UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

TESIS DOCTORAL

Duality on abelian topological groups: the Mackey problema

Dualidad en grupos abelianos topológicos : el problema Mackey

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Madrid, 2016

UNIVERSIDAD COMPLUTENSE DE MADRID
Facultad de Ciencias Matemáticas

Duality on abelian topological groups: The Mackey Problem.

(Dualidad en grupos abelianos topológicos: El Problema Mackey.)

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Memoria presentada para optar al grado de Doctor con mención de *Doctor Europeus* en Matemáticas, elaborada bajo la dirección de:

Prof. Lydia Außenhofer
Prof. Elena Martín-Peinador

Agradecimientos.

En primer lugar, por supuesto, a mi familia, para los cuáles es un honor que haya llegado a doctorarme.

También, tengo que agradecer a Elena y a Lydia, por estos seis años de mi vida que nunca podré olvidar. Por todas las conferencias en las que Elena ha estado dando su apoyo moral entre el público. A Lydia, por esa semana que pasamos en Eichstätt y dieron lugar a nuestro primer artículo conjunto.

También debo agradecer a los miembros del departamento por ese compañerismo del que hacen gala y por la posibilidad de iniciarme en ese arte que se llama docencia universitaria.

Por supuesto, debo agradecer al Ministerio de Educación por las becas concedidas en mis años de Licenciatura y posteriormente por los Proyectos MTM2013-42486-P and MTM2009-14409-C02-01, que me han permitido participar en varios congresos y viajar a Udine.

A Vaja le agradezco las largas y gratas conversaciones matemáticas en sus visitas a la Universidad Complutense. A Mª Jesús, Xabier y Salvador les debo también un gran apoyo en mis diversos viajes por la geografía española.

A Dikran, Anna, Guisepina y Hans, por haberme acogido en el Departamento de Informática y Matemática de la Universidad de Udine durante mi estancia en Italia. Especialmente a Dikran, por los viajes en tierras italianas que hicimos juntos y las matemáticas que aprendí allí. Tampoco debo olvidar a Marco, Joaquín, María y Merche, quienes fueron los primeros en conocerme dentro de la facultad con las Olimpiadas Matemáticas y Concursos de Primavera.

Al P. Emilio, debo reconocerle el mérito de ser el primero en darse cuenta que podría llegar lejos en matemáticas cuando aún andaba por Secundaria.

A Santi, Miguel y David por compartir aquéllos maravillosos años de licenciatura. A Santi, Jesús y Sergio, les agradezco que, sin ser doctores, se hayan leído mi tesis. A mis amigos, tengo que agradecerles que me aguanten completamente y sin dudarlo: María, Miguel, Vero, Álvaro, Estrella, Antonio, Alberto, va por vosotros. Paco y Ángel, esos hombres de negro que siempre están dispuestos a echar una mano.

Contents

Introduction.

The idea of a general theory of continuous groups is due to S. Lie, who developed his theory in the decade 1874-1884. Lie's work is the origin of both modern theory of Lie groups and the general theory of topological groups. However, the topological considerations that are nowadays essential in both theories, are not part of his work.

A topological point of view in the theory of continuous groups was first introduced by Hilbert. Precisely, in his famous list of 23 problems, presented in the International Congress of Mathematicians of 1900, held in Paris, the Fifth Problem boosted investigations on topological groups.

In 1932, Stefan Banach, defined in Thèorie des Opèrations Linèaires, the spaces that would be named after him as special cases of topological groups. Since then, both theories (Banach spaces and topological groups) had been developed in a different, but somehow parallel, way.

Several theorems for Banach, or even for locally convex spaces have been reformulated for abelian topological groups [BCMPT01], [BMPT04], [CHT96], [HGM99], [MPT04]. The main obstacle for this task is the lack of the notion of convexity for topological groups due to the lack of an outer operation. However, Vilenkin gave the definition of quasi-convex subset for a topological abelian group inspired in the Hahn-Banach Theorem. In order to deal with it, many convenient tools had to be developed and this opened the possibility of a fruitful treatment of topological groups. The notion of quasi-convexity depends on the topology, in contrast with convexity, which is a purely algebraic notion.

With quasi-convex subsets at hand, it was quite natural to define locally quasi-convex groups, which was also done by Vilenkin in [Vil51].

In the sequel, we will only deal with abelian groups, even if we do not explicitly mention it.

Duality theory for locally convex spaces was mainly developed in the mid of twentieth century, and by now it is a well-known rich theory.

There is a very natural way to extend it to locally quasi-convex abelian groups. Fix first the dualizing object as the unit circle of the complex plane $\mathbb{T}$. The continuous homomorphisms from a topological group into $\mathbb{T}$ play the role of the continuous linear forms and they are called continuous characters.

The set of continuous characters defined on a group G has a natural group structure provided by the group structure of $\mathbb{T}$. Thus, we can speak of the group of continuous characters of G, which is called the **dual group** of G and it is the analogous notion to the dual space in the topological vector space setting. The natural topology in the dual group is the compact open topology, because of the Pontryagin Duality Theorem which holds for locally compact groups, the most distinguished class of topological groups.

The research we have done in this memory deals with the following problem, which we have denominated the Mackey Problem:

The Mackey Problem: Let (G, τ) be an abelian topological group and let $G^\wedge$ be its dual group. Consider the set of all locally quasi-convex group topologies in G whose dual group coincides with $G^\wedge$. They will be called **compatible** topologies. It is known that there is a minimum for this set, which is the weak topology induced by $G^\wedge$. Does there exist a maximum element in this set?

Current state of the problem:

For an abelian topological group G, it is known that there is always at least one locally quasi-convex compatible topology, namely the weak topology, which is always precompact. In general, it is not known if the set of compatible locally quasi-convex group topologies has a maximum element; whenever it exists, it will be called the Mackey topology for (G, τ). The problem of finding the largest compatible locally quasi-convex topology for a given topological group G, with this degree of generality (i.e. in the framework of locally quasi-convex groups) was first stated in 1999 in [CMPT99]. Previously in 1964 Varopoulos [Var64] had studied the question for the class of locally precompact group topologies.

Partial answers are the following: if G is locally quasi-convex complete and metrizable then there exists the Mackey topology [CMPT99] and it coincides with the original one. If G is locally quasi-convex metrizable (but not complete) the original topology may not coincide with the Mackey topology [DMPT14].

If G is a locally compact abelian group, its original topology is locally quasi-convex and it coincides with the Mackey topology. It is proved in [ADMP15] that the set of all locally quasi-convex compatible group topologies has cardinal greater or equal than 3.

In the recent work [DNMP14], the Mackey Problem is also studied and a grading in the property of being a Mackey group is established. The results of the mentioned paper are based on the families of equicontinuous subsets of the dual group produced by each group topology. Fixed a group G, any group topology τ on G gives rise to a family $\varepsilon(\tau)$ of equicontinuous sets in the dual group. Furthermore, a locally quasi-convex topology on G can be recovered from the knowledge of its corresponding family of equicontinuous sets, and this is the route followed in [DNMP14].

Objectives:

The main purpose of this work is to study the Mackey Problem as stated above: We had the conjecture that the Mackey Problem had a negative answer and that a counterexample could be obtained by means of the group of integers. First, we proved that non-discrete linear topologies on $\mathbb{Z}$ are not Mackey [ABM12]. By doing this, we realized that to every linear topology on $\mathbb{Z}$, a sequence of natural numbers could be associated, so that the sequence characterizes the linear topology. Thus, the sequences of integer numbers started to play an important role in our work. Loosely speaking, we called D-sequences to those sequences associated to linear topologies on $\mathbb{Z}$. Through them, we succeed to obtain, for any linear topology on $\mathbb{Z}$, a finer locally quasi-convex compatible topology, a fact which proves that linear topologies on $\mathbb{Z}$ are not Mackey. Our considerations have led us to produce a wide range of examples and tools to work with topologies on the integers.

Further, the linear topologies on $\mathbb{Z}$ provided a family of metrizable, locally quasi-convex topologies which are not Mackey. This was a discovery after many former attempts to prove the opposite. In fact, following the behaviour of locally convex spaces, it should have been the counterpart of the following statement: every locally convex metrizable space carries the Mackey topology.

A second objective was to find examples of metrizable non-Mackey topologies in the class of locally quasi-convex groups. To that end, only non-complete topologies should be considered (indeed, it was proved in [CMPT99] that locally quasi-convex complete metrizable groups carry their Mackey topology).

In this sense, the class of countable groups gives a nice setting. Countable groups cannot be endowed with non-discrete complete metrizable topologies (by Baire Category Theorem). Hence, we have that every metrizable group topology on a countable group is a good candidate not to be Mackey. However, for bounded countable groups Trigos et al had given examples of metrizable Mackey topologies ([BTM03]).

After studying the problem in the group of the integers, we moved to other groups. The most natural group would be the group of the real line, but its usual topology (being locally compact) is Mackey. Hence, we study next if $\mathbb{Q}$ with the topology inherited as a subgroup of $\mathbb{R}$ is Mackey. This was an open problem for several years, without knowing how to prove/disprove it. By a result in [DNMP14] this problem can be reduced to study topologies on, specific, infinite torsion subgroups of the unit circle $\mathbb{T}$. These subgroups are closely related to the linear topologies studied on the group of the integers. Indeed, they are the dual groups of the integers endowed with linear topologies. Finally we obtained that the usual topology on $\mathbb{Q}$ is not a Mackey topology [BM14].

A special characteristic of bounded groups is that every locally quasi-convex topology in such a group must be linear. The converse holds for all sorts of groups. Therefore, the Mackey Problem in bounded groups can be supported on available properties of linear groups, together with the fact that "being a Mackey group" is a property which is hereditary for open subgroups.

Contents and results:

The first Chapter of this dissertation deals with the introduction to topological groups and some basic facts about them.

In Section 1.1 we recall the definition of topological group and give some basic examples. We also deal with the notion of quasi-convex subsets – a cornerstone in this work.

Section 1.2 is devoted to precompact topologies. We collect basic facts and results about them.

In Section 1.3 we define the topologies of uniform convergence on a set of homomorphisms. We give a suitable neighborhood basis for this topology and prove some results.

In Section 1.4 we deal with a family of complete group topologies. These topologies, introduced by Graev, are defined through the condition of being the finest group topology in which a fixed sequence of elements converges to the neutral element.

In this chapter we have included references to known results and given a proof whenever it is interesting enough.

Chapter 2 is devoted to D-sequences. They are sequences of natural numbers such that any term divides the following one. They generalize the sequence (p^n) and will be fundamental to describe linear topologies on the integers and to build infinite torsion subgroups of $\mathbb{T}$.

In Section 2.1 we give the definition of D-sequence and some special classes of them. The sequence of ratios of a D-sequence also plays an important role. We define some subsets of the set $\mathcal{D}$ of all D-sequences, which will be important through Chapters 4 and 5 ($\mathcal{D}_\infty, \mathcal{D}_\infty^\ell$). We also define the "union" of two subsequences of a D-sequence. Finally, we define the function $\mathbb{P}$ which is, roughly speaking, a generalization of the decomposition of a natural number in a product of prime numbers.

Section 2.2 is devoted to find suitable representations of integer numbers and real numbers taking a D-sequence as reference (Propositions 2.2.1 and 2.2.4, respectively). The 10-adic representation of integers and real numbers is generalized to D-sequences. This representation will help us to deal with neighborhoods of certain topologies of uniform convergence related to D-sequences (in the sense that will be explained in Chapters 4 and 5). We also include some examples to familiarize the reader with our representation. The coefficients of the representation of an integer number will be called the **b**-coordinates of that integer.

In Section 2.3 we assign to a D-sequence, $\mathbf{b}$, an infinite torsion subgroup $\mathbb{Z}(\mathbf{b}^\infty)$ of $\mathbb{T}$. These subgroups are a generalization of the Prüfer groups $\mathbb{Z}(\mathbf{p}^\infty)$. Certain subgroups $\mathbb{Z}(\mathbf{b}^\infty)$ will allow us to prove that the euclidean topology in $\mathbb{Q}$ is not Mackey.

According to the representation of numbers as sums of series based upon a D-sequence, $\mathbf{b}$, we define the set $\mathbb{Z}_\mathbf{b}$ of **b**-adic integers in Section 2.4. Later on, we shall see that it is the completion of $\mathbb{Z}$ endowed with the corresponding **b**-

adic topology. By Pontryagin-van Kampen Theorem, it becomes clear that $\mathbb{Z}_{\mathbf{b}} = \mathrm{Hom}(\mathbb{Z}(\mathbf{b}^{\infty}), \mathbb{T})$ (Remark 4.2.22).

Chapter 3 establishes the general framework for the Mackey Problem.

In Section 3.1 the historical background is set. It points out the path from the Mackey-Arens Theorem to the statement of the Mackey Problem for topological abelian group. Some recent results are also included.

Section 3.2 deals with the family of compatible (with a duality) group topologies. Some known results are included (compatible topologies share the same dually closed subgroups) and other new ones are proved. For bounded groups, we find several classes of subgroups that compatible topologies share (closed, dually closed, finite-index closed subgroups).

In Section 3.3 we prove that open subgroups of Mackey groups are Mackey. This result was stated in [Leo08], without a proof. It is posed the question whether the converse is also true. A partial answer is given in Chapter 6.

In Section 3.4 some alternative statements of the Mackey Problem for bounded groups are provided as a consequence of Theorem 3.2.5.

In Section 3.5 we state the $\mathcal{G}$-Mackey Problem, which is a generalization of the Mackey Problem. We give some results of independent interest.

Chapter 4 is the main chapter in this dissertation and deals with group topologies on the group of the integers. We, mainly, consider topologies of uniform convergence on a subset B of the dual group $\mathbb{T}$ related to a D-sequence. We also consider complete topologies on $\mathbb{Z}$.

Section 4.2 deals with precompact topologies on the integers. We collect some known results. In order to generalize p-adic topologies, we construct, by means of a D sequence $\mathbf{b}$, a linear (and precompact) topology on the integers, $\lambda_{\mathbf{b}}$, (Definition 4.2.5), which we have called $\mathbf{b}$-adic topology. First, we prove that any linear topology on the integers can be defined via a suitable D-sequence (Proposition 4.2.6).

In order to compare two different linear topologies, we use the function $\mathbb{P}$ defined in Chapter 2. In Proposition 4.2.9 we prove that two linear topologies $\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}$ satisfy $\lambda_{\mathbf{b}} \leq \lambda_{\mathbf{c}}$ if the associated images, through $\mathbb{P}$, satisfy $\mathbb{P}(\mathbf{b}) \leq \mathbb{P}(\mathbf{c})$. This allows us to claim that there exist 2^{ω} different linear topologies on the group of the integers (Lemma 4.2.11).

For a D-sequence, $\mathbf{b}$, we compute the dual group of $\mathbb{Z}$ endowed with the linear topology $\lambda_{\mathbf{b}}$. In Proposition 4.2.13 it is proved that $(\mathbb{Z}, \lambda_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

Since convergent sequences are enough to describe a metrizable topology, we study in Proposition 4.2.15 the null sequences in $\lambda_{\mathbf{b}}$. In particular, we characterize the convergence in a linear topology in terms of the $\mathbf{b}$-coordinates of the elements of the sequence.

In Theorem 4.2.21 we prove that for a fixed D-sequence the completion of $(\mathbb{Z}, \lambda_{\mathbf{b}})$ is isomorphic to $\mathbb{Z}_{\mathbf{b}}$.

Section 4.3 deals with topologies of uniform convergence on the group of integers.

Subsection 4.3.1 deals with topologies of uniform convergence which are, in some sense, originated by a D-sequence. More precisely, fixed a D-sequence $\mathbf{b} = (b_n)$, we define $\underline{\mathbf{b}} := \left\{\frac{1}{b_n} + \mathbb{Z}\right\}$ in $\mathbb{T}$. Since $\mathbb{T}$ can be identified with all the characters defined on $\mathbb{Z}$, the range of the sequence $\underline{\mathbf{b}}$ can be considered as a subset of $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$. Thus, we define $\tau_{\mathbf{b}}$ as the topology on $\mathbb{Z}$ of uniform convergence on the range of $\underline{\mathbf{b}}$. A neighborhood basis for $\tau_{\mathbf{b}}$ is described in Definition 4.3.3. In Proposition 4.3.4 we prove that $\tau_{\mathbf{b}}$ is metrizable and locally quasi-convex.

Proposition 4.3.8 gives us the possibility to compare the topologies $\tau_{\mathbf{b}}, \tau_{\mathbf{c}}$, where $\mathbf{c}$ is a subsequence of $\mathbf{b}$ under some extra conditions. This proposition will be the main tool to prove Theorem 4.3.31.

It is not easy to handle the topology $\tau_{\mathbf{b}}$. For this reason, we find a characterization on the $\mathbf{b}$-coordinates of an integer in order that it belongs to a fixed basic neighborhood. Combining Propositions 4.3.11 and 4.3.12 we can define a new neighborhood basis for the topology $\tau_{\mathbf{b}}$ which depends exclusively on the $\mathbf{b}$-coordinates of the integer numbers.

Since the topologies $\tau_{\mathbf{b}}$ are metrizable, it is important to have a description of the null sequences in $\tau_{\mathbf{b}}$, which is done in Proposition 4.3.15.

Related to a D-sequence $\mathbf{b}$, we have defined, so far, two topologies $\lambda_{\mathbf{b}}$ and $\tau_{\mathbf{b}}$. Proposition 4.3.17 states that these topologies are comparable. In fact, $\lambda_{\mathbf{b}} \leq \tau_{\mathbf{b}}$ holds. Since $\tau_{\mathbf{b}}$ is never precompact, it is also clear that $\lambda_{\mathbf{b}} \neq \tau_{\mathbf{b}}$.

Subsection 4.3.2 is devoted to the dual group of $(\mathbb{Z}, \tau_{\mathbf{b}})$ if $\mathbf{b} \in \mathcal{D}_\infty^\ell$. The main theorem of the subsection (Theorem 4.3.31) characterizes it.

In Lemma 4.3.19 we find a “special” subsequence of the D-sequence $\mathbf{b}$ which is $\tau_{\mathbf{b}}$-null. Naturally, this subsequence is not unique, but its existence will be important in the sequel.

In order to prove Theorem 4.3.31, which is a very interesting tool for the sequel, we require some technical properties, as expressed in Proposition 4.3.21, after some helping notation.

Proposition 4.3.22 is of independent interest. For a D-sequence, $\mathbf{b}$ and an element χ belonging to $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$, we find a condition on the $\mathbf{b}$-coordinates of $\chi(1)$. This will be helpful later to decide whether a character on $\mathbb{Z}$ is $\tau_{\mathbf{b}}$-continuous. As a Corollary, we obtain Theorem 4.3.24, which states that linear topologies on $\mathbb{Z}$ are not Mackey topologies. This result follows also as a corollary of Theorem 4.3.31, but chronologically we first obtained this result which precisely motivated the obtention of Theorem 4.3.31.

As an intermediate step for Theorem 4.3.31, we find a device to write the $\mathbf{c}$-coordinates of a real number in terms of its $\mathbf{b}$-coordinates, whenever $\mathbf{c}$ is a subsequence of $\mathbf{b}$.

Finally, Theorem 4.3.31 states that if $\mathbf{b} \in \mathcal{D}_\infty^\ell$, then $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

The D-sequences considered to this point belong to $\mathcal{D}_\infty^\ell$. In Subsection 4.3.3 we study duality for D-sequences which are not in $\mathcal{D}_\infty^\ell$ for any ℓ. To that end, we introduce the block-condition. Roughly speaking, a D-sequence satisfies the block condition if there exists an integer c and blocks of consecutive terms of the sequence of ratios with values less than or equal to c, such that the length of the blocks tends to infinity.

Although D-sequences satisfying the block condition have not been completely studied, we have obtained the following result: for a D-sequence $\mathbf{b}$ satisfying the block condition, the topology $\tau_{\mathbf{b}}$ is not compatible with the duality $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$ (see Theorem 4.3.36). This is in contrast with D-sequences being in $\mathcal{D}_\infty^\ell$.

In Proposition 4.3.37 we give an example of D-sequence which is not in $\mathcal{D}_\infty^\ell$ for any natural number ℓ and which does not satisfy the block condition either.

In Subsection 4.3.4 we construct a sequence of D-sequences and prove that the supremum of the topologies of uniform convergence on each subsequence does not coincide with the topology of uniform convergence on the "limit" of the sequence.

In Section 4.4, we define the third type of locally quasi-convex topology on the integers associated to a given D-sequence. For a D-sequence, $\mathbf{b}$ we collect all the subsequences of $\mathbf{b}$ which belong to $\mathcal{D}_\infty^\ell$ for some natural number ℓ. We denote this family by Δ. We define the new topology $\delta_{\mathbf{b}}$ as the supremum of all the topologies of uniform convergence associated to D-sequences in Δ (Definition 4.4.4).

In Proposition 4.4.3 it is proved that the family of topologies of uniform convergence associated to the elements of Δ is a directed family. This immediately leads to the fact that the topology $\delta_{\mathbf{b}}$ is compatible (Theorem 4.4.6).

In Proposition 4.4.7 the topologies $\tau_{\mathbf{b}}$ and $\delta_{\mathbf{b}}$ are compared.

In Subsection 4.4.1, we study some characteristics of the topology $\delta_{\mathbf{b}}$ if $\mathbf{b}$ has bounded ratios (that is, the sequence of ratios is bounded).

Theorem 4.4.8 is a powerful tool. If $\mathbf{b}$ denotes a basic D-sequence with bounded ratios, for a given null sequence in $\lambda_{\mathbf{b}}$, (a_n), we construct a new locally quasi-convex metrizable topology, compatible with $\lambda_{\mathbf{b}}$, thus finer than $\lambda_{\mathbf{b}}$ and such that (a_n) does not converge in this new topology. By this procedure of killing the null sequences in $\lambda_{\mathbf{b}}$, we obtain a family of topologies whose supremum S has no convergent sequence. Hence, it is a non-metrizable topology which is further locally quasi-convex and compatible. As a consequence, we obtain that if $\mathbf{b}$ has bounded ratios, then $\delta_{\mathbf{b}}$, which is finer than S is a non-metrizable compatible (with $\lambda_{\mathbf{b}}$) topology (Theorem 4.4.9).

Additionally, we prove that for a D-sequence $\mathbf{b}$, with bounded ratios, the group $(\mathbb{Z}, \delta_{\mathbf{b}})$ is not reflexive (Proposition 4.4.12).

Subsection 4.4.2 deals with characteristics of $\delta_{\mathbf{b}}$ if $\mathbf{b} \in \mathcal{D}_\infty^\ell$ for some ℓ. In Proposition 4.4.20, we find two D-sequences, $\mathbf{b}, \mathbf{c}$ satisfying that the topologies $\delta_{\mathbf{b}}$ and $\delta_{\mathbf{c}}$ are compatible but not comparable. In this specific example, neither $\delta_{\mathbf{b}}$, nor $\delta_{\mathbf{c}}$ can be the Mackey topology for the duality $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$.

In Section 4.5 we study complete topologies on the group of the integers. Since a D-sequence $\mathbf{b}$ converges to zero in $\lambda_{\mathbf{b}}$, it is in particular a T-sequence and there is a Graev-topology $\mathcal{T}_{\{b_n\}}$ associated to each D-sequence. Namely, the finest group topology on $\mathbb{Z}$ making $\mathbf{b}$ convergent. In particular, for $\mathbf{p} = (p^n)$, we prove that the corresponding Graev-topology is compatible with $\lambda_{\mathbf{p}}$ (Proposition 4.5.4). It is not locally quasi-convex, therefore non-reflexive (Proposition 4.5.5).

In Section 4.6 we collect the open-questions about topologies on the integers.

In Chapter 5, we move our attention to infinite torsion subgroups of the unit circle, namely $\mathbb{Z}(\mathbf{b}^\infty)$ for a suitable D-sequences. The main result in the chapter is that the group of rational numbers endowed with the topology inherited from the real line is not a Mackey group.

In Section 5.1 we consider two different locally quasi-convex group topologies on the group $\mathbb{Z}(\mathbf{b}^\infty)$. The first one (in Subsection 5.1.1) is the topology inherited from $\mathbb{T}$ (the "usual" topology) and the second one the topology of uniform convergence on $\mathbf{b}$ (Subsection 5.1.2), namely $\eta_\mathbf{b}$.

In Subsection 5.1.1, first we give a suitable neighborhood basis for the usual topology in $\mathbb{Z}(\mathbf{b}^\infty)$ and then give conditions on the $\mathbf{b}$-coordinates of an element in $\mathbb{Z}(\mathbf{b}^\infty)$ in order that it belongs to a specific neighborhood of the mentioned basis (Lemma 5.1.2).

In Proposition 5.1.3 we characterize the null sequences in the usual topology by means of the $\mathbf{b}$-coordinates of the terms of the sequence. This will be needed to compare the usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$ and $\eta_\mathbf{b}$.

In Subsection 5.1.2, we consider, for a D-sequence $\mathbf{b}$ in $\mathbb{Z}$, a new topology on the group $\mathbb{Z}(\mathbf{b}^\infty)$. Namely, the topology of uniform convergence on $\mathbf{b}$, which we have called $\eta_\mathbf{b}$. In Definition 5.1.6 we give a neighborhood basis for the topology $\eta_\mathbf{b}$. We observe that in this case we have identified the integers with its natural copy in $\mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T}) \cong \mathbb{Z}_\mathbf{b}$ obtained identifying an integer with the homomorphism "evaluation on it".

Next, we define a suitable neighborhood basis for $\eta_\mathbf{b}$ (Propositions 5.1.9 and 5.1.10).

In Proposition 5.1.12 null sequences in $\eta_\mathbf{b}$ are characterized in terms of the $\mathbf{b}$-coordinates of the elements of the sequence. This proposition allows us to compare the topology induced from the euclidean of $\mathbb{T}$ in $\mathbb{Z}(\mathbf{b}^\infty)$ with $\eta_\mathbf{b}$, which is done in Subsection 5.1.3 (Proposition 5.1.15).

Section 5.2, is devoted to the dual group of $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})$.

In Theorem 5.2.3 the dual group of $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})$ is computed. In particular, this dual group is $\mathbb{Z}$, which means that the usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$ and $\eta_\mathbf{b}$ are compatible topologies. Hence, the usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$ is not Mackey (Theorem 5.2.4).

By Corollary 7.5 in [DNMP14], which states that locally quasi-convex quotients of Mackey groups are Mackey, it is possible to prove that $\mathbb{Q}$ endowed with the topology inherited from the real line is not a Mackey group. This example is the first one known of a locally precompact metrizable topology which is not Mackey.

In Chapter 6 we consider the following question: Are there some especial families of abelian groups $\mathcal{G}$ in which the following claim holds: every metrizable locally quasi-convex group in $\mathcal{G}$ is a Mackey group? This question is inspired from the Mackey Theory in locally convex spaces. As said above, a metrizable locally convex space carries the Mackey topology, therefore is a Mackey space. For the case of abelian groups, we prove that this property characterizes precisely the bounded groups (Theorem 6.1.3). This is an interesting instance of how an algebraic feature can be characterized by means of a topological property.

The proof of Theorem 6.1.3 is divided in two parts: in Section 6.2 we prove that a locally quasi-convex metrizable topology on a bounded group is Mackey and in Section 6.3 we find for any unbounded group G a metrizable locally quasi-convex topology on G which is not Mackey.

For Section 6.2 we use some notions on dense subgroups and prove that any locally quasi-convex group topology on a bounded group is linear (Proposition 6.2.10). Hence a locally quasi-convex topology on a bounded group is Mackey if and only if it is $\mathcal{L}in$-Mackey (Mackey in the class of linear group topologies). We study the conditions of linear topologies on bounded groups to be $\mathcal{L}in$-Mackey.

In addition we prove that a locally quasi-convex topology on a bounded group whose dual has cardinality $< \mathfrak{c}$ must be precompact. As a consequence, we obtain that such topologies are Mackey.

At the end of the section we prove that a locally quasi-convex topology on a bounded group which is (a) metrizable or (b) whose dual group has cardinality $< \mathfrak{c}$ is a Mackey topology (Theorem 6.2.1).

Section 6.3 consists of the proof that for any unbounded group G, there exists a metrizable locally quasi-convex non-Mackey topology on G. It leans on the fact that any unbounded group has a subgroup of the form $\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty), \mathbb{Q}, \bigoplus_{n=1}^{\infty} \mathbb{Z}_{m_n}$ where (m_n) is an increasing sequence of natural numbers. At this point, we already know that $\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty)$ and $\mathbb{Q}$ can be endowed with a metrizable locally quasi-convex non-Mackey topology.

In Proposition 6.3.2 we prove that the mentioned groups $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{m_n}$ can also be endowed with a metrizable, locally quasi-convex non-Mackey topology, namely the topology inherited from the product $\prod_{n=1}^{\infty} \mathbb{Z}_{m_n}$. Finally we prove Theorem 6.3.1, taking into account that an open subgroup of a Mackey group is again Mackey.

In Section 6.4 we find some corollaries of Theorem 6.2.1 about topologies on bounded groups.

Conclusions:

The Mackey Problem for topological groups has proved to be a rich source for mathematical research. Although we do not present an answer to the main problem, we have proved several results of independent interest which could be used to solve a wide range of problems. Our techniques have led us to develop Numerical Analysis in $\mathbb{Z}$ and other groups. Very simple questions still remain open. For instance: Is there a Mackey topology in the duality $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^{\infty}))$ or in $(\mathbb{Q}, \mathbb{R})$?

Furthermore, the research carried out for this dissertation has motivated publications in indexed journals (see [ABM12] in Journal of Pure and Applied Algebra and [BM14] in Topology and its Applications). Sections 4.3 and 4.4, and Chapter 6 will be object of two future publications. Some results of Section 4.5 appeared in a publication (joint with Elena Martín-Peinador) in a special issue in honor of Professor Etayo. In addition, the techniques we have used can be of interest in other areas of mathematical research.

The possibility of finding families of metrizable locally quasi-convex topologies which are not Mackey leads to the thought that completeness or, at least, other condition is needed to find Mackey topologies. For bounded groups, we have completely solved the question: every metrizable locally quasi-convex topology on a bounded group is Mackey.

Furthermore, we have re-stated the Mackey Problem in bounded groups in several alternative ways. This is done in the hope that these alternative statements might help to solve the Mackey Problem in the class of bounded groups.

We have also proved, by different examples, that dense subgroups of Mackey groups need not be Mackey. For instance, linear topologies on $\mathbb{Z}$ are not Mackey. Since they are precompact, the completion of a linear group is a compact Hausdorf group, which clearly is a Mackey group that contains a dense non-Mackey subgroup. The same argument holds true for $\mathbb{Z}(\mathbf{b}^{\infty})$ endowed with the topology inherited from the euclidean topology in $\mathbb{T}$. The topology on $\mathbb{Q}$ inherited from $\mathbb{R}$

is not, either, Mackey. However an open subgroup of a Mackey group is again Mackey. The converse is proved in the class of bounded locally quasi-convex groups (Proposition 6.2.16). Other permanence properties of Mackey topology are not known. For instance, is the product of two Mackey groups again a Mackey group? This will be one of our first objectives for future work.

Introducción.

La idea de una teoría general sobre grupos continuos se debe a S. Lie, que desarrolló su teoría en la década 1874-1884. El trabajo de Lie es el origen tanto de la teoría moderna de los grupos de Lie, así como de la teoría general de grupos topológicos. Sin embargo, las consideraciones topológicas que son actualmente esenciales en ambas teorías no son parte de su trabajo.

Un punto de vista topológico en la teoría de grupos continuos fue introducida por primera vez por Hilbert. Precisamente, dentro de su famosa lista de 23 problemas, presentada en el Congreso Internacional de Matemáticos de 1900, celebrado en París, el Quinto Problema propició gran cantidad de investigaciones en grupos topológicos.

En 1932, Stefan Banach definió en Thèorie des Opèrations Linèaires, los espacios, que posteriormente recibirían su nombre, como casos especiales de grupos topológicos. Desde entonces, ambas teorías (espacios de Banach y grupos topológicos) se han desarrollado de una manera diferente, pero de algún modo paralela.

Varios teoremas en espacios de Banach, o incluso, espacos localmente convexos han sido reformulados para grupos abelianos topológicos. Por ejemplo: [BCMPT01], [BMPT04], [CHT96], [HGM99], [MPT04]. El mayor obstáculo para esta tarea es la ausencia de la noción de convexidad para grupos topológicos propiciada por la ausencia de una operación externa. Sin embargo, Vilenkin dio la definición de conjunto cuasi-convexo para un grupo topológico inspirada en el Teorema de Hahn-Banach. Para trabajar con esta noción, se requieren herramientas que tuvieron que desarrollarse y esto abrió la posibilidad de nuevos resultados para los grupos topológicos. La noción de cuasi-convexidad depende de la topología, en contraste con la convexidad que es una noción puramente algebraica.

Con la noción de conjuntos cuasi-convexos, fue bastante natural definir los grupos localmente cuasi-convexos, también hecho por Vilenkin en [Vil51].

En lo que sigue, todos los grupos que tratemos serán abelianos, aunque no se mencione explícitamente.

La teoría de la dualidad en espacios localmente convexos fue desarrollada principalmente a mediados del siglo XX, y es actualmente una teoría rica y conocida.

Existe una manera muy natural de extenderla a grupos abelianos localmente cuasi-convexos. Primero, fijamos el círculo unidad del plano complejo $\mathbb{T}$ como objeto dualizante. Los homomorfismos continuos de un grupo topológico en $\mathbb{T}$ juegan el papel de las formas lineales continuas y se denominan caracteres continuos.

El conjunto de los caracteres continuos definidos en un grupo G tiene una estructura natural de grupo proporcionada por la estructura de grupo de $\mathbb{T}$. Por tanto, podemos hablar del grupo de caracteres continuos de un grupo topológico G, que se llama **grupo dual** de G y es la noción análoga a la de espacio dual en el marco de los espacios vectoriales topológicos. La topología natural en el grupo dual es la topología compacto-abierta, dado que el Teorema de Dualidad de Pontryagin es cierto para grupos localmente compactos, la clase más distinguida de grupos topológicos.

La investigación que hemos realizado en esta memoria trata sobre el siguiente problema, que hemos denominado el Problema Mackey:

Problema Mackey: Sea (G,τ) un grupo abeliano topológico y sea $G^\wedge$ su grupo dual. Consideramos el conjunto de todas las topologías localmente cuasi-convexas en G cuyo grupo dual coincide con $G^\wedge$. Éstas se llamarán topologías **compatibles**. Se sabe que existe un mínimo en este conjunto, que es la topología débil inducida por $G^\wedge$. ¿Existe un elemento máximo en este conjunto?

Estado actual del problema:

Para un grupo abeliano topológico G, se sabe que siempre hay, al menos una topología localmente cuasi-convexa compatible, la topología débil, que además es siempre precompacta. En general, no se sabe si el conjunto de topologías localmente cuasi-convexas compatibles tiene un elemento máximo; cuando existe, se denominará la topología de Mackey para (G,τ). El problema de encontrar la topología localmente cuasi-convexa compatible más grande, con este grado de generalidad (es decir, en el marco de los grupos localmente cuasi-convexos) fue enunciado por primera vez en 1999 en [CMPT99]. Previamente, en 1964, Varopoulos [Var64] había estudiado la cuestión en la clase de las topologías de grupo localmente precompactas.

Conocemos las siguientes respuestas parciales: si G es un grupo topológico completo localmente cuasi-convexo y metrizable, entonces existe la topología de Mackey [CMPT99] y coincide con la original. Si G es localmente convexo y metrizable (pero no completo) la topología original podría no coincidir con la topología de Mackey [DMPT14].

Si G es un grupo localmente compacto abeliano, su topología original es localmente cuasi-convexa y coincide con la topología de Mackey. Se prueba en [ADMP15] que, en ese caso, el conjunto de topologías localmente cuasi-convexas tiene cardinal mayor o igual que 3

En el reciente trabajo [DNMP14], el Problema Mackey también es estudiado y se establece una graduación en la propiedad de ser un grupo de Mackey. Los resultados del artículo mencionado se basan en las familias de conjuntos equicontinuos del grupo dual producidas por cada topología de grupo. En particular, una topología localmente cuasi-convexa se puede recuperar a partir del conocimiento de la correspondiente familia de conjuntos equicontinuos, y esta es la ruta seguida en [DNMP14].

Objetivos:

El objetivo principal de este trabajo es estudiar el Problema Mackey enunciado anteriormente: Teníamos la conjetura de que el Problema Mackey tenía respuesta negativa y que se podría obtener un contraejemplo a partir del grupo de los enteros. Primero probamos que las topologías lineales no-discretas en $\mathbb{Z}$ no son Mackey [ABM12]. Probando esto, nos dimos cuenta de que a cada topología lineal en $\mathbb{Z}$, se le podía asociar una sucesión de números naturales, de tal modo que la sucesión caracterizaba la topología lineal. Por tanto, las sucesiones de números enteros comenzaron a jugar un papel importante en nuestro trabajo. En términos sencillos, llamamos D-sucesiones a las sucesiones asociadas a topologías lineales en $\mathbb{Z}$. A través de ellas, obtuvimos, para cada topología lineal en $\mathbb{Z}$, una topología localmente cuasi-convexa compatible más fina, un hecho que prueba que las topologías lineales en $\mathbb{Z}$ no son Mackey. Nuestras consideraciones nos han llevado a producir una amplia cantidad de ejemplos y herramientas para trabajar con topologías en los enteros.

Además, las topologías lineales en $\mathbb{Z}$ proporcionaron una familia de topologías metrizables localmente cuasi-convexas que no son Mackey. Esto fue todo un descubrimiento, después de muchos intentos anteriores, de probar lo contrario, basados en el hecho: todo espacio metrizable localmente convexo tiene la topología de Mackey.

Nos marcamos un segundo objetivo: encontrar ejemplos de topologías metrizables, en la clase de los grupos localmente cuasi-convexos, que no fueran Mackey. Para este fin, solamente debíamos considerar topologías no completas (en efecto, en [CMPT99] se probó que los grupos localmente cuasi-convexos, metrizables y completos llevan la topología de Mackey).

En este sentido, la clase de los grupos numerables podría proporcionar ejemplos ya quela única topología de grupo metrizable y completa que admiten es la discreta (debido al Teorema de Categoría de Baire). Por tanto, tenemos que las topologías de grupo metrizables en grupos numerables son buenos candidatos para no ser Mackey. Sin embargo, para grupos numerables acotados, en [BTM03] encontramos ejemplos de topologías metrizables que son Mackey.

Después de estudiar el problema en el grupo de los enteros pasmos a otros grupos. El grupo más natural habría sido el de los reales, pero su topología (siendo localmente compacta) es Mackey. Por lo tanto, estudiamos a continuación si $\mathbb{Q}$ con la topología heredada de $\mathbb{R}$ (como subgrupo) es Mackey.Esta pregunta permaneció abierta durante varios años. Por un resultado de [DNMP14] este problema se puede reducir al estudio de topologías en unos subgrupos infinitos de torsión del círculo unidad $\mathbb{T}$. Estos subgrupos están íntimamente relacionados con las topologías lineales estudiadas en el grupo de los enteros. En efecto, estos subgrupos de $\mathbb{T}$ coinciden con grupos duales de $\mathbb{Z}$ dotado de topologías lineales. Finalmente, obtuvimos que la topología en $\mathbb{Q}$ heredada de $\mathbb{R}$ no es Mackey [BM14].

Una característica especial de los grupos acotados es que toda topología localmente cuasi-convexa es lineal. El recíproco es cierto para cualquier tipo de grupo. Por lo tanto, el estudio del Problema Mackey en grupos acotados puede apoyarse en las propiedades de los grupos lineales, junto con el hecho de que “ser un grupo Mackey” es hereditario para subgrupos abiertos.

Contenidos and resultados:

El primer Capítulo trata sobre la introducción a los grupos topológicos.

En la Sección 1.1 recordamos la definición de grupo topológico y damos algunos ejemplos básicos. También tratamos la definición de conjunto cuasi-convexo – un pilar fundamental en este trabajo.

La Sección 1.2 está dedicada a topologías precompactas. Recogemos hechos básicos y resultados sobre ellas.

En la Sección 1.3 definimos la topología de convergencia uniforme en un conjunto de homomorfismos. Damos una base de entornos apropiada para esta topología y demostramos algunos resultados.

En la Sección 1.4 tratamos con una familia de topologías completas. Estas topologías, introducidas por Graev, vienen definidas por la condición de ser la topología de grupo más fina en la que una sucesión de elementos, fijada de antemano, converge al elemento neutro.

A lo largo de este capítulo hemos incluido referencias a resultados conocidos y dado una demostración si ésta resultaba interesante.

El Capítulo 2 está dedicado a las D-sucesiones. Estas sucesiones de números naturales tienen la particularidad de que cada término divide al siguiente. Generalizan la sucesión (p^n) y serán fundamentales para describir tanto las topologías lineales en los enteros como para construir subgrupos infinitos de torsión en $\mathbb{T}$.

En la Sección 2.1 damos la definición de D-sucesión y algunas clases especiales de ellas. La sucesión de los cocientes de una D-sucesión también tendrá un papel importante. Definimos algunos subconjuntos del conjunto $\mathcal{D}$ de todas las D-sucesiones, que serán importantes en los Capítulos 4 y 5 ($\mathcal{D}_\infty, \mathcal{D}_\infty^\ell$). También definimos la "unión" de dos subsucesiones de una D-sucesión. Finalmente, definimos la función $\mathbb{P}$ que es, sencillamente, una generalización de la descomposición de un número natural en el producto de sus factores primos.

La Sección 2.2 está dedicada a encontrar representaciones adecuadas tanto de los números enteros como de los números reales tomando una D-sucesión como referencia (Proposiciones 2.2.1 y 2.2.4, respectivamente). La representación 10-ádica de enteros y reales es generalizada a D-sucesiones. Esta representación nos ayudará a trabajar con entornos de ciertas topologías de convergencia uniforme relacionadas con D-sucesiones (en el sentido que será explicado en los Capítulos

4 y 5). También incluimos algunos ejemplos para que el lector se familiarice con nuestra representación. Hemos llamado **b**-coordenadas a los coeficientes de la representación de los enteros o reales en términos de la D-sucesión **b**.

En la Sección 2.3 asignamos a cada D-sucesión, **b**, un subgrupo infinito de torsión $\mathbb{Z}(\mathbf{b}^\infty)$ de $\mathbb{T}$. Estos subgrupos son una generalización de los grupos de Prüfer $\mathbb{Z}(\mathbf{p}^\infty)$. Algnos subgrupos específicos $\mathbb{Z}(\mathbf{b}^\infty)$ nos permitirán probar que la topología euclídea de $\mathbb{Q}$ no es Mackey.

En consonancia con la representación de los números enteros como suma de series respecto de una D-sucesión, **b**, definimos el conjunto $\mathbb{Z}_\mathbf{b}$ de los enteros **b**-ádicos en la Sección 2.4. Posteriormente, veremos que se corresponde con la complección del grupo de los enteros con la correspondiente topología **b**-ádica. Por el Teorema de Pontryagin-van Kampen, es claro que $\mathbb{Z}_\mathbf{b} = \mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T})$ (Remark 4.2.22).

El Capítulo 3 establece el marco general del Problema Mackey.

En la Sección 3.1 damos el marco histórico del problema. Se señala el camino seguido desde el Teorema de Mackey-Arens al enunciado del Problema Mackey para grupos abelianos topológicos. Algunos resultados recientes también están incluidos.

La Sección 3.2 trata con la familia de topologías de grupo compatibles con una dualidad. Hemos incluido resultados conocidos (las topologías compatibles tienen los mismos subgrupos dualmente cerrados) y los nuevos son demostrados. Para grupos acotados, encontramos varias clases de subgrupos que dos topologías compatibles deben compartir (cerrados, dualmente cerrados, cerrados de índice finito).

En la Sección 3.3 probamos que los subgrupos abiertos de grupos de Mackey son también de Mackey. Este resultado está enunciado en [Leo08], pero sin demostración. Se formula la pregunta de si el recíproco también es cierto. Una respuesta parcial se obtiene en el Capítulo 6.

