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TESIS DOCTORAL

**Backward Reasoning in problem-solving situations: a
multidimensional model**

**Razonamiento regresivo en situaciones de resolución de
problemas: un modelo multidimensional**

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Facultad de Ciencias Matemáticas



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PH.D. THESIS

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List of Abbreviations

AiC	Abstraction in Context
BR	Backward Reasoning characteristics
BRI	Backward Reasoning Indicators
FLIM	Finer Logic of Inquiry Model
GTL	Game Theory Logic
HIM	Hintikka's Interrogative Model
RBC	Recognizing/Building-with/Constructing model

Analysis models abbreviations:

- Backward Reasoning characteristics:

D	Breakdown
E	Cause-effect relationship research
G	Going backward
SF	Solution formulation
T	Transformative
X	Introduction of auxiliary elements

- Finer Logic of Inquiry Model:

A	Ascendant Modality
D	Descendant Modality
N	Neutral Modality

- Hintikka's Inquiry Model

answ	Answer to the interrogative move
Ast	Assertoric move
Ded	Deductive move
Def	Defining move
Init	Initial move
Int	Interrogative move
Int/a	Interrogative move and answer
Ru	Standard rules

- RBC model

B	Building-with epistemic action
C	Constructing epistemic action
R	Recognizing epistemic action

ABSTRACTS

English version

The increasing technological progress has highlighted the importance of problem-solving processes and skills connected to programming methods. Among them, backward reasoning is recognized as a critical issue in advanced mathematics education. This, together with the growing interest in recent years of game-based university education is at the base of this research project. Two objectives are established: on the one hand, to extend the epistemic model of backward reasoning, existing in the mathematical literature, to a cognitive and didactic one; on the other hand, to establish principles for the design of university teaching situations focused on backward reasoning. To reach these objectives, four design experiments using strategy games and mathematical problems are developed. These involved a total of 322 university students, from first year of bachelor to PhD, attending the Universidad Complutense de Madrid (Spain) and the Università di Torino (Italy). They are involved in scientific careers (Mathematics, Mathematics Engineering and Computer Science) and teacher training careers (future mathematics professors in secondary school). The research project is framed on qualitative studies based on a networking of two theories, the Game Theory Logic (Hintikka, 1999) and the Abstraction in Context theory (Dreyfus, et al., 2015), with subsequent hybridization through a fragment of the Commognition approach (Sfard, 2008). From the emerging theoretical framework, a multidimensional analysis tool is developed to analyse students' resolution protocols, video-recordings, and interview and identify backward reasoning moments. As research result, eleven Backward Reasoning Indicators (BRI), that represent the cognitive dimensions of backward reasoning, are pointed out. They allow to respond to both research objectives and to make some further didactic conclusions.

Key words:

Backward reasoning, Mathematical Problems, Strategy games, Mathematical thinking, Mathematical reasoning, Cognitive dimensions, Logic of Inquiry, Abstraction in Context, Commognition.

Spanish version

El creciente progreso tecnológico ha puesto de relieve la importancia de los procesos de resolución de problemas y los conocimientos técnicos relacionados con los métodos de programación. Entre ellos, el razonamiento regresivo se reconoce como una cuestión crítica en la enseñanza de las matemáticas avanzada. Esto, junto con el creciente interés en los últimos años de la educación universitaria basada en juegos, es la base de esta investigación. Se establecen dos objetivos: 1) ampliar el modelo epistémico de razonamiento regresivo, existente en la literatura matemática, a uno cognitivo y didáctico, y 2) establecer principios para el diseño de situaciones de enseñanza universitaria centradas en el razonamiento regresivo. Para lograr estos objetivos, se desarrollan cuatro *Design experiments* utilizando juegos de estrategia y problemas matemáticos. En ellos participaron un total de 322 estudiantes universitarios, desde el primer año de grado hasta el doctorado, procedentes de la Universidad Complutense de Madrid (España) y de la Università di Torino (Italia). Son estudiantes de las ramas científica y de ingeniería (Matemáticas, Ingeniería Matemática e Informática) y en la especialidad de formación de profesores (futuros profesores de matemáticas en la escuela secundaria). El proyecto de investigación se enmarca en estudios cualitativos basados en el *networking* de dos teorías, la Game Theory Logic (Hintikka, 1999) y la teoría de la Abstraction in Context (Dreyfus, et al., 2015), con la posterior *hybridization* a través de un fragmento del enfoque de la Commognition (Sfard, 2008). A partir del marco teórico emergente del estudio, se desarrolla una herramienta de análisis multidimensional para analizar los protocolos de resolución de los estudiantes, las grabaciones de vídeo, y entrevistar e identificar los momentos de razonamiento regresivo. Como resultado de la investigación, se señalan once Backward Reasoning Indicators (BRI), que representan las dimensiones cognitivas del razonamiento hacia atrás. Estos permiten responder a los objetivos de la investigación y sacar algunas conclusiones didácticas adicionales.

Palabras clave:

Razonamiento regresivo, Problemas matemáticos, Juegos de estrategia, Pensamiento matemático, Razonamiento matemático, Dimensiones cognitivas, Logic of Inquiry, Abstraction in Context, Commognition.

Italian version

Il crescente progresso tecnologico ha evidenziato l'importanza dei processi di problem-solving e delle competenze legate ai metodi di programmazione. Tra questi, il ragionamento regressivo è riconosciuto come un punto cruciale nell'insegnamento della matematica avanzata. Questo, insieme al crescente interesse degli ultimi anni per la formazione universitaria game-based, è alla base di questo progetto di ricerca. Si sono individuati due obiettivi: da un lato, estendere il modello epistemico del ragionamento regressivo, esistente nella letteratura matematica, ad un modello cognitivo e didattico; dall'altro, stabilire i principi per la progettazione di situazioni di insegnamento universitario incentrate sul ragionamento regressivo. Per raggiungere questi obiettivi, vengono sviluppati quattro *design experiments* utilizzando giochi di strategia e problemi matematici. Questi hanno coinvolto un totale di 322 studenti universitari, dal primo anno di triennale al dottorato di ricerca, frequentanti l'Universidad Complutense de Madrid (Spagna) e l'Università di Torino (Italia). Gli studenti sono immatricolati in facoltà scientifiche (Matematica, Ingegneria Matematica e Informatica) o in masters per la formazione insegnanti (futuri professori di matematica nella scuola secondaria). Il progetto di ricerca, di carattere qualitativo, si basa sul *networking* di due teorie, la Game Theory Logic (Hintikka, 1999) e la teoria dell'Abstraction in Context (Dreyfus, et al., 2015), con una successiva *hybridization* attraverso un frammento dell'approccio della Commognition (Sfard, 2008). Dal quadro teorico che emerge, viene sviluppato uno strumento di analisi multidimensionale per analizzare i protocolli di risoluzione degli studenti, le registrazioni video, le interviste e identificare i momenti di ragionamento regressivo. Come risultato della ricerca, vengono evidenziati undici Backward Reasoning Indicators (BRI), che rappresentano le dimensioni cognitive del ragionamento regressivo. Essi permettono di rispondere ad entrambi gli obiettivi della ricerca e di trarre ulteriori conclusioni didattiche.

Parole chiave:

Ragionamento regressivo, Problemi matematici, Giochi di strategia, Pensiero matematico, Ragionamento matematico, Dimensioni cognitive, Logic of Inquiry, Abstraction in Context, Commognition.

1

INTRODUCTION

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INTRODUCTION 1

Over the last century, increasing technological progress has highlighted the importance of skills connected to programming methods. One of the most used reasoning in problem solving and programming methods is the backward reasoning, which is also a critical issue in advanced mathematics education. This, together with the growing interest in recent years of game-based university education is at the base of this research project.

Its aim is to study the backward reasoning processes through the observation of university students (of mathematics and engineering courses), in order to provide useful tools that can be exploited for the development of consequent effective teaching techniques, suitable for the increasing need of technology skills.

1.1 Problem statement and justification

In mathematics, forward reasoning alone is not exhaustive to fulfil the tasks of solving problems. Besides deduction, the natural way for approaching a problem also requires inductive, abductive, and backward reasoning (Hintikka, 1999; Lakatos, 1976; Peirce, CP 2.623). They correspond also to different ways of reasoning and thinking in problem solving from a cognitive standpoint (for a general survey see Holyoak and Morrison, 2015). Great mathematicians like Pappus, Descartes, Leibniz, in their discussions about analysis and synthesis, emphasize this fact (Beaney, 2018; Peckhaus, 2002).

Backward reasoning is a modality of mathematical thinking involved in the method of analysis. It is used in problem solving discovery phases, and it has a fundamental importance also in programming methods (Mäenpää, 1993, 1998). It consists in developing a series of

logical steps beginning from the end of the problem (its claim) toward its premises. Proceeding through logical correspondences, something known is obtained. This method is a procedure that starts with the formulation of the problem and ends with the determination of the conditions for its solution (Hintikka and Remes, 1974). It is characterized by the insertion of new elements in the resolution and it has a component of creativity and discovery. Alone it does not solve or prove the problem, but it is the fundamental basis for developing a successive form of reasoning, necessary to prove the problem itself: the synthesis. In literature, backward reasoning is known by different denominations: regressive reasoning, regressive analysis, backward solution, etc. This process underlies different ways of proceeding in problem solving: working backward strategy, assuming the problem solved strategy, Reductio ad Absurdum, beginning at the end of the problem strategy, etc. Depending on the type of problem and the path of resolution or construction the solver chooses, one or more of those strategies arise.

A small example is shown to grasp the concept. Let us consider the following problem (Arzarello, 2014):

<p>Problem</p> <p>$f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function; $\lim_{x \rightarrow +\infty} f(x) = +\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$</p> <p><i>Prove that there is at least one point c such that $f(c) = 0$</i></p>
--

Fig. 1.1 - Example problem

A way to solve this problem is relating it with the Intermediate Value Theorem (IVT). A backward reasoning can consist in starting from the statement “there is at least one point c such that $f(c) = 0$ ”, and going to: “there is at least one point c in a interval $[x', x'']$ such that $f(c) = 0$ ”. This is the conclusion of the IVT theorem. At this point the step can be to reduce the assumptions of the problem in order to apply the IVT theorem. Observing this small example, which will be discussed in more detail in Chapter 3 (section 3.4), one can notice that backward reasoning does not make sense without its forward counterpart.

The concepts encountered at university level are more abstract and require advanced mathematical thinking (Dreyfus, 1990 and 2002; Tall, 2002), particularly those associated

with technology. The backward reasoning has proved to be extremely stimulating in various fields. As well as in mathematics, the combination of the two branches of analysis and synthesis, has been applied to several fields of artificial intelligence, theoretical computer science, and in programming methodology (Grosholz, Breger, 2000; Peckhaus, 2002). It has been also studied in the way medical doctors elaborate their diagnosis (ten Cate, Custers, and Durning, 2018) and consequently has been used in automated medical diagnosis, based on Data Mining, which strongly involve Artificial Intelligence and Machine Learning domains (Bishop, 2006; Goodfellow, Bengio, and Courville, 2016; Gorunescu and Belciug, 2018). In the Artificial Intelligence field, for example, one of the necessary concepts to design computer programs that produce reasoning rules for obtain data is the concept of “forward-backward chains”. The forward and backward chaining algorithms are defined in this way (Russell and Norvig, 2010):

“The forward-chaining algorithm determines if a single proposition q -the query- is entailed by a knowledge base of definite clauses. It begins from known facts (positive literals) in the knowledge base. If all the premises of an implication are known, then its conclusion is added to the set of known facts. [...] The backward-chaining algorithm, as its name suggests, works backward from the query. If the query q is known to be true, then no work is needed. Otherwise, the algorithm finds those implications in the knowledge base whose conclusion is q . If all the premises of one of those implications can be proved true (by backward chaining), then q is true.” (pp. 256-257)

Backward and forward reasoning are implied in these algorithms. To apply them in an efficient way it is necessary to understand the reasoning that underly them. In this regard, Sharma, Tiwari and Kelkar (2012) state that apply these rules is not “easy or intuitive”, and they add that “the best way to build efficient applications using rule engines is to take the time to learn how each approach works and use both techniques. [...] It's crucial to consider the intent of the rule, size of the dataset and performance requirements” (p. 273).

But, for many engineering and mathematics undergraduate students, learning the method of analysis in mathematics tertiary education is a critical issue (Antonini, 2011, Peckhaus, 2002, Wickelgren, 1974, Xu, Xing and Van Der Schaar, 2016). Students have the challenge

of incorporating it in different disciplines related to the design and production of products and services (such as Project Management, Systems Engineering and Design Science), but they don't have the necessary theoretical and methodological basis (Koskela and Kagioglou, 2006). Numerous authors (Barbero, 2015; Byers, 2007; Corbalán, 1994 and 1997; Hintikka and Remes, 1974) underline the difficulties in using and understanding the above reasoning as a general procedure. Studies at the university level focused on the diagnosis of difficulties in the use of backward reasoning, they point out that is more difficult to work backwards than to work forwards (Gómez-Chacón, 2017; Gómez-Chacón and Barbero, 2019). Studies focused on the heuristics related to backwards reasoning and on the students' difficulties showing a number of effects on the interplay cognition and affect in the mathematical thought, in particular on the creation of the solution, mathematical models and actions of discovery. A positive correlation between these heuristics and confidence emotion were also noted (Gómez-Chacón, 2017, Gómez-Chacón and Barbero, 2020).

Backward reasoning study has a great potential in Mathematics Education field; it can be used to improve student achievement and to help them to develop mathematical argumentation, inquiry and proof processes. It plays a central role in advanced mathematical thinking development, where the abstract processes are predominant. In fact, while the teaching of forward reasoning and deductive processes are typical of mathematical thinking, where reproductive routines must be followed, backward reasoning becomes crucial when more creative proving processes are involved (Dreyfus, 1990; Tall, 2002). This reasoning is part of the pragmatic aspects of problem solving but needs a high level of abstraction to be used. It is a form of reasoning based on looking at things in a fresh non routine way: objects involved are organized depends on the way things are seen.

From the second half of the twentieth century, problem solving, as a proposal for education and learning, was one of the most developed areas of mathematical education (Cellucci, 2017; Koichu and Leron, 2015; Mason, Burton and Stacey, 1982; Polya, 1945; Santos-Trigo and Moreno-Armella, 2013; Schoenfeld, 1985 and 1994) and several studies focused at university level (Koichu, 2008 and 2010; Lithner, 2000 and 2003). A more recent development theme has been the use of games in teaching (Dickey, 2007; Garris, Ahlers, and Driskell, 2002; Kapp, 2012; Kiili, 2005; Niman, 2014; Shute, Rieber and van Eck, 2011)

and in particular in mathematics (Barbero, 2015 and 2016; Barbero and Gómez-Chacón, 2018; Barbero, Gómez-Chacón and Arzarello, 2020; Barbero, Rubio and Gómez-Chacón, 2017; De Guzmán 1984; Delucchi, Gaiffi and Pernazza, 2012; Gómez-Chacón, 2005; Martignone & Sabena, 2014; Soldano, 2017).

Problem solving is one of the main skills needed in mathematics and engineering practice. The growing importance of this has led to considering strategy games as a key element of the educational process. In fact, these can be used to facilitate the learning of distinct aspects (processes, phases ...) of problem solving; therefore, are an important methodological instrument for his teaching (Gómez-Chacón, 1992; Koichu, 2010; Lithner, 2000).

The relationship between strategy games and problem solving is rooted in the fact that, in order to solve them, it is necessary to follow the same heuristic processes. Gómez-Chacón (1992) argues that both of their resolution phases coincide. This structure allows, in both cases, to use the same tools and the same reasoning processes necessary to the development of typical mathematical thinking processes. Some researches (Barbero, 2015; Soldano, 2017) have also shown how the processes involved in mathematical gaming situations strongly influence and guide the students during the discovery and justification stages of the resolution, stimulating them positively.

From a theoretical point of view, Hintikka's Game Theory Logic (1999) can provide an adequate epistemic framework for backward reasoning. In his approach he considers Game Theory and Wittgenstein's language games to support formal epistemic logic in mathematics. The main concept of his logic is the notion of truth that he introduces in a fresh top-down way (Hintikka, 1995), where the usual bottom-up definition of truth given by Tarski (Tarski, 1933; Tarski and Vaught, 1956) is reversed. Hintikka's new approach highlights the regressive way of proceeding in problem solving from an epistemological point of view. This theory is based on the idea that the processes of discovery have a question-answer nature.

An Interrogative Model of Inquiry, considered as a general theory of reasoning, emerge from the Game Theory Logic (Başkent, 2016; Brook, 2007; Hintikka, Halonen and Mutanen 2002). Several studies use this model for analysing the reasoning and the statements in

dialogical games, using it as a methodological approach in the production of knowledge (Harmaakorpi & Mutanen, 2008), or as a learning model where the main goal is shaping the learning strategy (Mutanen, 2010). There is a background in Mathematics Education field in the use of Hintikka's model as a tool for analysing dialogues between students during mathematical inquiry (Barbero and Gómez-Chacón, 2018; Barbero, Gómez-Chacón and Arzarello, 2020; Barrier 2008, Hakkarainen and Sintonen, 2002). A model adaptation, the *Finer Logic of Inquiry Model* (Arzarello 2014, Soldano 2017, Soldano and Arzarello, 2016), has been developed to characterize the cognitive dimension of reasoning through three cognitive modalities: ascending, neutral and descending.

This research project is part of a larger *Research Program in Mathematical education at University Level (INVEDUMAT)* of the Institute of Interdisciplinary Mathematics (IMI) in Spain, developed since 2013. And it is linked with the Teaching Innovation Project on *Multimedia learning scenarios in professional development of the novice university mathematics lecturers (ESCEMMAT-Univ)* (academic years 2018-2020) carried out at the Chair UCM Miguel de Guzmán (Complutense University of Madrid) (Gómez-Chacón, et al., 2019). The Teaching Innovation Project focuses on the introduction of inquiry-based teaching and learning methodology in university level lectures.

1.2 Research project

This research project was developed exploiting all the considerations mentioned above. A conscious integration of backward reasoning in mathematics university learning raises the need for articulation between epistemological and cognitive aspects. Two main objectives were set:

- 1. To extend the epistemic model of backward reasoning, existing in the mathematical literature, to a cognitive model.**
- 2. To establish principles for the design of university teaching situations focused on backward reasoning.**

Starting from these objectives, a first set of raw research questions emerged.

They will be better redefined basing on the in-depth literature research about backward reasoning and the networking-elaboration of the theoretical framework (see Chapter 4, section 4.7):

- How does backward reasoning develop at a cognitive level?
- How does it interact with other types of reasoning?
- Which didactic situations can promote this type of reasoning?

To answer these questions, the project work identified two specific objectives of research:

1. To characterize the meaning of backward reasoning in the mathematical instructional environments (especially at university level), explaining the systems of practices linked to backward reasoning in this context and delimiting the corresponding context elements, especially discursive ones.
2. To conduct a Design Experiment using some strategy games and mathematical problems in order to:
 - a. To reflect on the mathematical thinking processes that appear to be essential for the development of backward reasoning and its connection with forward reasoning. More precisely:
 - i. observing what kind of backward reasoning develops in strategy games and mathematical problems.
 - ii. investigating how the backward reasoning develops and how it articulates with forward reasoning.
 - b. To establish principles that can be useful for the development of backward reasoning in problem solving activities.

1.2.1 Research layout

The research project was organized in six phases:

1. *Familiarization with the theoretical framework*
 - a. In-depth literature review. Collecting precise information about the backward reasoning and all the type of reasoning directly related, in particular, forward reasoning and abductive reasoning (see Chapter 3).
 - b. Familiarization with the Mathematical Education work at the international level about the backward reasoning.
 - c. Problems and theoretical frameworks associated with methodological tools research.
 - d. Analysis of some existent protocols to verify the possible connection between the theoretical frameworks studied.
 - e. Refining the research questions (see below).
2. *First Design Experiment: Triangular Peg Solitaire*
 - a. Proposal of the strategy game Triangular Peg Solitaire (see Chapter 5) to students pursuing a BSc. in Mathematics.
 - b. Analysis of the resolution protocols with Hintikka's Interrogative Model (HIM) and Finer Logic of Inquiry Model (see Chapter 6).
 - c. Results re-elaboration in order to improve the strategy games proposal and the analysis model.
3. *Second Design Experiment: Maude Task*
 - a. Proposal of the programming Maude task (see Chapter 5) to students attending a MSc. in Computer Science.
 - b. Analysis of the resolution protocols with the interpretation of HIM through resolution context (see Chapter 7).
 - c. Results re-elaboration in order to improve the strategy games proposal and the analysis model.
 - d. Introduction of a new analysis tool starting from the networking of theories between Game Theory Logic and Abstraction in Context approach (see Chapter 4).
4. *Third Design Experiment: 3D Tick-Tack-Toe*

- a. Proposal of the strategy game 3D Tick-Tack-Toe to students attending a BSc. in Mathematics, a Master's in Mathematics Teacher Training for Secondary School, and the Engineering Mathematics, Statistics and Operations Research doctoral program (IMEIO).
 - b. Analysis of the resolution protocols through HIM and RBC-model (see Chapter 8).
 - c. Results re-elaboration in order to improve the analysis model.
 - d. Introduction of a new analysis tool starting from the hybridization of theories with the introduction of a fragment of Commognition theory (see Chapter 4).
5. *Fourth Design Experiment: Mathematical Problems*
- a. Proposal of four mathematical problems to undergraduate students attending a BSc. in Mathematics, and graduate students attending a Master's in Mathematics Teacher Training for Secondary School.
 - b. Analysis of the resolution protocols through HIM, RBC-model identifying objectification moments (see Chapter 9).
 - c. Results re-elaboration order to check the analysis model.
6. *Re-elaboration of the results obtained in the design experiment*
- a. Analysis of the resolution protocols of the first, second and third design experiment in order to integrate the analysis tools.
 - b. Results re-elaboration order to check find regularities along the four design experiments (see Chapter 10).

For each design experiment a quantitative analysis of the data was carried out through graphs and summary tables. The qualitative analysis was carried out through a multidimensional¹ analysis tool elaborated interconnecting the Hintikka's Interrogative Model (Hintikka, 1986), the RBC model (Dreyfus et al., 2015) and the identification of objectification

¹ In the creation of the analysis tool, different theoretical frameworks come into play and different aspects of the students' productions, such as video-recordings or resolution protocols, are analysed. These make the analysis tool have both multidimensional and multimodality aspects. In order to get discourse flowing we will only use the term multidimensional.

processes (Sfard, 2008) (see Chapter 4). In Chapter 4, we will discuss the rationale for using a networking (Prediger, Bikner-Ahsbabs, 2014) of such theories.

The conclusion of the research contains the reflections and considerations carried out in the last re-elaboration of the results. The re-elaboration allows to identify 11 Backward Reasoning Indicators (BRI), which properly describe the different processes in backward reasoning. The BRI and the observations during their construction allow to answer to the three refined research questions elaborated during the research project (at the end of the first phase):

1. *What is the epistemological and cognitive link between backward and forward reasoning?*
2. *How does the transition from backward reasoning to forward reasoning (and vice versa) take place?*
3. *Are there any non-playing situations that lead to backward reasoning?*

1.3 Dissertation overview

The dissertation is divided in three main parts and ten chapters:

Chapter 1: Introduction

Part I - Theoretical elements

- **Chapter 2: Mathematical reasoning.** It contains concepts and ideas, developed by other researches, which underly the research project. The differences between elementary and advanced mathematical thinking, the problem-solving environment and the heuristic analogy with strategy games are displayed.
- **Chapter 3: Backward reasoning in mathematics (Not to mention forward).** It contains the in-depth literature research about backward reasoning carried out in the

first phase of the research project. Starting from the historical concept of method of analysis, the main features of backward reasoning are characterized. Then, the relationship between backward and forward reasoning is highlighted, emphasizing the importance of the introduction of auxiliary elements in the resolution process. Subsequently the resolution strategies related to backward reasoning are shown. Finally, a refined definition of backward reasoning is given.

- **Chapter 4: Analysis tools of mathematical reasoning: Theoretical frameworks.** It contains the elaboration of the theoretical framework. Game Theory Logic (GTL), Abstraction in Context (AiC) theory and Commognition theory are shown. Networking strategy (*combining and coordinating*) between GTL and AiC and the hybridization of Commognition are illustrated. The chapter ends with the explanation of the refined research questions.

Part II - Design Experiments

- **Chapter 5: Research Design.** In this chapter the research context and the methodology used are explained. The chapter contains the explication of each design experiment settings and the a priori analysis of the tasks. The analysis tools are shown.
- **Chapter 6: Triangular Peg Solitaire analysis.** It consists in the quantitative analysis of the first design experiment and the qualitative analysis of a case study.
- **Chapter 7: Maude task analysis.** It consists in the quantitative analysis of the first design experiment and the qualitative analysis of a case study.
- **Chapter 8: 3D Tick-Tick-Toc analysis.** It consists in the quantitative analysis of the first design experiment and the qualitative analysis of four case study.
- **Chapter 9: Mathematical problems analysis.** It consists in the quantitative analysis of the first design experiment and the qualitative analysis of fourteen case study.

Part III – Results, Discussion and Conclusions

- **Chapter 10: Results: General discussion.** A comparison of design experiments analysis and obtained results is shown. Then the 11 Backward Reasoning Indicators (BRI) are highlighted, extending the epistemic model to a cognitive one.

- **Chapter 11: Conclusions.** The BRI and the conclusive results developed in Chapter 10 are used to answer to the three research questions and make some conclusion about the methodology used and the didactic consequences.

I

THEORETICAL ELEMENTS

PART I – THEORETICAL ELEMENTS

This first part of the dissertation consists of 3 chapters: *Mathematical reasoning*, *Backward reasoning in mathematics*, and *Analysis tools of mathematical reasoning: Theoretical frameworks*. In these chapters, all the theoretical notions that underly the development of the four design experiments and their analysis, the subject of Part II, are displayed.

In *Mathematical reasoning* (Chapter 2) some theoretical concepts that were considered necessary for the research project are shown. On the one hand, being part of the Teaching Innovation Project on *Multimedia learning scenarios in professional development of the novice university mathematics lecturers (ESCEMMAT-Univ)* (Gómez-Chacón, et al., 2019), was chosen to develop the design experiments involving university students. Then, some in-depth analysis on Advanced Mathematical Thinking researches were carried out. They are researches on mathematical thinking involving students from upper secondary school onwards, in particular, university students. On the other hand, backward reasoning is a typical thought of problem solving and programming methods. The researchers team chose to develop the design experiments tasks based on open-problems and strategy games. For this reason, the second part of the chapter is dedicated to these topics. In particular, two theme are highlighted: the notion of open-problem, on the basis of which the tasks have been chosen, and the procedural analogy between strategy games and problem solving, which allows to use the former in mathematics teaching for the learning of mathematical thinking and problem solving skills.

After presenting some general theoretical elements, *Backward reasoning in mathematics* (Chapter 3) displays the research carried out during phase 1 of the research project (Familiarization with the theoretical framework) and in particular the results of the in-depth literature review about backward reasoning, the focus of this research project, and all the type of directly related reasoning (forward reasoning and abductive reasoning). The chapter starts with a raw definition of backward reasoning to contextualize it. The second section concerns the historical definition elaborated by Pappus of Method of Analysis, of which backward reasoning is part. Here the controversies about the translation of Pappus' *Collectio* work and the close link between the analysis and synthesis methods are underlined. Then, a historical-philosophical overview of the ideas of mathematicians and philosophers, from

Ancient Greece to the present day, is displayed. From the analysis of those texts, the features of backward reasoning emerge (section 3.3.5). Then, backward reasoning is related to its forward counterpart and the importance of the introduction of auxiliary constructions in the process is pointed out. Later, the connexions between the backward and abduction reasoning are shown. Finally, backward reasoning is analysed in problem solving situations pointing out the resolution strategies that it underlies. The chapter ends with the rigorous definition of backward reasoning in the light of all the research elaborated in the previous sections.

In *Analysis tools of mathematical reasoning: Theoretical frameworks* (Chapter 4), the sequence of steps that led, over the three years, to the elaboration of the research project's theoretical framework is presented. As anticipated in the introduction, this has been developed with a networking of theories between the Game Theory Logic (Hintikka, 1999) and the Abstraction in Context theory (Dreyfus et al., 2015) and a subsequent hybridization with a fragment of Commognition approach (Sfard, 2008). The chapter begins with the definition of networking and hybridization strategies. Subsequently, the theories and the implementation of the strategies are shown. Finally, the chapter ends with the reworking and refining of the research questions in the light of the in-depth analysis of the theoretical elements displayed in the previous chapters and the theoretical framework elaborated in the preceding paragraphs.

For each chapter, a Table of Contents is shown to help the reader in approaching the chapter.

2

MATHEMATICAL REASONING

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MATHEMATICAL REASONING 2

It is well known that the vision of mathematics as a rigorous and deductive discipline dates back to Euclid. He formulated the first organic and complete representation of geometry in his fundamental work: *The Elements* (Στοιχεῖα: about 300 b.C.). The text is written in a rigorous hypothetical-deductive style, proving all its (465) theorems from a finite list of Postulates, Common notions (general logic principles), and Definitions: all the premises necessary to infer a theorem are so made explicit, sometimes employing some already proved properties that “apparently seem to have nothing to do with it” but which actually serve for its resolution. It is a model for the writing of mathematical texts that essentially still persists today.

But mathematics is not static and infallible; it develops by formulating conjectures, proposing and then analysing proofs with the formulation of examples and counterexamples. The axioms themselves have a conjectural and fallible character. The deductive transposition does not fully capture the sense of reasoning that is the way in which the concrete mathematical investigation proceeds (Byers, 2007). Formulas, definitions, typographical abbreviations do not follow the process of typical mathematical discovery reasoning, but they are convenient from a formal point of view. Euler before and Hilbert later, axiomatizing mathematics, do not highlight the activity of mathematical discovery, and say nothing about its evolution in their specific papers. Consequently, the textbooks never present the evolution that theories, theorems, and problems have had; they formulate, prove, and resolve them in a strictly deductive way (Peckhaus, 2002).

Lakatos (1976), in his work, initiates to crush the formal view of mathematics beginning to give more chances to the mathematics of discovery, and he affirms:

“According to formalists, mathematics is identical with formalised mathematics. But what can one discover in a formalised theory? [...] Informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations.” (Lakatos, 1976, p. 4)

The importance of discovery processes in mathematical reasoning has been recognized by researchers in Mathematics Education who take them into account from several perspectives. The field is very wide, and researchers deal with different aspects of the subject at different educational levels. In order to introduce this research project, it is considered important to briefly explore some aspects of mathematical reasoning at university level, problem solving and the use of strategy games in teaching.

2.1 **Advanced mathematical thinking**

Since 1985, with the birth of “Advanced Mathematical Thinking” Working Group at the International Group for the Psychology of Mathematics (PME) annual meeting, researchers in Mathematics Education field started to focus their attention on teaching and learning processes developed in environments with students from aged 16 and over, which was not previously contemplated. They realized that, as stated by Lakatos, in university lectures the notions are usually presented as a sequence of deductive steps, that contrasts with the mathematics they developed when doing research. Furthermore, they evidence that some university students had difficulties solving tasks because they are not accustomed to using the skills that are usually acquired through mathematical experience (Tall, 2002):

“There is a huge gulf between the way in which ideas are built cognitively and the way in which they are arranged and presented in a deductive order. This warns us that simply presenting a mathematical theory as a sequence of definitions, theorems and proofs (as happens in a typical university course) may show the logical structure of the mathematics, but it fails to allow for the psychological growth of the developing human mind.” (Tall, 2002, p. xiv)

They noticed that teaching processes changes from middle school to high school and even more to university. Teachers in primary education point out “the *synthesis* of knowledge, starting from simple concepts, building up from experience and examples to more general concepts” (Tall, 2002). However, in higher education general abstraction is usually the starting point of the lessons. Moreover, while the request that is made to pupils is to describe concepts and argue, students in higher grades are asked to define and prove concepts based on abstract entities. Advanced mathematical thinking, therefore, can be understood in two ways: as the elaboration of more advanced topics and as the implementation of more “complex processes, in which a large number of component processes interact in intricate ways” (Dreyfus, 2002).

Dreyfus (1990) states that an advanced mathematical concept is based on a structure formed by other (basics or not) concepts and their relationships. To understand it, it is necessary comprehend the entire structure and all the progression thought necessary for its construction. Different processes are involved in the construction of this network of knowledge, such as “abstraction, analyse, categorize, conjecture, define, formalize, generalize, proof or synthesize”. The same processes can be found in the works of lower level students, but these have a higher frequency in advanced mathematics. Processes of analysis and synthesis, in which backward reasoning is involved (Hintikka, 2012), are pointed out in advanced mathematical thinking due to their high level of abstraction.

A new concept can only be mastered at the end of an abstraction process. Abstraction begins with the mental representation of the concept in different forms. The second step generally consists in translating the representations one into the other by breaking away from the concrete mathematical situation in which they emerged. The third and final step is the incorporation of the concept into a complex mathematical structure. "If a student develops the ability to consciously make abstractions from mathematical situations, he has achieved an advanced level of mathematical thinking" (Dreyfus 2002). But to do it, he must be involved in discovery and knowledge development mathematical activities. These ideas about the abstraction processes are at the basis of the elaboration of the Abstraction in Context theory (Dreyfus et. al, 2015) which will be used to frame this research project, along the Game Theory Logic (Hintikka, 1999) and the Commognition approach (Sfard, 2008).

Several studies have been carried out over the years on the subject of advanced mathematical thinking (for a brief summary see Harel, Selden and Selden (2006) and Nardi (2017)). While some studies have a more general character (Artigue, Batanero and Kent, 2007; Jaworski, Robinson, Matthews, and Croft, 2012), or are developed on specific university subject (Carlson and Rasmussen, 2008; Castela and Romo Vazquez, 2011) other studies focus on different aspects of problem-solving, the theme of the next session. Among the others, for example, they focus on the study of difficulties in problem solving (Dringenberg and Purzer, 2018; Lithner, 2003; McNeill, Douglas, Koro-Ljungberg, Therriault and Krause (2016)), or problem-solving processes used in creating argument (Cellucci, 2017; Fukawa-Connelly, 2012; Koichu and Leron, 2015), or studies that are based on inquiry-based teaching and learning methodology at university level (Goodchild, 2014; Jaworski and Matthews, 2011; Laursen and Rasmussen, 2019; Rasmussen and Kwon, 2007; Rasmussen and Wawro, 2017)

2.2 Problem Solving and strategy games

Since the second half of the twentieth century, problem solving has been a subject dealt with by several authors in the field of Mathematics Education. It is considered the centre of the discipline, and it is even called "the heart of mathematics":

“The core of mathematics is problem solving. It has been rightly called the heart of mathematics because it is precisely this that attracts and continues to fascinate mathematicians of all ages” (Miguel De Guzmán in Corbalán (1994), p. 109, translated by the author).

In the following sections some problem solving notions are briefly shown. The procedural analogy with strategy games, which makes them a useful tool for learning, is highlighted.

2.2.1 Problem Solving

In Mathematics Education field, several authors (such as Polya (1945), Schoenfeld (1985), De Guzmán (1991), Lithner (2000), Santos-Trigo and Moreno-Armella (2013), Koichu

(2014), Liljedahl (2016); Cellucci (2017), etc.), have dealt with problem solving and the consequences of their use in teaching activities. In the didactic activity, in fact, different types of problems can be applied with different purposes. e.g. exercises in the application of mathematical concepts, modelling problems in order to mathematize a concrete situation, etc. Among them, a problem that allows to put into play different skills and knowledge is, what Arsac, Germain and Mante (1988) call, the “probleme ouvert” (open problem). An open problem must have three characteristics:

- A short statement, in order to be easily understood.
- The statement should not suggest either the methodology to be used to solve it or its solution. This allows the student to explore and walk different paths to reach the solution.
- The conceptual domain that underly it has to be appropriate to the student involved. The student can easily involve himself in the situation and obtain, at least, a first conjecture before the end of the activity.

The open problem resolution is not obvious, and the problem cannot be solved by direct application of any previously known mathematical results. The student has to get notions from his background, correlating and intertwining them in order to follow one of the multiple existing paths toward the solution (Corbalán, 1994 and 1997; Kiili, 2005). The purposes of open problem solving within a mathematics class are different (Savin-Baden and Major, 2004). They can be used, for example, for learning new knowledge in mathematical field or “to train students to ‘think creatively’ and/or ‘develop their solving abilities’” (Schoenfeld, 1992). The resolution structure of this kind of problems was defined by Polya in his work *How to solve it?* (1945); here, he specifies the four fundamental phases for problem solving development:

1. Understanding the problem
2. Devising a plan
3. Carrying out the plan
4. Looking back

During the first phase, the solver explores the problem observing the elements and thinking about the links between the problem and his own prior knowledge that can help in the resolution. The second and third phases can be repeated several times during resolution. As Garris et al. (2002) argue, thinking, solving and acquiring knowledge are not linear processes and do not happen in the same way for everyone. In problem solving, there are continuous changes between the design of the resolution plan, which includes the development of different strategies, and its implementation. The last phase concerns the comparison of the results obtained with the resolution context; it allows to evaluate the results obtained and verify that they are correct. It can be considered that the discovery processes occur in the first three phases of the resolution, while the last phase is mainly dedicated to the verification of the results obtained previously.

Solving a problem, different techniques and strategies are developed. Polya (1945) dedicates the most full-bodied part of his work to those, the heuristics notions:

“Heuristics or ‘ars inveniendi’ [art of finding] was the name of certain branch of study, not very clearly circumscribed, belonging to logic, or to philosophy, or to psychology, often outlined, seldom presented in detail, and as good as forgotten today. The aim of heuristic is to study the methods and rules of discovery and invention.” (Polya, 1945, p. 112)

Over the years, several authors, as such Schoenfeld (1979), Gascón Perez (1989), Corbalán (1994), have continued the research by reworking and defining more precisely different techniques of problem solving. From Corbalán’s (1994, 1997) works, eighteen problem solving strategies can be identified. A brief description follows their definition:

- Starting with something easy: solve a simpler problem
The solver faces with a complex problem that is beyond his capabilities, then a simpler problem is solved. Working on simpler cases, he may find patterns or regularities that can be generalized.
- Breaking the problem down into smaller problems
The solver decomposes the problem into a few smaller and easier problems. The combining of their resolutions lead to the solution of the initial problem.

- Reformulating the problem
The solver reformulates the initial problem with simpler data or in a way that is easier to solve.
- Solving similar problems (identify analogies)
The auxiliary problems introduced have a known solution related to the initial problem and/or a similar resolution structure.
- Start at the end, assume the problem solved
The author considers the two statements such as the description of a single strategy involving backward steps from the end of the problem. In the next chapter (Chapter 3, section 3.7) it will be shown that these are two distinct strategies: working backward strategy and beginning from the end of the problem strategy.
- Use an appropriate mathematical language: verbal, algebraic, graphic, numerical
This is a particularly important strategy. Finding a good way to express the procedures and results allows the solver to reproduce and manipulate them.
- Attempts and errors
The solver tries to develop a resolution option. If he hasn't come up with a solution, then he tries another way. It is a very common way to proceed, it leads to a solution, but it is not sure if that solution is the only one or the best.
- Making a systematic study of all cases
It is a research procedure that tries to minimize analysis time and avoid repetition of cases already analysed. With this strategy the solver examines all the situations.
- Analysing borderline cases
It consists of analysing the borderline cases of a problem and evaluating them to draw general conclusions.
- Experimenting and extracting patterns (practicing induction)
The solver starts with specific cases and tries to find a common rule. It allows to a generalization of the procedure that can be applied to all cases. This strategy may lead to a general formulation of the problem through a formula.
- Deduce consequences
It consists in reaching conclusions with logical deductive steps.
- Making conjectures
The solver makes explicit a possible conclusion from uncertain data.

- Reductio ad absurdum
It consists in denying a statement and following the reasoning to reach something incoherent and contradictory.
- Making diagrams, tables, drawings, graphic representations
These visual elements may help to develop the reasoning.
- Taking advantage of symmetry
Identifying a symmetry, the solver introduces visual reasoning in problem solving, even not necessarily in purely geometric contexts. It allows the globalization of thought processes, helping to bring together arithmetic and geometric procedures.
- Manipulate and manually experiment
It is a strategy used in problems where the solver has to build something or manipulate it.
- Pigeonhole principle
The principle can be resumed in this way: if there are m objects in n drawers ($m > n$), then at least one drawer must contain more than one object. It can be used to demonstrate unexpected results, such as “At least two people in Rome have the same number of hairs”.
- Follow a method, get organized
The solver organizes the problem resolution in order to proceed without impediment and, if necessary, to be able to review the work done.

Solvers apply one or more of these techniques in problem solving. They can be applied one at a time or in combination with the others; if the solver fails his goal, he possibly changes his strategy. In a previous study (Barbero, 2015), it was seen how changing strategy can help the solver to overcome the resolution difficulties. Solving problems is therefore useful for the development of mathematical reasoning and methods as well as for the acquisition of knowledge. The next session shows how also strategy games, thanks to the procedural analogy with problem solving, are beneficial for this purpose.

2.2.2 Strategy games: analogy with problem solving

Swan, in 2012, introduced the term “Gamification” to describe the process “of adding game mechanics to processes, programs and platforms that wouldn’t traditionally use such concepts. The goal is to create incentives and a more engaging experience.” This idea was quickly translated into teaching context especially for its enormous benefits.

“As we tried to solve the problems of higher education over lunch, it slowly dawned on me that Gamification could form the basis for shifting the value proposition offered by colleges and universities in a way that would embrace changing technologies and reflect the new economic realities.” (Niman, 2014, p. viii)

De Freitas & De Freitas (2013) have summarized the main advantages of gamification in the educational field highlighted in literature:

1. *Instant responses*: Gamification provides students with quick feedback that leads them to explore various options on their own and to consider error as part of the learning process (Kapp, 2012).

2. *Motivation*: The games were related to five different types of intrinsic reasons:

- The choice (between roles to be supported),
- The control (through the selection of requests and the choice of the completion order),
- The collaboration (through chat and group research),
- The challenge (through high-level content),
- The accomplishment (through levels, status, and skills) (Dickey, 2007).

3. *Focus on the learners*: Unlike traditional teaching methods, gamification seeks to stimulate student engagement through smart project choices and attention to content-focused on players (Jensen, 2012).

The same advantages illustrated by the authors can be found in the introduction of strategy games in educational activities (Delucchi, Gaiffi and Pernazza, 2012). Strategy games are board games or video games, solitaires or two (or more) players games. They have a fixed

set of rules that establish the objectives for all involved players. The players choose their own path to achieve the game goal. They develop tactics and strategies which generate procedures aimed at winning or at least at not losing. The choice of moves is based on all the game information available and all the knowledge or skills that everyone has. The goal of this type of games is the search for a winning strategy which generates a safe process to prevail, where luck plays a minimal role or even is completely absent in the process (Corbalán 1994, Gómez-Chacón, 1992).

The game activity proposes situations in which the research adopted to find the solution is very similar to that used when dealing with mathematical topics. There is a strong analogy between the design and implementation of strategy games and the problem solving (Barbero, 2015 and 2016; Corbalán, 1994 and 1997; Gómez-Chacón, 1988, 1990, 1992). Solving a game, in fact, the solver develops a logic reasoning and he thinks in a mathematical way (Barbero and Gómez-Chacón, 2018; Barbero, Gómez-Chacón and Arzarello, 2020).

The design of the most successful and interesting games is very close to modelling or simulation. Chess, for example, recreates a battle between two armies. These games are particularly attractive because they are models of idealized real and engaging situations. Besides that, Corbalán (1994) states that the design and resolution of the game have characteristics that follow the Euclidian axiomatic model. In fact, the latter starts from the assumption that there are a small number of self-evident results (axioms and postulates) and a series of fixed and explicit laws, starting from which the whole doctrinal body is formulated. Similarly, a game starts with the description of the initial conditions and some fixed rules; from that, the sequence of phases is developed. The movements that are realized can correspond to the distinct steps of the deduction, the partial strategies to the partial applications in mathematics, and the achievement of the general game strategy can be equivalent to new or consequent theorems. Winning a game or facing a game satisfactorily can therefore be equivalent to solving a problem in mathematics.

Moreover, the heuristic structures of problem solving and strategy games are similar. Both activate mental processes such as reading and interpreting data, representing, systematizing, formulating conjecture, selecting strategies and verifying of the strategies' effectiveness. They coincide at the level of thought processes activated for their solution. Precisely for this

reason many typical mathematical skills can be acquired through the game. (Gómez-Chacón, 1992)

Strategy games resolution can be schematized in a four-phases structure similar to the Polya four-phases structure explained above (De Guzmán (1994); Gómez-Chacón (1992)):

1. Familiarization with the game: before doing, try to understand
2. Initial exploration: research and design of strategies
3. Execution of the strategy: assessing whether the strategy leads to a conclusion
4. Reflection on the path followed: generalisation of the strategy developed

The analogy between the two resolution processes was identified by Gómez-Chacón (1992). The author states that:

- understanding what the problem requires corresponds to understanding the requirements of the game, she calls this phase "to read the problem or the rules of the game".
- both in problems and in games, strategies are sought, and conjectures are formulated, she calls this phase "to explore".
- subsequently, the resolution strategies are applied in both problems and games, she calls this phase "to carry out the strategy".
- finally, the phase of reflection on the generality of the strategies used in the problems corresponds to that of the strategy games, she calls this phase "to check the results"

The author summarizes this analogy (see table 2.1) by putting together the two heuristic processes and underlining the similarities between them. Moreover, by observing the strategy game resolution, the same problem solving strategies can be identified. During the analysis of the resolution protocols, the subdivision proposed by Gómez-Chacón will be used to split problem solving and strategy games resolution into phases.

Heuristic		
Problem Solving phases		Strategy Game phases
Understanding the problem	READ THE PROBLEM OR THE GAME RULES	Familiarization with the game
Devising a plan	EXPLORE	Initial exploration
Carrying out the plan	IMPLEMENT THE STRATEGY	Execution of the strategy
Looking back	CHECK THE RESULTS	Reflection on the path followed

Tab. 2.1 - Problem Solving and Strategy games heuristics (Gómez-Chacón, 1992)

The greater interest in strategy games for the research project development, rather than gamification of mathematical problems, lies in the fact that in games players strategic choices are triggered by typical implicit questions as “What can I do in this situation? What is better to do?” To answer them, they reflect both on moves already made and on possible moves, activating, in a natural way, backward ways of thinking (the backward reasoning will be discussed in the next Chapter) (Barbero, 2015; Corbalán, 1994 and 1997; De Guzmán 1984; Gómez-Chacón and Barbero, 2020). Brousseau (1998), using different terminology, illustrates this way of thinking in games during the famous discussion about the processes of students playing the “Race to 20” game. He demonstrates its utility in problem solving processes.

“The game is played by pairs of players. Each player of a pair tries to say “20” by adding 1 or 2 to the number given by the other. One of the pair starts by saying “1” or “2” (for example, “1”); the other continues by adding 1 or 2 to this number (“2” for example) and saying the result (which would be “3” in this example); the first person then continues by adding 1 or 2 to this number (“1” for example) and saying the result (which would be “4” in this example); and so on.” (Brousseau, 1998, p. 3)

The players use backward reasoning when, for example, they understand that if they want to reach “20” winning the game, they must reach before “17”, and before “14”, and “11”, and so on. This way of thinking is incentivized from the game structure itself. The solver is led

to ask himself "Where have I to be in order to reach 20?", starting a process of backwards reasoning that leads him to the solution of the problem.

Therefore, strategy games were chosen to come up beside mathematical problem in this research project. They provide a useful didactical tool to observe the epistemic dimension of backward reasoning and to compare its development in more strategic-centred contexts and in mathematical problems.

3

**BACKWARD REASONING
IN MATHEMATICS
(NOT TO MENTION FORWARD)**

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BACKWARD REASONING IN MATHEMATICS (NOT TO MENTION FORWARD)

3

3.1 Row definition

Different types of thinking are part of the backward reasoning. These reasoning processes start from the solution of the problem and develop a series of logical steps towards the premises, often not in a linear path. Once a proved premise has been obtained, the problem solution will be reached through the development of forward reasoning. In order to formally define this concept, an historical-philosophic research was performed, studying nowadays authors and going all the way back to ancient Greece. A formal backward reasoning definition (section 3.8) will be given only after illustrating all the in-depth literature research carried out.

3.2 The method of analysis

The typical mathematical thinking process that is used in the discovery phases is the backward reasoning. The origins of it date back in the Ancient Greece with the so-called “Method of Analysis”. Menn (2002) wrote about this method:

“The Method of Analysis had enormous prestige [in Ancient Greece] because it was seen as the basic method of mathematical discovery: not simply a way for a student to discover and assimilate for himself propositions already known to his teachers, but also a way for a mature geometer to discover previously unknown proposition.”
(pp. 194-195)

Precisely because of this great prestige, several authors, such as Proclus and other Platonists, attribute to Plato the invention of this method, but probably it was already used by Hippocrates of Chios (~430 BC). Greek mathematicians were rather reticent about the nature of the method, most likely because it did not provide a formally correct proof of a theorem or a problem to be solved (Hintikka, 2012; Hintikka and Remes, 1974). The method is not considered "rigorous" and does not serve to obtain a formal proof but is involved in the processes of discovery. Authors such as Descartes give it an esoteric character.

“It was synthesis alone that the ancient geometers usually employed in their writings. But in my view this was not because they were utterly ignorant of analysis, but because they had such a high regard for it that they kept it to themselves like a sacred mystery.”
(Descartes, *Philosophical Writings* (1985), [2]²)

Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his *Collectio* (~340 AD) he deals with the topic of Heuristics (methods to solve the problems). There he exemplifies the method of analysis and the method of synthesis, therefore making the development of this reasoning clearer. There are several translations of his work which differently affect the interpretation of the method and its study. Many authors as Polya (1945) or Jones (1986) translate the text of Pappus as follows:

“In analysis, we start from what is required, we take it for granted; and we draw consequence (ακολουθον) from it, and consequence from the consequence, till we reach a point that we can use as a starting point in synthesis. That is to say, in analysis

² “Definitions and Descriptions of Analysis” section in *The Stanford Encyclopaedia of Philosophy* (on-line version), recollects fragments of texts by 56 different authors for a total of more than 160 quotations. This is a supplement section to “Analysis” (Beaney, 2018). For each author, different excerpts of their works, focused on analysis and synthesis, are showed and ordered in a numbered list. For example, there are nine excerpts from Kant works numbered from 1 to 9: three from *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (1764), four from *Critique of Pure Reason* (1781, translation of 1997), and two from *Prolegomena to Any Future Metaphysics* (1783, translation of 1977). To cite these excerpts, or some parts of those, the author of this dissertation chose to show: the name of the author, the title of the work (in *Times New Roman Italic*), the date of publication or translation (if needed) and a number in brackets referring to the excerpt position in the list. So, for example, at the end of a quotation from the fifth excerpt of Kant works will appear: (Kant, *Critique of Pure Reason* (1997), [5]). The “Definitions and Descriptions of Analysis” section can be found at this link: <https://plato.stanford.edu/entries/analysis/s1.html>

we assume what is sought as already found (what we have to prove as true). We inquire from what antecedent the desired result could be derived; then we inquire again what could be the antecedent of that antecedent, and so on, until we come eventually upon something already known or admittedly true. This procedure we call analysis, as if to say *anapalin lysis*.” (personal re-elaboration from Polya, 1945 and Jones, 1986).

Where *anapalin lysis* (*αναπαλιν λύσις*) can be translated as: reduction backward, or solution backward, or regressive reasoning.

The essence of the method of analysis is to solving a problem begin from what is required in its formulation, then, developing logical consequence, until reach a point from which deductive steps are performed (using what Pappus calls the "method of synthesis"). Pappus's text continues with the description of two types of analysis, the analysis of "problems to prove" and the analysis of "problems to find" (following the translation of Polya).

“Analysis is of two kinds: the one is the analysis of the “problems to prove” and aims at establishing true theorems; the other is the analysis of the “problems to find” and aims at finding the unknown.” (Polya, 1945, p. 142)

The “problems to prove” are those problems where there is a statement and the solver wants to understand if this is true or false. The solver begins from this statement (we call it A) and derives a second statement (B), from B a third statement (C) and so on until reaching a certain statement (L) which is a definite knowledge. If L is false then also A will be false, but if L is true A will be true only if the solver proves it reversing the processes and deriving A from L.



Fig. 3.1 – Problem to prove path resolution

For example, consider this problem (De Guzmán, 1991):

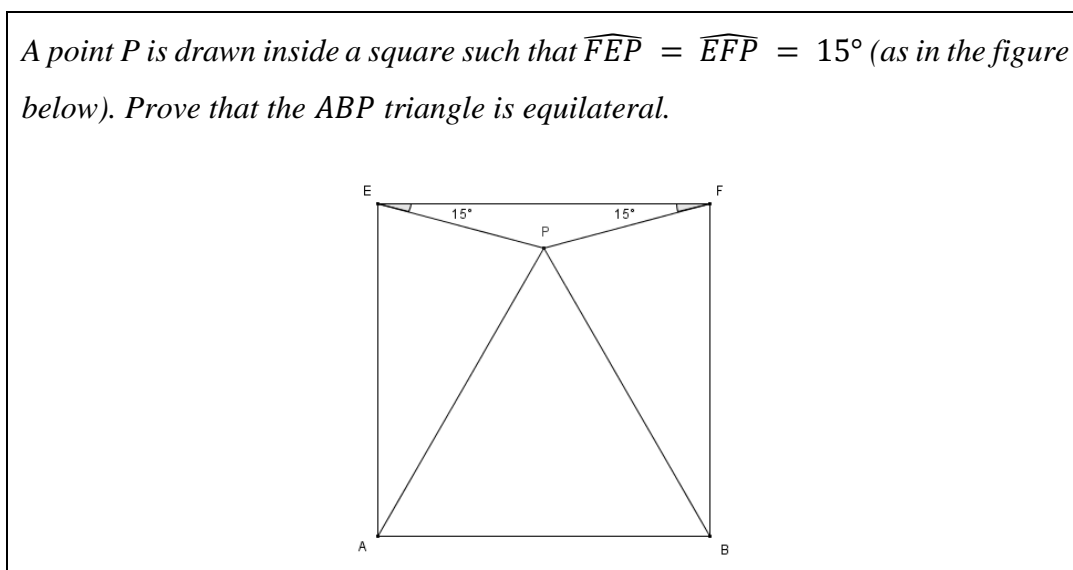


Fig. 3.2 - Mathematical problem to prove (De Guzman, 1991, p. 150)

A possible solution can be developed in the following way:

- Point P is uniquely determined.
- Inside the EFBA square we place a point Q so that the triangle ABQ is equilateral. Point Q is uniquely determined.
- Since the triangle QAB is equilateral, then its internal angles are congruent, and they measure 60° . Then the angles QAE and QBF are congruent and they measure 30° .
- Since the triangle QAB is equilateral, its sides are congruent; in particular the sides QA and QB are congruent to the side AB so they are congruent to EA and EB (sides of the square).
- Then the triangles QAE and QBF are congruent and isosceles. The angles QAE and QBF measure 30° . Then the QEA and QFB angles are congruent and measure 75° .
- Then, due to the difference of congruent angles, the FEQ and EFQ angles are congruent and measure 15° .
- As P and Q are uniquely determined points, then P and Q are the same point.
- So, APB is equilateral.

To prove this problem, the meaning of “equilateral triangle” is considered; then the reasoning proceeds logically from these found characteristic. An equilateral triangle has all

congruent sides and consequently congruent angles. Assuming the problem solved: an equilateral AQB triangle is considered, inside the square, and then a proof is developed until reach the hypothesis of the problem (QE and QF segments characteristics).

The “problems to find” are those questions in which the solver needs to look for a certain X which satisfy certain conditions. It imposes that X exists and derives an affirmation Y from X satisfying the same initial conditions, obtaining a subsequent affirmation (T) and so on until reaching a statement Z that derives from the previous and which satisfies the same starting conditions. Z is a statement that the solver can find, can develop, can derive with a known method. At this point, if Z exists then X also exists but one must prove it with some deductive logic steps, if Z does not exist then there is not even X and the problem has no solution.



Fig. 3.3 - Problem to find path resolution

An example is shown to better understand.

Find the measure of the sides of a right triangle such that the major side and the hypotenuse are respectively 7 and 8 cm larger than the minor side.

Fig. 3.4 - Mathematical problem to find

A possible solution can be developed in the following way:

- Assuming that a right triangle ABC, with data characteristics, exists and drawing it.

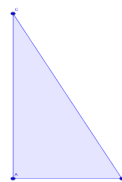


Fig. 3.5 - ABC triangle, problem to find example

- The vertices of the triangle are named, and the lesser side is identified with AB. It can be said that $AB = x$.

- Depending on the characteristics of the triangle drawn, the other sides may be expressed as combinations of the lesser side: $AC = x + 7$ and $BC = x + 8$.
- The triangle is supposed to be a right triangle. A consequence of its definition, the Pythagorean theorem, can be applied:

$$AB^2 + AC^2 = BC^2$$

$$x^2 + (x + 7)^2 = (x + 8)^2.$$

- Solving the equation, using the algebraic rules, the solver passes through a series of equivalent steps that allow the initial conditions to be maintained.
- Two solutions are obtained from the equation resolution:

$$x = -3 \vee x = 5$$

- Two AB conditions were found. These AB conditions satisfy the initial conditions.
- Since it is a geometrical triangle the measure of the side cannot be a negative value.
- Rejecting the negative solution and considering only the solution $x = 5$
- This allows to conclude that the measures of the three sides of the triangle ABC are

$$AB = 5, AC = 12 \text{ and } BC = 13$$

To solve this problem, firstly the sought triangle is supposed found and the solver draws it; then the reasoning proceeds logically from the triangle characteristics. The sides are expressed in algebraic language and the Pythagorean theorem is applied; in this way an initial condition for AB is found. Then the measures of the other sides are calculated. Starting from the end of the problem some characteristics are considered and developed; through this process an initial condition is found that lead to the problem solution.

As can be seen from the two examples shown, the translation of the word *ακολουθον* with “consequence” (see Polya and Jones translation above) does not fully express the characteristics of the method of analysis defined by Pappus because it seems to exclude all those regressive processes that emerge during the resolution and lead to obtain premises, initial conditions or hypotheses to the problem being solved.

3.2.1 The controversy of *ακολουθον*

As can be noticed, the word *ακολουθον* translated as “consequence” does not have all the meanings attributed to it by Pappus and gives to the method a forward directional character. Hintikka and Remes (1974) criticize this choice of lexicon adopted by most scholars and translate *ακολουθον* with “concomitant”, trying to point out that, starting from what is required by the problem, the logical steps that are going to be made are not logical consequence but are logical correspondences: they are statements that “go together with” the starting affirmation (A or X in the examples, see figures 3.2 and 3.3) of the resolution. During the reversal process, the solver can proceed with some logical deductive steps from one statement to another. The translation by Hintikka and Remes (1974) is:

“Now analysis is the way from what is sought—as if it were admitted—through its concomitants in order to reach something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards [*anapalin lysin*].” (p. 8)

In the description of the two analysis Hintikka and Remes name the resolution of a “problem to prove” *theoretical analysis* and the resolution of a “problem to find” *problematical analysis*. And it goes on:

“Now analysis is of two kinds. One seeks the truth, being called theoretical. The other serves to carry out what was desired to do, and this is called problematical. In the theoretical kind we suppose the thing sought as being and as being true, and then we pass through its concomitants (*ακολουθον*) in order, as though they were true and existent by hypothesis, to something admitted; then, if that which is admitted be true, the thing sought is true, too, and the proof will be the reverse of analysis. But if we come upon something false to admit, the thing sought will be false, too. In the problematic kind we suppose the desired thing to be known, and then we pass through its concomitants (*ακολουθον*) in order, as though they were true, up to something admitted. If the thing admitted is possible or can be done, that is, if it is what the

mathematicians call given, the desired thing will also be possible. The proof will again be the reverse of analysis. But if we come upon something impossible to admit, the problem will also be impossible.” (Hintikka and Remes, 1974, pp. 9-10)

The use of “concomitant” allows different types of logical steps to be included in the process. In addition to the logical steps that are a consequence of “what is sought”, all those that allow to advance in the resolution of the problem in different directions are included in the process. For example, the steps backwards, the steps towards the hypothesis of the problem or the abductive steps are included.

In summary, the following was considered the “Method of Analysis”: the practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical *correspondences* which allow to obtain something known or to be reached through other paths. The analytical method consists of a procedure that starts with the formulation of the problem and ends with the determination of the conditions for its solution.

3.2.2 Analysis vs Synthesis

After explaining what Analysis means, Pappus introduces the concept of Synthesis. Referring to the translation of Hintikka and Remes (1974):

“In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. And this we call synthesis.” (pp. 8-9)

Observing the previous lines, the two processes are closely related and there is no analysis method without the synthesis one. Solving a problem is therefore a combination of the two procedures. Peckhaus (2002) studies this analysis-synthesis scheme and affirms that “The analytical [is] [...] the procedure which starts with the formulation of the problem and ends with the determination of the conditions for its solution. The synthetical represents the way from the conditions to the actual solution of the problem. [...] This branch of the scheme is

deeply connected with the complementary [one].” Not only analysis can’t exist without synthesis but also “synthesis can’t be isolated and presupposes analysis.”

From this first definition of analysis and synthesis it can be observed that backward reasoning is closely intertwined with the forward one. An historical-philosophical study of philosophers and mathematicians works, from the Ancient to the Contemporary Age, is shown in next section in order to extrapolate the fundamental features of backward reasoning and to define it rigorously.

3.3 Theoretical perspective of backward reasoning

Different authors have addressed the issues of *analysis* by giving their own definition. An in-depth literature review allows to observe the evolution of the concept throughout history, and to identify some common features that characterise its backward reasoning component. Many references from Ancient Age, Modern Age and Contemporary Age, were found (see Note 1 in section 3.2).

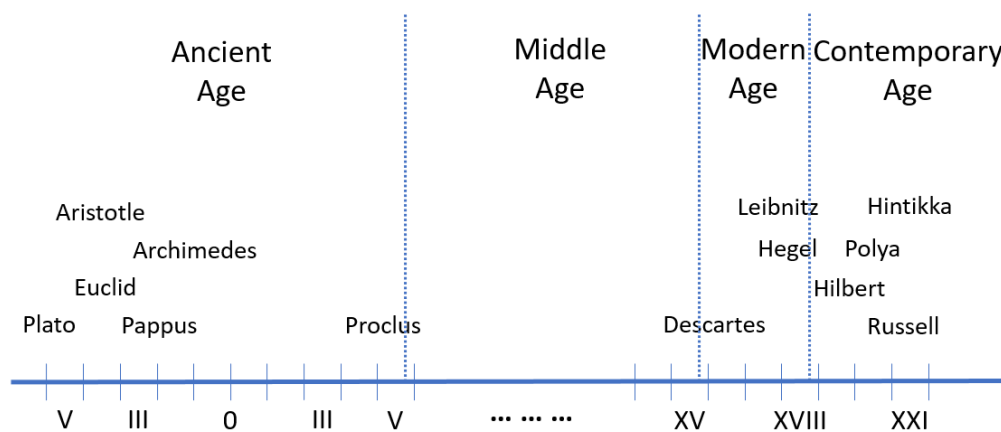


Fig. 3.6 - Some authors from Ancient, Modern and Contemporary Age

The authors of the Ancient Age (Aristotle, Plato, Pappus, Proclus, ...) mainly emphasize the regressive character of the analysis. They conceive analysis as the process of working backwards to find the principles of the problem; which can be taken as a basis to prove the problem itself (with the method of synthesis). In the Modern Age, authors such as Descartes,

Hegel, Leibniz, ..., conceive the analysis in its character of breakdown, that is to say as a process where a concept is broken down into its primary elements, which allow to make evident its logical structure. The authors of the Contemporary Age (Frege, Russell, Moore, Wittgenstein, ...) focus their attention on the analysis of statements and their translation into the correct logical form, they focus on what Beaney (2018) calls “transformative and interpretative dimension of analysis”. The three concepts are the different sides of the same construct. In fact, solving a problem consists in interpreting an entity, translating it in mathematical language, identifying its relevant elements and finding its principles. These processes lead to something known from which progressively move forward.

3.3.1 Ancient Age

As seen in the previous paragraphs, the method of analysis has its roots in the geometry of ancient Greece, in which it is used as a methodology to solve problems. The definition of Pappus (see section 3.2) influences all the authors of the Modern and Contemporary Ages. It focuses on the backward feature of the method, strongly relating it with the inverse process of synthesis. Different authors highlighted other characteristics.

3.3.1.1 Backward direction

Different authors, such as Alexander of Aphrodisias, Euclid, Proclus, refer to the method as the inverse of the synthesis. This entails going backward from the end of the problem to its beginning. By applying the method, the premises of a certain idea are sought. The method is interpreted by Aristotle as an instrument to follow the order of the questions to be used in the proof process.

“For geometers are said to analyze when, beginning from the conclusion they go up to the principles and the problem, following the order of those things which were assumed for the demonstration of the conclusion” (Alexander of Aphrodisias, *Commentary on Aristotle’s Prior Analytics* (1960), [1])

Plato, although he never uses the word "analysis", describes this procedure backwards in his dialogues:

“For we should remember that if a person goes on analyzing names into words, and inquiring also into the elements out of which the words are formed, and keeps on always repeating this process, he who has to answer him must at last give up the inquiry in despair ... But if we take a word which is incapable of further resolution, then we shall be right in saying that we at last reached a primary element, which need not be resolved any further.” (Plato, *Collected Dialogues* (1966), [1])

Proclus states that this procedure of searching for relationships from what is sought is developed on the way back to the basic principles. This path is prolonged as much as possible, not in an infinite search but until the bases of the problem are found, in order to show its complexity. And it makes an example of the possible methods found in Euclid's *Elements*, underlining the regressive character of the analysis:

“[Euclid’s *Elements*] contains all the dialectical methods: the method of division for finding kinds, definitions for making statements of essential properties, demonstrations for proceeding from premises to conclusions, and analysis for passing in the reverse direction from conclusions to principles.” (Proclus, *A Commentary on the First Book of Euclid's Elements* (1970), [4])

3.3.1.2 Breakdown

Despite the focus on the breakdown feature is very strong in Modern Age, some Ancient Age authors notice it. Proclus, Aristotle and Plato, talk about reducing something to its simplest components, or to extract the basic principles with which it is composed, to identify the properties that define it. To make this concept clearer, Aristotle refers to figures (“diagrams”) and their use in geometrical problems. Sometimes, the only thing that can be done is to decompose the figure into its basic components and to understand the connections between them.

“Sometimes it happens with diagrams; for there we can sometimes analyse the figure, but not construct it again.” (Aristotle, *Sophistical Refutations* (1928), [1])

3.3.1.3 A historical example from the Ancient Age

Pythagoras' theorem proof by Euclid (Elements, Book I, proposition 47) (translated by S.T. Heath, 1908, pp. 349-350) is showed here to point out the characteristics of the method of analysis highlighted in the previous paragraphs.

The Elements, Book I, proposition 47

In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

Let ABC be a right-angled triangle having the angle BAC right.

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square BE, and on BA, AC the squares GB, HC. [1.46]

Through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined.

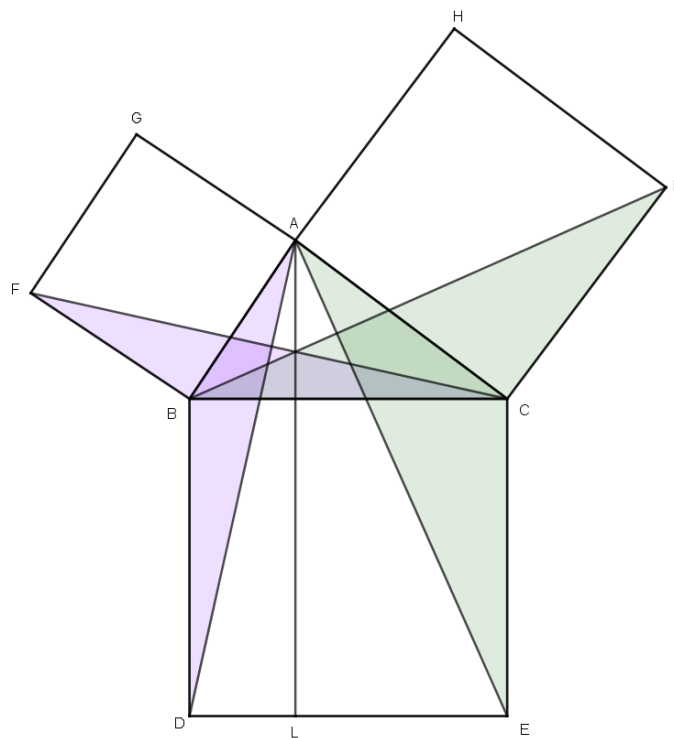


Fig. 3.7 - Euclid construction of Proposition 47, Book I

Then, since each of the angles BAC , BAG is right, it follows that with a straight line BA , and at the point A on it, the two straight lines AC , AG not lying on the same side make the adjacent angles equal to two right angles; therefore CA is in a straight line with AG . [Proposition 1.14]

For the same reason BA is also in a straight line with AH . And, since the angle DBC is equal to the angle FBA : for each is right: let the angle ABC be added to each; therefore, the whole angle DBA is equal to the whole angle FBC .

And, since DB is equal to BC , and FB to BA , the two sides AB , BD are equal to the two sides FB , BC respectively, and the angle ABD is equal to the angle FBC ; therefore the base AD is equal to the base FC , and the triangle ABD is equal to the triangle FBC . [Proposition I.4]

Now the parallelogram BL is double of the triangle ABD , for they have the same base BD and are in the same parallels BD , AL . [Proposition 1.41]

And the square GB is double of the triangle FBC , for they again have the same base FB and are in the same parallels FB , GC . [Proposition 1. 41]

[But the doubles of equals are equal to one another.]

Therefore, the parallelogram BL is also equal to the square GB .

Similarly, if AE , BK be joined, the parallelogram CL can also be proved equal to the square HC ; therefore, the whole square BE is equal to the two squares GB , HC . [Common Notion 2]

And the square BE is described on BC , and the squares GB , HC on BA , AC .

Therefore, the square on the side BC is equal to the squares on the sides BA , AC .

Therefore etc.

Q. E. D.

Euclid applies the method of analysis. Different characteristics can be noticed:

- Euclid starts with the end of the problem, assuming that there are three squares built on the sides of a right triangle;
- He builds auxiliary lines (FC , DA , AL , BK and AE) between particular points, so he obtains a complex figure;

- He analyses the simplest elements that make up the complex figure: quadrilaterals and triangles;
- The analysis of the elements of the figure allows to find relationships between angles, segments, triangles, areas;
- By investigating the properties of these figures, he deduces the relationships between the relevant areas until he proves the theorem.

This demonstration is a good example to observe the backward features of the method of analysis. Euclid starts from the end of the problem, breaking the figure down into parts and finding basic properties of the complex figure. The proof is based on preliminary theorems (congruence of triangles and areas between parallels) that have to be known before.

3.3.2 Modern Age

The concept of analysis in the Middle Age has been influenced by the ideas of the Ancient Age authors. According to Beaney (2018) knowledge was filtered through comments that were not always reliable. During the Renaissance, the original texts of the ancient Greeks began to be taken up again and interest in knowledge about analysis was revived.

In Modern Age, with the emergence of new mathematical techniques, the authors return to think about the concept of analysis, maintaining and developing the most ancient roots. This period is characterized for an in-depth exploration of knowledge about methodologies, in this sense, the method of analysis begins to be seen as a method of discovery. The focus on the breakdown character of analysis is dominant. More strength is given to the relationship between analysis and synthesis. The latter is seen as a method of testing; it involves forward processes from what has been discovered through analysis, to the problem goal. The two methods are complementary and have different purposes.

The text *La logique, ou l'art de penser, contenant, outre les règles communes, plusieurs observations nouvelles propres à former le jugement* (1662), by Antoine Arnauld and Pierre Nicole, mainly known as the Logic of Port Royal, emphasises the idea of analysis as a methodology:

“The art of arranging a series of thoughts properly, either for discovering the truth when we do not know it, or for proving to others what we already know, can generally be called method.

Hence there are two kinds of method, one for discovering the truth, which is known as analysis, or the method of resolution, and which can also be called the method of discovery. The other is for making the truth understood by others once it is found. This is known as synthesis, or the method of composition, and can also be called the method of instruction.” (Arnauld and Nicole, *Port-Royal Logic* (1964), [1])

The text distinguishes four main methods involved in the analysis: “seeking causes by their effects, seeking effects by their causes, finding the whole from the parts, and looking for another part from the whole and a given part” (Beaney, 2018). The first two refer to the backward character, while the last two refer to the breakdown character of analysis. These methods can be derived from the thirteenth rule of *Rules for the Direction of the Mind* by Descartes (1623-1929). Here Descartes highlight the breakdown character of analysis method:

“If we perfectly understand a problem we must abstract it from every superfluous conception, reduce it to its simplest terms and, by means of an enumeration, divide it up into the smallest possible parts.”

As for the authors of the Ancient Age, some characteristics are highlighted from the reading of the Modern Age texts.

3.3.2.1 Breakdown

As already emphasized, the authors of this historical period focus on the breakdown character of the analysis. The greatest exponent of that time is surely Descartes. In his greatest work *Le discours de la méthode* (1637), he states that he adopted four rules in his scientific work. Here it can be possible to see that the focus has shifted from the backward character to the breakdown character. The relationship between analysis and synthesis is maintained, and made more explicit:

“The first was never to accept anything as true if I did not have evident knowledge of its truth: that is, carefully to avoid precipitate conclusions and preconceptions, and to include nothing more in my judgements than what presented itself to my mind so clearly and so distinctly that I had no occasion to doubt it.

