible representation and also its corresponding Schur functor.

In the second section of this chapter we will discuss Bott's Theorem. First we will study the Littlewood-Richardson rule and then we will show two important special cases of this rule, the Pieri's formula. Finally we will give the algorithm of Bott's Theorem.

2.1 Schur Functors

For any finite-dimensional complex vector space V, we have the canonical decomposition

$$V \otimes V = S^2V \oplus \bigwedge^2 V$$

The group $GL(V)$ acts on $V \otimes V$ and this is the decomposition of $V \otimes V$ into a direct sum of irreducible $GL(V)-$representations. For the next power, we instead have

$$V \otimes V \otimes V = S^3V \oplus \bigwedge^3 V \oplus \text{another space.} \tag{2.1}$$

The theory of Schur functors will allow us to compute the missing space but it requires to establish first some definitions of irreducible representations of $\mathfrak{S}_d$, Young diagrams, ...

2.1.1 Irreducible representations and characters

Definition 2.1.1. A *representation* of a finite group G on a finite-dimensional complex vector space V (or just over an algebraically closed field) is a homomorphism $\rho : G \to GL(V)$ of G to the group of automorphisms of V. We call V itself a representation of G; in this way we will often suppress the symbol ρ and write $g \cdot v$ or gv for $\rho(g)(v)$. The dimension of V is called the *degree* of ρ. A *subrepresentation* of a representation V is a vector subspace W of V which is invariant under G.

We say that a representation V is *irreducible* if there is no proper nonzero invariant subspace W of V (this means that the only subrepresentations are 0 and V).

Example 2.1.2. Suppose $G = \mathfrak{S}_3$ the symmetric group of permutations of 3 elements. Suppose $g \in \mathfrak{S}_3$ and $v \in \mathbb{C}^3$. We have three natural irreducible representations,

- the trivial representation U, $g \cdot v = v$, which is one-dimensional.
- the alternating representation U', $g \cdot v = sgn(g) \cdot v$, which is also one-dimensional.
- the standard representation, which is not so easy to compute and is a two-dimensional representation. We have a natural permutation representation, in which $\mathfrak{S}_3$ acts on $\mathbb{C}^3$ by permuting coordinates. Explicitly, if $\{e_1, e_2, e_3\}$ is the standard basis, then consider $g \cdot e_i = e_{g(i)}$. But this representation is not irreducible. Since the line spanned by the sum $(1, 1, 1)$ of the basis vectors is invariant, let us consider the complementary subspace

$$V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}$$

  as the standard representation of $\mathfrak{S}_3$ that we were looking for.

In fact, each representation corresponds to one of the direct summand of the equality (2.1): U corresponds to the symmetric power, U' with the wedge power and finally V corresponds to the new space we built (we will give the precise construction by making use of the Schur functor at the end of Example 2.1.14 and in Example 2.1.16).

There is a remarkably effective tool for understanding the representations of a finite group G, called *character theory*. Let us start with the main definition.

Definition 2.1.3. If V is a representation of G, its *character* $\mathcal{X}_V$ is the complex-valued function on the group defined by the trace of g on V:

$$\mathcal{X}_V(g) = Tr(g|_V)$$

Note that $\mathcal{X}_V(1) = dim(V)$.

The character of a representation of a group G is really a function on the set of conjugacy classes in G. Hence, we express the basic information about the irreducible representations of a group in the form of a *character table*. This is a table with the conjugacy classes $[g]$ of G listed across the top, usually given by a representative g, with the number of elements in each conjugacy class over it; the irreducible representations of G listed on the left; and in the appropriate box, the value of the character $\mathcal{X}_V$ on the conjugacy class $[g]$.

Example 2.1.4. Let us compute the character table of $\mathfrak{S}_3$. We take the three conjugacy classes $[1]$, $[(12)]$ and $[(123)]$ and the three representations computed in Example 2.1.2, U (trivial representation), U' (alternating representation) and V (standard representation). The trivial representation U takes the values $(1, 1, 1)$ on the three conjugacy classes, whereas the alternating representation U' takes the values $(1, -1, 1)$. In order to compute these values for the standard representation, we provide the specific maps of the representations for each element $g \in \mathfrak{S}_3$.

$$\begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (1) & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (12) & \longmapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (123) & \longmapsto & \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \end{array}$$

$$\begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (13) & \longmapsto & \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (132) & \longmapsto & \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{ccc} \mathfrak{S}_3 & \longrightarrow & GL(V) \\ (23) & \longmapsto & \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \end{array}$$

We take an action on $\mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle$ that sends $g(e_i) = e_{g(i)}$. Consider $\mathbb{C}^3 / _{\langle e_1+e_2+e_3 \rangle} = \langle \bar{e_1}, \bar{e_2} \rangle$

Finally the character table of $\mathfrak{S}_3$ is:

	1 (1)	3 (12)	2 (123)
U	1	1	1
U'	1	-1	1
V	2	0	-1

We also notice that the values of the first column correspond to the dimension of each representation. Moreover, the values of the first row are always 1 since each of them corresponds to the trivial representation.

Let us remark the following:

Proposition 2.1.5. *(Proposition 2.30 of [19]) The number of irreducible representations of a group G is equal to the number of its conjugacy classes.*

2.1.2 Young diagrams

We recall the notation, main definitions and properties from [19] and [37]. Let $\mathfrak{S}_d$ be the symmetric group of permutations over d elements.

The number of irreducible representations of $\mathfrak{S}_d$ is the number of conjugacy classes, which coincides with the number $p(d)$ of partitions of d: $d = \lambda_1 + \ldots + \lambda_m$, $\lambda_1 \geq \ldots \geq \lambda_m \geq 1$.

Definition 2.1.6. To a partition λ is associated a *Young diagram* denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq 0$. It consists of a collection of boxes ordered in consecutive rows, where the $i-$th row has exactly λ_i boxes (the rows of boxes are aligned top-left). The number of boxes of λ is denoted by $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_m$.

For example, the Young diagram associated to $\lambda = (7, 4, 3, 1, 1)$ with $|\lambda| = 16$ corresponds to:

$$\begin{array}{c} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{array} \ydiagram{7,4,3,1,1} \tag{2.2}$$

Definition 2.1.7. The *conjugate partition* $\lambda' = (\mu_1, \dots, \mu_r) = \mu$ to a partition λ is defined by interchanging rows and columns in the Young diagram.

With the same λ as before we have that the conjugate partition, $\lambda' = (5, 3, 3, 2, 1, 1, 1) = \mu$, gives the following conjugate Young diagram:

$$\begin{array}{ll} \mu_1 & \square\square\square\square\square \\ \mu_2 & \square\square\square \\ \mu_3 & \square\square\square \\ \mu_4 & \square\square \\ \mu_5 & \square \\ \mu_6 & \square \\ \mu_7 & \square \end{array} \tag{2.3}$$

Young diagrams can be used to describe projection operators for the regular representation, which will then give the irreducible representations of $\mathfrak{S}_d$.

Definition 2.1.8. For a given Young diagram we can number the boxes. Any filling of λ with numbers is called a *Young tableau.*

Just to fix convention, for a given Young diagram, number the boxes consecutively (from left to right and top to bottom). Here we use all numbers from 1 to d in order to fill d boxes. More generally, a tableau can allow repetitions of numbers. Each filling describes a vector in $V^{\otimes d}$.

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & & \\ 9 & 10 & 11 & & \\ 12 & 13 & & & \\ 14 & & & & \end{array} \tag{2.4}$$

Given a tableau, say the canonical one shown, define two subgroups of the symmetric group (due to the filling, we can consider the elements of $\mathfrak{S}_d$ as permuting the boxes and the representations constructed will be isomorphic to the ones made with the canonical tableau):

$$P = P_\lambda = \{g \in \mathfrak{S}_d : g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in \mathfrak{S}_d : g \text{ preserves each column}\}$$

They depend on λ but also on the filling of λ. In the group algebra $\mathbb{C}\mathfrak{S}_d$, we introduce two elements corresponding to the subgroups and we set

$$a_\lambda = \sum_{g \in P} e_g \quad \text{and} \quad b_\lambda = \sum_{g \in Q} sgn(g) \cdot e_g$$

To see what a_λ and b_λ do, observe that if V is any vector space and $\mathfrak{S}_d$ acts on the $d-$th tensor power $V^{\otimes d}$ by permuting factors, the image of the element $a_\lambda \in \mathbb{C}\mathfrak{S}_d \to End(V^{\otimes d})$ is just the subspace

$$Im(a_\lambda) = S^{\lambda_1}V \otimes S^{\lambda_2}V \otimes \cdots \otimes S^{\lambda_k}V \subset V^{\otimes d}$$

where the inclusion on the right is obtained by grouping the factors of $V^{\otimes d}$ according to the rows of the Young tableau. Similarly, the image of b_λ on this tensor power is

$$Im(b_\lambda) = \bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \ldots \otimes \bigwedge^{\mu_r} V \subset V^{\otimes d}$$

where μ is the conjugate partition to λ. This is because $\mathfrak{S}_d$ acts on the $d-$th tensor power $V^{\otimes d}$ by permuting factors.

Definition 2.1.9. Finally, we set

$$c_\lambda = a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d$$

that is called the *Young symmetrizer* corresponding to λ.

Example 2.1.10. Let us compute c_λ in some particular examples:

- When $\lambda = (d)$ we have $c_{(d)} = a_{(d)} = \sum_{g \in \mathfrak{S}_d} e_g$ and the image of c_λ on $V^{\otimes d}$ is $S^d V$
- When $\lambda = \underbrace{(1, \ldots, 1)}_{d}$ we have $c_{(1,\ldots,1)} = b_{(1,\ldots,1)} = \sum_{g \in \mathfrak{S}_d} sgn(g) \cdot e_g$ and the image of $c_{(1,\ldots,1)}$ on $V^{\otimes d}$ is $\bigwedge^d V$
- Suppose now the following particular Young tableau of $\lambda = (2, 1)$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$$

First, let us compute the permutations that preserve rows and columns plus their corresponding a_λ and b_λ:

$$P = \{(1), (1,2)\} \text{ preserves rows} \Longrightarrow a_\lambda = e_{(1)} + e_{(12)} \text{ changes rows}$$

$$Q = \{(1), (1,3)\} \text{ preserves columns} \Longrightarrow b_\lambda = e_{(1)} - e_{(13)} \text{ changes columns}$$

Finally we compute c_λ:

$$c_\lambda = e_{(1)} + e_{(12)} - e_{(13)} - e_{(132)}$$

Theorem 2.1.11. *(Theorem 4.1 of [19]) Some scalar multiple of c_λ is idempotent, i.e., $c_\lambda^2 = n_\lambda c_\lambda$, and the image of c_λ (by right multiplication on $\mathbb{C}\mathfrak{S}_d$) is an irreducible representation V_λ of $\mathfrak{S}_d$. Every irreducible representation of $\mathfrak{S}_d$ can be obtained in this way for a unique partition.*

Note that this theorem gives a direct correspondence between conjugacy classes in $\mathfrak{S}_d$ and irreducible representations of $\mathfrak{S}_d$.

Example 2.1.12. We have computed in Example 2.1.2 the irreducible representations for $\mathfrak{S}_3$. It is not hard to work out which representations came from which Young diagram:

- for $\lambda = (3)$ we get the trivial representation $\mathbb{C}\mathfrak{S}_3 \cdot c_\lambda = U$ $\quad \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline\end{array}$
- for $\lambda = (1, 1, 1)$ we get the alternating representation $\mathbb{C}\mathfrak{S}_3 \cdot c_\lambda = U'$ $\quad \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$
- for $\lambda = (2, 1)$ we get the standard representation $\mathbb{C}\mathfrak{S}_3 \cdot c_\lambda = V$ $\quad \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{|c}{} \\ \cline{1-1}\end{array}$

2.1.3 Definition and first properties of Schur Functor

We continue with the same notations and arguments in Section 2.1.2, so, the symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$, say on the right, by permuting the factors

$$(v_1 \otimes \ldots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$$

This action commutes with the left action of $GL(V)$. We denote the image of c_λ on $V^{\otimes d}$ by $\mathbb{S}_\lambda V$.

$$\mathbb{S}_\lambda V = Im(c_\lambda|_{V^{\otimes d}})$$

which is again a representation of $GL(V)$.

Definition 2.1.13. We call the functor that associates $V \rightsquigarrow \mathbb{S}_\lambda V$ the *Schur functor* corresponding to λ.

Example 2.1.14. Let us see some examples of Schur functors.

- (*Symmetric power*) For the partition $\lambda = (d)$ corresponds to the functor $V \rightsquigarrow S^d V$

$$\mathbb{S}_{(d)}V = S^d V \Longrightarrow \underbrace{\boxed{}\boxed{}\boxed{}\cdots\boxed{}\boxed{}}_{d \text{ boxes}} \tag{2.5}$$

- (*Exterior power*) For the partition $\lambda = (\underbrace{1, \ldots, 1}_{d})$ corresponds to the functor $V \rightsquigarrow \bigwedge^d V$

$$\mathbb{S}_{\underbrace{(1,1,\ldots,1)}_{d}} V = \bigwedge^d V \Longrightarrow \left.\begin{array}{c} \boxed{} \\ \boxed{} \\ \boxed{} \\ \vdots \\ \boxed{} \\ \boxed{} \end{array}\right\} \text{d boxes} \tag{2.6}$$

- For the partition $\lambda = (2, 1)$ we have computed c_λ in Example 2.1.10:

$$c_\lambda = e_{(1)} + e_{(12)} - e_{(13)} - e_{(132)}$$

So the image of c_λ is the subspace of $V^{\otimes 3}$ spanned by all vectors:

$$c_\lambda(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes v_2 \otimes v_3 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_3 \otimes v_1 \otimes v_2$$

If we consider that $\bigwedge^2 V \otimes V$ is embedded in $V^{\otimes 3}$ by the map

$$\begin{array}{ccc} \bigwedge^2 V \otimes V & \longrightarrow & V \otimes V \otimes V \\ u_1 \wedge u_2 \otimes u_3 & \longmapsto & u_1 \otimes u_3 \otimes u_2 - u_2 \otimes u_3 \otimes u_1 \end{array} \tag{2.7}$$

then the image of c_λ is the subspace of $\bigwedge^2 V \otimes V$ spanned by all vectors

$$c_\lambda(v_1 \otimes v_2 \otimes v_3) = v_1 \wedge v_3 \otimes v_2 + v_2 \wedge v_3 \otimes v_1$$

Finally

$$\begin{array}{ccc} V \otimes V \otimes V & \xrightarrow{c_\lambda} & \mathbb{S}_\lambda V \\ v_1 \otimes v_2 \otimes v_3 & \longmapsto & v_1 \wedge v_3 \otimes v_2 + v_2 \wedge v_3 \otimes v_1 \end{array} \tag{2.8}$$

It is not hard to verify that these vectors span the kernel of the canonical map from $\bigwedge^2 V \otimes V$ to $\bigwedge^3 V$ that sends $u_1 \wedge u_2 \otimes u_3$ to $u_1 \wedge u_2 \wedge u_3$, so we have:

$$\mathbb{S}_\lambda V = Ker(\bigwedge^2 V \otimes V \to \bigwedge^3 V)$$

Hence, we have found the missing space in the decomposition mention at the beginning

$$V \otimes V \otimes V = S^3 V \oplus \bigwedge^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\otimes 2}$$

We can see this in another way:

$$V \otimes V \otimes V = (\bigwedge^2 V \oplus S^2 V) \otimes V = (\bigwedge^2 V \otimes V) \oplus (S^2 V \otimes V) = (\bigwedge^3 V \oplus \mathbb{S}_{(2,1)} V) \oplus (\mathbb{S}_{(2,1)} V \oplus S^3 V)$$

In order to give a general decomposition of $V^{\otimes d}$ into direct sum we state the following proposition:

Proposition 2.1.15. *(Theorem 6.3 (2) of [19]) Let m_λ be the dimension of the irreducible representation V_λ of $\mathfrak{S}_d$ corresponding to λ and $|\lambda| = d$. Then*

$$V^{\otimes d} \cong \bigoplus_\lambda \mathbb{S}_\lambda V^{\otimes m_\lambda}$$

Example 2.1.16. For $V \otimes V \otimes V = V^{\otimes 3}$ we have three possibles Young diagrams $(1, 1, 1)$, (3) and $(2, 1)$ associated to irreducible representations U (one dimensional), U' (one dimensional) and V (two dimensional). Hence,

$$V \otimes V \otimes V = \mathbb{S}_{(1,1,1)} V \oplus \mathbb{S}_{(3)} V \oplus \mathbb{S}_{(2,1)} = S^3 V \oplus \bigwedge^3 V \oplus (\mathbb{S}_{(2,1)} V)^{\otimes 2}$$

Note that some of the $\mathbb{S}_\lambda V$ can be zero if V has small dimension. Let us see concretely when this happens and how to compute the dimension of the Schur functor in the case it is not zero.

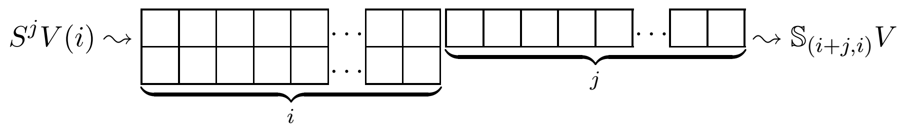
Proposition 2.1.17. *(Theorem 6.3 (1) of [19]) Let $k = dim(V)$:*

- *if $\lambda_{k+1} \neq 0$ then $\mathbb{S}_\lambda V$ is zero*
- *if $\lambda_{k+1} = 0$ then*

$$dim(\mathbb{S}_\lambda V) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

If the number of rows of λ is less than the dimension of V we are allowed to complete λ with zeros at the end.

Remark 2.1.18. Notice that the only Schur functor of a vector space V of dimension 2 are just symmetric powers with some particular twists. This will be computed for the universal bundle $\mathcal{Q}$ of rank 2 in Remark 4.1.25,



2.2 Bott's Theorem

The aim of this section is to compute the cohomology of Schur functors of the universal bundles of the form

$$\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee \qquad (\text{or} \qquad \mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee)$$

where $\lambda = (\lambda_0, \ldots, \lambda_k)$ and $\mu = (\mu_{k+1}, \ldots, \mu_n)$ (or $\eta = (\eta_1, \ldots, \eta_{n-k})$ and $\alpha = (\alpha_0, \ldots, \alpha_k)$). But first, let us give the Littlewood-Richardson rule which is one important property. We use this rule to decompose a tensor product of the form $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ where V is a vector space (or vector bundle). In Section 2.2.2 we will give two particular cases of this rule.

2.2.1 Littlewood-Richardson rule

One important property we want to discuss now is the way in which we decompose a tensor product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ of two Schur functors, with being λ a partition of d and μ a partition of n (we assume λ a conjugacy class of $\mathfrak{S}_d$ and μ a conjugacy class of $\mathfrak{S}_n$). The result is

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \cong \bigoplus_{\upsilon} N_{\lambda\mu\upsilon} \mathbb{S}_\upsilon V \tag{2.9}$$

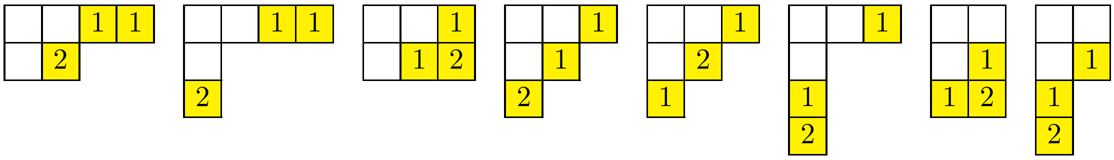
where the sum is over partitions υ with $|\upsilon| = d + n$, and $N_{\lambda\mu\upsilon}$ are called *Littlewood-Richardson numbers* that are determined by the *Littlewood-Richardson rule.* This equality corresponds to (6.7) of [19] and *Theorem* 2.3.4 of [37].

Remark 2.2.1. One remark is in order. The Young diagram associated to each Schur Functor musts fit inside a box with no more rows than the rank of V, otherwise we must remove this summand from the formula (2.9). Moreover, it has to be a partition, in other cases we remove it from the formula.

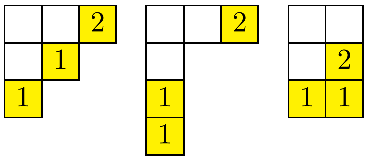
In order to compute the Littlewood-Richardson numbers we give some definitions:

Definition 2.2.2. If $\mu = (\mu_1, \ldots, \mu_r)$, a $\mu-$*expansion* of a Young diagram is obtained by first adding μ_1 boxes to the rows, without being two boxes in the same column, and putting the integer 1 in each of these μ_1 boxes. Then adding similarly μ_2 boxes with a 2 in each of them, continuing until finally μ_r boxes are added with the integer r. The expansion is called *strict* if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first t entries in this list (for any t between 1 and $|\mu| = \mu_1 + \ldots + \mu_r$), each integer p between 1 and $r-1$ occurs at least as many times as the next integer $p+1$.

Example 2.2.3. Consider $\lambda = (2,1)$ and $\mu = (2,1)$. Let us compute all the possibles strict $\mu-$expansions of λ.

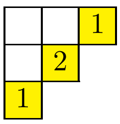
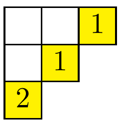


It is not difficult to see that the following $\mu-$expansions of λ are not strict:



Definition 2.2.4. We define the *Littlewood-Richardson numbers* $N_{\lambda\mu\upsilon}$ as the number of ways in which υ can be obtained as a strict $\mu-$extension of λ. Therefore, υ is also a Young diagram with as many boxes as λ and μ together.

Example 2.2.5. Let us compute $N_{\lambda\mu\upsilon}$ in a particular example. Consider $\lambda = (2,1)$, $\mu = (2,1)$ and $\upsilon = (3,2,1)$. Taking a look to Example 2.2.3, there are only two different ways in which we can obtain υ as a strict $\mu-$extension of λ.

$N_{(2,1)(2,1)(3,2,1)} = 2 \Longrightarrow$  and 

2.2.2 Pieri's Formula

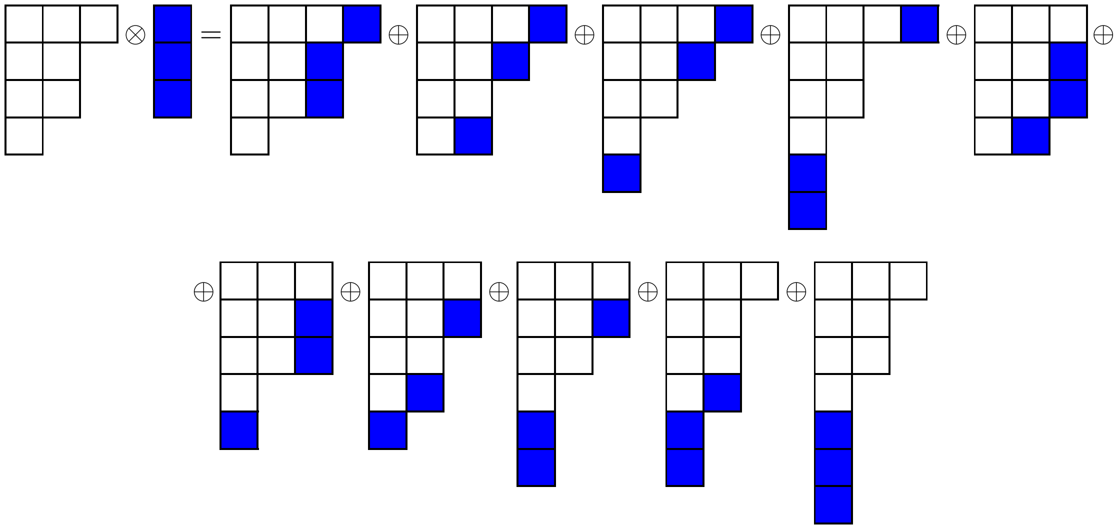
Two important special cases of the Littlewood-Richardson rule are easier to use. We call them *Pieri's Formula.*

Proposition 2.2.6. *(Formula 6.8 and 6.9 of [19])*

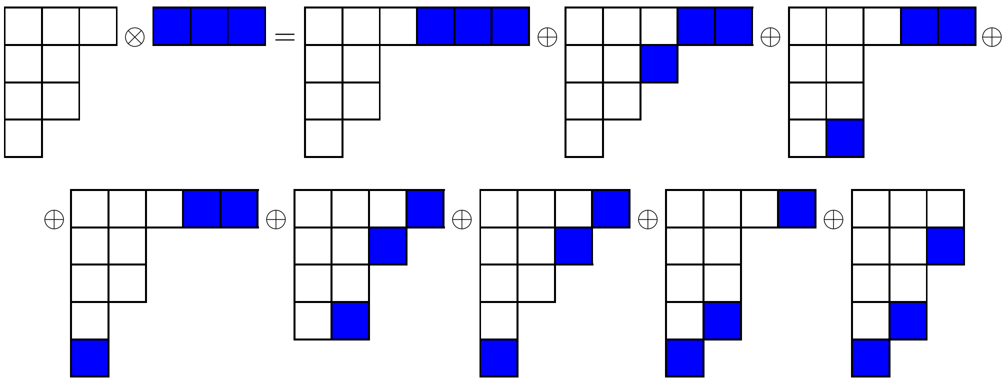
- $\mathbb{S}_\lambda V \otimes \bigwedge^k V \cong \oplus_\mu \mathbb{S}_\mu V$ *where the sum goes over all partitions* μ *whose Young diagram is obtained by adding* k *boxes to the Young diagram* λ*, without being two of them in the same row*
- $\mathbb{S}_\lambda V \otimes S^k V \cong \oplus_\mu \mathbb{S}_\mu V$ *where the sum goes over all partitions* μ *whose Young diagram is obtained by adding* k *boxes to the Young diagram* λ*, without being two of them in the same column*

Example 2.2.7. Let us give some examples of Pieri's formula.

- $\mathbb{S}_{(3,2,2,1)}V \otimes \bigwedge^3 V$ we have to add three boxes but we are not allowed to add two new boxes in the same row:



- $\mathbb{S}_{(3,2,2,1)}V \otimes S^3 V$ we have to add three boxes but we are not allowed to add two new boxes in the same column:



Remark 2.2.8. Let us state some precise properties by using Littlewood-Richardson rule and Pier's formula.

- $\mathbb{S}_\lambda V \otimes V = \mathbb{S}_{(\lambda_1+1,\lambda_2,\ldots,\lambda_k)}V \oplus \mathbb{S}_{(\lambda_1,\lambda_2+1,\ldots,\lambda_k)}V \oplus \ldots \oplus \mathbb{S}_{(\lambda_1,\lambda_2,\ldots,\lambda_k+1)}V$

- Problem of Plethysm: the general problem is to decompose the functor $\mathbb{S}_\lambda(\mathbb{S}_\mu V)$ into Schur functors,
$$\mathbb{S}_\lambda(\mathbb{S}_\mu V) = \bigoplus_{\{v\,:\,|v|=|\lambda|\cdot|\mu|\}} M_{\lambda\mu v}\mathbb{S}_v V$$

and also to decompose $\mathbb{S}_\lambda(V\otimes U)$ into tensor product of Schur functors,

$$\mathbb{S}_\lambda(V\otimes U)=\bigoplus_{|\lambda|=|\mu|=|\upsilon|} M'_{\lambda\mu\upsilon}\mathbb{S}_\mu V\otimes\mathbb{S}_\upsilon U.$$

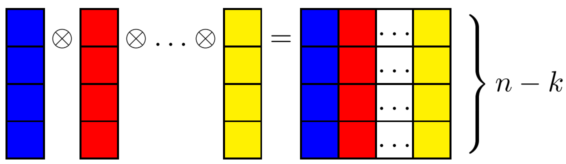
There are some particular cases already solved (for example Corollary 2.3.3 of [37] that gives the result for $S^m(V\otimes U)$ and $\bigwedge^m(V\otimes U)$) but the general problem is to find the multiplicities $M_{\lambda\mu\upsilon}$ and $M'_{\lambda\mu\upsilon}$.