En la Sección 3.4 encontramos algunas formulaciones equivalentes del Problema Mackey para grupos acotados como consecuencia del Teorema 3.2.5.

En la Sección 3.5 enunciamos el Problema $\mathcal{G}$-Mackey, que es una generalización del Problema Mackey. Damos algunos resultados interesantes.

El Capítulo 4 es el capítulo principal de esta tesis y trata sobre topologías de grupo en el grupo de los enteros. Consideramos, principalmente, topologías de convergencia uniforme en un subconjunto B del grupo dual $\mathbb{T}$ relacionado con una D-sucesión. También consideramos topologías de Graev en el grupo de los enteros.

En la Sección 4.2 recogemos algunos resultados conocidos sobre topologías precompactas en los enteros. Para generalizar las topologías p-ádicas, construimos, mediante una D-sucesión $\mathbf{b}$, una topología lineal (y precompacta) en los enteros, $\lambda_{\mathbf{b}}$, (Definición 4.2.5), a la que hemos llamado topología $\mathbf{b}$-ádica. En primer lugar, probamos que toda topología lineal en los enteros se puede definir mediante una D-sucesión adecuada (Proposition 4.2.6).

Para poder comparar diferentes topologías lineales, utilizamos la función $\mathbb{P}$ definida en el Capítulo 2. En la Proposición 4.2.9 probamos que dos topologías lineales $\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}$ satisfacen $\lambda_{\mathbf{b}} \leq \lambda_{\mathbf{c}}$ si las imágenes asociadas, mediante $\mathbb{P}$, cumplen $\mathbb{P}(\mathbf{b}) \leq \mathbb{P}(\mathbf{c})$. Esto nos permite asegurar que existen 2^{ω} topologías lineales diferentes en el grupo de los enteros (Lemma 4.2.11).

Para una D-sucesión, $\mathbf{b}$, calculamos el grupo dual $\mathbb{Z}$ dotado de la topología lineal $\lambda_{\mathbf{b}}$. En la Proposición 4.2.13 se prueba que $(\mathbb{Z}, \lambda_{\mathbf{b}})^{\wedge} = \mathbb{Z}(\mathbf{b}^{\infty})$.

Puesto que toda topología metrizableestá determinada por sus sucesiones convergentes, en la Proposición 4.2.15 estudiamos las sucesiones nulas de $\lambda_{\mathbf{b}}$. En particular, caracterizamos la convergencia en una topología lineal en términos de las $\mathbf{b}$-coordenadas de los elementos de la sucesión.

En el Teorema 4.2.21 probamos que para una D-sucesión, la complección de $(\mathbb{Z}, \lambda_{\mathbf{b}})$ es isomorfa a $\mathbb{Z}_{\mathbf{b}}$.

La Sección 4.3 trata sobre las topologías de convergencia uniforme en el grupo de los enteros.

La Subsección 4.3.1 aborda las topologías de convergencia uniforme que están, en cierto sentido, originadas por una D-sucesión. Más precisamente, fijada una D-sucesión $\mathbf{b} = (b_n)$, definimos $\underline{\mathbf{b}} := \left\{\frac{1}{b_n} + \mathbb{Z}\right\}$ en $\mathbb{T}$. Debido a que $\mathbb{T}$ puede ser identificado con todos los caracteres definidos en $\mathbb{Z}$, el rango de la sucesión $\underline{\mathbf{b}}$ puede ser considerado como un subconjunto de $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$. Por tanto, definimos $\tau_{\mathbf{b}}$ como la topología en $\mathbb{Z}$ de convergencia uniforme en el rango de $\underline{\mathbf{b}}$. Una base de entornos de 0 es de $\tau_{\mathbf{b}}$ es descrita en la Definición 4.3.3. En la Proposición 4.3.4 probamos que $\tau_{\mathbf{b}}$ es metrizable y localmente cuasi-convexa.

La Proposición 4.3.8 nos da la posibilidad de comparar las topologías $\tau_{\mathbf{b}}, \tau_{\mathbf{c}}$, siendo **c** una subsucesión de **b** con ciertas condiciones extra. Esta proposición será la herramienta principal para demostrar el Teorema 4.3.31.

No es sencillo manejar la topología $\tau_{\mathbf{b}}$. Por esta razón encontramos una caracterización de las **b**-coordenadas de un entero para que pertenezca a un entorno básico fijado. Combinando las Proposiciones 4.3.11 y 4.3.12 podemos definir una nueva base de entornos para la topología $\tau_{\mathbf{b}}$ que depende exclusivamente de las **b**-coordenadas de los números enteros.

Las sucesiones nulas de $\tau_{\mathbf{b}}$, se caracterizan en la Proposición 4.3.15.

Relacionadas con una D-sucesión **b**, hemos definido dos topologías $\lambda_{\mathbf{b}}$ y $\tau_{\mathbf{b}}$. La Proposición 4.3.17 afirma que estas dos topologías son comparables. De hecho, obtenemos que $\lambda_{\mathbf{b}} \leq \tau_{\mathbf{b}}$. Como $\tau_{\mathbf{b}}$ nunca es precompacta, también resulta claro que $\lambda_{\mathbf{b}} \neq \tau_{\mathbf{b}}$.

La Subsección 4.3.2 está dedicada al grupo dual de $(\mathbb{Z}, \tau_{\mathbf{b}})$ si $\mathbf{b} \in \mathcal{D}_{\infty}^{\ell}$. El Teorema principal de la subsección (Teorema 4.3.31) lo caracteriza.

En el Lema 4.3.19 encontramos una subsucesión “especial” de la D-sucesión **b** que es nula en $\tau_{\mathbf{b}}$. Naturalmente, esta subsucesión no es única, pero su existencia será importante en lo que sigue.

Para demostrar el Teorema 4.3.31, que es una herramienta formidable para el resto del capítulo, requerimos de algunas propiedades técnicas, tal y como expresamos en la Proposición 4.3.21, después de definir algunas nociones útiles.

La Proposición 4.3.22 es de gran interés. Para una D-sucesión, **b** y un elemento χ que pertenece a $(\mathbb{Z}, \tau_{\mathbf{b}})^{\wedge}$, encontramos una condición en las **b**-cordenadas de $\chi(1)$. Esto será útil posteriormente para decidir si un carácter de $\mathbb{Z}$ es $\tau_{\mathbf{b}}$-continuo. Como corolario obtenemos el Teorema 4.3.24, que afirma que las topologías lineales no-discretas en Z no son de Mackey. Este resultado también se obtiene como corolario del Teorema 4.3.31, pero, cronológicamente primero obtuvimos este resultado, que, precisamente, motivó la obtención del Teorema 4.3.31.

Como paso intermedio para el Teorema 4.3.31, encontramos un método para escribir las **c**-coordenadas de un número real en función de sus **b**-coordenadas, cuando **c** es una subsucesión de **b**.

Finalmente, el Teorema 4.3.31 afirma que si $\mathbf{b} \in \mathcal{D}_{\infty}^{\ell}$, entonces se tiene que $(\mathbb{Z}, \tau_{\mathbf{b}})^{\wedge} = \mathbb{Z}(\mathbf{b}^{\infty})$.

Las D-sucesiones que hemos considerado hasta ahora pertenecen a $\mathcal{D}_{\infty}^{\ell}$. En la Subsección 4.3.3 estudiamos la dualidad para D-sucesiones que no están en $\mathcal{D}_\infty^\ell$ para ningún ℓ. Para ese fin, introducimos la condición bloque. En términos sencillos, una D-sucesión satisface la condición bloque si existe un número entero c y bloques de términos consecutivos de la sucesión de cocientes con valores menores o iguales a c, cuyas longitudes tiendan a infinito.

A pesar de que las D-sucesiones que satisfacen la condición bloque no han sido completamente estudiadas, hemos obtenido el siguiente resultado: para una D-sucesión $\mathbf{b}$ que satisface la condición bloque, la topología $\tau_{\mathbf{b}}$ no es compatible con la dualidad $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$ (ver Teorema 4.3.36). Esta situación contrasta con las D-sucesiones pertenecientes a $\mathcal{D}_\infty^\ell$.

En la Proposición 4.3.37 damos un ejemplo de D-sucesión que no pertenece a $\mathcal{D}_\infty^\ell$ para ningún número natural ℓ pero que tampoco satisface la condición bloque.

En la Subsección 4.3.4 construimos una sucesión de D-sucesiones y probamos que el supremo de las topologías de convergencia uniforme en cada D-sucesión, no coincide con la topología de convergencia uniforme en la D-sucesión obtenisda como unión de todas las sucesiones.

En la Sección 4.4, definimos el tercer tipo de topologías localmente cuasi-convexa en los enteros asociada a una D-sucesión dada. Para una D-sucesión, $\mathbf{b}$, tomamos todas las subsucesiones de $\mathbf{b}$ que pertenecen a $\mathcal{D}_\infty^\ell$ para algún número natural ℓ. Denotamos a esta familia por Δ y definimos la nueva topología $\delta_{\mathbf{b}}$ como el supremo de todas las topologías de convergencia uniforme asociadas a las D-sucesiones en Δ (Definición 4.4.4).

En la Proposición 4.4.3 probamos que la familia de topologías de convergencia uniforme asociadas a elementos de Δ es una familia dirigida. Esto inmediatamente nos lleva a que la topología $\delta_{\mathbf{b}}$ es compatible (Teorema 4.4.6).

En la Proposición 4.4.7 las topologías $\tau_{\mathbf{b}}$ y $\delta_{\mathbf{b}}$ son comparadas.

En la Subsección 4.4.1, estudiamos algunas características de la topología $\delta_{\mathbf{b}}$ si $\mathbf{b}$ tiene cocientes acotados (es decir, la sucesión de cocientes es acotada).

El Teorema 4.4.8 es una herramienta importante. Si $\mathbf{b}$ denota a una D-sucesión básica con cocientes acotados, para una sucesión nula en $\lambda_{\mathbf{b}}$ dada, (a_n), construimos una nueva topología metrizable, localmente cuasi-convexa, compatible con $\lambda_{\mathbf{b}}$ (y por tanto más fina que $\lambda_{\mathbf{b}}$) y tal que (a_n) no converge en esta nueva topología. Por este procedimiento de eliminar las sucesiones nulas en $\lambda_{\mathbf{b}}$, obtenemos una familia de topologías cuyo supremo S no tiene sucesiones convergentes. Por lo tanto, es una topología no metrizable, que es, además, localmente cuasi-convexa y compatible. Como consecuencia, obtenemos que si $\mathbf{b}$ tiene cocientes acotados, entonces $\delta_{\mathbf{b}}$, que es más fina que S es una topología no metrizable y compatible con $\lambda_{\mathbf{b}}$ (Teorema 4.4.9).

Adicionalmente, probamos que para una D-sucesión $\mathbf{b}$ con cocientes acotados, el grupo $(\mathbb{Z}, \delta_{\mathbf{b}})$ no es reflexivo (Proposición 4.4.12).

La Subsección 4.4.2 sobre las características de $\delta_{\mathbf{b}}$ si $\mathbf{b} \in \mathcal{D}_\infty^\ell$ para algún ℓ. En la Proposición 4.4.20, encontramos dos D-sucesiones, $\mathbf{b}, \mathbf{c}$ que satisfacen que las topologías $\delta_{\mathbf{b}}$ y $\delta_{\mathbf{c}}$ son compatibles pero no comparables. En este ejemplo específico, ni $\delta_{\mathbf{b}}$, ni $\delta_{\mathbf{c}}$ pueden ser la topologías de Mackey para la dualidad $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$.

En la Sección 4.5 estudiamos topologías completas en el grupo de los enteros. Dado que cualquier D-sucesión $\mathbf{b}$ converge a 0 en $\lambda_{\mathbf{b}}$, es en particular una T-sucesión y, por tanto, hay una topología de Graev $\mathcal{T}_{\{b_n\}}$ asociada a cada D-sucesión. Esta es la topología de grupo en $\mathbb{Z}$ más fina que hace que $\mathbf{b}$ sea convergente. En particular, para $\mathbf{p} = (p^n)$, probamos que la correspondiente topología de Graev es compatible con $\lambda_{\mathbf{p}}$ (Proposición 4.5.4). Sin embargo, no es localmente cuasi-convexa, y por tanto no reflexiva (Proposición 4.5.5).

En la Sección 4.6 recogemos preguntas abiertas sobre topologías en los enteros.

El Capítulo 5, trata sobre subgrupos infinitos de torsión en el círculo unidad. El resultado principal de este capítulo es que el grupo de números racionales dotado de la topología euclídea no es un grupo de Mackey.

En la Sección 5.1 consideramos dos topologías de grupo localmente cuasi-convexas en el grupo $\mathbb{Z}(\mathbf{b}^\infty)$. La primera (en la Subsección 5.1.1) es la topología heredada de $\mathbb{T}$ (la topología "usual") y la segunda es la topología de convergencia uniforme en $\mathbf{b}$ (Subsección 5.1.2), a la que llamamos $\eta_{\mathbf{b}}$.

En la Subsección 5.1.1, primero damos una base de entornos apropiada para la topología usual en $\mathbb{Z}(\mathbf{b}^\infty)$ y, después, damos condiciones sobre las $\mathbf{b}$-coordenadas de un elemento de $\mathbb{Z}(\mathbf{b}^\infty)$ para que pertenezca a un entorno específico de la mencionada base (Lema 5.1.2).

En la Proposición 5.1.3 caracterizamos las sucesiones nulas en la topología usual por medio de las $\mathbf{b}$-coordenadas de los términos de la sucesión. Esto lo utilizaremos para comparar la topología usual en $\mathbb{Z}(\mathbf{b}^\infty)$ y $\eta_{\mathbf{b}}$.

En la Subsección 5.1.2, consideramos, para una D-sucesión $\mathbf{b}$ en $\mathbb{Z}$, una nueva topología en el grupo $\mathbb{Z}(\mathbf{b}^\infty)$. En particular, la topología de convergencia uniforme en $\mathbf{b}$, a la que llamamos $\eta_{\mathbf{b}}$. En la Definición 5.1.6 damos una base de entornos para la topología $\eta_{\mathbf{b}}$. Observamos que en este caso hemos identificado los enteros con su copia natural en $\mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T}) \cong \mathbb{Z}_{\mathbf{b}}$ obtenida identificando un entero con el homomorfismo "evaluación en él".

A continuación, definimos una base de entornos adecuada para $\eta_{\mathbf{b}}$ (Proposiciones 5.1.9 y 5.1.10).

En la Proposición 5.1.12 las sucesiones nulas en $\eta_{\mathbf{b}}$ son caracterizadas en función de las $\mathbf{b}$-coordenadas de los términos de la sucesión. Esta proposición nos permite comparar la topología inducida por la euclídea de $\mathbb{T}$ en $\mathbb{Z}(\mathbf{b}^\infty)$ con $\eta_{\mathbf{b}}$, lo cual se hace en la Subsección 5.1.3 (Proposición 5.1.15).

La Sección 5.2 está dedicada al grupo dual de $(\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{b}})$.

En el Teorema 5.2.3 el grupo dual de $(\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{b}})$ es calculado. En particular, este grupo dual es $\mathbb{Z}$, lo que significa que la topología usual en $\mathbb{Z}(\mathbf{b}^\infty)$ y $\eta_{\mathbf{b}}$ son topologías compatibles. Por lo tanto, la topología usual en $\mathbb{Z}(\mathbf{b}^\infty)$ no es Mackey (Teorema 5.2.4).

Por el Corolario 7.5 de [DNMP14], que afirma que los cocientes localmente cuasi-convexos de grupos Mackey son Mackey, es posible probar que $\mathbb{Q}$ equipado con la topología heredada de la recta real no es un grupo Mackey. Este es el primer ejemplo conocido de una topología localmente precompacta y metrizable que no es Mackey.

En el Capítulo 6 consideramos la siguiente pregunta: ¿Existe alguna familia especial de grupos abelianos $\mathcal{G}$ en los que la siguiente afirmación se cumpla: todo grupo metrizable y localmente cuasi-convexo en $\mathcal{G}$ es Mackey? Esta pregunta está inspirada por la Teoría de Mackey en espacios localmente convexos. Como hemos dicho anteriormente, un espacio metrizable y localmente convexo tiene la topología de Mackey, y por tanto, es un espacio Mackey. En el caso de grupos abelianos, probamos que esta propiedad caracteriza precisamente los grupos acotados (Teorema 6.1.3). Este es un ejemplo interesante de como una propiedad algebraica puede caracterizarse mediante una propiedad topológica.

La demostración del Teorema 6.1.3 se divide en dos partes: en la Sección 6.2

probamos que una topología metrizable localmente cuasi-convexa en un grupo acotado es Mackey y en la Sección 6.3 encontramos para cada grupo G no acotado una topología metrizable localmente cuasi-convexa en G que no es Mackey.

En la Sección 6.2 utilizamos algunas nociones en subgrupos densos y demostramos que toda topología localmente cuasi-convexa en un grupo acotado es lineal (Proposición 6.2.10). Por lo tanto, una topología localmente cuasi-convexa en un grupo acotado es Mackey si y sólo si es $\mathcal{L}in$-Mackey (Mackey en la clase de topologías de grupo lineales).

Adicionalemente, probamos que una topología localmente cuasi-convexa en un grupo acotado cuyo dual tenga cardinalidad $< \mathfrak{c}$ debe ser precompacta. Como consecuencia obtenemos que tales topologías son Mackey.

Al final de la Sección probamos que una topología localmente cuasi-convexa en un grupo acotado que sea (a) metrizable o (b) cuyo dual tenga cardinalidad $< \mathfrak{c}$ es una topología Mackey (Teorema 6.2.1).

La Sección 6.3 consiste en la demostración de que para todo grupo no acotado G, existe una topología metrizable localmente cuasi-convexa en G que no es Mackey. Se apoya en el hecho que todo grupo no acotado tiene un subgrupo de la forma $\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty), \mathbb{Q}, \bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$ siendo (m_n) una sucesión creciente de números naturales. En este punto, sabemos que $\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty)$ y $\mathbb{Q}$ pueden ser equipados con una topología metrizable localmente cuasi-convexa que no es Mackey.

En la Proposición 6.3.2 demostramos que los mencionados grupos $\bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$ también pueden ser dotados de una topología metrizable localmente cuasi-convexa que no es Mackey. En este caso es la topología heredada del producto $\prod_{n=1}^\infty \mathbb{Z}_{m_n}$. Finalmente probamos el Teorema 6.3.1, teniendo en cuenta que un subgrupo abierto de un grupo Mackey es, también, Mackey.

En la Sección 6.4 encontramos algunos corolarios del Teorema 6.2.1 sobre topologías en grupos acotados.

Conclusiones:

El Problema Mackey para grupos topológicos ha demostrado ser una fuente para investigación matemática. A pesar de no presentar una respuesta al problema principal, hemos probado varios resultados de gran interés que podrían ser usados para resolver una gran variedad de problemas. Nuestras técnicas nos han llevado a desarrollar Análisis Numérico en $\mathbb{Z}$ y otros grupos. Preguntas muy sencillas siguen abiertas. Por ejemplo: Existe la topología de Mackey en la dualidad

$(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$ o en $(\mathbb{Q}, \mathbb{R})$?

Además, la investigación desarrollada para esta tesis ha motivado publicaciones en revistas indexadas (ver [ABM12] en Journal of Pure and Applied Algebra y [BM14] en Topology and its Applications). Las Secciones 4.3 y 4.4, y el Capítulo 6 serán objeto de dos futuras publicaciones. Algunos resultados de la Sección 4.5 aparecieron en una publicación (conjunta con Elena Martín-Peinador) En un volumen especial en honor del Profesor Etayo.

Además, las técnicas que hemos utilizado pueden ser de interés en otras áreas de investigación matemática.

La posibilidad de encontrar familias de topologías metrizables localmente cuasi-convexas que no son Mackey hacen pensar que la completitus o, al menos, otra condición, es necesaria para encontrar topologías Mackey. Para grupos acotados, hemos resuelto completamente la cuestión: toda topología metrizable localmente cuasi-convexa en un grupo acotado es Mackey.

Además, hemos reformulado el Problema Mackey en grupos acotados de varias maneras alternativas. Esto se hace en la esperanza de que estas formulaciones alternativas pudieran ayudar a resolver el Problema Mackey en la clase de grupos acotados.

También hemos probado, por diferentes ejemplos, que los subgrupos densos de los grupos Mackey no son, necesariamente, Mackey. Por ejemplo, las topologías lineales en $\mathbb{Z}$ no son Mackey. Al ser precompactas, su compleción es un grupo Haussdorf y compacto, por tanto, un grupo Mackey, que contiene un subgrupo denso que no es Mackey. El mismo argumento es válido para $\mathbb{Z}(\mathbf{b}^\infty)$ dotado de la topología heredada de la euclídea de $\mathbb{T}$. La topología de $\mathbb{Q}$ heredada de $\mathbb{R}$ tampoco es Mackey. Sin embargo, un subgrupo abierto de un grupo Mackey es Mackey. El recíproco lo probamos para la clase de grupos acotados localmente cuasi-convexos (Proposición 6.2.16). Otras propiedades de permanencia de la topología de Mackey no son conocidas. Por ejemplo, ¿es el producto de dos grupos Mackey, de nuevo, Mackey? Este será uno de nuestros primeros objetivos para el trabajo futuro.

Chapter 1

Definitions, notation and basic results.

1.1 Topological groups.

In this section we set the main definitions that will be used along this dissertation. Some notation and basic results are also presented. We start with the definition of a topological group.

Definition 1.1.1 Let $(G, \cdot)$ be a group. Let τ be a topology on G. If the mappings

$$\cdot : G \times G \to G \qquad\qquad (g,h) \mapsto g \cdot h$$

and

$$\cdot^{-1} : G \to G \qquad\qquad g \mapsto g^{-1}$$

are continuous, then τ is said to be a **group topology** and (G, τ) is called **topological group**

The following examples show that the most usual groups are topological groups:

Example 1.1.2 The following pairs of groups and topologies are topological groups:

- The group of real numbers $\mathbb{R}$ endowed with its usual topology is a topological group.

- The group of the integers with the discrete topology is a topological group. In fact, any discrete group is a topological group.
- The unit circle of the complex plane endowed with the euclidean topology is a topological group.

Notation 1.1.3 *A group G endowed with the discrete topology will be denoted by G_d.*

Definition 1.1.4 We will consider the unit circle as

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} = \left(-\frac{1}{2}, \frac{1}{2}\right] + \mathbb{Z}.$$

We will consider in $\mathbb{T}$ the following neighborhood basis:

$$\mathbb{T}_m := \left[-\frac{1}{4m}, \frac{1}{4m}\right] + \mathbb{Z}.$$

We set

$$\mathbb{T}_+ := \left[-\frac{1}{4}, \frac{1}{4}\right] + \mathbb{Z} = \mathbb{T}_1.$$

Definition 1.1.5 A group topology is called **linear** if it has a neighbourhood basis at 0 consisting of (necessarily open) subgroups. We denote by $\mathcal{L}in$ the class of all linear topological groups.

The following lemma allows us to prove if a subset of the power set of the group G is a neighborhood basis for a group topology on G.

Lemma 1.1.6 (Chapter III, §1.2, Proposition 1 in [Bou66].) *Let G be an abelian topological group and let $\mathcal{U}$ be a non-empty subset of the power set of G satisfying:*

(i) $0 \in U$ for all $U \in \mathcal{U}$.

(ii) For all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $-V \subseteq U$.

(iii) For all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V + V \subseteq U$.

(iv) For every pair $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subseteq U \cap V$.

Then there exists a unique group topology $\mathcal{O}$ on G such that $\mathcal{U}$ is a neighbourhood basis at 0. $(G, \mathcal{O})$ is a Hausdorff space if and only if $\bigcap_{U \in \mathcal{U}} U = \{0\}$.

Theorem 1.1.7 (Birkhoff-Kakutani Theorem) *[MZ55]*
Let (G, τ) be a topological group. The following assertions are equivalent:

- *G is metrizable.*
- *G is T_2 and there exists a countable neighborhood basis at 0 for τ.*

Definition 1.1.8 Let $\mathcal{F}$ be a family of group topologies on a group G. We say that $\mathcal{F}$ **is a directed family** if for any $F_1, F_2 \in \mathcal{F}$ exists $F_3 \in \mathcal{F}$ such that $\sup\{F_1, F_2\} \leq F_3$.

Given a topological group (G, τ) it is natural to study the homomorphisms from G to some other group. The most natural candidate would be the group $\mathbb{R}$. However, if we consider G to be a compact group, the only continuous homomorphism from G to $\mathbb{R}$ is the trivial one (the image of a compact group through a continuous homomorphism is again a compact group and the only compact subgroups of $\mathbb{R}$ are the trivial ones). So we consider $\mathbb{T}$ instead of $\mathbb{R}$ as dualizing object.

Definition 1.1.9 Let (G, τ) be a topological group. Denote by $G^\wedge$ the set of continuous homomorphisms $\mathrm{CHom}(G, \mathbb{T})$. This set is a group when we consider the operation $(f + g)(x) = f(x) + g(x)$. In addition, if we consider the compact open topology on $G^\wedge$ it is a topological group. The topological group $G^\wedge$ is called the **dual group of G.**

Example 1.1.10 The following examples are very natural:

- $\mathbb{Z}^\wedge = \mathbb{T}$.
- $\mathbb{T}^\wedge = \mathbb{Z}$.
- $\mathbb{R}^\wedge = \mathbb{R}$.

Definition 1.1.11 Let (G, τ) be a topological group and $G^\wedge$ its dual group. Let ν be another group topology on G. We say that τ and ν are **compatible** if $(G, \nu)^\wedge = G^\wedge$.

Definition 1.1.12 Let (G, τ) be a topological group and $H \subset G^\wedge$. We say that H **separates the points of** G if for every $x \neq 0$, there exists $\chi \in H$ such that $\chi(x) \neq 0 + \mathbb{Z}$

Definition 1.1.13 Let (G, τ) be a topological group and $H \leq G$ a subgroup. We say that H is **dually closed** if for any $x \in G \setminus H$ there exists $\chi \in G^\wedge$ such that $\chi(H) = 0 + \mathbb{Z}$ and $\chi(x) \neq 0 + \mathbb{Z}$.

Lemma 1.1.14 ([Nob70]) *Every open subgroup is dually closed.*

Definition 1.1.15 Let (G, τ) a topological group. The **polar** of a subset $S \subset G$ is defined as

$$S^{\triangleright} := \{\chi \in G^\wedge \mid \chi(S) \subset \mathbb{T}_+\}.$$

Definition 1.1.16 For a dual group $G^\wedge$, the **pre-polar** of a subset $S \subset G^\wedge$ is defined as

$$S^{\triangleleft} := \{x \in G \mid \chi(x) \in \mathbb{T}_+ \text{ for all } \chi \in S\}.$$

Lemma 1.1.17 *Let (G, τ) be a topological group. The compact-open topology in $G^\wedge$ can be described as the family of sets*

$$\mathcal{U}_{G^\wedge}(0) = \{K^{\triangleright} \mid K \subset G \textit{ is compact }\}$$

taken as a neighborhood basis at 0.

Notation 1.1.18 *Put* $G^* := \mathrm{Hom}(G, \mathbb{T}) = \mathrm{CHom}(G_d, \mathbb{T})$.

We recall now two easy results about duality:

Proposition 1.1.19 *Let K be a compact abelian topological group. Then $K^\wedge$ is discrete.*

Proof:

$K^{\triangleright} = \{\chi \in K^\wedge \mid \chi(K) \subseteq \mathbb{T}_+\}$. Since $\chi(K)$ is a subgroup of $\mathbb{T}$,

$$\chi(K) \subset \mathbb{T}_+ \Longleftrightarrow \chi(K) = \{1\}.$$

Hence $K^{\triangleright} = \{\chi \in K^\wedge \mid \chi(K) = \{1\}\}$. Thus, $K^{\triangleright}$ consists only of the null character, 1. Hence $\{1\}$ is open, and $K^\wedge$ is discrete.

QED

Proposition 1.1.20 *Let G_d be a discrete topological group. Then* $\mathrm{Hom}(G, \mathbb{T})$ *is a compact subgroup of* $\mathbb{T}^G$.

Proof:

By Tychonoff Theorem, we know that $\mathbb{T}^G$ is compact. Therefore, it suffices to prove that $\mathrm{Hom}(G, \mathbb{T})$ is a closed subgroup of $\mathbb{T}^G$.

In order to prove that $\mathrm{Hom}(G, \mathbb{T})$ is closed in $\mathbb{T}^G$, we take $(\chi_j)_{j\in J}$ a net in $\mathrm{Hom}(G, \mathbb{T})$ converging to $\chi : G \to \mathbb{T}$. We must see that χ is a homomorphism.

Let $x, y \in G$. It suffices to show that $\chi(x + y) = \chi(x)\chi(y)$.

$\chi(x+y) = \lim_j \chi_j(x+y) = \lim_j(\chi_j(x)\chi_j(y)) = \lim_j \chi_j(x) \lim_j \chi_j(y) = \chi(x)\chi(y)$.

Hence $\mathrm{Hom}(G, \mathbb{T})$ is closed.

QED

Corollary 1.1.21 *Let* (G, τ) *be a discrete group. Then* $(G, \tau)^\wedge$ *is compact.*

Proof:

Since $\mathrm{Hom}(G, \mathbb{T}) = (G, \tau)^\wedge$ the result follows.

QED

Since $G^\wedge$ is a topological group we can consider its dual group. This new group is the bidual of G. Now we recall some facts on this group.

Definition 1.1.22 Let (G, τ) be a topological group. The group

$$G^{\wedge\wedge} := (G^\wedge)^\wedge$$

is called the **bidual** group of G.

Definition 1.1.23 The natural embedding

$$\alpha_G : G \to G^{\wedge\wedge},$$

is defined by

$$x \mapsto \alpha_G(x) : G^\wedge \to \mathbb{T},$$

where

$$\chi \mapsto \chi(x).$$

Proposition 1.1.24 *α_G is a homomorphism.*

Proof:

$\alpha_G(x+y)(\chi) = \chi(x+y) = \chi(x)\chi(y) = \alpha_G(x)(\chi)\alpha_G(y)(\chi) = (\alpha_G(x)\alpha_G(y))(\chi)$, for all $\chi \in G^\wedge$.

Thus, $\alpha_G(x+y) = \alpha_G(x)\alpha_G(y)$

QED

Definition 1.1.25 A topological group (G, τ) is **reflexive** if α_G is a topological isomorphism.

Proposition 1.1.26 *The canonical mapping α_G is injective if and only if $G^\wedge$ separates points; that is, for every $x \neq 0$, there exists $\chi \in G^\wedge$ such that $\chi(x) \neq 0 + \mathbb{Z}$.*

Proposition 1.1.27 *If G is metrizable, then α_G is continuous.*

The following highly nontrivial assertion was proved by Weyl and is the cornerstone for the duality Theorem of Pontryagin-Van Kampen.

Theorem 1.1.28 *Let G be a compact abelian topological group, then α_G is injective.*

An important tool in our subsequent work is the Pontryagin-Van Kampen theorem, which states the following.

Theorem 1.1.29 (Pontryagin-Van Kampen) *Let G be a locally compact abelian topological group. Then G is reflexive.*

We recall now the definition of quasi-convex subset of a topological group and some of their properties.

Definition 1.1.30 Let (G, τ) be a topological group and let $M \subset G$ be a subset. We say that M is **quasi-convex** if for every $x \in G \setminus M$, there exists $\chi \in G^\wedge$ such that $\chi(M) \subset \mathbb{T}_+$ and $\chi(x) \notin \mathbb{T}_+$. We say that τ is **locally quasi-convex** if there exists a neighborhood basis for τ consisting of quasi-convex subsets. We denote by *LQC* the class of all locally quasi-convex topological groups.

Example 1.1.31 The following groups are locally quasi-convex:

- $\mathbb{R}$ is locally quasi-convex, since $[-\frac{1}{n}, \frac{1}{n}]$ is quasi-convex for all n.
- $\mathbb{T}$ is locally quasi-convex, since $\{[-\frac{1}{4n}, \frac{1}{4n}] + \mathbb{Z} \mid n \in \mathbb{N}\}$ is a quasi-convex neighbourhood basis for 0 in $\mathbb{T}$. In particular, $\mathbb{T}_+$ is quasi-convex.

Proposition 1.1.32 *Let (G, τ) be a topological group and let $H \leq G$ be an open subgroup. If $M \subset H$ is quasi-convex, M is also quasi-convex in G.*

Properties 1.1.33 *Let (G, τ) be a topological group and $A \subset G$ be a quasi-convex set, then:*

(i) *A is symmetric.*

(ii) $0 \in A$.

(iii) $A = \bigcap_{\chi \in A^{\triangleright}} \chi^{-1}(\mathbb{T}_+) = (A^{\triangleright})^{\triangleleft}$

(iv) *A is closed.*

Proposition 1.1.34 *Let (G, τ) be a topological group and let $M \subset G$. Then $M^{\triangleright}$ is quasi-convex.*

Proof:

Take $\chi_0 \in G^\wedge \setminus M^{\triangleright}$; that is, $\exists x \in M$ such that $\chi_0(x) \notin \mathbb{T}_+$. $\alpha_G(x) \in G^{\wedge\wedge}$, and for all $\chi \in M^{\triangleright}$ we have $\alpha_G(x)(\chi) = \chi(x) \in \mathbb{T}_+$ but $\alpha_G(x)(\chi_0) = \chi_0(x) \notin \mathbb{T}_+$

QED

Proposition 1.1.35 *Let (G, τ) be a topological group; then $G^\wedge$ endowed with the compact open topology τ_{CO} is locally quasi-convex.*

Proof:

We recall that the family $\mathcal{K} = \{K^{\triangleright} \mid K \text{ is compact}\}$ is a neighbourhood basis for τ_{CO}.

We have just proved that the polar of any subset is quasi-convex, in particular $\mathcal{K}$ is a neighbourhood basis formed by quasi-convex sets.

Hence, $(G^\wedge, \tau_{CO})$ is locally quasi-convex.

QED

Proposition 1.1.36 *Let* $\varphi : (G, \tau) \to (H, \sigma)$ *be a continuous homomorphism and* $A \subset H$ *quasi-convex; then* $\varphi^{-1}(A) \subset G$ *is quasi-convex.*

Proof:

Let $x \notin \varphi^{-1}(A)$, then $\varphi(x) \notin A$.

Since A is quasi-convex, there exists $\chi \in H^\wedge$ such that $\chi\varphi(x) \notin \mathbb{T}_+$ and $\chi(A) \subset \mathbb{T}_+$. Consider $\Psi = \chi \circ \varphi : G \to \mathbb{T}$.

It is clear that

$$\Psi(\varphi^{-1}(A)) = \chi \circ \varphi \circ \varphi^{-1}(A) \subset \chi(A) \subset \mathbb{T}_+,$$

but

$$\Psi(x) = \chi(\varphi(x)) \notin \mathbb{T}_+.$$

QED

Proposition 1.1.37 *Let* $(A_i)_{i \in I} \subset G$ *be a family of quasi-convex sets. Then*

$$\bigcap_{i \in I} A_i$$

is quasi-convex.

Proof:

Let $x \notin \bigcap_{i \in I} A_i$, then there exists i_0 such that $x \notin A_{i_0}$.

Since A_{i_0} is quasi-convex, there exists $\varphi \in G^\wedge$ such that

$$\varphi(x) \notin \mathbb{T}_+$$

and

$$\varphi(A_{i_0}) \subset \mathbb{T}_+.$$

Since $\bigcap_{i \in I} A_i \subset A_{i_0}$, it follows $\varphi\left(\bigcap_{i \in I} A_i\right) \subset \varphi(A_{i_0}) \subset \mathbb{T}_+$.

QED

Proposition 1.1.38 *Let* $S \subset G^\wedge$. *Then* $S^{\triangleleft} := \bigcap_{\chi \in S} \chi^{-1}(\mathbb{T}_+)$ *is quasi-convex.*

Proof:

Since $\mathbb{T}_+$ is quasi-convex it follows that $\chi^{-1}(\mathbb{T}_+)$ is quasi-convex. Hence

$$\bigcap_{\chi\in S}\chi^{-1}(\mathbb{T}_+)$$

is quasi-convex.

QED

After giving some properties of quasi-convexity, we turn our attention to some notation:

Notation 1.1.39 *We denote by*

$$\mathbb{N}_0 := \mathbb{N}\cup\{0\}$$

and by

$$\mathbb{N}_{0,\infty} := \mathbb{N}\cup\{0,\infty\}.$$

Denote by $\mathbb{P} = (p_n)$ the increasing sequence of prime numbers.

Lemma 1.1.40 *Let $(m_n)\subset\mathbb{N}_{0,\infty}$ be a non-decreasing sequence in $\mathbb{N}_{0,\infty}$. Then m_n converges.*

We can also consider the product $\mathbb{N}_{0,\infty}^{\mathbb{N}}$ and define an order.

Definition 1.1.41 Let $(a_n),(b_n)\subset\mathbb{N}_{0,\infty}^{\mathbb{N}}$. We say that $(a_n)\leq(b_n)$ if $a_n\leq b_n$ for all $n\in\mathbb{N}$.

Lemma 1.1.42 *Let (a_n) be a non-decreasing (with the order defined above) sequence of elements of $\mathbb{N}_{0,\infty}^{\mathbb{N}}$. Then a_n is convergent.*

Now we recall the algebraic notation we will use in the sequel:

Notation 1.1.43 *Let G be a group.*

- *Let $M\subset G$ be a subset of G. We denote by $\langle M\rangle$ the subgroup of G generated by M.*

- *Let $x \in G$. We say that x* ***has order*** *m if $mx = 0$ and m is the least natural number satisfying this condition. If such m exists, we say that x is a* ***torsion element***
- *We say that G* ***has exponent*** *m if $mG = 0$ and m is the least natural number satisfying this condition. If such m exists, we say that G is* ***bounded****. Otherwise, we say that G is* ***unbounded****.*
- *We say that G* ***is torsion*** *if every element of G is torsion.*
- *We say that G* ***is a p-group*** *(for a prime number p) if for any element $x \in G$ there exists a natural number n satisfying $p^n x = 0$.*
- *A subgroup $H \leq G$ is said to have* ***finite index*** *if the number of equivalence classes of G/H is finite.*

Remark 1.1.44 It is clear that any bounded group is torsion and that any p-group is torsion. The converse implications are not true.

Proposition 1.1.45 *If G has exponent m, then $G^\wedge$ has exponent m.*

Proof: $m\chi(x) = \chi(x) \overset{(m)}{\cdots} \chi(x) = \chi(mx) = \chi(0) = 0 + \mathbb{Z}$ for all $\chi \in G^\wedge$. QED

We consider the following subsets of the poset $\mathcal{S}(G)$ of all subgroups of a topological abelian group $G = (G, \tau)$:

Notation 1.1.46 *Let (G, τ) be a topological group. We denote by:*

- *$\mathcal{O}(G, \tau)$ the set of all τ-open subgroups of G;*
- *$\mathcal{C}(G, \tau)$ the set of all τ-closed subgroups of G;*
- *$\mathcal{C}_{dua}(G, \tau)$ the set of all dually closed subgroups of G;*
- *$\mathcal{C}_{fi}(G, \tau)$ the set of all τ-closed subgroups of finite index of G;*
- *$\mathcal{C}_*(G, \tau)$ the set of all τ-closed subgroups H of G such that all larger $N \in \mathcal{S}(G)$ are still τ-closed;*
- *$\mathcal{D}(G, \tau)$ the set of all τ-dense subgroups of G;*

(if no confusion may arise we will use $\mathcal{O}(G)$, for $\mathcal{O}(G,\tau)$, and so on).

Clearly, $\mathcal{O}(G,\tau) \subseteq \mathcal{C}_*(G,\tau)$ since any subgroup of G that contains an open subgroup is open. We also take into consideration the following definition:

Definition 1.1.47 Denote by $\mathcal{D}en$ the class of topological abelian groups G with $\mathcal{D}(\mathcal{G}) \neq \{\mathcal{G}\}$, i.e., having proper dense subgroups.

The poset $\mathcal{D}en$ has been intensively studied in [CD99, CD14, Dik01]

1.2 Precompact topologies.

Definition 1.2.1 Let G be a group and $H \leq G^*$. The **weak topology** on G with respect to H is the coarsest group topology making the elements of H continuous. We denote it by $\sigma(G,H)$.

Definition 1.2.2 A topological group (G,τ) is **precompact** if for every zero-neighborhood V, there exists a finite subset $F \subset G$ such that $G = FV$. In the same form a precompact subset is defined. We say that (G,τ) is **locally precompact** if τ has a neighborhood basis consisting in precompact subsets. We denote by $\mathcal{L}pc$ the class of all locally precompact topologies.

Theorem 1.2.3 *[CR64, Theorem 1.2] Let (G,τ) be a topological group and $G^\wedge$ its dual group. The following assertions are equivalent:*

(a) (G,τ) is precompact.

(b) The topology τ coincides with the weak topology on G with respect to the family of its continuous homomorphisms $G^\wedge$.

Theorem 1.2.4 *[CR64, Theorem 1.3]*

Let G be an abelian group and let H be a subgroup of $\mathrm{Hom}(G,\mathbb{T})$ *that separates points of G, and τ_H the weak topology with respect to to the family H. Then the dual group $(G,\tau_H)^\wedge$ is precisely H.*

According to Theorems 1.2.3 y 1.2.4, the following procedure can be used to construct all the Hausdorff precompact topologies on G. Let H be a subgroup of G^*. The topology τ_H has as subbasis of neighborhoods of 0 the following family:

$$\mathcal{B} = \{\varphi^{-1}(V) \text{ such that } V \in \mathcal{N}_0(\mathbb{T}) \text{ and } \varphi \in H\}.$$

It is easy to check that τ_H is a group topology on G.

The following properties are easy to prove:

Properties 1.2.5 *Let $H \leq G^*$ and $\tau_H = \sigma(G, H)$. The following properties hold:*

- *Let $H_1 < H_2 \leq G^*$. Then $\tau_{H_1} < \tau_{H_2}$.*
- *The correspondence $H \to \tau_H$ between the set of subgroups of G^* and the set of precompact topologies on G is bijective.*
- *The subgroup H is countable if and only if τ_H is metrizable.*

Proposition 1.2.6 *[BCR85, Corollary 5.7]*

Let G be an infinite group satisfying that $|G| = \alpha$. Then there exist 2^{2^α} precompact topologies on G.

Following Van Douwen in [Dou90], an abelian group, G, endowed with the weak topology for the family $Hom(G, \mathbb{T})$ is denoted by $G^\#$. In that paper, it is proved that $G^\#$ has no nontrivial convergent sequences, if G is an infinite abelian group. In [BMP96, Lema 1], a direct proof of this fact for $\mathbb{Z}^\#$ is given.

1.3 Topologies of uniform convergence.

Let G be a group and $\mathcal{B} \subset \mathcal{P}(G^*)$, where $\mathcal{P}(G^*)$ is the power set of G^*. The **topology of uniform convergence on** $\mathcal{B}$, $\mathcal{U}_{\mathcal{B}}$, for the group G is defined in the following way: first define for $B \in \mathcal{B}$

$$V_{B,m} := \{g \in G : \chi(g) \in \mathbb{T}_m \text{ for all } \chi \in B\}$$

then

$$\{V_{B,m}\}_{B \in \mathcal{B}, m \in \mathbb{N}}$$

is a neighborhood basis for the topology of uniform convergence on $\mathcal{B}$.

The convergence of nets is defined as follows:

$$(g_\alpha)_{\alpha\in A} \stackrel{\mathcal{U}_B}{\to} g \Longleftrightarrow \varphi(g_\alpha) \to \varphi(g), \text{ for all } \varphi \in B \text{ uniformly.}$$

An alternative way to express this definition is the following: For every $\varepsilon > 0$, let $\mathbb{T}_\varepsilon \in \mathcal{N}_1(\mathbb{T})$, then, there exists $\alpha_0 \in A$ such that $\varphi(g_\alpha) \in \varphi(g) + \mathbb{T}_\varepsilon$, for every $\varphi \in B$ and $\alpha \geq \alpha_0$.

