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Homogenization and Shape Differentiation of Quasilinear Elliptic Equations

Homogeneización y diferenciación de formas de ecuaciones elípticas cuasilineales

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David Gómez Castro

Director

Jesús Ildefonso Díaz Díaz

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and Shape Differentiation
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David Gómez Castro

Director: Prof. Jesús Ildefonso Díaz Díaz

Dpto. de Matemática Aplicada &
Instituto de Matemática Interdisciplinar
Facultad de Matemáticas
Universidad Complutense de Madrid

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Quisiera dedicar esta tesis a mi abuelo, Ángel Castro,
con el que tantos momentos compartí de pequeño.

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The next paragraphs will oscillate between Spanish and English, for the convenience of the different people involved.

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Por último, a Cheri. Por todo.

Abstract

This thesis has been divided into two parts of different proportions. The first part is the main work of the candidate. It deals with the optimization of chemical reactors, and the study of the effectiveness, as it will explained in the next paragraphs. The second part is the result of the visit of the candidate to Prof. Häim Brezis at the Israel Institute of Technology (Technion) in Haifa, Israel. It deals with a particular question about optimal basis in L^2 of relevance in Image Proccesing, which was raised by Prof. Brezis.

The first part of the thesis, which deals with chemical reactors, has been divided into four chapters. It studies well-established models which have direct applications in Chemical Engineering, and the notion of "effectiveness of a chemical reactor". One of the main difficulties we faced is the fact that, due to the Chemical Engineering applications, we were interested in dealing with root-type nonlinearities.

The first chapter focuses on modeling: obtaining a macroscopic (homogeneous) model from a prescribed microscopic behaviour. This method is known as *homogenization*. The idea is to consider periodically repeated particles of a fixed shape G_0, at a distance ε, which have been rescaled by a factor a_ε. This factor is usually of the form $a_\varepsilon = C_0 \varepsilon^\alpha$, where $\alpha \geq 1$ and C_0 is a positive constant. The aim is to study the different behaviours as $\varepsilon \to 0$, when the particles are no longer considered. It was known that depending of this factor there are usually different behaviours as $\varepsilon \to 0$. First, the case of *big particles* and *small particles* are treated differently. The latter, which have been the main focus of this chapter, are divided into subcritical, critical and supercritical holes. Roughly speaking, there is a critical value α^* such that the behaviours $\alpha = 1$ (big particles), $1 < \alpha < \alpha^*$(subcritical particles), $\alpha = \alpha^*$ (critical particles) and $\alpha > \alpha^*$ (supercritical particles) are significantly different.

The main focus of the thesis has been in the cases $\alpha > 1$, although some new results for the case $\alpha = 1$ have been obtained (see [DGCT15; DGCT16]). In the subcritical cases we have significantly improved the regularity of the nonlinearities that are allowed, by applying uniform approximation arguments (see [DGCPS17d]). We also proved that, when the diffusion depends on the gradient (p-Laplacian type) with p greater than the spatial

dimension, no critical scales exists (see [DGCPS17b]). Also, this thesis includes some unpublished estimates which give a unified study of these cases, and provide some new insights. The newest and most relevant results in this sections are the ones obtained for the critical case. The state of the art in this field was dealing only with the case in which the shape of the particles (or holes), G_0, is a ball. In this direction we have shown that the case of maximal monotone graphs behaves as expected, providing a common roof for results with Neumann, Dirichlet and even Signorini boundary conditions (see [DGCPS17a; DGCPS17c]). In this case we have shown that the "strange" nonlinear term appearing in the homogeneous problem is always smooth, even when the microscopic problem is not. This behaviour can be linked to Nanotechnological properties of some materials. Furthermore, we have managed, for the time in the literature, to study the cases in which G_0 is not a ball (see [DGCSZ17]). In this last paper, the techniques of which are very new, gives some seemingly unexpected results, that answer the intuition of the experts. The results mentioned above were obtained by applying a modification of Tartar's oscillating test function method. The periodical unfolding method has also been applied by this candidate, in some unpublished work, and this was acknowledged by the authors of [CD16].

The contributions presented in this chapter improve many different works in the literature, and it has been presented in Table 1.1. The work in this chapter has been presented in the international congress ECMI 2016 (Spain, 2016) and Nanomath 2016 (France, 2016). At the time of presentation of this thesis a new paper dealing with the critical case and general shape of the particles has been submitted for publication.

The second chapter deals with a priori estimates for the effectiveness factor of a chemical reactor, which is a functional depending of the solution of the limit behaviour deduced in Chapter 1. This problems comes motivated, for example, by the application to waste water treatment tanks. Once we have obtained a homogeneous model, our aim is to decide which reactors are of optimizer in some classes, and also provide bounds for the effectiveness. In this direction, we have dealt with Steiner symmetrization, which allows us to compare the solution of any product type domain $\Omega_1 \times \Omega_2$, which would represent the chemical reactor, with a cylinder of circular basis, $B \times \Omega_2$, where $|B| = |\Omega_1|$. In this direction we have published two papers [DGC15b; DGC16] dealing with convex and concave kinetics, which extends the pioneering paper [ATDL96]. The work in this chapter has been presented in the following congresses: MathGeo 2013 (Spain), 10th AIMS Conference in Dynamical Systems, Differential Equations and Applications (Spain, 2013), Nanomath 2014 (Spain, 2014), Mini-workshop in honour of Prof. G. Hetzer (USA, 2016).

The third chapter deals with direct shape optimization techniques. This can be organized into two sections: shape differentiation and convex optimization. Shape differentiation is a technique that, given a initial shape Ω_0 characterizes the infinitesimal change of the solution of our homogeneous problem when we consider deformations $(I+\theta)(\Omega_0)$. In this directions two papers have been published. First, we studied the Fréchet differentiable case, that requires the kinetic to be twice differentiable (see [DGC15a]). This was a first step to the problem in which we were interested, the case of non-smooth kinetic. In this setting, the solution may even develop a dead core (see [GC17]). Another of the techniques of direct optimization we applied was the convex optimization of the domain G_0. If we only allow the admissible set of shapes G_0 to be convex sets, then we have some compactness results, that guaranty that there exist optimal sets in this family (see [DGCT16; DGCT15]). The work in this chapter has been presented in the 11th AIMS Conference on Dynamical Systems, Differential Equations and Applications (USA, 2017).

The fourth chapter deals with linear elliptic equations with a potential, $-\Delta u + Vu = f$, where the potential, V, "blows up" near the boundary. This kind of equations appear as a result of the shape differentiation process, in the non-smooth case. The problems with a transport term, $\nabla u \cdot b$, $-\Delta u + \nabla u \cdot b + Vu$ is studied, in collaboration with Profs. Jean-Michel Rakotoson and Roger Temam. Different results of existence, uniqueness and regularity of solutions of this equations are presented (see [DGCRT17]). One of this results is the fact that, shall the blow up of the potential V be fast enough, the condition $Vu \in L^1$ can act as a boundary condition for u. Some unpublished results are included in this chapter, which improve some of the results of [DGCRT17], in a limit case, by applying an extension of the argument in [DGCRT17] suggested recently by Prof. Brezis to this candidate, and which have not been published. The work in this chapter has been presented in the 11th AIMS Conference on Dynamical Systems, Differential Equations and Applications (USA, 2017). At the time of submission of this thesis a new paper improving the results of [DGCRT17] is under development.

The fifth chapter develops the second part of the thesis, and includes results obtained during the 2017 visit to Prof. Brezis (see [BGC17]). They improve some previous results by Brezis in collaboration with the group of Prof. Ron Kimmel. We showed that the basis of eigenvalues of $-\Delta$ with Dirichlet boundary conditions is the unique basis to approximate functions in H_0^1 in L^2 in an optimal way.

Besides the contributions in this thesis, this candidate has also developed other projects. The author published, jointly with Prof. Brú and Nuño, a paper [BGCN17] which involved the simulation of the fractional Laplacian in bounded domains, and their study by new statistical physics techniques. Besides this work, the candidate studied the modeling of Lithium-ion batteries, and is about to publish work on the well-posedness of the Newman model. This was presented in the 11th AIMS conference on Dynamical Systems (USA, 2017).

Resumen en castellano

Esta tesis se ha divido en dos partes de tamaños desiguales. La primera parte es la componente central del trabajo del candidato. Se encarga de la optimización de reactores químicos de lecho fijo, y el estudio de su efectividad, como se expondrá en los siguientes párrafos. La segunda parte es el resultado de la visita del candidato al Prof. Häim Brezis en el Instituto Tecnológico de Israel (Technion) en Haifa, Israel. Se entra en una pregunta concreta sobre bases óptimas en L^2, que es de importancia en Tratamiento de Imágenes, y que fue formulado por el Prof. Brezis.

La primera parte de la tesis, que estudio reactores químicos, se ha dividido en 4 capítulos. Estudia un modelo establecido que tiene aplicaciones directas en Ingeniería Química, y la noción de efectividad. Una de las mayores dificultades con la que nos enfrentamos es el hecho que, por las aplicaciones en Ingeniería Química, estamos interesados en reacciones de orden menor que uni (de tipo raíz).

El primer capítulo se centra en la modelización: obtener un modelo macroscópico (homogéneo) a partir de un comportamiento microscópico prescrito. A este método se le conoce como *homogeneización*. La idea es considerar partículas periódicamente repetidas, de forma fija G_0, a una distancia ε, y que han sido reescaladas por un factor a_ε. La expresión habitual de este factor es $a_\varepsilon = C_0\varepsilon^\alpha$, donde $\alpha \geq 1$ y C_0 es una constante positiva. El objetivo es estudiar los diferentes comportamientos cuando $\varepsilon \to 0$, y ya no se consideran las partículas. Primero, los casos de *partículas grandes* y *partículas pequeños* se tratan de formas distintas. Este segundo, que ha sido el central en esta tesis, se divide en subcrítico, crítico y supercrítico. En términos generales, existe un valor α^* tal que los comportamientos de los casos $\alpha = 1$ (partículas grandes), $1 < \alpha < \alpha^*$ (partículas subcríticos), $\alpha = \alpha^*$ (partículas críticos) y $\alpha > \alpha^*$ (partículas supercríticos) son significativamente distintos. El objetivo central de la tesis han sido los casos $\alpha > 1$, aunque se han obtenido también algunos resultados para el caso $\alpha = 1$ (ver [DGCT15; DGCT16]). En el caso subcrítico hemos mejorado significativamente la regularidad de las no-linealidades permitidas, por argumentos de aproximación uniforme (ver [DGCPS17d]). También hemos demostrado que, cuando la difusión depende del gradiente (operadores de tipo p-Laplaciano) con p mayor que la dimensión espacial, entonces no existen escalas críticas (ver [DGCPS17b]). Además, esta tesis incluye estimaciones no publicadas que dan un estudio unificado de estos casos, e introducen nuevas perspectivas. Los resultados más nuevos y más relevantes en estas secciones son los que se refieren al caso crítico. El estado del arte en este campo era lidiar sólo con el caso en que la forma de la partícula es una esfera. En esta dirección hemos mostrado que los grafos maximales monótonos se comporta como era de esperar, dando un techo común a los resultados con condiciones de frontera Neumann, Dirichlet e incluso Signorini (ver [DGCPS17a; DGCPS17c]). En este caso hemos demostrado que el término "extraño" en el problema homogéneo es siempre regular, incluso cuando la no-linealidad del problema microscópico no lo es. Este comportamiento se puede enlazar con propiedes Nanotecnológicas de algunos materiales. También hemos conseguido estudiar el caso en las partículas no son esferas, si no que tienen una forma más general (ver [DGCSZ17]). En este último artículo, que emplea las técnicas muy nuevas, se dan algunos resultados de aspecto aparentemente inesperado, pero que satisfacen la intuición de los expertos. Todo los resultados presentados en el texto precedente son obtenidos utilizando modificaciones del método de funciones test oscilantes de Tartar. El método de desdoble periódico (*periodical unfolding* en inglés) también ha sido usado por el candidato, en un trabajo sin publicar, y ha sido reconocido en los agradecimientos de [CD16]. Las contribuciones de este capítulo mejoran muchos trabajos previous, como se ha presentado en la Tabla 1.1. El trabajo de este capítulo se ha presentado en los congresos internacionales ECMI 2016 (Spain) y Nanomath 2016 (France).

El segundo capítulo trata sobre estimaciones a priori del factor de efectividad de las reacciones químicas: un funcional que depende de la solución del problema límite obtenido en el Capítulo 1. Este problema viene motivado, por ejemplo, por la aplicación en reactores de tratamiento de aguas residuales. Una vez que se ha obtenido el problema homogeneizado, nuestro objetivo es decidir qué reactores son optimizadores de este funcional, y dar cotas para la efectividad. En este sentido, hemos trabajado con la optimización de Steiner, que permite comparar reactores de la forma $\Omega_1 \times \Omega_2$ con reactores cilíndricos de la forma $B \times \Omega_2$, donde $|B| = |\Omega_1|$. En esta dirección se han publicado dos trabajos, [DGC15b; DGC16] lidiando con no-linealidades convexas y cóncavas. El trabajo de este capítulo se ha presentado en los siguientes congresos: MathGeo 2013 (España), 10th AIMS Conference in Dynamical Systems, Differential Equations and Applications (España, 2013), Nanomath 2014 (España), Mini-workshop in honour of Prof. G. Hetzer (USA, 2016).

El tercer capítulo trata con técnicas de optimización de formas directas. Se ha organizado en dos secciones: diferenciación de formas y optimización convexa. La diferenciación de formas es una técnica que, dado un una forma inicial Ω_0, caracteriza el cambio infinitesimal de la solución de nuestro problema homogeneizado cuando se considera una deformación $(I+\theta)(\Omega_0)$. En esta dirección se han publicado dos artículos. Primero hemos estudiado la diferenciabilidad en el sentido de Fréchet, que requiere que la cinética sea dos veces derivable (ver [DGC15a]). Éste fue un primer paso hacia el problema en que estábamos interesados, el caso no-suave. En este contexto, la solución puede desarrollar un *dead core* (ver [GC17]). Otra de las técnicas que hemos usado es la optimización convexa directa del dominio G_0. Si solo consideramos el conjunto admisible de formas G_0 dentro de la familia convexa, entonces podemos obtener la existencia de extremos (ver [DGCT16; DGCT15]). Este trabajo ha sido presentado en el 11th AIMS Conference in Dynamical Systems, Differential Equations and Applications (USA, 2017).

El cuarto capítulo trata con ecuaciones elípticas con un potencial, $-\Delta u + Vu = f$, donde el potencial, V, "explota" cerca del borde. Este tipo de ecuaciones aparecen como resultado del proceso de diferenciación de formas del Capítulo 3, en el caso en que aparece un dead core. El problema con un término de transporte, $\vec{b} \cdot \nabla u$, fue también estudiado. Se obtuvieron diferentes resultados de existencia, regularidad y unicidad de soluciones (ver [DGCRT17]). Uno de los resultados más sorprendentes es que si V explota suficientemente rápido, entonces la condición $Vu \in L^1$, que se suponía habitualmente como púramente técnica, se convierte en una condición de contorno Dirichlet homogénea. Los resultados expuestos en esta tesis se han presentado en el 11th AIMS Conference in Dynamical Systems, Differential Equations and Applications (USA, 2016). Se incluyen en esta tesis algunos resultados no publicados, sugeridos por Häim Brezis, que mejoran a los publicados en algunos casos. En el momento del déposito se está trabajando en un borrador que mejora, aún más, estos resultados.

El quinto capítulo desarrolla la segunda parte de la tesis, e incluye resultados obtenidos durante la visita en 2017 al Prof. Häim Brezis (ver [BGC17]). Se mejoran algunos resultados previos con el grupo de Ron Kimmel, sobre la existencia y unicidad de bases óptimas para representación de funciones H^1 en L^2.

Además de las contribuciones incluídas en esta tesis, el candidato ha desarrollado otros proyectos. El candidato ha publicado, conjuntamente con Antonio Brú y Juan Carlos Nuño, un artículo [BGCN17] que incluye la simulación numérica de un laplaciano fraccionario en un dominio acotado, y su estudio mediante técnicas de física estadística. Además de este trabajo se ha estudiado la modelización de baterías de ion-Litio, y se va a publicar un trabajo sobre la buena formulación del modelo de Newman (problema abierto desde 1971). Este último trabajo se presentó en el congreso 11th de la AIMS (USA, 2016).

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Notation

Roman Symbols

a_ε The size of particles in the homogenization process. Usually $a_\varepsilon = C_0\varepsilon^\alpha$

$B(x,R)$ Ball centered at x of radius R

$\mathscr{C}^n(\Omega)$ where $n \in \mathbb{N} \cup \{\infty\}$. Space of n times differentiable functions with continuous derivative.

$\mathscr{C}_c^n(\Omega)$ Space of n times differentiable functions with continuous derivative and compact support.

$\mathscr{C}_{per}^n(Q)$ where Q is an n-dimensional cube, space of functions that can be extended by periodicity to a function in $C^n(\mathbb{R}^n)$

$\mathscr{E}$ Effectiveness factor

$F(A)$ Space of all functions $f : A \to \mathbb{R}$

$g(u)$ Kinetic of the homogeneous problem for u. Chapters 2, 3 and 4

$\mathscr{A}$ $= \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$

$\mathscr{B}_0$ $= \left(\frac{n-p}{C_0(p-1)}\right)^{p-1}$

n Spatial dimension

$W^{1,p}(\Omega)$ Space of functions such that $u \in L^p(\Omega)$ and $\nabla u \in (L^p(\Omega))^n$.

$W_0^{1,p}(\Omega)$ Closure in $W^{1,p}(\Omega)$ of the space $\mathscr{C}_c^\infty(\Omega)$

$W^{1,p}(\Omega,\Gamma)$ where $\Gamma \subset \partial\Omega$. Closure in $W^{1,p}(\Omega)$ of the space of functions $f \in \mathscr{C}^\infty(\bar{\Omega})$ such that $\operatorname{supp} f \cap \Gamma = \emptyset$

$W^{1,p}_{loc}(\Omega)$ space of functions such that they are in $W^{1,p}(K)$, for every $K \subset \Omega$ compact

$W^{1,p}_{per}(Q)$ where Q is an n-dimensional, space of functions in $W^{1,p}$ that are periodic in Q, i.e. they can be extended by periodicity to a function in $W^{1,p}_{loc}(\mathbb{R}^n)$

Y In Chapter 1 it will represent a cube of side 1. Either, $Y = [0,1]^n$ or $(-\frac{1}{2}, \frac{1}{2})^n$ depending on the case

Greek Symbols

α Power such that $a_\varepsilon = C_0\varepsilon^\alpha$

$\beta(\varepsilon)$ Corrector order of the boundary term in homogenization. Usually $\beta(\varepsilon) = \varepsilon^{-\gamma}$. Chapter 1

$\beta(w)$ Kinetic of the homogeneous problem for w. Chapter 2, 3 and 4

$\beta^*(\varepsilon)$ Critical value of $\beta(\varepsilon)$. $\beta^*(\varepsilon) = a_\varepsilon^{1-n}\varepsilon^n$. Chapter 1.

$\partial\Omega$ Boundary of the set Ω

$\emptyset$ Empty set

ε Small parameter destined to go to zero

η Ineffectiveness factor

Ω Generic open set. Its smoothness will be specified on a case by case basis

ω_n Volume of the n-dimensional ball

∂f Partial derivative of function f

$\sigma(u)$ Kinetic of the problem for u. Chapter 1.

Superscripts

$*$ In Chapter 1 critical value. In Chapter 2 and onwards decreasing rearrangement

$\star$ Schwarz rearrangement

\# Steiner rearrangement

Subscripts

ε In functions or sets will indicate that it refers to the nonhomogeneous problem.

$*$	Nondescript rearrangement

Other Symbols

$\overline{\Omega}$	Closure of the set of Ω
$\Subset$	Compact subset
$\preceq$	Comparison of concentrations
$\emptyset$	Empty set
$\ll$	given sequences $(a_\varepsilon)_{\varepsilon>0}$, $(b_\varepsilon)_{\varepsilon>0}$. $a_\varepsilon \ll b_\varepsilon$ if $\lim_{\varepsilon\to 0} a_\varepsilon b_\varepsilon^{-1} = 0$
$\gg$	given sequences $(a_\varepsilon)_{\varepsilon>0}$, $(b_\varepsilon)_{\varepsilon>0}$. $a_\varepsilon \gg b_\varepsilon$ if $b_\varepsilon \ll a_\varepsilon$
$\sim$	given sequences $(a_\varepsilon)_{\varepsilon>0}$, $(b_\varepsilon)_{\varepsilon>0}$. $a_\varepsilon \sim b_\varepsilon$ if $\lim_{\varepsilon\to 0} a_\varepsilon b_\varepsilon^{-1} \in (0,+\infty)$

Acronyms / Abbreviations

CIF	Cauchy's Integral Formula
DCT	Dominated Convergence Theorem
div	Divergence operator. For a differentiable function $f:\mathbb{R}^n \to \mathbb{R}$ its definition is $\operatorname{div} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}$
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

Part I

Optimization of chemical reactors

Introduction

Rutherford Aris and the definition of effectiveness.

The monographs of R. Aris [Ari75] and Aris and Strieder [AS73] model the behaviour of chemical reactions in terms of partial differential equations. Their works are amongst the first to consider the microscopic behaviour of the system, and derive from it the macroscopic properties.

In their book, Aris and Strieder model a Chemical Reactor by an open set Ω. In it, they model the concentration of a Chemical Reactant by a spatial function $c = c(x)$. For the constituent equation they introduce an spatial diffusion term $\operatorname{div}(D_e \nabla c)$ and a reaction term $r(c)$ (the amount of reaction that is produced as a function of the amount of reactant). Their spatial model results:

$$\operatorname{div}(D_e \nabla c) = r(c) \text{ in } \Omega.$$

This equation alone is ill-posed, since there are many solutions of this problem. To fix a single one another equation needs to be considered. The author choose to allow a flux in the boundary of $\partial\Omega$. The full model results:

$$\begin{cases} \operatorname{div}(D_e \nabla c) = r(c) & \text{in } \Omega, \\ D_e \vec{n} \cdot \nabla c = k_c (c_f - c), & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where k_c is a permitivity constant of the boundary and c_f is a maximum concentration of the reactant admitted by the solvent.

One of the novelties in the mentioned book is the they also propose a model, which we will call *non homogeneous*, in which the reactor contains many microscopic particles, which they model by an open domain G of $\mathbb{R}^n$ (usually $n = 2$ or 3). In this model that the reaction is taking place on the boundary of the particles. The following model comes up:

$$\begin{cases} \operatorname{div}(D_e \nabla c) = 0, & \hat{\Omega} \\ D_e \vec{n} \cdot \nabla c = k_c (c_f - c), & \partial\Omega \\ D_e \vec{n} \cdot \nabla c = \hat{r}(c), & \partial G \end{cases} \tag{2}$$

where $\hat{\Omega} = \Omega \setminus G$ and G represents pellets and $\hat{r}$ is a reaction rate, possibly different from r. They represent the situation as Figure 1.

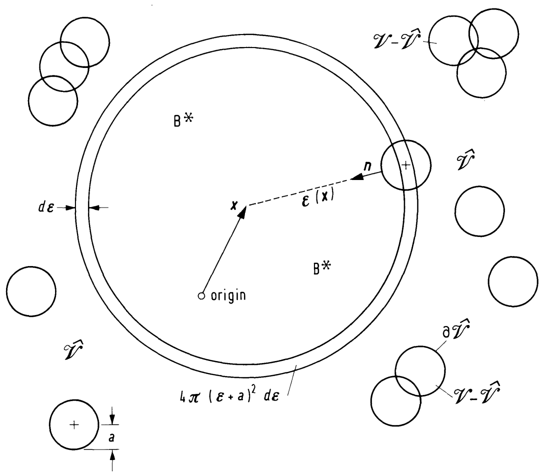


Fig. 1.5.1. Random spheres of radius a. The void region $\hat{\mathscr{V}}$, reaction zone $\mathscr{V} - \hat{\mathscr{V}}$, the interface $\partial\hat{\mathscr{V}}$, and the unit normal vector **n** to $\partial\hat{\mathscr{V}}$ are shown. The minimum distance from the point x in the void to the reactive interface is $\varepsilon(x)$. The regions about the point x of radius $\varepsilon + a$ from which spheres are excluded, and the adjacent shell of thickness $d\varepsilon$ in which at least one sphere center is found are also indicated.

Fig. 1 Domain as portrayed in [AS73]. In a small change of notation we have considered Ω instead of V and $\hat{\Omega}$ instead of $\hat{V}$.

Aris and Strieder define the effectiveness factor of the chemical reactor as

$$\mathscr{E} = \frac{1}{|\Omega| r(c_f)} \int_\Omega r(c)$$

for the homogeneous problem, where c is the solution of (1) and

$$\hat{\mathscr{E}} = \frac{1}{|\partial\hat{\Omega}| \hat{r}(c_f)} \int_{\partial\hat{\Omega}} \hat{r}(c)$$

for the non-homogeneous model, where c is the solution of (2).

Another novelty in the work of Aris and Strieder is that, albeit by naive methods, they show which model of type (1) we must consider once we if we consider a constituent equation of form (2) (and viceversa).

This effectiveness factor is a very relevant quantity. There is a lot of mathematical literature dedicated to it (see, e.g., [BSS84; DS95]). It will be the main quantity under investigation in Chapters 2 and 3, through very different techniques.

Amongst other things, they were interested in the behaviour of the effectiveness in the different domains and, in particular, in choosing domains of optimal effectiveness.

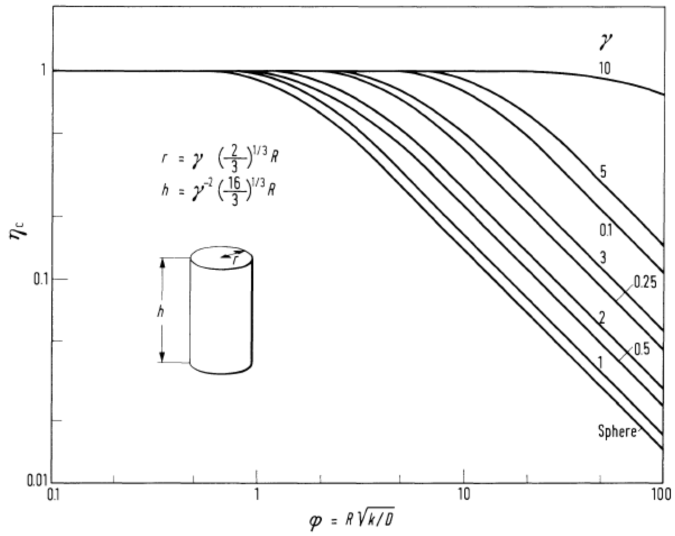


Fig. 4.5.1. Plot of effectiveness factor η_c for a cylinder vs. ϕ where R is the radius of a sphere having the same volume. (Note that this ϕ is $R\phi_c$).

Fig. 2 Image showing some (probably estimated) curves of the effectiveness factor $\mathscr{E}$ as a function of the shape parameters of the cylinder. Extract from [AS73]

Roughly speaking, the aim of Chapter 1 is to properly define the set $\hat{\Omega}$, to study in which sense we can pass from the equation over $\hat{\Omega}$ to the equation over Ω and in which sense we can pass from $\hat{\mathscr{E}}$ to $\mathscr{E}$.

A comment on the notation

The first part of this thesis applies techniques for different problems from different fields of Applied Mathematics, which involve different communities. Whenever possible, we have tried to be consistent along the paper, but in some cases this would have made reading more inconvenient for some of the specialists. In particular the use of σ, g and β changes between Chapter 1 and the rest.

In Chapter 1 we define w_ε as the solution of (1.9), and the $u_\varepsilon = 1 - w_\varepsilon$ as the solution of (1.12). Then, under some assumptions, we show that $\tilde{u}_\varepsilon$ to u, which is the solution of (1.180). This will be the relevant function studied in chapter 2 and onwards. In this setting we define the effectiveness as (1.14) and (1.15). Then, in Chapter 2 and onwards, w is the solution of (2.1), whereas $u = 1 - w$ is the solution of (2.2), and the effectiveness is defined as (2.3) and (2.4).

Chapter 1

Deriving macroscopic equations from microscopic behaviour: Homogenization

1.1 Formulation of the microscopic problem

Let us present the precise mathematical formulation of the problem we will be interested in, and that is directly motivated by the problem proposed by Aris.

1.1.1 Open domain with particles

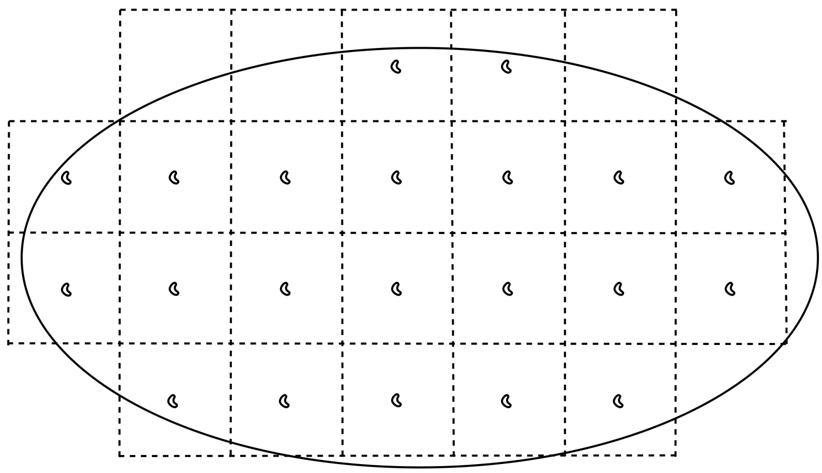


Fig. 1.1 The domain Ω_ε.

Let us set up the geometrical framework. Let $\Omega \subset \mathbb{R}^n$ be an open set (bounded and regular, for simplicity), and let the shape of a generic inclusion (in our setting a particle, but it applies also to the case of a hole) be represented by a domain G_0 be an open set homeomorphic to a ball such that $\overline{G_0} \subset Y = (0,1)^n$ (i.e., there exists a invertible continuous map $\Psi : U \to V$ between open sets of $\mathbb{R}^n$, U and V, where $G_0 \subset U$ and V contains the open ball of radius one, $\Psi(G_0)$ is the ball and Ψ^{-1} is continuous).

Considering the parameter $\varepsilon > 0$ the distance between the equispaced particles, we will typically be set in the following geometry

$$G_i^\varepsilon = \varepsilon i + a_\varepsilon G, \qquad i \in \mathbb{Z}^n \tag{1.1}$$

$$\Upsilon_\varepsilon = \left\{ i \in \mathbb{Z}^n : \overline{G_i^\varepsilon} \subset \Omega \right\}, \tag{1.2}$$

$$G^\varepsilon = \bigcup_{i \in \Upsilon_\varepsilon} G_i^\varepsilon, \tag{1.3}$$

$$S^\varepsilon = \bigcup_{i \in \Upsilon_\varepsilon} \partial G_i^\varepsilon, \tag{1.4}$$

$$\Omega_\varepsilon = \Omega \setminus \overline{G^\varepsilon}. \tag{1.5}$$

We will sometimes consider that

$$a_\varepsilon = C_0 \varepsilon^\alpha. \tag{1.6}$$

The parameter $\alpha \geq 1$ indicates the size of the particle relative to the repetition. We will also use the notion of periodicity box

$$Y_i^\varepsilon = \varepsilon i + \varepsilon Y. \tag{1.7}$$

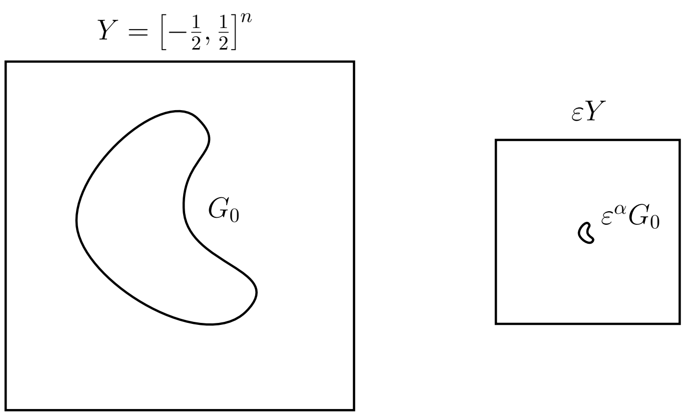


Fig. 1.2 The reference cell Y and the scalings by ε and ε^α, for $\alpha > 1$. Notice that, for $\alpha > 1$, $\varepsilon^\alpha G_0$ (for a general particle shaped as G_0) becomes smaller relative to εY, which scales as the repetition. In most of our cases G_0 will be a ball $B_1(0)$.

1.1.2 Governing equation

We will consider that there is no diffusion inside G^ε, but it can be seen as a catalyst agent, producing a reaction on its boundary

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla w^\varepsilon) = \hat{f}^\varepsilon & \Omega_\varepsilon, \\ A^\varepsilon \nabla w_\varepsilon \cdot n + \beta(\varepsilon)\hat{\sigma}(w_\varepsilon) = \hat{g}^\varepsilon & S_\varepsilon, \\ w^\varepsilon = 1 & \partial\Omega. \end{cases} \tag{1.8}$$

Here $\hat{\sigma}$ is the reaction kinetics, typically a nondecreasing function. Notice that $\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon$. The parameter $\beta(\varepsilon)$ modulates the intensity of the reaction, and its value shall be precised shortly.

We will also deal, in this problem, with nonlinear diffusion

$$\begin{cases} -\Delta_p w_\varepsilon = \hat{f}^\varepsilon & \Omega_\varepsilon, \\ \dfrac{\partial w_\varepsilon}{\partial \nu_p} + \beta(\varepsilon)\hat{\sigma}(w^\varepsilon) = \hat{g}^\varepsilon & S_\varepsilon, \\ w^\varepsilon = 1 & \partial\Omega, \end{cases} \tag{1.9}$$

where $p > 1$ and

$$-\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$$
$$\frac{\partial w}{\partial \nu_p} = |\nabla w|^{p-2}\nabla w \cdot n.$$

The quasilinear diffusion operator $-\Delta_p$ represents the cases in which the diffusion coefficient depends of $|\nabla w|$ (see [Día85] and the references therein). Notice that for $p = 2$ we get the usual, linear, Laplacian operator. However, for $p > 2$ the operator becomes degenerate (the diffusion coefficient vanishes when $|\nabla w| = 0$) and for $1 < p < 2$ the operator becomes singular (the diffusion coefficient is unbounded as $|\nabla w| \to 0$).

A change in variable Boundary condition $w = 1$ is not nice in terms of functional spaces. We will thus choose the change in variable

$$u_\varepsilon = 1 - w_\varepsilon. \tag{1.10}$$

With the change of variable in mind we can rewrite (1.8) as

$$\begin{cases} -\operatorname{div}(A^{\varepsilon}\nabla u^{\varepsilon}) = f^{\varepsilon} & \Omega_{\varepsilon}, \\ A^{\varepsilon}\nabla u_{\varepsilon}\cdot n + \beta(\varepsilon)\sigma(u_{\varepsilon}) = g^{\varepsilon} & S_{\varepsilon}, \\ u^{\varepsilon} = 0 & \partial\Omega, \end{cases} \tag{1.11}$$

and (1.9) as

$$\begin{cases} -\Delta_p u_{\varepsilon} = f^{\varepsilon} & \Omega_{\varepsilon}, \\ \dfrac{\partial u_{\varepsilon}}{\partial \nu_p} + \beta(\varepsilon)\sigma(u^{\varepsilon}) = g^{\varepsilon} & S_{\varepsilon}, \\ u^{\varepsilon} = 0 & \partial\Omega, \end{cases} \tag{1.12}$$

where

$$\sigma(u) = \hat\sigma(1) - \hat\sigma(1-u), \qquad f^{\varepsilon} = \hat\sigma(1) - \hat f^{\varepsilon}, \qquad g^{\varepsilon} = \hat\sigma(1) - \hat g^{\varepsilon}. \tag{1.13}$$

1.1.3 Effectiveness and ineffectiveness

As motivated by the definition of Aris we define the *effectiveness* of the non-homogeneous problem as

$$\mathscr{E}_{\varepsilon}(\Omega, G_0) = \frac{1}{|S_{\varepsilon}|}\int_{S_{\varepsilon}} \hat\sigma(w_{\varepsilon}). \tag{1.14}$$

Since in this chapter we will deal with u_{ε} rather than w_{ε} let us define the *ineffectiveness functional*

$$\eta_{\varepsilon}(\Omega, G_0) = \frac{1}{|S_{\varepsilon}|}\int_{S_{\varepsilon}} \sigma(u_{\varepsilon}). \tag{1.15}$$

We have that

$$\eta_{\varepsilon}(\Omega, G_0) = \sigma(1) - \mathscr{E}_{\varepsilon}(\Omega, G_0). \tag{1.16}$$

Hence, in terms of convergence and optimization, analyzing one of the functionals is exactly the same as analyzing the other one.