The second, to divide each of the difficulties I examined into as many parts as possible and as may be required in order to resolve them better.

The third, to direct my thoughts in an orderly manner, by beginning with the simplest and most easily known objects in the order to ascend little by little, step by step, to knowledge of the most complex, and by supposing some order even among objects that have no natural order of precedence.

And the last, throughout to make enumerations so complete, and reviews so comprehensive, that I could be sure of leaving nothing out.” (Descartes, *Discourse on the Method* (1985), in Beaney (2018))

Several authors highlight distinct processes involved in the breakdown of a concept:

- Subdivision of a complex entity into simple entities

“Since the object of mathematics in general is magnitude and that of geometry in particular extension, one can say that in mathematics in general our concepts of magnitude are unpacked and analyzed, while in geometry in particular our concepts of extension are unpacked and analyzed. [...] This truth also lay tangled up, as one might say, in the original concept of extension, but it escaped our attention and could not be distinctly known and distinguished until, through analysis, we unpacked all the parts of this concept and separated them from one another.” (Mendelssohn, *Philosophical Writings* (1997), [1])

- Separation of the general concept into basic concepts

“There are two ways in which one can arrive at a general concept: either by the arbitrary combination of concepts, or by separating out that cognition which has been rendered distinct by means of analysis.” (Kant, *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (1764), [1])

- Searching for entity's properties

“The true method of metaphysics is basically the same as that introduced by Newton into natural science and which has been of such benefit to it. Newton’s method maintains that one ought, on the basis of certain experience and, if need be, with the help of geometry, to seek out the rules in accordance with which certain phenomena of nature occur.” (Kant, *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (1764), [2])

- Searching for the causes of a phenomenon

“By this way of Analysis we may proceed from Compounds to Ingredients, and from Motions to the Forces producing them; and in general, from Effects to their Causes, and from particular Causes to more general ones, till the Argument end in the most general. This is the Method of Analysis: and the Synthesis consists in assuming the Causes discover’d, and establish’d as Principles, and by them explaining the Phænomena proceeding from them, and proving the Explanations.” (Newton, *Opticks* (1952), [1])

“And by this same method of resolving things into other things one will know what those things are, of which, when their causes are known and composed one by one, the causes of all singular things are known. We thus conclude that the method of investigating the universal notions of things is purely analytic.” (Hobbes, *Computatio sive Logica* (1981), [2])

As underlined by different authors such as Hegel, Kant, Mendelssohn, Newton, etc., through the process of breakdown a deeper understanding of the notions is reached. For a complete understanding, Kant (*Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (1764), [1]) suggests to comparing the different characteristics of concepts, at the end of the process of decomposition. These characteristics must be compared with the original concept and in different contexts.

“The concept has to be analysed; the characteristic marks which have been separated out and the concept which has been given have to be compared with each other in all

kinds of contexts; and this abstract thought must be rendered complete and determinate. [...] One starts with what is the most difficult: one starts with possibility, with existence in general, with necessity and contingency, and so on – all of them concepts which demand great abstraction and close attention.” (Kant, *Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality* (1764), [1])

Leibniz states that going towards the principles is something “necessary” for the resolution of a problem. He suggests two types of analysis: one in which intuition is involved and the second he calls “reductive”. The latter he said is “less known” but is necessary for the resolution of a problem. He relates the process of analysis to that of synthesis and, in the end, prefers to apply the latter insofar; according to him there are fewer difficulties in its use.

“Analysis, however, goes back to principles solely for the sake of a given problem, just as if nothing had been discovered previously, by ourselves or by others. It is better to produce a synthesis, since that work is of permanent value, whereas when we begin an analysis on account of particular problems we often do what has been done before. However, to use a synthesis which has been established by others, and theorems which have already been discovered, is less of an art than to do everything by oneself by carrying out an analysis; especially as what has been discovered by others, or even by ourselves, does not always occur to us or come to hand. There are two kinds of analysis: one is the common type proceeding by leaps, which is used in algebra, and the other is a special kind which I call ‘reductive’. This is much more elegant, but is less well-known. In practice, analysis is more necessary, so that we may solve the problems which are presented to us; but the man who can indulge in theorising will be content to practice analysis just far enough to master the art. For the rest, he will rather practise synthesis, and will apply himself readily only to those questions to which order itself leads him. For in this way he will always progress pleasantly and easily, and will never feel any difficulties, nor be disappointed of success, and in a short time he will achieve much more than he would ever have hoped for at the outset.” (Leibniz, *Of Universal Synthesis and Analysis* (1973), [1])

3.3.2.2 Analysis vs Synthesis

The strong relationship, already existing in the authors of the Ancient Age, between analysis and synthesis is retook in much more detail in the works of the Modern Age. The logic of Port-Royal underlines the fact that the process of analysis is necessary to discover the truth while that of synthesis is useful to explain what it was found with the analysis. As hinted above (see section 3.3.2) Arnauld and Nicole (*Port-Royal Logic* (1964), [2]), develop four different points talking about this relationship. In the first place, they affirm that in both cases the methods start from what they know and proceed to what they want to know.

“This is what we call analysis or resolution. We should notice, first, that in this method - as in the one called composition - we should practice proceeding from what is better known to what is less known. For there is no true method which could dispense with this rule.” ([2])

In the second point they underline the differences between the two methods by stating that in the method of analysis one arrives at general truths through a step-by-step procedure, starting from some truths; unlike what happens in synthesis.

“Second, it nevertheless differs from the method of composition in that these known truths are taken from a particular examination of the thing we are investigating, and not from more general things as is done in the method of instruction. [...] Instead we rose by stages to these general notions.” ([2])

In the third point they dwell on the introduction of "evident maxims" or axioms. They affirm in fact that, while in the synthesis method the process starts from the exposition of all the necessary axioms for the development of the resolution, in the analysis these axioms are introduced only if strictly necessary for the development of the problem.

“Third, in analysis we introduce clear and evident maxims only to the extent that we need them, whereas in the other method we establish them first, as we will explain below.” ([2])

In the fourth point, the authors emphasize that the substantial difference between the two methods is their direction. The method of analysis proceeds backwards, from the end of the problem to the premises while the method of synthesis develops in the opposite direction. They particularly point out this fact using two very clear examples: a mountain path that can be walked in both directions, mountain-valley or mountain-valley, and on the family tree that can be composed and read from the branches (the descendants) or from the trunk (the ancestors).

“Fourth and finally, these two methods differ only as the route one takes in climbing a mountain from a valley differs from the route taken in descending from the mountain into the valley, or as the two ways differ that are used to prove that a person is descended from St. Louis. One way is to show that this person had a certain man for a father who was the son of a certain man, and that man was the son of another, and so on up to St. Louis. The other way is to begin with St. Louis and show that he had a certain child, and this child had others, thereby descending to the person in question. This example is all the more appropriate in this case, since it is certain that to trace an unknown genealogy, it is necessary to go from the son to the father, whereas to explain it after finding it, the most common method is to begin with the trunk to show the descendants. This is also what is usually done in the sciences where, after analysis is used to find some truth, the other method is employed to explain what has been found.”

([2])

3.3.2.3 Introduction of auxiliary notions

Several authors have considered the auxiliary notions that may emerge in resolution processes: we have already met some of them in previous paragraphs. These notions do not appear explicitly in the formulation of the problem but reside in the solver’s acquired knowledge. Theorems or properties that are in his background emerge while facing situations in in which they are necessary. The new elements may be geometrical (in a construction), or new variables (in an analytical problem). The emergence of these auxiliary notions is characteristic of backward reasoning.

“The difference is that the primary notions which are presupposed for the demonstration of geometrical truths are readily accepted by anyone, since they accord with the use of our senses. Hence there is no difficulty there, except in the proper deduction of the consequences, which can be done even by the less attentive, provided they remember what has gone before. Moreover, the breaking down of propositions to their smallest elements is specifically designed to enable them to be recited with ease so that the student recalls them whether he wants to or not.” (Descartes, *Philosophical Writings* (1985), [2])

In this regard Kant makes a very explanatory example:

“Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze [zergliedern] and make distinct the concept of a straight line, or of an angle, or of the number three, but he will not come upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angles that together are equal to two right ones. Now he divides the external one of these angles by drawing a line parallel to the opposite side of the triangle, and sees that here there arises an external adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.” (Kant, *Critique of Pure Reason* (1997), [7])

3.3.2.4 Cause-Effect relationships

From the 17th century onwards, authors such as Descartes and Hobbes began to interpret the method of analysis not only by referring to its directional character but also to the processes of knowledge. Assuming that the method is an advantageous way of thinking, through an

interpretation of the knowledge being investigated, general concepts can be established in a methodical way.

“Seeing that the causes of all singulars are composed from the causes of universals or simples, it is necessary for those who are looking simply for scientific knowledge, which consists of the knowledge of the causes of all things insofar as this can be achieved, to know the causes of universals or those accidents which are common to all bodies, that is, to every material thing, before they know the causes of singular things, that is, of the accidents by which one thing is distinguished from another. Again, before the causes of those things can be known, it is necessary to know which things are universals. But since universals are contained in the nature of singular things, they must be unearthed by reason, that is, by resolution.” (Hobbes, *Computatio sive Logica* (1981), [2])

The method allows to show how things are discovered, and what are the connections between the background notions of the problem and the problem itself. “The analysis allows us to see how the effects depend on the causes” (Descartes, 1637, *Discourse on the Method*, in Beaney (2018)). For example, in order to heat a room, it is necessary to light a fireplace that needs wood to burn. The breakdown of the problem into basic notions allows to grasp the knowledge of the effects and the cause of each notion involved in the process.

3.3.2.5 A historical example from the Modern Age

A very explanatory text of these kind of processes is Descartes' *La Géométrie* (1637). It involves the breakdown of complex problems into simple problems, and the use of algebra to develop geometrical notions and solve geometrical problems. The introduction of analytical geometry allows to transform geometrical problems into arithmetic problems that are easier to solve through algebraic representations. The representation of an unknown geometrical entity (X) plays a central role in analysis. This is the idea of the ancients of taking an entity as something given and working backwards from there.

An explanatory example from Descartes' *Geometry*, the “general method of drawing a straight line making right angles with a curve at an arbitrarily chosen point upon it”, is

showed to point out the characteristics of the method of analysis highlighted in the previous paragraphs. Descartes states about this problem that “this is not only the most useful and most general problem in geometry [that I know], but even that I have ever desired to know (Descartes, *The Geometry*, translated by Smith and Latham, 1954, pp. 94-96). He starts the problem with a figure, then he constructs the line; in order to make it more understandable, in fig. 3.8b, the point B is shown, it was not included in the original drawing.

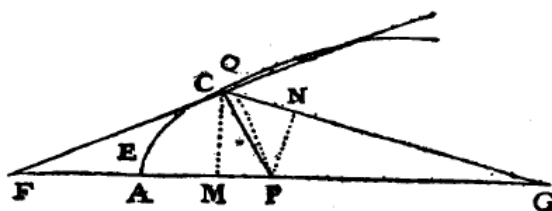


Fig. 3.8 - Descartes' construction (Descartes, 1954, p. 94)

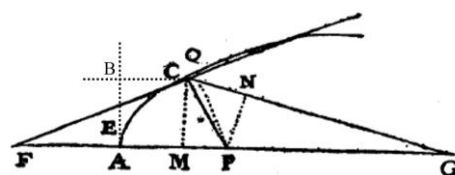


Fig. 3.8b - Descartes' construction with point B

Let CE be the given curve, and let it be required to draw through C a straight line making right angles with CE. Suppose the problem solved, and let the required line be CP. Produce CP to meet the straight line GA, to whose points the points of CE are to be related. Then, let $MA = CB = y$; and $CM = BA = x$. An equation must be found expressing the relation between x and y . I let $PC = s$, $PA = v$, whence $PM = v - y$. Since PMC is a right triangle, we see that s^2 , the square of the hypotenuse, is equal to $x^2 + v^2 - 2vy + y^2$, the sum of the squares of the two sides. That is to say, $x = \sqrt{s^2 - v^2 + 2vy - y^2}$ or $y = v + \sqrt{s^2 - x^2}$. By means of these last two equations, I can eliminate one of the two quantities x and y from the equation expressing the relation between the points of the curve CE and those of the straight line GA. If x is to be eliminated, this may easily be done by replacing x wherever it occurs by $\sqrt{s^2 - v^2 + 2vy - y^2}$, x^2 by the square of this expression, x^3 by its cube, etc., while if y is to be eliminated, y must be replaced by $v + \sqrt{s^2 - x^2}$, and y^2 , y^3 , ... by the square of this expression, its cube, and so on. The result will be an equation in only one unknown quantity, x or y .

Some characteristics of the backward reasoning developed in this problem solution are now highlighted:

- Assume the problem solved and draw the “normal line”, that is what is to be found in the solution of the problem.
- Assuming that the entity x exists.
- Interpret the geometrical entity in algebraic terms.
- Make auxiliary constructions of triangles. They allow applying known notions for the development and resolution of the problem.
- Breakdown the auxiliary construction to find properties.

This problem is a good example to observe the backward features of the method of analysis. Descartes starts supposing that the problem solution exists (x), then adds auxiliary elements and breaks down the construction. All the backward reasoning features identified are shown in this example.

3.3.3 Contemporary age

The scientific and philosophic research during the Modern Age, mainly focused on the analysis and its breakdown characteristics, contributing to debates between contemporary authors. Idealism and romanticism criticize this vision describing this type of analysis as destructing and degrading. On the contrary, Husserl and Russell, claim the breakdown characteristics and perform studies and research on this topic (Beaney, 2018). In general, authors of the Contemporary Age focus on the transformative and interpretative dimension of the analysis, in which the logical analysis plays a key role.

As for the authors of the Ancient and Modern Ages, some characteristics that emerge from reading the texts of the authors of the Contemporary Age are showed.

3.3.3.1 Backward direction

Several authors as Frege, Russell, Husserl focus their attention on mathematics foundations. Husserl is looking for a “radical foundation of mathematics”. To do so, he tries to isolate the

“essences” at the base of the concept, the elementary concepts, the roots from which all the elements can be rigorously deduced.

“It was my great teacher, Weierstrass, who during my university years gave birth in me, with his lectures on function theory, the interest for a radical foundation of mathematics. I became deeply sensitive to his efforts to transform analysis, which was a mixture of rational thought and irrational instinct and intuition, into a purely rational theory. He aimed to highlight the original roots, the elementary concepts and axioms on the basis of which the entire system of analysis could be constructed and deduced according to a completely rigorous and absolutely perspicuous method.” (Husserl, in Schuhmann (1997), p.7, author's translation)

3.3.3.2 Breakdown

Particularly bound to the decomposition characteristics of the analysis are the Neo-Kantian authors as Cassirer, who underlined the crucial role of the identification of a structure in the conceptual experience. Also analytic philosophers, such as Moore, highlight this characteristic. For example, in his first works, he uses the breakdown conception developed in Modern Age. He tries to reduce complex concepts to basic and constituent ones.

“It seems necessary, then, to regard the world as formed of concepts. These are the only objects of knowledge. They cannot be regarded fundamentally as abstractions either from things or from ideas; since both alike can, if anything is to be true of them, be composed of nothing but concepts. A thing becomes intelligible first when it is analysed into its constituent concepts.” (Moore, *The Nature of Judgement* (1993), [1])

These basic concepts are not definable but only perceivable because they are so simple that it is impossible to describe them. He, the describe when aa definition occurs and how to find the simple terms that compose the complex ones.

“My point is that ‘good’ is a simple notion, just as ‘yellow’ is a simple notion; that, just as you cannot, by any manner of means, explain to any one who does not already know it, what yellow is, so you cannot explain what good is. Definitions of the kind that I was asking for, definitions which describe the real nature of the object or notion

denoted by a word, and which do not merely tell us what the word is used to mean, are only possible when the object or notion in question is something complex. You can give a definition of a horse, because a horse has many different properties and qualities, all of which you can enumerate. But when you have enumerated them all, when you have reduced a horse to his simplest terms, then you no longer define those terms. They are simply something which you think of or perceive, and to any one who cannot think of or perceive them, you can never, by any definition, make their nature known.” (Moore, *Principia Ethica* (1903), [3])

3.3.3.3 Transformative and interpretative dimension

Not only backward and breakdown features are part of the analytical processes. It is an intrinsic characteristic of the process itself that is to interpret what is being analysed, inside a certain theoretical frame. To be analysed, the concept itself, sometimes, undergoes a transformation in order to be better interpreted. A striking example is the analytic geometry, where the geometrical elements are transformed and interpreted as algebraic elements. Here, a geometrical problem is transformed into a proposition through algebra and arithmetic. The development of the quantification theory allowed an easier “translation” (Beaney, 2018).

“For the mathematician, it is no more right and no more wrong to define a conic as the line of intersection of a plane with the surface of a circular cone than to define it as a plane curve with an equation of the second degree in parallel coordinates. His choice of one or the other of these expressions or of some other one is guided solely by reasons of convenience and is made irrespective of the fact that the expressions have neither the same sense nor evoke the same ideas. I do not intend by this that a concept and its extension are one and the same, but that coincidence in extension is a necessary and sufficient criterion for the occurrence between concepts of the relation that corresponds to identity [Gleichheit] between objects.” (Frege, *The Frege Reader* (1997), [1])

Frege attention is focused on the interpretation of the mathematical statements under the light of universal laws and logic. Frege states that the concepts are abstract entities that have their own logic structure, more or less complex, and a certain truth value. In order to be able

to study them a logic instrument that studies the connections in between these entities is needed. Hence, he creates the first predicate logic system that gives a notation to the quantification theory. The possibility to translate the concepts into logic language, allows a linguistic analysis of the expressions and implies a deep understanding of the expressions themselves.

“We have a simple sign with a long-established use. We believe that we can give a logical analysis [Zerlegung] of its sense, obtaining a complex expression which in our opinion has the same sense. We can only allow something as a constituent of a complex expression if it has a sense we recognize. The sense of the complex expression must be yielded by the way in which it is put together. That it agrees with the sense of the long established simple sign is not a matter for arbitrary stipulation, but can only be recognized by an immediate insight. No doubt we speak of a definition in this case too. It might be called an ‘analytic definition’ [‘zerlegende Definition’] to distinguish it from the first case. But it is better to eschew the word ‘definition’ altogether in this case, because what we should here like to call a definition is really to be regarded as an axiom.” (Frege, *The Frege Reader* (1997), [3])

Russell, on the other hand, defines the analysis as part of the discovery processes. He conceives two different types of analysis: the one that allows the comprehension of the components of a concept; and the one that allows the comprehension of the connections and combinations of those components.

“Analysis may be defined as the discovery of the constituents and the manner of combination of a given complex. The complex is to be one with which we are acquainted; the analysis is complete when we become acquainted with all the constituents and with their manner of combination, and know that there are no more constituents and that that is their manner of combination. We may distinguish formal analysis as the discovery of the manner of combination, and material analysis as the discovery of the constituents. Material analysis may be called descriptive when the constituents are only known by description, not by acquaintance.” (Russell, *Theory of Knowledge: The 1913 Manuscript* (1984), [10])

Afterwards he specifies that the analysis processes start from a set of knowledge with no clear logic interdependence. Though these processes, it is possible to reduce the knowledge to simple proposition logically bounded.

“The nature of philosophic analysis ... can now be stated in general terms. We start from a body of common knowledge, which constitutes our data. On examination, the data are found to be complex, rather vague, and largely interdependent logically. By analysis we reduce them to propositions which are as nearly as possible simple and precise, and we arrange them in deductive chains, in which a certain number of initial propositions form a logical guarantee for all the rest.” (Russell, *Our Knowledge of the External World* (1914), [16])

3.3.3.4 A historical example from the Contemporary Age

In order to better understand the backward direction of reasoning and highlight its characteristics, we focus on an example proposed by Polya (1945), a non-mathematical clarification of reasoning.

“A primitive man wishes to cross a creek; but he cannot do so in the usual way because the water has risen overnight. Thus, the crossing becomes the object of a problem; ‘crossing the creek’ is the x of this primitive problem. The man may recall that he has crossed some other creek by walking along a fallen tree. He looks around for a suitable fallen tree which becomes his new unknown, his y . He cannot find any suitable tree but there are plenty of trees standing along the creek; he wishes that one of them would fall. Could he make a tree fall across the creek? There is a great idea and there is a new unknown; by what means could he tilt the tree over the creek?

This train of ideas ought to be called analysis if we accept the terminology of Pappus. If the primitive man succeeds in finishing his analysis, he may become the inventor of the bridge and of the axe. What will be the synthesis? Translation of ideas into actions. The finishing act of the synthesis is walking along a tree across the creek.

The same objects fill the analysis and the synthesis; they exercise the mind of the man in the analysis and his muscles in the synthesis; the analysis consists in thoughts, the synthesis in acts. There is another difference; the order is reversed. Walking across the

creek is the first desire from which the analysis starts and it is the last act with which the synthesis ends.” (Poya, 1945, p. 145)

It is clear, from Polya’s words the backward reasoning direction. The man starts from the end of his problem, go across the creek and, with a series of steps, goes back to the beginning of his problem: where to chop a tree of the right dimensions in order to use it as a bridge above the creek. But also the transformative dimension appears in this example. In fact, representing the “thing sought” of the problem as x or y is the starting process to transform the problem in an algebraic one.

3.3.4 Another common feature throughout history

A common feature highlighted from several authors throughout history, is that analysis is a research process that allows to reach an understanding of the proposed problem, the understanding of its components and properties.

Plato on this point affirms that it is the understanding that allows to advance in the process until arriving at the basic principles of the ideas. Proclus insists on the fact that this method, with other mathematical processes, is employed in the mediation of ideas and in the understanding of concepts, it allows to proceed from what is best known to what is unknown. The analysis forms part of the dialectic and contributes to the intellectual processes that allow the understanding of mathematics:

“Being thus endowed and led towards perfection, mathematics reaches some of its results by analysis, others by synthesis, expounds some matters by division, others by definition, and some of its discoveries binds fast by demonstration, adapting these methods to its subjects and employing each of them for gaining insight into mediating ideas. Thus, its analyses are under the control of dialectic, and its definitions, divisions, and demonstrations are of the same family and unfold in conformity with the way of mathematical understanding. It is reasonable, then, to say that dialectic is the capstone of the mathematical sciences. It brings to perfection all the intellectual insight they contain, making what is exact in them more irrefutable, confirming the stability of

what they have established and referring what is pure and incorporeal in them to the simplicity and immateriality of Nous, making precise their primary starting-points through definitions and explicating the distinctions of genera and species within their subject-matters, teaching the use of synthesis to bring out the consequences that follow from principles and of analysis to lead up to the first principles and starting-points.” (Proclus, 1970, *A Commentary on the First Book of Euclid's Elements*, [2])

Hobbes underlines that the comprehension is possible only through the notion’s breakdown. Afterwards, it will be possible to recompose the acquired knowledge in order to comprehend the whole concept.

“The analytic method is needed for understanding the circumstances of the effect one by one; the synthetic method for putting together those things which, single in themselves, act as one.” (Hobbes, *Computatio sive Logica* (1981), [2])

On this point, Kant states that the breakdown method, led by intuition, allows to reach a clear general solution of the problem.

“In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question.” (Kant, *Critique of Pure Reason* (1997), [7])

Frege, supporter of the transformative and interpretative dimension, reconfirms that a logic analysis allows to clearly comprehend the meaning of an expression.

“The effect of the logical analysis of which we spoke will then be precisely this – to articulate the sense clearly.” (Frege, *The Frege Reader* (1997), [3])

Finally, Russel sees the analysis as the only method that allows to progress in the knowledge processes.

“I remain firmly persuaded, in spite of some modern tendencies to the contrary, that only by analysing is progress possible.” (Russell, *My Philosophical Development* (1959), [21])

This feature is very interesting from the point of view of a mathematics education research. This type of reasoning, in fact, comes into play in learning processes.

3.3.5 Characteristics of backward reasoning according to the historical-philosophical overview

Based on the analysis of backward reasoning meaning in different historical moments and by different mathematicians, that we presented in the previous historical-philosophical overview (sections 3.3.1-3.3.3), we have elaborated four epistemic reasoning dimensions that will be key to define the backward reasoning concept and are useful for the reading of the design experiments data (Chapter 5) These categories (*cause-effect relationships research, breakdown, introduction of auxiliary elements, and transformation and interpretation*) will be used during the resolution protocols analysis of the four design experiments (Chapters 6-10).

- *Direction vs. Cause-Effect.*

In Pappus' definition, the backward direction of this type of reasoning is highlighted, that is, going back from the end of the problem to the beginning of it. Applying the method, the premises of a certain idea are sought. In the 17th and 18th centuries this idea changes. Authors such as Arnauld and Nicole interpreted the reasoning as a search for a cause-effect relationship between ideas. This means to identify how the ideas are discovered and which are the connections between the background notions of the problem and the problem itself. In this sense, this process allows the knowledge of the development of the resolution and of the effects and causes of each notion involved in the process. Later on, authors such as Husserl and Frege collect the regressive conception of reasoning, focusing their attention on the search for the foundation of mathematics, which can be translate into the search for the basic principles of mathematics from the general concepts (Beaney, 2018; Peckhaus, 2002).

- ***Breakdown.***

According to Plato and Pappus, backward reasoning involves actions that allow the problem to be reduced to its simplest components. Extracting and investigating the principles that are at the basis of the task, allows to identify the properties that define it. This breakdown shows the relationships between the most complex objects and the simple ones. Aristotle, for example, underlines the fact that "sometimes, to solve a geometrical problem you can only analyse a figure", break it down into its basic components and understand the parts by which it is formed. The concept of breakdown is the focus of research carried out in the Middle Ages and of some authors of the Contemporary Age (Beaney, 2018).

- ***Auxiliary elements.***

Kant, Polya and Hintikka, and other authors, focus their attention on a fundamental part of the process: the introduction of new elements. Unlike the forward and deductive processes, in which the solver begins with all the bases and from these the consequences are elaborated, in the backward reasoning the notions appear and develop along the resolution in specific moments, according to the needs of the solver (Beaney, 2018; Hintikka&Remes, 1974).

- ***Transformation and interpretation.***

Already in the Middle Ages, with the birth of analytical geometry, and more in the Contemporary Age, with the birth of analytical philosophy, authors such as Descartes, Frege and Russell questioned the role of this type of reasoning in the interpretation and translation of concepts. Backward reasoning involves processes of transformation of entities, for instance those geometrics entities that are translated into algebraic expressions. (Beaney, 2018)

3.4 Backward reasoning vs Forward reasoning

To solve a problem, then, we use two types of reasoning that are combined, which can be named backward reasoning and forward reasoning. There are two questions that arise during

the resolution of a problem, as Ruesga Ramos et al. (2004) affirm: using backward reasoning the question that one poses to himself is “What should I consider to get ...?” while using forward reasoning is “What can I get when I have ..?”. The resolution of an ideal problem can be represented as follows (where the green arrows represent the steps of forward reasoning and the red ones the steps of backward reasoning):

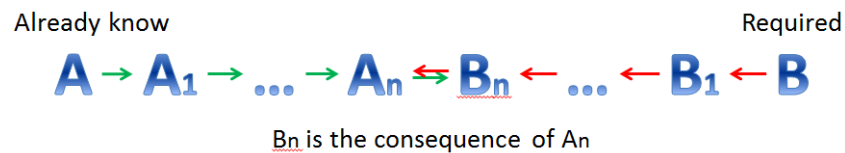


Fig. 3.9 - Ideal resolution problem scheme

If the solver knows A and he wants to demonstrate, or construct, or obtain B he can proceed as follows. He can use forward reasoning to move from A , something that is known, through a series of deductive logic chains until reach an A_n affirmation. Doing this process, he poses the first type of question, for examples he can proceed asking to himself “What can I get when I have A ?”. He finds the answer “ A_1 ” and so he can continue with “What can I get when I have A_1 ?”, and so on. Instead, he uses backward reasoning starting from B and retrograding in a series of logic chains to a B_n statement. In this case he can proceed asking himself “What should I consider, to get B ?”. The two statements that he obtains (A_n and B_n) are a consequence one of the other.

The possibility to reverse the logic chain process from B to B_n allows demonstrating the theorem $A \rightarrow B$ with a series of deductive, forward reasonings. This is an *ideal* example of reasoning, actually the forward and backward chains alternate during the resolution in a series of more complex logic steps. This combination of forward and backward reasoning is well expressed in the words of Arnauld and Nicole:

“This is the way to understand the nature of analysis as used by geometers. Here is what it consists in. Suppose a question is presented to them, such as whether it is true or false that something is a theorem, or whether a problem is possible or impossible; they assume what is at issue and examine what follows from that assumption. If in this examination they arrive at some clear truth from which the assumption follows necessarily, they conclude that the assumption is true. Then starting over from the end

point, they demonstrate it by the other method which is called *composition*. But if they fall into some absurdity or impossibility as a necessary consequence of their assumption, they conclude from this that the assumption is false and impossible.” (Arnauld and Nicole, *Port-Royal Logic* (1964), [2])

An example of a resolution problem is shown to better understand the A-B sequences. To do it the problem already mentioned in Introduction chapter (Chapter 1, section 1.1) will be considered (Arzarello, 2014) (in Figure 3.10 the problem assignment).

Problem

$f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function; $\lim_{x \rightarrow +\infty} f(x) = +\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$

Prove that there is at least one point c such that $f(c) = 0$

Fig. 3.10 - IVT problem

Given the problem, the initial link and the final link of the reasoning chain can be identified:

- A: “ $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function; $\lim_{x \rightarrow +\infty} f(x) = +\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ”
- B: “there is at least one point c such that $f(c) = 0$ ”.

Considering B, some questions arise: “How do you prove that there is at least one point c such that $f(c) = 0$?” “What characteristics do I have to consider in order to conclude that the point c exists?”. Instead, considering A, other kind of questions occur: “What consequences can I gain from the fact that the function is continuous?” “What does it mean that the function has two infinite limits?”

An expert mathematician can solve this problem immediately relating it with the Intermediate Value Theorem:

- B: there is at least one point c such that $f(c) = 0$
- B₁: there is at least one point c in an interval $[x', x'']$ such that $f(c) = 0$
- B₂: B₁ is the conclusion of the particular case of the Intermediate value theorem: $f: [x', x''] \rightarrow \mathbb{R}$ is a continuous function; $f(x') < 0 < f(x'')$ then there is some value c in $[x', x'']$ such that $f(c) = 0$.

- A: $\lim_{x \rightarrow +\infty} f(x) = +\infty$
- A₁: for any positive integer M , there is a value N so that for all $x > N$, $f(x) > M$
- A₂: exist x'' so that $f(x'') > M > 0$
- A: $\lim_{x \rightarrow -\infty} f(x) = -\infty$
- A₃: for any negative integer H , there is a value Q so that for all $x < Q$, $f(x) < H$
- A₄: exist x' so that $f(x') < H < 0$
- A₅: Joining A₂ and A₄: $f(x') * f(x'') < 0$
- A₆: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function so $f: [x', x''] \rightarrow \mathbb{R}$ is a continuous function
- A₇: Joining A₅ and A₆ is possible apply the particular case of the intermediate value theorem

At this point A₇ and B₂ are connected and it is possible say that they are a consequence of each other.

3.4.1 Ideal resolution vs Real resolution

The figure 3.8 shows the problem resolution ideal flow. Starting the problem from point A, the solver is able to reach point A_n through a series of forward steps. Afterwards, starting from point B, which is the end of the problem, he is able to reach point B_n through a series of backward steps. At this point the solver observes that B_n is a consequence of A_n and he is able to obtain B inverting the series of backward steps.

In general, a problem resolution does not follow a linear path. For instance, the IVT problem resolution described in the previous paragraph, as it was shown, might seem it could be summarised with the following scheme:



Fig. 3.11 - Linear flow IVT problem

Actually, the development of the final solution is not that linear, not even in this simple case. It would be possible to represent it with a tree, what Polya (1968) calls “the geometric

representation of the solution”. It is a visual representation where at the top of the tree there are the problem premises/hypothesis and at the bottom there is the solution. All the space between the premises and the solution is filled by the more or less linear chain of reasoning, the movement between the reasoning steps are represented with arrow: green for the forward steps and red for the backward ones. The tree should be travelled from the top to the bottom, and vice versa, performing a series of choices with a finite set of possibilities. Figure 3.11 shows the tree of the expert mathematician IVT problem resolution (it is relative to a resolution process auto-analysis of the author).

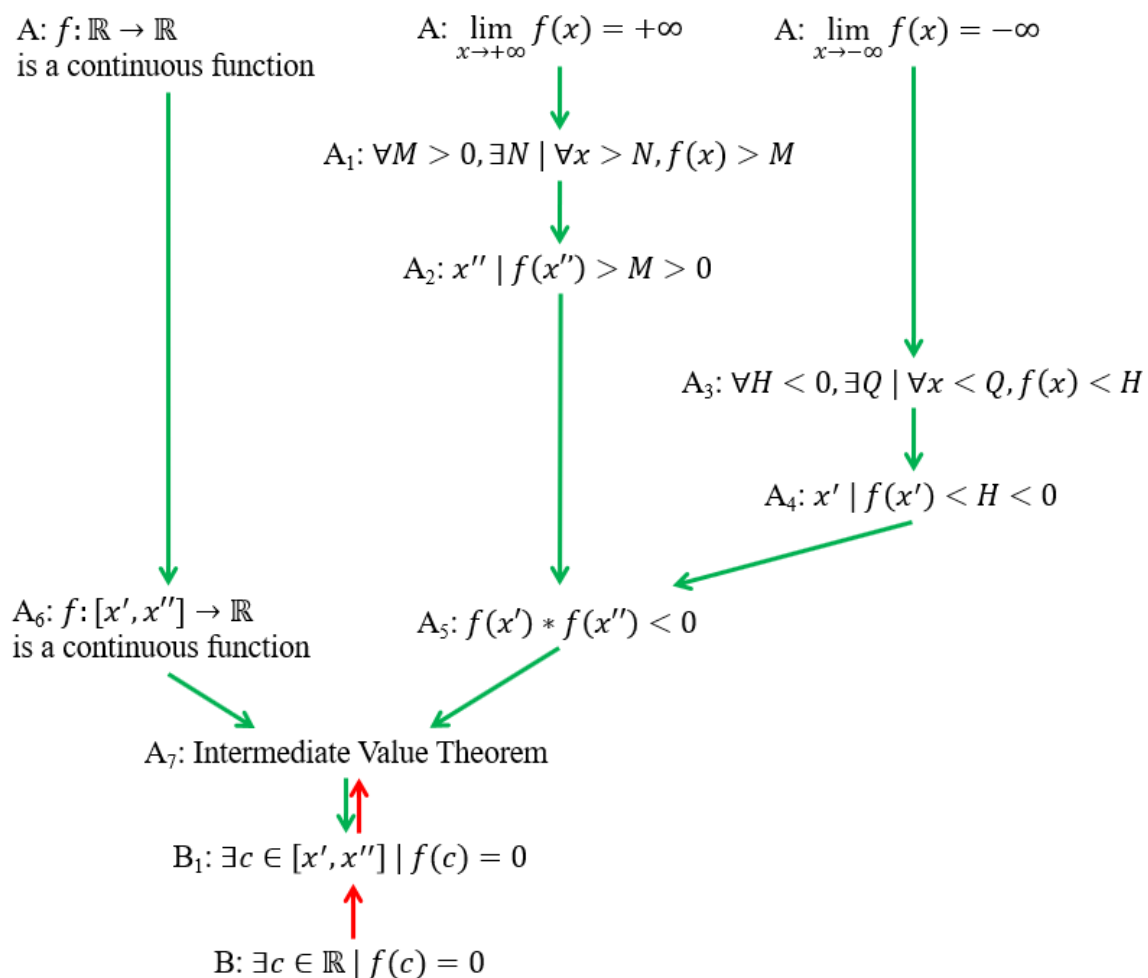


Fig. 3.12 - Tree scheme of IVT problem expert resolution

The resolution process might be very different for a not so expert solver, maybe more tangled and with more forward and backward steps. The same thing happens to expert solvers during the resolution of a complex problem. For instance, looking at the protocol of a student that is solving this same problem (it was used by Arzarello (2014) to illustrate the Finer Logic of

Inquiry Model) we observe that the student does not follow a linear path as an expert solver, instead develops a tangled path performing forward and backward steps during his resolution scheme.

The student solves this problem in this way:

- B: there is at least one point c such that $f(c) = 0$
- B₁: it seems that is possible to use the IVT
- A₁: there is at least one point c in an interval $[x', x'']$ such that $f(c) = 0$
- B₂: it is necessary to find $[x', x'']$
- A: $\lim_{x \rightarrow +\infty} f(x) = +\infty$
- A₂: for any positive integer M , there is a value x_M so that for all $x > x_M$, $f(x) > M$
- B₃: exist M
- A'₂: for any positive integer M , there is a value x_M so that for all $x > x_M$, $f(x) > M$
- B₄: $M > 0$
- A₃: exist $x' > x_M$ so that $f(x') > M > 0$
- A: $\lim_{x \rightarrow -\infty} f(x) = -\infty$
- A₄: exist $x'' < x_N$ so that $f(x'') < N < 0$
- A₅: Joining A₃ and A₄: $f(x') * f(x'') < 0$
- B'₂: the interval $[a, b]$ of the IVT is $[x', x'']$
- A'₃: exist $x' > x_M$ so that $f(x') > M > 0$
- A'₅: Joining A₃ and A₄: $f(x') * f(x'') < 0$
- A₆: $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function so $f: [x', x''] \rightarrow \mathbb{R}$ is a continuous function
- B'₁: now is possible to use the IVT
- A'₁: there is at least one point c in an interval $[x', x'']$ such that $f(c) = 0$

The student reasoning processes representation can be schematised as shown in figure 3.12, representing the linear flow, and in figure 3.12 representing the tree flow. The linear flow is a condensed representation of the student resolution.



Fig. 3.13 - Linear representation student's flow resolution (IVT problem)

The tree scheme obtained is a procedures tree much more complicated with respect to the one that should schematize the reasoning of an expert solver (see figure 3.11).

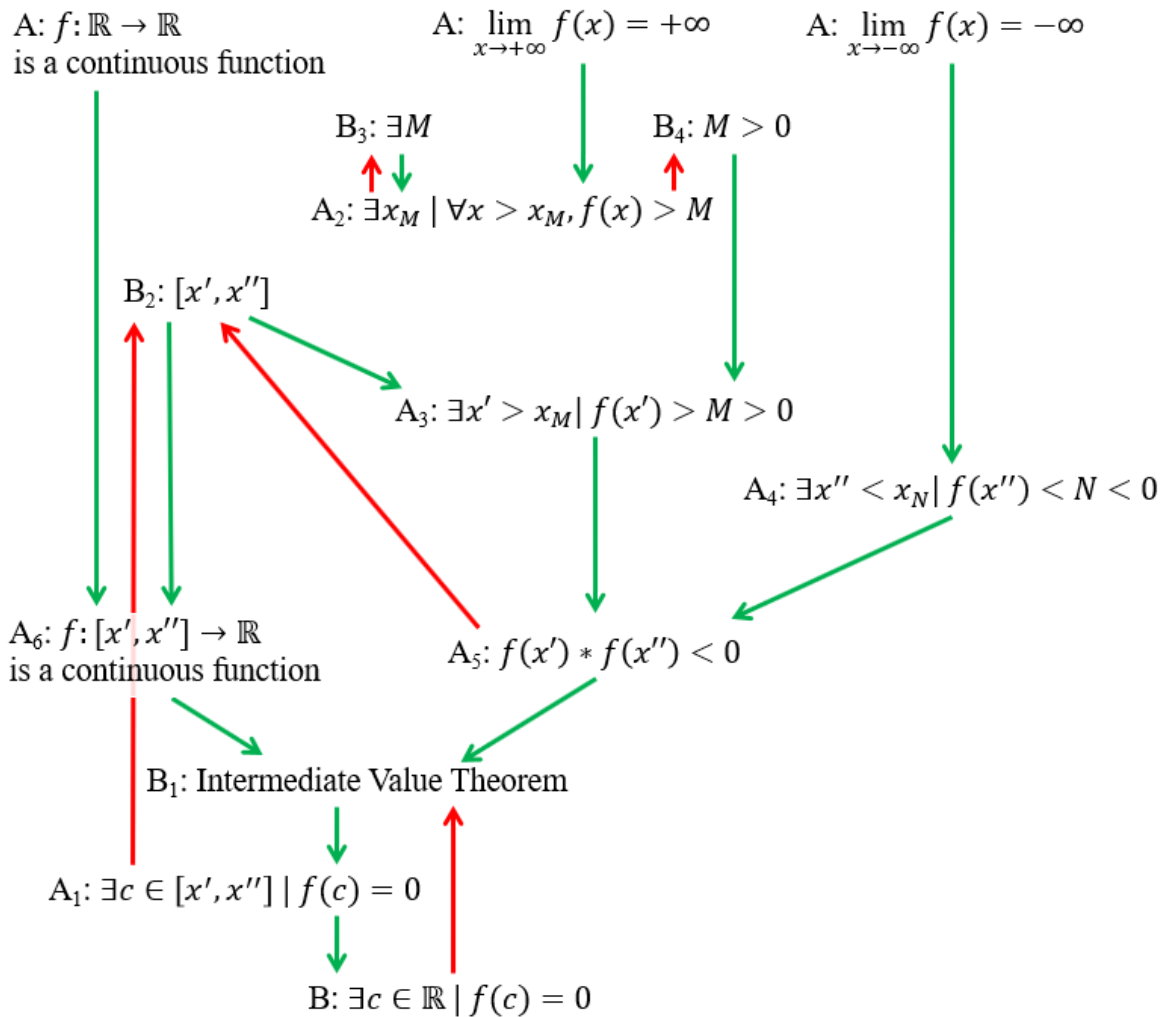


Fig. 3.14 - Student tree resolution (IVT problem)

The comparison between the two resolution trees allows to observe that the process of discovery is not linear but proceeds through different ramifications. In less experienced solvers these ramifications are more complicated than in experienced resolvers: backward and forward reasoning are deeply intertwined.

3.4.2 The importance of asking questions

Considering the previous example, the importance to ask the right questions, in order to solve the problem, emerges. The solver starts to reason about the element he is considering and asks himself questions, that generally are not verbally expressed. To solve a problem, specific elements are taken into consideration, such as: premises, problem solution, a schematic routine element characteristic of that certain problem type, etc. Hence, the questions might be different from a solver to the other depending on his knowledge; the useless details (for each solver) are neglected and the focus is moved on the important points of the resolution. Considering step B of the IVT problem, “*there is at least one point c such that $f(c) = 0$* ”. This statement *can* imply a series of subsequent statements such as: the function will have an upward/downward trend in a certain interval, the function is positive/negative in a certain interval, etc. The solver will be led to choose, among the several options, the one that is closer to his objective: demonstrate the given theorem.

In backward reasoning situations, the solver asks himself about the possible previous step or about the characteristics/properties of the solution. This is named by Solow (1990) “ask an abstract question”. He states that a well formulated abstract question should not contain any symbol or notation relative to the specific problem but should solely refer to the general knowledge. It is crucial to correctly formulate them in order to be able to solve the problem drawing information from the right knowledge. In relation to the IVT problem, it is said before that the expression “*there is at least one point c such that $f(c) = 0$* ”, can be triggered by a different knowledge. Relate it to the IVT theorem this means formulating the right question.

The answer process consists of two phases (Solow, 1990). An abstract phase: answering the question exploiting the knowledge in a general way. For example, saying “it is a particular case of IVT”. And a second phase with the application of the general knowledge to the specific problem. In the example, applying the steps to the hypothesis proposed.

The forward reasoning process develops in a forward way. It occurs when certain premises are combined to obtain some consequences, Hintikka e Hintikka (1982) represent this type

of reasoning with a scheme where p_1, p_2, \dots are the premises and c_1, c_2, \dots are the consequences (see figure 3.14).

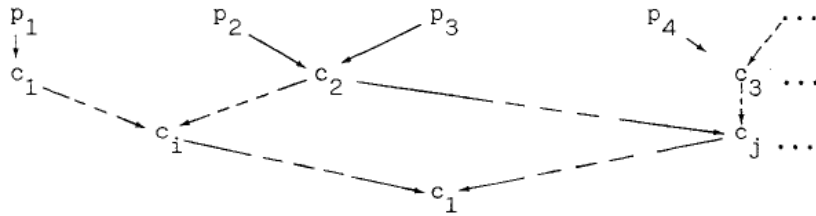


Fig. 3.15 - Deductive scheme (Hintikka and Hintikka, 1982, p 58)

However, this scheme cannot be applied to those phases of problem solving in which backward reasoning comes into play. Focusing on the importance of questions, they propose a new scheme (see figure 3.15) that manages to interpret those processes in which an unknown object/an auxiliary element is brought to reality through a question (q_1, q_2, \dots). This is typical of backward reasoning processes. In fact, the different phases of the process B_1, B_2, \dots described above, are not logical consequences of B. On the contrary, they are often the fundamental auxiliary elements introduced in the resolution. The role of the questions is, therefore, to activate that tacit knowledge that allows to make new elements come true. An appropriate question can then extract information from the subject's background knowledge and for example allow him to formulate premises for certain statements (left part of figure 3.15), or in combination with certain statements draw some conclusions (right part of figure 3.15); in the latter case the question arises as a result of an observation or an experiment.

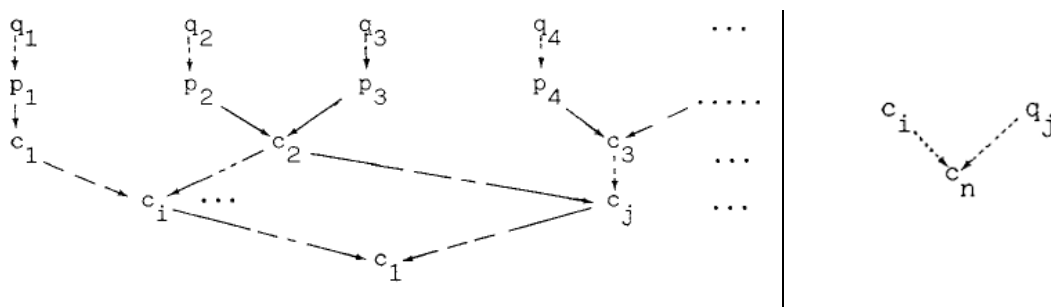


Fig. 3.16 - Questioning process schemes (Hintikka and Hintikka, 1982, p. 61) on the left questioning process to find some premises, on the right questioning process combining knowledge from a consequence

3.5 Auxiliary constructions

Hintikka and Remes (1974) wrote an entire chapter about the auxiliary constructions that can be useful to understand their role in backward processes, even if the authors focalised their attention in geometrical problems. Auxiliary constructions are one or more additional elements that come into play during a resolution process; in geometrical problems these elements are interconnected with the problem geometrical construction. To solve the problem, the solver has to reach a final construction from that, with the synthesis, he arrives to the solution. To create the final construction, (s)he travels intermediate steps: the auxiliary constructions come into play during this process, introducing new logical elements.

As mentioned in the previous paragraphs, the problem resolution happens through a combination of backward and forward processes. The possibility to resolve the problem is granted only if the backward steps can be inverted in some way. In fact, the problem is solved only when the premises can be connected to the objective with a series of logical step. Auxiliary constructions come into play in backward steps, where they are hypothesised. Solving a geometrical problem, the desired construction is assumed true and new interdepend elements are introduced. The new elements need to be reversible. The particularity of auxiliary constructions introduced in geometrical problems is that, even if they are added in relation to the problem construction, they can be independent, and they do not compromise the desired solution. For Hintikka e Remes (1974), this is the main distinction between backward and forward way of thinking: geometrical constructions emerge in backward steps and they are then used to solve/demonstrate the problem in forward steps.

The backward reasonings (whether of a backward, breakdown or transformative nature) involve all the geometrical elements including both the initial and the auxiliary constructions. Breakdown nature manifests itself when the solver divides the solution into simpler elements; backward directional nature manifests itself when the solver looks for the premises for the construction; while the transformative nature manifest itself when the solver transform the geometric language into an algebraic one.

The same discourse made on geometric problems can be extended to any kind of problem involving backward reasoning; depending on the problem the auxiliary elements may have a different nature from geometrical construction. For instance, the new element introduced may also involve auxiliary theorems that are introduced in order to connect the premises to the theorem objective. With regard to geometrical problems, Hintikka e Remes (1974) state that new theorems can be introduced as auxiliary elements, they involve more complex geometrical configurations with respect to those included in the problem premises. This way to introduce auxiliary constructions does not change problem resolution method.

The analytic method structure has often been considered as a heuristic method and has not been analysed in logical terms. Hintikka and Remes (1974) state that, given the not trivial nature of mathematic logical truth, the analytical method cannot be mechanized as a discovery procedure because of the necessity to introduce countless unpredictable auxiliary constructions.

3.6 **Abduction**

The previous paragraphs emphasize the fact that the ability to introduce new elements into an argumentation is typical of backward reasoning. C. S. Peirce (1932) explains how technically is possible to enrich the argumentation with these new hypotheses and individuals. He identified, besides classical deductive and inductive reasoning, a third type of reasoning, which he called abduction (or hypothesis). The same example made by Peirce is here used to explain it (CP 2.623):

Suppose we know that a bag is full of white beans. We see white beans in the corridor, and we say, "These beans probably come from that bag."

The argumentation can be schemed as follows:

- *We see White beans (A);*
- *We know that if the beans come from that bag then they are white ($C \rightarrow A$);*
- *Then we say that probably those beans come from that bag (C).*

In other words: we observe a fact A, we know that if a fact C would be true, certainly A would be true so it is reasonable to assume that C is true.

During abduction the solver considers the facts and seeks for a theory to explain them. Unlike deduction, that has a certainty character, it has a probabilistic nature. Among the features that characterize and distinguish this reasoning from the others, Peirce (1932) highlight that “abduction is the only kind of argument which starts a new idea.” The following scheme show the different nature of the two thinking.



Fig. 3.17 - Deduction and abduction reasoning schemes

Different scholars considered abduction, for example G. Polya (1945) taken into account these types of arguments and called them “heuristic syllogisms”. More recently other aspects of abduction are discussed in Magnani (2001), in particular he gives another interpretation of abduction. It is the process by which the solver can infer certain facts, laws, assumptions that make plausible certain statements, which explain or discover phenomena or observations (possibly new ones). For Magnani it is the reasoning process in which explanatory hypotheses can arise and can be evaluated. His different point of view is schematised in the following figure.

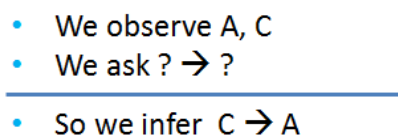


Fig. 3.18 - Magnani's abduction scheme

Abduction is a form of reasoning that is based on strategic principles and not on definitive rules and generally introduces a new knowledge into the subject, which was not accessible first: as such, it is not a deductive reasoning. From the results of some research (Arzarello et al., 1998; Arzarello et al., 2012; Soldano, 2017) it is evident that abduction is intertwined

both with the perception of the solvers during the process of research and with the process itself: together with induction and deduction, it becomes part of a complex process of investigation in which these components are integrated. The main effect of abduction is to change the way in which the solvers see the objects during the resolution: in the first phase they look for regularity and invariants, so a hypothesis (abduction) is produced and then they make “controlled experiments” to test the conjecture.

Abduction has an affinity with backward reasoning. Putting the two form of reasoning in relation, abduction can be one of the methods to infer possible premises within the backward process. But, as seen in the previous paragraphs, backward reasoning also involves other types of processes which determine the phenomenon in a wider way.

3.7 Backward reasoning in problem solving

The previous paragraphs show that backward reasoning is involved in all problem-solving phases of discovery. It allows to obtain something from which making deductive progresses. Hintikka and Remes (1974), studying an example of Pappus's work, propose a six parts subdivision of problem solving in which the method of analysis, and consequently backward reasoning, is implicated. They divide the process into three main parts, each of them is divided into two more specific components.

Part 1. The theorem or the problem to be solved

It is the first part of the resolution process. It can be, for example, a general "if-then" implication in the case of theorems to be demonstrate or statement of construction to be found. It is subdivided into:

- a. *That which is given*: constitutes the first part of the statement of a theorem (if sentence), includes the classification of certain mathematical objects useful for resolving the problem and the details of their relationships. Sometimes, in the case of constructions, “*that which is given*” is omitted (for example, in the statement

"construct a spiral" does not appear the first part of the statement, the classification of the mathematical objects useful to the resolution is omitted)

- b. *The thing sought*: constitutes the second part of the statement of a theorem to be proved, or an implication (then-clause sentence), in some cases it is alone, as in some constructions.

Part 2. Analysis in a broad sense

It is the part of the resolution in which backward reasoning is applied, it can also be divided into two components. Part c and part d have different importance depending on the type of analysis. For problematic analysis part c is more important than d, for theoretical analysis the opposite is valid:

- c. *Proper analysis*: it is the first part of problem resolution using backward thinking. It is characterized by the implementation of auxiliary constructions that the solver develops starting from the initial configuration in which the thing sought is true or exists, depending on the type of analysis. It is characterized also by the insertion of new mathematical objects necessary for the development of the problem. These objects are not present in the formulation of the problem (Part 1). At the end of this part the solver gets something known or that can achieve with another defined method.
- d. *The resolution*: it is the most important part of the resolution; in this part the solver seeks to achieve the independence of the result obtained in part c from the auxiliary configurations used. Thanks to this process, it is possible to justify that the steps taken in the analysis (part c) are reversible and therefore they can be transformed into synthesis (deductive) steps.

Part 3. Synthesis

It is the last part of the resolution. Deductive logical inferences are developed in order to reverse the passages of the analysis and get a justification for the resolution.

- e. *Construction*: in this part the solver considers the construction from which it deduces the resolution of the problem.

- f. *Proof*: in this part deductive logic inferences are developed to solve the problem.

The scheme proposed by the authors, similarly to those proposed by Polya, De Guzmán and Gómez-Chacón (see Chapter 2), allows to subdivide the resolution of a problem by focusing on the interaction between the different phases, in particular on the steps between analysis and synthesis. Some references to this division will be made in the presentation of the analysis results (see Chapter 6 to 10).

3.7.1 Strategies implies in backward reasoning

Already in the Ancient Age there is the idea that there are different strategies, resolution techniques that are below the concept of backward reasoning. Proclus (*A Commentary on the First Book of Euclid's Elements* (1970), [5]) defines three different techniques:

- *Method of analysis*: trace the result backwards, from what is required to an unknown principle.
- *Diaeresis method*: divide into parts what is proposed to be examined. Through this, it is possible to reach a starting point for proof development (that it evolves in a progressive way) by eliminating the irrelevant parts and establishing the basic principles.
- *Method of reduction to impossibility*: does not allow to show directly what is being looked for but tries to look for its contradiction. This allows indirectly to establish the truth of the starting point.

The author underlines the fact that, backward processes can be developed in a positive way, going backward, or negatively, through Reduction ad Absurdum.

Throughout history, several authors such as Polya, Schoenfeld and De Guzmán (see Chapter 2, section 2.2), studied problem solving. By looking at the different heuristic techniques developed throughout history it is possible to identify different resolution strategies which are supported by backward reasoning: the working backward strategy (Proclus's *method of analysis*), assuming the problem solved strategy, beginning at the end of the problem

strategy, the *diaeresis method*, and the Reduction ad Absurdum,. They are resolution strategies that involve “the thing sought”. To clarify the differences on strategies a definition and a visual example are shown. Supposing that the problem to be solved is:

Proof that $A \rightarrow B$

Fig. 3.19 – Problem: proof that $A \Rightarrow B$

Working backward strategy

The working backward strategy is the strategy of turning back, rather it consist in doing some steps backward in the process (represented by red lines). These steps can be done starting from B, the end of the problem, or during the process of resolution in combination with forward steps.



Fig. 3.20 – Working backward strategy solving problem “proof that $A \Rightarrow B$ ”

Beginning at the end of the problem

Beginning at the end of the problem, it means start form B. From here it is possible proceed progressively or regressively with logical steps.



Fig. 3.21 - Beginning at the end of the problem strategy solving problem “proof that $A \Rightarrow B$ ”

Assuming the problem solved

The strategy of assuming the problem solved consists in assuming that “ $A \Rightarrow B$ ” is true and making some forward or backward inferences from this supposition.



Fig. 3.22 – Assuming the problem solved strategy solving problem “proof that $A \Rightarrow B$ ”

Diaeresis method

This strategy consists in dividing into parts what is proposed to be examined, in this case B. This lead to reach its constituent parts.

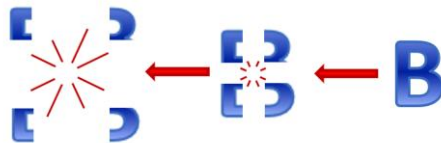


Fig. 3.23 – Diaeresis method strategy solving problem “proof that $A \Rightarrow B$ ”

Reductio ad Absurdum

Starting from the denial of the thesis, do some steps to reach something incoherent and contradictory. From here it is possible proceed progressively or regressively with logical steps. It is the negative counterpart of *beginning at the end of the problem strategy*.



Fig. 3.24 – Reductio ad Absurdum strategy solving problem “proof that $A \Rightarrow B$ ”

3.8 Definition

From the historical-philosophical overview of the analysis and synthesis processes, a new conceptualization of backward elements emerges. The backward reasoning can be defined by four different features: breakdown, cause-effect relationships research, transformative,

and introduction of auxiliary elements. These extrapolated categories (see section 3.3.5 of this chapter) allow to analyse the resolution protocols in a new way by identifying backward reasoning moments and classifying them. This classification will be useful to extend the epistemic model to a cognitive one. After analysing several literature elements, it is possible to give a backward reasoning definition.

Backward reasoning is a type of reasoning involved in creative and discovery processes. It is an essential part of the analysis method. In fact, it starts from the end of the problem and reaches something known through a series of *backward logical steps*. The backward reasoning, as though the analysis, is strongly bounded to synthesis processes. In fact, the logical steps done during backward reasoning need to be reversible, this means that they have to be travelled backwards during the synthesis processes in order to reach the solution. To start a backward path, it is crucial to ask relevant questions. These questions allow auxiliary elements interconnected to the end of the problem to emerge. The abduction can be a pathway by the introduction of these elements. The backward reasoning has a tripartite nature: backward, including the cause-effect relationships research, breakdown and transformative. These features allow to identify the moments in which backward reasoning arise.

In particular, backward reasoning can be identified in problem solving when:

- Solver's reasoning develops in a backward direction,
- Solver breaks down the problem in sub-problems,
- Solver divides an entity in parts,
- Solver searches for properties of an entity,
- Solver transform the mathematical language (for instance from geometrical language to algebraic one),
- Solver introduces new elements,
- Solver sorts problem phases by going backwards along the path followed,

- Solver uses working backward strategy, assuming the problem solved strategy, beginning at the end of the problem strategy, the *diaeresis method*, or the Reduction ad Absurdum.

These features and moments were used during student protocols analysis throughout the four design experiments. They allow the author to recognize backward reasoning moments and classify them according to the four features (breakdown, cause-effect relationships research, transformative and introduction of auxiliary elements).

4

ANALYSIS TOOLS OF MATHEMATICAL REASONING: THEORETICAL FRAMEWORKS

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ANALYSIS TOOLS OF MATHEMATICAL REASONING: THEORETICAL FRAMEWORKS

4

“Since mathematics learning and teaching is a multi-faceted phenomenon which cannot be described, understood, or explained by one monolithic theory alone, a variety of theories is necessary to grasp the complexity of the field” (Bikner-Ahsbahs, et al., 2014, p. 5)

In Mathematics Education field there are several theories rooted in different philosophical fields. These theories have been developed through time in order to understand teaching and learning complexity approaching them in a different way. Some phenomena are so complex and articulated that is not possible to completely analyse them referring to a single theory. One of these complex phenomena is the backward reasoning, which is at the centre of this research. In order to create a data analysis instrument able to observe, understand, describe and explain the reasoning, three theoretical frameworks had to be combined. To do so, two strategies were identified: the Networking of Theories and the Hybridization of Theories. These strategies allow to successfully link two or more theories without compromising the respective fundamental principles, methodologies and the paradigmatic research questions associated.

4.1 Networking and Hybridization

Since 2005, with the birth of the Working Group “Networking Theories” at the 4th Congress of European Research in Mathematics Education, researchers in Mathematics Education started to think about the possibility to use multiple theories in order to analyse the same phenomenon from different points of view. They tried to understand how the multiple resources, provided by each theory, could be combined to connect two or more apparently

distinct theories; this led to affirm that “the plurality of theoretical approaches can only become fruitful when different approaches and traditions come into a dialogue” (Prediger and Bikner-Ahsbabs, 2014, p. 117). To comprehend the possible twine, the researchers observed how different theories could give different points of view on the same dataset. They tried also to understand if it was possible that fundamental aspects identified by a certain theory could be identified through the analysis of the same data with a different theory. Hence they defined the theories networking as “the connecting strategies that respect on the one hand the pluralism and/or modularity of autonomous theoretical approaches but are on the other hand concerned with reducing the unconnected multiplicity of theoretical approaches in the scientific discipline” (Prediger and Bikner-Ahsbabs, 2014, p.119).

Following Lotman's ideas, Radford (2008, 2014) suggest that theories are immersed and developed in the so called Semiosphere: “a multi-cultural, heterogeneous, and dynamically changing space of conflicting views and meaning-making processes generated by theories and their different research cultures” (Radford, 2014, p.283). Here, the theories “live, move, evolve” and are inter-related through a dialog that contributes to the development of the Semiosphere itself. The theories network as such can be visualized as a set of connections, each involving at least two theories.

Radford (2008, 2014) states that each theory is characterized by a trio (P, M, Q) where P refers to the theory principles expressed in a certain language, M its methodology and Q the paradigmatic research questions associated to the methodology. Two theories that coexist in the Semiosphere can have more or less tight connections that do not only depend from their nature, but also by the specific research objectives of the project for which they are chosen as support. In order to identify these connections, specific research questions related to the research project are needed (Q') (tasks, problems, etc.). The research questions leading to the creation of a new methodology M'. M' is the result of the more or less strong connection between the two starting theories, it will allow to answer the questions Q'.

Prediger and Bikner-Ahsbabs (2014) define the “research practices that aim at creating a dialogue and establishing relationships between parts of theoretical approaches while respecting the identity of the different approaches” (p. 118). In doing it, they explicit all the different ways two (or more) theories can connect. The way to connect are then placed into

a graph that represent the connection degree of the two theories. The two ends of the line represent respectively the situation in which two theories are so distant that they have no common points (ignoring), and the situation in which the two theories are so connected that they could be referred as a single theory (unifying).

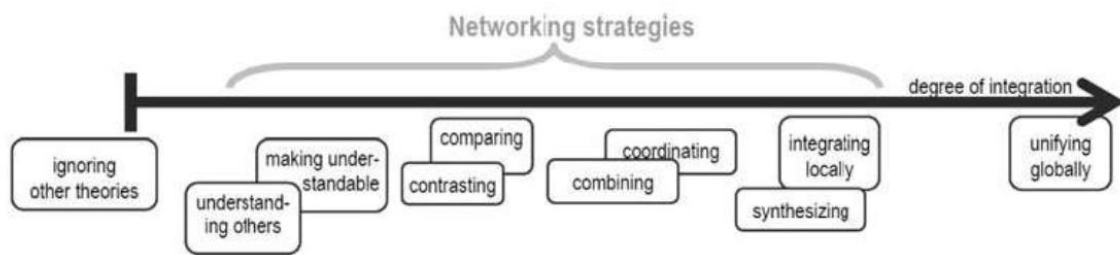


Fig. 4.1 - Strategies for connecting theoretical approaches (Bikner-Ahsbabs & Prediger, 2014, p.119)

Among all possible combinations, this dissertation will discuss the *combining and coordinating* strategy. This type of interaction is “mostly used for a networked understanding of an empirical phenomenon or a piece of data. ... [it] means looking at the same phenomenon from different theoretical perspectives as a method for deepening insights into the phenomenon” (Prediger e Bikner-Ahsbabs, 2014, p. 119).

Within the Semiosphere, another strategy, similar to the networking, is the hybridization. Through a hybridization, a specific part of a certain theory interact whit another theory in a coherent, operative and productive way. This method was conceived by Arzarello and presented for the first time during the 10th ARDM young researcher national seminary in 2016, afterwards it was applied by Taranto (2017) in her PhD thesis. The aim of hybridization is to integrate a theory that gives partially satisfactory answers to the research questions. Through hybridization, it is possible to introduce specific fragments of a theory T_1 into a wider and more consolidated theoretical framework (T_2). These fragments are adapted and integrated into T_2 which has now become a hybridized theory. It keeps a coherent methodological development allowing, at the same time, a better answers precision to the research questions that, differently from networking, remain the same.

Hybridization happens in three steps (Arzarello, 2016; Taranto, 2017): connection, interpretation and adaption. The hybridization process occurs primarily establishing a **connection** between two theories; it depends on theories structure and on the purpose of the

connection itself. Afterwards a fragment of the hybridizing theory (T_1) expressed in its original language is **interpreted** into the wider theory (T_2) and then expressed in the language of the latter. Finally, an **adaptation** of the hybridized theory (T_2) is needed in order to connect the interpreted fragment to the other components of the theory. The hybridization sometimes provokes a T_2 elements suppression, this suppression should not compromise the consistency of the hybridized theory.

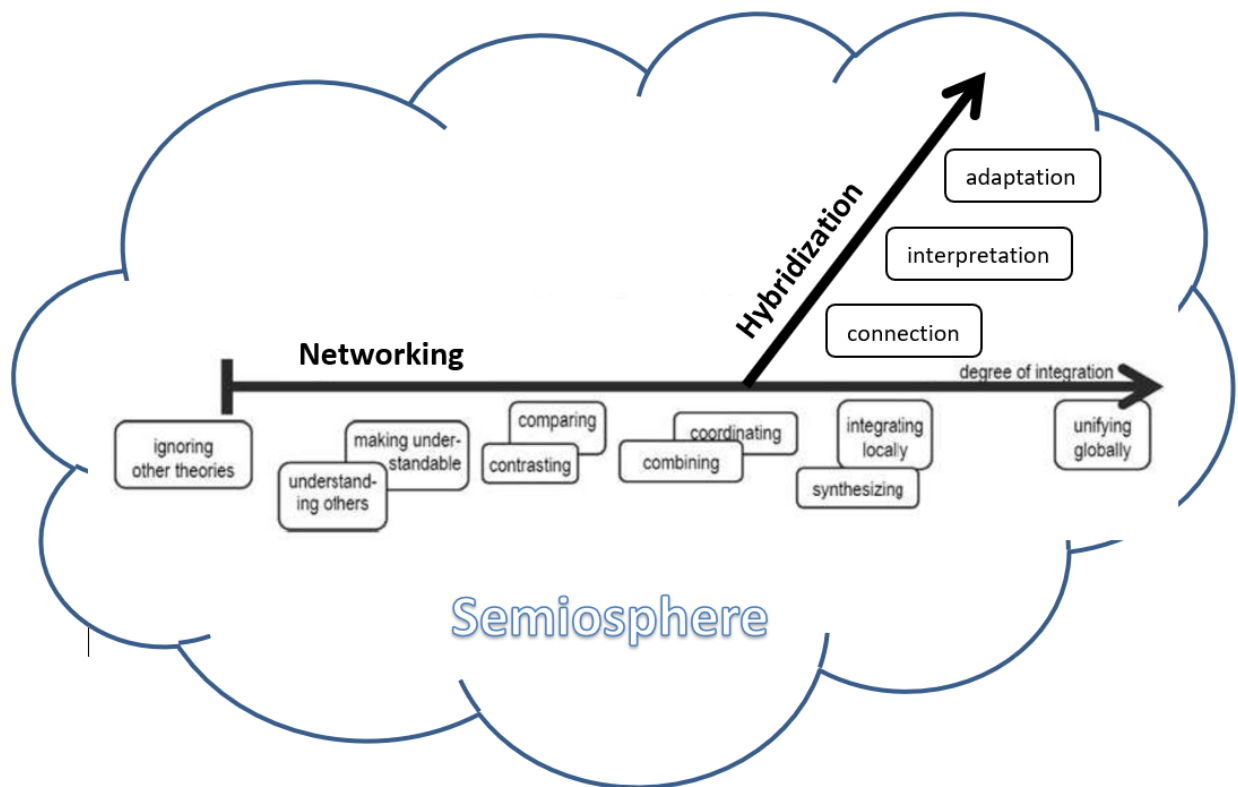


Fig. 4.2 - Integration of the Bikner-Ahsbals & Prediger (2014, p.119) plot with the hybridization strategies (Arzarello, 2016 and Taranto, 2017, p.41)

Hybridization, as the *combining and coordinating* theory, allows a deeper understanding of an empirical phenomenon (or certain data), but produces a different narrative. The *combining and coordinating* strategy allows to realize that it is impossible to grasp some aspect analysing the phenomenon with only one focal lens; networking provides consciousness over previously unknown aspects (*narration of unveiled identity*). On the other hand, the introduction of new fragments, through the hybridization, modifies the principles or the methodology itself of the hybridized theory; the fragment operationally

influences the hybridized theory in a more satisfactory way (*narration of the change of identity*).

The theoretical framework used to develop this research is the result of a networking, at a *combining and coordinating* level between the Game Theory Logic (Hintikka, 1999) and the Abstraction in Context (Dreyfus et al, 2015) approach, with a subsequent hybridization integrating a fragment of the Commognition (Sfard, 2008) point of view. To better understand how and why these strategies were chosen, the following paragraphs are structured following the theoretical framework construction path. The first section is about the Game Theory Logic, the first theory taken into consideration.

4.2 Game Theory Logic

In the study of reasoning, logic is often considered in its deductive nature remaining identified with the theory of reasoning. In classical logic, in fact, the interest is mainly focused on the formal relations between the propositions while all those pragmatic aspects, fundamental in the natural search, are excluded (Harmaakorpi & Mutanen, 2008). But reasoning is not only deductive. In a rational knowledge search, in the processes of argumentation, and also in mathematical demonstrations, both deductive and informal characters of reasoning appear (Lakatos 1976, Hintikka 1999).

After a continuous search begun in the 1970s, Jaakko Hintikka, a Finnish philosopher and professor of logic at Boston University, developed what he called *Logic of Inquiry* overcoming the static approach of habitual logical mathematical reasoning. The idea, already elaborated by ancient Greek philosophers (the so-called Socratic method), is building knowledge through a questioning process, implicit or explicit. The knowledge is the result of a research generated by a specific question. The philosopher introduces it as the “logic of question and answer” or rather as the logic of questions and answers sequences:

“This idea [of the LI] is as old as Socrates, and hence older than most of our familiar epistemology and logic. It is the idea of knowledge-seeking by questioning or, more accurately, of all rational knowledge-seeking as implicit or explicit questioning. I am

using the phrase "inquiry as inquiry" to express the idea. For what my leading idea is precisely an assimilation of all rational inquiry in the generic sense of searching for information or knowledge to inquiry in the etymological sense, that is, to a process of querying, or interrogation." (Hintikka, 1999, p. ix)

This question-answer idea captures the dynamics of a theory of discovery so relevant in mathematics teaching and learning as well as in research. This process, indeed, "reflects the characteristic structural aspects of our investigation and resolution activities" Hintikka (1996, p. 98). In fact, the statements that structure the typical argumentation coming out from the answers the researcher is able to develop in his inquiries. For this reason, Hintikka (1999) affirms that a purely deductive logic is inadequate for a scientific inquiry:

"Most philosophers have apparently assumed that for a scientific inquirer all the rock-bottom answers must be thought of as particular propositions. This assumption has led to the inductivist and to the hypothetic-deductive models of science. In reality, it is nevertheless totally unrealistic, as is illustrated among other things by the possibility of putting questions to nature in the form of experiments. An answer to an experimental question is typically a functional dependence between two variables, which can only be expressed in terms of dependent quantifiers, and hence not a particular proposition." (p. xi)

In his approach he considers Game Theory and Wittgenstein's language games (rule-governed human activities that mediate the descriptive meaning) to support formal epistemic logic in mathematics. He states, in fact, that the semantics of games are suitable to encode the "mathematicians' way of thinking and speaking". Introducing the rules of the game theory in a specific logical sense, Hintikka (1995, 1999) extends the framework of deductive logic to a wider coherent theoretical context: the Game Theory Logic (GTL), an epistemic logic huge different from the usual deductive logic because based on a question-answer process. The main feature of GTL consists in reviewing all the propositional and quantificational aspects of logic according to the functional method that derives precisely from the game theory.

With this theory, Hintikka succeeds in overcoming the excessive abstractedness of Tarski's definitions of logical truth (Tarski, 1933; Tarski and Vaught, 1956), which leaves the path of thought to reach the truth unexplained. Tarski's truth standard definition starts from its simplest (atomic) statement and proceeds recursively to the complex ones. For example, the truth of $A \wedge B$ is based on the truth of A and B. While, Hintikka's work, is focused in the research of "a path towards the formulation of a truth that, instead of proceeding recursively from atomic to complex formulas, reverses the approach and proceeds from the more complex ones to their simplest constituents." (Hintikka, 1998, pp. 28-29) In this direction, he introduces a top-down definition of truth (Hintikka, 1995) unlike the classical bottom-up view, highlighting the regressive way of proceeding in problem solving from an epistemological point of view.