If B separates points of G, then the topology is metrizable, otherwise, it is only pseudo-metrizable.

Let

$$\rho_B(x,y) := \sup_{\varphi\in B} |\varphi(x) - \varphi(y)| \,.$$

Then ρ_B is a traslation invariant metric on G. Indeed, it is clear that ρ_B is a metric and $\rho_B(x+t, y+t) = \sup_{\varphi\in B} |\varphi(x+t) - \varphi(y+t)| = \sup_{\varphi\in B} |\varphi(x) - \varphi(y)|$, since φ is a homomorphism.

Hence, the topology defined by ρ_B is a group topology.

If B separates points of G, a group topology $\mathcal{U}_B$ on G is obtained.

Proposition 1.3.1 *If B is dense in $\mathcal{H} := \mathrm{Hom}(G,\mathbb{T})$ endowed with the topology of pointwise convergence, τ_p, then the following equality holds:*

$$\sup_{\varphi\in B} |\varphi(x) - \varphi(y)| = \sup_{\varphi\in\mathcal{H}} |\varphi(x) - \varphi(y)| \,.$$

Proof:

Indeed, given $\psi \in \mathcal{H}$, there exists a net $(\varphi_\mathcal{B})_{\mathcal{B}\subset I} \subset B$ satisfying $\varphi_\mathcal{B} \to \psi$ in τ_p. In particular, for $x, y \in G$, this means that $\varphi_\mathcal{B}(x) \to \psi(x)$ and $\varphi_\mathcal{B}(y) \to \psi(y)$.

Fix $\varepsilon > 0$. Hence, there exists $\mathcal{B}_0 \in I$ satisfying that $|\varphi_\mathcal{B}(x) - \psi(x)| < \frac{\varepsilon}{2}$ and $|\varphi_\mathcal{B}(y) - \psi(y)| < \frac{\varepsilon}{2}$ if $\mathcal{B} \geq \mathcal{B}_0$. Then,

$$|\psi(x) - \psi(y)| = |\psi(x) - \varphi_\mathcal{B}(x) + \varphi_\mathcal{B}(x) - \varphi_\mathcal{B}(y) + \varphi_\mathcal{B}(y) - \psi(y)| \leq$$
$$\leq \frac{\varepsilon}{2} + |\varphi_\mathcal{B}(x) - \varphi_\mathcal{B}(y)| + \frac{\varepsilon}{2}$$

for $\mathcal{B}_0 \leq \mathcal{B}$. Hence, we have $|\psi(x) - \psi(y)| \leq \sup_{\varphi\in B} |\varphi(x) - \varphi(y)| = \rho_B(x,y)$. Hence, $\rho_B(x,y) = \rho_\mathcal{H}(x,y)$.

QED

Remark 1.3.2 If B is not dense in $(\mathrm{Hom}(G,\mathbb{T}),\tau_p)$, the topology $\mathcal{U}_B$ can be discrete (see Theorem 4.3.2 to get an example on the integers).

Proposition 1.3.3 *Let* $B = \mathrm{Hom}(G,\mathbb{T})$*. Then* $\mathcal{U}_B$ *is the discrete topology.*

Proof:

Since ρ_B is a metric, it suffices to show that there exists $\varepsilon > 0$ such that $\rho_B(0,x) \geq \varepsilon$ for every $x \in G$.

Since $\mathrm{Hom}(G,\mathbb{T})$ separates the points of G, there exists $\varphi \in \mathrm{Hom}(G,\mathbb{T})$ such that $\varphi(x) \neq 0$. It is clear that $n \cdot \varphi \in B$ for all $n \in \mathbb{N}$. By definition of ρ_B, we have that $\rho_B(0,x) \geq \sup_{n\in\mathbb{N}} |n\varphi(x)|$. Since $\{n\varphi(x) : n \in \mathbb{Z}\}$ is the subgroup of $\mathbb{T}$ generated by $\varphi(x)$, there exists n_0 satisfying $|n_0\varphi(x)| \geq \frac{1}{4}$. Hence $\rho_B(0,x) \geq \frac{1}{4}$ for every $0 \neq x \in G$ and the topology is discrete.

QED

1.4 T-sequences and complete topologies.

In this section we consider some special families of topologies which were introduced by Graev and deeply studied in the group of the integers by Protasov and Zelenyuk ([ZP90, PZ99]. We need first some definitions about sequences. All sequences considered will be without repeated terms.

Definition 1.4.1 Let G be a group and $\mathbf{g} = (g_n) \subset G$ a sequence of elements of G. We say that $\mathbf{g}$ is a T-sequence if there exists a **Hausdorff** group topology τ such that $g_n \xrightarrow{\tau} 0$. We denote by $\mathcal{T}_G$ the set of all T-sequences in G.

Lemma 1.4.2 *Let* $\mathbf{g} \subset G$ *be a* T*-sequence, then there exists the finest group topology* $\mathcal{T}_{\{g_n\}}$ *satisfying that* $g_n \xrightarrow{\mathcal{T}_{\{g_n\}}} 0$.

Next, we include a theorem of great interest from [PZ99]:

Theorem 1.4.3 *[PZ99, Theorem 2.3.11]*

Let (g_n) *be a* T*-sequence on a group* G*. Then, the topology* $\mathcal{T}_{\{g_n\}}$ *is complete.*

Applying the Baire Category Theorem, we can obtain the following corollary:

Corollary 1.4.4 *Let G be a countable group. For any T-sequence* $\mathbf{g}$*, the topology $\mathcal{T}_{\{g_n\}}$ is not metrizable.*

In Section 4.5, some results on the topology $\mathcal{T}_{\{g_n\}}$ are given.

In a more general framework, it is possible to define TB-sequences:

Definition 1.4.5 [BDMW03, Definition 1.2]

Let G be a group and $\mathbf{g} = (g_n) \subset G$ a sequence of elements of G. We say that $\mathbf{g}$ is a TB-sequence if there exists a **Hausdorff precompact** group topology τ such that $g_n \xrightarrow{\tau} 0$. The set of all TB-sequences in G is denoted by $\mathcal{TB}_G$.

Chapter 2

D-sequences.

2.1 Main definitions.

In this section we define a D-sequence, and give some families of D-sequences that will be of interest along this thesis.

Definition 2.1.1 A sequence of natural numbers $\mathbf{b} = (b_n)_{n\in\mathbb{N}_0} \subset \mathbb{N}^{\mathbb{N}_0}$ is called a ***D*-sequence** if it satisfies:

(1) $b_0 = 1$,

(2) $b_n \neq b_{n+1}$ for all $n \in \mathbb{N}_0$,

(3) b_n divides b_{n+1} for all $n \in \mathbb{N}_0$.

The following notation will be used in the sequel:

- $\mathcal{D} := \{\mathbf{b} : \mathbf{b} \text{ is a } D - \text{sequence}\}$.
- $\mathcal{D}_\infty := \{\mathbf{b} \in \mathcal{D} : \frac{b_{n+1}}{b_n} \to \infty\}$.
- $\mathcal{D}_\infty^\ell := \{\mathbf{b} \in \mathcal{D} : \frac{b_{n+\ell}}{b_n} \to \infty\}$.

We now present some examples of D-sequences and some easy remarks concerning the definition:

Example 2.1.2 (1) The sequence $(a^n)_{n\in\mathbb{N}_0} \in \mathcal{D} \setminus \mathcal{D}_\infty$ for every $a \in \mathbb{N} \setminus \{1\}$ is a D-sequence. In particular, let p be a prime number. Then (p^n) is a D-sequence.

(2) The sequence $((n+1)!)_{n\in\mathbb{N}_0} \in \mathcal{D}_\infty$.

(3) The sequence $\left(a^{n^2}\right)_{n\in\mathbb{N}_0} \in \mathcal{D}_\infty$ for every natural number $a > 1$ is a $\mathcal{D}$ sequence belonging to $\mathcal{D}_\infty$.

(4) The sequence (b_n) defined by $b_{2k} = p^{2^k}$ and $b_{2k+1} = p^{2^k+1}$ is a D-sequence in $\mathcal{D}^2_\infty \setminus \mathcal{D}_\infty$.

Remark 2.1.3 The following claims are clear from the definition of D-sequence:

(1) Obviously, every subsequence of a D-sequence $\mathbf{b}$ which contains b_0 is a D-sequence.

(2) Let $\mathbf{b}$ be a D-sequence. Inductively, since $\frac{b_{n+1}}{b_n} \geq 2$, we have $b_{n+k} \geq 2^k b_n$ for all $n, k \in \mathbb{N}_0$.

(3) For a D-sequence $\mathbf{b}$ we have $\displaystyle\sum_{n\geq j} \frac{1}{b_n} \leq \frac{2}{b_j} \qquad \forall\ j \in \mathbb{N}_0.$

Only (3) needs to be shown.

We have $\displaystyle\sum_{n\geq j} \frac{1}{b_n} = \sum_{k=0}^{\infty} \frac{1}{b_{j+k}} \overset{(2)}{\leq} \sum_{k=0}^{\infty} \frac{1}{b_j}\frac{1}{2^k} = \frac{1}{b_j}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{2}{b_j}$ for all $j \in \mathbb{N}_0$.

Remark 2.1.3(1) gives rise to the following notation:

Notation 2.1.4 *Let* $\mathbf{b}, \mathbf{b}'$ *be two D-sequences*

- *We will write* $\mathbf{b} \sqsubseteq \mathbf{b}'$ *if* $\mathbf{b}$ *is a subsequence of* $\mathbf{b}'$ *or if* $\mathbf{b} = \mathbf{b}'$. *In an analogous way, we say that* $x \in \mathbf{b}$ *if there exists* $n \in \mathbb{N}_0$ *such that* $x = b_n$
- *For* $\mathbf{b}, \mathbf{b}'$ *such that* $\mathbf{b}, \mathbf{b}' \sqsubseteq \mathbf{c}$, *for* $\mathbf{c} \in \mathcal{D}$, *we denote by* $(\mathbf{b} \sqcup \mathbf{b}')$ *the sequence obtained by an increasing rearrangement of the terms of* $\mathbf{b}$ *and* $\mathbf{b}'$. *Furthermore,* $(\mathbf{b} \sqcup \mathbf{b}')$ *is a D-sequence.*

For a given D-sequence, $\mathbf{b}$*, we define:*

- $\mathcal{D}_\infty(\mathbf{b}) := \{\mathbf{c} \sqsubseteq \mathbf{b} : \mathbf{c} \in \mathcal{D}_\infty\}$.
- $\mathcal{D}^\ell_\infty(\mathbf{b}) := \{\mathbf{c} \sqsubseteq \mathbf{b} : \mathbf{c} \in \mathcal{D}^\ell_\infty\}$.

Our interest on D-sequences stems from the fact that several topologies on $\mathbb{Z}$ can be associated to D-sequences. They will be defined in Chapter 4, where it will become clear that not only the D-sequence plays a fundamental role but also the sequence of the ratios and their properties. Next we fix the definitions and convenient properties.

Definition 2.1.5 Let $\mathbf{b}$ be a D-sequence. Define $(q_n)_{n\in\mathbb{N}}$ by $q_n := \frac{b_n}{b_{n-1}}$.

The following claims are straight-forward:

Remark 2.1.6 Let $\mathbf{b}$ be a D-sequence and let (q_n) be as in Definition 2.1.5. Then:

1. The sequence $\mathbf{b} \in \mathcal{D}_\infty$ if and only if $q_n \to \infty$.
2. In order to recover $\mathbf{b}$ from the sequence (q_n), we define $b_n := \prod_{k=1}^{n} q_k$.

Definition 2.1.7 Let $\mathbf{b}$ be a D-sequence. We say that

(a) $\mathbf{b}$ has **bounded ratios** if there exists a natural number N, satisfying that $q_n = \frac{b_n}{b_{n-1}} \leq N$.

(b) $\mathbf{b}$ is **basic** if q_n is a prime number for all $n \in \mathbb{N}$.

Basic D-sequences are maximal in the following sense: let $\mathbf{b}$ be a basic D-sequence. Let $\mathbf{b}'$ be another D-sequence satisfying $\mathbf{b} \sqsubseteq \mathbf{b}'$. Then $\mathbf{b} = \mathbf{b}'$. We set the following definition.

Definition 2.1.8 For a natural number m consider its decomposition in prime factors. Write $m = p_1^{m_1} \cdot p_2^{m_2} \cdots p_N^{m_N}$. Define $p : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}_{0,\infty}$ as

$$p(m) := (m_1, m_2, \ldots, m_N, 0, \ldots).$$

Analogously, let $\mathbf{b}$ be a D-sequence. Consider the function $\mathbb{P} : \mathcal{D} \to \mathbb{N}_{0,\infty}^{\mathbb{N}}$ defined by

$$\mathbb{P}(\mathbf{b}) := \lim_{n\to\infty} p(b_n)$$

and $\mathbb{P}_i(\mathbf{b}) = n_i$. It is clear that $\sum_{k=1}^{\infty} n_i = \infty$.

We say that the i-th prime number appears n_i times if $n_i = \mathbb{P}_i(\mathbf{b})$.

Proposition 2.1.9 *Let* $\mathbf{b}$ *be a D-sequence.Then* $p(b_n) < p(b_{n+1})$ *in the order induced in* $\mathbb{N}_{0,\infty}^{\mathbb{N}}$.

Remark 2.1.10 Let $\mathbf{b} \sqsubseteq \mathbf{c}$. Then $(p(b_n))_n \sqsubseteq (p(c_n))_n$.

Remark 2.1.11 Let $\mathbf{b} \sqsubseteq \mathbf{c}$ be two D-sequences. Then $\mathbb{P}(\mathbf{b}) = \mathbb{P}(\mathbf{c})$

To each D-sequence $\mathbf{b}$ a sequence (n_i) is associated. Conversely, a sequence (n_i) does not determine the sequence $\mathbf{b}$.

Remark 2.1.12 There exists a bijection between $Im(\mathbb{P}) = \{\mathbb{P}(\mathbf{b}) : \mathbf{b} \in \mathcal{D}\}$ and $\mathbb{N}_{\infty}^{\mathbb{N}}$. Hence $|Im(\mathbb{P})| = \left|\mathbb{N}_{\infty}^{\mathbb{N}}\right| = \omega^{\omega}$.

This notation will give us a device to compare linear topologies on $\mathbb{Z}$ just by comparing some D-sequences and their associated $\mathbb{P}(\mathbf{b})$. To that end, we define an order in $Im(\mathbb{P})$.

Definition 2.1.13 Let $\mathbf{b}, \mathbf{c}$ be D-sequences. Write $\mathbb{P}(\mathbf{b}) = (n_1, n_2, \dots, n_k, \dots)$ and $\mathbb{P}(\mathbf{c}) = (m_1, m_2, \dots, m_k, \dots)$.

- We say that $\mathbb{P}(\mathbf{b}) \leq \mathbb{P}(\mathbf{c})$ if $n_i \leq m_i$ for all $i \in \mathbb{N}$.
- We say that $\mathbb{P}(\mathbf{b}) < \mathbb{P}(\mathbf{c})$ if $\mathbb{P}(\mathbf{b}) \leq \mathbb{P}(\mathbf{c})$ and $\mathbb{P}(\mathbf{b}) \neq \mathbb{P}(\mathbf{c})$.

2.2 Numerical results concerning *D*-sequences.

In this section we present two auxiliary results that allow us to write integer and real numbers using a fixed D-sequence as basis. The first one, has a similar expression in the paper [Mar98, Lemma 1]. Here, we present a different proof. We will use Proposition 2.2.1 and Proposition 2.2.4 essentially in this thesis.

Proposition 2.2.1 *Let $\boldsymbol{b}$ be a D-sequence. Suppose that $q_{j+1} \neq 2$ for infinitely many j. For each integer number $L \in \mathbb{Z}$, there exists a natural number $N = N(L)$ and* ***unique*** *integers $k_0, \ldots, k_N$, such that:*

(1) $L = \sum_{j=0}^{N} k_j b_j$.

(2) $\left| \sum_{j=0}^{n} k_j b_j \right| \leq \frac{b_{n+1}}{2}$ *for all n.*

(3) $k_j \in \left(-\frac{q_{j+1}}{2}, \frac{q_{j+1}}{2} \right]$, *for $0 \leq j \leq N$.*

Proof:

As an auxiliary step, we define simultaneously sequences of integer numbers (L_i) and (k_i) satisfying for all i the following conditions:

(a) $L = L_0$ and $|L_{i+1}| \leq |L_i|$.

(b) k_i satisfies (3).

(c) $L_i = \frac{L - k_0 - k_1 b_1 - \ldots - k_{i-1} b_{i-1}}{b_i}$.

We prove that $L_i = 0$ for some $i \in \mathbb{N}$, and from here retrospectively, the proposed expression of L is reached.

First, we choose the unique integer $k_0 \in \left(-\frac{q_1}{2}, \frac{q_1}{2}\right]$ such that $q_1 \mid (L_0 - k_0)$. Define

$$L_1 := (L_0 - k_0)/q_1 \in \mathbb{Z}.$$

We compute now

$$|L_1| = \frac{|L_0 - k_0|}{q_1} \leq \frac{|L_0|}{q_1} + \frac{|k_0|}{q_1}.$$

We have two possibilities: either $L_1 = 0$ and the process is over ($k_i = 0$ for $i \geq 1$) or $L_1 \neq 0$. If $L_1 \neq 0$, then $L_0 \neq k_0$. Since $|k_0| \leq \frac{q_1}{2}$ and $|L_0| \geq \frac{q_1}{2}$, we have that $|L_0| \geq |k_0|$. Then we get

$$|L_1| \leq \frac{|L_0|}{q_1} + \frac{|L_0|}{q_1} = \frac{2\,|L_0|}{q_1} \leq |L_0|\,.$$

The last inequality may be an equality only if $q_1 = 2$.

Suppose we have found $L_0, \ldots, L_n, k_0, \ldots, k_{n-1}$ satisfying (a) and (b) and $L_n \neq 0$. We choose the unique integer $k_n \in \left(-\frac{q_{n+1}}{2}, \frac{q_{n+1}}{2}\right]$ such that $q_{n+1} \mid (L_n - k_n)$. Define

$$L_{n+1} := \frac{L_n - k_n}{q_{n+1}} \in \mathbb{Z}.$$

We compute now

$$|L_{n+1}| = \frac{|L_n - k_n|}{q_{n+1}} \leq \frac{|L_n|}{q_{n+1}} + \frac{|k_n|}{q_{n+1}}.$$

Two possibilities arise: either $L_{n+1} = 0$ and the process is over ($k_i = 0$ for $i \geq n+1$) or $L_n \neq 0$. If $L_{n+1} \neq 0$, then $L_n \neq k_n$. Since $|k_n| \leq \frac{q_{n+1}}{2}$ and $|L_n| \geq \frac{q_{n+1}}{2}$, we have that $|L_n| \geq |k_n|$. Then we get

$$|L_{n+1}| \leq \frac{|L_n|}{q_{n+1}} + \frac{|L_n|}{q_{n+1}} = \frac{2\,|L_n|}{q_{n+1}} \leq |L_n|\,.$$

The last inequality may be an equality only if $q_{n+1} = 2$.

Since $q_{n+1} \neq 2$ for infinitely many natural numbers and the sequence of $|L_n|$ decreases for infinitely many natural numbers (which is impossible, since $|L_n|$ is a sequence of non-negative integers) we obtain $L_n = 0$ for some n. So the process ends.

By construction and the fact $L_n = k_n + q_{n+1}L_{n+1}$ it is clear that

$$L_0 = L = k_0 + q_1 L_1 = k_0 + q_1(k_1 + q_2 L_2) = \cdots = \sum_{j=0}^{N} k_j b_j$$

the representation in (1) holds. Since

$$b_{j+1} \mid \left(L - \sum_{i=0}^{j} k_i b_i\right)$$

and

$$\sum_{i=0}^{j} k_i b_i \text{ is \textbf{the} unique integer in } \left(-\frac{b_{j+1}}{2}, \frac{b_j+1}{2}\right] \text{ satisfying this condition,}$$

(2) follows.

In order to prove the uniqueness of the sequence (k_n) assume by contradiction that there exist $(k_n) \neq (k'_n)$ satisfying that $k_n, k'_n \in \left(-\frac{q_{n+1}}{2}, \frac{q_{n+1}}{2}\right]$ for all n and

$$L = \sum_{n=0}^{N(L)} k_n b_n = \sum_{n=0}^{N'(L)} k'_n b_n.$$

Let $n_0 \in \mathbb{N}_0$ be the minimum index such that $k_n \neq k'_n$. Write

$$L = \sum_{n=0}^{n_0-1} k_n b_n + k_{n_0} b_{n_0} + \sum_{n=n_0+1}^{N(L)} k_n b_n = \sum_{n=0}^{n_0-1} k'_n b_n + k'_{n_0} b_{n_0} + \sum_{n=n_0+1}^{N'(L)} k'_n b_n.$$

Now

$$0 = L - L = \sum_{n=0}^{n_0-1} k_n b_n + k_{n_0} b_{n_0} + \sum_{n=n_0+1}^{N(L)} k_n b_n - \sum_{n=0}^{n_0-1} k'_n b_n - k'_{n_0} b_{n_0} - \sum_{n=n_0+1}^{N'(L)} k'_n b_n =$$

$$= \sum_{n=0}^{n_0-1} \underbrace{(k_n - k'_n)}_{=0} b_n + (k_{n_0} - k'_{n_0}) b_{n_0} + \sum_{n=n_0+1}^{\max\{N(L),N'(L)\}} (k_n - k'_n) b_n.$$

We turn our attention to $(k_{n_0} - k'_{n_0}) b_{n_0}$. Since $k_{n_0}, k'_{n_0} \in \left(-\frac{q_{n_0+1}}{2}, \frac{q_{n_0+1}}{2}\right]$, we have

$$-q_{n_0+1} < \underbrace{(k_{n_0} - k'_{n_0})}_{\neq 0} < q_{n_0+1}.$$

Hence $(k_{n_0} - k'_{n_0})$ is not multiple of q_{n_0+1} and $b_{n_0+1} \nmid (k_{n_0} - k'_{n_0}) b_{n_0}$. But

$$b_{n_0+1} \mid \sum_{n=n_0+1}^{\max\{N(L),N'(L)\}} (k_n - k'_n) b_n$$

and consequently $b_{n_0+1} \nmid (L - L) = 0$, which is a contradiction. Hence, $k_n = k'_n$ for all n.

QED

The sequence

$$(k_0, k_1, \cdots, k_{N(L)}, 0, \dots) \in \prod_{n=1}^{\infty} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$$

will be called the **b-coordinates of** L.

Lemma 2.2.2 *Let* $\mathbf{b}$ *be a D-sequence and* $(k_0, k_1, \dots)$ *the* $\mathbf{b}$*-coordinates of some integer number. Then* $\left| \sum_{s=0}^{n-1} \frac{k_s b_s}{b_n} \right| \leq \frac{1}{2}$ *for all* $n \in \mathbb{N}$.

Proof:

Consider the following computations:

$$\left|\sum_{s=0}^{n-1}\frac{k_s b_s}{b_n}\right| = \frac{\left|\sum_{s=0}^{n-1}k_s b_s\right|}{b_n} \overset{2.2.1}{\leq} \frac{\frac{b_n}{2}}{b_n} = \frac{1}{2}.$$

QED

We write here examples of the algorithm just described. One of them highlights the importance of the condition $q_{n+1} \neq 2$ for infinitely many natural numbers n.

Example 2.2.3 (1) We choose the sequence $(b_n) = ((n+1)!)$ and $L = 62$.

Step 1: $L_0 = 62$.

Find $k_0 \in \left(-\frac{q_1}{2}, \frac{q_1}{2}\right] = (-1, 1]$, such that $q_1 = 2 \mid (L_0 - k_0)$. So $k_0 = 0$.

Put $L_1 = \frac{L_0 - k_0}{q_1} = 31$.

Step 2: $L_1 = 31$.

Find $k_1 \in \left(-\frac{q_2}{2}, \frac{q_2}{2}\right] = \left(-\frac{3}{2}, \frac{3}{2}\right]$, such that $q_2 = 3 \mid (L_1 - k_1)$. So $k_1 = 1$.

Put $L_2 = \frac{L_1 - k_1}{q_2} = 10$.

Step 3: $L_2 = 10$.

Find $k_2 \in \left(-\frac{q_3}{2}, \frac{q_3}{2}\right] = (-2, 2]$, such that $q_3 = 4 \mid (L_2 - k_2)$. So $k_2 = 2$.

Put $L_3 = \frac{L_2 - k_2}{q_3} = 2$.

Step 4: $L_3 = 2$.

Find $k_3 \in \left(-\frac{b_4}{2b_3}, \frac{q_4}{2}\right] = \left(-\frac{5}{2}, \frac{5}{2}\right]$, such that $q_4 = 5 \mid (L_3 - k_3)$. So $k_3 = 2$.

Put $L_4 = \frac{L_3 - k_3}{q_4} = 0$. Hence $k_n = 0$ for all $n \geq 4$.

Final step: We rewrite $62 = L_0 = k_0 + b_1 L_1 = k_0 + b_1(k_1 + q_2 L_2) = k_0 + k_1 b_1 + b_2 L_2 = k_0 + k_1 b_1 + b_2(k_2 + q_3 L_3) = k_0 + k_1 b_1 + k_2 b_2 + b_3 L_3 = k_0 + k_1 b_1 + k_2 b_2 + b_3(k_3 + q_4 L_4) = k_0 + k_1 b_1 + k_2 b_2 + k_3 b_3 + b_4 \cdot 0 = k_0 + k_1 b_1 + k_2 b_2 + k_3 b_3 = 0 + 1(2!) + 2(3!) + 2(4!)$.

(2) We choose the sequence $(b_n) = (10^n)$ and $L = 683$. In this case, $q_n = 10$ and $k_n \in (-5, 5]$ for all n.

Step 1: $L_0 = 683$.

Find $k_0 \in (-5, 5]$ such that $10 \mid L_0 - k_0$. So $k_0 = 3$.

Put $L_1 = \frac{L_0 - k_0}{10} = 68$.

Step 2: $L_1 = 68$.

Find $k_1 \in (-5, 5]$ such that $10 \mid L_1 - k_1$. So $k_1 = -2$.

Put $L_2 = \frac{L_1 - k_1}{10} = 7$.

Step 3: $L_2 = 7$.

Find $k_2 \in (-5, 5]$ such that $10 \mid L_2 - k_2$. So $k_2 = -3$.

Put $L_3 = \frac{L_2 - k_2}{10} = 1$.

Step 4: $L_3 = 1$.

Find $k_3 \in (-5, 5]$ such that $10 \mid L_3 - k_3$. So $k_3 = 1$.

Put $L_4 = \frac{L_3 - k_3}{10} = 0$. So $k_n = 0$ if $n \geq 4$.

Final step: We write $683 = 3 + (-2)10 + (-3)10^2 + 1 \cdot 10^3$.

(3) We use the sequence $(b_n) = (2^n)$, which does not satisfy the condition $q_{n+1} \neq 2$ for infinitely many numbers. Hence $q_{n+1} = 2$ and $k_n \in \{0, 1\}$ for all $n \in \mathbb{N}_0$. We consider $L = -1$.

Step 1: $L_0 = -1$.

Find $k_0 \in \{0, 1\}$ such that $2 \mid (L_0 - k_0)$. So $k_0 = 1$.

Put $L_1 = \frac{L_0 - k_0}{2} = -1$.

In this case we get $L_{n+1} = L_n = -1$ and $k_n = 1$ for all n. Which leads to the **b**-coordinates:

$$-1 = (1, 1, 1, \dots, 1, \dots).$$

(4) We use the sequence $(b_n) = ((n + 1)!)$, and $L = 5$.

Step 1: $L_0 = 5$.

Find $k_0 \in \{0, 1\}$ such that $q_1 = 2 \mid (L_0 - k_0)$. So $k_0 = 1$.

Put $L_1 = \frac{L_0 - k_0}{q_1} = 2$.

Step 2: $L_1 = 2$.

Find $k_1 \in \{-1, 0, 1\}$ such that $q_2 = 3 \mid (L_1 - k_1)$. So $k_1 = -1$.

Put $L_2 = \frac{L_1 - k_1}{q_2} = 1$.

Step 3: $L_2 = 1$.

Hence $k_2 = 1$ and $L_3 = 0$.

In [BM10, Proposition 4.1.1.], a similar result is obtained but the expression given there is not unique. If we consider $L = 5$ and $b_n = (n+1)!$, we can write $5 = 1 \cdot 3! + (-1)2! + 1 \cdot 1! = 1 \cdot 3! + 0 \cdot 2! + (-1)1!$, and both expressions are admissible for the bounds in [BM10]. With the bounds in Proposition 2.2.1 the second expression is no longer correct.

Now, we describe a similar algorithm to represent a real number.

Proposition 2.2.4 *Let* **b** *be a D-sequence. Then any* $y \in \mathbb{R}$ *can be written uniquely in the form*

$$y = \sum_{n=0}^{\infty} \frac{\beta_n}{b_n} = \frac{\beta_0}{b_0} + \frac{\beta_1}{b_1} + \frac{\beta_2}{b_2} + \cdots + \frac{\beta_s}{b_s} + \cdots , \tag{1}$$

where $\beta_n \in \mathbb{Z}$ *and* $|\beta_{n+1}| \leq \frac{q_{n+1}}{2}$. *Further,*

$$-\frac{1}{2b_n} < y - \sum_{j=0}^{n} \frac{\beta_j}{b_j} \leq \frac{1}{2b_n}$$

holds for all $n \in \mathbb{N}_0$. *If* $y \in (-1/2, 1/2]$ *then* $\beta_0 = 0$.

Proof:

For $n = 0$ we choose the unique integer β_0 which satisfies $y - \beta_0 = y - \dfrac{\beta_0}{b_0} \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

Suppose that

$$-\frac{1}{2b_n} < y - \sum_{j=0}^{n} \frac{\beta_j}{b_j} \leq \frac{1}{2b_n}$$

holds for some $n \geq 0$. There is a unique $\beta_{n+1} \in \mathbb{Z}$ with

$$-\frac{1}{2} < b_{n+1}\left(y - \sum_{j=0}^{n} \frac{\beta_j}{b_j}\right) - \beta_{n+1} \leq \frac{1}{2}$$

which is equivalent to

$$-\frac{1}{2b_{n+1}} < y - \sum_{j=0}^{n+1} \frac{\beta_j}{b_j} \le \frac{1}{2b_{n+1}}.$$

Also, we obtain:

$$-\frac{q_{n+1}}{2} - \frac{1}{2} < b_{n+1}\left(y - \sum_{j=0}^{n} \frac{\beta_j}{b_j}\right) - \frac{1}{2} \le \beta_{n+1} < b_{n+1}\left(y - \sum_{j=0}^{n} \frac{\beta_j}{b_j}\right) + \frac{1}{2} \le \frac{q_{n+1}}{2} + \frac{1}{2}$$

and taking into account that β_{n+1} is an integer, we get $|\beta_{n+1}| \le \frac{b_{n+1}}{2b_n}$.

Let us show that the representation is unique: So, let

$$y = \sum_{n=0}^{\infty} \frac{\beta_n}{b_n} = \sum_{n=0}^{\infty} \frac{\widetilde{\beta}_n}{b_n}$$

where

$$-\frac{1}{2b_n} < y - \sum_{j=0}^{n} \frac{\beta_j}{b_j} \le \frac{1}{2b_n} \text{ and } -\frac{1}{2b_n} < y - \sum_{j=0}^{n} \frac{\widetilde{\beta}_j}{b_j} \le \frac{1}{2b_n}$$

holds for all $n \in \mathbb{N}_0$.

For $n = 0$ we get $-\frac{1}{2} < y - \beta_0 \le \frac{1}{2}$ and $-\frac{1}{2} < y - \widetilde{\beta}_0 \le \frac{1}{2}$, which shows that $\beta_0 = \widetilde{\beta}_0$, as $\beta_0, \widetilde{\beta}_0 \in \mathbb{Z}$.

Assume now that $\beta_j = \widetilde{\beta}_j$ for $0 \le j \le n$. Put

$$\overline{y} := y - \sum_{j=0}^{n} \frac{\beta_j}{b_j} = y - \sum_{j=0}^{n} \frac{\widetilde{\beta}_j}{b_j}.$$

We have

$$-\frac{1}{2} < b_{n+1}\overline{y} - \beta_{n+1} \le \frac{1}{2} \text{ and } -\frac{1}{2} < b_{n+1}\overline{y} - \widetilde{\beta}_{n+1} \le \frac{1}{2},$$

which implies $\beta_{n+1} = \widetilde{\beta}_{n+1}$ as β_{n+1} and $\widetilde{\beta}_{n+1}$ are integers. QED

The sequence

$$(\beta_0, \beta_1, \dots) \in \mathbb{Z} \times \prod_{n=1}^{\infty} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$$

will be called the **b-coordinates of** y.

2.3 The groups $\mathbb{Z}(\mathbf{b}^\infty)$.

Related to *D*-sequences, we can define now two different types of groups, which will be of great interest. The groups of the first type are countable subgroups of $\mathbb{T}$ (generalizing the Prüfer groups) and the second ones generalize the notion of p-adic integers. The latter group will be studied in Section 2.4.

For a prime number p, the Prüfer group, $\mathbb{Z}(\mathbf{p}^\infty)$, is defined as the subgroup of elements of the unit circle consisting of all p^n-th roots of the unit. In our context, this can be expressed as the elements $x+\mathbb{Z}$ which satisfy $xp^n+\mathbb{Z} = 0+\mathbb{Z}$ for some natural number $n \in \mathbb{N}_0$. As a quotient group of $\mathbb{R}$, it can be written as:

$$\mathbb{Z}(\mathbf{p}^\infty) = \left\langle \left\{ \frac{1}{p^n} : n \in \mathbb{N}_0 \right\} \right\rangle / \mathbb{Z}.$$

A *D*-sequence gives rise to similar subgroups of $\mathbb{T}$ as we write out next:

Definition 2.3.1 Let $\mathbf{b}$ be a *D*-sequence. We write

$$\mathbb{Z}(b_n) := \left\{ \frac{k}{b_n} + \mathbb{Z} : k = 0, 1, \dots, b_n - 1 \right\} \text{ and } \mathbb{Z}(\mathbf{b}^\infty) := \bigcup_{n \in \mathbb{N}_0} \mathbb{Z}(b_n).$$

As quotient groups, we can write

$$\mathbb{Z}(b_n) = \left\langle \left\{ \frac{1}{b_n} \right\} \right\rangle / \mathbb{Z} \text{ and } \mathbb{Z}(\mathbf{b}^\infty) = \left\langle \left\{ \frac{1}{b_m} : m \in \mathbb{N}_0 \right\} \right\rangle / \mathbb{Z}.$$

In other words, $\mathbb{Z}(b_n)$ consists of the elements whose order divides b_n and $\mathbb{Z}(\mathbf{b}^\infty)$ is generated by the elements having order b_n for some natural number $n \in \mathbb{N}_0$.

Remark 2.3.2 Let $\mathbf{b}$ be a *D*-sequence. Then:

- $\mathbb{Z}(b_0) \lneq \mathbb{Z}(b_1) \lneq \mathbb{Z}(b_2) \lneq \cdots \lneq \mathbb{Z}(\mathbf{b}^\infty)$.

The most natural topology for $\mathbb{Z}(\mathbf{b}^\infty)$ is the discrete one. In Section 5.1 topologies of uniform convergence on $\mathbb{Z}(\mathbf{b}^\infty)$ are defined.

Since $b_n \in \mathbb{N}$ for every *D*-sequence and $n \in \mathbb{N}_0$, we have that $\mathbb{Z}(\mathbf{b}^\infty) \subset \mathbb{Q}/\mathbb{Z}$ for every *D*-sequence. In special cases, we get the equality, as the following example shows:

Example 2.3.3 For $\mathbf{b} = ((n+1)!)$, we have that $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Z}(\mathbf{b}^\infty)$.

Proof:

We can identify $\mathbb{Z}(\mathbf{b}^\infty)$ with the quotient $\left\langle\left\{\frac{1}{b_n} : n \in \mathbb{N}_0\right\}\right\rangle/\mathbb{Z}$. Since $\mathbb{Q}$ is algebraically generated by the sequence $\left\langle\left(\frac{1}{n!}\right)\right\rangle$, the remark follows.

QED

Related to the boundedness of the **b**-coordinates of a real number, we have this result.

Remark 2.3.4 Let **b** be a D-sequence and $x \in \mathbb{R}$. Let $(\beta_0, \beta_1, \ldots)$ be the **b**-coordinates of x. Then the following assertions are equivalent:

- There exists $n_0 \in \mathbb{N}_0$ satisfying that $\beta_n = 0$ if $n \geq n_0$.
- $x + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$.

2.4 The group $\mathbb{Z}_{\mathbf{b}}$ of b-adic integers.

Let **b** be a D-sequence. For $n \in \mathbb{N}_0$ let s_n, r_n be the following functions:

$$s_n : \mathbb{Z}^{N_0} \to \mathbb{Z}^{\mathbb{N}_0} \text{ defined as } \mathbf{x} = (x_k) \mapsto s_n(\mathbf{x}) = (s_n(\mathbf{x})_k), \text{ where}$$

$$s_n(\mathbf{x})_k := \begin{cases} x_k & \text{if} \quad k \neq \{n, n+1\} \\ x_n + q_n & \text{if} \quad k = n \\ x_{n+1} - 1 & \text{if} \quad k = n+1 \end{cases}$$

and

$$r_n : \mathbb{Z}^{N_0} \to \mathbb{Z}^{\mathbb{N}_0} \text{ defined as } \mathbf{x} = (x_k) \mapsto r_n(\mathbf{x}) = (r_n(\mathbf{x})_k), \text{ where}$$

$$r_n(\mathbf{x})_k := \begin{cases} x_k & \text{if} \quad k \neq \{n, n+1\} \\ x_n - q_n & \text{if} \quad k = n \\ x_{n+1} + 1 & \text{if} \quad k = n+1 \end{cases}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^{\mathbb{N}_0}$. We define the following relation:

$$\mathbf{x} \sim_R \mathbf{y} \Leftrightarrow \begin{cases} \text{There exists a function } f \text{ which is a composition of} \\ r_n \text{ and } s_n \text{ satisfying that } \mathbf{y} = f(\mathbf{x}) \end{cases}$$

Proposition 2.4.1 *The relation $\sim_R$ is an equivalence relation.*

Proof:

Reflexive: It is clear that $\mathbf{x} \sim_R \mathbf{x}$, since $r_n \circ s_n = \mathrm{Id}$.

Symmetric: Suppose that $\mathbf{x} \sim_R \mathbf{y}$. Then there exists a function f satisfying that f is a composition of the functions s_n and r_n and $\mathbf{y} = f(\mathbf{x})$. Taking into account that $r_n^{-1} = s_n$, we have that f^{-1} is the composition of the functions in f in the same order and changing r_n by s_n and vice versa. Hence, $\mathbf{x} = f^{-1}(\mathbf{y})$ and $\mathbf{y} \sim_R \mathbf{x}$.

Transitive: Suppose that $\mathbf{x} \sim_R \mathbf{y}$ and $\mathbf{y} \sim_R \mathbf{z}$. Then, there exist two functions f, g which are composition of the functions s_n, r_n satisfying that $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{z} = g(\mathbf{y})$. It is clear that $f \circ g$ is composition of the functions s_n, r_n and $\mathbf{z} = f(g(\mathbf{x}))$. Hence, $\mathbf{x} \sim_R \mathbf{z}$.

QED

We can now consider the following definition:

Definition 2.4.2 Let $\mathbf{b}$ be a D-sequence. Define:

$$\mathbb{Z}_{\mathbf{b}} := \mathbb{Z}^{\mathbb{N}_0} / \sim_R$$

$\mathbb{Z}_{\mathbf{b}}$ endowed with the operation $[\mathbf{x}]_R + [\mathbf{y}]_R := [\mathbf{x} + \mathbf{y}]_R$ is a group. The inverse class is $-[\mathbf{x}]_R = [-\mathbf{x}]_R$.

We take the representant of each equivalence class in the product $\prod_\omega \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$. Per abuse of notation, we write $(k_0, k_1, \dots)$ instead of $[(k_0, k_1, \dots)]$ if $(k_0, k_1, \dots) \in \prod_\omega \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$.

Definition 2.4.3 For any D-sequence, we can consider the mapping $i : \mathbb{Z} \to \mathbb{Z}_{\mathbf{b}}$, defined by $k \mapsto i(k) = [(k, 0, 0, \dots)]_R \overset{Prop 2.2.1}{=} (k_0, k_1, \dots)$, where $(k_0, \dots, k_N, 0, \dots)$ are the $\mathbf{b}$-coordinates of k.

Since $i(k)+i(\ell) = [(k, 0, 0, \dots)]_R+[(\ell, 0, 0, \dots)]_R = [(k+\ell, 0, 0, \dots)]_R = i(k+\ell)$, the mapping i is a homomorphism. Since the $\mathbf{b}$-coordinates of an integer are unique, i is also injective. Hence, i is an embedding.

The key point for choosing the representant in $\prod_\omega \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$ is the following:

Lemma 2.4.4 *Let* $\mathbf{b}$ *be a D-sequence. Then the following assertions are equivalent:*

1. $i(\mathbb{Z}) = \bigoplus_{\omega} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$.

2. $q_n \neq 2$ *for infinitely many natural numbers n.*

Proof:

We will use the notation in Proposition 2.2.1.

2. $\Rightarrow$ 1. is a direct consequence of Proposition 2.2.1.

To prove the converse, we suppose that $q_n \neq 2$ for finitely many $n \in \mathbb{N}$. Fix n_0 such that $q_n = 2$ if $n \geq n_0$. We prove that the **b**-coordinates of $L = -b_{n_0}$ do not belong to $\bigoplus_{\omega} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$. Since $b_n \mid b_{n_0}$, if $n \leq n_0$, we have that $k_0 = k_1 = \cdots = k_{n_0-1} = 0$ and $L_n = -1$. A similar argument to the one in Example 2.2.3 (4) shows that $k_n = 1$ if $n \geq n_0$, obtaining that the **b**-coordinates of L do not belong to $\bigoplus_{\omega} \left(-\frac{q_n}{2}, \frac{q_n}{2}\right]$. QED

Remark 2.4.5 In Chapter 4, we will give a topology to $\mathbb{Z}_{\mathbf{b}}$. With that topology, the group $\mathbb{Z}_{\mathbf{b}}$ is the completion of some linear topology on the group of the integers. By this reason, we call $\mathbb{Z}_{\mathbf{b}}$ the group of **b-adic integers**. In addition, $\mathbb{Z}_{\mathbf{b}}$ will be a compact group, hence reflexive (by Pontryagin-van Kampen Theorem). This will be essential to compute $\mathbb{Z}(\mathbf{b}^\infty)^*$. If $\mathbf{b} = (p^n)$, then $\mathbb{Z}_{\mathbf{b}}$ are the p-adic integers.

Remark 2.4.6 Let $k = (k_0, k_1, \dots) \in \mathbb{Z}_{\mathbf{b}}$. For short, we will write

$$k = \sum_{n \in \mathbb{N}_0} k_n b_n.$$

The writing of the p-adic integers as

$$\sum_{n=0}^{\infty} a_n p^n \text{ where } a_0 \in \{0, 1, \dots, p-1\}$$

is used by van Dantzig in [Dan30, Page 117, Subsection 18.]. “Our” writing

$$\sum_{n=0}^{\infty} k_n b_n$$

is inspired by that from van Dantzig.

Chapter 3

The Mackey problem.

3.1 Historical background.

In the framework of Functional Analysis the Mackey-Arens Theorem establishes the following: given a vector space E, and a subspace F of $Lin(E)$ (the vector space of all linear functionals on E) there exists a finest locally convex topology on E, τ_M among all those locally convex vector space topologies which admit F as dual space. The contribution of Arens was to determine constructively the topology τ_M in the following terms: The Mackey topology τ_M on E can be defined as the topology of uniform convergence on the $\sigma(F, E)$-compact absolutely convex subsets of F.