- Problem of Plethysm for Grassmannian: in this particular case we already know,

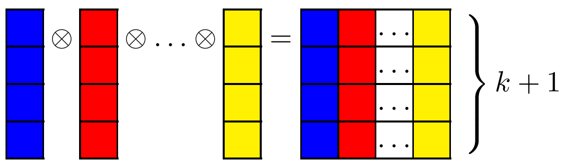
$$\mathbb{S}_\mu\mathcal{Q}=\mathbb{S}_\mu(\bigwedge^{k}\mathcal{Q}^\vee\otimes\bigwedge^{n-k}\mathcal{S}).$$

Remark 2.2.9. Some properties of Grassmannians $\mathbb{G}(k,n)$ by using Pieri's rule:

1. $\mathcal{O}(1)=\bigwedge^{n-k}\mathcal{S}=\mathbb{S}_{\underbrace{(1,\dots,1)}_{n-k}}\mathcal{S}\Rightarrow\mathcal{O}(r)=\underbrace{\mathcal{O}(1)\otimes\ldots\otimes\mathcal{O}(1)}_{r}=\underbrace{\bigwedge^{n-k}\mathcal{S}\otimes\ldots\otimes\bigwedge^{n-k}\mathcal{S}}_{r}=$ $\mathbb{S}_{\underbrace{(r,\dots,r)}_{n-k}}\mathcal{S}$ (we have to consider r columns with $n-k$ boxes)



2. $\mathcal{O}(1)=\bigwedge^{k+1}\mathcal{Q}=\mathbb{S}_{\underbrace{(1,\dots,1)}_{k+1}}\mathcal{S}\Rightarrow\mathcal{O}(r)=\underbrace{\mathcal{O}(1)\otimes\ldots\otimes\mathcal{O}(1)}_{r}=\underbrace{\bigwedge^{k+1}\mathcal{Q}\otimes\ldots\otimes\bigwedge^{k+1}\mathcal{Q}}_{r}=$ $\mathbb{S}_{\underbrace{(r,\dots,r)}_{k+1}}\mathcal{Q}$ (we have to consider r columns with $k+1$ boxes)



3. $\mathcal{O}(-1)=\bigwedge^{n-k}\mathcal{S}^\vee=\mathbb{S}_{\underbrace{(1,\dots,1)}_{n-k}}\mathcal{S}^\vee\Rightarrow\mathcal{O}(-r)=\underbrace{\mathcal{O}(-1)\otimes\ldots\otimes\mathcal{O}(-1)}_{r}=$ $=\underbrace{\bigwedge^{n-k}\mathcal{S}^\vee\otimes\ldots\otimes\bigwedge^{n-k}\mathcal{S}^\vee}_{r}=\mathbb{S}_{\underbrace{(r,\dots,r)}_{n-k}}\mathcal{S}^\vee$ (as in the first case)

4. $\mathcal{O}(-1) = \bigwedge^{k+1} \mathcal{Q}^\vee = \mathbb{S}_{\underbrace{(1,\ldots,1)}_{k+1}} \mathcal{Q}^\vee \Rightarrow \mathcal{O}(-r) = \underbrace{\mathcal{O}(-1) \otimes \ldots \otimes \mathcal{O}(-1)}_{r} =$

$= \underbrace{\bigwedge^{k+1} \mathcal{Q}^\vee \otimes \ldots \otimes \bigwedge^{k+1} \mathcal{Q}^\vee}_{r} = \mathbb{S}_{\underbrace{(r,\ldots,r)}_{k+1}} \mathcal{Q}^\vee$ (as in the second case)

2.2.3 Bott's algorithm

Now we want to compute the cohomology of some symmetric and skew-symmetric product of the universal bundles over the Grassmannians $\mathbb{G}(k,n)$, let us introduce an algorithm that calculates the cohomology of $\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee$ (or $\mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee$). Since rank of $\mathcal{Q}$ is $k+1$, and rank of $\mathcal{S}^\vee$ is $n-k$, let $\lambda = (\lambda_0, \ldots, \lambda_k)$ and $\mu = (\mu_{k+1}, \ldots, \mu_n)$ two partitions. We call

$$\nu_{(\lambda,\mu)} := \mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee.$$

By considering $\eta = (\eta_1, \ldots, \eta_{n-k})$ and $\alpha = (\alpha_{n-k+1}, \ldots, \alpha_{n+1})$ instead of λ and μ we can also consider

$$\nu_{(\eta,\alpha)} := \mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee.$$

We put always in the first place the bundle that has sections.

We extract here from Remark 4.1.5, Corollary 4.1.7 and Corollary 4.1.9 of [37] the following algorithm that gives the values of $H^j(\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee)$ for all $j \geq 0$.

Proposition 2.2.10. *Let $\nu := (\lambda_0, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_n)$ and suppose $j = 0$. We have two possibilities:*

1. *If ν is a partition, then $H^j(\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee) = \mathbb{S}_\nu \mathbb{K}^{n+1}$ and $H^i(\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee) = 0$ for all $i \neq j$*
2. *If ν is not a partition, then consider the first l such that $\nu_l < \nu_{l+1}$. Two possibilities can occur:*
   - *If $\nu_{l+1} - \nu_l = 1$, then $H^i(\mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee) = 0$ for all $i \geq 0$*
   - *If $\nu_{l+1} - \nu_l \neq 1$, then consider $\nu := (\nu_1, \ldots, \nu_{l-1}, \nu_{l+1} - 1, \nu_l + 1, \nu_{l+2}, \ldots, \nu_{n+1})$ and $j = j+1$, and go back to 1.*

If we want to compute $H^j(\mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee)$, at the beginning of Proposition 2.2.10 we have to consider $\nu_{(\eta,\alpha)} := (\eta_1, \ldots, \eta_{n-k}, \alpha_{n-k+1}, \ldots, \alpha_{n+1})$. Now let us rewrite the algorithm for this case:

Proposition 2.2.11. *Let $\nu := (\eta_1, \ldots, \eta_{n-k}, \alpha_{n-k+1}, \ldots, \alpha_{n+1})$ and suppose $j = 0$. We have two possibilities:*

1. *If ν is a partition, then $H^j(\mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee) = \mathbb{S}_\nu \mathbb{K}^{n+1}$ and $H^i(\mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee) = 0$ for all $i \neq j$*
2. *If ν is not a partition, then consider the first l such that $\nu_l < \nu_{l+1}$. Two possibilities can occur:*
   - *If $\nu_{l+1} - \nu_l = 1$, then $H^i(\mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee) = 0$ for all $i \geq 0$*
   - *If $\nu_{l+1} - \nu_l \neq 1$, then consider $\nu := (\nu_1, \ldots, \nu_{l-1}, \nu_{l+1} - 1, \nu_l + 1, \nu_{l+2}, \ldots, \nu_{n+1})$ and $j = j+1$, and go back to 1.*

2.2.4 Some particular computations

If we want to apply the above algorithm, we need to transform the universal bundles and their combinations in an expression of the form $\nu_{(\lambda,\mu)} = \mathbb{S}_\lambda \mathcal{Q} \otimes \mathbb{S}_\mu \mathcal{S}^\vee$ or $\nu_{(\eta,\alpha)} = \mathbb{S}_\eta \mathcal{S} \otimes \mathbb{S}_\alpha \mathcal{Q}^\vee$. We give the computation of the cohomology for some particular bundles over the Grassmannian of lines $\mathbb{G}(1,n)$ that will be needed in Chapter 3.

Remark 2.2.12. We recall that, if $i \leq j$, there is a decomposition,

$$S^i\mathcal{Q} \otimes S^j\mathcal{Q} = S^{i+j}\mathcal{Q} \oplus (S^{i+j-2}\mathcal{Q})(1) \oplus (S^{i+j-4}\mathcal{Q})(2) \oplus \ldots \oplus (S^{j-i}\mathcal{Q})(i) \tag{2.10}$$

Proposition 2.2.13. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1,n)$. Then:*

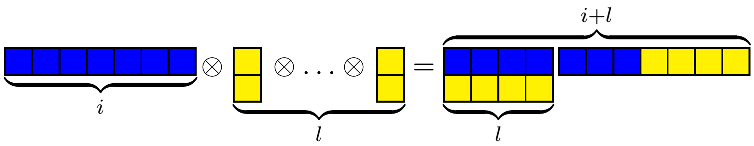
(i) $H^j(S^i\mathcal{Q}(l)) = 0$ *for* $l \geq 0$, $i \geq 0$ *and* $j > 0$

(ii) $H^j(S^i\mathcal{Q}(-l)) = 0$ *for* $l > 0$, $i \leq n-2$ *and* $j < 2n-2$

Proof. For (i) we have that:

$$S^i\mathcal{Q}(l) = S^i\mathcal{Q} \otimes \mathbb{S}_{(l,l)}\mathcal{Q} = \mathbb{S}_{(i+l,l)}\mathcal{Q}$$

The last equality comes from Pieri's formula.



So, we have to consider $\nu = (i+l, l, \underbrace{0, \ldots, 0}_{n-1})$. Since $l \geq 0 \Rightarrow i+l > l$ (for all $i \geq 0$), then ν is a partition. Applying Bott's algorithm we obtain:

$$\begin{cases} H^0(S^i\mathcal{Q}(l)) = \mathbb{S}_{(i+l,l)}\mathbb{K}^{n+1} \\ H^j(S^i\mathcal{Q}(l)) = 0 \quad \text{for all} \quad j > 0 \end{cases}$$

For (ii) we have now:

$$S^i\mathcal{Q}(-l) = S^i\mathcal{Q} \otimes \mathbb{S}_{(l,\ldots,l)}\mathcal{S}^\vee$$

So, we have to consider $\nu = (i, 0, l, \ldots, l)$. We discuss two cases:

- if $l \geq i+n+1$ then we have to make $2n-2$ transposition to get a partition $\nu' = (l-2, \ldots, l-2, i+n-1, n-1)$ is a partition. Hence,

$$\begin{cases} H^{2n-2}(S^i\mathcal{Q}(-l)) = \mathbb{S}_{(l-2,\ldots,l-2,i+n-1,n-1)}\mathbb{K}^{n+1} \\ H^j(S^i\mathcal{Q}(-l)) = 0 \quad \text{for} \quad j < 2n-2 \end{cases}$$

- if $0 < l < i+n+1$ then by making some particular transpositions there always exists t such that $\nu_{t+1} - \nu_t = 1$. Hence $H^j(S^i\mathcal{Q}(-l)) = 0$ for all j.

Moreove, since $H^0(S^i\mathcal{Q}(l)) = 0$ if $l \geq 0$ then by using Serre duality,

$$H^0(S^i\mathcal{Q}(l)) = H^{2n-2}(S^i\mathcal{Q}^\vee(-l-n-1)) = H^{2n-2}(S^i\mathcal{Q}(-l-n-1-i))$$

Hence $H^{2n-2}(S^i\mathcal{Q}(-l-n-1-i)) = 0$ also if $-l > n+i+1$. □

Proposition 2.2.14. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1,n)$. Then,*

$$H^{n-1}(S^i\mathcal{Q}(-n-r)) \neq 0 \quad \textit{for} \quad r = 0,1,\ldots,i-n+1 \quad \textit{and} \quad i \geq n-1.$$

Proof. Let us discuss three different cases.

1. First we study $i = n-1$. There is only one possible twist.

   - $S^{n-1}\mathcal{Q}(-n) = S^{n-1}\mathcal{Q}^\vee(-1) = \mathbb{S}_{(n,1)}\mathcal{Q}^\vee \Rightarrow \nu = (0,\ldots,0,n,1)$
   By applying $n-1$ transposition we obtain $\nu' = (1,\ldots,1)$ and so,

$$\begin{cases} H^{n-1}(S^{n-1}\mathcal{Q}(-n)) = \mathbb{S}_{(1,\ldots,1)}\mathbb{K}^{n+1} \\ H^j(S^{n-1}\mathcal{Q}(-n)) = 0 \quad \text{for} \quad j \neq n-1 \end{cases}$$

2. Now we study $j = n$. There are two possible twists.

   - $S^n\mathcal{Q}(-n) = S^n\mathcal{Q}^\vee = \mathbb{S}_{(n,0)}\mathcal{Q}^\vee \Rightarrow \nu = (0,\ldots,0,n,0)$
   By applying $n-1$ transposition we obtain $\nu' = (1,\ldots,1,0)$ and hence,

$$\begin{cases} H^{n-1}(S^n\mathcal{Q}(-n)) = \mathbb{S}_{(1,\ldots,1,0)}\mathbb{K}^{n+1} \\ H^j(S^n\mathcal{Q}(-n)) = 0 \quad \text{for} \quad j \neq n-1 \end{cases}$$

   - $S^n\mathcal{Q}(-n-1) = S^n\mathcal{Q}^\vee(-1) = \mathbb{S}_{(n+1,1)}\mathcal{Q}^\vee \Rightarrow \nu = (0,\ldots,0,n+1,1)$
   By applying $n-1$ transposition we obtain $\nu' = (2,1,\ldots,1)$ and hence,

$$\begin{cases} H^{n-1}(S^n\mathcal{Q}(-n-1)) = \mathbb{S}_{(2,1,\ldots,1)}\mathbb{K}^{n+1} \\ H^j(S^n\mathcal{Q}(-n-1)) = 0 \quad \text{for} \quad j \neq n-1 \end{cases}$$

3. Now consider the remaining cases $i = n+k$, $k \geq 1$. Here we use also the identifications of Remark 2.2.9. There are $k+2$ possible twists.

   - We can study the first $k+1$ cases together ($t = 0,1,\ldots,k$).
   $S^{n+k}\mathcal{Q}(-n-t) = S^{n+k}\mathcal{Q}^\vee(k-t) \Rightarrow \nu = (k-t,\ldots,k-t,n+k,0)$
   By applying $n-1$ transposition we obtain $\nu' = (k+1,k-t+1\ldots,k-t+1,0)$ and hence,

$$\begin{cases} H^{n-1}(S^{n+k}\mathcal{Q}(-n-t)) = \mathbb{S}_{(k+1,k-t+1\ldots,k-t+1,0)}\mathbb{K}^{n+1} \\ H^j(S^{n+k}\mathcal{Q}(-n-t)) = 0 \quad \text{for} \quad j \neq n-1 \end{cases}$$

- $S^{n+k}\mathcal{Q}(-n-(k+1))) = S^{n+k}\mathcal{Q}^\vee(-1) \Rightarrow \nu = (0,\dots,0,n+k+1,1)$
  By applying $n-1$ transposition we obtain $\nu' = (k+2,1,\dots,1)$ and hence,

$$\begin{cases} H^{n-1}(S^{n+k}\mathcal{Q}(-n-(k+1))) = \mathbb{S}_{(k+2,1,\dots,1)}\mathbb{K}^{n+1} \\ H^j(S^{n+k}\mathcal{Q}(-n-(k+1))) = 0 \quad \text{for} \quad j \neq n-1 \end{cases}$$

Notice that for any other twists there always exists t such that $\nu_{t+1} - \nu_t = 1$ for the corresponding ν in each case, hence all the cohomology vanish. □

Remark 2.2.15. We could have proved that $S^i\mathcal{Q}(l)$ does not have intermediate cohomology for $i \geq n-1$ by using the Eagon-Northcott complexes. For example, the exact sequence $(R^\vee_{n-1})$ of Example 1.2.2 produces a nonzero element in $Ext^{n-1}(\mathcal{O}(1), S^{n-1}\mathcal{Q}(-n+1)) = H^{n-1}(S^{n-1}\mathcal{Q}(-n))$. In fact this is the only nonzero intermediate cohomology of $S^{n-1}\mathcal{Q}$, while (R_i) shows that the only nonzero intermediate cohomology of $S^i\mathcal{Q}$ with $i \geq n-1$ is $H^{n-1}(S^j\mathcal{Q}(-n-k))$, with $k = 0,1,\dots,i-n+1$.

Proposition 2.2.16. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1,n)$. Then $S^i\mathcal{Q}$ is simple (i.e. $Hom(S^i\mathcal{Q}, S^i\mathcal{Q}) = \mathbb{K}$).*

Proof. Since $Hom(S^i\mathcal{Q}, S^i\mathcal{Q}) = H^0(S^i\mathcal{Q} \otimes S^i\mathcal{Q}^\vee) = H^{2n-2}(S^i\mathcal{Q}^\vee \otimes S^i\mathcal{Q}(-n-1))^*$ we apply (2.10).

$$S^i\mathcal{Q}^\vee \otimes S^i\mathcal{Q}(-n-1) = S^i\mathcal{Q} \otimes S^i\mathcal{Q}(-i-n-1) =$$

$$S^{2i}\mathcal{Q}(-i-n-1) \oplus S^{2i-2}\mathcal{Q}(-i-n) \oplus S^{2i-4}\mathcal{Q}(-i-n+1) \oplus \dots \oplus S^2\mathcal{Q}(-n-2) \oplus \mathcal{O}(-n-1)$$

From Proposition 2.2.13 and Proposition 2.2.14 we get that the only summand with the cohomology of order $2n-2$ non zero is $\mathcal{O}(-n-1)$. By Theorem 1.3.4 we know that $H^{2n-2}(\mathcal{O}(-n-1)) = \mathbb{K}$. Hence $Hom(S^i\mathcal{Q}, S^i\mathcal{Q}) = \mathbb{K}$. □

Lemma 2.2.17. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1,n)$ and $k \in \{1,\dots,n-2\}$. Then,*

(i) $H^{2n-2}(S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}(-n+1)) = 0$

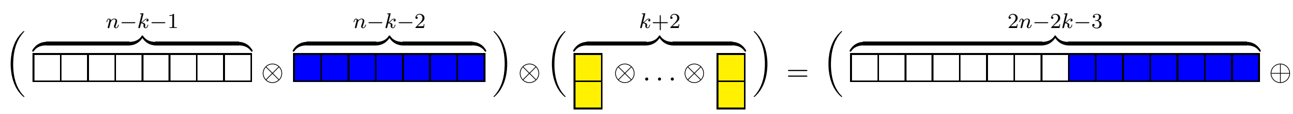
(ii) $H^{n-1}(S^{n-k-1}\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}(-k)) = 0$

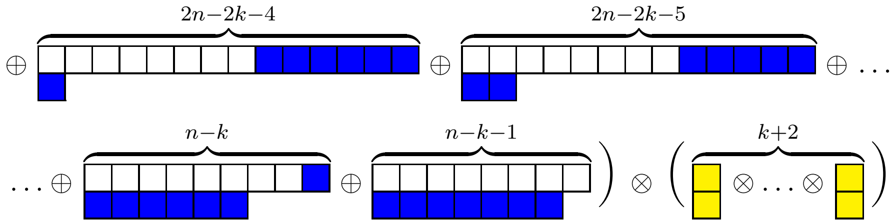
Proof. For (i) we use Pieri's formula to transform this vector bundle into sum of some particular Schur functors.

$$S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}(-n+1) = S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee(-k-2) =$$

$$= S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee \otimes \underbrace{\mathcal{O}(-1) \otimes \dots \otimes \mathcal{O}(-1)}_{k+2} = S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee \otimes \underbrace{\bigwedge^2 \mathcal{Q}^\vee \otimes \dots \otimes \bigwedge^2 \mathcal{Q}^\vee}_{k+2}$$

In terms of Young diagrams this can be expressed as:





The last thing we have to do is to add $k+2$ boxes in each row (since we cannot add two boxes in the same column). Hence these diagrams corresponds to the following:

$$(2n-k-1, k+2),\ (2n-k-2, k+3),\ (2n-k-3, k+4),\ \dots,\ (n+2, n-1),\ (n+1, n) \quad (2.11)$$

So, $S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}(-n+1) = \oplus_\lambda \mathbb{S}_\lambda \mathcal{Q}^\vee$ where $\lambda = (r, s)$ is as in (2.11).

Now we want to compute Bott's algorithm to $\mathbb{S}_\lambda \mathcal{Q}^\vee$ for these λ. Let us apply Proposition 2.2.11 to elements of the form $\nu = (\underbrace{0, \dots, 0}_{n-1}, r, s)$. Making $n-1$ changes we get $\nu = (r-(n-1), \underbrace{1, \dots, 1}_{n-1}, s)$. Since $r - s \geq 1$, when we apply the changes to s we obtain at some point:

$$\nu = (r-(n-1), 1 \dots, 1, 2, 2, \dots, 2)$$

Hence, if we take the first l such that $\nu_l < \nu_{l+1}$ we have that $\nu_{l+1} - \nu_l = 1$. Furthermore, $H^j(\mathbb{S}_\lambda \mathcal{Q}^\vee) = 0$ for all $j \geq 0$. In particular,

$$H^{2n-2}(S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}(-n+1)) = 0.$$

For (ii) we do not need to use Bott's algorithm since by formula (2.10):

$$S^{n-k-1}\mathcal{Q}^\vee(-1) \otimes S^{k-1}\mathcal{Q}(-k) = S^{n-k-1}\mathcal{Q}(-n+k) \otimes S^{k-1}\mathcal{Q}(-k) = S^{n-k-1}\mathcal{Q} \otimes S^{k-1}\mathcal{Q}(-n) =$$

$$= S^{n-2}\mathcal{Q}(-n) \oplus S^{n-4}\mathcal{Q}(-n+1) \oplus S^{n-6}\mathcal{Q}(-n+2) \oplus \dots$$

As we say in Proposition 2.2.13, $H^j(S^i\mathcal{Q}(l))$ has no intermediate cohomology for $i \leq n-2$. In particular:

$$H^{n-1}(S^{n-k-1}\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}(-k)) = H^{n-1}(S^{n-2}\mathcal{Q}(-n) \oplus S^{n-4}\mathcal{Q}(-n+1) \oplus S^{n-6}\mathcal{Q}(-n+2) \oplus \dots) = 0$$

$\square$

Lemma 2.2.18. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1, n)$ and $k \in \{1, \dots, n-2\}$. Then,*

(i) $H^{n-1}(S^k\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee(-1)) = 0$

(ii) $H^0(S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}) = 0$

Proof. For (i) we consider,

$$S^k\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee(-1) = S^k\mathcal{Q} \otimes S^{n-k-2}\mathcal{Q}(-n+1) =$$
$$= S^{n-2}\mathcal{Q}(-n+1) \oplus S^{n-4}\mathcal{Q}(-n+2) \oplus S^{n-6}\mathcal{Q}(-n+3) \oplus \dots$$

We notice that, since $H^j(S^i\mathcal{Q}(l))$ has no intermediate cohomology for $i \leq n-2$,

$$H^{n-1}(S^k\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee(-1)) = H^{n-1}(S^{n-2}\mathcal{Q}(-n+1) \oplus S^{n-4}\mathcal{Q}(-n+2) \oplus S^{n-6}\mathcal{Q}(-n+3) \oplus \dots) = 0$$

For (ii) we transform this condition with the Serre duality in order to obtain a Schur functor of the form $\mathbb{S}_\alpha\mathcal{Q}^\vee$:

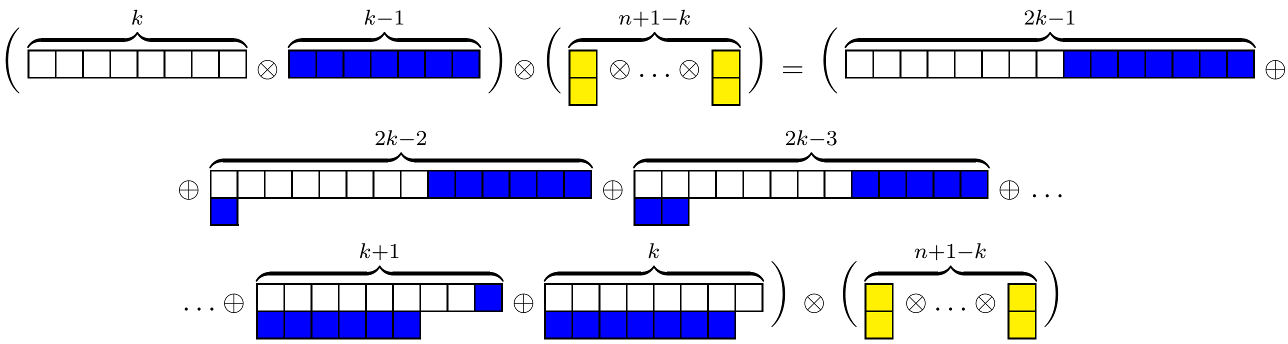
$$H^0(S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}) = H^0(S^k\mathcal{Q} \otimes S^{k-1}\mathcal{Q}(-k)) = H^{2n-2}(S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}^\vee(k-n-1))^*$$

One notice that $k-n-1 < 0$ always since $k \in \{1, \dots, n-2\}$. Now, let us express this vector bundle as follows:

$$S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}^\vee(k-n-1) = S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}^\vee \otimes \underbrace{\mathcal{O}(-1) \otimes \dots \otimes \mathcal{O}(-1)}_{n+1-k} =$$

$$= S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}^\vee \otimes \underbrace{\bigwedge^2 \mathcal{Q}^\vee \otimes \dots \otimes \bigwedge^2 \mathcal{Q}^\vee}_{n+1-k}$$

As in Example 2.2.17, we use Pieri's formula:



The last thing we have to do is to add $n+1-k$ boxes in each row (since we cannot add two boxes in the same column). Hence these diagrams corresponds to the following:

$$(n+k, n+1-k),\ (n+k-1, n+2-k),\ (n+k-2, n+3-k),\ \dots,\ (n+2, n-1),\ (n+1, n) \quad (2.12)$$

So, $S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q} = \oplus_\lambda \mathbb{S}_\lambda\mathcal{Q}^\vee$ where $\lambda = (r, s)$ is as in (2.12).

By applying the same argument as in Lemma 2.2.17 we obtain that, $H^j(\mathbb{S}_\lambda\mathcal{Q}^\vee) = 0$ for all $j \geq 0$. In particular,

$$H^{2n-2}(S^{n-k-1}\mathcal{Q}^\vee \otimes S^{n-k-2}\mathcal{Q}(-n+1))^* = 0$$

Hence,

$$H^0(S^k\mathcal{Q}^\vee \otimes S^{k-1}\mathcal{Q}) = 0$$

$\square$

Lemma 2.2.19. *Consider $\mathcal{Q}$ the quotient bundle of rank 2 over the Grassmannian of lines $\mathbb{G}(1,n)$ and $k \in \{1,\ldots,n-2\}$. Then,*

(i) $H^{n-1}(S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}(-n)) = H^{n-1}(S^{n-1}\mathcal{Q}(-n)) = \mathbb{S}_{(1,\ldots,1)}\mathbb{K}^{n+1} = \bigwedge^{n+1}\mathbb{K}^{n+1}$ *and its dimension is equal to* 1

(ii) $H^j_*(S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}) = 0$ *for* $j \neq n-1$

(iii) $H^j_*(S^k\mathcal{Q} \otimes S^i\mathcal{Q}) = 0$ *for* $i \neq n-k-1$ *for* $j = 1,2,\ldots,2n-3$

Proof. By applying (2.10) to $S^k\mathcal{Q} \otimes S^i\mathcal{Q}(l)$ we get,

$$S^k\mathcal{Q} \otimes S^i\mathcal{Q}(l) = (S^{i+k}\mathcal{Q} \oplus S^{i+k-2}\mathcal{Q}(1) \oplus \ldots) \otimes \mathcal{O}(l)$$

Notice that for $i = n-k-1$ we have that,

$$S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}(l) = (S^{n-1}\mathcal{Q} \oplus S^{n-3}\mathcal{Q}(1) \oplus \ldots) \otimes \mathcal{O}(l)$$

By Proposition 2.2.14 we know that,

$$\begin{cases} H^{n-1}(S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}(-n)) = H^{n-1}(S^{n-1}\mathcal{Q}(-n)) = \mathbb{S}_{(1,\ldots,1)}\mathbb{K}^{n+1} \\ H^j_*(S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}) = 0 \quad \text{for} \quad j \neq n-1 \\ H^j_*(S^k\mathcal{Q} \otimes S^i\mathcal{Q}) = 0 \quad \text{for} \quad i \neq n-k-1 \quad \text{for} \quad j = 1,2,\ldots,2n-3 \end{cases}$$

Clearly the dimension of the determinant representation is 1 (we could have also used the formula that appears in Proposition 2.1.17 to compute it). ☐

2.3 Remarks about Schur functors

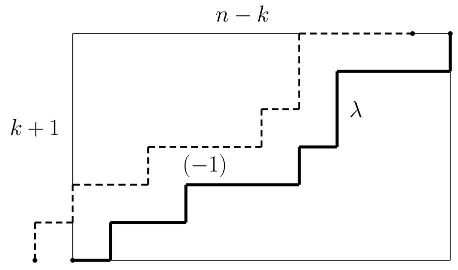
Although not needed for our results, we want to point out two interesting results related to Schur functors.