1.1.4 Maximal monotone operators. *A common roof*

In some contexts, it is desirable to substitute the condition

$$\frac{\partial u_{\varepsilon}}{\partial \nu_p} + \beta(\varepsilon)\sigma(u^{\varepsilon}) = 0 \text{ on } S_{\varepsilon} \tag{1.17}$$

by the Dirichlet boundary condition

$$u_\varepsilon = 0 \text{ on } S_\varepsilon, \tag{1.18}$$

or even the case of Signorini type boundary condition (also known as boundary obstacle problem)

$$\begin{cases} u_\varepsilon \geq 0 & S_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma_0(u_\varepsilon) \geq 0 & S_\varepsilon, \\ u_\varepsilon \left(\partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma_0(u_\varepsilon)\right) = 0 & S_\varepsilon. \end{cases} \tag{1.19}$$

There is a unified presentation of this theory, using (1.9). The idea is to use *maximal monotone operators* (see [Bré73] or [Bro67])

Definition 1.1. Let X be a Banach space and $\sigma : X \to \mathscr{P}(X')$. We say that σ is a monotone operator if, for all $x, \hat{x} \in X$,

$$\langle x - \hat{x}, \xi - \hat{\xi} \rangle_{X \times X'} \geq 0 \qquad \forall \xi \in \sigma(x), \hat{\xi} \in \sigma(\hat{x}). \tag{1.20}$$

We define the domain of σ as

$$D(\sigma) = \{x \in X : \sigma(x) \neq \emptyset\}. \tag{1.21}$$

Here $\emptyset$ is the empty set. We say that σ is a maximal monotone operator if there is no other monotone operator $\widetilde{\sigma}$ such that $D(\sigma) \subset D(\widetilde{\sigma})$ and $\sigma(x) \subset \widetilde{\sigma}(x)$ for all $x \in X$.

It can be shown that any maximal monotone operator in $\mathbb{R}$ is given by a monotone functions, the jumps of which are filled by a vertical segment. It is immediate to prove the following:

Proposition 1.1. *Let $\sigma \in \mathscr{C}(\mathbb{R})$ be nondecreasing. Then σ is a maximal monotone operator.*

Furthermore

Proposition 1.2. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function, and let $(x_n)_n$ be its set of discontinuities. Then, the function*

$$\widetilde{\sigma}(x) = \begin{cases} \sigma(x) & x \in \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}, \\ [\sigma(x_n^-), \sigma(x_n^+)] & x = x_n \textit{ for some } n \in \mathbb{N}, \end{cases} \tag{1.22}$$

is a maximal monotone operator.

Boundary condition (1.18) can be written in terms of maximal monotone operators as (1.17) with

$$\sigma(x) = \begin{cases} \emptyset & x < 0, \\ \mathbb{R} & x = 0, \\ \emptyset & x > 0. \end{cases} \tag{1.23}$$

One the other hand, (1.19) can be written as (1.17) with

$$\sigma(x) = \begin{cases} \emptyset & x < 0, \\ (-\infty, 0] & x = 0, \\ \sigma_0(x) & x > 0. \end{cases} \tag{1.24}$$

Of course, the use of maximal monotone operators escapes the usual framework of classical solutions of PDEs. We will present the definition of weak solutions for this setting in Section 1.4.1.

Another advantage of maximal monotone operators is the simplicity to define inverses. For $\sigma : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ we define its inverse in the sense of maximal monotone operators as the map $\sigma^{-1} : \mathbb{R} \to \mathscr{P}(\mathbb{R})$ given by

$$\sigma^{-1}(s) = \{x \in \mathbb{R} : s \in \sigma(x)\}. \tag{1.25}$$

It is a trivial exercise that σ^{-1} is also a maximal monotone operator.

1.2 An introduction to homogenization

The main idea of this theory is to consider an inhomogeneous setting -be it due to some oscillating term in the equation or because of the domain itself- and decide which homogeneous equation can approximate the result in a “mean field approach” in order to “remove” these obstacles. Usually, studying the heterogeneous medium is not feasable, whereas the homogeneous equation can be easily undestood.

In order to fix notations, let us define some Sobolev spaces. For Ω a smooth set we define the space

$$W^{1,p}_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} : \text{ for all } K \subset \Omega \text{ compact }, f \in W^{1,p}(K)\}. \tag{1.26}$$

For $\Gamma \subset \partial\Omega$ we define

$$W^{1,p}(\Omega,\Gamma) = \overline{\{f \in \mathscr{C}^\infty(\Omega) : f = 0 \text{ on } \Gamma\}}^{W^{1,p}}. \tag{1.27}$$

Some particular cases deserve their own notation:

$$\begin{aligned} W_0^{1,p}(\Omega) &= W^{1,p}(\Omega,\partial\Omega), \\ H^1(\Omega,\Gamma) &= W^{1,2}(\Omega,\Gamma), \\ H_0^1(\Omega) &= W_0^{1,2}(\Omega). \end{aligned}$$

Finally, for a cube Q, we define

$$W_{per}^{1,p}(Q) = \{f \in W^{1,p}(Q) : f \text{ can be extended by periodicity to } W_{loc}^{1,p}(\mathbb{R}^n)\}. \tag{1.28}$$

The theory of homogenezation is very broad and has been adapted to deal with many problem, from fluid mechanics to Lithium-ion batteries (see, e.g., [BC15]). Many books can be found which aim to give an introduction to this extensive and technical field (see, e.g., [BLP78; SP80; CD99; Tar10]).

In this Chapter we aim to give a comprehensive study of the problem in which the domain Ω contains some inclusions (or holes). This kind of problem, for which the literature is quite extensive, is known sometimes as the problem of "open domain with holes".

1.2.1 Some first results

To illustrate, in a very simple example, how some of the ideas work, let us go back to one the earliest results in homogenization. The idea behind the following example is a G-convergence argument (owed to Spagnolo [Spa68]).

Example 1.1. Let $a : \mathbb{R} \to \mathbb{R}$ be a $[0,1]$-periodic function such that $0 < \alpha \le a \le \beta$, $f \in L^2(0,1)$ and $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$. We consider the one dimensional problem

$$\begin{cases} -\dfrac{d}{dx}\left(a^\varepsilon \dfrac{du_\varepsilon}{dx}\right) = f \quad x \in (0,1), \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases} \tag{1.29}$$

By multiplying by u_ε and integrating, we have that the sequence u_ε is bounded in $H_0^1(0,1)$, and therefore

$$u_\varepsilon \rightharpoonup u_0$$

in $H_0^1(0,1)$ and, by the same argument

$$a^\varepsilon \nabla u_\varepsilon = \xi_\varepsilon \rightharpoonup \xi_0$$

is convergent in $H^1(0,1)$ (since $f \in L^2$) and, in the limit

$$\begin{cases} -\dfrac{d}{dx}\left(\xi^0\right) = f & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.30}$$

holds. It is well-known that, for $h \in L^2(0,1)$, $h\left(\frac{\cdot}{\varepsilon}\right) \rightharpoonup \int_0^1 h$ in $L^2(0,1)$. Hence, up to a subsequence,

$$\nabla u_\varepsilon = \frac{1}{a^\varepsilon}\xi^\varepsilon \rightharpoonup \int_0^1 \frac{1}{a(x)} dx \cdot \xi^0 \tag{1.31}$$

in $L^2(0,1)$. Hence $\xi^0 = \frac{1}{\int_0^1 \frac{1}{a(x)} dx} \frac{du_0}{dx}$ and thus u_0 satisfies

$$\begin{cases} -\dfrac{d}{dx}\left(\dfrac{1}{\int_0^1 \frac{1}{a(x)} dx} \dfrac{du}{dx}\right) = f & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.32}$$

The term $a_0 = \frac{1}{\int_0^1 \frac{1}{a(x)} dx}$ is sometimes known as *effective diffussion coefficient*. This concludes this example.

One of the many works in homogenization in dimension higher than one is due to J.L Lions [Lio76], which contains a compendium of different references (e.g. [Bab76]). The focus of this work is the problem of oscillating coefficients

$$A^\varepsilon u_\varepsilon = f, \qquad A^\varepsilon v = \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla v\right) \tag{1.33}$$

where $A = (a_{ij})$ is a matrix, $a_{ij} = a_{ji} \in L^\infty([0,1]^n)$ and are extended by periodicity. This models the behaviour of a periodical two phase composite (a material formed by the inclusion of two materials with different properties). This work is, no doubt, based on previous results, for example by Spagnolo (see, e.g., [Spa68]) on the limit behaviour of problems $-\operatorname{div}(A_k u_k)$ as $A_k \to A_\infty$.

The different approaches are very well presented in [BLP78] and [CD99].

1.2.2 Different techniques of homogenization

Here we will briefly present some of the most relevant methodologies applied in homogenization. Most of them have been applied to our problem, as we will see later.

Multiple scales method One of the possibilities in dealing with the limit consists on considering an expansion -which is known as asymptotical expansion- of the solutions as

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \cdots, \tag{1.34}$$

and deriving the behaviour from there. This method, which is now known as *multiple scales method* is still very much in use to these days (see, e.g., [Día99; BC15]).

This kind of argument work in two steps. First, a formal deduction of the good approximation and a later rigorous proof. In particular, they use repeatedly the computation that, if $v = v(x, \xi)$ then

$$\frac{\partial}{\partial x_i}(v(x, \varepsilon^{-1}x)) = \frac{\partial v}{\partial x_i}(x, \varepsilon^{-1}x) + \varepsilon^{-1}\frac{\partial v}{\partial \xi_i}(x, \varepsilon^{-1}x). \tag{1.35}$$

Substituting (1.34) into $-\operatorname{div}(A^\varepsilon u^\varepsilon) = f$ and gathering terms the is seen that

$$u_\varepsilon(x) = u_0(x) + \varepsilon\hat{\xi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla u_0 + \varepsilon^2\hat{\theta} : D^2 u_0 + \cdots, \tag{1.36}$$

and the equations for u_0, $\hat{\xi}$ and $\hat{\theta}$ can be found explicitly. The second part of this kind of argument is to estimate the convergence. It can be shown that

$$\left\| u_\varepsilon(x) - \left(u_0(x) + \varepsilon\hat{\xi}\left(\frac{x}{\varepsilon}\right) \cdot \nabla u_0 + \varepsilon^2\hat{\theta} : D^2 u_0\right)\right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}. \tag{1.37}$$

Detailed examples can be found, e.g., in [CD99, Chapter 7], [BLP78] or, for the case of the elasticity equation, [OSY92].

The Γ-convergence method This method introduce by De Giorgi [DF75] and later developed in [DD83; Dal93]. The essential idea behind the Γ-convergence method is to study the problem in its energy form and study the conditions under which convergence of the energies implies convergence of their minimizers, the solutions of the elliptic problems. Here we present some results extracted from [Dal93].

Definition 1.2. Let X be a topological space. The Γ-lower limit and Γ-upper limit of a sequence (F_n) of functions $X \to [-\infty, \infty]$ are defined as follows

$$\left(\Gamma - \liminf_{n\to+\infty} F_n\right)(x) = \sup_{U\in\mathcal{N}(x)} \liminf_{n\to+\infty} \inf_{y\in U} F_n(y) \tag{1.38}$$

$$\left(\Gamma - \limsup_{n\to+\infty} F_n\right)(x) = \sup_{U\in\mathcal{N}(x)} \limsup_{n\to+\infty} \inf_{y\in U} F_n(y) \tag{1.39}$$

where $\mathcal{N}(x) = \{U \subset X \text{ open } : x \in U\}$. If there exists $F : X \to [-\infty, +\infty]$ such that $F = \Gamma - \liminf_{n\to+\infty} F_n = \Gamma - \limsup_{n\to+\infty} F_n$ then (F_n) we say that F_n Γ-converges to F, and we denote it as

$$F = \Gamma - \lim_{n\to+\infty} F_n. \tag{1.40}$$

For the length of this section we will note

$$F' = \Gamma - \liminf_{n\to+\infty} F_n, \tag{1.41}$$

$$F'' = \Gamma - \limsup_{n\to+\infty} F_n. \tag{1.42}$$

The results that make this technique interesting for us are the following:

Theorem 1.1. *Suppose that (F_n) are equi-coercive in X. Then F' and F'' are coercive and*

$$\inf_{x\in X} F'(x) = \liminf_{n\to+\infty} \inf_{x\in X} F_n(x). \tag{1.43}$$

Proposition 1.3. *Let x_n be a minimizer of F_n in X and assume that $x_n \to x$ in X. Then*

$$F'(x) = \liminf_{n\to\infty} F_n(x_n), \qquad F''(x) = \limsup_{n\to\infty} F_n(x_n). \tag{1.44}$$

In the context of homogenization we are mainly interested in the behavior of functionals

$$F_\varepsilon(u,A) = \begin{cases} \displaystyle\int_A f\left(\frac{x}{\varepsilon}, u(x), Du(x)\right) & u \in W^{1,p}(A), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.45}$$

where $p > 1$.

Assume $f = f(y, Du)$ satisfies the following

i) For every $x \in \mathbb{R}^n$ the function $f(x,\cdot)$ is convex of class $\mathscr{C}^1$.

ii) For every $\xi \in \mathbb{R}^n$ is measurable and Y-periodic (where Y is the unit cube).

iii) There exists $c_i \in \mathbb{R}$ $i = 0, \cdots, 4$ such that $c_1 \geq c_0 > 0$ and

$$c_0|\xi|^p \leq f(x,\xi) \leq c_1|\xi|^p + c_2, \tag{1.46}$$

$$\left|\frac{\partial f}{\partial \xi}(x,\xi)\right| \leq c_3|\xi|^{p-1} + c_4. \tag{1.47}$$

Then, let

$$f_0(\xi) = \inf_{v \in W^{1,p}_{per}(Y)} \int_Y f(y, \xi + Dv(y))dy. \tag{1.48}$$

Then for every sequence $\varepsilon_n \to 0$ we have that F_{ε_n} Γ-converges to F_0 the functional defined by

$$F_0(u,A) = \begin{cases} \int_A f_0(Du) & u \in W^{1,p}(A), \\ +\infty & \text{otherwise}. \end{cases} \tag{1.49}$$

Function f_0 can be characterized in a more practical manner.

Proposition 1.4. *We have that*

$$f_0(\xi) = \int_Y f(y, Dv(y))dy, \tag{1.50}$$

where u is the unique function that

$$\begin{cases} v \in W^{1,p}_{loc}(\mathbb{R}^n), \\ Dv \text{ is } Y-\text{periodic}, \\ \int_Y Dv = \xi, \\ \operatorname{div}(D_\xi f(y,Dv)) = 0 \text{ in } \mathbb{R}^n \text{ in the sense of distributions}. \end{cases} \tag{1.51}$$

Applying this method we can obtain the same result as in Example 1.1 in a way that can generalized to higher dimension

Example 1.2. Let $n = 1$ and $\Omega = Y = (0,1)$. Let us consider solutions of problem for the operator $-\operatorname{div}(a(\frac{x}{\varepsilon})u^\varepsilon)$. We consider the energy function

$$f(x,\xi) = a\left(\frac{x}{\varepsilon}\right)|\xi|^2. \tag{1.52}$$

Let us characterize v given by Proposition 1.4. Since $(av')' = 0$ we have that that $v' = \frac{a_0}{a}$, where k is a constant. Taking into account that $\int_0^1 v' = \xi$ we deduce that

$$a_0 = \frac{1}{\int_0^1 \frac{1}{a(x)} dx} \xi.$$

Therefore

$$\begin{aligned} f_0(\xi) &= \int_0^1 a(y)|v'(y)|^2 dy = \int_0^1 a(y) \frac{a_0^2}{a(y)^2} dy |\xi|^2 = a_0^2 \int \frac{1}{a(y)} dy |\xi|^2 \\ &= a_0 |\xi|^2, \end{aligned}$$

where the effective diffusion is given by the coefficient

$$a_0 = \frac{1}{\int_0^1 \frac{1}{a(x)} dx}.$$

Hence, as we previously showed in Example 1.1, the limit of this kind $-\operatorname{div}(a(\frac{x}{\varepsilon})u^\varepsilon)$ is $-\operatorname{div}(a_0 u)$. Depending on the space X we consider, we can fix one type of boundary condition or another. This completes this example.

In [Dal93] examples in higher dimensions are presented. This method was applied to our cases of interest, with some modification, by [Kai89] and [Gon97]. The details of this last paper (which is extremely synthetic and skips most of the computations) are given in Appendix 1.A.

The two-scale convergence method The two-scale method was introduced by Nguetseng [Ngu89] and later developed by Allaire [All92; All94]. The central definition of the theory is the following:

Definition 1.3. Let (v_ε) be a sequence in $L^2(\Omega)$. We say that the sequence v_ε *two scale converges* to a function $v_0 \in L^2(\Omega \times Y)$ if, for any function $\psi = \psi(x,y) \in \mathscr{D}(\Omega; \mathscr{C}^\infty_{\text{per}}(Y))$ one has

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) = \frac{1}{|Y|} \int_\Omega \int_Y v_0(x,y) \psi(x,y). \tag{1.53}$$

By taking $\psi = \psi(x)$ in the previous definition it is immediate that

$$v_\varepsilon \rightharpoonup V^0 = \frac{1}{|Y|} \int_Y v_0(\cdot, y) dy \tag{1.54}$$

weakly in $L^2(\Omega)$. The key point of this theory is to study the convergence of functions of the type $\psi(x, \frac{x}{\varepsilon})$, and then apply them suitably to the weak formulation.

Tartar's method of oscillating test functions This method is due to L. Tartar (see [Tar77; Tar10; MT97]). The idea behind it is to consider the appropriate weak formulation and select suitable test functions φ^ε, with properties that, in the limit, reveal weak formulation of the homogeneous problem.

This is, basically, the general method applied to obtain the results of this thesis. As well shall see, its is not a straightforward recipy, and the choice of test function and their analysis can become a very hard task. Many detailed examples will be given in the following text. Perhaps the most illustrative, due to its simplicity, is Section 1.5.7.

One of the main difficulties that rise with this method in domains with particles or holes is the need of a common functional space, since $u_\varepsilon \in L^p(\Omega_\varepsilon)$. This leads to the construction of extension operators $P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega)$, that will be discussed in Section 1.5.1.

The periodical unfolding method The periodical unfolding method was introduced by Cioranescu, Damlamian and Griso in [CDG02; CDG08]. It consists on transforming the solution to a fixed domain $\Omega \times Y$. The case of particles (or holes) was considered in [CDGO08; CDDGZ12; CD16]. The latter paper acknowledges the contribution of the author of this thesis.

Let us present the reasoning in domains with particles (or holes). The idea is to decompose every point in Ω as a sum

$$x = [x]_Y + \{x\}_Y \tag{1.55}$$

where $[x]_Y$ is the unique element in $\mathbb{Z}^n$ such that $x - [x]_Y \in [0,1)^n$. That is, we have that $[\cdot]_Y$ is constant over Y_ε^j.

We define the operator

$$\mathscr{T}_{\varepsilon,\delta} : \varphi \in L^2(\Omega) \mapsto \mathscr{T}_{\varepsilon,\delta}(\varphi) \in L^p(\Omega \times \mathbb{R}^n)$$

as

$$\mathscr{T}_{\varepsilon,\delta}(\varphi)(x,z) = \begin{cases} \varphi\left(\varepsilon \left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon\delta z\right) & (x,z) \in \hat{\Omega}_\varepsilon \times \dfrac{1}{\delta} Y, \\ 0 & \text{otherwise}, \end{cases} \tag{1.56}$$

where

$$\hat{\Omega}_\varepsilon = \text{interior}\left(\bigcup_{\substack{\xi\in\mathbb{Z}^n:\\ \varepsilon(\xi+Y)\subset\Omega}} \varepsilon(\xi+\overline{Y})\right). \tag{1.57}$$

Notice that $\mathscr{T}_{\varepsilon,\delta}(\varphi)(x,z)$ is piecewise constant in x. The boundary of G_ε^j corresponds to $\hat{\Omega}_\varepsilon \times \partial G_0$.

The big advantage of this approach is that it removes the need to construct extension operators. Therefore, it allows to considers non-smooth shapes of G_0. This method as shown very good result, and the properties of $\mathscr{T}_{\varepsilon,\delta}(u_\varepsilon)$ are well understood, at least in the non-critical cases.

1.3 Literature review of our problem

Let us do a comprehensive review of the literature. Let us understand why. Recall the definition (1.2), (1.3) and (1.5). In a moderate abuse of notation, which will not lead to confusion let us use the notations:

- If $a \in \mathbb{R}$, $|a|$ will indicate its absolute value.
- If $A \subset \mathbb{R}^n$ is a set of dimension m (i.e. $m = n$ if the domain is open, $m = n-1$ if $A = \partial U$ for U open, etc.) then $|A|$ will indicate its m-dimensional Lebesgue measure.
- If A is a finite set, then $|A|$ will indicate its cardinal (i.e. the number of the elements).

We also introduce two notations. Given $(a_\varepsilon)_{\varepsilon>0}, (b_\varepsilon)_{\varepsilon>0}$ sequence of real number we define

$$a_\varepsilon \sim b_\varepsilon \quad \equiv \quad \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{b_\varepsilon} \in (0,+\infty) \tag{1.58}$$

$$a_\varepsilon \ll b_\varepsilon \quad \equiv \quad \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{b_\varepsilon} = 0. \tag{1.59}$$

First, we estimate $|\Upsilon_\varepsilon|$. Since

$$\left| \Omega - \bigcup_{j\in\Upsilon_\varepsilon} (\varepsilon j + \varepsilon Y) \right| \to 0, \tag{1.60}$$

we have that

$$\frac{|\Omega|}{|\Upsilon_\varepsilon|\varepsilon^n} = \frac{|\Omega|}{|\Upsilon_\varepsilon||\varepsilon Y|} \to 1, \tag{1.61}$$

as $\varepsilon \to 0$. Hence

$$|\Upsilon_\varepsilon| \sim \varepsilon^{-n}. \tag{1.62}$$

We can therefore compute

$$\begin{aligned} |G_\varepsilon| &= \left| \bigcup_{j\in\Upsilon_\varepsilon} (\varepsilon j + a_\varepsilon G_0) \right| \quad &(1.63)\\ &= |\Upsilon_\varepsilon||a_\varepsilon G_0| \\ &\sim \varepsilon^{-n} a_\varepsilon^n \\ &\sim (a_\varepsilon \varepsilon^{-1})^n. \quad &(1.64) \end{aligned}$$

We see that the the case $a_\varepsilon \sim \varepsilon$ is different from the case $a_\varepsilon \ll \varepsilon$. In the first case, known, as case of *big particles*, $|G_\varepsilon|$ has a positive volume in the limit. We will show that this volumetric presence affects the kind of homogenized diffusion. The diffusion coefficient becomes a function of G_0. However, it has been shown that when $a_\varepsilon \ll \varepsilon$, case known as case of *small particles*, we have no volumetric contribution, and, as we will see, the diffusion is not affected.

Under some conditions, it has been reported that the case $a_\varepsilon \ll \varepsilon$ presents a critical scale a_ε^* that separates different behaviour (the precise values and behaviours will be given later):

- In the subcritical case $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$ the nature of the kinetic is preserved
- In the critical case $a_\varepsilon \sim a_\varepsilon^*$ the nature of the kinetic changes. This effect is known as the appearance of an *strange term*.
- in the supercritical case $a_\varepsilon \ll a_\varepsilon^*$ the problem behaves, in the limit, as the case $\sigma \equiv 0$. This case is not very relevant, as we will see in Section 1.5.7.

All of this problems have undergone extensive work, and Table 1.1 presents a detailed literature review, focusing on the case studied in terms of a_ε and the regularity of σ. We hope this table puts the contributions of the author to this field in context.

1.3.1 Homogenization with big particles $a_\varepsilon \sim \varepsilon$

Dirichlet boundary conditions on the particles. The first work in this direction is [CS79]. It deals with problem (1.12) with $p = 2$ but fixing the value of u_ε in G_j^ε to a constant not necessarily zero rather than the Neumann boundary condition. This paper introduces the extension operator

$$P_\varepsilon : \{u \in H^1(\Omega_\varepsilon) : u = \text{ const. on } \partial G_i^\varepsilon,\ u = 0 \text{ on } \partial\Omega\} \to H_0^1(\Omega) \tag{1.65}$$

such that

$$\|P_\varepsilon v\|_{H^1(\Omega)} \le C\|v\|_{H^1(\Omega_\varepsilon)}. \tag{1.66}$$

The advantage for the perforated domain problem is that $P_\varepsilon u_\varepsilon$ are all defined in the same space $H_0^1(\Omega)$, and therefore the convergence is easy to establish. In this paper the authors study several structures for matrix A^ε in (1.11). The authors show that $P_\varepsilon u_\varepsilon \rightharpoonup u$, the solution of

$$\begin{cases} \operatorname{div}(a_0 \nabla u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{1.67}$$

where a_0 is an effective diffusion matrix that depends of G_0. The nature of this result is similar to Example (1.1).

Neumann boundary conditions on the particles Later [CD88a] dealt with problem (1.12) with $p = 2$ (equivalently (1.12) with $A^\varepsilon = I$) in the case $\sigma(u) = au$ and $g^\varepsilon(x) = g\left(\frac{x}{\varepsilon} - j\right)$ in ∂G_ε^j. Their approach is an asymptotic expansion (this method is known as multiple scales method, see [SP80]).

If $\int_{\partial T} g(y)dS = 0$ they consider

$$u^\varepsilon(x) = u_0(x,y) + \varepsilon u_1(x,y) + \cdots \tag{1.68}$$

where $y = \frac{x}{\varepsilon}$. Otherwise they perform the expansion

$$u^\varepsilon(x) = \varepsilon^{-1} u_{-1}(x,y) + u_0(x,y) + \varepsilon u_1(x,y) + \cdots. \tag{1.69}$$

The result is that $P_\varepsilon u_\varepsilon \rightharpoonup u$, the solution of

$$\sum_{i,j} q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{|\partial T|}{|Y|} au = \begin{cases} f & \text{if } \int_{\partial T} g(y)dS = 0, \\ \int_{\partial T} g(y)dS & \text{otherwise}, \end{cases} \tag{1.70}$$

where q_{ij} is given as

$$q_{ij} = \delta_{ij} + \frac{1}{|Y \setminus T|} \int_{Y \setminus T} \frac{\partial \chi_j}{\partial y_i} dy, \tag{1.71}$$

and χ_i are the solutions of the so-called *cell problems*:

$$\begin{cases} -\Delta\chi_i = 0 & \text{in } Y \setminus T, \\ \frac{\partial(\chi_i + y_i)}{\partial \nu} = 0 & \text{on } \partial T, \\ \chi_i & Y\text{-periodic}. \end{cases} \tag{1.72}$$

The surprising conclusion is that $u_\varepsilon \to u_0$ in the first case, but $\frac{1}{\varepsilon} u^\varepsilon \to u_{-1}$ in the second.

The nonlinear problem was later studied by Conca, J.I. Díaz, Timofte and Liñán in [CDT03; CDLT04]. Their technique, which involves oscillating test functions, requires the introduction of extension operators

$$P_\varepsilon : \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\} \to H_0^1(\Omega) \tag{1.73}$$

such that $(P_\varepsilon u)|_{\Omega_\varepsilon} = u$. This techniques allows for a fairly general class of nonlinearities σ, but not all can be considered. In particular, they consider the following cases $\sigma = \sigma(x, v)$

$$\left| \frac{\partial \sigma}{\partial v}(x, v) \right| \leq C(1 + |v|^q), \qquad 0 \leq q < \frac{n}{n-2} \tag{1.74}$$

or

$$|\sigma(x, v)| \leq C(1 + |v|^q), \qquad 0 \leq q < \frac{n}{n-2}. \tag{1.75}$$

The result is, naturally, that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the solution of

$$\begin{cases} -\operatorname{div}(a_0(G_0)\nabla u) + \frac{|\partial G_0|}{|Y \setminus G_0|}\sigma(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.76}$$

and

$$a_0(G_0) = (q_{ij}) \tag{1.77}$$

is the effective diffusion matrix, where q_{ij} are given by (1.71).

The same results as in [CDLT04] were obtained in [CDZ07] applying the unfolding method (developed for this case in [CDZ06]). The advantage of the unfolding method is that it reduces the regularity constraints on ∂T.

1.3.2 Homogenization with sub-critical small particles $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$

The first result in this direction can be found in [CD88b]. The difficulty of this case is to understand the behaviour of the integrals $\beta(\varepsilon)\int_{S_\varepsilon}\cdot$ as $\varepsilon \to 0$. The theory in [CD88b] is rather comprehensive for the case $p = 2$. In their work the authors detect that, for $p = 2$, there exists values a_ε^*, depending on the spatial dimension, such that the behaviour changes. The term *critical case* already appears in this text. However, they are unable to specify what happens in this case.

The work in understanding the behavior of the boundary integral under the different situation of p and regularity of σ has been incremental over the last decades (see table 1.1). In Section 1.5.2 we present a unified approach that covers most of the results for the sub-critical cases. Through approximation techniques, developed in [DGCPS17d] and briefly presented in Section 1.5.5, the non smooth cases can be treated.

1.3.3 Homogenization with critical small particles

Dirichlet boundary condition First, Hruslov dealt with the Dirichlet homogeneous boundary condition on the holes [Hru72] (see the higher order case in [Hru77]) in a rather convoluted an functional way. In 1997 a measure theoretic analysis dealt the appearance of "strange term" in [CM97]. This later paper was much easier to understand.

The Neumann boundary condition The linear Neumann boundary condition was studied first in the homogeneous setting: [Hru79; Kai89; Kai90; Kai91]. The linear setting, $\sigma(u) = \lambda u$, was studied later in [OS96] (see also [OS95]).

In [Kai89; Kai91] (see also [Kai90]) an analysis of the nonlinear critical and subcritical cases is made by Kaizu, who is unable to properly characterize the change of the nonlinearity in the critical case. The first paper to properly study this case, and characterize the nature of the new ("strange") nonlinear function, is [Gon97], which applies the technique of Γ-convergence when G_0 is a ball, smooth σ and $n = 3$. The very surprising result is the following

Theorem 1.2. *Let G_0 be a ball and assume*

i) $n = 3$, $a_\varepsilon = \varepsilon^\alpha$ $\alpha \le 3 = \alpha^*$, $\beta(\varepsilon) = \varepsilon^{-\gamma} \sim \beta^*(\varepsilon)$. *That is* $\gamma = 2\alpha - 3$

ii) $\frac{\partial \sigma}{\partial u} \ge C > 0$

iii) There exists a nonnegative function $a \in \mathscr{C}_0^\infty(\Omega)$ such that

$$\rho(x,u) = 2\int_0^u \sigma(x,s)ds + a(x) \geq 0. \tag{1.78}$$

Then, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ where u is the solution of the problem

$$\begin{cases} -\Delta u + 4\pi C(x,u) = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{1.79}$$

and

$$C(x,u) = \begin{cases} \sigma(x,u) & 2 < \alpha < 3, \\ H(x,u) & \alpha = \gamma = 3, \end{cases} \tag{1.80}$$

being H is the (unique) solution of functional equation

$$H(s) = \sigma(s - H(s)). \tag{1.81}$$

However, there are a few steps that are not clearly justified in the paper. The computations of this paper have been detailed and explained in this thesis (as unpublished material) and correspond to Section 1.A.

The case of general n, non smooth σ and $-\Delta_p$ were later studied in detail. This different problems introduce a different number of difficulties, and are a larger part of the work developed by the author of this thesis. For many years, the only case of G_0 that was understood was a ball:

- The usual Laplacian in $\mathbb{R}^n$ for $n = 3$: [Gon97]
- The usual Laplacian in $\mathbb{R}^n$ for $n \geq 2$: [ZS11] [ZS13]
- The p-Laplace operator and $2 < p < n$: [SP12] [Pod10] [Pod12] [Pod15].
- n-Laplacian for $n \geq 2$: The critical size of holes in the case $p < n$ is $a_\varepsilon = \varepsilon^{\frac{n}{n-p}}$. Naturally, this critical exponent α_p^* blows up as $p \to n$. As it turns out, a critical case also exists for the case $p = n$, and this was studied in [PS15].
- Roots and Heaviside type nonlinearity: [DGCPS16] .
- Signorini boundary conditions: [DGCPS17a]

- Maximal monotone operators and $1 < p < n$: [DGCPS17c]. This result covers all the previous cases under a common roof.

In 2017, the author of this thesis jointly with J.I. Díaz, T.A. Shaposhnikova and M.N. Zubova [DGCSZ17], considered (for the first time in the literature) the case of G_0 not a ball. The structure of the limit equation is unprecedented in the literature.

			Dirichlet	$\sigma = 0$	$\sigma = \lambda u$	$0 < k_1 \le \sigma \le k_2$ $\sigma(x,u) \in \mathscr{C}^1$	$\lvert\sigma'(s)\rvert \le C\lvert u\rvert^{p-1}$ $(\sigma(x,u) - \sigma(x,v)) \ge C\lvert u-v\rvert^p$	$\lvert\sigma'(u)\rvert \le C(1+\lvert u\rvert^r)$ or $\lvert\sigma(u)\rvert \le C(1+\lvert u\rvert^r)$	Signorini	σ m.m.g.
Big particles	$p=2$	$\alpha = 1$	[CS79]	[CD88a]		[CDLT04] [CDZ07] (see also [CDZ06])				
Small non-critical particles	$p=2$	$1 < \alpha < \frac{n}{n-p}$		[CD88b]	[OS96]	[Gon97] $(N=3)$ [ZS11][ZS13]	[SP12]	[Kai89],[DGCPS17d]	[JNRS14]	[Kai91]
	$2 < p < n$							[DGCPS17d]		
	$1 < p < 2$			[Pod15]						
	$p = n$	$\frac{a_\varepsilon}{e^{-\varepsilon^{-\frac{n}{n-1}}}} \to 0$		[Podolskii and Shaposhnikova (to appear)]						
	$p > n$	$\alpha > 1$		[DGCPS17b]						
Small critical particles	$p=2$	$\alpha = \frac{n}{n-p}$ G_0 a ball	[Hru72] [CM97]	[OS96]		[Gon97] $(N=3)$ [ZS11][ZS13]	[SP12]	[Kai89]	[JNRS11]	[Kai91] , [DGCPS17c]
	$2 < p < n$							$(\sigma = \lvert u\rvert^{q-1} u)$	[GPPS15]	[DGCPS17c]
	$1 < p < 2$		[DGCPS17c]				[DGCPS16]	[DGCPS17a]		
	$p=2$	$\alpha = \frac{n}{n-p}$ G_0 not a ball		[DGCSZ17]						
	$p = n$	$\frac{a_\varepsilon}{e^{-\varepsilon^{-\frac{n}{n-1}}}} \to C \neq 0$		[PS15]						

Table 1.1 Schematic representation of bibliography for the homogenization problems (1.11), (1.12). Where α is present $a_\varepsilon = C_0 \varepsilon^\alpha$. Gray background represents new results introduced by this thesis.