The introduction of locally quasi-convex topologies on abelian groups [Vil51] paved the way to give a counterpart of Mackey-Arens Theorem for groups. This was tried for the first time in [CMPT99]. It became clear that the "Mackey-Arens Theorem for groups" is not a counterpart of the result from Functional Analysis. The main questions left open in [CMPT99] can be expressed as follows:

Problem 3.1.1 (Mackey Problem)

1. *Let G be and abelian and let H be a subgroup of the group* $\mathrm{Hom}(G, \mathbb{T})$*, is there a finest locally quasi-convex topology among all those which admit H as dual group?*

2. *If it exists, can it be described as the topology of uniform convergence on the $\sigma(H,G)$-compact quasi-convex subsets of H?*

Inspired on the result of Arens, we will call ***the Arens topology*** *to the topology of uniform convergence on all $\sigma(H,G)$-compact and quasi-convex subsets of H.*

Definition 3.1.2 Let (G,H) be a duality. **The Mackey topology on** G **for** (G,H) is defined as the supremum of all locally quasi-convex compatible topologies, provided it is again compatible.

Definition 3.1.3 If (G,τ) is a topological group, **the Mackey topology for** (G,τ) is the Mackey topology with respect to $(G,G^\wedge)$.

So far, it is not proved that every topological group has a Mackey topology. As proved in [CMPT99] the Mackey topology is the topology of uniform convergence on a family of subsets of $G^\wedge$, which are $\sigma(G^\wedge,G)$-compact and quasi-convex. Thus, whenever the Arens topology is compatible, it coincides with the Mackey topology.

Definition 3.1.4 A topological group (G,μ) is a **Mackey group** if μ is the Mackey topology with respect to the duality $(G,G^\wedge)$.

A negative answer to the second question in Problem 3.1.1 was provided in [BTM03, Examples 4.2 and 4.5]. The authors present a topological group G which happens to be precompact and metrizable, and which does not admit any other locally quasi-convex compatible topology. Therefore, G has a Mackey topology: its original one is already the finest locally quasi-convex compatible topology. In the mentioned example G, the Arens topology is discrete (therefore, distinct from the original) and non-compatible (the original one, being metrizable and non-discrete has non-continuous characters). Thus, the Mackey topology on G, does not coincide with the Arens topology. In Proposition 4.3.7, we give another example of discrete Arens topology.

Remark 3.1.5 In [Leo08], a topological group G is told to be an Arens group if the Arens topology is compatible with the original topology.

Although a general answer to the Mackey Problem is not so far available, in [CMPT99] it is proved that in some particular cases, the original topology is the Mackey topology.

Theorem 3.1.6 *[CMPT99]*

Let (G, τ) be a locally quasi-convex topological group. If G satisfies one of these conditions:

1. *metrizable and complete.*

2. *Baire separable.*

3. *Čech-complete (in particular locally compact).*

4. *pseudocompact.*

Then τ is the Mackey topology and coincides with the Arens topology.

In locally convex spaces, the condition of completeness is not necessary to get a Mackey topology. In fact, the following can be proved:

Theorem 3.1.7 *[KN63, Corollary 22.3, page 210]*

Every metrizable locally convex topological vector space is a Mackey space.

An interesting question that arose after the publication of [CMPT99] is if this particular theorem is true for topological groups. That is: Let (G, τ) be a metrizable locally quasi-convex abelian group. Is τ the Mackey topology?

It became clear that the answer was negative, as the following example shows:

Example 3.1.8 The following groups are examples of metrizable locally quasi-convex groups which fail to be Mackey:

1. ([DMPT14]) Let X be a connected compact metrizable group. Then, the group $c_0(X)$ endowed with the topology induced from the product topology on $X^{\mathbb{N}}$ is a metrizable locally quasi-convex group which is not a Mackey group.

2. (Theorem 4.3.24) Every non-discrete linear topology on the group of the integers is a metrizable locally quasi-convex group topology which is not Mackey.

3. (Theorem 5.2.4) Every infinite torsion subgroup of $\mathbb{T}$ endowed with the topology inherited from $\mathbb{T}$ is a metrizable locally quasi-convex group which is not Mackey.

4. (Theorem 5.2.8 and [DNMP14], independently) The group of rational numbers, endowed with the topology inherited from the real line is a **non-precompact** metrizable locally quasi-convex group topology which is not Mackey.

Although several authors have worked to give a solution to the Mackey Problem (Außenhofer, Díaz-Nieto [DNMP14], Dikranjan, de Leo [Leo08], Gabriyelyan, Lukács, Trigos [BTM03]), so far, there is not a definite answer for the existence of a Mackey topology on an abelian group. Our particular aim has been to solve the conjecture in the negative, by means of producing in the group of the integers $\mathbb{Z}$ a family of locally quasi-convex compatible topologies, whose supremum -which is known to be locally quasi convex- is not a compatible topology on $\mathbb{Z}$.

The main difficulty when facing the Mackey Problem is that we cannot ensure that the supremum of two compatible topologies is compatible. However, with some additional conditions we can solve this situation.

Proposition 3.1.9 *[CMPT99, Proposition 1.14]*

Let G be an abelian group and let $\mathcal{F} := \{\tau_i\}_{i\in I}$ be a family of group topologies on G and define $\tau := \sup_{i\in I}\{\tau_i\}$. If $\mathcal{F}$ is a ***directed family****, then*

$$(G,\tau)^\wedge = \bigcup_{i\in I}(G,\tau_i)^\wedge.$$

3.2 Compatible group topologies.

The compatible topologies on a topological group have some common features that we describe in this section. For particular classes of groups (like bounded groups) they are even stronger.

Proposition 3.2.1 *Let G be an abelian group. If τ, τ' are compatible topologies on G, they share the same dually closed subgroups. That is, $C_{dua}(G, \tau) = C_{dua}(G, \tau') = C_{dua}(G, \tau^+)$.*

Proof: The assertion trivially follows from these two facts:

(a) both topologies have the same weak topology $\sigma(G, G^\wedge)$;

(b) a subgroup H of G is dually closed (with respect to any of these two topologies) precisely when it is $\sigma(G, G^\wedge)$-closed.

QED

Lemma 3.2.2 *In a linearly topologized group (G, τ) every closed subgroup is dually closed. Hence $C_{dua}(G, \tau) = C(G, \tau)$*

Proof: Let H be a closed subgroup of a linearly topologized group G. Then the closure $\bigcap_i (H + U_i)$ of H must coincide with H, where the intersection is taken over a base (U_i) of open subgroups of G. Each $H + U_i$ is an open subgroup, hence dually closed [BCMP94, Lemma 2.2]. So it remains to use the fact that the intersection of dually closed subgroups must be dually closed. QED

Remark 3.2.3 Observe that every closed, finite-index subgroup is also open. Thus, the family $C_{fi}(G, \tau)$ is precisely the family of open finite index subgroups. The Bohr topology of a bounded discrete group has as neighborhood basis the family of all finite index subgroups, i.e., coincides with the profinite topology as we prove next (for more detail see [DGB11]).

Lemma 3.2.4 *Let (G, τ) be a bounded topological abelian group. Then τ^+ is a linear topology which has as neighborhood base at* 0 *the family $C_{fi}(G, \tau)$.*

Proof: First recall that for every $\chi \in (G, \tau)^\wedge$ the subgroup $\chi(G)$ of $\mathbb{T}$ is finite. Hence, each τ^+-neighborhood contains an open subgroup of the form $\bigcap_i \chi_i^{-1}(\{0\})$ where χ_i runs in a finite subset of $G^\wedge$. On the other hand, being τ^+ precompact, every open subgroup must have finite index. Thus, the open subgroups are exactly the closed ones of finite index. QED

Theorem 3.2.5 *Let G be a bounded group and τ, τ' linear topologies on G. The following assertions are equivalent:*

(a) *τ is compatible with τ'.*

(b) $\tau^+ = \tau'^+$.

(c) $C(G, \tau) = C(G, \tau')$.

(d) $C_{dua}(G, \tau) = C_{dua}(G, \tau')$.

(e) $C_*(G, \tau) = C_*(G, \tau')$.

(f) $C_{fi}(G, \tau) = C_{fi}(G, \tau')$.

Proof: Clearly (a) and (b) are equivalent. The equivalence $(c) \Leftrightarrow (d)$ follows from Lemma 3.2.2.

$(a) \Rightarrow (d)$ is Proposition 3.2.1.

$(c) \Rightarrow (e)$ is straightforward.

$(e) \Rightarrow (f)$ is due to the fact that the subgroups satisfying (e) are closed and the property of having finite index is an algebraic property (hence not depending on the topology).

$(f) \Rightarrow (b)$ is Lemma 3.2.4. QED

3.3 Mackey topology and open subgroups.

In this section we prove that the Mackey topology is "hereditary" for open subgroups. In other words, an open subgroup of a Mackey group is also a Mackey group in the induced topology. To this end we recall next a standard extension of a topology from a subgroup to the whole group.

Definition 3.3.1 Let G be an abelian group and H a subgroup. Consider on H a group topology τ. We define the extension $\overline{\tau}$ of τ as the topology on G which has as a basis of neighborhoods at zero the neighborhoods of (H, τ).

If the topology τ on H is metrizable, the topology $\overline{\tau}$ on G is metrizable. Analogously, if τ is locally quasi-convex, then $\overline{\tau}$ is also locally quasi-convex (indeed, a subset which is quasi-convex in H is quasi-convex in G, due to the fact that H is an open subgroup in $(G, \overline{\tau})$).

Lemma 3.3.2 *Let H be an open subgroup of the topological group* (G, τ)*. If* τ *is the Mackey topology, then the induced topology* $\tau_{|H}$ *is also Mackey. As a consequence, if* ν *is a metrizable, locally quasi-convex* non-Mackey *group topology on H, then there exists a metrizable locally quasi-convex-group topology on G which is not Mackey, namely* $\overline{\nu}$*.*

Proof:

In order to prove that $(H, \tau_{|H})$ is also a Mackey group take a locally quasi-convex topology ν in H such that $(H, \nu)^\wedge = (H, \tau_{|H})^\wedge$. Define the topology $\overline{\nu}$ extension of ν.

Let us prove that $(G, \overline{\nu})^\wedge = (G, \tau)^\wedge$. Take $\phi \in (G, \overline{\nu})^\wedge$. Then $\phi_{|H}$ is ν-continuous, and, since ν and $\tau_{|H}$ are compatible, $\phi_{|H}$ is also a character of $(H, \tau_{|H})$. Since H is τ-open, ϕ is τ-continuous. Thus $(G, \overline{\nu})^\wedge \leq (G, \tau)^\wedge$. The same argument can be used to prove the converse inequality, taking into account that H is an open subgroup in $\overline{\nu}$. So $\overline{\nu}$ and τ are compatible topologies.

Since (G, τ) is Mackey, we obtain that $\overline{\nu} \leq \tau$ and this inequality also holds for their restrictions to H, $\nu \leq \tau_{|H}$. Hence $(H, \tau_{|H})$ is Mackey.

Let ν be a metrizable locally quasi-convex group topology on H which is not Mackey and $\overline{\nu}$ its extension to G. Clearly, H is $\overline{\nu}$-open, and $\overline{\nu}$ is a metrizable locally quasi-convex topology on G. If $\overline{\nu}$ were Mackey, by our previous arguments, ν should be Mackey as well, a contradiction.

QED

We do not know if the converse of Lemma 3.3.2 holds in general:

Question 3.3.3 Let (G, τ) be a topological group and $H \leq G$ an open subgroup. If $\tau_{|H}$ is the Mackey topology, is τ the Mackey topology for G?

In Section 6.2 we prove that the answer to Question 3.3.3 is positive for bounded groups.

3.4 Alternative statements for the Mackey Problem.

In this section we collect some statements of the Mackey Problem for bounded groups using Theorem 3.2.5. In these statements no reference to the dual group is made.

Problem 3.4.1 *Let (G, τ) be a bounded locally quasi-convex topological group and $C_1(G, \tau)$ be the family of locally quasi-convex group topologies having the same dually closed subgroups as τ. Does there exist a maximum in the $C_1(G, \tau)$?*

Problem 3.4.2 *Let (G, τ) be a bounded locally quasi-convex topological group and $C_2(G, \tau)$ be the family of locally quasi-convex group topologies having the same closed subgroups as τ. Does there exist a maximum in the $C_2(G, \tau)$?*

Problem 3.4.3 *Let (G, τ) be a bounded locally quasi-convex topological group and $C_3(G, \tau)$ be the family of locally quasi-convex group topologies ν satisfying $C_*(G, \tau) = C_*(G, \nu)$. Does there exist a maximum in the $C_3(G, \tau)$?*

Problem 3.4.4 *Let (G, τ) be a bounded locally quasi-convex topological group and $C_4(G, \tau)$ be the family of locally quasi-convex group topologies ν satisfying $C_{fi}(G, \tau) = C_{fi}(G, \nu)$. Does there exist a maximum in the $C_4(G, \tau)$?*

In addition, determining if the supremum of two compatible topologies is compatible can be restated as: Let τ, τ' be two group topologies having the same dually closed subgroups. Is $C_{dua}(G, \tau) = C_{dua}(G, \sup\{\tau, \tau'\})$?

3.5 $\mathcal{G}$-Mackey problem.

The Mackey Problem stated above is a specialization of a more general problem. In order to formulate it we need the following:

Definition 3.5.1 Let $\mathcal{G}$ be a class of abelian topological groups and let (G, τ) be a topological group in $\mathcal{G}$. Let $C_{\mathcal{G}}(G, \tau)$ denote the family of all $\mathcal{G}$-topologies ν on G compatible with τ. We say that $\mu \in C_{\mathcal{G}}(G, \tau)$ is the **$\mathcal{G}$-Mackey topology** for $(G, G^\wedge)$ (or the **Mackey topology for (G, τ) in $\mathcal{G}$**) if $\nu \leq \mu$ for all $\nu \in C_{\mathcal{G}}(G, \tau)$.
If $\mathcal{G}$ is the class of locally quasi-convex groups LQC, we will simply say that the topology is Mackey.

Problem 3.5.2 *Let $\mathcal{G}$ be a class of topological abelian groups. Does there exist a maximum in the family $C_{\mathcal{G}}(G,\tau)$, i.e., does the $\mathcal{G}$-Mackey topology exist?*

For the particular case $\mathcal{G} = LQC$, we get the original Mackey problem stated in [CMPT99]. Hence, we will say indifferently Mackey topology and LQC-Mackey topology to refer to the latter.

Theorem 3.5.3 *[Var64]*

*Let τ be a **metrizable** locally precompact group topology. Then τ is $\mathcal{L}pc$-Mackey.*

Corollary 3.5.4 *The group of rational numbers endowed with the topology inherited from the real line is $\mathcal{L}pc$-Mackey.*

It is a well known fact that for an LCA group (G,τ) every τ^+- compact subset $M \subset G$ is also τ-compact. This was proved by Glicksberg in 1962, and independently by Varopoulos. Later on in [BMP96] it is proved that the property holds for nuclear groups, a class of groups which embraces the LCA-groups and is closed under the formation of products, countable sums, subgroups and quotients. So there are many groups for which the property holds. This justifies the following definition:

Definition 3.5.5 An abelian topological group (G,τ) is said to be a Glicksberg group if every τ^+- compact subset $M \subset G$ is also τ-compact. Let $\mathcal{GL}$ be the class of all Glicksberg groups.

Denote by $\mathcal{GL}(G,\tau)$ the set of Glicksberg group topologies on G which are compatible with τ. The topologies in $\mathcal{GL}(G,\tau)$ share the same family $\mathbb{K}$ of compact sets.

Proposition 3.5.6 *Every metrizable Glicksberg group is $\mathcal{GL}$-Mackey.*

Proof:

Let (G,τ) be a metrizable group in $\mathcal{GL}$ and let $\nu \in \mathcal{GL}(G,\tau)$. We must prove that $\nu \leq \tau$. To that end pick a ν-closed subset $C \subset G$. For every compact subset $K \in \mathbb{K}$ the set $C \cap K$ is closed in K. Now, since (G,τ) is metrizable in particular is a k-space and therefore C is closed in τ.

QED

Chapter 4

The group of integers

4.1 Different topologies on the integers.

The most natural group topology on the group of the integers is the discrete one: in fact, it is the topology inherited from the usual topology on the real line. Examples of linear non-discrete group topologies on the integers are the p-adic topologies, which are precompact.

Any infinite group which is compact and T_2 or metrizable and complete must be uncountable, by the Baire Category Theorem [HR63, Appendix B]. Hence, there cannot exist a compact Hausdorff topology on $\mathbb{Z}$, neither a complete metrizable non-discrete group topology on $\mathbb{Z}$. Along this chapter, it will become clear that there exist in $\mathbb{Z}$:

(1) A family of $2^{\mathfrak{c}}$ precompact (non-compact) Hausdorff group topologies.

(2) A family of metrizable group topologies, which is neither complete nor precompact.

(3) A family of non metrizable, but complete group topologies.

Remark 4.1.1 The topologies of families (1) and (2) are not reflexive topologies. Indeed, if they were reflexive and metrizable, they should be complete as well (indeed, this derives from the important fact that the dual of a metrizable group is a k-space ([Cha98]). Since the dual of a k-space is complete, the result follows). But we have just stated that it is impossible to have a metrizable and complete topology on $\mathbb{Z}$. However, there exist non-discrete reflexive topologies on $\mathbb{Z}$. Indeed, in [Gab10], an example of a reflexive topology on the group of the integers is given. This example belongs to the third family. Despite this example, not every topology in family (3) is reflexive, since some of them are not locally quasi-convex (Proposition 4.5.5).

4.2 Precompact topologies, p-adic topologies and b-adic topologies.

In this section we study a family of linear topologies, which turn out to be precompact as well (since every subgroup of $\mathbb{Z}$ has finite index). The cardinality of this family is 2^{ω} (see Remark 4.3.25). The convergent sequences and the dual groups are characterized (see Propositions 4.2.15 and 4.2.13).

First we present this proposition in order to know when a precompact topology on $\mathbb{Z}$ is *MAP*:

Proposition 4.2.1 *Let H be an infinite subgroup of $\mathbb{T}$, then H is dense and separates points of $\mathbb{Z}$.*

We collect now some results on precompact topologies on $\mathbb{Z}$:

Theorem 4.2.2 *[Rac02, Theorem 10]*

Let (u_n) be a sequence of integer numbers. If $\frac{u_{n+1}}{u_n} \geq n+1$, then there exists a precompact group topology τ of weight $\mathfrak{c}$ on $\mathbb{Z}$ such that $u_n \to 0$ in $(\mathbb{Z}, \tau)$.

In [BDMW03], this condition is weakened and some other interesting results on this direction are obtained.

Theorem 4.2.3 *[BDMW03, Corollary 3.2]*

For every sequence (u_n) in $\mathbb{Z} \setminus \{0\}$ with $\frac{u_{n+1}}{u_n} \to \infty$ there exists a precompact group topology of weight $\mathfrak{c}$ such that $u_n \to 0$.

Proposition 4.2.4 *[BDMW03, Corollary 3.4]*

Let (u_n) be a sequence in $\mathbb{Z} \setminus \{0\}$ such that $\frac{u_{n+1}}{u_n}$ is bounded. Then every precompact group topology for which $u_n \to 0$ must be metrizable.

Now we describe the p-adic topologies: let p be a prime number. Then the family:

$$\mathcal{B} := \{p^n\mathbb{Z} : n \in \mathbb{N}_0\}$$

is a neighborhood basis of 0 for a group topology on $\mathbb{Z}$ (it is a direct consequence of Lemma 1.1.6), which is called the p-adic topology and will be denoted by $\lambda_{\mathbf{p}}$. Since $p^n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ for all n and p, these topologies are linear topologies. Clearly $\mathbb{Z}/(p^n\mathbb{Z})$ is finite and hence the topology $\lambda_{\mathbf{p}}$ is precompact.

We generalize these topologies just by replacing the sequence (p^n) by a D-sequence.

Definition 4.2.5 The family

$$\mathcal{B}_{\mathbf{b}} := \{b_n\mathbb{Z} : n \in \mathbb{N}_0\}$$

is a neighborhood basis of 0 for a linear group topology on $\mathbb{Z}$, which will be called **b**-adic topology and is denoted by $\lambda_{\mathbf{b}}$.

Now, we prove that every linear topology on $\mathbb{Z}$ can be obtained as a **b**-adic topology.

Proposition 4.2.6 *The following statements are equivalent:*

(1) λ is a non-discrete Hausdorff linear group topology on $\mathbb{Z}$.

(2) There exists a sequence $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}_0}$ in $\mathbb{Z}$ with $b_0 = 1$, $b_n \neq b_{n+1}$ and $b_n|b_{n+1}$ for all $n \in \mathbb{N}_0$ such that $\{b_n\mathbb{Z} : n \in \mathbb{N}_0\}$ is a neighborhood basis at 0 in $(\mathbb{Z}, \lambda)$. That is, there exists a D-sequence $\mathbf{b}$ such that $\lambda = \lambda_{\mathbf{b}}$.

(3) λ is precompact and linear.

Proof:

(1) $\Rightarrow$ (2) Let $\mathcal{U} = (U_i)_{i\in I}$ be a neighborhood basis at 0 consisting of subgroups. Since λ is not discrete, for every $i \in I$ there exists $m_i \in \mathbb{N}$ such that $U_i = m_i\mathbb{Z}$. Further, $\{m_i : i \in I\}$ is countable, hence there exists (b_n) satisfying $\{U_i : i \in I\} = \{b_n\mathbb{Z} : n \in \mathbb{N}_0\}$ for suitable $b_n \in \mathbb{N}$. Without loss of generality, we may assume that $b_0 = 1$ and that $b_n\mathbb{Z} \supsetneq b_{n+1}\mathbb{Z}$ for all $n \in \mathbb{N}$, which is equivalent to $b_n|b_{n+1}$ and $b_n \neq b_{n+1}$.

(2) $\Rightarrow$ (1) follows easily from Proposition 1 , Chapter III, § 1.2 in [Bou66]. The induced topology is Hausdorff, since $x \in b_n\mathbb{Z}$ for all $n \in \mathbb{N}$ if and only if b_n divides x for all $n \in \mathbb{N}$ if and only if $x = 0$.

The equivalence between (1) and (3) is clear.

QED

Remark 4.2.7 The topology $\lambda_{\mathbf{b}}$ generated by a D-sequence is metrizable by the Birkhoff-Kakutani theorem (see e.g. 3.3.12 in [AT08]).

Proposition 4.2.8 *Let* $\mathbf{b}$ *be a D-sequence. Then there exists* $\mathbf{c} \sqsubseteq \mathbf{b}$ *satisfying:*

1. $\mathbf{c} \in \mathcal{D}_\infty$.
2. $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}}$

Proof:

We shall define recursively a subsequence $\mathbf{c}$ with the stated condition. Fix $c_0 = b_0 = 1$ and $c_1 = b_1$. Given $c_0, \dots, c_n$ we define c_{n+1} in the following way: For i such that $c_n = b_i$ there exists $j \in \mathbb{N}_0$ such that $\frac{b_j}{b_i} > \frac{c_n}{c_{n-1}}$; or equivalently, $\frac{b_j}{c_n} > \frac{c_n}{c_{n-1}}$. Define $c_{n+1} := b_j$. This implies that $\frac{c_{n+1}}{c_n} > \frac{c_n}{c_{n-1}}$. Since $\frac{c_n}{c_{n-1}} \in \mathbb{N}$, we have $\frac{c_{n+1}}{c_n} \to \infty$.

Since $\mathbf{c}$ is a subsequence of $\mathbf{b}$ we have that $\{c_n\mathbb{Z} : n \in \mathbb{N}_0\} \subseteq \{b_n\mathbb{Z} : n \in \mathbb{N}_0\}$ and hence $\lambda_{\mathbf{c}} \leq \lambda_{\mathbf{b}}$. Conversely, for $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $n_m \in \mathbb{N}$ such that $c_m = b_{n_m} > b_n$ and hence $b_n | b_{n_m} = c_m$, which implies $c_m\mathbb{Z} \subseteq b_n\mathbb{Z}$. This shows that $\lambda_{\mathbf{c}} \geq \lambda_{\mathbf{b}}$.

QED

Now, we can compare different linear topologies. To compare $\lambda_{\mathbf{b}}$ and $\lambda_{\mathbf{c}}$ we compare the functions $\mathbb{P}(\mathbf{b})$ and $\mathbb{P}(\mathbf{c})$.

Proposition 4.2.9 *Let* $\mathbf{b}, \mathbf{c}$ *be two D-sequences. Then:*

1. $\mathbb{P}(\mathbf{b}) \leq \mathbb{P}(\mathbf{c})$ *if and only if* $\lambda_{\mathbf{b}} \leq \lambda_{\mathbf{c}}$.
2. $\mathbb{P}(\mathbf{b}) = \mathbb{P}(\mathbf{c})$ *if and only if* $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}}$.
3. $\mathbb{P}(\mathbf{b}) < \mathbb{P}(\mathbf{c})$ *if and only if* $\lambda_{\mathbf{b}} < \lambda_{\mathbf{c}}$.

Proof:

We only need to prove that if $\mathbb{P}(\mathbf{b}) \le \mathbb{P}(\mathbf{c})$ then $\lambda_{\mathbf{b}} \le \lambda_{\mathbf{c}}$.

Fix $n \in \mathbb{N}_0$ and consider the basic neighborhood $b_n\mathbb{Z}$ of $\lambda_{\mathbf{b}}$. Write $p(b_n) = (n_1^{(n)}, n_2^{(n)}, \dots, n_k^{(n)} \dots)$. Since $p(b_n) < \mathbb{P}(\mathbf{b})$, we have in particular that $p(b_n) < \mathbb{P}(\mathbf{c}) = (m_1, m_2, \dots, m_k, \dots)$. Since b_n is a natural number, there exists N_0 such that $n_k^{(n)} = 0$ if $k \ge N_0$. For $k = 1, \dots, N_0 - 1$, find i_k satisfying $n_k^{(n)} \le p_k(c_{i_k})$, where $p_k(m)$ is the k-th component of $p(m)$. Define $i := \max\{i_k : 1 \le k \le N_0 - 1\}$. Now, we have that $p(c_i) > p(b_n)$. Since c_i and b_n are natural numbers, this means that $b_n \mid c_i$. Hence, $b_n\mathbb{Z} \supset c_i\mathbb{Z}$. Consequently, we get $\lambda_{\mathbf{b}} \le \lambda_{\mathbf{c}}$.

QED

Since any linear topology on $\mathbb{Z}$ is determined by a basic D-sequence, we compute the cardinality of basic D-sequences. To that end, the following propositions are presented:

Proposition 4.2.10 *For every D-sequence* $\mathbf{b}$ *there exists a basic D-sequence* $\mathbf{c}$ *satisfying* $\mathbf{b} \sqsubseteq \mathbf{c}$ *and* $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}}$.

Proof:

For each $n \in \mathbb{N}_0$, the natural number $q_{n+1} := \frac{b_{n+1}}{b_n}$ can be written as a product of prime factors $q_{n+1} = p_1 \cdot \ldots \cdot p_m$. We interpolate after b_n the terms $b_n \cdot p_1, b_n \cdot p_1 \cdot p_2, \dots, b_n \cdot p_1 \cdots p_m = b_{n+1}$, and call the resulting supersequence $\mathbf{c}$. By Proposition 4.2.8, $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}}$.

QED

Hence, by Remark 2.1.11, it suffices to consider basic D-sequences in order to describe the cardinality of the family of linear topologies.

Lemma 4.2.11 *Following Proposition 4.2.10, the following assertions hold:*

(1) *Let* $\mathbf{b}, \mathbf{c}$ *be two basic D-sequences. The topologies* $\lambda_{\mathbf{b}}$ *and* $\lambda_{\mathbf{c}}$ *coincide if and only if each prime number p_i appears n_i times in the sequence* $\left(\frac{b_{n+1}}{b_n}\right)$ *and n_i times in the sequence* $\left(\frac{c_{n+1}}{c_n}\right)$ *where* $n_i \in \{0, 1, \dots, \infty\}$.

(2) *The cardinality of the family of linear topologies on* $\mathbb{Z}$ *is* 2^{ω}.

Proof:

Two linear topologies $\lambda_{\mathbf{b}}, \lambda_{\mathbf{c}}$ coincide if and only if $\mathbb{P}(\mathbf{b}) = \mathbb{P}(\mathbf{c})$. This means that there exist a bijection between linear topologies on $\mathbb{Z}$ and the family $\{\mathbb{P}(\mathbf{b}) : b \in \mathcal{D}\}$. Hence, the cardinality of both sets coincide, and there exist 2^ω different linear topologies on $\mathbb{Z}$.

QED

In order to study the structure of the dual of $(\mathbb{Z}, \lambda_{\mathbf{b}})$ we introduce the following notation:

Notation 4.2.12 *For every $\boldsymbol{b} \in \mathcal{D}$ and every $n \in \mathbb{N}_0$, the formula*

$$\xi_n^{\boldsymbol{b}} : \mathbb{Z} \to \mathbb{T},\ k \mapsto \frac{k}{b_n} + \mathbb{Z}.$$

defines a character on $\mathbb{Z}$. If no confusion can arise, we simply write ξ_n instead of $\xi_n^{\boldsymbol{b}}$. Each ξ_n has order b_n. Of course, $\langle\{\xi_n^{\boldsymbol{b}} :\ n \in \mathbb{N}\}\rangle = \bigcup_{n\in\mathbb{N}}\langle\xi_n^{\boldsymbol{b}}\rangle = \mathbb{Z}(\mathbf{b}^\infty)$ holds.

Proposition 4.2.13 *For $\boldsymbol{b} = (b_n) \in \mathcal{D}$, the equality*

$$(\mathbb{Z}, \lambda_{\boldsymbol{b}})^\wedge = \langle\{\xi_n^{\boldsymbol{b}} :\ n \in \mathbb{N}\}\rangle = \mathbb{Z}(\mathbf{b}^\infty)$$

holds. In particular, $\lambda_{\mathbf{b}}$ is the weak topology relative to the group $\mathbb{Z}(\mathbf{b}^\infty)$.

Proof:

Fix $n \in \mathbb{N}$ and $\chi \in (b_n\mathbb{Z})^{\triangleright}$. Since $\chi(b_n\mathbb{Z})$ is a subgroup contained in $\mathbb{T}_+$, $\chi(b_n\mathbb{Z}) = \{0 + \mathbb{Z}\}$. For $x \in \mathbb{R}$ such that $\chi(1) = x + \mathbb{Z}$, $\chi(b_n) = b_n x + \mathbb{Z} = 0 + \mathbb{Z}$ and hence there exists $k \in \mathbb{Z}$ such that $x = \frac{k}{b_n}$. So $(b_n\mathbb{Z})^{\triangleright} \subseteq \langle\xi_n^{\mathbf{b}}\rangle$. In particular, this set is finite.

Now we show that $(\mathbb{Z}, \lambda_{\mathbf{b}})^\wedge \subseteq \langle\{\xi_n^{\mathbf{b}} :\ n \in \mathbb{N}\}\rangle$ holds. Fix $\chi \in (\mathbb{Z}, \lambda_{\mathbf{b}})^\wedge$. Since χ is continuous, there exists a neighborhood $b_n\mathbb{Z}$ such that $\chi(b_n\mathbb{Z}) \subseteq \mathbb{T}_+$. As shown above, this implies that $\chi \in \langle\xi_n^{\mathbf{b}}\rangle \subseteq \langle\{\xi_m^{\mathbf{b}} :\ m \in \mathbb{N}\}\rangle$.

Conversely, fix $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. The kernel of the homomorphism $\chi = k\xi_n^{\mathbf{b}} : \mathbb{Z} \to \mathbb{T}$, $l \mapsto \frac{kl}{b_n} + \mathbb{Z}$ contains the neighborhood $b_n\mathbb{Z}$ and therefore, χ is continuous. Since k and n were arbitrary, the assertion follows. By Proposition 4.2.6, $(\mathbb{Z}, \lambda_{\mathbf{b}})$ is precompact. This implies that $\lambda_{\mathbf{b}}$ is the weak topology induced by $\langle\{\xi_n^{\mathbf{b}} :\ n \in \mathbb{N}\}\rangle$.

QED

Taking into account Proposition 4.2.1 and the fact that $\mathbb{Z}(\mathbf{b}^\infty)$ is infinite, we get the following result:

Corollary 4.2.14 *Let* $\mathbf{b}$ *be a D-sequence. Then, the topology* $\lambda_{\mathbf{b}}$ *is MAP.*

Since $\lambda_{\mathbf{b}}$ is a metrizable group topology, the null sequences are enough to describe them. So we characterize them next:

Proposition 4.2.15 *Let* $(b_n)_{n\in\mathbb{N}_0}$ *be a D-sequence and consider a sequence of integers* $(l_j)_{j\in\mathbb{N}}$*. Then the following conditions are equivalent:*

(1) $l_j \to 0$ *in* $\lambda_{\boldsymbol{b}}$*.*

(2) For every $n \in \mathbb{N}$, $\xi_n^{\boldsymbol{b}}(l_j) \to 0 + \mathbb{Z}$.

(3) For every $n \in \mathbb{N}$ *there exists* j_n *such that* $b_n | l_j$ *for all* $j \geq j_n$.

Proof:

(1) $\Longleftrightarrow$ (2) holds because of Proposition 4.2.13.

(1) $\Longleftrightarrow$ (3): (l_j) converges to 0 in $\lambda_{\mathbf{b}}$ if and only if for every $n \in \mathbb{N}$ there exists j_n such that $l_j \in b_n\mathbb{Z}$ for all $j \geq j_n$. This condition is equivalent to $b_n | l_j$ for all $j \geq j_n$.

QED

Corollary 4.2.16 *Let* $\mathbf{b}$ *be a D-sequence. Then* $\mathbf{b} \overset{\lambda_{\mathbf{b}}}{\to} 0$. *In particular, we have* $p^n \overset{\lambda_{\mathbf{p}}}{\to} 0$.

As a consequence of Propositions 4.2.6 and 4.2.16, we obtain

$$\mathcal{D}_\infty \subseteq \mathcal{D} \subseteq \mathcal{TB}_{\mathbb{Z}} \subseteq \mathcal{T}_{\mathbb{Z}}.$$

We turn on now to the completion of $(\mathbb{Z}, \lambda_{\mathbf{b}})$. From [HR63, Section 10.2], it can be deduced the following proposition

Proposition 4.2.17 *The completion of the precompact metrizable abelian group* $(\mathbb{Z}, \lambda_{\mathbf{b}})$ *coincides with the compact metrizable abelian group of* $\mathbf{q}$*-adic integers* $\Delta_{\mathbf{q}}$*, where* $q_n = \frac{b_{n+1}}{b_n}$, $n = 0, 1, \ldots$ *(see [HR63, Section 10.2] for the definition of* $\Delta_{\mathbf{q}}$*). This group is isomorphic to* $\mathbb{Z}_{\mathbf{b}}$*.*

To prove that $\Delta_{\mathbf{q}}$ is isomorphic to $\mathbb{Z}_{\mathbf{b}}$, we will give a group topology $\Lambda_{\mathbf{b}}$ to $\mathbb{Z}_{\mathbf{b}}$, satisfying that the restriction to $\Lambda_{\mathbf{b}|i(\mathbb{Z})}$ is exactly $i(\lambda_{\mathbf{b}})$ (for the definition of i, see Definition 2.4.3). Then, we will prove that $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ is complete. Since the completion of an abelian topological group is unique (up to isomorphism) the result will follow.

Definition 4.2.18 Let $\mathbf{b}$ be a D-sequence. Define

$$U_n := \{\mathbf{x} \in \mathbb{Z}_{\mathbf{b}} : x_0 = \cdots = x_{n-1} = 0\}$$

for $n \geq 1$. Put $U_0 = \mathbb{Z}_{\mathbf{b}}$.

Proposition 4.2.19 *Let* $\mathbf{b}$ *be a D-sequence. Then the family* $\{U_n\}_{n\in\mathbb{N}_0}$ *is a neighborhood basis of* 0 *for a linear metrizable group topology on* $\mathbb{Z}_{\mathbf{b}}$*. We will denote this topology by* $\Lambda_{\mathbf{b}}$*.*

Proof:

$\Lambda_{\mathbf{b}}$ is a Hausdorff group topology by Lemma 1.1.6. By Birkhoff-Kakutani Theorem, it is metrizable. Since U_n is a subgroup of $\mathbb{Z}_{\mathbf{b}}$, the topology is linear.

QED

Remark 4.2.20 Since $i(b_n\mathbb{Z}) = i(\mathbb{Z}) \cap U_n$, we have that $\Lambda_{\mathbf{b}|i(\mathbb{Z})} = i(\lambda_{\mathbf{b}})$.

Theorem 4.2.21 *Let* $\mathbf{b}$ *be a D-sequence. Then* $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ *is the completion (up to isomorphism) of* $(\mathbb{Z}, \lambda_{\mathbf{b}})$*.*

Proof:

First we prove that $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ is complete. Since $\Lambda_{\mathbf{b}}$ is a metrizable group topology it suffices to see that any Cauchy sequence converges and its limit belongs to $\mathbb{Z}_{\mathbf{b}}$. Let $(k^{(n)})_n \subset \mathbb{Z}_{\mathbf{b}}$ be a Cauchy sequence. This means that for every $n \in \mathbb{N}_0$, there exists $n_0 \in \mathbb{N}$ satisfying that $k^{(m)} - k^{(m')} \in U_n$ if $m, m' \geq n_0$.

In particular, for $m' = n_0$, this means that $k^{(m)} \in k^{(n_0)} + U_n$. This means that there exists $(0, \overset{(n)}{\ldots}, 0, \ldots) = l_m \in U_n$ with $k^{(m)} = k^{(n_0)} + l_m$. By the definition of the operation in $\mathbb{Z}_{\mathbf{b}}$, this is equivalent to $k_0^{(n_0)} = k_0^{(m)}, k_1^{(n_0)} = k_1^{(m)}, \ldots, k_{n-1}^{(n_0)} = k_{n-1}^{(m)}$. Hence, for all $n \in \mathbb{N}_0$, there exists n_0 satisfying that $k_n^{(m)} = k_n^{(n_0)}$ if $m \geq n_0$.

Define $k = (k_n)$, where $\left(-\frac{q_{n+1}}{2}, \frac{q_{n+1}}{2}\right] \ni k_n := \lim_{n\to\infty} k_n^{(m)}$. By construction, $k \in \mathbb{Z}_{\mathbf{b}}$ and $k^{(n)} \to k$ in $\Lambda_{\mathbf{b}}$. Hence $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ is complete.

Now we see that $i(\mathbb{Z}, \lambda_{\mathbf{b}})$ is dense in $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$. Indeed, let $k = (k_0, k_1, \dots) \in \mathbb{Z}_{\mathbf{b}}$. Define the sequence $k^{(n)} = (k_0, k_1, \dots, k_{n-1}, 0, 0, \dots)$. It is clear that $k^{(n)} \in i(\mathbb{Z})$ and $k - k^{(n)} \in U_n$. Hence, $i(\mathbb{Z})$ is dense in $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$. Up to isomorphism, $(\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ is the completion of $(\mathbb{Z}, \lambda_{\mathbf{b}})$.

QED

Remark 4.2.22 It is known that the completion of a precompact group is compact. Hence, $G = (\mathbb{Z}_{\mathbf{b}}, \Lambda_{\mathbf{b}})$ is a compact group. Since $H = (\mathbb{Z}, \lambda_{\mathbf{b}})$ is a metrizable group, which is dense in G, we get that $G^\wedge \cong H^\wedge \cong \mathbb{Z}(\mathbf{b}^\infty)_d$ (see [Auß99, 4.5] or [Cha98]). Taking dual groups in this expression, we get

$$\mathbb{Z}(\mathbf{b}^\infty)^* \cong G^{\wedge\wedge} \cong G(= \mathbb{Z}_{\mathbf{b}}).$$

The last isomorphism is due to the fact that compact groups are reflexive. The fact $\mathbb{Z}(\mathbf{b}^\infty)^* \cong \mathbb{Z}_{\mathbf{b}}$ will be used in Chapter 5 to prove that $\mathbb{Z}(\mathbf{b}^\infty)$ with the topology inherited from the unit circle is not Mackey.

Now, we give some results related to non-linear precompact topologies and to the Bohr topology on $\mathbb{Z}$.

Remark 4.2.23 In [Rac02] it is proved that there exist two families $\mathcal{A}_1$ and $\mathcal{A}_2$ of $2^{\mathfrak{c}}$ precompact topologies in $\mathbb{Z}$ -pairwise non-homeomorphic- satisfying that the topologies of $\mathcal{A}_1$ have no non-trivial convergent sequences, and the ones of $\mathcal{A}_2$ have convergent sequences. Taking into account that each metrizable topology in an infinite group has convergent sequences, it is clear that the topologies in $\mathcal{A}_1$ are not metrizable.

Let $\mathfrak{P}$ be the family of all Hausdorff precompact topologies on $\mathbb{Z}$.

The family $\mathfrak{P}$ (ordered by $\subseteq$) has a maximum, namely the weak topology induced by all the characters of $\mathbb{Z}$, $\tau_{\mathbb{T}}(= \mathbb{Z}^\#)$.

Proposition 4.2.24 *Let $m \geq 2$ be a natural number. Then $m\mathbb{Z}$ is an open subgroup in $\mathbb{Z}^\#$.*

Proof:

Let $\mathbf{b} = (m^n)_{n\in\mathbb{N}_0}$. Then $m\mathbb{Z}$ is, by definition, an open subgroup of $\lambda_{\mathbf{b}}$. Since $\lambda_{\mathbf{b}} < \sigma(\mathbb{Z}, \mathbb{T})$, it follows that $m\mathbb{Z}$ is open in $\mathbb{Z}^\#$.

QED

In Section 4.3, we prove that non-discrete linear topologies on $\mathbb{Z}$ are not *LQC*-Mackey, since we find finer compatible group topologies on $\mathbb{Z}$ (see Proposition 4.3.17 and Theorem 4.3.24).

4.3 Topologies of uniform convergence on $\mathbb{Z}$.

The topologies of uniform convergence on the group $\mathbb{Z}$ are given by different families in $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$. Identifying $\mathrm{Hom}(\mathbb{Z}, \mathbb{T})$ with $\mathbb{T}$, we simply have to consider families of subsets in $\mathbb{T}$. We will mainly concentrate on families of cardinality 1, say we will consider the topology of uniform convergence on $\{B\}$ for $B \subset \mathbb{T}$. We shall write τ_B instead of $\tau_{\{B\}}$. The topology τ_B admits the following neighborhood basis of 0:

$$V_{B,m} := \{z \in \mathbb{Z} : \chi(z) + \mathbb{Z} \in \mathbb{T}_m \text{ for all } \chi \in B\} = \bigcap_{\chi \in B} \chi^{-1}(\mathbb{T}_m).$$

As seen in Section 1.3, τ_B is discrete whenever B is dense in $\mathbb{T}$. We prove in Theorem 4.3.2 that it can be discrete also for a non-dense subset B. It is surprising also that the sets of the form $B = \{\frac{1}{b_n} + \mathbb{Z}\}$, where (b_n) is a D-sequence give rise to topologies τ_B related with the corresponding **b**-adic topology $\lambda_{\mathbf{b}}$. This fact has allowed us to prove that linear topologies on $\mathbb{Z}$ are not Mackey. We will study this in Subsection 4.3.1.

By the choice of the basis of τ_B, the following is clear:

Proposition 4.3.1 *Let $B \subset \mathbb{T}$. Then, the following holds* $(\mathbb{Z}, \tau_B)^\wedge \geq \langle B \rangle$.