To explain the complexity of the Game Theory Logic, Hintikka (1999) defines two types of rules that govern any goal-oriented activity or rather the activities that can be conceptualized as a game in the sense of the mathematical games theory: the definitory rules, that is the rules of inference, and the strategic rules, or argumentative moves. The strategic rules are those movements that a player makes to achieve an optimal strategy. They allow a general organization of reasoning thanks to the strategic skills that they are bringing into play. They determine which of definitory rules is the best and which order must be applied, to achieve the goal as quickly and as best as possible. He explains (Hintikka, 1999) the role that each rule plays in what he calls "the Game of Logic":

"The so-called rules of inference are definitory rules, not strategic ones. At each stage of a deductive argument, there are normally several propositions that can be used as premises of valid deductive inferences. The so-called rules of inference will tell you which of these alternative applications of the rules of inference are admissible. They do not say anything as to which of these rule applications one ought to make or which ones are better than others. For that purpose, you need rules of an entirely different kind, viz. strategic rules. The so-called rules of inference are merely permissive. They are rules for avoiding fallacies. They are not "laws of thought" either in the sense that they would tell us how people actually draw inferences or in the sense that they would tell us how we ought to draw inferences." (p. 3)

The philosopher, resuming the idea of Wittgenstein's language-games, defines game-theoretical semantics (GTS) all the two-player games of verification and falsification that approach formal and natural language. The two players involved were called initial verifier and initial falsifier. The verifier tries to show that the statement considered is true and, at the same time, the falsifier tries to prove that it is false. Hintikka demonstrate that every semantic game end after a finite number of moves, with a winning player. Hintikka (1995) defines the notion of truth, focus of the GTL, basing it on the notion of winning strategy:

“The statement S is true in the environment M if and only if there exists a winning strategy for the initial verifier in the game $G(S)$ when played on M .” (p. 234)

A mathematical statement can be interpreted through the notion of game-theoretical semantics. To illustrate this, Hintikka (1998) uses the words of Ian Stewart, but he considers the phrase “it’s like a game” not like a metaphor but like a real game between two players Epsilon and Delta:

“A function $f(x)$ approaches a limit L at the x tendency to a value of itself, given a positive number ε the difference $|f(x) - L|$ is less than ε whenever $|x - a|$ is less than a number δ , which depends on ε . It's like a game: "Tell me how much you want $f(x)$ close to L ; then I'll tell you how x is to be close to. Player Epsilon tells how close he likes; Then Delta is free to look for value according to his own desire. If Delta always has a winning strategy, then $f(x)$ tends to the limit L .” (Stewart in Hintikka, 1998, p. 29)

Hintikka not only focuses on formal games, independent of experience (indoor games), but, on the contrary, he points out that his GTL semantics constitutes the meeting point, between external reality and the mathematical knowledge of this reality. The game of knowledge research involves manipulations of extra-linguistic objects beyond manipulating the symbols of the language (outdoor games).

To clarify the type of reasoning involved in its logic, Hintikka (1999), like Peirce (1932), distinguishes two types of reasoning step: Corollarial and Theoretic. Corollarials are trivial steps in reasoning: they are logical consequence of what was demonstrated or happened

before, these steps are developed working with known notions. Instead, Theoretic are those non-trivial steps of reasoning in which new elements are introduced; in this case, some new objects, not mentioned in the initial statement, must be introduced by performing auxiliary constructions. On the one hand, the problems characterized by corollarial deduction can be solved by considering only the configurations of the objects actually mentioned in the initial statement. On the other hand, it is the theoretical deductions that make possible new lines of thought in the argumentation.

Any goal-oriented activity is an interrogative game in which the questions have a central methodological role. Even the process of seeking new knowledge is a questioning process. The questions that emerge can be qualified (Hintikka, 1999) in types: the propositional questions ($a \vee b?$) and the so-called "wh-questions" ($\exists x/S(x)?$). The introduction of new individuals in the argument can be given through examples of existence or answers to the wh-questions.

The useful information to answer the questions comes from a known external source, what Hintikka (1999) calls the *oracle* (or the *nature*). The oracle is an entity that is supposed to be a source of true information. In knowledge acquisition processes, the learner (what Hintikka name the *inquirer*) can consult it at any time assuming that the given answers are always sincere. The oracle can be assume different aspects: "The oracle can be the databased stored in the memory of a computer, a witness in a court of law, or one's tacit knowledge partly based on one's memory" (Hintikka, 1999, p. 34). The different moves in the dialogical game between the inquirer and the oracle constitute the Interrogative Model (Hintikka, 1984).

4.2.1 The interrogative model

During his long research about the Game Theory Logic, Hintikka (1996, 1998, 1999) develops some categories for the analysis of reasoning processes in the process of inquiry: the previous paragraph showed the differences between definitory and strategic rules, the corollarial and theoretic types of reasoning, and the qualification of questions. These categories can be used as analytical elements for the understanding of how the resolution of

a problem is produced in their dimension of “outdoor games” and “indoor games”. The dialogical game that takes place in reasoning, which is essentially dialogues of questions and answers between two players, can be described by the following moves elaborated by Hintikka (1984).

a) *Initial move*

Each player raises a conjecture or a thesis to be demonstrated. After an initial move the players follow the line of reasoning (expressed in the dialogue) until one of the two achieves its objective. At each stage, each player can decide which next move he will make (see moves b) - e))

b) *Deductive move*

Players develop a series of deductive steps according to defined rules.

c) *Interrogative move*

One player raises a question to the other and the second player can give an answer or deny the assumptions made. This entails an analysis of the issues (concepts, conjectures, etc. ...).

d) *Assertoric move*

A player exposes a new thesis that is connected with the previous statements (previous lines of thought). The other player has two options: agree or disagree with the above.

e) *Defining move*

An explicit definition is assumed.

This model can be used to analyse the paths of thought in a resolution process. The player, in this case, is alone and plays against the oracle.

Several studies use this or similar models (Barrier, Durand-Guerrier and Blossier, 2009; Başkent, 2016; Brook, 2007; Hintikka, et al., 2002), emerging from the Game Theory Logic, for analyse the reasoning and the statements in dialogical games (Harmaakorpi & Mutanen, 2008; Mutanen, 2010). There is a background in Mathematics Education field in the use of Hintikka’s models of inquiry as tool for analysing dialogues between students during mathematical research in validation situations (Barrier 2008; Hakkarainen and Sintonen, 2002).

4.2.2 The finer logic of inquiry model

The models of inquiry were adapted for the analysis of the interaction between the investigative and the deductive component in the problems resolution, under the denomination of Finer Logic of Inquiry Model (FLIM) (Arzarello 2014; Soldano 2017; Barbero, Gómez-Chacón, 2018). The FLIM derives from the Logic of Inquiry and the Game Theory Logic of Hintikka and Saada-Robert (1989) psychological model for solving mathematical problems. The Saada-Robert model focuses on the distinction between two phases of the resolution: investigate why things are like this and verify this investigation. The FLIM model is useful to analyse interactions between strategic and deductive components in student resolution protocols. The model is structured in two components: The Inquiry and the Deductive Component.

The IC (Inquiry Component) is a phase in the resolution where the subjects involved alternates a series of questions, explorations and answers, according to Hintikka's "Logic of Inquiry". At this stage, the subject is deeply involved in the activity and its purpose is to reach the goal of the problem by verifying the conjecture that comes from the exploration of the problem.

The explorations during the process of resolution can be of two types: a *real exploration* with the aim of analysing and understanding the situation in which the subject is involved and a *control exploration* with the aim of verifying the ideas or conjectures that have arisen during the course of the activity.

These sequences of actions taken by the subject are characterized by three different modalities: Ascendant, Neutral and Descendant modality. The ascendant modality moving the subject's mind from the exploration of the problem to the formation of a conjecture concerning the formation of an idea following a research and analysis of the situation. The descendant one characterizes the shift from a conjecture or a guiding idea to the realization of a research about the problem. The purpose of the descending modality is to find equivalence between the subject of thought (the conjecture, the guiding idea) and the worked object (the problem and its resolution). Arzarello calls "neutral modality" the modality that marks change between an ascendant and a descendant one. The actions that can be observed

by the subject in the Inquiry Component can then be summarized as follows: Question, Affirmation, Conjecture, Exploration, Control and Formulating a Resolution Plan.

The DC (Deductive Component) is a phase of activity where the subject is not directly involved in the search for conjectures and in their verification. This second component is characterized by the use of a logical language to formulate formally the truths found during the search phase. We can add two modalities of action in the DC: Detached modality and Logical control. Detached modality is characteristic of those phases of the resolution in which the subject's point of view is clearly different from the activity performed; the language at this stage assumes a strong logical and formal connotation. Logical control, on the other hand, is a specific descending modality where the guiding idea is the logical structure while the work object is a specific action-control phrase. We can add two actions in the DC: Deductive steps and Logical chains

The two components are often not distinct from each other, and when a problem is solved, the subject often moves from one component to the other. We can say therefore that the typical structure they assume is nested in this way: (IC ~ (DC ~ (IC ...))) with IC ~ DC it the passing from one to another.

4.3 Networking of GTL with AiC

In order to study backward reasoning both from an epistemic and a cognitive point of view, the first step has been to frame the research with the Game Theory Logic (GTL) elaborated by Hintikka (1999). As shown in the previous paragraph, GTL, with the new top-down semantic, seems to approach backward reasoning structure suggesting to the reader a reverse strategy. The GTL allows to start examining backward reasoning at an epistemic level interpreting it through the game theory rules. The models elaborated starting from the Game theory Logic, the Interrogative Model (Hintikka, 1984, 1999) and the Finer Logic of Inquiry Model (Arzarello, 2014), allow to focus on logical-strategic aspects of backward reasoning and only partially on the cognitive aspects. Models permit to identify characteristic questions

emerging within the problem solution but not enough for a deeper analysis from a cognitive point of view.

Any kind of reasoning produced during the resolution of a problem is strongly connected to the actions that were accomplished or will be accomplished during the resolution itself. Each reasoning to reach a certain objective is immediately followed by an action (or vice versa each action is followed by a reasoning) that is strongly bounded to the previous actions. The reasoning so developed takes into account what already happened during the resolution and sometimes tries to anticipate what will happen. For each action performed, a certain type of knowledge is activated; it is a processing of the previous ones or an anticipation of the future ones. In order to grasp these reasoning aspects and deepen the bounding between the epistemic and cognitive level, the Abstraction in Context Theory (AiC) (Hershkowitz & al., 2001) is introduced. The theoretical model allows to study the epistemic actions during the problem resolution and to elaborate deeper backward reasoning analysis. Such as the FLIM, RBC-model (Dreyfus & al., 2015), created starting from AiC, involve both epistemic and cognitive aspects, but it is more complex allowing to grasp aspects that could not be unveiled with the other models. Thanks to this model it is possible to identify several reasoning cognitive chains.

The *combining and coordinating* of the two theories can be translated into the combined use of the analysis models derived. The Hintikka Interrogative Model (HIM) looks at backward reasoning from a logical-strategic point of view, while the RBC-model (RBC) focuses on the epistemic-cognitive aspect.

4.4 Abstraction in Context

The Abstraction in Context theoretical approach was born, at the beginning of 2000's, in order to respond to a series of questions, such as "What did students learn and consolidate, and how? What mathematical concepts and strategies remain with them?", emerged during a research about the development of innovative curricula for schools performed by a group of researchers in Mathematics Education mainly based in Israel (Hershkowitz & al., 2001,

Dreyfus et al., 2015). This and other researches (such as Dreyfus and Kidron (2014) or Schwarz et al. (2009)) of the last twenty years have been concretized in the development of the Abstraction in Context theory (AiC) which aims to provide a theoretical and methodological approach on the processes of learning mathematical knowledge. “Theoretically, AiC attempts to bridge between cognitive and situated theories of abstraction, as well as between constructivist and activity-oriented approaches. Methodologically, AiC proposes tools that allow the researcher to infer learners’ thought processes” (Dreyfus and Kidron, 2014).

The focus of the theory is twofold: on the one hand there are the abstraction processes, key processes of mathematics; on the other hand, there are the contexts in which these processes develop. The two parts are strongly interconnected, and it is not possible to disconnect the abstraction process from its context, the later influences the individual processes of knowledge construction (Dreyfus and Kidron, 2014).

Two theories, belonging to different traditions, are taken into consideration by the researchers during the AiC development: the Freudenthal and Davydov theories (the latter, belonging to the Vygotsky tradition). The researchers take the mathematization concept and particularly the vertical mathematization by Freudenthal (Schwarz et al., 2009, Dreyfus and Kidron, 2014). Mathematization is an activity that allows the student to manipulate and develop mathematics. The mathematization is vertical: mathematical ideas reorganization process that allows the creation of new mathematical meanings is based on constructs and mathematical meanings previously developed by other mathematicians. During vertical mathematization, connections between mathematical concepts and strategies are identified reorganizing all the elements in a creation process that integrate and expand the existent mathematical knowledge. This process allows not only to reach a deep level of mathematical knowledge but also to let it evolve. The vertical mathematization idea is strictly connected to the complex abstraction idea of Davydov. He considers the scientifically knowledge as the cultivation of a certain way of thinking that allows the creation of new concepts through the connection of ideas. Abstraction is an ascension process that leads to the development of a consistent idea through the reorganization of a series of undeveloped concepts. The AiC, based on these views, develops the definition of abstraction “as a process of vertically

reorganizing some of the learner's previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner"(Dreyfus et al., 2015).

The second point AiC theory focuses on is the context in which the learner is immersed. It can refer to different types of context (Dreyfus and Kidron, 2014):

- Social context, including the relations between other students and professor.
- Historical context, referring to the previous mathematical experiences of the learner or of the mathematical community.
- Learning context, including curriculum, social norms and technological instruments used.

Researchers (Dreyfus et al., 2015) consider the Activity Theory, "which has an underlying constructivist philosophy" as adequate theoretical framework to consider the cognitive processes developed in the different contexts. They state that all activity results naturally transform themselves into artefacts for the next activities: this allows the abstraction development identification through the observation of the following activities. It is possible to trace this development identifying a series of actions: the epistemic actions.

The concept of epistemic action was introduced by Kirsh and Maglio (1992) to indicate those physical actions that allow problems to be solved more quickly and easily. These actions are used to change the work setting of the problem being solved so that this change helps to acquire useful information for the resolution that is hidden or difficult to compute mentally. These actions are intended to simplify mental processes. Epistemic actions differ from the actions that the authors (Kirsh and Maglio, 1994) define as pragmatic. The latter are instead those actions that bring the solver physically closer to the goal. Pontecorvo and Girardet (1993) use the same concept in historical research contexts. They explain that epistemic actions develop within argumentative processes and in particular are at the basis of interpretative activities. The actions involve high level methodological and metacognitive procedures and include the explanation of those procedures used for the interpretation of particular events. They also underline the close connection of actions to social interaction as

well as to individual reasoning and state that operationally they need an argumentative cognitive language and setting.

The Israeli group of researchers (Hershkowitz & al., 2001) resume the concept within the Mathematics Education field and define epistemic actions as those actions in which knowledge is used or constructed. They consider it as a model of analysis within their research in mathematics education. The researchers consider epistemic actions as observable manifestations of the mental processes that take place in the student (Dreyfus and Kidron, 2014). These processes, generally unveiled, can be identified through the analysis of the student's verbalisation or their physical actions. Dreyfus considers, like the other authors, that epistemic actions develop in the processes in which knowledge is built. They focus their researches on the process of abstraction and consider it as composed of three different epistemic actions: Recognizing, Building-with and Constructing. These three actions allow them to describe the process operationally through analysis model: the RBC model.

4.4.1 RBC model

"The central theoretical construct of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analysed by means of three observable epistemic actions: recognizing (R), building-with (B), and constructing (C)" (Dreyfus and Kidron, 2014, p. 89). Each action performed by the solver, during the task resolution, can be classified and described tracing its evolution. The three epistemic action are therefore detailly defined:

Recognizing (R). It consists of recognizing knowledge previously acquired as relevant for the task resolution.

This action occurs when the solver understands that a certain construct is relevant/connected to the mathematical situation of the ongoing activity. The acknowledgement can occur by specialization or by analogy. The acknowledgement by specialization is when an element, present in the problem or in the solution, is recognized as relevant. The acknowledgement by analogy occurs when the solver needs to apply theorems, properties, etc. that are present

in his background but that are not necessarily directly related to the problem itself. Some examples are the introduction of new geometrical elements in a new construction; new variables of an analytic problem; or when the “recognition of a familiar mathematical structure occurs and the solver realizes that the structure is inherent in a given mathematical situation” (Hershkowitz et al, 2001, p.115). Researchers use the word “re-cognizing” to highlight that it is not the first time that the construct “enters the mind” of the solver.

Building-with (B). It consists of combining a set of knowledge with the aim of achieving a specific goal.

The action occurs when the solver combines knowledge already present in the task resolution. He combines structures, concepts and ideas emerged during the resolution and that could have already been used or analysed during previous activities. During this action there are not new emerging knowledge but only a richer and more complex reworking of the knowledge in possess of the solver. Anyhow this restructuration allows the solver to get closer to the objective “such as solving a problem, understanding and explaining a situation, or reflecting on a process. For these purposes, students may appeal to strategies, rules, or theorems” (Hershkowitz et al, 2001, p. 116).

Constructing (C). It consists in assembling and integrating the previous knowledge with the aim of producing a new construct.

This action is the most important step in the abstraction process. The knowledge that is at solver’s hand are reworked, reconnected and restructured in order to build new knowledge, a new method, a new strategy or a new concept. When an element “enters the mind” of the solver for the first time, the researchers refer to that element as “cognized” by the solver.

These three actions are not totally independent one from the other, they are nested. The recognition actions are nested inside the building-with actions. The solver before gathering knowledge, explicitly or implicitly recognizes it. In the same way, the recognizing and building-with actions are nested inside the construction actions. In order to develop new knowledge, the solver relies on previous knowledge that is highlighted by the recognizing and building-with actions. This knowledge collection produces a new construct that is a

bigger construct with respect to the sum of all the previous knowledge. This is reason why this model is also named Nested Epistemic Action model. (Dreyfus and Kidron, 2014).

The construct developed during the third epistemic action is not necessarily completely understood by the solver. Often this construct is strongly dependent on the context, hence it is difficult for the solver to acquire it and reuse it in other contexts. The new constructs acquisition process is named *Consolidation* by the Israeli researchers (Dreyfus et al., 2015). It is during this process that the solver becomes aware of the new knowledge built during the abstraction. He becomes more and more aware and begins to use it with more confidence even in contexts different from the one where it was developed. The consolidation is when knowledge spontaneously emerges in an activity different from the one where it was developed.

4.5 Hybridization with Commognition

As specified in Chapter 3, the focus of the research project is on backward reasoning, but it does not exist without its forward counterpart. The Hintikka Interrogative Model (HIM) allows to consider backward/forward reasoning from a logical-strategic point of view, while the RBC-model (RBC) does that also from an epistemic-cognitive point of view.

Thanks to the *combining and coordinating* of the GLT with the AiC, the reasoning analysis proceeds through the observation of logical/strategic and epistemic actions of the students involved during the design experiment. This approach highlighted that backward reasoning is a fundamental part of the new objects' creation processes during the resolution. The analysed actions are a manifestation of the interpersonal discourse and of the students' thoughts during tasks resolution. Unfortunately, the analysis available thanks to the two networked theories, does not allow any linguistic scrutiny, which could add a powerful tool for analysing reasoning processes. We have so chosen to hybridize the currently got theory with some components of the Commognition perspective (Sfard, 2008), centred on the mathematical discourse and its transformations. It makes available a fresh theoretical

perspective apt to complete our analysis of backward mathematical reasoning in a more satisfactory way.

Specifically, only a fragment of the Commognition perspective will be considered: the *objectification processes*. Investigating the episodes of backward reasoning in different contexts (chapters 6-9) we have realized that what happens there and is described through the lenses of the networked theories AiC + GLT still needs another tool to be properly investigated: the objectification lens from Commognition. The networked AiC + GLT theories are so hybridized with the objectification component: the result is a very meaningful and satisfactory understanding of backward processes. The design of the analysis model derived from the interconnection of all these theories will be discussed in the next part of this dissertation (see Chapter 5, section 5.2.2)

4.6 Commognition

Starting from the assumption that the human thought is a communication form, principally inspired by the Wittgenstein e Vygotskij work, after decades of research, Anna Sfard publishes in 2008 the monograph *Thinking as communicating: Human development, the growth of discourses, and mathematizing*, where she illustrates the *Commognition theory*.

Taking from Vygotskij the idea of collective performances learning, the author states that also thinking occurs through collective performances. It develops itself through interpersonal and mostly intrapersonal communication. Hence, she defines thinking as “the individualized form of the activity of communicating. Indeed, it is self-communication – a person’s communication with oneself. This self-communication does not have to be in any way audible or visible, nor does it have to be in words” (Sfard, 2009, p.174). Sfard considers thinking and communications as the two sides of the same coin and considers them as the same entity. Combining the words communication and cognition creates the term Commognition. It “is the focal notion of the approach to learning grounded in the assumption

that thinking can be usefully conceptualized as one's communication with oneself" (Sfard, 2018, p.1³).

Not all communication types are the same, they differ both in rules and in objects to which they are referred. The discourse is that type of communication accessible only to certain people while it is inaccessible for others. Each person, depending on his knowledge, can participate only to some discourses but not to other. The language in knowledge communication includes a "finite set of arbitrary symbols and a set of rules to regulate the manipulation of the beforementioned symbols". If these symbols or rules are not known, it is impossible to participate to the discussion. Each discourse, in fact, is characterized by (Sfard, 2009):

- Specific *key words* used with certain rules;
- *Visual mediators* that allow to identify objects in the discussion and coordinate the communication;
- *Routines*, repetitive patterns developed by the speakers;
- *Endorsed narratives*, narrative set within the discourse approved and confirmed to be truth by the discourse community.

Therefore, thinking means to participate to the development of a certain type of discussion that happens during an interpersonal or intrapersonal interaction; thinking in a mathematical way is equivalent to participating to the development of a historic-mathematical discourse. The mathematical discourse (Sfard, 2018, p.2), like any other, is based on:

- "Specific key words ("line", "point", "set", "function", etc.);
- Visual mediators (numbers, graphs, algebraic symbols);
- Routines (define, prove, deduce);
- Endorsed narratives, approved by the mathematician's community along the years (theorems, definitions, computational rules)."

Each discourse, as already described, is developed around a series of objects characteristic of the discourse itself. The mathematical discourse develops around mathematical objects; they have been created by the participants to the discourse, by mathematicians, hence

³ The page refers to the online edition. link-springer-com-443.webvpn.jxutcm.edu.cn

mathematics is an autopoietic system, namely it generates its same objects with its discourses; in this sense, it is different from any other scientific system: for example, in Physics the discourse develops about existing objects and the names in its discourse, like mass, force, etc. concerns them and the relationships between them. The introduction of mathematical entities into the discourse is what Sfard calls objectification. The objectification process can be recognized within the mathematical discourse because it corresponds to the appearance of at least one of the following discursive devices (Sfard, 2018):

- *Saming*: introduction of a name in common to things that were not interrelated at the beginning, but that can be equivalent in certain contexts (examples: quadratic function, x^2 , parabola).
- *Encapsulating*: replace a discourse on separate objects with one relative to a single entity (example: function, set).
- *Reifying*: replace a discourse about a process with a discourse about an object (example: from “when I add 5 to 7, I get 12” \Rightarrow to “the sum of 5 and 7 is 12”).

Once the object has been introduced in one or more levels, the alienation process begins, it leads to the use of the object in an impersonal way granting its existence independently from the discourse itself.

From the Commognition point of view (Presmeg, 2016; Sfard, 2008, 2009, 2018) it is fundamental to consider what happens during the knowledge acquisition process. The learning, interpreted like a collective phenomenon, develops through a specific type of communication: a determined discourse developed with a determined language. A notion is learned when the learner is able to produce articulate discussions using the new constructs in a proper way and with the appropriate meaning. The crucial step during the learning process is the passage from the interpersonal to the intrapersonal communication. The subject processes the discussions developed with others and transforms them into intrapersonal communication: only at this point he can use them to interact with the external world in an active way to fulfil his needs. Therefore, understanding mathematics means mastering a mathematical notion, a key word, a routine or a narrative, so that is possible to handle a complex discourse with mathematicians' community. Once a notion is mastered, the discourse changes, and this corresponds to learning that notion (Sfard, 2009). To evaluate the learning achievement is necessary to examine how discourses change.

Sfard (2008) distinguishes between two learning levels: the object-level and the meta-level. On the one hand, the object-level learning occurs during activities in which no expert in mathematical discourse intervenes. In this type of learning new narratives are constructed by deducing them from those already endorsed. On the other hand, the meta-level learning develops when the learner interacts with the experts. The learning occurs when the apprentice encounters a discourse incommensurable with his own. This causes a *commognitive conflict*, a situation in which “communication occurs across incommensurable discourses” (Sfard, 2008). To overcome this conflict the learner starts to imitate the expert performances, and doing so, he develops some routine. In this phase, the learner uses mathematical object and develops a discourse, but cannot judge if the mathematical narrative produced are endorsed. Then, the process of learning proceeds through a *de-ritualization*, in which gradually the learner starts to participate at the mathematical discourse in a more conscious way. He transforms the routines in explorations.

As widely illustrated in the mentioned works of A. Sfard, typical examples of this transition happen when students pass from Arithmetic to Algebra; from Naturals to Integers, then to Rationals, and finally to Reals; from finite to infinite sets; etc. Sfard points also out that this transition from object to meta-object levels happens not only at an ontogenetic scale (a student in his school career) but also at a phylogenetic level: in fact all such transitions correspond to relevant progresses in the history of mathematical thought (Sfard, 2008, p. 535; Caspi and Sfard, 2012, p.46).

We will see that the two forms of reasoning (backward and forward) that we see in our students when facing the different problems in our experimentation constitute two contrasting discursive forms that make explicit a commognitive conflict between the forward and the backward approach. In fact, in them the same words are used, but within two different discourses structure, which depend on the two modalities (forward/backward).

Backward and forward reasoning, when they occur together, possibly producing some abductive modality (section 3.6), are the sign of a commognitive conflict (generally intrapersonal) which is on the way of solution. To say it better: as seen previously (Chapter 3), the forward reasoning is not a feasible heuristic to produce an effective result, be it a new knowledge, like a statement, or a proof of it. It is the backward process that makes accessible

to problem solver the way to look at the relationships between the involved object so to make accessible a proof reversing them and passing from a backward (ascending) modality to a forward (descending) one. These reversing processes have been studied in our teaching experiments and their modalities have been carefully investigated with our networked model AiC + GLT, as it will be described in the following chapters 6-9. It is this reversing process that allows to find the proof of a statement, or even a new result, and the reverse action can develop because of the backward forms of reasoning. In fact, as discussed in Chapter 3, reversing reasoning is a modality which changes the relationships between the components of discourse and makes accessible a solution, which at first sight was inaccessible. Backward reasoning makes commensurable the forward modality of discourse, which would otherwise remain only at a ritualized form of proving. Using the framework of Sfard's commognition, it can produce a transition to an exploratory modality.

Also the history of backward reasoning, widely discussed in Chapter 3, shows the long way required before that the two incommensurable discourses could be 'tamed' in the centuries: the masterpiece of Descartes, who was able to objectify synthetic and analytic discourses in the language of algebra (Chapter 3), and the results of Hintikka, who could objectify them at a logical level showing the 'duality' of the logic of inquiry with standard deductive logic (Chapter 4), illustrate the great efforts that were historically necessary to arrive at this objectification settlement.

At a phylogenetic level, its complexity and difficulties are illustrated by the enormous quantity of research papers about the difficulties encountered by students in facing the variety of incommensurable mathematical discourses they meet at school, particularly when they learn algebra or proofs: for example, the Section 3 (more than 350 pages: Mathematical Processes and Context) in the recent *Compendium* edited by Jinfa Cai (2017) are concerned with this matter (for an approach to the cognitive difficulties related to proving see: van Lambalgen & Stenning, 2008).

Of course, as Sfard points out, this transition generally requires the contribution of an expert, who supports the students in this delicate task. With respect to this issue, two important observations must be made: they are a consequence both of known results about games (see Chapter 2) and of our findings.

First, in game solution processes, backward forms of reasoning are very ‘natural’ because of the game: in a sense the exploratory modality which is supported by backward reasoning is produced because of the game context. Going beyond the Sfard’ framework, we can say that game contexts facilitate to overcome the incommensurability of the two discourses, that of the logic of inquiry (Chapter 4) and that of the deductive logic.

Second, it is exactly the production of backward reasoning that can help to bridge the gap between the two modalities. As it has been recalled above, according to Sfard, the switch between two incommensurable discourses is marked by a process of objectification essentially through some discursive devices (*saming, encapsulating, reifying*). This is the main reason for the hybridization of our networked theory. As we will describe in the following chapters, the hybridized theory will allow to give a precise description of a finer structure of backward reasoning, which evolves in time within different contexts. Precisely, the linguistic analysis through the hybridized component from the Commognition theory will allow to point out the objectification processes within this finer structure. Such a result will confirm the hypothesis about the backward reasoning as a construct that allows and facilitates to overcome the incommensurability between the inquiry and the deductive forms of reasoning. This last point will be discussed in Chapter 11.

4.7 Research Questions

The theoretical framework, elaborated through networking and hybridization, allows to frame the research project in a more complete way, so that the main objectives can be achieved. It also permits for a more refined development of the raw research questions (see Chapter 1). The main objectives, stated in the introductory chapter, are two:

- developing a cognitive model of backward reasoning, extending the existing epistemic model;
- establishing principles that can be used for the design of teaching situations focused on backward reasoning.

In order to achieve the two objectives, the research has begun with an in-depth study of the literature about the backward reasoning phenomenon. Analysing the texts of mathematicians and philosophers, from the ancient Greeks to contemporary ones (Beaney, 2018), it was possible to highlight specific epistemic characteristics that determine the backward reasoning. It was also noted that, it is closely related to its counterpart. Several authors underline the fact that backward reasoning it does not exist without forward reasoning.

In order to achieve the first objective of the research project it is necessary to understand how this backward reasoning develops at a cognitive level. Given the strong link with forward reasoning it is necessary to understand how the two are interconnected. Therefore, the first two research questions have been formulated.

1. What is the epistemological and cognitive link between backward and forward reasoning?

2. How does the transition from backward reasoning to forward reasoning (and vice versa) take place?

Previous research on the subject (Barbero, 2015) had confirmed that this type of reasoning develops naturally in strategy games. With the idea, in the future, of developing teaching situations focused on learning this phenomenon, it was necessary to ask whether there were any non-game situations that could be identified as favourable for the development of backward reasoning and its learning. From these observations the third research question arise.

3. Are there any non-playing situations that lead to backward reasoning?

In order to answer to these first three research questions, it was necessary to build a multidimensional theoretical framework that would take into account the different aspects of backward reasoning, allowing an epistemological, logical, and cognitive analysis. The analysis model thus created makes it possible to observe the backward reasoning and to characterize it on a cognitive level. Wanting to establish principles for the design of future teaching activities based on backward reasoning, a reflection on the results of the experimentation is necessary.

In order to answer to the three research questions four design experiments have been developed. In all of them, it has been used the analysis model created from the networking and the hybridization of theory explained in previous paragraphs. The methodology used and the development of the analysis will be the theme of the second part of this dissertation.

II

DESIGN EXPERIMENTS

PART II – DESIGN EXPERIMENTS

This second part of the dissertation consists of 5 chapters: *Research design*, *Triangular Peg Solitaire analysis*, *Maude Task analysis*, *3D Tick-Tack-Toe analysis*, and *Mathematical Problems analysis*. After the presentation of the research design, the four design experiments analysis and results are displayed. The general discussion of those results and the conclusions are developed in Part III.

The first part of *Research design* (Chapter 5) is about the research context. In this part the academic path of the students involved in the four design experiments and the context of practice where the design experiments were developed are shown. Then, in the second part of the chapter, the methodology use for the entire research project is explained. Firstly (section 5.2.1), for each design experiment, the proposed task, with its a priori analysis, is shown and the task's settings (type of working –alone, in pairs, in group–, video-recording tools, PC suites, etc.) are highlighted (data collection tools). Then (section 5.2.2), the design of the multidimensional tool for analysis is displayed. Here, the aggregation of the analysis models deriving from the different theories involved in the theoretical framework is shown. Later (section 5.2.3), an explanation of how the analysis will be displayed in the following chapters is showed to make it easier to read. Finally (section 5.2.4), the reliability and validation criteria of data analysis is displayed.

Triangular Peg Solitaire analysis (Chapter 6), *Maude Task analysis* (Chapter 7), *3D Tick-Tack-Toe analysis* (Chapter 8), and *Mathematical Problems analysis* (Chapter 9) show the analysis and results of the four design experiments. The chapters are organized in a quite similar way. After a summary of the task proposed in the design experiment, the first section (x.1, with x the number of the chapter) is about the analysis of the whole group of involved students. Here, the identified backward reasoning moments are highlighted pointing out the percentages of use in the students' group. Then, in second section, one or more case studies are shown to deepening the backward reasoning analysis. For each identified backward reasoning moments, at least one example is shown with an in-depth analysis of a protocol excerpt (or a transcription excerpt depending on which one is analysed); this excerpt can be part of a case study (in section x.2) or, if there is no example in the case study, in a separate sub-section in section x.1. For each chapter the display modality is explained. All the

excerpts, protocols and transcriptions are analysed with the multidimensional analysis model developed in Chapter 5. Later, in last section, a discussion of whole group and case study analysis is developed highlighting some results. Since the fourth design experiment involve four mathematical problems, in Chapter 9 there are four sections dedicated to each problem. Therefore, the subdivision is: in section 9.1 the whole group analysis, in section 9.2-9.5 the case studies of each problem and in section 9.6 the discussion.

As in previous chapters, for each one, a Table of Contents is shown to help the reader in approaching the chapter.

5

RESEARCH DESIGN

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RESEARCH DESIGN 5

This research project is based on a mixed methodology that involves a quantitative and qualitative study. Four design experiments were carried out with university students, solving two strategy games and four mathematical problems. Their written productions (solved tasks and resolution protocol), direct observations during the session, and video-recordings were analysed. The study was then deepened through some case studies. Among all those developed, one or two case studies, for each design experiment, have been chosen for the writing of this dissertation.

5.1 Research context

The whole research involved 207 university students attending the Universidad Complutense de Madrid (Spain) and 115 university students from the Università di Torino (Italy), for a total of 322 students divided as follows:

- 227 Undergraduate students (128 from Spain and 99 from Italy). They attend the Bachelor's Degree in Mathematics.
- 87 Master students (71 from Spain and 16 from Italy). In Spain, 48 students attend the Master's in Mathematics Teacher Training for Secondary School, and 23 students attend the Master in Computer Science. In Italy, all the students attend the Master's in Mathematics.
- 8 PhD students from Spain. They attend the Engineering Mathematics, Statistics and Operations Research (IMEIO) doctoral program.

Education in Spain and in Italy is compulsory from 6 to 16 years of age. Both educational systems are divided into seven stages: Kindergarten (Scuola dell'Infanzia/Educación Infantil), Primary School (Scuola Primaria/Educación Primaria), Lower Secondary School (Scuola Secondaria di primo grado/Educación Secundaria Obligatoria), Upper Secondary School (Scuola Secondaria di secondo grado/Bachillerato), Bachelor (Laurea triennale/Grado), Master (Laurea Magistrale/Máster) and PhD (Dottorato di ricerca/Doctorado). The number of ages dedicated to each stage varies between the two countries. The following chart shows the different educational levels of the two countries with their respective names. They are put into a timeline where each notch corresponds to a year, so that similarities and differences between the two educational systems are showed. In both countries, starting from upper secondary education, the educational offer is very varied. The chart shows a standard example of an educational career leading to a PhD.

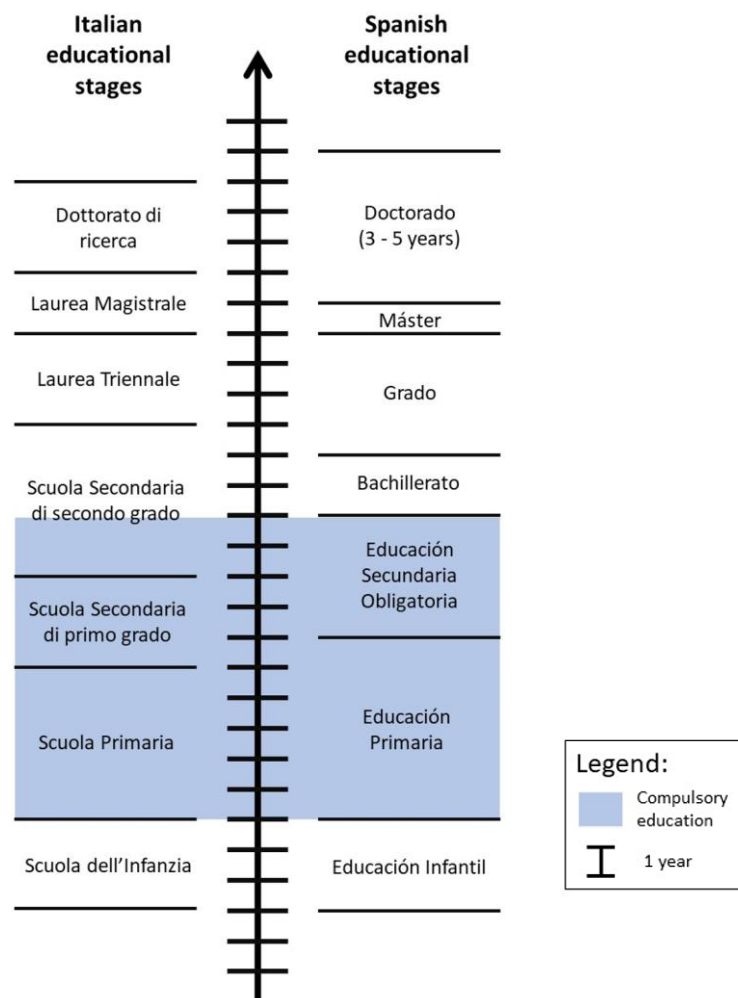


Fig. 5.1- Spanish and Italian educational systems

The students enrolled in the research can be divided into three groups according to which level they reach in their academic careers. This subdivision will be useful to illustrate the different contexts in which each design experiment was developed; it was necessary to do it, due to the differences between Spanish educational stages and Italian ones.

- *First level group*: these students attend the first years of their university career. They are building basic knowledge about the different aspects of mathematics: calculus, geometry, algebra, etc. To this group belong 99 students of the first year of the Bachelor's degree in Mathematics in Italy and 30 students from the first to the third year of the Bachelor's degree in Mathematics in Spain.
- *Second level group*: these students attend the final year of the bachelor's degree or the master's degree. They have a consolidated knowledge about the different aspects of mathematics. To this group belong 96 students from final year of the Bachelor's degree in Mathematics in Spain and all the Master students (87).
- *Third level group*: these students (8) attend the PhD in Spain. They have an in-depth Mathematical background with an emphasis on the subjects that are necessary for advanced applications in the field.

Each student participated in one single design experiment except for 18 final-year bachelor's degree students from Spain who participated in the first and the third design experiment.

5.1.1 First design experiment: Triangular Peg Solitaire

The first design experiment involved 50 Spanish students aged between 21 and 24 years:

- 44 students (28 women and 16 men) who belong to the second level group;
- 6 students (2 women and 4 men) who belong to the first level group.

The students were involved in the course of Matemáticas para la Enseñanza (Mathematics for Teaching). This course is part of the educational offer for students in the fourth (last) year of the bachelor's degree in Mathematics. It is an optional course within the curriculum, for this reason in the group there were also six students enrolled in the third year. All the bachelor's and master's degree courses in Mathematics, in the Universidad Complutense de

Madrid, are structured in the same way: the lessons of each subject are divided according to specific themes; for each topic, the teacher is required to propose some task as a practice session. The design experiment was carried out during the final practice session of the topic “Problem Solving”, where the teacher proposed to solve the strategy game “Triangular Peg Solitaire” (see section 5.2.1.1 in this chapter).

The performances of the students involved was heterogeneous, this first group did not present specific characteristics such as to be highlighted.

5.1.2 Second design experiment: Maude task

The second design experiment involved 23 Spanish students (2 women, 21 men), aged between 22 and 23 years; the students belong to the second level group. The students were involved in the course “Auditory and Quality Assurance”, taught to Master students in Computer Science. Part of the lectures for Quality Assurance consist of specifying and verifying properties in Maude software (Clavel, et al., 2007) (see section 5.2.1.2.1 in this chapter). This part of the subject requires students to implement programming assignments of growing complexity. The third task proposed (out of 5) was the Triangular Peg Solitaire in Maude (see section 5.2.1.2 in this chapter). The students received a summary of the assignment and had to implement it in Maude software.

The performances of the students involved was heterogeneous except for one student. This student (Student-E in Chapter 7) has a 5-year bachelor’s degree in Mathematics and Computer Science. She has in-depth Computer Science background, so she can help other students of the group to solve the task acting in an expert role.

5.1.3 Third design experiment: 3D Tick-Tack-Toe

The third design experiment involved 185 students from Spain and Italy aged between 18 and 28 years:

- 8 Spanish PhD students (2 women and 6 men), who belong to third level group;
- 63 Spanish students (30 women and 33 men) who belong to the second level group;

- 114 students (62 women and 52 men) who belong to the first level group; 99 of them are Italians.

For each group of students, the research team proposed to solve the strategy game “3D Tick-Tack-Toe” (see section 5.2.1.3 in this chapter) in a specific practice session.

The PhD students were involved in the PhD course “Didactic tools to design, manage and analyse university teaching processes” offered by the doctoral program. The task was part of a practice session connected with the topic “Useful tools to analyse the works of students”. From the second level group, 35 students were involved in the course Matemáticas para la Enseñanza (Mathematics for Teaching), which is part of the educational offer of the fourth year of bachelor’s degree in Mathematics. As the Triangular Peg Solitaire in the first design experiment, the task was proposed like practice session. 28 students belonging to the second level group were involved in the course Pensamiento matemático y resolución de problemas (Mathematical Thinking and Problem Solving); it is part of the educational offer of the Master’s in Mathematics Teacher Training for Secondary School. The task was proposed by the research team like practice session at the end of the topic “Problem solving”. The Italian students were involved in the course Introduzione al Pensiero Matematico (Introduction to Mathematical Thinking). This course is part of the educational offer for students in the first year of the bachelor’s degree in Mathematics in Torino. The researchers proposed the task as final practice session of the entire course. The Spanish students belonging to the first level group were volunteers.

The performances of the students involved was heterogeneous, each group of students did not present specific characteristics such as to be highlighted.

5.1.4 Fourth design experiment: Mathematical Problems

The fourth design experiment involved 82 students from Spain and Italy aged between 18 and 25 years:

- 73 students (36 women and 37 men) who belong to the second level group; 16 of them are Italians;

- 9 Spanish students (2 women and 7 men) who belong to the first level group.

For each group of students, the research team proposed to solve four mathematical problems (see section 5.2.1.4 in this chapter) in a specific practice session.

From the second level group: 37 students were involved in the course *Matemáticas para la Enseñanza* (Mathematics for Teaching), which is part of the educational offer of the fourth year of bachelor's degree in Mathematics; and 20 students were involved in the course *Pensamiento matemático y resolución de problemas* (Mathematical Thinking and Problem Solving), which is part of the educational offer of the Master's in Mathematics Teacher Training for Secondary School. As the 3D Tick-Tack-Toe in the third design experiment, the task was proposed like practice session. The Italian students were involved in the course *Didattica della Matematica 2* (Mathematics Education 2). This course is part of the educational offer for students enrolled in the master's degree in Mathematics in Torino. The researchers proposed the task as final practice session of the entire course. The Spanish students belonging to the first level group were involved in the course *Matemáticas Básicas* (Basic Mathematics), which is part of the educational offer of the first year of bachelor's degree in Mathematics.

The performances of the students involved was heterogeneous, each group of students did not present specific characteristics such as to be highlighted.

5.2 Methodology

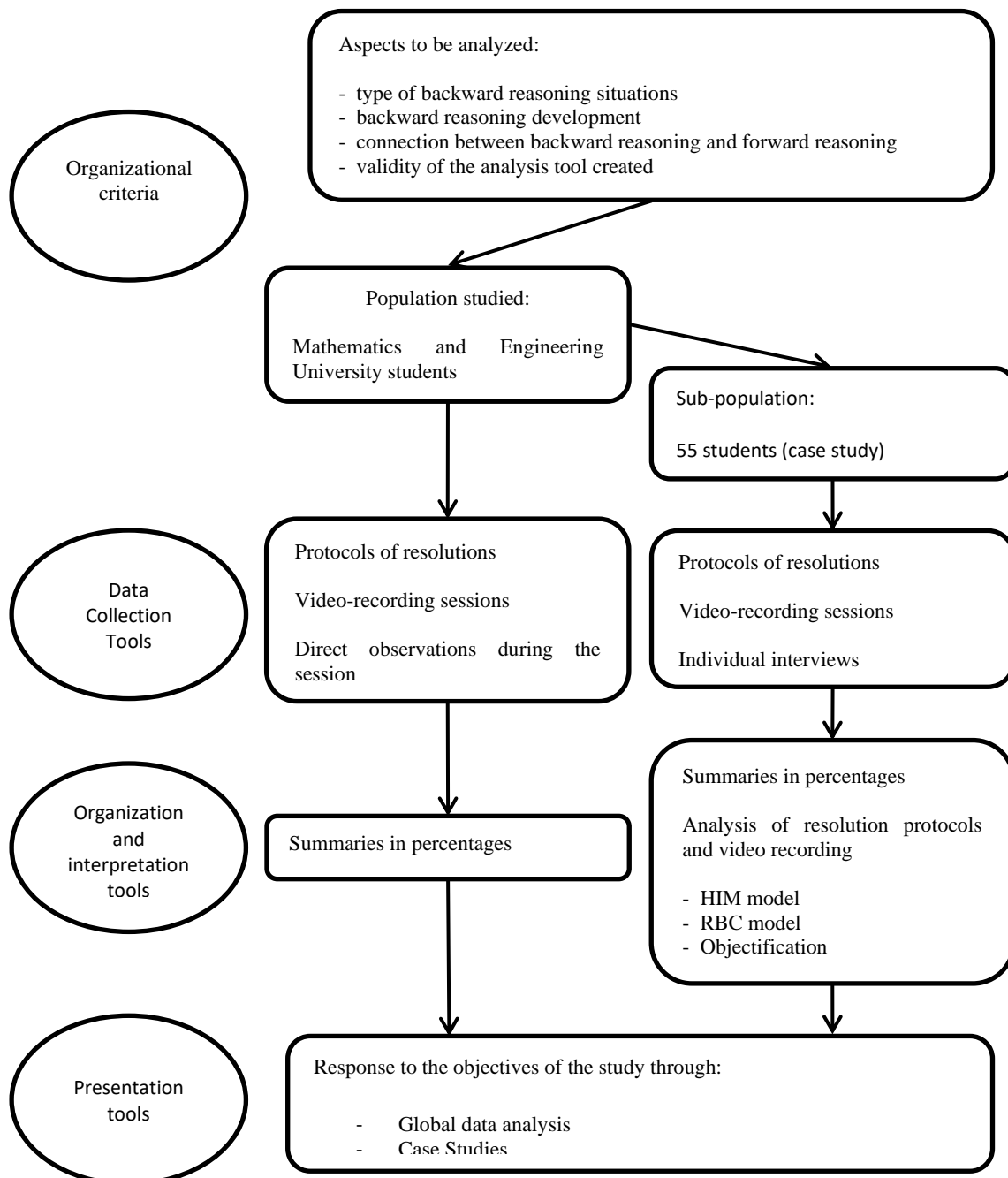
This research project is based on a qualitative and quantitative study. Four different teaching experiments conducted between Spain and Italy, over the course of two academic years, were carried out using the Design Based Research methodology (Cobb, et al., 2003). The Design Based Research is a form of “engineering research” that allows a systematic study of forms of learning, whose main goal is the development of (new and not new) theories about different aspects of learning processes. It is characterized by successive research cycles that permit the evolution, improvement and redesign of research over time, and the consequent possibility of methodological innovation thanks to the iterative character and the

rapid feedbacks about the effects of the experiment. This idea was used to develop four design experiments that are evolutionary cycles in a longitudinal way; we improve the investigation throughout different student level and contexts. It allows to better characterize elements and categories of backward reasoning and to observe its dynamic behaviour.

The design experiments were developed after focusing our research attention on the phenomena of backward reasoning and of its relationships with forward reasoning: consequently, we collected the existing literature about them. Each design experiment allows to have different vision of the theme by using new resources. The work is organized taking into account the intellectual starting point of the students involved in the experiment. The possibility to analyse multiple data allows us to properly develop the investigated ideas.

In each design experiment a different task is proposed to a certain group of students. Each task consists of solving an open problem (a strategy games, a programming activity, or a mathematical open problem) (see Chapter 2 for the open problem definition). At the same time, students are required to produce a resolution protocol by writing down observations about the reasoning processes that they developed along the resolution. The researcher observes the development of the problem in the natural educational context, focusing on particular cases (Bassegy, 1999). Each study was deepened through some case studies: the students who were chosen are key representatives of each group. The data are collected by combining different sources: direct observations during the session, the recordings from the cameras, and the documents (solved tasks and resolution protocol).

In the elaboration of the investigative tools four levels were followed: Data Collection Tools, Organization of the information criteria, Information and interpretation tools, and Presentation Tools.



5.2.1 Design of the data collection tools

In order to observe and study in depth the phenomenon of backward reasoning several data were collected by combining different sources: the written production (solved tasks and individual resolution protocol), the video-recordings from the cameras placed during the sessions, and the individual interviews.

The strategy games and the mathematical problems that were proposed, were chosen based on some fundamental features necessary to the research: the possibility to use backward reasoning in their resolution and in particular the need to use auxiliary construction or novelty elements. To reach this, the selected games and problems have a strong visual component and a geometric development.

In order to study backward reasoning from different point of view, four problem situations were developed. Starting from the assumption that strategy games allow for the natural development of backward reasoning, the first task proposed was the Triangular Peg Solitaire (Gomez-Chacon, 1992); it is a strategy game with a two-dimensional board. This game was chosen because, despite having a geometric board (Deza, Onn, 2002; Vallot, 1841-2), it is disconnected from the mathematical content which forces the student to use their mathematical knowledge acquired in their university degree. The student focuses on finding a winning strategy; the geometric properties of the board influence the strategic choices but do not “distract” him from the resolution. An informal environment also allows the student to feel free to make mistakes, to start over, to express doubts and difficulties. The task proposed in the second design experiment concerned the implementation of the Triangular Peg Solitaire game in Maude programming language. To solve it, students have to understand the game in order to, first, interpret it at a mathematical level and then, implement it in the programming language. After the implementation, solving the game becomes simply writing a logical rule in computational language. In third design experiment the 3D Tick-Tack-Toe game (Berlekamp, Conway, and Guy, 1982) was proposed. The task asks, in addition to solving the game, to express mathematically the properties of the board and its elements. This is a mixed task in which the game resolution and the mathematical problem solving are intertwined. Unlike the Maude task, this requires the mathematical interpretation of the board in three dimensions but not a computational contextualization. Finally, the last

design experiment task concerned the resolution of four mathematical problems. These were chosen to cover different aspects of mathematics. The first problem concerns the graphical representation of functions, the second is a geometric problem, the third is a construction problem and the last is a combinatorial problem that develops from a three-dimensional geometric visualization. The evolution of the tasks has gone hand in hand with the development of the analysis model that will be explained in detail in the 5.2.2 section.

These different types of tasks and mathematical problems were chosen to observe how mathematical knowledge affects the development of backward reasoning. The researchers assume that when mathematical knowledge is involved, it affects the resolution. Depending on the task, students are required to work more at the heuristic level (as for example for the first and third task) or more at the mathematical level (second, third and fourth task), with the formalisation of mathematical knowledge.

For each design experiment, the proposed tasks and the specific data collection setting will be explained.

5.2.1.1 Triangular Peg Solitaire

The game proposed in the first design experiment, the Triangular Peg Solitaire (Gómez-Chacón, 1992), was chosen so that it could be solved by using backward reasoning and strategies in problem solving. The choice of this particular game was dictated by the fact that:

1. The game is not so common and can be played in a paper and pencil modality.
2. The game board is 2-dimensional. It has a geometrical shape that affects strategic choices.

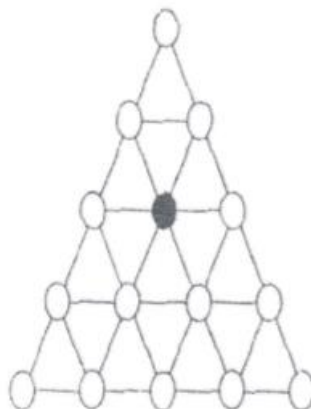
5.2.1.1.1 Data collection settings

The Triangular Peg Solitaire game, a variation of the most famous Peg Solitaire (Berlekamp, Conway, Guy, 1982), was proposed as shown in the figure 5.2. Being a solitary game, each

student worked on his own during a 2 hours session. Students were allowed to use paper balls to simulate the game. No video-recordings were made during the practice session.

Triangular Peg Solitaire

The Triangular Peg Solitaire requires a board with 15 boxes as shown in the figure.



These are the rules:

1. Place the pegs in all boxes, except in the one marked in black.
2. The player can move as many pegs as are the chances of jumping a peg, adjacent to an empty box (along the line); at the same time he "eats" and retreats from the board the peg that was jumped. All pegs will move in this way. Pegs can move around the board.

Target: The player wins when there is only one peg on the board.

Solve the game by finding the winning strategy. Detail your entire thinking process using the resolution protocol technique.

Fig. 5.2 - Triangular Peg Solitaire Task

Based on the analysis of the resolution protocols, two students, belonging to the second level group, were selected for an in-depth case study. This was done through an interview that aimed to understand better the students' choices that appeared in their protocol, the use of

backward reasoning and their mental processes. The interview, 30 minutes long, was structured with a series of questions that provided detailed information on the progress of the resolution protocols and on their difficulties. The choice of cases was done in this way. After the quantitative and qualitative analysis of the resolution protocols was done, six representative protocols were selected. The criteria for choosing these protocols are that they had to have some characteristics: existence of moments of backward reasoning use; existence of different backward reasoning strategies; and they were to be good protocols with a detailed development. Then two of them were chosen to the interviews. They were emblematic for representing the whole group: one used graphics to visualize the resolution process while the other processed thoughts only in words. The first student protocol was chosen for the development of this dissertation.

5.2.1.1.2 A priori analysis

The Triangular Peg Solitaire game with 15 holes (Gómez-Chacón, 1992, Bell, 2008) is a variation of the most famous Peg Solitaire with 33 holes (Berlekamp, Conway, Guy, 1982) showed in figure 5.3a; in both boards, the only movement allowed is showed in figure 5.3b.

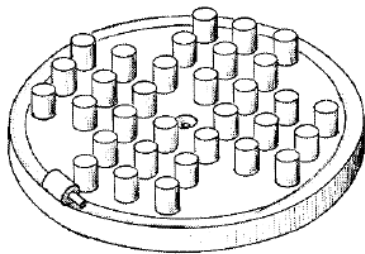


Fig. 5.3a - Peg Solitaire Board

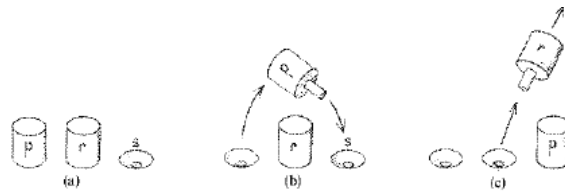


Fig. 5.3b - Peg Solitaire jump

In order to analyse the game, it is better to give a notation to the 15 boxes of the board. There are several ways to associate a notation to the board, for example using a Cartesian notation (see figure 5.4a) or numbering the boxes progressively (see fig. 5.4b); the same peg movement is expressed, for example, by a5-c3 or by 1-6 depending on which one it is used.

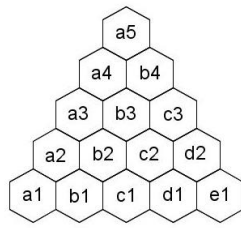


Fig. 5.4a - Peg Solitaire Board

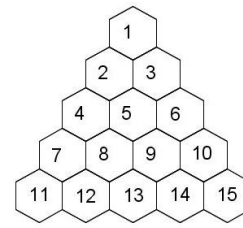


Fig. 5.4b - Peg Solitaire jump

Using the Cartesian notation, a possible solution of the task proposed is: b1-b3, d2-b2, d1-b1, a4-c2, b4-d2, a2-a4, e1-c3, c3-c1, b1-d1, a5-a3, a3-c1, d1-b1, a1-c1. This solution is represented in figure 5.5 (Bell, 2008).



Fig. 5.5 - Possible Solitaire Peg Solution starting with the empty hole in position b3

Bell (2008) shows that, starting with the empty hole in b3 it is possible to solve the problem reaching the positions c1, a3 and c3. All the other positions are impossible to achieve.

Some strategies are useful to solve the game:

- Studying the properties of the board and the boxes:
 - o The board has three axes of symmetry. Whatever solution is found, there will be other 5 different solutions symmetrical to it.
 - o Pegs located in the blue area of the figure 5.6a can move in two directions. Pegs in the green area can move in four directions;
 - o Pegs that are in a position of a certain colour in the figure 5.6b can only move within the positions with the same colour.

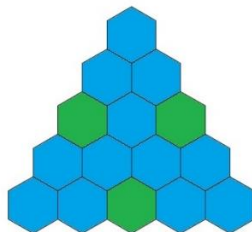


Fig. 5.6a - Peg Solitaire Board

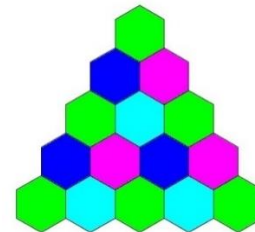


Fig. 5.6b - Peg Solitaire jump

- Working backward strategy: going backward in the jumps (see figure 5.7a, own redraft of Berlekamp, Conway, Guy (1982) original image), use the rule backwards.
- Beginning at the end of the problem: beginning the resolution with only one peg in the board, for example like in the fig 5.7b. Using this and the previous rule together, a similar game is created. The aim of this analogous game is filling the board using the rules backwards, except for one hole that remains empty.

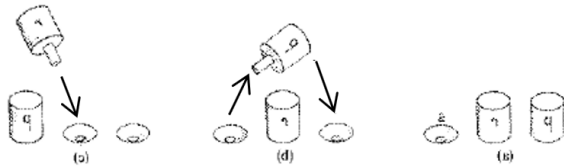


Fig. 5.7a - Peg Solitaire Board

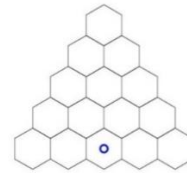


Fig. 5.7b - Peg Solitaire jump

- Solving a simpler problem: it can have two meanings:
 - o starting with simpler configurations, i.e. with fewer pegs on the board,
 - o solving the game in a smaller board (for example with 10 holes).
- Breaking down the problem: breaking down the board into parts.
- Extracting patterns: starting with a particular configuration and trying to find a common rule that allows to achieve the solution.
- Attempts and errors: making an attempt and see if it is possible to get the solution; if it does not work, make a different attempt. This strategy is not advisable for this game, there are many possible movements, the risk is taking a long time to reach a solution.

5.2.1.2 Maude task

The task proposed in the second design experiment, the implementation in programming language of the Triangular Peg Solitaire, was chosen in order to extend the first design experiment. The backward reasoning is necessary to solve this task.

5.2.1.2.1 Data collection settings

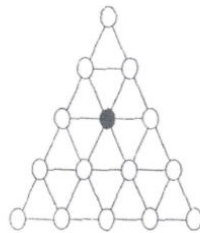
The second design experiment was carried out with students from the Master's degree in Computer Science enrolled in the course Auditory and Quality Assurance. Part of the lectures for Quality Assurance consist of specifying and verifying properties in Maude. Maude is a language design that support membership equational logic and rewriting logic.

It is used for a wide range of applications. Membership equational logic (Meseguer, 1997) is a variation of many-sorted equational logic that includes the concept of *membership axiom*, which allows specifiers to define the sort of a given term by means of equations and other membership axioms. Rewriting logic (Meseguer, 1992) extends equational logic by introducing the notion of rewrites, corresponding to transitions between states. That is, while equations are interpreted as equalities and therefore are symmetric, rewrites denote changes, which can be irreversible and hence they are non-symmetric. In the case of Maude, the underlying equational logic is membership equational logic.

During the practice sessions the students have to implement programming assignments of growing complexity. The third task (out of 5) was the following: to implement in Maude language program the Triangular Peg Solitaire game. The task given to the student is shown in figure 5.8.

Triangular Peg Solitaire

The Triangular Peg Solitaire is a 1-player game that can be played on different sized boards. Initially, all positions except one contain pegs, while a winning board contains exactly one peg. To reach this configuration, pegs can "jump" over others, "eating" the pegs they jump over, as long as an empty position is available after that peg, like in Checkers. We will work with the triangle board, as shown in the figure below.



Exercise 1 Define a datatype for representing a Triangular Peg Solitaire. We are particularly interested in supporting boards of different sizes.

Exercise 2 Implement jumps using rewrite rules.

Exercise 3 Define an initial board and use the **search** command to find: (a) any solution; (b) a "perfect" solution. A perfect solution consists of a board with a single peg in the central position, as shown in the figure above.

Fig. 5.8 - Maude task

Students are generally free to choose to work in pairs or singularly during practice sessions. The research team choose not to vary this routine; from the whole group (23 students) 6 of them worked in pairs. Students have a 2-hour lab session to solve the task. Each group (either 1 student or a pair) has a desktop computer with Maude software installed. Two pairs got their work recorded by cameras.

The two pair recorded by camera were chosen for an in-depth case study. In this dissertation will be displayed one of this two. The choice was done because the pair, during the resolution, interact with the Student-E, the student with a double Degree in Mathematics and Computer Science who acts with an expert role. The interactions of the three students allowed to observe the backward reasoning in a different way.

For this design experiment, the research team involved the instructor of the subject as a professor-researcher. The methods for obtaining the data are direct observations during the lab session, the recordings from the cameras, and the students' written productions (solved tasks and the resolution protocols).

5.2.1.2.2 A priori analysis

For the a priori analysis development, the team relied on the professor-researcher involved in the design experiment, he is an expert in the field of Computer Science. Solving the task requires defining the appropriate data structures in membership equational logic, as presented in the previous section. Such a specification must be flexible enough to support boards of different sizes while being specific enough to support a simple definition of movement. We suggest using three sorts: (built-in) Boolean values standing for the presence of a peg (true) or the absence of a peg (false). Secondly, Row, standing for lists of Boolean values and representing the rows. Thirdly, Board, standing for lists of rows representing the board. In this specification, we will define a subsort relation between Boolean and Row and Row and Board, respectively. Our equational theory will be completed with the sorts previously mentioned, the corresponding constructor (empty syntax for composition of Boolean values and commas for composing rowsm both of them defined as associative and with unit element), and the definition of a function $|L|$ computing the elements of the list L.

The rewriting logic specification extends the one above with rules defining the possible movements. Horizontal moves can be defined in a straightforward way as shown below:

rl [horizontal-left-to-right] : true true false => false false true .

This indicates that, given two consecutive pegs followed by an empty position, we can obtain a new configuration where the leftmost peg "jumped" over the one in the middle, hence resulting in a configuration where both the leftmost and the middle positions are empty and the rightmost one contains a peg. The movement from right to left is analogous. Next, we describe a diagonal movement left-to-right, bottom-up. Assuming we have three consecutive rows, the first one with a space (preceded by some pegs/spaces L1 in the same row and followed by L1') and the next two rows containing pegs (preceded by some pegs/spaces in the corresponding rows, L2 and L3, and followed by pegs/spaces, L2' and L3', respectively), we simulate the "jump" made by the peg in the bottom as follows:

crl [ab-arr-izq-der] : L1 false L1',

L2 true L2',

L3 true L3'

=> L1 true L1',

L2 false L2',

L3 false L3'

if | L1 | == | L2 | and | L2 | == | L3 | .

where we have added the appropriate conditions to check that the length of L1, L2, and L3 is the same. Now, given an initial configuration **init**, we can check whether a solution is reachable by using

search init =>* B:Board s.t. isSolution(B:Board) .

where \Rightarrow^* indicates that 0 or more rewriting steps are allowed and is Solution is an auxiliary function defined by means of equations that returns **true** if the board given as argument contains exactly one peg.

5.2.1.3 3D Tick-Tack-Toe

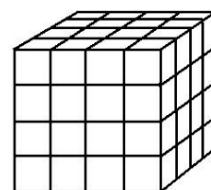
The game proposed in the third design experiment is the 3D Tick-Tack-Toe (Gardner, 1988), also known as “3D Tic-Tac-Toe” or “Qubic” (Allis, 1994; Golomb and Hales, 2002). It was chosen so that it could be solved by using backward reasoning and strategies. It is the three-dimensional version of the Three-in-a-Skate game, also known as Three-in-a-row, Tick-Tack-Toe, Tic-Tac-Toe or Noughts-and-Crosses. The choice of this particular game was dictated by the fact that:

1. The game board is 3-dimensional. The students have to develop more visualization skills to identify the winning lines.
2. The game is a two-player game so the students can interact.
3. Interpret the game in a mathematical way is not trivial.

5.2.1.3.1 Data collection settings

The game was proposed as follow.

3D Tick-tack-toe



The 3D Tick-tack-toe is a three-dimensional version of the classic Three in a Skate game.

The game board is a 4x4x4 cube, be made up of 64 small cubes.

3D Tick-Tack-Toe is a two players game. One player can use "crosses" marks and the other "zeros" marks. Players move alternately by occupying with the own mark any empty cube.⁴

Target: To place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.

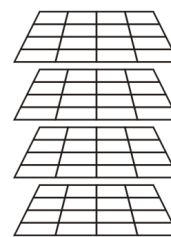
How to represent a cube?

Three dimensions: 4 squares with dimension 4x4 one on top of the other

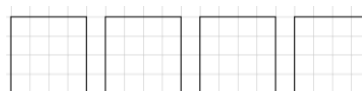
Three-dimensions representation:




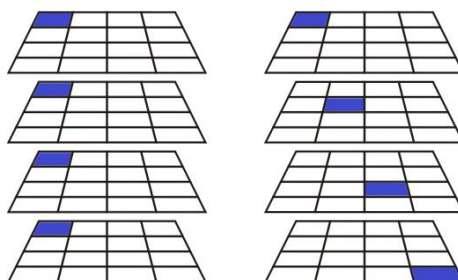
Two-dimensions representation:



or

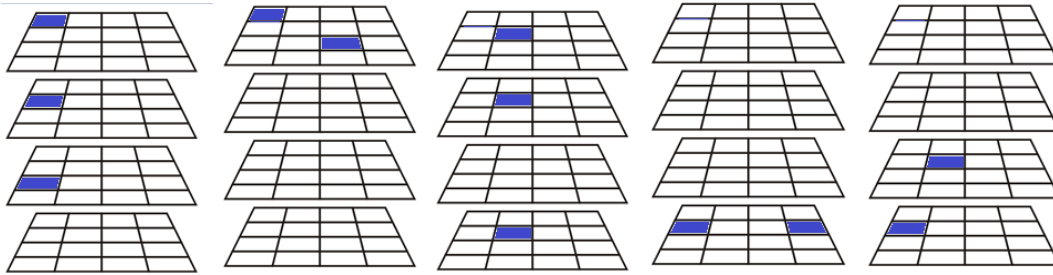


 Winning lines can be formed in all three dimensions! Here are two examples:

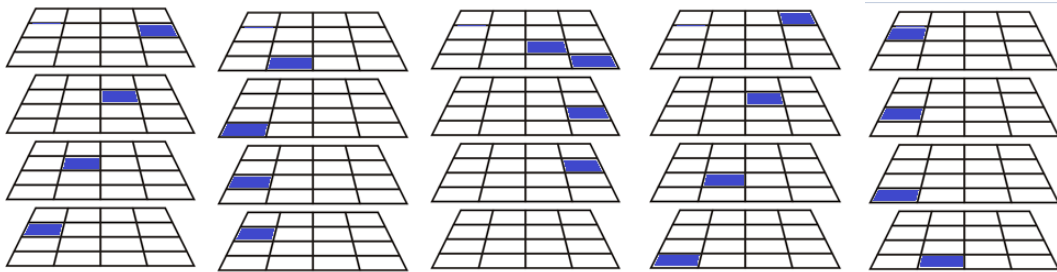


⁴ Since it may not be clear where the marks can be placed during the game, the students who asked for clarification were explained that each of the 64 small cubes is part of the game. So, they can place the marks in the 56 outer and 8 inner small cubes.

1. Complete the following winning lines



2. Indicate which are winning lines and which are not



3. Solve the game by finding the winning strategy. Detail your entire thinking process using the resolution protocol technique.

4. Express mathematically (formula, pattern, routine, ...) the relationships that can happen between the dimensions of the game board and the winning lines.

These empty boards can help you to solve the game.

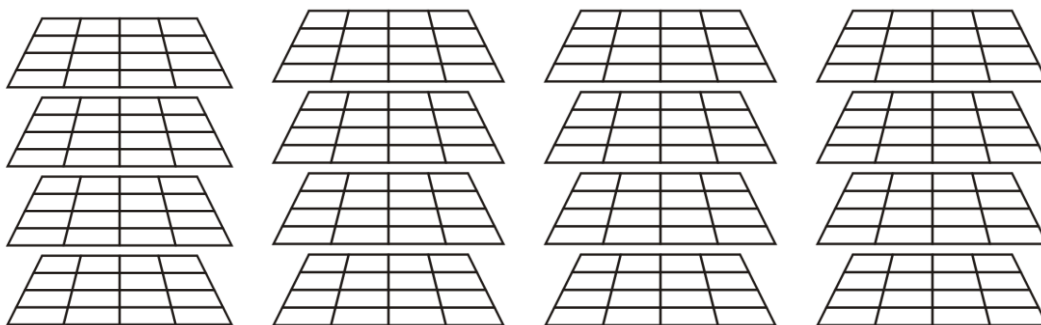


Fig. 5.9 - 3D Tick-Tack-Toe task

Being a two players game, the students worked in pair during a 2 hours session. Students were allowed to use 2D paper board given by the researcher to simulate the game. 6 pair have been video-recorded during the practice session. Due to the limited time committed to the PhD course the 8 PhD students enrolled in the design experiment solved the task in one lesson hour, then they completed it singularly in their homework time, and finally they discuss it in in the next lesson. During the latter practice session, they discussed among themselves about the resolution processes they have followed.

Based on the analysis of the resolution protocols, four students belonging to the first level group, eight students belonging to the second level group and four PhD student (third level group) were selected for an in-depth case study. The PhD students develop a deeper mathematical formalization: they are emblematic students of the entire research group. A video-recording of the discussion session was collected with this aim. During the discussion session, one hour, arose some questions that provided detailed information on the progress of the resolution protocols and on their difficulties. In this dissertation two PhD case studies will be displayed.

5.2.1.3.2 A priori analysis

The 3D Tick-Tack-Toe (Allis, 1994; Golomb and Hales, 2002) is a variation of the most famous Three in a Skate game (Gardner, 1988). This is a two-person game played on a n^k board (i.e. a k-dimensional hypercube of side n) with $n=4$ and $k=3$. Several editions of the 4^3 game are commercially available, also as apps that can be installed on the pc (like the free app “Tic-Tac-Toe Universe 4D” edited by Trump Software). It is a complex game and Patashnik, using a combination of human expert knowledge and a standard search algorithm, solved it for the first time in 1977 (Patashnik, 1980).

For this reason, the resolution of the game was not expected during the design experiment. The researchers were interested in the strategies and the reasoning that emerge during the attempt of resolution. There are several strategies that are useful to solve the game:

- Studying the properties of the board and the boxes:
 - o The board has 9 axes of symmetry and 9 planes of symmetry.

- 7 winning lines pass through each blue box, 4 winning lines pass through the green boxes. The blue boxes are favourable positions.
- Working backward strategy: going backward in the movements of the game from a desired winning line to the present configuration.
- Beginning at the end of the problem: search for the last winning move (Figures 10b and 10c). It is the configuration in which a player has two almost-complete lines at the same time (Fig. 10b). The opponent is forced to block one of the two lines and the player wins by completing the other one (Fig.10c).
- Solving a simpler problem: it can have two meanings:
 - starting with an on-going configuration, i.e. with fewer pegs on the board;
 - solving the game in a smaller board (for example a cube of dimension 3^3).
- Breaking down the problem: breaking down the board into parts.
- Extracting patterns: starting with a particular configuration and trying to find a common rule that allows to achieve the solution.

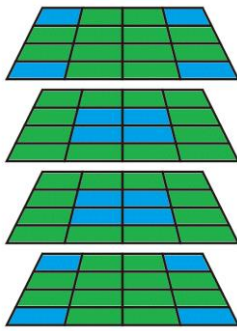


Fig. 5.10a - Favourable positions

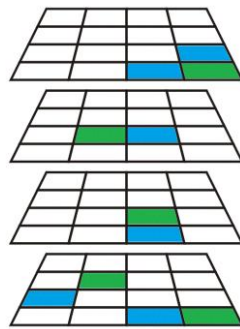


Fig. 5.10b - Winning configuration for blue player

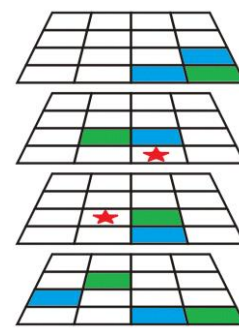


Fig. 5.10c - Green player is forced to put the token in one of the boxes with a star

- Attempts and errors: making an attempt and see if it is possible to get the solution; if it does not work, make a different attempt. This strategy is not advisable for this game, there are many possible movements, the risk is taking a long time to reach a solution.