1.4 A unified theory of the case of small particles $a_\varepsilon \ll \varepsilon$

1.4.1 Weak formulation

When $\sigma : \mathbb{R} \to \mathbb{R}$ we usually define a weak solution of (1.12) as a function $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ such that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla v + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon) v = \int_{\Omega_\varepsilon} f^\varepsilon v + \beta(\varepsilon) \int_{S_\varepsilon} g^\varepsilon v \tag{1.82}$$

for all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. In Section 1.1.4 we introduced the concept of maximal monotone operator. When σ is a maximal monotone operator this definition is no longer valid, since $\sigma(u_\varepsilon(x))$ may be multivalued. We change the previous equation by

Definition 1.4. We say that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of (1.12) if there exists $\xi \in L^p(S_\varepsilon)$ such that $\xi(x) \in \sigma(u_\varepsilon(x))$ for a.e. $x \in S_\varepsilon$ and

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla v + \beta(\varepsilon) \int_{S_\varepsilon} \xi v = \int_{\Omega_\varepsilon} f^\varepsilon v + \beta(\varepsilon) \int_{S_\varepsilon} g^\varepsilon v, \tag{1.83}$$

for all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

Uniqueness of this kind of solution is a direct consequence of the monotonicity of σ (see, e.g., [Día85]). However, it is not easy to show directly that there exist solutions of (1.12) in this sense. The energy formulation is much better for this task.

1.4.2 Energy formulation

Let us start by considering the usual case $\sigma : \mathbb{R} \to \mathbb{R}$. In this setting it is standard to define the energy functional over $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as

$$J_\varepsilon(v) = \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla v|^p + \beta(\varepsilon) \int_{S_\varepsilon} \Psi(v) - \int_{\Omega_\varepsilon} f^\varepsilon v - \beta(\varepsilon) \int_{S_\varepsilon} g^\varepsilon v, \tag{1.84}$$

where $\Psi(s) = \int_0^s \sigma(\tau)$. It is common to say that the energy formulation for (1.12) is

$$J_\varepsilon(u_\varepsilon) = \min_{v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)} J_\varepsilon(v). \tag{1.85}$$

For smooth σ it can be shown that the unique solution u_ε of (1.12) is the unique minimizer of this functional. One possible way to give a meaning to (1.12) is to use this formulation. For that we recall the concept of the subdifferential:

Definition 1.5. Let X be a Banach space and $J : X \to (-\infty, +\infty]$ be a convex function. We usually define the domain of J as

$$D(J) = \{x \in X : J(x) \neq +\infty\}. \tag{1.86}$$

We define the subdifferential of J as the map $\partial J : X \to \mathscr{P}(X')$ given by

$$\partial J(x_0) = \{\xi \in X' : J(x) - J(x_0) \geq \langle \xi, x - x_0 \rangle \ \forall x \in X\}. \tag{1.87}$$

As is turns out, for every maximal monotone operator σ defined in $\mathbb{R}$ there exists a convex function Ψ such that $\sigma = \partial\Psi$. Furthermore, this Ψ can be chosen uniquely under the extra condition $\Psi(0) = 0$.

This energy formulation connects directly with the concept of weak solution. The subdifferential $A_\varepsilon = \partial J_\varepsilon$ is given by the set of dual elements $\widehat{\xi}$ such that

$$\langle \widehat{\xi}, w \rangle = \int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla w + \beta(\varepsilon) \int_{S_\varepsilon} \xi w - \int_{\Omega_\varepsilon} f^\varepsilon w - \beta(\varepsilon) \int_{S_\varepsilon} g^\varepsilon w, \tag{1.88}$$

where $\xi(x) \in \sigma(v(x))$ for a.e. $x \in S_\varepsilon$ (see, e.g., [Lio69]).

The weak formulation of (1.12) is, precisely,

$$A_\varepsilon u_\varepsilon \ni 0. \tag{1.89}$$

1.4.3 Formulation as functional inequalities

Let us prove the equivalence between the weak and energy formulations:

Lemma 1.4.1 (Chapter 1 in [ET99])**.** *Let X be a reflexive Banach space, $J : X \to (-\infty, +\infty]$ be a convex functional $A = \partial J : X \to \mathscr{P}(X')$ be its subdifferential. Then the following are equivalent:*

i) u is a minimizer of J,

ii) $u \in D(A)$ and $0 \in Au$.

If either hold, then

iii) For every $v \in D(A)$ and $\xi \in Av$

$$\langle \xi, v - u \rangle \geq 0. \tag{1.90}$$

Furthermore, assume that J is Gâteaux-differentiable on X and A is continuous on X then iii)) *is also equivalent to* i)).

Remark 1.1. Naturally, if there is uniqueness of iii) then the i)-iii) are also equivalent.

Remark 1.2. One should not confuse condition iii) with the -very similar- Stampacchia formulation (see e.g. [Día85]). For a bilinear form a and a linear function G the Stampacchia formulation is

$$a(u, v-u) \geq G(v-u) \tag{1.91}$$

for all v in the correspondent space, whereas in formulation iii) we have $a(v, v-u)$.

From Lemma 1.4.1 we can extract some characterizing equations of the weak solution, which will be useful later.

Proposition 1.5. *Let u_ε be a minimizer of J_ε. Then*

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla(v-u_\varepsilon) + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) \geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon), \tag{1.92}$$

there exists $\xi \in \sigma(v(x))$ such that

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot (v-u_\varepsilon) + \beta(\varepsilon) \int_{S_\varepsilon} \xi (v-u_\varepsilon) \geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon), \tag{1.93}$$

and

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot (v-u_\varepsilon) + \beta(\varepsilon) \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) \geq \int_{\Omega_\varepsilon} f(v-u_\varepsilon), \tag{1.94}$$

hold for all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

Proof. Let us assume that u_ε is a minimizer of J_ε. Considering characterization iii) of Lemma 1.4.1 we have that

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla w + \beta(\varepsilon) \int_{S_\varepsilon} \xi w \geq \int_{\Omega_\varepsilon} f^\varepsilon w \tag{1.95}$$

for some ξ such that $\xi(x) \in \sigma(u_\varepsilon(x))$. Since Ψ is convex and $\sigma = \partial\Psi$ we have that

$$\Psi(v) - \Psi(u_\varepsilon) \geq \xi(v-u_\varepsilon). \tag{1.96}$$

Hence, (1.92) is proved.

Equation (1.93) can be obtained by considering the Brézis-Sibony characterization of the weak of (1.12) (see Lemme 1.1 of [BS71] or Theorem 2.2 of Chapter 2 in [Lio69]). Finally, let us prove (1.94). Consider the map $x \in \mathbb{R}^n \to |x|^p \in \mathbb{R}$. It is a convex map with derivative $D|x|^p = p|x|^{p-2}x$. Hence, for $a, b \in \mathbb{R}^n$ we have that

$$|a|^p - |b|^p \geq p|b|^{p-2}b \cdot (a-b). \tag{1.97}$$

Hence

$$|b|^p - |a|^p \leq p|b|^{p-2}b \cdot (b-a). \tag{1.98}$$

Considering $b = \nabla v$ and $a = \nabla u_\varepsilon$ we have that

$$|\nabla v|^p - |\nabla u_\varepsilon|^p \leq p|\nabla v|^{p-2}\nabla v \cdot (v - \nabla u_\varepsilon). \tag{1.99}$$

Taking into account this fact and that u_ε is a minimizer of J_ε we have that

$$0 \leq J(v) - J(u_\varepsilon) = \frac{1}{p}\int_{\Omega_\varepsilon} (|\nabla v|^p - |\nabla u_\varepsilon|^p) + \beta(\varepsilon)\int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) - \int_{\Omega_\varepsilon} f(v - u_\varepsilon) \tag{1.100}$$

$$\leq \frac{1}{p}\int_{\Omega_\varepsilon} (|\nabla v|^p - |\nabla u_\varepsilon|^p) + \beta(\varepsilon)\int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) - \int_{\Omega_\varepsilon} f(v - u_\varepsilon). \tag{1.101}$$

Thus, we have obtained (1.94). $\square$

Under some conditions, one can show that these Variational Inequalities are, in fact, equivalent to the definition of weak and energy solutions. Since we will not need this, we give no further details here.

1.5 Existence and uniqueness of solutions

To prove the existence of solutions we can use Convex Analysis to prove the existence of minizers of u_ε, or consider a very strong theorem. To state in its broadest generality we introduce (following Brezis, see [Bre68]) the definition

Definition 1.6. Let V be a reflexive Banach space. We say that $A : V \to V'$ is a *pseudo monotone operator* if it is bounded and it has following property: if $u_j \rightharpoonup u$ in V and that

$$\limsup_{j\to+\infty} \langle T(u_j), u_j - u\rangle \leq 0,$$

then, for all $v \in X$,

$$\liminf_{j\to+\infty} \langle T(u_j), u_j - v\rangle \geq \langle T(u), u-v\rangle. \tag{1.102}$$

We can now state the theorem

Theorem 1.3 ([Bre68], also Theorem 8.5 in [Lio69]). *Let $A : V \to V'$ be a pseudo-monotone operator and φ a proper convex function lower semi-continuous such that*

$$\begin{cases} \text{there exist } v_0 \text{ such that } \varphi(v_0) < \infty \text{ and} \\ \dfrac{(Au, u - v_0) + \varphi(u)}{\|u\|} \to \infty, \text{ as } \|u\| \to \infty. \end{cases} \tag{1.103}$$

Then, for $f \in V'$, there exists a unique solution of the problem

$$(A(u) - f, v-u) + \varphi(v) - \varphi(u) \geq 0, \qquad \forall v \in V. \tag{1.104}$$

Uniqueness is a routine task. Let us give a sketch of proof, when σ is a maximal monotone operator and $p \geq 2$. Assume that $u_\varepsilon^1, u_\varepsilon^2$ satisfy (1.83). Considering the difference between the two formulations

$$\int_{\Omega_\varepsilon} (|\nabla u_\varepsilon^1|^{p-2}\nabla u_\varepsilon^1 - |\nabla u_\varepsilon^2|^{p-2}\nabla u_\varepsilon^2)\cdot \nabla v + \beta(\varepsilon)\int_{S_\varepsilon} (\xi^1 - \xi^2) v = 0. \tag{1.105}$$

Taking $v = u_\varepsilon^1 - u_\varepsilon^2$, since $(\xi^1 - \xi^2)(u_\varepsilon^1 - u_\varepsilon^2)$ we have that

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon^1 - u_\varepsilon^2)|^p \leq 0. \tag{1.106}$$

There $u_\varepsilon^1 - u_\varepsilon^2$ is a constant. This constant is 0, due to the boundary condition. This concludes the proof.

We provide a complete proof of existence and uniqueness Considering the weak formulation the following result is immediate.

Proposition 1.6 ([DGCPS17d]). *Let $p > 1$. Then, for every $\varepsilon > 0$ there exists a unique weak solution of (1.12) $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Furthermore, there exists a constant C independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1} \leq C(\|f^\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|g^\varepsilon\|_{L^\infty(S_\varepsilon)}). \tag{1.107}$$

Some extra information can be given about the pseudo-primitive $\Psi(u_\varepsilon)$.

Proposition 1.7 ([DGCPS17c]). *There exists a unique $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ weak solution of* (1.92). *Besides, there exists $K > 0$ independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|\Psi(u_\varepsilon)\|_{L^1(S_\varepsilon)} \leq K. \tag{1.108}$$

1.5.1 Extension operators

In order to introduce a definition of "convergence" we will need to construct an extension operator so that all solutions are extended to a common Sobolev space. If we do this correctly we will be able to take advantage of the compactness properties of this common space.

Let $A \subset B$. We say that P is an extension operator if $P : F(A) = \{f : A \to \mathbb{R}\} \to F(B)$ and has the property that $P(f)|_A = f$. Let $p > 1$. We will say that a family of linear extension operator

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega) \tag{1.109}$$

is uniformly bounded if there exists a constant $C > 0$, independent of ε, such that

$$\|P_\varepsilon u\|_{W_0^{1,p}(\Omega)} \leq C \|u\|_{W_0^{1,p}(\Omega_\varepsilon)} \quad \forall u \in W^{1,p}(\Omega_\varepsilon). \tag{1.110}$$

A family of operators with this property, for $1 \leq p < +\infty$, was constructed in [Pod15]. The idea is to apply the following theorem

Theorem 1.4 (Theorem 7.25 in [GT01]). *Let Ω be a $C^{k-1,1}$ domain in $\mathbb{R}^n$, $k \geq 1$. Then (i) $\mathscr{C}^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$, $1 \leq p < +\infty$ and (ii) for any open set $\Omega' \supset\supset \Omega$ there exists a linear extension operator $E : W^{k,p}(\Omega) \to W_0^{k,p}(\Omega')$ such that $Eu = u$ in Ω and*

$$\|Eu\|_{W^{k,p}(\Omega')} \leq C \|u\|_{W^{k,p}(\Omega)} \tag{1.111}$$

where $C = C(k, \Omega, \Omega')$.

We consider a large ball B such that $Y \Subset B$ and the linear extension operator

$$E : W^{1,p}(Y \setminus G_0) \to W^{1,p}(B) \tag{1.112}$$

such that

$$\|Eu\|_{W^{1,p}(B)} \leq C_0 \|u\|_{W^{1,p}(Y \setminus G_0)}. \tag{1.113}$$

In particular,

$$\|\nabla Eu\|_{L^p(B)} \leq C_1 \|\nabla u\|_{L^p(Y \setminus G_0)}. \tag{1.114}$$

Let us scale it down by a_ε:

$$E_{\varepsilon,j} : W^{1,p}((\varepsilon j + a_\varepsilon Y) \setminus G_\varepsilon^j) \to W^{1,p}(Y, G_0) \xrightarrow{E} W^{1,p}(B) \to W^{1,p}(\varepsilon j + a_\varepsilon B). \tag{1.115}$$

Notice that, rather than $Y_\varepsilon^j \setminus G_\varepsilon^j$ we are considering the a_ε-rescale of Y. By a simple change in variable we observe that

$$\|\nabla E_{\varepsilon,j} u\|_{L^p(\varepsilon j + B)} \le C_1 \|u\|_{L^p((\varepsilon j + a_\varepsilon Y)\setminus G_\varepsilon^j)}. \tag{1.116}$$

Let $u \in W_0^{1,p}(\Omega_\varepsilon)$. Let us consider extend by 0 outside Ω, i.e.

$$\widetilde{u}(x) = \begin{cases} u(x) & x \in \Omega_\varepsilon, \\ 0 & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.117}$$

We then define

$$P_\varepsilon u(x) = \begin{cases} E_{\varepsilon,j}\widetilde{u}(x) & x \in \varepsilon j + a_\varepsilon Y, j \in \Upsilon_\varepsilon, \\ u(x) & \text{otherwise.} \end{cases} \tag{1.118}$$

This is well defined, since the sets $\varepsilon j + a_\varepsilon Y$ does not overlap for ε small. It is clear that P_ε is linear, $P_\varepsilon u = u$ in Ω_ε and, by considering the sum over the space decomposition, we have the uniform bound (1.110), so $P_\varepsilon u \in W^{1,p}(\Omega)$. Since the boundary behaviour has not been modified, $P_\varepsilon u_\varepsilon \in W_0^{1,p}(\Omega)$. We can conclude

Lemma 1.5.1. *Let $G_0 \in \mathscr{C}^{0,1}$ such that $G_0 \Subset Y$. Then, there exists a uniformly bounded family of linear extension operators* (1.109).

1.5.1.1 Extension operators and Poincaré constants

We will use the existence of a Poincaré constant for $W_0^{1,p}(\Omega)$, $C_{p,\Omega}$, such that

$$\|v\|_{L^p(\Omega)} \le C_{p,\Omega} \|\nabla v\|_{L^p(\Omega)}, \qquad \forall v \in W_0^{1,p}(\Omega). \tag{1.119}$$

This constant $C_{p,\Omega}$ is known to exist for every domain Ω bounded. However, it is not trivial to show that all domain Ω_ε have a common constant. The following result is very often used in the literature but it is seldom stated. In [DGCPS17b] we took the time to prove it.

Theorem 1.5 ([DGCPS17b])**.** *Let $p > 1$. If there exists a sequence of uniformly bounded extension operators in $W_0^{1,p}$ then there exists a uniform Poincaré constant for $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$,* *in the sense that*

$$\|u\|_{L^p(\Omega_\varepsilon)} \le C\|\nabla u\|_{L^p(\Omega_\varepsilon)} \qquad \forall u \in W_0^{1,p}(\Omega_\varepsilon) \text{ and } \varepsilon > 0, \tag{1.120}$$

where C does not depend of ε. In particular, let

$$\|\nabla P_\varepsilon u\|_{L^p(\Omega)} \le K_p\|\nabla u\|_{L^p(\Omega_\varepsilon)} \quad \forall u \in W_0^{1,p}(\Omega_\varepsilon), \tag{1.121}$$

hold and $C_{p,\Omega}$ be a Poincaré constant for $W_0^{1,p}(\Omega)$. Then, $C = K_p C_{p,\Omega}$.

1.5.1.2 Convergence of the extension

Hence, the solution u_ε can be extended, and $P_\varepsilon u_\varepsilon$ is a bounded sequence in $W^{1,p}(\Omega)$. Thus, it has a weak limit. The focus of the theory of homogenization is to characterize the equation satisfied by the limit function.

1.5.2 On treating the boundary measure and the appearance of a critical case

Treating the sequence of integrals $\int_{S_\varepsilon}$ is a delicate business. Before we begin their study rigorously, we will start by providing some intuitive (informal) computations.

1.5.2.1 An informal approach

Let us focus first on (1.11). Its weak formulation reads

$$\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla \varphi + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(u_\varepsilon)\varphi = \int_{\Omega_\varepsilon} f^\varepsilon \varphi + \beta(\varepsilon)\int_{S_\varepsilon} g^\varepsilon \varphi \tag{1.122}$$

Let us see how the coefficient $\beta(\varepsilon)$ is decisive for the limit behaviour. First, we should keep in mind that (1.61). For a continuous function $\varphi \in \mathscr{C}^1(\Omega)$ we have, since $|\partial G_i^\varepsilon| = a_\varepsilon^{n-1}|\partial G_0|$ that

$$\beta(\varepsilon)\int_{S_\varepsilon} \varphi = \beta(\varepsilon)\sum_{i\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^i} \varphi = \beta(\varepsilon)\sum_{i\in\Upsilon_\varepsilon}\left(\varphi(x_\varepsilon^i)|\partial G_\varepsilon^i| + \int_{\partial G_\varepsilon^i} \varphi'(\xi_\varepsilon^i(x))(x - x_\varepsilon^i)\right) \tag{1.123}$$

$$= \sum_{i\in\Upsilon_\varepsilon}\left(\beta(\varepsilon)\varphi(x_\varepsilon^i)a_\varepsilon^{n-1}|\partial G_0| + \alpha_\varepsilon^i\right). \tag{1.124}$$

If

$$\beta(\varepsilon)a_\varepsilon^{n-1} \sim \varepsilon^n, \tag{1.125}$$

we have (almost) a Riemann sum in the first term, except for the term $Y_\varepsilon^i \cap \partial\Omega \neq \emptyset$ (but this part has no contribution as $\varepsilon \to 0$). We check immediately that

$$\begin{aligned}
|\alpha_\varepsilon^i| &\leq \|\varphi'\|_\infty 2 \operatorname{diam}(G_\varepsilon^i)|\partial G_i^\varepsilon|\beta(\varepsilon) \leq \|\varphi'\|_\infty 2 \operatorname{diam}(G_\varepsilon^i)|\partial G_0|\beta(\varepsilon)a_\varepsilon^{n-1} \\
&= 2\|\varphi'\|_\infty \operatorname{diam}(G_0)\beta(\varepsilon)a_\varepsilon^n \\
&= 2\|\varphi'\|_\infty \operatorname{diam}(G_0)a_\varepsilon\varepsilon^n.
\end{aligned}$$

Hence, we can expect that

$$\beta(\varepsilon)\int_{S_\varepsilon}\varphi \to \begin{cases} C\displaystyle\int_\Omega \varphi & \beta(\varepsilon) \sim \beta^*(\varepsilon), \\ 0 & \beta(\varepsilon) \ll \beta^*(\varepsilon), \\ +\infty & \beta(\varepsilon) \gg \beta^*(\varepsilon), \end{cases} \tag{1.126}$$

where, recalling (1.125),

$$\beta^*(\varepsilon) = a_\varepsilon^{1-n}\varepsilon^n \tag{1.127}$$

as $\varepsilon \to 0$. Notice that

$$|S_\varepsilon| = \left|\bigcup_{j\in\Upsilon_\varepsilon}(\varepsilon j + \partial(a_\varepsilon G_0))\right| = |\Upsilon_\varepsilon||\partial(a_\varepsilon G_0)| \sim \varepsilon^{-n}a_\varepsilon^{n-1} \sim \frac{1}{\beta^*(\varepsilon)}. \tag{1.128}$$

Therefore, up to constants, this is an average

$$\beta^*(\varepsilon)\int_{S_\varepsilon} \text{ should behave like } \frac{1}{|S_\varepsilon|}\int_{S_\varepsilon}. \tag{1.129}$$

If there is any good behaviour, the only expectable result is that

$$\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} \to \frac{1}{|\Omega|}\int_\Omega. \tag{1.130}$$

This is true, at least, for constant functions.

Remark 1.3. In particular, if we consider the case $a_\varepsilon = C_0\varepsilon^\alpha$, $\beta(\varepsilon) = \varepsilon^{-\gamma}$ and $\beta^*(\varepsilon) = \varepsilon^{-\gamma^*}$ we can expect that

$$\gamma^* = \alpha(n-1) - n. \tag{1.131}$$

Thus, if $\beta(\varepsilon)$ is too small we cannot expect any reaction term in the limit equation, hence (in some sense) it becomes uninteresting. If $\beta(\varepsilon)$ is too large then there the reaction dominates the diffusion, we write

$$\beta(\varepsilon)^{-1}\beta^*(\varepsilon)\int_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla\varphi+\beta^*(\varepsilon)\int_{S_\varepsilon}\sigma(u_\varepsilon)\varphi=\beta(\varepsilon)^{-1}\beta^*(\varepsilon)\int_{\Omega_\varepsilon}f\varphi+\beta^*(\varepsilon)\int_{S^\varepsilon}g^\varepsilon\varphi \tag{1.132}$$

and we see the diffusion term disappear in the effective equation.

This good intuitions are not always true. In Theorem 1.6 we will see some assumptions under which this intuitions hold.

1.5.2.2 A trace theorem for $a_\varepsilon G_0$ in εY

The most difficult part of this analysis is the study of the boundary measure $\int_{S_\varepsilon}$, as well as the unexpected properties of the diffusion in the critical case. The following estimate will be fundamental to our study. The proof can be found for $p=2$ and $n\geq 2$ in [CD88b] and a different proof [OS96] in the case of balls. Here we extend the proof in [CD88b] to the case of $p>1$ and $n\geq 2$. Some of the following results were for $1<p<n$ were presented in [Pod15].

In the following pages we present a unified analysis of the different cases, similar to that of [CD88b], but including the cases $p\neq 2$.

Lemma 1.5.2. *Let $u\in W^{1,p}(Y_\varepsilon)$, $p>1$. Then*

$$\int\limits_{a_\varepsilon G_0}|u|^p\leq Ca_\varepsilon^{n-1}\left(\varepsilon^{-n}\int\limits_{Y_\varepsilon}|u|^p+\tau_\varepsilon\int\limits_{Y_\varepsilon}|\nabla u|^p\right) \tag{1.133}$$

where

$$\tau_\varepsilon\sim\begin{cases}a_\varepsilon^{p-n} & p<n,\\ \ln\left(\dfrac{\varepsilon}{a_\varepsilon}\right)^{p-1} & p=n,\\ \varepsilon^{p-n} & p>n,\end{cases} \tag{1.134}$$

and C is a constant independent of ε and u.

Proof. Let

$$B_\varepsilon=B(0,\varepsilon)\setminus\overline{(a_\varepsilon G_0)} \tag{1.135}$$

and let $\varphi \in \mathscr{C}^\infty(\overline{B_\varepsilon})$. Since G_0 is star shaped then we can represent it in polar coordinates as

$$\partial G_0 = \{(\rho,\theta) : \rho = \Phi(\theta), \theta \in \Theta\}, \tag{1.136}$$

where $\Theta = [0,2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]^{n-2}$.

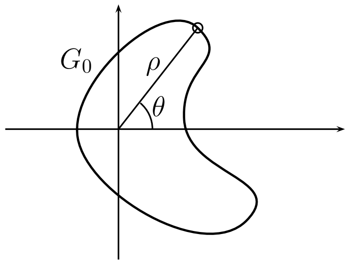


Fig. 1.3 The domain G_0 and its representation in polar coordinates.

The Jacobian can be written as $\rho^{n-1}J(\theta)$. Let us write u in polar coordinates as $\chi(\rho,\theta) = u(x)$. Then, as in [CD88b],

$$\int_{a_\varepsilon G_0} |u|^p dx = a_\varepsilon^{n-1} \int_\Theta |\chi(a_\varepsilon \Phi(\theta),\theta)|^p J(\theta) F(\theta) d\theta, \tag{1.137}$$

where

$$F(\theta) = \prod_{i=1}^{n} \sqrt{\Phi(\theta)^2 + \left(\frac{\partial \Phi}{\partial \theta_i}\right)^2}. \tag{1.138}$$

We write, for any $\rho > a_\varepsilon \Phi(\theta)$ and $\theta \in \Theta$

$$\chi(a_\varepsilon \Phi(\theta),\theta) = \chi(\rho,\theta) - \int_{a_\varepsilon \Phi(\theta)}^{\rho} \frac{\partial \chi}{\partial t}(t,\theta) dt.$$

For $p > 1$, due to convexity

$$|\chi(a_\varepsilon \Phi(\theta),\theta)|^p \leq 2^{p-1} |\chi(\rho,\theta)|^p + 2^{p-1} \left| \int_{a_\varepsilon \Phi(\theta)}^{\rho} \frac{\partial \chi}{\partial t}(t,\theta) dt \right|^p.$$

On the other hand

$$\left| \int_{a_\varepsilon \Phi(\theta)}^{\rho} \frac{\partial \chi}{\partial t}(t,\theta) dt \right|^p \leq \left| \int_{a_\varepsilon \Phi(\theta)}^{\rho} \frac{\partial \chi}{\partial t}(t,\theta) t^{\frac{n-1}{p}} t^{-\frac{n-1}{p}} dt \right|^p$$

$$\leq \left(\int_{a_\varepsilon \Phi(\theta)}^{\rho} t^{-\frac{n-1}{p-1}} dt \right)^{p-1} \left(\int_{a_\varepsilon \Phi(\theta)}^{\rho} \left| \frac{\partial \chi}{\partial t}(t,\theta) \right|^p t^{n-1} dt \right).$$

Taking

$$b_1 = \min_{\theta \in \Theta} \Phi(\theta) \qquad b_2 = \max_{\theta \in \Theta} \Phi(\theta)$$

we get

$$|\chi(a_\varepsilon \Phi(\theta), \theta)|^p \leq 2^{p-1} |\chi(\rho, \theta)|^p + 2^{p-1} \tau_\varepsilon \left(\int_{a_\varepsilon \Phi(\theta)}^{\rho} \left| \frac{\partial \chi}{\partial t}(t,\theta) \right|^p t^{n-1} dt \right),$$

where

$$\tau_\varepsilon = \left(\int_{a_\varepsilon b_1}^{\rho} t^{-\frac{n-1}{p-1}} dt \right)^{p-1}.$$

Integrating over B_ε we obtain

$$\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\rho} |\chi(a_\varepsilon\Phi(\theta),\theta)|^p\rho^{n-1}J(\theta)F(\theta)d\rho d\theta$$

$$\leq 2^{p-1}\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\rho} |\chi(\rho,\theta)|^p\rho^{n-1}JFd\rho d\theta$$

$$+2^{p-1}\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\varepsilon}\tau_\varepsilon\left(\int\limits_{a_\varepsilon\Phi(\theta)}^{\rho}\left|\frac{\partial\chi}{\partial t}(t,\theta)\right|^p t^{n-1}dt\right)\rho^{n-1}JFd\rho d\theta$$

$$\leq 2^{p-1}\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\rho} |\chi(\rho,\theta)|^p\rho^{n-1}JFd\rho d\theta$$

$$+2^{p-1}\tau_\varepsilon\tau_{2,\varepsilon}\int\limits_{\Theta}\tau_\varepsilon\left(\int\limits_{a_\varepsilon\Phi(\theta)}^{\rho}\left|\frac{\partial\chi}{\partial t}(t,\theta)\right|^p t^{n-1}dt\right)JFd\rho d\theta,$$

where

$$\tau_{2,\varepsilon}=\int_{b_1a_\varepsilon}^{\varepsilon}\rho^{n-1}d\rho. \tag{1.139}$$

We can estimate the integral we wanted by

$$\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\rho} |\chi(a_\varepsilon\Phi(\theta),\theta)|^p\rho^{n-1}J(\theta)F(\theta)d\rho d\theta$$

$$\geq \tau_{3,\varepsilon}\int\limits_{\Theta}\int\limits_{a_\varepsilon\Phi}^{\rho} |\chi(a_\varepsilon\Phi(\theta),\theta)|^pJ(\theta)F(\theta)d\rho d\theta$$

$$=\tau_{3,\varepsilon}\|\varphi\|^p_{L^p(\partial(a_\varepsilon G_0))},$$

where

$$\tau_{3,\varepsilon}=\int_{b_2a_\varepsilon}^{\varepsilon}\rho^{n-1}d\rho. \tag{1.140}$$

Since $\tau_{3,\varepsilon}^{-1}\tau_{2,\varepsilon}$ is bounded we can conclude the estimate on $\|u\|_{L^p(\partial(a_\varepsilon G_0))}$. On the other hand

$$\tau_\varepsilon^{\frac{1}{p-1}} = \begin{cases} \dfrac{1}{1-\frac{n-1}{p-1}}\left(\varepsilon^{1-\frac{n-1}{p-1}} - (b_1 a_\varepsilon)^{1-\frac{n-1}{p-1}}\right) & p \neq n, \\ \ln\left(\dfrac{\varepsilon}{b_1 a_\varepsilon}\right) & p = n, \end{cases} \sim \begin{cases} a_\varepsilon^{1-\frac{n-1}{p-1}} & p < n, \\ \ln\left(\dfrac{\varepsilon}{a_\varepsilon}\right) & p = n, \\ \varepsilon^{1-\frac{n-1}{p-1}} & p > n, \end{cases} \tag{1.141}$$

which concludes the proof. □

Remark 1.4. It is not surprising that the $W^{1,n}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ behaves differently. For example, radial solution solution of $\Delta_n u = 0$ in $\mathbb{R}^n$ includes $\ln|x|$, whereas for any other p-Laplacian radial solutions are of power type.

From this point forward we will assume that

$$G_0 \text{ is star-shaped.} \tag{1.142}$$

1.5.3 Behaviour of $\int_{S_\varepsilon}$ and appearance of a_ε^*

Define function $M_\varepsilon(x)$ as the unique Y_ε - periodic built through the solution of the boundary value problem

$$\begin{cases} \Delta_p m_\varepsilon = \mu_\varepsilon & x \in Y_\varepsilon = \varepsilon Y \setminus \overline{a_\varepsilon G_0}; \\ \partial_{\nu_p} m_\varepsilon = 1 & x \in \partial(a_\varepsilon G_0) = S_\varepsilon^0; \\ \partial_{\nu_p} m_\varepsilon = 0 & x \in \partial Y_\varepsilon \setminus S_\varepsilon^0; \end{cases}, \qquad \int_{Y_\varepsilon} m_\varepsilon(x)dx = 0. \tag{1.143}$$

where μ_ε is a constant defined so as to satisfy the integrability condition

$$\mu_\varepsilon = \frac{\varepsilon^{-n} a_\varepsilon^{n-1} |\partial G_0|}{1 - (a_\varepsilon \varepsilon^{-1})^n |G_0|}. \tag{1.144}$$

That is

$$M_\varepsilon(x) = m_\varepsilon(x - P_\varepsilon^j), \qquad x \in Y_\varepsilon^j. \tag{1.145}$$

The aim of this section is to prove the following result

Theorem 1.6. *Assume that $a_\varepsilon \ll \varepsilon$,*

$$\beta(\varepsilon)\|M_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1} \to 0 \tag{1.146}$$

as $\varepsilon \to 0$ *and let*

$$\beta_0 = \lim_{\varepsilon \to 0} \mu_\varepsilon \beta(\varepsilon). \tag{1.147}$$

Then, for all sequence $v_\varepsilon \in L^p(\Omega)$ *such that* $v_\varepsilon \to v$ *in* $L^p(\Omega)$ *we have that*

$$\beta(\varepsilon) \int\limits_{S_\varepsilon} v_\varepsilon dS \to \beta_0 \int\limits_{\Omega} v dx \tag{1.148}$$

as $\varepsilon \to 0$.

Function m_ε has the nice property of allowing us to write, for any test function $\varphi \in W^{1,p}(Y_\varepsilon)$,

$$-\int\limits_{Y_\varepsilon} |\nabla m_\varepsilon|^{p-2} \nabla m_\varepsilon \nabla \varphi dx + \int\limits_{S_\varepsilon^0} \varphi ds = \mu_\varepsilon \int\limits_{Y_\varepsilon} \varphi dx. \tag{1.149}$$

We will use the following fact

Lemma 1.5.3. *Let* $p > 1$. *Then*

$$\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^{p-1} \leq C a_\varepsilon^{n-1} \left(\varepsilon^{-n+p} + \tau_\varepsilon\right)^{\frac{1}{p}}. \tag{1.150}$$

Proof. Setting in (1.149) $\varphi = m_\varepsilon$ and the definition of $m_\varepsilon(x)$, we obtain

$$\begin{aligned}
\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^{p^2} &\leq \left(\left| \int_{S_\varepsilon^0} m_\varepsilon ds \right| + \mu_\varepsilon \left| \int_{Y_\varepsilon} m_\varepsilon dx \right| \right)^p \\
&\leq \left(\int_{S_\varepsilon^0} |m_\varepsilon| ds + \mu_\varepsilon \times 0 \right)^p \\
&\leq \left(\left(\int_{S_\varepsilon^0} 1^{p'} \right)^{\frac{1}{p'}} \left(\int_{S_\varepsilon^0} |m_\varepsilon|^p \right)^{\frac{1}{p}} \right)^p \\
&\leq \left(\int_{S_\varepsilon^0} 1 ds \right)^{p-1} \|m_\varepsilon\|_{L^p(S_\varepsilon^0)}^p \\
&\leq C_1 a_\varepsilon^{(n-1)(p-1)} \|m_\varepsilon\|_{L^p(S_\varepsilon^0)}^p \leq \\
&\leq C_2 a_\varepsilon^{(n-1)(p-1)} a_\varepsilon^{n-1} \left(\varepsilon^{-n} \|m_\varepsilon\|_{L^p(Y_\varepsilon)}^p + \tau_\varepsilon \|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^p \right) \\
&\leq C_3 a_\varepsilon^{p(n-1)} \left(\varepsilon^{-n+p} + \tau_\varepsilon \right) \|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)}^p
\end{aligned} \tag{1.151}$$

which concludes the proof. □

We have that

$$\frac{a_\varepsilon^{p(n-1)}\varepsilon^{-n+p}}{a_\varepsilon^{n(p-1)}} = (a_\varepsilon \varepsilon^{-1})^{p-n} \tag{1.152}$$

which in the case $a_\varepsilon = C_0 \varepsilon^\alpha$ results in

$$\frac{a_\varepsilon^{p(n-1)}\varepsilon^{-n+p}}{a_\varepsilon^{n(p-1)}} = C\varepsilon^{(p-n)(\alpha-1)}. \tag{1.153}$$

Using the previous estimates we get:

Corollary 1.1. *Let $p > 1$. Then*

$$\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)} \leq \begin{cases} C a_\varepsilon^{\frac{n}{p}} & p < n, \\ C a_\varepsilon \ln\left(\dfrac{\varepsilon}{a_\varepsilon}\right)^{\frac{1}{n}} & p = n, \\ C a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}} & p > n. \end{cases} \tag{1.154}$$

This allows us to write the following result:

Corollary 1.2. *Let $a_\varepsilon \ll \varepsilon$. Then, since $|\Upsilon_\varepsilon| \sim \varepsilon^{-n}$,*

$$\|\nabla M_\varepsilon\|_{L^p(\cup_j Y_\varepsilon^j)} \leq \begin{cases} C(a_\varepsilon \varepsilon^{-1})^{\frac{n}{p}} & 1 < p \leq n, \\ C(a_\varepsilon \varepsilon^{-1}) \ln(a_\varepsilon^{-1}\varepsilon)^{\frac{1}{n}} & p = n, \\ C(a_\varepsilon \varepsilon^{-1})^{\frac{n-1}{p-1}} & p > n. \end{cases} \tag{1.155}$$

Corollary 1.3. *Let $v_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Then,*

$$\beta(\varepsilon) \int_{S_\varepsilon} v_\varepsilon = \rho_\varepsilon + \beta(\varepsilon)\mu_\varepsilon \sum_{j \in \Upsilon_\varepsilon} \int_{Y_\varepsilon^j} v_\varepsilon dx \tag{1.156}$$

where

$$0 \leq \rho_\varepsilon \leq C\beta(\varepsilon) \|M_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1}, \tag{1.157}$$

and C depends only on Ω and $\|v_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)}$.