Proof:

For $\varphi \in B$ it is clear that $\varphi(V_{B,m}) \subseteq \mathbb{T}_+$. Hence $B \in (\mathbb{Z}, \tau_B)^\wedge$. Since $\langle B \rangle$ is the smallest subgroup containing B, we get $\langle B \rangle \leq (\mathbb{Z}, \tau_B)^\wedge$

QED

The topology of uniform convergence on B can be discrete under certain conditions, as the following theorem shows:

Theorem 4.3.2 *Let $\{x_n\}$ be a strictly decreasing sequence in $\left(0, \frac{1}{2}\right]$, where $x_n \to 0$, such that $(\frac{x_n}{x_{n+1}})_{n\in\mathbb{N}}$ is bounded. Let $B = \{x_n + \mathbb{Z} \mid n \in \mathbb{N}\} \subset \mathbb{T}$. Then τ_B is discrete.*

Proof:

Since $(\frac{x_n}{x_{n+1}})$ is bounded, there exists $m \in \mathbb{N}$, with $m > 1$ such that $\frac{x_n}{x_{n+1}} \leq m$, $\forall n \in \mathbb{N}$. Choose $k \in \mathbb{Z}$ where $k \in \bigcap\limits_{\chi\in B} \chi^{-1}(\mathbb{T}_m)$. Since $\mathbb{T}_m$ is symmetric we can pick $k \geq 0$.

First, we prove $k \leq \frac{1}{4x_1}$. By contradiction, suppose that $k > \frac{1}{4x_1}$. Since $(x_n)_{n\in\mathbb{N}}$ is strictly decreasing, there exists a unique $n \in \mathbb{N}$ such that $\frac{1}{4x_n} < k \leq \frac{1}{4x_{n+1}}$. Multiplying by x_{n+1} we obtain that $\frac{1}{4}\frac{x_{n+1}}{x_n} < kx_{n+1} \leq \frac{1}{4}$. Since m is a bound for $\frac{x_n}{x_{n+1}}$ we get $\frac{1}{4m} < kx_{n+1} \leq \frac{1}{4}$. Hence $kx_{n+1} + \mathbb{Z} \notin \mathbb{T}_m$. Choosing $B \ni \chi : k \mapsto k \cdot x_{n+1} + \mathbb{Z}$ we get that $\chi(k) \notin \mathbb{T}_m$. Which contradicts $k \in \bigcap\limits_{\chi\in B} \chi^{-1}(\mathbb{T}_m)$.

Hence, we get

$$\bigcap_{\chi\in B} \chi^{-1}(\mathbb{T}_m) \subset \mathbb{Z} \cap \left[-\frac{1}{4x_1}, \frac{1}{4x_1}\right],$$

which is a finite subset. Now, fix $l \in \mathbb{N}$ such that $4lx_1 > 1$. Considering

$$j \in \bigcap_{\chi\in B} \chi^{-1}(\mathbb{T}_{lm}),$$

we get $j, 2j, \ldots, lj \in \bigcap_{\chi\in B} \chi^{-1}(\mathbb{T}_m) \subset \left[-\frac{1}{4}, \frac{1}{4}\right] \cap \mathbb{Z}$. Equivalently: $j \in \left[-\frac{1}{4lx_1}, \frac{1}{4lx_1}\right] \cap \mathbb{Z}$, by the choice of l this means that $j = 0$. Thus, $\bigcap_{\chi\in B} \chi^{-1}(\mathbb{T}_{lm}) = \{0\}$. Hence τ_B is discrete.

QED

4.3.1 Topologies of uniform convergence on $\mathbb{Z}$ induced by a D-sequence.

A D-sequence $\mathbf{b}$ in $\mathbb{Z}$ induces in a natural way a sequence in $\mathbb{T}$, by means of the inverses modulo $\mathbb{Z}$. Namely, for a D-sequence, $\mathbf{b} = (b_n)_{n\in\mathbb{N}_0}$, define

$$\underline{\mathbf{b}} := \left(\frac{1}{b_n} + \mathbb{Z} : n \in \mathbb{N}_0\right) \subset \mathbb{T}.$$

Denote by $\tau_{\mathbf{b}}$ the topology of uniform convergence on $\underline{\mathbf{b}}$. As we will see in this section it is related to the $\mathbf{b}$-adic topology $\lambda_{\mathbf{b}}$.

We describe next a suitable neighborhood basis for $\tau_{\mathbf{b}}$.

Definition 4.3.3 Let $\mathbf{b}$ be a D-sequence. Set

$$V_{\mathbf{b},m} := \left\{ z \in \mathbb{Z} : \frac{z}{b_n} + \mathbb{Z} \in \mathbb{T}_m \text{ for all } n \in \mathbb{N}_0 \right\}.$$

Clearly, the family $\{V_{\mathbf{b},m}\}_{m\in\mathbb{N}}$ is a neighborhood basis of 0 for the topology $\tau_{\mathbf{b}}$.

Proposition 4.3.4 *Let* $\mathbf{b}$ *be a D-sequence. Then:*

- *The topology* $\tau_{\mathbf{b}}$ *is Hausdorff.*
- *The topology* $\tau_{\mathbf{b}}$ *is metrizable.*
- *The topology* $\tau_{\mathbf{b}}$ *is locally quasi-convex.*

Proof:

For the first statement, we consider the following: If k is an integer belonging to $\bigcap_{m\in\mathbb{N}} V_{\mathbf{b},m}$, then $\frac{k}{b_m} + \mathbb{Z} = 0 + \mathbb{Z}$ for all $m \in \mathbb{N}$, hence b_m divides k for all $m \in \mathbb{N}$ and so $k = 0$.

By the Birkhoff-Kakutani theorem, we obtain that $\tau_{\mathbf{b}}$ is metrizable.

Finally, it remains to be shown that the sets $V_{\mathbf{b},m}$ are quasi-convex. This is an easy consequence of the following facts: Intersections and inverse images under continuous homomorphisms of quasi-convex sets are quasi-convex (e.g. (6.2) in [Auß99]). It is straightforward to check that the sets $\mathbb{T}_m$ are quasi-convex. Hence the assertion follows.

QED

By Theorem 4.3.2, the following is clear:

Corollary 4.3.5 *If* $\mathbf{b}$ *is a D-sequence with bounded ratios, then* $\tau_{\mathbf{b}}$ *is discrete.*

Proposition 4.3.6 *[DL14, Theorem F]*

Let $\mathbf{b}$ *be a D-sequence. If* $q_n \geq 8$ *for every* $n \in \mathbb{N}$*, then* $\{0\} \cup \{\pm\frac{1}{b_n}\}$ *is a quasi-convex sequence in* $\mathbb{T}$.

Proposition 4.3.7 *The Arens topology in* $(\mathbb{Z}, \mathbb{Z}(2^\infty))$ *is discrete.*

Proof:

Let $(b_n) = (8^n)_{n\in\mathbb{N}_0}$ and $S = \{0\} \cup \{\pm\frac{1}{b_n} + \mathbb{Z}\} \subset \mathbb{T}$. By Proposition 4.3.6, S is quasi-convex in $\mathbb{T}$ and in $\mathbb{Z}(2^\infty)$ and it is clearly compact. By Corollary 4.3.5, $\tau_{\mathbf{b}}$ is discrete. Since $\tau_{\mathbf{b}}$ is coarser than the Arens topology, we get that the Arens topology is discrete.

QED

The following proposition shows that the application

$$\mathcal{D} \ni \mathbf{b} \longmapsto \tau_{\mathbf{b}}$$

is not injective.

Proposition 4.3.8 *Let* $\mathbf{b}$ *be a D-sequence. Let* $\mathbf{c} \sqsubseteq \mathbf{b}$ *with* $c_0 = b_0$. *For any* $n \in \mathbb{N}$ *let* j_n *be the unique natural number such that* $c_{j_n} < b_n \leq c_{j_n+1}$. *If*

$$\sup\left\{\frac{c_{j_n+1}}{b_n} \ : \ n \in \mathbb{N}\right\} < \infty$$

then $\tau_{\mathbf{b}} = \tau_{\mathbf{c}}$.

Proof:

Since $\mathbf{c}$ is a subsequence of $\mathbf{b}$, we have that $\tau_{\mathbf{c}} \leq \tau_{\mathbf{b}}$.

Conversely, let $L := \sup\left\{\frac{c_{jn+1}}{b_n} \ : \ n \in \mathbb{N}\right\}$. For $m \in \mathbb{N}$, we define $m' = mL$. We shall prove that $V_{\mathbf{c},m'} \subseteq V_{\mathbf{b},m}$. For that, fix $k \in V_{\mathbf{c},m'}$.

If $b_n \in \mathbf{c}$, then $\frac{k}{b_n} + \mathbb{Z} \in \mathbb{T}_{m'} \subseteq \mathbb{T}_m$.

If $b_n \notin \mathbf{c}$, we choose the unique index j_n such that $c_{j_n} < b_n < c_{j_n+1}$. As $\frac{k}{b_n} = \frac{k}{c_{jn+1}}\frac{c_{jn+1}}{b_n}$ and $\frac{c_{jn+1}}{b_n}$ is a natural number $\leq L$ and $\frac{k}{c_{jn+1}} + \mathbb{Z} \in \mathbb{T}_{m'}$, we obtain $\frac{k}{b_n} + \mathbb{Z} \in \mathbb{T}_m$, as desired.

QED

Using the conditions in Definition 4.3.3 it is hard to test if an integer belongs to $V_{\mathbf{b},m}$. Hence, it is important to find milder conditions, which can help to characterize the integers belonging to $V_{\mathbf{b},m}$ for a fixed m. To this end, the following propositions are presented.

Lemma 4.3.9 *Let* $\mathbf{b}$ *be a D-sequence. Let* $k = \sum_{s=0}^{N(k)} k_s b_s$ *be an integer, and let* $k_0, \ldots, k_{N(k)}$ *be the* $\mathbf{b}$*-coordinates of* k. *Let* N *denote* $\max\{n-1, N(k)\}$. *Then the following assertions are equivalent:*

(1) $k \in V_{\mathbf{b},m}$.

(2) $\displaystyle\sum_{s=0}^{N} \frac{k_s b_s}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ *for all* $n \in \mathbb{N}$.

Proof:

By Proposition 2.2.1 we write $k = \displaystyle\sum_{s=0}^{N(k)} k_s b_s$.

Then, $\displaystyle\frac{k}{b_n} = \sum_{s=0}^{N(k)} \frac{k_s b_s}{b_n}$, and $\displaystyle\frac{k}{b_n} + \mathbb{Z} = \sum_{s=0}^{N(k)} \frac{k_s b_s}{b_n} + \mathbb{Z}$.

Since $b_n \mid b_m$ if $m \geq n$ and $k_m \in \mathbb{Z}$, we get

$$\sum_{s=0}^{N(k)} \frac{k_s b_s}{b_n} + \mathbb{Z} = \sum_{s=0}^{N} \frac{k_s b_s}{b_n} + \mathbb{Z}.$$

As $\frac{k}{b_n} + \mathbb{Z} \in \mathbb{T}_m$, the assertion follows.

QED

The following corollary is the characterization we were seeking for.

Theorem 4.3.10 *Let* $\mathbf{b}$ *be a D-sequence and* $k \in \mathbb{Z}$ *and let* $k_0, \dots, k_{N(k)}$ *be its* $\mathbf{b}$*-coordinates. Then the following assertions are equivalent:*

(1) $k \in V_{\mathbf{b},m}$.

(2) $\displaystyle\left|\sum_{s=0}^{n-1} \frac{k_s b_s}{b_n}\right| \leq \frac{1}{4m}$ *for all* $n \in \mathbb{N}_0$.

Proof:

The proof is an immediate consequence of Lemma 4.3.9 and Lemma 2.2.2.

QED

Next we obtain a bound for the coordinates of the elements in $V_{\mathbf{b},m}$ in terms of the sequence (q_n)

Proposition 4.3.11 *Let* $\mathbf{b}$ *be a D-sequence and* $k \in \mathbb{Z}$. *Let* $(k_0, k_1, \dots)$ *be the* $\mathbf{b}$*-coordinates of* k. *If* $k \in V_{\mathbf{b},m}$, *then* $|k_n| \leq \frac{3q_{n+1}}{8m}$ *for all* $n \in \mathbb{N}_0$.

Proof:

For $n = 0$, we have $\left|\frac{k_0 b_0}{b_1}\right| \overset{4.3.10}{\leq} \frac{1}{4m} \leq \frac{3}{8m}$. Hence $|k_0| \leq \frac{3q_1}{8m}$.

Suppose that for some $n \geq 1$, we have $|k_n| > \frac{3q_{n+1}}{8m}$. Then,

$$\left|\sum_{s=0}^{n-1} \frac{k_s b_s}{b_{n+1}}\right| = \left|\sum_{s=0}^{n} \frac{k_s b_s}{b_{n+1}} - \frac{k_n}{q_{n+1}}\right| \overset{4.3.10}{>} \frac{3}{8m} - \frac{1}{4m} = \frac{1}{8m}.$$

Since

$$\frac{\left|\sum_{s=0}^{n-1} k_s b_s\right|}{b_n} = \frac{\left|\sum_{s=0}^{n-1} k_s b_s\right|}{b_{n+1}} q_{n+1} > \frac{1}{8m} 2 = \frac{1}{4m},$$

we get a contradiction with Theorem 4.3.10.

QED

Proposition 4.3.12 *Let* **b** *be a D-sequence. Choose* $k_0, k_1, \ldots, k_N$ *such that*

$$|k_n| \leq \frac{q_{n+1}}{8m} \text{ for } 0 \leq n \leq N.$$

Let k be the integer whose **b***-coordinates are* $(k_0, \ldots, k_N, 0, \ldots)$. *Then*

$$k \in V_{\mathbf{b},m}.$$

Proof:

Let $0 \leq n \leq N + 1$, then

$$\left|\sum_{p=0}^{n-1} \frac{k_p b_p}{b_n}\right| \leq \sum_{p=0}^{n-1} \left|\frac{k_p b_p}{b_n}\right| = \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} \frac{|k_p|}{q_{p+1}} \leq \frac{1}{8m} \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n}$$

holds.

Since $\frac{1}{q_n} \leq \frac{1}{2}$ for all $0 \leq n \leq N$, we get $\frac{b_{n-k}}{b_n} \leq \frac{1}{2^k}$.

In the first equality, we consider the change $k = n - 1 - p$ and obtain

$$\sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} = \sum_{k=0}^{n-1} \frac{b_{n-k}}{b_n} \leq \sum_{k=0}^{n-1} \frac{1}{2^k} \leq \sum_{k=0}^{\infty} 2^{-k} = 2.$$

Hence,

$$\left|\sum_{p=0}^{n-1} \frac{k_p b_p}{b_n}\right| \leq \frac{1}{8m} \sum_{p=0}^{n-1} \frac{b_{p+1}}{b_n} \leq \frac{1}{4m}. \quad (*)$$

For $n > N+1$, it is clear that

$$\left|\sum_{p=0}^{n-1} \frac{k_p b_p}{b_n}\right| = \left|\sum_{p=0}^{N} \frac{k_p b_p}{b_{N+1}}\right| \frac{b_{N+1}}{b_n} < \left|\sum_{p=0}^{n-1} \frac{k_p b_p}{b_{N+1}}\right| \leq \frac{1}{4m}.$$

The last inequality follows from (*) in the case $n = N+1$.

By Theorem 4.3.10, it follows that $k \in V_{\mathbf{b},m}$.

QED

As a consequence of Proposition 4.3.11 and Proposition 4.3.12, it is possible to define a new neighborhood basis of 0 for the topology $\tau_{\mathbf{b}}$.

Notation 4.3.13 *Define*

$$B_{\mathbf{b},\varepsilon} := \left\{k = \sum k_n b_n \,:\, (k_n) \in \mathbb{Z}^{\mathbb{N}_0}, |k_n| \leq \varepsilon q_{n+1} \text{ for all } n\right\}.$$

These $B_{\mathbf{b},\varepsilon}$ turn out to be useful as the following proposition shows:

Proposition 4.3.14 *Let* $\mathbf{b}$ *be a D-sequence, and* $V_{\mathbf{b},m}$ *and* $B_{\mathbf{b},\varepsilon}$ *as defined above. Then for all* $m \in \mathbb{N}$ *we have:*

$$B_{\mathbf{b},\frac{1}{8m}} \subset V_{\mathbf{b},m} \subset B_{\mathbf{b},\frac{3}{8m}}.$$

Consequently, the family $\{B_{\mathbf{b},\frac{1}{n}}\}_{n\in\mathbb{N}_0}$ *is a neighborhood basis of* 0 *for the topology* $\tau_{\mathbf{b}}$.

In particular,

$$\left(B_{\mathbf{b},\frac{3}{8m}}\right)^{\triangleright} \subset (V_{\mathbf{b},m})^{\triangleright} \subset \left(B_{\mathbf{b},\frac{1}{8m}}\right)^{\triangleright}$$

We characterize now the null sequences in $\tau_{\mathbf{b}}$ in terms of the $\mathbf{b}$-coordinates. We obtain as a corollary that the starting sequence $\mathbf{b}$ need not converge in $\tau_{\mathbf{b}}$.

Proposition 4.3.15 *Let* $(l_j)_{j\in\mathbb{N}}$ *be a sequence of integers. Then the following conditions are equivalent:*

(1) $l_j \to 0$ *in* $\tau_{\boldsymbol{b}}$.

(2) For every $m \in \mathbb{N}$*, there exists* $j_m \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *and all* $j \geq j_m$ $\quad \underbrace{\frac{l_j}{b_n}}_{=\xi_n^{\boldsymbol{b}}(l_j)} + \mathbb{Z} \in \mathbb{T}_m$ *holds.*

(3) $(\xi_n^{\boldsymbol{b}}(l_j))_{j\in\mathbb{N}}$ *converges uniformly in n to* $0 + \mathbb{Z} \in \mathbb{T}$.

Proof:

The sequence (l_j) converges to 0 in $\tau_{\mathbf{b}}$ if and only if for every $m \in \mathbb{N}$, there exists j_m such that for all $j \geq j_m$ we have $l_j \in V_{\mathbf{b},m}$. The last condition is equivalent to $\xi_n^{\mathbf{b}}(l_j) \in \mathbb{T}_m$ for all $n \in \mathbb{N}$ and all $j \geq j_m$. This shows the equivalence between (1) and (3). Since $\xi_n^{\mathbf{b}}(l_j) = \frac{l_j}{b_n} + \mathbb{Z}$, also the equivalence with (2) is clear.

QED

Corollary 4.3.16 *For* $\boldsymbol{b} \in \mathcal{D}$*, the following statements are equivalent:*

(1) $\boldsymbol{b} \in \mathcal{D}_\infty$.

(2) $b_j \xrightarrow{\tau_{\boldsymbol{b}}} 0$.

Proof:

(1) $\Longrightarrow$ (2): Fix $m \in \mathbb{N}$. Since $\frac{1}{q_{j+1}} \to 0$ in $\mathbb{R}$, there exists j_m such that $\frac{1}{q_{j+1}} \leq \frac{1}{4m}$ for all $j \geq j_m$. Choose $j \geq j_m$.

If $n \leq j$, then we have $\frac{b_j}{b_n} + \mathbb{Z} = 0 + \mathbb{Z} \in \mathbb{T}_m$.

If $n > j$, then $\left|\frac{b_j}{b_n}\right| \leq \frac{b_j}{b_{j+1}} = \frac{1}{q_{j+1}} \leq \frac{1}{4m}$; which implies that $\frac{b_j}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ for all $n \in \mathbb{N}$ and $j \geq j_m$.

Combining these facts, we conclude from Proposition 4.3.15 that (b_j) converges to 0 in $\tau_{\mathbf{b}}$.

(2) $\Longrightarrow$ (1): Assume that (b_j) converges to 0 in $\tau_{\mathbf{b}}$. This implies that for given $m \in \mathbb{N}$ there exists $j_m \in \mathbb{N}$ such that $\frac{b_j}{b_n} + \mathbb{Z} \in \mathbb{T}_m$ holds for all $j \geq j_m$ and all $n \in \mathbb{N}$. In particular $\frac{b_j}{b_{j+1}} + \mathbb{Z} \in \mathbb{T}_m$ for all $j \geq j_m$. This is equivalent to $\frac{1}{q_{j+1}} \leq \frac{1}{4m}$, so (1) follows.

QED

Proposition 4.3.17 *For every D-sequence* $\mathbf{b}$*, the topology* $\tau_{\boldsymbol{b}}$ *is strictly finer than* $\lambda_{\boldsymbol{b}}$*. In other words,*

$$\mathrm{id} : (\mathbb{Z}, \tau_{\boldsymbol{b}}) \to (\mathbb{Z}, \lambda_{\boldsymbol{b}})$$

is a non-open continuous isomorphism.

Proof:

Since both topologies are metrizable, it is sufficient to consider sequences. So let $(l_j)_{j\in\mathbb{N}}$ be a sequence which converges to 0 in $\tau_{\mathbf{b}}$. According to Proposition 4.3.15, $\xi_n^{\mathbf{b}}(l_j)$ converges uniformly in n to $0+\mathbb{Z}$, in particular it converges pointwise. So it is a consequence of Proposition 4.2.15 that (l_j) converges to 0 in $\lambda_{\mathbf{b}}$. Hence $\lambda_{\mathbf{b}} \leq \tau_{\mathbf{b}}$. Since $\{\frac{1}{b_n}+\mathbb{Z}\}$ is an infinite subset of $\mathbb{T}$, $\tau_{\mathbf{b}}$ is not precompact (see [Leo08, Proposition 7.43]). Hence $\lambda_{\mathbf{b}} \neq \tau_{\mathbf{b}}$.

QED

Since $\mathrm{id} : (\mathbb{Z},\tau_{\mathbf{b}}) \to (\mathbb{Z},\lambda_{\mathbf{b}})$ is continuous, every $\lambda_{\mathbf{b}}$-continuous character is $\tau_{\mathbf{b}}$-continuous, hence we obtain:

Corollary 4.3.18 $(\mathbb{Z},\lambda_{\boldsymbol{b}})^\wedge$ *is a subgroup of* $(\mathbb{Z},\tau_{\boldsymbol{b}})^\wedge$.

4.3.2 Duality if $\mathbf{b} \in \mathcal{D}_\infty^\ell$.

In this subsection we state a significant result: linear topologies on $\mathbb{Z}$ are not Mackey topologies (Theorem 4.3.24). It was obtained in collaboration with Lydia Außenhofer and has already been published ([ABM12]). The techniques used there have allowed to prove Theorem 4.3.31, which states the following: if $\mathbf{b}$ is a D-sequence which belongs to $\mathcal{D}_\infty^\ell$, for some natural number ℓ, then the dual group $(\mathbb{Z},\tau_{\mathbf{b}})^\wedge$ coincides with $\mathbb{Z}(\mathbf{b}^\infty) = (\mathbb{Z},\lambda_{\mathbf{b}})^\wedge$, so $\tau_{\mathbf{b}}$ is compatible with $\lambda_{\mathbf{b}}$. The lemmas and propositions involved in the proof are interesting by themselves.

First we establish a family of $\tau_{\mathbf{b}}$-convergent subsequences of $\mathbf{b}$ which will be used in the proof of the main theorem.

Lemma 4.3.19 *Let* $\boldsymbol{b}$ *be a D-sequence. For every sequence of indices* $(n_j)\subseteq\mathbb{N}$ *such that* $q_{n_j+1} \xrightarrow{j\to\infty} \infty$, *the subsequence* $(b_{n_j})_{j\in\mathbb{N}}$ *of* $\boldsymbol{b}$ *is a null sequence in* $\tau_{\boldsymbol{b}}$.

Proof:

Fix $m\in\mathbb{N}$. Since $\dfrac{1}{q_{n_j+1}} \to 0$ in $\mathbb{R}$, there exists $j_m\in\mathbb{N}$ such that $\dfrac{1}{q_{n_j+1}} < \dfrac{1}{4m}$ for all $j\geq j_m$. Fix $j\geq j_m$ and $n\in\mathbb{N}$.

If $n\leq n_j$, then $\frac{b_{n_j}}{b_n}+\mathbb{Z} = 0+\mathbb{Z}\in\mathbb{T}_m$.

If $n>n_j$, then $\left|\frac{b_{n_j}}{b_n}\right| \leq \frac{b_{n_j}}{b_{n_j+1}} = \frac{1}{q_{n_j+1}} < \frac{1}{4m}$, which implies that $\dfrac{b_{n_j}}{b_n}+\mathbb{Z}\in\mathbb{T}_m$.

This shows that for all $j\geq j_m$ we have $b_{n_j}\in V_{\mathbf{b},m}$ and hence $b_{n_j} \xrightarrow{\tau_{\mathbf{b}}} 0$.

QED

Notation 4.3.20 *Fix $\ell \in \mathbb{N}$ and a D-sequence $\boldsymbol{b}$. Let $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ have the representation $x = \sum_{n\geq 1} \frac{\beta_n}{b_n}$ as in Proposition 2.2.4. Then*

$$b_j x + \mathbb{Z} = \sum_{n\geq j+1} b_j \frac{\beta_n}{b_n} + \mathbb{Z} = \underbrace{b_j \frac{\beta_{j+1}}{b_{j+1}} + \cdots + b_j \frac{\beta_{j+\ell+1}}{b_{j+\ell+1}}}_{:= e_j^{[\ell]}} + \underbrace{b_j \sum_{n\geq j+\ell+2} \frac{\beta_n}{b_n}}_{:= \varepsilon_j^{[\ell]}} + \mathbb{Z}.$$

Proposition 4.3.21 *With the above introduced notations the following assertions hold:*

(1) $\left|e_j^{[\ell]}\right| \leq 1 - \frac{1}{2^{\ell+1}}$.

(2) If $\beta_{j+1} \neq 0$, *then* $\left|e_j^{[\ell]}\right| \geq \frac{b_j}{2^\ell b_{j+1}}$.

(3) $\left|\varepsilon_j^{[\ell]}\right| \leq \frac{b_j}{b_{j+\ell+1}}$.

Proof:

(1) $$\left|e_j^{[\ell]}\right| \leq b_j \sum_{i=1}^{\ell+1} \frac{\left|\beta_{j+i}\right|}{b_{j+i}} \overset{2.2.4}{\leq} b_j \sum_{i=1}^{\ell+1} \frac{1}{2b_{j+i-1}} = \frac{1}{2}\left(\frac{b_j}{b_j} + \ldots + \frac{b_j}{b_{j+\ell}}\right) \leq$$
$$\leq \frac{1}{2}\left(1 + \ldots + \frac{1}{2^\ell}\right) = \frac{1}{2}\left(2 - \frac{1}{2^\ell}\right) = 1 - \frac{1}{2^{\ell+1}}.$$

(2) Let $\beta_{j+1} \neq 0$.

Then we have $$\left|e_j^{[\ell]}\right| \geq b_j \left(\frac{\left|\beta_{j+1}\right|}{b_{j+1}} - \sum_{i=2}^{\ell+1} \frac{\left|\beta_{j+i}\right|}{b_{j+i}}\right) \overset{2.2.4}{\geq} b_j \left(\frac{1}{b_{j+1}} - \sum_{i=2}^{\ell+1} \frac{1}{2b_{j+i-1}}\right) =$$
$$= b_j \left(\frac{1}{b_{j+1}} - \frac{1}{b_{j+1}} \sum_{i=2}^{\ell+1} \frac{b_{j+1}}{2b_{j+i-1}}\right) = \frac{b_j}{b_{j+1}}\left(1 - \frac{b_{j+1}}{2b_{j+1}} - \ldots - \frac{b_{j+1}}{2b_{j+\ell}}\right)$$
$$\geq \frac{b_j}{b_{j+1}}\left(1 - \frac{1}{2} - \ldots - \frac{1}{2^\ell}\right) = \frac{b_j}{2^\ell b_{j+1}}.$$

(3) $\left|\varepsilon_j^{[\ell]}\right| \leq b_j \sum_{i\geq j+\ell+2} \frac{|\beta_i|}{b_i} \overset{2.2.4}{\leq} b_j \sum_{i\geq j+\ell+2} \frac{1}{2b_{i-1}} = \frac{b_j}{2} \sum_{i\geq j+\ell+1} \frac{1}{b_i} \leq \frac{b_j}{2} \cdot \frac{2}{b_{j+\ell+1}} = \frac{b_j}{b_{j+\ell+1}}$.

QED

The following proposition will permit us to relate the elements, χ, of $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$ with the **b**-coordinates of the representative of $\chi(1)$ in $\left(-\frac{1}{2}, \frac{1}{2}\right]$.

Proposition 4.3.22 *Let* $\mathbf{b} \in \mathcal{D}_\infty^\ell$*, for some natural number* ℓ*, let* $\chi \in (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$ *and* $\chi(1) = x + \mathbb{Z}$*, with* $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$*. Write* $x = \sum_{n\in\mathbb{N}} \frac{\beta_n}{b_n}$ *as in Proposition 2.2.4 and let* $A := \{j \in \mathbb{N} : \beta_j \neq 0\}$*. Then* $\sup\left\{q_j : j \in A\right\} < \infty$.

Proof:

Suppose that $\sup\left\{q_j : j \in A\right\} = \infty$. This immediately implies that A is infinite. Hence it is possible to choose a strictly increasing sequence (j_n) in A such that $q_{j_n} \overset{n\to\infty}{\longrightarrow} \infty$. Observe already here, that $j_n \in A$ implies $\beta_{j_n} \neq 0$ and hence, by Proposition 4.3.21 (2), we have that $\left|e_{j_n-1}^{[\ell]}\right| \geq \frac{b_{j_n-1}}{2^\ell b_{j_n}}$.

<u>Claim:</u> $e_{j_n-1}^{[\ell]} \overset{n\to\infty}{\longrightarrow} 0$.

Proof of the claim: Since, by Proposition 4.3.21 (1), we know that $\left|e_j^{[\ell]}\right| \leq 1 - \frac{1}{2^\ell}$, it suffices to show that $e_{j_n-1}^{[\ell]} + \mathbb{Z} \longrightarrow 0 + \mathbb{Z}$. By Lemma 4.3.19, $b_{j_n-1} \overset{n\to\infty}{\longrightarrow} 0$ in $\tau_{\mathbf{b}}$. The continuity of χ implies that $\chi(b_{j_n-1}) = b_{j_n-1}x + \mathbb{Z} \longrightarrow 0 + \mathbb{Z}$. By Proposition 4.3.21 (3), we have $\left|\varepsilon_j^{[\ell]}\right| \leq \frac{b_j}{b_{j+\ell+1}}$; since $\mathbf{b}$ belongs to $\mathcal{D}_\infty^\ell$, the sequence of ratios

$$\left(\frac{b_j}{b_{j+\ell+1}}\right)$$

converges to 0. Consequently,

$$\left|\varepsilon_j^{[\ell]}\right| \to 0$$

follows. This implies that $e_{j_n-1}^{[\ell]} + \mathbb{Z} = \chi(b_{j_n-1}) - \varepsilon_{j_n-1}^{[\ell]} + \mathbb{Z} \to 0 + \mathbb{Z}$ and consequently, the sequence $e_{j_n-1}^{[\ell]} \to 0$ in $\mathbb{R}$.

Since $\chi \in (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$, there exists $m \in \mathbb{N}$ such that $\chi \in V_{\mathbf{b},m}^\triangleright$. In order to obtain a contradiction, we want to find $k \in V_{\mathbf{b},m}$ such that $\chi(k) = xk + \mathbb{Z} \notin \mathbb{T}_+$.

Let $N := 2^\ell \cdot 4 \cdot m$. By the claim and since $\mathbf{b} \in \mathcal{D}_\infty^\ell$, there exists n_0 such that:

(a) $e_{j_n-1}^{[\ell]} < \dfrac{1}{10N}$ for all $n \geq n_0$.

(b) $\dfrac{b_{j+\ell}}{b_j} > 5 \cdot 2^\ell$ for all $j \geq j_{n_0}$.

We fix $n \geq n_0$ and define $k_{j_n-1} := \left\lfloor \dfrac{1}{2N\left|e_{j_n-1}^{[\ell]}\right|} \right\rfloor \operatorname{sign}(e_{j_n-1}^{[\ell]})$. We have the following inequalities

$$k_{j_n-1} e_{j_n-1}^{[\ell]} = \left\lfloor \frac{1}{2N \mid e_{j_n-1}^{[\ell]} \mid} \right\rfloor \left|e_{j_n-1}^{[\ell]}\right| \leq \frac{1}{2N}$$

and

$$k_{j_n-1} e_{j_n-1}^{[\ell]} \geq \left(\frac{1}{2N\left|e_{j_n-1}^{[\ell]}\right|} - 1 \right) \left|e_{j_n-1}^{[\ell]}\right| = \frac{1}{2N} - \left|e_{j_n-1}^{[\ell]}\right| \overset{(a)}{\geq} \frac{2}{5N}.$$

Further, by Proposition 4.3.21 (3) and (2),

$$\left|k_{j_n-1} \varepsilon_{j_n-1}^{[\ell]}\right| \leq \frac{1}{2N\left|e_{j_n-1}^{[\ell]}\right|} \frac{b_{j_n-1}}{b_{j_n+\ell}} \leq \frac{1}{2N} \frac{b_{j_n-1}}{b_{j_n+\ell}} \frac{b_{j_n} \cdot 2^\ell}{b_{j_n-1}} \overset{(b)}{<} \frac{1}{10N}.$$

Combining these estimates, we get:

$$\frac{3}{10N} = \frac{2}{5N} - \frac{1}{10N} < k_{j_n} \cdot (e_{j_n-1}^{[\ell]} + \varepsilon_{j_n-1}^{[\ell]}) < \frac{1}{2N} + \frac{1}{10N} = \frac{3}{5N}$$

for all $n \geq n_0$. We choose a subset J of $\{j_n - 1 : \ n \geq n_0\}$ containing exactly $2^\ell \cdot 5 \cdot m$ elements and define

$$k := \sum_{j \in J} k_j b_j.$$

Since

$$\left|\frac{k_j b_j}{b_{j+1}}\right| \leq \frac{1}{2N\left|e_j^{[\ell]}\right|} \frac{b_j}{b_{j+1}} \leq \frac{1}{2N} \frac{2^\ell b_{j+1}}{b_j} \frac{b_j}{b_{j+1}} = \frac{2^\ell}{2N} = \frac{1}{8m},$$

which is equivalent to

$$\left|k_j\right| \leq \frac{q_{j+1}}{8m},$$

so we obtain that $k \in V_{\mathbf{b},m}$ (by Proposition 4.3.12) and

$$\chi(k) = \sum_{j \in J} k_j b_j x + \mathbb{Z} = \sum_{j \in J} k_j \left(e_j^{[\ell]} + \varepsilon_j^{[\ell]}\right) + \mathbb{Z}.$$

Finally, it is clear that

$$\frac{1}{4} < \frac{3}{8} = \frac{3}{10}\frac{2^{\ell}\cdot 5\cdot m}{2^{\ell}\cdot 4\cdot m} = \frac{3\,|J|}{10N} < \sum_{j\in J} k_j(e_j^{[\ell]} + \varepsilon_j^{[\ell]}) < \frac{3\,|J|}{5N} = \frac{3}{5}\frac{2^{\ell}\cdot 5\cdot m}{2^{\ell}\cdot 4\cdot m} = \frac{3}{4}.$$

This implies that $\chi(k) = \sum_{j\in J} k_j x + \mathbb{Z} \notin \mathbb{T}_+$ and so we obtain the desired contradiction.

QED

For the particular case $\ell = 1$ we obtain the following theorem (already proved in [ABM12, Theorem 4.4]).

Theorem 4.3.23 *If* $\mathbf{b} \in \mathcal{D}_\infty$ *then* $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

Proof:

Let $\mathbf{b} \in \mathcal{D}_\infty$. By Proposition 4.3.17, it is clear that $\mathbb{Z}(\mathbf{b}^\infty) \leq (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$. Suppose that there exists a continuous character $\chi \in \mathbb{T} \setminus \mathbb{Z}(\mathbf{b}^\infty)$. Then the **b**-coordinates of $\chi(1)$ satisfy that $\beta_n \neq 0$ for infinitely many n. Take the set A as in the proof of Proposition 4.3.22. Then, $\sup\left\{q_j : j \in A\right\}$ is unbounded, since A infinite and $\mathbf{b} \in \mathcal{D}_\infty$. Hence, by Proposition 4.3.22 we obtain that $\chi \notin (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$.

QED

As a corollary of Theorem 4.3.23 we can prove that linear non-discrete Hausdorff topologies on $\mathbb{Z}$ are not Mackey.

Theorem 4.3.24 *Let* λ *be a non-discrete Hausdorff linear topology on* $\mathbb{Z}$*, then* λ *is not LQC-Mackey. In particular, the p-adic topologies are not LQC-Mackey.*

Proof:

Since λ is not discrete, $\lambda = \lambda_{\mathbf{b}}$ for some $\mathbf{b} \in \mathcal{D}$ (Proposition 4.2.6). According to Proposition 4.2.8, there exists $\mathbf{c} \in \mathcal{D}_\infty$ such that $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}}$. By Theorem 4.3.23, we have $(\mathbb{Z}, \tau_{\mathbf{c}})^\wedge = (\mathbb{Z}, \lambda_{\mathbf{c}})^\wedge = (\mathbb{Z}, \lambda_{\mathbf{b}})^\wedge$. Since $\lambda_{\mathbf{b}} = \lambda_{\mathbf{c}} \overset{4.3.17}{<} \tau_{\mathbf{c}}$, the linear topology $\lambda_{\mathbf{b}}$ is not a *LQC*-Mackey topology.

QED

On one hand it was already known that precompact topologies on a countable **bounded** group were Mackey and on the other hand it was expected that a metrizable locally quasi-convex should be Mackey. As a matter of fact this is what happens in the realm of locally convex spaces. A family of examples of metrizable locally quasi-convex groups which are not Mackey was obtained simultaneously in [DMPT14].

Combining Lemma 4.2.11 (2) and Proposition 4.3.24, we get

Remark 4.3.25 The linear topologies on $\mathbb{Z}$ constitute a family of metrizable precompact (in particular locally quasi-convex) non-Mackey group topologies of cardinality 2^{ω}.

Next we finding the relationships between the coefficients of Proposition 2.2.4 of a D-sequence and the ones of its subsequences.

Notation 4.3.26 *Let* $\mathbf{b} = (b_n)$ *be a D-sequence and let* $\mathbf{c} = (c_n)$ *be a subsequence of* $\mathbf{b}$ *satisfying* $c_0 = 1$. *For* $x \in \mathbb{R}$ *let*

$$x = \sum_{j\geq 0} \frac{\beta_j}{b_j} = \sum_{j\geq 0} \frac{\gamma_j}{c_j}$$

be the unique representations of Proposition 2.2.4. Furthermore, for any $j \in \mathbb{N}_0$ *there exists a unique* n_j *such that* $b_{n_j} = c_j$.

Proposition 4.3.27 *Let* $\mathbf{b}$, $\mathbf{c}$, (β_j), (γ_j) *as in Notation 4.3.26. For our purpose set* $n_{-1} = -1$. *Then for all* $j \geq 0$ *we get* $\gamma_j = c_j \sum_{k=n_{j-1}+1}^{n_j} \frac{\beta_k}{b_k}$.

Proof: We proceed inductively:
Let $j = 0$: It is clear by Proposition 2.2.4 that $\beta_0 = \gamma_0$.
Assume now that the assertion holds for $\gamma_0, \dots, \gamma_{j-1}$.

By Proposition 2.2.4, we have that

$$-\frac{1}{2b_{n_j}} < x - \sum_{k=0}^{n_j} \frac{\beta_k}{b_k} \leq \frac{1}{2b_{n_j}}$$

$$-\frac{1}{2c_j} < x - \sum_{k=0}^{j} \frac{\gamma_k}{c_k} \leq \frac{1}{2c_j}.$$

The first two inequalities are equivalent to:

$$-\frac{1}{2} < b_{n_j}x - b_{n_j} \sum_{k=0}^{n_j} \frac{\beta_k}{b_k} \leq \frac{1}{2} \tag{4.1}$$

Analogously, for the second pair, we get:

$$-\frac{1}{2} < c_j x - c_j \sum_{k=0}^{j} \frac{\gamma_k}{b_k} \leq \frac{1}{2} \tag{4.2}$$

From equations (4.1) and (4.2) we obtain:

$$-1 < b_{n_j} x - b_{n_j} \sum_{k=0}^{n_j} \frac{\beta_k}{b_k} - c_j x + c_j \sum_{k=0}^{j} \frac{\gamma_k}{c_k} < 1$$

Since $c_j = b_{n_j}$, we have

$$-1 < c_j \sum_{k=0}^{j} \frac{\gamma_k}{c_k} - b_{n_j} \sum_{k=0}^{n_j} \frac{\beta_k}{b_k} < 1$$

Since $\frac{c_j}{c_k} \in \mathbb{Z}$ if $j \geq k$ and $\frac{b_{n_j}}{b_k} \in \mathbb{Z}$ if $n_j \geq k$, we have that $c_j \sum_{k=0}^{j} \frac{\gamma_k}{c_k} - b_{n_j} \sum_{k=0}^{n_j} \frac{\beta_k}{b_k} = 0$.

Or, equivalently, $\sum_{k=0}^{j} \frac{\gamma_k}{c_k} = \sum_{k=0}^{n_j} \frac{\beta_k}{b_k}$. Since $\gamma_m = c_m \sum_{k=n_{m-1}+1}^{n_m} \frac{\beta_k}{b_k}$ for $m \leq j-1$, we get from the previous equality that $\frac{\gamma_j}{c_j} = \sum_{k=n_{j-1}+1}^{n_j} \frac{\beta_k}{b_k}$.

QED

Lemma 4.3.28 *Let* **b** *be a D-sequence and* $k_0 \leq k_1$ *natural numbers. If* $\sum_{k=k_0}^{k_1} \frac{\beta_k}{b_k} = 0$*, where* $|\beta_k| \leq \frac{b_k}{2b_{k-1}}$*, then* $\beta_{k_0} = \beta_{k_0+1} = \cdots = \beta_{k_1} = 0$.

Proof:

If $k_0 = k_1$, the result is trivial. If $k_1 > k_0$, we have that $0 = b_{k_1} \sum_{k=k_0}^{k_1} \frac{\beta_k}{b_k} = \beta_{k_1} + b_{k_1} \sum_{k=k_0}^{k_1-1} \frac{\beta_k}{b_k}$. Since $q_{k_1} \mid b_{k_1} \sum_{k=k_0}^{k_1-1} \frac{\beta_k}{b_k}$, it is clear that $q_{k_1} \mid \beta_{k_1}$. As $|\beta_k| \leq \frac{q_k}{2}$, this yields $\beta_{k_1} = 0$. Hence $\sum_{k=k_0}^{k_1-1} \frac{\beta_k}{b_k} = 0$. Iterating, we obtain that $\beta_{k_0} = \beta_{k_0+1} = \cdots = \beta_{k_1} = 0$.

QED

Corollary 4.3.29 *Let* $\mathbf{b}$, $\mathbf{c}$, (β_n), (γ_n), (n_j) *as in Notation 4.3.26. Then* $\gamma_j = 0$ *if and only if* $\beta_{n_{j-1}+1} = \cdots = \beta_{n_j} = 0$.

Proof:

According to Proposition 4.3.27, we have $\gamma_j = c_j \sum_{k=n_{j-1}+1}^{n_j} \frac{\beta_k}{b_k}$. So it is sufficient to apply Lemma 4.3.28.

QED

Notation 4.3.30 *Let* $\mathbf{b}$ *be a D-sequence. Let* $\mathbf{b} = \mathbf{b}^{(0)} \sqsupset \mathbf{b}^{(1)} \sqsupset \cdots \sqsupset \mathbf{b}^{(\ell-1)}$*, where* $\mathbf{b}^{(j)} = \left(b_n^{(j)}\right)_n$*. For each* $1 \leq k \leq \ell - 1$*, and each* $j \in \mathbb{N}$*, there exists a unique* $n_j^{(k)} \in \mathbb{N}$ *such that* $b_j^{(k)} = b_{n_j^{(k)}}$ *if* $\mathbf{b}^{(j)} = \left(b_n^{(j)}\right)_n$*. For a fixed* $y \in \mathbb{R}$*, we can write* $y = \sum_{j\geq 0} \frac{\beta_j}{b_j} = \sum_{j\geq 0} \frac{\beta_j^{(1)}}{b_j^{(1)}} = \cdots = \sum_{j\geq 0} \frac{\beta_j^{(\ell)}}{b_j^{(\ell)}}$ *as in Proposition 2.2.4. Write* $q_j^{(k)} = \frac{b_j^{(k)}}{b_{j-1}^{(k)}}$.