It is easier for the students answer to the last question of the task: the mathematical formula that connect the number of winning lines to the board dimension. The number of winning lines of a n^k dimension board is:

$$\frac{(n + 2)^k - n^k}{2}$$

Golomb and Hales (2002) give a very interesting intuition proof of this statement:

Given the n^k hypercube think about an $(n+2)^k$ hypercube: this hypercube embed the n^k hypercube; n^k hypercube is extended one unit farther in each direction in each of the k dimensions. [For example, the 3^2 hypercube (Three in a Skate board) extension is the 5^2 hypercube, see figure 5.11a and 5.11b]

Extend the n^k winning lines of a unit in each direction. Each line ends in two border boxes of the $(n+2)^k$ hypercube. Each border box of the enlarged hypercube has only one winning line that passes through it.

Every border box is at the end of a winning line, so the $(n + 2)^k$ hypercube border boxes are in two-to-one correspondence with the winning lines. There are $(n + 2)^k - n^k$ boarder boxes in the enlarged hypercube. So, there are $\frac{(n+2)^k - n^k}{2}$ winning lines in the n^k hypercube.

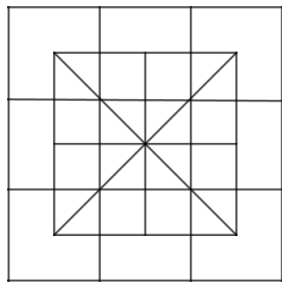


Fig. 5.11a – 3^2 hypercube with winning lines

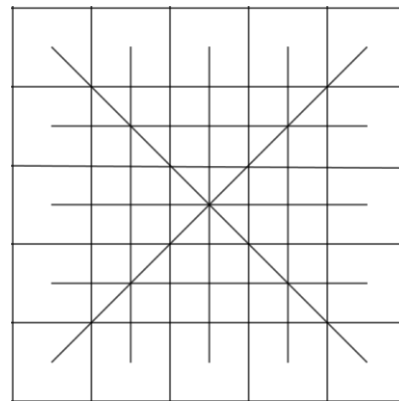


Fig. 5.11b - 5^2 hypercube with 3^2 extended winning lines

5.2.1.4 Mathematical problems

The problems proposed in the fourth design experiment were chosen so that they could be solved by using backward reasoning. The choice of these four problems in particular was dictated by the fact that:

1. The visualization component (which was seen to influence backward reasoning in the first three design experiments) is predominant in their resolution
2. The four problems bring into play different mathematical skills in different mathematical fields. The first problem is related to the study of function graphs. The second problem is a geometric problem that can be traced back to an algebraic equation. The third problem is a construction problem. The fourth problem is a combinatorial calculation problem.

5.2.1.4.1 Data collection settings

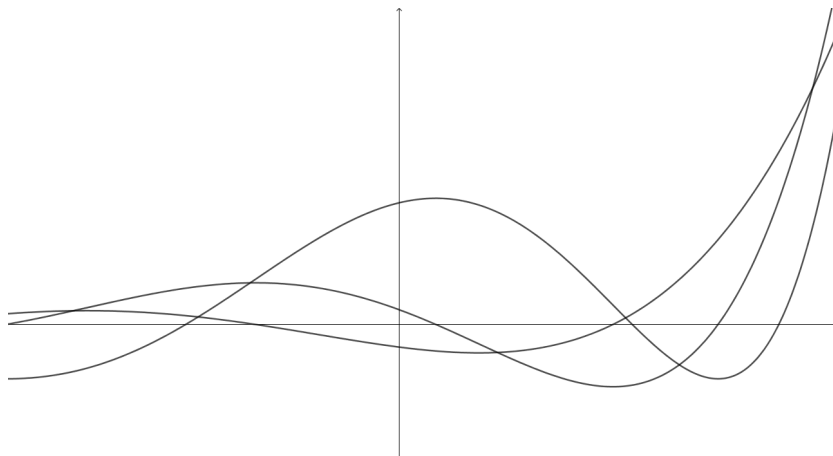
The researchers proposed the task with the four mathematical problems as follow.

Backward Reasoning Problems

Problem 1: Functions

The drawing below shows the graph of three functions.

- A function f
- The derivative of function f
- The primitive of the function



1. Identify the graph of each function by explaining in detail your entire thinking process using the resolution protocol technique.
2. Describe a general method for solving these types of problems.

Problem 2: Triangle and Circle

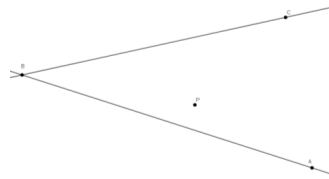
Among all the isosceles triangles inscribed in a circumference, look for that of maximum area.

Solve the problem. Detail your entire thinking process using the technique of resolution protocols.

Problem 3: Geometrical Construction

Given an \widehat{ABC} angle and a P point inside the angle, construct a QT segment, using only a ruler and compass, so that it passes through P and QP is twice PT.

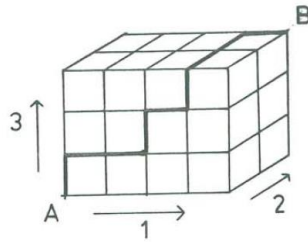
Note: the Q point belongs to BA and the T point belongs to BC.



Solve the construction problem. Detail your entire thinking process using the resolution protocol technique.

Problem 4: Paths

How many 9-section paths, that link point A with point B, are there? Each section must necessarily be travelled in the directions indicated "1", "2" or "3".



Solve the problem. Detail your entire thinking process using the resolution protocol technique.

Fig. 5.12 - Mathematical Problems task

Each Spanish student enrolled in this design experiment worked on his own during a 2 hours session. No video-recordings were made during the practice session.

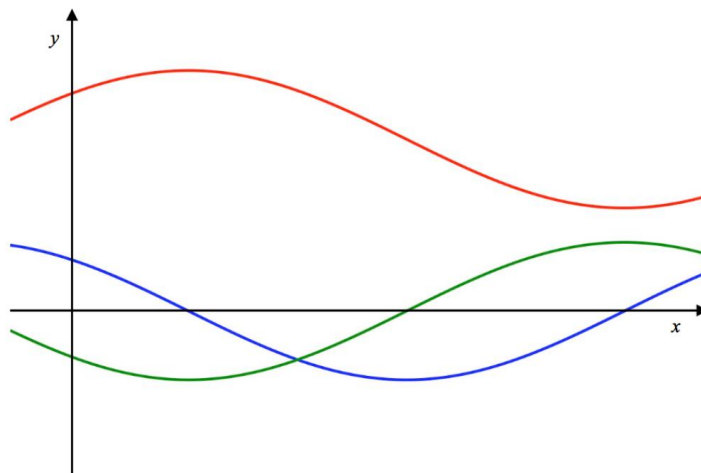
The 16 Italian students involved in this design experiment solved a quite different task. They worked in groups of four students on the task showed in figure 5.13, during a 2-hour session. A video-recording of each group was made. Although the task is slightly different, the resolution processes that emerge are comparable with those of the first problem of the task proposed to the Spanish students.

Problem 0

The drawing below shows the graph of three functions.

- A function f
- The derivative of function f
- The primitive of the function

Identify the graph of each function.



Lorenzo and Francesca want to know how solve the problem they found in the textbook (Problem 0). Rather than settle for just the answer, however, they want a method to use to solve other problems like this one, so they can be prepared for similar questions during the exam.

Think of a method that Lorenzo and Francesca can use to solve problems like the one they found in the book. Your method must work, not only for this problem, but also for problems similar to this one, such as those of the next tabs.

Write a letter to Lorenzo and Francesca in which:

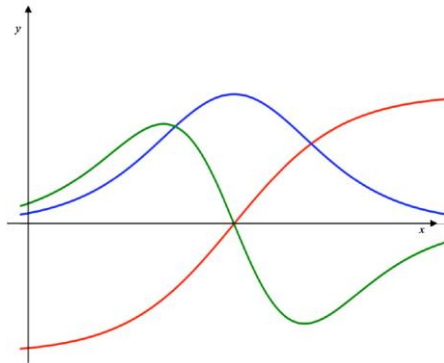
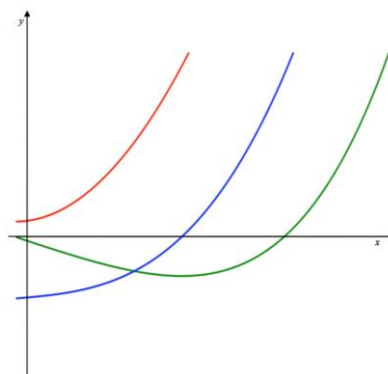
- (1) describe your method,
- (2) explain why it works (under which assumptions, if any),
- (3) show how to use it to solve problem 0 and the following problems (1, 2, 3).

Problems 1, 2, and 3 assignment

The drawing below shows the graph of three functions.

- A function f
- The derivative of function f
- The primitive of the function

Identify the graph of each function.

Problem 1**Problem 2**

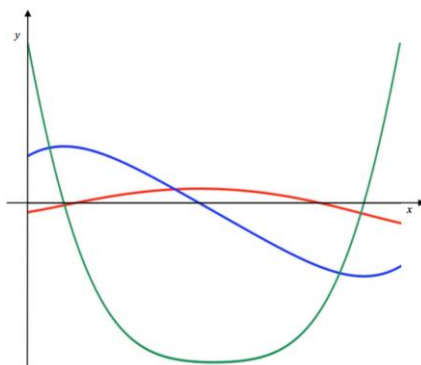
Problem 3

Fig. 5.13 - Italian students' Function Problem task

Based on the analysis of the resolution protocols, 16 Spanish students and three of the Italian groups (12 students), all belonging to the second level group, were selected for an in-depth case study. The choice of cases was based on some protocols' characteristics: existence of moments of backward reasoning use; existence of different backward reasoning strategies; and they were to be protocols with key information and a detailed development. In this dissertation one Italian group and 7 Spanish students' protocols will be shown.

5.2.1.4.2 A priori analysis

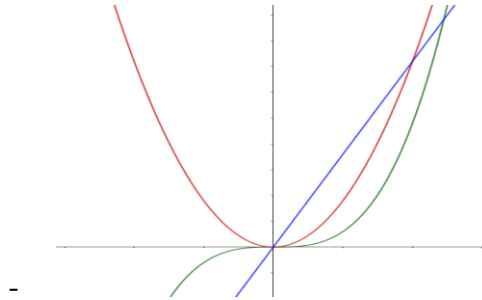
The four problems can be solved using different strategies that are identified in the next sessions. The first problem is an elaboration of Yoon, Thomas, and Dreyfus (2011). While reading several texts on problem solving, the Circle and Triangle problem, the Construction problem and the Paths problem were encountered in Gascón Pérez (1989). For each problem, a solution is proposed.

5.2.1.4.2.1 Functions Problem

There are several strategies that are useful to solve the Function Problem:

- Studying the properties of the graphs:
 - Identifying the maximum and minimum points and the zeroes of each graph.
 - Identifying the monotonicity intervals and the positivity intervals of each graph.
- Working backward strategy: construct the graphs of a specific function, his primitive and his derivative

- Suppose the problem solved: identify the three graphs as primitive, function and derivative and verify that the properties of the graphs match.
- Solving a simpler problem: focus on just two graphs.
- Breaking down the problem: breaking down the graphs according to specific intervals and studying the properties of the graph inside them.
- Solving an analogous problem: the problem can be the same one with three known graphs, for example those in figure 5.14.



- Fig. 5.14 - Quadratic function representation (red graph) with one of his primitive function (green graph) and his derivative function (blue graph)
- Attempts and errors: making an attempt, for example conjecturing the problem solution, and see if it is feasible get the solution; if it does not work, make a different attempt.
- Use a different language: expressing graphs in analytical language.
- Analyse borderline cases: analyse the graph 1 in the first part of Italian students' task.

A possible solution to the Spanish version of the function problem is now proposed:

Function Problem: The drawing below shows the graph of three functions. A function f , the derivative of function f , and the primitive of the function f . Identify the graph of each function.

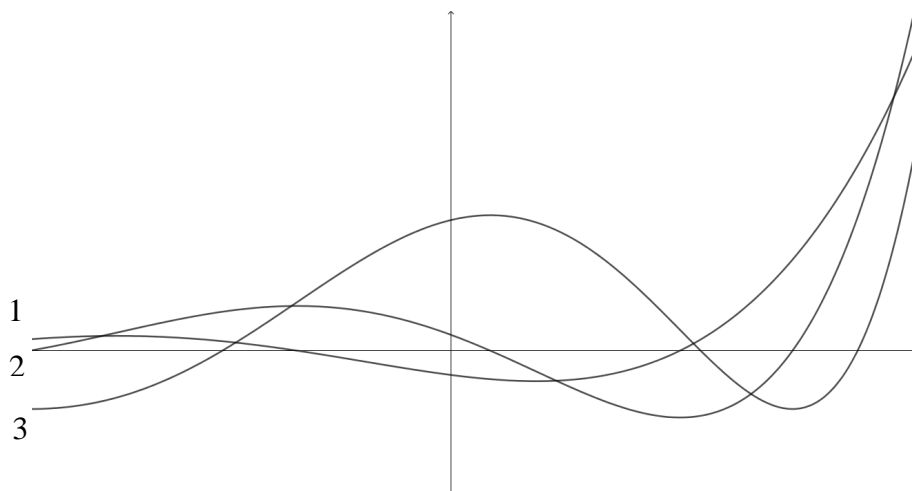


Fig. 5.15 - Graphs of the Spanish Function Problem task

In order to identify the three functions, maxima, minima and zeros of the three graphs are identified. A derivative of a function has zeros in correspondence of a maximum (or a minimum) of its function. In addition, if the function increases in an interval, his derivative is positive; if it decreases, his derivative is negative. In correspondence of the maximum and the minima of the graph 3, the graph 2 has zeros. When graph 3 increase, graph 2 is positive. In correspondence of the maximum and the minimum of graph 2, graph 1 has zeros. When graph 2 increase, graph 1 is positive. Then, the primitive F is graph 3, the function f is graph 2 and the derivative f' is graph 1.

A general method for the solution of this task is shown. It is displayed as indications for a possible solver:

1. Identify maxima, minima and zeros of the three graphs.
2. Chose a graph: naming it g .
3. Observe which of the other two graphs has zeros in correspondence of maxima and minima of g .
 - a. If no function has zeros in correspondence of maxima and minima of g , then $g=f'$.
 - b. If there is a function such that in correspondence of the maxima and minima of g it has zeros, then name it h . Observe the increasing and decreasing intervals of g and the intervals where h is positive and negative.
 - i. If graph h is positive when g increases and is negative when g decreases, then $h=g'$.
 - ii. If the conditions of the point 4.a. do not occur, then: $g=f'$.

Now two pathways open depending on which result is found in point 3: (1) $g=f'$ or (2) $h=g'$.

Supposing that result (1) is obtained, then the method follows this way:

1. Chose a graph different from g : naming it k , the remaining graph is named t .
2. Observe which of the other two graphs has zeros in correspondence of maxima and minima of k .
 - a. If the graph is g and it is positive when k increases and is negative when k decreases, then $k=t'$. Then the result is $t=F$, $k=f$, $g=f'$.

- b. If the graph is t and it is positive when k increases and is negative when k decreases, then $t=k'$. Then the result is $k=F$, $t=f$ and $g=f'$.

Supposing that result (2) is obtained, then the method follows this way:

1. Consider the remaining function. Name it q .
2. Observe which of the other two graphs has zeros in correspondence of maxima and minima of q .
 - a. If the graph is h and it is positive when q increases and is negative when q decreases, then $q=h'$. Then the result is $q=F$, $h=f$ and $g=f'$.
 - b. If there is no function such that in correspondence of maxima and minima of q it has zeros, or the increasing/decreasing intervals of q do not match with the positivity/negativity interval of the other graphs, then $q=f'$. Then the result is $h=F$, $g=f$ and $q=f'$.

5.2.1.4.2.2 Circle and Triangle Problem

There are several strategies that are useful to solve the Circle and Triangle Problem:

- Studying the properties of the geometric configuration with an isosceles triangle inscribed in a circle, adding some geometric elements connected with the configuration: for example, the radius of the circle or the height of the triangle.
- Suppose the problem solved: suppose that the sought triangle is the equilateral triangle and reason starting from that assumption.
- Solving a simpler problem: calculate the area of a specific inscribed triangle.
- Breaking down the problem: breaking down configuration to study its property.
- Use a different language, expressing the configuration elements in algebraic or analytical language.
- Solving an analogous problem: the problem can be “calculate the maximum of a specific function”.
- Attempts and errors: making an attempt, for example express the relationship between the configuration elements, and see if it is feasible to get the solution; if it does not work, make a different attempt.
- Analyse borderline cases: analyse the area of specific triangle.

A very interesting resolution, developed by Gascón Pérez (1989), is now proposed:

Circle and Triangle Problem: Among all the isosceles triangles inscribed in a circumference, look for that of maximum area.

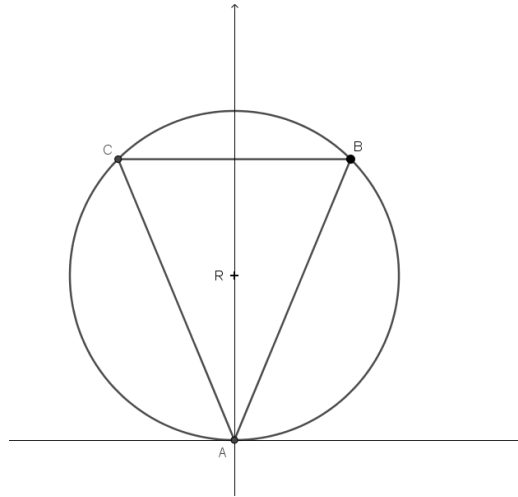


Fig. 5.16 - Triangle and Circle configuration

Considering the configuration of the problem in a Cartesian plane. The circle is tangent to the x-axis in O, origin of the plane; it has radius R and centre $(0, R)$. The family of isosceles triangles inscribed in the circle is that in which the vertex between the two congruent sides is O (see figure 5.16).

The circle equation is: $C: x^2 + (y - R)^2 = R^2$

A point $P: (x_P, y_P)$ which belongs to the curve C [$P \in C$] has coordinates:

$$x_P = \pm \sqrt{R^2 - (y - R)^2}$$

$$y_P = y$$

The problem is reduced to finding the coordinates of the point B which maximize the area of the triangle. $B: (x_B, y_B)$ is the vertex, in the I quadrant, of the general triangle of the family. It has coordinates:

$$x_B = \sqrt{R^2 - (y_B - R)^2}$$

$$y_B = y$$

The area of each triangle of the family is:

$$A = \frac{b * h}{2} = \frac{BC * AH}{2}$$

Expressing the area in general terms:

$$A = \frac{2x_B * y_B}{2} = x_B * y_B = \sqrt{R^2 - (y - R)^2} * y$$

$$A(y) = \sqrt{2Ry - y^2} * y = \sqrt{2Ry^3 - y^4}$$

The expression $\sqrt{2Ry^3 - y^4}$ identify the area of each triangle of the family. The maximum of the function $A(y) = \sqrt{2Ry^3 - y^4}$ is the sought value.

The first derivative of $A(y)$ is:

$$A'(y) = \frac{\frac{1}{2} * 6Ry^2 - 4y^3}{\sqrt{2Ry^3 - y^4}} = \frac{3Ry^2 - 2y^3}{\sqrt{2Ry^3 - y^4}}$$

To find the maximum of the function $A'(y) = 0$ is calculated.

$$A'(y) = \frac{3Ry^2 - 2y^3}{\sqrt{2Ry^3 - y^4}} = 0$$

$$3Ry^2 - 2y^3 = 0$$

$$\begin{cases} y^2 = 0 \\ 3R - 2y = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ y = \frac{3}{2}R \end{cases}$$

$y = 0$ is not a valid value for the resolution, it corresponds to a minimum of $A(y)$. The value $y = \frac{3}{2}R$ correspond to a maximum of $A(y)$.

The searched B coordinates are:

$$x_B = \sqrt{R^2 - (y - R)^2} = \sqrt{R^2 - \left(\frac{3}{2}R - R\right)^2} = \sqrt{R^2 - \frac{1}{4}R^2} = \sqrt{\frac{3}{4}R^2} = \frac{\sqrt{3}}{2}R$$

$$y_B = \frac{3}{2}R$$

The size of the side AB is:

$$AB = \sqrt{\left(\frac{\sqrt{3}}{2}R\right)^2 + \left(\frac{3}{2}R\right)^2} = \sqrt{\frac{3}{4}R^2 + \frac{9}{4}R^2} = \sqrt{3R^2} = \sqrt{3}R$$

The size of the side BC is:

$$BC = 2 * x_B = 2 * \frac{\sqrt{3}}{2}R = \sqrt{3}R$$

Which correspond to the sizes of the sides of an equilateral triangle.

5.2.1.4.2.3 Geometrical construction Problem

There are several strategies that are useful to solve the Geometrical Construction Problem:

- Studying the properties of the final geometric configuration (see figure 5.23)
- Working backward strategy: overturn a known construction to obtain the sought segment (see solution proposed below)
- Suppose the problem solved: suppose that the sought segment is construct, draw it and observe the configuration.
- Breaking down the problem: breaking down configuration to study its property.
- Solving an analogous problem: find an analogous construction.
- Attempts and errors: making an attempt, for example add some elements to the configuration to try to observe something known, and see if it is feasible get the solution; if it does not work, make a different attempt.
- Analyse borderline cases: analyse the of a segment perpendicular to of of the side.

A possible resolution is now proposed, it the construction process of the trisection segment is overturned:

Geometrical Construction Problem Given an \widehat{ABC} angle and a P point inside the angle, construct a QT segment, using only a ruler and compass, so that it passes through P and QP is twice PT . Note: the Q point belongs to BA and the T point belongs to BC .

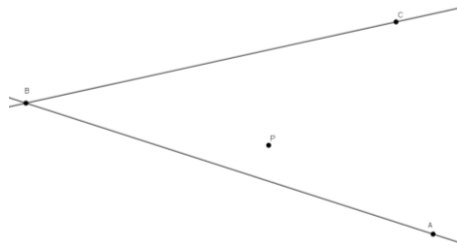


Fig. 5.17 - Starting configuration of Geometrical Construction Problem

Construct the perpendicular line to AB (side of the angle) passing through P.

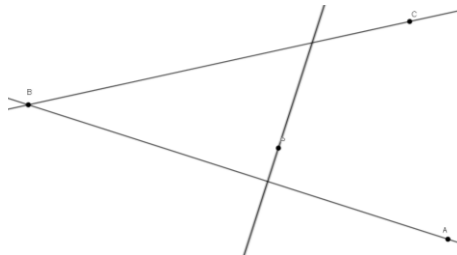


Fig. 5.18 - First step: construct the perpendicular line to AB passing through P

Construct the circle with centre P and passing through the intersection point between the perpendicular line and the side AB.

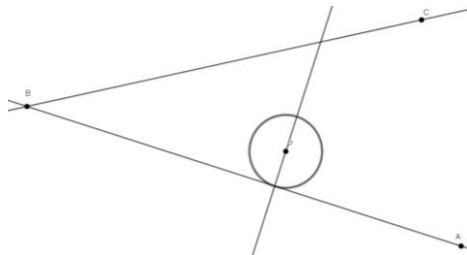


Fig. 5.19 - Second step: construct the circumference with centre in P and passing through the intersection point between perpendicular line (step 1) and side AB

Construct a second circle. It has the centre in the intersection point between the first circle and the perpendicular line. It passes through P.

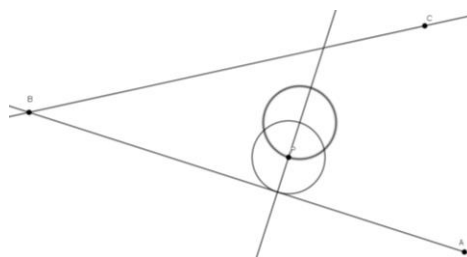


Fig. 5.20 - Third step: construct the circumference with centre in the intersection point between perpendicular line (step 1) and first circumference (step 2), and passing through P

Construct the parallel line to AB that pass through the intersection point between the second circumference and the perpendicular line (different from P). This line cut the side BC of the angle in a point T.

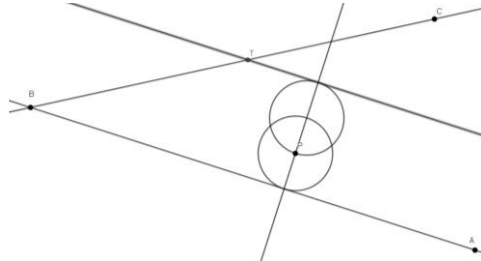


Fig. 5.21 - Fourth step: construct the parallel line to AB passing through the intersection point between perpendicular line (step 1) and second circumference (step 3) (different from P). It cut side BC in T.

Construct the line TP. The intersection point between TP line and the side AB is the sought point Q.

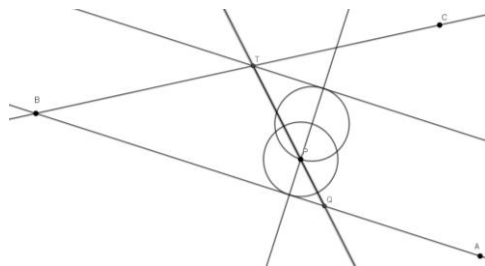


Fig. 5.22 - Fifth step configuration of Geometrical Construction Problem

The sought segment is QT

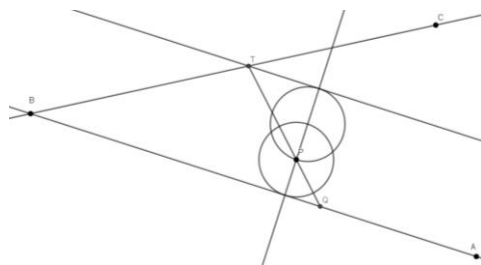


Fig. 5.23 - Final configuration of Geometrical Construction Problem

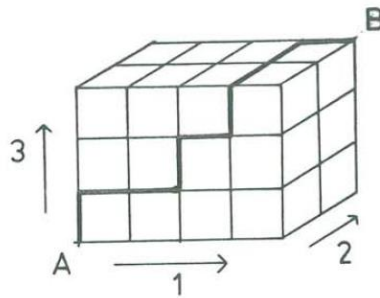
5.2.1.4.2.4 Paths Problem

The Paths Problem is a combinatorial problem, there are several strategies that are useful to solve it:

- Studying the properties of the paths' configuration.
- Breaking down the problem: it can develop in two ways
 - o breaking down the configuration of the paths to study their property.
 - o Breaking down the parallelepiped and solve the problem "piece by piece".
- Solving an analogous problem: recognize the problem like a combinatorial one.
- Solve a simple problem: reduce the size of the parallelepiped.
- Attempts and errors: making an attempt, for example group the paths elements, and see if it is feasible get the solution; if it does not work, make a different attempt.
- Analyse borderline cases: analyse paths that travel on the side of the parallelepiped or on its the faces.
- Make a systematic study of all cases: count all the paths one by one.

A possible resolution is now proposed:

Paths Problem How many 9-section paths, that link point A with point B, are there? Each section must necessarily be travelled in the directions indicated "1", "2" or "3".



There are 9 sections: 4 sections to the right, 2 sections to the bottom and 3 upwards sections.

This problem can be interpreted as a combinatorial problem. In particular, the sought number can be found calculating the number of permutations with repetitions. There are three set of repeating elements: a, b and c. a is the set with right-sections, it has 4 elements; b is the set with bottom-sections, it has 2 elements; and c is the set with upwards sections, it has 3 elements. The total number of elements is 9.

$$P_n^{(a,b,c)} = \frac{9!}{a! b! c!}$$

$$P_9^{(4,3,2)} = \frac{9!}{4! 3! 2!} = 3 * 2 * 7 * 6 * 5 = 1260$$

5.2.1.5 An overview

The following table want to recapitulate the different research settings and the characteristics of the tasks to better understand the evolution of the research project.

Design experiment		Task type		Data collection settings				Students				
Number	Task	Strategy game	Mathematical	Work in group	Work alone	PC suite	Video-recordings	First level	Second level	Third level	Total	Case studies
1	Triangular Solitaire	✓	-	-	✓	-	✓	6	44	-	50	6
2	Maude task	✓	✓	✓	✓	✓	✓	-	23	-	23	5*
3	3D Tick-Tack-Toe	✓	✓	✓	✓	-	✓	114	63	8	185	16**
4	Mathematical Problems	-	✓	-	✓	-	✓	9	73	-	82	28***

Tab. 5.1 - Design experiment settings

* in Maude task two pairs, one of them collaborating with a classmate.

** in 3D-Tick-Tack-Toe task 12 students worked in pair and 4 PhD students worked alone.

*** in Mathematical problem task 16 Spanish students worked alone and 3 Italian groups (4 students each).

In order to illustrate the results in this dissertation will be displayed:

- Triangular Peg Solitaire task: one case study (1 student);
- Maude task: one case study (3 students involved);
- 3D Tick-Tack-Toe task: 2 case studies (2 students);
- Mathematical problems task: 8 case studies (4 Italian and 7 Spanish students).

5.2.2 Design of data analysis tools

The data analysis was carried out throughout the entire research project in accordance with the following steps:

1. Resolution protocols quantitative analysis for each design experiment;
2. Case studies resolution protocols qualitative analysis for each design experiment;
3. General comparative analysis of the design experiments results.

5.2.2.1 Quantitative data analysis

For each design experiment, a quantitative analysis of the resolution protocols was made. The aim of this analysis was: understand how often specific moments of backward reasoning and strategies are developed in the various tasks.

The quantitative analysis was made, firstly, listing all the different backward reasoning moments and backward reasoning strategies observed in the protocols, and then, analysing the frequency of those items in the design experiment. This allows to establish the percentage of occurrences of backward reasoning moments in the group of students.

5.2.2.2 Qualitative data analysis

The qualitative analysis was developed using different analysis model deriving from the theories that frame this research project. The analysis of the resolution protocols underwent an evolution during the research project. In fact, the theoretical framework, that support the research project, has evolved in the transition from one design experiment to another. This evolution has led to the development of the final theoretical framework exposed in Chapter

4. This is the result of a networking between the Game Theory Logic (Hintikka, 1999) and the Abstraction in Context theory (Dreyfus et al., 2015) and the hybridization of this network with Commognition perspective (Sfard, 2008).

5.2.2.2.1 Analysis models

Two analysis models derive from the Game Theory Logic: the Hintikka's Interrogative Model (1986) and the Finer Logic of Inquiry Model (Soldano, 2017), a refinement of the former. These models allow to classify each line of thought according to a specific characteristic defined by the model. Through Hintikka Interrogative Model (figure 5.24a) the dialogical game between the two players involved (the student and the task if he works alone, or the different students if they work in groups) is described according to specific move. This classification allows to describe the entire resolution protocol in a logic and strategic way. After the analysis of the first group of protocols (related to the task Triangular Peg Solitaire), it emerged that some lines of thought could not be classified according to the model. They were those lines related to the application of the rules of the game. It was therefore decided to integrate the model by adding an element: *Standard rules* (see figure 5.24b). the new item was also useful for subsequent cases.

Standard rules: Players apply the standard rules of the environment in which they are immersed in order to get closer to the goal of the task.

Hintikka's Interrogative Model
Initial Move
Deductive Move
Interrogative Move
Assertoric Move
Defining Move

Fig. 5.24a - Hintikka's Interrogative Model

Modified Hintikka's Interrogative Model
Initial Move
Deductive Move
Interrogative Move
Assertoric Move
Defining Move
Standard Rules

Fig. 5.24b - Hintikka's Interrogative Model
modified by the researchers

Through the Finer Logic of Inquiry Model, it is possible to classify each line of thought according to three different group of characteristics: general observable actions, specific observable actions and cognitive modalities (see figure 5.25). Since this model was used only for written protocols analysis each line of thought was classified according to the specific observable actions and the cognitive modalities.

Finer Logic of Inquiry Model		
Observable actions		Modalities
<i>General</i>	<i>Specific</i>	
Verbal	Question	Ascendant
Handwritten	Affirmation	Neutral
Gestures	Conjecture	Descendant
Others (gaze, ...)	Exploration	Detached
Silent	Control	Logical Control
	Plan formulation	Deductive
	Deductive step	
	Logical chain	

Fig. 5.25 - Finer Logic of Inquiry Model

One analysis model derives from the Abstraction in Context Theory: the RBC-model. With AiC theory, the concept of *epistemic action* was introduced. The resolution protocols were divided into epistemic action, according to its definition. It is a better way to identify the protocol lines; before, the protocols were divided according to the student's sentences. Each epistemic action is classified like *recognizing*, *building-with* or *constructing* (see figure 5.26a).

The fourth type of analysis that was developed was the analysis of the discourse, in the Commognition perspective. All the objectification moments involved backward reasoning throughout the resolution protocol were identified; for each of them, the discursive devices used were identified (see figure 5.26.b).

RBC-Model
Recognizing
Building with
Constructing

Fig. 5.26a – RBC-Model (Abstraction in Context)

Objectification devices
Saming
Encapsulating
Reifying

Fig. 5.26b – Discursive devices of the objectification process

5.2.2.2.2 Settings of analysis models

Thirty-five resolutions protocols and nine video-recordings were analysed with the analysis models. The protocols were split in resolution phases, where necessary, according to Polya (1945) subdivision of problem solving; the video-recordings were split into "episodes", in which a major goal is pointed out. Backward reasoning moments have been identified in each resolution protocol or video-recording episode.

Then, all the sentences are divided into epistemic actions, and each of them is classified using the analysis model categories. The epistemic action involved in backward reasoning moments were classified also with the categories extrapolated from the analysis of the literature on the phenomenon. Each of them can be associated to one or more backward reasoning characteristics identified in Chapter 3. They are shown in the following table.

Backward Reasoning
Direction backward
Breakdown
Cause-Effect Relationship research
Transformative
Introduction of auxiliary elements
Solution formulation

Fig. 5.27 - Backward reasoning characteristics

In addition to the features highlighted in Chapter 3 (breakdown, cause-effect relationship research, transformative and introduction of auxiliary elements) it was necessary to

introduce “going backward” and “solution formulation”. The first refers to those moments when students go backward in strategy games, developing the steps of the game in reverse way. The second refers to the creation of the solution object at the end of the backward process; it will then be verified in the following phases.

The analysis settings with the theoretical models, following the theoretical framework, evolved during the research project. The protocol of the first design experiment (Triangular Peg Solitaire) was analysed using the Hintikka’s Interrogative Model (HIM) and the Finer Logic of Inquiry Model (FLIM). The analysis with these two models was a priori thought to be sufficient to characterize epistemologically and cognitively the backward reasoning.

In the second design experiment (Maude task) three resolution context are involved. In fact, the task asks to implement in programming language the Triangular Peg Solitaire; the students have to transform game rules in computational object. To do so they pass through three different resolution contexts: informal context (related to the game), mathematical context (related to purely mathematic representations) and computational context (codification in Maude programming language). The symbols, diagrams, and words used by the students are observed to provide evidence of the context involved in their works. To analyse the episodes of the video-recording of the section, it was chosen to interpret the HIM in relation to the three resolution contexts.

Realizing that it was not possible to characterize exhaustively the cognitive dimension of backward reasoning, a deeper theoretical investigation was done, and the Abstraction in Context theory was networked with the Game Theory Logic. In the third design experiment (3D Tick-Tack-Toe) the HIM was used in combination with the RBC-model, which is considered more suitable for a cognitive approach. In this experiment the analysis of the resolution contexts, considered less relevant, was abandoned. The FLIM analysis was abandoned too, considering that, for the objective of the research project, the model doesn't seem to give particularly significant results compared to the analysis with the HIM (see Chapter 10, section 10.1.1.1 for details).

Realizing that the results of the analysis lacked a linguistic characterization, in the fourth design experiment (Mathematical Problems), the interpretation of Commognition was added to complete the analysis framework.

At this point, the analysis model was considered more satisfactory: the HIM allows to characterize a logic-strategic dimension of backward reasoning, the RBC-model an epistemological-cognitive dimension and the Commognition a linguistic-cognitive dimension.

Considering the final multidimensional analysis model, a second analysis on the protocols of the first, second and third design experiments was developed to complete the framework. For example, the resolution protocols of the first experiment were first analysed according to the Hintikka's Interrogative Model in combination with the Finer Logic of Inquiry Model; later this analysis was integrated with the Abstraction in Context model and the Commognition approach.

Hintikka's Interrogative Model	RBC-Model	Objectification devices
Initial Move	Recognizing	Saming
Deductive Move	Building with	Encapsulating
Interrogative Move	Constructing	Reifying
Assertoric Move		
Defining Move		
Standard Rules		

Fig. 5.28 – Multidimensional analysis model

The analysis evolution can be summarized in this table:

n	Design Experiments	Models of Analysis				
		First analysis			Second analysis	
		HIM	AiC	Commognition	AiC	Commognition
1	Triangle Solitaire	✓			✓	✓
2	Maude task	✓			✓	✓
3	3D Tick-Tack-Toe	✓	✓			✓
4	Mathematical Problems	✓	✓	✓		

Tab. 5.2 – Design experiments analysis

At the end of the second analysis of the four design experiments data, a comparative analysis of the results was made, obtaining more general information that allowed to categorize the backward reasoning at a cognitive level (see Chapter 10).

5.2.3 Structure of the data analysis

In the next chapters a posteriori analysis of the four design experiments is shown. In the analysis of student' productions, the methodology described in this chapter is used.

For every design experiment a global analysis is firstly presented. Examples of backward reasoning moments, extrapolated from the study group, are shown with the respective percentages of use. For each moment type a representative example, part of a student protocol, is shown. The excerpt extrapolated is analysed as the case studies (see below the data presentation modalities).

In the second part of each chapter, the case studies are presented. In each chapter, it is specified what type of source was used for the analysis: whether the resolution protocol or video-recording. The cases are chosen from the highest-level group involved in the design experiment. For example, in the first design experiment the students involved belonged to the first level group and the second level group: the case study belongs to the second level

group. For each case study, the final analysis with the complete multidimensional analysis model (see section 5.2.2) is shown. The abbreviations used for the analysis models are displayed in the following table.

Backward Reasoning		Hintikka's Inquiry Model		RBC model	
Characteristics	Abb.	Moves	Abb.	Epistemic actions	Abb.
Breakdown	D	Initial Move	Init	Recognizing	R
Cause-Effect Relationship	E	Deductive Move	Ded	Building-with	B
Transformative	T	Interrogative Move	Int	Constructing	C
Introduction of auxiliary elements	X	Answer to the interrogative move	answ (or Int/a)*		
Solution formulation	SF	Assertoric Move	Ast		
Direction backward	G	Defining Move	Def		
		Standard Rules	Ru		

*where the Interrogative move and its answer are on the same line

Tab. 5.3 - Abbreviations of the multidimensional analysis model

The analysis is shown using a table that display in each column: the line protocol number, the name of the student involved (if the case study is about a student group), the lines of protocol (or the transcription of the video-recording), the backward reasoning character, the Hintikka's Interrogative Model moves, and epistemic action classification from Abstraction in Context model (RBC model). The table is accompanied by a brief comment in which the choices made for the classification are justified; in the same comment the discourses devices used in the moments in which the backward reasoning appears (objectification) are highlighted. If the line cannot be classified according to the analysis model, a dash is written in the corresponding box (-).

An example of protocol line of single case study:

Line	Protocol	BR	HIM	RBC
1.1	I seem to remember that the usual tick-tack-toe has a winning strategy.	-	I	R

Tab. 5.4 - Single case study analysis example

An example of protocol line of group case study. In section “Transcription” the *times new roman italic* is used for the sentences spoken by students, the times new roman regular for the gestures made by students who influence the resolution of the problem. For each group study, the transcription of the video-recording is analysed.

Line	Student	Transcription	BR	HIM	RBC
1.3	Fe	<i>Its derivative should grow...</i> [she points graph 1]	E	I	R

Tab. 5.5 - Group case study analysis example

In the first design experiment the Finer Logic of Inquiry Model analysis is shown too. For this reason, the protocol is shown entirely before the table analysis; the table contains the number of lines protocol, the backward reasoning characteristics, the HIM moves, the specific actions made (FLIM) by students, the cognitive modalities developed (FLIM), and the epistemics action classification (see table 5.7). Below the table, a short commentary justifies the choices made for the analyses with the multidimensional model (+ FLIM) and specifies the discursive devices used by the student. The following table displays the abbreviations used for the FLIM analysis.

Finer Logic of Inquiry Model			
Actions	Abb.	Modalities	Abb.
Question	No need for an abbreviation	Ascendant	A
Affirmation		Neutral	N
Conjecture		Descendant	D
Exploration		Logical Control	LC
Control		Detached	DT
Plan formulation		Deductive	DD
Deductive step			
Logical chain			

Tab. 5.6 - FLIM abbreviations

Line	BR	HIM	FLIM		RBC
			Actions	Modalities	
15	-	Ass	Plan formulation	N	B

Tab. 5.7 - First design experiment case study example

5.2.4 Reliability and validation of data analysis

The analysis was iterative and inductive to discover and explore themes, categories, patterns and relationships (Cohen, Manion, & Morrison, 2011) in the backward reasoning; it was based on natural environment of lessons activities. To ensure the reliability and validation of this research project's results (Lincoln & Guba, 1985), its analysis process also included discussion and comparisons with researchers in order to minimize biases and ensure accuracy. In the analysis, methodologically, the triangulation of sources and judges (thesis directors and professors involved in the experimentation) was considered. The combining of different perspectives on the same empirical context, allows to validate and make reliable data (Jensen, 2002). Specifically, some parts of the data were discussed at the level of experts in a broader scientific community: for example, case study data from the first design experiment were discussed at the *Conference of the International Network for Didactic Research in University Mathematics (INDRUM)* in 2018 (Barbero and Gómez-Chacón,

2018) and at the *Congress of the European Society for Research in Mathematics Education (CERME)* in 2019 (Gómez-Chacón and Barbero, 2019); while in the *Mathematics* journal (Barbero, Gómez-Chacón and Arzarello, 2020) some results from the third design experiment were presented.

6

**TRIANGULAR PEG SOLITAIRE
ANALYSIS**

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TRIANGULAR PEG SOLITAIRE ANALYSIS

6

In this chapter the results of the analysis of the first design experiment are shown. Briefly the design experiment settings are summarized in table 6.1.

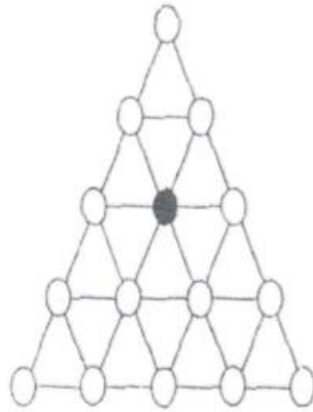
Task type		Data collection settings				Students				
Strategy game	Mathematical	Work in group	Work alone	PC suite	Video-recordings	First level	Second level	Higher level	Total	Case studies
✓	-	-	✓	-	✓	6	44	-	50	6

Tab. 6.1 - First design experiment settings

The task proposed in this design experiment is the Triangular Peg Solitaire, as shown in figure 6.1. Although the strategy game has a triangular board and a mathematical structure and is solved according to the resolution phases similar to those of mathematical problems, students are not asked to investigate these features of the game but only to solve it. Fifty students from first (6) and second (44) level group were involved in the design experiment (see Chapter 5). Their resolution protocol will be considered together. Six case studies (from second level group) were carried out analysing in depth the resolution protocols and interviewing (with video-recordings) two of them, but only one is shown in this dissertation (Student-M, section 6.2). A short presentation of Student-M in section 6.2 clarify this choice.

Triangular Peg Solitaire

The Triangular Peg Solitaire requires a board with 15 boxes as shown in the figure.



These are the rules:

3. Place the peg in all boxes, except in the one marked in black.
4. The player can move as many pegs as are the chances of jumping a peg, adjacent to an empty box (along the line); at the same time he "eats" and retreats from the board the peg that was jumped. All pegs will move in this way. Pegs can move around the board.

Target: The player wins when there is only one peg on the board.

Solve the game by finding the winning strategy. Detail your entire thinking process using the resolution protocol technique.

Fig. 6.1 - Triangular Peg Solitaire Task

The chapter is structured in the following way. Firstly, the analysis of the whole group is presented; the moments of backward reasoning development are shown through an in-depth analysis of some excerpts (section 6.1). Then a case study is displayed (section 6.2); it consists in the in-depth analysis of the protocol of Student-M, integrating it with some elements deriving from the interview did with her. Finally, a general discussion is developed (section 6.3).

6.1 Analysis of the whole group

In relation to the level of the students involved in the experiment, no major resolution differences were found. Analysing the 50 resolution protocols, three moments in which backward reasoning is developed are identified:

1. Reverse Game. The student starts with only one peg on the board and takes the steps backward. 33 students use this strategy (66%).
2. Attempt to remove a specific peg. The students imagine the possible path (backwards) to remove a specific peg from the board. If they succeed, they continue; if they fail, they declare that have lost the game and restart from the beginning. 28 students use this strategy (56%).
3. Search for the final movements. The students search for a configuration with 3-4 pegs on the board in a regressive way. This configuration can be a possible intermediate configuration to reach in a progressive way. 15 students use this strategy (30%).

In the following sections, for each moment of backward reasoning recognized in the group, an example will be presented. The shown excerpts belong to the second level group students and have been translated from Spanish by the author. Each protocol has been divided into phases according to the Polya's subdivision: familiarisation, exploring and carrying out the strategy (phase in which strategies are repeatedly developed and applied), results verification. The excerpts reported below belong to the second phase: exploring and carrying out the strategy. According to the Finer Logic of Inquiry Model classification they belong to the Inquiry Component. Each protocol is divided in lines, each figure is associated with a line (for example: figure 15 is associated to line 15). Within the excerpt the lines (or the figures) where the backward reasoning is identified are put in *times new roman italic* (except when the student simply goes backward with the steps, without applying any strategy). Each part of the excerpt has a short comment to identify the characteristics according to each analysis model (HIM, FLIM and RBC). At the end of each excerpt the analyses are summarized in a table. In the final comment a short description of the backward reasoning is made, and it is associated to the discursive devices (see Chapter 4 and 5) of the objectification (Commognition approach).

6.1.1 Reverse the game

Thirty-three students of the group, after applying different resolution strategies, decide to try to solve the game starting with a single peg on the board and applying the rules of the game backwards. These students are applying two specific resolution strategies together: working backward and start from the end of the problem strategy. This is the case of student T36, she belongs to the second level group.

The T36 student applies the strategy and makes two attempts: one starting from position 5, the other starting from position 1. At this point he realizes that he can't start from any position, so he analyses all the possible "last moves" of the game from the various positions and divides the board into boxes with different characteristics. In this process he goes to try to break down the board into positions with different characteristics, looking for the properties of each position. Finally, he will decide that the most favourable position to start applying the working backward strategy is one of the three positions on the board that are located at the midpoint of the sides of the triangle.

T36 resolution protocol excerpt

Line 14 I can't think of where I might be missing, so I'm going to try the reverse game.

Line 15 Now I'm going to name the holes that have pegs on them (5) and the rest of the numbers are holes.

Figure 15

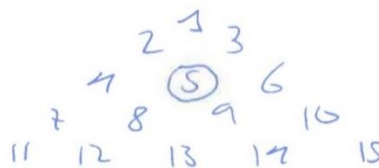


Fig. 6.2 – Figure 15 (Student T36 protocol)

Line 16 Starting from 5 I can't make it, I have 4 holes left.

Line 17 I'm starting from a corner, for example 1. I can't make it either, I don't think it's helping me.

In this first part of the excerpt, the student starts the resolution by deciding to work backwards; then she defines the notation she will use and tries twice (descendant modality)

to obtain a solution. By making control (ascendant modality) over the failure. From the RBC point of view, she recognizes the backward reasoning strategy and a specific notation. Then she builds(-with) the backward path by arriving at two conclusions of failure.

-
- Line 18* I'm already more familiar with the game. Let's think about the math behind the game by analysing each type of position.
- Line 19* Corners, position 1, 11, 15. If we have a piece in any of these positions, for example 1, we can only take it out if we have pieces in 2 or 3 and not in 4 or 6 respectively.
- Line 20* Sides: 2, 4, 7, 12, 13, 15, 10, 6, 3 for example 2.
- Line 21* Wait, it can't work the same as those adjacent to a corner.
- Line 22* I separate pieces 2, 7, 12, 14, 10, 3. You can jump if there are pieces in 4 or 5 and not in 7 or 8 respectively.
- Line 23* Pieces in 4, 13, 6. you can jump if there are pieces in 2, 7, 8, 5 and not in 1, 11, 13, 16 respectively.
- Line 24* These pieces look more favourable.
- Line 25* I'm going to draw these last ones which can be more messy.

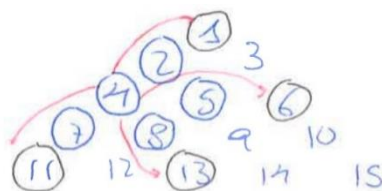


Figure 25

Fig. 6.3 - Figure 25 (Student T36 protocol)

At this stage, the student decides to break down the board, analysing it in an interrogative process probably driven by the question “what moves allow me to remove a peg from a certain position?” Exploring the game (descendant modality), she extrapolates information about the types of positions (ascendant modality), making control over her actions observing the board (ascendant modality). Finally, she assumes that positions 4, 13 and 6 are the most favourable, defining them graphically. From the RBC point of view, she recognizes the importance of analysing the board and its geometric characteristics. Through reasoning on possible movements, she constructs a classification of positions.

The analysis of this protocol excerpt is summarized in table 6.2.

Protocol	BR	HIM	FLIM		RBC
			Actions	Modalities	
L14	G+X	Init	Plan elaboration	N	R
L15 – F	G	Def	Affirmation	N	R
L16	G	Ru	Exploration + Control	D+A	B+C
L17	G	Ru	Exploration + Control	D+A	B+C
L18	D	Init	Plan elaboration	N	R
L19	D+G	Int/answ	Exploration	D+A	B+C
L20	D+G	Int/answ	Exploration	D+A	B+C
L21	D	Int/answ	Control	A	R
L22	D+G	Int/answ	Exploration	D+A	B+C
L23	D+G	Int/answ	Exploration	D+A	B+C
L24	D	Ast	Conjecture	N	R
L25 – F	G	Def	Control	N	B

Tab. 6.2 - Student T36 excerpt analysis

The backward reasoning develops within the questioning process breaking down the problem. The strategy of going backward is introduced like an auxiliary element. Supposedly before line 16 and line 17 the student has made an assertoric move conjecturing the last piece on the board in position 5 and then in position 1. Supposedly before line 21, line 22 and line 23 the student asks himself “How do I get to that positions?”. The *saming* discourse device is used in lines 19, 20, 22 to put together the holes that have the same geometric characteristics.

6.1.2 Attempt to remove a specific peg

Twenty-eight students of the group apply backward reasoning during the forward resolution process. They set themselves the goal of releasing a specific position of the board starting from the configuration in which they arrived at that moment. They use working backward

strategy and start from the end of the problem strategy, considering the sub-problem “remove this specific peg”. This is the case of student T43, she belongs to the second level group.

Student T43 is solving the game forward and stops to think about the configuration she arrived at. She immediately observes that she cannot win the game because she has one peg left isolated in position 15 (corner of the board). Then she stops to think about whether there is any series of movements that allow her to free position 15. Thinking backward, she understands that in order to free position 15, she will have to place a peg again in one of the positions adjacent to position 15, so that she can make a jump and free the corner. She then goes back to thinking about the previous configuration and understands that, with 2 movements, she can occupy the position adjacent to box 15 and then move the peg. She then makes a similar reasoning in order to remove peg 1 from the other corner of the board.

T43 resolution protocol excerpt

Line 20 This is getting me nowhere because now it's impossible to finish, I have to eat 15.

Figure 20

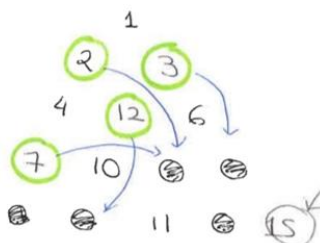


Fig. 6.4 - Figure 20 (student T43 resolution)

Line 21 So, my idea now is going to be "never leave the corner alone". In fact, [I don't have to leave] any peg [alone]. But I'll set the target in the corners first.

Line 22 I'm going back thinking about the initial move.

Line 23 I am able to find a move, or a set of them, that will free me a corner that is now in danger. Defining “in danger” that it can never have a hole, unless one of the pegs covers one of the holes around it, again.

The student arrives at a configuration (figure 20.1) with some forward steps. She makes control over the resolution, and understands that she has to remove the peg in 15 position if

she wants to continue (ascendant modality). She sets herself the goal of ‘not leaving the corners isolated’ (Assertoric move). She explores, by flipping the game, from position 15 to her configuration (descendant modality) until she determines which movements allow her to remove the peg in 15 position (ascendant modality). From RBC point of view, she recognizes that she has to quit the peg in 15 position, then she explores (building-with) until reach a configuration to achieve her objective (constructing).

Line 24 I go back to the previous configuration [to figure 20.1].

Line 25 I'll move seven and then three, so we have:

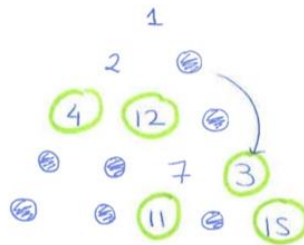


Figure 25

Fig. 6.5 - Figure 25 (Student T43 resolution)

Line 26 ..so, now I can move 15 and take off a corner.

Line 27 I move 15.

Line 28 The idea now is to move 2 and then move 7 so I can move 1, taking off the third corners.

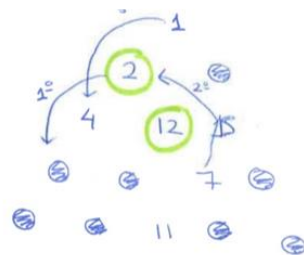


Figure 28

Fig. 6.6 - Figure 28 (Student T43 resolution)

She decides to go back at the starting configuration and apply the found moves (descendant modality), she understands that he can remove the peg in position 15 (ascendant modality), and she do it. A similar reasoning for the peg in position 1 (ascendant modality) is briefly explained. From the RBC point of view, a series of building-with actions lead the student to construct the desired move.

The analysis of this protocol excerpt is summarized in the table 6.3.

Protocol	BR	HIM	FLIM		RBC
			Actions	Modalities	
L 20 – F	-	Ru	Exploration + Control	A	R
L 21	-	Ast	Plan elaboration	N	R
L 22	G+E	Ru	Exploration	D	B
L 23	G+E	Int/answ	Control	A	C
L 24	-	Init	Plan elaboration	N	B
L 25 – F	-	Ru	Exploration	D	B
L 26	-	Answ	Control	A	B
L 27	-	Ru	Exploration	N	C
L 28 – F	G+E	Int/answ	Plan formulation	A	R

Tab. 6.3 - Student T43 excerpt

The backward reasoning develops within the questioning process searching from some moves that lead to quit a specific peg (cause-effect relationship). Supposedly before line 21 and 28 the student asks himself “How do I quit the peg in that position?”. The encapsulating discourse device is used in lines 2 and 25, while the student considers the two step like one only path towards the end of her sub-problem. Before line 25 she probably applied the same routine that she developed in line 22. The *reifying* discourse device is used to pass from talking about moving throughout the board to talking about the move that “will free me a corner” (line 23).

An example of the case in which the student imagines the possible path (backwards) and he notice that he lost the game can be seen in the case study resolution. Similar to the previous case, some students apply backward reasoning during the forward resolution process and anticipate the game's failure. By controlling their future moves they realize that they can't succeed in solving the task because they can no longer remove a specific peg (or group of pegs).

6.1.3 Search for the final movements

Two different moments can be traced back to the search for the possible final movement. The first, when the students build step by step the final movements; and the second, when they search for a graphic pattern that can help them in their final movements. Of the 15 students that search for the final movements, 9 build step by step them, while 6 students search for a graphic pattern.

6.1.3.1 Step by step construction

Nine students of the group, applying backward step in the reverse game, search for a sequence of movements leading to the solution. They set themselves sub-targets: identify the possible configurations to win the game with one movement, then with two, then with three, and so on. This is the case of student T43. She belongs to the second level group.

Student T43 has decided to apply the working backward strategy and wants to look for the winning combination of movements starting from the last one. She starts with the configuration in which, with a single movement, she wins the game; then, she moves on to the configuration where doing two movements she wins the game. The student notes that there are some impossible configurations, i.e. configurations from which it is not possible to win. She will have to take this into account when overturning the problem: if the configuration she reaches is 3 pegs lined up, she cannot win. The student then makes a study of the possible cases in which with a configuration consisting of 3 pegs she can win. At this point, she passes to 4 pegs, then to 5.

T43 resolution protocol excerpt

Line 38 I'm going to think going backward.

Line 39 How do the pegs have to be, in order to win in a move?

Line 40 This is clear: stuck together.

Figure 40



Fig. 6.7 - Figure 40 (student T43 resolution)

Line 41 How do the pieces have to be, in order to win in two moves?

Line 42 I mean, I have two moves, and the first one must lead to the two pegs stuck together.

Line 43 One possibility is this:

Figure 43

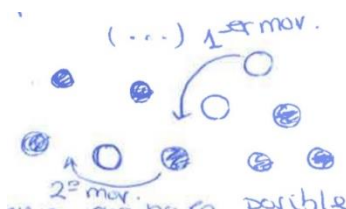


Fig. 6.8 - Figure 43 (Student T43 resolution)

The student starts thinking about going backwards in a systematic way, through an explicit question-answer process. Each answer is given by an exploration (descending modality) with a subsequent control of the result obtained (ascendant modality). In line 42, she defines the properties of the pegs in a two-moves configuration based on the previous explorations (ascendant modality). From the RBC point of view, she builds(-with) some game's information to construct the steps one at a time, while the definition is a recognizing action.

Line 44 I notice that, it's not possible that if there are three left, they're lined up.

Line 45 Because then, we can't win. In the first move they'd be separated, and you can't win anymore.

The student does an exploration in a forward way, putting together the three pegs and trying to go until the end of the game. She controls the moves giving an explication for the first affirmation (ascendant modality). From RBC point of view, she recognizes the failing configuration.

Line 46 Another possibility with 2 movements is

Figure 46

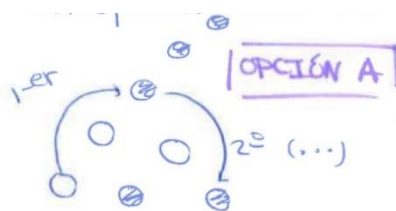


Fig. 6.9 - Figure 46 (Student T43 resolution)

Line 47 Is equivalent to the other right-hand corner.

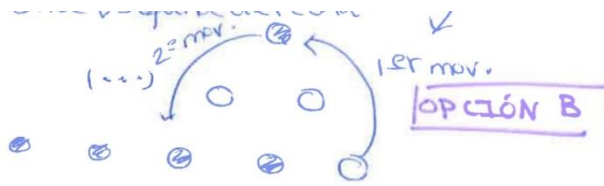


Figure 47

Fig. 6.10 - Figure 47 (Student T43 resolution)

Line 48 These possibilities leave the final piece in the middle of the line below.

Line 49 I'll try to keep going until here.

Line 50 Starting with these 3 final-pegs, what subsequent movements are there?

Line 51 For example, this:

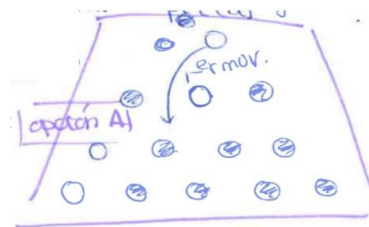


Figure 51

Fig. 6.11 - Figure 51 (Student T43 resolution)

Line 52 Or the equivalent:

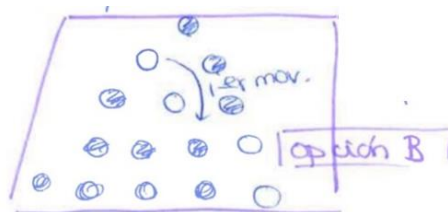


Figure 52

Fig. 6.12 - Figure 52 (Student T43 resolution)

Line 53 We think now in 4 movements, in the movement after the 4 previous pegs.

The student keeps on answering to the question formulated in Line 41. Each answer is given by an exploration (descending modality) with a subsequent control of the result obtained (ascending modality). At the end of the question-answers process she notes that all configurations have the end point in common. Then she conjectures that the position 13 is the final position (ascendant modality). She continues formulating questions, increasing the number of movements. The structure of the reasoning is the same. From the RBC point of

view, the student observes the properties of the game (building-with), first, to construct the answers to the questions she asks herself, the conjecture emerges from a recognizing action.

The analysis of this protocol excerpt is summarized in the table 6.4.

Protocol	BR	HIM	FLIM		RBC
			Actions	Modalities	
L 38	G+X	Ass	Plan formulation	N	B
L 39	G+E	Int	Question	N	B
L 40 – F	G+E	Answ	Affirmation	D+A	C
L 41	G+E	Int	Question	N	B
L 42	-	Def	Affirmation	A	R
L 43 – F	G+E	Answ	Exploration	D+A	C
L 44	-	Int/answ	Affirmation	N	R
L 45	-	Int/answ	Control	A	R
L 46 – F	G+E	Ru	Exploration	D+A	B
L 47 – F	G+E	Ru	Exploration	D+A	B
L 48	-	Ast	Conjecture	A	R
L 49	-	Ru	Exploration	N	B
L 50	G+E	Int	Question	N	B
L 51 – F	G+E	Answ	Exploration	D+A	C
L 52 – F	G+E	Answ	Exploration	D+A	C
L 53	G+E	Int	Question	N	B

Tab. 6.4 - Student T43 excerpt

The backward reasoning develops within the questioning process searching from some moves that lead to the final peg (cause-effect relationship). The strategy of going backward is introduced like an auxiliary element. In this case the student claims the questions she asks herself in order to solve the problem. The *saming* discourse device is used identify the configurations that have the same geometric characteristics.

6.1.3.2 Search for a graphic pattern

Six students of the group apply backward reasoning in the search for a sequence of movements leading to the solution. They set themselves as sub-targets to identify possible configurations to win the game that have a geometric “regularity”. This is the case of student T50. She belongs to the second level group.

Student T50 has decided to apply the working backward strategy and wants to look for the winning combination of movements starting from the last one. After a series of steps and considerations she decides to start with the last peg in position 13. She then illustrates graphically the 4 options that allow her to win the game with a single movement. So, she starts studying the two pegs in blue. She lists the possible movements that can be made to reach this configuration and identifies a geometric pattern of the possible movements highlighting them with colours. Then she considers all the possible options to reach this configuration using only one jumping peg.

T50 resolution protocol excerpt

- Line 22 Commenting in class with other classmates, they have the last ball in the position 13, as in my case.
- Line 23 Although there are more possibilities, I will start from that ball to retrieve the last 3 movements.
- Line 24 I try with coloured papers. I have 4 possibilities in the first movement.

Figure 24

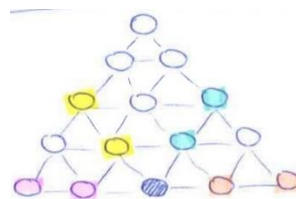


Fig. 6.13 - Figure 24 (Student T50 resolution)

The student starts conjecturing that the last peg is in position 13, then she plans to go backward starting from there. She searches for possible moves backwards (descendant

modality) and she identifies four, making a control on her moves (ascendant modality). From the RBC point of view, she recognizes the working backward strategy, then she explores (building with), to construct a first possibilities schema.

Line 25 We see that the blue and orange are symmetrical, and the yellow and purple also.

Line 26 Let's consider the case of the blue balls.

Line 27 Ball 1 has three possible movements while ball 2 only has two.

Figure 27

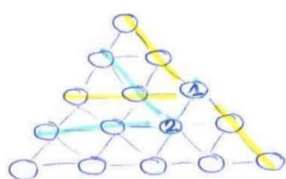


Fig. 6.14 . Figure 26 (Student T50 resolution)

Line 28 We crossed out $3 \times 2 = 6$ possible positions for 3 movements

Line 29 But we should remove the option where the two lines cross each other.

Line 30 This would be the option if we had to move 2 different balls, but we have to consider the case that you move only 1 and the other movement is with one of the balls of the 2 new positions.

The student recognizes the symmetry of the option and decides to consider the blue case. She explores the configuration (descendant modality) and she construct the geometric configuration of the moves (ascendant modality). In a detached modality she explores the configuration identifying the number of possibilities (breakdown). Then she states that she has to quit 2 option because they are not part of her sub-problem (ascendant modality).

The analysis of this protocol excerpt is summarized in the table 6.5.

Protocol	BR	HIM	FLIM		RBC
			Actions	Modalities	
L 22	-	Ast	Conjecture	N	R
L 23	G	Init	Plan formulation	N	R
L 24	G+E	Int/answ	Exploration + Control	D+A	B+C
L 25	-	Def	Affirmation	N	R
L 26	-	Init	Plan formulation	N	B
L 27	G+E	Int/answ	Exploration + Control	D+A	C
L 28	D	Int/answ	Exploration + Control	DT	R
L 29	-	Ast	Affirmation	A	B
L 30	-	Def	Control	A	B

Tab. 6.5 - Student T50 excerpt

The backward reasoning develops within the questioning process searching from some moves that lead to the final peg (cause-effect relationship). Supposedly before line 24 the student ask herself “What are the possibilities for the first movement?”, and before line 27 “what are the possible movements of the blue pegs?” e and before 28 “how many possibilities are there?”. Asking to the last question she breaks down the geometric configuration in figure 27.1. The *saming* discourse device is used to identify the configurations that have the same geometric characteristics, the *encapsulating* device to identify the geometric configuration putting together the different moves of the blue pegs. The reasoning of T50 is similar to T43 reasoning but differs in the part of visualization and identification of possible movements.

6.2 Student-M case study: visual elements in the resolution

As seen from the global analysis there are several moments in which the backward reasoning develops. A case study is displayed in this section. A student (Student-M) has combined backward reasoning with different strategies and auxiliary constructions: drawings, graphical representations. The student uses drawings and graphic representations that help her during the resolution process. She performs continuous control over its own resolution process. She is able to slightly modify the strategy or even change it completely to reach the solution. After the decision to deepen this case study, an interview was made with the student. Student-M indicates difficulties in creating the solution. In fact, on the one hand, she states that she didn't know what systematic actions she had to take, and, on the other hand, she could not trace the problem back to something known or some mathematical pattern.

For analysis purposes, Student-M's protocol was divided into the following phases: familiarisation, exploring and carrying out the strategy, results verification. According to the Finer Logic of Inquiry Model, this student's protocol is mainly characterised by the inquiry component. This begins with the first part of the protocol, corresponding to the *familiarization phase*. The entire protocol has been translated from Spanish by the author. Each protocol is divided in lines, each figure is associated with a line, one or more (for example: figure 15 is associated to line 15; figure 5-7 is associated to line 5, line 6 and line 7). Within the excerpt the lines (or the figures) where the backward reasoning is identified are put in *times new roman italic* (except when the student simply goes backward with the steps, reversing the game, without applying any strategy). Each part of the excerpt has a short comment to identify the characteristics according to each analysis model (HIM, FLIM and RBC), the backward reasoning characteristics and the discursive devices used by the student. A partial first analysis of this case study was presented at INDRUM2018 conference (Barbero and Gómez-Chacón, 2018).

Student M protocol

Line 1 To accomplish the exercise, I'm going to number the holes on the board in order to leave a trace of the movements I'm doing. At the beginning, all the holes are filled except number 5.

Figure 1

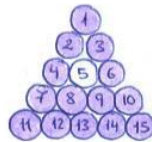


Fig. 6.15 - Figure 1 (Student-M resolution protocol)

Line 2 I observe that you can only start with two movements 14-9-5 or 12-8-5.

Line 3 Since this is an equilateral triangle, I think it does not matter what the starting movement is because they should lead to "symmetrical" solutions.

Line 4 I'll start to do it roughly.

Line 5 The steps I'll take are: 14-9-5; 7-8-9; 12-13-14; 2-4-7; 11-7-4; 10-9-8; 3-6-10.

Line 6 At this point, I note that the only way to eliminate 1 would be to move 8-5-3.

Figure 5-7

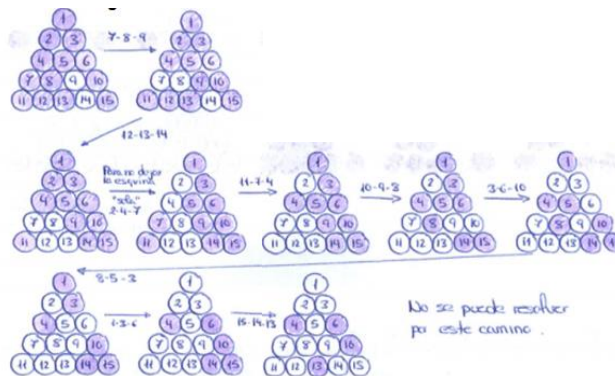


Fig. 6.16 - Figure 5-7 (Student-M resolution protocol)

Line 7 Here I notice that [with these movements] the game cannot be solved because the 4 cannot be eliminated and the remaining pegs cannot eliminate each other.

This excerpt part corresponds to the *familiarization phase*. The student explores the game (descendant modality) making a control on the moves (ascendant modality) that allows her to recognize the board symmetry and the way to eliminate peg in position 1 and in position

4. From RBC point of view Student-M through building-with actions explores the game, recognizing the symmetry and the ways to eliminate specific pegs. Then, she constructs the moves that allows her to continue the game. Lines 6 and 7 correspond to the attempt to remove a specific peg, the first successful and the second failed. She declares that she has lost the game, and then restarts from the beginning. In these lines the *reifying* device is used. She talks about moves she is doing, and then she starts to talk about the game that can't be won.

-
- Line 8* I realise that I can try to go backwards, that is, starting with just one peg in one position and undo the jumps trying to fill the board except for a hole.
- Line 9* Looking at the board, I think that maybe the fact that the last piece stays on the board (the peg from which I start to move backwards), in a position that you can come up with many jumps, facilitates the strategy.
- Line 10* These places are positions 4, 6 and 13 because you can reach them with 4 jumps.

Figure 10

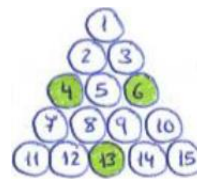


Fig. 6.17 - Figure 10 (Student-M resolution protocol)

This is the first part of the second phase: *Explore and carry out the strategy*. The student introduces a new element in the resolution: solve the game by starting from the end of the problem. Through an interrogative process, she explores the problem searching the best place to put the last peg (descendant modality), and she recognizes that each position can be achieved with several jumps (ascendant modality). She conjectures that, the best positions where to start are 4, 6 and 13. From RBC point of view she recognizes the working backward strategy in his background then she builds(-with) notion of the games to construct the possible game solution. The backward reasoning, in this part, starts with the introduction of an auxiliary element (the working backward strategy) and continues with the breakdown of the position of the board to recognize the favourable. The conjecture in line 10 correspond to the solution formulation. The *saming* device is used to put together the holes that have the

same geometric characteristics; while the *encapsulating* one to identify the position 4, 6 and 13 as favourable positions.

Line 11 To fill up the board I will have to do 13 moves, because there are 15 holes, an initial peg and an empty final hole.

Line 12 Let's start only with peg 13.

Figure 12-19

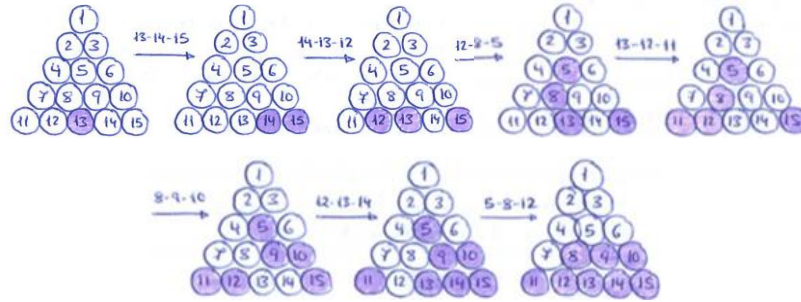


Fig. 6.18 - Figure 12-20 (Student-M resolution protocol)

Line 13 Let's start with the reason for the various steps:

13-14-15: I want to start filling the corners as soon as possible because these holes are the hardest to fill up (the peg is in hole 15 and I will not move it anymore).

Line 14 14-13-12: Random movement.

Line 15 12-8-5: I want to leave hole 12 free to get to the next step at corner 11.

Line 16 8-9-10: I want to leave hole 8 free to retrieve peg 12 (to fill 13 and 14) in the next step, so I can complete it later [the row].

Line 17 12-13-14: I want to complete the row below.

Line 18 5-8-12: I want to complete the row below.

Line 19 Here I already notice that I do not reach the solution because I will never fill the top corner due to the absence of a peg in the 3rd row; I should do 11-7-4 leaving corner 11 without a peg [so that the top corner will be filled].

The student starts going backward. She states that she starts with peg in position 13. Through an interrogative process, the student explores the game (descendant modality) making a control on the moves (ascendant modality) that allows her to fill the board in a particular way: filling it row by row. From RBC point of view, she recognizes the position that must necessarily fill and constructs the move. Lines 13-19 correspond to some attempts to fill

specific positions, the last one is a failed attempt. Line in line 6 and 7, in these lines the *reifying* device is used.

- Line 20 I think trying to fill the centre was not a good strategy...
- Line 21 ... so now I'm going to try to fill the outside of the triangle, that is, [I'll try to] undo the jumps to the corners and sides. (Playing normally would involve jumping to the centre avoiding corners and sides if possible.).
- Line 22 I also get stuck [on the fact] that by eating pegs or undoing the jumps, the movements that are made are triangular.
- Line 23 So, I will try to fill the smaller triangles contained in the big triangle.