First, by using the language of Young diagrams and Schur functors one can state the following proposition from [18] which is a kind of generalization of the Eagon-Northcott introduced in Section 1.2. This proposition gives the corresponding extension $Ext^{n-k}(\mathbb{S}_\lambda\mathcal{Q}, \mathbb{S}_{\lambda^*}\mathcal{Q}(-1))$ for a particular Young diagram λ with $\lambda_1 = n-k$.

Proposition 2.3.1. *(Proposition 5.3 of [18])*
Let $\lambda \in Y_{n,k}$ be a diagram with $\lambda = (n-k, \lambda_2, \ldots, \lambda_{k+1})$ (where $Y_{n,k}$ is the set of Young diagrams inscribed in a rectangle of size $(k+1) \times (n-k)$). Consider $\lambda^* = (\lambda_2, \ldots, \lambda_{k+1})$. Then there exists the following long exact sequence for the tautological subbundle $\mathcal{Q}^\vee$ of rank $k+1$,

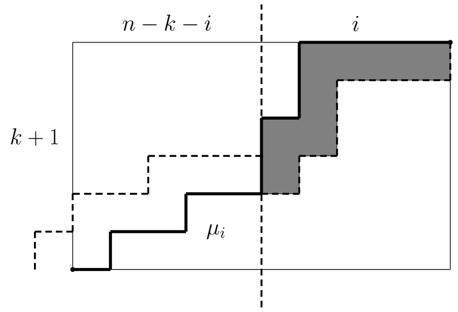
$$0 \longrightarrow \mathbb{S}_{\lambda^*}\mathcal{Q}(-1) \longrightarrow \bigwedge^{\upsilon_{n-k}} V \otimes \mathbb{S}_{\mu_{n-k}}\mathcal{Q} \longrightarrow \bigwedge^{\upsilon_{n-k-1}} V \otimes \mathbb{S}_{\mu_{n-k-1}}\mathcal{Q} \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigwedge^{\upsilon_2} V \otimes \mathbb{S}_{\mu_2}\mathcal{Q} \longrightarrow \bigwedge^{\upsilon_1} V \otimes \mathbb{S}_{\mu_1}\mathcal{Q} \longrightarrow \mathbb{S}_\lambda\mathcal{Q} \longrightarrow 0 \qquad (2.13)$$



for some $0 \leq \upsilon_i < n$ *and* $\mu_i \in Y_{n,k}$ *where* υ_i *and* μ_i *are defined in a nice combinatorial way. Given* $\lambda \in Y_{n,k}$ *with* $\lambda_1 = n-k$*, let us draw a stripe of width* 1 *as shown:*

Now, μ_i *is pictured in the following figure by the solid line. One takes the path* λ *going from left to right and "jumps" upward on the path* $\lambda'(-1)$ *in the point with abscissa* $n-k-i$*.*



The number υ_i *is the number of boxes one needs to remove from* λ *in order to get* μ_i*. These are pictured in gray on the previous figure.*

Secondly, if we consider the multideterminant of a multilinear form $V \otimes \ldots \otimes V \longrightarrow \mathbb{K}$ (see for example [34] for the general definition of multideterminant) by decomposing this multilinear form as summands of Schur functors (Proposition 2.1.15) we proof that the multideterminant vanishes for all but two Schur functors. The concrete statement is the following, see [36]:

Theorem 2.3.2. *When* $A \in \mathbb{S}_\lambda V^{\otimes m_\lambda} \subseteq V^{\otimes p}$*,* m_λ *is the dimension of the irreducible representation* V_λ *of* $\mathfrak{S}_d$ *corresponding to* λ*,* $|\lambda| = p$ *and* $p \geq 2$*,* $Det(A)$ *can be nonzero only for* $\lambda = (p)$ *(corresponding to the symmetric power* $\mathrm{Sym}^p V$*) and* $\lambda = (p-1, 1)$*, where* $\mathbb{S}^\lambda V$ *(resp.* $S_\lambda V$*) is the* Σ_p*-module (resp.* $GL(V)-$*module) associated to* λ*.*

Chapter 3

Characterization of Universal Bundles

In this chapter we will give the main result of this thesis. Our starting point is the following splitting criterion that appears in [7]:

Theorem 0.0.3. *Let F be a vector bundle over the Grassmannian of lines $\mathbb{G}(1,n)$. Then F splits if and only if*

a. $H^j_*(F \otimes S^j\mathcal{Q}) = 0 \quad j \in \{1, 2, \dots, n-3, n-2\}$

b. $H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad (i,j) \in \{(0, 2n-3), (1, 2n-4), \dots, (n-3, n), (n-2, n-1)\}$

Remark 3.0.3. We can try to imitate the tecnique of [6] and see which of those conditions is not satisfied by $\mathcal{Q}$ and try to see whether the remaining ones characterize vector bundles of the form $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$. Observe that, by Lemma 2.2.19, $\mathcal{Q}$ satisfies all the conditions of this splitting criteria except one:

- $H^j_*(\mathcal{Q} \otimes S^j\mathcal{Q}) = 0 \quad j = 1, 2, \dots, n-3, n-2$
- $H^j_*(\mathcal{Q} \otimes S^i\mathcal{Q}) = 0 \quad (i,j) \in \{(0, 2n-3), (1, 2n-4), \dots, (n-3, n)\}$
- $H^{n-1}_*(\mathcal{Q} \otimes S^{n-2}\mathcal{Q}) \neq 0$

However, also $F = S^k\mathcal{Q}(l)$ with $k \leq n-2$, satisfies all the hypothesis of Theorem 0.0.3 but $H^{n-1}_*(F \otimes S^{n-2}\mathcal{Q}) = 0$. So, we have to add more conditions. In fact, this is what we are going to do during this chapter. Once we have the characterization for the direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$,..., $S^{k-1}\mathcal{Q}$ the idea is to remove one particular hypothesis and add a few more. In this way we obtain the characterization of direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$,...,$S^k\mathcal{Q}$ with $k \leq n-2$.

3.1 Graphical representation of the results

Here we will introduce a graphical representation of the results that we will obtain in the rest of the chapter.

Remark 3.1.1. We can express graphically the conditions of the form $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ that appear in the previous theorem as the points (i, j) in the following diagram.

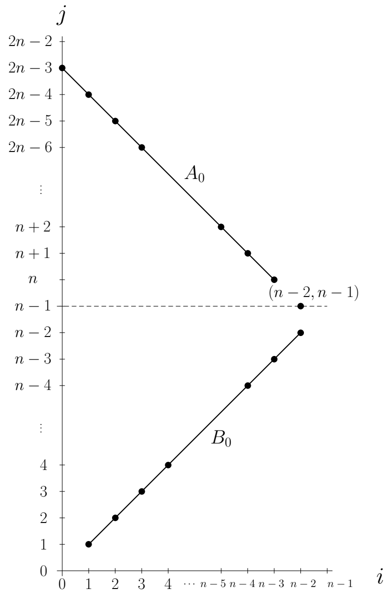


Figure 3.1: Hypothesis for Theorem 0.0.3

Definition 3.1.2. Let us define the segments A_0 and B_0 appearing in the previous figure as follows:

- $A_0 = \{(0, 2n-3), (1, 2n-4), (2, 2n-5), \ldots, (n-5, n+2), (n-4, n+1), (n-3, n)\}$
- $B_0 = \{(1, 1), (2, 2), (3, 3), \ldots, (n-4, n-4), (n-3, n-3), (n-2, n-2)\}$

Remark 3.1.3. Notice that the point $(n-2, n-1)$ is the continuation of the segment A_0 but we put it separately since it is the condition we need to remove to get the next characterization.

Remark 3.1.4. We have also expressed with a dashed line the $n-1$ order of the cohomology of $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ because all the cohomologies we remove in each step are in this line.

Remark 3.1.5. We can compare graphically this splitting criteria with the one made by G. Ottaviani (Theorem 0.0.2). For this purpose let us define the following segments.

- $L_0 = \{(0, 2n-3), (1, 2n-4), \ldots, (n-3, n), (n-2, n-1)\}$
- For $k = 1, \ldots, n-2$ we define $L_k = \{(0, 2n-3-k), (1, 2n-4-k), \ldots, (n-k-2, n-1), (n-k-1, n-2)\} = \{(j, 2n-3-k-j) \quad \text{for} \quad j = 0, 1, \ldots, n-k-1\}$
- $R_0 = \{(1, 1), (2, 2), \ldots, (n-3, n-3), (n-2, n-2)\}$
- For $k = 1, 2, \ldots, n-2$ we define $R_k = \{(0, k), (1, k+1), \ldots, (n-3-k, n-3), (n-2-k, n-2)\} = \{(j, k+j) \quad \text{for} \quad j = 0, 1, \ldots, n-k-2\}$

All the cohomological vanishings of Theorem 0.0.2 consist in:

$$H^j_*(F \otimes S^i \mathcal{Q}) = 0 \quad \text{where} \quad (i,j) \in \begin{cases} L_0 \cup L_1 \cup \ldots \cup L_{n-2} \\ R_0 \cup R_1 \cup \ldots \cup R_{n-2} \end{cases}$$

These points correspond to the following figure. Notice that $L_0 = A_0$ and $R_0 = B_0$.

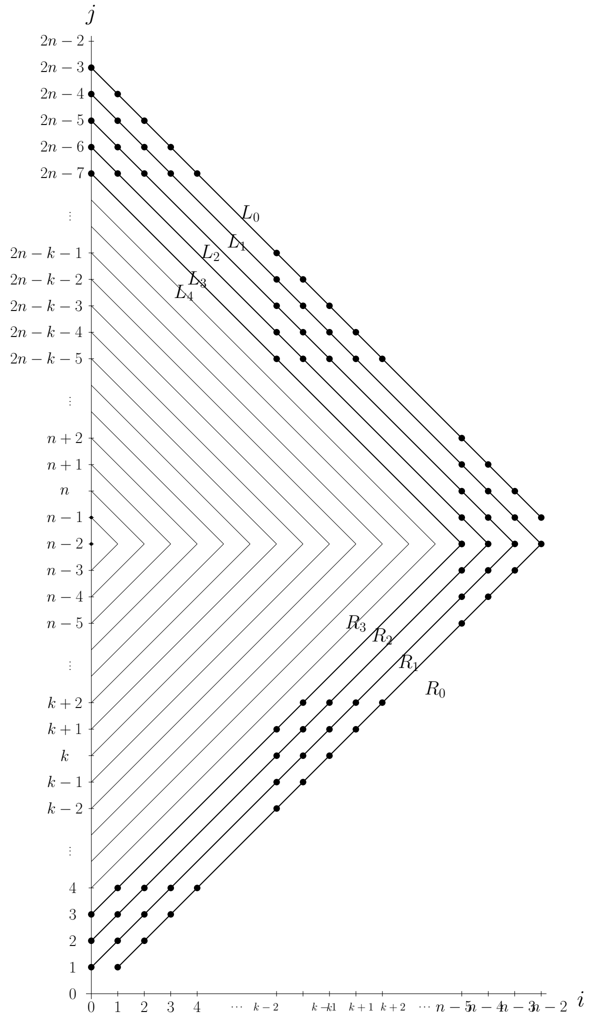


Figure 3.2: Hypothesis for Theorem 0.0.2

We will prove in Theorem 3.3.1 that the characterization of $F = (\oplus \mathcal{O}(l_{i_0})) \bigoplus (\oplus \mathcal{Q}(l_{i_1}))$ will be the vanishing of $H^j_*(F \otimes S^i \mathcal{Q})$ for (i,j) in A_0, B_0 and the points that are shown in the following figure.

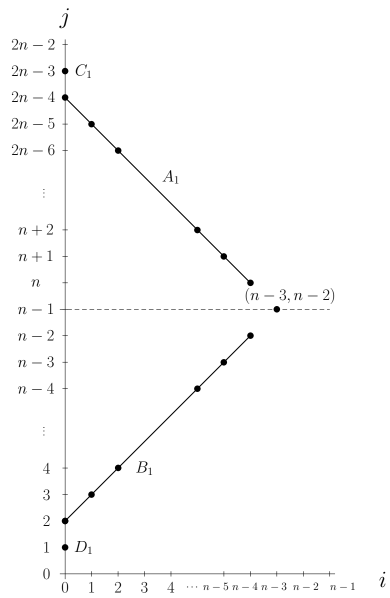


Figure 3.3: Hypothesis to be added to characterize $F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$

Hence, if F satisfies $H^j(F \otimes S^i\mathcal{Q}) = 0$ for (i, j) as in conditions A_0 and B_0 of Figure 3.1 and as in all the conditions in Figure 3.3 then we get the characterization for $F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$. All these conditions can be expressed in Figure 3.4.

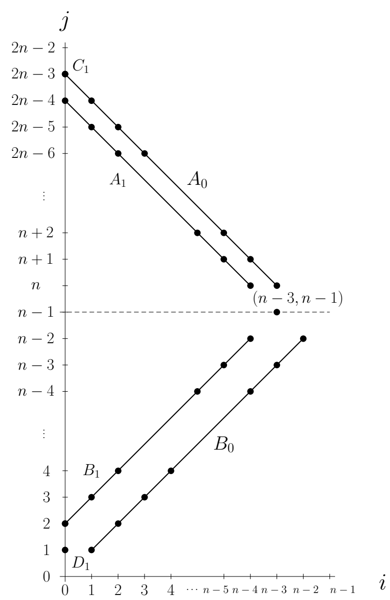


Figure 3.4: Hypothesis needed to characterize $F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$

Remark 3.1.6. Notice that $C_1 \subseteq A_0$ but we highlight it in Figure 3.3 and 3.4 it since in the iteration process we will need the segments C_k not contained in any other defined segment.

Remark 3.1.7. Let us give one more step. Observe that $H_*^{n-1}(S^2\mathcal{Q} \otimes S^{n-3}\mathcal{Q}) \neq 0$ (by Lemma 2.2.19), so we need to remove the condition corresponding to the point $(n-3, n-1)$ to get the next characterization. Notice that this point is in the intersection of the lines containing A_1 and B_1 but we put it separately since we remove it.

We will prove in Theorem 3.3.1 that the characterization of

$$F = (\oplus\mathcal{O}(l_{i_0})) \bigoplus (\oplus\mathcal{Q}(l_{i_1})) \bigoplus (\oplus S^2\mathcal{Q}(l_{i_2}))$$

will be the vanishing of $H_*^j(F \otimes S^i\mathcal{Q})$ for (i, j) as in conditions A_0, B_0, A_1, B_1, C_1 and D_1 of Figure 3.4 and the points that are shown in the following figure.

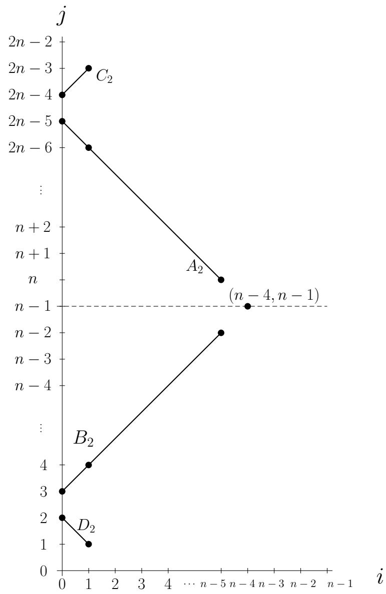


Figure 3.5: Hypothesis to be added to characterize $F = (\oplus\mathcal{O}(l_{i_0})) \bigoplus (\oplus\mathcal{Q}(l_{i_1})) \bigoplus (\oplus S^2\mathcal{Q}(l_{i_2}))$

Remark 3.1.8. Notice that the point $(n-4, n-1)$ is in the intersection of the lines containing A_2 and B_2 but, as before, we put it separately since is the condition we need to remove to get the next characterization ($H_*^{n-1}(S^3\mathcal{Q} \otimes S^{n-4}\mathcal{Q}) \neq 0$ by Lemma 2.2.19).

Hence, if F satisfies the vanishings of $H^j(F \otimes S^i\mathcal{Q})$ for (i, j) as in Figure 3.4 (except $(n-3, n-1)$) and all the conditions in Figure 3.5 then we get the characterization for

$$F = (\oplus\mathcal{O}(l_{i_0})) \bigoplus (\oplus\mathcal{Q}(l_{i_1})) \bigoplus (\oplus S^2\mathcal{Q}(l_{i_2}))$$

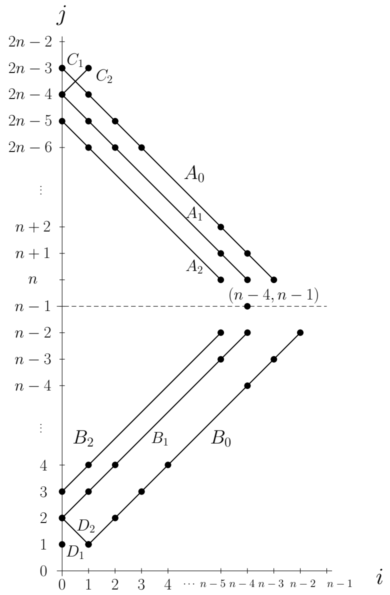


Figure 3.6: Hypothesis needed to characterize $F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus(\oplus S^2\mathcal{Q}(l_{i_2}))$

In Theorem 3.3.1 we will iterate this process to obtain a characterization of direct sums of twists of symmetric products of the universal bundle

$$F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus\cdots\bigoplus(\oplus S^k\mathcal{Q}(l_{i_k}))$$

until $k = n-2$.

For that, we will need to give the definition of the segments A_k, B_k, C_k, D_k. Notice that we have already introduced A_0 and B_0 in the first part of this chapter.

Definition 3.1.9. Let us define the following segments for $1 \leq k \leq n-2$:

- $A_k = \{(0, 2n-k-3), (1, 2n-k-4), (2, 2n-k-5), \ldots, (n-k-4, n+1), (n-k-3, 1)\}$
- $B_k = \{(0, k+1), (1, k+2), (2, k+3), \ldots, (n-k-4, n-3), (n-k-3, n-2)\}$
- $C_k = \{(0, 2n-k-2), (1, 2n-k-1), (2, 2n-k), \ldots, (k-2, 2n-4), (k-1, 2n-3)\}$
- $D_k = \{(0, k), (1, k-1), (2, k-2), \ldots, (k-2, 2), (k-1, 1)\}$

Observe that A_0 can be defined in the same way as the A_k with $k = 0$ whereas B_0 cannot be defined as the B_k with $k = 0$.

Remark 3.1.10. Notice that $A_{n-2} = \emptyset$ and $B_{n-2} = \emptyset$. Hence, in the case that $k \geq n-2$ we are not adding any cohomological condition with the notation of A_k and B_k.

Moreover, all the conditions corresponding to the points of the lines C_k and D_k has no sense for $k \geq n-1$ because of the way they are defined. For example, if $k = n-1$ we get that

$C_{n-1} = \{(0, n-1), (1, n), (2, n+1), \dots, (n-3, 2n-4), (n-2, 2n-3)\}$ and $D_{n-1} = \{(0, n-1), (1, n-2), (2, n-3), \dots, (n-3, 2), (n-2, 1)\}$. The point $(0, n-1)$ of C_{n-1} and D_{n-1} corresponds to the condition $H^{n-1}_*(F) = 0$ and this cannot be true if we want to characterize $F = (\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus\dots\bigoplus(\oplus S^k\mathcal{Q}(l_{i_k}))$ with $k = n-1$ since $H^{n-1}_*(S^{n-1}\mathcal{Q}) \neq 0$ (see Proposition 2.2.14).

Remark 3.1.11. Let us show graphically which are the points of the segments defined in Definition 3.1.9 (that correspond to the cohomological conditions $H^j_*(F \otimes S^i\mathcal{Q}) = 0$). We will also draw the point $(n-k-2, n-1)$ since we will use it in the lemmas and proposition of the following section.

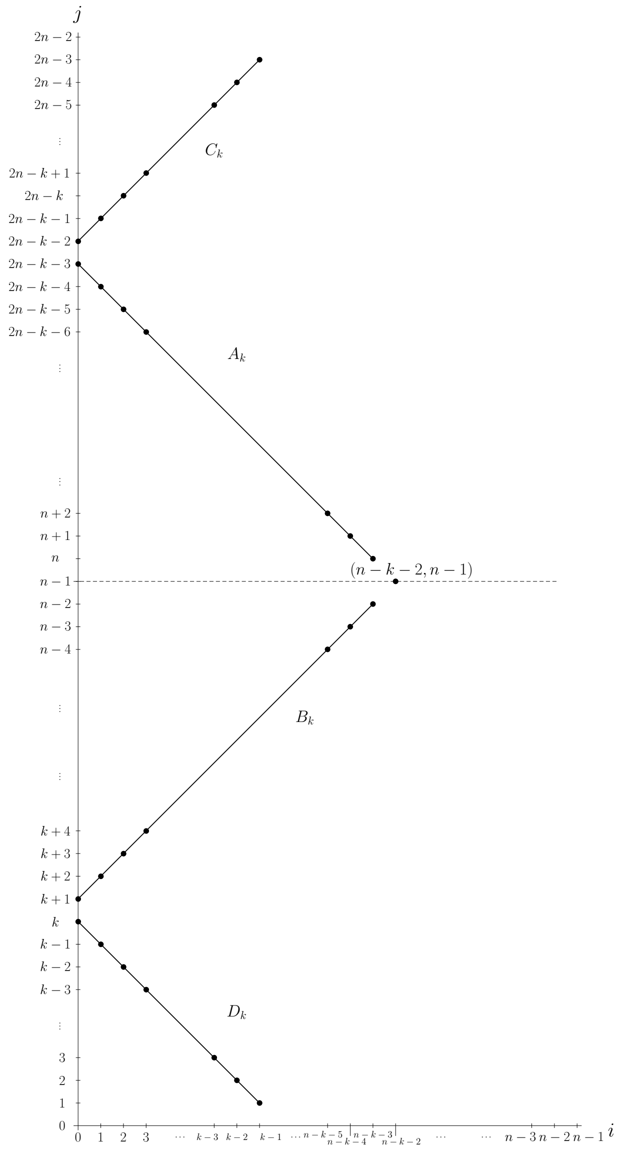


Figure 3.7: A_k, B_k, C_k, D_k and the point $(n-k-2, n-1)$

Remark 3.1.12. One can easily see that the points corresponding to:

$$(i,j) \in \begin{cases} A_0 \cup A_1 \cup \ldots \cup A_{k-1} \\ B_0 \cup B_1 \cup \ldots \cup B_{k-1} \\ C_1 \cup C_2 \cup \ldots \cup C_{k-1} \\ D_1 \cup D_2 \cup \ldots \cup D_{k-1} \end{cases}$$

are inside of the set of points:

$$(i,j) \in \begin{cases} A_0 \cup A_1 \cup \ldots \cup A_k \\ B_0 \cup B_1 \cup \ldots \cup B_k \\ C_1 \cup C_2 \cup \ldots \cup C_k \\ D_1 \cup D_2 \cup \ldots \cup D_k \end{cases}$$

This observation will be useful during the proof of the main theorem. Specifically, these last conditions plus the point $(n-k-2, n-1)$ form the hypothesis of the main theorem. Let us show them graphically in the following figure. One starts drawing the lines corresponding to A_k, B_k, C_k, D_k and fill the corresponding spaces with lines until the ones drawn in Figure 3.4 (except condition $(n-3, n-1)$).

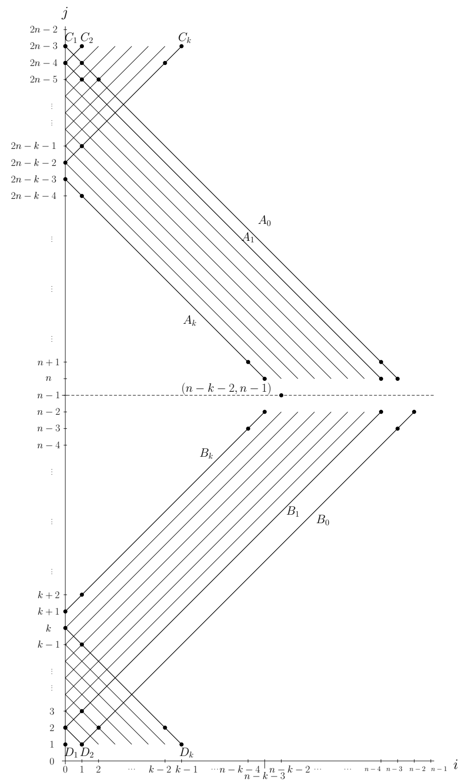


Figure 3.8: Conditions to characterize direct sum of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$

Remark 3.1.13. Finally, let us give graphically all the cohomological conditions we need for the characterization of direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, $\ldots$, $S^k\mathcal{Q}$ when $k = n-3$ and $k = n-2$ (i.e. for the last two characterizations we can make) in order to give a complete idea of what kind of hypothesis we will need for each characterization (from $k=0$ to $k=n-2$). As we have observed in Remark 3.1.10, to get the classification when $k = n-2$, we just have to add the line C_{n-2} and D_{n-2} to the points we already have for the classification when $k = n-3$, since $A_{n-2} = B_{n-2} = \emptyset$.

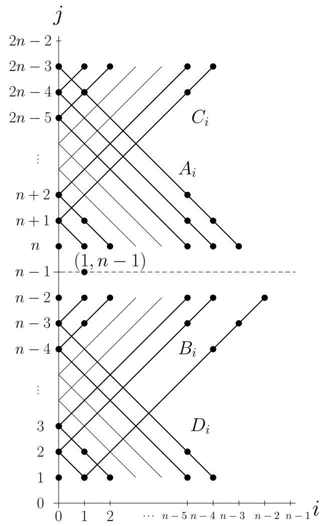


Figure 3.9: Conditions to characterize direct sum of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^{n-3}\mathcal{Q}$

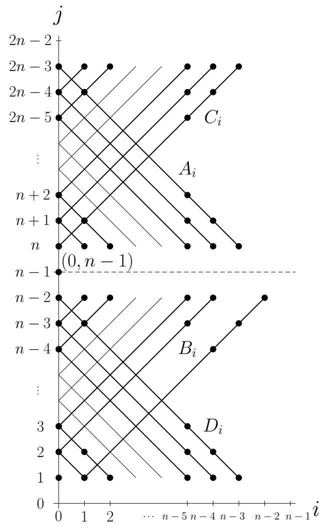


Figure 3.10: Conditions to characterize direct sum of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^{n-2}\mathcal{Q}$

3.1.1 Relation between cohomological vanishings

For each k there exists the complex (R_k) glued together with $(R^\vee_{n-k-1})\otimes\mathcal{O}(-1)$ from Example 1.2.2,

$$0 \longrightarrow S^{n-1-k}\mathcal{Q}^\vee(-1) \longrightarrow V^*\otimes S^{n-2-k}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^*\otimes S^{n-3-k}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-2-k} V^*\otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-1-k} V^*\otimes\mathcal{O}(-1) \longrightarrow \bigwedge^{k} V\otimes\mathcal{O} \longrightarrow \bigwedge^{k-1} V\otimes\mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{2} V\otimes S^{k-2}\mathcal{Q} \longrightarrow V\otimes S^{k-1}\mathcal{Q} \longrightarrow S^k\mathcal{Q} \longrightarrow 0$$

It is glaringly clear that one cohomological condition $H^s_*(F\otimes S^r\mathcal{Q})=0$ (that will be represented with a blue dot in the figures of this section) can be obtained from the vanishings of some particular cohomological conditions related with this complex. Depending on which of the two parts of the complex we are focused we obtain two different relations corresponding to Lemma 3.1.14 (if we fix $r\in\{1,2,\dots,k\}$) and Lemma 3.1.15 (if we fix $r\in\{1,2,\dots,n-k-1\}$).

Lemma 3.1.14. *Fix $k\in\{1,2,\dots,n-3,n-2\}$, $r\in\{1,2,\dots,k\}$ and consider $max\{k-r,1\}\le s\le n-r+k-2$. Suppose F is a vector bundle over the Grassmannian of lines $\mathbb{G}(1,n)$. If the following condition hold:*

$$H^j_*(F\otimes S^i\mathcal{Q})=0 \quad \textit{for} \quad (i,j)\in P_1\cup P_2\cup P_3$$

where P_1, P_2 and P_3 are defined as the following segments and represented in Figure 3.11:

- $P_1=\{(k,s-k+r+1),(k-1,s-k+r+2),(k-2,s-k+r+3),\dots,(r+3,s-2),(r+2,s-1),(r+1,s)\}$
- $P_2=\{(r-1,s),(r-2,s+1),(r-3,s+2),\dots,(2,s+r-3),(1,s+r-2),(0,s+r-1)\}$
- $P_3=\{(0,s+r),(1,s+r+1),(2,s+r+2),\dots,(n-k-3,s+r+n-3-k),(n-k-2,s+r+n-2-k),(n-k-1,s+r+n-1-k)\}$

then $H^s_*(F\otimes S^r\mathcal{Q})=0$.