For the proof of Theorem 4.3.31 we will follow these steps:

- We assume that there is $\chi \in (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge \setminus \mathbb{Z}(\mathbf{b}^\infty)$ and set $\chi(1) = x + \mathbb{Z}$ with $\left(-\frac{1}{2}, \frac{1}{2}\right] \ni x = \sum_{j=0}^{\infty} \frac{\beta_j}{b_j}$.
- First, a sequence $(N_m)_{m\in\mathbb{N}_0}$ of natural numbers which satisfies that $q_{N_m} \to \infty$ and $N_{m+1} - N_m \leq \ell$ is constructed.
- We construct subsequences $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(\ell-1)}$ of $\mathbf{b} = \mathbf{b}^{(0)}$ satisfying $\tau_{\mathbf{b}^{(i)}} = \tau_{\mathbf{b}^{(i+1)}}$ for $0 \leq i \leq \ell - 1$. **Important:** The elements b_{N_m} will never be eliminated when passing to a subsequence, and we will denote them by the term "barrier-points".
- We construct mappings d_k $(0 \leq k \leq \ell - 1)$ which measure how far away an element $j \in \mathbb{N}$ with $\beta_j^{(k)} \neq 0$ (in the representation of x w.r.t. $\mathbf{b}^{(k)}$) is from the set $\{N_m : m \in \mathbb{N}\}$.
- We obtain that the sequence $\mathbf{b}^{(\ell-1)}$ satisfies that every $b_j^{(\ell-1)}$ with $\beta_j^{(\ell-1)} \neq 0$ belongs to $\{b_{N_m} : m \in \mathbb{N}_0\}$.
- Assuming that the character does not belong to $\mathbb{Z}(\mathbf{b}^\infty)$ we obtain a contradiction with Proposition 4.3.22.

Theorem 4.3.31 *If* $\mathbf{b} \in \mathcal{D}_\infty^\ell$*, then* $(\mathbb{Z}, \tau_\mathbf{b})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

Proof:

It is clear that $\mathbb{Z}(\mathbf{b}^\infty) \leq (\mathbb{Z}, \tau_\mathbf{b})^\wedge$. In order to prove the opposite inclusion, we assume there is $\chi \in (\mathbb{Z}, \tau_\mathbf{b})^\wedge \setminus \mathbb{Z}(\mathbf{b}^\infty)$. Let $\chi(1) = x + \mathbb{Z}$ with $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. This means $x \notin \mathbb{Z}(\mathbf{b}^\infty)$.

We construct a strictly increasing sequence of non-negative integers $(N_m)_{m\geq 0}$ satisfying that

(a) $N_0 = 0$,

(b) $N_{m+1} - N_m \leq \ell$, and

(c) $q_{N_m} \to \infty$

We put $N_0 = 0$ and suppose that $N_0, \ldots, N_m$ have been chosen with the above properties. Then choose $N_{m+1} \in \{N_m + 1, \ldots, N_m + \ell\}$ such that

$$q_{N_{m+1}} = \max\left\{q_{N_m+j} : 1 \leq j \leq \ell\right\}.$$

Since

$$q_{N_{m+1}} \geq \sqrt[\ell]{q_{N_m+\ell} \cdot q_{N_m+\ell-1} \cdot \ldots \cdot q_{N_m+1}} = \sqrt[\ell]{\frac{b_{N_m+\ell}}{b_{N_m}}}$$

and the last term of this inequality tends to ∞, the first one also does. So condition (c) is fulfilled. We write $\mathbb{B} := \{N_m : m \in \mathbb{N}_0\}$.

Now we will choose subsequences $\mathbf{b}^{(\ell)} \sqsubseteq \mathbf{b}^{(\ell-1)} \sqsubseteq \cdots \sqsubseteq \mathbf{b}^{(1)} \sqsubseteq \mathbf{b}^{(0)} = \mathbf{b}$ such that $\tau_{\mathbf{b}^{(i)}} = \tau_{\mathbf{b}^{(i-1)}}$ for all $1 \leq i \leq \ell$. Suppose that $\mathbf{b}^{(0)} = \mathbf{b}, \ldots, \mathbf{b}^{(k)}$ are already constructed satisfying $\tau_\mathbf{b} = \tau_{\mathbf{b}^{(1)}} = \ldots = \tau_{\mathbf{b}^{(k)}}$. We define

$$A_k = \{j \in \mathbb{N} : \beta_j^{(k)} \neq 0\} \text{ and } A_k' = A_k \setminus \{j : n_j^{(k)} \in \mathbb{B}\}.$$

We define $\mathbf{b}^{(k+1)}$ as the sequence obtained after eliminating the elements $b_j^{(k)}$ for $j \in A_k'$ from $\mathbf{b}^{(k)}$. Since the elements corresponding the barrier, i.e. b_{N_m} are never eliminated, condition (b) of the sequence (N_m) implies that no more than $\ell - 1$ consecutive elements of $\mathbf{b}^{(k)}$ are eliminated.

Since $\chi \in (\mathbb{Z}, \tau_\mathbf{b})^\wedge = (\mathbb{Z}, \tau_{\mathbf{b}^{(k)}})^\wedge$, we have that

$$\sup\left\{q_j^{(k)} : j \in A_k\right\} =: L_k < \infty,$$

by Proposition 4.3.22. Hence, the quotients $\dfrac{b_{j'}^{(k)}}{b_j^{(k)}}$, where $j \in A'_k$ and $b_{j'}^{(k)}$ is the minimum element of $\mathbf{b}^{(k)}$ which is greater or equal than $b_j^{(k)}$ and $j' \notin A'_k$, are bounded by $L_k^{\ell-1}$. Hence, by Proposition 4.3.8, we have that $\tau_{\mathbf{b}^{(k)}} = \tau_{\mathbf{b}^{(k+1)}}$.

We want to show $A_{\ell-1} \subseteq \mathbb{B}$.
Therefore, we consider for $0 \leq k \leq \ell - 1$ the function

$$d_k : A_k \longrightarrow \mathbb{N}_0,\ j \longmapsto \min\{i - n_j^{(k)} :\ i \in \mathbb{B}\ \wedge\ i \geq n_j^{(k)}\}$$

which measures the distance from an element of the sequence $\mathbf{b}^{(k)}$ to the barrier-points.

Recall, that $b_{n_j^{(k)}} = b_j^{(k)}$. Of course, $d_k(j) = 0$ if and only if $n_j^{(k)} \in \mathbb{B}$. So the above assertion is equivalent to $d_{\ell-1}(A_{\ell-1}) = \{0\}$. This will follow once shown that

$$\forall 0 \leq k \leq \ell - 1 \quad d_k(A_k) \subseteq \{0, 1, \ldots, \ell - k - 1\} \qquad (*).$$

For $k = 0$ and $0 \neq j \in A_0$ there is a unique $m \in \mathbb{N}_0$ such that $N_m < j \leq N_{m+1}$. Then

$$d_0(j) = \min\{i - j :\ i \in \mathbb{B}\ \wedge\ i \geq j\} = N_{m+1} - j < N_{m+1} - N_m \leq \ell,$$

which implies $d_0(j) \leq \ell - 1$. If j=0, then $d_0(0) = 0$, since $N_0 = 0$

Assume now that the assertion $(*)$ has been shown for $0 \leq k \leq k_0$ and suppose that $k_0 + 1 \leq \ell - 1$. Suppose that A'_{k_0+1} is not empty; fix $j \in A'_{k_0+1}$. There are unique m_j and $m_{j-1} \in \mathbb{N}_0$ such that $b_j^{(k_0+1)} = b_{m_j}^{(k_0)}$ and $b_{j-1}^{(k_0+1)} = b_{m_{j-1}}^{(k_0)}$. (We do not assume $m_{j-1} + 1 \neq m_j$ (see below) neither $n_{j-1}^{(k_0+1)} \notin \mathbb{B}$ nor $\beta_{j-1}^{(k_0+1)} \neq 0$.)

According to Proposition 4.3.27,

$$\frac{\beta_j^{(k_0+1)}}{b_j^{(k_0+1)}} = \sum_{m=m_{j-1}+1}^{m_j} \frac{\beta_m^{(k_0)}}{b_m^{(k_0)}}.$$

Since $j \in A'_{k_0+1}$, we have $\beta_j^{(k_0+1)} \neq 0$ and $n_j^{(k_0+1)} \notin \mathbb{B}$. According to Corollary 4.3.29 there exists $m \in \{m_{j-1} + 1, \ldots, m_j\}$ such that $\beta_m^{(k_0)} \neq 0$. If $m = m_j$ then $b_m^{(k_0)} = b_{m_j}^{(k_0)} = b_j^{(k_0+1)}$ would have been eliminated by passing to the subsequence $\mathbf{b}^{(k_0+1)}$, a contradiction. Further, there exists $\mu \in \mathbb{N}_0$ such that $b_{N_\mu} < b_j^{(k_0+1)} < b_{N_{\mu+1}}$. It follows

$$b_{N_\mu} \leq b_{m_{j-1}}^{(k_0+1)} < b_{m_j}^{(k_0+1)} \leq b_{N_{\mu+1}},$$

since all elements between $b^{(k_0)}_{m_{j-1}}$ and $b^{(k_0)}_{m_j}$ have been eliminated. This implies

$$b_{N_\mu} \leq b^{(k_0+1)}_{j-1} = b^{(k_0)}_{m_j-1} < b^{(k_0)}_{m} < b^{(k_0)}_{m_j} = b^{(k_0+1)}_{j} < b_{N_{\mu+1}}.$$

Thus

$$d_{k_0+1}(j) = N_{\mu+1} - n^{(k_0+1)}_j < N_{\mu+1} - n^{(k_0)}_m = d_{k_0}(m) \leq \ell - k_0 - 1$$

by induction hypothesis. Hence we have $d_{k_0+1}(j) \leq \ell - (k_0+1) - 1$, which was to show.

Now we know that $A_{\ell-1} \subseteq \mathbb{B}$ and since $\chi \notin \mathbb{Z}(\mathbf{b})$, the set $A_{\ell-1}$ is infinite. Fix a sequence $(j_\nu)_{\nu\in\mathbb{N}_0}$ in $A_{\ell-1}$. We have

$$q^{(\ell-1)}_{j_\nu} = \frac{b^{(\ell-1)}_{j_\nu}}{b^{(\ell-1)}_{j_\nu-1}} = \frac{b_{n^{(\ell-1)}_{j_\nu}}}{b_{n^{(\ell-1)}_{j_\nu-1}}} \geq \frac{b_{n^{(\ell-1)}_{j_\nu}}}{b_{n^{(\ell-1)}_{j_\nu}-1}} = q_{n^{(\ell-1)}_{j_\nu}}.$$

Since $\left(q_{n^{(\ell-1)}_{j_\nu}}\right)_{\nu\in\mathbb{N}_0}$ is a subsequence of $(q_{N_m})_{m\in\mathbb{N}_0}$ which tends to infinity by condition (c), the sequence $\left(q^{(\ell-1)}_{j_\nu}\right)_{\nu\in\mathbb{N}_0}$ cannot be bounded. This contradicts Proposition 4.3.22, so we get $\chi \notin (\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$.

QED

Remark 4.3.32 Theorem 4.3.23 can also be obtained from Theorem 4.3.31.

4.3.3 Duality if $\mathbf{b} \notin \mathcal{D}^\ell_\infty$.

In subsection 4.3.2, some dual groups have been studied and a necessary condition is given in order to obtain compatible topologies (Theorem 4.3.31). It would be nice to check that the condition of that theorem is also sufficient. So far, we have not been able to prove the converse of Theorem 4.3.31. In this subsection, we will prove that some D-sequences give rise to topologies of uniform convergence which are not discrete, but are neither compatible with the linear topology. To that end, some notation is given and a new condition (the block condition) will appear in a natural way. It is proved that for a D-sequence $\mathbf{b}$ satisfying the block condition, the topology $\tau_{\mathbf{b}}$ is not compatible with $\lambda_{\mathbf{b}}$. An example of D-sequence not belonging to $\mathcal{D}^\ell_\infty$ and not satisfying either the block condition is given in Proposition 4.3.37.

Notation 4.3.33 *For a natural number $c \geq 2$ and a D-sequence* $\mathbf{b}$ *we define*

$$A^{(c)} := \{j \in \mathbb{N}_0 : \ q_{j+1} \leq c\}.$$

If $\mathbb{N}_0 \setminus A^{(c)}$ is finite then $\left(q_{j+1}\right)$ is a bounded sequence and hence $\tau_{\mathbf{b}}$ is discrete (by Theorem 4.3.2). Assuming that $\mathbb{N}_0 \setminus A^{(c)}$, is infinite, we put $\mathbb{N}_0 \setminus A^{(c)} = \{c_i : \ i \in \mathbb{N}_0\}$ where $(c_i)_{i \in \mathbb{N}_0}$ is a strictly increasing sequence. We have

$$A^{(c)} = A_0^{(c)} \cup \bigcup_{i=0}^{\infty} \{c_i + 1, \ldots, c_{i+1} - 1\}$$

where

$$A_0^{(c)} = \begin{cases} \{0, \ldots, c_0 - 1\} & : \quad 0 \in A^{(c)} \\ \emptyset & : \quad 0 \notin A^{(c)} \end{cases}.$$

Further, we put $A_i^{(c)} := \{c_i + 1, \ldots, c_{i+1} - 1\}$.

Remark 4.3.34 Let $\mathbf{b}$ be a D-sequence and $\ell \in \mathbb{N}$. The following assertions are equivalent:

$$\begin{aligned} \mathbf{b} \notin \mathcal{D}_\infty^\ell \quad &\iff \quad \exists c \geq 2 : \ \frac{b_{n+\ell}}{b_n} \leq c \text{ for infinitely many } n \\ &\iff \quad \exists c \geq 2 : \ \left|A_n^{(c)}\right| \geq \ell \text{ for infinitely many } n \end{aligned}$$

and hence

$$\mathbf{b} \notin \textstyle\bigcup_{\ell \in \mathbb{N}} \mathcal{D}_\infty^\ell \quad \iff \quad \forall \ell \in \mathbb{N} \ \exists c \geq 2 : \ \left|A_n^{(c)}\right| \geq \ell \text{ for infinitely many } n.$$

In the special case that c does not depend on ℓ, we have the following notation and negative assertion:

Notation 4.3.35 *Let* $\mathbf{b}$ *be a D-sequence. With the above notation, suppose that*

$$\textit{there exists } c \geq 2 \textit{ such that } \sup \left|A_n^{(c)}\right| = \infty.$$

Then we say that $\mathbf{b}$ *satisfies the* ***block condition****.*

Theorem 4.3.36 *Let* $\mathbf{b}$ *be a D-sequence. We apply the notation introduced above. If* $\mathbf{b}$ *satisfies the block condition, then $\mathbb{Z}(\mathbf{b}^\infty)$ is strictly contained in $(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge$. In other words, there exist $\tau_{\mathbf{b}}$-continuous characters which are not $\lambda_{\mathbf{b}}$-continuous.*

Proof:

Let $c \geq 2$ such that $\sup \left|A_n^{(c)}\right| = \infty$. Fix m such that $\frac{3}{8m} < \frac{1}{c}$. For $i \in A^{(c)}$ one has $q_{i+1} \leq c < \frac{8m}{3}$. Suppose that $\mathbb{N}_0 \setminus A^{(c)}$ is infinite and $\sup |A_n^c| = \infty$. We shall find $\chi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{T})$, such that $\chi \in V_{\mathbf{b},m}^{\triangleright}$, but $\chi \notin \mathbb{Z}(\mathbf{b}^\infty)$, being thus a $\tau_{\mathbf{b}}$-continuous homomorphism, which is not $\lambda_{\mathbf{b}}$-continuous. By Lemma 4.3.14, we have

$$V_{\mathbf{b},m} \subseteq \left\{ \sum k_j b_j : \left|k_j\right| \leq \frac{3q_{j+1}}{8m} \right\}.$$

Let $k = \sum_{j=0}^{N} k_j b_j \in V_{\mathbf{b},m}$. By the choice of m, we have $k_j = 0$ for all $j \in A^{(c)}$. Indeed, $\dfrac{|k_j| b_j}{b_{j+1}} \geq \dfrac{|k_j|}{c} \geq |k_j| \cdot \dfrac{3}{8m}$ (equivalently $\left|k_j\right| \geq \left|k_j\right| \frac{3q_{j+1}}{8m}$) for all $j \in A^{(c)}$. For $i \in \mathbb{N}_0$,we define $N_i := \max A_{(i)}^{(c)}$ and $n_i := \min A_{(i)}^{(c)}$. We compute

$$\frac{1}{b_{N_i+1}} k + \mathbb{Z} = \frac{1}{b_{N_i+1}} \sum_{j<N_i} k_j b_j + \mathbb{Z}.$$

Now

$$\left| \frac{1}{b_{N_i+1}} \sum_{j<N_i} k_j b_j \right| \overset{k_j=0 \text{ for all } j\in A^{(c)}}{=} \left| \frac{1}{b_{N_i+1}} \sum_{j<n_i} k_j b_j \right| = \left| \frac{k_1 b_1}{b_{N_i+1}} + \cdots + \frac{k_{n_i-1} b_{n_i-1}}{b_{N_i+1}} \right| =$$

$$= \left| \frac{k_1 b_1}{b_{n_i}} + \cdots + \frac{k_{n_i-1} b_{n_i-1}}{b_{n_i}} \right| \frac{b_{n_i}}{b_{N_i+1}} = \left| \frac{k_1 b_1}{b_2} \frac{b_2}{b_{n_i}} + \cdots + \frac{k_{n_i-1} b_{n_i-1}}{b_{n_i}} \right| \frac{b_{n_i}}{b_{N_i+1}} \leq$$

$$\leq \frac{3}{8m} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n_i-2}} \right) \frac{b_{n_i}}{b_{N_i+1}} \leq \frac{3}{4m} 2^{-\left|A_i^{(c)}\right|+1}.$$

Choose $(A_i^{(c)})_{i\in\mathbb{N}}$ such that $A_i^{(c)} \neq A_j^{(c)}$ if $i \neq j$ and $\mid A_i^{(c)} \mid \geq 2i+1$. Put

$$x := \sum_{i\in\mathbb{N}} \frac{1}{b_{N_i+1}}.$$

Since we have written χ as an infinite sum, it holds $\chi \notin \langle \frac{1}{b_n} + \mathbb{Z} \rangle$. Now, for k as above, we have

$$\chi(k) = xk + \mathbb{Z} = \sum_{i\in\mathbb{N}} \frac{1}{b_{N_i+1}} k + \mathbb{Z}.$$

Then

$$d(xk, \mathbb{Z}) \leq \frac{3}{4m} \sum_{i\in\mathbb{N}} 2^{-\left|A_i^{(c)}\right|+1} \leq \frac{3}{4m} \left(\frac{1}{4} + \frac{1}{16} + \dots \right) = \frac{1}{4m}.$$

Hence, $xk + \mathbb{Z} \in \mathbb{T}_m$ for all $k \in V_{\mathbf{b},m}$.

QED

We construct now a D-sequence $\mathbf{b}$ such that $\mathbf{b} \notin \mathcal{D}^{\ell}_{\infty}$ for any ℓ but $\mathbf{b}$ does not satisfy the block condition, which we proves that $\bigcup_{\ell\in\mathbb{N}_0} \mathcal{D}^{\ell}_{\infty}$ and the set of D-sequences that satisfy the block condition do not define a partition of $\mathcal{D}$.

Proposition 4.3.37 *Let*

$$q = (q_n)_{n\in\mathbb{N}} = (\underbrace{2}, \underbrace{2,3}, \underbrace{2,3,5}, \underbrace{2,3,5,7,11}, 2, \dots, p_{n-1}, \underbrace{2,3,\dots,p_n}, \dots),$$

where p_n is the n-th prime number. Consider

$$b_n = \prod_{i=1}^{n} q_i \text{ for } n \in \mathbb{N}_0.$$

Then $\mathbf{b}$ *is a D-sequence which satisfies*

(1) *For all $\ell \in \mathbb{N}$, we have* $\mathbf{b} \notin \mathcal{D}^{\ell}_{\infty}$.

(2) *The sequence* $\mathbf{b}$ *does not satisfy the block condition.*

Proof:

First we note that the k-th 2 appears at the index $\binom{k+1}{2} + 1 = k(k+1)/2 + 1$.

(1) Fix $\ell \in \mathbb{N}$, define $M_n := \frac{(n+1)n}{2} + \ell + 1$ and $m_n := \frac{(n+1)n}{2} + 1$. Then, the quotient

$$\frac{b_{M_n}}{b_{m_n}} = 2 \cdot 3 \cdots p_{\ell} \text{ if } n \geq \ell + 1.$$

Hence $\frac{b_{n+\ell}}{b_n} \nrightarrow \infty$, which is equivalent to $\mathbf{b} \notin \mathcal{D}^{\ell}_{\infty}$.

(2) Consider the terms introduced in Notation 4.3.33. Fix $c \geq 2$. There exists a unique $n \in \mathbb{N}$ such that $p_n \leq c$ and $p_{n+1} > c$. For $i \geq 1$, we have

$$A_i^{(c)} = \left\{ \frac{(i+n+2)(i+n+1)}{2} + 1, \cdots, \frac{(i+n+2)(i+n+1)}{2} + n \right\}.$$

This means that $\left|A_i^{(c)}\right| = n$ if $i \geq 1$. Hence, $\sup\left\{\left|A_i^{(c)}\right|\right\} = \sup\left\{\left|A_0^{(c)}\right|, n\right\} < \infty$. Consequently, the sequence $\mathbf{b}$ does not satisfy the block condition.

QED

4.3.4 Supremum of topologies of uniform convergence on D-sequences.

In this subsection we prove that the supremum of a countable family of topologies of uniform convergence, each considered on a sequence $\mathbf{b}^{[n]}$, need not be the topology of uniform convergence on the union of all the sequences $\sqcup\mathbf{b}^{[n]}$. For that we consider the basic D-sequence $\mathbf{p} = (p^n)$, and we take subsequences $\mathbf{b}^{[n]} \sqsubseteq \mathbf{p}$. The case of the supremum of a finite family of topologies of uniform convergence related to D-sequences is studied in Section 4.4 (Proposition 4.4.2).

We define the following sequences of integers. The first sequence is formed by the sequence of all the square powers of a prime number p. The following steps will "cover" the gaps between the even powers and the odd ones. In each step we add a new power of p, however no term between an odd power and an even power will be permitted.

$\mathbf{b}^{[0]} = (1, p, p^4, p^9, p^{16}, p^{25}, p^{36}, p^{49}, p^{64}, p^{81}, p^{100}, p^{121}, \dots)$

$\mathbf{b}^{[1]} = (1, p, p^4, p^5, p^9, p^{16}, p^{17}, p^{25}, p^{36}, p^{37}, p^{49}, p^{64}, p^{65}, p^{81}, p^{100}, p^{101}, p^{121}, \dots)$

$\mathbf{b}^{[2]} = (1, p, p^4, p^5, p^6, p^9, p^{16}, p^{17}, p^{18}, p^{25}, p^{36}, p^{37}, p^{38}, p^{49}, p^{64}, p^{65}, p^{66}, p^{81}, p^{100}, p^{101}, p^{102}, p^{121}, \dots)$

$\mathbf{b}^{[3]} = (1, p, p^4, p^5, p^6, p^7, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{121}, \dots)$

$\mathbf{b}^{[4]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{121}, \dots)$

$\mathbf{b}^{[5]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{121}, \dots)$

$\mathbf{b}^{[6]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{81}, p^{100}, p^{101}, p^{102},$

$p^{103}, p^{104}, p^{105}, p^{106}, p^{121}, \ldots)$

$\mathbf{b}^{[7]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{23}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{43}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{71}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{106}, p^{107}, p^{121}, \ldots)$

$\mathbf{b}^{[8]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{23}, p^{24}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{43}, p^{44}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{71}, p^{72}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{106}, p^{107}, p^{108}, p^{121}, \ldots)$

$\mathbf{b}^{[9]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{23}, p^{24}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{43}, p^{44}, p^{45}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{71}, p^{72}, p^{73}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{106}, p^{107}, p^{108}, p^{109}, p^{121}, \ldots)$

$\mathbf{b}^{[10]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{23}, p^{24}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{43}, p^{44}, p^{45}, p^{46}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{71}, p^{72}, p^{73}, p^{74}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{106}, p^{107}, p^{108}, p^{109}, p^{110}, p^{121}, \ldots)$

$\ldots$

Finally we define the sequence

$$\mathbf{b}^{[\infty]} := \bigsqcup_{n\in\mathbb{N}} \mathbf{b}^{[n]}.$$

$\mathbf{b}^{[\infty]} = (1, p, p^4, p^5, p^6, p^7, p^8, p^9, p^{16}, p^{17}, p^{18}, p^{19}, p^{20}, p^{21}, p^{22}, p^{23}, p^{24}, p^{25}, p^{36}, p^{37}, p^{38}, p^{39}, p^{40}, p^{41}, p^{42}, p^{43}, p^{44}, p^{45}, p^{46}, p^{47}, p^{48}, p^{49}, p^{64}, p^{65}, p^{66}, p^{67}, p^{68}, p^{69}, p^{70}, p^{71}, p^{72}, p^{73}, p^{74}, p^{75}, p^{76}, p^{77}, p^{78}, p^{79}, p^{80}, p^{81}, p^{100}, p^{101}, p^{102}, p^{103}, p^{104}, p^{105}, p^{106}, p^{107}, p^{108}, p^{109}, p^{110}, p^{111}, p^{112}, p^{113}, p^{114}, p^{115}, p^{116}, p^{117}, p^{118}, p^{119}, p^{120}, p^{121}, \ldots)$

Furthermore, by the definition of $\tau_{\mathbf{b}^{[n]}}$ it is trivial that $\tau_{\mathbf{b}^{[n]}} \leq \tau_{\mathbf{b}^{[n+1]}}$ for all $n \in \mathbb{N}$. Since $\mathbf{b}^{[n]} \in \mathcal{D}_\infty^{n+1}$, we get the following:

Corollary 4.3.38 (of Theorem 4.3.31) *For every natural number $n \in \mathbb{N}_0$, we have $(\mathbb{Z}, \tau_{\mathbf{b}^{[n]}})^\wedge = \mathbb{Z}(\mathbf{p}^\infty)$.*

Further, we get the surprising result:

Theorem 4.3.39 *The following assertion holds:* $\sup_{n\in\mathbb{N}_0}\{\tau_{\mathbf{b}^{[n]}}\} \neq \tau_{\mathbf{b}^{[\infty]}}$.

Proof:

If both topologies were identical, their dual groups would coincide. Since $\tau_{\mathbf{b}^{[n]}} < \tau_{\mathbf{b}^{[n+1]}}$, Proposition 3.1.9 can be applied to obtain that the dual group $(\mathbb{Z}, \sup\{\tau_{\mathbf{b}^{[n]}}\})^\wedge$ is $\mathbb{Z}(\mathbf{p}^\infty)$. Since $\mathbf{b}^{[\infty]}$ satisfies the block condition, by Proposition 4.3.36, we know that $(\mathbb{Z}, \tau_{\mathbf{b}^{[\infty]}})^\wedge \gneqq \mathbb{Z}(\mathbf{p}^\infty)$. Hence, both topologies are different.

QED

4.4 The topology $\delta_{\mathbf{b}}$.

In this section we introduce a new topology $\delta_{\mathbf{b}}$ related to a basic D-sequence $\mathbf{b}$. It appeared in a natural way: once proved that the $\mathbf{b}$-adic topology $\lambda_{\mathbf{b}}$ on $\mathbb{Z}$ is not Mackey, we had the conjecture that a Mackey topology on the duality $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$ did not exist. For this reason, we constructed a family of locally quasi-convex compatible topologies, finer than $\lambda_{\mathbf{b}}$, whose supremum we suspected was not compatible. Had this been true, we would already have the non-existence of the Mackey topology for this concrete duality. However, we proved that it is again compatible.

We defined the topology of uniform convergence on the family of all the subsequences of $\mathbf{b}$ belonging to $\mathcal{D}_\infty^\ell$, which we have called $\delta_{\mathbf{b}}$. As we prove in the main theorem of this section, $\delta_{\mathbf{b}}$ is a compatible topology. Clearly, it is also locally quasi-convex since it is the supremum of locally quasi-convex topologies.

First, we prove a lemma concerning D-sequences.

Lemma 4.4.1 *Let* $\mathbf{c}$ *be a basic D-sequence ant let* $\mathbf{b}, \mathbf{b}' \sqsubseteq \mathbf{c}$ *be two D-sequences satisfying* $\mathbf{b} \in \mathcal{D}_\infty^n$ *and* $\mathbf{b}' \in \mathcal{D}_\infty^m$ *for some natural numbers* n, m. *Then* $\mathbf{b} \sqcup \mathbf{b}'$ *is a D-sequence that satisfies* $\mathbf{b} \sqcup \mathbf{b}' \in \mathcal{D}_\infty^{n+m}$.

Proof:

The first assertion is clear since $\mathbf{b} \sqcup \mathbf{b}' \sqsubseteq \mathbf{c}$.

Now we prove the second one. Let $N \in \mathbb{N}$. Since $\mathbf{b} \in \mathcal{D}^n_\infty$ and $\mathbf{b}' \in \mathcal{D}^m_\infty$ there exist $j_n, j_m \in \mathbb{N}$ such that

$$\frac{b_{j+n}}{b_j} \geq N \text{ and } \frac{b'_{k+m}}{b'_k} \geq N \text{ if } j \geq j_n \text{ and } k \geq j_m.$$

Put $B := \max\{b_{j_n}, b'_{j_m}\}$. Let j_0 be the index of B in $\mathbf{b} \sqcup \mathbf{b}'$. We must prove that

$$\frac{(\mathbf{b} \sqcup \mathbf{b}')_{j+m+n}}{(\mathbf{b} \sqcup \mathbf{b}')_j} \geq N$$

if $j \geq j_0$. We consider the set $C := \{(\mathbf{b} \sqcup \mathbf{b}')_j, \ldots, (\mathbf{b} \sqcup \mathbf{b}')_{j+m+n}\}$. Since $\mid C \mid = m + n + 1$, it must contain at least $n + 1$ elements of $\mathbf{b}$ or $m + 1$ elements of $\mathbf{b}'$.

If C contains $n + 1$ elements of $\mathbf{b}$, denote by b_{k_1} the first one and by $b_{k_{n+1}}$ the $(n + 1)$-th. Then

$$\frac{(\mathbf{b} \sqcup \mathbf{b}')_{j+m+n}}{(\mathbf{b} \sqcup \mathbf{b}')_j} \geq \frac{b_{k_{n+1}}}{b_{k_1}}.$$

Since $k_1 \geq j_0$, we get that $\frac{b_{k_{n+1}}}{b_{k_1}} \geq N$. Analogously, if C contains $m + 1$ elements of $\mathbf{b}'$, we denote by b'_{k_1} the first of these elements and by $b'_{k_{m+1}}$ the $(m + 1)$-th. Then

$$\frac{(\mathbf{b} \sqcup \mathbf{b}')_{j+m+n}}{(\mathbf{b} \sqcup \mathbf{b}')_j} \geq \frac{b'_{k_{m+1}}}{b'_{k_1}}.$$

Since $k_1 \geq j_0$, we get that $\frac{b'_{k_{m+1}}}{b'_{k_1}} \geq N$. Hence, $\mathbf{b} \sqcup \mathbf{b}' \in \mathcal{D}^{m+n}_\infty$.

QED

Proposition 4.4.2 *Let* $\mathbf{b}$ *and* $\mathbf{b}'$ *be subsequences of the D-sequences* $\mathbf{c}$ *starting with* 1. *Then* $\sup\{\tau_{\mathbf{b}}, \tau_{\mathbf{b}'}\} = \tau_{\mathbf{b} \sqcup \mathbf{b}'}$.

Proof:

The topology $\sup\{\tau_{\mathbf{b}}, \tau_{\mathbf{b}'}\}$ has a basis of neighborhood of 0 given by $V_{\mathbf{b},n} \cap V_{\mathbf{b}',n}$ for $n \in \mathbb{N}$. Then we consider:

$$V_{\mathbf{b} \sqcup \mathbf{b}',n} \subseteq V_{\mathbf{b},n} \cap V_{\mathbf{b}',n} \subseteq V_{\mathbf{b} \sqcup \mathbf{b}',n},$$

which clearly implies the result.

QED

Proposition 4.4.3 *Let* $\mathbf{b}$ *be a basic D-sequence. Let* Δ *be the set of all subsequences* $\mathbf{c}^\alpha \sqsubseteq \mathbf{b}$*, satisfying* $\mathbf{c}^\alpha \in \mathcal{D}_\infty^\ell$ *for some* ℓ*. Then the set of topologies* $\{\tau_{\mathbf{c}^\alpha} : \alpha \in \Delta\}$ *is a directed set.*

Proof: We must prove that for $\alpha, \beta \in \Delta$, there exists $\gamma \in \Delta$ such that $\tau_{\mathbf{c}^\gamma} \geq \tau_{\mathbf{c}^\alpha}, \tau_{\mathbf{c}^\beta}$. Since $\alpha, \beta \in \Delta$, there exist $n, m \in \mathbb{N}$ such that $\mathbf{c}^\alpha \in \mathcal{D}_\infty^n$ and $\mathbf{c}^\beta \in \mathcal{D}_\infty^m$. By Proposition 4.4.2, we have that $\sup\{\tau_{\mathbf{c}^\alpha}, \tau_{\mathbf{c}^\beta}\} = \tau_{\mathbf{c}^\alpha \sqcup \mathbf{c}^\beta}$. By Lemma 4.4.1, we know that $\mathbf{c}^\alpha \sqcup \mathbf{c}^\beta \in \mathcal{D}_\infty^{n+m}$. Hence $\mathbf{c}^\gamma := \mathbf{c}^\alpha \sqcup \mathbf{c}^\beta \in \Delta$ satisfies the desired conditions. Then $\{\tau_{\mathbf{c}^\alpha} : \alpha \in \Delta\}$ is a directed set.

QED

Definition 4.4.4 Let $\mathbf{b}$ be a basic D-sequence. Let Δ be the set of all subsequences, $\mathbf{c}^\alpha$, of $\mathbf{b}$, satisfying $\mathbf{c}^\alpha \in \mathcal{D}_\infty^\ell$ for some ℓ. We define

$$\delta_{\mathbf{b}} := \sup\{\tau_{\mathbf{c}^\alpha} : \mathbf{c}^\alpha \in \Delta\}.$$

Remark 4.4.5 The topology $\delta_{\mathbf{b}}$ is locally quasi-convex, since it is the supremum of a family of locally quasi-convex topologies.

We now compute the dual of $(\mathbb{Z}, \delta_{\mathbf{b}})$.

Theorem 4.4.6 *Let* $\mathbf{b}$ *be a basic D-sequence. Then* $(\mathbb{Z}, \delta_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

Proof: Let Δ be as in Definition 4.4.4. By Proposition 3.1.9, and since $\{\tau_{\mathbf{c}^\alpha} : \alpha \in \Delta\}$ is a directed set, we get that

$$(\mathbb{Z}, \delta_{\mathbf{b}})^\wedge = \bigcup_{\alpha \in \Delta} (\mathbb{Z}, \tau_{\mathbf{c}^\alpha})^\wedge.$$

By Theorem 4.3.31, we have that $(\mathbb{Z}, \tau_{\mathbf{c}^\alpha})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$ for all $\alpha \in \Delta$. Hence $(\mathbb{Z}, \delta_{\mathbf{b}})^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

QED

Now, we compare $\delta_{\mathbf{b}}$ and $\tau_{\mathbf{b}}$ for a basic D-sequence:

Proposition 4.4.7 *Let* $\mathbf{b}$ *be a basic D-sequence. Then*

(1) $\delta_{\mathbf{b}} \leq \tau_{\mathbf{b}}$

(2) $\delta_{\mathbf{b}} = \tau_{\mathbf{b}}$ *if and only if* $\mathbf{b} \in \mathcal{D}_{\infty}^{\ell}$ *for some* $\ell \in \mathbb{N}$.

Proof:

For item (1), it suffices to remark that $\tau_{\mathbf{c}^{\alpha}} < \tau_{\mathbf{b}}$ for any subsequence $\mathbf{c}^{\alpha}$ of $\mathbf{b}$. The equality is only obtained if $\mathbf{b} \in \mathcal{D}_{\infty}^{\ell}$ for some ℓ.

QED

4.4.1 $\delta_{\mathbf{b}}$ if b has bounded ratios.

If $\mathbf{b}$ has bounded ratios, we obtain a family of metrizable locally quasi-convex compatible topologies on $\mathbb{Z}$ by killing the null sequences in $\lambda_{\mathbf{b}}$ one by one. The supremum of this family of topologies is again locally quasi-convex. It is compatible with $\lambda_{\mathbf{b}}$, hence non-discrete, and it has no null sequences. As a further consequence, we prove that for a basic D-sequence with bounded ratios the only $\delta_{\mathbf{b}}$-convergent sequences are the eventually constant ones. In the next proposition we fix a $\lambda_{\mathbf{b}}$-null sequence, γ, and give a device to obtain a new locally quasi-convex compatible metrizable group topology on $\mathbb{Z}$, τ_{γ}, such that γ does not converge in τ_{γ}. Loosely speaking, we have eliminated the convergent sequence γ by passing from $\lambda_{\mathbf{b}}$ to τ_{γ}.

Theorem 4.4.8 *Let* $\mathbf{b}$ *be a basic D-sequence with bounded ratios and let* $(x_n) \subset \mathbb{Z}$ *be a non-quasiconstant sequence such that* $x_n \xrightarrow{\lambda_{\mathbf{b}}} 0$. *Then there exists a metrizable locally quasi-convex compatible group topology* $\tau (= \tau_{\mathbf{c}}$ *for some subsequence* $\mathbf{c}$ *of* $\mathbf{b})$ *on* $\mathbb{Z}$ *satisfying:*

(a) τ *is compatible with* $\lambda_{\mathbf{b}}$.

(b) $x_n \overset{\tau}{\nrightarrow} 0$.

(c) $\lambda_{\mathbf{b}} < \tau$.

Proof:

We define τ as the topology of uniform convergence on a subsequence $\mathbf{c}$ of $\mathbf{b}$, which we construct by means of the claim:

Claim:

Since $\mathbf{b}$ *is a D-sequence with bounded ratios, there exists* $L \in \mathbb{N}$ *such that* $q_{n+1} \leq L$ *for all* $n \in \mathbb{N}$.

We can construct inductively two sequences $(n_j), (m_j) \subset \mathbb{N}$ *such that*

$$n_{j+1} - n_j > j \text{ and } \frac{x_{m_j}}{b_{n_j+1}} + \mathbb{Z} \notin \mathbb{T}_L.$$

Proof of the claim:

$j = 1.$

Choose $n_1 \in \mathbb{N}$ such that there exists x_{m_1}, satisfying $b_{n_1} \mid x_{m_1}$ but $b_{n_1+1} \nmid x_{m_1}$. By Proposition 2.2.1, we write

$$x_{m_1} = \sum_{i=0}^{N(x_{m_1})} k_i b_i.$$

The condition $b_{n_1} \mid x_{m_1}$ implies $k_i = 0$ if $i < n_1$ and $b_{n_1+1} \nmid x_{m_1}$ implies $k_{n_1} \neq 0$.

Hence,

$$\frac{x_{m_1}}{b_{n_1+1}} + \mathbb{Z} = \frac{\sum_{i=0}^{N(x_{m_1})} k_i b_i}{b_{n_1+1}} + \mathbb{Z} = \frac{\sum_{i=0}^{n_1} k_i b_i}{b_{n_1+1}} + \mathbb{Z} = \frac{k_{n_1}}{q_{n_1+1}} + \mathbb{Z}.$$

We have that

$$\frac{1}{2} \geq \frac{|k_{n_1}|}{q_{n_1+1}} \geq \frac{1}{L}.$$

Therefore,

$$\frac{x_{m_1}}{b_{n_1+1}} + \mathbb{Z} \notin \mathbb{T}_L = \left[-\frac{1}{4L}, \frac{1}{4L}\right] + \mathbb{Z}.$$

$j \Rightarrow j+1.$

Suppose we have n_j, m_j satisfying the desired conditions. Let $n_{j+1} \geq n_j + j$ be a natural number such that there exists $x_{m_{j+1}}$ satisfying $b_{n_{j+1}} \mid x_{m_{j+1}}$ and $b_{n_{j+1}+1} \nmid x_{m_{j+1}}$. By Proposition 2.2.1, we write

$$x_{m_{j+1}} = \sum_{i=0}^{N(x_{m_{j+1}})} k_i b_i.$$

The condition $b_{n_{j+1}} \mid x_{m_{j+1}}$ implies that $k_i = 0$ if $i < n_{j+1}$ and $b_{n_{j+1}+1} \nmid x_{m_{j+1}}$ implies $k_{n_{j+1}} \neq 0$.

Then,

$$\frac{x_{m_{j+1}}}{b_{n_{j+1}+1}} + \mathbb{Z} = \frac{\sum_{i=0}^{N(x_{m_{j+1}})} k_i b_i}{b_{n_{j+1}+1}} + \mathbb{Z} = \frac{\sum_{i=0}^{n_{j+1}} k_i b_i}{b_{n_{j+1}+1}} + \mathbb{Z} = \frac{k_{n_{j+1}}}{q_{n_{j+1}+1}} + \mathbb{Z}.$$

From
$$\frac{1}{2} \geq \frac{\left|k_{n_{j+1}}\right|}{q_{n_{j+1}+1}} \geq \frac{1}{q_{n_{j+1}+1}} \geq \frac{1}{L},$$
we get that
$$\frac{x_{m_{j+1}}}{b_{n_{j+1}+1}} + \mathbb{Z} \notin \mathbb{T}_L.$$

This ends the proof of the claim.

We continue the proof of Theorem 4.4.8. Consider now $\mathbf{c} = \left(b_{n_j+1}\right)_{j\in\mathbb{N}}$. Let
$$\underline{\mathbf{c}} = \left\{\frac{1}{b_{n_j+1}} + \mathbb{Z} : j \in \mathbb{N}\right\}$$
and let $\tau = \tau_{\mathbf{c}}$ be the topology of uniform convergence on $\underline{\mathbf{c}}$. Then:

(1) By Proposition 4.3.4, τ is metrizable and locally quasi-convex. By Proposition 4.3.17, we have $\lambda_{\mathbf{b}} < \tau$.

(2) By the claim, we have proved that $x_{m_j} \overset{j\to\infty}{\not\to} 0$ in τ. This implies that $x_n \not\to 0$ in τ.

(3) Since $n_{j+1} \geq n_j + j$, we get that $n_{j+1} - n_j \geq j$. Thus, $\dfrac{b_{n_j+1}}{b_{n_{j+1}+1}} \leq \dfrac{1}{2^j} \to 0$. By Theorem 4.3.31, we have that $(\mathbb{Z},\tau)^\wedge = \mathbb{Z}(\mathbf{b}^\infty)$.

QED

Theorem 4.4.9 *Let* $\mathbf{b}$ *a basic D-sequence with bounded ratios. Then the topology* $\delta_{\mathbf{b}}$ *has **no** nontrivial convergent sequences. Hence, it is not metrizable.*

Proof:

Since every $\delta_{\mathbf{b}}$-convergent sequence is $\lambda_{\mathbf{b}}$-convergent, we consider only $\lambda_{\mathbf{b}}$-convergent sequences. Let $(x_n) \subseteq \mathbb{Z}$ be a sequence such that $x_n \overset{\lambda_{\mathbf{b}}}{\to} 0$. By Proposition 4.4.8, there exists a subsequence $(b_{n_k})_k$ satisfying that $\tau_{(b_{n_k})}$ is a locally quasi-convex topology, $(b_{n_k}) \in \mathcal{D}_\infty$ and $x_n \overset{\tau_{b_{n_k}}}{\not\to} 0$. Since $\tau_{b_{n_k}} \leq \delta_{\mathbf{b}}$, we know that $x_n \overset{\delta_{\mathbf{b}}}{\not\to} 0$. Hence, the only convergent sequences in $\delta_{\mathbf{b}}$ are the trivial ones. Since $\delta_{\mathbf{b}}$ has no nontrivial convergent sequences it is non-metrizable. By Theorem 4.4.6, this topology is not discrete.