Figure 23



Fig. 6.19 - Figure 23 (Student-M resolution protocol)

The student, making a control on her moves, recognizes that the previous strategy was not good (ascendant modality). Then she plans to fill the board from the outside to the inside. immediately she recognizes that the movements of the pegs have a triangular shape (ascendant modality). She decides to breakdown the board in triangles. From RBC point of view, these actions are recognizing actions. The backward reasoning with its breakdown character start in line 23 with the introduction of the auxiliary subdivision of the board shown in Figure 23.

- Line 24 First, I will fill the lower right triangle.

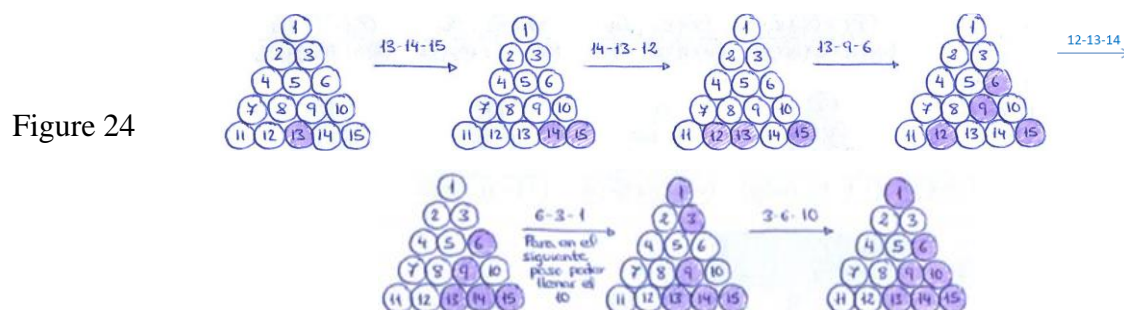


Fig. 6.20 - Figure 24 (Student-M resolution protocol)

Line 25 Now I'm going to fill the upper triangle; to do so (Since I do not want to remove the peg I placed in position 1), I have to get some pegs in the 4th row that, undoing the jump fills the 2nd and 3rd row. I undo the jump with the 9.

Line 26 Now you have to fill the lower left triangle.

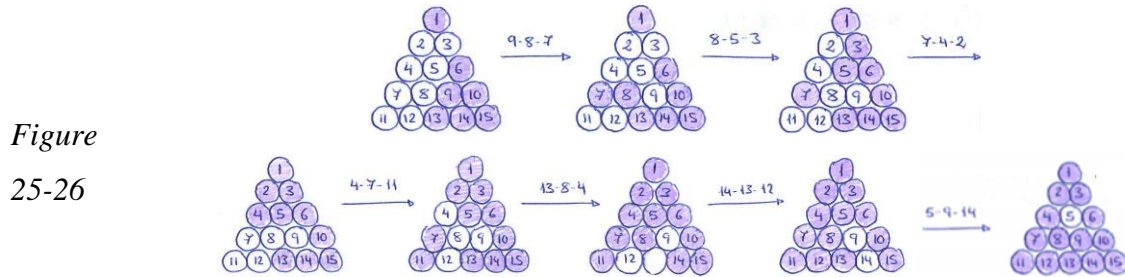


Fig. 6.21 - Figure 25-26 (Student-M resolution protocol)

Through an interrogative process, she explores the game (descendant modality) making a control on the moves (ascendant modality) that allows her to fill all the triangles deriving from the breakdown of the board (it is explicit in line 25). She finds a game winning path. From RBC point of view Student-M through building-with actions explores the game, recognizes the positions to fill, and then constructs the moves to do it. These lines correspond to sequence of attempt to put a peg in a specific position, the reasoning is the same as in the case of attempt to remove a peg from a specific position. In these lines the *encapsulating* device is used. She encapsulates the movements to create a set of moves with the same goal: to fill a triangle. The backward reasoning appears both with a breakdown and a search for cause-effect relationships character.

Line 27 Now I just have to write the jumps in the correct order

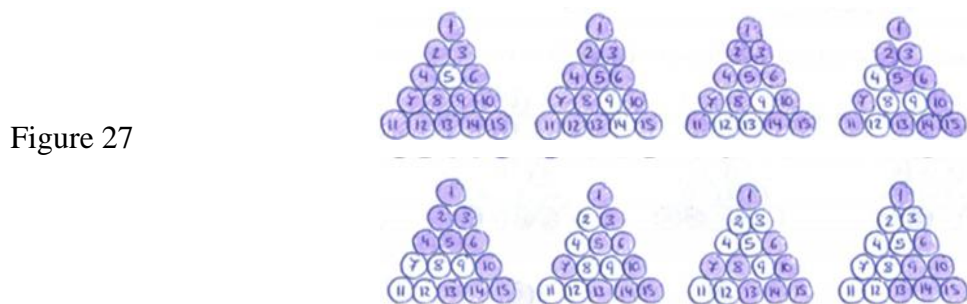


Figure 27

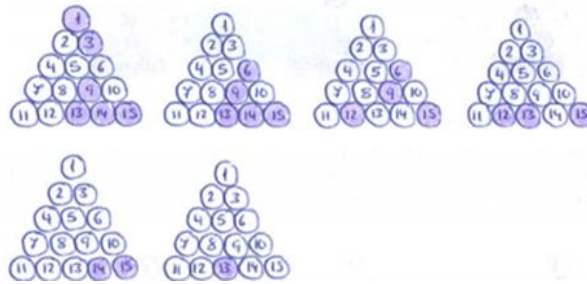


Fig. 6.22 - Figure 27 (Student-M resolution protocol)

This last part corresponds to the *results verification* phase. From the FLIM point of view this line correspond to the Deductive Component. The student reverses the winning path found in the previous steps validating it (detached modality), by writing and graphically representing the steps taken to reach the solution. It is the construction of the final solution.

The following table shows the summary of the analysis on Student-M resolution protocol. As for the previous tables in the chapter, each column corresponds to a type of analysis. In addition, the table has been subdivided according to the resolution phases: familiarization, exploring and carrying out the strategy and result verification.

Protocol	BR	HIM	FLIM		RBC
			Actions	Modalities	
<i>Familiarization</i>					
L 1 – F	-	Init	Exploration	D	B
L 2	-	Ru	Control	A	R
L 3	-	Ast	Affirmation	N	R
L 4	-	Init	Exploration	D	B
L 5 – F	-	Ru	Exploration	D	B
L 6 – F	G+E	Int/ans	Exploration + Control	D+A	R+C
L 7 – F	G+E	Int/ans	Exploration + Control	D+A	R+C
<i>Explore and carry out the strategy</i>					
L 8	X	Init	Plan formulation	N	R
L 9	D	Int/answ	Exploration + Control	D+A	B

L 10	FS	Ast	Conjecture	N	C
L 11	-	Ru	Exploration	D	B
L 12	G	Init	Plan formulation	N	R
L 13	G+E	Int/answ	Exploration + Control	D+A	R+C
L 14	-	Ru	Exploration	D	B
L 15	G+E	Int/answ	Exploration + Control	D+A	R+C
L 16	G+E	Int/answ	Exploration + Control	D+A	R+C
L 17	G+E	Int/answ	Exploration + Control	D+A	R+C
L 18	G+E	Int/answ	Exploration + Control	D+A	R+C
L 19	G+E	Int/answ	Exploration + Control	D+A	R+C
L 20	-	Ast	Control	A	R
L 21	G	Init	Plan formulation	N	R
L 22	-	Ast	Control	A	R
L 23	D+X	Init	Plan formulation	N	R
L 24	G+E+D	Int/answ	Exploration + Control	D+A	B+R+C
L 25	G+E+D	Int/answ	Exploration + Control	D+A	B+R+C
L 26	G+E+D	Int/answ	Exploration + Control	D+A	B+R+C
<i>Results verification</i>					
L 27	-	Ru	Control	DT	C

Tab. 6.6 – Student-M resolution protocol

It is possible to observe the analysis in two ways, with a global approach and with an approach focused on backward reasoning moments.

6.2.1 Discussion on Student-M resolution

Starting with the global approach, the FLIM analysis shows that the first two resolution phases are characterised by a continuous alternation of explorations/control actions and plan formulations actions together with an alternation of descending and ascending modalities. A routine that can be established regarding the use of modalities is $A \sim N \sim D \sim (A \sim N \sim D \sim (A \sim \dots))$. The neutral modality marks the transition between A and D and

it is characterised by the incorporation of auxiliary constructions as generating tools of new knowledge (epistemic transaction).

The analysis with the RBC model shows a series of chains. In the first phase there are two B-R-C chains that characterise the first attempt of resolution. The first one allows to remove the peg from the position 1, the second to recognize that the game was lost. The second phase starts with a R-B-C chain that allows the student to formulate a conjecture: the favourable positions to start reverse game are 4, 6 and 13. Then some sequences of B-R-C chains (even if B is hidden in the process of exploration) characterize the second attempt: to fill the board row by row. After a series of recognizing action, in which the triangle board decomposition is introduced, three B-R-C chains appears again leading the student-M to the solution. The backward reasoning appears through these chains. The R-B-C chain is characterized by the introduction of auxiliary elements and a breakdown that lead to construct the solution (peg in position 13). The B-R-C chains are characterized by the research of cause-effect relationships, the last three also by the board breakdown. The entire protocol can be schematized through the RBC model in the following way.

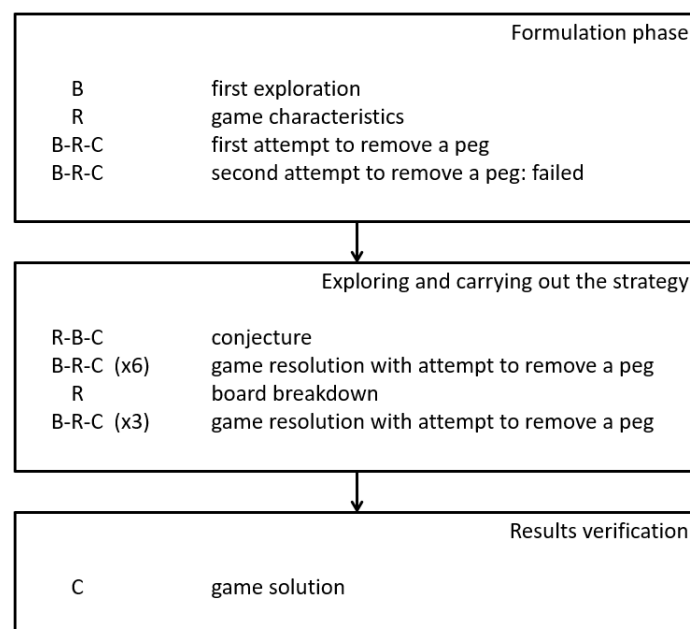


Fig. 6.23 - RBC flow (Student-M resolution protocol)

Focusing on backward reasoning moments, the second resolution phase involves the continuous use of the working backward strategy. Student-M modifies the strategy slightly

by adding new elements in the resolution, the board subdivision into rows and triangles. These subdivisions are fundamental to reach the solution. Crucial points of backward reasoning are reached in the ascending modality (see in Table 6.6 the lines 6 and 7 in the first phase, and 9, 13, 15-19, and 24-26 in second phase) where ideas occur. Backward reasoning develops during interrogative moves, probably formulating question like these: “how do I get the peg out of that position?” or “what are the best positions to start the reverse game?”. The question is formulated with reference to the game general knowledge and allows student-M to eliminate irrelevant details and focus on the important aspects for solving the specific problem. The new ideas emerge from the questioning process.

The analysis of the discursive devices allows to notice that:

- When the backward reasoning appears in attempt to put/quit a peg in/from a specific position, the student first push herself to the final state that she wants to reach, then she goes backward and changing again direction, through a *reifying* device, in forward way, she constructs the solution.
- When the backward reasoning appears in attempt to put/quit a peg in/from a specific position, and in correspondence there is a board breakdown, the student does a similar backward and forward movement but using the *encapsulating* device.
- When the backward reasoning appears in conjecture formulation, the *saming* and the *encapsulating* devices are used.

6.3 Discussion

The global analysis of the group has identified three different backward reasoning moments: reversing the game and analysing the board to find the best place to put the first peg; attempting to remove a specific peg (or its ‘dual version’ in the reverse game: attempting to put a specific peg); searching for the final movements. In each moment, backward reasoning occurs mainly in interrogative moves (HIM analysis) and in ascendant modality (FLIM analysis). The RBC flow connected to these moments is characterized by R-B-C and B-R-C chains.

When students reverse the game and look for the best place to put the first peg, they work regressively breaking down the board. These moments (section 6.1.1 and conjecture formulation in Student-M resolution) are characterized by R-B-C chains in which the students recognize an important element (geometric characteristics of the game) and through a series of reasoning (building-with) come to identify the starting position (constructing). The *saming* discursive device appears in these moments.

When students attempt to remove a specific peg, backward reasoning appears in its characteristic of cause-effect relationship research. These moments can be found in Student-M resolution in the exploration phase, and in her last two attempts to solve the game in a reverse way. B-R-C chains characterize all cases: the students explore the game, then recognize a specific position and later construct a move to quit (or put) the peg. The *reifying* discursive device appears, except for the last Student-M attempt to solve the game; in this case, she also breaks down the board in triangles, so the *encapsulating* discursive device appears.

In Student-M protocol the search for final movements doesn't appear. In these moments, students search for cause-effect relationships that can help to progress in the movements. Through the global analysis two development modalities emerge: step by step construction of the final movements and the research for graphic patterns in the final movements. The first one is characterised by B-R-C chains and *saming* discursive device. The students explore the game in a reverse way until recognize a geometric property, then they construct the reverse step. The second development modality is characterised by R-B-C chains and *saming* and *encapsulating* discursive devices. The difference is based on the fact that, in this case, the student focus is on the geometric properties that are immediately identified, then, with a series of reasoning (building-with), the pattern is constructed.

7

MAUDE TASK ANALYSIS

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MAUDE TASK ANALYSIS

7

In this chapter the results of the analysis of the second design experiment are shown. Briefly the design experiment settings are summarized in table 7.1.

Task type		Data collection settings				Students				
Strategy game	Mathematical	Work in group	Work alone	PC suite	Video-recordings	Firs level	Second level	Third level	Total	Case studies
✓	✓	✓	✓	✓	✓	-	23	-	23	5*

* Two pairs and a student that interact with one of them.

Tab. 7.1 - Second design experiment settings

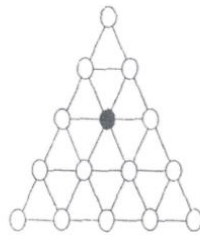
The task proposed in this design experiment is the implementation of the Triangular Peg Solitaire in Maude programming language, as shown in figure 7.1. To solve this task the students have to activate heuristic, mathematical and computational knowledge. Twenty-three students from second level group were involved in the design experiment (see Chapter 5). Two case studies were carried out analysing in depth the resolution protocols and video-recordings. Only one of them is shown in this dissertation (a pair composed by Student-P and Student-D, and their interactions with Student-E, section 7.2).

In this chapter, first, the analysis of the whole group of students is presented; a categorisation of learning difficulties of the participants is shown. Then one case study is displayed; it consists in an in-depth analysis of two “episodes” of a video-recording through the multidimensional analysis model. In the first episode two students (Student-P and Student-

D) are solving the first exercise of the task. In the second, they interact with a student acting in the role of expert (Student-E) that explain the resolution of the second exercise: she has an in-depth Computer Science background. As the resolution protocols of the group were very schematic, the moments of backward reasoning were identified and illustrated within the case study. Finally, a general discussion is developed (section 7.3).

Triangular Peg Solitaire

The Triangular Peg Solitaire is a 1-player game that can be played on different sized boards. Initially, all positions except one contain pegs, while a winning board contains exactly one peg. To reach this configuration, pegs can "jump" over others, "eating" the pegs they jump over, as long as an empty position is available after that peg, like in Checkers. We will work with the triangle board, as shown in the figure below.



Exercise 1 Define a datatype for representing a Triangular Peg Solitaire. We are particularly interested in supporting boards of different sizes.

Exercise 2 Implement jumps using rewrite rules.

Exercise 3 Define an initial board and use the **search** command to find: (a) any solution; (b) a "perfect" solution. A perfect solution consists of a board with a single peg in the central position, as shown in the figure above.

Fig. 7.1 - Maude task

7.1 Analysis of the whole group: difficulties categorisation

Based on classroom observations and video-recordings, five students' difficulties were identified. These can be classified into two main categories: factual difficulties and methodological difficulties. The first category includes both incorrect propositional knowledge and experimental errors. Namely, a student may have false opinions or may carry out incorrect practices. These are, in a certain sense, closed and local errors: the opinions are

true or false and the practices are done correctly or incorrectly. Instead, the second category, methodological difficulties, is related to learning steps. This means that the difficulties identification must be done reflecting on the whole learning process. The same implementation in Maude language can be appropriate or not according to the learning phase where it occurs. Methodological difficulties are identified when the students implement basic commands when they can use more sophisticated commands already learned.

The most frequent factual difficulties are:

Completeness difficulties: in Maude programming language, functions are complete by definition (that is, they return a value for all arguments, although these results might be erroneous). They can be defined by specifying their equational properties of associativity, commutativity, etc. Maude language generates an algorithm so that each function's argument is matched to its properties. This allows to distinguish cases in a clearer way but also might prevent functions or rules from being applied: this *partial* nature is sometimes overlooked by students, who fail to specify all cases.

Behaviour difficulties: students have difficulties in distinguishing between the static and dynamic behaviour of the system. This is exemplified when the students use an equational theory for specifying state transitions, or they use rules for describing how data structures behave.

On the other hand, methodological difficulties are:

Description difficulties: Standard languages have a fixed catalogue of data structures (e.g. basic types -natural numbers, Strings, Boolean values- and structured types -lists, maps, and trees-) and new data structures are created by combining them. Instead, Maude language provides a more flexible syntax. These difficulties appear when students try to reduce their data structures to those they already know from standard languages, making subsequent implementations in Maude language complicated.

Estimation difficulties: Maude allows to define transitions both in a symbolic way (terms with variables standing for several states) and in an explicit way (defining the specific state). Since the explicit way is easier to implement it is often chosen by students. But though the

explicit way the number of rules grows a lot, even for medium systems, making the transition definition unfeasible. Those students are not able to correctly estimate the size of the problem, and they fail even if the data structures are correctly defined.

Transference difficulties: these difficulties appear when students are not capable of applying mathematical concepts while programming, like equational axioms such as commutativity or associativity.

The factual difficulties have a direct influence on the execution of programs, making them buggy. While methodological difficulties make the path of resolution more complex and requires more attempts to be solved; in the end, a solution can still be achieved. For example, a description difficulty can be, due to lack of experience, using a list instead of a set; both lead students to a solution, even if the former is more complex. The same occurs in estimation difficulty (the problem is still solvable, but it takes longer) and transference difficulty (when students lack experience, they need to make up for it by putting in more effort). In the case study, the moments in which these types of difficulties appear were highlighted and it was observed how the students overcame them.

7.2 Case Study

A case study is displayed in this section. Two “episodes” of the video-recording from the pair composed of Student-P and Student-D are analysed. This is an emblematic pair for the whole group: throughout their resolution several difficulties of those identified with the analysis of the study group emerge. They solved the task talking to each other, which made it possible to analyse their thought processes that were emphasized in the speech; the interaction with student-E allows a greater deepening of the use of backward reasoning in the explanation moments. The students’ reasoning develops throughout three contexts: the informal context (related to the game), the mathematical context (related to purely mathematic representations) and the computational context (related to the codification in Maude language). From the Commognition (Sfard, 2008) perspective, each of the three contexts is related to a specific discourse with different characteristics that can help to

identify it: *key words*, *visual mediators*, *routines*, and *endorsed narratives*. For example, when the students are involved in the informal context they talk about pegs, board, pegs movements, etc.; when they are in the mathematical context they talk about natural numbers, pairs of numbers, cartesian notation, etc.; when they are in the computational context they talk about list, operators, structures, etc. The symbols, diagrams, and words used by the students are emphasized to provide evidence of the context in their work. The two selected episodes concern the first two exercises of the task. In the first episode, the students try to specify the data structures of the assignment (Exercise 1); their reasoning develops across the three contexts shown above. In the second episode, Student-E joins the discussion. As said before, she is a student who has a deep knowledge of the subject and she adopts an expert role to explain to her classmates the second exercise resolution.

Transcriptions refer to minutes 0.00-6.30 (episode 1 from video-recording 1) and 39.00-50.00 (episode 2 from video-recording 2) from the recording. The entire transcriptions were translated from Spanish by the author. Each transcription is divided in lines, each figure made by the students is associated with a line (for example: figure 39 is associated to line 39). Within the excerpt, each line (or figure) is associated to three characteristics: the first is related to backward reasoning (if it appears), the second to HIM model and the third to RBC model. Each part of the excerpt has a short comment to identify the characteristics according to each analysis model (HIM and RBC), the backward reasoning characteristics and the discursive devices used by the student. Section 7.2.1 refers to the first episode, here evolution of the students' reasoning in the transition between resolution contexts is shown; section 7.2.2 refers to the second episode, here the evolution of Student-E reasoning is shown, while helping students to complete the exercise by resolving their difficulties.

7.2.1 Episode 1: Development of reasoning across the contexts

As indicated in previous paragraph, in this section the analysis of the first episode, using the multidimensional tool, is shown. The transcription of Video-recording 1, minutes 0.00-6.30 is considered. After reading the task text, Student-D and Student-P begin to solve the first exercise and implement data to represent the Triangular Peg Solitaire board.

Episode 1 transcription

Line	Stud.	Transcription	BR	HIM	RBC
L 1	D	<i>... A structure for Solitaire: a list? Or what?</i>	-	Init	R
L 2	P	<i>It is "board".</i>	-	Ru	B
L 3	D	<i>I put "board", right? It's like those examples of pairs, where is the paired exercise that we did the other day? Because a board has two positions, in the end.</i>	D	Int/ answ	R
L 4	P	<i>How was it created? Was it a list..?</i>	-	Int	B
L 5	D	<i>...and the pair, they were a list and the pair.</i>	D	answ	B
L 6	D	<i>We have got a pair that is going to be two naturals... and this is the position. And what would be missing... is knowing whether the position was captured or not. Did I understand correctly? [..] We have to make a type of data to represent a board: many pairs, a list of pairs. And every pair with the naturals as we have.</i>	FS	Ast	C

[...] They read the task again.

The initial move is done to achieve the Maude implementation of an object list that represents the Triangular Peg Solitaire board. Then, an interrogative move occurs when they ask themselves whether there is a similar exercise that they have previously solved. And in Lines 3-5 (again by means of an interrogative move) they look for analogies while taking into account the elements that are considered in creating the list. With an assertoric move (Line 6) Student-D proposes a new (sub)conjecture in relation to the initial move: the elements of the list that they are going to build represent the position and the "state of the peg" (if it is taken or not). From RBC point of view the students recognize the analogous problem and the object "list" as a possible structure useful to solve the exercise. They elaborate notions from the previous exercise (building-with) and then they make explicit the structure that must have the list to solve the task. The backward reasoning is used to break down the analogous problem solution and identify the components of the list; a solution is

formulated in rough form. The *encapsulating* discursive device appears in line 6 where the student start to talk about list of pairs. The students start from the informal context and goes directly to the computational talking about lists. Then they refer to the problem previously carried out by naming both the lists and the pairs in a speech halfway between mathematical and computational context. Then they move on to the explanation of a possible abstract data structure. The Student-D discourse (Line 6) is a discourse that intertwine the three contexts, he talks about pairs and natural numbers (mathematical), peg state (informal), and he relates them translated into computational language by talking about pairs lists.

Line	Stud.	Transcription	BR	HIM	RBC
L 7	P	<i>But you have to also take into account the peg, right?</i>	-	Int	R
L 8	D	<i>I'm going to do it with pairs and nothing else, and a Boolean or something like that ... that is, we are going to define an operator for the empty list, instead of "nil" we put "sel" ... or would it be worth having a list?</i>	D	Def	B

[...] They read the task again.

L 9	D	<i>then a list of naturals and two positions.</i>	D	Int	B
L 10	P	<i>no, but it's not like the pair because the pair has two naturals and it turns out to be something like this... [Student-D: ¿What?] ...That is, you have nat nat and you get a pair, I don't know, is it okay? You have to draw two numbers, right?</i>	FS	answ	C
L 11	D	<i>A pair that has two positions and we represent it in that way and then we use ...</i>	D	Int	R
L 12	P	<i>...the peg...</i>	D	Int	R
L 13	D	<i>Ah.. Whether it's taken or not, okay. It's true, we have to take something.</i>	D	answ	B

L 14	P	<i>I do not know if we can define a Boolean like we have done.</i>	D	Int	B
L 15	D	<i>Well, but let's put a.. [Student-P: a 0 or 1] a 0 or 1 and that's it.</i>	FS	answ	C

In this second part of the excerpt a concrete data structure is developed. Student-D starts to implement the program (Line 7) and decides to put a pair of naturals (position) and a Boolean (state of the peg). With a definitory move (Line 8) he defines an operator for the empty list, called “**sel**”, used later as identity for an associative constructor (unit element of list composition). After, with a series of questions/answers (interrogative moves - Lines 9-15) they construct the pair of naturals and they add a third element to the couple: 0 or 1 that represents the "state". They get a list of lists. In these lines the use of backward reasoning is essential. Students focused on the objective of creating the list, looking for the necessary elements/backgrounds for its formal construction. Throughout a breakdown they formulate the solution into steps: first the pair of naturals that represent the board position and then the Boolean that represent the peg state. From the RBC point of view, they recognize that it is necessary put the peg state in the implementation and then with a series of building-with they construct firstly the pairs of natural and then the Boolean. The *encapsulating* discursive device appears: the students talk about the pair instead of naturals and then they talk about Boolean instead of different peg status. The students are in the computational context. They go back to the informal context to capture some information (the state of the peg) and then return again to the computational context expressing the information in terms of Boolean and 0, 1.

7.2.1.1 General considerations

As specified in Section 7.1, a methodological "description difficulty" occurs here. Students choose to solve the static part of the practice using lists instead of sets. Throughout the dialogue, some difficulties can be also observed in representing ideas and in following the syntax of the language. Students continue using contextual or structural analogy to return to previous exercises as a way of inspiration and to get ideas for resolution. In the minutes that

follow this excerpt, we observe how they modify the code several times before achieving something correct, that has no errors in code lines and that the program can parse.

The behaviour routine is summarised in the diagram in figure 7.2. It shows the moments of backward reasoning, the strategic movements according to the HIM categories and the contexts of representation.

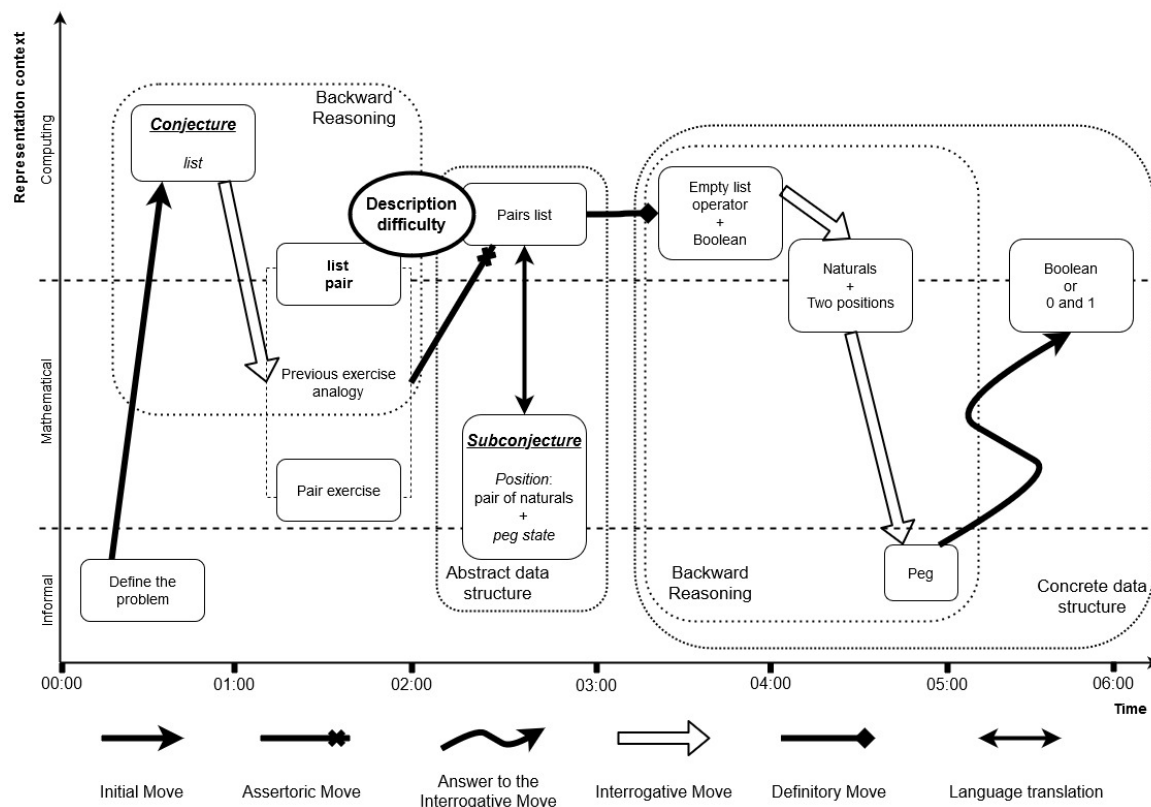


Fig. 7.2 – Student-P and Student-D reasoning flow diagram (episode 1)

Students reasoning often crosses from one context to another. Through the initial move, to specify a conjecture, they pass from the informal context to the computational one. In this case it is evident that the interrogative move characterises the steps from the computational context to the informal one, while the assertive move and the answers to the questions go in the opposite direction. It is also observed how backward reasoning characterises the situations of interrogative moves while the students, to implement different parts of the program, think about the solution they want to obtain. In this first part, the students' difficulties are focused on the step between different contexts and on syntax issues.

The analysis with the RBC model shows a series of chains. Three R-B-C chains appear: the first one leads to conjecture the list, the second and the third to construct it. The backward reasoning appears through the first and the third chains. The breakdown lead to the construction of the rough solution after the first chain and the list after the second and the third phase.

The analysis of the discursive devices allows to notice that when the backward reasoning appears breaking down the conjectured list the students use the *encapsulating* device to replace the discourse on specific parts of the program with discourse on entities that include them. For example, from naturals to pairs and from pairs to list.

7.2.2 Episode 2: Dynamism of an expert's reasoning

In this section the analysis of the second episode, using the multidimensional tool, is shown. The transcription of Video-recording 2, minutes 39:00-50:00 is considered. After solving the first exercise, Student-D and Student-P start to solve the second exercise, that asks to implement the jump of pegs with rules. The students, after a long period of time, do not achieve the solution; at that time a classmate (Student-E) offers help. Reasoning with them, Student-E explains how they can represent peg jumping and solve the exercise. In this excerpt, several difficulties of students P and D emerge; Student-E tries to solve them. The main discourse is developed by Student-E, the classmates intervene punctually. As for episode 1, the resolution contexts in which their reasoning is developed are highlighted.

Episode 2 transcription

Line	Stud.	Transcription	BR	HIM	RBC
L 1	E	<i>Let's imagine that I have a board here [she draws a triangle and writes (1, 1, boole)], this is the row number [she indicates the first 1] and the position of the row it occupies [she indicates the second 1].</i>	-	Init	R
L 2	D	<i>Row and column, yes, yes</i>	-	Init	R

Student-E makes a summary of the previous reasoning used to solve the exercise 1 and draws a triangle on the paper. She depicts the board and the triple that stands for the position and the state of the peg. From the RBC point of view these lines are a recognizing of the previous solved exercise.

L 3	E	<p><i>To move... Let's imagine that what we want to do [the movement that] goes like this, okay? [she draws an arrow to represent the diagonal movement from bottom-left to top-right, see figure 7.3] Then we need to have to start ... these are our three pairs, okay? [she draws three open parentheses] Let's imagine that these represent this, this and this... [she draws three little balls on the diagonal of the triangle, see figure 7.3]</i></p>	E	Init	R
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The objective that she proposes is to mathematically represent the jump of the pegs along the diagonal of the triangle (see figure 7.3): it is an initial move.

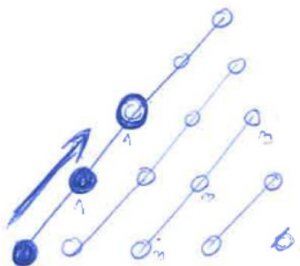


Fig. 7.3 - Student-E's first goal: represent the jump indicated with the arrow

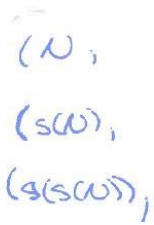
She starts the reasoning by highlighting the construction she wants to achieve and schematically representing the starting positions on paper. She is using backward reasoning with its character of cause-effect relationship research. This is a recognizing epistemic action. In these first three lines Student-E starts defining the static representation of the problem and then she highlights the target of the exercise, both discourses include informal and mathematical context elements.

Line	Stud.	Transcription	BR	HIM	RBC
L 4	E	<i>That's important: if you define this as a list, which result to be all pairs, you define the commutative constructor so that Maude can turn it over to fit...</i>	X+E	Ast	R
L 5	P	<i>Ok, a commutative constructor...</i>	-	Ru	B
L 6	E	ctor and you put a space and C-O-M-M	X	Ru	B
L7	D	<i>But, in the board?</i>	-	Int	B
L8	E	<i>In the board. When you define the list on the board... If you are going to see it as a list, put them commutative so that the program can turn everything around to fit all the patterns because</i>	E	Int/ answ	R
L9	P	<i>No, but it isn't associative!</i>	-	Ast	R
L10	E	<i>So, the program can turn it around. Because imagine that you have $N, N, S(N), S(S(N))$ [N stands for 'natural number', while $S()$ stands for 'successor', the expression $S(N)$ means successor of N] which is what you are going to want, if you do not give it that is commutative he will not be able to mix these so that this N will be here [she "scrolls" the "N" with an arrow to the right] and these three [she indicates $N, S(N)$ and $S(S(N))$] may be together to apply it.</i>	E	Int/ answ	B+C

With an assertoric move (lines 4 and 6) Student-E introduce a necessary auxiliary element so that the program can work: the commutative constructor. Then, she tries to overcome (Line 8) the methodological "description difficulty" specified in the previous section. Student-E takes into account that she is working with sets and that, when she is going to implement a jump, she must have to use three consecutive positions (three positions on the board that are aligned) that are not necessarily consecutive in the list that represent the board. For this reason, Student-E needs to implement the commutative constructor. To do it, she uses the backward reasoning, taking into account the data structure that she has and looking

for necessary elements needed for the implementation of her objective. Reasoning backward (Line 10) she explains what might happen if the commutativity of the list is not taken into account. Unlike Student-E, Student-P and Student-D do not clearly understand the properties of the list that they are building to solve the task and claim that it is not associative (transference difficulty). From RBC point of view Student-E recognize the commutative constructor and through a series of building-with actions she explains why it is important, until use it to construct the desired sequence. The *reifying* discursive device appears where the Student-E goes from talking about adding parts to the program to make it work in a certain way to talk about the program itself that "can turn the positions around". In these lines, the discourse develops between the mathematical (when she talks about naturals and successors) and the computational context (when she talks about the commutative constructor).

Student-E's reasoning carries on with the mathematical representation of the state of the pegs involved in the jump. Firstly, she defines the initial state of the pegs; to do it, she determines the "row value" and the "column value" of the peg position.

Line	Stud.	Transcription	BR	HIM	RBC
L 11	E	<i>In order to make this movement you need three pairs. If these are the rows, this row will be $S(S(N))$ where N is a natural number, it will have $S(N)$, and this will have an N.</i> [while she writes]			
F 11		 <p style="text-align: center;"><i>Fig. 7.4 - Figure 11 (episode 2 transcription)</i></p>	D	Def/ Ded	R+B C
L 12	E	<i>Ok, since we are in a triangle, your column will be the same in all cases.</i>	-	Def	R

Student-E defines, in the mathematical context, the three positions of the pegs that she considers: definitory and deductive moves overlap. She is defining the three pairs but at the same time she uses the rules of addition to represent (through deductive reasoning) in a general way three consecutive number. She is breaking down the representation of the jump that she did when she solved the exercise to explain it step by step to her classmates. From the RBC point of view the student recognize a certain notation and builds(-with) it to construct the first three value of the initial pegs state for the jump. The *saming* discursive device is used to give the notation to all different three position that have the same direction in the board.

Line	Stud.	Transcription	BR	HIM	RBC
L 13	D	<i>Will the column be the same? (Student-E: Yes) the one on this peg? When I go up?</i>	-	Int	B
L 14	E	<i>Yes, because you realise that it starts here [she indicates the peg in the corner of the triangle bottom-left, see figure 7.3] and if the movement that you are going to see, for example this one ... [she indicates the movement that she is considering] it is the first column always. And when you move to the right you are always moving the peg to the right passing through each row...</i>	-	Def	B

In the logic of the investigation, Student-D's move is interrogative while Student-E answers the question with a definitory move explaining why "the column will be the same". From RBC point of view these are building-with because the students are relating their notions.

Student-D asks for the representation of the column value if the peg positions considered are three aligned along the other side of the triangle board (see figure 7.5).

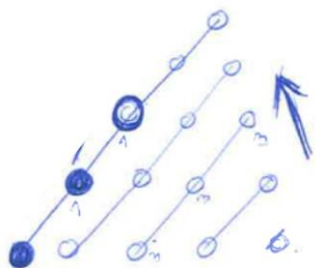



Fig. 7.5 – the “contrary movement”

Line	Stud.	Transcription	BR	HIM	RBC
L 15	D	<i>In that movement [see figure 7.3], because on the contrary [movement highlighted in figure 7.5] no. On the contrary, the column will not be the same ...</i>	-	Int	R
L 16	E	<i>... when you do this too [she means the movement from top-right to bottom-left] ...</i>	-	answ	B
L 17	D	<i>No, I mean when you take the peg from this corner to here ... [he indicates the diagonal movement from bottom-right to top-left, fig. 7.5]</i>	-	Int	B
L 18	E	<i>On the contrary you are going to have to look ... let's imagine ... it will not be a successor of N ... on the contrary, you will not have the same column but you will have the same sum ... this inverse of what you have in the first ... Row minus column. Because here.... Let's imagine that this is 1, 2, this is 1, 2, 3, [she indicates the positions of the second and third row starting from the vertex above] then row 2 ... 2-2 is 0 ... 3-3 is 0 ... Here on this side [she indicates the right-left diagonal movement] you are going to have to use these ... and on this side the real ones. The movements like this and like this [she indicates the left and right movements on the right diagonal] you do them with the positions and</i>	D	Def/ Ded	B+C

		<i>the movements like this and like this</i> [she indicates the right-left diagonal back and forth movements] <i>with the sums.</i>			
L 19	P	<i>But I mean ... and if you take the peg like that, it's the same as eating that one, right?</i>	-	Int	R
L 20	E	<i>This is horizontal. That is ... You have to implement this, this, this and this and then this and this ..</i> [she draws the six movements]	-	Def	B+C
F 20		 <p><i>Fig. 7.6 - Figure 20 (Episode 2 transcription)</i></p>	-		
L 21	P	<i>Six!</i>	-	Int	R
L 22	D	<i>Of course, we thought that with only a movement we had it.</i>	-	answ	R

Student-P and Student-D carry on a series of interrogative moves asking questions to Student-E. In Line 18, Student-E develops a move that is defintory and deductive at the same time. She defines the initial state of the jump along the other diagonal as that jump which involves the positions that have the difference between row and column number constant, to overcome Students P and D description difficulties she explains it through the successors. This definition supposes again a use of the backward reasoning: Student-E wants to use three positions in the same right-left diagonal and looks for the basic elements that characterize them. We noticed a factual "completeness difficulty" when Student-P and Student-D realize that it was necessary to represent six different jumps. From the RBC point of view Student-D recognize the "contrary movement", then Student-E building(-with) the notions constructs the rule to represent the column value of the contrary moments. Later Student-P recognize the horizontal movement and again Student-E puts together the previous knowledges (building-with) to identify the six movements (constructing). The *saming* discursive device is used to give the notation for the column value, putting together positions

that have the same geometrical properties, then through the *encapsulating* device all the different jumps in the board are reduced to six representative movements. To explain the mathematical resolution of the task, Student-E moves from the mathematical to the informal context and then she goes back to the mathematical one. This linking the two contexts helps students P and D to understand the task.

Student-E's reasoning continues with the definition of the jump. Starting from the mathematical representation that she already carried out, she generalizes the positions on the board until reach the global state for the peg jumps.

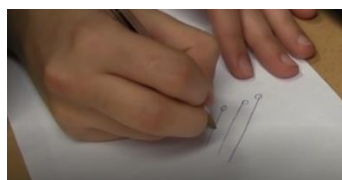


Fig. 7.7 - Student-E drawing diagonals


Line	Stud.	Transcription	BR	HIM	RBC
L 23	E	<i>Then...imagine that we have this board and our board is like this... [she draws the board underlining the diagonals, see fig 7.8]</i>	-	Init	R
L 24	E	<i>Ok, our diagonals are these. Here we have a peg.. and we want to implement this movement ... the one from the bottom up. Ok, if we want to implement this movement with these pegs is that... this is true, this is true and this is false... [she indicates the three pieces of the movement from bottom to top, see Figure 24] Have you labelled the empty position "false"?</i>	D	Int/ answ	B
F 24		<p style="text-align: center;"><i>Fig. 7.8 - Figure 24 (Episode 2 transcription)</i></p>			

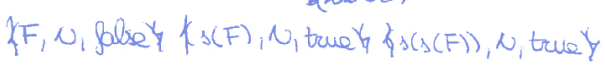
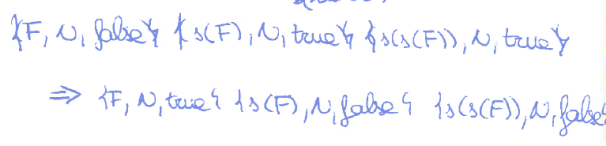
L 25	D	<i>Yes</i>	-	answ	B
L 26	P	<i>Well we have not put it, but it is what we think...</i>	-	-	-
L 27	E	<i>Okay, so, we want to make this move from here ... let's put ... [...] this is ... if the rows start 1, 2, 3, 4, 5..ok? [She numbers the rows from top to bottom] In this we have the 5... [she indicates the black peg below in figure 7.8]</i>	D	Def	R
L 28	P	<i>Wait in this ... the 1 is this, this or this? [he indicates the three corners of the triangle] I got involved ... Ah this is 1 and this is row 2 ... ok, ok.</i>	-	Int	R
L 29	E	<i>So, if you have this row ... [she indicates the first diagonal in figure 7.6] this is going to be 5, this is going to be 4 and this is going to be 3. [she indicates the three pegs and names the numbers of the rows to which they belong] And it is 1, 1 and 1. [she indicates the diagonal number to which they belong] If you wanted to move in this one ... [she indicates the third diagonal] you will have 3, 3 and 3. always moving on the diagonal you will have the same number here... [she indicates the second position of the definition by triple] then here you are going to put N, N and N at three.</i>	D	answ/ Def/ Ded	B+C
L 30	D	<i>With a constant?</i>	E	Int	R
L 31	E	<i>You declare a variable N and that's it. If you put "var" and N of natural at three it's ok.</i>	E	Def	B+C
L 32	E	<i>And what you have to put then is that this has to be true, this has to be true and this has to be false. Ok? This is your initial board for you to do this movement. In all cases you will need it.... wherever you are. Well and instead of having 3, 4 and 5</i>	D	answ/ Def/ Ded	B+C

		<i>which is the example here, you need... (D: Three different numbers?) No: F, $S(F)$ and $S(S(F))$.</i>			
L 33	D	<i>And the successor, how we indicate it?</i>	-	Int	R
L 34	E	<i>It is in the naturals. When it is in the naturals it can put 0, tiny s of N of natural that is $N+1$ and you can put the ones that you want “S of S”.</i>	E	Def	B+C
L 35	E	<i>Then your global state will always be, if F is natural. I'll have var F N nat and then I'll have rl diagonal ... top left.</i>	D	Def	B+C
L 36	D	<i>You have defined pairs here, right?</i>	-	Int	R

A series of interrogative moves continue to appear. In lines 29 and 32 Student-E defines the three initial positions on the board so a jump can be made. These definitions respect some deductive steps following the rules of addition. The backward reasoning appears in Student-E discourses. She always has her objective in mind, and she breaks down the elements she needs to be able to implement the peg jump. In line 30 and 33 some difficulties emerge: the first one is a description difficulty (Student-D asks for using a constant instead of a variable), while the second is a basic difficulty (Student-D doesn't know that the successor is a basic operation with natural numbers). Here the backward reasoning is used to explain what basic elements are needed. From RBC point of view, she recognizes the board and a notation, then through a series of building with she constructs the general row value and then the general column value (lines 29 and 32) until reach (constructing) the global computational state for the peg jumps. In lines 30-31 and then in lines 33-34 Student-D asks a question, recognizing a difficulty and Student-E answer to the question builds(-with) some notions together to overcome the difficulty by defining the notions that Student-D doesn't have (constructing). The *saming* discursive device is used to put together positions that have the same geometrical properties, then, through the *encapsulating* device, all the different jumps with the same geometrical properties are represented in a general way. In this transcription part there is an evolution of the discourse throughout the contexts, from the informal, through the mathematical until reach the computational with the definition of the global state.

In the last part Student-E finally defines, in computational language, the pegs jump.

Line	Stud.	Transcription	BR	HIM	RBC
L 37	E	<i>We need to start with a ... let's imagine, we have that this is Solitaire...</i>	-	Init	R
L 38	D	Solitaire and little more, right? ...On our board...	-	Int	R
L 39 F 39	E	<i>The good thing about the functional modules is that they try to fit in... Let's see, the good thing about the functional modules is that if you have defined a Solitaire as many pairs together, they will try to fit what you want here. [she indicates the pairs in the list] So they're going to pick this up... [she draws parentheses to include different triple of the lists, see Figure 9]</i>  <i>Fig. 7.9 - Figure 29 (Episode 2 transcription)</i>	E	answ/ Def	B+C
L 40	E	<i>...and you do not have to define that there is also a Solitaire. In the functional modules you have to put an s back to indicate that there are more things, but not in this. So, what you have to do here is simply ... How did you declare this? Have you put brackets, you have put something or are they just parentheses and ... that's it?</i>	-	Int	B
L 41	D	<i>Without parentheses or anything, just hyphen, comma, hyphen, comma...</i>	-	Int/ answ	R
L 42	E	<i>Maybe if you put a bracket or something ... you'll get confused less, yes. If you define these three with a parenthesis it [the program] is going to get less confused.</i>	E	answ	B+C

L 43	E	Well ... you are going to put this ... [she writes, see figure 7.8]	-	Def	C
F 43		 <p>Fig. 7.10 - Figure 43 (Episode 2 transcription)</p>			
L 44	E	You can also put some parentheses so that it does not fail. And then you make the implication and if you have this, [she indicates what she just wrote] then your final state will be that the first ones do not change and how this has been eaten (D: let's change the Booleans, right?) it is going to be true, false, false. [she writes, see figure 7.9]	-	Def	C
F 44		 <p>Fig. 7.11 - Figure 44 (Episode 2 transcription)</p>			

In the last part of the transcript, Student-E explains the importance of the commutative constructor in more technical terms, it is necessary to be able to make jumps. In line 39 Student-E puts onto paper the set structure she had visualised in her mind. This drawing represents the different pairs defining the board positions (the hyphens) and the operator that joins together the pairs three by three (the brackets). Taking into account the nature of system modules, which work on subterms, she realises it is enough to define the rule taking into account only the positions involved in the jump. That is, the board positions that are involved and the Boolean change from the initial position (true, true, false) to the final position (false, false, true). Student-E uses backward reasoning to answer to the questions that continue to emerge from her classmates. In line 41 a behaviour difficulty emerges; brackets and parentheses are necessary to not create confusion in the program. From RBC point of view, Student-E recognize the structure already constructed and then through a series of building-with she constructs the peg jump in computational terms. In lines 38-39 and 40-41, like in

the previous part, Student-D asks a question, recognizing a difficulty and Student-E answer to the question builds(-with) some notions together to overcome the difficulty by defining the notions that Student-D doesn't have (constructing). The *reifying* discursive device is used when Student-E starts to talk about the functional modules and then the program; the discourse change from "you put" or "you define" to "[the functional modules] are going to pick this up" or "[the program] is going to get less confused". In this last part, the students discourse remains in computational context.

7.2.2.1 General considerations

During the resolution, the movement inference modalities throughout the three contexts are indicative. The diagram in figure 7.12 represents the logical reasoning dynamism of Student-E who explains to Student-P and Student-D how to implement the jump on the board. Difficulties expressed by Student-P and Student-D are denoted by circles with a bold border. They are both methodological and factual: transference, basic, description, completeness and behaviour. Difficulties are not focussed on a single context, although they are more concurrent in the mathematical one. The greatest difficulties are observed in the passage between one context to another, particularly between the mathematical and computational one. Those emerge in the evolution of mathematical properties and concepts underlying the program development and in the necessary semiotic of signs and registers of representation typical of Maude programming language.

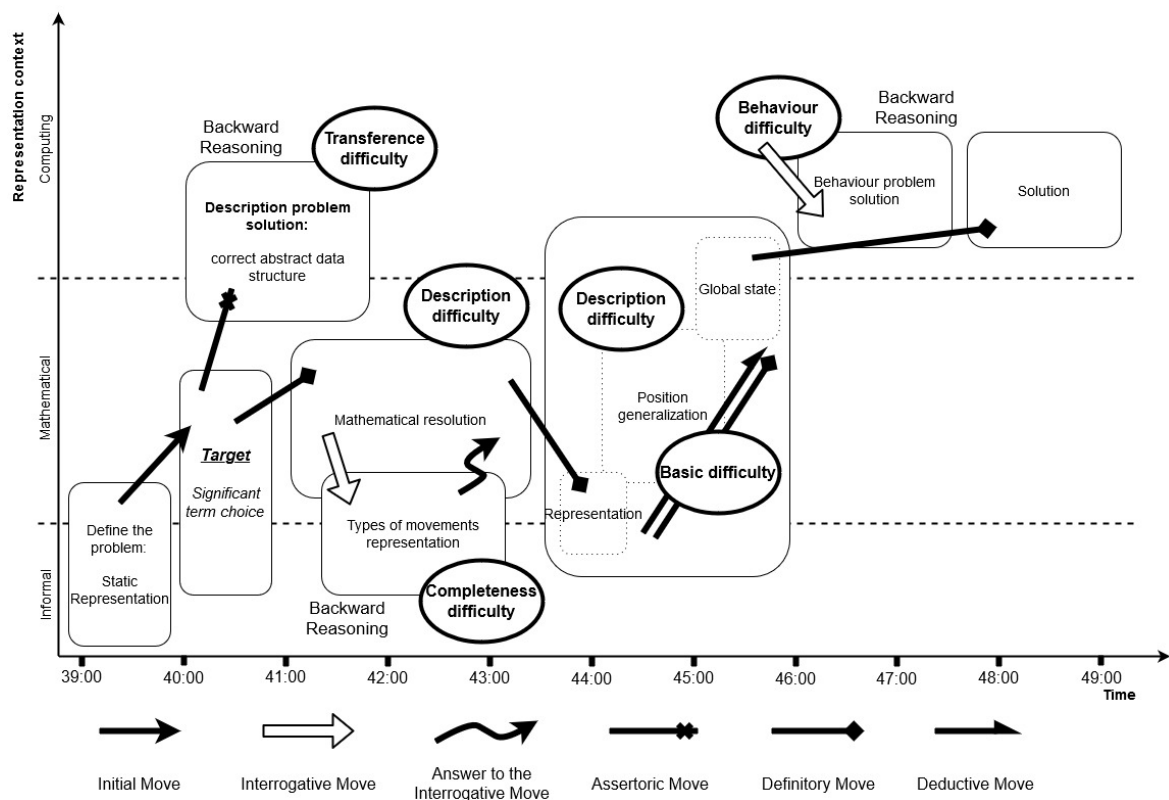


Fig. 7.12 – Student-E reasoning diagram. In bold circles her classmates' difficulties (episode 2)

Student-E, as an expert, would have developed the reasoning in linear form by applying forward reasoning. Having to interface with the difficulties of her classmates, she is forced to go back in her reasoning to explain the basis and premises of what she is saying (for example in lines 31 and 34). She uses also backward reasoning to reorganize her knowledge. In this way she can explain the resolution process to students P and D; this can be seen for example in lines 11 and 18 when she anticipates how the final structure will have to be (which she has already seen by solving the task before).

The analysis with the RBC model shows a series of chains. First with a recognizing Student-M makes a summary of exercise 1, then with a R-B-C chain she explains the importance of the commutative constructor. Then she starts to develop the mathematical representation of the pegs' jumps. After recognizing the Boolean values, though four R-B-C chains she develops the row and column values. Interspersed with those there are three R-B-C chains related to overcoming difficulties. The computational representation is characterized by two R-B-C chains, one related to the global state expression and the second to the solution.

Nested in the latter there are two R-B-C chains related to overcoming difficulties. The entire episode can be schematized through the RBC model in the following way.

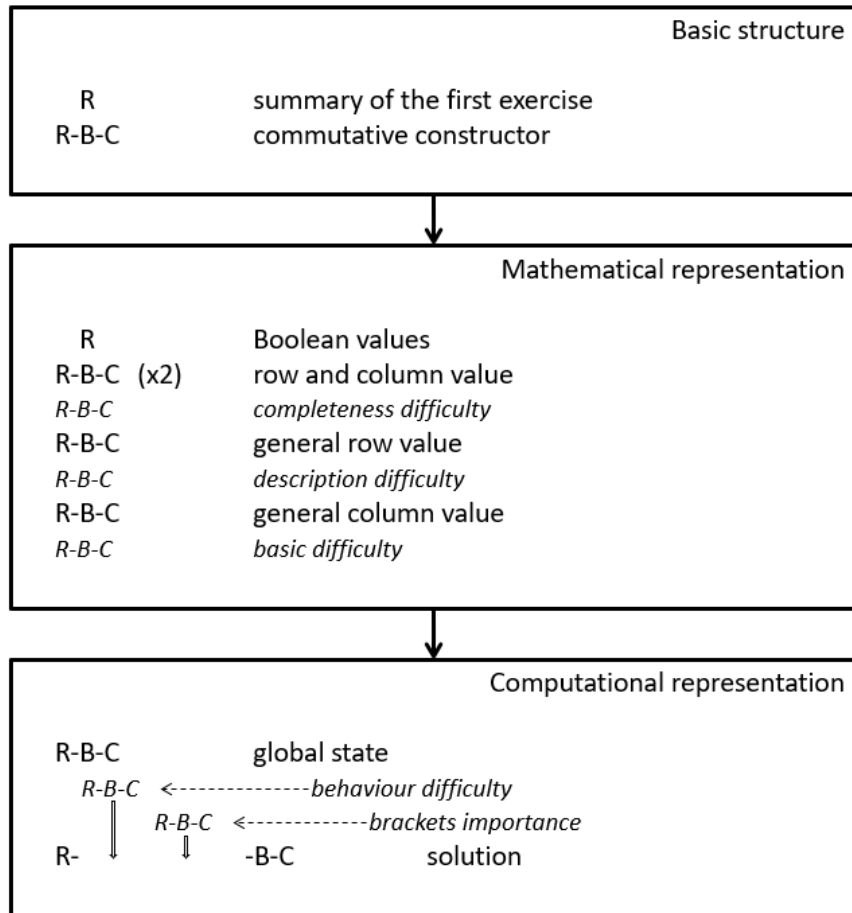


Fig. 7.13 - RBC flow (Student E resolution, episode 2)

The analysis of the discursive devices allows to notice that:

- When the backward reasoning appears in situation in which the Student-E refers to the operation of the program, a *reifying* device appears. In these situations, the student thinks about how the program acts and explains the elements necessary to achieve the desired behaviour. She first pushes herself to the final state that she wants to reach, then she goes backward, and, in forward way, she defines the necessary elements.
- When the backward reasoning appears breaking down the conjectured pegs' jump that Student-E have in her mind, the *saming* and *encapsulating* device appears to

associate different movements of pegs, according to their geometrical characteristics, and to consider them as an entity that can be formally defined in mathematical and computational language.

7.3 Discussion

The global analysis of the group has identified five different students' difficulties. Two are factual (completeness and behaviour) and three are methodological (description, estimation and transference). The in-depth analysis of the two chosen episodes has allowed to observe that the difficulties are generated in the transitions between the contexts.

The second episode analysis has highlighted the fact that an expert explanation generally develops in a forward linear way (Tall, 2002), but to help the novices, she considers the discovery processes involved in the creation of the explained knowledge. In fact, facing the difficulties of her classmates, Student-E, as a mediator of knowledge, takes into account the nature of her discovery reasoning to overcome them. To do it, she uses backward reasoning like an "ordering device" (Peckhaus, 2002), returning to the informal context and explaining the processes developed during her resolution. This reasoning helps to connect more intuitive aspects with the mathematical and computational ones, constructing productive paths for the explanation of concepts and for the transition between contexts, where the novices' difficulties are focused. These transitions are not linear but proceed with back and forth movements (confirming previous studies (Gómez-Chacón, et al., 2016)).

The movement of reasoning between different contexts is essential to reach the solution. The mathematical context mediates the transitions between the others. In this context other types of reasoning occur beyond deductive reasoning. The backward reasoning is used in its character of breakdown to extrapolate all the elements of the final computational formulation and anchor them to their informal and mathematical representation. This allow to progress towards the computational resolution. This analysis highlights the need for mathematical computational knowledge in the training of engineers.

The RBC flow is characterized by R-B-C chains. In the first episode they appear because the students recognize a familiar computational structure, the list (taking it from previous practice sessions), and through a series of building-with actions they construct the structure elements related to the Solitaire task. In the second episode the R-B-C chains appear in two distinct moments: when Student-E constructs element for exercise 2 resolution, and when she tries to overcome her classmates' difficulties. In the first moment she introduces an element, important for the resolution, and then through a series of building-with actions she constructs notions useful for the resolution. In the second moment is the classmate that recognize a difficulty (asking a question) and she overcome it constructing, for example, the motivation for the use of specific commands.

It is emphasised that backward reasoning occurs mainly in interrogative moves (HIM analysis). This reasoning is used in two principal moments: when the students manage to find elements necessary for the construction/definition of computational commands (the list in the first episode and the jump configuration in the second), and when the students think about the program final behaviour and introduce specific elements in the implementation (the commutative constructor). The first moment is characterized by the appearance of *saming* and *encapsulating* device that help in the solution formulation after a breakdown, while the second moment is characterized by the *reifying* discursive device used to define necessary elements after a forth-back-forth movement to the sought final state.

8

3D TICK-TACK-TOE ANALYSIS

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3D TICK-TACK-TOE ANALYSIS

8

In this chapter the results of the analysis of the third design experiment are shown. Briefly the design experiment settings are summarized in table 8.1.

Task type		Data collection settings				Students				
Strategy game	Mathematical	Work in group	Work alone	PC suite	Video-recordings	First level	Second level	Third level	Total	Case studies
✓	✓	✓	✓	-	✓	114	63	8	185	16*

*Twelve students working in pair and four PhD students working alone.

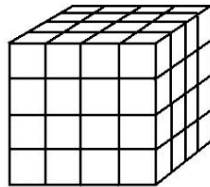
Tab. 8.1 - Third design experiment settings

The task proposed in this design experiment is the 3D Tick-Tack-Toe, as shown in figure 8.1. To solve this task, the students have to activate both heuristic and mathematical knowledge. In fact, the development of a winning strategy and a mathematical formula are asked by the task. The students involved were 185: 114 from the first, 63 from the second and 8 from the third level group (see Chapter 5). Sixteen students were involved in the case studies, analysing in depth the resolution protocols and video-recordings, for a total of ten case studies: three from first, three from second and four from third level group. Only two case studies belonging to the third level group (PhD students) are shown in this dissertation (see section 8.2).

First, the analysis of the whole group (185 students) is presented; the moments of backward reasoning development are shown through an in-depth analysis of the excerpts (section 8.1). Then, two case studies are displayed (section 8.1): they were chosen for the detailed mathematical formalization beside the strategic resolution. An in-depth analysis of the protocols of the two PhD students was made integrating it with some elements deriving from the discussion session developed during the PhD course. Finally, a general discussion is developed (section 8.3).

3D Tick-tack-toe

The 3D Tick-tack-toe is a three-dimensional version of the classic Three in a Skate game.



The game board is a 4x4x4 cube, be made up of 64 small cubes.

3D Tick-Tack-Toe is a two players game. One player can use "crosses" marks and the other "zeros" marks. Players move alternately by occupying with the own mark any empty cube.

Target: To place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.

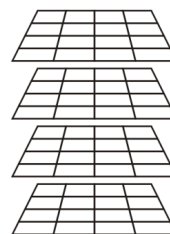
How to represent a cube?

Three dimensions: 4 squares with dimension 4x4 one on top of the other

Three-dimensions
representation:




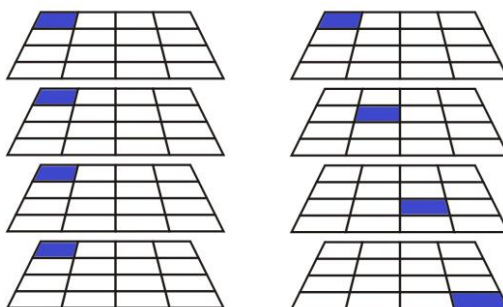
Two-dimensions
representation:



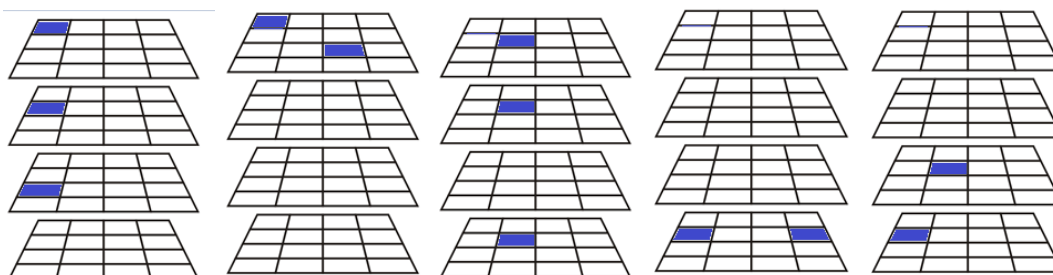
or



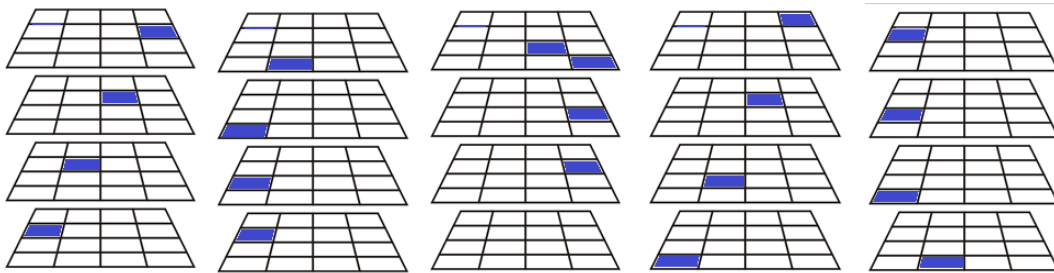
 Winning lines can be formed in all three dimensions! Here are two examples:



1. Complete the following winning lines



2. Indicate which are winning lines and which are not



3. Solve the game by finding the winning strategy. Detail your entire thinking process using the resolution protocol technique.

4. Express mathematically (formula, pattern, routine, ...) the relationships that can happen between the dimensions of the game board and the winning lines.

These empty boards can help you to solve the game.

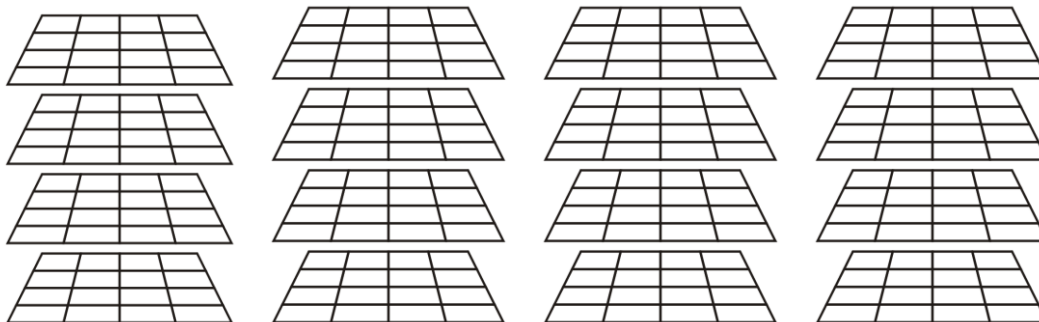


Fig. 8.1 – 3D Tick-Tack-Toe task

8.1 Analysis of the whole group

Analysing the 185 resolution protocols, five moments in which backward reasoning is developed are identified:

- Analyse the winning lines. The student considers the winning lines and classify them according to their geometric properties. 91 students use this strategy (49%)

- Define the favourable positions. The student evaluates how many winning lines pass through each box by identifying the ones where the most lines pass through. 130 students use this strategy (70%)
- Search for the final movements. The student studies the final movement that leads to the goal. It identifies it in the configuration in which the winner has two half-finished winning lines at the same time. This forces the opponent to put a token in a specific position, blocking only one of the two lines. 100 students use this strategy (54%)
- Block the opponent. The student decides where to place the token by identifying the opponent's possible winning lines and predicting his possible movements. He places the token in the best place to block the rival. Students who work in this way make a similar argument when they put their tokens thinking about the best position to complete their own winning lines. 136 students use this strategy (74%)
- Develop a mathematical formula. The student, analysing the geometric properties of the winning lines, develops a mathematical formula that links the number of winning lines to the board size. 62 students use this strategy (34%)

There are several differences between the academic level groups (for their definition see chapter 5, section 5.1) involved in the design experiment. In fact, looking at the percentages of development of these strategies for each group, the following data are obtained.

Backward reasoning moments	Students level group		
	First	Second	Third
Classify winning lines	32%	75%	87.5%
Favourable positions	61%	87%	62.5%
Final movements	57%	54%	12.5%
Block the opponent	68%	90%	12.5%
Mathematical formula	12%	71%	50%

Tab. 8.2 - Percentage backward reasoning moments divided by academic level groups

Although the size of the academic level groups is very different, since they are composed of 114, 63 and 8 students respectively, it is possible to make considerations by observing the percentages shown in the table.

The first difference that can be noticed is between the first two level groups and the third one. The PhD students, belonging to the third level group, worked alone, finished solving the task at home and then they discussed its resolution together; while the other students solved the task in pairs, playing together against each other. PhD students didn't play against a real opponent but imagined doing so (some used an internet simulation encountered on their own on internet pages) to explore the game. Probably, this made them focus more on the analysis of the winning lines and the subsequent development of the mathematical formula, while the strategic moves typical of the game (block the opponent and develop the final movement) were less considered.

The second difference that can be noticed is between the first and second level group. The students of the first level group focus their attention on solving the game, leaving aside the development of the mathematical formula. Only 32% of them analyse the geometric arrangement of the winning lines, which is a preparatory step for the mathematical formulation required in task exercise 4. Students of the second level group have solved the task in a more complete way by focusing on both the strategic and the mathematical part.

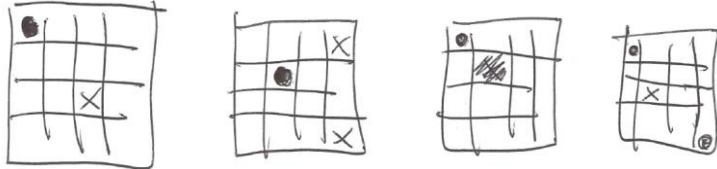
In the following section, two examples of backward reasoning development in “search for the final movement” and “block the opponent” moments will be presented. The first example excerpt belongs to third level group, while the second one belongs to second level group. Both were translated from Spanish by the author. Examples of other backward reasoning moments can be found in case studies in section 8.2. In particular: “analyse the winning lines” and “define the favourable positions” are in-depth analysed in section 8.2.2.3.1 (Case study 2), and “develop a mathematical formula” is in-depth analysed in section 8.2.1.3.1 (Case study 1).

8.1.1 Search for the final movement

One-hundred students of the whole group go to the end of the problem to figure out what is the last winning configuration. These students search for a configuration in which one player is forced to put a token in a specific position so that the other player wins. This is the case of student D179, he is a PhD student specialized in differential geometry.

The student, after defining the winning lines, tries to identify the final key movement, i.e. the movement that allows one of the two players to win the game, whatever move his opponent makes.

D179 resolution protocol excerpt

Line	Protocol	BR	HIM	RBC
3.1	I try to establish key positions for one of the two players that make it possible for him to win in 2 moves.	E	Init	B
3.2	These types of moves are those in which a player has two possible lines to complete, each one missing a single token.	X	Int/a	R
3.3	Example:  <p><i>Fig. 8.2 - Figure 3.3 (Student D179 resolution protocol)</i></p>	E	Def	B
3.4	Move where you move the next "X", the player who places the next "O" will win...	-	Def	C
3.5	... because he has the diagonal and a 3-dimensional column where a token is missing.	E	Int/a	B

The student decides to define the final position in which one of the two players has managed to align 3 tokens in two separate winning lines at the same time, so that his opponent is forced to place a token in one of the two free positions. In this way the opponent blocks a winning line and the player can win completing the other one. The action is guided by the implicit question "what's the last movement that allows to win?". A B-R-C chain characterise this excerpt. After exploring the situation (building-with), the student recognizes (line 3.2) the winning configuration. At this point the student exemplifies his definition with a drawing representative of the situation (B, line 3.3). He then expresses the rule defining the key movement (line 3.4) with a constructing action and justifies it (line 3.5) with a building-with action. From the point of view of backward reasoning this part of the protocol is

characterized by the research of cause-effect relationships, in fact the student is searching for some winning final moves. The introduction of the key movement definition can be considered an auxiliary element (line 3.2) that emerge from the explorations. The student pushes his thinking to the end of the game and explores the possible final combinations, and then goes back and runs them forward to win (line 3.4). The *reifying* discursive device appears when the sentence subject becomes the final configuration.

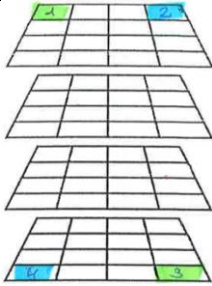
8.1.2 Block the opponent

Of the whole group, 136 students, after exploring the game, decide to try to solve it in a more systematic way. These students search to put their tokens in the best place to follow their sub-goals: block the opponent winning lines or create the best winning line for themselves. This is the case of student D119 and D120, they belong to the second level group and they are working in pair.

The students apply the strategy and make four steps: they are working together playing an imaginary game. They try to develop the best reasoning for each step of the resolution: put the token in the best place for creating its own winning lines and for blocking the opponent.

D119 and D120 resolution protocol excerpt

Line	Protocol	BR	HIM	RBC
L 21	I put in the corner because it gives us six winning combinations. (green token)	E	Int/ answ	B+R +C
L 22	The other player puts in the other corner to have 5 combinations and take one away from the opponent. (blue token)	E	Int/ answ	B+R +C
L 22	If the first player places in another plane, he wins two possible solutions. But if it's a top corner, he wins 3 lines.	E	Int/ answ	B
L 23	He places in position 3 (see picture [8.3], green token), to close a combination to the blue player.	E	Int/ answ	B+R +C

F 21-24	 <p data-bbox="411 546 1070 577">Fig. 8.3 - Figure 21-24 (D119 and D120 resolution protocol)</p>			
L 24	The blue player reasons in a similar way and places in position 4 (see picture).	E	Int/ answ	B+R +C

The students solve the game, analysing each move in an interrogative process driven by the questions “where is the best place to place the token?” and “what is the best place to block the opponent?” In each step the students proceed in this way: they observe the game situation (building-with), they recognize the useful winning lines and the best place to put the token and then they put the token (constructing). They go forward until they recognize the winning lines, then they come back in reasoning to choose the best position, and later they go forward again putting the token. The backward reasoning process is a cause-effect relationship research, in fact, students search for the best place to put the token, to get the desired effect (block the opponent or develop a winning line). The *reifying* discursive device appears when students pass from talk about movements to talk about the box, e.g. “the corner gives us....”.

8.2 Case studies

From the global analysis, five backward reasoning moments were identified. To see the connection between strategic and mathematical reasoning and the mathematical development, was decided to deepen the analysis on these moments through sixteen case studies. Here, two cases are displayed. The chosen students (for this dissertation) belong to the third level group. Being PhD students, they solved the game by developing an individual resolution protocol; this has allowed for more in-depth study and reflection on each student's personal reasoning.

Each student structured the protocol slightly differently. Student-A structured his protocol according to 35 points. Each point corresponds to a problem solving resolution phase, the numbering seems to correspond to a series of strategic steps, in chronological order, leading to the game resolution. Student-B structured his resolution protocol temporally, indicating about every 10 minutes the passage of time in his resolution. Student-C structured the resolution protocol according to 9 points. Each point seems to correspond to a more or less long phase of the resolution. Student-D, again, structured the protocol temporally, indicating every 15 minutes the passage of time. The entire protocols have been translated from Spanish by the author.

Each protocol has been divided according to the epistemic (and pragmatic) actions developed. To understand the different structure of the protocols, for each section the codification of the protocol lines is explained. To each protocol line is associated a characteristic of backward reasoning (column BR), if present, and two characteristics of the multidimensional analysis tool (column HIM and RBC) (see Chapter 5, section 5.2.2). Each protocol part has a short comment to identify the characteristics according to each analysis model (HIM and RBC), the backward reasoning characteristics and the discursive devices used by the student.