Proof. It is enough to use the Eagon-Northcott complex (R_k) glued together with $(R^\vee_{n-k-1})\otimes\mathcal{O}(-1)$ from Example 1.2.2:

$$0 \longrightarrow S^{n-1-k}\mathcal{Q}^\vee(-1) \longrightarrow V^*\otimes S^{n-2-k}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^*\otimes S^{n-3-k}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-2-k} V^*\otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-1-k} V^*\otimes\mathcal{O}(-1) \longrightarrow \bigwedge^{k} V\otimes\mathcal{O} \longrightarrow \bigwedge^{k-1} V\otimes\mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{k-r+1} V\otimes S^{r-1}\mathcal{Q} \longrightarrow \bigwedge^{k-r} V\otimes S^{r}\mathcal{Q} \longrightarrow \bigwedge^{k-r-1} V\otimes S^{r+1}\mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^2 V \otimes S^{k-2}\mathcal{Q} \longrightarrow V \otimes S^{k-1}\mathcal{Q} \longrightarrow S^k\mathcal{Q} \longrightarrow 0$$

and apply cohomology to each short exact sequence. Observe that the range for s is chosen to avoid that no j coordinate in $P_1 \cup P_2 \cup P_3$ is neither 0 nor $2n-2$. ☐

We represent graphically the points of Lemma 3.1.14 (P_1, P_2 and P_3) and also the point (r, s) which corresponds to the new cohomological vanishing.

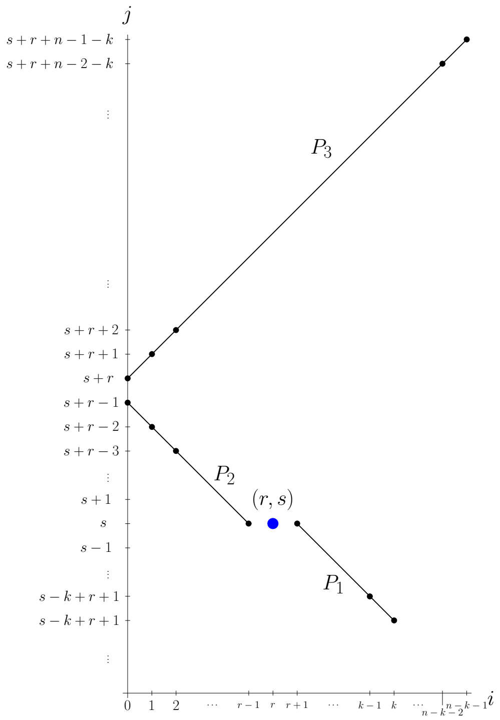


Figure 3.11: Segments P_1, P_2 and P_3

Lemma 3.1.15. *Fix $k \in \{1, 2, \dots, n-3, n-2\}$, $r \in \{1, 2, \dots, n-1-k\}$ and consider $1+r+k \le s \le min\{2n-3, n-1+k+r\}$. Suppose F is a vector bundle over the Grassmannian of lines $\mathbb{G}(1, n)$. If the following condition hold:*

$$H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{for} \quad (i,j) \in P'_1 \cup P'_2 \cup P'_3$$

where P_1, P_2 and P_3 are defined as the following segments and represented in Figure 3.12:

- $P'_1 = \{(n-1-k, n-k-r+s-2), (n-k-2, n-k-r+s-3), (n-k-3, n-k-r+s-4), \dots, (r+3, s+2), (r+2, s+1), (r+1, s)\}$
- $P'_2 = \{(r-1, s), (r-2, s-1), (r-3, s-2), \dots, (2, s-r+3), (1, s-r+2), (0, s-r+1)\}$
- $P'_3 = \{(0, s-r), (1, s-r-1), (2, s-r-2), \dots, (k-2, s-r-k+2), (k-1, s-r-k+1), (k, s-r-k)\}$

then $H^s_*(F \otimes S^r\mathcal{Q}) = 0$.

Proof. We consider the same complex as in the proof of Lemma 3.1.14,

$$0 \longrightarrow S^{n-1-k}\mathcal{Q}^\vee(-1) \longrightarrow V^* \otimes S^{n-2-k}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^* \otimes S^{n-3-k}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-r-2} V^* \otimes S^{r+1}\mathcal{Q} \longrightarrow \bigwedge^{n-k-r-1} V^* \otimes S^r\mathcal{Q} \longrightarrow \bigwedge^{n-k-r} V^* \otimes S^{r-1}\mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-2-k} V^* \otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-1-k} V^* \otimes \mathcal{O}(-1) \longrightarrow \bigwedge^{k} V \otimes \mathcal{O} \longrightarrow \bigwedge^{k-1} V \otimes \mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{2} V \otimes S^{k-2}\mathcal{Q} \longrightarrow V \otimes S^{k-1}\mathcal{Q} \longrightarrow S^k\mathcal{Q} \longrightarrow 0$$

and apply cohomology to each short exact sequence. Again the range for s in the hypothesis avoids any j to be 0 or $2n-2$. $\square$

As before we represent graphically the points of Lemma 3.1.15 (P_1', P_2' and P_3') and also the point (r, s).

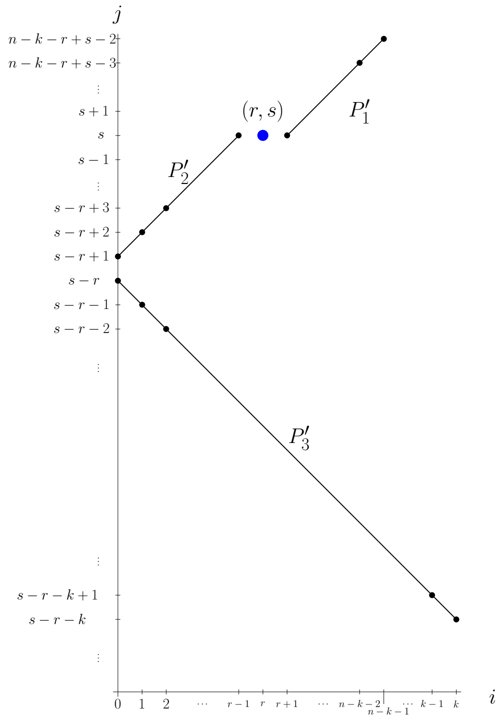


Figure 3.12: Segments P_1', P_2' and P_3'

Remark 3.1.16. The case $r = n-1$ will be very useful in Remark 4.3.4 since with this method we can obtain equivalent conditions to the one corresponding to $(n-1, n-2)$ by using the Eagon-Northcott complex (R_{n-1}). More precisely, these equivalent conditions are:

- $P_3 = \{(0, s+n-2), (1, s+n-3), \ldots, (n-3, s+1), (n-2, s)\}$
- $P_4 = \{(0, s+n-1)\}$

represented in the following figure.

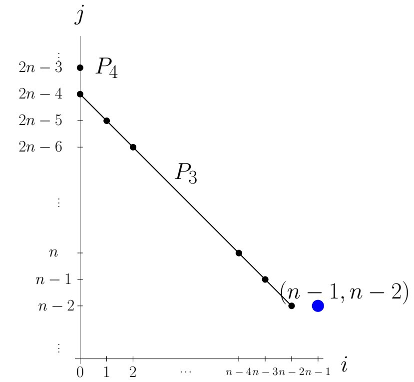


Figure 3.13: Segments P_3 and P_4

Remark 3.1.17. Notice that if in Lemma 3.1.14 we have that $r = k$ there will be only two pieces in Figure 3.11, corresponding to P_2 and P_3. In the case of Lemma 3.1.15, if $r = n-k-1$ there will be only two pieces in Figure 3.12, corresponding to P_2' and P_3'.

Remark 3.1.18. Observe that we cannot give this kind of relation when $r = 0$ since we always need to know that $H^s_*(F) = 0$ to prove that $H^s_*(F) = 0$ if we use the same technique as before.

Remark 3.1.19. Notice that since the number of requested conditions of Lemma 3.1.14 and 3.1.15 is always n we cannot use these arguments to simplify the conditions that appear in Figure 3.8.

The following remark will be very useful to compare Theorem 0.0.2 with the splitting criterion that we will obtain in Theorem 4.3.9 with the techniques of derived categories.

Remark 3.1.20. By applying Lemma 3.1.14 (with $k = r$ and $2 \leq r \leq n-2$) to the conditions of Theorem 0.0.2 we also obtain the vanishings of $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ where:

$$(i, j) \in \{(2,1), (3,2), (4,3), \ldots, (n-4, n-5), (n-3, n-4), (n-2, n-3)\} = R_{-1}$$

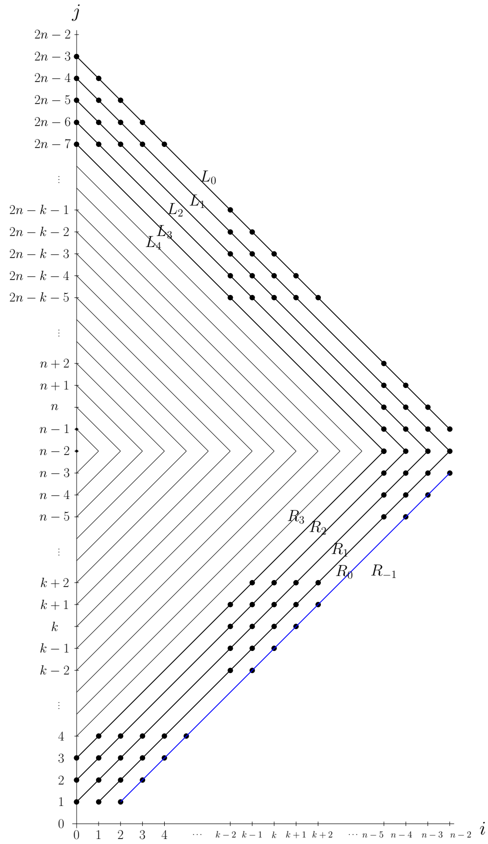


Figure 3.14: Extended version of Theorem 0.0.2

3.2 Previous Lemmas

Before giving the main result, let us give a general idea of the technique we will use during this chapter. To characterize when a vector bundle F decomposes as $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$ from the characterization of $\oplus\mathcal{O}(l_{i_0})$ we can assume, after twisting F, $H^{n-1}(F\otimes S^{n-2}\mathcal{Q}(-n))\neq 0$. Otherwise we could use Theorem 0.0.3. Then we use a particular Eagon-Northcott complex to transform $H^{n-1}(F\otimes S^{n-2}\mathcal{Q}(-n))$ into $H^0(F\otimes\mathcal{Q}^\vee)$. By Serre duality we have also $H^{n-1}(F^\vee\otimes S^{n-2}\mathcal{Q}^\vee(-n-1))\neq 0$ and again, by a particular Eagon-Northcott complex, we transform it into $H^0(F^\vee\otimes\mathcal{Q})$. Hence, we obtain the maps $\mathcal{Q}\longrightarrow F$ and $F\longrightarrow\mathcal{Q}$ and we will show that their composition is the identity of $\mathcal{Q}$ or a multiple of it. Hence $\mathcal{Q}$ is a direct summand of F and we proceed by induction on $rank(F)$.

Once we have done this, we try to characterize when F decomposes as $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus(\oplus S^2\mathcal{Q}(l_{i_2}))$. But this time we can assume $H^{n-1}(F\otimes S^{n-3}\mathcal{Q}(-n))\neq 0$. By using Eagon-Northcott complexes and Serre duality we obtain the maps $S^2\mathcal{Q}\longrightarrow F$ and $F\longrightarrow S^2\mathcal{Q}$ and again we will prove their composition to be the identity or a multiple of it, which allows to complete the proof.

In general we use induction on k to characterize $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus\dots\bigoplus(\oplus S^k\mathcal{Q}(l_{i_k}))$ from the characterization of $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus\dots\bigoplus(\oplus S^{k-1}\mathcal{Q}(l_{i_{k-1}}))$ and for this purpose we assume $H^{n-1}(F\otimes S^{n-k-1}\mathcal{Q}(-n))\neq 0$ (since $H^{n-1}(S^k\mathcal{Q}\otimes S^{n-k-1}\mathcal{Q}(-n))\neq 0$ as we have observed in Lemma 2.2.19). And again we will get suitable maps $S^k\mathcal{Q}\longrightarrow F$ and $F\longrightarrow S^k\mathcal{Q}$.

In the Lemmas of this section we will see how to produce maps from the cohomology groups. We will keep the notation of Definition 3.1.9.

Lemma 3.2.1. *Fix $k\in\{1,\dots,n-2\}$. Then there exists a natural map*

$$H^0(F\otimes S^k\mathcal{Q}^\vee)\xrightarrow{\psi_1}H^{n-1}(F\otimes S^{n-k-1}\mathcal{Q}(-n))$$

Moreover, if the following conditions hold:

$$H^j_*(F\otimes S^i\mathcal{Q})=0\quad\textit{with}\quad(i,j)\in B_k\cup D_k\cup(n-k-2,n-1)$$

the map ψ_1 is a surjective map.

Proof. Since $k\leq n-2$ we can consider the Eagon-Northcott complex $(R^\vee_{n-1-k})\otimes\mathcal{O}(-k-1)$, glued together with $(R_k)\otimes\mathcal{O}(-k)$ from Example 1.2.2 and using Remark 1.1.15:

$$0\longrightarrow S^{n-k-1}\mathcal{Q}(-n)\longrightarrow V^*\otimes S^{n-k-2}\mathcal{Q}(-n+1)\longrightarrow\bigwedge^2 V^*\otimes S^{n-k-3}\mathcal{Q}(-n+2)\longrightarrow\dots$$

$$\dots\longrightarrow\bigwedge^{n-k-2}V^*\otimes\mathcal{Q}(-k-2)\longrightarrow\bigwedge^{n-k-1}V^*\otimes\mathcal{O}(-k-1)\longrightarrow\bigwedge^{k}V\otimes\mathcal{O}(-k)\longrightarrow\bigwedge^{k-1}V\otimes\mathcal{Q}(-k)\longrightarrow\dots$$

$$\dots\longrightarrow\bigwedge^2 V\otimes S^{k-2}\mathcal{Q}(-k)\longrightarrow V\otimes S^{k-1}\mathcal{Q}(-k)\longrightarrow S^k\mathcal{Q}(-k)\longrightarrow 0\tag{3.1}$$

Now we tensorize it by F and take all the short exact sequences with their corresponding cokernels $(K^\vee_j)$. Then apply cohomology. Let us start from left to right.

$$\begin{tikzcd}
H^{n-2}(F\otimes S^{n-k-1}\mathcal{Q}(-n)) \arrow[r] & V^*\otimes H^{n-2}(F\otimes S^{n-k-2}\mathcal{Q}(-n+1)) \arrow[r] & H^{n-2}(F\otimes K^\vee_{n-k-2}(-k-1)) \arrow[dll, "\varphi_1"'] \\
H^{n-1}(F\otimes S^{n-k-1}\mathcal{Q}(-n)) \arrow[r] & V^*\otimes \underbrace{H^{n-1}(F\otimes S^{n-k-2}\mathcal{Q}(-n+1))}_{\text{point }(n-k-2,n-1)} \arrow[r] & H^{n-1}(F\otimes K^\vee_{n-k-2}(-k-1))
\end{tikzcd}$$

Moreover, if $H^{n-1}_*(F\otimes S^{n-k-2}\mathcal{Q})=0$, then φ_1 is a surjective map. Let us repeat the same argument. By using the first part of (3.1) we construct φ_j for $j=2,\dots,n-k-2$.

$$\begin{tikzcd}
H^{n-j-1}(F\otimes K^\vee_{n-k-j}(-k-1)) \arrow[r] & \bigwedge^j V^*\otimes H^{n-j-1}(F\otimes S^{n-k-j-1}\mathcal{Q}(-n+j)) \arrow[r] & H^{n-j-1}(F\otimes K^\vee_{n-k-j+1}(-k-1)) \arrow[dll, "\varphi_j"'] \\
H^{n-j}(F\otimes K^\vee_{n-k-j}(-k-1)) \arrow[r] & \bigwedge^j V^*\otimes \underbrace{H^{n-j}(F\otimes S^{n-k-j-1}\mathcal{Q}(-n+j))}_{\text{point }(n-k-j-1,n-j)\in B_k} \arrow[r] & H^{n-j}(F\otimes K^\vee_{n-k-j+1}(-k-1))
\end{tikzcd}$$

We use the middle part of (3.1) for the maps φ_{n-k-1} and φ_{n-k}.

$$\begin{tikzcd}
H^k(F \otimes K_1^\vee(-k-1)) \arrow[r] & \bigwedge^{n-k-1} V^* \otimes H^k(F(-k-1)) \arrow[r] & H^k(F \otimes \bigwedge^{n-k-1} \mathcal{S}(-k-1)) \arrow[dll, "\varphi_{n-k-1}"'] \\
H^{k+1}(F \otimes K_1^\vee(-k-1)) \arrow[r] & \bigwedge^{n-k-1} V^* \otimes \underbrace{H^{k+1}(F(-k-1))}_{\text{point } (0,k+1) \in B_k} \arrow[r] & H^{k+1}(F \otimes \bigwedge^{n-k-1} \mathcal{S}(-k-1))
\end{tikzcd}$$

$$\begin{tikzcd}
H^{k-1}(F \otimes \bigwedge^k \mathcal{S}^\vee(-k)) \arrow[r] & \bigwedge^k V \otimes H^{k-1}(F(-k)) \arrow[r] & H^{k-1}(F \otimes K_1'(-k)) \arrow[dll, "\varphi_{n-k}"'] \\
H^k(F \otimes \bigwedge^k \mathcal{S}^\vee(-k)) \arrow[r] & \bigwedge^k V \otimes \underbrace{H^k(F(-k))}_{\text{point } (0,k) \in D_k} \arrow[r] & H^k(F \otimes \bigwedge^{n-k-1} \mathcal{S}(-k-1))
\end{tikzcd}$$

We repeat now for φ_j with $j = n-k+1, \dots, n-2$ by using the third part of (3.1), and finally for φ_{n-1}.

$$\begin{tikzcd}
H^{n-j-1}(F \otimes K'_{k+j-n}(-k)) \arrow[r] & \bigwedge^{n-j} V \otimes H^{n-j-1}(F \otimes S^{k+j-n}\mathcal{Q}(-k)) \arrow[r] & H^{n-j-1}(F \otimes K'_{k+j-n+1}(-k)) \arrow[dll, "\varphi_j"'] \\
H^{n-j}(F \otimes K'_{k+j-n}(-k)) \arrow[r] & \bigwedge^{n-j} V \otimes \underbrace{H^{n-j}(F \otimes S^{k+j-n}\mathcal{Q}(-k))}_{\text{point } (k+j-n, n-j) \in D_k} \arrow[r] & H^{n-j}(F \otimes K'_{k+j-n+1}(-k))
\end{tikzcd}$$

$$\begin{tikzcd}
H^0(F \otimes K'_{k-1}(-k)) \arrow[r] & V \otimes H^0(F \otimes S^{k-1}\mathcal{Q}(-k)) \arrow[r] & H^0(F \otimes S^k\mathcal{Q}(-k)) \arrow[dll, "\varphi_{n-1}"'] \\
H^1(F \otimes K'_{k-1}(-k)) \arrow[r] & V \otimes \underbrace{H^1(F \otimes S^{k-1}\mathcal{Q}(-k))}_{\text{point } (k-1,1) \in D_k} \arrow[r] & H^1(F \otimes S^k\mathcal{Q}(-k))
\end{tikzcd}$$

By the composition of all φ_j for $j = 1, \dots, n-1$ we have finally construct the map ψ_1

$$H^0(F \otimes S^k\mathcal{Q}(-k)) \xrightarrow{\psi_1} H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n))$$

Moreover, if all the cohomologies of the statement vanish then ψ_1 is a surjective map. $\square$

Remark 3.2.2. Notice that the cohomological condition corresponding to the point $(n-k-2, n-1)$ makes no sense if $k > n-2$. This is why Theorem 3.3 will be valid only for $k \leq n-2$.

Remark 3.2.3. For the next lemma we need the conditions of the form $H^j_*(F^\vee \otimes S^i\mathcal{Q}) = 0$ for some (i, j). More precisely, the conditions we need are:

- $H^{n-2}_*(F^\vee \otimes S^{n-k-3}\mathcal{Q}^\vee) = H^{n-3}_*(F^\vee \otimes S^{n-k-4}\mathcal{Q}^\vee) = \dots$
  $\dots = H^{k+3}_*(F^\vee \otimes S^2\mathcal{Q}^\vee) = H^{k+2}_*(F^\vee \otimes \mathcal{Q}^\vee) = H^{k+1}_*(F^\vee) = 0$
- $H^k_*(F^\vee) = H^{k-1}_*(F^\vee \otimes \mathcal{Q}^\vee) = H^{k-2}_*(F^\vee \otimes S^2\mathcal{Q}^\vee) = \dots$
  $\dots = H^3_*(F^\vee \otimes S^{k-3}\mathcal{Q}) = H^2_*(F^\vee \otimes S^{k-2}\mathcal{Q}) = H^1_*(F^\vee \otimes S^{k-1}\mathcal{Q}) = 0$
- $H^{n-1}_*(F^\vee \otimes S^{n-k-2}\mathcal{Q}^\vee) = 0$

For simplicity, by using the Serre duality we transform them into another ones without $F^\vee$:

- $H^n_*(F \otimes S^{n-k-3}\mathcal{Q}) = H^{n+1}_*(F \otimes S^{n-k-4}\mathcal{Q}) = \dots$
  $\dots = H^{2n-5-k}_*(F \otimes S^2\mathcal{Q}) = H^{2n-4-k}_*(F \otimes \mathcal{Q}) = H^{2n-3-k}_*(F) = 0$
- $H^{2n-2-k}_*(F) = H^{2n-1-k}_*(F \otimes \mathcal{Q}) = H^{2n-k}_*(F \otimes S^2\mathcal{Q}) = \dots$
  $\dots = H^{2n-5}_*(F \otimes S^{k-3}\mathcal{Q}) = H^{2n-4}_*(F \otimes S^{k-2}\mathcal{Q}) = H^{2n-3}_*(F \otimes S^{k-1}\mathcal{Q}) = 0$
- $H^{n-1}_*(F \otimes S^{n-k-2}\mathcal{Q}) = 0$

These correspond to the segments A_k, C_k and the point $(n-k-2, n-1)$.

Lemma 3.2.4. *Fix $k \in \{1, \dots, n-2\}$. Then there exists a natural map*

$$H^0(F^\vee \otimes S^k\mathcal{Q}) \xrightarrow{\psi_2} H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1))$$

Moreover, if the following conditions hold:

$$H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{with} \quad (i, j) \in A_k \cup C_k \cup (n-k-2, n-1)$$

the map ψ_2 is a surjective map.

Proof. Argue in the same way as before. Since $k \leq n-2$ we can consider the Eagon-Northcott complex $(R^\vee_{n-1-k}) \otimes \mathcal{O}(-1)$ glue together with (R_k) from Example 1.2.2:

$$0 \longrightarrow S^{n-k-1}\mathcal{Q}^\vee(-1) \longrightarrow V^* \otimes S^{n-k-2}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^* \otimes S^{n-k-3}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-2} V^* \otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-k-1} V^* \otimes \mathcal{O}(-1) \longrightarrow \bigwedge^k V \otimes \mathcal{O} \longrightarrow \bigwedge^{k-1} V \otimes \mathcal{Q} \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^2 V \otimes S^{k-2}\mathcal{Q} \longrightarrow V \otimes S^{k-1}\mathcal{Q} \longrightarrow S^k\mathcal{Q} \longrightarrow 0 \tag{3.2}$$

This time we tensorize it by $F^\vee$ and take all the short exact sequences with their corresponding cokernels $(K^\vee_j)$ as before. ∎

Notation 3.2.5. We want to remark the following isomorphism:

$$\mathbb{K} = H^{2n-2}(\mathcal{O}(-n-1)) \cong H^{2n-2}(S^{n-k-1}\mathcal{Q}(-n) \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)).$$

We call the following perfect pairing:

$$\begin{array}{rcl} H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) & \xrightarrow{\phi} & H^{2n-2}(\mathcal{O}(-n-1)) \\ (\alpha, \beta) & \longmapsto & \phi((\alpha, \beta)) \end{array}$$

the *Serre pairing.*

Now we are ready to state the following proposition in which we prove that the composition of the maps $S^k\mathcal{Q} \longrightarrow F$ and $F \longrightarrow S^k\mathcal{Q}$ corresponds to the identity or with a multiple of it.

Proposition 3.2.6. *Fix $k \in \{1, \dots, n-2\}$ and consider α and β be non zero elements such that the image of the Serre pairing,*

$$\begin{array}{rcl} H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) & \xrightarrow{\phi} & H^{2n-2}(\mathcal{O}(-n-1)) \\ (\alpha, \beta) & \longmapsto & \phi((\alpha, \beta)) \end{array}$$

is non zero. Suppose that the following conditions hold:

$$H^j_*(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{with} \quad (i,j) \in A_k \cup B_k \cup C_k \cup D_k \cup (n-k-2, n-1)$$

Then we can lift α to $\alpha' \in H^0(F \otimes S^k\mathcal{Q}^\vee)$ and β to $\beta' \in H^0(F^\vee \otimes S^k\mathcal{Q})$ by the natural maps,

$$\begin{array}{rcl} H^0(F \otimes S^k\mathcal{Q}^\vee) & \xrightarrow{\psi_1} & H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n)) \\ \alpha' & \longmapsto & \alpha \end{array}$$

$$\begin{array}{rcl} H^0(F^\vee \otimes S^k\mathcal{Q}) & \xrightarrow{\psi_2} & H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \\ \beta' & \longmapsto & \beta \end{array}$$

of Lemma 3.2.1 and Lemma 3.2.4, such that, regarding $\alpha' \in Hom(S^k\mathcal{Q}, F)$ and $\beta' \in Hom(F, S^k\mathcal{Q})$, their composition is a nonzero multiple of the identity map $S^k\mathcal{Q} \longrightarrow S^k\mathcal{Q}$.

Proof. By Lemma 3.2.1 and Lemma 3.2.4 we know that we can lift α to α' by ψ_1 and by ψ_2 we can lift β to β'. We would like also to lift ϕ to the composition map $Hom(S^k\mathcal{Q}, F) \times Hom(F, S^k\mathcal{Q}) \longrightarrow Hom(S^k\mathcal{Q}, S^k\mathcal{Q})$. To do this, let us identify $H^{2n-2}(\mathcal{O}(-n-1))$ with $Hom(S^k\mathcal{Q}, S^k\mathcal{Q})$. We will do this by using the same complexes we used for the maps ψ_1 and ψ_2 since $k \leq n-2$.

If we tensorize now the complex (3.1) by $S^{n-k-1}\mathcal{Q}^\vee(-1)$ instead of by F and repeat the same argument as in Lemma 3.2.1, but starting from $2n-2$ as the higher order of the cohomology, we obtain that the following map is an isomorphism:

$$H^{n-1}(S^k\mathcal{Q}^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \underset{\psi_3}{\xrightarrow{\simeq}} H^{2n-2}(S^{n-k-1}\mathcal{Q}(-n) \otimes S^{n-k-1}\mathcal{Q}^\vee(-1))$$

The cohomology vanishings we need are the ones given in Lemma 2.2.17.

Now let us consider the complex (3.2) tensorized by $S^k\mathcal{Q}^\vee$ instead of by $F^\vee$ from Lemma 3.2.4. This time we start from the cohomology of order $n-1$ and obtain that the following map is an isomorphism:

$$H^0(S^k\mathcal{Q}^\vee \otimes S^k\mathcal{Q}) \xrightarrow[\psi_4]{\simeq} H^{n-1}(S^k\mathcal{Q}^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1))$$

As before, the cohomology vanishings we need are the ones given in Lemma 2.2.18.