QED

Corollary 4.4.10 *Let* $\mathbf{p} = (p^n)$*. Then* $\delta_{\mathbf{p}}$ *has no nontrivial convergent sequences.*

As mentioned in Remark 4.2.10, for any D-sequence we can construct a basic supersequence, which gives rise to the same linear topology. Hence, it suffices to consider basic D-sequences.

The following lemma is easy to prove, even for a topological space instead of a topological group.

Lemma 4.4.11 *If a countable topological group has no nontrivial convergent sequences, then it does not have infinite compact subsets either.*

Proposition 4.4.12 *Let* $\mathbf{b}$ *be a basic D-sequence with bounded ratios, then the group* $G = (\mathbb{Z}, \delta_{\mathbf{b}})$ *is not reflexive. In fact, the bidual* $G^{\wedge\wedge}$ *can be identified with* $\mathbb{Z}$ *endowed with the discrete topology.*

Proof:

The dual group of $(\mathbb{Z}, \delta_{\mathbf{b}})$ has supporting set $\mathbb{Z}(\mathbf{b}^\infty)$. The $\delta_{\mathbf{b}}$-compact subsets of $\mathbb{Z}$ are finite by Lemma 4.4.11, therefore, the dual group carries the pointwise convergence topology. Thus, $G^\wedge$ is exactly $\mathbb{Z}(\mathbf{b}^\infty)$ with the topology induced by the euclidean of $\mathbb{T}$. By [Auß99, 4.5], the group $\mathbb{Z}(\mathbf{b}^\infty)$ has the same dual group as $\mathbb{T}$, namely, $\mathbb{Z}$ with the discrete topology. We conclude that the canonical mapping α_G is an open non-continuous isomorphism.

QED

Corollary 4.4.13 *Since* $(\mathbb{Z}, \delta_{\mathbf{b}})^{\wedge\wedge}$ *is discrete* $\alpha_{(\mathbb{Z},\delta_{\mathbf{b}})}$ *is not continuous. However, it is an open algebraic isomorphism.*

Proposition 4.4.14 *Let* $\mathbf{b} \in \mathcal{D}^\ell_\infty$ *and* $G_\gamma := (\mathbb{Z}, \tau_{\mathbf{b}})$*. Then* α_{G_γ} *is a non-surjective embedding from* G_γ *into* $G_\gamma^{\wedge\wedge}$*.*

Proof:

Since τ_γ is metrizable, α_{G_γ} is continuous. The fact that τ_γ is locally quasi-convex and $\mathbb{Z}(\mathbf{b}^\infty)$ separates points of $\mathbb{Z}$ imply that α_{G_γ} is injective and open in its image [Auß99, 6.10]. On the other hand, α_{G_γ} is not onto, for otherwise G_γ would be reflexive. However, a non-discrete countable metrizable group cannot be reflexive. Indeed, the dual of a metrizable group is a k-space ([Auß99, Cha98]) and the dual group of a k-space is complete. Hence, the original group must be complete as well. By Baire Category Theorem, the only metrizable complete group topology on a countable group is the discrete one.

QED

Remark 4.4.15 If $\mathbf{b}$ is a basic D-sequence with bounded ratios, then $\mathcal{D}^k_\infty(\mathbf{b}) \neq \mathcal{D}^{k+1}_\infty(\mathbf{b})$, for all $k \in \mathbb{N}$.

Remark 4.4.16 If $\mathbf{b}$ is a basic D-sequence with bounded ratios, then $\tau_{\mathbf{b}}$ is discrete.

4.4.2 $\delta_{\mathbf{b}}$ if $\mathbf{b} \in \mathcal{D}^\ell_\infty$ for some ℓ.

In this subsection we give some properties of the topology $\delta_{\mathbf{b}}$ if $\mathbf{b} \in \mathcal{D}^\ell_\infty$ for some natural number ℓ. In addition, we answer the following question in the negative:

Question 4.4.17 [L. Außenhofer]

Let $\mathbb{Z}(\mathbf{b}^\infty) \leq \mathbb{T}$ be a subgroup generated by a D-sequence. Suppose that there exist two basic D-sequences $\mathbf{b}, \mathbf{c}$ such that $\mathbb{Z}(\mathbf{b}^\infty) = \mathbb{Z}(\mathbf{c}^\infty)$. Is $\delta_{\mathbf{b}} = \delta_{\mathbf{c}}$?

Example 4.4.18 We denote by ρ the basic D-sequence satisfying that $q_n = p_n$, where p_n is the n-th prime number. The following facts can be easily proved:

- The sequence ρ has unbounded ratios.
- We have that $\mathcal{D}^k_\infty(\rho) = \mathcal{D}_\infty(\rho)$, for any natural number k.
- By Proposition 4.4.7, we have $\delta_\rho = \tau_\rho$.
- The topology δ_ρ is metrizable. Hence, it has nontrivial convergent sequences and the analogous of Proposition 4.4.8 does not hold for λ_ρ.

It is natural to generalize this example to a more general framework. In fact, it is easy to check the following:

Remark 4.4.19 Let $\mathbf{b}$ be a basic D-sequence such that $\mathbf{b} \in \mathcal{D}^\ell_\infty$. The following facts can be easily proved:

- The sequence $\mathbf{b}$ has unbounded ratios.

- We have that $\mathcal{D}^k_\infty(\mathbf{b}) = \mathcal{D}^\ell_\infty(\mathbf{b})$, for any natural number $k \geq \ell$.
- By Proposition 4.4.7, we have $\delta_\mathbf{b} = \tau_\mathbf{b}$.
- The topology $\delta_\mathbf{b}$ is metrizable. Hence, it has nontrivial convergent sequences and the analogous of Proposition 4.4.8 does not hold for $\lambda_\mathbf{b}$.

Now, Question 4.4.17 is answered:

Proposition 4.4.20 *Let* $\mathbf{b}, \mathbf{c}$ *be the D-sequences defined as follows:*

$$\bullet b_{3n+1} = 2 \cdot b_{3n} \qquad \bullet b_{3n+2} = p_{n+1} b_{3n+1} \qquad \bullet b_{3n+3} = 3 \cdot b_{3n+2}$$

and

$$\bullet c_{3n+1} = 3 \cdot c_{3n} \qquad \bullet c_{3n+2} = p_{n+1} c_{3n+1} \qquad \bullet c_{3n+3} = 2 \cdot c_{3n+2},$$

where p_n is the n-th prime number.

Then $\mathbb{Z}(\mathbf{b}^\infty) = \mathbb{Z}(\mathbf{c}^\infty)$ *and* $\delta_\mathbf{b} \neq \delta_\mathbf{c}$.

Proof:

Since $b_{3n} = c_{3n}$ it is clear that $\mathbb{Z}(\mathbf{b}^\infty) = \mathbb{Z}(\mathbf{c}^\infty)$.

For the second statement, note that $\mathbf{b}, \mathbf{c} \in \mathcal{D}^3_\infty$. Hence $\delta_\mathbf{b} = \tau_\mathbf{b}$ and $\delta_\mathbf{c} = \tau_\mathbf{c}$. To prove that $\delta_\mathbf{b} \neq \delta_\mathbf{c}$, we consider (b_{3n+1}) and (c_{3n+1}). By Lemma 4.3.19, it is clear that $b_{3n+1} \to 0$ in $\tau_\mathbf{b}$ and $c_{3n+1} \to 0$ in $\tau_\mathbf{c}$. Now we prove that $b_{3n+1} \nrightarrow 0$ in $\tau_\mathbf{c}$. For that, we consider the quotients $\frac{b_{3n+1}}{c_{3n+1}} + \mathbb{Z} = \frac{2b_{3n}}{3c_{3n}} + \mathbb{Z} \notin \mathbb{T}_+$ for all n. Hence, the sequence $b_{3n+1} \nrightarrow 0$ in $\tau_\mathbf{c}$. Analogously, $\frac{c_{3n+1}}{b_{3n+1}} + \mathbb{Z} = \frac{3c_{3n}}{2b_{3n}} + \mathbb{Z} \notin \mathbb{T}_+$ holds for all n and $c_{3n+1} \nrightarrow 0$ in $\tau_\mathbf{b}$ follows. Hence $\delta_\mathbf{b} = \tau_\mathbf{b}$ and $\delta_\mathbf{c} = \tau_\mathbf{c}$ are not even comparable.

QED

Corollary 4.4.21 *Let* $\mathbf{b}, \mathbf{c}$ *be the D-sequences defined in Proposition 4.4.20. Then* $\delta_\mathbf{b}$ *and* $\delta_\mathbf{c}$ *are not Mackey topologies.*

Proof:

If one of them, say $\delta_\mathbf{b}$ were the Mackey topology, then $\delta_\mathbf{c}$ should be coarser than $\delta_\mathbf{b}$, but they are not comparable.

QED

4.5 Complete topologies on $\mathbb{Z}$.

In this section we collect some known results and then give some new ones concerning duality and reflexivity are given.

Proposition 4.5.1 *[PZ99, Theorem 2.2.1]*
If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ *then* (a_n) *is a* T*-sequence in* $\mathbb{Z}$.

This proposition provides a condition to prove that the following sequences are T-sequences: $(b_n) = (2^{n^2})$, $(\rho_n) = (p_1 \cdot \cdots \cdot p_n)$.

Proposition 4.5.2 *[PZ99, Theorem 2.2.3]*
If $lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ *and* r *is a transcendental number then* (a_n) *is a* T*-sequence in* $\mathbb{Z}$.

Let $\mathbf{b}$ be a D-sequence. It is clear that $\mathbf{b}$ is a T-sequence (by Proposition 4.2.16). Hence, the topology $\mathcal{T}_{\{a_n\}}$ can be constructed on $\mathbb{Z}$. By Theorem 1.4.3, it is a complete topology.

The following properties of $\mathbb{Z}_{\{a_n\}}$ are interesting by themselves. The first one solves a problem by Malykhin and the second one states that the dual group is metrizable and a k-space.

Remark 4.5.3 Let (a_n) be a T-sequence on $\mathbb{Z}$. Then:

(1) $\mathbb{Z}_{\{a_n\}}$ is sequential, but not Frechet-Urysohn ([ZP90, Theorem 7 and Theorem 6]).

(2) $\mathbb{Z}_{\{a_n\}}$ is a k_ω-group ([PZ99, Corollary 4.1.5]).

Proposition 4.5.4 *The dual group of* $\mathbb{Z}_{\{p^n\}}$ *coincides algebraically and topologically with the dual group of* $(\mathbb{Z}, \lambda_p)$*; that is, the dual group of* $\mathbb{Z}_{\{p^n\}}$ *is* $\mathbb{Z}(\mathbf{p}^\infty)$ *endowed with the discrete topology.*

Proof:
From [DGT14, Fact 1.14] the dual group of $\mathbb{Z}_{\{p^n\}}$ coincides algebraically with the dual of $(\mathbb{Z}, \lambda_{\mathbf{p}})$. This means $\mathbb{Z}^\wedge_{\{p^n\}} = \mathbb{Z}(\mathbf{p}^\infty)$. Item (2) in Remark 4.5.3 implies that $\mathbb{Z}^\wedge_{\{p^n\}}$ is metrizable and complete. By Baire Category Theorem and since $\mathbb{Z}^\wedge_{\{p^n\}}$

is countable, we get that it is discrete. Hence, it coincides topologically with $(\mathbb{Z}, \lambda_{\mathbf{p}})^\wedge$.

QED

Using Proposition 4.5.4 we prove that $\mathbb{Z}_{\{p^n\}}$ is not locally quasi-convex.

Proposition 4.5.5 *Let p be a prime number. The group $\mathbb{Z}_{\{p^n\}}$ is not locally quasi-convex, and consequently it is not reflexive.*

Proof:

Write $G := \mathbb{Z}_{\{p^n\}}$. Suppose that G is locally quasi-convex. By [Auß99, 6.10], the canonic mapping α_G is injective and open. By Remark 4.5.3 (2), the group G is a k-group, which implies that α_G is continuous. Hence α_G is a topological embedding from G into $G^{\wedge\wedge}$. By Proposition 4.5.4, the dual group $G^\wedge$ is discrete and consequently $G^{\wedge\wedge}$ is compact. The embedding α_G allows us to state that G is precompact. By Theorem 1.2.3, there exists a unique precompact group topology on $\mathbb{Z}$ having $\mathbb{Z}(\mathbf{p}^\infty))$ as dual group, namely the p-adic topology λ_p. Hence, we have $G = (\mathbb{Z}, \lambda_{\mathbf{p}})$, this contradicts the fact that $\mathbb{Z}_{\{p^n\}}$ is not metrizable. Consequently, the group $\mathbb{Z}_{\{p^n\}}$ is not locally quasi-convex.

QED

4.6 Open questions and problems.

We collect here some open questions related to topologies on $\mathbb{Z}$ and the corresponding dual groups. Most of them are related to the structure of the dual groups of $\mathbb{Z}$ endowed with different topologies of uniform convergence.

Open question 4.6.1 Does there exist the Mackey topology for the duality $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^\infty))$?

Open question 4.6.2 [Tkachenko]

Does there exists a precompact topology ν on $\mathbb{Z}$ such that $(\mathbb{Z}, \nu)$ is a reflexive group?

Open question 4.6.3

Let $\mathbf{b}$ be a D-sequence. Does there exist a general algorithm to find all the D-sequences $\mathbf{c}$ satisfying that $\tau_{\mathbf{b}} = \tau_{\mathbf{c}}$?

Open question 4.6.4

Let $\mathbf{b}$ be a D-sequence satisfying $\mathbf{b} \in \mathcal{D}^{\ell}_{\infty}$ for some natural number ℓ. Is $\tau_{\mathbf{b}}$ never the Mackey topology?

Open question 4.6.5

Let $\mathbf{b} \in \mathcal{D}^{\ell}_{\infty}$ be a D-sequence. How can the compact-open topology of $\mathbb{Z}(\mathbf{b}^{\infty})$ as the dual group of $(\mathbb{Z}, \tau_{\mathbf{b}})$ be described?

Open question 4.6.6

Let $\mathbf{b}$ be a D-sequence which satisfies that $\tau_{\mathbf{b}}$ is not discrete. Is $(\mathbb{Z}, \tau_{\mathbf{b}})^{\wedge}$ reflexive for some D-sequence?

Open question 4.6.7

Let $\mathbf{b}$ be a D-sequence satisfying the block condition. Determine $(\mathbb{Z}, \tau_{\mathbf{b}})^{\wedge}$. Determine $(\mathbb{Z}, \delta_{\mathbf{b}})^{\wedge}$.

Open question 4.6.8

Let $\mathbf{b}$ be the sequence defined in Proposition 4.3.37, is algebraically $(\mathbb{Z}, \tau_{\mathbf{b}})^{\wedge} = \mathbb{Z}(\mathbf{b}^{\infty})$?

Open question 4.6.9

Let $\mathbf{b}, \mathbf{c}$ be as in Proposition 4.4.20. Determine $(\mathbb{Z}, \sup(\delta_{\mathbf{b}}, \delta_{\mathbf{c}}))^{\wedge}$. Equivalently, determine if $\sup(\delta_{\mathbf{b}}, \delta_{\mathbf{c}})$ is a compatible topology.

Open question 4.6.10

Let $\mathbf{b}$ be a basic D-sequence and consider $\mathbb{Z}(\mathbf{b}^{\infty})$. Give sufficient conditions on $\mathbf{b}$ so that for any $\mathbf{c}$ satisfying that $\mathbb{Z}(\mathbf{b}^{\infty}) = \mathbb{Z}(\mathbf{c}^{\infty})$, the topologies $\delta_{\mathbf{c}}$ and $\delta_{\mathbf{b}}$ coincide. Is $\delta_{\mathbf{b}}$, furthermore, the Mackey topology for $(\mathbb{Z}, \mathbb{Z}(\mathbf{b}^{\infty}))$?

In fact, we know that there exist such sequences. For example, the sequence $\mathbf{p} = (p^n)$ satisfies the condition.

Open question 4.6.11

Let (a_n) and (b_n) two D-sequences (in particular, they are T-sequences). Determine if $\mathbb{Z}_{\{a_n\}} = \mathbb{Z}_{\{b_n\}}$.

Open question 4.6.12

Determine if the locally quasi-convex modification of $\mathcal{T}_{\{a_n\}}$ is a Mackey topology.

Open problem:

Every proposition and theorem of this chapter depends strongly on the choice of considering D-sequences. This allows us to consider the following problem: If we consider a general sequence of natural numbers $\mathbf{b} \notin \mathcal{D}$ satisfying that $b_n \to \infty$, find results similar to the ones obtained in this chapter.

All the discussion in this chapter deals with the group of integer numbers, but $\mathbf{b}$-adic topologies can be extended to any group. Hence we can consider the following problem:

Open problem:

Let G be an abelian group. Study the duality properties of $\mathbf{b}$-adic topologies. Are $\mathbf{b}$-adic topologies Mackey for G?

Chapter 5

The unit circle and its subgroups.

This chapter is devoted to study some topologies on the group $\mathbb{T}$ and the group $\mathbb{Z}(\mathbf{b}^\infty)$ for a D-sequence $\mathbf{b}$. As a consequence of this study we prove that the group of rational numbers endowed with the topology inherited from the topology of the real line is not Mackey. The results in this chapter are similar to the ones obtained in Chapter 4.

5.1 Two different topologies on $\mathbb{Z}(\mathbf{b}^\infty)$

In this section we fix a D-sequence $\mathbf{b} = (b_n)$ and consider two different topologies on the group $\mathbb{Z}(\mathbf{b}^\infty)$. The first one is the usual topology τ_U, inherited from $\mathbb{T}$, and the second one is the topology of uniform convergence on a suitable sequence of $\mathbb{Z}_\mathbf{b} = \mathbb{Z}(\mathbf{b}^\infty)^*$. It will be denoted by $\eta_\mathbf{b}$. The topology $\eta_\mathbf{b}$ is strictly finer than τ_U.

5.1.1 The usual topology on $\mathbb{Z}(\mathbf{b}^\infty)$.

We describe some properties of the usual topology on $\mathbb{T}$, that lead to convenient expressions of the topology induced on $\mathbb{Z}(\mathbf{b}^\infty)$. We start by choosing the following basis of neighborhoods of 0.

Notation 5.1.1 *For a fixed $n \in \mathbb{N}_0$, we write*

$$U_n := \left(\left(-\frac{1}{2b_n}, \frac{1}{2b_n}\right) + \mathbb{Z}\right) \cap \mathbb{Z}(\mathbf{b}^\infty),$$

then the family $\{U_n\}_{n\in\mathbb{N}_0}$ is a neighborhood basis of 0 for the topology τ_U.

Lemma 5.1.2 *Let $n \in \mathbb{N}$ and $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ and let $(\beta_0, \beta_1, \dots)$ be its* **b***-coordinates. Then the following conditions are equivalent:*

1. $x + \mathbb{Z} \in U_n$.

2. $\beta_1 = \cdots = \beta_n = 0$

Proof: $1 \Rightarrow 2$ Suppose that there exists $j_0 \leq n$ such that $\beta_{j_0} \neq 0$. Without loss of generality j_0 is minimal with this property. We can write

$$x = \frac{\beta_{j_0}}{b_{j_0}} + \sum_{k=j_0+1}^{\infty} \frac{\beta_k}{b_k}.$$

Consequently

$$|x| \geq \left|\frac{\beta_{j_0}}{b_{j_0}}\right| - \left|\sum_{k=j_0+1}^{\infty} \frac{\beta_k}{b_k}\right| = \frac{|\beta_{j_0}|}{b_{j_0}} - \left|x - \sum_{k=0}^{j_0} \frac{\beta_k}{b_k}\right| \overset{2.2.4}{\geq} \frac{1}{b_{j_0}} - \frac{1}{2b_{j_0}} = \frac{1}{2b_{j_0}} \geq \frac{1}{2b_n}.$$

This contradicts the fact $x \in U_n$.

The fact $2 \Rightarrow 1$ follows directly from Proposition 2.2.4.

QED

Proposition 5.1.3 *Let $x_n \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $x_n + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$. For each n let $(\beta_0^{(n)}, \beta_1^{(n)}, \dots)$ be the* **b***-coordinates of x_n. Then the following assertions are equivalent:*

1. $x_n + \mathbb{Z} \to 0 + \mathbb{Z}$ *in* τ_U.

2. $|x_n| \to 0$ *in* $\mathbb{R}$.

3. *For any $k \in \mathbb{N}$ there exists n_k such that $\beta_1^{(n)} = \cdots = \beta_k^{(n)} = 0$ if $n \geq n_k$.*

Proof:

Since $|x| \leq \frac{1}{2}$, it is clear that 1. and 2. are equivalent. The equivalence between 1. and 3. is a consequence of Lemma 5.1.2.

QED

5.1.2 Topologies of uniform convergence on $\mathbb{Z}(\mathbf{b}^\infty)$.

In order to construct topologies of uniform convergence on $\mathbb{Z}(\mathbf{b}^\infty)$, the group $\mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T})$ must be considered. Hence, on $\mathbb{Z}(\mathbf{b}^\infty)$ we can consider the topology of uniform convergence on a fixed subset B of $\mathbb{Z}$, as a subset of $\mathbb{Z}_{\mathbf{b}} = \mathrm{Hom}(\mathbb{Z}(\mathbf{b}^\infty), \mathbb{T})$. In this chapter we will take the range of our sequence $\mathbf{b}$ as the subset B and denote by $\eta_{\mathbf{b}}$ the corresponding topology of uniform convergence on B.

Using Proposition 2.2.1, we can state the following:

Proposition 5.1.4 *If* $\mathbf{b}$ *is a D-sequence such that* $q_{j+1} \neq 2$ *for infinitely many* j*, each element* $k \in \mathbb{Z}_{\mathbf{b}}$ *admits a representation*

$$k = \sum_{n\in\mathbb{N}} k_n b_n$$

where $k_n \in \left(-\frac{q_{n+1}}{2}, \frac{q_{n+1}}{2}\right] \cap \mathbb{Z}$*. Hence*

$$k \cdot t := \sum_{n\in\mathbb{N}_0} k_n b_n t$$

is well-defined for every $t \in \mathbb{Z}(\mathbf{b}^\infty)$ *and* $k \in \mathbb{Z}_{\mathbf{b}}$.

Proof:

Fix $k \in \mathbb{Z}_{\mathbf{b}}$. If $k \in \mathbb{Z}$, the conclusion is true by Proposition 2.2.1. Therefore we can suppose that $k \in \mathbb{Z}_{\mathbf{b}} \setminus \mathbb{Z}$. Since $\mathbb{Z}_{\mathbf{b}}$ is metrizable and $\mathbb{Z}$ is dense in $\mathbb{Z}_{\mathbf{b}}$, there exists a Cauchy sequence $(\ell_n)_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $\ell_n \to k$. By Proposition 2.2.1, for each n, write

$$\ell_n = \sum_{m=0}^{N(\ell_n)} k_m^{(n)} b_m,$$

where $k_m^{(n)} \in \left(-\frac{q_{m+1}}{2}, \frac{q_{m+1}}{2}\right]$ for $0 \leq m \leq N(\ell_n)$. Put $k_m^{(n)} = 0$ if $m > N(\ell_n)$. We will prove that $k_m^{(n)} \overset{n\to\infty}{\to} k_m$, where $k_m \in \left(-\frac{q_{m+1}}{2}, \frac{q_{m+1}}{2}\right] \cap \mathbb{Z}$, for all $m \in \mathbb{N}$.

Indeed, if $m = 0$, there exists n_0 such that $\ell_p - \ell_q \in b_1\mathbb{Z}$ if $p, q \geq n_0$. Hence, we can write

$$k_0^{(p)} - k_0^{(q)} + \left(\sum_{m=1}^{N(\ell_p)} k_m^{(p)} b_m - \sum_{m=1}^{N(\ell_q)} k_m^{(q)} b_m \right) \in b_1\mathbb{Z}.$$

Taking into account that $b_1 | b_m$ if $m \geq 1$, we get that $k_0^{(p)} - k_0^{(q)} \in b_1\mathbb{Z}$. Since $k_0^{(p)}, k_0^{(q)} \in \left(-\frac{q_1}{2}, \frac{q_1}{2}\right] \cap \mathbb{Z}$, we conclude that $k_0^{(p)} = k_0^{(q)}$ if $p, q \geq n_0$. So $k_0^{(n)} \to k_0$.

Now, suppose that the assertion is true for $0, 1, \ldots m-1$. Let us prove the assertion for m. Since (ℓ_n) is a Cauchy sequence there exists $n_m(\geq n_{m-1})$ such that $\ell_p - \ell_q \in b_{m+1}\mathbb{Z}$. Equivalently,

$$\left(\sum_{n=0}^{m-1} k_n^{(p)} b_n - \sum_{n=0}^{m-1} k_n^{(q)} b_n\right) + k_m^{(p)} b_m - k_m^{(q)} b_m + \left(\sum_{n=m+1}^{N(\ell_p)} k_n^{(p)} b_n - \sum_{n=m+1}^{N(\ell_q)} k_n^{(q)} b_n\right) \in b_{m+1}\mathbb{Z}.$$

Since, by hypothesis, $k_n^{(p)} = k_n^{(q)}$ for $n = 0, 1, \ldots, m-1$ and $b_{m+1} \mid b_n$ if $n \geq m+1$, we get $k_m^{(p)} b_m - k_m^{(q)} b_m \in b_{m+1}\mathbb{Z}$. Using number theory, we get that $k_m^{(p)} - k_m^{(q)} \in \frac{b_{m+1}}{b_m}\mathbb{Z}$. Since $k_m^{(p)}, k_m^{(q)} \in \left(-\frac{q_{m+1}}{2}, \frac{q_{m+1}}{2}\right] \cap \mathbb{Z}$, we get that $k_m^{(p)} = k_m^{(q)}$ if $p, q \geq n_m$.

In conclusion, we can write

$$k = \sum_{m \in \mathbb{N}_0} k_m b_m$$

where

$$k_m := \lim_{n \to \infty} k_m^{(n)}.$$

QED

Remark 5.1.5 The condition $q_{j+1} \neq 2$ for infinitely many j in Proposition 5.1.4 is not essential. As pointed out in Example 2.2.3, if the condition holds, the expression of integer numbers is finite, whereas elements of $\mathbb{Z}_{\mathbf{b}} \setminus \mathbb{Z}$ have infinite expansions. If the conditions does not hold, negative integers would have infinite expansions as well (see Example 2.2.3).

A neighborhood basis of 0 for $\eta_{\mathbf{b}}$ can be described as follows:

Definition 5.1.6 Let $\mathbf{b}$ be a D-sequence and

$$W_{\mathbf{b},m} := \{t \in \mathbb{Z}(\mathbf{b}^\infty) : tb_n \in \mathbb{T}_m \text{ for all } n \in \mathbb{N}_0\}.$$

Then $\{W_{\mathbf{b},m}\}_{m \in \mathbb{N}}$ is a neighborhood basis of 0 for the topology $\eta_{\mathbf{b}}$.

Remark 5.1.7 Since $b_0 = 1$, we have that $W_{\mathbf{b},m} \subset \{t \in \mathbb{Z}(\mathbf{b}^\infty) : t \in \mathbb{T}_m\} = \mathbb{T}_m \cap \mathbb{Z}(\mathbf{b}^\infty)$.

Let us now give necessary and/or sufficient conditions in order that an element $x \in \mathbb{Z}(\mathbf{b}^\infty)$ belongs to $W_{\mathbf{b},m}$, for a fixed m.

Proposition 5.1.8 *Let $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $t := x+\mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$; let $(\beta_0, \beta_1, \dots)$ be the $\mathbf{b}$-coordinates of x. Then for a fixed $m \in \mathbb{N}$ the following assertions are equivalent:*

1. $t \in W_{\mathbf{b},m}$.
2. $\left|\sum_{k=n+1}^{\infty} \frac{b_n \beta_k}{b_k}\right| \leq \frac{1}{4m}$ *for all* $n \in \mathbb{N}$.

Proof:

By the definition of $W_{\mathbf{b},m}$, we have that $x \in W_{\mathbf{b},m}$ if and only if $xb_n + \mathbb{Z} \in \mathbb{T}_m$ for all n. This means that

$$\sum_{k \geq 1} \frac{\beta_k}{b_k} b_n + \mathbb{Z} \in \mathbb{T}_m$$

and taking into account that $\frac{b_n}{b_k} \in \mathbb{Z}$ if $n \geq k$, we have equivalently:

$$b_n \sum_{k=n+1}^{\infty} \frac{\beta_k}{b_k} + \mathbb{Z} \in \mathbb{T}_m \text{ for all } n.$$

Since by Proposition 2.2.4

$$-\frac{1}{2b_n} < \sum_{k=n+1}^{\infty} \frac{\beta_k}{b_k} \leq \frac{1}{2b_n}$$

the last condition is equivalent to

$$\left|\sum_{k=n+1}^{\infty} \frac{\beta_k b_n}{b_k}\right| \leq \frac{1}{4m}.$$

QED

Proposition 5.1.9 *Let $x \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $t := x+\mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$; let $(\beta_0, \beta_1, \dots)$ be the $\mathbf{b}$-coordinates of x. If for a fixed $m \in \mathbb{N}$, we have $|\beta_k| \leq \frac{q_k}{8m}$ for all $k \in \mathbb{N}$, then $t \in W_{\mathbf{b},m}$.*

Proof:

We compute

$$\left|\sum_{k=n+1}^{\infty} \frac{b_n \beta_k}{b_k}\right| \leq \sum_{k=n+1}^{\infty} \frac{b_n}{b_{k-1}} \frac{b_{k-1} |\beta_k|}{b_k} = \sum_{k=n+1}^{\infty} \frac{b_n}{b_{k-1}} \frac{|\beta_k|}{q_k} \leq \frac{1}{8m} \sum_{k=n+1}^{\infty} \frac{b_n}{b_{k-1}} \leq$$

$$\leq \frac{1}{8m}\sum_{n\geq 0}\frac{1}{2^n} = \frac{1}{4m}.$$

By Proposition 5.1.8, we have that

$$x = \sum_{k\geq 1}\frac{\beta_k}{b_k} \in W_{\mathbf{b},m}.$$

QED

Proposition 5.1.10 *Let $x \in (-\frac{1}{2}, \frac{1}{2}]$ be such that $t := x+\mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$; let $(\beta_0, \beta_1, \dots)$ be the* **b***-coordinates of x. If for a fixed $m \in \mathbb{N}$ we have $t \in W_{\mathbf{b},m}$, then*

$$|\beta_k| \leq \frac{3q_k}{8m} \text{ for all } k \in \mathbb{N}.$$

Proof:

Clearly, we have that

$$\frac{\beta_k}{q_k} = \frac{\beta_k b_{k-1}}{b_k} = b_{k-1}\left(\sum_{n=k}^{\infty}\frac{\beta_n}{b_n} - \sum_{n=k+1}^{\infty}\frac{\beta_n}{b_n}\right).$$

Hence

$$\frac{|\beta_k|}{q_k} \leq \left|\sum_{n=k}^{\infty}\frac{b_{k-1}\beta_n}{b_n}\right| + \frac{1}{q_k}\left|\sum_{n=k+1}^{\infty}\frac{b_k\beta_n}{b_n}\right| \overset{5.1.8}{\leq} \frac{1}{4m} + \frac{1}{2}\cdot\frac{1}{4m} = \frac{3}{8m},$$

as desired.

QED

In order to find natural bounds for the polar sets $W^{\triangleright}_{\mathbf{b},m}$, we introduce a new neighborhood basis of 0 for $\eta_{\mathbf{b}}$.

Notation 5.1.11 *Define*

$$C_{\mathbf{b},\varepsilon} := \left\{x = \sum \frac{\beta_n}{b_n} \text{ such that } |\beta_n| \leq \varepsilon q_n \text{ for all } n\right\}.$$

For all $m \in \mathbb{N}$ we have:

$$C_{\mathbf{b},\frac{1}{8m}} \subset W_{\mathbf{b},m} \subset C_{\mathbf{b},\frac{3}{8m}}.$$

Consequently,

$$\left(C_{\mathbf{b},\frac{3}{8m}}\right)^{\triangleright} \subset (W_{\mathbf{b},m})^{\triangleright} \subset \left(C_{\mathbf{b},\frac{1}{8m}}\right)^{\triangleright}.$$

We can now characterize convergence of sequences in $\eta_{\mathbf{b}}$.

Proposition 5.1.12 *Let $x_n \in \left(-\frac{1}{2}, \frac{1}{2}\right]$ be such that $x_n + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$; let $(\beta_0^{(n)}, \beta_1^{(n)}, \dots)$ the $\mathbf{b}$-coordinates of x_n. Then, the following assertions are equivalent:*

1. *$x_n + \mathbb{Z} \to 0 + \mathbb{Z}$ in $\eta_{\mathbf{b}}$.*
2. *For all $m \in \mathbb{N}$ there exists n_m such that $\left|\beta_k^{(n)}\right| \leq \frac{q_k}{8m}$ if $n \geq n_m$.*

Observe that for a D-sequence, $\mathbf{b} \in \mathcal{D}$, the naturally defined sequence $\left\{\frac{1}{b_n} + \mathbb{Z}\right\}$ is not always convergent. From Proposition 5.1.12, we obtain the following two assertions.

Corollary 5.1.13 *Let $\mathbf{b} \in \mathcal{D}_\infty$. Then $\frac{1}{b_n} + \mathbb{Z} \to 0 + \mathbb{Z}$ in $\eta_{\mathbf{b}}$.*

Corollary 5.1.14 *Let $\mathbf{b}$ be a D-sequence. Then the following conditions are equivalent:*

1. *There exists L such that $q_{n+1} \leq L$ for all n.*
2. *The topology $\eta_{\mathbf{b}}$ in $\mathbb{Z}(\mathbf{b}^\infty)$ is discrete.*

Proof:

First we prove that (1) implies (2). To that end, we shall see that $W_{\mathbf{b},L} = \{0\}$. Let $x \in W_{\mathbf{b},L}$. By 2.2.4, let $(\beta_0, \beta_1, \dots)$ be the $\mathbf{b}$-coordinates of x. By Proposition 5.1.10, we have that $|\beta_n| \leq \frac{3q_n}{8L}$. This means

$$|\beta_n| \leq \frac{3}{8} \cdot \frac{q_n}{L} \leq \frac{3}{8}.$$

Since β_n must be an integer, we have that $\beta_n = 0$ for all n. Equivalently, we get that $x = 0$. Thus, $\eta_{\mathbf{b}}$ is discrete.

Now we prove that (2) implies (1). By contradiction, assume that (1) does not hold. There exists (n_j) such that

$$q_{n_j+1} \to \infty.$$

Define the sequence

$$x_j := \frac{1}{b_{n_j+1}}.$$

We will prove that $x_j \to 0$ in $\eta_{\mathbf{b}}$. Indeed, fix $m \in \mathbb{N}$. Since

$$\frac{1}{q_{n_j+1}} \to 0,$$

there exists n_{j_m} such that

$$\frac{1}{q_{n_j+1}} \leq \frac{1}{4m} \text{ if } n_j \geq n_{j_m}.$$

Choose $n_j \geq n_{j_m}$.

If $n \leq n_j$ we have $b_n x_j + \mathbb{Z} = b_n \frac{1}{b_{n_j+1}} + \mathbb{Z}$. In fact, we get $\left|\frac{b_n}{b_{n_j+1}}\right| \leq \left|\frac{b_{n_j}}{b_{n_j+1}}\right| = \frac{1}{q_{n_j+1}} \leq \frac{1}{4m}$.

If $n > n_j$ we have that $b_n x_j = b_n \frac{1}{b_{n_j+1}} \in \mathbb{Z}$.

Hence $b_n x_j \in \mathbb{T}_m$ for all n. Equivalently, $x_j \in \mathbb{T}_m$ if $n_j \geq n_{j_m}$ and $x_j \to 0$ in $\eta_{\mathbf{b}}$. Consequently, $\eta_{\mathbf{b}}$ is not discrete.

QED

5.1.3 Comparing $\eta_{\mathbf{b}}$ and τ_U.

This subsection is devoted to prove that $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U) \lneq (\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{b}})$. We will essentially use Propositions 5.1.3 and 5.1.12, where the respective criteria for convergence of sequences are established.

Proposition 5.1.15 *Let* **b** *be a D-sequence. Then* $\eta_{\mathbf{b}}$ *is strictly finer than* τ_U.

Proof:

Since $W_{\mathbf{b},n} \subset \mathbb{T}_n \cap \mathbb{Z}(\mathbf{b}^\infty)$, we have that $\tau_U \leq \eta_{\mathbf{b}}$. In order to prove that they are different, we find a sequence which converges in τ_U but does not converge in $\eta_{\mathbf{b}}$. Define

$$x_n := \frac{1}{b_n}\left\lfloor \frac{b_n}{2b_{n-1}} \right\rfloor.$$

Since $|x_n| \leq \frac{1}{2b_{n-1}}$, it is clear that $|x_n| \to 0$. Hence $x_n + \mathbb{Z} \to 0 + \mathbb{Z}$ in τ_U. But $\beta_n^{(n)} = \left\lfloor \frac{b_n}{2b_{n-1}} \right\rfloor$. We have two cases:

If $q_n = 2$, we have

$$\frac{\left|\beta_n^{(n)}\right|}{q_n} = \frac{1}{2}$$

If $q_n \geq 3$, we have

$$\frac{\left|\beta_n^{(n)}\right|}{q_n} > \left(\frac{q_n}{2} - 1\right)\frac{1}{q_n} = \frac{1}{2} - \frac{1}{q_n} \geq \frac{1}{6}.$$

Hence $x_n + \mathbb{Z} \notin W_{\mathbf{b},2}$.

QED

5.2 Computing the dual group of $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})$.

The main result of this section is Theorem 5.2.3 which states that for $\mathbf{b} \in \mathcal{D}_\infty$, the dual group of $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})$ is $\mathbb{Z}$. Therefore, $\eta_\mathbf{b}$ and τ_U are compatible topologies on $\mathbb{Z}(\mathbf{b}^\infty)$. For the proof we need a computational lemma and some considerations. First note that due to Proposition 5.1.15, the identity $i : (\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b}) \to (\mathbb{Z}(\mathbf{b}^\infty), \tau_U)$ is a continuous homomorphism and the dual homomorphism $i^\wedge : (\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge \to (\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge$ is also continuous. By [Auß99, 4.5] and [Cha98] it is known that a dense subgroup of a metrizable group has topologically the same dual as the whole group. Thus, $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge \cong \mathbb{T}^\wedge \cong \mathbb{Z}$, and consequently $\mathbb{Z} \leq (\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge$. We must now prove that $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge \leq \mathbb{Z}$. We first describe how an element of $\mathbb{Z}_\mathbf{b}$ acts in $\mathbb{Z}(\mathbf{b}^\infty)$. For $k \in \mathbb{Z}_\mathbf{b}$, write $k = \sum_{n\geq 0} k_n b_n$ where $k_n \in \left(-\frac{q_{n+1}}{2}, \frac{q_{n+1}}{2}\right]$. We define the homomorphism associated to k in the following way:

$$\chi_k : \mathbb{Z}(\mathbf{b}^\infty) \to \mathbb{T},$$

where

$$\chi_k(x + \mathbb{Z}) = kx + \mathbb{Z} = \sum_{n\geq 0} k_n b_n x + \mathbb{Z}.$$

Now we prove that $xk + \mathbb{Z}$ is well-defined. Indeed, since $x + \mathbb{Z} \in \mathbb{Z}(\mathbf{b}^\infty)$, there exists $N \in \mathbb{N}$ such that $xb_N \in \mathbb{Z}$; thus, the sum $\sum_{n\geq 0} k_n b_n x + \mathbb{Z}$ can be considered as the finite sum $\sum_{n=0}^{N} k_n b_n x + \mathbb{Z}$ and the element $\chi_k(x) = xk + \mathbb{Z}$ is well-defined. Our strategy is to find for any $k \in \mathbb{Z}_\mathbf{b} \setminus \mathbb{Z}$ a suitable $x \in \mathbb{Z}(\mathbf{b}^\infty)$, satisfying certain numerical conditions which force χ_k not to belong to $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge$. To this end we give a technical lemma.

Notation 5.2.1 *Let $k \in \mathbb{Z}_{\mathbf{b}}$. By 5.1.4, write $k = \sum_{n=0}^{\infty} k_n b_n$, where $|k_n| \leq \frac{q_{n+1}}{2}$. We define, for $j \in \mathbb{N}$,*

$$e_j := \frac{k_{j-1}b_{j-1}}{b_j} + \frac{k_{j-2}b_{j-2}}{b_j}$$

and

$$\varepsilon_j := \sum_{n=0}^{j-3} \frac{k_n b_n}{b_j}.$$

It is clear that $1/b_j k + \mathbb{Z} = (e_j + \varepsilon_j) + \mathbb{Z}$.

Lemma 5.2.2 *Let $k \in \mathbb{Z}_{\mathbf{b}}$ and e_j, ε_j as in 5.2.1. The following assertions hold:*

1. *$|e_j| \leq \frac{3}{4}$.*
2. *If $k_{j-1} \neq 0$, then $|e_j| \geq \frac{1}{2q_j}$.*
3. *$|\varepsilon_j| \leq \frac{b_{j-2}}{2b_j}$.*

Proof:

1. $|e_j| \leq \dfrac{|k_{j-1}|\, b_{j-1}}{b_j} + \dfrac{|k_{j-2}|\, b_{j-2}}{b_j} = \dfrac{|k_{j-1}|\, b_{j-1}}{b_j} + \dfrac{|k_{j-2}|\, b_{j-2}}{b_{j-1}} \cdot \dfrac{b_{j-1}}{b_j} \leq \dfrac{1}{2} + \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{3}{4}.$
2. Suppose $k_{j-1} \neq 0$.
   Then $|e_j| = \left|\dfrac{k_{j-1}b_{j-1}}{b_j} + \dfrac{k_{j-2}b_{j-2}}{b_j}\right| \geq \left|\dfrac{k_{j-1}b_{j-1}}{b_j}\right| - \left|\dfrac{k_{j-2}b_{j-2}}{b_j}\right| = |k_{j-1}| \left|\dfrac{b_{j-1}}{b_j}\right| - \left|\dfrac{k_{j-2}b_{j-2}}{b_{j-1}}\right| \left|\dfrac{b_{j-1}}{b_j}\right| \geq \dfrac{1}{q_j} - \dfrac{1}{2q_j} = \dfrac{1}{2q_j}.$
3. $|\varepsilon_j| \leq \dfrac{1}{b_j} \left|\sum_{n=0}^{j-3} k_n b_n\right| \overset{2.2.1(2)}{\leq} \dfrac{b_{j-2}}{2b_j}.$

QED

We state now the main theorem in this chapter, which will be the cornerstone to prove that $(\mathbb{Q}, \tau_U)$ is not a Mackey group. Therefore, we will have an example of a metrizable, non-complete, **non-precompact**, locally quasi-convex group, which is not Mackey.

Theorem 5.2.3 *Let* $\mathbf{b} \in \mathcal{D}_\infty$. *Then* $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge = (\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge = \mathbb{Z}$.

Proof:

The fact $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge \leq (\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge$ has been considered at the beginning of this section.

We must prove that $(\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge \leq (\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge$. Assume, by contradiction that there exists $k \in \mathbb{Z}_\mathbf{b} \setminus \mathbb{Z}$ such that $\chi_k \in (\mathbb{Z}(\mathbf{b}^\infty), \eta_\mathbf{b})^\wedge$.