8.2.1 Case study 1: focus on mathematical formulation

The protocol of Student-A was chosen for the first case study; he is a PhD student specialized in Complex and Algebraic Geometry. Despite having solved the game from both strategic and mathematical point of view, most of the protocol is focused on the game mathematization. Some results of this case study were presented in *Mathematics* journal (Barbero, Gómez-Chacón and Arzarello, 2020).

Student-A has structured his protocol according to 35 points. It is a highly elaborate protocol, each line was coded according to a pair of values (x.y): where the first number (x) corresponds to the student's notation, the second number (y) corresponds to the subdivision made by the researcher. The protocol can be divided into two large parts: the resolution of the game in two dimensions and the resolution of the game in three dimensions. From line

1.1 to line 11.2 the student solves the 2D game, i.e. he solves the classic Three in a Skate played in a 3x3 grid. From line 12.1 to line 35.5 the student thinks about the 3D game. In both parts of the protocol the student's goals are two: to find a winning strategy for the game and to develop a mathematical formula that links the number of winning lines to the size of the board.

8.2.1.1 Part 1: 2D game resolution

Part 1 can still be divided according to the goal the student is trying to achieve: first he focuses on the winning strategy development (Lines 2.1-8.2), then he develops the mathematical formula (Lines 9.1-11.2). The student starts recognizing an analogous game. He remembers having already seen the 2D game and tries to remember its strategy characteristics. This analogy with the previous game will support the student's entire resolution process. As it refers to an analogy between game contexts, we call it “contextual analogy”.

Lines	Protocol	BR	HIM	RBC
1.1	I seem to remember that the usual tick-tack-toe has a winning strategy.	-	Init	R
1.2	Is that so? I'm going to explore the case of 3 marks in 2D.	-	Int	B

8.2.1.1.1 Part 1.1: Winning strategy development

The student starts to solve the 2D game and looks for a winning strategy. He explores the 2D case until he conjectures the existence of a winning strategy.

Line	Protocol	BR	HIM	RBC
2.1	It seems logical to put the first mark in the centre.	-	Ast	B
2.2	This way I cover more ground.	D+E+X	Int/a	B+R

Firstly, he puts the first token in the centre. He justifies this action by saying that "this way I cover more ground". It can be assumed that, not explicitly, the student is identifying the winning lines of the game by combining his geometric knowledge and his game memories; then, he recognizes pattern in which he identifies the favourable square: the centre. These two actions, identifying the winning lines and recognizing a pattern in the squares can be classified, according to the RBC model as building-with and recognizing (line 2.2). According to the HIM, these actions can be considered as an answer to an implicit question: "what is the best position I can occupy?" The backward reasoning is involved; Student-A breaks down the winning lines to create a pattern (auxiliary element) and he search for the best place to put the token through a cause-effect relationship research. Researchers suppose this development because a similar reasoning is made more explicit later in the resolution.

Line	Protocol	BR	HIM	RBC
2.3	I try several counter-response strategies.	-	Ru	B
3.1	If your opponent responds by putting a mark in the middle of an edge, you can always win.	-	Ast	C
4.1	If the opponent puts it in a corner, he can force me to draw.	-	Ast	C

Student-A formulates a first conjecture (constructing) regarding the winning strategy, combining the strategic knowledge he has acquired from the exploration. The conjecture is divided into two parts, depending on the opponent's move.

At this point, the student decides to introduce and defines a notation for the squares of the game (line 4.2), this is based on the combination of knowledge derived from the explorations, the geometric properties of the square and the introduction of a pattern in the squares. The notation proposed by the student divides the squares not only according to their position on the board but also according to the amount of winning lines passing through each square. This can be classified as breakdown for backward reasoning processes.

Line	Protocol	BR	HIM	RBC
4.2	I decide to name the boxes: Vertices, Edges and Centre.	D	Def	C
5.1	That way, whoever starts always wins or draws.	-	Ded	B
5.2	Interesting. There is no winning strategy.	-	Ast	B
6.1	I try the other way around, starting by playing somewhere else.	-	Ru	B
6.2	By symmetry, if the opponent responds in the centre, he wins.	-	Ast	R+C
6.3	The centre seems the key.	-	Ast	C

The student continues to think about the winning strategy. Through a series of building-with, or deductive move, combines the strategic knowledge acquired until this point and states that whoever starts, if he plays the best, either wins or at most draws, never loses. At this point he elaborates the second conjecture on the winning strategy. This is based on the symmetry of the two-player game (R) and, like the previous one, can be classified as constructing (RBC) or Assertoric (HIM) (line 6.2). Although it is not explicit, again, in order to formulate the conjecture, the student has in mind the pattern of favourable boxes elaborated previously (recognizing). The *encapsulating* discursive device is used: student-A replaces the discourse about several strategies with the definition of the winning strategy.

Line	Protocol	BR	HIM	RBC
7.1	Except for the opponent's mistakes, the way to win is to control the centre and two corners, and that the opponent only controls one corner.	-	Def	B
7.2	It seems that this position is advantageous.	-	Ast	B
8.1	I play with a few more examples.	-	Ru	B
8.2	Indeed, the key is to control as many lines as possible.	-	Def	B

As last step, the student elaborates the general strategy by combining the two conjectures previously explained (Def, B, line 7.1, 8.2).

In this first part of the protocol the student's strategic thinking seems to follow a chain of epistemic actions B-R-C, until the first conjecture is made explicit, and then another chain B-R-C, until the second conjecture. The combination of the two conjectures (B) allows the student to formulate the final conjecture. Elements of backward reasoning are shown in this part of the protocol. For example, in the initial exploratory phase when the student reasons about the connections of the board with the winning lines.

8.2.1.1.2 Part 1.2: Mathematical formula development

In this second part, the student, starting from the memory of the game and his previous explorations of the winning strategy, begins to think about the mathematical formula that links the number of winning lines to the size of the board.

Line	Protocol	BR	HIM	RBC
9.1	I start to quantify how many lines I cancel out the opponent with each move.	D	Int	B
9.2	The centre cancels out more than any (4), vertices 3 and edges 2. That justifies the heuristics.	X	answ	R
9.3	Avaricious strategy seems to work.	-	Ast	R

The student reasons starting from the winning lines. He decides to quantify the winning lines he takes away from the opponent. To do this, he combines the geometric knowledge and identifies, explicitly, a pattern in the position of the squares. As in the previous case, these two actions are classified according to RBC as building-with (line 9.1) and recognizing (line 9.2), for HIM as interrogative move and answer, and from the point of view of backward reasoning as breakdown and introduction of auxiliary elements. By doing this, Student-A seeks confirmation of the strategic conjecture he formulated in previous part of the protocol.

9.4	I try a few more examples and, ...	-	Ru	B
9.5	...in fact, it seems that behaving avariciously allows you to win or draw.	-	Ast	B

The student continues to play and tries new combinations, until, once again, he reformulates the winning strategy he has already explained (Ast, B, line 9.5).

Line	Protocol	BR	HIM	RBC
10.1	I count how many lines there are.	D	Init	B
10.2	I calculate 8: 3 in each of the two directions and two diagonals.	D	Ded	C

At this point he reclaim the initial reasoning (Init, B, line 10.1): there is a correspondence between Lines 9.1 and 10.1, the student wants to calculate the number of winning lines. Student-A counts the number of winning lines following a mathematical scheme (Ded, C, line 10.2). He divides the winning lines according to groups of parallel lines: by "the two directions" he refers to the winning lines parallel to the sides of the board. In this fragment there is backward reasoning in its breakdown feature: in fact, the student considers the winning lines, counts them and divides them according to their geometric characteristics. The *saming* and *encapsulating* discursive device are used: the first to put together the lines according to their geometric characteristics, the second to identify the scheme.

Line	Protocol	BR	HIM	RBC
10.3	I realize that I always try to subtract as many active lines from the opponent. It seems like a formal strategy.	-	Ast	B
11.1	I try to test it rigorously, but I can only exhaust the cases.	-	Ru	B
11.2	I don't see a pattern.	-	Ast	B

At this point, through a building-with action, he verifies that the number of winning lines is correct. In doing so, he expresses his difficulties in not being able to find a pattern that can help him to identify the lines in a mathematical way. He constantly controls his work by checking the number of winning lines on a case by case basis.

In this part, like the previous one, a B-R-C sequence leads the student to the mathematical decomposition of the winning lines number. This is interrupted by the need to reformulate

again the winning strategy (B). At the end of the chain, the student makes control over his actions. As before, elements of backward reasoning are present in the discovery process that leads the student to the formulation of the breakdown.

8.2.1.2 Part 2: 3D game resolution

The student switches to solving the game in three dimensions. This part of the protocol can still be divided into three smaller parts according to the goal the student is trying to achieve: in the first part he focuses on the development of a general mathematical formula (Lines 12.1-19.3), in the second part on the verification of the general mathematical formula (Lines 20.1-26.4), and in the third on the development of the winning strategy (Lines 27.1-35.5). The first epistemic action that is identified in this second part corresponds to line 12.2: it is an initial move and a recognizing action. The student recognizes an analogy between the three-dimensional game context and the two-dimensional game context. Thanks to this contextual analogy, he transfers the strategic and mathematical knowledge acquired in the previous parts to the new phase. This knowledge will remain as a background to the game resolution and allow the student to achieve his goals.

Line	Protocol	BR	HIM	RBC
12.1	I decide to move on to the 3D case.	-	Init	B
12.2	The previous strategy suggests me to count lines.	-	Init	R

8.2.1.2.1 Part 2.1: General mathematical formula development

In this part, the student focuses on explaining a general mathematical formula that links the number of winning lines to the size of the board. He begins to reason by relying on the knowledge learned in the previous phases of the resolution, in particular, in the numerical breakdown of the winning lines number. In this part of the protocol the use of backward reasoning is predominant.

Line	Protocol	BR	HIM	RBC
12.3	I make a few drawings to test.	-	Init	B
12.4	There are 10 lines in each plan parallel to the axes ...	D	Int/a	B
12.5	... and there are 12 planes parallel to the axes.	D	Int/a	B
12.6	I lack the “diagonal lines” as in the example. They seem more complicated.	X	A	R
13.1	I'm starting to do numerology: $10 = 4 * 2 + 2$...	D	Int/a	C
13.2	...which is broken down as the number of pawns times dimension of the plane plus two diagonals.	X	Int/a	C
13.3	Will it be general?	-	Int	B
14.1	I realize that $12 = 4 * 3$...	D	Int/a	C
14.2	...that seems to follow the previous pattern.	X	Int/a	R

The student begins the 3D game resolution by focusing on the latest knowledge learned in 2D game resolution. He focuses on counting the winning lines. He divides the board (a cube) according to its geometric sub-spaces (the planes); he then considers, for each plan, the winning lines. The student answers the implicit question “In each plane, how many lines are there?” by combining (B, line 12.4-5) different geometric and game knowledge. Then, he recognizes a structure in the arrangement of the winning lines (Ast, R, line 12.6): there are 10 winning lines in each plane and the "diagonal lines". Combining the winning line arrangement and alpha-numeric knowledge, he subdivides the line number clarifying a linear combination (C, lines 13.1-14.1). It can be interpreted as he is answering the implicit question "How can I break down the winning lines number?". The student is alternating breakdown phases with the introduction of auxiliary elements. The first one whenever he operates on the winning lines by breaking them down according to their geometric characteristics (lines 12.5 and 13.1), the second one when he makes the linear combination explicit (line 13.2). The *saming*, *encapsulating* and *reifying* discursive device are used: the first to put together the lines according to their geometric characteristics, the second to identify the scheme, the third when the sentence subject becomes ‘the number’.

Line	Protocol	BR	HIM	RBC
14.3	Hope.	-	-	-
14.4	It looks like a nice combinatorial problem.	X	Ast	R
15.1	It reminds me of geometry calculations on finite fields.	X	Ast	R
15.2	I think about shooting over there, but I realize that there are cyclic lines that come out on one side and appear on the opposite side.	D	Ast	R
15.3	These movements are not allowed.	E	Ast	B
15.4	I could rule them out, but it seems too complicated.	E	Ast	B
15.5	I abandon this strategy.	-	Ast	C

A series of analogies characterize this protocol excerpt. They emerge through an assertoric move. The student compares the structure of the problem firstly to a combinatory problem, then to the finite fields. He recognizes that there are differences between the characteristics of the finite fields line and those in the game (lines 14.4-15.2). Later, he combines his knowledge in both fields to conclude that some finite fields lines are missing in the game, and that they could be removed to structurally match the two problems (lines 15.3-15.4). However, he abandons the idea considering it too complicated. Although he has not continued on the finite fields path, this analogy, that we call “structural analogy” remains in the resolution. It allows to get the solution. The student introduces in the resolution the auxiliary element "finite field", through the breakdown he obtains its properties, and when he notices that there are some characteristics that do not match, he is making cause-effect reasoning (E, lines 15.3, 15.4).

Line	Protocol	BR	HIM	RBC
16.1	I think of a recursive pattern.	X+T	Ast	R
16.2	I guess n pieces in d dimensions (the usual case is $(n, d) = (3, 2)$ and this is $(n, d) = (4, 3)$).	-	Ded	B
16.3	Maybe the number of straight lines follows a pattern.	FS	Int	C
16.4	$L(n, d) = cnt(n, d) * L(n, d - 1) + Diagonals$	-	Ast	B

The student introduces another element through an assertoric move: the recursive pattern. The recognition of knowledge occurs through a structural analogy (line 16.1). The student identifies a possible structure of the mathematical formula he is looking for. He constructs the recursive formula on the basis of the knowledge previously acquired in the resolution, in particular those related to the breakdown of the number 10. The student, by introducing the formula to line 16.4, makes explicit the solution of his sub-problem: finding a mathematical formula that links the number of winning lines to the size of the board. The formula he proposes is not a “clean” mathematical formula. In fact, some elements of the formula still remain unknown from a mathematical point of view: he has to explain the value of the constant (that depends on n and d) and the value of the diagonals. He is working with backward reasoning in its transformative characteristic.

Line	Protocol	BR	HIM	RBC
17.1	The constant must be the number of planes parallel to the axes.	D+T	Int/a	B
17.2	As in the previous case, these have to be nd , ...	D+T	Def	C
17.3	... then I refine my formula to $L(n, d) = nd * L(n, d - 1) + Diagonals$	-	Ru	B
18.1	Diagonals don't seem that simple.	D+T	Int	B
18.2	I start to play with the example of the cube and the plane.	-	Ru	B
18.3	They seem to join opposite vertices of opposite faces.	D+T	answ	B
18.4	Will it be general?	-	Int	B
19.1	I calculate that a hypercube has 2^d vertices, which gives me two faces with 2^{d-1} vertices.	FS	Def	C
19.2	Thus, if my previous observation is correct, the formula is $L(n, d) = nd * L(n, d - 1) + 2^{d-1}$	-	Ru	B

After having conjectured the existence of a general recursive formula, the student clarifies the values of the constant and diagonals. In order to do so, he puts together previous game and mathematics knowledge (B, line 17.1). He is able to formulate first the constant value

(C, line 17.2) and then the diagonals one (C, line 19.1). To do this, the student breaks down the "raw" formula and analyses part by part the missing elements (D, lines 17.1-18.3). Doing it, he reaches the explanation of the general mathematical formula (FS) and the creation of his sub-problem solution. The HIM allows to identify a series of interrogative moves alternating with defining moves. The *reifying* discursive device is used during the formula development: the sentence subjects become the formula and its elements.

In this third part of the resolution protocol, in order to arrive at the formulation of the "raw" mathematical formula, the student has passed through two B-R-C chains. The first led him to the mathematical decomposition of the number of winning lines (in a plane), the second to conjecture the existence of a general mathematical formula and to express it in "raw" form. The second chain B-R-C is not linear. In fact, it is "interrupted" by the structural analogies present in lines 15. Then the student expresses the formula through two B-C chains. The backward reasoning is predominant. It strongly characterizes the discovery processes that lead the student to the creation of the solution element: the mathematical formula. In this case we notice that some characteristics of the reasoning manifest themselves in a momentary manner, such as the introduction of auxiliary elements, while other processes, such as the breakdown or the transformative features, that leads through the analysis of the "raw" formula to the final mathematical formulation, are more protracted in time.

8.2.1.2.2 Part 2.2: Formula verification

The fourth part of the resolution protocol concerns the verification of the general formula set out in the third part. The student builds the general formula in a forward way starting from the analytical representation of the winning lines.

Line	Protocol	BR	HIM	RBC
20.1	I try it on low cases.	-	Ru	B
20.2	It works.	-	Ast	B
20.3	I think about proving it by induction, but I still have the problem of diagonals.	-	Init	R
20.4	I abandon the idea.	-	Ast	B

21.1	I try with $d = 4, n = 3$ to check the formula.	-	Int	B
21.3	I have a problem to represent it, but I decide to follow the representation of the previous exercise and arrange 3 juxtaposed cubes.	-	Init	C

The student starts to test the formula using small n and d values, so that, with his acquired game knowledge and counting the lines case by case, he can easily verify the formula (Init, B, line 20.2). Then he thinks about proving the correctness of the general formula through an induction (R, line 20.3) but he has difficulties in representing diagonals and abandons the idea. Later, he tries the formula in the case of a $3 \times 3 \times 3 \times 3$ hypercube (B, line 21.1), also in this case he has some representation problems, but observing the board proposed by the researchers he succeeds in representing it (C, line 21.2). The interrogative move guides the reasoning, the student is wondering how to prove the formula validity.

Line	Protocol	BR	HIM	RBC
21.4	I realize that what I'm actually doing is setting the first coordinate and moving the others.	-	Def	R
21.5	Maybe, is this the recursive pattern?	-	Int	B
22.1	I realize that the key is in which vector is the director of the line.	-	Ast	R

The student encounters an analogy in the analytical structure of the problem (structural analogy) and associates each winning line to a vector director (Ast, R, line 22.1).

22.2	If it has a zero, that coordinate remains fixed and, in reality, it is a line of the recursive case.	-	Ded	B
23.1	Aha! That seems to be the way to proceed to the end.	-	Ast	B
23.2	I realize that the vector director has to have the form $v = (\varepsilon_1, \dots, \varepsilon_d)$ with $\varepsilon_i = \pm 1, 0$.	-	Def	C
23.3	That seems to be the key to count.	-	Ast	B

24.1	If any $\varepsilon_i = 0$, then the line belongs to a parallel hyperplane.	-	Ded	B
24.2	There are d positions to put the zero and, in that case, it can start at any of the n hyperplanes.	-	Def	B
24.3	That justifies the $nd * L(n, d - 1)$. Perfect.	-	Ded	C
25.1	If all $\varepsilon_i = \pm 1$, then I realize that they have to go from vertex to vertex.	-	Ded	B
25.2	I also realize that the original vertex is fixed by v , since it is $(\delta_1, \dots, \delta_d)$ with $\delta_i = (n - n\varepsilon_i)/2$.	-	Def	B
25.3	That would give me 2^d diagonals, as many as v possibilities.	-	Ded	C

Through the winning lines representation as vectors director, the student is able to construct rigorously the mathematical formula found previously. The epistemic actions of building-with and constructing alternate, as defining and deductive moves. At this point, the student realizes that there is something wrong in the reasoning, because he expected a different number of diagonals.

Line	Protocol	BR	HIM	RBC
26.1	It doesn't work even in low cases.	-	Ast	C
26.2	I realize that v and $-v$ give the same straight line.	-	Def	R
26.3	Light. There is an action of \mathbb{Z}_2 and that gives me the desired 2^{d-1} diagonals. Proven conjecture.	-	Ded	C

The student manages to overcome the error thanks to the introduction (R, line 26.2) of another element from previous knowledge: the actions of \mathbb{Z}_2 . He identifies the structure within its resolution and easily obtains the desired number of diagonals (C, line 26.3).

The backward reasoning is absent in this part, which is characterized only by synthesis processes. The excerpt is characterized by a series of B-C chains interrupted in some points by recognizing actions. The recognizing allows the introduction of structural analogies useful to the identification and overcoming of difficulties. The *reifying* discursive device is

used in the entire part: the sentence subject becomes the analytical elements that Student-A introduces in the discourse.

8.2.1.2.3 Part 2.3: Strategy development in 3D game

In the fifth part of the protocol the student reclaims thinking about the 2D winning strategy (Part 1.1). The student begins to think about the strategy already encountered and tries to apply it to the 3D case.

Line	Protocol	BR	HIM	RBC
27.1	I begin to think that this strategy is better.	-	Ast	R
27.2	First, I think of $(n, d) = (4, 2)$. I do a few tests.	-	Ru	B
27.3	In that case it is much more difficult to argue for forced movements.	-	Ast	R
28.1	In any case, it seems that the criterion of nullifying as many lines to the opponent works quite well.	-	Init	R
28.2	I realize that now there is no centre, so there is no preferred movement.	D	Ded	B
29.1	All the boxes, except the ones in the centre of the edges annul 3 lines (two straight lines and one diagonal). That could explain why there are so many good movements.	-	Def	B

The student, after having tried a couple of times the strategy in a 4x4 board, realizes that in this case the central position is not unique. This means that there is no preferred position on the board in order to win. The epistemic actions that appears is building-with where the student reasons by composing previously acquired knowledge. There are deductive and defining move in taking up the reasoning of parts 1.1 and 2.1.

Line	Protocol	BR	HIM	RBC
30.1	I think of the centre.	-	Init	B
30.2	I realize that it exists only when n is odd.	E	Int/a	R
31.1	If n is odd, we have a centre.	E	Ded	C

31.2	All (2^{d-1}) diagonals and d parallel planes pass through the centre, then it seems the best option.	E	Int/a	C
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The student thinks about the existence of the central box and recognizes that this box is unique only when n is odd (R, line 30.2). Then, the student thinks about a hypercube of size (n, d) . He reasons about the characteristics of the central box when it is unique (C, line 31.1-31.2). From the point of view of backward reasoning, the student is trying to make explicit cause and effect relationships (E, lines 30.2-31.2) between the dimensions of the board and the characteristics of the middle boxes. These reasonings always start from the geometrical considerations on the winning lines. The interrogative move dominates the excerpt, the implicit question is always the same "what is the best position for the tokens?"

Line	Protocol	BR	HIM	RBC
31.3	Through the boxes that some diagonal passes, if it's not the centre, only one passes and then $d + 1$ lines pass.	D	Int/a	B
31.4	Through the others, d lines pass.	D	Ded	B
31.5	I decide to call these 3 possibilities as before (centres, vertices and generalized edges).	X	Def	R
32.1	Is there any quick criterion to know if a diagonal passes through a box?	-	Int	R
32.2	In the case of odd n it seems that yes, because the center is $C = ((n - 1)/2, \dots, (n - 1)/2)$ and then through A passes a diagonal if and only if $A - C = s(\pm 1, \dots, \pm 1)$ for an integer s .	E	Int/a	C
32.3	That shows me again that only one diagonal passes through each vertex that is not a centre.	-	answ	C

The student continues to think about the board boxes characteristics. He explains how many lines pass through each type of box (B, lines 31.3-4) and then he extrapolates a pattern (R, Line 31.5). At this point, he searches a mathematical criterion to understand if a diagonal

passes through a specific box. This is the first time within the protocol that the student clearly explains a question that he asks himself. Then, he gives an answer to his question for the n odd case (C, line 32.2). The student breaks down the board (D) to extrapolate a pattern (X, line 31.5), and then makes explicit the relationship between squares and diagonal winning lines (E, line 32.2). The *encapsulating* discursive device appears when Student-A consider all the winning lines passing through the centre box.

Line	Protocol	BR	HIM	RBC
33.1	I test the case with even n .	-	Ru	B
33.2	In this case I don't find a simple pattern to determine if a diagonal passes through a box, beyond the definition itself.	-	Int/a	B
33.3	I realize that it's not really very expensive to revise it directly, so you can follow this criterion.	-	Ru	B
34.1	I conclude that a good strategy is to put in the centre [the mark], if you can't [put the mark] in a vertex and, if you can't in an edge; as long as you don't win or you don't have to prevent them from winning.	-	Ast	C
34.2	I realize that the strategy of placing in the corners, in the classic case, precisely sought to exhaust the vertexes.	-	Int/a	R
35.1	I remember again that it is a greedy strategy. I wonder if it will be optimal.	-	Ast	R
35.2	I think about using decision trees and alpha beta pruning. I don't see it clear. I think the best thing to do is a simulation if I have time.	X	Ast	R

The student explains the winning strategy (Ast, C, line 34.1) for the $4 \times 4 \times 4$ board after having reasoned on the relationship between winning lines (B, line 33.3). Again, explicit references to the previously solved 2D case can be noted. Later, student-A explicitly asks himself whether the strategy he has encountered is good. In order to be able to answer this question, he assumes that he can use verification techniques learned in his university career (Ast, R, line 35.3). He is not able to apply immediately the auxiliary techniques introduced and due

to lack of time he does not finish the task. The *encapsulating* discursive device is used: student-A replaces the discourse about several strategies with the definition of the good strategy.

In this last part of the resolution protocol, the student goes through two B-R-C chains to identify the relationships between the favourable positions and the winning lines. With a building-with action the student verifies the strategy formulated in a specific case, and then through a B-C sequence he arrives at the general strategy formulation. The sequences are interspersed with explicit questions; he wonders whether the criteria and the strategy are general. Through a structural analogy, the student proposes a way to verify the generality of the strategy. He will not do this verification for lack of time. Again, backward reasoning appears in the discovery phases. Compared to the other parts of the resolution protocol, here it is used more for the search of cause-effect relationships between the position of the boxes and the winning lines.

8.2.1.3 Case study 1 discussion

From the HIM point of view, the backward reasoning develops when the student asks a question during the path towards the formation of ideas and conjectures after a phase of exploration. This is found for example in lines 12.2-14.2 in part 2.1 of the protocol where the student refers to notions learned previously during the resolution of the 2D game. Therefore, he answers the question "What is the total number of lines in the 3D game?". A good question allows the subject to formulate premises for certain statements, or in combination with certain statements to draw some conclusions. It can be clearly observed, again in Part 2.1 of the Protocol, (between lines 16.1 and 19.2). The student alternates interrogative moves followed by an answer and defining moves (an elaboration of the answers). Through this process the student identifies all the formula terms.

From the RBC point of view, the resolution protocol is characterized by two different types of chains: B-R-C and B-C. B-R-C chains characterize the discovery processes while B-C chains are predominant in the processes of verification or construction of mathematical

concepts, in this case the general formula. By observing the resolution protocol in its five parts it can be said that:

- In Part 1.1, the student applies two B-R-C chains to formulate a conjecture on the 2D game winning strategy (lines 2.2-6.3);
- In Part 1.2, the student applies a B-R-C chain to obtain a mathematical breakdown of the number of 2D game winning lines (lines 9.1-10.2) which will be used for the mathematical formulation of the relations between winning lines and the board size;
- In Part 2.1, through two chains B-R-C (lines 12.4-16.4) he conjectures the existence of a general formula, and with two chains B-C (lines 17.1-19.2) he explains the mathematical formula;
- In Part 2.2, he verifies the mathematical formula, by constructing it starting from the premises, through four B-C chains (lines 20.2-26.3);
- In Part 2.3, he explains the relations between the boxes and the winning lines of the 3D case through two B-R-C chains (lines 28.2-32.2) and then he explains the winning strategy with a B-C chain (lines 33.3-34.1).

The sequence of chains of the whole protocol can be schematised (figure 8.3) following the five parts subdivision, dotted lines indicate the transfer of knowledge from previous phases.

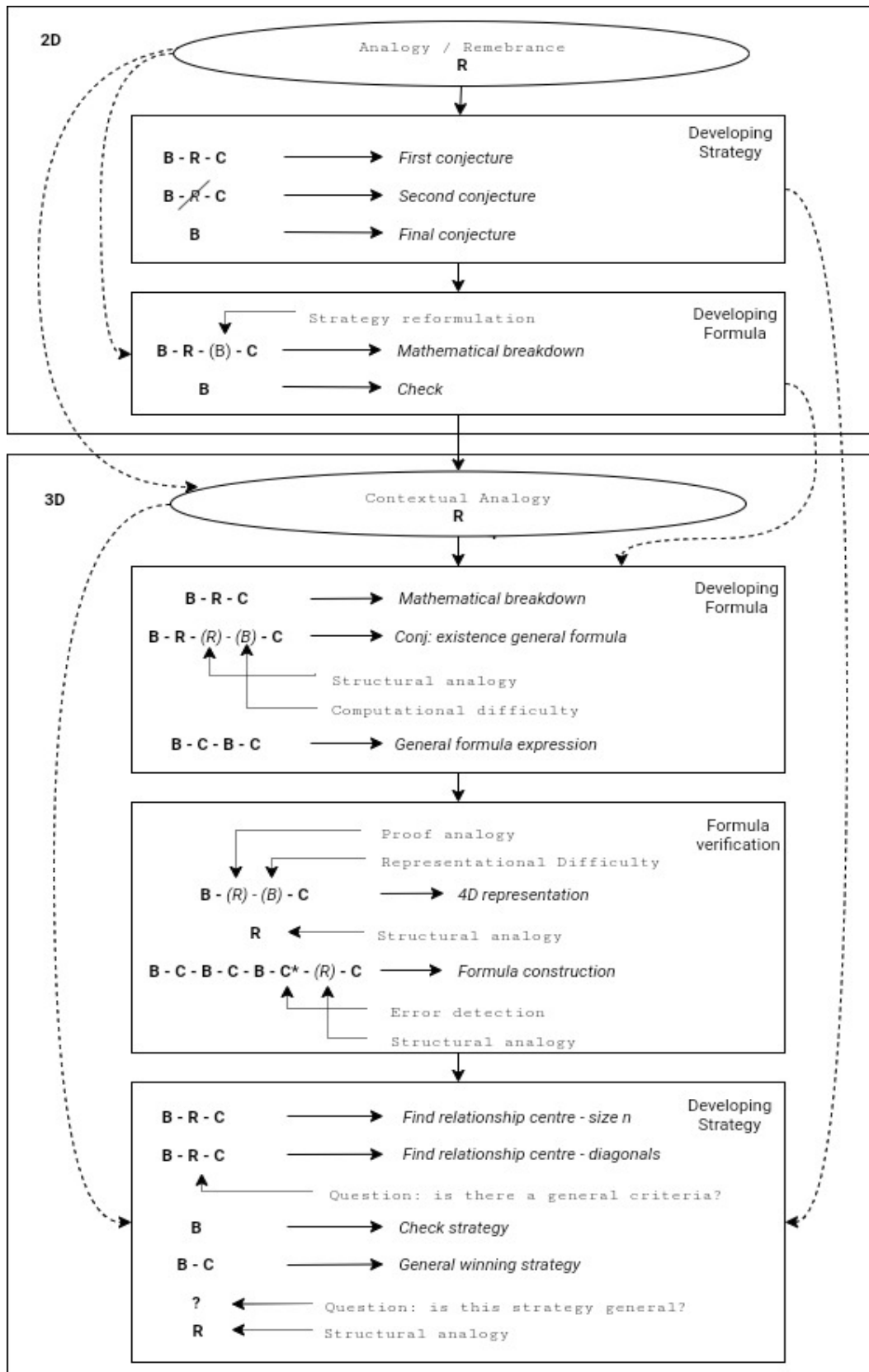


Fig. 8.4 - RBC flow (Student-A resolution protocol)

Within the resolution protocol there is an alternation between forward and backward reasoning. Elements of backward reasoning can be recognized throughout the entire protocol but are concentrated in those parts where the strategy (Part 1.1 and 2.3) or the mathematical formula (Part 1.2 and 2.1) is developed. It is predominant in part 2.1, where the student explains the general mathematical formula, while it is absent in part 2.2, where the student verifies the formula. The backward reasoning never appears without its forward counterpart. It develops mainly in correspondence with the B-R-C sequences.

Three different backward reasoning moments can be identified along the resolution: the favourable positions definition, the winning lines classification and the mathematical formula creation. Adding to the analysis framework the discursive devices, the three moments can be distinguished:

- When the backward reasoning develops during favourable positions definition, the *encapsulating* device appears (part 1.1 and 2.3). The student breaks down the board and search for a pattern useful for the winning strategy. B-R-C chains appear.
- When the backward reasoning develops during the winning lines classification, the *saming* and *encapsulating* devices appear (part 1.2 and 2.1). The student put together the winning lines that have the same geometric characteristics and encapsulate them in a scheme. In part 2.1 this moment anticipates the elaboration of the mathematical formula. In this latter case also the *reifying* device appears. B-R-C chains appear.
- When the backward reasoning appears in the mathematical formula elaboration in its breakdown and transformative characteristics, the *reifying* device is used. B-R-C and B-C chains appear.

In the next section the backward reasoning development during the mathematical formula creation is explored. The definition of winning lines and favourable positions will be discussed in more detail in section 8.2.2.3.1, during the case study 2 discussion.

8.2.1.3.1 In-depth analysis: Mathematical formula development

Analysing specifically part 2.1 of the protocol, in which Student-A develops the general formula, different backward reasoning characteristics emerge. In fact, the reasoning develops both in the general formula conjecture and in its mathematical formulation.

In the conjecture formulation, backward reasoning develops within B-R-C chains. The focus of the reasoning is the total number of winning lines; Student-A, starting from the end of the problem, i.e. the determination of the winning lines, makes a series of logical steps first finding a board subdivision, then a boxes pattern, later the number of lines per plane, and finally he subdivides the number found according to the winning lines characteristics starting a transformative process that leads to the mathematical formula creation. The winning lines analysis is a basic step for the formula development. Through the recognizing action the student introduces new elements in the resolution, whether they are specific to the problem (e.g. the pattern) or mathematical constructs that are structurally analogous to the problem (e.g. finite fields). All the discursive devices appear in this phase, the *saming* during the board subdivision, the *encapsulating* in the pattern creation and then the *reifying* in the next moments. At the end of this process a “raw” mathematical formula is constructed.

Once the “raw” mathematical formula has been obtained, the student focuses on it and starting from it, the end of his sub-problem. He analyses it element by element until the desired expression is obtained. The backward reasoning develops within B-C chains. The action, in this case, proceeds on the manipulation of representations. The breakdown and transformative characteristics, that portray this process, are not continuous, as it might seem, but are interspersed with moments of forward reasoning. In fact, the student derives the expression of the constant and diagonals from the notions learned previously in the resolution and then makes them explicit in a forward way. The *reifying* discursive device characterize these moments.

8.2.2 Case study 2: mathematization with a visual approach

The protocol of Student-B was chosen for the second case study; he is a PhD student specialized in Applied Mathematics and Numerical Systems. Student-B structured the protocol according to the timing of the resolution. Every 10 minutes (more or less) he marked the protocol with a line indicating the elapsed time from the beginning of the resolution. The student worked for about 60 minutes defining the winning lines and developing a general formula for a cubic board. Each line of the protocol has been coded according to a pair of values (x.y): where the first number (x) corresponds to the student's subdivision (1. refers to the first 10 minutes, 2. from minute 10 to 20, 3. from minute 20 to 25, 4. from 25 to 40, and 5. from minute 40 to 60), and the second number (y) corresponds to the subdivision made by the researcher.

The student only refers to the game in three dimensions. Looking at the whole protocol, the protocol can be divided into two main parts: the analysis of the game, with the definition of the winning lines, the favourable positions and a hint of resolution strategy, and the formulation of a general mathematical expression that represents the number of winning lines in a cubic board. From line 1.1 to line 4.4 the student analyses the game. From line 5.1 to line 5.16 the student develops the mathematical formula.

8.2.2.1 Part 1: game analysis

This protocol part starts with the game simulation for about 10 minutes. Then the student defines the winning lines (lines 2.1-2.23), he defines favourable positions (lines 3.1-3.12) and later a hint of winning strategy (4.1-4.2).

Lines	Protocol	BR	HIM	RBC
1.1	Since we don't have time restriction, we play some games randomly to familiarize ourselves with the problem.	-	Ru	B
<u>Time</u>	<u>10 minutes</u>			

8.2.2.1.1 Part 1.1: Winning lines definition

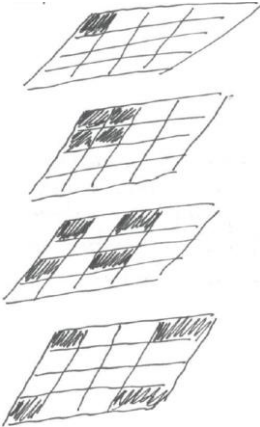
After simulating the game for about ten minutes, the student begins to think about the winning lines and counts them systematically. To do this, he divides the board according to the cube geometry and defines the winning lines according to their geometric characteristics.

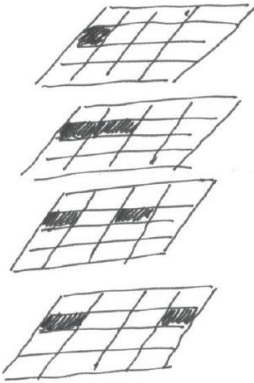
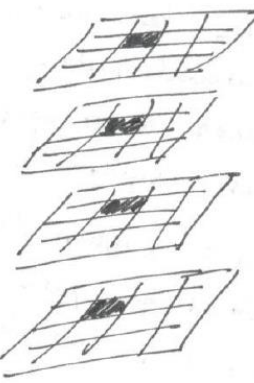
Line	Protocol	BR	HIM	RBC
2.1	We started answering some questions: Winning lines.	-	Init/Int	B
2.2	In each plane: 4 columns + 4 rows + 2 diagonals = 10 winning lines	D+X	answ	B+R
2.3	4 planes * 10 = 40 horizontally winning lines	-	Ded	C
2.4	It is clear that the winning lines for each plane are maintained.	-	Ast	B

The student starts by asking himself how many winning lines are there. He counts the winning lines on each of the four planes into which he has divided the board. The student identifies the winning lines by combining his geometry knowledge and recognizes in the winning lines a pattern. He subdivides the lines according to their geometric characteristics: columns, rows and diagonals (for each plane). These two actions can be classified, according to the RBC, as building-with action and recognizing action (line 2.2). The winning lines, i.e. the end of the problem, are involved; identifying them can be classified as breakdown while recognizing a pattern as an introduction of auxiliary elements (line 2.2). The student, with a constructing action (line 2.3) combines the acquired knowledge and constructs the winning lines number (in the 4 planes). The *saming*, *encapsulating* and *reifying* devices appear: the first to put together the lines according to their geometric characteristics, the second to identify the winning line set, the third when the sentence subject becomes ‘the winning lines for each planes’.

Line	Protocol	BR	HIM	RBC
2.5	We separate 3 types: corners, edge that is not corners and interiors.	D	Int/a	C

Then, the student decides to introduce a notation for the types of winning lines he has yet to count. This is based on the combination of knowledge derived from the explorations and knowledge about cube geometric properties. Student-B proposes a notation to split the winning lines according to their position on the board, in particular, according to their starting point on the upper plane: starting from the corner, the centre or the side of the 4x4 square. This can be classified as a breakdown in terms of backward reasoning processes. As in the previous excerpt (lines 2.1-2.4), *saming* and *encapsulating* devices appear.

Line	Protocol	BR	HIM	RBC
2.6	We must add now the verticals lines.	D	answ	B
2.7	It is not so simple to visualize them, for this reason we began to draw.	X	Int/a	R
2.8	We begin by the corners.	D	answ	R
2.9	In each corner can only be reached: the vertical line that passes through it, the two “lateral diagonals” and the diagonal towards the opposite corner.	D	Int/a	B
2.10	Corners:  <i>Fig. 8.5 - Figure 2.10 (Student B resolution protocol)</i>	X	Def	R
2.11	There can't be more.	-	Ast	B
2.12	4 lines * 4 corners = 16	-	Ded	C
2.13	We look for the parts of the edge that are not corner. For each position only passes: the vertical line that passes through it and the line that is adding a horizontal position horizontally.	D	Int/a	B

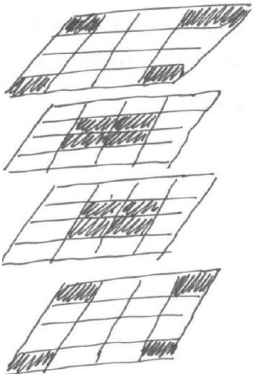
2.14	 <p><i>Fig. 8.6 - Figure 2.14 (Student-B resolution protocol)</i></p>	X	Def	R
2.15	We see that there are 8 positions in the first plane.	D	Def	B
2.16	8 positions * 2 lines = 16 lines	-	Ded	C
2.17	By symmetry, the 8 positions of the first plane are equivalent to those of the last.	D	Int/a	B
2.18	So, they are only taken into account once.	-	answ	B
2.19	It is clear that for the interiors there is only one line, the vertical one.	D	Int/a	B
2.20	<p>Interiors points:</p>  <p><i>Fig. 8.7 - Figure 2.20 (Student-B resolution protocol)</i></p>	X	Def	R
2.21	4 interior positions * 1 line = 4 winning lines	-	Ded	C
2.22	We add them all together and we review each step to make sure that we don't forget to count any line neither we don't count the same line several times.	-	Ded	B
2.23	Total: $40 + 16 + 16 + 4 = 76$ winning lines	-	Ded	C
<u>Time</u>	<u>20 minutes</u>			

The interrogative reasoning process is based on a series of implicit questions of the type "how many winning lines, with these characteristics, are there?". The student continues to reason on the winning lines following a recurring pattern of reasoning: through a building-with action he combines the geometric and strategic knowledge acquired in the explorations (lines 2.9, 2.13, 2.19) and through a recognizing action (lines 2.10, 2.14, 2.20) he identifies, according to the different geometric characteristics, the winning lines pattern. At this point he elaborates, through a constructing action (lines 2.12, 2.16, 2.21) the number of winning lines for each starting point: angle, side or centre of the upper square. At this point, by means of a building-with action he adds together the numbers found previously (line 2.22) defining the correct number of winning lines: 76 (C, 2.23). There is a combination of breakdown processes with the introduction of auxiliary elements. The *saming*, *encapsulating* and *reifying* devices appear: the first to put together the lines according to their geometric characteristics, the second to identify the pattern, and the third when the student write down the mathematical addition.

In this first part, the student strategic thinking follows a series B-R-C chains. These allow him to define the number of winning lines for each horizontal plane, and the number of winning lines starting from each corner, side and middle boxes of the cube's upper plane. The subdivision that the student proposes and the count that he develops allow him to count every possible winning line of the board. The combination of the values found in each chain (B) allows the student to formulate the total number of winning lines. Elements of backward reasoning are shown: breakdown and auxiliary elements introduction.

8.2.2.1.2 Part 1.2: Favourable position definition

After graphically defining the winning lines and counting them, Student-B begins to think about favourable positions. First, he defines what he thinks are the favourable positions through a recognizing action.

Line	Protocol	BR	HIM	RBC
3.1	We start to think about what "favourable positions" means, then we see which are the most favourable. The favourable positions would be those through which more winning lines pass.	-	Init	R
3.2	Clearly there are 16 favourable positions: 8 corners in the first and fourth plane, and 8 interiors in the second and third plane.	D	Int/a	B+C
3.3	 <p><i>Fig. 8.8 - Figure 3.3 (Student-B resolution protocol)</i></p>	X	Def	R

Through a recognizing action (line 3.1) the student identifies favourable positions such as the corners of the upper and lower planes and the central positions of the middle planes. To do so, he asks which types of boxes might have the desired characteristics, he composes the knowledge acquired in the game resolution, he recognizes a pattern in the boxes of the board (line 3.3) and he constructs a pattern (line 3.4). Then, he justifies his choice of favourable positions. The student use backward reasoning introducing (X), through these actions, the solution of the sub-problem "favourable positions".

Line	Protocol	BR	HIM	RBC
3.5	Through each of the corners, 7 winning lines pass..	-	Def	B
3.6	...(3 horizontal and 4 vertical).	D	Int/a	B
3.7	$7 * 8 = 56$ winning lines	-	Ded	C
3.8	Through each of the interiors of the planes 2 and 3 other 7 [winning lines] pass..	-	Def	B

3.9	...(3 horizontal and 4 vertical) and..	D	Int/a	B
3.10	...again $7 * 8 = 56$ winning lines..	-	Ded	C
3.11	...(several coincide with the previous ones).	-	Def	B
3.12	Through the other boxes, pass at most 4 winning lines for each one.	-	Def	B+C
<u>Time</u>	<u>25 minutes</u>			

The reasoning is guided by interrogative moves corresponding to the question "how many lines pass through each box?". Through a series of building-with and constructing actions the student justifies his choice of favourable positions. Combining geometrical representations and acquired knowledge, the student identifies the boxes where several winning lines pass through (lines 3.5, 3.8 and 3.12) and, through a constructing action, clarifies how many winning lines pass through those boxes (lines 3.7, 3.10 and 3.12). From the point of view of backward reasoning the student continues to be involved in the decomposition of the board (D) according to its geometric and strategic characteristics. The *saming*, and *encapsulating* devices appear: the first to put together the lines according to their geometric characteristics, the second to identify the boxes pattern.

In this second part of the protocol, after a first moment in which the student introduces the favourable positions with a recognizing action, the student's strategic thinking follows a series of B-C chains. These allow him to justify his definition of favourable positions and their scheme. Elements of backward reasoning are displayed in this part. Breakdown is used in justifying phases, when the student applies the geometric and strategic knowledge by breaking down the board.

8.2.2.1.3 Part 1.3: Winning strategy development

After defining the winning lines and favourable positions of the board, the student plays, again, a series of simulated games using an online platform (independently encountered).

Line	Protocol	BR	HIM	RBC
4.1	I re-read the previous point and I play some games (online) to become more familiar with the key movements.	-	Ru	B
4.2	It is clear that, there are two types of key movements: those that are destined to reach the 16 key positions and those that are destined to prevent the opposite from doing a winning line.	-	Ast	R
<u>Time</u>	<u>40 minutes</u>			

The student, through an action of recognizing, defines what, according to him are the "key movements" that is those movements that allow him to win the game. He then conjectures a first idea of necessary actions to obtain a winning strategy. To do this, the student selects, among all the information he has collected during the resolution, the most important and relevant information in order to solve the game and summarizes it in line 4.2. Unfortunately, he will not go in-depth into the strategy.

8.2.2.2 Part 2: Mathematical formula development

In this second part of the resolution protocol, the student uses the knowledge previously acquired to explain a general mathematical formula that links the number of winning lines to the size of the board. He then retraces the steps of the winning lines formulation considering a cubic (size n) board.

Line	Protocol	BR	HIM	RBC
5.1	The step of reasoning developed for $n = 4$, which we have followed before, helps us to begin.	-	Init	R
5.2	We'll have n plans with n boxes each.	D+T	Int	B
5.3	It is clear that horizontally we have n winning columns n winning rows and two diagonals.	D+T	ans	B
5.4	$2n + 2$ winning lines for each plane	-	Ded	C
5.5	$n(2n + 2)$ horizontally winning lines.	-	Ded	C

5.6	Vertically we separate by 3: corners, edge without being corner and interior.	D+T	Def	C
5.7	Corner: It is clear that there are still 4 corners in the 1st and last planes and...	D+T	Int	B
5.8	...therefore 4 lines pass through \rightarrow 16 lines.	-	Ded	C
5.9	Edge without corner: Two lines pass through each box and there are $n - 2$ boxes on each side of the plane:	D+T	Int	B
5.10	$\rightarrow 8(n - 2)$ winning lines.	-	Ded	C
5.11	Inside: Only one line passes through each box and..	D+T	Int	B
5.12	.. there are $(n - 2)(n - 2)$ boxes	-	Ded	C
5.13	Total vertically = $16 + 8(n - 2) + (n - 2)^2 = (n + 2)^2$	-	Ded	B
5.14	Total = total horizontally + total vertically = $n(2n + 2) + (n + 2)^2 = 2n^2 + 2n + n^2 + 4n + 4 = 3n^2 + 6n + 4$	-	R	B
5.15	We check the result:	-	R	B
5.16	$3(4)^2 + 6(4) + 4 = 48 + 24 + 4 = 76$	-	R	B
<u>Time</u>	<u>60 minutes</u>			

The student acknowledges (R, line 5.1) the importance of the strategic steps developed previously and retakes the reasoning process of part 1.1 step by step. The mathematical formula development is totally based on that process and on the decomposition of the board already carried out. The student combines the previously acquired knowledge (B, lines 5.2, 5.3, 5.7, 5.11) to construct (C, lines 5.4, 5.5, 5.8, 5.10, 5.12) at each step the general number of winning lines (in the plane, starting from an angle, side or centre of the upper plane) for a generic cube of size n^3 . At the end of this excerpt, with a building-with action (lines 5.13-5.16) the student puts together the previously encountered formulas to make explicit the general formula for a size n cube. Subsequently he verifies its correctness for the 4^3 case. The backward reasoning in its breakdown and transformative characteristics is shown: the student considers the winning lines, counts them and subdivides them according to their geometric characteristics, transforming them in each step until reach a general formula. The *saming*, *encapsulating* and *reifying* discursive device are used: the first to put together the

lines according to their geometric characteristics, the second to identify the scheme, the third when the sentence subject becomes 'the number'.

In this fourth part, in order to get to the mathematical formula, the student has passed through a series of (R)-B-C chains. The R corresponds to the recognizing of the previous schema, developed in part 1.1, that underly the formula development; it is not made explicit, but it is clearly in student's mind. In this part, the backward reasoning strongly characterizes the generalization processes that lead the student to the creation of the solution: the mathematical formula. The breakdown process lasts over time and characterizes the whole part of the formulation.

8.2.2.3 Case study 2 discussion

From the HIM point of view, the backward reasoning develops in interrogative moves. This is found for example in lines 2.1-2.23 in part 1.1 where the student does a series of steps to answer the question "how many winning lines, with these characteristics, are there?". The information obtained from the answers to this question allowed the student to answer the more general question: "how many winning lines, with these characteristics, are there in a size n cube?" and consequently make explicit a general formula (Part 2). Like Student-A, Student-B alternates interrogative moves followed by an answer and defining moves (an elaboration of the answers).

From the RBC point of view, the resolution protocol is characterised by three different types of chains: B-R-C, R-B-C and B-C. The B-R-C and R-B-C chains characterize the discovery processes while the B-C chains are predominant in the processes of verification, in this case the favourable position schema. Observing the resolution protocol in its four parts it can be said that:

- In Part 1.1 the student applies four B-R-C chains (lines 2.2-2.21) to define the number of winning lines. With the first chain he defines the number of winning lines in the four planes into which the board is divided; with the second, third and fourth chains he defines the winning lines starting from specific boxes of the upper plane: the corners, the side, and the four middle boxes;

- In Part 1.2, the student applies a B-R-C chain (lines 3.2-3.4) to define what for him are the favourable positions, and with a series of B-C chains (lines 3.5-3.12) he justifies this choice;
- In Part 1.3 the student defines what is the necessary strategic movements to solve the game, no particular chain is identified, only recognizing actions;
- In Part 2, with a series of chains (R)-B-C (lines 5.2-5.12) the student constructs the general mathematical formula; to do so he bases himself on the pattern he generated a few minutes before in Part 1.1.

The sequence of chains of the entire protocol can be schematized as follow, the dotted line indicates the transfer of knowledge from the process of defining winning lines to the mathematical formula development phase.

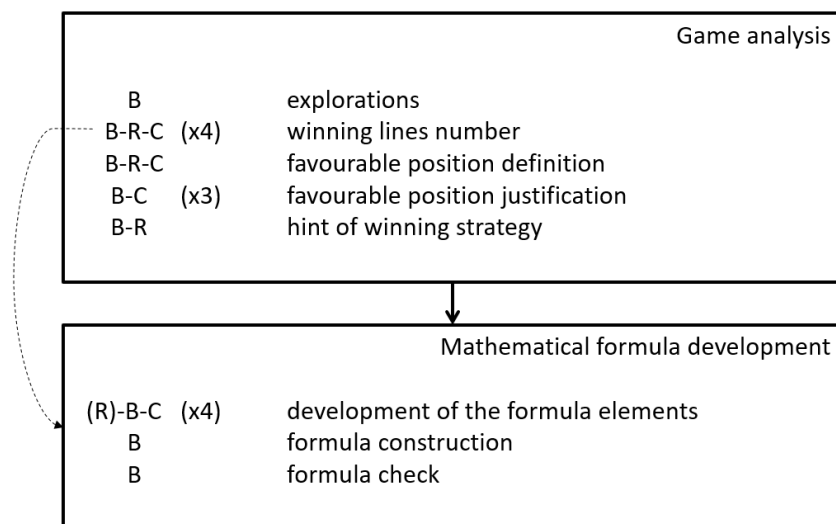


Fig. 8.9 - RBC flow (Student-B resolution protocol)

Within the resolution protocol there is an alternation between forward and backward reasoning. Three different backward reasoning moments can be identified along the resolution: the winning lines classification, the favourable positions definition, and the mathematical formula creation.

In the first part of the protocol where the winning lines and the favourable positions are defined, backward reasoning appears in its breakdown feature interspersed with the introduction of some auxiliary elements. In fact, the student works on the winning lines by

breaking down the board and grouping the lines according to their geometric properties. New patterns are introduced to visualize the winning lines and specific positions on the board. The *saming* and *encapsulating* devices appear (part 1.1 and 1.2). The student put together the winning lines that have the same geometric characteristics and encapsulate them in a scheme. In part 1.1 also the *reifying* device appears, while the student expresses the winning line in formulas. Reasoning develops both in correspondence of the B-R-C and B-C chains.

Where the mathematical formula is developed, the backward reasoning appears in its breakdown and transformative characteristics. The reasoning process continues over time and involves the previously created winning lines pattern. *Saming*, *encapsulating*, and *reifying* devices are used to develop the mathematical formula; (R)-B-C chains appears, R doesn't appear explicitly but corresponds to the recognizing of the previous schema, developed in part 1.1, that underly the formula development.

In the next sections the backward reasoning development during the winning lines and favourable positions definition is explored. The mathematical formula development was discussed in section 8.2.1.3.1.

8.2.2.3.1 In-depth analysis: Winning lines analysis

Analysing specifically part 1.1 of the protocol, in which Student-B analyses the winning lines, the breakdown characteristics and the introduction of auxiliary elements emerge. Student-B splits his reasoning in four steps that correspond to four B-R-C chains. Starting from the winning lines, he subdivides them according to their geometric characteristics. For each group, he recognizes a pattern that is introduced through a drawing (lines 2.10, 2.14 and 2.20). Reasoning on the pattern and previous knowledge, he builds a mathematical relationship to calculate the number of winning lines in a systematic way. Through the recognizing action the student introduces new elements in the resolution: the patterns. All the discursive devices appear in this phase, the *saming* during the winning lines grouping, the *encapsulating* in the pattern creation and then the *reifying* in mathematical relationships elaboration. As in the case of student-A, the analysis of the winning lines is a basic step for the formula development; in this case the student clearly separates the two moments by developing one at the beginning and the other at the end of the protocol.

8.2.2.3.2 In-depth analysis: Favourable position definition

Student-B develops the favourable position definition in part 1.2 of the protocol, where the breakdown characteristics and the introduction of auxiliary elements emerge. Starting from the winning lines, he subdivides them according to their geometric characteristics and he identifies a boxes pattern. Then he explains how he came to the decision to define those specific boxes as favourable. The *saming* device appears during the winning lines grouping, while the *encapsulating* one in the pattern identification. The peculiarity of the student-B protocol is that, firstly, he defines the favourable positions introducing a pattern, and then, he justifies his choice. Therefore, a B-R-C chain appears for the introduction of the definition and three B-C chains appear for its justification. However, observing the favourable position development in other protocols (see for example student-A, part 1.1), the chains formed is of B-R-C type. In fact, generally, the students explore the game, recognize how many winning lines pass through each square and construct the pattern of favourable positions. Student-B did the same reasoning path of other students, but he shows it in reverse way. From this excerpt, it can be seen that, while the chains of actions leading to an element discovery are of B-R-C type, the ones for its justification are of type B-C.

8.3 Discussion

From the HIM point of view, the backward reasoning develops when the student asks a question during the path towards the formation of ideas and conjectures after a phase of exploration. The role of the questions is, therefore, to activate that tacit knowledge that allows new elements to become reality (Hintikka and Hintikka, 1982). It is essential to ask an appropriate question (Solow, 1990) to extract information from the subject's background of knowledge. This is found for example in part 2.1 of the Case study 1 where the student refers to notions learned previously during the resolution of the 2D game. Formulating a good question allows the subject to formulate premises for certain statements, or in combination with certain statements to draw some conclusions. It can be clearly observed, in Part 2 of the Case study 2, where the student alternates interrogative moves followed by an answer and defining moves to elaborate a mathematical formula.

The global analysis of the group has identified five different backward reasoning moments: analyse the winning lines, define the favourable positions, search for the final movements, block the opponent and develop a mathematical formula. In each moment, backward reasoning occurs mainly in interrogative moves (HIM analysis). While the second, third and fourth moments are mainly related to the search for a winning strategy and the last to the mathematization of the game, the first moment is necessary both for the winning strategy search and for the mathematization. The RBC flow connected to these moments varies and it is characterized by R-B-C, B-R-C and B-C chains. Backward reasoning appears in protocol discovery phases while is absent in verifying phases (as in part 2.2 of the Case study 1). While B-R-C and R-B-C chains characterize the discovery moments, B-C chains are typical of transformative and verifying processes. Some recognizing actions, such as the board breakdown or the introduction of a recursive pattern, are concept recognitions belonging from previous resolution parts or students' background. These recognitions occur after a contextual or a structural analogy. During these analogies, the students remember geometrical concepts previously identified or studied in the university career. They help them to identify patterns and proceed with the resolution (Barbero, Gómez-Chacón and Arzarello, 2020).

When students analyse the winning lines, they develop backward reasoning in its breakdown feature, introducing also some auxiliary elements (like patterns). These moments are characterized by B-R-C chains; *saming*, *encapsulating* and *reifying* discursive devices appear. In fact, students group the winning lines according to their geometric characteristics, then they recognize a pattern and later they make explicit the winning lines number for each group. In task resolution, this moment is useful for the winning strategy search and is preparatory for the mathematical formula expression. This is evident in the two case studies presented, although the episodes are not always developed in sequence.

A similar development can be seen during the favourable positions' definition. In fact, also in this case the backward reasoning is characterized by breakdown moments and introduction of auxiliary elements. In these moments B-R-C chains appears together with *saming* and *encapsulating* discursive devices. In fact, students subdivide winning lines

according to their geometric characteristics and then identify a boxes pattern. If the students need to justify their choice, then B-C chains appear.

When students search for the final movement (i.e. the movement that allows one of the two players to win the game, whatever move his opponent makes) or try to block the opponent, backward reasoning appears in its characteristic of cause-effect relationship research and some auxiliary elements are introduced. These are moments strictly connected with the winning strategy development. B-R-C chains characterize both cases: the students explore the game, then recognize a specific configuration and later construct a move (to win or to block the rival). The *reifying* discursive device appears.

The moments in which students develop a mathematical formula have a more complex nature. They are based on the winning lines analysis, but, depending on when the analysis takes place, there are two different types. If the analysis takes place during the formula construction, B-R-C chains are generated; the students group the winning lines, identify a pattern and then transform the acquired information into a formula. If the analysis takes place before the construction of the formula, then R-B-C chains are generated. In fact, the students firstly identify the pattern generated by the analysis of the winning lines, and then, with a series of arguments, they construct the mathematical formula. In both cases *saming*, *encapsulating* and *reifying* discursive devices appear. If the generated formula is in a “raw” state, then some manipulations are required. This process is characterized by B-C chains and *reifying* discursive device. In all these moments the backward reasoning appears in its breakdown and transformative features.

9

**MATHEMATICAL PROBLEMS
ANALYSIS**

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MATHEMATICAL PROBLEMS ANALYSIS

9

In this chapter the results of the analysis of the fourth design experiment are shown. Briefly the design experiment settings are summarized.

Task type		Data collection settings				Students				
Strategy game	Mathematical	Work in group	Work alone	PC suite	Video-recordings	First level	Second level	Third level	Total	Case studies
-	✓	✓	✓	-	✓	9	73*	-	82*	28**

* including 16 Italian students

** 16 Spanish students working alone and 3 Italian groups (4 students each)

Tab. 9.1 - Fourth design experiment settings

The task proposed in this design experiment is composed by four mathematical problem, as shown in figure 9.1. Of the 82 involved students 66 were Spanish and 16 were Italian. The Italian students belong from the second level group (see Chapter 5, section 5.1) and solved the slightly different task, comparable to the first problem of the Spanish task. They solved the problem in group of four student each; they were video recorded. The Spanish students solved the entire task with four problems; they worked alone. There are no differences between the Spanish students belonging to the first and to the second level group. The resolution protocols will be considered jointly. Nineteen case studies were carried out in this design experiment: 3 Italian groups (12 students: 4 students each group) and 16 Spanish students. Only a selection of one Italian group and seven Spanish students are shown in this

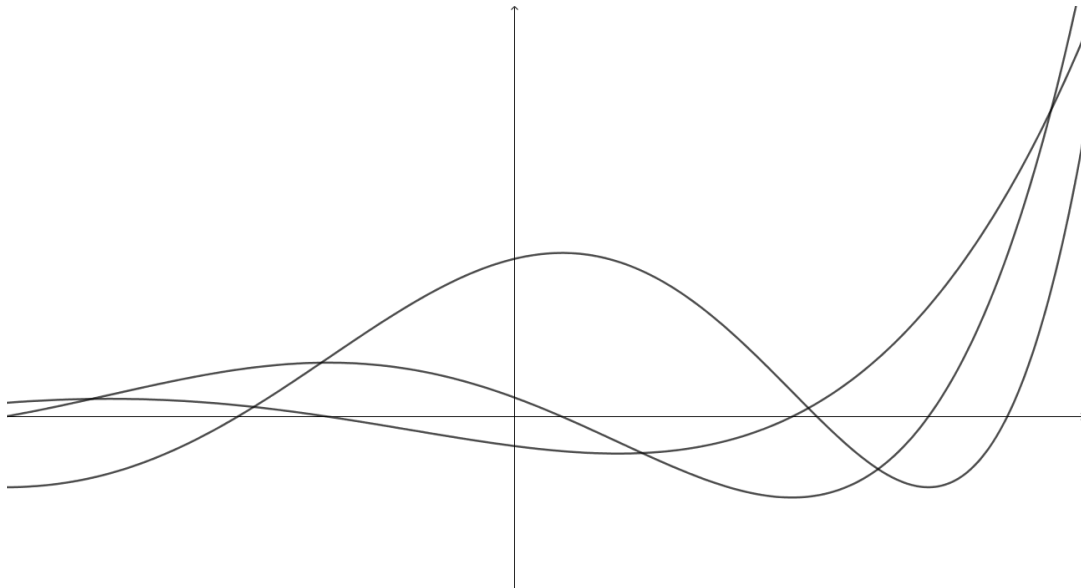
chapter to exemplify the analysis work done and the obtained results. A short presentation at the beginning of sections 9.2, 9.3, 9.4 and 9.5 clarify each choice.

Backward Reasoning Problems

Problem 1: Functions

The drawing below shows the graph of three functions.

- A function f
- The derivative of function f
- The primitive of the function



1. Identify the graph of each function by explaining in detail your entire thinking process using the resolution protocol technique.
2. Describe a general method for solving these types of problems.

Problem 2: Triangle and Circle

Among all the isosceles triangles inscribed in a circumference, look for that of maximum area.

Solve the problem. Detail your entire thinking process using the technique of resolution protocols.

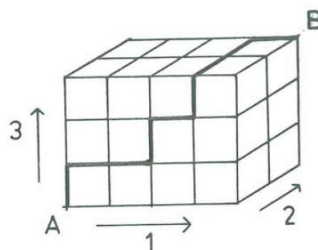
Problem 3: Geometrical Construction

Given an \widehat{ABC} angle and a P point inside the angle, construct a QT segment, using only a ruler and compass, so that it passes through P and QP is twice PT. The Q point belongs to BA and the T point belongs to BC.

Solve the construction problem. Detail your entire thinking process using the resolution protocol technique.

Problem 4: Paths

How many 9-section paths, that link point A with point B, are there? Each section must necessarily be travelled in the directions indicated "1", "2" or "3".



Solve the problem. Detail your entire thinking process using the resolution protocol technique.

Fig. 9.1 - Mathematical Problems Task

The chapter is structured in the following way. Firstly, the analysis of the whole group is presented (section 9.1); unlike the previous chapters, the moments of backward reasoning development, shown through an in-depth analysis, appear in sections dedicated to each problem (sections 9.2-9.5). In section 9.2, a case study from Functions problem is displayed; it consists in the in-depth analysis of two episodes (from video-recording) of Group 2 (Italian students) resolution. In section 9.3 a case study from Circle and Triangle problem (Student-G resolution protocol) is displayed. In section 9.4, an excerpt from Student-L resolution protocol and two case studies (Student T and Student-F resolution protocols) from Geometrical Construction problem are shown. In section 9.5, two excerpts from Student-H and Student-V resolution protocols, and a case study (Student-N resolution protocol) from Paths problem are displayed. All the excerpts and case studies exemplify, in a complete way, the backward reasoning moments relative to each problem. Finally, a general discussion is developed (section 9.3).

9.1 Analysis of the whole group

Analysing the 66 Spanish resolution protocols, and the 4 Italian groups data, seven moments in which backward reasoning is developed are identified throughout the four proposed mathematical problems. Since there were no major differences between the level groups involved in the design experiment, the students are considered together. For each problem, the backward reasoning moments are specified.

Functions Problem

1. Supposing identified the function and its derivative. The students start the problem supposing that two graphs represent the function and its derivative and analyse the cases to discover if this hypothesis is true or not. From the Spanish group, 34 students use this strategy (51%). From the Italian group 12 students (3 groups) use his strategy (75%).

Circle and Triangle Problem

2. Analysing the geometric configuration of the problem. The students observe the geometric configuration (the isosceles family triangle inscribed in the circle) and recognize some known geometric configuration, adding, if necessary, auxiliary elements. 51 students use this strategy (77%).
3. Expressing the relationships among the geometric configuration in algebraic language. The students represent some configuration elements in algebraic language constructing an algebraic formula, then they work on this formula to solve the problem. 44 students use this strategy (67%).

Construction problem

4. Analysing the solution of the problem (the sought geometric configuration). The students observe the final geometric configuration, explore it, and add some auxiliary elements with the aim of achieving a known configuration. 49 students use this strategy (74%).
5. Identifying an analogy between the result of the problem and a known theorem. The students recognize the final configuration as a specific element of a known theorem or configuration, then they reverse it to find the sought configuration. 28 students use this strategy (42%). In particular, 12 students (18%) recognize the sought segment as the inverse of the segment trisection configuration (Thales theorem) and 16 students (24%) recognize the sought segment as a median of the triangle which has one of its vertexes in T and barycentre P.
 - ➔ Both strategies are used by 16 students (24%), first they explore the final configuration and then they recognize de known theorem.

Paths problem

6. Analysing the generic path. The students identify a generic path and represent it in an algebraic way. Analysing it, they count the paths in a systematic way, or translate the problem to a combinatorial one applying their knowledge about permutations with repetition. 24 students use this strategy (36%).

7. Identifying the combinatorial problem. The students identify the problem like a combinatorial one and the number of paths like the number of permutations with repetition. 30 students use this strategy (45%).

In the following sections some case studies are shown to exemplify the problem resolutions. For each moment of backward reasoning recognized in the group, an example will be analysed in-depth within the case studies sections. In particular, each backward reasoning moment is shown in a specific section:

- Supposing identified the function and its derivative (Functions) in section 9.2.3.1;
- Analysing the geometric configuration of the problem, and Expressing the relationships among the geometric configuration in algebraic language (Circle and Triangle) in sections 9.3.1.1 and 9.3.1.2;
- Analysing the sought geometric configuration and Identifying an analogy between the result of the problem and a known theorem (Construction) in sections 9.4.1.1 and 9.4.3.1;
- Analysing the generic path, and Identifying the combinatorial problem (Paths) in sections 9.5.1.3 and 9.5.3.1.

9.2 Case study of the Functions Problem: Group 2

As seen from the global analysis of the group there is one moment in which the backward reasoning generally develops: suppose the function and its derivative be identified. But the backward reasoning sometimes also can support students to overcome difficulties. To exemplify these moments, two episodes of an Italian group case study are displayed in this section. As explained in Chapter 5, the first problem was solved by the Italian and Spanish student groups in two slightly different ways. The Italian students worked in groups of four, while the Spanish students worked alone. Even if the two tasks are slightly different, the processes of reasoning to solve them are comparable.

Of the three case studies carried out with Italian students and the five case studies carried out with Spanish students, all belonging to the second level group, it was chosen to show the

case of Italian Group 2, composed of four students: Student-Fe, Student-Ma, Student-Fra and Student-Si. Talking about Italian case studies, while Groups 1 and 3 had no difficulty in solving the task, group 2 defined the relationship between function and its derivative in an incorrect way: when the function increases, its derivative increases too. Therefore, two episodes of the Group 2 video-recording are shown. In the first episode, which corresponds to 00:00-03:00 minutes of video-recording, the students read the task and solve the first problem. In the second episode, (from 49:00 to 1:14:00 minutes) they understand that they have not done the right reasoning during the first episode, hence they correct their solution. The Groups 1 and 3 solved the problem in a similar way to the one shown in the first episode but applying the right definition. The Spanish case studies develop the resolution in a similar way too. One of them (Student-H) recognize the graphs as polynomial function charts and associates them to polynomial functions. Despite considering the analytical expression of the functions, he develops a reasoning that similar other students' one.

Students from Group 2 use different auxiliary constructions (drawings, graphical representations) as a support to their resolution processes and can so overcome the difficulties they met. They make a control over their own resolution process to identify the error. After the correction phase (episode 2) the students rewrite the method used and verify that it works with the other examples proposed by the task. For convenience, during the analysis, the functions are named 1, 2 and 3 as they named them during the activity (Figure 9.2). The entire transcription has been translated from Italian by the author. It is divided in lines, the students use some gestures to explain their reasoning, so, some picture from the video-recording are added to clarify the discourse.

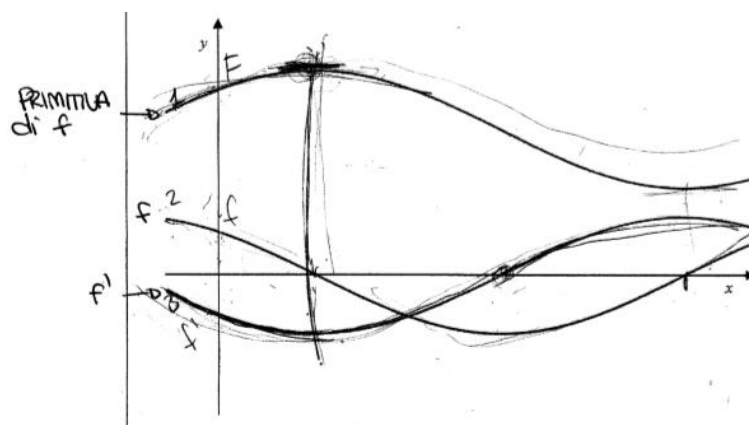


Fig. 9.2 - Group 2 sheet

9.2.1 Episode 1: problem solution

They start talking almost together, trying to solve the problem. Students Fe and Ma begin their reasoning, the former immediately puts her hands on the paper and begins to indicate the functions she is talking about. Some seconds later, also Student-Si enters the conversation. All of them point and move along the functions' graphs with their fingers while they talk about increasing and decreasing. Initially they refer only to the leftmost part of the functions: they do not consider them globally.

Episode 1 transcription

Line	Stud.	Transcript	BR	HIM	RBC
1.1	Fe	<i>If this was the function...</i> [points graph 1]	FS	Init	R
1.2	Ma	<i>If this was f...</i> [she refers to the function pointed by Fe]	FS	Init	R
1.3	Fe	<i>Its derivative should increase...</i>	E	Init	R
1.4	Ma	<i>Exactly...</i>	-	-	-
1.5	Fe	<i>But these ones are decreasing</i> [points at graph 2 and 3] <i>I mean...this one is increasing a little bit...</i> [points at graph 2] <i>but not enough</i> [she moves along graph 1 and shows the difference of increase between graph 1 and 2]	E	Int/a	B
1.6	Ma	<i>Ehm no... in fact...so it is not right...let's try to start with another one...</i>	-	Ast	C

With a recognizing action, Student-Fe (lines 1.1 and 1.3) identifies the property between the function and its derivative: if the function increases, its derivative increases as well. It will be seen later on how, this erroneous property, will be carried forward until the same student recognizes the error at minute 1:01:45 of video-recording 1. In line 1.5 the students suppose that function 1 is f and observe the graphs looking for a possible derivative of the function (building-with), but they cannot find a function that suites the requested characteristics (constructing). This can be interpreted as an answer to the implicit question: "what characteristics must the derivative of f have?" In this case the students are applying the

backward reasoning supposing that function 1 is function f (supposing the problem solved strategy) and are looking for its derivative (cause-effect relationship research). Afterwards they suppose that one of the other two functions has to be the f that satisfy the mentioned property (line 1.7). The *reifying* discursive device appears when the students stop talking about increasing and decreasing and say that graph 1 is f .