Putting all together we get the following commutative diagram:

$$\begin{tikzcd}
H^{n-1}(F\otimes S^{n-k-1}\mathcal{Q}(-n))\times H^{n-1}(F^\vee\otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \arrow[r, "\phi"] & H^{2n-2}(S^{n-k-1}\mathcal{Q}(-n)\otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \\
H^0(F\otimes S^k\mathcal{Q}^\vee)\times H^{n-1}(F^\vee\otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \arrow[r] \arrow[u, "\psi_1\times id"] \arrow[ur, phantom, "\circlearrowleft"] & H^{n-1}(S^k\mathcal{Q}^\vee\otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \arrow[u, "\psi_3"', "\simeq"] \\
H^0(F\otimes S^k\mathcal{Q}^\vee)\times H^0(F^\vee\otimes S^k\mathcal{Q}) \arrow[r, "\phi'"] \arrow[u, "id\times\psi_2"] \arrow[ur, phantom, "\circlearrowleft"] & H^0(S^k\mathcal{Q}^\vee\otimes S^k\mathcal{Q}) \arrow[u, "\psi_4"', "\simeq"]
\end{tikzcd}$$

where ψ_1 and ψ_2 are surjective maps and ψ_3 and ψ_4 are isomorphisms.

Hence, the composition of the morphisms $\alpha' \in Hom(S^k\mathcal{Q}, F)$ and $\beta' \in Hom(F^\vee \otimes S^k\mathcal{Q})$ is a non zero map $S^k\mathcal{Q} \longrightarrow S^k\mathcal{Q}$. Since $S^k\mathcal{Q}$ is simple (see Proposition 2.2.16) this composition is necessarily a multiple of the identity. $\square$

Remark 3.2.7. We can identify the hypothesis of Proposition 3.2.6 with the points (i,j) in Figure 3.7.

3.3 Main Theorem

We prove now the main theorem of this thesis.

Theorem 3.3.1. *Fix $k \in \{0,1,\ldots,n-2\}$ an let F be a vector bundle over the Grassmannian of lines $\mathbb{G}(1,n)$. Then F is a direct sum of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$ if and only if the following conditions hold:*

$$H^j_*(F\otimes S^i\mathcal{Q}) = 0 \quad \textit{for} \quad (i,j) \in \begin{cases} A_0 \cup A_1 \cup \ldots \cup A_k \\ B_0 \cup B_1 \cup \ldots \cup B_k \\ C_1 \cup C_2 \cup \ldots \cup C_k \\ D_1 \cup D_2 \cup \ldots \cup D_k \\ (n-k-2, n-1) \end{cases}$$

Proof. We use double induction. First we make induction on k, the case $k=0$ being Theorem 0.0.3. We suppose now the theorem true for $k-1$ and we want to prove it for k. Since $k \le n-2$ we can now apply induction on $m := \sum_l h^{n-1}(F\otimes S^{n-k-1}\mathcal{Q}(l))$ (by Remark 1.3.7 we know that m is a finite number). When $m=0$ we are in the hypothesis of the theorem when replacing k with $k-1$. Hence by induction hypothesis, F can be expressed as the direct sum of twist of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, $\ldots$, $S^{k-1}\mathcal{Q}$.

Assume now $m > 0$ and that we know the result for $m-1$. In particular, $H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(l)) \neq 0$ for some l. We want to show that then $S^k\mathcal{Q}(-n-l)$ is a direct summand of F (see Lemma 2.2.19). Since the result is independent on the twist, we can assume $l = -n$. We take a non zero element $\alpha \in H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n))$. Since the Serre pairing:

$$H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \xrightarrow{\phi} H^{2n-2}(\mathcal{O}(-n-1))$$

is perfect we can take $\beta \in H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1))$ such that $\phi(\alpha, \beta) \neq 0$.

Since $k \leq n-2$, by Proposition 3.2.6 we can lift α to $\alpha' \in Hom(S^k\mathcal{Q})$ and β to $\beta' \in Hom(F, S^k\mathcal{Q})$ such that the composition of α' and β' is a nonzero multiple of the identity map $S^k\mathcal{Q} \longrightarrow S^k\mathcal{Q}$. Thus we can write $F = S^k\mathcal{Q} \oplus F'$ for some F'. Clearly F' satisfies all the hypothesis of the theorem (since F does). Moreover, since $h^{n-1}(S^k\mathcal{Q} \otimes S^{n-k-1}\mathcal{Q}(-n)) = 1$ by Lemma 2.2.19 we have $h^{n-1}(F' \otimes S^{n-k-1}\mathcal{Q}(-n)) = m - 1$.

By induction hypothesis F' can be expressed as direct sums of twists of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$ and hence the same holds for F as wanted.

□

Remark 3.3.2. As we can observe, there are some repeated conditions in the statement of the Theorem 3.3.1. Graphically these repetitions coincide with the intersections between segments.

3.4 Remarks about the Main Theorem

3.4.1 Some remarks about sharpness

Some remarks are in order.

Remark 3.4.1. As we have remarked the condition $H^{n-1}_*(F \otimes S^{n-k-2}\mathcal{Q}) = 0$ corresponding to the point $(n-k-2, n-1)$ is necessary, since for $F = S^{k+1}\mathcal{Q}$ satisfies all the other conditions (see Lemma 2.2.19).

Remark 3.4.2. Let us discuss why we cannot include $S^{n-1}\mathcal{Q}$ (or in general $S^k\mathcal{Q}$ with $k > n-2$) as a direct summand of F in the characterization of the main theorem.

If we are in the hypothesis of the characterization for

$$(\oplus\mathcal{O}(l_{i_0})) \bigoplus (\oplus\mathcal{Q}(l_{i_1})) \bigoplus \ldots \bigoplus (\oplus S^{n-2}\mathcal{Q}(l_{i_{n-2}}))$$

we need to remove one cohomological condition of order $n-1$ to get the characterization of

$$(\oplus\mathcal{O}(l_{i_0})) \bigoplus (\oplus\mathcal{Q}(l_{i_1})) \bigoplus \ldots \bigoplus (\oplus S^{n-2}\mathcal{Q}(l_{i_{n-2}})) \bigoplus (\oplus S^{n-1}\mathcal{Q}(l_{i_{n-1}}))$$

The only possibility is $H^{n-1}_*(F) = 0$. Let us suppose $H^{n-1}_*(F) \neq 0$ (since $H^{n-1}(S^{n-1}\mathcal{Q}(-n)) \neq 0$, see Lemma 2.2.19). Now we should construct the map ψ_1 as in Lemma 3.2.1.

$$H^0(F \otimes S^{n-1}\mathcal{Q}^\vee) \xrightarrow{\psi_1} H^{n-1}(F(-n))$$

To do this we consider the complex $(R_{n-1})\otimes\mathcal{O}(-n-1)$ and apply cohomology. The first cohomology that must vanish to construct ψ_1 is $H^{n-1}_*(F)=0$, but this cannot occur since $H^{n-1}_*(S^{n-1}\mathcal{Q})\neq 0$.

Moreover, if one removes the condition $H^{n-1}_*(F)$ from the hypothesis of the theorem one expect to characterize, with the remaining conditions, the direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, …, $S^{n-2}\mathcal{Q}$, $S^{n-1}\mathcal{Q}$, $S^n\mathcal{Q}$, … since any other symmetric product of order greater or equal than $n-1$ satisfies all the conditions required for $k=n-2$ except $H^{n-1}_*(F)=0$.

On the other hand, notice that the statement of the main theorem has no sense for $k\geq n-1$. We have already mentioned that $A_k=B_k=\emptyset$ for $k\geq n-2$. Observe also that conditions corresponding to C_k and D_k for $k\geq n-1$ generate a cohomological condition of order $n-1$ but this cannot be one of the hypothesis since $H^{n-1}_*(S^i\mathcal{Q})\neq 0$ when $i\geq n-1$.

3.4.2 Comparison between characterizations

We can compare our theorem with the characterizations that are already made by other authors. For instance, we can compare the splitting criteria we assume for $k=0$ (Theorem 0.0.3, by E. Arrondo and F. Malaspina) with the one made by G. Ottaviani in the particular case of $\mathbb{G}(1,n)$ (Theorem 0.0.2). At the end of Chapter 4 we will also obtain a splitting criteria for the Grassmannian of lines and we will compare these theorems with it. Let us now recall the statement of Theorem 0.0.2:

> Let F be a vector bundle over $\mathbb{G}(1,n)$. Then F splits if and only if the following conditions hold for $i=0,1,\dots,n-2$:
>
> $$H^j_*(F\otimes S^i\mathcal{Q})=0 \quad \text{with} \quad i\leq j<2n-2-i \quad \text{and} \quad j>0.$$

Remark 3.4.3. Although the original conditions for this splitting criteria are given as the vanishings of some cohomologies of the bundles F, $F\otimes\mathcal{Q}$, $F\otimes\mathcal{Q}\otimes\mathcal{Q}$, …, $F\otimes\mathcal{Q}\underbrace{\otimes\ldots\otimes}_{n-2}\mathcal{Q}$ by using the well-known expression (2.2.13) we easily achieve that these conditions are equivalent to the ones of Theorem 0.0.2, which are represented in Figure 3.2.

As we observed in Remark 3.1.5, Theorem 0.0.2 has much more conditions than Theorem 0.0.3 (represented in Figure 3.1).

On the other hand, H. Knörrer gave a characterization of aCM bundles over quadrics (see [28]), which in the particular case of $\mathbb{G}(1,3)$ reads as;

Theorem 3.4.4. *Let F be a vector bundle over $\mathbb{G}(1,3)$. Then F can be expressed as $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))\bigoplus(\oplus\mathcal{S}(l_{i_2}))$ if and only if $H^j_*(F)=0$ for $j=1,2,3$.*

Two immediate corollaries of this are the following (see also [8]).

Theorem 3.4.5. *Let F be a vector bundle over $\mathbb{G}(1,3)$. Then F can be expressed as $(\oplus\mathcal{O}(l_{i_0}))\bigoplus(\oplus\mathcal{Q}(l_{i_1}))$ if and only if,*

- $H^j_*(F)=0$ *for* $j=1,2,3$

- $H^1_*(F \otimes \mathcal{Q}) = 0$

Observe that this is exactly what we get in Theorem 3.3.1 for $n = 3$.

Theorem 3.4.6. *Let F be a vector bundle over $\mathbb{G}(1,3)$. Then F can be expressed as $\oplus\mathcal{O}(l_{i_0})$ if and only if,*

- $H^j_*(F) = 0$ *for* $j = 1, 2, 3$
- $H^j_*(F \otimes \mathcal{Q}) = 0$ *for* $j = 1, 2$

Now this characterization has two more conditions than the one given in Theorem 0.0.3 for $n = 3$, and coincides with Theorem 0.0.2.

We finally compare Theorem 0.0.4 of E. Arrondo and B. Graña with the characterization of sums of line bundles and twists of $\mathcal{Q}$ that we got in Theorem 3.3.1. For that, we particularize that resulf for $k = 1$ and $n = 4$:

Theorem 3.4.7. *(Theorem 3.3.1 for $k = 1$, $n = 4$) Let F be a vector bundle in $\mathbb{G}(1,4)$. Then F can be expressed as direct sum of twist of line bundles and $\mathcal{Q}$ if and only if the following conditions hold:*

- $H^1_*(F) = H^2_*(F) = H^4_*(F) = H^5_*(F) = 0$
- $H^1_*(F \otimes \mathcal{Q}) = H^2_*(F \otimes S^2\mathcal{Q}) = H^3_*(F \otimes \mathcal{Q}) = H^4_*(F \otimes \mathcal{Q}) = 0$

Representing graphically the conditions of Theorem 0.0.4 and 3.4.7, we notice that our charac-

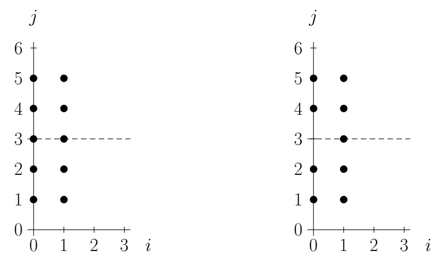


Figure 3.15: Theorem 0.0.4 vs Theorem 3.4.7

terization has one condition less than the one made by E. Arrondo and B. Graña. This is because our starting point was Theorem 0.0.3 instead of Theorem 0.0.2

3.4.3 Generalization to other varieties

Observe that the main underlying idea in the proof of our result is to interpret Serre's duality as a composition pairing of extensions $Ext^i(F, S^k\mathcal{Q}\otimes\omega_{\mathbb{G}(1,n)})\times Ext^{2n-2-i}(S^k\mathcal{Q}, F) \to Ext^{2n-2}(S^k\mathcal{Q}, S^k\mathcal{Q}\otimes$

$\mathcal{O}(-n-1))$, in which this last space $Ext^{2n-2}(S^k\mathcal{Q}, S^k\mathcal{Q}\otimes\mathcal{O}(-n-1))$ is one-dimensional and generated by the gluing of (3.2) and its dual twisted by $\mathcal{O}(-n-1+k)$. The starting point, which is the splitting Theorem 0.0.3, comes in fact from the case $k=0$.

Hence, in order to generalize all this to an arbitrary $N-$dimensional variety X we need to find a suitable extension generating $Ext^N(\mathcal{O}, \omega_X)$. For example, if X is an arbitrary Grassmannian we have the following:

Proposition 3.4.8. *The one-dimensional vector space*

$$H^{(k+1)(n-k)}(\omega_{\mathbb{G}(k,n)}) = H^{(k+1)(n-k)}(\mathcal{O}(-n-1)) = Ext^{(k+1)(n-k)}(\mathcal{O}, \mathcal{O}(-n-1))$$

is generated by the following exact sequence over $\mathbb{G}(k,n)$*:*

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow V\otimes\mathbb{S}_{(n-k,\dots,n-k,n-k-1)}\mathcal{Q}^\vee(-k-1) \longrightarrow \textstyle\bigwedge^2 V\otimes\mathbb{S}_{(n-k,\dots,n-k,n-k-2)}\mathcal{Q}^\vee(-k-1) \longrightarrow \dots$$

$$\dots \longrightarrow \textstyle\bigwedge^{n-k-1} V\otimes\mathbb{S}_{(n-k,\dots,n-k,1)}\mathcal{Q}^\vee(-k-1) \longrightarrow \bigwedge^{n-k} V\otimes\mathbb{S}_{(n-k,\dots,n-k)}\mathcal{Q}^\vee(-k-1) \longrightarrow \dots$$

$$\vdots$$

$$\dots \longrightarrow V\otimes\mathbb{S}_{(n-k,n-k,n-k-1)}\mathcal{Q}^\vee(-3) \longrightarrow \bigwedge^2 V\otimes\mathbb{S}_{(n-k,n-k,n-k-2)}\mathcal{Q}^\vee(-3) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V\otimes\mathbb{S}_{(n-k,n-k,1)}\mathcal{Q}^\vee(-3) \longrightarrow \bigwedge^{n-k} V\otimes\mathbb{S}_{(n-k,n-k)}\mathcal{Q}^\vee(-3) \longrightarrow$$

$$\longrightarrow V\otimes\mathbb{S}_{(n-k,n-k-1)}\mathcal{Q}^\vee(-2) \longrightarrow \bigwedge^2 V\otimes\mathbb{S}_{(n-k,n-k-2)}\mathcal{Q}^\vee(-2) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V\otimes\mathbb{S}_{(n-k,1)}\mathcal{Q}^\vee(-2) \longrightarrow \bigwedge^{n-k} V\otimes\mathbb{S}_{(n-k)}\mathcal{Q}^\vee(-2) \longrightarrow$$

$$\longrightarrow V^*\otimes S^{n-k-1}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^*\otimes S^{n-k-2}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V^*\otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-k} V^*\otimes\mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

Proof. We start with $(R^\vee_{n-k})\otimes\mathcal{O}(-1)$ of Example 1.2.2:

$$0 \longrightarrow S^{n-k}\mathcal{Q}^\vee(-1) \longrightarrow V^*\otimes S^{n-k-1}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^*\otimes S^{n-k-2}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V^*\otimes\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-k} V^*\otimes\mathcal{O}(-1) \longrightarrow \bigwedge^{n-k}\mathcal{S}(-1) \longrightarrow 0$$

We have the identification $\bigwedge^{n-k} \mathcal{S}(-1) = \mathcal{O}$. We want to glue a complex on the left, hence we identify $S^{n-k}\mathcal{Q}^\vee = \mathbb{S}_{(n-k)}\mathcal{Q}^\vee$ and use the dual of (2.13) of Proposition 2.3.1, tensorized by $\mathcal{O}(-2)$. So $\lambda^* = (n-k)$ and $\lambda = (n-k, n-k)$. We compute also $\mu_i = (n-k, n-k-i)$ and $\upsilon_i = i$:

$$0 \longrightarrow \mathbb{S}_{(n-k,n-k)}\mathcal{Q}^\vee(-2) \longrightarrow V \otimes \mathbb{S}_{(n-k,n-k-1)}\mathcal{Q}^\vee(-2) \longrightarrow \bigwedge^2 V \otimes \mathbb{S}_{(n-k,n-k-2)}\mathcal{Q}^\vee(-2) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V \otimes \mathbb{S}_{(n-k,1)}\mathcal{Q}^\vee(-2) \longrightarrow \bigwedge^{n-k} V \otimes \mathbb{S}_{(n-k)}\mathcal{Q}^\vee(-2) \longrightarrow \mathbb{S}_{(n-k)}\mathcal{Q}^\vee(-1) \longrightarrow 0$$

Now we want to glue anther complex to the left, hence we use the dual of (2.13) again tensorized by $\mathcal{O}(-3)$. So $\lambda^* = (n-k, n-k)$ and $\lambda = (n-k, n-k, n-k)$. We compute also $\mu_i = (n-k, n-k, n-k-i)$ and $\upsilon_i = i$:

$$0 \longrightarrow \mathbb{S}_{(n-k,n-k,n-k)}\mathcal{Q}^\vee(-3) \longrightarrow V \otimes \mathbb{S}_{(n-k,n-k,n-k-1)}\mathcal{Q}^\vee(-3) \longrightarrow \bigwedge^2 V \otimes \mathbb{S}_{(n-k,n-k,n-k-2)}\mathcal{Q}^\vee(-3) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V \otimes \mathbb{S}_{(n-k,n-k,1)}\mathcal{Q}^\vee(-3) \longrightarrow \bigwedge^{n-k} V \otimes \mathbb{S}_{(n-k,n-k)}\mathcal{Q}^\vee(-3) \longrightarrow \mathbb{S}_{(n-k,n-k)}\mathcal{Q}^\vee(-2) \longrightarrow 0$$

If we continue with the same argument, gluing $k+1$ complexes of this form, we will get the complex we want. The last part we must add corresponds with the complex dual of (2.13) where $\lambda^* = (\underbrace{n-k, \dots, n-k}_{k})$, $\lambda = (\underbrace{n-k, \dots, n-k}_{k+1})$, $\mu_i = (\underbrace{n-k, \dots, n-k, n-k-i}_{k})$ and $\upsilon_i = i$ tensorized by $\mathcal{O}(-k-1)$:

$$0 \longrightarrow \mathbb{S}_{(\underbrace{n-k, \dots, n-k}_{k+1})}\mathcal{Q}^\vee(-k-1) \longrightarrow V \otimes \mathbb{S}_{(n-k,\dots,n-k,n-k-1)}\mathcal{Q}^\vee(-k-1) \longrightarrow \bigwedge^2 V \otimes \mathbb{S}_{(n-k,\dots,n-k,n-k-2)}\mathcal{Q}^\vee(-k-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-k-1} V \otimes \mathbb{S}_{(n-k,\dots,n-k,1)}\mathcal{Q}^\vee(-k-1) \longrightarrow \bigwedge^{n-k} V \otimes \mathbb{S}_{(n-k,\dots,n-k)}\mathcal{Q}^\vee(-k-1) \longrightarrow \mathbb{S}_{(\underbrace{n-k, \dots, n-k}_{k})}\mathcal{Q}^\vee(-k) \longrightarrow 0$$

Finally, we have the identification:

$$\mathbb{S}_{(\underbrace{n-k, \dots, n-k}_{k+1})}\mathcal{Q}^\vee(-k-1) = \mathcal{O}(-n+k) \otimes \mathcal{O}(-k-1) = \mathcal{O}(-n-1)$$

Hence we obtain a complex that begins in $\mathcal{O}(-n-1)$ and ends in $\mathcal{O}$ of length $(k+1)(n-k)$ as we wanted. $\square$

Observe that, for $k > 1$, we get Schur functors of $\mathcal{Q}^\vee$ that are not symmetric powers. This makes a precise writing of the conditions of a splitting theorem much more complicated to write down. This is why we decided to concentrate on the case $k = 1$, which makes easier to illustrate our general method.

Chapter 4

Derived categories

A natural way to obtain cohomological characterizations of vector bundles is the use of derived categories (specifically Beilinson's Theorem). As we have shown, a first natural case is a splitting criterion. In this chapter we will find such a criterion for the Grassmannian of lines and we will compare with the ones we already know, showing that our starting criterion of [7] is the most effective.

4.1 Main definitions and properties

In this section we will define the concepts of derived category and derived functors. This will be used only to put in the right general context the Beilinson's Theorem. Hence we are not giving all the details, although we will for sure give the notion of shift functor and mapping cone for the definition of derived category and the notion of "having enough injectives" and injective resolution for the definition of (right) derived functor.

We will suppose in all the chapter that $\mathcal{A}$ is an abelian category. One can check the following main definitions and properties from [24]. And for further information take a look to [10], [14]

4.1.1 Derived categories

We briefly recall the definition of abelian category and category of complexes $Kom(\mathcal{A})$ of an abelian category $\mathcal{A}$..

Definition 4.1.1. An *abelian category* is a category $\mathcal{A}$ such that for each $A, B \in Ob(\mathcal{A})$, $Hom(A, B)$ has a structure of an abelian group, and the composition law is linear (finite direct sums exist, every morphism has a kernel and a cokernel, every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel, and finally every morphism can be factored into an epimorphism followed by a monomophirsm).

Definition 4.1.2. A *complex* $A^\bullet$ in $\mathcal{A}$ consists of a diagram of objects and morphisms in $\mathcal{A}$ of the form

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \ldots$$

satisfying $d^i \circ d^{i-1} = 0$ or, equivalently, $Im(d^{i-1}) \subset Ker(d^i)$, for all $i \in \mathbb{Z}$. A morphism $f : A^\bullet \longrightarrow B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ is given by a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \ldots \\
\ldots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \ldots
\end{tikzcd}$$

Hence, the *category of complexes* $Kom(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category whose objects are complexes $A^\bullet$ in $\mathcal{A}$ and whose morphisms are morphisms of complexes. If the objects A^i are specified only in a certain range, for example when $i \geq 0$, then we set $A^i = 0$ for all other i.

Now, let us define the operation of shifting one place to the left and changing the sign of the differential of a complex $A^\bullet$ (*shift functor*) and also of a morphism of complexes.

Definition 4.1.3. Let $A^\bullet \in Kom(\mathcal{A})$ be a given complex. Then $A^\bullet[1]$ is the complex with $(A^\bullet[1])^i := A^{i+1}$ and differential $d^i_{A[1]} := -d^{i+1}_A$.

The shift $f[1]$ of a morphism of complexes $f : A^\bullet \to B^\bullet$ is the complex morphism $A^\bullet[1] \to B^\bullet[1]$ given by $f[1]^i := f^{i+1}$.

Corollary 4.1.4. *The shift functor $T : Kom(\mathcal{A}) \to Kom(\mathcal{A})$, such that $A^\bullet \longmapsto A^\bullet[1]$ defines an equivalence of abelian categories*

More precisely, the inverse functor T^{-1} is given by $A^\bullet \longmapsto A^\bullet[-1]$, where $A^\bullet[k]^i = A^{k+i}$ and $d^i_{a[k]} = (-1)^k d^{i+k}_A$ for any $k \in \mathbb{Z}$.

Let us recall now the notion and some properties of the cohomology of a complex.

Remark 4.1.5. The cohomology $H^i(A^\bullet)$ of a complex $A^\bullet$ is the quotient

$$H^i(A^\bullet) := \frac{Ker(d^i)}{Im(d^{i-1})} \in \mathcal{A},$$

i.e. $H^i(A^\bullet) = Coker(Im(d^{i-1}) \to Ker(d^i))$. A complex $A^\bullet$ is acyclic if $H^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$. Any morphism of complexes $f : A^\bullet \to B^\bullet$ induces natural homomorphisms

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet).$$

If $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ is a short exact sequence of complexes, then there are natural maps $\delta^i : H^i(C^\bullet) \longrightarrow H^{i+1}(A^\bullet)$ giving rise to a long exact sequence

$$\ldots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C^\bullet) \xrightarrow{\delta^i} H^{i+1}(A^\bullet) \longrightarrow \ldots$$

Now let us define the homotopy category of complexes $K(\mathcal{A})$. This category is very important since its objects are the same as the objects of the derived category of $\mathcal{A}$ (and also the same as $Kom(\mathcal{A})$).

Definition 4.1.6. Two morphisms of complexes

$$f, g : A^\bullet \to B^\bullet$$

are called *homotopically equivalent*, $f \sim g$, if there exits a collection of homomorphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i.$$

The *homotopy category of complexes* $K(A)$ is the category whose objects are the objects of $Kom(\mathcal{A})$, i.e. $Ob(K(\mathcal{A})) = Ob(Kom(\mathcal{A}))$, and the morphisms $Hom_{K(\mathcal{A})}(A^\bullet, B^\bullet) := Hom_{Kom(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$

Remark 4.1.7. The collection of morphisms, $h = (h^i)$ is called *homotopy operator.* If $f \sim g$, then f and g induce the same morphism $H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ on the cohomology objects, for each i.

Now comes the precise definition of the *derived category.* To do this we have to give the objects of $D(\mathcal{A})$ (which are complexes) and the morphism between these complexes.

Definition 4.1.8. We set the *objects of* $D(\mathcal{A})$ as

$$Ob(D(\mathcal{A})) := Ob(K(\mathcal{A})) = Ob(Kom(\mathcal{A})).$$

and the set of *morphism* $Hom_{D(\mathcal{A})}$ between two complexes $A^\bullet$ and $B^\bullet$ viewed as objects in $D(\mathcal{A})$ is the set of all equivalence classes of diagrams of the form

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism. By a *quasi-isomorphism* we mean a morphism $f : A^\bullet \to B^\bullet$ in $K(\mathcal{A})$ which induces an isomorphism in cohomology (note that two homotopic morphisms induce the same maps in cohomology).

Two such diagrams are equivalent if they are dominated in the homotopy category $K(\mathcal{A})$ by a third one of the same sort, i.e. there exists a commutative diagram in $K(\mathcal{A})$ of the form

$$\begin{tikzcd} & & C^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dotted, bend right, "\text{qis}"'] & & \\ & C_1^\bullet \arrow[dl] \arrow[drrr] & & C_2^\bullet \arrow[dlll] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

In this way we have defined objects and morphisms of our category $D(\mathcal{A})$, but we still have to check a number of properties. In particular we have to define the composition of two morphism.

Definition 4.1.9. If two morphisms

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

are given, we want the *composition* of both be given by a commutative (in the homotopy category $K(\mathcal{A})$) diagram of the form

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd} \tag{4.1}$$

Remark 4.1.10. There are two obvious problems: one has to ensure that such a diagram exists an that it is unique up to equivalence.

Both things hold true with the help of the *mapping cone*. Although we introduce this notion we are not going to explain the previous problems because they are a bit tedious.

Definition 4.1.11. Let $f : A^\bullet \to B^\bullet$ be a complex morphism. Its *mapping cone* is the complex $C(f)$ with

$$C(f)^i = A^{i+1} \oplus B^i \text{ and the differentials } d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$$

There exist two natural complex morphisms

$$\tau : B^\bullet \to C(f) \text{ and } \pi : C(f) \to A^\bullet[1]$$

given by the natural injection $B^i \to A^{i+1} \oplus B^i$ and the natural projection $A^{i+1} \oplus B^i \to A^\bullet[1]^i = A^{i+1}$, respectively.

$$\begin{tikzcd} A^\bullet \arrow[d, "f"] & \dots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \dots \\ B^\bullet \arrow[d, "\tau"] & \dots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] \arrow[d] & B^i \arrow[r, "d_B^i"] \arrow[d] & B^{i+1} \arrow[r, "d_B^{i+1}"] \arrow[d] & \dots \\ C(f)^\bullet \arrow[d, "\pi"] & \dots \arrow[r] & A^i \oplus B^{i-1} \arrow[r, "d_{C(f)}^{i-1}"] \arrow[d] & A^{i+1} \oplus B^i \arrow[r, "d_{C(f)}^i"] \arrow[d] & A^{i+2} \oplus B^{i+1} \arrow[r, "d_{C(f)}^{i+1}"] \arrow[d] & \dots \\ A^\bullet[1] & \dots \arrow[r] & A^i \arrow[r, "d_A^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] & A^{i+2} \arrow[r, "d_A^{i+2}"] & \dots \end{tikzcd}$$

Remark 4.1.12. The composition $B^\bullet \to C(f) \to A^\bullet[1]$ is trivial and the composition $A^\bullet \to B^\bullet \to C(f)$ is homotopic to the trivial map. In fact, $B^\bullet \to C(f) \to A^\bullet[1]$ is a short exact sequence of complexes. In particular, we obtain the long exact cohomology sequence

$$\ldots \longrightarrow H^i(A^\bullet) \longrightarrow H^i(B^\bullet) \longrightarrow H^i(C(f)) \longrightarrow H^{i+1}(A^\bullet) \longrightarrow \ldots$$

Also, by construction any commutative diagram can be completed as follows,

$$\begin{tikzcd} A_1^\bullet \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_1^\bullet \arrow[r] \arrow[d] & C(f_1) \arrow[r] \arrow[d, dotted] & A_1^\bullet[1] \arrow[d] \\ A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r] & C(f_2) \arrow[r] & A_2^\bullet[1] \end{tikzcd}$$

It will be crucial for defining the composition of morphisms in the derived category.