First we claim that $e_j \to 0$ in $\mathbb{R}$. Indeed, since χ_k is continuous and $\frac{1}{b_j} + \mathbb{Z} \to 0$ in $\eta_\mathbf{b}$, we have that

$$\chi_k\left(\frac{1}{b_j} + \mathbb{Z}\right) = \frac{k}{b_j} + \mathbb{Z} = e_j + \varepsilon_j + \mathbb{Z} \to 0 + \mathbb{Z}.$$

From $|\varepsilon_j| \to 0$ in $\mathbb{R}$, it follows that $e_j + \mathbb{Z} \to 0 + \mathbb{Z}$. By Lemma 5.2.2(1), we obtain that $e_j \to 0$ in $\mathbb{R}$.

Define $A := \{j \in \mathbb{N} : k_{j-1} \neq 0\}$. The set A is infinite due to the fact that $k \notin \mathbb{Z}$. Since $\chi_k \in (\mathbb{Z}, \eta_\mathbf{b})^\wedge$ there exists $m \in \mathbb{N}$ such that $\chi_k \in W_{\mathbf{b},m}^\triangleright$. We find an element $x \in W_{\mathbf{b},m}$ such that $xk + \mathbb{Z} \notin \mathbb{T}_+$, which will provide the contradiction sought.

Since $\mathbf{b} \in \mathcal{D}_\infty$ and by the above claim, there exists j_0 such that for $j \geq j_0$ the following inequalities hold:

(a) $|e_j| < \dfrac{1}{160m}$

(b) $\left|\dfrac{1}{q_{j-1}}\right| < \dfrac{1}{10}$

For $j \in A \cap \{j \geq j_0\}$, we define

$$\beta_j := \left\lfloor \frac{1}{16m|e_j|} \right\rfloor sign(e_j).$$

Since $k_{j-1} \neq 0$, we can apply Lemma 5.2.2 (2). We have that

$$\beta_j e_j = \left\lfloor \frac{1}{16m|e_j|} \right\rfloor |e_j|,$$

which means that $\beta_j e_j \leq \frac{1}{16m}$. Furthermore, we get

$$\beta_j e_j \geq \left(\frac{1}{16m|e_j|} - 1\right)|e_j| = \frac{1}{16m} - |e_j| > \frac{1}{16m} - \frac{1}{160m} = \frac{9}{160m}.$$

On the other hand, we have that

$$\left|\beta_j\varepsilon_j\right| \overset{5.2.2(3)}{\leq} \frac{1}{16m\left|e_j\right|}\cdot\frac{b_{j-2}}{2b_j} \overset{5.2.2(2)}{\leq} \frac{1}{16m}\cdot\frac{2b_j}{b_{j-1}}\cdot\frac{b_{j-2}}{2b_j} < \frac{1}{160m}.$$

Combining these estimates, we get,

$$\frac{8}{160m} = \frac{9}{160m} - \frac{1}{160m} < \beta_j(e_j+\varepsilon_j) < \frac{1}{16m} + \frac{1}{160m} = \frac{11}{160m}.$$

We choose a subset I of $A \cap \{j \geq j_0\}$ containing $8m$ elements, and define

$$x := \sum_{j\in I}\frac{\beta_j}{b_j}.$$

Since

$$\left|\frac{\beta_j}{q_j}\right| \leq \frac{1}{16m\left|e_j\right|}\cdot\frac{1}{q_j} \overset{5.2.2(2)}{\leq} \frac{1}{16m}\cdot 2q_j\cdot\frac{1}{q_j} = \frac{1}{8m},$$

we have that $x \in W_{\mathbf{b},m}$ (by Proposition 5.1.9). Now

$$\chi_k(x+\mathbb{Z}) = \sum_{j\in I}\frac{\beta_j}{b_j}k + \mathbb{Z} = \sum_{j\in I}\beta_j(e_j+\varepsilon_j) + \mathbb{Z}.$$

From the above estimates, we get that

$$\frac{1}{4} < \frac{2}{5} = \frac{8}{160m}8m < \sum_{j\in I}\beta_j(e_j+\varepsilon_j) < 8m\frac{11}{160m} = \frac{11}{20} < \frac{3}{4},$$

which implies that $\chi_k(x+\mathbb{Z}) \notin \mathbb{T}_+$. Hence $\chi_k \notin (\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{b}})^\wedge$.

QED

Theorem 5.2.4 *Let* $\mathbf{b} \in \mathcal{D}$. *Then* $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)$ *is not a Mackey group.*

Proof:

Since $\mathbf{b} \in \mathcal{D}$, there exists a subsequence $\mathbf{c}$ of $\mathbf{b}$ such that $\mathbb{Z}(\mathbf{b}^\infty) = \mathbb{Z}(\mathbf{c}^\infty)$ and $\mathbf{c} \in \mathcal{D}_\infty$. By Theorem 5.2.3, we have that $\eta_{\mathbf{c}}$ is a topology on $\mathbb{Z}(\mathbf{b}^\infty)$ satisfying $\tau_U < \eta_{\mathbf{c}}$ and $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)^\wedge = (\mathbb{Z}(\mathbf{b}^\infty), \eta_{\mathbf{c}})^\wedge$.

QED

Corollary 5.2.5 *The Prüfer group* $\mathbb{Z}(\mathbf{p}^\infty)$ *endowed with the topology inherited from* $\mathbb{T}$ *is not a Mackey group.*

Corollary 5.2.6 *The group $\mathbb{Q}/\mathbb{Z}$ endowed with the euclidean topology is not a Mackey group.*

Remark 5.2.7 From Theorem 5.2.4 we can assure that there exist $\mathfrak{c}$-many non-isomorphic, non-Mackey topological groups of the form $\mathbb{Z}(\mathbf{b}^\infty)$. In fact, $\mathbb{Z}(\mathbf{b}^\infty)$ and $\mathbb{Z}(\mathbf{c}^\infty)$ are isomorphic if and only if $\mathbb{P}(\mathbf{b}) = \mathbb{P}(\mathbf{c})$.

We turn now our attention to $(\mathbb{Q}, \tau_U)$.

Theorem 5.2.8 *The group $(\mathbb{Q}, \tau_U)$ is not Mackey.*

Proof:

Suppose that $(\mathbb{Q}, \tau_U)$ is Mackey. By Corollary 7.5 in [DNMP14], its quotient $(\mathbb{Q}/\mathbb{Z})$ with the quotient topology should also be a Mackey group. It is clear that the quotient topology coincides with the topology inherited from $\mathbb{T}$. Take $\mathbf{b} = ((n+1)!)$. By Example 2.3.3, the group $\mathbb{Z}(\mathbf{b}^\infty)$ coincides with $\mathbb{Q}/\mathbb{Z}$. By Theorem 5.2.4, we have that $(\mathbb{Z}(\mathbf{b}^\infty), \tau_U)$ is not a Mackey group and consequently $(\mathbb{Q}, \tau_U)$ is neither a Mackey group.

QED

Open question 5.2.9 Does there exist a Mackey topology for the duality $(\mathbb{Q}, \mathbb{R})$?

Chapter 6

The Varopoulos Paradigm.

6.1 Introduction.

In the previous chapters we have studied compatible topologies on some specific groups: the group of the integers and some subgroups of the unit circle of the complex plane. In this Chapter we deal with general abelian groups and inspired by Theorem 3.1.7 we study the following problem:

Problem 6.1.1 *Let G be an abelian group. Let τ be a metrizable locally quasi-convex group topology on G. Is τ Mackey?*

The answer, in general, is no (see Theorems 4.3.24, 5.2.4 and 5.2.8). However in [BTM03] we find examples of metrizable locally quasi-convex group topologies which are Mackey.

So we study the following problem:

Problem 6.1.2 *Characterize the abelian groups G satisfying that every metrizable locally quasi-convex group topology on G is Mackey.*

In [Var64] Varopoulos proved that a metrizable locally precompact topological abelian group carries the finest locally precompact compatible group topology. That is, a metrizable locally precompact group is Mackey in the class of locally precompact groups (or $\mathcal{L}pc$-Mackey). Thus we have defined the following class of groups: A group G as above is said to satisfy the **Varopoulos Paradigm** ($G \in$ **VP**) if for every metrizable locally quasi-convex group topology τ in G, we get that τ is Mackey.

The aim of this Chapter is proving that for an abelian topological group G the topological condition *every metrizable locally quasi-convex group topology on G is Mackey* is equivalent to the algebraic property *G is bounded*.

Indeed, we prove the following theorem:

Theorem 6.1.3 *Let G be an abelian group. The following assertions are equivalent:*

(i) *G is bounded.*

(ii) *Every metrizable locally quasi-convex group topology on G is Mackey.*

The proof of Theorem 6.1.3 is given in the following two sections as well as some partial results of independent interest. For instance, it is proved that locally quasi-convex topologies on groups of finite exponent are linear (in [AG12, Proposition 2.1] it is proved, using different arguments, that a group topology on a bounded group is locally quasi-convex if and only if it is linear).

6.2 Proofs on bounded groups: $(i) \Rightarrow (ii)$ in Theorem 6.1.3.

The aim of this section is proving $(i) \Rightarrow (ii)$ in Theorem 6.1.3. To this end we will prove a deeper result, namely:

Theorem 6.2.1 *Let G be a bounded abelian group and τ a locally quasi-convex topology on G. If τ satisfies one of the following conditions:*

(i) *τ is metrizable;*

(ii) $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$.

Then (G,τ) is Mackey.

Additionally, we get some results which are of independent interest. For example, every non-discrete metrizable group of prime exponent has a proper dense subgroup.

In order to prove Theorem 6.2.1 under hypothesis (ii), we prove first that linear groups (G, τ) satisfying $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$ are precompact (Lemma 6.2.12). Immediately we get this corollary:

Corollary 6.2.2 *Let (G, τ) be a metrizable locally quasi-convex group topology on a bounded group which is not precompact. Then $\left|(G, \tau)^\wedge\right| \geq \mathfrak{c}$.*

Remark 6.2.3 The boundedness condition in Theorem 6.2.1 is necessary. Even in the realm of precompact groups, there are examples of groups satisfying (i) or (ii) which are not Mackey. Indeed, a precompact group of weight $< \mathfrak{c}$ need not be a Mackey group. For instance, the linear topologies on $\mathbb{Z}$ are metrizable precompact and never Mackey (see Theorem 4.3.24). A milder condition (torsion instead of bounded) is not sufficient either. The groups $\mathbb{Z}(\mathbf{p}^\infty)$ are metrizable precompact and torsion, but they are not Mackey groups (see Corollary 5.2.5).

Lemma 6.2.4 *If a metrizable topological abelian group (G, τ) has a closed subgroup H with $G/H \in \mathfrak{Den}$, then $G \in \mathfrak{Den}$ as well.*

Proof:

Let $q : G \to G/H$ be the quotient map and let D be a proper dense subgroup of G/H. Then $D' = q^{-1}(D)$ is a proper subgroup of G, so it suffices to check that D' is dense. Let N be the closure of D'. Then N is a closed subgroup of G containing $H = \ker q$. By [DPS90, Lemma 4.3.2], $q(D')$ is a closed subgroup of G/H. Since q is surjective, $q(D') = D$. As D was assumed to be dense, we conclude that $D = G/H$, a contradiction.

QED

Lemma 6.2.5 *Let G be a topological abelian group of prime exponent whose topology is not finer than the Bohr topology of the discrete group G_d. Then G belongs to $\mathfrak{Den}$.*

Proof:

According to Remark 3.2.3, the Bohr topology of the discrete group G_d coincides with the pro-finite topology of G. Hence, our hypothesis implies that there exists a subgroup of finite index H that is not open in G. Since H is of finite index, this implies that H is not closed either. Hence, H is a proper dense subgroup of its closure N in G. Since G has exponent p, there is a splitting $G = N \times F$, where F is a finite subgroup of G. Since N is a closed subgroup of finite index in G, it is also open. Then G has the product topology, being F discrete. Thus $H \times F$ is a proper dense subgroup of G.

QED

Fact 6.2.6 *[Dou90, Gli62]*
The Bohr topology of a discrete group has no nontrivial null sequences.

This fact combined with Lemma 6.2.5 gives the following immediate corollary:

Corollary 6.2.7 *Every non-discrete metrizable group of prime exponent belongs to $\mathcal{D}en$.*

Proof:
Let (G, τ) be a metrizable group of prime exponent which is not discrete. Denote by $\tau^{\#}$ the Bohr topology of the discrete group G_d. Since τ is metrizable, it has nontrivial convergent sequences that cannot converge in $\tau^{\#}$ due to Fact 6.2.6. Therefore $\tau^{\#} \not\subseteq \tau$. Now apply Lemma 6.2.5

QED

Lemma 6.2.8 *If (G, τ) is a metrizable bounded topological abelian p-group, then $G \notin \mathcal{D}en$ if and only if the subgroup pG is open.*

Proof:
Suppose that the subgroup pG is open and assume that D is a dense subgroup of G. We have to prove that $D = G$. The density of D implies $D + pG = G$, as pG is a neighborhood of zero. Since it is also a subgroup, by an induction argument on n, we obtain $D + p^nG = G$ for every $n \in \mathbb{N}$. Indeed, $D + pG = G$ implies

$pD + p^2G = pG$. This implies $D + p^2G = D + pD + p^2G = D + pG = G$. Now picking an $n \in \mathbb{N}$ with $p^nG = 0$, we get $D = G$.

Conversely, assume that $G \notin \mathcal{D}en$, and the subgroup pG is not open. Since G has no proper dense subgroups, every subgroup of index p of G is closed. Now, the intersection of these subgroups is precisely pG, so we deduce that pG is a closed subgroup. Let $q : G \to G/pG$ be the canonical map. As G/pG is a non-discrete metrizable group, by Corollary 6.2.7, $G/pG \in \mathcal{D}en$. By Lemma 6.2.4, also $G \in \mathcal{D}en$, a contradiction.

QED

Lemma 6.2.9 *If $\mathcal{O}(G,\tau) = \mathcal{C}_*(G,\tau)$ for a linearly topologized abelian group, then (G,τ) is $\mathcal{L}in$-Mackey.*

Proof:

For every linear compatible topology τ' one has

$$\mathcal{O}(G,\tau') \subseteq \mathcal{C}_*(G,\tau') = \mathcal{C}_*(G,\tau) = \mathcal{O}(G,\tau)$$

(the first equality follows from linearity as τ and τ' have the same closed subgroups by Proposition 3.2.1 and Lemma 3.2.2). In other words, for every linear compatible topology τ' one has $\mathcal{O}(G,\tau') \subseteq \mathcal{O}(G,\tau)$. Since these are linear topologies, this proves that $\tau' \subseteq \tau$. Therefore, (G,τ) is $\mathcal{L}in$-Mackey.

QED

Proposition 6.2.10 *Let (G,τ) be a locally quasi-convex bounded group. Then τ is linear.*

Proof:

Suppose that $mG = 0$. Then $G^* = \mathrm{Hom}(G,\mathbb{T})$ satisfies $mG^* = 0$ as well, so we can assume without loss of generality that $G^* = \mathrm{Hom}(G,\mathbb{Z}_m)$.

Since τ is locally quasi-convex, it is the topology of uniform convergence on a family $\mathcal{K} \subset \mathcal{P}(G^*)$ (see for example [DNMP14]). For $K \in \mathcal{K}$ and $n \geq m$ let

$$V_{n,K} := \left\{ g \in G : x(g) \in \left[-\frac{1}{4n}, \frac{1}{4n}\right] + \mathbb{Z} \text{ for all } x \in K \right\}.$$

Then the neighborhoods $\{V_{n,K} : K \in \mathcal{K}, n \geq m\}$ form a neighborhood basis of τ. Since $x(g) \in \mathbb{Z}_m$ for all $g \in G, x \in G^*$, it follows that $V_{n,K}$ is the annihilator of K. Hence, the neighborhood basis consists in subgroups and τ is linear.

QED

A stronger version of Proposition 6.2.10 can be found also in [AG12, Proposition 2.1]. We include here a different proof.

Lemma 6.2.11 *Let (G, τ) be a linear topological group. If $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$, then τ is precompact.*

Proof:

Suppose that τ is not precompact. Then there exists an open subgroup $U \leq G$ satisfying that $|G/U| \geq \omega$. Since U is open, the quotient G/U is discrete. Hence $(G/U)^\wedge$ is compact and infinite, therefore $\left|(G/U)^\wedge\right| \geq \mathfrak{c}$. Since $\left|G^\wedge\right| \geq \left|(G/U)^\wedge\right|$ the inequality $\left|(G, \tau)^\wedge\right| \geq \mathfrak{c}$ follows.

QED

The above lemma cannot be inverted. Indeed, the direct sum of $\mathfrak{c}$-many copies of $\mathbb{Z}_m$ endowed with the topology inherited from the product $\mathbb{Z}_m^{\mathbb{R}}$ is precompact and linear, but its dual group has cardinality $\mathfrak{c}$.

Lemma 6.2.12 *Let (G, τ) be a locally quasi-convex topological group satisfying:*

(i) G is bounded.

(ii) $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$.

Then (G, τ) is precompact.

Proof:

By (i) and Proposition 6.2.10, τ is linear. Since $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$ and τ is linear, by Lemma 6.2.11, τ is precompact.

QED

Remark 6.2.13 Both conditions (i) and (ii) in Lemma 6.2.12 are necessary. Indeed, consider a D-sequence $\mathbf{b} \in \mathcal{D}_\infty$. The topology$\tau_{\mathbf{b}}$ is locally quasi-convex and satisfies that $\left|(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge\right| < \mathfrak{c}$ but it is not precompact. On the other hand, the group $G = \bigoplus_\omega \mathbb{Z}_4$ endowed with the discrete topology is bounded, locally quasi-convex with $\left|G^\wedge\right| = \mathfrak{c}$, but it is not precompact.

Next, we want to show that a metrizable locally quasi-convex bounded group is Mackey. For the proof we need the following Proposition which is interesting on its own:

Proposition 6.2.14 *Let (G, τ) be a torsion, metrizable non-discrete abelian group. Then there exists a homomorphism $f : G \to \mathbb{T}$ which is not continuous.*

Proof: Since G is metrizable and τ is not the discrete topology, we can choose a null sequence $(a_n)_{n\in\mathbb{N}}$ which satisfies $a_n \neq a_m$ whenever $n \neq m$ and $a_n \neq 0$ for all $n \in \mathbb{N}$. Since every finitely generated torsion group is finite we may additionally assume that a_{n+1} does not belong to the subgroup generated by $a_1, \ldots, a_n$ for every $n \in \mathbb{N}$.

We are constructing inductively a homomorphism $f_n : \langle a_1, \ldots, a_n\rangle \to \mathbb{T}$ in the following way:

Let $f_1 : \langle a_1\rangle \to \mathbb{T}$ be a homomorphism which satisfies $f_1(a_1) \notin \mathbb{T}_+$. This is possible, since $a_1 \neq 0$.

Suppose that $f_n : \langle a_1, \ldots, a_n\rangle \to \mathbb{T}$ is a homomorphism, which satisfies $f_n(a_k) \notin \mathbb{T}_+$ for all $1 \leq k \leq n$.
If $\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle = \{0\}$, we can define a homomorphism

$$f_{n+1} : \langle a_1, \ldots, a_{n+1}\rangle \to \mathbb{T}$$

which extends f_n and satisfies $f_{n+1}(a_{n+1}) \notin \mathbb{T}_+$.
If $\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle \neq \{0\}$, we choose the minimal natural number $k \in \mathbb{N}$ such that $ka_{n+1} = k_1a_1 + \ldots + k_na_n$ for suitable $k_1, \ldots, k_n \in \mathbb{Z}$. By assumption, $2 \leq k < \mathrm{ord}(a_{n+1})$. We define $f_{n+1}(a_{n+1})$ such that $f_{n+1}(a_{n+1}) \notin \mathbb{T}_+$ and $f_{n+1}(ka_{n+1}) = f_n(k_1a_1 + \ldots + k_na_n)$ and obtain a homomorphism $f_{n+1} : \langle a_1, \ldots, a_{n+1}\rangle \to \mathbb{T}$ which extends f_n.

In this way, we obtain a homomorphism $\widetilde{f} : \langle\{a_n : \ n \in \mathbb{N}\}\rangle \to \mathbb{T}$ which can be extended to a homomorphism $f : G \to \mathbb{T}$. By construction, f satisfies:

$$\forall n \in \mathbb{N} \ f(a_n) \notin \mathbb{T}_+.$$

So f is not continuous. QED

Proof of Theorem 6.2.1:

Let τ be a locally quasi-convex topology on a bounded group G. We have to prove that (G,τ) is a Mackey group provided that: (i) τ is metrizable; or (ii) $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$.

(i) Suppose that τ is metrizable. Let τ' be another locally quasi-convex group topology on G compatible with τ. According to 6.2.10, τ and τ' are linear topologies. We have to show that $\tau' \subseteq \tau$. So fix a τ'-open subgroup H of G and denote by $q : G \to G/H$ the canonical projection. As H is open in (G,τ'), it is also dually closed in (G,τ') and according to 3.2.1 also in (G,τ). Let $f : G/H \to \mathbb{T}$ be an arbitrary homomorphism. Then $f \circ q \in (G,\tau')^\wedge = (G,\tau)^\wedge$. Let $\hat{\tau}$ denote the Hausdorff quotient topology on G/H induced by τ. Of course, $(G/H,\hat{\tau})$ is a metrizable bounded group. Since every homomorphism $f : (G/H,\hat{\tau}) \to \mathbb{T}$ is continuous, we obtain that G/H is discrete by the above Proposition. This implies that H is τ-open. Hence, $\tau' \subseteq \tau$.

(ii) Now, suppose that $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$. Let τ' be another locally quasi-convex group topology on G, which is compatible with τ. By Lemma 6.2.12, τ and τ' are precompact topologies. Since τ and τ' are compatible, they must coincide $\tau = \tau'$. Therefore, (G,τ) is Mackey under the hypothesis (*ii*).

QED

In Section 6.4 we prove some corollaries of Theorem 6.2.1.

Finally, we answer Question 3.3.3 for the case of bounded groups.

Lemma 6.2.15 *Let (G,τ) be a bounded locally quasi-convex topological group. Let $H \leq G$ be a τ-open subgroup. Let τ' be a locally quasi-convex topology on G, which is compatible with τ. Then $\tau_{|H}$ and $\tau'_{|H}$ are compatible.*

Proof:

Note first that τ, τ', $\tau_{|H}$ and $\tau'_{|H}$ are linear topologies (Proposition 6.2.10). Since H is τ-open, it is τ-closed. By Theorem 3.2.5, H is τ'-closed as well. We shall prove that the $\tau_{|H}$-closed subgroups coincide with the $\tau'_{|H}$-closed ones. Then, by Theorem 3.2.5, both topologies are compatible.

Let $N \leq H$ be a $\tau_{|H}$-closed subgroup. In particular, N is τ-closed. Since τ and τ' are compatible, by Theorem 3.2.5, N is τ'-closed. Since H is τ'-closed, N is $\tau'_{|H}$-closed.

Now, pick $N \leq H$ a $\tau'_{|H}$-closed subgroup. Since H is τ'-closed, N is τ'-closed. By Theorem 3.2.5, N is τ-closed. Since H is τ-open, N is $\tau_{|H}$-closed.

QED

Proposition 6.2.16 *Let (G,τ) be a bounded topological abelian group and $H \leq G$ an open subgroup. If $(H,\tau_{|H})$ is Mackey, then (G,τ) is Mackey.*

Proof:

Let τ' be a locally quasi-convex group topology on G which is compatible with τ. By Lemma 6.2.15, $\tau_{|H}$ and $\tau'_{|H}$ are compatible. Since $\tau_{|H}$ is Mackey, we have that $\tau'_{|H} \leq \tau_{|H}$. Being H open, it follows that $\tau' \leq \tau$. On the other hand, being $\tau_{|H}$ locally quasi-convex and H an open subgroup, then τ is locally quasi-convex and, consequently, the Mackey topology on G.

QED

6.3 Proofs on unbounded groups: $(ii) \Rightarrow (i)$ in Theorem 6.1.3.

The target of this section is proving Theorem 6.3.1, which is $(ii) \Rightarrow (i)$ in Theorem 6.1.3 as well.

Theorem 6.3.1 *If G is an unbounded group, then $G \notin$ **VP**, i.e., there exists a metrizable locally quasi-convex topology on G which is not Mackey.*

The proof of Theorem 6.3.1 is based in the following facts:

(i) The groups $\mathbb{Z}, \mathbb{Z}(\mathbf{p}^\infty)$ and $\mathbb{Q}$ have a metrizable locally quasi-convex non-Mackey topology (Theorem 4.3.24, Corollary 5.2.5 and Theorem 5.2.8, respectively).

(ii) Every unbounded group G has a subgroup H which is isomorphic to one of the following: ([Fuc70])

- $H \cong \mathbb{Z}$.
- $H \cong \mathbb{Z}(\mathbf{p}^\infty)$ where p is a prime number.
- $H \cong \mathbb{Q}$.

– $H \cong \bigoplus_{n=1}^{\infty} \mathbb{Z}_{m_n}$, where (m_n) is a sequence with $\lim m_n = \infty$.

(iii) For any sequence (m_n) with $m_n \to \infty$ the product topology of the group $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{m_n}$ is metrizable and non-Mackey (Proposition 6.3.2), as a consequence $\bigoplus_{n=1}^{\infty} \mathbb{Z}_{m_n} \notin \mathbf{VP}$.

(iv) Since an unbounded group G contains one of the above groups, by means of Lemma 3.3.2, we will have that G admits a metrizable non-Mackey topology.

We now we prove (iii)

Proposition 6.3.2 *Let (m_n) be a sequence of natural numbers with $m_n \to \infty$ and $G = \bigoplus \mathbb{Z}_{m_n}$. Then the metrizable locally quasi-convex topology τ on G inherited from the product $\prod_{\omega} \mathbb{Z}_{m_n}$ is not Mackey.*

Proof:

The idea of the proof is to fix appropriately a null sequence $\mathbf{c} \subset G^\wedge = \bigoplus_{\omega} \mathbb{Z}_{m_n}^\wedge$. Define $\tau_{\mathbf{c}}$ as the topology of uniform convergence on the range of $\mathbf{c}$, $\tau_{\mathbf{c}}$ is non-precompact, and strictly finer than τ, but still compatible. Hence, τ is not Mackey.

Observe that $G^* = \prod \mathbb{Z}_{m_n}^\wedge$ and $G^\wedge = \bigoplus_{\omega} \mathbb{Z}_{m_n}^\wedge$. Clearly, $\mathbb{Z}_{m_n}^\wedge \cong \mathbb{Z}_{m_n}$, and for convenience we make the following identifications:

$$\mathbb{Z}_{m_n}^\wedge = \left(-\frac{m_n}{2}, \frac{m_n}{2}\right] \cap \mathbb{Z}$$

and

$$\mathbb{Z}_{m_n} = \left\{\frac{k}{m_n} : k \in \left(-\frac{m_n}{2}, \frac{m_n}{2}\right] \cap \mathbb{Z}\right\}. \tag{6.1}$$

We can assume without loss of generality that (m_n) is non-decreasing, since re-arranging the sequence (m_n) leads to isomorphic groups.

For $x = (x_1, x_2, \dots) \in G$ and $\chi = (\chi_1, \chi_2, \dots) \in G^\wedge$, we have $\chi(x) = \sum_{n=1}^{\infty} \chi_n x_n + \mathbb{Z}$.

For $n \in \mathbb{N}$, let $(e_n) = (0, \dots, 0, 1, 0, \dots) \in G^\wedge$, with 1 in n-th position. Define $\mathbf{c} := \{\pm e_n : n \in \mathbb{N}\} \cup \{0\} \subset G^\wedge$. Identifying $\mathbb{T}$ with $(-\frac{1}{2}, \frac{1}{2}]$, we let $\mathbb{T}_m = \{x \in \mathbb{T} : |x_n| \leq \frac{1}{4m}\}$ and

$$V_m := \{x \in G : e_n(x) \in \mathbb{T}_m \ \forall n \in \mathbb{N}\} = \left\{x \in G : |x_n| \leq \frac{1}{4m} \ \forall n \in \mathbb{N}\right\}. \tag{6.2}$$

The family $\{V_m : m \in \mathbb{N}\}$ is a neighborhood basis of 0 for the topology $\tau_{\mathbf{c}}$ of uniform convergence on $\mathbf{c}$.

Writing the elements $x = (x_n)$ of G, with $x_n = \frac{k_n}{m_n}$ (as in (6.1)), one has $V_m = \left\{x \in G : |k_n| \leq \frac{m_n}{4m} \text{ for all } n\right\}$.

To conclude the proof of the proposition we need the following lemma:

Lemma 6.3.3 $(G, \tau_{\mathbf{c}})^\wedge = \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$.

Proof:

The fact $\bigoplus_\omega \mathbb{Z}_{m_n}^\wedge \leq (G, \tau_{\mathbf{c}})^\wedge$ is straightforward. Indeed, let $\chi \in \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$. Then there exists $n_1 \in \mathbb{N}$ such that $\chi_n = 0$ if $n \geq n_1$. Let $m = m_{n_1}$, hence $x_1 = \cdots = x_{n_1-1} = 0$ for all $x \in V_m$. Then $\chi(V_m) = \{0_{\mathbb{T}}\}$.

Let $\chi \in \prod_\omega \mathbb{Z}_{m_n}^\wedge \setminus \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$. Then the set $A := \{n \in \mathbb{N} : \chi_n \neq 0\}$ is infinite. Our aim is to prove that $\chi \notin (G, \tau_{\mathbf{c}})^\wedge$. We argue by contradiction, assuming that $\chi \in (G, \tau_{\mathbf{c}})^\wedge$. Then $\chi \in V_m^\triangleright := \{\phi \in (G, \tau_{\mathbf{c}})^\wedge : \phi(V_m) \subset \mathbb{T}_+\}$ for some $m \in \mathbb{N}$. Hence, we must find $x \in V_m$ with $\chi(x) \notin \mathbb{T}_+$.

To this end, we pick

$$n_0 := \min\left\{n \in \mathbb{N} : \frac{m_n}{4m} \geq \frac{5}{2}\right\}. \tag{6.3}$$

Equivalently, $\frac{1}{m_n} \leq \frac{1}{10m}$ if $n \geq n_0$.

We describe next in (a) some elements of V_m, which will be suitable for our purposes and in (b) a condition that holds for the components of χ.

Claim:

(a) *If $x_n = 0$ for $n < n_0$ and $x_n \in \{0, \pm\frac{1}{m_n}\}$ for $n \geq n_0$, then $x = (x_n) \in V_m$.*

(b) $\frac{|\chi_n|}{m_n} \leq \frac{1}{4}$ *for all $n_0 \leq n \in A$.*

Proof:

(a)Indeed, since $\frac{1}{m_n} \leq \frac{1}{m_{n_0}} \leq \frac{1}{10m}$ if $n \geq n_0$, from (6.2) it follows that $x \in V_m$.

(b) By item (a), we have that $x = (0, \dots, \overset{(n)}{\frac{1}{m_n}}, \dots) \in V_m$. Since $\chi \in V_m^{\triangleright}$, it follows that $|\chi(x)| = \frac{|\chi_n|}{m_n} \leq \frac{1}{4}$. QED

To continue the proof of Lemma 6.3.3, define, for each $k \in \left\{0, 1, \dots, \left\lfloor \frac{m_{n_0}}{2} \right\rfloor\right\}$, the set

$$A_k := \left\{ n \in A : \frac{k}{m_{n_0}} < \frac{|\chi_n|}{m_n} \leq \frac{k+1}{m_{n_0}} \right\}. \tag{6.4}$$

Then $\left\{A_k : 0 \leq k \leq \frac{m_{n_0}}{2}\right\}$ is a finite partition of A and at least one A_k is infinite, as A is infinite. Depending in k, two cases will be considered, building in each case an element $x \in V_m$ with $\chi(x) \notin \mathbb{T}_+$.

<u>Case 1: $k \geq 1$.</u>

Since A_k is infinite, one can find $n_1 < n_2 < \cdots < n_{m_{n_0}}$ in A_k. Define

$$x_n := \begin{cases} \frac{1}{m_{n_i}} \operatorname{sign}(\chi_{n_i}) & \text{when } n = n_i \\ 0 & \text{otherwise} \end{cases}$$

By the definition of A_k, for $1 \leq i \leq m_{n_0}$ we have $\frac{k}{m_{n_0}} < \chi_{n_i} x_{n_i}$. Thus,

$$\sum_{i=1}^{m_{n_0}} \chi_{n_i} x_{n_i} > k \geq 1.$$

Taking into account that each summand in the above sum satisfies $0 < \chi_n x_n \leq \frac{1}{4}$ by Claim 6.3(b), we conclude that there exists a natural $N \leq m_{n_0}$ such that

$$\frac{1}{4} < \sum_{i=1}^{N} \chi_{n_i} x_{n_i} < \frac{3}{4}.$$

Define

$$x'_n := \begin{cases} \frac{1}{m_{n_i}} \operatorname{sign}(\chi_{n_i}) & \text{when } n = n_i \text{ for some } 1 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}$$

By Claim 6.3(a), $x' \in V_m$ and by construction $\chi(x') \notin \mathbb{T}_+$.

<u>Case 2: $k = 0$.</u>

That is: A_0 is infinite. Choose $J \subset A_0$ with $|J| = 2m$. For $n \in J$, define

$$j_n := \frac{\left\lfloor \frac{m_n}{4m|\chi_n|} \right\rfloor \operatorname{sign}(\chi_n)}{m_n}.$$

First, note that for all $n \in J$ one has $j_n \neq 0$, as $\frac{m_n}{4m|\chi_n|} \geq \frac{m_{n_0}}{4m} > 2$, where the last inequality is due to (6.3). On the other hand,

$$|j_n| \leq \left| \frac{\frac{m_n}{4m|\chi_n|}}{m_n} \right| = \frac{1}{4m\,|\chi_n|}. \tag{6.5}$$

This gives $|j_n| \leq \frac{1}{4m}$, as $|\chi_n| \geq 1$. Define

$$x_n := \begin{cases} j_n & \text{when } n \in J \\ 0 & \text{otherwise} \end{cases}$$

Then, $x = (x_n) \in V_m$, by Claim 6.3 (a).

In addition,

$$|x_n| \geq \frac{\frac{m_n}{4m|\chi_n|} - 1}{m_n} = \frac{1}{4m\,|\chi_n|} - \frac{1}{m_n}. \tag{6.6}$$

Combining (6.5) and (6.6), we get, for $n \in J$:

$$\frac{3}{20m} = \frac{1}{4m} - \frac{1}{10m} \leq \frac{1}{4m} - \frac{1}{m_{n_0}} \overset{n \in A_0}{\leq} \frac{1}{4m} - \frac{|\chi_n|}{m_n} \overset{(6.6)}{\leq} \chi_n x_n \overset{(6.5)}{\leq} \frac{1}{4m}.$$

Applying these inequalities to $\chi(x) = \sum_{n \in J} \chi_n x_n$ gives

$$\frac{1}{4} < \frac{3}{10} = \frac{3}{20m}\,|J| \leq \sum_{n \in J} \chi_n x_n = \chi(x) \leq \frac{1}{4m}\,|J| = \frac{1}{2}.$$

This implies that $\chi(x) \notin \mathbb{T}_+$, a contradiction in view of $x \in V_m$.

In both cases this proves that $\chi \notin (G, \tau_{\mathbf{c}})^\wedge$. QED

This ends up the proof of Proposition 6.3.2. QED

Proof of Theorem 6.3.1:

We have to prove that for any unbounded abelian group G, there exists a metrizable locally quasi-convex group topology which is not Mackey. We divide the proof in three cases, namely:

(a) G is non reduced.

(b) G is non torsion.

(c) G is reduced and torsion.

(a) If G is non reduced, let $d(G)$ be its maximal divisible subgroup. Then $d(G) \cong \bigoplus_{\alpha} \mathbb{Q} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}(p_i^{\infty})^{(\beta_i)}$ [Fuc70, Theorem 21.1].

If $\alpha \neq 0$, then we there exists $H \leq G$, where $H \cong \mathbb{Q}$. Since, there exists a metrizable non-Mackey topology on $\mathbb{Q}$ (Theorem 5.2.8), applying Lemma 3.3.2, there is a metrizable topology on G which is not Mackey.

If $\alpha = 0$, then there exists p_i satisfying $\mathbb{Z}(p_i^{\infty}) \leq d(G) \leq G$. Since there exists a metrizable, non-Mackey topology on $\mathbb{Z}(p_i^{\infty})$ (Corollary 5.2.5), there exists (by Lemma 3.3.2) a metrizable topology on G which is not Mackey.

(b) If G is non torsion, there exists $x \in G$, such that $\langle x \rangle \cong \mathbb{Z}$. Equip $\mathbb{Z}$ with the **b**-adic topology, which is metrizable but not Mackey (Theorem 4.3.24). Applying Lemma 3.3.2, we get the result.

(c) We can write G as the direct sum of its primary components, that is $G = \bigoplus_p G_p$. Consider two cases:

$(c.1)$ $G_p \neq 0$ for infinitely many prime numbers p. Then, for some infinite set Π of prime numbers,we have $\bigoplus_{p \in \Pi} \mathbb{Z}_p \hookrightarrow G$. By Lemma 6.3.2, there exists a metrizable locally quasi-convex group topology on $\bigoplus_{p \in \Pi} \mathbb{Z}_p$ which is not Mackey. Hence, by Lemma 3.3.2, there exists a metrizable locally quasi-convex group topology which is not Mackey on G.

$(c.2)$ $G_p \neq 0$ for finitely many p. Then, one of them is unbounded, say G_p. Let $B := \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}^{(\alpha_n)}$ be a basic subgroup of G_p, i.e., B is pure and G_p/B is divisible [Fuc70]. Let us recall the fact that purity of B means that $p^n B = p^n G \cap B$ for every $n \in \mathbb{N}$.

We aim to prove that B is unbounded. Assume for contradiction that B is bounded, so $p^n B = 0$ for some $n \in \mathbb{N}$. Then $\{0\} = p^n B = p^n G \cap B$. By divisibility of G_p/B, one has $(p^n G_p + B)/B = p^n(G_p/B) = G_p/B$; which means that $p^n G_p + B = G_p$. Since $p^n G_p \cap B = 0$, this is equivalent to $G_p = p^n G_p \oplus B$. Then $p^n G_p \cong G_p/B$

is a divisible subgroup of the reduced group G_p. Hence $p^n G_p \cong G_p/B = 0$ and $G_p = B$ is bounded, a contradiction.

Since B is unbounded there exists a sequence $(n_i) \subseteq \mathbb{N}$ with $\alpha_{n_i} \neq 0$. Consequently, $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{p^{n_i}} \leq B \leq G$. By Lemma 6.3.2, there exists a metrizable locally quasi-convex group topology on $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{p^{n_i}}$ which is not Mackey. Hence, by Lemma 3.3.2, there exists a metrizable locally quasi-convex group topology which is not Mackey on G.

QED

Remark 6.3.4 Since we have found families of metrizable non-Mackey topologies on $\mathbb{Z}$, we can build families of metrizable non-Mackey topologies on any unbounded non-torsion group.

6.4 Some corollaries of Theorem 6.2.1.

From Theorem 6.2.1 (and some extra results from [ADMP15]), we compute the cardinality of the family $C(G, \tau)$ for a metrizable bounded group, as expressed next:

Corollary 6.4.1 *Let (G, τ) be a metrizable, locally quasi-convex and bounded group. Then:*

1. $|C(G, \tau)| = 1$ *if and only if τ is precompact. In this case, (G, τ) is precompact and Mackey.*

2. *Otherwise $|C(G, \tau)| \geq 2^{\mathfrak{c}}$. In this case, τ is the only metrizable topology in $C(G, \tau)$.*

Proof:

If $|C(G, \tau)| = 1$ it is clear that τ is precompact (for any dual pair $(G, G^\wedge)$ there exists one precompact topology). Conversely, if τ is precompact, and Mackey (by Theorem 6.2.1), it is clear that $|C(G, \tau)| = 1$.

The proof of 2. relies on the fact that τ is linear (Proposition 6.2.10). Assume that (G, τ) is not precompact. Then, there exists a τ-open subgroup U such that G/U is infinite. Since U is open, G/U is discrete. By Theorem 3.10 in

[ADMP15], the family $C((G/U)_d)$ can be embedded in $C(G,\tau)$. We denote by $\mathcal{F}il_{G/U}$ the set of all filters on G/U and we recall that $\left|\mathcal{F}il_{G/U}\right| = 2^{2^{|G/U|}}$. By Theorem 4.5 in [ADMP15], we have that $|C((G/U)_d)| = \left|\mathcal{F}il_{G/U}\right|$, since G/U has infinite rank (being an infinite bounded group). So

$$|C(G,\tau)| \geq |C((G/U)_d)| = 2^{2^{|G/U|}} \geq 2^{2^{\omega}} = 2^{\mathfrak{c}}$$

holds. It is also clear from Theorem 6.2.1, that τ is the only metrizable topology in $C(G,\tau)$.

QED

Corollary 6.4.1 solves Conjecture 7.6 in [ADMP15] for bounded groups.

Item (*ii*) in Theorem 6.2.1, also allows us to compute the cardinality of the family $C(G,\tau)$ if (G,τ) is bounded and $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$, as follows:

Corollary 6.4.2 *Let* (G,τ) *be a locally quasi-convex topology on a bounded group such that* $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$. *Then* $|C(G,\tau)| = 1$.

The corollary will automatically follow from the fact that locally quasi-convex topologies on bounded groups satisfying $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$ are precompact.

In Theorem 6.2.1, there is a connection between metrizability and Mackeyness on precompact topologies. However, there are bounded Mackey groups which are not metrizable:

Corollary 6.4.3 *Every bounded precompact topological group* (G,τ) *satisfying that* $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$ *is Mackey.*

Corollary 6.4.3 is also obtained in [BTM03, Example 4], while corollaries 6.4.2 and 6.4.3 can be found also in [Leo08, Prop.8.57]. In all these cases the arguments are different from ours.

Lemma 6.4.4 *Let G be a countable group. Under Martin Axiom, there exist* $2^{\mathfrak{c}}$ *precompact topologies of weight* $\mathfrak{c}$ *on G.*

Proof:

Indeed, there exist $2^{2^{|G|}} = 2^{\mathfrak{c}}$ precompact -pairwise different- topologies on G. We compute the cardinality of precompact topologies with weight $< \mathfrak{c}$. Consider a cardinal $\kappa < \mathfrak{c}$. Since every precompact topology of weight κ is determined exactly by a subgroup of G^* of cardinality κ, there exist $|G^*|^\kappa$ precompact topologies of weight κ. Now, $|G^*| = \mathfrak{c}$, hence $|G^*|^\kappa = \mathfrak{c}^\kappa = (2^\omega)^\kappa = 2^\kappa \overset{M.A.}{=} \mathfrak{c}$. Consequently, there exist $2^\mathfrak{c}$ precompact topologies of weight $\mathfrak{c}$ on G.

QED

The following question arises:

Question 6.4.5 Let G be a countable group. How many of the precompact topologies of weight $\mathfrak{c}$ on G are Mackey? How many of them fail to be Mackey?

Although we don't know the answer to this question, we give the following partial answer:

Corollary 6.4.6 *Let G be a bounded countably infinite group. Then, there exist $\mathfrak{c}$ precompact topologies of weight $\mathfrak{c}$ which are not Mackey.*

Proof:

Since G is bounded and countably infinite, there exists a natural number L such that $G = \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$, where $m_n < L$ for all $n \in \mathbb{N}$. Hence, we can write $G = G_1 \times G_2$ where both G_1 and G_2 are infinite. Since G_1 is infinite, there exists a family of $\mathfrak{c}$ precompact topologies $\{\tau_i\}$ on G_1 whose weight is $< \mathfrak{c}$. Consider the discrete topology d in G_2 and in G the product topology, namely $\mathcal{T}_i = \tau_i \times d$. The topology $\mathcal{T}_i$ is metrizable (hence, by Theorem 6.2.1, Mackey) but not precompact (for each i). In addition, $\left|(G, \mathcal{T}_i)^\wedge\right| = \mathfrak{c}$. Hence the topologies $\mathcal{T}_i^+$ are precompact of weight $\mathfrak{c}$ and not Mackey.

QED

Question 6.4.7 Does there exist a precompact bounded group of weight $\mathfrak{c}$ which is Mackey?

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