Proof.

$$
\begin{aligned}
\beta(\varepsilon)\int\limits_{S_\varepsilon} v_\varepsilon &= \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j} div(|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j v_\varepsilon) = \\
&= \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j} |\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j \nabla v_\varepsilon dx + \\
&\quad + \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j} (\Delta_p M_\varepsilon^j) v_\varepsilon dx = \\
&= \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j} |\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j \nabla v_\varepsilon dx + \\
&\quad + \beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon\int\limits_{Y_\varepsilon^j} v_\varepsilon dx \qquad (1.158)
\end{aligned}
$$

Using Hölder's inequality

$$
\beta(\varepsilon)\int\limits_{\Omega_\varepsilon} |\nabla M_\varepsilon|^{p-1}|\nabla v_\varepsilon| dx \le C\beta(\varepsilon)\|M_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{p-1}. \qquad (1.159)
$$

which concludes the proof. □

This is the reason why critical scales appear in the homogenization process for $p \le n$ and none can appear when $p > n$. The critical case occurs when $\rho_\varepsilon \not\to 0$. In particular, if $\rho_\varepsilon \to C \neq 0$ (where ρ_ε is the quantity given by (1.156)) then the critical case rises, as we will see in Section 1.5.8.

Proof of Theorem 1.6. Due to Corollary 1.3 and

$$
\left|\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j} v_\varepsilon - \int_\Omega v_\varepsilon\right| \le \|v_\varepsilon\|_{L^p(\Omega)}\left|\Omega\setminus\bigcup_{j\in\Upsilon_\varepsilon} Y_\varepsilon^j\right| \to 0, \qquad (1.160)
$$

which completes the proof. □

Also, this explicit computation explains the *a priori* strange formula for the critical scales. Consider the good scaling $\beta^*(\varepsilon)$.

Corollary 1.4. *We have that*

$$\beta^*(\varepsilon)\|M_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \le \begin{cases} C a_\varepsilon^{\frac{p-n}{p}} \varepsilon^{\frac{n}{p}} & p < n, \\ C\varepsilon \ln\left(a_\varepsilon^{-1}\varepsilon\right)^{\frac{n-1}{n}} & p = n, \\ C\varepsilon & p > n. \end{cases} \tag{1.161}$$

Remark 1.5. The right hand side of (1.161) is rather significant. As will see immediately in Theorem 1.6, the fact that this right hand goes to 0 as $\varepsilon \to 0$ is a sufficient condition for the integrals to behave nicely in the limit, and thus will see later that we are in the subcritical case. A priori, these estimates need to be sharp. However, as will see in Section 1.5.4, it is sharp, and a_ε^* is the value such that the RHS of (1.161) converges to a constant.

Remark 1.6. If $a_\varepsilon = C_0\varepsilon^\alpha$ and $\beta(\varepsilon) = \varepsilon^{-(\alpha(n-1)-n)}$, then the result implies that

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} v_\varepsilon dS \to C_0^{n-1}|\partial G_0| \int_\Omega v dx \tag{1.162}$$

as $\varepsilon \to 0$ if $\alpha < \frac{n}{n-p}$. If $\alpha > \frac{n}{n-p}$ then

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} v_\varepsilon dS \to 0. \tag{1.163}$$

1.5.3.1 $L^p - L^q$ estimates for S_ε

It is obvious that there are $L^p - L^q$ estimates for S_ε, in the sense that, if $0 < p < q$, for every $\varepsilon > 0$ there exists a constant C_ε such that

$$\left(\int_{S_\varepsilon} |v|^p\right)^{\frac{1}{p}} \le C_\varepsilon \left(\int_{S_\varepsilon} |v|^q\right)^{\frac{1}{q}} \qquad \forall v \in L^q(S_\varepsilon). \tag{1.164}$$

The interesting question is whether we can do this with uniform constant C_ε. The fact is that such results are true, but we have to be careful with the choice of constants. We will use this in the following sections.

Lemma 1.5.4. *Let $0 < p < q$. Then, there exists C, independent of ε, such that*

$$\left(\beta^*(\varepsilon)\int_{S_\varepsilon} |v|^p\right)^{\frac{1}{p}} \le C\left(\beta^*(\varepsilon)\int_{S_\varepsilon} |v|^q\right)^{\frac{1}{q}} \qquad \forall v \in L^q(S_\varepsilon). \tag{1.165}$$

1.5.4 The critical scales a_ε^*

It has been long present in the literature that the critical size of holes in the case $1 < p < n$ is

$$a_\varepsilon^* = \varepsilon^{\frac{n}{n-p}}. \tag{1.166}$$

We will see in Sections 1.5.6, 1.5.7 and 1.5.8 that the three situations are entirely different. This critical value aligns precisely with estimate (1.161). As indicated in Remark 1.5, a_ε^* is the value such that the RHS of (1.161) converges to a constant. If we write $a_\varepsilon = C_0\varepsilon^\alpha$, this critical exponent $\alpha^* = \frac{n}{n-p}$ blows up as $p \to n$.

As it turns out, a critical case also exists for the case $p = n$, and this was studied in [PS15]. The critical choice, as presented in that paper, is the one that satisfies

$$\beta(\varepsilon) a_\varepsilon^{n-1} \varepsilon^{-n} \to C_1^2, \tag{1.167}$$

$$\frac{1}{\beta(\varepsilon)^{\frac{1}{n-1}} a_\varepsilon \ln \frac{4a_\varepsilon}{\varepsilon}} \to -C_2^2, \tag{1.168}$$

where $C_1, C_2 \neq 0$. Again, estimate (1.161) is sharp. Although this a bit more convoluted. Equation (1.167) only indicates $\beta(\varepsilon) \sim \beta^*(\varepsilon)$. Let us read (1.168) carefully

$$\begin{aligned} 1 &\sim -\frac{1}{\beta(\varepsilon)^{\frac{1}{n-1}} a_\varepsilon \ln \frac{4a_\varepsilon}{\varepsilon}} \\ &\sim \frac{1}{\beta(\varepsilon)^{\frac{1}{n-1}} a_\varepsilon \ln \frac{\varepsilon}{4a_\varepsilon}} \end{aligned}$$

Since $D_\varepsilon \sim 1$ is equivalent to $\frac{1}{D_\varepsilon} \sim 1$ we have that

$$\begin{aligned} 1 &\sim \beta(\varepsilon)^{\frac{1}{n-1}} a_\varepsilon \ln \frac{\varepsilon}{a_\varepsilon} \\ &\sim a_\varepsilon^{-1} \varepsilon^{\frac{n}{n-1}} a_\varepsilon \ln \frac{\varepsilon}{a_\varepsilon} \\ &\sim \varepsilon^{\frac{n}{n-1}} \ln \frac{\varepsilon}{a_\varepsilon}. \end{aligned}$$

This is exactly what we anticipated in (1.161). Again, a_ε^* is the value such that the RHS of (1.161) converges to a constant. We can give the critical scale explictly

$$a_\varepsilon^* = \varepsilon e^{-\varepsilon^{-\frac{n}{n-1}}}. \tag{1.169}$$

We point out that this critical scale is not of the form $a_\varepsilon^* = \varepsilon^\alpha$, but rather the convergence to 0 of a_ε^* is much faster as $\varepsilon \to 0$.

For $p > n$ we deduce that there exists no critical scale. In [DGCPS17b] the authors first noted that no critical scale in the realm $a_\varepsilon = \varepsilon^\alpha$ may exist. The estimates in this thesis go much further. Since, for $p > n$, the RHS of (1.161) always converges to 0 we can guaranty that no a_ε critical may exist. In this sense we can say that, for $p > n$, $a_\varepsilon^* = 0$. With this notation, the cases $p > n$ in Theorem 1.7 are a direct improvement of the results in [DGCPS17b].

To summarize, going forward we will keep in mind that the critical scale is precisely

$$a_\varepsilon^* = \begin{cases} \varepsilon^{\frac{n}{n-p}} & 1 < p < n, \\ \varepsilon e^{-\varepsilon^{-\frac{n}{n-1}}} & p = n, \\ 0 & p > n. \end{cases} \tag{1.170}$$

1.5.5 Double approximation arguments

In the process of homogenization is typically more convenient to work with a function σ that is as smooth as possible. Many authors have only stated their results for such smooth σ. Since the central theme of this thesis deals with root-type σ, it was one of our aims to develop a framework to extend the result to general σ. A natural way to do this, which has been successful in the past, is to consider uniform approximations. This is the subject of this section.

The following comparison results are obtained in [DGCPS17d]. They allow us to extend the results proved for σ smooth to the case of σ non as smooth.

Lemma 1.5.5 ([DGCPS17d])**.** *Let $\sigma, \hat\sigma$ be continuous nondecreasing functions such that $\sigma(0) = 0$ and $u, \hat u$ be their respective solutions of* (1.12) *with $f^\varepsilon = f \in L^{p'}(\Omega)$ and $g^\varepsilon = 0$. Then, there exist constants C depending on p, but independent of ε, such that*

i) If $1 < p < 2$

$$\|\nabla(u_\varepsilon - \hat u_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \le C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma - \hat\sigma\|_{\mathscr{C}(\mathbb{R})} \left(\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{2-p} + \|\nabla \hat u_\varepsilon\|_{L^p(\Omega_\varepsilon)}^{2-p}\right)^{\frac{2}{p}}. \tag{1.171}$$

ii) If $p \ge 2$ then

$$\|\nabla(u_\varepsilon - \hat u_\varepsilon)\|_{L^p(\Omega_\varepsilon)}^{p-1} \le C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma - \hat\sigma\|_{\mathscr{C}(\mathbb{R})}. \tag{1.172}$$

We need to study a sufficiently large family of functions σ so that the uniform $\mathbb{R}$ approximation by smooth functions is possible. The following condition seems to fit our purposes:

$$|\sigma(t)-\sigma(s)| \leq C(|t-s|^{\alpha}+|t-s|^{p}) \qquad \forall t,s \in \mathbb{R}, \tag{1.173}$$

for some $0<\alpha\leq 1$ and $p\geq 1$. It represents "local Hölder" continuity, in the sense that there is no need for the function to be differentiable. On the other hand, as $|s-t|\to+\infty$, the function σ behaves like a power, and then σ can be non sublinear at infinity.

We have the following approximation result:

Lemma 1.5.6. *Let $\sigma\in\mathscr{C}(\mathbb{R})$, nondecreasing and there exists $0<\alpha\leq 1$, $p>1$ such that* (1.173) *holds. Then, for every $0<\delta<\frac{1}{4C}$ there exists $\sigma_\delta\in\mathscr{C}(\mathbb{R})$ (piecewise linear) such that*

$$\|\sigma_\delta-\sigma\|_{\mathscr{C}(\mathbb{R})}\leq\delta, \tag{1.174}$$

$$0\leq\sigma_\delta'\leq D\delta^{1-\frac{1}{\alpha}}, \tag{1.175}$$

where D depends only on C,α,p.

The idea now is to consider the solution $u_{\varepsilon,\delta}$ of problems

$$\begin{cases}-\Delta_p u_{\varepsilon,\delta}=f^{\varepsilon} & \Omega_\varepsilon,\\ \partial_{\nu_p}u_{\varepsilon,\delta}+\sigma_\delta(u_{\varepsilon,\delta})=g^{\varepsilon} & S_\varepsilon,\\ u_{\varepsilon,\delta}=0 & \partial\Omega.\end{cases} \tag{1.176}$$

The process is the following. Pass to the limit as $\varepsilon\to 0$ for δ fixed, and characterize the limit of the solution $P_\varepsilon u_{\varepsilon,\delta}$ as $\varepsilon\to0$ to some function u_δ. Then study the limit of function u_δ as $\delta\to0$ to a certain function $\widehat{u}$, the equation for which can be obtained through standard theory. The idea is to construct uniform bounds, that allow us to show $\widehat{u}$ is the limit of $P_\varepsilon u_\varepsilon$. We will see an example of application of this reasoning in the following section.

1.5.6 Homogenization of the subcritical cases $a_\varepsilon^*\ll a_\varepsilon\ll\varepsilon$

In this section we will study the limit behaviour for

$$1<p<+\infty \qquad a_\varepsilon^*\ll a_\varepsilon\ll\varepsilon. \tag{1.177}$$

Due to the definitions of β_0 (see (1.147)) and μ_ε (see (1.144)) we have that

$$\beta_0 = |\partial G_0| \lim_{\varepsilon \to 0} \beta(\varepsilon)\beta^*(\varepsilon)^{-1}. \tag{1.178}$$

The aim of this subsection will be to prove the following result:

Theorem 1.7. *Let $1 < p < n$, $f^\varepsilon = f \in L^{p'}(\Omega)$,$g^\varepsilon = g \in W^{1,\infty}(\Omega)$, $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, $\sigma \in \mathscr{C}(\mathbb{R})$ nondecreasing such that $\sigma(0) = 0$ and*

$$|\sigma(v)| \le C(1 + |u|^{p-1}). \tag{1.179}$$

Then the following results hold:

i) Let $\beta_0 < +\infty$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is the unique solution of

$$\begin{cases} -\Delta_p u + \beta_0 \sigma(u) = f + \beta_0 g & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{1.180}$$

ii) Let $\beta_0 = +\infty$, $g = 0$ and $\sigma \in \mathscr{C}^1$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and u satisfies

$$u(x) \in \sigma^{-1}(0) \tag{1.181}$$

a.e. in Ω. In other words, $\sigma(u(x)) = 0$ for a.e. $x \in \Omega$. In particular, if σ is strictly increasing then $u = 0$.

The regularity of σ will be the key difficulty of our approach. As mentioned in the previous section, let us first study the smooth case.

Remark 1.7. When $a_\varepsilon = C_0\varepsilon^\alpha$ and $\beta_\varepsilon = \varepsilon^{-\gamma^*}$ then it is easy to compute that

$$\beta_0 = |\partial G_0| C_0^{n-1}. \tag{1.182}$$

This coefficient is obtained in all cases.

Smooth kinetic Just the estimates on the normal derivatives allows to homogenize the noncritical case directly if σ is a uniformly Lipschitz continuous function, since in that case the sequence $\sigma(u_\varepsilon)$ in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is bounded, and we can pass to the limit in the standard weak formulation. However, a further analysis allows to do so in many different settings.

Even the case of σ monotone nondecreasing such that $\sigma(0) = 0$ and σ' locally bounded is easy to understand. Then, we can pass to the limit in formulation (1.93). If we consider test functions $v \in W_0^{1,\infty}(\Omega)$, then $\sigma(v) \in W_0^{1,\infty}(\Omega)$. From the definition of (1.12) it is immediate that $\|\nabla u_\varepsilon\|_{W_0^{1,p}(\Omega)}$ is bounded and hence that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega_\varepsilon)$. Then, if $\beta(\varepsilon) \sim \beta^*(\varepsilon)$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot (v-u)dx + \beta_0 \int_\Omega \sigma(v)(v-u) \geq \int_{\Omega_\varepsilon} fvdx. \tag{1.183}$$

The case $\beta(\varepsilon) \gg \beta^*(\varepsilon)$ can be studied in a even easier way. In the weak formulation we get

$$\beta^*(\varepsilon)\beta(\varepsilon)^{-1} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p \nabla u_\varepsilon \cdot \nabla v + \beta^*(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon)v = \beta^*(\varepsilon)\beta(\varepsilon)^{-1} \int_{S_\varepsilon} fv \tag{1.184}$$

for any $v \in W^{1,p}(\Omega_\varepsilon, \delta\Omega)$. Then, at least for $\sigma \in W^{1,\infty}(\mathbb{R})$ monotone nondecreasing such that $\sigma(0) = 0$, as $\varepsilon \to 0$

$$\int_\Omega \sigma(u_\varepsilon)v = 0. \tag{1.185}$$

Hence $\sigma(u_\varepsilon) = 0$. It is important to remark that in the previous literature the limits were identified to $u_\varepsilon = 0$, but this is only because σ is required to be strictly increasing (see Table 1.1).

Non smooth kinetic The case of $\sigma \in \mathscr{C}(\mathbb{R})$, nondecreasing and $\sigma(0) = 0$ and the case $\beta(\varepsilon)\beta^*(\varepsilon)^{-1} \to 0$ can be treated thanks to the approximation Lemma 1.5.5. In essence

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{W^{1,p}} \leq C\|\sigma - \sigma_\delta\|_\infty^\alpha \tag{1.186}$$

for some power $\alpha > 0$, where σ_δ is smooth. Hence, as $\varepsilon \to 0$, $P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta$ in $W^{1,p}(\Omega)$, where u_δ is the solution of

$$\begin{cases} -\Delta_p u_\delta + \beta_0 \sigma_\delta(u_\delta) = f + \beta_0 g & \Omega, \\ u_\delta = 0 & \partial\Omega. \end{cases} \tag{1.187}$$

Furthermore, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$, and the uniform comparison holds in the limit

$$\|u - u_\delta\|_{W^{1,p}} \leq C\|\sigma - \sigma_\delta\|_\infty^\alpha. \tag{1.188}$$

It is easy to show that, as $\delta \to 0$, we have that $u_\delta \rightharpoonup \widehat{u}$ in $W^{1,p}(\Omega)$, the solution of (1.180). As we pass $\delta \to 0$ in (1.188) we deduce that $u = \widehat{u}$.

1.5.7 Homogenization of the supercritical case $a_\varepsilon \ll a_\varepsilon^*$

As mentioned before this case is not very relevant. The proof is very simple. Here we present briefly the proof by Shaposhnikova and Zubova in [ZS13].

Theorem 1.8. *Let $\sigma \in W^{1,\infty}(\mathbb{R})$ and let us us consider u_ε the solution of* (1.12) *for $p = 2$, where $f^\varepsilon = f \in L^2(\Omega)$ and $g^\varepsilon = 0$. Let $a_\varepsilon \ll a_\varepsilon^*$. Then, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u is the unique solution of*

$$\begin{cases} -\Delta u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{1.189}$$

Remark 1.8. Notice that the previous result is independent of $\beta(\varepsilon)$. If $g \not\equiv 0$ then, due to Proposition 1.6 we should consider only the cases $\beta(\varepsilon) \ll \beta^*(\varepsilon)$ and $\beta(\varepsilon) \sim \beta^*(\varepsilon)$.

Proof. We already know that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ independently of $\beta(\varepsilon)$, due to Proposition 1.6, since $g^\varepsilon = 0$. Let

$$K_0 = \max_{y \in G_0} |y|.$$

Let us define, for $j \in \Upsilon_\varepsilon$, functions $\psi_\varepsilon^j \in C_0^\infty(\Omega)$ such that $0 \le \psi_\varepsilon^j \le 1$ and

$$\psi_\varepsilon^j(x) = \begin{cases} 0 & \text{if } |x - P_\varepsilon^j| \ge 2K_0 a_\varepsilon, \\ 1 & \text{if } |x - P_\varepsilon^j| \le K_0 a_\varepsilon, \end{cases} \qquad |\nabla \psi_\varepsilon^j| \le K a_\varepsilon^{-1}, \tag{1.190}$$

and let

$$\psi_\varepsilon = \sum_{j \in \Upsilon_\varepsilon} \psi_\varepsilon^j. \tag{1.191}$$

It is clear that $\psi_\varepsilon = 1$ in G_ε^j. Due to the size of the support, it is also easy to check that

$$\psi_\varepsilon \to 0 \quad \text{in } H^1(\Omega). \tag{1.192}$$

Let $\varphi \in H_0^1(\Omega)$. Taking $\varphi(1 - \psi_\varepsilon)$ as a test function in the weak formulation of (1.12) for $p = 2$, we have that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\varphi(1 - \psi_\varepsilon)) + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon)\varphi(1 - \psi_\varepsilon) = \int_{\Omega_\varepsilon} f\varphi(1 - \psi_\varepsilon). \tag{1.193}$$

Since $(1 - \psi_\varepsilon) = 0$ on S_ε we have that

$$\beta(\varepsilon) \int_{S_\varepsilon} \sigma(u_\varepsilon)\varphi(1 - \psi_\varepsilon) = 0. \tag{1.194}$$

On the other hand $\varphi(1-\psi_\varepsilon) \to \varphi$ in H^1 we have that the limit as $\varepsilon \to 0$ of equation (1.193) is

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi. \tag{1.195}$$

This completes the proof. □

Remark 1.9. As it is clearly seen in the proof, the information of the limit weak formulation is revealed by the choice of a specific sequence of test function. The auxiliary function ψ_ε oscillate, by construction, with the repetition of particle. This is precisely why this method is known as *oscillating test function*.

In the following sections we will present the results obtained by the author in the critical cases.

1.5.8 Homogenization of the critical case when G_0 is a ball and $1 < p < n$

In this section we will study the behaviour for

$$1 < p < n \qquad a_\varepsilon = C_0 \varepsilon^\alpha \qquad \beta(\varepsilon) = \varepsilon^{-\gamma} \qquad \alpha = \frac{n}{n-p} \qquad \gamma = \alpha(n-1) - n.$$

In this cases, the limit behaviour is the solution of the following problem:

$$\begin{cases} -\Delta_p u + \mathscr{A} |H(u)|^{p-2} H(u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{1.196}$$

where

$$\mathscr{A} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p} \omega_n \tag{1.197}$$

and H is the solution of the functional equation

$$\mathscr{B}_0 |H(x,s)|^{p-2} H(x,s) = \sigma(x, s - H(x,s)) - g(x) \tag{1.198}$$

$$\mathscr{B}_0 = \left(\frac{n-p}{C_0(p-1)}\right)^{p-1} \tag{1.199}$$

where $g^\varepsilon(x) = \beta(\varepsilon) g(x)$.

As it can be seen in Table 1.1 there are many previous works in this direction. The term $|H(u)|^{p-2}H(u)$ is usually refered to as *strange term* in homogenization. Since σ and H are different functions it can be said that the nature of the reaction changes. This change of behaviour between the critical and subcritical cases has driven some researchers to make a connection between this critical case and the unexpected properties of well-studied elements when presented in nanoparticle form. For example, while a presentation as of gold as microparticles is inert (behaviour at microscale and macroscale coincide), some studies have shown that gold nanoparticles are, in fact, catalysts (see [Sch+02]).

1.5.8.1 Weak convergence

This case is the trickiest. In this direction we first studied the case of power type reactions $\sigma(u) = |u|^{q-1}u$, where $0 \le q < 1$. The case $q = 0$ corresponds to the case of the Heaviside functions (which needs to be understood in the sense of maximal monotone operators). In this direction we published [DGCPS16]. The results and techniques applied in these cases were later generalized in [DGCPS17c], that deals with a general maximal monotone operator and $1 < p < n$. We dealt firstly with the case $g = 0$, $\sigma = \sigma(u)$ and G_0 is a ball.

The good setting for this equation is the energy setting, and we consider the definition of solution (1.94).

As its turns out, equation (1.198), which can be rewritten for maximal monotone operators as

$$\mathscr{B}_0 |H(s)|^{p-2} H(s) \subset \sigma(s - H(s)), \tag{1.200}$$

has the following nice property

Lemma 1.5.7. *Let σ be a maximal monotone operator. Then* (1.200) *has a unique solution H, a nondecreasing nonexpansive function $\mathbb{R} \to \mathbb{R}$ (i.e. $0 < H' \le 1$ a.e.).*

In fact, function H can be written in the following way

$$H(r) = (I + \sigma^{-1} \circ \Theta_{n,p})^{-1}(r), \tag{1.201}$$

where

$$\Theta_{n,p}(s) = \mathscr{B}_0 |s|^{p-2} s. \tag{1.202}$$

and $\mathscr{B}_0$ is given by (1.199)

We proved the following

Theorem 1.9 ([DGCPS17c])**.** *Let $n \ge 3$, $1 < p < n$, $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$ and G_0 be a ball. Let σ be any maximal monotone operator of $\mathbb{R}^2$ with $0 \in \sigma(0)$ and let $f \in L^{p'}(\Omega)$. Let $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be the (unique) weak solution of problem* (1.12) *where $f^\varepsilon = f$ and $g^\varepsilon = 0$. Then there exists an extension $\tilde{u}_\varepsilon$ of u_ε such that $\tilde{u}_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$ where*

$u \in W_0^{1,p}(\Omega)$ *is the (unique) weak solution of the problem* (1.196) *associated to the function* H, *defined by* (1.201).

We seek to apply oscillating test functions of the form $v_\varepsilon = v - H(v)W_\varepsilon$, where v is a test function of the limit problem. For this we define the auxiliary problem

$$W_\varepsilon = \begin{cases} w_\varepsilon(x - P_\varepsilon^j) & x \in B_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \\ 0 & x \notin \cup_j B_\varepsilon^j, \\ 1 & x \in \cup_j G_\varepsilon^j, \end{cases} \tag{1.203}$$

where P_ε^j is the center of $Y_\varepsilon^j = \varepsilon j + \varepsilon[-\frac{1}{2}, \frac{1}{2}]^n$, $B_\varepsilon^j = B\left(P_\varepsilon^j, \frac{\varepsilon}{4}\right)$, $G_\varepsilon^j = B\left(P_\varepsilon^j, a_\varepsilon\right)$ and w_ε is the solution of

$$\begin{cases} -\Delta_p w_\varepsilon = 0 & a_\varepsilon < |x| < \dfrac{\varepsilon}{4}, \\ w_\varepsilon = 0 & |x| = \dfrac{\varepsilon}{4}, \\ w_\varepsilon = 1 & |x| = a_\varepsilon. \end{cases} \tag{1.204}$$

This function can be computed explicitly

$$w_\varepsilon(x) = \frac{|x|^{-\frac{n-p}{p-1}} - \left(\frac{\varepsilon}{4}\right)^{-\frac{n-p}{p-1}}}{a_\varepsilon^{-\frac{n-p}{p-1}} - \left(\frac{\varepsilon}{4}\right)^{-\frac{n-p}{p-1}}}. \tag{1.205}$$

The profile of this radial function can be seen in Figure 1.4

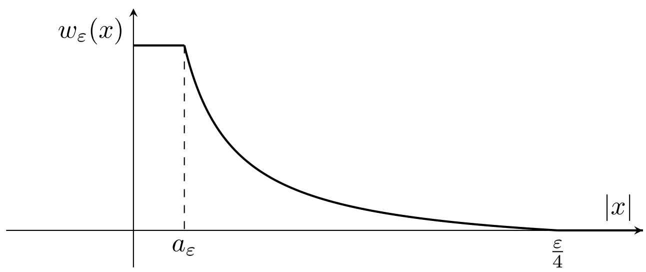


Fig. 1.4 Function w_ε

For $1 \le q \le p$

$$\int_\Omega |\nabla W_\varepsilon|^q dx \le K\varepsilon^{\frac{n(p-q)}{n-p}}, \tag{1.206}$$

hence

$$W_\varepsilon \to 0 \qquad \begin{cases} \text{strongly in } W_0^{1,q}(\Omega) \text{ if } 1 \le q < p, \\ \text{weakly in } W_0^{1,p}(\Omega). \end{cases} \tag{1.207}$$

The second statement may not seem obvious. However, in $W^{1,p}$ the norm is bounded, and hence there must exist a weak limit. This limit must coincide with the $W^{1,q}$ limits for $q<p$, and therefore must be 0.

Theorem 1.10 (Theorem 1.2 in [DGCPS17c]). *Let $1<p<n$ and $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be a sequence of uniformly bounded norm, $v \in \mathscr{C}_c^\infty(\Omega)$, $h \in W^{1,\infty}(\Omega)$ and let*

$$v_\varepsilon = v - hW_\varepsilon. \tag{1.208}$$

Then

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)dx\right) = \lim_{\varepsilon\to 0}\left(I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon}\right), \tag{1.209}$$

where

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u_\varepsilon)dx \tag{1.210}$$

$$I_{2,\varepsilon} = -\varepsilon^{-\gamma}\mathscr{B}_0 \int_{S_\varepsilon} |h|^{p-2}h(v-h-u_\varepsilon)dS \tag{1.211}$$

$$I_{3,\varepsilon} = -A_\varepsilon \varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} |h|^{p-2}h(v-u_\varepsilon)dS, \tag{1.212}$$

and A_ε is a bounded sequence. Besides, if $\tilde{u}_\varepsilon$ is an extension of u_ε and $\tilde{u}_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ then, for any $v \in W_0^{1,p}(\Omega)$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)dx = \int_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)dx. \tag{1.213}$$

Applying the convexity inequality

$$\Psi(v-H(v)) - \Psi(u_\varepsilon) \le \mathscr{B}_0|H(v)|^{p-2}H(v) \tag{1.214}$$

(this is the reason why $\mathscr{B}_0|H(v)|^{p-2}H(v) \in \sigma(v-H(v))$) and hence the good choice is $h=H(v)$. Thus, we show that we have $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W^{1,p}(\Omega)$ and u satisfies that, at least for

$v \in W^{1,\infty}(\Omega)$,

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot f(v-u) + \mathscr{A} \int_\Omega |H(v)|^{p-2} H(v)(v-u) \geq \int_\Omega fv, \tag{1.215}$$

with $\mathscr{A}$ given by (1.197) which is enough to conclude the result.

1.5.8.2 Some examples

A relevant case in the applications corresponds to the Signorini type boundary condition (1.19), which can be written with the maximal monotone operator (1.24), given

$$H(s) = \begin{cases} H_0(s) & s \geq 0, \\ s & s < 0, \end{cases} \tag{1.216}$$

where

$$\mathscr{B}_0 |H_0(s)|^{p-2} H_0(s) = \sigma_0(s - H_0(s)), \quad s > 0. \tag{1.217}$$

This result was obtained previously in [JNRS14] by *ad hoc* techniques. In [DGCPS17c], we provide it as a corollary of a more general theory.

1.5.8.3 Strong convergence with correctors

It was known in the literature that, at least for smooth σ,

$$\|u - (u_\varepsilon - H(u_\varepsilon)W_\varepsilon)\|_{W^{1,p}(\Omega_\varepsilon, \partial\Omega)} \to 0 \tag{1.218}$$

as $\varepsilon \to 0$. Nonetheless, it seems that no one had noticed that W_ε converges strongly to 0 in $W^{1,q}$ for $q < p$. From this fact, we deduce immediately that

$$\|u - u_\varepsilon\|_{W^{1,q}(\Omega_\varepsilon, \partial\Omega)} \to 0, \qquad \text{for } q < p. \tag{1.219}$$

In the case of Signorini boundary conditions we proved the strong convergence (with the corrector term for $q = p$ and without it when $q < p$), which, for the case $1 < p < 2$, was published in [DGCPS17a].

1.5.9 Homogenization of the critical case when G_0 is not a ball and $p=2$

For many years, several authors have tried to find a functional equation similar to (1.198) for the case in which G_0 is not a ball. In [DGCSZ17] we proved that such equation does not exist in a strict sense. Nonetheless, a equation of form (1.196) still holds, but with a more complicated function H.

Let G_0 be diffeomorphic to a ball, $p=2$ and $a_\varepsilon = C_0\varepsilon^{\alpha^*}$. Then, for any given constant $u \in \mathbb{R}$, we define $\widehat{w}(y;G_0,u)$, for $y \in \mathbb{R}^n \setminus G_0$, as the solution of the following one-parametric family of auxiliary external problems associated to the prescribed asymmetric geometry G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$:

$$\begin{cases} -\Delta_y \widehat{w} = 0 & \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y}\widehat{w} - C_0\sigma(u-\widehat{w}) = 0, & \text{if } y \in \partial G_0, \\ \widehat{w} \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{1.220}$$

We will prove in Section 4 that the above auxiliary external problems are well defined and, in particular, there exists a unique solution $\widehat{w}(y;G_0,u) \in H^1(\mathbb{R}^n \setminus \overline{G_0})$, for any $u \in \mathbb{R}$.

Definition 1.7. Given G_0, we define $H_{G_0} : \mathbb{R} \to \mathbb{R}$ by means of the identity

$$\begin{aligned} H_{G_0}(u) &:= \int_{\partial G_0} \partial_{\nu_y}\widehat{w}(y;G_0,u)\,\mathrm{d}S_y \\ &= C_0 \int_{\partial G_0} \sigma(u-\widehat{w}(y;G_0,u))\,\mathrm{d}S_y, \quad \text{for any } u \in \mathbb{R}. \end{aligned} \tag{1.221}$$

Remark 1.10. Let $G_0 = B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. We can find the solution of problem (1.220) in the form $\widehat{w}(y;G_0,u) = \frac{\mathscr{H}(u)}{|y|^{n-2}}$, where, in this case, $\mathscr{H}(u)$ is proportional to $H_{B_1(0)}(u)$. We can compute that

$$\begin{aligned} H_{G_0}(u) &= \int_{\partial G_0} \partial_\nu \widehat{w}(u,y)\,\mathrm{d}S_y \\ &= \int_{\partial G_0} (n-2)H_{G_0}(u)\,\mathrm{d}S_y \\ &= (n-2)\mathscr{H}(u)\omega(n), \end{aligned}$$

where $\omega(n)$ is the area of the unit sphere. Hence, due to (1.221), $\mathscr{H}(u)$ is the unique solution of the following functional equation

$$(n-2)\mathscr{H}(u) = C_0\sigma(u-\mathscr{H}(u)). \tag{1.222}$$

In this case, it is easy to prove that H is nonexpansive (Lipschitz continuous with constant 1). As mentioned before, this equation has been considered in many papers (see [DGCPS17c] and the references therein).

In [DGCSZ17] we proved several results on the regularity and monotonicity of the homogenized reaction $H_{G_0}(u)$ below. Concerning the convergence as $\varepsilon \to 0$ the following statement collects some of the more relevant aspects of this process:

Theorem 1.11. *Let $n \geq 3$, $a_\varepsilon = C_0\varepsilon^{-\gamma}$, $\gamma = \frac{n}{n-2}$, σ a nondecreasing function such that $\sigma(0) = 0$ and that satisfies* (1.223).

$$|\sigma(s)-\sigma(t)| \leq k_1|s-t|^\alpha + k_2|s-t| \quad \forall s,t \in \mathbb{R}, \quad \textit{for some } 0 < \alpha \leq 1. \tag{1.223}$$

Let u_ε be the weak solution of (1.12) *with $p = 2$, $f^\varepsilon = f \in L^2(\Omega)$ and $g^\varepsilon = 0$. Then there exists an extension to $H_0^1(\Omega)$, still denoted by u_ε, such that $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$ as $\varepsilon \to 0$, where $u_0 \in H_0^1(\Omega)$ is the unique weak solution of*

$$\begin{cases} -\Delta u_0 + C_0^{n-2} H_{G_0}(u_0) = f & \textit{in } \Omega, \\ u_0 = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1.224}$$

Remark 1.11. Since $|H_{G_0}(u)| \leq C(1+|u|)$ it is clear that $H_{G_0}(u_0) \in L^2(\Omega)$.

Lemma 1.5.8. *H_{G_0} is a nondecreasing function. Furthermore:*

i) If σ satisfies (1.223), *then so does H_{G_0}.*

ii) If $\sigma \in \mathscr{C}^{0,\alpha}(\mathbb{R})$, then so is H_{G_0}.

iii) If $\sigma \in \mathscr{C}^1(\mathbb{R})$, then H_{G_0} is locally Lipschitz continuous.

iv) If $\sigma \in W^{1,\infty}(\mathbb{R})$, then so is H_{G_0}.