Line	Stud.	Transcript	BR	HIM	RBC
1.7	Si	<i>So, one of these two has to be the f...</i> [repeatedly points at graph 2 and 3]	FS	Ast	R
1.8	Fe	<i>Here decreases</i> [looks at Si's pen moving on the sheet along graph 2, meanwhile places two finger so one on graph 2 and the other one on graph 3 pointing the two functions that are comparing]	E	Int/a	B
1.9	Ma	<i>Of course, say that it's this...</i> [point at graph 2 and moves the pen along the function] <i>let's take...let's consider this as f...</i> {Fe: <i>it might be</i> } <i>here it is decreasing... this one is decreasing</i> [points at graph 3] <i>and this one is increasing</i> [points at graph 1]...	E	Int/a	B
1.10	Fe	<i>Yes, this one cannot be its derivative...</i> [points at graph 1, then goes back to its earlier finger position, continuing to confront graph 2 and 3]	E	Ast	C
1.11	Fra	<i>Then this one might be its primitive...because it is increasing...</i> [she enters the discussion pointing at graph 1]	E	Int/a	B
1.12	Ma	<i>Exactly...This one could be...if this was f</i> [pointing at graph 2], <i>this could be its primitive</i> [points at graph 1] <i>and this one its derivative</i> [point at graph 3] ...	E	Ast	C

The students continue the reasoning moving the focus on the second function and repeating the previously performed logical steps (B, lines 1.8-1.9. In this case, like before, they understand that function 1 does not satisfy the required characteristics so they hypothesise, by exclusion, that it is function 2's primitive (C, line 1.10). Afterwards they justify with

increase and decrease the just formulated hypothesis (B, line 1.11) and conjecture the problem solution (C, line 1.12). From a backward reasoning point of view, the two students continue supposing that one of the functions is f and keep on searching its derivative (cause-effect relationships). Student-Si begins to have a doubt about the relationship between function and derivative, she's trying to make a control. The *reifying* discursive device appears when the students stop talking about increasing and decreasing and say that “this one could be...”.

Line	Stud.	Transcript	BR	HIM	RBC
1.13	Fe	<i>Because here it reaches a points where it has a minimum [points a graph 2] and here it is nullified [points at graph 3]... {Ma: here it is zero...[she also points at graph 3]}... and in fact here it decreases and the derivative too...[points at graph 2 and then at graph 3] right?</i>	D+E	Int/a	R+B
1.14	Si	<i>But, in theory, here it increases and here decreases [pointing at graph 2 and 3 around their point of incidence]</i>	D+E	Ast	R
1.15	Fra	<i>--Inaudible words addressed to Ma-- [meanwhile with the hand vertically positioned she does an iconic gesture (McNeill, 2005) (Fig. 9.3a) that unequivocally represent the tangent, afterwards gets closer to graph 2 keeping the hand in the same position (Fig. 9.3b)]</i>	X	Int/a	R



Fig. 9.3a - Fra's gesture



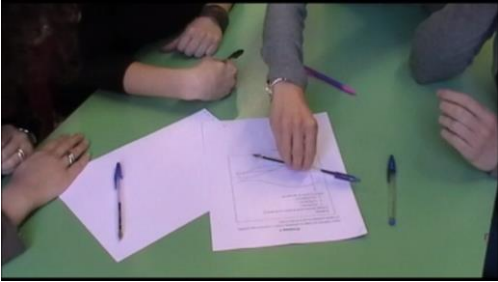
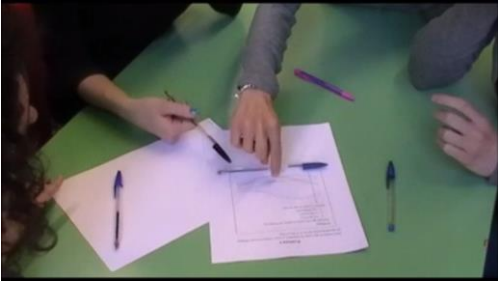
Fig. 9.2b - Fra's gesture closer to the graph

1.16	Ma	<i>Exactly...because if you have the tangent...you know what we use to do during the lessons with the</i>	X	Int/a	R
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		<i>derivative and the tangent, right?</i> [she refers to a tangent surfing activity seen during classes with the professor] <i>if you had...</i>			
1.17	Fra	<i>You see here... take a point like this...tatata...then here</i> [she goes along graph 2 with the pen tip, the onomatopoeic tatata represents the pen movement among the graph]	-	Int/a	B
1.18	Ma	<i>Here it goes down ...here goes up again...technically here could be...</i> [she refers to Fra pen movement, she also retraces with her pen Fra pen's path]	-	Int/a	B
1.19	Fra	<i>Here there could be a zero...a minimum...</i> [repeats the same gesture used by Fe in the same point, draws an imaginary line on the minimum point of graph 2]	E	Int/a	B
1.20	Ma	<i>Exactly...exactly... but now analysing...yes it is f...</i> [she moves along graph 2 with her pen until reaching the minimum point] <i>hence it should be like this</i> [she repeats the same gesture that Fe and Fra did] <i>but here</i> [points at the zero point of graph 3 in correspondence to the minimum point of graph 2]... <i>so, do you think that this is f...</i> [she goes along graph 2 again]	FS	Ast	C
1.21	Fra	<i>Mmm...</i> (she assents to Ma's question)	FS	Ast	C

Student-Fe, through a recognizing action (line 1.13), starts observing the functions' remarkable points and looks for possible relations between maxima, minima and zeros (B, line 1.13) trying to justify their conjecture. At this point, Student-Fra introduces the tangent notion with a gesture (R, line 1.15) immediately supported by Student-Ma who refers to an activity performed during classes (R, line 1.16). Therefore the students introduce a new element in the resolution, taking an activity previously performed as an example, a point that moves along the function and the movement that the tangent does along the function itself (represented by a pen moving in the air). They explore the three functions graphs (B, line

1.17-1.21) trying to understand what happens to tangent while the point moves on the function identified as f (graph 2). From a backward reasoning point of view, the students are trying to verify their conjecture about the problem solution: to do that they introduce auxiliary elements (X, lines 1.15 and 1.16) that came mainly from activities already performed during their university career, such as the moving point and the tangent. They also break down the functions to analyse all their characteristics points, searching for other cause-effect relationships. The *reifying* discursive device appears when the students stop talking about the point that goes up or down and say that graph 2 is f .

Line	Stud.	Transcript	BR	HIM	RBC
1.22	Ma	<p><i>So, you have to think about the tangent</i> [she lays the pen on the sheet like it was the tangent and moves it along the function]</p>  <p><i>Fig. 9.4 - Ma moving the pen</i></p>	E	Int/a	B
1.23	Fr	<p><i>Yes, is that one...</i></p>	E	answ	B
1.24	Ma	<p><i>Here the tangent is zero</i> [she has reached the minimum point of graph 2 with her pen, she places the pen horizontally] <i>Hence f...</i></p>  <p><i>Fig. 9.5 - Ma moving the pen</i></p>	E	Int/a	B
1.25	Si	<p><i>I think that f is the one below...</i> [she refers to graph 2]</p>	FS	Ast	C

1.26	Ma	Yes...			
1.27	Si	<i>Because if we consider this point as...</i> [she also repeats the same gesture and draws an imaginary tangent with the pen on the <i>minimum</i> point of graph 2] {Ma: <i>minimum...</i> } <i>minimum</i> {Ma: <i>...of f</i> }, <i>correspond to this that is zero...</i> [points at graph 3].	E	Int/a	C
1.28	Ma	<i>Then it is right...</i>	FS	Ast	C
1.29	Fr	<i>So, this is f...</i> [points at graph 2]	FS	Ast	C
1.30	Ma	<i>So, this is f, this is the derivative and this the primitive...</i> [points at graph 2, graph 3 and then at graph 1]	FS	Ast	C
1.31	Si	<i>This is f</i> [point at graph 2], <i>this is its derivative</i> [points at graph 3] ... <i>Exactly...</i>	FS	Def	C
1.32	Fr	<i>These two should be right...and the primitive...</i>	FS	Def	C
1.33	Ma	<i>And by exclusion the other one...</i>	FS	Def	C

The students focus on the tangent line movement, previously introduced with a series of explorations (B, lines 1.22-1.24) they get to identify a relation between maxima/minima of function and zeros of its derivative, when the function tangent is parallel to the abscissa axis (C, line 1.27) concluding that their conjecture is correct, even if it was based on a wrong relation. By exclusion (C, lines 1.30-1.33) they state that the function that is out of the discussion should be the function primitive: function 1 is, by exclusion, the primitive, function 2 is the function f and function 3 is the derivative. The students continue searching cause-effect relationships to justify their initial conjecture, then they formulate the solution.

In this episode, in order to get to the problem solution, the student has passed through a series of R-B-C chains. The R corresponds to the recognizing of the possible problem solution in first three chains, while the function tangent line in the last ones. In this episode, the backward reasoning is characterized by the cause-effect relationships research that is a process that lasts over time. The *reifying* discursive device characterises this excerpt.

9.2.2 Episode 2: overcoming the difficulty

This episode is split into two parts. The first refers to the recognition of the difficulty encountered in episode 1, while the second refers to its overcoming.

9.2.2.1 Part 1: recognizing the difficulty

After having written their method on the sheet, remarking the relationship increasing-increasing, decreasing- decreasing between function and its derivative, Student-Fe takes the floor to try to overcome the difficulty.

Line	Stud.	Transcript	BR	HIM	RBC
2.1	Fe	<i>We need to start from this one</i> [points at graph 1] ...	FS	Def	R
2.2	Ma	<i>Like f..</i>	FS	Def	R
2.3	Fe	<i>Exactly... the primitive... F, no? Is such that its derivative is f</i> [points at graph 2], <i>meaning that in theory you see the two functions</i> [repeatedly points at graph 1 and 2], <i>or...</i>	FS	Def	R
2.4	Ma	<i>I did not understand...</i>	-	Int	B
2.5	Fe	<i>This one is the F, right?</i> [points at graph 1] <i>And so this is its derivative</i> [points at graph 2] <i>and so you can say F and F'</i> [points at graph 1 and then at graph 2], <i>and then f and f'</i> [points first at graph 2 and then at graph 3]...	X	Def	R
2.6	Ma	<i>Of course, you do the opposite. So, you do the procedure to find its primitive...</i> [repeats Fe's gestures]	-	Def	R
2.7	Fe	<i>Exactly...</i>	-	-	-

Student-Fe starts reasoning from the conjecture about the previously explicated functions, changing her point of view. She tries to explain the relations between the graphs not as primitive-function-derivative but as function-derivative (graph 1 and graph 2) and function-

derivative again (graph 2 and graph 3) (lines 2.3 and 2.5). Hence, she introduces a new relation (X) between the functions through a Recognizing action.

Line	Stud.	Transcript	BR	HIM	RBC
2.12	Fra	<i>This is the f [points at graph 2] because it is our function, but the primitive is F because F' is equal to f...</i>	D	Int/a	B
2.13	Fe	<i>Exactly... This one [points at graph 1] is F and this one [points at graph 2] is {Fr: F' ...} f... [they say it at the same time] <i>Exactly, f is equal to F'.</i></i>	D	Int/a	B
2.14	Ma	<i>f is equal to F'...</i> [she writes it at the bottom of the page as she says it]	-	Def	C
2.15	Fe	<i>Exactly... so, in this way you can see this one [pointing at graph 1] as f and that one [pointing at graph 2] as f' prime...</i>	-	Def	R
2.16	Si	<i>I understood but...</i>	-	Int/a	B

The students start an interrogative process, reasoning, with a series of building-with, about the idea newly introduced by Student-Fe. Student-Fe concludes her reasoning explicating the correspondences between the primitive's derivative and function f (B, line 2.13). Meanwhile, Student-Ma writes on the protocol the relation $F' = f$ (C, line 2.14). At this point they start to recognize that their reasoning is not correct, or rather that the reasoning is based on a wrong property. They are breaking down the problem solution to analyse its characteristics, their focus on the function algebraic expressions and not on their graphs' properties like in Episode 1. The *saming* discursive device appears: speaking together they realize that the graph that represents f is the derivative of F graph.

Line	Stud.	Transcript	BR	HIM	RBC
2.17	Ma	<i>No! Something is wrong about increasing and decreasing...</i>	-	Int	R
2.18	Si	<i>The slope is different...</i>	-	Int	R
2.19	Ma	<i>Also, because this one [points at graph 2] is not f'. I mean, this one as f [points at graph 1] and you should look at this one... [points at graph 2]</i>	-	Def	B
2.20	Si	<i>No, she means... thinking about the function and its derivative we could do...</i>	-	Int	B
2.21	Ma	<i>We could say that this is the function [points at graph 1] and this one is f' [points at graph 2] {Si: and this is its derivative... [points together with Ma]}. But we wouldn't have [the right graphs relationship] with F', we would have it with f and f' ... [Ma is pointing the writing "$F'=f'$"] This one is F'...</i>	-	Int	B
2.22	Fra	<i>You mean that this one [points at graph 1] is the primitive... it is F and from here f [points at graph 2] which is its derivative... [Ma is pointing them too, there is perfect synchronization between their gestures]</i>	D	Int	B
2.23	Fe	<i>Exactly...</i>	-	-	-
2.24	Fra	<i>... do you think there is something wrong with this one? [points at graph 2]</i>	-	Int	B
2.25	Fe	<i>No, no, it works... but... we found that this one is f [points at graph 2] and this one is f' [points at graph 3] ... but...</i>	-	Int	B
2.26	Ma	<i>You mean that we can't...</i>	-	-	-
2.27	Fe	<i>Why don't we start [our reasoning] from this one [points at graph 1] and find this one [points at graph 2] which is its derivative? And then I must check that this one [points at graph 3] is f derivative...</i>	FS	Ast	C

The students continue, with a series of building-with, to explore the problem and try to understand the relation just introduced by Student-Fe. Meanwhile, Student-Fe tries to convince the others about the truth of her reasoning, reformulating the problem and highlighting the derivability relation between the given functions (C, line 2.27). They continue breaking down the problem solution. The *saming* discursive device appears again: Student-Fe has highlighted that the new pointed out reasoning is valid, both considering the graphs in pairs (function-derivative and function-derivative) and the three graphs together (primitive-function-derivative).

Line	Stud.	Transcript	BR	HIM	RBC
2.28	Ma	<i>But wait... it is not its derivative... {Fra: but this one increases and this one decreases... [first points at graph 1 and then at graph 2]} it is the integral... this one should be the integral right?? This derivative [points at graph 1] ... if you derive this one [points at graph 1] you obtain this one [points at graph 2] ...</i>	D	Int	B
2.29	Fe	<i>Mmm..</i>	-	-	-
2.30	Ma	<i>But you must derive it... I mean... wait... I understood...</i>	-	Def	R
2.31	Si	<i>That's why I said that finding the primitive is more complicated... because for example if you look at increase and decrease [points at the initial part of graph 1 and 2] it doesn't work...</i>	D	Ast	C
2.32	Ma	<i>They never work...</i>	-	Def	R
2.33	Fra	<i>But she was saying ... What if this is a function [points at graph 1] and this one is its derivative [points at graph 2]... this one must be... it must work... is this what you mean, right?</i>	D	Int	B
2.34	Fe	<i>Yes... it must work...</i>	-	Def	C
2.35	Fra	<i>But it doesn't work...</i>	-	Def	C


The students keep on going back to the previously introduced knowledge trying to put it together to understand the relation between functions introduced by Student-Fe, that recognize as correct (B, line 2.28). At this point Student-Si, making a more specific control of the problem resolution, recognizes that, applying the wrong property, all the relations do not make sense anymore (C, line 2.31). Afterwards the other students agree with her.

In this first part, the student has passed through a series of R-B-C chains (and a B-R-C one) to understand Student-Fe's reasoning and recognize the difficulty appeared in episode one. The R corresponds to the introduction of Student-Fe's idea. The backward reasoning is characterized by the breakdown of the problem solution considering the analytical expression of the functions F , f and f' . The *saming* discursive device characterises this excerpt.

9.2.2.2 Part 2: overcoming the difficulty

They are confused, so they decide to make an example. They represent $y = x^2 + 2x$, obtaining derivative and primitive and they draw the three graphs. Then, they do the same thing with $y = x^2$. After a couple of minutes analysing the represented graphs, they introduce again the concept of tangent to the curve referring to its analytical expression.

Line	Stud.	Transcript	BR	HIM	RBC
3.1	Ma	<p><i>Because we know that the derivative in a point is the tangent line slope... I mean</i></p> $y = f(x_0) + f'(x_0)(x - x_0)$ <p><i>I mean, this is the tangent, and the slope is the derivative ... [while speaking she writes on paper the tangent line formula]</i></p>	X	Def	R+B
3.2	Fe	<i>Yes... in fact the slope is increasing ... [with the hand gestures the tangent moving along graph 1]</i>	D	Int	C
3.3	Ma	<i>So, in each point ...</i>	-	-	-

3.4	Fra	<i>I mean... this one... the one you drawn before...</i> [she redraws the parabola $y = x^2$ (Figure 9.5)] 	-	Int	B
3.5	Ma	<i>Let's do the normal one... yes... I mean... normal...</i> [she means $y = x^2$ and its derivative]	-	Int	B
3.6	Fra	<i>In a point... here decreases, decreases, decreases...</i> [she is drawing the left Branch of the curve]	D	Def	C
3.7	Ma	<i>I think we said it wrong... it is the tangent... right... is the tangent... not the graph...</i>	-	Int	R

Trying to understand why the reasoning they developed until that point does not work, Student-Ma introduces the tangent line formula. Then, she highlights the line properties and in particular that its slope corresponds to the value of the first derivative in the tangent point. In line 3.1 the recognizing action emerges, it corresponds to the recognition of the tangent line object, and a building-with, explicating its properties. At this point Students Fe and Fra start to link the tangent movement to the movement of the point on the graph with a constructing action (line 3.2 and 3.6). From a backward reasoning point of view, the tangent line is introduced as an auxiliary element, then the students break down the tangent, analysing each elements of the analytical formula. The students refer to the performed activity and imagine the tangent as a line moving simultaneously with the point on the graph and analyse the problem. Also, the recognized relation between graph's increasing/decreasing and tangent movement is not physical expressed with a drawing or a text, but it stays in the imaginary form as an hand (or a pen) movement in the air in-between the students or close to the graph. The *saming* discursive device appears in highlighting the relationships between movements.

Line	Stud.	Transcript	BR	HIM	RBC
3.8	Ma	<i>We have that in each point... the derivative in each point...</i> [points the parabola derivative just drawn, in particular she points on the derivative part correspondent to the points marked on the curve, its negative part] <i>it gives us the slope of the tangent line...</i> [draws the tangent line in a point] <i>so from the value of the derivative point</i> [points at the derivative] <i>we have the tangent slope...</i> [remarks the already drawn tangent line] <i>right?</i>	D	Def	B
3.9	Fra	<i>So, if here increases and then decreases, is that one...</i> [moves the pen along the parabola graph]	-	Int	B
3.10	Ma	<i>So, if the derivative...</i> [remarks the derivative] <i>if each point of the derivative</i> [draws the point on the derivative in correspondence to the tangent drawn on the parabola] <i>it gives me the tangent line slope... if this... but it is negative here...</i> [it points at the derivative, the point just drawn] <i>it is negative without any doubt...</i>	X	Int	R
3.11	Si	<i>Because we took two x... [...] as positive...but if x is negative...this is positive</i> [points at $y = x^2$]	D	Int	B
3.12	Ma	<i>Exactly... it is what I was thinking...so here it is negative...</i> [remarks the part of the derivative graph below the abscissas axis] <i>and, as a consequence, here the tangent</i> [remarks the tangent] <i>will have a... [inaudible] it is decreasing... and so... maybe it works but we did it in a much more clumsy way... I mean, you cannot say...</i>	D	Int	B
3.13	Ma	<i>Maybe it is not decreasing</i> [remarks with the pencil the left branch of the parabola and the negative part of the derivative]... <i>because it is not true that if this</i>	X	Def	R

		<i>one decreases, this one decreases</i> [keeps on pointing to the parabola and derivative] <i>at the level of...I mean... if this one is negative... right...if this one is negative</i> [remarks again the derivative, the tone is much brighter] <i>our function decreases...</i>			
3.14	Ma	<i>Like here</i> [she moves on the problem 1 page (Figure 9.2)] <i>ok... maybe it works here... f decreases</i> [moves the pen along graph 2] <i>and this one is negative</i> [moves the pen along graph 3]... <i>f starts increasing and this one is positive...</i>	-	Ast	B
3.15	Fe	<i>Yes, that is right...</i>	-	-	-
3.16	Ma	<i>Ok...</i> [she moves again on their drawing (Figure 9.5)] <i>I think it is not about increase and decrease...the function decreases, my derivative is negative...</i> [remarks again both line and parabola] <i>it is not decreasing... my function increases, the derivative is positive... the function decreases, the derivative is negative</i>	FS	Ast	C
3.17	Ma	[she goes back to the problem page] <i>Stationary point...</i> [points at the zero of graph 3, there is a fusion, she is speaking about a stationary point, referring to the minimum of function 2 but she is pointing to the corresponding zero of the derivative which is function 3]	-	Int	B
3.18	Ma	[...] [she moves again on their drawing] <i>...my function increases, and the derivative is positive... it is not exactly increase-increase, decrease-decrease... It is about positivity and it works like this because here it is decreasing and it is negative, here it is increasing, and it is positive</i> [points again both parabola and line]	FS	Def	C

In this last part of the episode, the students are finally able to overcome the difficulty and reformulate the property that links a function and its derivative, hence they can justify the problem solution previously given in a correct way. Student-Ma is the one who develops the reasoning: her colleagues participate to the process but at a lower level of involvement. She starts with a building-with, in which she summarizes the properties and the relations explicit so far (line 3.8). The turning point can be seen at line 3.10 (and then 3.13) when, looking at the relations between the function, tangent line, slope and derivative, she recognizes that the derivative is negative when the function decreases. It is precisely the recognizing of the term “negative”, does what was not explicit until that moment: it allows her to formulate the correct relation increasing/positive of a function and its derivative (line 3.16). Then, with a building-with action and a constructing she verifies her idea saying it aloud. From a backward reasoning point of view, the term “negative” is introduced as an auxiliary element and the students continue to breakdown the problem, analysing each elements of the graphs. The *saming* discursive device appears in highlighting the relationships between function and derivative and between function’s stationary points and derivative’s zeros. In this last case the relationship is observed by the fusion between Student-Ma's gesture and her utterance, namely when she points to the zero of the derivative’s graph saying “stationary point”.

In this second part, the students have passed through a series of R-B-C chains to overcome their difficulty. To do it, they recognize the relationships decreasing function-negative derivative. Then, with a B-C chain they verify their conjecture. The backward reasoning is characterized by the breakdown, firstly of the analytical expression of the tangent line, and then of the graphs. The *saming* discursive device characterises this excerpt.

9.2.3 Function problem discussion

From the HIM point of view, the backward reasoning develops through interrogative moves. The students develop a process of question and answer to replay to the question “what characteristics must the derivative of f have?”. The information obtained from the answers to this question allowed the student to solve the problem. They alternate interrogative moves

followed by answers and defining moves (an elaboration of the answers) or assertoric ones (Hintikka, 1984, p. 277).

From the RBC point of view, the two episodes are characterised by three different types of chains: B-R-C, R-B-C and B-C. The B-R-C and R-B-C chains characterize the discovery processes while the B-C chains feature the verification one. Observing the transcriptions:

- In episode 1 the students apply five R-B-C chains (lines 1.1-1.33) to find the solution. With the first three chains they make attempt to identify the graphs. With the last to they recognize the graphs relevant points and they formulate the solution;
- In first part of episode 2, the students apply three R-B-C chains and a B-R-C chain (lines 2.1-2.35) to introduce and understand Student-Fe's point of view. In the first two R-B-C chains, Student-Fe introduces her idea, with a B-R-C and then a R-B-C chain, Student-Ma (and the other classmates) understands the difficulty in episode 1;
- In second part of episode 2, with three R-B-C chains and a B-C chain (lines 3.1-3.18) the students overcome the difficulty with the help of the analytical expression of the tangent line and they verify the result (B-C chain);

The sequence of chains of the two episodes can be schematized as follow. The sign "[...]" indicates a resolution time not shown in the transcripts.

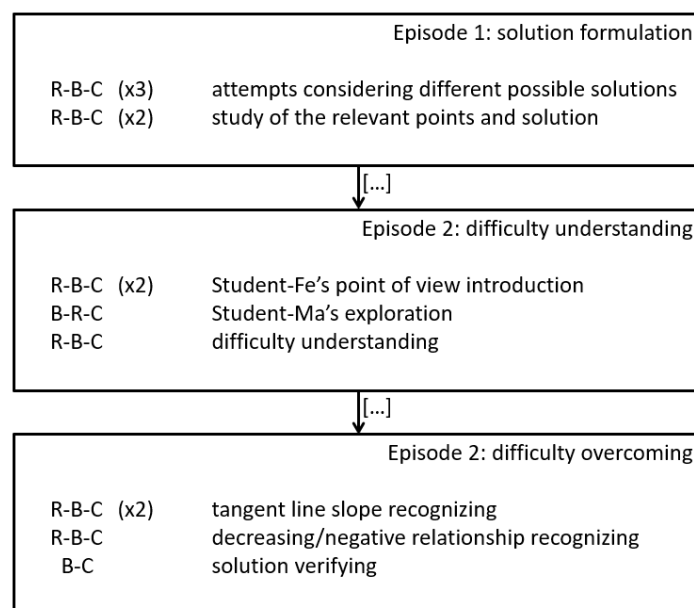


Fig. 9.7 - RBC flow (Group 2 transcription)

Within the transcription there is an alternation between forward and backward reasoning. Two different backward reasoning moments can be identified along the resolution: suppose that one graph is f and search for its derivative, and analyse problem elements to overcome the difficulty. In the next sections the backward reasoning development during the two moments is explored.

9.2.3.1 In-depth analysis: Suppose identified the function and its derivative

When the students suppose that a graph represent the function f and search for its derivative the backward reasoning develops searching cause-effect relationships. The students, in fact, work on the supposed solved problem and hypothesize that two specific graphs are the function and its derivative. Then, they search for some elements to confirm this relationship. The *reifying* discursive device appears when there is a shift in the subject discourse from talking about the graphs' property and the graphs movements to talking about the graphs and the functions themselves. Five R-B-C chains appear, the students recognize a possible solution and building(-with) some notion can validate it or understand that it is not.

9.2.3.2 In-depth analysis: Analyse the elements to overcome the difficulty

In the second episode the students recognize and overcome the difficulty that appear in episode 1. To do it they breakdown the problem solution in three different ways. Firstly, they analyse the analytical expression of f and F and recognize that $F' = f$. Then, they analyse the tangent line analytical expression elements until recognize that the slope in a point is the value of the derivative in this point. Later, they breakdown the graphs until recognize that the right relationship between function and derivative is when the function is increasing/decreasing the derivative is positive/negative. New elements are introduced to overcome the difficulty, the first one introduced by Student-Fe is that $F' = f$, the second one is the tangent line analytical expression and the third, the key point to overcome the difficulty, the word "negative". The *saming* discursive device appears: the students recognize the similarity of the involved elements. The reasoning develops in correspondence of six R-B-C chains and one B-R-C chain. The first two R-B-C chains develop during the introduction of Student-Fe's point of view: in the first one $F' = f$ is recognized and in the second one there is a shift from the primitive-function-derivative idea to the function-

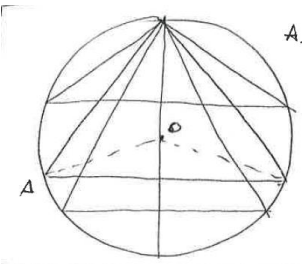
derivative and the function-derivative one. Then, a B-R-C chain appear; it is the moment in which the classmates understand Student-Fe's reasoning. Though an R-B-C chain they understand that their reasoning doesn't work. The next two R-B-C chains allow to recognize the relationship between tangent line slope and derivative, while the last one to understand the right property.

9.3 Case study of the Circle and Triangle Problem: Student-G

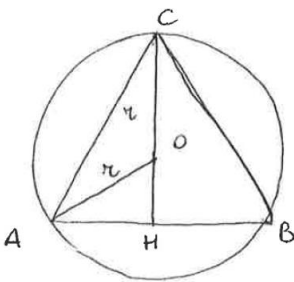
As seen from the global analysis there are two moments in which the backward reasoning develops: analysing the geometric configuration of the problem and expressing the relationships among the geometric configuration in the algebraic language. A case study is displayed in this section: Student-G protocol. This protocol shows both moments and is written in great detail. The student uses drawings and graphic representations firstly to explore the problem and then to solve it. They help her during the resolution process and to hypothesize that the sought triangle is the equilateral one. She traces the problem back to a geometrical known problem and has no doubt about which methods to apply in order to find the triangle of maximum area.

The entire protocol has been translated from Spanish by the author. The protocol is divided in lines, each figure is associated with a line (for example: figure 1 is associated to line 1). Each part of the excerpt has a short comment to identify the backward reasoning characteristics, the characteristics according to both analysis model (HIM and RBC), and the discursive devices used by the student.

Student-G protocol

Line	Protocol	BR	HIM	RBC
1	I draw several isosceles triangles inscribed in a circle. Fig. 1  <i>Fig. 9.8 - Figure 1 (Student-G resolution protocol)</i>	X	Init	R+B
2	I notice that I can describe the height of the triangle from its base as the ray of the circle plus an x quantity.	D+X+T	Int/a	C

The student starts the resolution by representing the proposed problem (R). She inserts within the represented circumference several isosceles triangles with a common vertex and the relative height of the base that belongs to the same diameter of the circumference (B). By doing so, in this exploratory phase, she understands that she can represent the height of the triangle as the radius of the circumference plus a certain variable amount x (C). Some auxiliary elements are added to the initial configuration and the backward reasoning appears in its breakdown and transformative features. In fact, the student analyses the geometric configuration pinpointing some key elements that she identifies as the unknown x . The *saming* discursive device appears: the student identifies a method to represent the segment “height” in a different way.

3	I draw an isosceles triangle inscribed in a circle Fig. 3  <i>Fig. 9.9 - Figure 3 (Student-G resolution protocol)</i>	D+T	Init	R
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4	I draw the height CH passing through O, the centre of the circle, and cutting the side AB in two equal parts.	X	Int/a	B
5	$\Rightarrow AH = HB = \frac{1}{2}AB$ and $CH = r + OH$	-	Def	C
6	I write the formula for the area of the triangle...	X+D	Def	R
7	$Area = \frac{AB * CH}{2}$	-	Def	C
8	...to see if I can find a way to write the area based on the OH segment, that I name x .	D+T	Int/a	B

At this point she draws a second geometric construction in which there is a single isosceles triangle inscribed in the circumference. She inserts in this construction two elements: the height, the radius of the circumference (R). Reasoning on the elements present in the construction (B), she represents the relations between the elements in algebraic form (C). Then, she introduces the formula of the area (R+C). The student, in fact, has the idea of representing the area of the triangle as a function of a single variable (B). Considering it, the student can then study the maximum and identify the corresponding value, thus solving the problem.

Line	Stud.	Transcript	BR	HIM
9	I consider the triangle AOH , it is a rectangle triangle, and I calculate AH with the Pythagorean theorem.	D+X	Ast	R+B
10	$AH = \sqrt{r^2 - OH^2} = \sqrt{r^2 - x^2}$	-	Def	B
11	Now I can write the area as:	-	Def	B
12	$Area = \frac{AB * CH}{2} = \frac{2AH * (r + x)}{2}$ $= \frac{\cancel{2}\sqrt{r^2 - x^2} * (r + x)}{\cancel{2}}$ $= \sqrt{r^2 - x^2} * (r + x)$	D+T	R	B+C
13	Now I would have to calculate the maximum of $\sqrt{r^2 - x^2} * (r + x)$ considering that $-r < x < r$	-	Ast	R

She considers now the triangle AOH (R). Applying Pythagoras' theorem to AOH, she can represent the side AH in function of the other two (B). At this point she replaces in the area formula the values of the base and the height as a function of the radius of the circle and the quantity x (B) and she manipulates the values until she gets the reduced formula. She states that she wants to calculate the maximum of the found function bearing in mind that the value of the variable x must be between the values $-r$ and r (R). In the last two excerpts the backward reasoning appears again in its breakdown and transformative features, and some auxiliary elements are added (like the Pythagorean theorem). In fact, the student analyses the geometric configuration and represents some elements with variables (the radius, for example). The *saming*, the *encapsulating* and the *reifying* discursive devices appear: the student identifies a way to represent some elements, she encapsulates the different elements considering a triangle and then she analytically represents the geometric elements.

Line	Stud.	Transcript	BR	HIM
14	In reality, what I'm doing is looking for a formal justification of what I suspect: that the maximum area triangle is the equilateral.	FS	Ast	R
15	If the triangle is equilateral: $OH = r * \sin 30^\circ = \frac{r}{2}$ $x = \frac{r}{2}$	FS	Def	B+C

At this point she makes a digression from the resolution of the problem and makes explicit the fact that, after the first initial exploration, she has informally conjectured that the sought triangle is the equilateral one (R). She therefore represents the OH value for the equilateral triangle inscribed in the circumference and, with a series of calculations (B), identifies the final sought value of the variable: $x=r/2$ (C).

Line	Stud.	Transcript	BR	HIM
16	I come back to the area formula: $\sqrt{r^2 - x^2} * (r + x)$	D+T	Int/a	B
17	$\frac{d}{dx} [\sqrt{r^2 - x^2} * (r + x)]$ $= (+1)\sqrt{r^2 - x^2}$ $+ (r + x) \left[\frac{1}{2} (r^2 - x^2)^{-\frac{1}{2}} (-2x) \right] =$ \dots $= \frac{(r + x)[r - x - x]}{\sqrt{r^2 - x^2}} = \frac{(r + x)(r - 2x)}{\sqrt{r^2 - x^2}}$	-	R	B+C
18	I'm looking for the maximum $\frac{d}{dx} f(x) = 0$	D+T	Int/a	B
19	$\frac{(r + x)(r - 2x)}{\sqrt{r^2 - x^2}} = 0$	-	R	B
20	$r + x = 0 \quad x = -r \quad \text{but} \quad -r < x < r$ $r - 2x = 0 \quad x = \frac{r}{2}$	-	Int/a	C
21	$x = \frac{r}{2} \Rightarrow$ Equilateral triangle	-	Def	R

Then she continues with the problem solving and calculates the area function derivative (B) and simplifies its expression (C). At this point, to meet the maximum of the function, she sets the derivative expression equal to zero and with a series of calculations (B), she obtains two results (C): $x=-r$ and $x=r/2$. She then recognizes that the first solution has no meaning for the problem and that $x=r/2$ is the sought result (R). The backward reasoning appears in its breakdown and transformative features: the student manipulates the formulas. The *reifying* discursive devices appears: she manipulates analytical elements that represent the geometric ones.

9.3.1 Circle and Triangle problem discussion

From the HIM point of view, the backward reasoning develops in interrogative moves. The student answers to the question “what are the important elements for the resolution?” or “how can I represent the triangle area?”. The information obtained from the answers to this question allowed the student to solve the problem. They alternate interrogative moves followed by defining moves.

From the RBC point of view, the protocol is characterised by two different types of chains: R-B-C, during discovery phases, and B-C, during formula manipulations. Observing the protocol, the student applies:

- One R-B-C chain (lines 1-2) to explore the geometrical configuration with the family triangles;
- Three R-B-C chains (lines 3-13) to develop the area formula. The first to identify the segments values, the second the area and the third the final formula;
- One R-B-C chain (line 14-15) to conjecture the problem solution;
- Two B-C chains (lines 16-21) to manipulate the formula and identify the sought value.

The sequence of chains can be schematized as follow.

Circle and Triangle	
R-B-C	first exploration
R-B-C (x3)	formula area developing
R-B-C	sought solution identification
B-C	formula manipulation

Fig. 9.10 - RBC flow (Student-G Circle and Triangle Problem resolution protocol)

Within the transcription there is an alternation between forward and backward reasoning. Two different backward reasoning moments can be identified along the resolution: analysing the geometric configuration of the problem, and expressing the relationships among the geometric configuration in algebraic language. In the next sections the backward reasoning development during the two moments is explored.

9.3.1.1 In depth-analysis: Analysing the geometric configuration

When the students are analysing the geometric configuration of a problem, the backward reasoning develops in its breakdown feature. Some auxiliary elements are added when they consider it appropriate. In the lines when a geometric element is identified with an unknown value, the backward reasoning appears in its transformative feature. It happens when, like in this case study, the two identified backward reasoning moments overlap. There are other cases where the moments are clearly distinct. The students work on the geometric configuration of the problem analysing it and identifying the relevant elements. In this case study the *saming*, the *encapsulating* and the *reifying* discursive devices appear: the student identifies a way to represent some elements, she encapsulates the different elements considering an entity, and then, she analytically represents it. When the two backward reasoning moments are not overlapped, the *reifying* discursive device doesn't appear. The R-B-C chains appear in this backward reasoning moment, the students recognize an element, and with a series of reasoning they construct the formula.

9.3.1.2 In-depth analysis: Expressing the geometric configuration in algebraic language

This backward reasoning moment can be split into two situations: the formula expression and the formula manipulation. Even if in this protocol the formula development situation coincides with the previous backward reasoning moment, there are some cases when the two moment do not overlap. Both situations are characterized by the transformative and breakdown features. In the first situation, *saming*, *encapsulating* and *reifying* discursive devices appear: the students recognize some geometric relations between elements, they identify them as an entity, and they represent in algebraic language. The R-B-C chains appear: the students identify a geometric element, then reasoning about the relationships with other entities they represent it in an algebraic way. The second situation is characterized by the *reifying* discursive device: the sentences subjects are the geometrical elements represented in algebraic language. B-C chains appears: the students do some calculations (building-with) to construct the required formula.

9.4 Cases studies of the Geometrical construction Problem

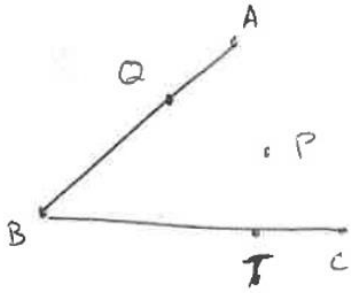
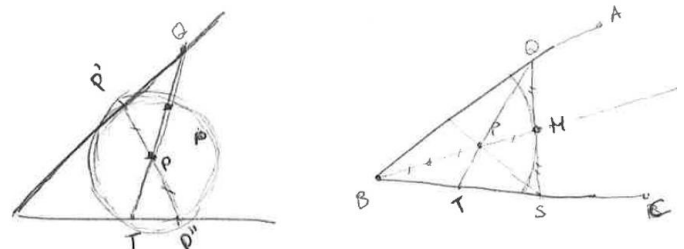
As seen from the global analysis, there are two moments in which the backward reasoning develops: analysing the solution of the problem (the sought geometric configuration), and identifying an analogy between the result of the problem and a known theorem. To exemplify these moments, an excerpt from a case study and two case studies are shown in this section: Student-L, Student-T and Student-F. Student-L solves the problem starting with the analysis of the sought geometric configuration. The first part of his protocol was chosen to exemplify the first moment of backward reasoning. During the second moment of backward reasoning, as said in the group analysis, the students traced the problem back to two different known theorems/constructions: the segment sought as the median of a triangle with barycentre P, and the segment sought as that met in the construction of the segment trisection. To exemplify this two moments Student-T and Student-F protocols are shown.

The entire protocols have been translated from Spanish by the author. The protocols are divided in lines, each figure is associated with a line (for example: figure 1 is associated to line 1). Each part of the excerpt has a short comment to identify the backward reasoning characteristics, the characteristics according to both analysis model (HIM and RBC), and the discursive devices used by the student.

9.4.1 Student-L: Exploring the sought geometric configuration

Student-L was chosen because is an emblematic case that represent the backward reasoning moment “exploring the sought geometric configuration”. He observes the final configuration in different ways adding several auxiliary elements trying to reconduct it to a known problem. For each of them he represents graphically the geometric configurations. The first lines of the resolution protocol will be presented.

Student-L protocol excerpt

Line	Protocol	BR	HIM	RBC
1	Firstly, I made a draw to get an overview. Fig. 1  <i>Fig. 9.11 - Figure 1 (Student-L resolution protocol)</i>	-	Init	R
2	I realized that I didn't understand the problem.	-	-	R
3	Then I tried to make relationships and simple drawings with the compass and the ruler to see if I could get any interesting results. Fig. 3.1 and 3.2  <i>Fig. 9.12a – Figure 3.1 (Student-L resolution protocol)</i> <i>Fig. 9.12b - Figure 3.2 (Student-L resolution protocol)</i>	X+D	Int/a	B
4	Later, I realized that the segment QT is a median, and P is the point where the medians are cut, so the request is fulfilled.	D	Ast	R+C

The student starts the resolution by representing the initial situation (R). An exploration phase follows the first representation. Here the student tries to add elements to the final construction (the angle with the sought segment) to look for possible relationships (B). He is trying to answer to the implicit question “what known configuration could lead me to the result?” In particular, he introduces some circumferences of centre P, some lines through P, then he draws the BP line and other segments according to the angle. He then realizes that

the median of a triangle has the same characteristics as the segment QT considering P as its barycentre (R). He then finds the request configuration (C).

9.4.1.1 In depth-analysis: Exploring the sought geometric configuration

The backward reasoning appears in its breakdown feature. Some auxiliary elements are added to the final configuration to reach a known one. The students work on the geometric configuration and analyse it adding some auxiliary elements. The *encapsulating* and the *reifying* discursive devices appear: the student studying the different elements of the configuration put them together in a known entity (for example the triangle BQS identifying the known theorem/configuration), and then the sentence subject change from the student to the geometric elements. A B-R-C chain characterize the excerpt: the student explores the problem until recognizes a known configuration, then he constructs it.

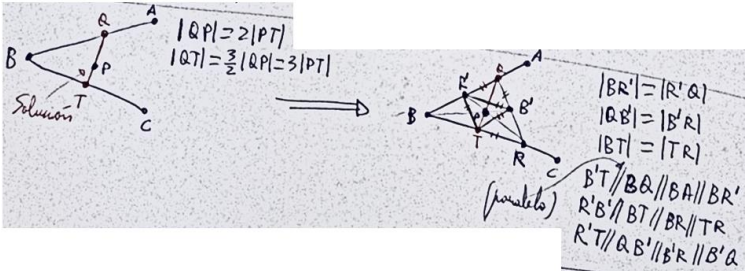
9.4.2 Student-T and Student-F: Identifying an analogy between the result of the problem and a known theorem

Student-T and Student-F were chosen to exemplify this backward reasoning moment due to the great details of their protocols. Both students start their protocols highlighting the backward reasoning use in their resolution. As stated in the global analysis this moment is applied in two different ways. In fact, the students trace the sought segment back to two different known problems/configurations: the median of a triangle with barycentre P, and the segment trisection (or Thales theorem).

9.4.2.1 Student-T: QT as a median of a triangle

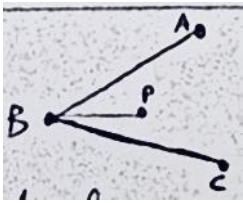
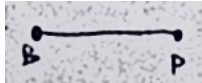
Student-T was chosen to represent the median case. As said before, the resolution protocol is divided in lines; in this case, the figures are more relevant for the resolution, so some of them are not associated with a line of text but are represented independently.

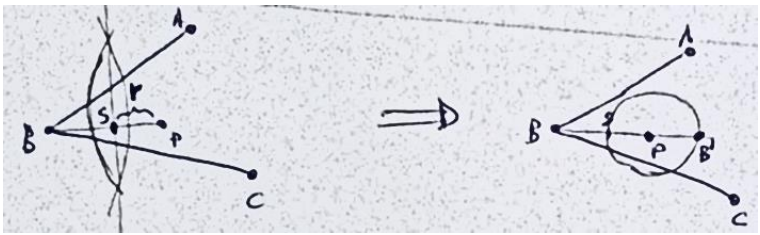
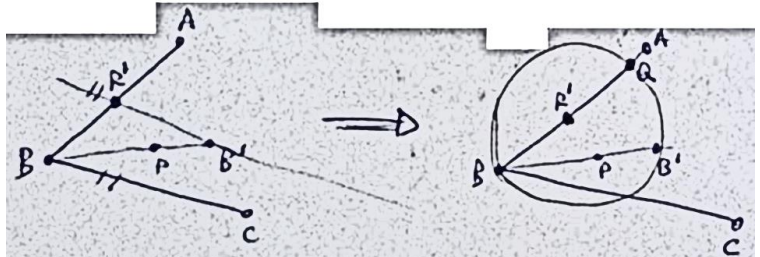
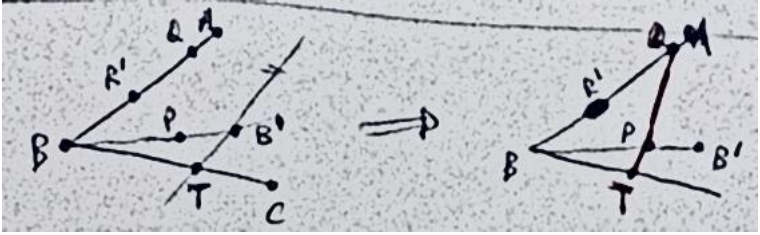
Student-T resolution protocol

Line	Protocol	BC	HIM	RBC
1	If we suppose the problem solved and we take into account that the barycentre of a triangle is the point where the three medians intersect each other and that the distance from it [the point] to each vertex is 2/3 of the length of the median that begins at that vertex and that the distance from the midpoint of the opposite side is 1/3 of the length of the median..	-	Init	R
2	...then we can suppose that point P is the barycentre of a certain triangle with vertices Q, B and R.	E	Ast	B
3	R is a point belonging to BC (T is the midpoint of the BR segment), QT is the median starting at the vertex Q.	E	Int/a	B
4	 <p style="text-align: center;">Fig. 9.13 - Figure line 4 (Student-L resolution protocol)</p>	X	Answ	C

The interrogative process seems to be sustained by the question “Which elements should I take into account to construct a triangle that has the sought segment as a median and barycentre P?” The student starts the resolution by recognizing the analogy with an element of a known problem (R). With a series of reasoning he identifies the elements that characterize the known problem and connects them with the Construction problem (B). Then, he graphically represents the sought configuration inside the known problem, identifying some peculiar characteristics (C). The backward reasoning appears in its cause-effect relationships research feature. The student searches for elements that can help him to get the known configuration: hence some auxiliary elements are constructed to reach it (line 4). The *saming*, *encapsulating* and *reifying* discursive devices appear: the student associates

some elements of the problem configuration with known problem elements, then he considers different elements as an entity (the sought triangle), later the sentences subject shift from the student to the elements.

Line	Protocol	BR	HIM	RBC
5 Fig. 5	Therefore, having:  <i>Fig. 9.14 - Figure 5 (Student-L resolution protocol)</i>	-	Int/a	B
6 Fig. 6	Given the segment, we divide the BP segment into 2 equal parts as follows:  <i>Fig. 9.15 - Figure 6 (Student-L resolution protocol)</i>	E	Int/a	R
7	we point the compass at point B with a “distance” [opening of the compass] that we want to be more than half that the segment. (In case of doubt, we can take the distance BP and draw a circle.)	-	Answ	B
8	Then with the same radius we draw a circle with P like a centre.	-	answ	B
9	Then, we join the two cut-off points of both circles and we obtain a segment that cut BP at point S, such that $ BS = r = SP $ (S is the middle point of BP)	-	answ	B
10	Once this is done, we draw a circle with a radius r and the centre P; we extend the segment BP until it cut the circle at a different point than S, we call it B’.	-	Answ	B

11	 <p data-bbox="422 506 1043 539"><i>Fig. 9.16 - Figure line 11 (Student-L resolution protocol)</i></p>	-	Int/a	C
12	<p data-bbox="347 573 1118 770">Then we draw a parallel line to BC that passes through B' and we call the point R' where this line cut BA. Then we draw a circle with centre B' and radius BR' and we call Q the point, different from B, where BA cuts the circle.</p>	E	Int/a	R+B
13	 <p data-bbox="422 1088 1043 1122"><i>Fig. 9.17 - Figure line 13 (Student-L resolution protocol)</i></p>	-	Int/a	C
14	<p data-bbox="347 1155 1118 1294">Then, we draw a parallel line to BA, that passes through B' and the point that cuts with BC we name it T. Then we join Q and T and we get the solution.</p>	E	Int/a	R+B
15	 <p data-bbox="422 1583 1043 1617"><i>Fig. 9.18 - Figure line 15 (Student-L resolution protocol)</i></p>	-	Int/a	C



The student constructs the known problem to solve this one. He starts recognizing the initial configuration and adding some element (B) to construct a specific circle. The he recognizes some auxiliary elements to construct, and adding other ones (B), he constructs the point Q and T. The backward reasoning appears again in its cause-effect relationships research feature. The student still searches for elements that help him to construct the known

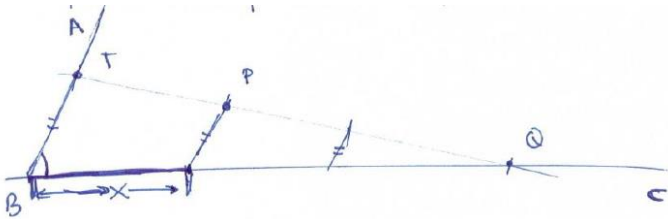
configuration, while he adds some auxiliary elements. The *encapsulating* discursive devices appears while Student-L identifies entities in the configurations to pinpoint useful points.

9.4.2.2 Student-F: QT as a trisected segment

Student-F was chosen to represent the segment trisection case. The student reverses the segment trisection construction, As said before, the resolution protocol is divided in lines; in this case, the figure are more relevant for the resolution, so some of them are not associated with a line of text but are represented independently.

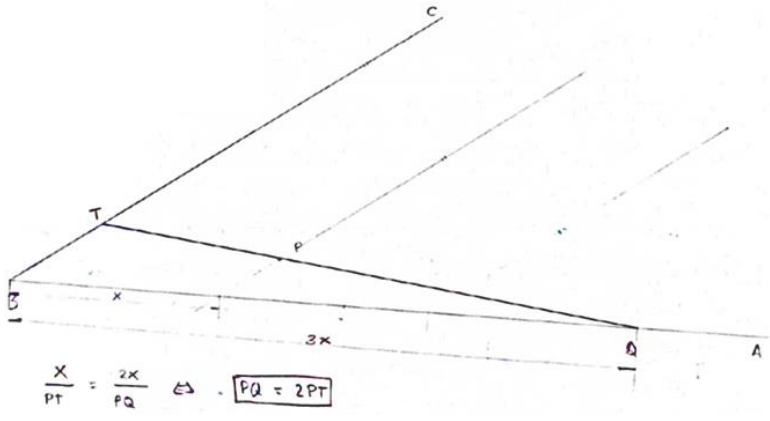
Student-F resolution protocol

Line	Protocol	BC	HIM	RBC
1	Using the backward reasoning strategy, we assume that we have a segment QT that goes through P, such that QP is twice as PT.	-	Init	R+B
2	 <p><i>Fig. 9.19 - Figure line 2 (Student-F resolution protocol)</i></p>	-	Init	C
3	In this way we realize that by tracing a horizontal line through Q, we can arbitrarily choose our point B which is the point where we will construct the angle ABC	E	Int/a	R
4	We arbitrarily chose B.	-	answ	B
Fig. 4	 <p><i>Fig. 9.20 - Figure 4 (Student-F resolution protocol)</i></p>			
5	And so, applying Thales' theorem, we have our segment BA and BC where we already knew that there were Q and T.	E+X	Int/a	R+B

6	 <p data-bbox="421 495 1027 524">Fig. 9.21 - Figure line 6 (Student-F resolution protocol)</p>	-	answ	C
---	--	---	------	---

The student starts the resolution by assuming to have the sought segment (R): by highlighting some elements on the segment (B), he represents it (C). He recognizes that tracing two lines he can represent the final configuration. He then, with a series of reasoning (B) start to draw it (C). Moreover, he recognizes that it is a Thales theorem application. The reasoning is sustained by the implicit question “Which elements should I take into account to construct the sought trisected segment?” The backward reasoning appears in its cause-effect relationships research feature. The student search for elements that help him to construct the final configuration. The parallel lines to BA are added as auxiliary element (line 5-6) to understand the analogy with Thales theorem. The *saming* and *encapsulating* discursive devices appear: the student associates some elements of the problem configuration with known problem elements, and considers different geometric elements as an entity that allows him to recognize the analogy.

Line	Protocol	BR	HIM	RBC
7	In this way, by reversing the process, we realize that the key is in Thales' theorem to divide a segment into three equal parts and thus find x , thanks to the parallel trough P and the side AB.	E	Ast	R
8	We then return to our step to carry out the geometric construction. Given the angle ABC and the point P.	E	Int/a	B
9	We draw a parallel line to BC through P	-	answ	C
10	→ we find the segment we are looking for to applicate Thales' Theorem	E	Int/a	R
11	We plotted $3x$ with a compass that will give us the point Q	-	Def	B+C

12	When we join Q and P, we obtain T because we are joining two segments by parallel lines...	E	Int/a	R+B
13	<p>...and thus, the condition of Thales' Theorem is fulfilled and $PQ = 2PT$.</p> <p>Fig. 8-13</p>  <p>Fig. 9.22 – Figure 8-13: the sought construction (Student-F resolution protocol)</p>	FS	Def	C

He then develops the sought construction step by step. The student recognizes that he has to reverse the steps to reach the final construction. Reasoning on the newly identified construction (B) he draws the parallel to BC passing through P. Then he recognizes that it can be considered as part of the bundle of straight lines necessary to apply Thales theorem. With a series of reasoning (B), he constructs the point Q. Student-F recognizes that tracing the QP line he obtains the sought construction. Observing the construction, he recognizes that he can apply Thales theorem, so highlighting the found relationship between QP and PT (C). The backward reasoning appears again in its cause-effect relationships research feature. The student searches for elements that help him to construct the known configuration, while he adds some auxiliary elements. The *encapsulating* discursive devices appears while Student-L identifies entities in the configurations to recognize useful points.

9.4.3 Discussion on Student-T and Student-F resolution

From the HIM point of view, the backward reasoning develops in interrogative moves. The student answers to questions like “Which elements should I take into account to construct a

triangle that has the sought segment as a median and barycentre P?” or “Which elements should I take into account to construct the sought trisected segment?”. The information obtained from the answers to this question allowed the student to solve the problem. They alternate interrogative moves followed by defining moves.

From the RBC point of view, the protocols are characterised by R-B-C chains. Observing the protocols, the students apply:

- Student-T
 - One R-B-C chain (lines 1-4) to trace the problem back to a known one;
 - Three R-B-C chains (lines 5-15) to develop the sought configuration.
- Student-F
 - Two R-B-C chain (line 1-6) to trace the problem back to a known one;
 - Three R-B-C chains (lines 7-13) to develop the sought configuration.

The sequence of chains can be schematized as follow.

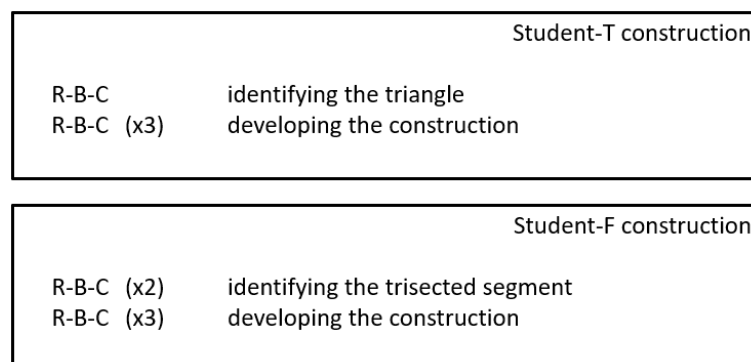


Fig. 9.23 - RBC flow (Student-T and Student-F Construction Problem resolution protocol)

Within the protocol there is an alternation between forward and backward reasoning. In the next sections the backward reasoning development during the analogy identification is explored.

9.4.3.1 In-depth analysis: Identifying an analogy between the result of the problem and a known theorem

The backward reasoning appears in its cause-effect relationships research feature. The students search for necessary elements to construct the analogous configuration identified at the reasoning beginning. Some auxiliary elements are added to reach it. The moment is divided into two situations: tracing back the problem to a known one and reverse the known construction to reach the solution. The *saming*, *encapsulating* and *reifying* discursive devices appear while students trace back the problem to a known one. Then, during the reverse construction only the *encapsulating* device appears. In the first situation the students associate some elements of the problem configuration with known problem elements, then he considers different elements as an entity, later (sometimes) the sentences subject shift from the student to the configuration elements. In the second situation the students identify entities in the configurations to recognize useful component. A series of R-B-C chain characterize the protocols: the students recognize a known configuration or a useful element, then, making some reasoning, they construct the sought configuration.

9.5 Case studies of the Paths Problem

As seen from the global analysis there are two moments in which the backward reasoning develops: analysing the generic path, and identifying the combinatorial problem. Two protocol excerpts and a case study are displayed in this section. Student-H and Student-V excerpt are used to show the first backward reasoning moment: analysing the generic path. This moment leads to two different strategies: recognizing a combinatorial problem or counting the paths in a systematic way. Student-N protocol is exemplificative to show the second backward reasoning moment: identifying the combinatorial problem. The protocols are written in great detail.

The entire protocols have been translated from Spanish by the author. The protocols are divided in lines, each figure is associated with a line (for example: figure 1 is associated to line 1). Each excerpt and each part of the protocol have a short comment to identify the

backward reasoning characteristics, the characteristics according to both analysis model (HIM and RBC), and the discursive devices used by the student.

9.5.1 Student-H and Student-V: Analysing the generic path

Student-H and Student-V were chosen to exemplify this backward reasoning moment due to the great details of their protocols. Both students identify the generic path and represent it in an algebraic way. The first one realizes that the problem is a combinatorial one, while the latter do a systematic calculation to all cases.

9.5.1.1 Student-H: Analysing the generic path to recognize the combinatorial problem

Student-H was chosen to exemplify the behaviours that lead to the awareness of the problem combinatorial structure. He does not use drawings in his protocol. He represents the problem in an algebraic way, reconducting it to a combinatorial one.

Student-H resolution protocol excerpt

Line	Protocol	BR	HIM	RBC
1	It occurs to me to extract an element that is a recurring item in the box (or the parallelepiped): it is a unitary cube extracted from the corners.	-	Init	R
2	For each cube I have three possibilities of movement.	D	Int/a	R
3	We have 24 unitary cubes that compose the box (or parallelepiped). We will take these cubes as the set of elements on which we will see the possible combinations of 9 sections of the path constituted by n sections in direction 1, m sections in direction 2 and k sections in direction 3.	D+T	Int/a	B
4	As we know, it has to comply that the sum of all the sections must be equal to 9. So: $n + m + k = 9$	-	Ast	C

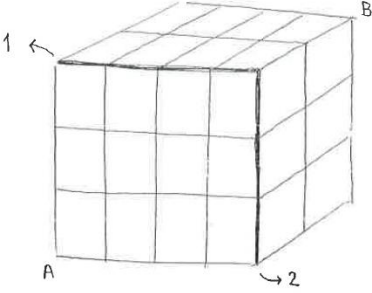
5	For each cube we calculate the possibilities of advancing to the next cube taking into account that order matters.	D	Int/a	B
6	So, I have to do it in the given number of movements (9) from A to B, then it can't go beyond n movements in direction 1, m movements in direction 2 and k movements in direction 3.	D+T	Def	R
7	Then I calculate permutations with repetition of three possible movements inside the cube to go to the next cube.	FS	Ast	C

The student starts the resolution by recognizing a basic element with its characteristics (R). Exploring the problem representation (B), he constructs the generic path in an algebraic way. Then, he explores the generic path (B), recognizing some properties (R). Later, he recognizes the combinatorial nature of the problem. He is trying to answer to the implicit question “how can I represent the paths in a way that can be mathematical manipulated and calculated?” The backward reasoning appears in its breakdown and transformative features. The student analyses the generic path representing it in an algebraic way. The *saming*, *encapsulating* and *reifying* discursive devices appear: the student recognizes some properties for each cube, encapsulate the movements in a generic path and change the subject sentences from him to the generic path.

9.5.1.2 Student-V: Analysing the generic path to calculate the paths in a systematic way

Student-V was chosen to exemplify the behaviours that lead to the systematically count of all cases. The student uses drawings and graphic representations firstly to explore the problem and then to count the path systematically. They help him during the resolution process.

Student-V resolution protocol excerpt

Line	Protocol	BR	HIM	RBC
1	How many paths of length 9 are there from A to B?	-	Int	B
2	We can only move $\uparrow, \nearrow, \rightarrow$	D	Init	R
3	The first thing I do is to name the top and side edges with 1 and 2 respectively.	-	Def	R
4	In this way there will be two types of paths that go from A to B, taking into account which edge he reached first. The two options are A-1-B or A-2-B.	D	Int/a	B+C
Fig. 4	 <p style="text-align: center;"><i>Fig. 9.24 - Figure 4 (Student-H resolution protocol)</i></p>			
5	To go from A to 1 the minimum of movements is three, so the maximum from 1 to B is 6 movements.	D	Int/a	B+R
6	The minimum to go from 1 to B is two movements, so the maximum from A to 1 will be 7 movements.	D	Int/a	B+R
7	Let's play with these combinations.	-	Ast	C

The student starts the resolution by explicating the question that sustain all the discourse (B). He explores the problem identifying the possible movements representing the initial situation and naming the parallelepiped sides (R). Then, exploring the problem he constructs two general paths (depending on which side they pass through) (C). Exploring the problem, he recognizes some characteristics of the generic paths (B+R) and he decides to count systematically the paths number (C). The backward reasoning appears in its breakdown feature. The student analyses the movements and represent two generic paths. The *saming*,

encapsulating discursive devices appear: the student recognizes same properties for each cube, and he encapsulates the movements in two generic paths.


9.5.1.3 In depth-analysis: Analysing the generic path

The backward reasoning appears in its breakdown feature. The transformative feature appears if the student transforms the generic path in algebraic language. The algebraic transformation leads Student-H to recognize the combinatorial structure of the problem, while the geometric representation of the generic path leads Student-V to systematically count all cases. The *saming* and *encapsulating* devices appear while the students recognize some common properties, and encapsulate the movements in the generic path. The *reifying* discursive devices appears when the students represent the generic path in algebraic language. A R-B-C and a B-R-C chains characterize both excerpts: the students recognize some elements, explores the problem, and then construct the generic path. Then, they explore again the problem until recognize some useful properties of the generic path that lead them to organize the next steps following the combinatorial structure of the problem (Student-H) or calculating the paths number in a systematic way.

9.5.2 Student-N: Identifying the combinatorial problem

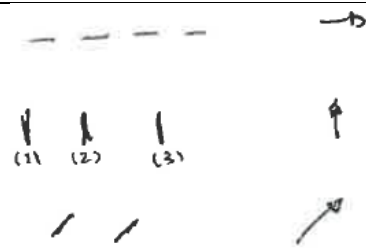
Student-N was chosen to exemplify this backward reasoning moment due to the great details of his protocol. He uses graphics representations to help himself solve the problem. He identifies the analogy with the combinatorial problem, and he solves it analysing the structure and identifying the permutations with repetition rule.

Student-N resolution protocol

Line	Protocol	BR	HIM	RBC
1	Let's look at the sections	D	Init	B
Fig.1	 <p><i>Fig. 9.25 - Figure 1 (Student-N resolution protocol)</i></p>			

2	I can think of many ways to count them, but none of them are good enough to keep me from getting caught up in the process. There is not a way that fits on the sheet.	-	Int/a	B
3	I've realized that it's a combinatorial problem in a totally random way. I think it's going to be a good answer.	X	Int/a	R
4	$\frac{9!}{4! * 3! * 2!} = \frac{9 * 8 * 7 * \cancel{6} * 5}{\cancel{3} * 2 * 2} = 9 * 7 * 5 * 4$ <p>This is the answer that looks like the correct one.</p>	X	Ast	C

The student starts the resolution by exploring the problem and breaking down the parallelepiped in basic parts (B). Observing the assignment and the drawing sections, he recognizes that the problem has a combinatorial structure. He, then, conjectures the solution (C). The student answers the implicit question “How can I represent the paths sections?” The backward reasoning appears in its breakdown feature, it is possible to note that some auxiliary elements are added to the resolution, like the combinatorial structure and the permutation with repetition formula. The *encapsulating* discursive devices characterize the recognizing of the combinatorial structure.

Line	Protocol	BR	HIM	RBC	
5	We have 9 segments:	-	Init	R	
6	4 like these 3 like these 2 like these		D	Int/a	B
7	There are 9 way to order them however you have to be careful...	-	answ	C	
8	No! It's 9!	-	answ	C	

The student recognizes that the generic path is subdivided in 9 segments, then he exemplifies the possible direction with a draw (B). Observing the segments, he conjectures that there are $9!$ combinations of the segments (he doesn't consider the repetitions, he is constructing the permutation with repetition formula step by step) (C). The backward reasoning appears again in its breakdown feature, Student-N breaks down the generic path in sections. The *encapsulating* discursive devices characterize the recognizing of the combinatorial structure, the *reifying* device appears when the subject change and become the permutations number.

Line	Protocol	BR	HIM	RBC
9	However, it doesn't matter if you use the segment ⁽²⁾ or the segment ⁽³⁾ so we have to remove those "reordering" as they are indistinguishable.	D+E	Int/a	R+B
10	Then $\frac{9!}{4!*3!*2!}$ it's the solution	-	Def	C

In the last part, Student-N recognizes the segments repetitions (R), and with a series of reasoning (B), he constructs the final combinatorial formula and the solution (C). The backward reasoning appears in its breakdown and cause-effect relationships features, the student analysing the generic path, realizes that to obtain the right value he has to quit the repeating cases. The *encapsulating* discursive devices characterize the recognizing of the combinatorial structure, the *reifying* device appears when the subject change and become the permutations number.

9.5.3 Discussion on Student-N resolution

From the HIM point of view, the backward reasoning develops in interrogative moves. The student answers questions like "How can I represent the paths sections?". The information obtained from the answers to this question allowed the student to solve the problem. They alternate interrogative moves followed by defining moves.

From the RBC point of view, the protocols are characterised by R-B-C and B-R-C chains. Observing the protocols, the students apply:

- One B-R-C chain (line 1-4) that lead the student to recognize the problem combinatorial structure;
- Two R-B-C chain (lines 5-10) that allow the student to construct the combinatorial formula.

The sequence of chains can be schematized as follow.

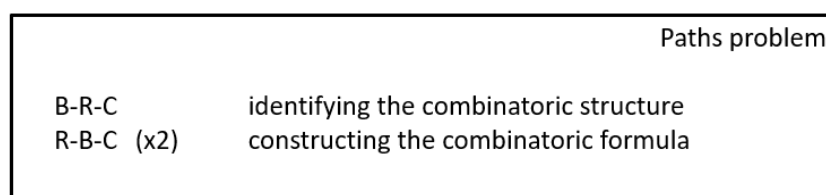


Fig. 9.26 - RBC flow (Student-T and Student-F Construction Problem resolution protocol)

Within the protocol there is an alternation between forward and backward reasoning. In the next sections the backward reasoning development during the combinatorial problem recognition is explored.

9.5.3.1 In-depth analysis: Identifying the combinatorial problem

The backward reasoning appears in its breakdown and cause-effect relationships research features. The students analyse the problem structure and then they search for necessary elements to construct the combinatorial formula. Some auxiliary elements are added to reach it. The *encapsulating* and *reifying* discursive devices appear while students recognize the problem combinatorial structure, they shift the subject sentences to impersonal ones. B-R-C chain appear to identify the combinatorial problem, instead, while the problem structure is recognized yet, the R-B-C chains appear. The students, in fact, explore the problem until recognize the combinatorial structure and conjecture the formula. Then, they recognize some sections properties, and, with a series of reasoning, they construct the final formula.

9.6 Discussion

The global analysis of the group has identified seven different backward reasoning moments through the four mathematical problem: Supposing identified the function and its derivative (Functions problem); Analysing the geometric configuration of the problem, and Expressing the relationships among the geometric configuration in algebraic language (Circle and Triangle problem); Analysing the sought geometric configuration and Identifying an analogy between the result of the problem and a known theorem (Construction problem); Analysing the generic path, and Identifying the combinatorial problem (Paths problem).

From the HIM point of view, this design experiment confirms what highlighted in the previous ones: the backward reasoning develops in interrogative moves. The students, in fact, ask (implicit or explicit) questions during the discovering processes making some interrogative moves. The ideas and conjectures emerge after a phase of exploration. The RBC flow connected to the backward reasoning moments varies and it is characterized by R-B-C, B-R-C and B-C chains. While B-R-C and R-B-C chains characterize the discovery moments, B-C chains are typical of transformative processes. The Paths problem is more characterized by B-R-C chains than the other problems. This could be due to the fact that students have had difficulties in recognizing its combinatorial nature. This led them to explore more the problem using different heuristics.

The nature of the problems leads students to behave differently using this type of reasoning. It is possible to distinguish four different behaviour throughout the four mathematical problems: supposing identified some problem elements, analysing the configuration, developing an algebraic formula, and identifying analogies with known problems. Each of them is discussed in the next sessions.

9.6.1 Supposing identified some problem elements

This behaviour characterises the first problem, concerning the study of the three functions graphs. It presupposes the knowledge of the relations between function and its derivative. When using backward reasoning, the students logically interact with the graphs assuming that two of them are related to each other as a function and its derivative. Then, they proceed

with the verification of the conjecture through what Arzarello and Sabena (2011) call “logic of not”:

“The strategy [...] is similar to the one of a chemist, who in the laboratory has to detect the nature of some substance. He knows that the substance must belong to one of three different categories (a, b, c) and uses suitable reagents to accomplish his task. For example, he knows that if a substance reacts in a certain way to a certain reagent it may be of type a or b but not c, and so on.” (p. 197)

The students, in fact, suppose the problem solved (generally for a pair of graphs) and analyse the relationships between the graphs and then verify their validity (cause-effect relationships research). The study of the relations takes place through the cause-effect relationship research with some moments of breakdown, where the logical relations and the graphs are analysed. The graph is then used to validate the conjecture developed previously: the students search for some elements to confirm the identified relationship.

These moments are characterized by B-R-C chains; *saming*, *encapsulating* and *reifying* discursive devices appear. In fact, students group the winning lines according to their geometric characteristics, then they recognize a pattern and later they make explicit the winning lines number for each group. In task resolution, this moment is useful for the winning strategy search and is preparatory for the mathematical formula expression. This is evident in the two case studies presented, although the episodes are not always developed in sequence.

9.6.2 Analysing the configuration

This behaviour characterises the second, third and fourth problems. It can be expressed join three different backward reasoning moments: Analysing the geometric configuration of the problem (Circle and Triangle problem), Analysing the sought geometric configuration (Construction problem); and Analysing the generic path (Paths problem).

The graphic representation strongly characterises these problems resolution, that starts with the representation of the geometric configuration. If it doesn't have enough elements to find some useful relationships or analogies, some students introduce various auxiliary constructions. This is an attempt to identify some regularities or properties known to them that would lead them to the solution of the problem. Then, through the breakdown of the construction elements, students are able to observe relations between segments that are then encapsulated in a more complex entity. Through this it is possible to express the problem in algebraic way or reconduct the problem to a known construction. The analysis of the given construction allows to search for rules/relations between elements and to formulate hypotheses that lead the solution. During the resolution development, new elements of different kinds are introduced: some graphic elements, that help in the identification of the relationships, or some theoretical elements (such as Pythagoras' theorem), that help in the elaboration of the relationships.

The RBC flow observed involves R-B-C and B-R-C chains. When the students relate the configuration with a known problem, the R-B-C chain appears. The students recognize some known elements and after some reasoning steps they reach a more complex configuration. When students can't relate the problem with a known one, the B-R-C chain appears. The students' research starts from considering different graphic configurations until finding the right ones that reveal the relationships sought for the development of new configurations. The *encapsulating* discursive device is always involved in the resolution, and sometimes is preceded by the *saming* one. The *reifying* device appears only when the students go toward a mathematical formula development.

9.6.3 Developing an algebraic formula

This behaviour characterises the second problem, concerning the study of the isosceles triangle with maximum area inscribed in a circle. Even if it is possible to solve the problem in a geometric way, passing through analytical or algebraic expression is easier. The latter was the way chosen by the students. In these processes backward reasoning is strictly

connected with its forward counterpart. The mathematical formula hides both reasoning natures.

The students explore the geometric configuration and transform its elements in algebraic language. Studying the relationships between them, the students can represent a mathematical formula and calculate its maximum value. Some students suppose/understand that the triangle with the maximum area inscribed in the circumference is the equilateral one, so they know to which value have to arrive after the formula manipulations. The verification of the conjectures develops on the formal-algorithmic plane.

As specified in section 9.3.1.2, this process is subdivided in two moments, both characterized by the transformative and breakdown features: expressing the mathematical formula and manipulate it. The main difference between the two moments are the chains development: R-B-C for the first moments and B-C for the second. The students identify geometric elements, then, reasoning about the relationships within the geometric construction, they represent it in an algebraic way. Later, the students manipulate the formula. The first part of the process is characterised by *saming*, *encapsulating* and *reifying* discursive devices, while the second one only by the *reifying* one. While in the first part in the second the students recognize some geometric relations, they identify entities, and then represent them in algebraic language, in the second only algebraic language transformation appears.

9.6.4 Identifying analogies with known problems

The fourth behaviour was found in the third and in the fourth mathematical problem. It corresponds to Identifying an analogy between the result of the problem and a known theorem (Construction problem), and Identifying the combinatorial problem (Paths problem).

During the resolution of these problems, some students identified a possible known auxiliary construction that would help them to solve the problem. To reach it, the introduction of auxiliary elements is crucial. It makes possible to have different geometric elements available which, analysed allow us to formulate hypotheses on the problem. The cause-effect

relationships research feature characterised these moments. The students, in fact, reaching the construction, search for previous necessary elements.

The RBC flow observed involves R-B-C chains for the Construction problem and R-B-C and B-R-C chains for the Paths one. In fact, when the students relate the resolution with a known problem, the R-B-C chain appears, while the B-R-C chains characterises the next exploration moments. The analogy identification is generated by a recognizing of some known elements/problems/configurations. The reasoning steps allow the students to reach some useful elements or a combinatorial formula. In these moments *saming*, *encapsulating* and *reifying* discursive device appear.