Proposition 4.1.13. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone that comes with the two natural morphisms $\tau : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$. Then there exists a complex morphism $g : A^\bullet[1] \to C(\tau)$ which is an isomorphism in $K(\mathcal{A})$ and such that the following diagram is comutative in $K(\mathcal{A})$:*

$$\begin{tikzcd} B^\bullet \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d] & A^\bullet[1] \arrow[r] \arrow[d, dotted, "g"] & B^\bullet[1] \arrow[d] \\ B^\bullet \arrow[r] & C(f) \arrow[r] & C(\tau) \arrow[r] & B^\bullet[1] \end{tikzcd}$$

If we consider a quasi-isomorphism $f : A^\bullet \to B^\bullet$ and an arbitrary morphism $g : C^\bullet \to B^\bullet$ by using the mapping cone there exits a commutative diagram in $K(\mathcal{A})$

$$\begin{tikzcd} C_0^\bullet \arrow[r, "qis"] \arrow[d] & C^\bullet[1] \arrow[d, "g"] \\ A^\bullet \arrow[r, "f", "qis"'] & B^\bullet \end{tikzcd}$$

Furthermore the composition of roofs as proposed by (4.1) exists and is well-defined.

Now let us state the last definition we want to recall.

Definition 4.1.14. A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $K(\mathcal{A})$ (respectively in $D(\mathcal{A})$) is called *distinguished* if it isomorphic in $K(\mathcal{A})$ (respectively in $D(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1] \tag{4.2}$$

with f a complex morphism.

Proposition 4.1.15. *Distinguished triangles given as in (4.2) and the natural shift functor for complexes $A^\bullet \mapsto A^\bullet[1]$ make the homotopy category of complexes $K(\mathcal{A})$ and the derived category $D(\mathcal{A})$ of an abelian category into a triangulated category (additive category $\mathcal{D}$ with a shift functor and a class of distinguished triangles, satisfying some properties).*

Moreover, the natural functor $Q_A : K(\mathcal{A}) \to D(\mathcal{A})$ is an exact functor of triangulated categories.

The following definition gives the notion of *bounded derived categories.*

Definition 4.1.16. Let $Kom^*(\mathcal{A})$, with $* = +, -, b$ be the category of complexes $A^\bullet$ with $A^i = 0$ for $i \ll 0$, $i \gg 0$, respectively $|i| \gg 0$.

By dividing out first by homotopy equivalence and then by quasi-isomorphisms one obtains the categories $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ with $* = +, -, b$.

We are mostly interested in the bounded derived category of coherent sheaves $D^b(Coh(X))$.

4.1.2 Derived Functors

It is convenient to recall first some definitions of covariant functors.

Definition 4.1.17. A covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ from an abelian category to another is *additive* if for any two objects A, A' in $\mathcal{A}$ the induced map $Hom(A, A') \longrightarrow Hom(F(A), F(A'))$ is a homomorphism of abelian groups. F is *left exact* if it is additive and for every short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{A}$, the sequence

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact in $\mathcal{B}$. If we write a 0 on the right instead of the left, we say that F is *right exact.* If it is both left and right exact, we say that it is *exact.* If only the middle part $F(A') \longrightarrow F(A) \longrightarrow F(A'')$ is exact, we say that F is *exact in the middle.*

Remark 4.1.18. For a contravariant functor we make analogous definitions. For example, $F : \mathcal{A} \longrightarrow \mathcal{B}$ is *left exact* if it is additive, and for every short exact sequence as above, the sequence

$$0 \longrightarrow F(A'') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact in $\mathcal{B}$.

A short exact sequence often gives rise to a long exact sequence. Suppose given a covariant left exact functor between two abelian categories $F : \mathcal{A} \to \mathcal{B}$. Consider the following exact sequence in $\mathcal{A}$.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Applying F yields the exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

In order to continue this sequence to the right to form a long exact sequence we need to introduce the concept of *right derived functor of* F. If $\mathcal{A}$ is nice enough, there is one canonical way for doing so.

For all $i \geq 0$ there exists the functor $R^iF : \mathcal{A} \to \mathcal{B}$ and the above sequence continues like this:

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1F(A) \longrightarrow R^1F(B) \longrightarrow R^1F(C) \longrightarrow R^2F(A) \longrightarrow \dots$$

From this we see that F is an exact functor if and only if $R^1F = 0$.

The right derived functors of F measure "how far" is F from being exact.

Definition of derived functor

For a covariant left-exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories one construct the *right derived functor*

$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

as follows:

We assume that $\mathcal{A}$ contains enough injectives. In particular, we will use the equivalence $\iota : K^+(\mathcal{I}_{\mathcal{A}}) \longrightarrow D^+(\mathcal{A})$ naturally induced by the functor $Q_A : K^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A})$. By ι^{-1} we denote a quasi-inverse of ι given by choosing a complex of injective objects quasi-isomorphic to any given complex that is bounded below. Thus, we have the diagram,

$$\begin{tikzcd}
K^+(\mathcal{I}_{\mathcal{A}}) \arrow[r, hook] \arrow[dr, "\iota"] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"'] \\
 & D^+(\mathcal{A}) \arrow[ul, bend left, "\iota^{-1}"] & D^+(\mathcal{B})
\end{tikzcd}$$

Here, $K(F)$ is the functor that maps $(\ldots \to A^{i-1} \to A^i \to A^{i+1} \to \ldots)$ to $(\ldots \to F(A^{i-1}) \to F(A^i) \to F(A^{i+1}) \to \ldots)$ which is well-defined for the homotopy categories.

Definition 4.1.19. The *right derived functor* of F is the functor

$$RF := Q_{\mathcal{B}} \circ K(F) \circ \iota^{-1} : D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

Definition 4.1.20. Let $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ be the right derived functor of a covariant left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any complex $A^\bullet \in D^+(\mathcal{A})$ one defines:

$$R^iF(A^\bullet) := H^i(RF(A^\bullet)) \in \mathcal{B}$$

The induced additive functors $R^iF : \mathcal{A} \to \mathcal{B}$ are the *higher derived functors* of F.

Remark 4.1.21. Let us take a deeper look to understand better the concept of R^iF. Start with an object A of an abelian category $\mathcal{A}$ and choose once and all an injective resolution $I^\bullet$ of A. Since there are enough injectives, we can construct a long exact sequence of the form

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

where I^i are injectives (injective resolution) of A.

Applying the functor F to this sequence and chopping off the first term we obtain the chain complex

$$0 \longrightarrow F(I^0) \xrightarrow{\phi_0} F(I^1) \xrightarrow{\phi_1} F(I^2) \xrightarrow{\phi_2} \dots$$

This is not in general an exact sequence anymore, but we can compute its cohomology at the $i-$th position

$$Ker(\phi_i)/Im(\phi_{i-1})$$

and we call the result $R^iF(A)$.

4.1.3 Generators of $D^b(\mathcal{A})$

Now we are interested in giving the generators of the derived category of the Grassmannians of lines which will be used at the end of this section to give some splitting criteria. But first, let us define when a class of objects of $D^b(\mathcal{A})$ generate it.

Definition 4.1.22. A class of objects C *generates* $D^b(\mathcal{A})$ if the smallest full triangulated subcategory containing the objects of C is equivalent to $D^b(\mathcal{A})$.

If C is a set, we will also call C a *generating set* in the sequel.

Unravelling this definition, one finds that this is equivalent to saying that, up to isomorphism, every object in $D^b(\mathcal{A})$ can be obtained by successively enlarging C through the following operations: taking finite direct sums, shifting in $D^b(\mathcal{A})$ (i.e. applying the shift functor), and taking a cone $C(f)$ of a morphism $f : A \to B$ between objects already constructed (this means we complete f to a distinguished triangle $A \xrightarrow{f} B \longrightarrow C(f) \longrightarrow A[1]$).

The main result for the derived category of the projective space is due to A. Beilinson (see [9]):

Theorem 4.1.23. *The derived category $D^b(\mathbb{P}^n)$ is generated equivalently by the objects*

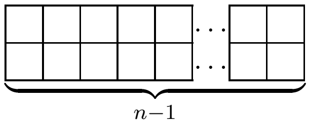
$$\{\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}\} \text{ or by } \{\Omega^n(n), \dots, \Omega^1(1), \mathcal{O}\}.$$

Generated means the minimal category closed under shift and mapping cones. The category is thus finitely generated, and we can somehow think of $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$ as an orthonormal basis.

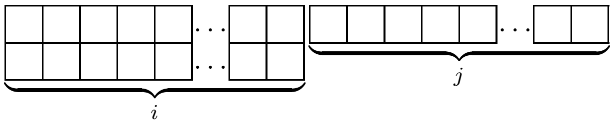
The above result has a generalization for Grassmannians given by M. M. Kapranov (see [26]):

Theorem 4.1.24. *The derived category of the Grassmannians $D^b(\mathbb{G}(k, n))$ is generated by $\{\mathbb{S}_\lambda \mathcal{Q}\}$ where $\mathcal{Q}$ is the universal bundle of rank $k+1$ and λ goes over all the partitions that fit inside a box of dimension $(k+1) \times (n-k)$.*

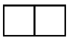
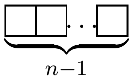
Remark 4.1.25. Let us compute all the generators of $D^b(\mathbb{G}(1,n))$. We have to calculate all the partitions λ that fits in a box of dimension $2 \times (n-1)$:

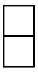
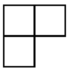
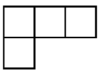


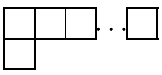
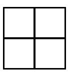
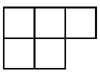
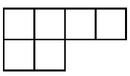
We have in mind also that the following Young diagram corresponds to the $j-$symmetric power of $\mathcal{Q}(i)$:

$S^j\mathcal{Q}(i) \rightsquigarrow$ 

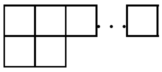
Let us enumerate the possibles diagrams and their corresponding $\mathbb{S}_\lambda\mathcal{Q}$:

- $\lambda = \emptyset \Longrightarrow \mathbb{S}_\lambda\mathcal{Q} = \bigwedge^0 \mathcal{Q} = \mathcal{O}$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(1)}\mathcal{Q} = \mathcal{Q}$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(2)}\mathcal{Q} = S^2\mathcal{Q}$
- $\vdots$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1)}\mathcal{Q} = S^{n-1}\mathcal{Q}$

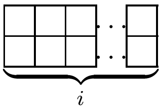
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(1,1)}\mathcal{Q} = \bigwedge^2 \mathcal{Q} = \mathcal{O}(1)$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(2,1)}\mathcal{Q} = \bigwedge^2 \mathcal{Q} \otimes \mathcal{Q} = \mathcal{O}(1) \otimes \mathcal{Q} = \mathcal{Q}(1)$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(3,1)}\mathcal{Q} = \bigwedge^2 \mathcal{Q} \otimes S^2\mathcal{Q} = \mathcal{O}(1) \otimes S^2\mathcal{Q} = S^2\mathcal{Q}(1)$

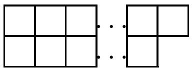
  $\vdots$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1,1)}\mathcal{Q} = \bigwedge^2 \mathcal{Q} \otimes S^{n-2}\mathcal{Q} = \mathcal{O}(1) \otimes S^{n-2}\mathcal{Q} = S^{n-2}\mathcal{Q}(1)$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(2,2)}\mathcal{Q} = \mathcal{O}(2)$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(3,2)}\mathcal{Q} = \mathcal{O}(2) \otimes \mathcal{Q} = \mathcal{Q}(2)$
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(4,2)}\mathcal{Q} = \mathcal{O}(2) \otimes S^3\mathcal{Q} = S^3\mathcal{Q}(2)$

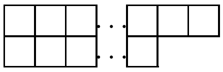
  $\vdots$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1,2)}\mathcal{Q} = \mathcal{O}(2) \otimes S^{n-3}\mathcal{Q} = S^{n-3}\mathcal{Q}(2)$

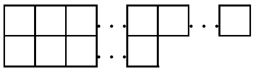
$\vdots$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(i,i)}\mathcal{Q} = \mathcal{O}(i)$

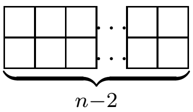
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(i+1,i)}\mathcal{Q} = \mathcal{O}(i) \otimes \mathcal{Q} = \mathcal{Q}(i)$

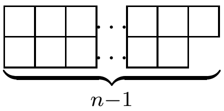
- $\lambda =$  $\Longrightarrow \mathbb{S}_{(i+2,i)}\mathcal{Q} = \mathcal{O}(i) \otimes S^2\mathcal{Q} = S^2\mathcal{Q}(i)$

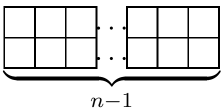
$\vdots$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1,i)}\mathcal{Q} = \mathcal{O}(i) \otimes S^{n-1-i}\mathcal{Q} = S^{n-1-i}\mathcal{Q}(i)$

$\vdots$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-2,n-2)}\mathcal{Q} = \mathcal{O}(n-2)$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1,n-2)}\mathcal{Q} = \mathcal{O}(n-2) \otimes \mathcal{Q} = \mathcal{Q}(n-2)$

- $\lambda =$  $\Longrightarrow \mathbb{S}_{(n-1,n-1)}\mathcal{Q} = \mathcal{O}(n-1)$

Finally we obtain that the generators of $D^b(\mathbb{G}(1,n))$ are

$$\{\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1), \mathcal{Q}, \mathcal{Q}(1), \dots, \mathcal{Q}(n-2), S^2\mathcal{Q}, S^2\mathcal{Q}(1), \dots, S^2\mathcal{Q}(n-3), \dots$$
$$\dots, S^j\mathcal{Q}, S^j\mathcal{Q}(1), \dots, S^j\mathcal{Q}(n-1-j), \dots, S^{n-2}\mathcal{Q}, S^{n-2}\mathcal{Q}(1), S^{n-1}\mathcal{Q}\}$$

4.1.4 Exceptional collections

The aim of this part is to give a strong complete exceptional collection in $D^b(Coh(\mathbb{G}(1,n)))$ (which will be similar to the generators of $D^b(Coh(\mathbb{G}(1,n)))$ computed in Remark 4.1.25). We give the definitions step by step.

Definition 4.1.26. Any object $E \in D^b(\mathcal{A})$ is called *exceptional* if

$$Ext^l(E, E) = Hom(E, E[l]) = \begin{cases} \mathbb{C} & if\, l = 0 \\ 0 & if\, l \neq 0 \end{cases}$$

An $n-$tuple $(E_0, \dots, E_n)$ of exceptional objects in $D^b(\mathcal{A})$ is called *exceptional collection* if

$$Ext^l(E_i, E_j) = Hom(E_i, E_j[l]) = 0 \text{ for all } 0 \leq j \leq i \leq n \text{ and for all } l \in \mathbb{Z}$$

If in addition $Ext^l(E_i, E_j) = Hom(E_i, E_j[l]) = 0$ for all $0 \leq j,\ i \leq n$ and for all $l \neq 0$ we call $(E_0, \dots, E_n)$ a *strongly* exceptional collection. The sequence is *full* (or *complete*) if $E_0, \dots, E_n$ generates $D^b(\mathcal{A})$.

In the case all E_i's are sheaves exceptionality condition amounts to the following

$$Ext^p(E_i, E_j) = 0 \quad \text{for } i > j$$

$$Ext^p(E_i, E_i) = 0 \quad \text{for } p > 0$$

$$Hom(E_i, E_i) = \mathbb{C}$$

Definition 4.1.27. A set of exceptional objects $\{E_0, \dots, E_n\}$ in $D^b(\mathcal{A})$ that generates $D^b(\mathcal{A})$ and such that $Ext^l(E_j, E_i) = 0$ for all $0 \leq i,\ j \leq n$ and all $l \neq 0$ will be called a *full strongly exceptional set.*

Example 4.1.28. The collection $(\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots, \mathcal{O}(n))$ is a full strongly exceptional collection of coherent sheaves on $\mathbb{P}^n$ and $(\Omega^n(n), \Omega^{n-1}(n-1), \Omega^{n-2}(n-2), \dots, \Omega^1(1), \mathcal{O})$ is also a full strongly exceptional collection of coherent sheaves on $\mathbb{P}^n$.

There exists such a full strongly exceptional collection for Grassmannian due to M. M. Kapranov (see [26]):

Theorem 4.1.29. *Let $\mathbb{G}(k, n)$ be the Grassmannian of $k-$dimensional subspaces of an $n-$dimensional vector space V, and let $\mathcal{Q}^\vee$ be the tautological rank $k+1$ subbundle over $\mathbb{G}(k, n)$. Then the bundles $\mathbb{S}_\lambda \mathcal{Q}^\vee$ where λ runs over the set of Young diagrams with no more than $k+1$ rows and no more than $n-k$ columns, are all exceptional, have no higher extension groups between each other and generates $D^b(Coh(\mathbb{G}(k, n))$.*

Moreover, $Hom(\mathbb{S}_\lambda \mathcal{Q}^\vee, \mathbb{S}_\mu \mathcal{Q}^\vee) \neq 0$ if and only if $\lambda_i \geq \mu_i$ for all $i = 1, \dots, k+1$. Thus these $\mathbb{S}_\lambda \mathcal{Q}^\vee$ form a full strongly exceptional collection in $D^b(Coh(\mathbb{G}(k, n))$ when appropriately ordered).

Now we can generalize the idea of exceptional collection to the $m-$block exceptional collection.

Definition 4.1.30. An exceptional collection $(F_0, F_1, \dots, F_m)$ of objects of $D^b(\mathcal{A})$ is a *block* if

$$Ext^i(F_j, F_k) = 0$$

for any i and $j \neq k$.

An $m-$*block collection of type* $(\alpha_0, \alpha_1, \dots, \alpha_m)$ of objects of $D^b(\mathcal{A})$ is an exceptional collection

$$(\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m) = (E_1^0, E_2^0, \dots, E_{\alpha_0}^0, E_1^1, E_2^1, \dots, E_{\alpha_2}^1, \dots, E_1^m, \mathcal{E}_2^m, \dots, E_{\alpha_m}^m)$$

such that all the subcollections $\mathcal{E}_j = (E_1^j, E_2^j, \dots, E_{\alpha_j}^j)$ are blocks.

Note that an exceptional collection $(E_0, E_1, \dots, E_m)$ is an $m-$block of type $(1, 1, \dots, 1)$.

Remark 4.1.31. We know that $\mathbb{G}(1, n)$ has two full and strong exceptional $(2n-1)-$block collections, $\{\mathcal{E}_0, \dots, \mathcal{E}_{2n-2}\}$ and $\{\mathcal{F}_0, \dots, \mathcal{F}_{2n-2}\}$, where we define:

$$\mathcal{F}_r = \{\mathbb{S}_\lambda \mathcal{Q}^\vee,\ |\lambda| = 2n-2-r\} = \{F_{a_r+1}, F_{a_r+2}, \dots, F_{a_{r+1}}\}$$

$$\mathcal{E}_r = \{\mathbb{S}_{\lambda'} \mathcal{S}^\vee,\ |\lambda| = 2n-2-r\} = \{E_{a_r+1}, E_{a_r+2}, \dots, E_{a_{r+1}}\}$$

considering λ a partition that fits inside a box of dimension $(n-1) \times (2)$ and λ' its conjugate partition (hence, λ' musts fit inside a box of dimensions $(2) \times (n-1)$).

We give with more detail the bundles of the $\mathcal{F}_r$ since its corresponding Schur functor is always a symmetric power. For the case of $\mathcal{E}_r$ there are some bundles that can only be expressed with the language of Schur functors. Moreover, the conditions we will need for the splitting criteria will involve the bundles from $\mathcal{F}_r$ and only $\mathcal{E}_0 = \{\mathcal{O}(-2)\}$.

Let us give explicitly some of the bundles from $\mathcal{F}_r$:

- $r = 0 \Rightarrow \mathcal{F}_0 = \{\mathcal{O}(-n+1)\}$
- $r = 1 \Rightarrow \mathcal{F}_1 = \{\mathcal{Q}^\vee(-n+2)\}$
- $r = 2 \Rightarrow \mathcal{F}_2 = \{S^2\mathcal{Q}^\vee(-n+3), \mathcal{O}(-n+2)\}$
- $r = 3 \Rightarrow \mathcal{F}_3 = \{S^3\mathcal{Q}^\vee(-n+4), \mathcal{Q}^\vee(-n+3)\}$
- $r = 4 \Rightarrow \mathcal{F}_4 = \{S^4\mathcal{Q}^\vee(-n+5), S^2\mathcal{Q}^\vee(-n+4), \mathcal{O}(-n+3)\}$
- $r = 5 \Rightarrow \mathcal{F}_5 = \{S^5\mathcal{Q}^\vee(-n+6), S^3\mathcal{Q}^\vee(-n+5), \mathcal{Q}^\vee(-n+4)\}$

  $\vdots$
- $r = n-1$ *(odd)* $\Rightarrow \mathcal{F}_{n-1} = \{S^{n-1}\mathcal{Q}^\vee, S^{n-3}\mathcal{Q}^\vee(-1), S^{n-4}\mathcal{Q}^\vee(-2), \dots$

  $\dots, S^3\mathcal{Q}^\vee(-[\frac{n-1}{2}]-1), \mathcal{Q}^\vee(-[\frac{n-1}{2}])\}$
- $r = n-1$ *(even)* $\Rightarrow \mathcal{F}_{n-1} = \{S^{n-1}\mathcal{Q}^\vee, S^{n-3}\mathcal{Q}^\vee(-1), S^{n-4}\mathcal{Q}^\vee(-2), \dots$

  $\dots, S^2\mathcal{Q}^\vee(-\frac{n-1}{2}-1), \mathcal{O}(-\frac{n-1}{2})\}$

  $\vdots$
- $r = 2n-8 \Rightarrow \mathcal{F}_{2n-8} = \{S^6\mathcal{Q}^\vee, S^4\mathcal{Q}^\vee(-1), S^2\mathcal{Q}^\vee(-2), \mathcal{O}(-3)\}$
- $r = 2n-7 \Rightarrow \mathcal{F}_{2n-7} = \{S^5\mathcal{Q}^\vee, S^3\mathcal{Q}^\vee(-1), \mathcal{Q}^\vee(-2)\}$

- $r = 2n-6 \Rightarrow \mathcal{F}_{2n-6} = \{S^4\mathcal{Q}^\vee, S^2\mathcal{Q}^\vee(-1), \mathcal{O}(-2)\}$
- $r = 2n-5 \Rightarrow \mathcal{F}_{2n-5} = \{S^3\mathcal{Q}^\vee, \mathcal{Q}^\vee(-1)\}$
- $r = 2n-4 \Rightarrow \mathcal{F}_{2n-4} = \{S^2\mathcal{Q}^\vee, \mathcal{O}(-1)\}$
- $r = 2n-3 \Rightarrow \mathcal{F}_{2n-3} = \{\mathcal{Q}^\vee\}$
- $r = 2n-2 \Rightarrow \mathcal{F}_{2n-2} = \{\mathcal{O}\}$

And now some of the bundles from $\mathcal{E}_r$:

- $r = 0 \Rightarrow \mathcal{E}_0 = \{\mathcal{O}(-2)\}$
- $r = 1 \Rightarrow \mathcal{E}_1 = \{\bigwedge^{n-2}\mathcal{S}^\vee(-1)\}$
- $r = 2 \Rightarrow \mathcal{E}_2 = \{\bigwedge^2\mathcal{S}^\vee, \mathbb{S}_{(2,\dots,2)}\mathcal{S}^\vee\}$
- $r = 3 \Rightarrow \mathcal{E}_3 = \{\bigwedge^3\mathcal{S}^\vee, \mathbb{S}_{(2,1)}\mathcal{S}^\vee\}$

  $\vdots$
- $r = 2n-5 \Rightarrow \mathcal{E}_{2n-5} = \{\bigwedge^3\mathcal{S}^\vee, \mathbb{S}_{(2,1)}\mathcal{S}^\vee\}$
- $r = 2n-4 \Rightarrow \mathcal{E}_{2n-4} = \{\bigwedge^2\mathcal{S}^\vee, S^2\mathcal{S}^\vee\}$
- $r = 2n-3 \Rightarrow \mathcal{E}_{2n-3} = \{\mathcal{S}^\vee\}$
- $r = 2n-2 \Rightarrow \mathcal{E}_{2n-2} = \{\mathcal{O}\}$

Let us put all of them together in a table to obtain a global idea:

r	$\|\lambda\|$	λ	$\mathbb{S}_\lambda \mathcal{Q}^\vee$	$\mathcal{F}_i$	λ'	$\mathbb{S}_{\lambda'} \mathcal{S}^\vee$	$\mathcal{E}_i$
$r = 2n-2$	$\|\lambda\| = 0$	0	$\mathcal{O}$	$\mathcal{F}_{2n-2}$	0	$\mathcal{O}$	$\mathcal{E}_{2n-2}$
$r = 2n-3$	$\|\lambda\| = 1$	(1)	$\mathcal{Q}^\vee$	$\mathcal{F}_{2n-3}$	(1)	$\mathcal{S}^\vee$	$\mathcal{E}_{2n-3}$
$r = 2n-4$	$\|\lambda\| = 2$	(2)	$S^2\mathcal{Q}^\vee$	$\mathcal{F}_{2n-4}$	$(1,1)$	$\wedge^2\mathcal{S}^\vee$	$\mathcal{E}_{2n-4}$
		$(1,1)$	$\mathcal{O}(-1)$		(2)	$S^2\mathcal{S}^\vee$	
$r = 2n-5$	$\|\lambda\| = 3$	(3)	$S^3\mathcal{Q}^\vee$	$\mathcal{F}_{2n-5}$	$(1,1,1)$	$\wedge^3\mathcal{S}^\vee$	$\mathcal{E}_{2n-5}$
		$(2,1)$	$\mathcal{Q}^\vee(-1)$		$(2,1)$	$\mathbb{S}_{(2,1)}\mathcal{S}^\vee$	
$r = 2n-6$	$\|\lambda\| = 4$	(4)	$S^4\mathcal{Q}^\vee$	$\mathcal{F}_{2n-6}$	$(1,1,1,1)$	$\wedge^4\mathcal{S}^\vee$	$\mathcal{E}_{2n-6}$
		$(3,1)$	$S^2\mathcal{Q}^\vee(-1)$		$(2,1,1)$	$\mathbb{S}_{(2,1,1)}\mathcal{S}^\vee$	
		$(2,2)$	$\mathcal{O}(-2)$		$(2,2)$	$\mathbb{S}_{(2,2)}\mathcal{S}^\vee$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$r = 3$	$\|\lambda\| = 2n-5$	$(n-1, n-4)$	$S^3\mathcal{Q}^\vee(-n+4)$	$\mathcal{F}_3$	$(2,\dots,2,1,1,1)$	$\wedge^{n-4}\mathcal{S}^\vee(-1)$	$\mathcal{E}_3$
		$(n-2, n-3)$	$\mathcal{Q}^\vee(-n+3)$		$(2,\dots,2,1)$	$\mathbb{S}_{(2,\dots,2,1)}\mathcal{S}^\vee$	
$r = 2$	$\|\lambda\| = 2n-4$	$(n-1, n-3)$	$S^2\mathcal{Q}^\vee(-n+3)$	$\mathcal{F}_2$	$(2,\dots,2,1,1)$	$\wedge^{n-3}\mathcal{S}^\vee(-1)$	$\mathcal{E}_2$
		$(n-2, n-2)$	$\mathcal{O}(-n+2)$		$(2,\dots,2)$	$\mathbb{S}_{(2,\dots,2)}\mathcal{S}^\vee$	
$r = 1$	$\|\lambda\| = 2n-3$	$(n-1, n-2)$	$\mathcal{Q}^\vee(-n+2)$	$\mathcal{F}_1$	$(2,\dots,2,1)$	$\wedge^{n-2}\mathcal{S}^\vee(-1)$	$\mathcal{E}_1$
$r = 0$	$\|\lambda\| = 2n-2$	$(n-1, n-1)$	$\mathcal{O}(-n+1)$	$\mathcal{F}_0$	$(2,\dots,2)$	$\mathcal{O}(-2)$	$\mathcal{E}_0$

4.2 Beilinsons's theorem

In the first part of this section we are going to recall Beilinson's theorem for the projective space. In the second part of this section we give its generalization to a general Grassmannian. We will not go through details that could be found in [15].