1.6 Homogenization of the effectiveness factor

We conlude the theoretical results in this chapter by presenting some results on the convergence of the effectiveness factor. They can be found in [DGCT16].

Here, as in [CDLT04], we consider the following regularity assumptions:

$$|g'(v)| \leq C(1+|v|^q), \qquad 0 \leq q < \frac{n}{n-2}, \tag{1.225}$$

and we consider the strictly increasing and uniformly Lipschitz condition:

$$0 < k_1 \leq g'(u) \leq k_2. \tag{1.226}$$

We proved the following result, which seems to fulfill the intuitions expressed by Aris in his many works on the subject:

Theorem 1.12 ([DGCT16])**.** *Assume that* $p = 2$*,* $a_\varepsilon = C_0 \varepsilon^\alpha$*,* $1 \leq \alpha < \frac{n}{n-2}$ *and*

- *If* $\alpha = 1$*,* (1.225)*,*
- *If* $1 < \alpha < \frac{n}{n-2}$*,* (1.226)*.*

Then

$$\frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u_\varepsilon) \to \frac{1}{|\Omega|} \int_\Omega \sigma(u) \qquad \text{as } \varepsilon \to 0. \tag{1.227}$$

This result was later improved in

Theorem 1.13 ([DGCPS17d])**.** *Let* $p > 1$*,* $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$*,* $\beta \sim \beta^*$ *and* σ *be continuous such that* $\sigma(0) = 0$*. Let* u_ε *and* u *be the solutions of* (1.12) *and* (1.180)*. Lastly, assume either:*

i) σ *is uniformly Lipschitz continuous* ($\sigma' \in L^\infty$)*, or*

ii) $\sigma \in \mathscr{C}(\mathbb{R})$ *and there exists* $0 < \alpha \leq 1$ *and* $q > 1$ *such that we have* (1.173) *and*

$$(\sigma(t) - \sigma(s))(t-s) \geq C|t-s|^q, \qquad \forall t, s \in \mathbb{R}. \tag{1.228}$$

Then (1.227) *holds.*

Remark 1.12. As we have seen, the behaviour of $\int_{S_\varepsilon}$ in the critical case is more convoluted. Thus, a convergence of type (1.227) should not be expected. However, results of similar nature, applying the strong convergence with corrector (1.218) are work under development.

1.7 Pointwise comparison of solutions of critical and non-critical solutions

Since we do not have a natural definition of effectiveness in the critical case, the claim that it is “more effective” than the non critical case -a claim that is often made in the Nanotechnology community- is difficult to test. However, we know that, for the non critical cases the effectiveness is increasing with the value of w. Thus, we have studied whether we can find a pointwise comparison of the critical and noncritical limits.

Assume in (1.9) that $\hat{g}_\varepsilon = 0$. Then (1.12) becomes

$$\begin{cases} -\Delta_p u_\varepsilon = 0 & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma}\hat{\sigma}(1) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega. \end{cases} \tag{1.229}$$

Notice that the presence of $w_\varepsilon = 1$ on $\partial\Omega$ is translated to a source in S_ε for u_ε. When $a_\varepsilon \sim a_\varepsilon^*$, the strange term H is the solution of

$$\mathscr{B}_0|H(s)|^{p-2}H(s) = \sigma(s - H(s)) - \hat{\sigma}(1). \tag{1.230}$$

Then w_ε converges weakly in $W^{1,p}(\Omega)$ to w_{crit}, the solution of

$$\begin{cases} -\Delta_p w_{\text{crit}} + \mathscr{A}|h(w_{\text{crit}})|^{p-2}h(w_{\text{crit}}) = 0 & \Omega, \\ w_{\text{crit}} = 1 & \partial\Omega, \end{cases} \tag{1.231}$$

and h is given by

$$|h(w)|^{p-2}h(w) = \hat{\sigma}(1) - |H(1-w)|^{p-2}H(1-w). \tag{1.232}$$

In the noncritical cases, $a_\varepsilon^* \ll a_\varepsilon = C_0\varepsilon^\alpha \ll \varepsilon$, we know that an extension of w_ε converges weakly in $W^{1,p}(\Omega)$ to $w_{\text{non-crit}}$, the solution of

$$\begin{cases} -\Delta_p w_{\text{non-crit}} + \hat{\mathscr{A}}\hat{\sigma}(w_{\text{non-crit}}) = 0 & \Omega, \\ w_{\text{non-crit}} = 1 & \partial\Omega, \end{cases} \tag{1.233}$$

with $\hat{\mathscr{A}} = C_0^{n-1}|\partial G_0|$.

We showed, first in [DGC17] under restricted assumptions and later in [DGCPS17c], that a pointwise comparison holds. We stated the following theorem:

Theorem 1.14 ([DGCPS17c]). *Let $n \geq 3, p \in [2,n)$, $a_\varepsilon \sim a^*_\varepsilon$, $f^\varepsilon = 0$ and $\hat{\sigma} \in \mathscr{C}(\mathbb{R})$ non decreasing such that $\sigma(0) = 0$ Then, we have that*

$$w_{crit} \geq w_{non\text{-}crit}. \tag{1.234}$$

The critical case produces a pointwise " better" reaction.

1.8 Some numerical work for the case $\alpha = 1$

To obtain explicit numerical solutions of the different homogeneous and nonhomogeneous problems COMSOL Multiphysics was applied[1]. Also, using the LiveLink tool, it allows to create a Matlab code that we have used to generalize the construction of the obstacles in our domains.

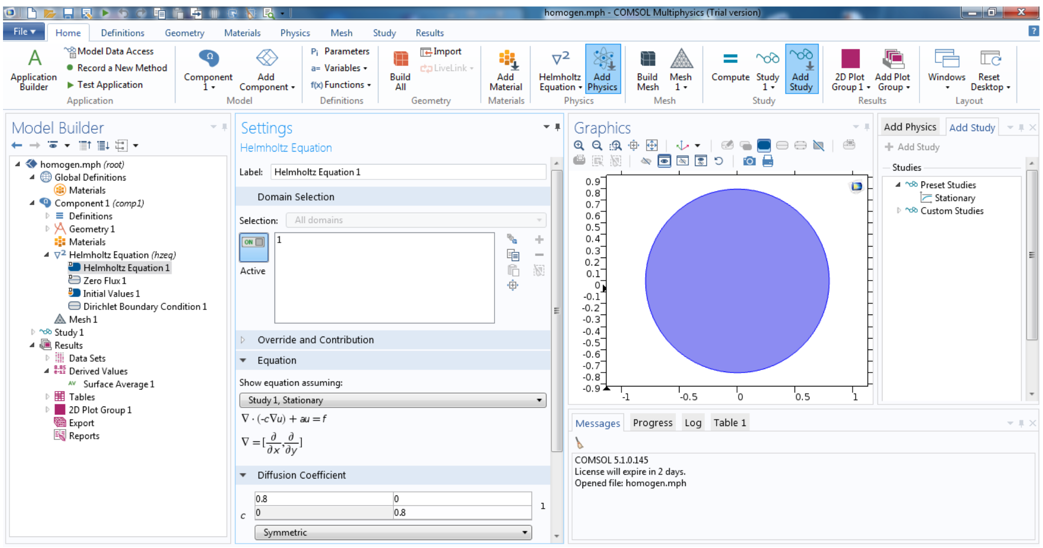
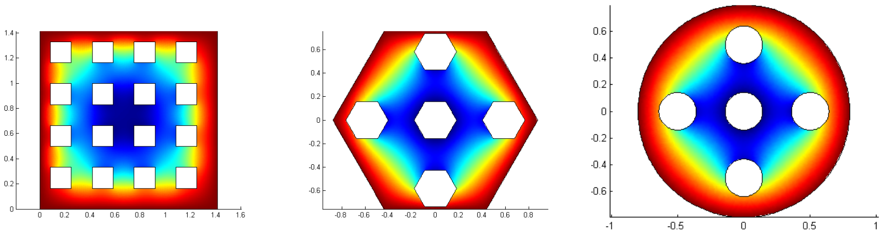


Fig. 1.5 Interfase of the COMSOL software

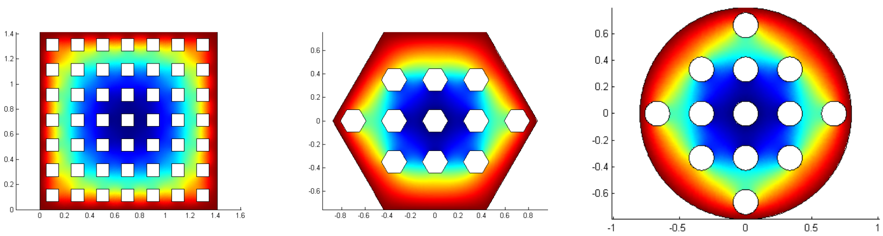
[1] The author wishes to thank Carlos Arechalde Pérez, Pablo Cañones Martín, Denis Coccolo Góngora, Nadia Loy and Amarpreet Kaur for their work during the IX Modelling Week UCM 2015, where the images were produced under the guidance of this author.

1.8.1 Numerical solutions of the non-homogeneous problem

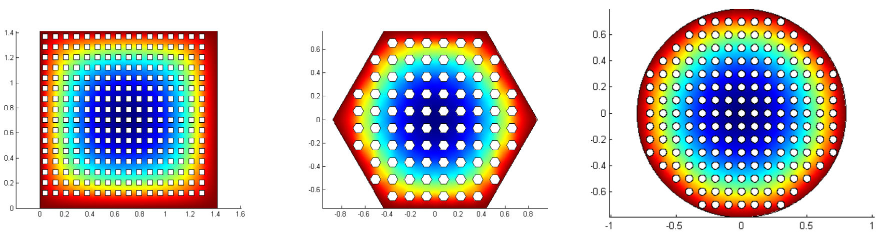
We have simulated that each pellet is inside a periodicity cell. The input parameter of the function is ε, the side of this periodicity cell, that has four times the area of the pellet. Figure 1.6 shows what happened if we change the value of ε



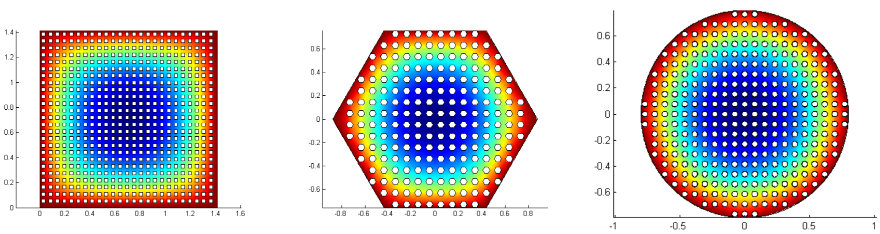
(a) $\varepsilon = \frac{1}{3}$



(b) $\varepsilon = \frac{1}{5}$



(c) $\varepsilon = \frac{1}{12}$

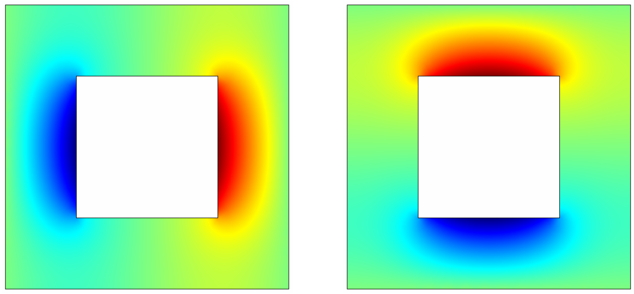


(d) $\varepsilon = \frac{1}{18}$

Fig. 1.6 Level set of the solution of (1.8) for $A^{\varepsilon} = I$, $\sigma(u) = u$ and $a_{\varepsilon} = \varepsilon$. Different values of ε, of domain Ω and G_0 are presented

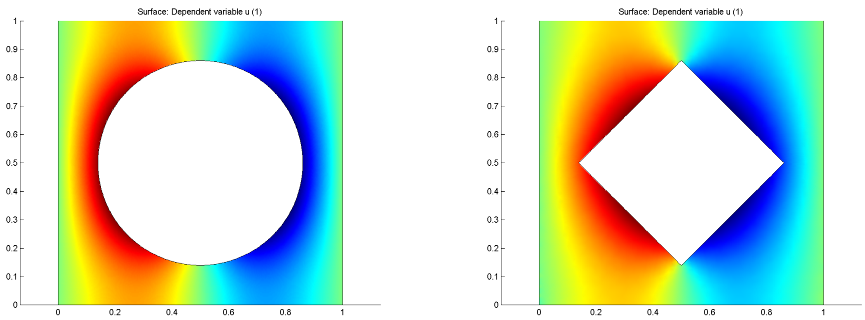
1.8.2 Numerical solutions of the cell problem

The cell problem has solutions which are rather characteristic. If the domain G_0 is symmetric with respect to the axis so is the solution. Due to the periodic boundary conditions, is usually not easy to simulate the solution with "black box" software.



(a) χ_1 solution (b) χ_2 solution

Fig. 1.7 Level set of the solutions of (1.72) for G_0 a square



(a) Circular inclusions (b) Square inclusions

Fig. 1.8 Two obstacles T, and the level sets of the solution of the cell problem (1.72)

1.8.3 Numerical solutions of the homogeneous problem

From all of the nonhomogeneous simulations, the most interesting results are obtained for the smallest ε, because we can see that diffusion in this case is pretty similar to the homogenized problem, as we expected, because of the theoretical results.

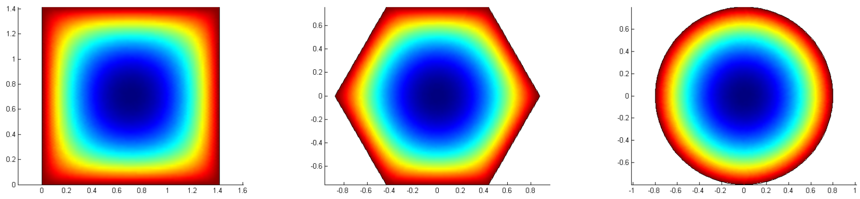
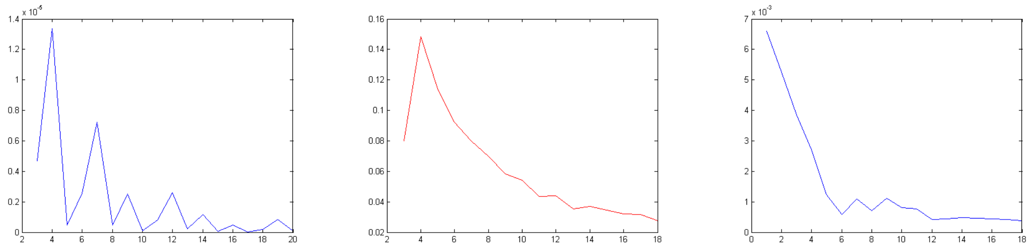


Fig. 1.9 Level sets of the solution of the homogenized problem (1.76), corresponding to the different cases in Figure 1.6

1.8.4 Approximation of the numerical solutions

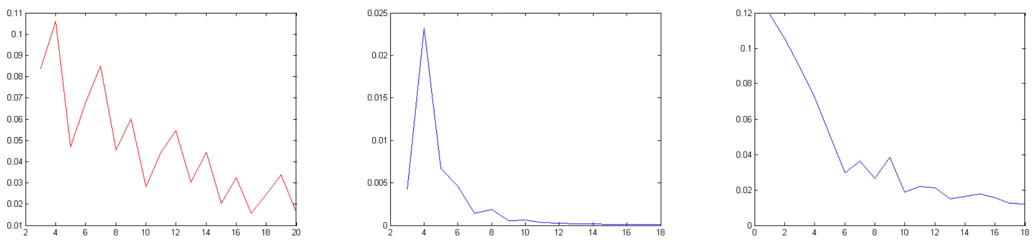
The L^2 convergence is guarantied by the theoretical results.



(a) Squares inside square (b) Hexagons inside hexagon (c) Circles inside circle

Fig. 1.10 L^2 norm convergence of $\widetilde{u}_\varepsilon \to u$

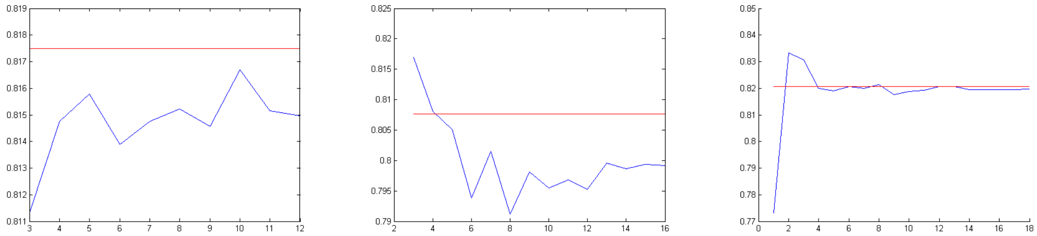
Nonetheless, the L^∞ convergence has never been proven in the theorical setting. However, the numerical solutions seem to converge.



(a) Squares inside square (b) Hexagons inside hexagon (c) Circles inside circle

Fig. 1.11 L^∞ norm convergence of $\widetilde{u}_\varepsilon \to u$

1.8.5 Convergence of the effectiveness



(a) Squares inside square (b) Hexagons inside hexagon (c) Circles inside circle

Fig. 1.12 Convergence efectiveness result: Red line shows the value of non homogeneous problem. Blue line shows the convergence of the homogeneous problem as a function of the value $n = \frac{1}{\varepsilon}$. Notice the order of magnitude in the graphs.

Appendix 1.A Explanation of [Gon97]

The first paper to properly characterize the change of nonlinear kinetic is [Gon97], which applies the technique of Γ-convergence when G_0 is a ball. However, there are a few steps that are not clear (at least to a part of the community). We will try to clarify them in this section.

1.A.1 A Γ-convergence theorem

First, we introduce a Γ-convergence theorem proved in [Gon97] under *ad-hoc* assumptions. More general statements of similar nature can be found in [Dal93].

Theorem 1.15 ([Gon97])**.** *Let X_ε, X be Hilbert spaces and Φ_ε and Φ be functionals in these spaces. Let us assume that Φ_ε satisfy the following conditions:*

i) There exists $\theta > 0$ such that

$$\Phi_\varepsilon(u+v) \geq \Phi_\varepsilon(u) + L_\varepsilon(u;v) + C\|v\|_\varepsilon^\theta \qquad \forall u,v \in X_\varepsilon \tag{1.235}$$

holds, where L_ε is the linear functional with respect to v given by the Fréchet differential of Φ_ε at a point u,

ii) $u_\varepsilon \in X_\varepsilon$ is a minimizer of Φ_ε such that

$$\Phi_\varepsilon(0) \geq \Phi_\varepsilon(u^\varepsilon) \geq C_1\|u_\varepsilon\|_\varepsilon^2 - C_2 \tag{1.236}$$

where C_1 and C_2 are constants.

Suppose there exists a set $M \subset X$ that is everywhere dense in X and operators $P_\varepsilon : X_\varepsilon \to X, R_\varepsilon : M \to X_\varepsilon$ satisfying the conditions

i) $\|P_\varepsilon w^\varepsilon\| \leq C\|w^\varepsilon\|_\varepsilon, \forall w^\varepsilon \in X_\varepsilon$

ii) $P_\varepsilon R_\varepsilon w \to w$ weakly in X as $\varepsilon \to 0$, for every $w \in M$

iii) $\lim_{\varepsilon\to 0} \Phi_\varepsilon R_\varepsilon w = \Phi w$, for all $w \in M$

iv) For any $\gamma^\varepsilon \in X_\varepsilon$ such that $P_\varepsilon\gamma_\varepsilon \to \gamma$ weakly in X and any $w \in M$

$$\overline{\lim_{\varepsilon\to 0}} |L_\varepsilon(R_\varepsilon w;\gamma_\varepsilon)| \leq \Psi(\|w\|)\|\gamma\| \tag{1.237}$$

where $\Psi(t)$ is a continuous function of $t \geq 0$.

Then $P_\varepsilon u^\varepsilon \to u$ weakly in X as $\varepsilon \to 0$ and u is a minimizer of Φ.

With Goncharenko's notation P_ε is a sort of extension operator (at least asymptotically), where R_ε is an adaptation operator.

1.A.2 Proof of Theorem 1.2

The result stated by Goncharenko is Theorem 1.2. Let us prove this result.

The choice of spaces is, naturally,

$$X = H_0^1(\Omega) \qquad X_\varepsilon = H^1(\Omega_\varepsilon, \partial\Omega) \qquad M = C_c^2(\Omega). \tag{1.238}$$

Let us define

$$V_\varepsilon = \left\{ v \in X_\varepsilon : \frac{\partial v}{\partial n} + \sigma^\varepsilon(v) = 0, S_\varepsilon \right\}.$$

The main argument of the paper is to show the Γ-convergence of the energy functional. We define the energy function to be minimized by the solutions:

$$\Phi_\varepsilon(v^\varepsilon) = \int_{\Omega^\varepsilon} (|\nabla v^\varepsilon|^2 + 2f^\varepsilon v^\varepsilon)\mathrm{d}x + \int_{S_\varepsilon} \varepsilon^{-\gamma}\rho(v^\varepsilon)\mathrm{d}\Gamma.$$

It is easy to check that Φ_ε satifies the hypothesis of Theorem 1.15. The technique of the proof by Goncharenko passes by the construction of the following operator

$$R_\varepsilon : M = \mathscr{C}^2(\Omega) \to V^\varepsilon$$

which imposes the boundary condition to any C^2 functions. Let $\varphi(t)$ be continuous ($0 \le \varphi \le 1$) and such that

$$\varphi(t) = \begin{cases} 1 & t \le \frac{3}{2}. \\ 0 & t \ge 2. \end{cases}$$

Let us suppose there is only one particle, a ball of center 0. Let us say we want a behaviour of the type

$$R_\varepsilon w \sim \begin{cases} w(x_i) & \varepsilon^\alpha - scale, \\ w(x) + F_\varepsilon w(x) & \varepsilon - scale, \\ w(x) & 1 - scale, \end{cases}$$

for some operator F_ε, to be defined later. Then we can define R_ε in three parts

$$R_\varepsilon w = w(x) + (w(0) - w(x))\varphi\left(\frac{|x|}{\varepsilon^\alpha}\right) + F_\varepsilon w(x)\varphi\left(\frac{4|x|}{\varepsilon}\right),$$

each one of them captures the behaviour in three different zones. Since the particle is a ball we can try with a operator F_ε that yields radial functions: $F_\varepsilon w(x) = F_\varepsilon w(|x|)$.

Let us impose the condition $R_\varepsilon w \in V^\varepsilon$. The definition of this function in the paper comes out of the blue to serve its purpose. We wanted here to explain the rationale behind this choice. First, in a neighbourhood of $\{|x| = \varepsilon^\alpha\} = G_\varepsilon^0$ we find that

$$\varphi\left(\frac{|x|}{\varepsilon^\alpha}\right) = 1,$$

so in this neighborhood

$$R_\varepsilon w(x) = w(x_0) + F_\varepsilon w(x), \qquad \varepsilon^\alpha < |x| < \frac{3}{8}\varepsilon^\alpha.$$

Therefore

$$\nabla R_\varepsilon w(x) = (F_\varepsilon w)'(|x|)\frac{x}{|x|}$$

and, on $\{|x| = \varepsilon^\alpha\}$, we have

$$\frac{\partial R_\varepsilon w}{\partial n} + \varepsilon^{-\gamma}\sigma(R_\varepsilon w(x)) = (F_\varepsilon w)'(\varepsilon^\alpha) + \varepsilon^{-\gamma}\sigma(w(0) + F_\varepsilon w(\varepsilon^\alpha)).$$

where, we remind that $\gamma = 2\alpha - 3$. Hence, it will be useful to take $F_\varepsilon w$ a function such that $\phi' = -\phi^2$. This is, precisely $\phi(s) = \frac{A}{s}$. If $F_\varepsilon w(|x|) = -\frac{A^\varepsilon}{|\varepsilon|}$ (so that the derivative later on has a nice sign) we have that

$$A^\varepsilon \varepsilon^{-2\alpha} = \varepsilon^{-2\alpha+3}\sigma(w(0) - A^\varepsilon \varepsilon^{-\alpha}).$$

Hence

$$A^\varepsilon \varepsilon^{-3} = \sigma(w(0) - A^\varepsilon \varepsilon^{-\alpha}).$$

Now take $B^\varepsilon = A^\varepsilon \varepsilon^{-3}$ then

$$B^\varepsilon = \sigma(w(0) + B^\varepsilon \varepsilon^{3-\varepsilon}),$$

in the limit the equation would become

$$\begin{cases} B^0 = \sigma(w(0)) & \alpha < 3, \\ B^0 = \sigma(w(0) - B^0) & \alpha = 3. \end{cases}$$

The previous reasoning explains the following choices. When $\alpha = \gamma = 3$ let us take $A^\varepsilon = A\varepsilon^3$ where A is the solution of the implicit equation

$$A = \sigma(w(0) - A).$$

Since H is the solution of (1.81) we have that

$$A = H(w(0)).$$

On the other hand, for $\alpha < 3$ let us choose simply $A = \sigma(w(0))$.

Eventually

$$R_\varepsilon w = w(x) + (w(0) - w(x))\varphi\left(\frac{|x|}{\varepsilon^\alpha}\right) - \frac{A^\varepsilon}{|x|}\varphi\left(\frac{4|x|}{\varepsilon}\right).$$

It is important that we do not loose the notion that $0 \le A^\varepsilon \le \varepsilon^3$. Eventually, since the particles are balls located at the points x_i

$$R_\varepsilon w(x) = w(x) + \sum_{i=1}^{N(\varepsilon)} (w(x_i) - w(x))\varphi\left(\frac{|x - x_i|}{\varepsilon^\alpha}\right) - \sum_{i=1}^{N(\varepsilon)} \frac{A_i^\varepsilon}{|x - x_i|}\varphi\left(\frac{4|x - x_i|}{\varepsilon}\right),$$

where

$$A_i^\varepsilon = A_i\varepsilon^3, \qquad A_i = \begin{cases} \sigma(w(x_i)) & \alpha < 3, \\ H(w(x_i)) & \alpha = 3. \end{cases} \tag{1.239}$$

The deduction of the estimates of the convergence were also not detailed in Goncharenko's paper. We give the details in the following lines. We have that

$$\nabla R_\varepsilon w = \nabla w + \sum_{i=1}^{N(\varepsilon)} (-\nabla w(x))\varphi\left(\frac{|x - x_i|}{\varepsilon^\alpha}\right) + \sum_{i=1}^{N(\varepsilon)} (w(x_i) - w(x))\varphi'\left(\frac{|x - x_i|}{\varepsilon^\alpha}\right)\frac{x - x_i}{|x - x_i|\varepsilon^\alpha} \tag{1.240}$$

$$+ \sum_{i=1}^{N(\varepsilon)} \frac{A_i^\varepsilon}{|x - x_i|^2}\varphi\left(\frac{4|x - x_i|}{\varepsilon}\right) - \sum_{i=1}^{N(\varepsilon)} \frac{A_i^\varepsilon}{|x - x_i|}\varphi'\left(\frac{4|x - x_i|}{\varepsilon}\right)\frac{x - x_i}{|x - x_i|\varepsilon}. \tag{1.241}$$

When integrating the squares of the above (which is easily done is spherical coordinates) only a couple of terms survive: for sure the terms without φ and, from the ones with φ only the most singular one, the term

$$\frac{A_i^\varepsilon}{|x-x_i|^2}\varphi\left(\frac{4|x-x_i|}{\varepsilon}\right)$$

is to be significant. Let us see how. First we integrate

$$\begin{aligned}\int_{\Omega_\varepsilon}\left|\frac{A_i^\varepsilon}{|x-x_i|^2}\varphi\left(\frac{4|x-x_i|}{\varepsilon}\right)\right|^2\mathrm{d}x &= A_i^2\varepsilon^6\int_{\varepsilon^\alpha}^{C\varepsilon}\frac{1}{r^4}\varphi\left(\frac{4r}{\varepsilon}\right)r^2\mathrm{d}r\\ &= A_i^2\varepsilon^5\int_{\varepsilon^{\alpha-1}}^{1}\frac{1}{s^2}\varphi(4s)\mathrm{d}s \sim A^2\varepsilon^{6-\alpha}.\end{aligned}$$

The other terms go to 0 as $\varepsilon^p, p>3$ and so

$$\int_{\Omega_\varepsilon}|R_\varepsilon w|^2\mathrm{d}x = \int_{\Omega_\varepsilon}|w|^2+\sum_{i=1}^{N(\varepsilon)}C\varepsilon^p = \int_{\Omega_\varepsilon}|w|^2+C\varepsilon^{p-3}, \qquad p>3,$$

and

$$\int_{\Omega_\varepsilon}|\nabla R_\varepsilon w|^2\mathrm{d}x = \int_{\Omega_\varepsilon}|\nabla w|^2+\sum_{i=1}^{N(\varepsilon)}A_i\varepsilon^{3-\alpha}+C\varepsilon^{p-3}, \qquad p>3,$$

since $A_i = H(w(x_i))$ as the author says

$$\begin{aligned}\Phi_\varepsilon(R_\varepsilon w) &= \int_{\Omega^\varepsilon}(|\nabla w|^2++2f^\varepsilon R_\varepsilon w)\mathrm{d}x+\sum_{i=1}^{N(\varepsilon)}A_i^2\varepsilon^{6-\alpha}\\ &\quad+\int_{S_\varepsilon}\varepsilon^{-\gamma}\rho(R_\varepsilon w)\mathrm{d}\Gamma+E(\varepsilon,w)\\ &= \int_{\Omega^\varepsilon}(|\nabla w|^2+2f^\varepsilon R_\varepsilon w)\mathrm{d}x+\sum_{i=1}^{N(\varepsilon)}A_i^2\varepsilon^3\\ &\quad+\varepsilon^{-\gamma}\sum_{i\in\Upsilon_\varepsilon}\int_{G_i^\varepsilon}\rho(w(x_i)-A_i\varepsilon^{\varepsilon-\alpha})\mathrm{d}\Gamma+E(\varepsilon,w).\end{aligned}$$

where $E(\varepsilon,w)$ goes to 0. Classical integration results guaranty the convergence of the Riemann sum. When $\alpha=3$

$$\sum_{i=1}^{N(\varepsilon)}A_i^2\varepsilon^{6-\alpha} = \sum_{i=1}^{N(\varepsilon)}H(w(x_i))^2\varepsilon^3 \to \int_\Omega H(w(x))^2.$$

Notice that, if $\alpha < 3$, then

$$\sum_{i=1}^{N(\varepsilon)} A_i^2 \varepsilon^{6-\alpha} = \varepsilon^{3-\alpha} \sum_{i=1}^{N(\varepsilon)} \sigma(w(x_i))^2 \varepsilon^3 \to 0.$$

On the other hand

$$\varepsilon^{-\gamma} \sum_{i \in \Upsilon_\varepsilon} \int_{G_i^\varepsilon} \rho(w(x_i) - A_i \varepsilon^{\varepsilon-\alpha}) \mathrm{d}\Gamma \to 4\pi \begin{cases} \displaystyle\int_\Omega \rho(x) & \alpha < 3, \\ \displaystyle\int_\Omega \rho(x - H(x)) & \alpha = 3. \end{cases} \tag{1.242}$$

Thus, the Γ-limit is

$$\Phi(w) = \int_\Omega |\nabla w|^2 + 2fw + \begin{cases} \rho(x) & 2 < \alpha < 3, \\ H(w(x))^2 + \rho(x - w(x)) & \alpha = 3. \end{cases} \tag{1.243}$$

Hence, we see the appearance of the "strange term" for $\alpha = 3$. If $\alpha < 3$ then we do not have this term, as it is noted on the paper. Surprisingly, note that the strange comes out of the diffusion operator. Applying Theorem 1.15 we conclude the proof of Theorem 1.2.

Chapter 2

Optimizing the effectiveness: symmetrization techniques

From this chapter on, due to the common practice of notation in this fields, which does not coincide with the practice the homogenization community will use the notation that follows for the homogeneous problem derived in the previous section:

$$\begin{cases} -\Delta w + \beta(w) = \hat{f} & \text{in } \Omega, \\ w = 1 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

As we will play now with different domains Ω we will denote this solution w_Ω. By introducing the change in variable $u = 1 - w$ the problem can be reformulated as

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $g(u) = \beta(1) - \beta(1-u)$ and $f = \beta(1) - \hat{f}$. In this case we write the effectiveness factor as:

$$\mathscr{E}(\Omega) = \frac{1}{|\Omega|} \int_\Omega \beta(w_\Omega) dx \tag{2.3}$$

and the ineffectiveness $\eta(\Omega) = \beta(1) - \mathscr{E}(\Omega)$ as

$$\eta(\Omega) = \frac{1}{|\Omega|} \int_\Omega g(u_\Omega) dx. \tag{2.4}$$

Roughly, we aim to find extremal sets Ω which maximize and minimize this functional, applying rearrangement techniques.

2.1 Geometric rearrangement: Steiner and Schwarz

As stated by Polya and Szegö [PS51] *symmetrization* is a geometric operation invented by Jakob Steiner[1] (see [Ste38] for the original reference). The original idea of Steiner symmetrization, as presented in [PS51] is purely geometrical: Considering a body B and a plane construct another body B^* such that:

- it is symmetrical with respect to the plane and
- for every line perpendicular to the plane, the intersections between it and the bodies B and B^* have the same lengths.

This process is shown to not increase the surface area, and to maintain the volume unchanged (this is simply a consequence of Fubini-Tonelli's theorem). By taking different planes we can deduce that, for fixed volume, a convex domain is a minimizer of surface area.

Later H. Schwarz[2] applied a similar method, but in which symmetrization was taken with respect to a line. As Steiner did with is symmetrization, Schwarz proved that Schwarz symmetrization leaves the volumen unchanged but diminishes (in the sense that it never increases) the surface area. In particular, Schwarz rearrangement can be obtained as a limit of Steiner symmetrizations. This was done for convex bodies in [PS51, p. 190], [Lei80, p. 226], and for non convex bodies in [BLL74].

For some reason, the original definition of Schwarz symmetrization was diffused in the literature, as noted by Kawohl in [Kaw85, p. 16]:

> Polya and Szegö distinguish between Schwarz and point symmetrization. Their definition of "symmetrization of a set with respect to a point" coincides with our [in his book] definition of Schwarz symmetrization and is commonly refered to as Schwarz symmetrization [Ban80b; Lio80; Mos84].

We will present later the definition as it commonly used nowadays. Let us start by saying a few words about isoperimetric inequalities.

[1] Jakob Steiner (18 March 1796 – 1 April 1863) was a Swiss mathematician who worked primarily in geometry.

[2] Karl Hermann Amandus Schwarz (25 January 1843 – 30 November 1921) was a German mathematician, known for his work in complex analysis. Do not confuse with Laurent-Moïse Schwartz (5 March 1915 – 4 July 2002), a French mathematician. The later pioneered the theory of distributions, which gives a well-defined meaning to objects such as the Dirac delta function.

2.2 Isoperimetric inequalities

As mentioned in the previous section, one of the most classical result obtained via symmetrization techniques is the isoperimetric inequality:

> From all n-dimensional bodies of a given volume, the n-ball is the one of least surface.

In the plane the solution was believed to be the circle from the time of Kepler. However the first succesful attempt towards proving this result mathematically in dimension 2 was made by Steiner in 1838 (see [Ste38]). This isoperimetic inequality in the plane can be written as follows. Let Ω be a smooth domain in $\mathbb{R}^2$, let A be its surface area and L its perimeter. Then

$$4\pi A \leq L^2. \tag{2.5}$$

Of course, equality holds for the circle. The surprising fact is that holds *only* when the domain is a circle.

Since this initial result there has been substancial research in this direction. For example, Hurwitz in 1902 applied Fourier series (see [Hur01]) and, in 1938, E. Schmidt made a proof using the arc length formula, Green's theorem and the Cauchy-Schwarz inequality (see [Sch39]).