III

RESULTS, DISCUSSION AND CONCLUSIONS

PART III – RESULTS, DISCUSSION AND CONCLUSIONS

This last part of the dissertation consists of 2 chapters: *Results: general discussion*, and *Conclusions*. In the previous parts, the theoretical elements, the research design, and the design experiments analysis have been discussed. In *Results: general discussion* (Chapter 10) a comparison of design experiments analysis and results is developed; from it, eleven indicators of the structure of backward reasoning emerge: the Backward Reasoning Indicators (BRI). Following the observations around the BRI, the conclusions about the research project objective and its methodological and didactical dimensions are discussed in the last chapter (*Conclusions*, Chapter 11).

As in previous chapters, for each one, a Table of Contents is shown to help the reader in approaching the chapter.

10

RESULTS: GENERAL DISCUSSION

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RESULTS: GENERAL DISCUSSION 10

In chapters 6, 7, 8 and 9 the students' productions of the four design experiments were analysed using a multidimensional analysis tool, obtained through the networking of GTL (Hintikka, 1999) with AiC theory (Dreyfus, et Al., 2015) and hybridizing the so produced theory with the Commognition approach (Sfard, 2008). The new framework allowed to identify the crucial backward reasoning moments and students' productions were consequently analysed and classified. At the end of each chapter some specific results, drawn from those analyses, were pointed out.

In this chapter, a general discussion of the outcomes obtained in the four design experiments is developed. Commonalities and differences between them are carefully considered: the main result of this overall analysis is the characterization of the backward reasoning typologies through a list of eleven basic indicators.

To achieve them, a comparative analysis of the design experiments results is developed following three different approaches:

- *Global approach (vertical diachronic analysis)*: the attention is focused on each resolution protocol as a whole and a recurring pattern is sought *along* the different analysis models by comparing them and observing where and how the backward reasoning intervenes.
- *Local approach (horizontal synchronic analysis)*: the attention is focused on single lines of the protocols in which the backward reasoning appears. Structural similarities are looked for *across* the different analysis models and, according to them, a structural multidimensional classification is elaborated (table 10.8).

- *Strategic approach (local + global)*: the attention is focused on the way backward reasoning is used strategically by students to solve the proposed problem; the connections with the two previous approaches are put forward.

Each approach is illustrated and discussed separately in the chapter:

- Section 10.1. Through the global approach the dynamic nature of backward reasoning is highlighted. Three different discussions are developed: first, the results concerning the FLIM model (Chapter 6) in the first design experiment and its abandonment in next ones (section 10.1.1); then, the particular case of Maude task design experiment (10.1.2); finally, the different types of process chains identified through the RBC model flow (section 10.1.3).
- Section 10.2. The local approach allows to observe backward reasoning in some specific resolution moments. Also in this case three discussions are developed: the first one concerning the interrogative phases (section 10.2.1), the second one about the introduction of auxiliary elements (section 10.2.2), and the third one about the time duration of backward reasoning processes (section 10.2.3).
- Section 10.3. Through the strategic approach the backward reasoning moments are grouped according to three different types of students behaviours: when they go backward while solving a problem (section 10.3.1), when they encapsulate objects (section 10.3.2), and when they use algebra (section 10.3.3).

In each section some excerpts from Chapters 6-9 data are recalled in order to exemplify the carried-out observations.

In the final part of the chapter (section 10.4) a discussion of the results is developed giving a characterization of the backward reasoning cognitive features and the eleven Backward Reasoning Indicators are finally pointed out. This issue can be considered the main general result of the entire research project. The discussion concerning how it allows answering the research questions of this investigation (see Chapter 4) will be developed in next chapter.

10.1 Global approach

The global approach to the interpretation of data in the different design experiments has led to point out a first group of three articulated features which characterize the backward reasoning process. The diachronic analysis of the resolution protocols made it possible to obtain a "film sequence" of the backward reasoning process in which it was possible to identify repeating pattern. In this section the different results obtained during the trials will be explained. Some examples of this patterns are displayed to better identify the features obtained.

10.1.1 A-N-D chains: lack of specificity

The analysis with the Hintikka's Interrogative Model (HIM) (Hintikka, 1999), developed along chapter 6 to 9, for all the design experiments, allowed to make a general distinction between what is strategic and backward and what is not; the other analysis lenses, instead, allowed to obtain more detailed data on the features of backward reasoning. Each student, as a player involved in the resolution, develops a series of moves to solve the game (or the problem) that are identified and characterized from an epistemological and strategic point of view through HIM. The sequence of moves used strongly depends on the type of person involved in the game. Therefore, no recurrent sequences have been identified in the data obtained from the analysis with the HIM, confirming what Hintikka and Remes (1974) said about the analytical method: it can not be mechanized as a discovery procedure because of the necessity to introduce countless unpredictable auxiliary constructions.

Depending on the situation and the person involved, different sequences of HIM moves appear, they are not comparable with each other. On one hand, the characteristics of the task to be solved influences the sequences. On the other hand, they depend on the personal paths of the solver's thought: the type of reasoning he uses, the analogies found and the subject's ability to focus attention on visual-spatial elements or algebraic ones. This classification allows to have a general picture of the specific situation for a specific subject, but not of a global pattern applicable to different subjects enrolled in different situations. The analyses in chapter 9 show that two or more students, dealing with the same situation, draw up the

protocol in a different way; common elements were found through a synchronic analysis, for example that backward reasoning develops mainly in questioning movements (see section 10.2.1 below).

To make up for this lack of HIM, during the first design experiment, it was decided to try to use the Finer Logic of Inquiry Model (Soldano, 2017) in conjunction with the HIM. From the FLIM point of view, it is possible to see a series of chains of actions and chains of cognitive modalities that are already repeated within the same resolution protocol. The cognitive analysis of the case study shows that the first two resolution phases (where the backward reasoning appears) are characterised by a continuous alternation of explorations and plan formulations together with an alternation of descending and ascending modalities. It is so possible to detect typical routines in solution processes, represented by successions like $A \sim N \sim D \sim (A \sim N \sim D \sim (A \sim \dots))$, where a neutral modality (N) marks the transition between an ascending (A) and a descending (D) modality and is possibly accompanied by the incorporation of auxiliary constructions as generating tools of new knowledge. For example, in Triangular Peg Solitaire task (chapter 6), the subdivision of the board into rows and then into triangles is fundamental to reach the solution: student-M, the case study student, modifies the strategy slightly by adding new elements in the resolution (board subdivision into rows and triangles). Crucial points of backward reasoning are reached in the ascending modality (see Table 6.6 in Chapter 6), where main ideas generally occur.

Analysis with the FLIM model allows to model student's cognitive movement in a logical concatenated way. The strategic aspects are more dominant in the ascending and descending modality, while the epistemic ones are prevailing in the neutral modality. These results confirm those obtained by Soldano (2017) (with upper secondary school students in geometry): the ascending modality characterises the backward way of thinking, while descending is the cognitive modality that characterises the forward way of reasoning.

Through this tool it's possible to emphasise that backward reasoning involves auxiliary intuition elements that are necessary to achieve the solution; these aspects are developed by looking at the consequence and looking for the premises. At a phenomenological level, this method allows to analyse the development of cognitive modality movements to reach the solution, but it doesn't distinguish between the strategic principles that are used. For this

reason, it has been chosen not to further deepen the study of backward reasoning using this analysis tool. To advance in the development of the analysis model, the HIM was first interpreted within a task involving three resolution contexts, and then the HIM was coordinated with the RCB-model based on Abstraction in Context theory (Dreyfus et al., 2015).

10.1.2 Maude task: a different interpretation of HIM

During the second design experiment, the protocols were first analysed according to Hintikka's interrogative model. In this design experiment, an interpretation of the HIM within three resolution contexts is carried out. In fact, the task, consisting of implementing the Triangular Peg Solitaire game in Maude software, involves the informal context (the context of the game), the mathematical context, and the computational context. Based on triangulation of the data from the information sources (resolution protocols, video recording and direct observation during the session), two main categories of difficulties were identified in the group composed of 15 students of the master's degree in computer engineering: factual mistakes and methodological mistakes. They have been classified like: *Completeness problems and Behaviour problems* as for factual mistakes, *Description problems, Estimation problems and Transference problems*, as for methodological errors (see Chapter 7). These difficulties have therefore been highlighted in the analysis of the resolution protocols. In particular, two excerpts were analysed in detail: one in which two students participate in the discussion, the other in which the intervention of an experienced student takes place.

10.1.2.1 Interconnections between contexts

The analysis of excerpts showed that the backward reasoning develops mainly in interrogative moves (see section 10.2.1 to further details). Focusing on the interpretation of the HIM within the three resolution contexts present in the task, it was noted that the interrogative moves develop in the transition from one context to another, in particular from the computational context to the mathematical one or from the computational context to the

informal one (the game context) passing through the mathematical context. The backward reasoning appears in the creation of computational elements in Maude programming language and it is essential. Students focused on the objective of creating the element (a list or a rewriting equation), looking for the necessary elements/backgrounds for its formal construction. They start looking at the computational context and then they go backwards through the mathematical one until the informal context. After finding the necessary elements in the informal context they translate them into the mathematical context and then they implement them within the computational one.

When learning rewriting logic, the transference of reasoning between different contexts, informal, mathematical and computational, is essential. The initial moves (formulation of theses, conjectures or objectives) are generated mainly in the informal context to finish in the computational/mathematical one. The resolution proceeds with a back and forth movements (confirming what Gómez-Chacón, et al., 2016 affirms) between the context until the solution is reached in the computational one. The movements towards the mathematical context facilitate the transitions between the others. The transition through the mathematical context is necessary to firstly develop a system of signs, and secondly to understand the interaction among the different elements involved. The mathematical knowledge, through the use of algebraic and logical properties, allows to “translate” game properties into computational elements. The backward reasoning, which is based on the return of reasoning to the informal context, helps to connect more intuitive aspects with the mathematical and computational context. The major difficulties, highlighted by students, are generated right in the transitions between the contexts.

10.1.2.2 The ordering device

The global analysis of the second excerpt from the second design experiment case study, highlighted how backward reasoning is used in its character of “ordering device” (Pekhaus, 2002). In this excerpt a student (Student-E), with a more advanced knowledge and a role of expert, interacts with two classmates, that worked in pair, helping them in the task resolution.

Generally, the explanation by an expert is developed in deductive terms; the expert mentions the premises, applies some deduction rules (for example modus ponens) and reaches a conclusion. This is given by the fact that the solution discovery process, with the characteristic back and forth movements, is not made explicit. The interpretation of the HIM through the three context shows the reasoning of student-E. Globally, she starts from the informal context, then she passes through the mathematical context, until reaching the computational one. But the reasoning is not linear. Facing the classmates' difficulties, Student-E's exposure evolves and moves away from pure deduction. In fact, during the explanation, some classmates factual and methodological obstacles came out, and, to solve them, she has to go back and forth through the contexts making explicit what they do not see. In these moments, Student-E shows her own construction process of knowledge, highlighting the movements that she used in her discovering process. The backward reasoning is used like an ordering device: she reasons with interrogative-backward moves highlighting the succession of phases in the resolution. She does a mediation between her knowledge and her resolution process. In this way her classmates understand the task resolution. Backward reasoning is used as a communicative tool to interpret how the understanding of a concept occurs in novice's thinking.

10.1.3 Processes chains

The analysis with the AiC model made it possible to identify a series of chains of epistemic actions that have been highlighted in the previous chapter. The resolution protocols are characterized by three different types of chains: B-R-C, R-B-C and B-C. B-R-C and R-B-C chains appear in the discovering processes while B-C chains are predominant in the processes of verification or construction of mathematical concepts. The B-R-C chains are typical of discovering processes that cannot be traced back to previous problems. R-B-C chains, on the other hand, are typical of discovering processes where analogy comes into play.

10.1.3.1 B-R-C and R-B-C chains: the power of analogy in discovering processes

In the previous chapter, the AiC model was used for the elaboration of diagrams representing the evolution of task resolution. The different tasks were divided into phases because of the analysis. Observing the phases that characterize the discovering processes in which backward reasoning is involved, the B-R-C chains can be seen in correspondence of inquiry moments. These chains in fact characterize the moments when students are faced with a problem they have never seen before, or that they don't recognize at that moment. This leads students to have a moment of exploration at the beginning of the process (building-with); then, they recognize a concept, or a structure, in their background that can be useful for the resolution. This finally leads them to the construction of a new concept. See, for example, the first part of the 3D Tick-Tack-Toe resolution protocol of student-A (Chapter 8, section 8.2.1) when focusing on counting the winning lines in 2D board. He starts to explore the board property according to its geometric characteristics, then he recognizes a pattern in the squares board, and finally, he calculates the inning line numbers based on the geometric properties of the boxes.

“Line 9.1 I start to quantify how many lines I cancel out the opponent with each move.

Line 9.2 The centre cancels out more than any (4), vertices 3 and edges 2. That justifies the heuristics.

[...]

Line 10.2 I calculate 8: 3 in each of the two directions and two diagonals.”

Protocol Student-A	Backward reasoning	HIM	AiC
9.1	Breakdown	Interrogative	Building-with
9.2	Auxiliary Elements	Assertoric	Recognizing
...
10.2	Breakdown	Deductive	Constructing

Tab. 10.1 – Student A (3D Tick-Tack-Toe)

The sequence starts with an action in which the student builds(-with) knowledge about the game learned up to that point. Then the student recognizes a pattern in the boxes on the board. By putting together the previous notions and the pattern, he constructs the new concept: the number of winning lines on the board. He will use this notion in the following sequences until he reaches the mathematical formula. He is working on the winning lines that are at the end of the problem. The backward reasoning emerges in the breakdown of the board geometric properties and in the introduction of the auxiliary scheme. The breakdown is extended over time while the introduction of the auxiliary scheme is a momentary application.

A second example taken from the problem section (Chapter 9, section 9.3) shows a different behaviour. It is an excerpt taken from student-G resolution from the triangle and circle problem. The student starts solving the problem by representing the data. Then she inserts within the represented circumference several isosceles triangles building(-with) some notions and observing the drawings. Finally, she represents the height of the triangle as the radius of the circumference plus a certain unknown quantity x (see fig. 1.1). This phase characterizes the discovering processes in which backward reasoning is involved: the R-B-C chains can be seen in correspondence of a moment where an analogy with previous problems comes into play.

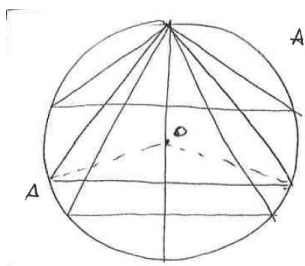


Figure 1.1

- Line 1.1 I draw several isosceles triangles inscribed in a circle.
- Line 1.2 I notice that I can describe the height of the triangle from its base as the ray of the circle plus an x quantity.”

Protocol Student-G	Backward reasoning	HIM	AiC
Figure 1.1	Breakdown	Interrogative	Recognizing
Line 1.1	Auxiliary Elements	Interrogative	Building-with
Line 1.2	Auxiliary Elements	Assertoric	Constructing

Tab. 10.2 - Student G (Triangle & Circle Problem)

The sequence starts with the recognizing of the configuration of the problem. It's not the first time the student sees a geometric problem: so, she immediately starts drawing different configurations (Figure 1.1). Then she builds(-with) some previous concepts and explorations until she notices a relationship between some elements of the figure. It's a discovering phase different from that in the previous example. Here the background notions arise before the exploratory phase and they are put into play immediately. After they have been included in the resolution, the background concepts are manipulated basing on the problem and new concepts, useful for the resolution, are constructed.

While B-R-C chains are typical in game discovering phases, R-B-C chains are typical in problem discovering phases. The approach to solving games is different from that of problems. Both, games and problems, had never been seen before by students. But, while they had to explore games from the beginning, without any previous notion, in solving problems, the analogy with previously seen problems or with notions belonging to the mathematical background was immediately triggered.

This statement is justified also by the analysis of the Paths Problem. This problem created some difficulties for most students, because it was hardly recognized it as a combinatorial problem. These students, therefore, began to explore the problem without activating knowledge in the background, as they had done for games. Some students solved the problem by following the different explorations, others, later, recognized that the problem was a combinatorial one by activating the background knowledge. The example of the student-N (Chapter 9, section 9.5.2) is an example of this second case. In the following excerpt, representing the protocol discovering phase, there are two chains in sequence, a B-R-C and then an R-B-C chain.

“Line 3.1 Let's look at the sections

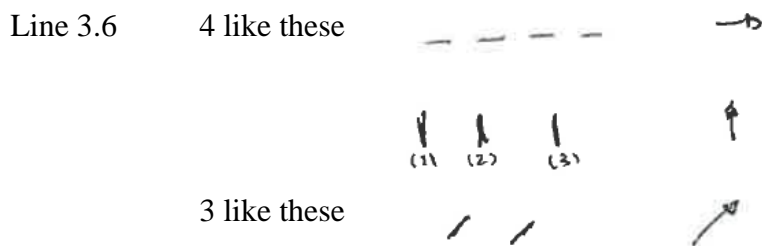


Line 3.2 I can think of many ways to count them, but none of them are good enough to keep me from getting caught up in the process. There is not a way that fits on the sheet.

Line 3.3 I've realized that it's a combinatorial problem in a totally random way. I think it's going to be a good answer.

Line 3.4 $\frac{9!}{4!*3!*2!} = \frac{9*8*7*6*5}{3*2*2} = 9 * 7 * 5 * 4$ This is the answer that looks like the correct one.

Line 3.5 We have 9 segments:



2 like these

Line 3.7 There are 9 way to order them however you have to pay attention...

Line 3.8 No! It's 9!

Protocol Student-N	Backward reasoning	HIM	AiC
Lines 3.1-2	Breakdown	Interrogative	Building-with
Line 3.3	Auxiliary Elements	Interrogative	Recognizing
Line 3.4	Auxiliary Elements	Assertoric	Constructing
Line 3.5		Initial	Recognizing
Line 3.6	Breakdown	Interrogative	Building-with
Lines 3.7-8		Answer	Constructing

Tab. 10.3 – Student-N (Paths Problem)

This sequence starts with an exploration of the problem. The student doesn't recognize the problem combinatorial structure. He starts breaking down the problem in small parts (Building-with). Then he recognises the combinatorial structure and conjectures a solution (constructing). This first part is characterised by a B-R-C chain: the student is exploring a problem that he has never seen before and that he can not relate with other problems he solved in the past. After identifying the problem like a combinatorial one, he starts to solve it using the notions in his background. First of all, he recognizes that there are 9 segments in the path. Then, he represents the groups into which these segments are split (building-with). And finally, he gives a first result. This second part is characterised by a R-B-C chain. The student is working at the problem by analogy. He is trying to reconduct its resolution to a combinatorial problem whose solution he knows.

We can say that, when the solver recognises the problem structure, he produces a sequence of R-B-C chains. In fact, he typically produces a sequence of actions like this:

1. recognition of the structure of the problem and identification of an analogy with a previous solved problem;
2. search-exploration-manipulation of the objects-data-elements-concepts given by the problem based on the identified analogy;
3. manipulation of the objects at point 2. in order to build a new element-concept.

On the contrary, if he is not able to identify the problem the produced sequence is of type B-R-C and the sequence of his/her actions is typically:

1. search-exploration-manipulation of the objects-data-elements-concepts given by the problem;
2. recognition of an element-concept useful for the resolution;
3. manipulation of the objects at point 1. with the new element introduced (or focusing on the element recognised) in order to build a new element-concept.

These findings correspond to Polya's analysis of analogy (1945 and 1954): he defines the analogy as that resolution strategy in which the solver connects the problem to be solved to a similar one, i.e. to a problem that has certain similar characteristics. He also identifies three ways to apply the analogy: using the method of the analogous problem, using the solution of the analogous problem, or using both. The R-B-C chains appear when a student uses this type of strategy. For example, the student G (above) has recognized the problem as geometric and she started by drawing the configuration, a method learned in her educational career; the student-N (above) has traced the problem back to a combinatorial one and he has solved it using its method and solution.

But that's not all. This type of chains appears also when there are certain elements of the problem that can trigger some memories in the student. This is the case of the Functions Problem. The data of the task make students immediately take some notions from their background: that is the definition of derivative and the relationships between derivative and function. The following excerpt shows the first R-B-C chain of the Group 2 solution process (Chapter 9, section 9.2.1).

- Lines 1.1-3 Fe *If this was the function...* [points graph 1] *Its derivative should increase...*
- Line 1.4 Fe *But these ones are decreasing* [points at graph 2 and 3] *I mean...this one is increasing a little bit...*[points at graph 2] *but not enough* [she moves along graph 1 and shows the difference of increase between graph 1 and 2]
- Line 1.5 Ma *Ehm no... in fact...so it is not right...let's try to start with another one...*

Protocol Group 2	Backward reasoning	HIM	AiC
Lines 1.1-3	Cause-Effect Relationship	Initial	Recognizing
Line 1.4	Cause-Effect Relationship	Interrogative	Building-with
Line 1.5	-	Assertoric	Constructing

Tab. 10.4 - Group 2 (Function Problem)

This sequence starts with the definition of the relationship between the function and its derivative. The students recognise some elements in the problem's data that make immediately explicit the relationship between function and derivative, which are present in their background. Then they reflect on this relationship applied to the problem graphs and make explicit their first conjecture: the graph 2 function is not the derivative of the graph 1 function.

10.1.3.2 B-C chains

B-C chains appears in both discovery and verification phases of the resolution. These chains are typical of the latter phases, while they appear only in some specific points in the first one. It is possible to notice some moments of the resolution protocols discovery phases where the B-C chains exist and are in correspondence of backward reasoning. In verification phases, instead, only forward reasoning and B-C chains are developed. Some examples are shown in the following sections to clarify the differences between these chains and the B-R-C and R-B-C chains.

10.1.3.2.1 B-C chains: manipulation processes

In the protocols analysed, two types of processes involved backward reasoning and B-C chains: processes of construction of mathematical concepts, and processes of transformation and manipulation from geometric to algebraic language.

There is a clear example of construction of mathematical concepts in the protocol of Student-B (3D Tick-Tack-Toe) (see chapter 8, section 8.2.2). In this excerpt he is constructing the

general mathematical formula that expresses the number of winning lines for a size n cubic board.

“Line 5.2 We'll have n planes with n boxes each.

Line 5.3 It is clear that horizontally we have n winning columns n winning rows and two diagonals.

Line 5.4 $2n+2$ winning lines for each plane

Line 5.5 $n(2n+2)$ horizontally winning lines.

Line 5.6 Vertically, we separate them [the winning lines] by 3: corners, edge without being corner and interior.

Line 5.7 Corners:

It is clear that there are still 4 corners in the 1st and in the last planes and...

Line 5.8 ...therefore 4 lines pass through the planes \Rightarrow 16 lines.

Line 5.9 Edge without corner:

Two lines pass through each box and...

Line 5.10 there are $n-2$ boxes on each side of the plane \Rightarrow $8(n-2)$ winning lines.

Line 5.11 Inside:

Only one line passes through each box and..

Line 5.12 .. there are $(n-2)(n-2)$ boxes

Line 5.13 Total vertically = $16 + 8(n - 2) + (n - 2)^2 = (n + 2)^2$

Line 5.14 Total = total horizontally + total vertically =

$$= n(2n + 2) + (n + 2)^2 = 2n^2 + 2n + n^2 + 4n + 4 = 3n^2 + 6n + 4$$

Protocol Student-B	BR	HIM	AiC
Line 5.2	Breakdown	Interrogative	Building-with
Line 5.3	Breakdown	Answer	Building-with
Lines 5.4-5		Deductive	Constructing
Line 5.6	Breakdown	Definitory	Constructing

Line 5.7	Breakdown	Interrogative	Building-with
Line 5.8		Deductive	Constructing
Line 5.9	Breakdown	Interrogative	Building-with
Line 5.10		Deductive	Constructing
Line 5.11	Breakdown	Interrogative	Building-with
Line 5.12		Deductive	Constructing
Line 5.13		Deductive	Building-with
Lines 5.14		Rules	Building-with Constructing

Tab. 10.5 – Student-B (3D Tick-Tack-Toe)

After having conjectured the existence of a general formula, the student constructs it basing his reasoning on the previously developed scheme. The student, in fact, has classified the winning lines according to their geometric properties. He is breaking down the winning lines and associating a certain “n-dependent value” to each group of lines. For each group, he builds-with his knowledge and he constructs a formula. At the end, he puts together the obtained formulas to construct the final general mathematical formula. The representations are manipulated in order to construct the sought mathematical object. The actions of Building-with and Constructing are performed by forming five B-C chains. Each chain is related to a group of winning lines except the last one that is related to the general formula. A very similar example can be found in the excerpt of the Student-A (3D Tick-Tack-Toe) resolution protocol considered in section 10.2.3.

B-C chains also appear during the transformation processes from geometric to algebraic language and in the algebraic language manipulation processes. An example can be found in the excerpt of student-G (Circle and Triangle Problem, Chapter 9, section 9.3). The student, after breakdown the figure and identifying a specific triangle, manipulates the values of the segments in an algebraic form.

Line 16 I come back to the area formula: $\sqrt{r^2 - x^2} * (r + x)$

Line 17
$$\frac{d}{dx} [\sqrt{r^2 - x^2} * (r + x)]$$

$$= (+1)\sqrt{r^2 - x^2} + (r + x) \left[\frac{1}{2} (r^2 - x^2)^{-\frac{1}{2}} (-2x) \right] =$$

...

$$= \frac{(r + x)[r - x - x]}{\sqrt{r^2 - x^2}} = \frac{(r + x)(r - 2x)}{\sqrt{r^2 - x^2}}$$

Line 18 I'm looking for the maximum $\frac{d}{dx} f(x) = 0$

Line 19
$$\frac{(r + x)(r - 2x)}{\sqrt{r^2 - x^2}} = 0$$

Line 20 $r + x = 0 \quad x = -r \quad \text{but} \quad -r < x < r$

$$r - 2x = 0 \quad x = \frac{r}{2}$$

Protocol Student-G	BR	HIM	AiC
Line 16	Breakdown Transformative	Interrogative	Building-with
Line 17	-	Rules	Building-with Constructing
Line 18	Breakdown Transformative	Interrogative	Building-with
Line 19	-	Rules	Building-with
Line 20	-	Interrogative	Constructing

Tab. 10.6 - Student-G (Triangle and Circle Problem)

This excerpt shows the manipulation of algebraic representations of geometric objects. The algebraic transformations allow the student to reach his goal that is to identify a relation between some segments of the configuration: this determines the triangle searched for. The actions of Building-with and Constructing follow one another by forming two B-C chains. The first chain is related to the resolution of the derivative of the function “area of the triangle”; the second one is related to its maximization.

When the backward reasoning develops in correspondence of manipulative processes, the B-C chains appear. Backward reasoning, in these cases, is not continuous but it is interrupted

by moments of forward reasoning. While backward moments correspond to Building-with actions, forward moments correspond to Constructing actions. The solver transforms known notions in a backward way, then he uses them to progressively build new concepts. Mathematical concepts are strongly involved when these chains appear.

10.1.3.2.2 B-C chains in verifying processes: no backward reasoning

Polya (1945) calls "Looking back" the phase where the results, that have been obtained in the previous phases, are verified. It corresponds to the last part of the problem solving (see Chapter 2, section 2.2.1). Looking at the problem, according to the subdivision proposed by Hintikka and Remes (1974) (see Chapter 3, section 3.7), this correspond to the third phase, the synthesis phase; here, deductive logical inferences are developed in order to reverse the passages of the analysis and get a justification for the resolution. In this part of the protocol, backward reasoning is absent. In fact, this part is characterized only by forward processes.

Due to the nature of the proposed problems, there are not long excerpts protocols related to this resolution phase. In the Triangle and Circle problem, for example, an algebraic language is used. The nature of algebraic manipulation implies the reversibility of the formulas, and, therefore, includes in the same process the phase of verification of the results. It is possible to found small parts in some protocols concerning the games and the Function problem.

In the few excerpts related to this phase, B-C chains are interrupted in some points by Recognizing actions that allow to introduce structural analogies. Through these, difficulties and errors made in the resolution are detected and overcome. There is a short verifying phase in the protocol of Group 3 (Functions Problems). Group is not shown in the Chapter 9 analysis but solves the problem in a very similar way to Group 2 (Chapter 9, section 9.2). In this excerpt the students verify that the functions identification, developed in the previous phases, is correct.

Line 1.14 Pa *We name the functions $F, f e f'$*

Line 1.15 Gi *We check them: f must be a derivative of the primitive, right? ...*

Line 1.16 Pa *Here it is increasing [she points along the graph 1] until here..*

Line 1.17 Gi *And here it is positive [she points along the graph 2]*

- Line 1.18 Pa *Here it is decreasing* [she points the graph 1]
- Line 1.19 Gi *Here it is negative* [she points the graph 2] *and here it is increasing* [she points the graph 1] *and here it is positive* [she points the graph 2], *ok, all right. Maybe we draw..* [with his hand she goes through an imaginary vertical line that passes through the notable points of the functions]
- Line 1.20 Al *Yes... at the minimum point the derivative is equal to zero, we're good.*
- Line 1.21 Gi *So it's right... and then f' is the derivative of f , because this one* [she points the graph 2] *decreases up to here* [she goes through the function and she stops at the minimum point] *and the other one is worth zero, and then it grows and the other one is positive.*

Protocol Group 3	BR	HIM	AiC
Line 1.14	-	Definitory	Recognizing
Line 1.15	-	Interrogative	Recognizing
Lines 1.16-18	-	Answer	Building-with
Line 1.19	-	Answer	Building-with Constructing
Line 1.20	-	Answer	Building-with
Line 1.21	-	Answer	Constructing

Tab. 10.7 - Group 3 (Function Problem)

Two B-C chains confirm the result obtained in the previous phases. In the first one the students verify that f is the derivative of F , while in the second one that f' is the derivative of f . The reasoning is forward. The students put together their knowledge to construct the mathematical concepts step by step and thus verify them. A very clear example of a similar process it can be found in Chapter 8, section 8.2.1.2.2. It is the part 2.2 of the Student-A (3D Tick-Tack-Toe) resolution protocol. Here he verifies the mathematical formula that emerge in the previous phase, justifying it formally. He builds(-with) some notions to formally construct the general formula step by step.

10.2 Local approach

The local approach to the interpretation of data in the different design experiments has led to point out on a second group of three articulated features which characterize backward reasoning. The synchronic analysis of the resolution protocols made it possible to obtain some "photos" of the backward reasoning process in which it was possible to identify common characteristics. In this section the different results obtained during the trials will be explained. Some examples of this "photos" are displayed to better identify the features obtained.

10.2.1 The interrogative phases

The local approach of data observation in the first design experiment allowed to identify a correspondence between the use of backward reasoning and the interrogative moves (HIM analysis). Furthermore, it was noted that this type of reasoning develops mainly in the exploratory phases of the protocol when the subject is in ascending cognitive modality (FLIM analysis). This can be easily observed from a horizontal reading of the summary table 6.6 (Chapter 6, section 6.2) of student-M Triangular Peg Solitaire resolution protocol, a small extract of the table is reported: line 6 of the protocol.

“Line 6 *At this point, I note that the only way to eliminate [the peg in position] I would be to move 8-5-3.*”

Protocol Stud-M	BR	HIM	FLIM		AiC
		Moves	Actions	Modalities	
...	
Line 6	Going backward Cause-effect relat.	Interrogative	Exploration	Ascendant	Recognizing
...	

Tab. 10.8 – Student-M (Triangular solitaire)

Focusing on the interpretation of this excerpt according to HIM and FLIM, this line has been classified as an interrogative move, during an exploratory phase, in ascending cognitive

modality. As specified in Chapter 3, this type of interrogative actions is closely connected to a question formulation. This inquiry, explicitly or not, is developed by the solver just before the interrogative action. In this specific example the question, implicit, is: "How can I remove peg 1?" This type of question is associated with the general "What should I consider to get ...?" (Ruesga Ramos, 2004) that are identified in the literature as specific to backward reasoning development (see Chapter 3, section 3.4)

Despite the different analysis developed in the evolution of the research project, other examples are identified in each kind of protocols. For example, in the second design experiment the students P and D, during the implementation of the pegs position in Maude software (Chapter 7, section 7.2.1), focused on the objective of creating the list, looking for the necessary elements and backgrounds for its formal construction.

“Line 10 Student-P: [...] *You have to draw two numbers, right?*”

Line 11 Student-D: *A pair that has two positions and we represent it in that way and then we use..*

Line 12 Student-P: *...the peg...*

Line 13 Student-D: *Ah.. Whether it's taken or not. [...]*”

Protocol Excerpt 1	BR	HIM Moves	Context	AiC
		...		
Line 10	Solution formulation	Interrogative	Mathematical	Recognizing
Line 11 – 13	Breakdown	Answer	Informal	
...

Tab. 10.9 - Excerpt 1 (Maude task)

In this case the lines, according to HIM, have been classified as interrogative move. The students were in an exploratory phase of the resolution. The questioning action transits from the computational context to the informal context through the mathematical one. The

interrogative movement can be broken down into two parts: the explicit question asked by student-P in line 10 and the answer in the following lines.

Given the large number of protocol lines classified in a similar way along the four design experiments, it can be said that backward reasoning emerges during exploratory phases in correspondence of questioning processes in which the student formulates conjectures or explicit ideas. The backward reasoning develops when one player asks a question to the other (or to the oracle – the game or the problem) who answers it. Cognitively it is part of the path towards the formation of ideas and conjectures after a phase of exploration (ascendant modality – FLIM analysis). It is essential to ask an appropriate question (Solow, 1990) to extract information from the subject's background of knowledge. A good question allows the subject to formulate premises for certain statements, or in combination with certain statements to draw some conclusions.

10.2.2 Recognizing auxiliary elements

Already in the in-depth study of the literature (see Chapter 3) the importance of introducing auxiliary elements during the resolution of problems where backward reasoning is predominant had emerged. The analysis of the resolution protocols of the four design experiments shows that the introduction of auxiliary elements is crucial to achieve the solution; these unknown objects are brought to reality by looking at the consequences and looking for the premises.

During the first analysis of the first design experiment it was noted that the backward reasoning appeared in correspondence with the introduction of new ideas in the resolution. For example, in line 9 of the protocol, the student-M (Chapter 6, section 6.2) is reasoning regressively since the end of the problem. She introduces here a classification of the board positions according to how many jumps the pegs can make. This new element allows to distinguish "normal" from "favourable" positions for the beginning of the backward resolution. This is a key point for the resolution and the achievement of the solution.

“Line 9 Looking at the board, I think that maybe the fact that the last piece stays on the board (the peg from which I start to move backwards), in a position that you can come up with many jumps, facilitates the strategy. These places are positions 4, 6 and 13 because you can get to them with 4 jumps.”

Protocol Stud-M	BR	HIM	FLIM		AiC
		Moves	Actions	Modalities	
...	
9	Breakdown	Interrogative	Exploration	Ascendant	Recognizing
...	

Tab. 10.10 - Student M (Triangular Solitaire)

The introduction of the AiC model has allowed a refinement of the analysis. Thanks to the subdivision of the protocol into epistemic actions it was possible to identify more finely the backward reasoning and classify it with respect to the dimensions identified in literature (see Chapter 3). With the local data interpretation approach it was possible to focus on moments of introduction of the auxiliary elements and to understand how they affect the resolution. See, for example, the student-A (Chapter 8, section 8.2.1.2.1) in the second part of the 3D Tick-Tack-Toe resolution when focusing on counting the winning lines. He divides the board (a cube) according to its geometric subspaces (the planes that make it up) considering, for each plan, the winning lines.

“Line 12.3 I make a few drawings to test.

Line 12.4 There are 10 lines in each plan parallel to the axes ...

Line 12.5 ... and there are 12 planes parallel to the axes.

Line 12.6 I lack the “diagonal lines” as in the example. They seem more complicated.

Line 13.1 I'm starting to do numerology: $10=4*2+2$...

Line 13.2 ...which is broken down as the number of pawns per dimension of the plane plus two diagonals.

Line 13.3 Will it be general?

Line 14.1 I realize that $12=4*3$...

Line 14.2 ...that seems to follow the previous pattern.”

Protocol Student-A	BR	HIM	AiC
Line 12.4	Breakdown	Interrogative	Building-with
Line 12.5	Breakdown	Interrogative	Building-with
Line 12.6	Auxiliary elements	Assertoric	Recognizing
Line 13.1	Breakdown	Interrogative	Constructing
Line 13.2	Auxiliary elements	Interrogative	Recognizing
Line 13.3	-	-	-
Line 14.1	Breakdown	Interrogative	Constructing
Line 14.2	Auxiliary elements	Interrogative	Recognizing

Tab. 10.11 - Student A (3D Tick-Tack-Toe)

The analysis allows to identify two backward reasoning components involved in this excerpt: breakdown and introduction of auxiliary elements. The student is developing an exploratory phase through an interrogative process. He is answering the implicit general question "How many lines are there on the board?" and in particular "How many lines are there on each plane?" and "How can I divide them according to their geometric properties?". In these few lines of the protocol two auxiliary elements are introduced: the diagonal lines of the board (line 12.6) and the linear combination of board elements to represent the number of winning lines (lines 13.2 and 14.2). The recognition of the structure in the arrangement of the winning lines is fundamental for the resolution of the problem, this will allow, in the following lines, to elaborate a general mathematical formula.

The introduction of auxiliary elements characterizes the actions in which backward reasoning is involved. At the cognitive level, the action of Recognizing is fundamental for the introduction of new elements in the resolution, whether they are specific to the problem (like the classification of board positions) or they are mathematical constructs structurally analogous to the problem (like the geometrical subdivision of the triangular board). These

elements are the result of the recognition of previous knowledge in the background of the solver that is related to the problem he/she is solving. The role of the questions is to activate that tacit knowledge that allows new elements to become reality (Hintikka and Hintikka, 1982).

10.2.3 Breakdown and Cause-Effect Relationship: two processes that last over time

There is a substantial difference in the time duration of the backward reasoning processes depending on which characteristic is implicated. This comparison could be made by looking at the number of identical (i.e. classified in the same way) epistemic actions that are involved in a certain process. If there are three or more identical epistemic actions in sequence, it can be said that this process lasts over time. On the other hand, if there is a maximum of two identical epistemic actions in sequence, then it can be said that this process is momentary applied.

In table 10.11 of the previous paragraph, excerpt from the Student-A resolution protocol, it is possible to see that the introduction of auxiliary elements has a momentary character. In fact, it is manifested in a single epistemic action (line 12.6) or in two epistemic actions in sequence (lines 13.2 and 14.2). The breakdown and the research of cause-effect relationship, on the contrary, last over time. An example of breakdown is found in another excerpt of the same resolution protocol (Student-A, 3D Tick-Tack-Toe, Chapter 8, section 8.2.1.2.1). Here, the student breaks down the raw formula, that he found in a previous phase, until he reaches the general mathematical formula.

16.3 Maybe the number of straight lines follows a pattern.

16.4
$$L(n, d) = cnt(n, d) * L(n, d - 1) + Diagonals$$

17.1 The constant must be the number of planes parallel to the axes.

17.2 As in the previous case, these have to be nd , ...

17.3 ... then I refine my formula to

$$L(n, d) = nd * L(n, d - 1) + Diagonals$$

- 18.1 Diagonals don't seem that simple.
- 18.2 I start to play with the example of the cube and the plane.
- 18.3 They seem to join opposite vertices of opposite faces.
- 18.4 Will it be general?
- 19.1 I calculate that a hypercube has 2^d vertices, which gives me two faces with 2^{d-1} vertices.
- 19.2 Thus, if my previous observation is correct, the formula is

$$L(n, d) = nd * L(n, d - 1) + 2^{d-1}$$

Protocol Student-A	BR	HIM	AiC
Line 16.3	Solution	Interrogative	Constructing
Line 16.4	Formulation	Answer	Building-with
Line 17.1	Breakdown	Interrogative	Building-with
Line 17.2	Breakdown	Definitory	Constructing
Line 17.3	-	Rules	Constructing
Line 18.1	Breakdown	Interrogative	Building-with
Line 18.2	-	Rules	Building-with
Line 18.3	Breakdown	Answer	Building-with
Line 18.4	-	Interrogative	Building-with
Line 19.1	Solution	Definitory	Constructing
Line 19.2	formulation	Rules	Constructing

Tab. 10.12 - Student A (3D Tick-Tack-Toe)

The student develops the breakdown of the raw formula, analysing it element by element and manipulating it, during four epistemic actions in sequence. The breakdown characteristic is not continuous, as it might seem, but it is intercut with moments of forward reasoning (lines 17.3, 18.2 and 18.4). The student alternates interrogative moves followed by an answer that can be a definition (line 19.1).

It is possible to observe the phenomenon of research of cause-effect relationships in the following excerpt (Group 1, Function Problem, Chapter 9, section 9.2). The students use backward reasoning assuming the problem solved, i.e. assuming that one of the proposed

graphs is the function that they are looking for. Then, they search for relations between the graphs in order to justify the initial conjecture.

Line 1.9 Ma *It has to be. Suppose this is it.* [she points at the plot 2 and makes the pen slide along the function] *let's take... let's think of this as f... {Fe: Could be...} Here it's decreasing... this one's decreasing* [she points at the plot 3] *and this one's increasing* [she points at the plot 1]...

Line 1.10 Fe *Yes, this could not be its derivative...*[she points at the plot 1, then returns with the fingers to the previous position, continuing to compare the plot 2 and the plot 3]

Line 1.11 Fra *This could be her primitive, so... because she grows up...* [she points at the plot 1]

Line 1.12 Ma *That's right. This could be... If this was f* [she points plot 2], *this could be its primitive* [she points plot 1] *and this could be its derivative* [she points plot 3].

Protocol Group 1	BR	HIM	AiC
Line 1.9	Cause-Effect Relationship	Assertoric + Interrogative	Building-with
Line 1.10	Cause-Effect Relationship	Assertoric + answer	Constructing
Line 1.11	Cause-Effect Relationship	Interrogative	Building-with
Line 1.12	Cause-Effect Relationship	Assertoric	Constructing

Tab. 10.13 - Group 1 (Mathematical Problems: Functions)

The students looking for the solution through a process of elimination. it takes place observing and comparing certain graphs properties. The recognition, and subsequent conjecture, of the solution is not immediate. Also in this case, the backward reasoning does not develop continuously. Within the protocol lines, in fact, forward actions can be identified when students are justifying their current assumptions.

While the actions of recognizing are fundamental for processes in which new elements are introduced, actions of building-with and constructing mainly characterize the processes of

breakdown and research for cause-effect relationship. Putting concepts together and creating new concepts are activities that develop through different moments; they are not instantaneous actions but involve thinking processes that last over time.

10.3 Strategic approach

With the strategic approach it is possible to identify some recurring episodes in which backward reasoning is used. Seventeen different types of episodes have been identified. Here they are briefly summarized:

- Triangular Peg Solitaire:
 - Starting with a single peg on the board and proceed backwards
 - Finding a strategy to eliminate a specific peg on the board from an existing intermediate configuration
 - Studying the possible final movements leading to victory
- Maude task
 - Finding the basic elements to implement a peg state or a jump
 - Identifying a structural element of the programming language to obtain a specific behaviour
- 3D Tick-Tack-Toe
 - Analysing the winning lines
 - Defining the favourable positions
 - Blocking the opponent
 - Finding the final winning configuration
 - Constructing the mathematical formula from the configuration of the winning lines
- Mathematical Problems
 - Functions problem
 - Assuming the problem solved by identifying the graphs and then verifying the conjecture
 - Triangle and Circle problem

- Analysing the geometric configuration of the problem
- Expressing the relationships among the geometric configuration in algebraic language
- Construction problem
 - Analysing the solution of the problem (sought geometric configuration)
 - Identifying an analogy between the result of the problem and a known theorem
- Paths problem
 - Analysing the generic path
 - Identifying the combinatorial problem

The linguistic analysis bases on the commognitive framework of Sfard (2008), especially on her devices, like *reifying*, *encapsulating*, *saming*, which feature the processes of objectification (see Chapter 4, section 4.6). This lens allowed to better interpret the backward reasoning at a cognitive level, identifying the moments of objectification in the texts and utterances produced by the students. In general, the discourses developed while solving games are different from those developed in mathematical problem solving: in fact the related narratives and routines are different. From the one hand, in games the discourse is strategic, and its components of course are closely related to the game keywords (pegs, board, positions, strategies, movements, etc.). In problem solving, on the other hand, a mathematical discourse is developed. In the Maude Task and in the 3D Tick-Tack-Toe the two discourses are intertwined, since the request of the tasks is of a mixed type. Moreover, in the Maude task a third type of discourse is active, related to the request of implementing the game in a computational language. Here, three types of closely intertwined language come into play: strategic, mathematical and computational. One of the requirements of the 3D game is to find a mathematical formula. Due to this, the strategical discourse is interweaved with the mathematical one. We observe also that in all cases also the visual mediators Sfard (2018) are active; hence we have all the discourse characteristics pointed out by Sfard (*ibid.*): keywords, visual mediators, routines, and endorsed narratives.

Analysing the seventeen moments identified in the different design experiments, three types of recurring linguistic structures were found: they are typical of backward reasoning, and will be illustrated in the following paragraphs.

10.3.1 Going backward

The first linguistic structure is found in all those backward reasoning situations where the solver supposes the problem solved and tries to concretize the sought solution. This kind of reasoning is found both in games and in mathematical problems.

In the resolution of the triangular solitaire, this type of reasoning is found when the focus is on removing a specific peg from the board: "The only way to eliminate 1 would be to move 8-5-3" (Student-M protocol). In the 3D Tick-Tack-Toe game, it appears when a player tries to block the opponent by placing the token in a specific position: he imagines where the opponent could complete a line and anticipates his moves by placing the token in this place. In the Maude task, it is found when the students know how the final behaviour of the software is, so they pay attention to use a specific structural element during the implementation. This is useful to not obtain incorrect results or bugs.

In mathematical problems this kind of pattern emerges in functions problem, construction problem and paths problem. In functions problem, it is found when students assume the problem solved and try to justify their conjecture. In the construction problem, it appears when students identify a known structure in the problem and associate it with a previously solved problem. Applying the known problem (or applying the inverse construction in the case of Thales) the students complete the construction as required. In the path problem, it emerges, instead, when students recognize the problem as a combinatorial problem and then apply the appropriate combinatorial rules.

The structure of reasoning that students develop in all these moments is similar. It seems that the students try to apply what we call a 'reverse process of objectification', starting from the end. The process starts making explicit the object which must be built. Then a mathematical discourse is developed around the identified object, or around elements closely related to it,

until the final actual objectification. This generally happens through a *reifying* discursive device, and, in some cases, it can be done through an *encapsulating* device (or a combination of the two). In this type of cognitive structure, thought is pushed towards the final solution of the problem, which is introduced as an auxiliary object. Then, starting from there, the discourse develops following the usual processes of objectification that lead to the final actual achievement of the object sought.

10.3.2 Encapsulating object

The second structure that is identified is recurrent in backward reasoning moments, where the breakdown characteristics are predominant. The solver starts from the end of the problem and analyses the situation by breaking it down. This type of linguistic structure is also found in both games and mathematical problems.

Both in the resolution of the Triangular Peg Solitaire and in that of the 3D Tick-Tack-Toe, this type of structure is found when the solver looks for a possible final configuration of the game; this can be, for example, the "L" configuration of the last three moves of the solitaire game, or the winning configuration of the 3D game, where a player have two lines with a single empty square at the same time. In 3D game appear also when the solver analyses the winning lines or defines the favourable positions. This structure also emerges in the Maude task when the students are searching for the basic elements necessary to implement a peg state or a jump.

In mathematical problems this structure appears in the Triangle and Circle problem, in the Construction problem and in the Paths problem. In these tasks, some students analyse the final configuration of the problem, being it a geometric construction (as in the first two problems) or the paths configuration in the last problem.

The structure of reasoning is characterized by a process of objectification in which the *encapsulating* discursive device appears. The discourse is developed around final configuration of the problem that is made up of different objects. These are encapsulated in

a single entity and the discourse moves over the created object. Generally, before being encapsulated, objects are related through the *saming* device.

10.3.3 The power of algebra

The last structure that is identified is the one related to the processes of language transformation. In Triangle and Circle problem this transformation occurs from the geometric language to the algebraic one, while in 3D Tick-Tack-Toe it occurs between the strategic and the algebraic language.

This type of structure is more articulated. In the Triangle and Circle Problem the students, after adding a series of auxiliary elements in the geometric construction, develop a mathematical discourse about a specific part of the geometric configuration. They relate some elements of the configuration (the segments) considering them as part of an entity (a triangle). At this point they express the elements and their relations through an algebraic equation.

A similar process is also found in the resolution part about the mathematical formula development in the 3D Tic-Tack-Toe. Here, in fact, students analyse the winning lines relating them to each other and dividing them into groups according to their geometric characteristics. At this point, the lines are part of a schematic structure, a pattern, that allows the mathematical formula to emerge.

The structure of reasoning that is developed is composed by three progressive objectification levels that lead to the creation of a mathematical formula. The first moment is characterized by the appearance of the *saming* device, through which the relations between the elements are identified; in the second the *encapsulating* device emerges, where an auxiliary entity involving the elements is identified; the third moment is characterised by the use of the *reifying* device, where the mathematical formula emerges.

10.4 Backward Reasoning Indicators

The three different data interpretation, which are shown in this chapter, allow to point out similarities of backward reasoning along the four different design experiments. In particular, the *Local approach* points out similarities of backward reasoning which can be found through a synchronic investigation. This approach gives snapshots of the backward reasoning and highlights their structural similarities and differences. The *Global approach* points out similarities of backward reasoning which can be found through a diachronic investigation. This approach gives “the movie” of the backward reasoning along the time. Finally, the *Strategic approach* puts together the two previous analyses and elaborates the complete structural features of backward reasoning.

These three different interpretations allow to find eleven different indicators for backward reasoning. Globally they can be called the Backward Reasoning Indicators (BRI). The indicators characterise the backward reasoning from a cognitive point of view. They can be summarized as follow:

1. Auxiliary elements emerge during recognizing moments;
2. While the introduction of auxiliary element is a momentary application, the process of breakdown and the research of cause-effect relationship lasts over time;
3. Backward reasoning develops in ascendant interrogative strategic moves;
4. When backward reasoning develops in inquiry processes, B-R-C chains are produced, but not conversely;
5. If backward reasoning develops when an analogy come into play, then R-B-C chains are produced, but not conversely;
6. When backward reasoning develops in manipulation processes B-C chains are produced, but not conversely;
7. Backward reasoning is an ordering device useful for the explanation of the resolution steps;
8. Backward reasoning develops in the transition between resolution contexts;
9. A reverse objectification characterizes the moments were the problem is supposed to be solved;
10. The *encapsulating* discursive device characterises the moments of breakdown;

11. A three-phases objectification characterises the moments where formulas are constructed; in this case, a sequence composed by *saming*, *encapsulating* and *reifying* discursive device appears.

These indicators represent the cognitive dimensions of backward reasoning with which it is possible to integrate and to extend the existing epistemic model that emerge from the literature historical-philosophical analysis (see Chapter 3).

11

CONCLUSIONS

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CONCLUSIONS 11

In this chapter, taking into account the objectives and the limits we set at the beginning of this research project and basing on the results of the design experiments carried out, the final conclusions are presented.

In previous chapter, the identification of backward reasoning moments, pointing out the theoretical framework traits of each one, allow to highlight eleven features that characterize backward reasoning: the Backward Reasoning Indicators (BRI). From those emerged observations, firstly some elements are identified to answer the three research questions and discuss the achievement of research project objectives. Secondly, some points of the methodology used and the strategies (networking and hybridization) for theoretical frameworks development are discussed. Finally, some didactic conclusions and the future implications, of them and of this work, are pointed out.

11.1 The research questions: conclusions

Within this dissertation significant results about the main objectives of the research project have been achieved. In particular, the results obtained from the identification of the Backward Reasoning Indicators (BRI) in the previous chapter allow:

1. To extend the epistemic model of backward reasoning, existing in the mathematical literature, to a cognitive model.
2. To establish principles that can be used for the design of teaching situations, not only at university level, focused on backward reasoning.

In order to achieve the first objective of the research project two research questions have been formulated, and one to reach the second objective: each of them will be recalled in the discussion below. The observations that emerged during the identification of BRI are now organized to answer the three research questions highlighted at the beginning of the work (see Chapter 4).

11.1.1 Objective 1

To extend the epistemic model of backward reasoning, existing in the mathematical literature, to a cognitive model

To achieve the first objective, first of all, a historical-philosophical analysis of the phenomenon has been made, identifying the main epistemic features identified in literature by exploring the writings of the authors from ancient Greece to our days. Four main features that determine backward reasoning were then highlighted and thus constitute the epistemic model of reasoning: breakdown, cause-effect relationships research, transformative, and introduction of auxiliary elements. Subsequently, after the four design experiments, the Backward Reasoning Indicators (BRI) were identified. BRI have been defined as indicators that allow to see cognitive manifestations of backward reasoning. In order to define BRI, some observations have been made regarding both backward and forward reasoning. For this reason, the BRI, and in particular the observations below, can be useful to interpret the links between the two types of reasoning and answer the first and the second research questions.

1. What is the epistemological and cognitive link between backward and forward reasoning?

Backward reasoning does not exist without its forward counterpart. In fact, the research has confirmed what is affirmed in literature (Beaney, 2018; Hintikka and Remes, 1974; Ruesga-Ramos et al., 2004, Peckhaus, 2002): after the application of a certain number of backward steps, it is necessary to reverse the process in order to move forward towards the solution of the problem. The forward steps are the concretization of the reasoning that are done in a backward way.

The two reasonings coexist and are intertwined in the resolution discovery phases, while only forward reasoning emerges in verification phases. The objective of the discovery phases is the construction of the solution of the problem. In this phase the two reasoning modalities appear alternately, both contributing to the creation of the solution object. During the verification phases, the backward reasoning disappears and only the forward reasoning remains: through it, the solution check is elaborated.

From the perspective of Hintikka's Logic of Inquiry (1999), the knowledge is built through a questioning process. The phases of discovery are characterized by an alternation of *interrogative* moments in which the backward reasoning appears and *answer* moments in which the forward one emerges. The strategic logical moves that characterize the answers to the questioning process are *deductive* and *definitory*. The alternation of questioning phase to *deductive-defining* phases characterizes the construction process of the solution. The last step of this process is characteristically forward; the same type of reasoning will be used for its verification.

At the cognitive level, the processes of knowledge construction occur through three types of chains: R-B-C, B-R-C and B-C. The R-B-C chains are characteristics of tasks in which the solver recognises a known structure and an analogy has come out. The chains of type B-R-C, on the other hand, are characteristics of the problems that the solver cannot relate to others already seen before. The alternation of backward and forward reasoning develops within the chains. If during the resolution of an unknown problem an analogy emerges, then the chains change from B-R-C to R-B-C. Instead, B-C chains are typical of manipulative processes and verification phases. While in the former there is also backward reasoning, the latter are typical of forward reasoning.

2. How does the transition from backward reasoning to forward reasoning (and vice versa) take place?

In order to answer this question, three different moments in which backward reasoning develops are highlighted: when the solver supposes the problem solved, when the solver breaks down the problem and when the solver develops a mathematical formula.

When the solver supposes the problem solved, backward reasoning develops during a reverse process of objectification (see section 10.1.3.1 in this chapter). The process starts with the *recognizing* of the solution object. Then a backward step is made toward the starting configuration, finally a forward step is made to reach the solution object. This process can be articulated with a series of forward and backward reasoning, in which the final step is always a forward one. At a cognitive level, the process is characterized by *building-with* actions while the final step is a *constructing* action.

When the solver breaks down the problem the backward reasoning process is not continuous. To each backward step, which the solver takes during the problem analysis, corresponds a forward step, as a direct consequence of the first one. The forward steps concretise the backward steps. After the forward step, the solver continues to break down the problem until he reaches his goal. At a cognitive level, the process is characterized by a sequence of *building-with* and *constructing* actions in which the final step is a *constructing* one. *Recognizing* actions appear when auxiliary elements are introduced. The *encapsulating* discourse device appears in this process of objectification.

In situations where the solver develops a mathematical formula, however, backward and forward reasoning coexist. As seen in the results discussion, the transformation itself into algebraic language is a backward process. The process of objectification is characterised by the use of three different devices: *saming*, *encapsulating* and *reifying*. The algebraic formula, moreover, allows to express backward reasoning in a more compact way: each backward step is followed by a forward step that consists in the manipulation of the algebraic elements itself. The compression of the mathematical discourse allows "to say more with less" (Sfard, 2018): within the produced formula there are all the elements needed to proceed in the resolution.

11.1.2 Objective 2

To establish principles that can be used for the design of teaching situations, not only at university level, focused on backward reasoning

The observations that emerged during the identification of BRI allow to establish some principles that can be used for the design of university teaching situations focused on backward reasoning:

- Backward reasoning emerges in discovery phases, in particular, it develops in ascendant *interrogative* strategic moves.
- *Recognizing* and introducing auxiliary elements is part of the development of backward reasoning.
- *Recognizing* the structure of the problem push to suppose the problem solved.
- The existence of a complex configuration promotes the emergence of breakdown moments.

These principles could be used to design university teaching situations focused on backward reasoning. The activities enrolled in this teaching situations could be structured through the proposal of open problems, in which the discovery phases are favoured. The different tasks, put in succession in a ‘crescendo’ of difficulties, could be related to each other in order to promote similarities. Subsequent tasks could be based on the knowledge introduced in the first tasks, which would facilitate the introduction of auxiliary elements and the emergence of analogies. The types of problems proposed could have similar characteristics to those used in this research project. These observations, therefore, allow to answer the third research question.

3. Are there any non-playing situations that lead to backward reasoning?

The starting idea of this research was that backward reasoning develops naturally in strategy games. After carrying out this project it can be said that: yes, as expected, there are also non-playing situations that lead to backward reasoning.

Based on this research project at least three types of mathematical problems are recognized useful for this aim:

- Problems that have a certain initial geometric configuration.

This type of problem forces the solver to analyse the geometric construction using backward reasoning. For example, geometric problems, where the task is concerning the validity of a certain configuration, or construction problems are suitable for this purpose.

- Problems that have a structural analogy with known problems

This type of problem leads the solver to assume the problem solved. The analogy with the known problem, in fact, supports a *recognizing* in the solver that can trigger backward reasoning. Routines are set in motion.

- Problems of algebraic transformation

Having to create, or transform, a mathematical formula pushes the solver to make backward type of reasoning. In these problems the student is pushed to identify an “*x*” representing a final element of the problem, or the desired unknown variable. The subsequent creation of the formula and the decisions on its development characterize the backward reasoning interconnected with the forward one.

11.2 Conclusions on the methodology used

In relation to the methodology used, some conclusions about the methodological design used, the data collection instruments, and the units and the categories of analysis developed, are pointed out.

The methodological design was conceptualized through the creation of a networking (and hybridization) of theories. It gives the theoretical foundation and the methodological analysis tools for the design experiments. In fact, to meet the demands of the research project, an analysis model was created combining three different theories: The Game Theory Logic (Hintikka, 1999), the Abstraction in Context theory (Dreyfus, et al. 2015), and the Commognition perspective (Sfard, 2008). The model was created combining the two analysis model provided by the GTL and AiC theories (Hintikka’s Interrogative Model

(HIM) and RBC model) with the idea of objectification developed by Sfard (2018) (see Chapter 5).

This model allows an epistemic, cognitive and strategic analysis of the texts and the utterances of the students with particular attention to the processes of objectification that emerge from the discourse. The multidimensional model allows to observe a specific phenomenon that develops in a certain task from different points of view. To achieve the aims of this research project this model has been used with the focus on backward reasoning and its epistemic dimensions emerging from the literature. However, the model can be employed to analyse other types of phenomena that may occur during the resolution of a task. The analysis of a phenomenon through the model can be the basis for the subsequent development of teaching and learning activities.

As mentioned above (see Chapter 4), the creation of this model occurred during the years of research. For the first design experiment, in fact, only the GTL and the two derived models, the HIM and the FLIM, were used to frame the research. In the second design experiment the HIM model was interpreted according to the resolution contexts, in the third one the AiC theory was introduced and the RBC model derived from it. Finally, in the fourth design experiment the discursive analysis of objectification processes was inserted, and all design experiments have been analysed again with the full multidimensional analysis model. The evolution is due to the fact that shortcomings have been found. The HIM allows to consider backward/forward reasoning from a logical-strategic point of view (see Chapter 10, section 10.2), the RBC-model does that also from an epistemic-cognitive point of view (see Chapter 10, section 10.1 and 10.2), while the processes of objectification study (through the Commognition) allows a linguistic analysis that lead to the classification of backward reasoning moments according to three main features (see Chapter 10 section 10.3).

Some parts of the data, the HIM and FLIM interconnections and the RBC model, were tested at the expert level in the wider scientific community. Two case studies, related to the first and third design experiments, were presented at the International Network for Didactic Research in University Mathematics in 2018 and 2020 (Barbero and Gómez-Chacón, (2018), and Barbero, Gómez-Chacón and Arzarello (2020)).

The different data collection instruments were a fundamental contribution in order to give a wide range of levels to the research project. Firstly, the use of strategy games, mathematical problems and hybrid tasks (games with mathematical or computational interpretation), allow to observe the backward reasoning in different contexts and identify different cognitive behaviours in each of them that have some common points. The distinction between mathematical and heuristic-strategic knowledge has been discriminatory for the identification of different cognitive chains. Secondly, the different students' academic level (from the first year of bachelor to the PhD) and their different academic paths (mathematics, computer sciences, engineering, and future mathematics teachers) allow to obtain different type of students written productions, utterances based on the their background, they influence the emersion of backward reasoning moments. Thirdly, the different type of task's settings, in particular, the recommendation of solve them alone, in pairs or in groups, allows different type of interactions between students and the emersion of fruitful dialogues. Through these interactions it is possible to observe that backward reasoning develops also in overcoming difficulties moments (Chapter 7, section 7.2.2 and Chapter 9, section 9.2.2) and in explanation processes (Chapter 7, section 7.2.2). Finally, using different data collection tools (source triangulation, Jensen (2002)) like resolution protocols, video-recording, interviews, etc. a wide range of data were collected. These data provide different advantages, for example, on the one hand, in video-recording it is possible to observe and in-depth analyse difficulty moments, while in resolution protocols they almost don't appear; on the other hand, resolution protocols allow a global analysis of the entire resolution process, while the video-recordings is more difficult to analyse in detail and need to select some episodes. All these data collection instruments, developed in the four design experiments, allow to better understand the elements and categories of backward reasoning and its dynamic nature.

The units and categories of analysis chosen have proved effective for the data extrapolation. Epistemic action was chosen as the basic unit of analysis. This proved to be useful because it allows to sub-divide the resolution protocol or the video-recording episode into basic elements that can be easily classified and analysed using the multidimensional analysis model created. It also clearly identifies the activated knowledge and consequently the students' reasoning steps and thought processes. The identification of backward reasoning

moments was effective to classify them and to study cognitive behaviour during their development. As far as video-recording is concerned, the selected episodes allowed to observe the backward reasoning also in processes of explanation and overcoming difficulties, not clearly observed in resolution protocols analysis. Finally, the aggregation of the results obtained from the four design experiments analysis, according to the three approaches (global, local and strategic, see Chapter 10), made it possible to identify the common points of the various analysis and to create the BRI. In particular, the strategic approach made it possible to group the moments of backward reasoning into three broad categories according to the discursive analysis of their development (see Chapter 10, section 10.3).

A final point concerns the rationale of the objectification processes (in the sense of Sfard commognitive framework) detected within the processes of backward reasoning. In section 4.6 it has been hypothesized that backward reasoning can allow and facilitate to overcome the incommensurability between the inquiry and the deductive forms of reasoning supporting specific objectification processes. Strategy games, from this point of view, help to overcome this conflict: they allow for the natural development of backward reasoning and consequently to develop suitable explorations and not only routines (Sfard, 2008). The examples discussed in chapters 6-8 show this. Backward reasoning can be present also in some mathematics problems, as it has been seen in Chapter 9.

Basing on the structure analysis of backward reasoning given by the BRI indicators, it is possible to properly highlight the relationship between backward reasoning and objectification. In particular, the last three BRI indicators concern exactly this point and allow to give a precise description of a finer structure of backward reasoning evolution in time within different contexts and of the modality, according to which it can happen. They are the result of the linguistic analysis through the hybridized component from the Commognition.

Specifically, Indicator 11 (“A three-phases objectification characterises the moments where formulas are constructed; in this case, a sequence composed by *saming*, *encapsulating* and *reifying* discursive device appears”) concerns the construction of formulas for solving the problem: the three discursive devices involved in this process show that in this case the

objectification process fits exactly with that described by Sfard for the algebrization processes (Caspi & Sfard, 2012. p. 50-51). This a concrete instantiation of what in section 4.6 has been called the ‘masterpiece of Descartes’: the construction of the algebraic formulas exploits the analytic approach through the three discursive devices. They correspond exactly to the R-B-C chains. This can be seen for example in Circle and Triangle and Geometric Construction problems. Figures 9.10 and 9.24 illustrate the number of R-B-C chains (Indicator 4) in the construction process (resp. 5, and 4 or 5 depending on the examined student). They develop in parallel with *reifying+saming+encapsulating* processes (Indicator 11): they are discussed in sections 9.3.1/2 and 9.4.3.1. This shows that the objectification process of Indicator 11 perfectly fits with the Indicator 4. Jointly they allow to point out two aspects of backward reasoning: its structure and its concomitant support for objectification. In this case the switching from backward to forward reasoning (Indicator 9) is embodied in the algebraic formula, so that the synthetic part is easily developed since it belongs to the background knowledge of students. The exploration, in the sense used by Sfard for this term, has been possible because of the R-B-C structure that has supported the corresponding algebraic objectification.

The discourse is a bit different when the *encapsulating* component is missing, as in Function problem. First of all, it is worthwhile noticing that this problem is formulated in a manner that inhibits the use of formulas and obliges students to reason logically basing on their knowledge of elementary Calculus. This could explain why the *encapsulating* device is missing. Here a parallel evolution between the R-B-C structure and the *reifying* and *saming* devices can still be observed: however, it is slower than in previous examples, possibly because the problem is more difficult compared to the geometric ones. But the most interesting fact is that, differently from the dynamics of the previous examples, there is a sort of unbalance between the appearance of Indicator 4 and that of the discursive device indicators. In episode 1, where the students face difficulties, there are exclusively (4) *reifying* moments and (5) R-B-C chains; then, when the difficulties are being overcome, there are (6) *saming* moments and (7) R-B-C chains with a last B-C chain (when students verify their result). Here it is the commognitive lens that underlines the dynamic evolution. The discursive devices path marks a clean evolution through the transition from *reifying* to

saming devices: it is analogous to the evolution that we find in algebraic objectification processes of younger students, not yet fond of the algebraic discourse (Caspi & Sfard, 2012).

Two things must be underlined here. First, the analogy with the algebraic evolution in young students highlights how the backward reasoning processes are deeply linked with objectification processes. Second, this synergy between backward reasoning (in the examples it was the R-B-C chain, but in others, especially when the context is that of games, it could be also a B-R-C chain) and the path towards objectification shows that there is a remarkable correlation between the two. It confirms the hypothesis that backward reasoning is a construct, which allows and facilitates to overcome the incommensurability between the inquiry and the deductive forms of reasoning.

Possibly these two constructs may be an epiphenomenon of some deeper construct, which only further research could point out.

11.3 Didactic conclusions

In addition to the teaching principles identified in the answer to the second objective and the third research question, which allow a possible operational conversion of the research results, during the analysis of the second design experiment an interesting result has emerged: the backward reasoning is involved in processes of explanation (see Chapter 7). This confirms what Peckhaus (2002) already stated: “Method is “the art of arranging a series of thought”, i. e., an ordering device, and ordering is the basic feature of both, discovery and presentation.”

Studying backward reasoning could also be useful for those who, like teachers, are required to explain a problem-solving process or a mathematical proof. The teacher, in fact, during the explanation, does not retrace the steps of the discovery process that led to the solution of the problem (or the theorem) but exposes the resolution in a linear way. The usefulness of understanding backward reasoning lies in two reasons. On the one hand, the teacher can have a better understanding of the resolution processes and how that should be rearranged to obtain a linear sequence to expose to his students. On the other hand, being aware that the

natural process to obtain the result is not linear, allows him to pay more attention to the difficulties his students may have in understanding the passages. In this way, it would be easier to rework the explanation after the students have pointed out difficulties or even anticipate them.

Also what has been discussed above about objectification can have important didactical consequences. In fact, from the perspective of Commognition (Sfard, 2008), when the learner interacts with a person who is already adept in the new discourse, he is involved in a meta-level learning situation. The meta-level learning occurs when the learners encounter a discourse incommensurable with their own. As described in Chapter 4, this may cause a commognitive conflict, a situation in which “communication occurs across incommensurable discourses” (Sfard, 2008). In these situations, the backward reasoning allows the teacher to go back and forth in the discourse making it possible for the student to overcome the commognitive conflict (see Chapter 4, section 4.6). Through backward reasoning, the previously incommensurable discourse is revealed. The first step of this type of learning are imitation rituals. Little by little, the student takes over the routines until he becomes aware of his knowledge and can use it autonomously in mathematical discourse. In this perspective, the backward reasoning, like ordering device, mediates the de-ritualization processes. These happen when the loosely assembled routines used in backward modalities start relating each other and merge into a highly consolidated discourse.

11.4 Limitations of the study

Studying students' reasoning is difficult because it is impossible to analyse it directly, but only through its manifestations (students' texts and utterances). Even if the identification of epistemic actions (see Chapter 5) has facilitated this task, written productions and task resolution video-recordings can never fully represent the reasoning. As mentioned above (see Chapter 2) the reasoning is personal and depends both on the person involved and the proposed task.

While trying to propose different types of tasks and mathematical problems to students, there are many others that could involve different moments of backward reasoning. Surely the choice of mathematical problems has limited the study of the phenomenon to specific elements. The formulation of the task also influences the reasoning. For example, the second mathematical problem (Circle and Triangle Problem) (see Chapter 9), despite being an open problem (Arsac et al. 1988), has been proposed in a version that requires a specific triangle. Probably propose the problem in an "exploratory" form (for example: What characteristics the family of isosceles triangles inscribed in a circle have?) would have generated other types of reasoning. Another thing to take into account is that tasks could be proposed as tasks using technologies and digital mathematics environments, such as dynamic geometry environments (DGE).

Another limitation of the study to underline is the fact that this was done mainly with university students of the Faculty of Mathematics (with the exception of the second design experiment which involved students of the Master in Computer Science) and not in a homogeneous way in the different experiments, both as regards the level of study of the students involved and as regards the distribution between Spanish and Italian students. This is due to the availability of the faculty professors to grant lesson hours for the experiments and to the limited number of people involved in the research who have to analyse the whole set of data. Although it is not homogeneous, the different tasks and participant ages and academic level allow a longitudinal study through different level and contexts to better characterize elements and categories of backward reasoning and to observe its dynamic behaviour. It would be interesting to see which patterns of backward reasoning emerge realizing the design experiments with students from different engineering faculties, scientific faculties and with future primary teachers.

Finally, the theoretical framework that was built through networking (GTL and AiC) and hybridization (Commognition) made it possible to observe the phenomenon from different points of view although limited to those three theories. As seen in chapter 3, with the historical-philosophical analysis of backward reasoning, different approaches highlight different characteristics of the same phenomenon. The same thing can happen with the

theories that underly the construction of the multidimensional model of analysis. Different theories can give different points of view and thus provide multiple cognitive interpretations.

11.5 Future implications

The above results leave open a number of questions that could be the subject of future considerations, both from the point of view of research and of teaching at university level. Some of these are outlined below.

At least three possible further research and two didactic implications emerge at the end of this study. Some open question for future consideration can be:

I. To overcome the limitations of the study

This may be done by designing experiments involving students with different background and level (for example including upper secondary school students too) and proposing various type of task involving digital environment too.

II. To use classification and regression tree to analyse backward reasoning elements

A study, focused on the perceptions of mathematical students on the use of backward reasoning was carried out during this research using classification and regression trees (Gómez-Chacón and Barbero, 2019). It is a data mining approach employed to model the behaviour of a variable of interest in terms of logical condition. With this method it is possible develop 'IF-THEN' rule (Breiman et al., 1984) that can link the backward reasoning to specific student's skills and resolution elements.

III. To search an automatic approach to diagnosis

The analysis of BRI, and in particular the identified epistemic chains of actions, may be useful for developing learning or data mining algorithms. If the learning mechanisms of a human and a computer are known, it is possible to compare them and observe how they can favour each other. The meta-learning field in machine learning (Grabczewski, 2014), study algorithms that try to learn about algorithms, to find a more general mechanism to solve

problems. With a diagnostic tool similar to the one developed in this dissertation, it might be possible to try to improve machine learning algorithms, or an automatic approach to the elaboration of medical diagnosis.

From the didactic point of view, also further didactic studies are possible:

IV. To explore educational situations focusing on the teacher.

As seen before, the backward reasoning is involved in explanation processes. A further research would be useful to better understand how this type of reasoning comes into play in the teacher's thought processes: this can be useful to improve explanations and interactions with students and to anticipate their difficulties as much as possible.

V. To design and realize a course (maybe on-line) that allows the development of backward reasoning.

Following the above outlined principles, it is possible to create and elaborate materials that help improving the development of backward reasoning in problem solving activities. This course could be structured so that it can be proposed to secondary and undergraduate students with tasks of different levels and activities related to the course of study. Given the positive outcome of the experimentation with strategy games, and their usefulness to work on the epistemic and cognitive dimension of backward reasoning, some of these (or other carefully chosen ones) could be used in the proposed activities.

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