4.2.1 Beilinson's theorem for projective spaces

Let us state Beilinson's theorem (see [9]):

Theorem 4.2.1. *Let $X = \mathbb{P}^n$, denote by $p, q : X \times X \to X$ the two projections and by Δ the diagonal in $X \times X$. For $\mathcal{F}, \mathcal{E} \in Coh(X)$ let us put $\mathcal{F} \boxtimes \mathcal{E} := p^*\mathcal{F} \otimes q^*\mathcal{E}$. Let $F \in Coh(X)$, $t \in \mathbb{Z}$. Then,*
***A.** The diagonal Δ has the following resolution on $X \times X$:*

$$0 \longrightarrow \Omega^n(n) \boxtimes \mathcal{O}(-n) \xrightarrow{u_n} \ldots \to \Omega^1(1) \boxtimes \mathcal{O}(-1) \xrightarrow{u_1} q^*\mathcal{O} \xrightarrow{u_0} q^*\mathcal{O}|_\Delta \longrightarrow 0 \tag{4.3}$$

***B.** There exists a complex of vector bundles $L^\bullet(t)$,*

$$0 \longrightarrow L^{-(n-1)} \longrightarrow L^{-(n-2)} \longrightarrow \ldots \longrightarrow L^0 \longrightarrow L^1 \longrightarrow \ldots \longrightarrow L^{n-2} \longrightarrow L^{n-1} \longrightarrow 0$$

on X such that:

1. *$L^\bullet(t) \sim F(t)$ in the derived category $D^b(Coh(X))$. In particular:*

$$H^k(L^\bullet(t)) = \frac{Ker(d_k)}{Im(d_{k-1})} = \begin{cases} F(t) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

2. *$L^k(t) = \bigoplus_{j+k=i} X^i_j(t)\,, \quad X^i_j(t) = \Omega^j(j) \otimes H^i(F(t-j))$*

3. *the maps $v^i_j(t,s) : X^i_j(t) \to X^{i-s+1}_{j-s}$ with $s \in \mathbb{Z}$ induced by the differentials $L^k \to L^{k+1}$ are zero for $s \leq 0$*

Remark 4.2.2. The complex $L^\bullet$ is minimal in the sense that each natural morphisms of the form $\Omega^p(p) \to \Omega^p(p)$ that we get from the morphisms d_k are zero. This is what means the point (3.) of the theorem.

Remark 4.2.3. Notice also that this resolution is just the Koszul complex associated to the map $\Omega^1(1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}$, indeed,

$$\bigwedge^p (\Omega^1(1) \boxtimes \mathcal{O}(-1) = \bigwedge^p \Omega^1(1) \boxtimes S^p\mathcal{O}(-1) = \Omega^p(p) \boxtimes \mathcal{O}(-p)$$

In order to study this in depth it is suitable to give a graphic representation that imitates the spectral sequence above.

Let us consider the square diagram:

$$
\begin{array}{ccccc}
\Omega^n(n) & \Omega^{n-1}(n-1) & \dots & \Omega^1(1) & \mathcal{O} \\
\Omega^n(n) & \Omega^{n-1}(n-1) & \dots & \Omega^1(1) & \mathcal{O} \\
\vdots & \vdots & & \vdots & \vdots \\
\Omega^n(n) & \Omega^{n-1}(n-1) & \dots & \Omega^1(1) & \mathcal{O}
\end{array}
\tag{4.4}
$$

Consider now the table of cohomology:

$$
\begin{array}{ccccc}
H^n(F(-n)) & H^n(F(-n+1)) & \dots & H^n(F(-1)) & H^n(F) \\
H^{n-1}(F(-n)) & H^{n-1}(F(-n+1)) & \dots & H^{n-1}(F(-1)) & H^{n-1}(F) \\
\vdots & \vdots & & \vdots & \vdots \\
H^0(F(-n)) & H^0(F(-n+1)) & \dots & H^0(F(-1)) & H^0(F)
\end{array}
\tag{4.5}
$$

The terms of the complex $L^\bullet$ that appear in Beilinson's theorem are obtained by taking the direct sum of the terms in the diagonal. We proceed with caution considering each bundles $\Omega^p(p)$ that appears in (4.4) with its corresponding integer that appears in (4.5). The following diagram may help:

$\Omega^n(n) \otimes H^n(F(-n))$ (X_n^n)	$\Omega^{n-1}(n-1) \otimes H^n(F(-n+1))$ (X_2^n)	$\dots$	$\Omega^1(1) \otimes H^n(F(1))$ (X_1^n)	$\mathcal{O} \otimes H^n(F)$ (X_0^n)
$\Omega^n(n) \otimes H^{n-1}(F(-n))$ (X_n^{n-1})	$\Omega^{n-1}(n-1) \otimes H^{n-1}(F(-n+1))$ (X_2^{n-1})	$\dots$	$\Omega^1(1) \otimes H^{n-1}(F(1))$ (X_1^{n-1})	$\mathcal{O} \otimes H^{n-1}(F)$ (X_0^{n-1})
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$\Omega^n(n) \otimes H^0(F(-n))$ (X_n^0)	$\Omega^{n-1}(n-1) \otimes H^0(F(-n+1))$ (X_2^0)	$\dots$	$\Omega^1(1) \otimes H^0(F(1))$ (X_1^0)	$\mathcal{O} \otimes H^0(F)$ (X_0^0)

For the particular case in which F is a sheaf on $\mathbb{P}^2$ (we use $\Omega^2(2) \cong \mathcal{O}(-1)$) we get the diagram:

$$
\begin{array}{ccc}
\mathcal{O}(-1) & \Omega^1(1) & \mathcal{O} \\
\mathcal{O}(-1) & \Omega^1(1) & \mathcal{O} \\
\mathcal{O}(-1) & \Omega^1(1) & \mathcal{O}
\end{array}
$$

and the table of cohomology:

$$
\begin{array}{ccc}
H^2(F(-2)) & H^2(F(-1)) & H^2(F) \\
H^1(F(-2)) & H^1(F(-1)) & H^1(F) \\
H^0(F(-2)) & H^0(F(-1)) & H^0(F)
\end{array}
$$

and the following diagram with all the information:

$\mathcal{O}(-1) \otimes H^2(F(-2))$	$\Omega^1(1) \otimes H^2(F(-1))$	$\mathcal{O} \otimes H^2(F)$
$\mathcal{O}(-1) \otimes H^1(F(-2))$	$\Omega^1(1) \otimes H^1(F(-1))$	$\mathcal{O} \otimes H^1(F)$
$\mathcal{O}(-1) \otimes H^0(F(-2))$	$\Omega^1(1) \otimes H^0(F(-1))$	$\mathcal{O} \otimes H^0(F)$

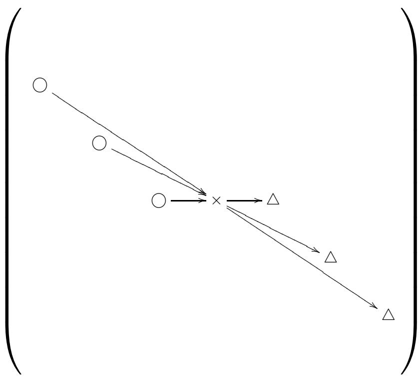
Finally, by taking the sum of the diagonals of this last diagram it provide the following complex:

$$0 \longrightarrow \mathcal{O}(-1) \otimes H^0(F(-2)) \xrightarrow{d_{-2}} \mathcal{O}(-1) \otimes H^1(F(-2)) \oplus \Omega^1(1) \otimes H^0(F(-1)) \xrightarrow{d_{-1}}$$

$$\xrightarrow{d_{-1}} \mathcal{O}(-1) \otimes H^2(F(-2)) \oplus \Omega^1(1) \otimes H^1(F(-1)) \oplus \mathcal{O} \otimes H^0(F) \xrightarrow{d_0}$$
$$\xrightarrow{d_0} \Omega^1(1) \otimes H^2(F(-1)) \oplus \mathcal{O} \otimes H^1(F) \xrightarrow{d_1} \mathcal{O} \otimes H^2(F) \longrightarrow 0$$

Its successive groups of cohomology are 0, 0, F, 0, 0, i.e. the complex $L^\bullet$ is always exact except in the middle point.

Remark 4.2.4. Let us observe one important remark. The minimality of the complex $L^\bullet$ implies that there are no vertical arrows in the diagram. Graphically, to each summand marked with $\times$ arrive only the arrows indicated with $\bigcirc$ and start arrows only to the summands indicated with $\triangle$.



If the summands marked with $\bigcirc$ and $\triangle$ are zero then the term indicated by $\times$ is:

- zero if it is outside the main diagonal
- direct summand of F if it is in the main diagonal

We will use the perspective of this remark for the theorems of Section 4.3.

4.2.2 Beilinson's theorem for Grassmannians

As in the projective space, the main point is to have a resolution of the diagonal. This is what the following result provides.

Theorem 4.2.5. *(Kapranov) Let X be the Grassmannian variety $\mathbb{G}(k,n)$ of $k-$dimensional subspaces of an $n-$dimensional vector space V. If $\mathcal{S}$ and $\mathcal{Q}$ are the universal bundles (see Definition 1.1.9), then the diagonal of $X \times X$ is the zero locus of a natural section of $\mathcal{Q}^\vee \times \mathcal{S}^\vee$. Therefore the following Koszul complex is a resolution of the diagonal of $X \times X$:*

$$0 \to \bigwedge^{(k+1)(n-k)} (\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) \to \dots \bigwedge^{2} (\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) \to \mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0 \qquad (4.6)$$

Remark 4.2.6. The main difference now is that the wedge products of $\mathcal{Q}^\vee \times \mathcal{S}^\vee$ are much more complicated and decompose into many pieces. Specifically:

$$\bigwedge^k (\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) = \bigoplus_{|\lambda|=k} \mathbb{S}_\lambda \mathcal{Q}^\vee \otimes \mathbb{S}_{\lambda'} \mathcal{S}^\vee$$

where the sum goes over all Young tableau with k cells, λ' is the conjugate Young tableau and $\mathbb{S}_\lambda$ is the Schur functor associated to the tableau λ (see Definition 2.1.6, 2.1.7, 2.1.8 and 2.1.13). These Schur functors correspond exactly to the $\{\mathcal{E}_r\}$ and $\{\mathcal{F}_r\}$ computed in Remark 4.1.31.

We can state now Beilinson's Theorem for Grassmannians:

Theorem 4.2.7. *(M. M. Kapranov, [25]) Let $X = \mathbb{G}(1, n)$. Denote by $p, q : X \times X \to X$ the two projections and by Δ the diagonal in $X \times X$. For $\mathcal{F}, \mathcal{G} \in Coh(X)$ let us put $\mathcal{F} \boxtimes \mathcal{G} := p^*\mathcal{F} \otimes q^*\mathcal{G}$. Let $F \in Coh(X)$, $t \in \mathbb{Z}$. Then:*

- *the diagonal Δ has the following resolution on $X \times X$:*

$$0 \to \oplus_{i=a_0+1}^{a_1} E_i \boxtimes F_i \to \oplus_{i=a_1+1}^{a_2} E_i \boxtimes F_i \to \ldots \to \oplus_{i=a_{2n-2}+1}^{a_{2n-1}} E_i \boxtimes F_i \to \mathcal{O}_\Delta \to 0$$

  with the notation of Remark 4.1.31.

- *there exits a complex of vector bundles $L^\bullet(t)$ on X,*

$$0 \longrightarrow L^{-(2n-3)} \longrightarrow L^{-(2n-4)} \longrightarrow \ldots \longrightarrow L^0 \longrightarrow L^1 \longrightarrow \ldots \longrightarrow L^{2n-4} \longrightarrow L^{2n-3} \longrightarrow 0$$

  such that:

  - *$L^\bullet(t) \sim F(t)$ in the derived category $D^b(Coh(X))$. In particular*

$$H^k(L^\bullet(t)) = \begin{cases} F(t) & \text{if } k = 0 \\ 0 & \text{if } k \neq 0 \end{cases}$$

  - *$L^k(t) = \oplus_{j+k=i} X_j^i(t)$ where $X_j^i(t) = \oplus_{k=a_{2n-2-j}+1}^{a_{2n-1-j}} E_k \otimes H^i(F_k \otimes F(t))$*
  - *the maps $v_j^i(t, s) : X_j^i(t) \to X_{j-s}^{i-s+1}(t)$ with $s \in \mathbb{Z}$ induced by the differentials $L^k \to L^{k+1}$ are zero for $s \leq 0$*

Using the notation given in Theorem 4.2.7 we construct the Beilinson diagram for Grassmannians of lines. In the notation of the theorem this is shown in Table 4.1. By replacing some of the bundles $E_i \in \mathcal{E}_r$ and $F_i \in \mathcal{F}_r$ from Remark 4.1.31 we get Table 4.2.

The elements of the complex $L^\bullet$ are just the sum of each diagonal in the Beilinson's diagram (Table 4.1or Table 4.2):

$$0 \longrightarrow X_{2n-2}^0 \longrightarrow X_{2n-3}^0 \oplus X_{2n-2}^1 \longrightarrow X_{2n-4}^0 \oplus X_{2n-3}^1 \oplus X_{2n-4}^2 \longrightarrow \ldots \longrightarrow X_0^1 \oplus X_2^1 \oplus \ldots \oplus X_{2n-3}^{2n-4} \oplus X_{2n-2}^{2n-3} \longrightarrow$$

$$\longrightarrow X_0^0 \oplus X_1^1 \oplus \ldots \oplus X_{2n-3}^{2n3} \oplus X_{2n-2}^{2n2} \longrightarrow X_0^1 \oplus X_1^2 \oplus \ldots \oplus X_{2n-4}^{2n-3} \oplus X_{2n-3}^{2n-2} \longrightarrow \ldots \longrightarrow X_0^{2n-3} \oplus X_1^{2n-2} \longrightarrow X_0^{2n-2} \longrightarrow 0$$

$\oplus_{i=a_0+1}^{a_1} H^{2n-2}(F\otimes F_i)\otimes E_i$ (X_{2n-2}^{2n-2})	$\dots$	$\oplus_{i=a_{2n-4}+1}^{a_{2n-3}} H^{2n-2}(F\otimes F_i)\otimes E_i$ (X_2^{2n-2})	$\oplus_{i=a_{2n-3}+1}^{a_{2n-2}} H^{2n-2}(F\otimes F_i)\otimes E_i$ (X_1^{2n-2})	$\oplus_{i=a_{2n-2}+1}^{a_{2n-1}} H^{2n-2}(F\otimes F_i)\otimes E_i$ (X_0^{2n-2})
$\oplus_{i=a_0+1}^{a_1} H^{2n-3}(F\otimes F_i)\otimes E_i$ (X_{2n-2}^{2n-3})	$\dots$	$\oplus_{i=a_{2n-4}+1}^{a_{2n-3}} H^{2n-3}(F\otimes F_i)\otimes E_i$ (X_2^{2n-3})	$\oplus_{i=a_{2n-3}+1}^{a_{2n-2}} H^{2n-3}(F\otimes F_i)\otimes E_i$ (X_1^{2n-3})	$\oplus_{i=a_{2n-2}+1}^{a_{2n-1}} H^{2n-3}(F\otimes F_i)\otimes E_i$ (X_0^{2n-3})
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$\oplus_{i=a_0+1}^{a_1} H^{2}(F\otimes F_i)\otimes E_i$ (X_{2n-2}^{2})	$\dots$	$\oplus_{i=a_{2n-4}+1}^{a_{2n-3}} H^{2}(F\otimes F_i)\otimes E_i$ (X_2^{2})	$\oplus_{i=a_{2n-3}+1}^{a_{2n-2}} H^{2}(F\otimes F_i)\otimes E_i$ (X_1^{2})	$\oplus_{i=a_{2n-2}+1}^{a_{2n-1}} H^{2}(F\otimes F_i)\otimes E_i$ (X_0^{2})
$\oplus_{i=a_0+1}^{a_1} H^{1}(F\otimes F_i)\otimes E_i$ (X_{2n-2}^{1})	$\dots$	$\oplus_{i=a_{2n-4}+1}^{a_{2n-3}} H^{1}(F\otimes F_i)\otimes E_i$ (X_2^{1})	$\oplus_{i=a_{2n-3}+1}^{a_{2n-2}} H^{1}(F\otimes F_i)\otimes E_i$ (X_1^{1})	$\oplus_{i=a_{2n-2}+1}^{a_{2n-1}} H^{1}(F\otimes F_i)\otimes E_i$ (X_0^{1})
$\oplus_{i=a_0+1}^{a_1} H^{0}(F\otimes F_i)\otimes E_i$ (X_{2n-2}^{0})	$\dots$	$\oplus_{i=a_{2n-4}+1}^{a_{2n-3}} H^{0}(F\otimes F_i)\otimes E_i$ (X_2^{0})	$\oplus_{i=a_{2n-3}+1}^{a_{2n-2}} H^{0}(F\otimes F_i)\otimes E_i$ (X_1^{0})	$\oplus_{i=a_{2n-2}+1}^{a_{2n-1}} H^{0}(F\otimes F_i)\otimes E_i$ (X_0^{0})

Table 4.1: X_i^j of Theorem 4.2.7

$H^{2n-2}(F\otimes\mathcal{O}(-n+1))\otimes\mathcal{O}(-2)$	$\dots$	$H^{2n-2}(F\otimes S^2\mathcal{Q}^\vee)\otimes\bigwedge^2\mathcal{S}^\vee\bigoplus H^{2n-2}(F\otimes\mathcal{O}(-1))\otimes S^2\mathcal{S}^\vee$	$H^{2n-2}(F\otimes\mathcal{Q}^\vee)\otimes\mathcal{S}^\vee$	$H^{2n-2}(F\otimes\mathcal{O})\otimes\mathcal{O}$
$H^{2n-3}(F\otimes\mathcal{O}(-n+1))\otimes\mathcal{O}(-2)$	$\dots$	$H^{2n-3}(F\otimes S^2\mathcal{Q}^\vee)\otimes\bigwedge^2\mathcal{S}^\vee\bigoplus H^{2n-3}(F\otimes\mathcal{O}(-1))\otimes S^2\mathcal{S}^\vee$	$H^{2n-3}(F\otimes\mathcal{Q}^\vee)\otimes\mathcal{S}^\vee$	$H^{2n-3}(F\otimes\mathcal{O})\otimes\mathcal{O}$
$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$H^{2}(F\otimes\mathcal{O}(-n+1))\otimes\mathcal{O}(-2)$	$\dots$	$H^{2}(F\otimes S^2\mathcal{Q}^\vee)\otimes\bigwedge^2\mathcal{S}^\vee\bigoplus H^{2}(F\otimes\mathcal{O}(-1))\otimes S^2\mathcal{S}^\vee$	$H^{2}(F\otimes\mathcal{Q}^\vee)\otimes\mathcal{S}^\vee$	$H^{2}(F\otimes\mathcal{O})\otimes\mathcal{O}$
$H^{1}(F\otimes\mathcal{O}(-n+1))\otimes\mathcal{O}(-2)$	$\dots$	$H^{1}(F\otimes S^2\mathcal{Q}^\vee)\otimes\bigwedge^2\mathcal{S}^\vee\bigoplus H^{1}(F\otimes\mathcal{O}(-1))\otimes S^2\mathcal{S}^\vee$	$H^{1}(F\otimes\mathcal{Q}^\vee)\otimes\mathcal{S}^\vee$	$H^{1}(F\otimes\mathcal{O})\otimes\mathcal{O}$
$H^{0}(F\otimes\mathcal{O}(-n+1))\otimes\mathcal{O}(-2)$	$\dots$	$H^{0}(F\otimes S^2\mathcal{Q}^\vee)\otimes\bigwedge^2\mathcal{S}^\vee\bigoplus H^{0}(F\otimes\mathcal{O}(-1))\otimes S^2\mathcal{S}^\vee$	$H^{0}(F\otimes\mathcal{Q}^\vee)\otimes\mathcal{S}^\vee$	$H^{0}(F\otimes\mathcal{O})\otimes\mathcal{O}$

Table 4.2: X_i^j of Theorem 4.2.7 with notation of Remark 4.1.31

4.3 Applications of Beilinson's theorem

Using Beilinson's theorem, V. Ancona and G. Ottaviani obtained a splitting criteria not just for locally free sheaves but for coherent sheaf in general. Moreover, apart from line bundles there also appears what they call g.skyscraper sheaf (see [2] for the definition).

Theorem 4.3.1. *Let F be a coherent sheaf on $\mathbb{P}^n$. Suppose that for some $t \in \mathbb{Z}$, $H^i(F(t-i-1)) = 0$ for $0 \leq i < n$. Then F contains $\mathcal{O}(-t)^{h^0(F(t))}$ as direct summand.*

Theorem 4.3.2. *Let F be a coherent sheaf on $\mathbb{P}^n$ such that $H^i_*(F) = 0$ for $0 < i < n$. Then F is a direct sum of line bundles and of a g.skyscraper sheaf.*

Our goal is to obtain the corresponding splitting criteria for the Grassmannian of lines by using Beilinson's theorem (Theorem 4.2.7) and Kapranov's resolution for Grassmannian of lines (4.6). To get this analogue of Theorem 4.3.1 we have to impose the elements under the diagonal of Table 4.2 to vanish (with their corresponding twists). In order to make these vanishings easier to understand we define the following segments of points.

Definition 4.3.3. Consider the following set of points:

- $M = \{(0, 2n-3), (1, 2n-4), (2, 2n-5), (3, 2n-6), \dots, (n-5, n+2), (n-4, n+1), (n-3, n), (n-2, n-1), (n-1, n-2)\}$
- $M_k = \{(0, 2n-3-2k), (1, (2n-3-2k)-1), (2, (2n-3-2k)-2), (3, (2n-3-2k)-3), \dots, (n-2-2k, n-1), (n-1-2k, n-2)\}$
- $N = \{(2,1), (3,2), (4,3), (5,4), \dots, (n-4, n-5), (n-3, n-4), (n-2, n-3), (n-1, n-2)\}$
- $N_k = \{(0, 1+2(k-1)), (1, (1+2(k-1))+1), (2, (1+2(k-1))+2), (3, (1+2(k-1))+3), \dots, (n-4-2(k-1), n-3), (n-3-2(k-1), n-2)\}$

for $k \in \{1, 2, \dots, [\frac{n-1}{2}]\}$.

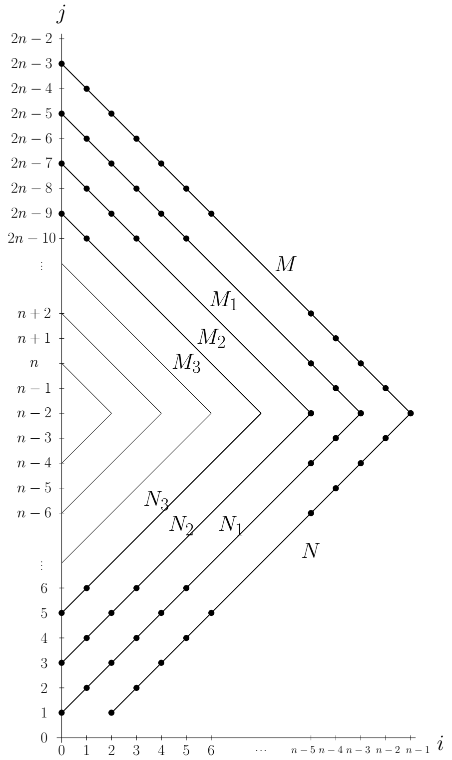


Figure 4.1: Segments: M, M_k, N, N_k

As in Chapter 3, the points (i, j) correspond to the vanishings of $H^j_*(F \otimes S^i\mathcal{Q}) = 0$.

Remark 4.3.4. The point $(n-1, n-2)$ corresponds to the vanishing of $H^{n-2}_*(F \otimes S^{n-1}\mathcal{Q}) = 0$. Since we do not want the order of the symmetric power greater than $n-2$ we change this condition by making use of Remark 3.1.16 for another ones that are equivalent and defined as M' in the following definition.

Definition 4.3.5. Consider the following set of points:

- $M_0 = \{(0, 2n-3), (1, 2n-4), (2, 2n-5), (3, 2n-6), \ldots, (n-5, n+2), (n-4, n+1), (n-3, n)\}$
- $M' = \{(0, 2n-4), (1, 2n-5), (2, 2n-6), (3, 2n-7), \ldots, (n-4, n), (n-3, n-1), (n-2, n-2)\}$
- $M_k = \{(0, 2n-3-2k), (1, (2n-3-2k)-1), (2, (2n-3-2k)-2), (3, (2n-3-2k)-3), \ldots, (n-2-2k, n-1), (n-1-2k, n-2)\}$
- $N_0 = \{(2, 1), (3, 2), (4, 3), (5, 4), \ldots, (n-4, n-5), (n-3, n-4), (n-2, n-3)\}$
- $N_k = \{(0, 1+2(k-1)), (1, (1+2(k-1))+1), (2, (1+2(k-1))+2), (3, (1+2(k-1))+3), \ldots, (n-4-2(k-1), n-3), (n-3-2(k-1), n-2)\}$

for $k \in \{1, 2, \ldots, [\frac{n-1}{2}]\}$.

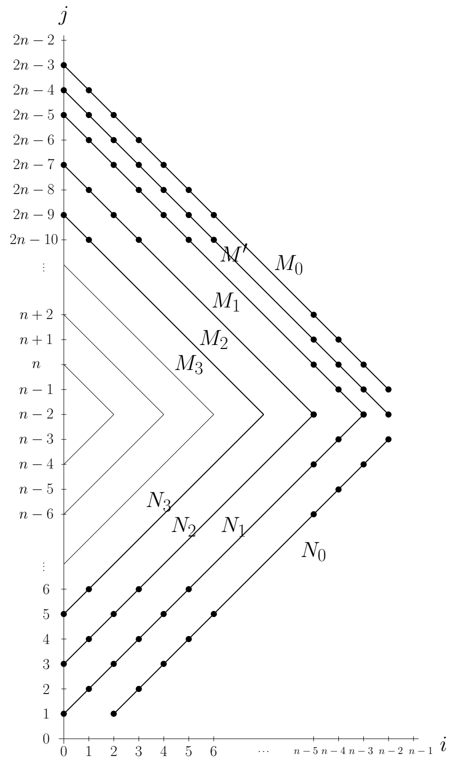


Figure 4.2: Hypothesis of Theorem 4.3.9

Theorem 4.3.6. *Let F be a vector bundle over $\mathbb{G}(1,n)$. Suppose that for some $t \in \mathbb{Z}$ we have the following vanishings:*

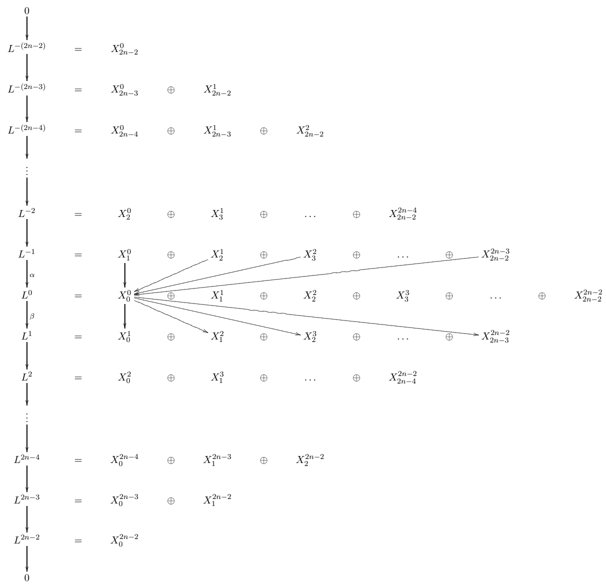
$$H^j(F \otimes S^i \mathcal{Q}^\vee(t - \frac{j-i+1}{2})) = 0 \quad \textit{for} \quad (i,j) \in \begin{cases} (1,0) \\ M_0 \cup M' \cup M_1 \cup \ldots \cup M_{[\frac{n-1}{2}]} \\ N_0 \cup N_1 \cup \ldots \cup N_{[\frac{n-1}{2}]} \end{cases}$$

Then F contains $\mathcal{O}(t) \otimes H^0(F(t))$ as a direct summand.