A generalization of the isoperimetric inequalities is already well known. It can be written in the following terms:

Theorem 2.1 (Federer, 1969 [Fed69])**.** *Let $S \subset \mathbb{R}^n$ be such that $\overline{S}$ has finite Lebesgue measure. Then*

$$n\omega_n^{\frac{1}{n}} L^n(\overline{S})^{\frac{n-1}{n}} \leq M_*^{n-1}(\partial S) \tag{2.6}$$

where ω_n is the volume of the n-ball, M_*^n is the Minkowski content and L^n is the Lebesgue measure.

In [Tal16; Ban80a; Rak08; BK06] the reader can find a survey on the study isoperimetric inequalities.

This geometrical inequality is equivalent to a result which is of interest to the specialist in Partial Differential Equations: the Sobolev inequality

$$n\omega_n^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla u|, \qquad \forall u \in W_c^{1,1}(\mathbb{R}^n). \tag{2.7}$$

2.3 From a geometrical viewpoint to rearrangement of functions

Once a particular type of rearrangement Ω_* of a set Ω is understood there is a natural way to define the rearrangement of a function $u : \Omega \to \mathbb{R}$. Consider the level sets:

$$\Omega_c = \{x \in \Omega : u(x) \geq c\}, \qquad c \in \mathbb{R}. \tag{2.8}$$

One can define the rearrangement u_* of u as:

$$u_* : \Omega_* \to \mathbb{R}, \qquad u_*(x) = \sup\{c \in \mathbb{R} : x \in (\Omega_c)_*\}. \tag{2.9}$$

Over the years, several other different types of rearrangements have been developed, an applied with success to different types of problems, with particularly good results in geometry and function theory, specially in PDEs. A catalogue of this techniques can be found in [Kaw85] (although there are many others excelent references, e.g., [Rak08; Ban80b]).

As noted in [PS51] both Steiner and Schwarz symmetrization reduce the Dirichlet integral of functions vanishing in the boundary. Informally

$$\int_{\Omega_*} |\nabla u_*|^2 \leq \int_{\Omega} |\nabla u|^2, \qquad \text{if } u = 0 \text{ on } \partial\Omega \tag{2.10}$$

This immediately appeals to the imagination of the PDE specialist. What seemed as a purely geometrical tool becomes a functional one.

2.4 The coarea formula

Symmetrization is the art of understanding the level set. The following result, known as the coarea formula, allows us to make consider level sets as domain of integration in Fubini-Tonelli theorem-like change of variable. For smooth functions it follows directly as a change of variables. A more general form it was stated by Federer in [Fed59] for Lipschitz functions and for bounded variation functions by Fleming and Rishel in [FR60]. We present the result as it appears in Federer's book [Fed69].

Theorem 2.2. *Let u be a Lipschitz function. Then for all $g \in L^1(\Omega)$ then*

$$\int_\Omega g(x)|\nabla u(x)|dx = \int_{-\infty}^{+\infty} \left(\int_{u^{-1}(t)} g(x) dH_{n-1}(x) \right) dt \tag{2.11}$$

where H_n is n-dimensional Hausdorff measure.

The usual formulation is the particular case $g \equiv 1$. This formula, jointly with the isoperimetric inequality, gives a proof of the Sobolev inequality for $W^{1,1}(\mathbb{R}^n)$ given by (2.7).

2.5 Schwarz rearrangement

2.5.1 Decreasing rearrangement

For the purpose of this thesis we will focus mainly on two types of rearrangements: Schwarz and Steiner rearrangements. In particular we will be interested in studying this rearrangement as a tool in studying the Laplace operator, and other operators in divergence form as a first step for the consideration of problem (2.1). First, we introduce the (modern) definition of Schwarz symmetrization

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$. We define the Schwarz rearrangement of Ω as

$$\Omega^\star = B(0,R), \qquad \text{such that } |\Omega^\star| = |\Omega|. \tag{2.12}$$

where $B(0,R)$ as ball centered at 0 of radius R.

The process of symmetrization for this kind of problems was introduced by Faber [Fab23] and Krahn [Kra25; Kra26] in their proof of the Rayleigh's conjecture, which can be stated in the following terms

Theorem 2.3 (Rayleigh-Faber-Krahn)**.** *Let*

$$\lambda(\Omega) = \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}. \tag{2.13}$$

Then

$$\lambda(\Omega) \geq \lambda(\Omega^\star). \tag{2.14}$$

In modern terms $\lambda(\Omega)$ is, of course, known as the first eigenvalue for the Laplace operator, and $\lambda(\Omega)$ can be though as the smallest real number such that

$$\begin{cases} -\Delta u = \lambda u, & \Omega, \\ u = 0, & \partial\Omega, \end{cases} \tag{2.15}$$

has a nontrivial solution. The nontrivial solutions of this problem are known as eigenfunction. They will be used extensively in Part II.

Definition 2.2. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *distribution function of* u, $\mu : [0,+\infty) \to [0,+\infty)$, as

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}| \tag{2.16}$$

and the *decreasing rearrangement of* u, $u^* : [0,+\infty) \to \mathbb{R}$ as

$$u^*(s) = \sup\{t \geq 0 : \mu(t) > s\}. \tag{2.17}$$

Definition 2.3. We introduce the Schwarz rearrangement $u^\star$ of u as

$$u^\star(x) = u^*(\omega_n |x|^n) \tag{2.18}$$

where ω_n represents the volume of the n-dimensional unit ball.

2.5.2 The three big inequalities and one big equation

There are several inequalities involving these rearrangements which will be of a great importance for us.

The Hardy-Littlewood-Polya inequality is very import since it allows to bound products in L^2. It can be stated as follows

Theorem 2.4 (Hardy-Littlewood-Polya, 1929 [HLP29])**.** *Let Ω be a measurable subset of $\mathbb{R}^n$ and f, g be non negative measurable functions. Then*

$$\int_\Omega fg \leq \int_0^{|G|} f^* g^*. \tag{2.19}$$

The second remarkable inequality is Riesz's inequality . It is very useful in order to make a priori comparisons with Green's kernel

Theorem 2.5 (Riesz, 1930 [Rie30]). *Let Ω be a measurable subset of $\mathbb{R}^n$ and f,g be non negative measurable functions. Then*

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} f(x)g(x-y)h(y)dxdy \leq \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} f^\star(x)g^\star(x-y)h^\star(y)dxdy. \tag{2.20}$$

Both results, and further techniques, were compiled in one of the references text in the subject [HLP52].

Only a few years after the first appearance of these two results, in 1945, Polya and Szegö publish [PS45], were they introduce the following inequality, to prove that the capacity of a condenser diminishes or remains unchanged by applying the process of Schwarz symmetrization.

Theorem 2.6 (Polya-Szego [PS45]). *Let $u:\mathbb{R}^n\to\mathbb{R}^+$ in $W^{1,p}(\mathbb{R}^n)$ where $1\leq p<\infty$. Then*

$$\int_{\mathbb{R}^n}|\nabla u^\star|^p \leq \int_{\mathbb{R}^n}|\nabla u|^p. \tag{2.21}$$

This result was also useful in the proof of the Choquard conjecture by Lieb [Lie77].

The collaboration between Polya and Szegö continued in time, and they updated [PS51] over several editions. This is a reference text in isoperimetric inequalities and the use of different rearrangements in Mathematical Physics.

The final inequality could be one the most important in the theory, because it is used to convert the original PDE for u to a PDE for μ.

Theorem 2.7 ([BZ87]). *Let $u\in W^{1,p}$ for some $1\leq p<\infty$. Then the following holds:*

i) μ *is one-to-one.*

ii) $u^*\circ\left(\frac{\mu(t)}{\omega_n}\right)^{\frac{1}{n}}=Id.$

iii) We can decompose μ *as*

$$\mu(t)=|\{x\in\Omega:|\nabla u|=0,u(x)>t\}|+\int_t^{+\infty}\int_{u^{-1}(t)}|\nabla u|^{-1}dH^{n-1}ds. \tag{2.22}$$

iv) For almost all t,

$$-\infty < \frac{d\mu}{dt} \leq \int_{\{x\in\Omega:u(x)=t\}} \frac{-1}{|\nabla u|} dH^{n-1}, \qquad a.e.\ t \in \mathbb{R}. \tag{2.23}$$

Equality holds in the previous item if $|\nabla u| \neq 0$.

v) For almost all t, $\frac{d\mu}{dt} < 0$.

2.5.3 Concentration and rearrangement

Even though the stronger results show that we have a pointwise comparison $u^\star \leq v$ this is not the case in general. However, there is a property much nicer in terms of rearrangements: the *concentration*. This term, which appears frequently in the mathematical literature, must not be confused with the chemical concept of concentration. As will see from the following definition they are entirely different.

Definition 2.4. Let Ω be an open set of $\mathbb{R}^n$ and let $\psi \in L^1(\Omega_1), \phi \in L^1(\Omega_2), |\Omega_1| = |\Omega_2|$. We say that the concentration of ϕ is less or equal than the concentration of ψ, and we denote this by $\phi \preceq \psi$ if

$$\int_0^t \phi^*(s)ds \leq \int_0^t \psi^*(s)ds, \qquad \forall t \in [0, |\Omega|]. \tag{2.24}$$

Equivalently,

$$\int_{B_r(0)} \phi^\star(x)dx \leq \int_{B_r(0)} \psi^\star(x)dx. \tag{2.25}$$

The following lemma is a very important result. It allows to understand the importance of convex functions in symmetrization.

Lemma 2.5.1. *Let* $y, z \in L^1(0,M)$*, y and z nonnegative. Suppose that y is nonincreasing and*

$$\int_0^t y(s)ds \leq \int_0^t z(s)ds, \qquad \forall t \in [0,M]. \tag{2.26}$$

Then, for every continuous non decreasing convex function Φ *we have*

$$\int_0^t \Phi(y(s))ds \leq \int_0^t \Phi(z(s))ds, \qquad \forall t \in [0,M]. \tag{2.27}$$

Applying this result it is possible to obtain the following properties (see [HLP52; HLP29; RF88; CR71; ATL89]).

Proposition 2.1. *Let Ω_1, Ω_2 be Borel sets in $\mathbb{R}^{N_1}, \mathbb{R}^{N_2}$ respectively such that $|\Omega_1| = |\Omega_2|$ and $\phi_i \in L^1_+(\Omega_i)$ (i.e. it is is in L^1 and non-negative). Then, the following are equivalent:*

i) $\phi_1 \preceq \phi_2$

ii) $F(\phi_1) \preceq F(\phi_2)$ *for every nondecreasing, convex function F on $[0,+\infty)$ such that $F(0) = 0$.*

iii) For all $\psi \in L^1 \cap L^\infty(\Omega_1)$

$$\int_{\Omega_1} \varphi_1 \psi \leq \int_0^{|\Omega_2|} \varphi_2^* \psi^* = \int_{\Omega_2^\star} \varphi_2^\star \psi^\star \tag{2.28}$$

iv) For all ψ nonincreasing on $(0, |\Omega_1|)$, $\psi \in L^1 \cap L^\infty(0, |\Omega_1|)$

$$\int_0^{|\Omega_1|} \varphi_1^* \psi \leq \int_0^{|\Omega_2|} \varphi_2^* \psi. \tag{2.29}$$

2.5.4 Schwarz symmetrization of elliptic problems

It is precisely in 1962, in a book in honor of Polya edited by Szegö [Wei62] that Weinberger extends the results by Faber and Krahn to obtain isoperimetric results for the Dirichlet problem with general elliptic self adjoint operator

$$L = \sum_{i,j=1}^{N} \partial_{x_i} \left(a_{ij}(x) \partial_{x_j} \right). \tag{2.30}$$

In 1976 Bandle [Ban76a] gives a pointwise estimates of the decreasing rearrangements of the solution of $-\Delta u + \alpha u + 1$ with Dirichlet boundary condition. In 1978, in [Ban78], she gives estimates on the Green kernel. In the same year Alvino and Trombetti [AT78] present result similar to Weinberger's for degenerate (non elliptic) equations.

In 1979 Talenti [Tal79] apply Schwarz symmetrization techniques, to improve upon the results of Weinberger and Bandle. He focuses on non linear elliptic equations of the form

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = 0 & \Omega, \\ u = h & \partial\Omega, \end{cases} \tag{2.31}$$

under the hypothesis

i) There exists $A : [0,+\infty) \to \mathbb{R}$ convex such that :

i) $a(x,u,\xi)\cdot\xi \geq A(|\xi|)$

ii) $A(r)/r \to 0$ as $r \to 0$

ii) g is measurable and

$$(g(x,u)-g(x,0))u \geq 0 \tag{2.32}$$

iii) $h \in L^\infty(\Omega)$ and

$$\int_\Omega A(|\nabla h|) < \infty. \tag{2.33}$$

In the case $h \equiv b$ a constant Talenti introduces the "rearranged problem"

$$\begin{cases} -\operatorname{div}(\frac{A(|\nabla v|)}{|\nabla v|^2}\nabla v) = f^\star & \Omega^\star, \\ v = b & \partial\Omega^\star. \end{cases} \tag{2.34}$$

He manages to prove, for the first time in literature, that we can compare pointwise $u^\star$ with a different solution (which is easier to compute analytically). In fact

$$u^\star \leq v, \qquad a.e.\ \Omega^\star \tag{2.35}$$

Notice that, if $a(x,u,\xi) = \xi$, the operator is the usual Laplace operator and $A(\xi) = \xi^2$.

In 1980 P.L. Lions [Lio80] provides a simpler proof of this result in the linear case with $h \equiv 0$ and extends the estimates to operator in the form $A = Lu + c$ were L is second order elliptic and $c : \Omega \to \mathbb{R}$. He compares the problems

$$\begin{cases} -\Delta u + cu = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v + (c^+)^{\star\star} v - (c^-)^\star v = f^\star & \Omega^\star \\ v = 0 & \partial\Omega^\star \end{cases} \tag{2.36}$$

where $\phi^{\star\star}(x) = \phi^{**}(\omega_n|x|)$ and $\phi^{**}(s) = \phi^*(|\Omega| - s)$. This last function is known as the increasing rearrangement of ϕ. Analogously to the Schwarz rearrangement $\phi^{\star\star}$ represent represents the unique radial increasing function with the same distribution function as $|\phi|$ (it can be defined by analogy to the Schwarz rearrangement).

Lions shows that,

$$\int_\Omega F(|u|) \leq \int_{\Omega^\star} F(v) \tag{2.37}$$

for all function $F : \mathbb{R}_+ \to \mathbb{R}_+$ positive, increasing and convex. Besides, if $c \leq 0$, we have that

$$u^\star \leq v, \qquad a.e.\ \Omega^\star. \tag{2.38}$$

At the end of this paper Lions indicates that some nonlinear cases are immediately covered by the linear case, by simply taking a sequence of solutions $-\Delta u_{n+1} = g(u_n)$ and applying the comparison, since $g(u_n)^\star = g(u_n^\star)$. Around the same time, in 1979, Chiti in [CP79] (see [Chi79] in Italian) proved a similar result, by using a limit of simple functions. However, his result was presented as a Orlicz norm result, rather than as a PDE result.

The ideas behind both [Tal79] and [Lio80] is explained very elegantly in 1990 by Talenti in [Tal90]. By applying the Hardy-Littlewood-Polya inequality, the isoperimetric inequality in $\mathbb{R}^n$ and the coarea formula one the boundary value problem can be rewritten in terms of a ODE for the distribution function

$$n\omega_n^{\frac{1}{n}}\mu(t)^{1-\frac{1}{n}}B\left(n\omega_n^{\frac{1}{n}}\frac{\mu(t)^{1-\frac{1}{n}}}{\mu'(t)}\right) \leq \int_0^{\mu(t)} g(s)ds. \tag{2.39}$$

Returning to a chronological order, in 1984, in the more general context of Mathematical Physics, Mossino publishes a book [Mos84] which contains a number of interesting statements on elliptic problems. However, none of the results are necessary to the interest of this Thesis.

In 1985 Díaz in [Día85] polishes some of the previously presented rearrangement techniques, in order to obtain estimates for free boundaries that rise in problem (2.40), when g is not a Lipschitz function. The results are extended to the p-Laplace operator and the regularity hypothesis are simplified. The following theorem is stated:

Theorem 2.8. *Let u, v be the solutions of problems*

$$\begin{cases} -\operatorname{div}(A(x,u,\nabla u)) + g(u) = f_1 & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{2.40}$$

and

$$\begin{cases} -\Delta_p v + g(v) = f_2 & \Omega, \\ v = 0 & \partial\Omega, \end{cases} \tag{2.41}$$

where

i) A is a Caratheodory function such that $A(x,u,\xi) \geq |\xi|^p$

ii) g is continuous non decreasing such that $g(0) = 0$

iii) $f_1 \in L^{p'}(\Omega)$ such that $f_1 \geq 0$

iv) $f_2 \in L^{p'}(\Omega^\star)$ such that $f_2 = f_2^\star$

v) $\int_0^t f_1^* \leq \int_0^t f_2^*$ for all $t \in [0,|\Omega|]$

Then

$$\int_0^t g(u^*) \leq \int_0^t g(v^*), \qquad t \in [0,|\Omega|]. \tag{2.42}$$

In particular, for any convex nondecreasing real function Φ

$$\int_\Omega \Phi(g(u)) \leq \int_{\Omega^\star} \Phi(g(v)). \tag{2.43}$$

Notice that, in this more general setting, the result is not as strong as in the case withouth the absortion. We do not have a pointwise comparison of $u^\star$ and v. Behind the proof is the following lemma

Lemma 2.5.2. *Let u be the solution of* (2.40). *Then u^* is absolutely continuous in* $[0,|\Omega|]$ *and*

$$-\frac{du^*}{ds}(s) \leq \left(\frac{1}{n\omega_n^{\frac{1}{n}} s^{\frac{n-1}{n}}} \right)^{\frac{p}{p-1}} \left[\int_0^s f_1^*(\theta)d\theta - \int_0^s g(u^*(\theta))d\theta \right]. \tag{2.44}$$

Let v be the solution of (2.41)*, then equality holds if g_2 is radial.*

In the proof of this lemma there are two main ingredients. The first is a general statement on the distribution function.

Lemma 2.5.3. *Let $z \in W_0^{1,p}(\Omega)$, $z \geq 0$. Then if $\mu(t) = |\{x \in \Omega : z(x) > t\}|$ one has*

$$n\omega_n^{\frac{1}{n}} \mu(t)^{\frac{n-1}{n}} \leq \left(-\frac{d\mu}{dt}(t) \right)^{\frac{1}{p}} \left(-\frac{d}{dt} \int_{\{z(x)>t\}} |\nabla z(x)|^p dx \right)^{\frac{1}{p}}. \tag{2.45}$$

The second one is a particular computation on an integral of the solution of problem (2.40).

Lemma 2.5.4. *Let $u \in W_0^{1,p}(\Omega)$ be a nonnegative solution of* (2.40). *Then the function*

$$\Psi(t) = \int_{\{u(x)>t\}} A(x,u,\nabla u)\cdot \nabla u dx \tag{2.46}$$

is a decreasing Lipschitz continuous function of $t \in [0,+\infty)$, and the inequality

$$0 \leq -\frac{d\Psi}{dt}(t) \leq \int_0^{\mu(t)} f_1^*(s)ds - \int_0^{\mu(t)} g(u^*(s))ds. \tag{2.47}$$

The last assertion of the theorem is a consequence of the Lemma 2.5.1.

Finally in 1990 Alvino, Trombetti and Lions, in [ATL90], prove the result for a general operator in the form

$$Lu = -\operatorname{div}(A(x)\nabla u) + \nabla(b(x)u) + d(x)\cdot\nabla u + c(x)u \tag{2.48}$$

under weak restriction on the operators. For completeness, and under the general operator $-\operatorname{div}(a(x,u,\nabla u))$, jointly with Ferone in [AFTL97], they extend the Polya-Szegö inequality and some of the previous lemmas to a more general context. As a corollary of their analysis they manage to obtain an comparison of the form $u^\star \leq C_n v$.

2.5.5 Schwarz rearrangement of parabolic problems

The application of rearrangement techniques to parabolic equations was considered for the first time in 1976 by C. Bandle in [Ban76b]. This result was announced in 1975 in a *Comptes Rendus* note (see [Ban75]). The compared problems in this case are

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x) & \Omega \times (0,\infty), \\ u = 0 & \partial\Omega \times [0,+\infty), \\ u(\cdot,0) = u_0 & \overline{\Omega} \end{cases}, \quad \text{and} \quad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = f^\star(x), & \Omega^\star \times (0,\infty), \\ v = 0 & \partial\Omega^\star \times [0,+\infty), \\ v(\cdot,0) = v_0^\star & \overline{\Omega^\star}, \end{cases} \tag{2.49}$$

where Ω has to be piecewise analytic. As described in the text:

> The proof is based on a differential inequality and uses very much a system of curvilinear coordinates defined by the level surfaces of $u(x,t)$. The introduction of those coordinates requires a strong assumption on the regularity of $u(x,t)$.

Let $\mu, \tilde{\mu}$ be the distribution functions of u and v respectively. By defining

$$H(a,t) = \int_0^t \mu(t,s)ds, \qquad \tilde{H}(a,t) = \int_0^a \tilde{\mu}(t,s)ds. \tag{2.50}$$

Then

$$\begin{cases} -\dfrac{\partial H}{\partial t} + p(a)\dfrac{\partial^2 H}{\partial a^2} + \displaystyle\int_0^{\cdot} f^*(s)ds \geq 0 \\ H(0,t) = 0, \\ \dfrac{\partial H}{\partial a}(0,t) = \max\limits_{\overline{\Omega}} u(x,t), \\ \dfrac{\partial H}{\partial a}(|\Omega|,t) = 0, \end{cases} \qquad \begin{cases} -\dfrac{\partial \tilde{H}}{\partial t} + p(a)\dfrac{\partial^2 \tilde{H}}{\partial a^2} + \displaystyle\int_0^{\cdot} f^*(s)ds = 0 \\ \tilde{H}(0,t) = 0, \\ \dfrac{\partial \tilde{H}}{\partial a}(0,t) = \max\limits_{x\in\overline{\Omega^\star}} v(x,t), \\ \dfrac{\partial \tilde{H}}{\partial a}(|\Omega|,t) = 0. \end{cases} \tag{2.51}$$

From this we can conclude that $u \preceq v$.

After this initial result, that worked only for smooth classical solutions some generalizations appeared. In 1982 J. L. Vazquez showed similar results for the porous medium equation: $u_t - \Delta\varphi(u) = f$ (see [Váz82]). Mossino and Rakotoson in 1986 (see [MR86]) obtained a similar result under weaker regularity by considering the directional derivative of the rearrangement, a technique that first appeared in [MT81].

A generalization comes in 1992, by Alvino, Trombetti and P.L. Lions in [ALT92], by applying the techniques in [MR86], simple properties of the fundamental solution and semigroup theory (in particular the Trotter formula which will be introduced in Section 2.7.3), which allows to reduce the regularity condition on the data and the solution. They obtain the expect comparison between the problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(A(x,t)\nabla u) + a(x,t)u = f(x,t) & \Omega \times (0,T), \\ u = 0 & \partial\Omega \times (0,T), \\ u = u_0 & \Omega \times \{0\}, \end{cases} \tag{2.52}$$

under the assumptions

$$\begin{cases} a_{ij}, a \in L^\infty(\Omega \times (0,T)), f \in L^2(\Omega \times (0,T)), u_0 \in L^2(\Omega), \\ \exists \nu \in L^\infty(0,T), \alpha > 0, \forall \xi \in \mathbb{R}^n : \qquad \xi^t A(x,t)\xi \geq \nu(t)|\xi|^2 \geq \alpha|\xi|^2, \end{cases} \tag{2.53}$$

and the problem

$$\begin{cases} \frac{\partial v}{\partial t} - \nu(t)\Delta v + \{a_1 - a_2\}v = g(x,t) & \Omega^\star \times (0,T), \\ v = 0 & \partial\Omega^\star \times (0,T), \\ v = u_0^\star & \Omega^\star \times \{0\}. \end{cases} \tag{2.54}$$

under the assumptions

$$\begin{cases} a_i \in L^\infty_+(\Omega^\star \times (0,T)), v_0 \in L^2_+(\Omega^\star), g \in L^2_+(\Omega^\star \times (0,T)), \\ a_2, -a_1, g \text{ are spherically symmetric, nonincreasing} \\ \text{ with respect to } |x|, \text{ for almost all } t \in (0,T), \end{cases} \tag{2.55}$$

As stated in [ALT92] the result reads:

Theorem 2.9. *Let u be the solution of* (2.52) *under hypothesis* (2.53) *and v the solution* (2.54) *under hypothesis* (2.55)*. Assume further that,*

$$\begin{cases} u_0 \preceq v_0, \quad a^-(t) \preceq a_2(t), \quad f(t) \prec q(t), \\ \int_0^r (a_1)^{**} \le \int_0^r (a^{**})_+, \text{ for all } r \in [0,|\Omega|], \text{ for a.e. } t \in (0,T) \end{cases} \tag{2.56}$$

Then, for all $t \in [0,T]$,

$$u(t) \preceq v(t) \tag{2.57}$$

(in the sense of Definition 2.4).

2.6 A differentiation formula

Most results in PDEs using symmetrizaqtion techniques pass by the consideration of differentiation formulas of the following function

$$H(s,y) = \int\limits_{\{u(x,y)>t\}} f(x,y)dx. \tag{2.58}$$

In 1998 Ferone and Mercaldo in [FM98] state a second order differentiation formula for rearrangements (citing works in Steiner rearrangement, [ATDL96], which we will present in the following section)

Theorem 2.10. *Let $\Omega = \Omega' \times (0,h)$ u be a nonnegative function in $W^{2,p}(\Omega)$, were $p > n+1$ and let f be Lipschitz in $\overline{\Omega}$. Assume that*

$$|\{x \in \Omega : |\nabla_x u| = 0, u(x,y) \in (0, \sup u(\cdot,y))\}| = 0, \qquad \forall y \in (0,h). \tag{2.59}$$

Then we have

i) For any $y \in (0,h)$, H is differentiable with respect to s for a.e. $s \geq 0$ and

$$\frac{\partial H}{\partial t}(t,y) = -\int\limits_{u(x,y)=t} \frac{f(x,y)}{|\nabla_x u|} dH^{n-1}(x). \tag{2.60}$$

ii) For every fixed s, H is differentiable with respect to y and

$$\frac{\partial H}{\partial y}(s,y) = \int\limits_{u(x,y)>t} \frac{\partial f}{\partial y}(x,y)dx + \int\limits_{u(x,y)=t} \frac{\partial u}{\partial y}(x,y)\frac{f(x,y)}{|\nabla_x u|} dH^{n-1}(x). \tag{2.61}$$

From this we can extract the following corollary, which is of capital importance in symmetrization.

Corollary 2.1. *Let $u \in W^{1,\infty}(\Omega \times (0,h))$ be nonnegative. Then*

$$\int\limits_{u(x,y)>u^*(s,y)} \frac{\partial^2 u}{\partial y^2}(x,y)dx = \frac{\partial^2}{\partial y^2}\int_0^s u^*(\sigma)d\sigma - \int_{u(x,y)=u^*(s,y)} \frac{\left(\frac{\partial u}{\partial y}\right)^2}{|\nabla_x u|} dH^{n-1}(x) \tag{2.62}$$

$$+ \left(\int\limits_{u(x,y)=u^*(s,y)} \frac{\frac{\partial u}{\partial y}(x,y)}{|\nabla_x u|} dH^{n-1}(x) \right)^2 \left(\int\limits_{u(x,y)=u^*(s,y)} \frac{1}{|\nabla_x u|} dH^{n-1}(x) \right)^{-1}. \tag{2.63}$$

2.7 Steiner rearrangement

The idea of the Schwarz rearrangement (in the modern definition) is to consider radially decreasing functions. A smart analysis of the pros and cons of performing this symmetrization is presented in [ATDL96]:

> On the one hand, these results [on the Schwarz symmetrization] make the problem of determining a priori estimates easier by turning it into a one-dimensional

problem; on the other hand, by this symmetrization process, the differential problem may lose properties that arise from the symmetry of the data with respect to a group of variables. In order to preserve this kind of symmetry, it is usefull to check whether comparison results hold when a partial symmetrization such as Steiner symmetrization is used.

In 1992 Alvino, Díaz, Lions and Trombetti introduce a new definition of Steiner symmetrization, which differs slightly from the one in [PS51]. We will follow this new definition. We point out that the new definition (which we give in precise terms below) can be obtained, as it was the case in of Schwartz symmetrization, as a limit of Steiner symmetrization perpendicular to a hyperplane, in the sense presented in [PS51].

The idea behind this (new) Steiner rearrangement is to symmetrize radially only in some variables, and therefore only works in product domains $\Omega = \Omega' \times \Omega''$.

Definition 2.5. Let

$$\Omega = \Omega' \times \Omega'' \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{2.64}$$

We usually refer to the variables in Ω' as x, and to the variables in Ω'' as y. We define the Steiner rearrangement of Ω with respect to the variables x as

$$\Omega^{\#} = B(0,R) \times \Omega'' \quad \text{ where } |B(0,R)| = |\Omega'|, \tag{2.65}$$

where $B(0,R) \subset \mathbb{R}^{n_1}$ is the ball centered at 0 of radius R.

Remark 2.1. Notice that

$$\Omega^{\#} = (\Omega')^{\star} \times \Omega''. \tag{2.66}$$

We can define the functional rearrangement as follows:

Definition 2.6. Let $\Omega = \Omega' \times \Omega'' \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $u : \Omega' \to \mathbb{R}$ be a measurable function. We define the *distribution function of* u, $\mu : [0,+\infty) \times \Omega'' \to [0,+\infty)$, as

$$\mu(t,y) = |\{x \in \Omega : |u(x,y)| > t\}|, \tag{2.67}$$

and the *decreasing rearrangement of* u, $u^* : [0,+\infty) \times \Omega'' \to \mathbb{R}$ as

$$u^*(s,y) = \sup\{t \geq 0 : \mu(t,y) > s\}. \tag{2.68}$$

Finally we introduce the *Steiner rearrangement of* u as

$$u^{\#}(x,y) = u^*(\omega_{n_1}|x|^{n_1}, y), \qquad (x,y) \in \Omega^{\#}, \tag{2.69}$$

where ω_n represents the volume of the n-dimensional ball.

Remark 2.2. Notice that

$$u^\#(x,y) = (u(\cdot,y))^\star(x), \qquad (x,y) \in (\Omega')^\star \times \Omega'' = \Omega^\#. \tag{2.70}$$

Naturally, the Steiner rearrangement shares, for every y, the same properties as the Schwarz rearrangement.

2.7.1 Steiner rearrangement of elliptic equations

As announced in [ADLT92], in [ATDL96] Alvino, Trombetti, Díaz and P.-L. Lions prove the following result

Theorem 2.11. *Let*

$$\begin{aligned} Lu = &- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x,y)\frac{\partial u}{\partial x_i}\right) - \sum_{h,k=1}^{m} \frac{\partial}{\partial y_k}\left(b_{hk}(x,y)\frac{\partial u}{\partial y_h}\right) \\ &- \sum_{i=1}^{n}\sum_{h=1}^{m} \frac{\partial}{\partial y_h}\left(c_{ih}(y)\frac{\partial u}{\partial x_i}\right) - \sum_{i=1}^{n}\sum_{h=1}^{m} \frac{\partial}{\partial x_i}\left(d_{hi}(y)\frac{\partial u}{\partial y_h}\right) \end{aligned} \tag{2.71}$$

and let u be a weak solution of the Dirichlet problem

$$\begin{cases} Lu = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{2.72}$$

We assume the following:

i) Coefficients $a_{ij}, b_{hk}, c_{ih}, d_{hl}$ and f belong to $L^\infty(\Omega)$,

ii) (ellipticity condition) there exists $\nu > 0$ such that, for every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m$ and a.e. $(x,y) \in \Omega$

$$\begin{aligned} &\sum_{i,j=1}^{n} a_{ij}(x,y)\xi_i\xi_j + \sum_{h,k=1}^{m} b_{hk}(y)\eta_h\eta_k \\ &\qquad + \sum_{i=1}^{n}\sum_{h=1}^{m} c_{ih}(y)\xi_i\eta_h + \sum_{i=1}^{n}\sum_{h=1}^{m} d_{hi}(y)\xi_i\eta_h \geq |\xi|^2 + \nu|\eta|^2, \end{aligned} \tag{2.73}$$

iii) $\Omega = \Omega' \times \Omega''$ open, bounded subset of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Let v be the weak solution of the problem

$$\begin{cases} -\Delta_x v - \sum_{h,k=1}^{m} \frac{\partial}{\partial y_k}\left(a_{hk}(x,y)\frac{\partial u}{\partial y_h}\right) = f & \Omega^{\#}, \\ v = 0 & \partial\Omega^{\#}. \end{cases} \tag{2.74}$$

Then we have, for any $y \in \mathbb{R}^m$

$$\int_0^s u^*(\sigma,y)d\sigma \le \int_0^s v^*(\sigma,y)d\sigma. \tag{2.75}$$

The proof of this result is highly technical. It uses, as it was the cases in previous results in the literature, the differential geometry behind the level sets, considering specially the case of C^1 solutions. Besides, in the effort of taking about the most general elliptic operator, the presence of subscripts i,j,h,k makes the work quite baroque.

In 2001 Chiacchio and Monetti in [CM01] (see also [Chi04]) introduce lower order terms to same equation. They deal with operators in the form:

$$Lu = -\Delta u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(y)u) - \sum_{j=1}^{m} \frac{\partial}{\partial y_j}(\tilde{b}_j(y)u) + \sum_{i=1}^{n} d_i(y)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{m} \tilde{d}_j(y)\frac{\partial u}{\partial y_j} + c(y)u \tag{2.76}$$

Later, in 2009, Chiacchio studies the eigenvalue problem (see [Chi09]).

2.7.2 Steiner rearrangement of linear parabolic problems

By applying [FM98] the following result can be deduced immediately. It is not written formally in any known paper, however, it is mentioned in [Chi04] and [Chi09].

Proposition 2.2. *Let u and v be the weak solutions of*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = f & \Omega \times (0,T), \\ u = 0 & \partial\Omega \times (0,T), \\ u(0) = u_0 & \Omega, \end{cases} \quad \textit{and} \quad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = f^{\#} & \Omega^{\#} \times (0,T), \\ v = 0 & \partial\Omega^{\#} \times (0,T), \\ v(0) = v_0 & \Omega^{\#}, \end{cases} \tag{2.77}$$

and let

$$U(t,s,y) = \int_0^s u^*(t,\sigma,y)d\sigma, \qquad V(t,s,y) = \int_0^s v^*(t,\sigma,y)d\sigma. \tag{2.78}$$

Then, there exists $g(s) \geq 0$ such that

$$U_t - g(s)U_{ss} - \Delta_y U \leq \int_0^s f^*(\sigma, y)d\sigma, \qquad V_t - g(s)V_{ss} - \Delta_y V = \int_0^s f^*(\sigma, y)d\sigma. \quad (2.79)$$

Hence we deduce that

Proposition 2.3. *For a.e. $y \in \Omega''$ and a.e. $t > 0$*

$$u_0(\cdot, y) \preceq v_0(\cdot, y) \implies u(t, \cdot, y) \preceq v.(t, \cdot, y). \quad (2.80)$$

2.7.3 The Trotter-Kato formula

In order to treat our problem we will apply the Neveu-Trotter-Kato theorem, that characterizes the convergence of the semigroup in terms of the convergence of its generators. The abstract statement can be found in [Bré73].

Theorem 2.12. *Let (A^n) and A be maximal monotone operators such that $\overline{D(A)} \subset \cap_n \overline{D(A^n)}$. Let S_n and S be the semigroups generated by $-A^n$ and $-A$ respectively. The following properties are equivalent:*

i) For every $x \in \overline{D(A)}$, $S_n(\cdot)x \to S(\cdot)x$ uniformly in compact subsets of $[0, +\infty)$.

ii) For every $x \in \overline{D(A)}$ and every $\lambda > 0$, $(I + \lambda A^n)^{-1}x \to (I + \lambda A)^{-1}x$.

There is an important corollary to this theorem, that allows us to study the semigroup of an operator given by a sum of operators as the sequential application of the semigroup of each of these operators.

Proposition 2.4 ([Bré73, Proposition 4.4 (p. 128)])**.** *Let A, B be univoque maximal monotone operators such that $A + B$ is maximal monotone. Let S_A, S_B, S_{A+B} be the semigroups associated to $-A, -B, -(A+B)$. Let C be a closed convex subset of $\overline{D(A)} \cap \overline{D(B)}$ such that $(I + \lambda A)^{-1}(C) \subset C$ and $(I + \lambda B)^{-1}C \subset C$. Then, for every $x \in C \cap \overline{D(A) \cap D(B)}$,*

$$\left[S_A\left(\frac{\cdot}{n}\right) S_B\left(\frac{\cdot}{n}\right) \right]^n x \to S_{A+B}(\cdot)x \quad (2.81)$$

uniformly in every compact subset of $[0, +\infty)$.

2.7.4 Steiner rearrangement of semilinear parabolic problems

In [DGC15b] and [DGC16] J.I. Díaz and myself apply Proposition 2.3, the Trotter-Kato theorem (in the form of Proposition 2.4) and explicit comparisons of the pointwise ODE

$$u_t + g(u) = f \tag{2.82}$$

to obtain the following symmetrization results. The proofs can be found in the corresponding papers collected in the Appendix.

Theorem 2.13 ([DGC15b]). *Let g be concave, verifying*

$$\int_0^\tau \frac{d\sigma}{g(\sigma)} < \infty, \qquad \forall \tau > 0. \tag{2.83}$$

Let $h \in W^{1,\infty}(0,T)$, such that $h(t) \geq 0$ for all $t \in (0,T)$, $f \in L^2(0,T : L^2(\Omega))$ with $f \geq 0$ in $(0,T)$ and let $u_0 \in L^2(\Omega)$ be such that $u_0 \geq 0$. Let, $u \in C([0,T] : L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$ and $v \in C([0,T] : L^2(\Omega^\#)) \cap L^2(\delta, T : H_0^1(\Omega^\#))$, for any $\delta \in (0,T)$, be the unique solutions of

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \Delta u + h(t) g(u) = f(t), & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

$$(P^\#) \begin{cases} \frac{\partial v}{\partial t} - \Delta v + h(t) g(v) = f^\#(t), & \text{in } \Omega^\# \times (0,T), \\ v = 0, & \text{on } \partial\Omega^\# \times (0,T), \\ v(0) = v_0, & \text{on } \Omega^\#, \end{cases}$$

where $v_0 \in L^2(\Omega^\#)$, $v_0 \geq 0$ is such that

$$\int_0^s u_0^*(\sigma, y) d\sigma \leq \int_0^s v_0^*(\sigma, y) d\sigma, \qquad \forall s \in [0, |\Omega'|].$$

Then, for any $t \in [0,T]$ and $s \in [0, |\Omega'|]$

$$\int_0^s u^*(t, \sigma, y) d\sigma \leq \int_0^s v^*(t, \sigma, y) d\sigma.$$

Theorem 2.14 ([DGC16]). *Let β be a concave continuous nondecreasing function such that $\beta(0) = 0$. Give $T > 0$ arbitrary and let $f \in L^2(0,T : L^2(\Omega))$ with $f \geq 0$ in $(0,T)$ and let $w_0 \in L^2(\Omega)$ be such that $0 \leq w_0 \leq 1$. Let $w \in C([0,T] : L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$ and*

$z \in C([0,T] : L^2(\Omega^\#)) \cap L^2(\delta, T : H_0^1(\Omega^\#))$, for any $\delta \in (0,T)$, be the unique solutions of

$$(P)\begin{cases} \frac{\partial w}{\partial t} - \Delta w + \lambda\beta(w) = f(t) & \text{in } \Omega \times (0,T), \\ w = 1 & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0 & \text{on } \Omega, \end{cases}$$

$$(P^\#)\begin{cases} \frac{\partial z}{\partial t} - \Delta z + \lambda\beta(z) = f^\#(t), & \text{in } \Omega^\# \times (0,T), \\ z = 1, & \text{on } \partial\Omega^\# \times (0,T), \\ z(0) = z_0, & \text{on } \Omega^\#, \end{cases}$$

where $z_0 \in L^2(\Omega^\#)$, $0 \leq z_0 \leq 1$ is such that

$$\int_s^{|\Omega'|} z_0^*(\sigma, y)d\sigma \leq \int_s^{|\Omega'|} w_0^*(\sigma, y)d\sigma, \qquad \forall s \in [0, |\Omega'|] \text{ and a.e. } y \in \Omega''.$$

Then, for any $t \in [0,T]$, $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$

$$\int_s^{|\Omega'|} z^*(t, \sigma, y)d\sigma \leq \int_s^{|\Omega'|} w^*(t, \sigma, y)d\sigma. \tag{2.84}$$

In terms of the comparison of the effectiveness we have the following consequence:

Corollary 2.2. *In the assumptions of Theorem 2.16, for any $t \in [0, +\infty)$ we have*

$$\int_{\Omega^\#} \beta(z(t,x))dx \leq \int_\Omega \beta(w(t,x))dx. \tag{2.85}$$

We include now an unpublished alternative proof to one given in [DGC15b]. Let us define S_A as the solution of

$$\begin{cases} u_t + Au = 0, & (0,T) \times \Omega \\ u = 0, & x \in \partial\Omega \\ u = u_0, & t = 0 \end{cases} \tag{2.86}$$

and $S_A(t)u_0 = u(t)$. If we solve in $\Omega^\#$ the semigroup operator will be called $S_A^\#$. For the remainder of the text $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, and if $A = \beta$ then A represents the Nemitskij operator associated to β in the sense that $Au = \beta \circ u$.

Proposition 2.5. *Let $f \in L^2(\Omega)$ and $u_0 \preceq v_0$. Then, for a.e. $t \in (0,T)$*

$$S_{-\Delta-f}(t)u_0 \preceq S^{\#}_{-\Delta-f^{\#}}(t)v_0. \tag{2.87}$$

Proof. We proceed as in [Ban76b]. We write a parabolic inequality for $U(t,\sigma,y)$, whereas for V the equality holds and the result follows from the comparison principle. □

Proposition 2.6. *Let β be convex and $u_0 \preceq v_0$. Then, for a.e. $t \in (0,T)$*

$$S_\beta(t)u_0 \preceq S^{\#}_\beta(t)v_0. \tag{2.88}$$

Proposition 2.7. *Let $f \in L^2(\Omega)$ and $u_0 \preceq v_0$. Then, for a.e. $t \in (0,T)$*

$$S_{-\Delta+\beta-f}(t)u_0 \preceq S^{\#}_{-\Delta+\beta-f^{\#}}(t)v_0 \tag{2.89}$$

Proof. The Trotter-Kato formula applies

$$\left(S_{-\Delta-f}\left(\frac{t}{n}\right)S_\beta\left(\frac{t}{n}\right)\right)^n u_0 \to S_{-\Delta+\beta-f}(t)u_0, \tag{2.90}$$

$$\left(S^{\#}_{-\Delta-f^{\#}}\left(\frac{t}{n}\right)S^{\#}_\beta\left(\frac{t}{n}\right)\right)^n v_0 \to S^{\#}_{-\Delta+\beta-f^{\#}}(t)v_0. \tag{2.91}$$

Thus, by applying the previous propositions

$$\left(S_{-\Delta-f}\left(\frac{t}{n}\right)S_\beta\left(\frac{t}{n}\right)\right)^n u_0 \preceq \left(S^{\#}_{-\Delta-f^{\#}}\left(\frac{t}{n}\right)S^{\#}_\beta\left(\frac{t}{n}\right)\right)^n v_0. \tag{2.92}$$

as $n \to +\infty$ in $L^2(\Omega)$ and $L^2(\Omega^{\#})$ uniformly in t for $t \in [0,T]$. Hence, the comparison holds in the limit. □

Proof of Theorem 2.16. Let us first assume that $f \in \mathscr{C}([0,T];L^2(\Omega))$. Let us split $[0,T]$ in n parts $t_k^n = \frac{k}{n}T$ and let f^n be a piecewise constant function, given by

$$f^n(t) = f(t_k^n) \text{ for } t \in [t_k^n, t_{k+1}^n).$$

Take the mild solution u^n. Since f^n is piecewise constant, a simple induction argument shows that we can take the semigroups piecewise, for $t \in [t_k^n, t_{k+1}^n)$

$$u^n(t) = S_{-\Delta+\beta-f(t_k^n)}(t-t_k^n)S_{-\Delta+\beta-f(t_{k-1}^n)}\left(\frac{T}{n}\right)\cdots S_{-\Delta+\beta-f(0)}\left(\frac{T}{n}\right)u_0 \tag{2.93}$$

Applying the same reasoning for v_n we get, for $t \in [t_k^n, t_{k+1}^n)$

$$v^n(t) = S^{\#}_{-\Delta+\beta-f(t_k^n)^{\#}}(t-t_k^n)\, S^{\#}_{-\Delta+\beta-f(t_{k-1}^n)^{\#}}\left(\frac{T}{n}\right)\cdots S^{\#}_{-\Delta+\beta-f(0)^{\#}}\left(\frac{T}{n}\right) v_0 \tag{2.94}$$

Applying the formulation (2.93) and (2.94) and Proposition 2.7 finite times we have that, for a.e. $t \in [0,T]$

$$u^n(t) \preceq v^n(t). \tag{2.95}$$

It is easy to check that $f^n \to f$ in $L^2((0,T)\times\Omega)$ and $(f^n)^{\#} = (f^{\#})^n \to f^{\#}$ in $L^2((0,T)\times\Omega^{\#})$ (see, for example, [Rak08]). Due to the properties of the equations we have that $u_n \to u$ in $L^2((0,T)\times\Omega)$ and $v_n \to v$ in $L^2((0,T)\times\Omega^{\#})$. Therefore, the comparison (2.95) holds also in the limit, which concludes the proof for $f \in \mathscr{C}([0,T];L^2(\Omega))$. For $f \in L^2((0,T)\times\Omega)$ we take a sequence of functions $(f^n) \in \mathscr{C}([0,T]\times\Omega)$, $f^n \to f$ in $L^2((0,T)\times\Omega)$. Due to the continuity of $\cdot^{\#} : L^2(\Omega\times(0,T)) \mapsto L^2(\Omega^{\#}\times(0,T))$ the result follows. □

2.7.5 Steiner symmetrization of semilinear elliptic problems

In [DGC15b] and [DGC16] the proof of the semilinear elliptic problem is done by passing to the limit in the parabolic problem. If the comparison holds for every time it holds for the limit elliptic problem. The details can be found in the indicated papers.

Theorem 2.15 ([DGC15b]). *Let g be concave, verifying*

$$\int_0^\tau \frac{d\sigma}{g(\sigma)} < \infty, \qquad \forall \tau > 0. \tag{2.96}$$

Let $f \in L^2(\Omega)$ with $f \geq 0$. Let $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega^{\#})$ be the unique solutions of

$$(P)\ \begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

$$(P^{\#})\ \begin{cases} -\Delta v + g(v) = f^{\#}, & \text{in } \Omega^{\#}, \\ v = 0, & \text{on } \partial\Omega^{\#}. \end{cases}$$

Then, for any $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$.

$$\int_0^s u^*(\sigma,y)d\sigma \leq \int_0^s v^*(\sigma,y)d\sigma.$$

Theorem 2.16 ([DGC16]). *Let β be a concave continuous nondecreasing function such that $\beta(0)=0$. Let $f \in L^2(\Omega)$ with $f \geq 0$, $0 \leq w_0 \leq 1$. Let $w \in H^1(\Omega)$ and $z \in H^1(\Omega^\#)$ be the unique solutions of*

$$(P) \begin{cases} -\Delta w + \lambda \beta(w) = f & \text{in } \Omega, \\ w = 1 & \text{on } \partial\Omega, \end{cases}$$

$$(P^\#) \begin{cases} -\Delta z + \lambda \beta(z) = f^\# & \text{in } \Omega^\#, \\ z = 1 & \text{on } \partial\Omega^\#, \end{cases}$$

Then, for any $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$

$$\int_s^{|\Omega'|} z^*(\sigma, y) d\sigma \leq \int_s^{|\Omega'|} w^*(\sigma, y) d\sigma. \tag{2.97}$$

2.8 Other kinds of rearrangements

2.8.1 Relative rearrangement

Another rearrangement technique which has brought a lot of results in the last century is known as relative rearrangement. We will use its properties in Chapter 4.

Let us define this rearrangement

Definition 2.7. Let u be a measurable function, $v \in L^1(\Omega)$ and, for $s \in [0, |\Omega|]$

$$w(s) = \begin{cases} \displaystyle\int_{Q(s)} v(x)dx, & \text{if } |P(s)| = 0, \\ \displaystyle\int_{Q(s)} v(x)dx + \int_0^{s-|Q(s)|} (v|_{P(s)})^*(\sigma)d\sigma, & \text{if } |P(s)| \neq 0. \end{cases} \tag{2.98}$$

where

$$P(s) = \{x \in \Omega : u(x) = u^*(s)\}, \tag{2.99}$$

$$Q(s) = \{x \in \Omega : u(x) > u^*(s)\}. \tag{2.100}$$

The relative rearrangement of v with respect to u as

$$v_u^*(s) = \frac{dw}{ds}(s). \tag{2.101}$$

It is known that if $v \in L^p(\Omega)$ then $\|v_u^*\|_{L^p(0,|\Omega|)} \leq \|v\|_{L^p(\Omega)}$. Many other properties can be found in [Rak88].

One of such results is the following:

Theorem 2.17 (Rakotoson and Temam [RT90])**.** *Let Ω be a bounded connected set, $\partial\Omega$ of class C^2. Let $f \in L^1(\Omega)$ and $u \in W^{1,1}(\Omega)$. For almost every $t \in (essinf(u), esssup(u))$ (where essinf and esssup are the essential infimum and supremum) we have*

$$\frac{d}{dt}\int_{\{x\in\Omega: u(x)>t\}} f(x)dx = \mu'(t) f_u^*(\mu(t)). \tag{2.102}$$

Another application of such kind of rearrangement is the obtention of a L^∞ bound for semilinear equations given by Leray-Lions type operators (see [Rak87]). The results can be generalized to weighted spaces, and similar results are obtained (see [RS97; RS93a; RS93b]).

This kind of technique has also been used in Chapter 4 to obtain estimates of very weak solutions.

We complete this chapter by saying a few words on weighted rearrangements.

Gaussian rearrangement

The previous approaches to rearrangement yields good results in bounded domains. However, unbounded domains are not covered, since $\Omega^\star$ would be the whole of $\mathbb{R}^n$. A rearrangement based on the the Gaussian distribution

$$\varphi(x) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|x|}{2}\right) \tag{2.103}$$

can be applied. It has been used in several papers [BBMP02; Di 03; CO04] with good results.

The idea is to define a weighted measure to substitute the Lebesgue measure

$$|\Omega|_\varphi = \int_\Omega \varphi \in [0,1]. \tag{2.104}$$

We this definition in mind the can the rearrangement of a set as

$$\Omega_\varphi^\sharp = \{x = (x_1, \cdots, x_n) : x_1 > a\} \text{ such that } |\Omega_\varphi^\sharp| = |\Omega|. \tag{2.105}$$

For a function $u : \Omega \to \mathbb{R}$ we define

$$\mu_\varphi(t) = |\{x \in \Omega : |u(x)| > t\}|_\varphi, \tag{2.106}$$
$$u_\varphi^*(s) = \inf\{t \geq 0 : \mu(t) \leq s\}, s \in [0,1] \tag{2.107}$$

and $u_\varphi^\sharp : \Omega_\varphi^\sharp \to \mathbb{R}$ as the only function such that

$$|\{x : \Omega_\varphi^\sharp : |u_\varphi^\sharp(x)| > t\}|_\varphi = |\{x \in \Omega : |u(x)| > t\}|_\varphi. \tag{2.108}$$

Naturally $u_\varphi^\sharp(x) = u_\varphi^\sharp(x_1) = u^*(k(x_1))$. This rearrangement is well suited to compare elliptic problem in which the coefficient are of Gaussian-type. Let us state here the result for the parabolic problem

Theorem 2.18 ([CO04]). *Let u be an analytical solution of*

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{\varphi}\operatorname{div}(\varphi \nabla u) = f & Q = \Omega \times (0,T), \\ u = u_0 & t = 0, \\ u = 0 & \partial\Omega. \end{cases} \tag{2.109}$$

where $f \in L^2(Q, \varphi)$ and $u_0 \in H_0^1(\Omega, \varphi)$. Let v be the symmetrized solution

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{1}{\varphi}\operatorname{div}(\varphi \nabla v) = f_\varphi^\sharp & \Omega_\varphi^\sharp \times (0,T), \\ v = (u_0)_\varphi^\sharp & t = 0, \\ v = 0 & \partial\Omega. \end{cases} \tag{2.110}$$

Then,

$$\int_0^s u_\varphi^*(t,\sigma)d\sigma \leq \int_0^s v_\varphi^*(t,\sigma)d\sigma \quad \forall s \in [0,1]. \tag{2.111}$$

Chapter 3

Shape optimization

3.1 Shape differentiation

The main goal of this section is to analyze the differentiability, with respect to the domain Ω, of the *effectiveness factor* (2.3)

$$\mathscr{E}(\Omega) = \frac{1}{|\Omega|}\int_\Omega \beta(w_\Omega)dx. \tag{3.1}$$

For convenience, we will sometimes refer to the ineffectiveness (2.4). This will pass by the differentiation of functions w_Ω (resp. u_Ω) defined by (2.1) (resp. (2.2)).

This kind of problem falls within the family of problems already considered by Hadamard [Had08] and it has been studied by several authors in the literature (see, e.g., [MS76; Pir12; Sim80] and the references therein). In the most general formulation this family of problems may be associated to the general boundary value problem:

$$\begin{cases} A(u(D)) = f, & \text{in } D, \\ B(u(D)) = g, & \text{on } \partial D \end{cases} \tag{3.2}$$

and the question is to study the differentiability with respect to D of a functional which can by given generally as

$$J(D) = \int_D C(u_D)dx,$$

where A, B, C are operators or functions that may contain also some derivatives of u_D and D is a domain belonging to a certain class.

As mentioned before, our aim is to study the differentiability of functional (2.4). We consider a fixed open bounded regular domain of $\mathbb{R}^n$, Ω_0, and study its deformations given by a "small" function $\theta : \mathbb{R}^n \to \mathbb{R}^n$, so that the new domain is $\Omega = (I+\theta)\Omega_0$, where I is the identity function

$$I(x) = x. \tag{3.3}$$

We consider, as it is the case in chemistry catalysis, g and f such that $0 \le w_\Omega, u_\Omega \le 1$. Besides the above mentioned references we recall here the articles [Der80] for a linear problem with a Dirichlet constant boundary condition and [MPM79] where a semilinear equation arising in combustion was considered (corresponding, in our formulation to take $g(u) = -e^u$).

First we studied in [DGC15a] the case in which g (and β) are smooth (in $W^{2,\infty}(\mathbb{R})$). Then, in [GC17] we studied some non smooth cases. In particular, due to its importance in Chemical Engineering, we will discuss the case of root type non linearities. This is much more difficult, since for this kind on nonlinearity a dead core appears, and therefore the non-differentiable point of the nonlinearity might be in the range of the solution.

3.1.1 Fréchet derivative when $\beta \in W^{2,\infty}$

In order to obtain properties in the sense of derivatives, we consider two approaches, mimicking the approach in Differential Geometry. We first consider the global differentiability of solutions (as it was done in the linear cases in [HP05; All07] and for abstract problems in [Sim80]), which unfortunately requires derivatives in spaces of very regular functions, and then we take advantage of the differentiation along curves (the approach followed in [SZZ91]).

Let us recall that, for $\Omega \subset \mathbb{R}^n$, u_Ω is the unique solution of (2.2) (we assume that the formulation of the problem leads to the uniqueness of solutions). Let Ω_0 be a fixed smooth domain. We will work in the family of deformations

$$\Omega_\theta = (I+\theta)\Omega_0 \tag{3.4}$$

where $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$. We will consider the Lagrangian representation of u_{Ω_θ} as

$$u_\theta = u_{\Omega_\theta}, \tag{3.5}$$

and the Eulerian representations

$$\widehat{u}_\theta = u_\theta \circ (I+\theta). \tag{3.6}$$

Notice that

$$u_\theta : \Omega_\theta \to \mathbb{R} \qquad \widehat{u}_\theta : \Omega_0 \to \mathbb{R}.$$

It turns out that $\widehat{u}_\theta$ simplifies the study of the differentiability of u_Ω and the functional $\eta(\Omega)$ with respect to Ω.

Our proof relies heavily on the Implicit Function Theorem. The application of this theorem requires an uniform choice of functional space, which would require some additional informations on u. This kind of difficulties in the functional setting is well portrayed in [Bre99].

For the nonlinearity g we shall consider the following assumptions:

Assumption 3.1. g is nondecreasing.

Assumption 3.2. The Nemitskij operator for g (which we will denote again by g in some circumstances, as a widely accepted abuse of notation)

$$\begin{aligned} G : H^1(\Omega) &\to L^2(\Omega) \qquad (3.7)\\ u &\mapsto g \circ u \qquad (3.8)\end{aligned}$$

is well defined and is of class $\mathscr{C}^m$ for some $m \geq 1$.

We recall that Assumption 3.2 immediately implies that $[DG](v)\varphi = g'(v)\varphi$ for $\varphi, v \in H^1(\Omega)$ and that, if G is of class $\mathscr{C}^k$, with $k > 1$ then necessarily $g(s) = as + b$ for some $a, b \in \mathbb{R}$ (see, e.g., [Hen93]).

Our first result collects some general results on the differentiability of the solution u_Ω with respect to Ω:

Theorem 3.1 ([DGC15a])**.** *Let g satisfy Assumption 3.1 and 3.2. Then, the map*

$$\begin{aligned} W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\to H_0^1(\Omega_0)\\ \theta &\mapsto \widehat{u}_\theta\end{aligned}$$

(where $\widehat{u}_\theta$ is defined by (3.6)*) is of class $\mathscr{C}^l$ in a neighbourhood of 0 if $f \in H^k(\mathbb{R}^n)$ where $l = \min\{k,l\}$. Furthermore, the application*

$$\begin{array}{rcl} u : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) & \to & L^2(\mathbb{R}^n) \\ \theta & \mapsto & u_\theta \end{array}$$

(where u_θ is given by (3.5) *and extended by zero outside Ω_θ) is differentiable at* 0*. In fact $u'(0) : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \to H^1(\Omega_0)$ and*

$$u'(0)\theta + \nabla u_{\Omega_0} \cdot \theta \in H_0^1(\Omega_0).$$

Remark 3.1. Since the function is only differentiable at 0 we will simplify write u' to represent $u'(0)$.

One of the easiests ways to characterize the global derivative is, as usually, to compute the directional derivatives.

Definition 3.1. We will say that Φ is a curve of deformations of Ω_0 if

$$\Phi : [0,T) \to W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$$

is such that $\det\Phi(\tau) > 0$ and $\Phi(0) = I$.

Assumption 3.3. We will say that θ is a curve of small perturbations of the identity if $\Phi(\tau) = I + \theta(\tau)$ is a curve of deformations and

i) $\theta : [0,T) \to W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$ is differentiable at 0 (from the right)

ii) $\theta(0) = 0$.

Sometimes we will consider higher order derivatives too. We will refer to θ or Φ indistinctively, since they relate by $\Phi(\tau) = I + \theta(\tau)$. It will be common that we consider the curve of deformations

$$\Phi(\tau) = I + \tau\theta, \tag{3.9}$$

for a fixed deformation $\theta \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$. In this notation we will admit the abuse of notation $\theta(\tau) = \tau\theta$, where naturally both elements are different, but this should not lead to confusion.

In this terms, the above theorem leads to:

Corollary 3.1. *Let Φ be a a curve of deformations of class $\mathscr{C}^k$. Then $\tau \mapsto v_{\theta(\tau)}$ is of class $\mathscr{C}^l$ with $l = \min\{m,k\}$.*

Our second result concerns the characterization of u'. We have:

Theorem 3.2. *Let g satisfy Assumption 3.1 and 3.2. Let θ be a curve satisfying assumptions 3.3. Then u is differentiable along Φ at least at 0. That is, the directional derivative $\frac{d}{d\tau}(u \circ \Phi)$ exists, and it is the solution u' of the linear Dirichlet problem*

$$\begin{cases} -\Delta u' + \lambda g'(u_{\Omega_0})u' = 0 & \text{in } \Omega_0, \\ u' = -\nabla u_{\Omega_0} \cdot \theta'(0) & \text{on } \partial\Omega_0. \end{cases} \tag{3.10}$$

We point out that the above result shows, in other terms, for $\theta \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, that $u'(0)\theta$ is the unique weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u' + \lambda g'(u_{\Omega_0})u' = 0, & \text{in } \Omega_0, \\ u' = -\nabla u_{\Omega_0} \cdot \theta, & \text{on } \partial\Omega_0. \end{cases} \tag{3.11}$$

As consequence we have:

Corollary 3.2. *The function $u' : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \to H^1(\Omega_0)$ is continuous. In fact, since due to Assumption 3.2, the solution u of (2.2) verifies $u \in W^{2,p}(\Omega_0)$ for any $p \in [1,+\infty)$, then for any $q \in [1,+\infty)$*

$$|u'(0)(\theta)|_q \leq c|\nabla u \cdot \theta|_{L^p(\partial\Omega_0)} \leq c|\theta|_\infty |\nabla u_{\Omega_0}|_{L^p(\partial\Omega_0)} \tag{3.12}$$

$$\leq c(p)|\theta|_\infty |u_{\Omega_0}|_{W^{2,p}(\Omega_0)}. \tag{3.13}$$

Concerning the differentiability of the effectiveness factor functional we have:

Theorem 3.3. *On the assumptions of Theorem 3.1, let*

$$\hat{\eta}(\theta) = \int_{(I+\theta)\Omega_0} g(u_{(I+\theta)\Omega_0})dx. \tag{3.14}$$

Then η is of class $\mathscr{C}^m$ in a neighbourhood of 0. It holds that

$$\hat{\eta}^{(m)}(0)(\theta_1,\cdots,\theta_m) = \int_{\Omega_0} \frac{d^n}{d\theta_n \cdots d\theta_1}(g(\widehat{u}_\theta)J_\theta)\, dx. \tag{3.15}$$

Its first derivative can be expressed in terms of u

$$\hat{\eta}'(0)(\theta) = \int_{\Omega_0} \left(g'(u_{\Omega_0})u' \ + \mathrm{div}(g(u_{\Omega_0})\theta) \right) dx, \tag{3.16}$$

and, if $\partial\Omega_0$ is Lipschitz,

$$\hat{\eta}'(0)(\theta) = \int_{\Omega_0} g'(u_{\Omega_0})u' \, dx + g(0) \int_{\partial\Omega_0} \theta \cdot n \, dS, \tag{3.17}$$

where $u' = u'(0)(\theta)$.

As a direct consequence we get:

Corollary 3.3. *On the assumptions of Theorem 3.1, it holds that*

$$\eta'(\theta) = \frac{1}{|\Omega_0|} \left(\int_{\Omega_0} g'(u_{\Omega_0})u' \, dx - \eta(0) \int_{\partial\Omega_0} \theta \cdot n \, dS \right).$$

Corollary 3.4. *On the assumptions of Theorem 3.1, if Φ is a volume preserving curve of deformations then*

$$\eta'(\theta) = \frac{1}{|\Omega_0|} \int_{\Omega_0} g'(u_{\Omega_0})u' \, dx.$$

We point out that if g is Lipschitz (i.e. $g \in W^{1,\infty}(\mathbb{R})$) then we get that

$$|\eta(\theta) - \eta(0)| = |\eta'(0)(\lambda\theta)| \le c|g'|_\infty |u|_{W^{2,p}} |\theta|_\infty.$$

The details of the proof of the results in this section can be found in [DGC15a]. They will be ommited here to speed-up the presentation of results.

3.1.1.1 Functional setting: Nemitskij operators and the implicit function theorem.

Let us formalize what we mean by a *shape functional*. At the most fundamental level it should be a function defined over a set of domain, that is defined over a subset $\mathfrak{C}$ of $\mathscr{P}(\mathbb{R}^n)$. Since we want to differentiate this functional we, at the very least, need to define proximity, that is a way to define the *neighbourhood of a set*. As it is usual in the literature of shape optimization we work over the set of weakly differentiable bounded deformations with bounded derivative, i.e. over the Sobolev space $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 3.2. We say that a functional $I : \mathfrak{C} \subset \mathscr{P}(\mathbb{R}^n) \to \mathbb{R}$ is defined on a neighbourhood of $\Omega_0 \subset \mathbb{R}^n$ if there exists U a neighbourhood of 0 on $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ such that I is defined over $\{(Id+\theta)(\Omega_0) : \theta \in U\}$. We say that I is differentiable at Ω_0 if the application

$$\begin{array}{rcl} W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n) & \to & \mathbb{R} \\ \theta & \mapsto & I((Id+\theta)(\Omega_0)) \end{array}$$

is differentiable at 0.

We present a sufficient condition so that Assumption 3.2 holds. This is widely used in the context of partial differential equations, but as far as we know no reference is known besides it being an exercise in [Hen93]. That being the case we provide a proof[1]. Other conditions, mainly on the growth of g can be considered so that Assumption 3.1 and 3.2 hold.

Lemma 3.1.1. *Let $g \in W^{2,\infty}(\mathbb{R})$. Then the Nemitskij operator* (3.8) *(in the sense $L^p(\Omega) \to L^2(\Omega)$) is of class $\mathscr{C}^1$ for all $p > 2$. In particular, Assumption 3.2 holds.*

Proof. Let us define G the Nemitskij operator defined in (3.8). Consider it $G : L^p(\Omega) \to L^2(\Omega)$ for $p \geq 2$. We first have that, for $L = \max\{\|g\|_\infty, \|g'\|_\infty, \|g''\|_\infty\}$

$$\|G(u)-G(v)\|_{L^2}^2 = \int_\Omega |g(u)-g(v)|^2 dx \leq L \int_\Omega |u-v|^2 dx$$

so that G is continuous. For $p > 2$ let $\varphi \in \mathscr{C}^\infty(\Omega)$ we compute

$$\|g(u+\varphi)-g(u)-g'(u)\varphi\|_{L^2}^2 = \int_\Omega |g'(\xi(x))-g'(u(x))|^2 |\varphi(x)|^2 dx$$

for some function $\xi(x)$, between $u(x)$ and $u(x)+\varphi(x)$, due to the intermediate value theorem. We have that

$$\begin{aligned} |g'(\xi(x))-g'(u((x))| &\leq L|\xi(x)-u(x)| \leq L|\varphi(x)| \\ |g'(\xi(x))-g'(u(x))| &\leq 2L \\ |g'(\xi(x))-g'(u(x))| &\leq L2^{1-\alpha}|\varphi(x)|^\alpha, \quad \forall \alpha \in (0,1). \end{aligned}$$

Therefore,

$$\|g(u+\varphi)-g(u)-g'(u)\varphi\|_{L^2}^2 \leq L^2 2^{2-2\alpha} \int_\Omega |\varphi(x)|^{2+2\alpha} dx.$$

Let $2 < p < 4$ then we have that $p = 2+2\alpha$ with $0 < \alpha < 1$. We then have that

$$\|g(u+\varphi)-g(u)-g'(u)\varphi\|_{L^2} \leq L2^{1-\alpha} \|\varphi(x)\|_{L^p}^{1+\alpha},$$

[1] This candidate is thankful to Prof. J.M. Arrieta for the details of the proof.

which proves the Fréchet differenciability. For $p > 4$ we have that $L^p(\Omega) \hookrightarrow L^3(\Omega)$. Furthermore, for any given dimension n we can use the Sobolev inclusions $H^1(\Omega) \hookrightarrow L^p(\Omega)$ with $p > 2$, proving the desired differenciability. □

Some other well-known results are quoted now:

Theorem 3.4. *Let $g \in W^{1,p}(\mathbb{R}^n)$. Then the map*

$$\begin{aligned} \mathfrak{G} : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\rightarrow L^p(\mathbb{R}^n) && (3.18)\\ \theta &\mapsto g \circ (I+\theta) && (3.19) \end{aligned}$$

is differentiable in a neighbourhood of 0 *and*

$$\mathfrak{G}'(0) = (\nabla g) \circ (I+\theta).$$

Theorem 3.5 ([HP05, Lemme 5.3.3.])**.** *Let*

$$\begin{aligned} g : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\rightarrow L^p(\mathbb{R}^n),\\ \Psi : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\rightarrow W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \end{aligned}$$

continuous at 0 *with* $\Psi(0) = I$*,*

$$\begin{aligned} W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\rightarrow L^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n;\mathbb{R}^n) && (3.20)\\ \theta &\mapsto (g(\theta),\Psi(\theta)) && (3.21) \end{aligned}$$

differentiable at 0*, with* $g(0) \in W^{1,p}(\mathbb{R}^n)$ *and*

$$g'(0) : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) \rightarrow W^{1,p}(\mathbb{R}^n)$$

is continuous. Then the application

$$\begin{aligned} \mathfrak{G} : W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n) &\rightarrow L^p(\mathbb{R}^n) && (3.22)\\ \theta &\mapsto g(\theta) \circ \Psi(\theta) && (3.23) \end{aligned}$$

is differentiable at 0 *and*

$$\mathfrak{G}'(0) = g'(0) + \nabla g(0) \cdot \Psi'(0).$$

To conclude this section we state a classical result, the Implicit Function Theorem. This result is typically a direct consequence of the Inverse Function Theorem. In the Banach space setting this result is originally due to Nash and Moser (see [Nas56; Mos66]). In the detailed survey [Ham82] the author points towards Zehnder [Zeh76] as one of the first presentations as implicit function theorem.