Proof. Suppose $t = 0$. From the Theorem 4.2.7 and with the notation of Remark 4.1.31 we define the vector bundles of the complex $L^\bullet(t)$ as follows:

$$L^k(t) = \bigoplus_{j+k=i} X^i_j(t) \quad \text{where} \quad X^i_j(t) = \bigoplus_{k=a_{2n-2-j}+1}^{a_{2n-1-j}} E_k \otimes H^i(F_k \otimes F(t))$$

We can give the complex $L^\bullet$ as follows:



We have pointed out the maps we use:

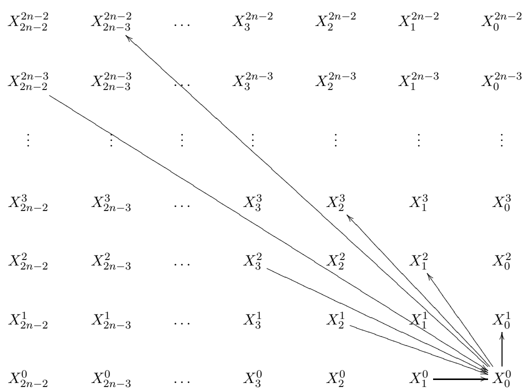
$$X_1^0,\, X_2^1,\, X_3^2, \dots, X_{2n-2}^{2n-3} \xrightarrow{\phi} X_0^0$$

$$X_0^0 \xrightarrow{\psi} X_0^1,\, X_1^2,\, X_2^3, \dots, X_{2n-3}^{2n-2}$$

The maps ϕ are zero by hypothesis (since all the elements $X_1^0,\, X_2^1,\, X_3^2, \dots, X_{2n-2}^{2n-3}$ vanish). These maps correspond to v_i^{i-1} for $0 \leq i < 2n-2$ which end at X_0^0.

The maps ψ are zero by hypothesis of Beilinson's Theorem (correspond to the maps $v_0^0(0,s) : X_0^0 \to X_{-s}^{-s+1}$ of Theorem 4.2.7).

Let us give a different diagrams of the elements X_i^j to understand how these maps and the complex work:



Since the elements under the diagonal of this last diagram vanish we have that all the maps that get to X_0^0 are zero, and also, all the maps that start from X_0^0. Hence, $H^0(L^\bullet) = Ker(\beta)/Im(\alpha)$ and we know part of the maps α and β. Moreover, part of the kernel of β consists in X_0^0. So, X_0^0 is a direct summand of $H^0(L^\bullet)$. Finally, $H^0(L^\bullet)$ coincides with F by Beilinson's theorem.

Since X_0^0 contains $H^0(F \otimes \mathcal{O}) \otimes \mathcal{O}$ as a direct summand we can conclude that $H^0(F \otimes \mathcal{O}) \otimes \mathcal{O}$ is a direct summand also of $H^0(L^\bullet(0)) = F$, as wanted.

□

Remark 4.3.7. Notice that for $(i, j) = (1, 0)$ the corresponding condition is $H^0(F \otimes \mathcal{Q}^\vee(t)) = 0$. This condition plays an important role since it does not appear in the conditions of Theorem 4.3.9 but we prove it holds with all the conditions we have in the hypotheses.

Remark 4.3.8. As the reader probably noticed, Theorem 4.3.6 still holds if F is just a coherent sheaf and not a vector bundle (locally free sheaf). But, since we are interested in comparing this new criterion with the one we already have we state it just for vector bundles.

Finally we can state the splitting criteria for the Grassmannian of lines by using the technique of derived categories where the vanishings will be now for all twist.

Theorem 4.3.9. *Let F be a vector bundle over $\mathbb{G}(1, n)$. Suppose that:*

$$H_*^j(F \otimes S^i\mathcal{Q}) = 0 \quad \textit{for} \quad (i, j) \in \begin{cases} M_0 \cup M' \cup M_1 \cup \ldots \cup M_{[\frac{n-1}{2}]} \\ N_0 \cup N_1 \cup \ldots \cup N_{[\frac{n-1}{2}]} \end{cases}$$

Then F is a direct sum of line bundles.

Proof. We make induction on the rank of F. Suppose $h^0(F) \neq 0$ and $h^0(F(-1)) = 0$. We consider the Eagon-Northcott complex $(R_{n-2}^\vee) \otimes F(-2)$ glue together with $(R_1) \otimes F(-1)$ from Example 1.2.2. Let us start from $(R_{n-2}^\vee) \otimes F(-2)$:

$$0 \to S^{n-2}\mathcal{Q}^\vee \otimes F(-2) \to V \otimes S^{n-3}\mathcal{Q}^\vee \otimes F(-2) \to \bigwedge^2 V \otimes S^{n-4}\mathcal{Q}^\vee \otimes F(-2) \to \ldots$$

$$\ldots \to \bigwedge^{n-4} V \otimes S^2\mathcal{Q}^\vee \otimes F(-2) \to \bigwedge^{n-3} V \otimes \mathcal{Q}^\vee \otimes F(-2) \to \bigwedge^{n-2} V \otimes F(-2) \to \bigwedge^{n-2} \mathcal{S} \otimes F(-2) \to 0$$

Since,

$$H_*^{n-1}(F \otimes S^{n-2}\mathcal{Q}) = H_*^{n-2}(F \otimes S^{n-3}) = H_*^{n-3}(F \otimes S^{n-4}\mathcal{Q}) = \ldots$$
$$\ldots = H_*^3(F \otimes S^2\mathcal{Q}) = H_*^2(F \otimes \mathcal{Q}) = H_*^1(F) = 0$$

(this means $H_*^j(F \otimes S^i\mathcal{Q}) = 0$ for $(i, j) \in N_1 \cup (n-2, n-1))$ we obtain that $H^1(\bigwedge^{n-2} \mathcal{S} \otimes F(-2)) = 0$. Now we use the identification $\mathcal{S}^\vee \simeq \bigwedge^{n-2} \mathcal{S}(-1)$. and the complex $(R_1) \otimes F(-1)$:

$$0 \to F \otimes \mathcal{S}^\vee(-1) \to V \otimes F(-1) \to F \otimes \mathcal{Q}(-1) \to 0$$

Since $H^0(F(-1)) = 0$ and $H^1(F \otimes \mathcal{S}^\vee(-1)) = 0$ finally we get $H^0(F \otimes \mathcal{Q}(-1)) = 0$.

Hence we are in the hypothesis of Theorem 4.3.6 where we have taken $t = 0$. Furthermore, we can express F as follows:

$$F = \oplus \mathcal{O}^{h^0(F)} \oplus F'$$

where the rank of F' is smaller than the rank of F. Applying iteration on the rank of F' we get our result. $\square$

Remark 4.3.10. The splitting criterion of Theorem 4.3.9 (represented in Figure 4.2) implies the criterion by Ottaviani. Indeed, the hypotheses of Ottaviani's criterion (Theorem 0.0.2) are represented also by Figure 3.14 by Remark 3.1.20. We notice that $M_0 = L_0$, $M' = L_1$, $M_1 = L_2$, $M_2 = L_4$, ..., $M_k = L_{2k}$. The same happens with the segments R_i and N_i. In this case $N_1 = R_1$, $N_2 = R_3$,..., $N_k = R_{2k-1}$ and $N_0 = R_{-1}$. Hence, all the conditions of Theorem 4.3.9 are contained in the hypothesis of Theorem 0.0.2.

Remark 4.3.11. We can also compare the characterization of Theorem 4.3.9 (represented in Figure 4.2) with the characterization made by Arrondo and Malaspina (Theorem 0.0.3, represented in Figure 3.1).

In Section 3.1 we define the segments A_0 and B_0 such that the conditions $H^j_*(E \otimes S^i \mathcal{Q}) = 0$ for $(i, j) \in A_0 \cup B_0 \cup (n-2, n-1)$ correspond to the hypothesis of Theorem 0.0.3. If we compare with Figure 4.2 we notice that $A_0 \cup (n-2, n-1)$ coincides exactly with M_0 but the points of B_0 do not coincide with any of N_i. In the figure we show with a blue line the conditions B_0 and $A_0 \cup (n-2, n-1)$. However, we see that Theorem 0.0.3 only needs $2n-2$ conditions, while Theorem 4.3.9 requires much more conditions. The reason is implicit in Remark 4.2.6, because each of the $2n-2$ terms of the resolution of the diagonal is made of many different pieces, each of them imposing a different condition.

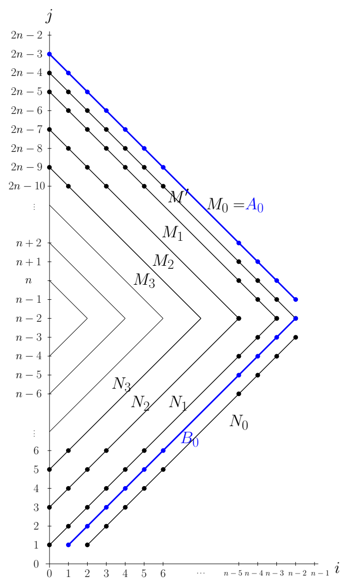


Figure 4.3: Theorem 0.0.3 vs Theorem 4.3.9

Chapter 5

Appendix: Cohomology calculator in SAGE

We briefly explain the implementation in SAGE of Bott's algorithm, made in collaboration with J. Caravantes.

We call Bott bundles to those homogeneous vector bundles on flag manifolds that can be decomposed as a direct sum of homogeneous vector bundles associated to irreducible representations. We can use Bott's algorithm to compute the cohomology of such bundles. This code allows the user to create, sum and twist of Bott bundles on flag manifolds. Data such as rank and the full cohomology can be computed.

Let us give part of the code of this package. First of all, we include some algorithms that will be just useful for the main classes and methods.

Remark 5.0.12. Notice that in the pakage we use the notation $\mathcal{Q}$ for the universal bundle of rank $n-k$ over $\mathbb{G}(k,n)$ and $\mathcal{S}$ for the universal bundle of rank $k+1$ over $\mathbb{G}(k,n)$.

In first place, we give Bott's algorithm for the general linear group as described in Proposition 2.2.3 (**def bott(alpha)**):

- INPUT (alpha): a sequence of nonnegative integers representing the concatenated partitions of the Schur functors of the quotients of adjacent tautological subbundles.
- OUTPUT ($[i, nu]$):
  - i is positive if and only if there is a nonvanishing cohomology group. Such group is H^i by Bott's Theorem, and it is the Schur Functor associated to the partition nu of the vector space of dimension equal to the length of alpha.
  - $i = -1$ if all cohomology groups vanish. In this case, nu has no relevant information.

```
def bott(alpha):
```

```
    n = len(alpha) - 1
    nu = alpha
    i = 0
    goon = True
    while goon:
        j = 0
        while j<=n-1 and nu[j] >= nu[j+1]:
            j += 1
        if j == n:
            return [i, nu]
        if nu[j]+1 == nu[j+1]:
            return [-1, nu]
        nu[j], nu[j+1] = nu[j+1]-1, nu[j]+1
        i+= 1
```

Now, given a partition l, we compute the dimension of the Schur functor of a vector space of dimension len(l) with the formula given in Proposition 2.1.17 (**def schur_dimension(l)**):

- INPUT (l): a partition
- OUTPUT (d): The dimension of the Schur functor associated to l of an l-dimensional vector space

```
def schur_dimension(l):
    n = len(l)
    d = 1
    for i in range(1, n):
        for j in range(i+1, n+1):
            d *= (j-i+l[i-1]-l[j-1]) / (j-i)
    return d
```

Now we simplify a redundant decomposition of a Bott bundle (**def simplify_decomposition(dec)**).

- INPUT (dec): A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers.
- OUTPUT: A list of pairs as in the INPUT where the second entries of the pairs are pairwise different. The function just sums up the multiplicities of the same list to simplify the data.

```
def simplify_decomposition(dec):
    i = 0
    while i < len(dec) - 1:
        j = i + 1
        while j < len(dec):
```

```
            if dec[i][1] == dec[j][1]:
                dec[i][0] = dec[i][0] + dec[j][0]
                dec.pop(j)
            else:
                j += 1
        i += 1
    return dec
```

We multiply a decomposition (of a Bott bundle) times an integer (**def multiply_decomposition(n, dec)**).

- INPUT (n, dec):
  - n: A nonnegative integer
  - dec: A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers
- OUTPUT: A list of pairs as in the INPUT where the all the multiplicities are multiplied by 'n'.

```
def multiply_decomposition(n, dec):
 return [[n*i[0], i[1]] for i in dec]
```

We have some more function on process like these:
1. def plethism_of_irred(lmbd, alpha, a, n, s): Apply Schur functor to an irreducible Bott bundle. Mainly based on [[30], Formula (8.9)].

- INPUT (lmbd, alpha, a, n, s)
  - lmbd: A partition
  - alpha: A list of integers: the concatenated Schur functors on the quotioent of adjacent tautological subbundles of the flag manifold. The resulting bundle is the twist of all this Schur functors of bundles
  - a: A list of nonnegative integers (the dimensions of the projective subspaces conforming the flags).
  - n: A nonnegative integer (the dimension of the ambient projective space of the flags).
  - s: It should be SymmetricFunctions(QQ).schur()
- OUTPUT: A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers.

2. def plethysmization(lmbd, dec, a, n, s): Tool to compute the decomposition of the Schur functor of a Bundle. Based on [[30], Formula (8.8)] and function plethism_of_irred.

- INPUT: (lmbd, dec, a, n, s):
  - lmbd: A partition
  - dec: A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers.
  - a: A list of nonnegative integers (the dimensions of the projective subspaces conforming the flags).
  - n: A nonnegative integer (the dimension of the ambient projective space of the flags).
  - s: It should be SymmetricFunctions(QQ).schur()
- OUTPUT: A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers.

Now we introduce the class we use for reducible bundles. (It intends to be defined for arbitrary flag manifolds.)

First, we define the **class BottBundle** that represents homogeneous vector bundles on a given homogeneous space (flag manifold) that can be decomposed as a sum of irreducible homogeneous vector bundles whose cohomology can be computed by means of Bott's algorithm (Proposition 2.2.3).

Such irreducible bundles are twists of Schur functors on the (duals of) quotients of tautological subbundles that are adjacent in the sequence (see [37] and [19]):

- ATTRIBUTES:
  - space: The “FlagManifold“ on which the bundle is defined
  - decomposition: A list whose elements are pairs of a positive integer (multiplicity) and a list of nonnegative integers (irreducible “BottBundle“)
  - aspect: A simpler way to represent the bundle that keeps the way it was constructed.
- METHODS:
  - _add_ : Direct sum of vector bundles “self“ and “other“ that are objects of BottBundle class, with the same attribute “space“.
  - _mul_: Ttwists “self“ times “other“ (tensor product) that are objects of BottBundle class, with the same attribute “space“.
  - rank: Returns the rank of “self“.
  - cohomology: Computes the whole sheaf cohomology of “self“. (Gives a list l of integers such that l[i] is the diension of the i-th cohomology space of the bundle. The length of l is equal to the index of the last nonvanishing cohomology space (+1). Therefore, if the length is zero, all cohomology vanishes.)
  - schur: Performs a given partition's schur functor of “self“ (on process).
  - sym: A given integer symmetric power of “self“ (on process).

    – wedge: A given number exterior power of “self“ (on process).

- INPUT (space): A value of type FlagManifold.
- OUTPUT: The zero bundle on “space“.

```
class BottBundle():
   def __init__(self, space):
        self.decomposition = []
        self.space = space
        self.aspect = '0'
   def __repr__(self):
        cadena = 'Homogeneous rank-' + str(self.rank()) + ' vector
        bundle on ' + self.space.__repr__()+'.\n' + 'It is the sum
        of the following irreducible homogeneous vector bundles:' +
        '\n'
        for i in self.decomposition:
            particiones = ''
            ant = 0
            contador = 1
            for d in self.space.a:
                particiones = particiones + '    Schur functor of
                partition ' + str(i[1][ant:d+1]) + ' of the dual of
                the ' + str(contador) + 'st/nd/rd/th  quotioent of
                tautological subbundles,\n'
                contador += 1
                ant = d + 1
            particiones = particiones + '    Schur functor of
            partition ' + str(i[1][ant:self.space.n+1]) + ' of the
            dual of the last quotioent of tautological subbundles,\n'
            cadena = cadena + '\n' + str(i[0]) + ' time(s) the
            tensor product of: \n' + particiones
        return cadena
   def __add__(self, other):
        if self.space <> other.space:
            print 'Error: Cannot sum bundles on different varieties'
            return
        result = BottBundle(self.space)
        result.aspect = self.aspect + '(+)' + other.aspect
        result.decomposition =
         simplify_decomposition(deepcopy(self.decomposition) +
         deepcopy(other.decomposition))
        return result
   def __mul__(self, other):
        if self.space <> other.space:
```

```
        print 'Error: Cannot twist bundles on different varieties'
        return
    import sage.libs.lrcalc.lrcalc as lrcalc
    n = self.space.n
    a = copy(self.space.a)
    r = len(a)
    result = BottBundle(self.space)
    result.aspect = '(' + self.aspect + ')(x)(' + other.aspect + ')'
    for i in self.decomposition:
        for j in other.decomposition:
            LR_Result = lrcalc.mult(i[1][0:a[0]+1],
            j[1][0:a[0]+1], a[0]+1)
            SimpleMult = []
            for l in LR_Result.items():
                SimpleMult.append([l[1]*i[0]*j[0], l[0][:]])
            for s in SimpleMult:
                for index in range(len(s[1]), a[0]+1):
                    s[1].append(0)
            for k in range(1, r):
                LR_Result = lrcalc.mult(i[1][a[k-1]+1:a[k]+1],
                j[1][a[k-1]+1:a[k]+1], a[k]-a[k-1])
                NewList = []
                for l in LR_Result.items():
                    for s in SimpleMult:
                        aux = deepcopy(s[1])
                        aux.extend(l[0][:])
                        NewList.append([l[1]*s[0],aux])
                SimpleMult = NewList
                for s in SimpleMult:
                    for index in range(len(s[1]), a[k]+1):
                        s[1].append(0)
            NewList = []
            LR_Result = lrcalc.mult(i[1][a[r-1]+1:n+1],
            j[1][a[r-1]+1:n+1], n-a[r-1])
            for l in LR_Result.items():
                for s in SimpleMult:
                    aux = deepcopy(s[1])
                    aux.extend(l[0][:])
                    NewList.append([l[1]*s[0], aux])
            SimpleMult = NewList
            for s in SimpleMult:
                for index in range(len(s[1]), n+1):
                    s[1].append(0)
            result.decomposition.extend(SimpleMult)
```

```
                result.decomposition =
                simplify_decomposition(result.decomposition)
        return result
    def rank(self):
        TotalRank = 0
        for d in self.decomposition:
            PartialRank = 1
            previousa = -1
            for a in self.space.a:
                PartialRank *= schur_dimension(d[1][previousa+1:a+1])
                previousa = a
            PartialRank *=
            schur_dimension(d[1][previousa+1:self.space.n+1])
            TotalRank += d[0] * PartialRank
        return TotalRank
    def cohomology(self):
        n = len(self.decomposition[0][1])
        B = []
        for el in self.decomposition:
            lista = deepcopy(el[1])
            aux = bott(lista)
            aux.append(el[0])
            B.append(aux)
        B.sort()
        m = B[len(B)-1][0]
        Cohom = []
        for ind in range(0, m+1):
            Cohom.append(0)
        for el in B:
            if el[0] <> -1:
                Cohom[el[0]] += el[2] * schur_dimension(el[1])
        for i in range(0, m+1):
            if Cohom[i] <> 0:
                print "h^", i, " = ", Cohom[i]
        if len(Cohom) == 0:
            print 'All cohomology vanishes'
        else:
            print 'All remaining cohomology is zero'
        return Cohom
```

Now we give some child classes of the BottBundle class that only work for Grassmannian.

- UniversalSubbundleDual(BottBundle): A child class of “BottBundle“ to easily initialize the dual to the universal subbundle of an instance of “Grassmannian“

```
class UniversalSubbundleDual(BottBundle):
    def __init__(self, fm):
        self.decomposition = [[1,[1]+[0]*(fm.n)]]
        self.space = fm
        self.aspect = 'S_1^*'
```

- UniversalSubbundle(BottBundle): A child class of “BottBundle“ to easily initialize the universal subbundle of an instance of “Grassmannian“

```
class UniversalSubbundle(BottBundle):
   def __init__(self, fm):
        self.decomposition = [[1, [1]*(fm.a[0])+[0]+[1]*(fm.n-fm.a[0])]]
        self.space = fm
        self.aspect = 'S_1'
```

- UniversalQuotientDual(BottBundle): A child class of “BottBundle“ to easily initialize the dual to the universal quotient bundle of an instance of “Grassmannian“

```
class UniversalQuotientDual(BottBundle):
   def __init__(self, fm):
        self.decomposition =
        [[1,[0]*(fm.a[0]+1)+[1]+[0]*(fm.n-fm.a[0] -1)]]
        self.space = fm
        self.aspect = 'Q_1^*'
```

- UniversalQuotient(BottBundle): A child class of “BottBundle“ to easily initialize the universal quotient bundle of an instance of “Grassmannian“

```
class UniversalQuotient(BottBundle):
    def __init__(self, fm):
        self.decomposition = [[1, [1]*(fm.n)+[0]]]
        self.space = fm
        self.aspect = 'Q_1'
```

- LineBundleGrass(BottBundle): A child class of “BottBundle“ to easily initialize line bundles of an instance of “Grassmannian“

```
class LineBundleGrass(BottBundle):
    def __init__(self, grass, k):
        if k > 0:
            self.decomposition = [[1,
            [k]*(grass.a[0]+1)+[0]*(grass.n-grass.a[0])]]
        elif k < 0:
            self.decomposition = [[1,
            [0]*(grass.a[0]+1)+[-k]*(grass.n-grass.a[0])]]
        else:
            self.decomposition = [[1, [0]*(grass.n+1)]]
        self.space = grass
        self.aspect = 'O(' + str(k) + ')'
```

Now we introduce the classes referring the homogeneous spaces (**class FlagManifold**). This is the class containing all quotients of the general linear group by a parabolic subgroup.

- ATTRIBUTES:
  - a: A list of increasing integers (dimensions of the subspaces of the flag)
  - n: Positive integer (dimension of the ambient projective space of the flag)
- METHODS:
  - is_grassmannian: Determines wether “self“ is a grassmannian.
  - is_projective_space: Determines wether “self“ is a projective space.
- INPUT
  - a: A list of increasing integers (dimensions of the subspaces of the flag)
  - n: Positive integer (dimension of the ambient projective space of the flag)
- OUTPUT: The space of flags of subspaces of dimensions given by “a“ in the projective space of dimension “n“

```
class FlagManifold():
    def __init__(self, a, n):
        self.a = a
        self.n = n
    def __repr__(self):
        return 'the space of flags of subspaces of dimensions ' +
        str(self.a) + ' in the ' + str(self.n) + '-dimensional
        projective space'
   def is_grassmannian(self):
        if len(self.a)==1:
```

```
            return True
        else:
            return False
    def is_projective_space(self):
        if self.is_grassmannian():
            if self.a[0]==0:
                return True
            else:
                return False
        else:
            return False
```

Now we give the particular child class of FlagManifold (**class Grassmannian(FlagManifold)**), to easily initialize Grassmannians.

- METHODS:
  - O: Creates a multiple of the hyperplane section
  - Q: Creates the universal quotient bundle
  - Q_dual: Creates the dual of the universal quotient bundle
  - S: Creates the universal subbundle
  - S_dual: Creates the dual of the universal subbundle
  - Om: Creates the cotangent bundle
  - T: Creates the tangent bundle
- INPUT:
  - k: A list of increasing integers (dimension of the projective subspace)
  - n: Possitive integer (the dimension of the ambient projective space of the flag)
- OUTPUT: The grassmannian of “k“-dimensional projective subspaces in the projective space of dimension “n“.

```
class Grassmannian(FlagManifold):
    def __init__(self,k,n):
        self.a = [k]
        self.n = n
        print 'You have defined the grassmannian of ' +
        str(self.a[0]) + '-dimensional subspaces of the ' +
        str(self.n) + '-dimensional projective space'
        print 'Consider the uiversal short exact sequence:'
        print '0 --> S --> O^' + str(self.n+1) + ' --> Q --> 0'
        print 'where rk(S)=' + str(self.a[0]+1) + ' and rk(Q)=' +
```

```
        str(self.n-self.a[0]) + '.'
        print 'These bundles can be defined with methods S() and
       Q().'
        print 'Their duals can be obtained with methods SDual() and
       Qdual()'
        print 'Method O(k) gives the k power of the hyperplane
        section.'
        print 'Method Om() gives the cotangent bundle.'
        print 'Method T() gives the tangent bundle.'
    def __repr__(self):
        return 'the grassmannian of ' + str(self.a[0]) +
        '-dimensional subspaces of the ' + str(self.n) +
        '-dimensional projective space'
   def O(self,k):
        return LineBundleGrass(self, k)
   def Q(self):
        return UniversalQuotient(self)
   def S(self):
        return UniversalSubbundle(self)
   def Q_dual(self):
        return UniversalQuotientDual(self)
   def S_dual(self):
        return UniversalSubbundleDual(self)
   def Om(self):
        return self.S()*self.Q_dual()
   def T(self):
        return self.S_dual()*self.Q()
```

Example 5.0.13. Let us finally give an example. We will define $\mathbb{G} = \mathbb{G}(1,4)$ and the universal bundles $\mathcal{Q}$ and $\mathcal{S}$. Then we construct a bundle comming from twist and sum of Bott bundles $\mathcal{Q}$ and $\mathcal{S}$ and compute its cohomology.

```
sage: G = Grassmannian(1,4)
    You have defined the grassmannian of 1-dimensional subspaces of
    the 4-dimensional projective space
    Consider the uiversal short exact sequence:
    0 --> S --> O^5 --> Q --> 0
    where rk(S)=2 and rk(Q)=3.
    These bundles can be defined with methods S() and Q().
    Their duals can be obtained with methods SDual() and Qdual()
    Method O(k) gives the k power of the hyperplane section.
    Method Om() gives the cotangent bundle.
    Method T() gives the tangent bundle.
```

```
sage: Q = G.Q()          #creates the universal bundle Q
    Homogeneous rank-3 vector bundle on the grassmannian of
    1-dimensional subspaces of the 4-dimensional projective space.
    It is the sum of the following irreducible homogeneous vector
    bundles:

    1 time(s) the tensor product of:
        Schur functor of partition [1, 1] of the dual of the
        1st/nd/rd/th quotioent of tautological subbundles,
        Schur functor of partition [1, 1, 0] of the dual of the last
        quotioent of tautological subbundles,

sage: S = G.S()          #creates the universal bundle S
    Homogeneous rank-2 vector bundle on the grassmannian of
    1-dimensional subspaces of the
    4-dimensional projective space.
    It is the sum of the following irreducible homogeneous vector
    bundles:

    1 time(s) the tensor product of:
        Schur functor of partition [1, 0] of the dual of the
        1st/nd/rd/th quotioent of tautological subbundles,
        Schur functor of partition [1, 1, 1] of the dual of the last
        quotioent of tautological subbundles,

sage: E = S*Q + Q        #Twist and sum of Bott bundles
    Homogeneous rank-9 vector bundle on the grassmannian of
    1-dimensional subspaces of the
    4-dimensional projective space.
    It is the sum of the following irreducible homogeneous vector
    bundles:

    1 time(s) the tensor product of:
        Schur functor of partition [2, 1] of the dual of the
        1st/nd/rd/th quotioent of tautological bundles,
        Schur functor of partition [2, 2, 1] of the dual of the last
        quotioent of tautological bundles,

    1 time(s) the tensor product of:
        Schur functor of partition [1, 1] of the dual of the
        1st/nd/rd/th quotioent of tautological bundles,
        Schur functor of partition [1, 1, 0] of the dual of the last
        quotioent of tautological bundles,
```

```
sage: E.cohomology()    #Computing the cohomology of a Bott bundle
     h^0 = 5
     All remaining cohomology is zero
     [5]
```

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