Theorem 3.6 (Implicit Function Theorem)**.** *Let X, Y and Z be Banach spaces and let U,V be neighbourhoods on X and Y, respectively. Let $F : U \times V \to Z$ be continuous and differentiable, and assume that $D_yF(0,0) \in \mathscr{L}(Y,Z)$ is bijective. Let us assume, further, that $F(0,0) = 0$. Then there exists W neighbourhood of 0 on X and a differentiable map $f : W \to Y$ such that $F(x, f(x)) = 0$. Furthermore, for x and y small, $f(x)$ is the only solution y of the equation $F(x,y) = 0$. If F is of class $\mathscr{C}^m$ then so is f.*

3.1.1.2 Differentiation of solutions

For the reader convenience we repeat here the general result in [Sim80]:

Theorem 3.7. *Let D be a bounded domain such that ∂D be a piecewise $\mathscr{C}^1$ and assume that D is locally on one side of ∂D. Let u_0 be the solution of (3.2). Let us use the notation $\mathscr{C}^k = \mathscr{C}^k(\mathbb{R}^n,\mathbb{R}^n)$ and $k \geq 1$. Assume that*

$$u(\theta) \in W^{m,p}((I+\theta)D) \tag{3.24}$$

and that for every open set D' close to D (for example $D' = (I+\theta)D$ for small θ in the norm of $\mathscr{C}^k$), $A,B,C : W^{m-1,p}(D') \to \mathscr{D}'(D)$ are differentiable and

$$\begin{cases} A : W^{m,p}(D') \to \mathscr{D}'(D') \\ B : W^{m,p}(D') \to W^{1,1}(D') \\ C : W^{m,p}(D') \to L^1(D') \end{cases} \tag{3.25}$$

and

$$\begin{aligned} \mathscr{C}^k &\to W^{m,p} && (3.26)\\ \theta &\mapsto u(\theta)\circ(I+\theta) && (3.27) \end{aligned}$$

is differentiable at 0. Then:

i) The solution u is differentiable in the sense that

$$u : \mathscr{C}^k \to W^{m-1,p}_{loc}(D) \text{ is differentiable}$$

and the derivative (i.e. the local derivative u' in the direction of τ) satisfies

$$\frac{\partial A}{\partial u}(u_0)u' = 0, \text{ in } D. \tag{3.28}$$

ii) If

$$\begin{cases} \theta \mapsto B(u(\theta)) \circ (I+\theta) \text{ is differentiable at 0 into } W^{1,1}(D), \\ \qquad \text{(i.e. with the } W^{1,1}(D) \text{ topology in the image set)} \\ B(u_0) \in W^{2,1}(D), \\ g \in W^{2,1}(\mathbb{R}^n) \end{cases} \tag{3.29}$$

then u' satisfies

$$\frac{\partial B}{\partial u}(u_0)u' = -\tau \cdot n \frac{\partial}{\partial n}(B(u_0) - g). \tag{3.30}$$

iii) If

$$\begin{cases} \theta \mapsto C(u(\theta)) \circ (I+\theta) \text{ is differentiable at 0 into } L^1(D), \\ C(u_0) \in W^{1,1}(D), \end{cases} \tag{3.31}$$

then $\theta \mapsto J(\theta)$ is differentiable and its directional derivative in the direction of τ is:

$$\frac{\partial J}{\partial \theta}(0)\tau = \int_D \frac{\partial C}{\partial u} u' \, dx + \int_{\partial D} \tau \cdot n C(u_0) \, dS. \tag{3.32}$$

3.1.1.3 Differentiation under the integral sign

We shall follow some reasonings similar to the ones presented in [HP05]. Let us define $\Omega_\tau = \Phi(\tau, \Omega_0)$ and consider a function f such that $f(\tau) \in L^1(\Omega_\tau)$. We take interest on the map

$$I : \mathbb{R} \rightarrow \mathbb{R} \tag{3.33}$$

$$\tau \mapsto \int_{\Omega_\tau} f(\tau, x) \, dx = \int_{\Omega_0} f(\tau, \Phi(\tau, y)) J(\tau, y) \, dy \tag{3.34}$$

where $f(\tau, x) = f(\tau)(x)$ and the Jacobian

$$J(\tau, y) = \det(D_y \Phi(\tau, y)).$$

Theorem 3.8. *Let Φ satisfy Assumption 3.3, f such that*

$$\begin{aligned} f:[0,T) &\to L^1(\mathbb{R}^n) \\ \tau &\mapsto f(\tau) \end{aligned}$$

is differentiable at 0 *and, besides, it satisfies the spatial regularity at* $\tau = 0$

$$f(0) \in W^{1,1}(\mathbb{R}^N).$$

Then, $\tau \mapsto I(\tau) = \int_{\Omega_\tau} f(\tau)$ *is differentiable at* 0 *and*

$$I'(0) = \int_{\Omega_0} f'(0) + \operatorname{div}\left(f(0) \frac{\partial \Phi}{\partial \tau}(0) \right).$$

If Ω_0 *is an open set with Lipschitz boundary then*

$$I'(0) = \int_{\Omega_0} f'(0) + \int_{\partial\Omega_0} f(0) n \cdot \frac{\partial \Phi}{\partial \tau}(0).$$

In simpler terms, under regularity it holds that

$$\left.\frac{\partial}{\partial \tau}\right|_{\tau=0} \left(\int_{G_\tau} f(\tau,x)dx \right) = \int_{\Omega_0} \left\{ \frac{\partial f}{\partial \tau}(0,x) + \operatorname{div}\left(f(0,x) \frac{\partial \Phi}{\partial \tau}(0,x) \right) \right\} dx. \tag{3.35}$$

We have some immediate consequences of Theorem 3.8

Lemma 3.1.2. *Let* $g \in W^{1,1}(\mathbb{R}^N)$ *and* $\Psi : [0,T) \to W^{1,\infty}$ *be continuous at* 0 *such that* $\Psi : [0,T) \to L^\infty$ *is differentiable at* 0*, and let* Z *be its derivative. Then*

$$\begin{aligned} G:[0,T) &\to L^1(\mathbb{R}^n) &(3.36)\\ \tau &\mapsto g \circ \Psi(\tau) &(3.37)\end{aligned}$$

is differentiable at 0 *and* $G'(0) = \nabla g \cdot Z$.

Lemma 3.1.3 (Differentiation under the integral sign)**.** *Let* E *be a Banach space and*

$$\begin{aligned} f:E\times\Omega &\to \mathbb{R} &(3.38)\\ (v,y) &\mapsto f(v,y) &(3.39)\end{aligned}$$

such that

$$\begin{aligned} \tilde{f} : E &\to L^1(\Omega) && (3.40)\\ v &\mapsto f(v,\cdot) && (3.41) \end{aligned}$$

is differentiable at v_0. Let

$$\begin{aligned} F : E &\to \mathbb{R} && (3.42)\\ v &\mapsto \int_\Omega f(v,y)dy && (3.43) \end{aligned}$$

Then F is differentiable at v_0 and

$$DF(v) = \int_\Omega (D_v \tilde{f})(v)(y).$$

3.1.2 Gateaux derivative when $\beta \in W^{1,\infty}$

Once the case $\beta \in W^{2,\infty}(\mathbb{R})$ is understood, let us focus on the less smooth case $\beta \in W^{1,\infty}(\mathbb{R})$. In this case, we can only prove that the shape derivative exists in the Gateaux sense (which is weaker than the Fréchet sense).

Theorem 3.9. *Let $\theta \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $\beta \in W^{1,\infty}(\mathbb{R})$ be nondecreasing such that $\beta(0) = 0$ and $f \in H^1(\mathbb{R}^n)$. Then, the applications*

$$\begin{aligned} \mathbb{R} &\to L^2(\Omega_0)\\ \tau &\mapsto u_{(I+\tau\theta)\Omega_0} \circ (I+\tau\theta), \end{aligned}$$

and

$$\begin{aligned} \mathbb{R} &\to L^2(\mathbb{R}^n)\\ \tau &\mapsto u_{(I+\tau\theta)\Omega_0} \end{aligned}$$

are differentiable at 0*. Furthermore, $\frac{du_\tau}{d\tau}|_{\tau=0}$ is the unique solution of* (3.11).

In this chapter we will be particularly interested in the case in which β' only has blow-up at $w = 0$. Let us define

$$v = \frac{dw_\tau}{d\tau}\Big|_{\tau=0}. \tag{3.44}$$

We can rewrite (3.11) in terms of w

$$\begin{cases} -\Delta v + \beta'(w_{\Omega_0}) v = 0 & \Omega, \\ v + \nabla w_{\Omega_0} \cdot \theta = 0 & \partial\Omega. \end{cases} \tag{3.45}$$

Remark 3.2. In most cases, the process of homogenization developed in Chapter 1 leads to an homogeneous equation (2.1) in which β is the same as the function in the microscopic problem, and thus it is natural that β be singular at 0. However, it sometimes happens that the limit kinetic is different. In the homogenization of problems with particles of critical size (see [DGCPS17c]) it turns out that the resulting kinetic in the macroscopic homogeneous equation (2.1) satisfies $\beta \in W^{1,\infty}$, even when the original kinetic of the microscopic problem was a general maximal monotone graph.

3.1.2.1 From $W^{2,\infty}$ to $W^{1,\infty} \cap \mathscr{C}^1$

Let us show that the shape derivative is continuously dependent on the nonlinearity, and thus that we can make a smooth transition from the Fréchet scenario presented in [DGC15a] to our current case. For the rest of the paper we will use the notation:

Lemma 3.1.4. *Let $f \in L^2(\mathbb{R}^n)$, $\beta \in W^{1,\infty}(\mathbb{R})$ be a nondecreasing function such that $\beta(0) = 0$ and let $\beta_n \in W^{2,\infty}(\mathbb{R})$ nondecreasing such that $\beta_n(0) = 0$. Let w_n be the unique solution of*

$$\begin{cases} -\Delta w_n + \beta_n(w_n) = f & \Omega_0, \\ w_n = 1 & \partial\Omega_0. \end{cases} \tag{3.46}$$

Then

$$\|w_n - w\|_{H^1(\Omega_0)} \le C \|\beta_n - \beta\|_{L^\infty(\mathbb{R})} \tag{3.47}$$

$$\|w_n - w\|_{H^2(\Omega_0)} \le C(1 + \|\beta'\|_{L^\infty(\mathbb{R})}) \|\beta_n - \beta\|_{L^\infty(\mathbb{R})}. \tag{3.48}$$

Furthermore, let $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ and v_n be the unique solution of

$$\begin{cases} -\Delta v_n + \beta_n'(w_n) v_n = 0 & \Omega_0, \\ v_n + \nabla w_n \cdot \theta = 0 & \partial\Omega_0. \end{cases} \tag{3.49}$$

Then, if $\beta_n \to \beta$ in $W^{1,\infty}(\mathbb{R})$,

$$v_n \rightharpoonup v \text{ in } H^1(\Omega_0). \tag{3.50}$$

Remark 3.3. In (3.47) we used the notation

$$\|\beta_n - \beta\|_{L^\infty} = \sup_{x\in\mathbb{R}} |\beta_n(x) - \beta(x)|.$$

It doesn't mean that either β_n or β are $L^\infty(\mathbb{R})$ functions themselves, but rather that their difference is pointwise bounded. In fact, this bound is destined to go 0 as $n \to +\infty$.

3.1.3 Shape derivative with a dead core

We can prove that the shape derivative in the smooth case has, under some assumptions, a natural limit when β is not smooth.

In some cases in the applications (see, e.g., [Día85]) we can take β so that $\beta'(w_{\Omega_0})$ has a blow up. It is common, specially in Chemical Engineering, that $\beta'(0) = +\infty$ and

$$N_{\Omega_0} = \{x \in \Omega_0 : w_{\Omega_0}(x) = 0\}$$

exists and has positive measure (see [Día85]). This is region is known as a dead core. In this case $\beta'(w_{\Omega_0}) = +\infty$ in N_{Ω_0}. Due to this fact, the natural behaviour of the weak solutions of (3.45) is $v = 0$ in N_{Ω_0}. We have the following result

Theorem 3.10. *Let β be nondecreasing, $\beta(0) = 0$, $\beta'(0) = +\infty$,*

$$\beta \in \mathscr{C}(\mathbb{R}) \cap \mathscr{C}^1(\mathbb{R}\setminus\{0\}),$$

and assume that $|N_{\Omega_0}| > 0$, $\theta \in W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$ and $0 \le f \le \beta(1)$. Then, there exists v a solution of

$$\begin{cases} -\Delta v + \beta'(w_{\Omega_0})v = 0 & \Omega_0 \setminus N_{\Omega_0}, \\ v = 0 & \partial N_{\Omega_0}, \\ v = -\nabla w_{\Omega_0}\cdot\theta & \partial\Omega_0, \end{cases} \tag{3.51}$$

in the sense that $v \in H^1(\Omega_0)$, $v = 0$ in N_{Ω_0}, $v = -\nabla w_{\Omega_0}\cdot\theta$ in $L^2(\partial\Omega_0)$, $\beta'(w_{\Omega_0})v^2 \in L^1(\Omega_0)$ and

$$\int_{\Omega_0\setminus N_{\Omega_0}} \nabla v \nabla\varphi + \int_{\Omega_0\setminus N_{\Omega_0}} \beta'(w) v\varphi = 0 \tag{3.52}$$

for every $\varphi \in W_c^{1,\infty}(\Omega_0\setminus N_{\Omega_0})$. Furthermore, for $m \in \mathbb{N}$, consider β_m defined by

$$\beta_m'(s) = \min\{m, \beta'(s)\}, \qquad \beta_m(0) = \beta(0) = 0,$$

and let w_m, v_m be the unique solutions of (3.46) *and* (3.49). *Then,*

$$v_m \rightharpoonup v, \quad in\ H^1(\Omega_0), \tag{3.53}$$

where v is a solution of (3.51).

The uniqueness of solutions of (3.51) when $\beta'(w_{\Omega_0})$ blows up is by no means trivial. Problem (3.51) can be written in the following way:

$$-\Delta v + V(x)v = f \tag{3.54}$$

where $V(x) = \beta'(w_{\Omega_0}(x))$ may blow up as a power of the distance to a piece of the boundary. This kind of problems are common in Quantum Physics, although their mathematical treatment is not always rigorous (cf. [Día15; Día17]).

In the next section we will show some estimates on $\beta'(w_{\Omega_0})$. Let us state here some uniqueness results depending on the different blow-up rates.

When the blow-up is subquadratic (i.e. not *too* rapid), by applying Hardy's inequality and the Lax-Migram theorem, we have the following result (see [Día15; Día17]).

Corollary 3.5. *Let N_{Ω_0} have positive measure and $\beta'(u(x)) \leq Cd(x, N_{\Omega_0})^{-2}$ for a.e. $x \in \Omega_0 \setminus N_{\Omega_0}$. Then the solution v is unique.*

The study of solutions of problem (3.54) in Ω_0 when $V \in L^1_{loc}(\Omega_0)$ was carried out by many authors (see [DR10; DGCRT17] and the references therein). Existence and uniqueness of this problem in the case $V(x) \geq Cd(x, \partial\Omega_0)^{-r}$ with $r > 2$ was proved in [DGCRT17]. Applying these techniques one can show that

Corollary 3.6. *Let N_{Ω_0} have positive measure and $\beta'(w(x)) \geq Cd(x, N_{\Omega_0})^{-r}, r > 2$ for a.e. $x \in \Omega_0 \setminus N_{\Omega_0}$. Then the solution v is unique.*

Similar techniques can be applied to the case $\beta'(w(x)) \geq Cd(x, N_{\Omega_0})^{-2}$. This will be the subject of a further paper[2].

[2] At the time of writing this thesis, there is a draft of such a paper by this candidate jointly with J.I. Díaz and J.M. Rakotoson

3.1.4 Estimates of w_{Ω_0} close to N_{Ω_0}

Let us study the solution w_{Ω_0} on the proximity of the dead core and the blow up behaviour of $\beta'(w_{\Omega_0})$. First, we recall a well-known example

Example 3.1. Explicit radial solutions with dead core are known when $\beta(w) = |w|^{q-1}w$ $(0 < q < 1)$, Ω_0 is a ball of large enough radius and f is radially symmetric. In this case it is known that N_{Ω_0} exists, has positive measure and

$$\frac{1}{C}d(x,N_{\Omega_0})^{-2} \leq \beta'(w_{\Omega_0}) \leq Cd(x,N_{\Omega_0})^{-2}.$$

For the details see [Día85].

In fact, we present here a more general result to study the behaviour in the proximity of the dead core, based on estimates from [Día85].

Proposition 3.1. *Let $f = 0$, β be continuous, monotone increasing such that $\beta(0) = 0$, w be a solution of* (2.1) *that develops a dead core N_{Ω_0} of positive measure and assume that $\partial N_{\Omega_0} \in \mathscr{C}^1$. Define*

$$G(t) = \sqrt{2}\left(\int_0^t \beta(\tau)d\tau + \alpha t\right)^{\frac{1}{2}}, \quad \text{where } \alpha = \max\left\{0, \min_{x\in\partial\Omega_0} H(x)\frac{\partial w}{\partial n}(x)\right\}, \tag{3.55}$$

and assume that $\frac{1}{G} \in L^1(\mathbb{R})$. Then

$$w_{\Omega_0}(x) \leq \Psi^{-1}(d(x,N_{\Omega_0})), \quad \text{in a neighbournood of } N_{\Omega_0}, \tag{3.56}$$

where $\Psi(s) = \int_0^s \frac{dt}{G(t)}$.

Example 3.2 (Root type reactions)**.** Let $f = 0$, $\beta(s) = \lambda|s|^{q-1}s$ with $0 < q < 1$ and let Ω_0 be a convex set such that N_{Ω_0} exists and satisfies that $\partial N_{\Omega_0} \in \mathscr{C}^1$. Then

$$w_{\Omega_0}(x) \leq Cd(x,N_{\Omega_0})^{\frac{2}{1-q}}. \tag{3.57}$$

Furthermore

$$\beta'(w_{\Omega_0}(x)) \geq Cd(x,N_{\Omega_0})^{-2}. \tag{3.58}$$

Remark 3.4. The regularity assumptions on ∂N_{Ω_0} are by no means trivials. Examples can be constructed in which this does not hold. However, there are many cases of relevance the applications in which this regularity holds.

3.2 Convex optimization of the homogenized solutions

This section includes results published in [DGCT15] [DGCT16].

For the homogenized problem, we have the following optimality result:

Theorem 3.11. *Let* $1 \le \alpha < \frac{n}{n-2}$, $0 < \theta < |Y|$, C, D *be fixed proper subsets of* Y *and* $\tilde{\varepsilon} > 0$. *Let us assume that*

$$G_0 \text{ satisfies the uniform } \tilde{\varepsilon}\text{-cone property.} \tag{3.59}$$

We define

$$\begin{aligned} U_{adm} &= \{\overline{C} \subset G_0 \subset \overline{D} : G_0 \text{ satisfies } (3.59) \text{ and } |G_0| = \theta\}, \\ C_\theta(D) &= \{G_0 \subset D : G_0 \text{ is open, convex and } |G_0| = \theta\}. \end{aligned}$$

Then, at fixed volume $\theta \in (0, |Y|)$, *there exists a domain of maximal (and minimal) effectiveness for the homogenized problem (see Chapter 1) in the class of* $G_0 \in U_{adm} \cap C_\theta(D)$.

For small (non-critical) holes, we can characterize the optimizer shape in the class of fixed volume.

Theorem 3.12. *For the case* $1 < \alpha < \frac{n}{n-2}$, *the ball is the domain* G_0 *of maximal effectiveness for a set volume in the class of star-shaped* C^2 *domains with fixed volume.*

Remark 3.5. It is a curious fact that Theorem 3.12 is opposed to the homogenization with respect to the exterior domain Ω. In this context, when Ω is a ball has least effectivity, as can be shown by rearrangement techniques (see [Día85]). In the context of product domains, $\Omega = B \times \Omega''$ is the least effective on the class $\Omega = \Omega' \times \Omega''$ for set volume, at least for convex or concave kinetics as presented in Chapter 2 (see [DGC15b; DGC16; KS80]).

Through standard procedures in weak solution theory, one easily gets several results (see, e.g., [Bré71a]).

3.2.1 Some auxiliary results for convex domains

For the optimization, we will restrict ourselves to a general enough family of domains, but in which we can define a topology which makes the family to be compact. It is well known (see, for example, [Pir84]) that the following result holds true.

Theorem 3.13 ([Pir84]). *The class of closed subsets of a compact set D is compact in $\mathscr{P}(\mathbb{R}^n)$ for the Hausdorff convergence.*

A proof for the continuity of the effective diffusion $a_0(G_0)$ (given by (1.77)) under the Hausdorff distance in U_{adm} can be found in [HD95].

Lemma 3.2.1 ([HD95]). *If U_{adm} is compact with respect to the Hausdorff metric and if $(G_0^n) \subset U_{adm}$, $G_0^m \to G_0$ as $m \to \infty$, $G_0 \in U_{adm}$, then $a_0(G_0^m) \to a_0(G_0)$ in $\mathscr{M}_n(\mathbb{R})$, where a_0 is the effective diffusion matrix given by* (1.77).

The behaviour of the measure $|Y \setminus G_0|$ is slightly more delicate (we include a commentary even though, in our case, this will be constant). A distance with a definition similar to Hausdorff metric is the Hausdorff complementary distance

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^n} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|.$$

It has the following property: given open domains $(\Omega_m)_m, \Omega$, such that $d_{H^c}(\Omega_m, \Omega) \to 0$ as $m \to \infty$ then $\liminf_m |\Omega_m| \geq |\Omega|$. However, lower semicontinuity of the measure of the boundary ($|\partial G_0|$) is, in general, false (see [HD95] for some counterexamples). Nevertheless, the set of **convex** domains has a number of very interesting properties (see [Van04]).

Lemma 3.2.2 ([Van04]). *The topological spaces $(C_\theta(D), d_H)$ and $(C_\theta(D), d_{H^c})$ are equivalent.*

The continuity of the boundary measure is provided by the following result, proved in [BG97].

Lemma 3.2.3 ([BG97]). *Let $(\Omega_m), \Omega \in C_\theta(D)$. If $\Omega_1 \subset \Omega_2$, then $|\partial\Omega_1| \leq |\partial\Omega_2|$. Moreover, if $\Omega_m \overset{d_H}{\to} \Omega$, then $|\Omega_m| \to |\Omega|$ and $|\partial\Omega_m| \to |\partial\Omega|$, as $m \to \infty$.*

For the continuity of solutions with respect to G_0, we need the following theorem on the continuity of the associated Nemitskij operators of g (see, for example, [Dal93] and [Lio69]).

Lemma 3.2.4 ([Lio69]). *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

$$|g(x, v)| \leq C(1 + |v|^q) \tag{3.60}$$

holds true for $q = \frac{r}{t}$ with $r \geq 1$ and $t < \infty$. Then, the map

$$L^r(\Omega) \to L^t(\Omega) \qquad v \mapsto g(x, v(x))$$

is continuous in the strong topologies.

Lemma 3.2.5. *Let $\mathscr{A}$ be the set of elliptic matrices and let g satisfy* (3.60) *for some* $0 \le q \le \frac{n}{n-2}$. *Let $u(A,\lambda)$ be the unique solution of*

$$\begin{cases} -\operatorname{div}(A\nabla u) + \lambda g(u) = f, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega, \end{cases}$$

Then, the application

$$\mathscr{A} \times \mathbb{R}_+ \to H^1(\Omega) \qquad (A,\lambda) \mapsto u(A,\lambda),$$

is continuous in the weak topology.

Proof. Let us define $G(u) = \int_0^u g(s)ds$ and

$$J_{A,\lambda}(v) = \frac{1}{2}\int_\Omega (A\nabla v)\cdot\nabla v + \int_\Omega \lambda G(v) - \int_\Omega fv.$$

We know that $u(A,\lambda)$ is the unique minimizer of this functional. Let $A_m \to A$ and $\lambda_m \to \lambda$ be two converging sequences. It is easy to prove that $u_m = u(A_m,\lambda_m)$ is bounded in $H^1(\Omega)$ and, up to a subsequence, $u_m \rightharpoonup u$ in H^1 as $m \to \infty$. Therefore, $\int_\Omega (A\nabla u)\cdot\nabla u \le \liminf_m \int_\Omega (A_m\nabla u_m)\cdot\nabla u_m$. We can apply Theorem 3.2.4 to show that $G(u_m) \to G(u)$ in L^1 as $n \to \infty$ (see details for a similar proof, for example, in [CDLT04]) and we have that $u = u(A,\lambda)$. $\square$

Corollary 3.7. *The map $(I,\lambda) \mapsto u$, where I is the identity matrix, is continuous in the weak topology of H^1.*

Corollary 3.8. *In the hypotheses of Lemma 3.2.5, the maps $(A,\lambda) \mapsto \int_\Omega g(u(A,\lambda))$ and $(I,\lambda) \mapsto \int_\Omega g(u(I,\lambda))$ are continuous.*

3.3 Some numerical work for the case $\alpha = 1$

The following work is part of [DGCT15].

There exists a large literature on the computation and behaviour of the homogenized coefficient $a_0(G_0)$, both from the mathematics and the engineering part (see, e.g., [ABG09], [HD95], [Kri03]). In these papers, one can find power series techniques and numerical analysis, generally for spherical obstacles. As it is common in the literature (e.g. [ABG09]), we use the commercial software COMSOL. As said on the introduction, in Nanotechnology, however, it is a common misconception that the measure of the surface alone, $|\partial G_0|$, is a good indicator of the effectiveness of the obstacle.

Considering obstacles with some symmetries (for $N = 2$ it is sufficient that they are invariant under a 90^0 rotation) in general, it is well known that

$$a_0(G_0) = \alpha(G_0)I, \tag{3.61}$$

where $\alpha(G_0)$ is a scalar (see, for example, [ABG09], [Kri03]) and I is the identity matrix in $\mathscr{M}_N(\mathbb{R})$. In this case, it can be easily proved that the effectiveness is an decreasing function of

$$\lambda(G_0) = \frac{|\partial G_0|}{\alpha(G_0)|Y \setminus G_0|} \tag{3.62}$$

(it is a direct consequence of the comparison principle, see [Día85]). In fact, this is the only relevant parameter (once $g(u)$ is fixed) of the equation (1.76). The behaviour of the effectiveness with respect to the coefficient λ can also be numerically computed:

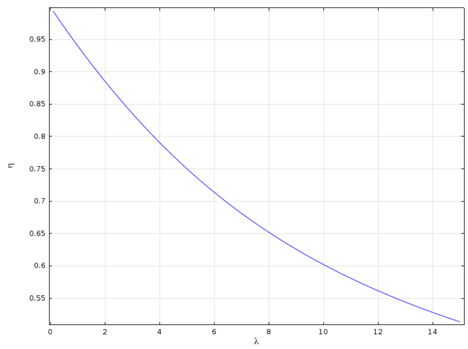
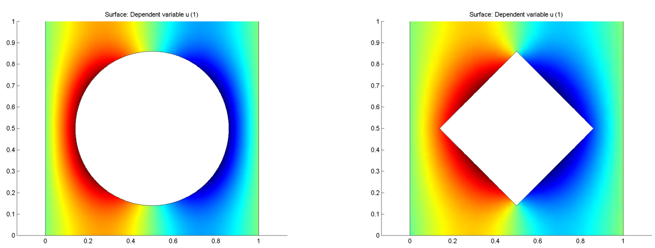


Fig. 3.1 Plot of η as a function of λ when Ω is a 2D circle.

Let us consider, in dimension two for simplicity, the following obstacles:



(a) Circular particle

(b) Square particle

Fig. 3.2 Two types of particle G_0, and the level sets of the solution of the cell problem (1.72)

We can numerically compute the homogenized diffusion coefficient $a_0(G_0)$ via a parametric sweep on the size of the particle.

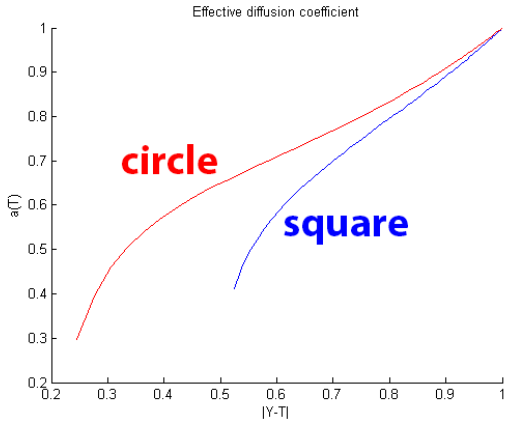
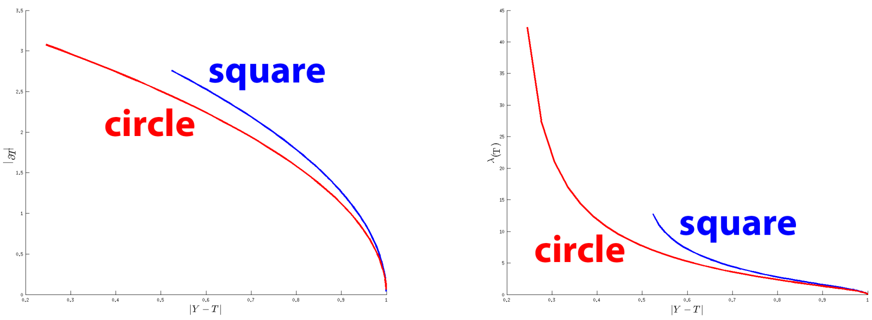


Fig. 3.3 The effective diffusion coefficient $\alpha(G_0)$ as a function of $|Y \setminus G_0|$.

Now, we can couple this with direct computations of $|\partial G_0|$ and compare the behaviour of both indicators.



(a) Classical coefficient $|\partial G_0|$.

(b) New coefficient $\lambda(G_0)$

Fig. 3.4 Coefficients $|\partial G_0|$ and $\lambda(G_0)$ as a function of $|Y \setminus G_0|$.

Chapter 4

Very weak solutions of problems with transport and reaction

In studying shape differentiation when the nonlinear kinetic term $\beta(u)$ is non smooth we find that we need to understand problems of the form

$$-\Delta u + \beta'(u_0)u = f, \tag{4.1}$$

where $\beta'(u_0)$ blows up in the proximity of the boundary of the dead core. Since $\beta'(u_0)$ is a priori known (before we study u) we can define $V(x) = \beta'(u_0)$. The expected behaviour is, in the blow up case that $V(x) \sim d(x, \partial\Omega)^{-\alpha}$ where $\alpha > 0$. Thus, we become interested in the study of the problem

$$-\Delta u + V(x)u = f. \tag{4.2}$$

4.1 The origin of very weak solutions

The notion of very weak solution with data f such that $fd(\cdot, \partial\Omega) \in L^1(\Omega)$ first appears in an unpublished paper by Brézis [Bré71b], and was later presented in [BCMR96]. Let

$$\delta(x) = d(x, \partial\Omega), \qquad x \in \Omega. \tag{4.3}$$

If $u \in \mathscr{C}^2(\overline{\Omega})$ is a solution of the following problem

$$\begin{cases} -\Delta u = f & \Omega, \\ u = u_0 & \partial\Omega, \end{cases} \tag{4.4}$$

then, integrating twice by parts we obtain that

$$-\int_\Omega u\Delta\varphi = \int_\Omega f\varphi - \int_{\partial\Omega} u_0 \frac{\partial\varphi}{\partial n} \tag{4.5}$$

for every $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. A fortiori, even if u is not of class $\mathscr{C}^2$, since, for this test functions, $\frac{\varphi}{\delta} \in L^\infty(\Omega)$, the problem (4.5) is well formulated for data f and u_0 such that $f\delta \in L^1(\Omega)$ and $u_0 \in L^1(\partial\Omega)$. Equation (4.5) is known as the very weak formulation of problem (4.4).

The surprising new result introduced by Brézis in 1971 is

Theorem 4.1 ([Bré71b]). *Let f be measurable such that $\delta f \in L^1(\Omega)$ and let $u_0 \in L^1(\partial\Omega)$. There exists a unique $u \in L^1(\Omega)$ such that* (4.5) *for all $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. Furthermore, there exists a constant $C > 0$ such that*

$$\|u\|_{L^1(\Omega)} \leq C(\|\delta f\|_{L^1(\Omega)} + \|u_0\|_{L^1(\partial\Omega)}) \tag{4.6}$$

Moreover, u satisfies that

$$-\int_\Omega |u|\Delta\rho + \int_{\partial\Omega} |u_0| \frac{\partial\rho}{\partial n} \leq \int_\Omega f\rho sign(u) \tag{4.7}$$

for all $\rho \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$, where

$$\mathrm{sign}(s) = \begin{cases} 1 & s > 0, \\ 0 & s = 0, \\ -1 & s < 0, \end{cases} \tag{4.8}$$

He goes further in a second result that states the following:

Theorem 4.2. *Let f be measurable in Ω such that $\delta f \in L^1(\Omega)$, $u_0 \in L^1(\partial\Omega)$ and β monotone nondecreasing and continuous. Then there exists a unique $u \in L^1(\Omega)$ such that $\delta\beta(u) \in L^1(\Omega)$ that satisfies*

$$-\int_\Omega \delta\varphi + \int_\Omega \beta(u)\varphi = \int_\Omega f\varphi - \int_{\partial\Omega} u_0 \frac{\partial\varphi}{\partial n} \tag{4.9}$$

for all $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. *Furthermore, if* u *and* $\hat{u}$ *are two solutions corresponding to* $f, \hat{f}, u_0, \hat{u}_0$ *then*

$$\|u - \hat{u}\|_{L^1(\Omega)} + \|\delta\beta(u) - \delta\beta(\hat{u})\|_{L^1(\Omega)} \leq C(\|\delta f - \delta \hat{f}\|_{L^1(\Omega)} + \|u_0 - \hat{u}_0\|_{L^1(\partial\Omega)}) \quad (4.10)$$

where C *depends only on* Ω.

The theory of very weak solutions developed in the 20$^{\text{th}}$-century focused on the use of weighted Lebesgue spaces. Let $L^0(\Omega)$ be the space of measurable functions in Ω, $\mu \in L^0(\Omega)$ and $1 \leq p \leq +\infty$. We define the weighted L^p space as

$$L^p(\Omega, \mu) = \left\{ f \in L^0(\Omega) : \int_\Omega |f|^p \mu < +\infty \right\}. \quad (4.11)$$

However, a more modern theory will require the definition of some interpolation spaces, known as Lorentz spaces, which allow for sharp regularity results, and have nice embedding and duality properties.

4.2 Lorentz spaces

In order to get sharper results of regularity we introduce some interpolation spaces. Lorentz defined the following spaces in [Lor50; Lor51].

Definition 4.1. Given $0 < p, q \leq \infty$ define

$$\|f\|_{(p,q)} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & q < +\infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & q = +\infty, \end{cases}$$

and $L^{(p,q)}(\Omega) = \{f \text{ measurable in } \Omega : \|f\|_{(p,q)} < +\infty\}$.

There is an alternative definition of the Lorentz spaces, which is the one we have considered.

Definition 4.2. Let $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$. Let $u \in L^0(\Omega)$. We define

$$||u||_{p,q} = \begin{cases} \left[\int_{\Omega_*} \left[t^{\frac{1}{p}} |u|_{**}(t) \right]^q \frac{dt}{t} \right]^{\frac{1}{q}} & q < +\infty, \\ \sup\limits_{0<t\leq|\Omega|} t^{\frac{1}{p}} |u|_{**}(t) & q = +\infty, \end{cases} \qquad \text{where } |u|_{**}(t) = \frac{1}{t} \int_0^t |u|_*(s) ds.$$

We define

$$L^{p,q}(\Omega) = \{f \text{ measurable in } \Omega : \|u\|_{p,q} < +\infty\}.$$

These spaces are equal, and their norms equivalent, to the previously defined Lorentz spaces.

Proposition 4.1 (Corollary 1.4.1 in [Rak08]). *Let* $1 < p \le +\infty, 1 \le q \le +\infty$. *Then*

$$L^{p,q}(\Omega) = L^{(p,q)}(\Omega)$$

with equivalent quasi-norms.

The functionals $\|\cdot\|_{(p,q)}$ do not, in general, satisfy the triangle inequality. However, $L^{p,q}$ is a quasi-Banach space. The following properties are known

Proposition 4.2 ([Gra09]). *i) If* $0 < p \le \infty$ *and* $0 < q < r \le +\infty$ *then* $L^{(p,q)} \subset L^{(p,r)}$.

ii) $L^{(p,p)} = L^p$ *for all* $p \ge 1$.

iii) Let $1 \le p, q < \infty$. *Then* $(L^{(p,q)}(\Omega))' = L^{(p',q')}(\Omega)$.

iv) If $q < r < p$ *then* $L^{(p,\infty)}(\Omega) \cap L^{(q,\infty)}(\Omega) \subset L^r(\Omega)$ *(even for* Ω *unbounded)*

v) If Ω *is bounded and* $r < p$, *then* $L^{(p,\infty)}(\Omega) \subset L^r(\Omega)$

For the convenience of the reader we include an inclusion diagram for $1 \le q \le r < p < +\infty$ and Ω bounded:

$$\begin{tikzcd}
 & & & L^{(\infty,\infty)} = L^\infty \arrow[dlll, hook] \arrow[d, hook] \\
L^{(p,1)} \arrow[r, hook] \arrow[d, hook] & L^{(p,q)} \arrow[rr, hook] \arrow[d, hook] & & L^{(p,\infty)} = \text{weak } L^p \arrow[dl, hook] \arrow[d, hook] \\
L^{(r,1)} \arrow[r, hook] & L^{(r,q)} \arrow[r, hook] & L^{(r,r)} = L^r \arrow[r, hook] & L^{(r,\infty)}.
\end{tikzcd}$$

4.3 Modern theory of very weak solutions

Even though very weak solutions have been studied in many different contexts (see, e.g. [MV13]) the papers most linked with the research in this thesis corresponds to [DR09; DR10]. In [DR09] the regularity of very weak solutions

$$\begin{cases} -\Delta u = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.12}$$

is studied, which can be formulated as (4.5) when $u_0 = 0$, and solutions and its gradients are shown to be in Lorentz spaces. For the reader's convenience a brief description of this was given in Section 4.2.

Later, in [DR10] the authors tackle the problem

$$\begin{cases} -\Delta u + Vu = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.13}$$

where $V \geq -\lambda_1(\Omega)$, the first eigenvalue of the Laplacian, which can be written in very weak formulation as

$$\begin{cases} Vu \in L^1(\Omega, \delta), \\ -\displaystyle\int_\Omega u\Delta\varphi + \int_\Omega Vu\varphi = \int_\Omega f\varphi \quad \forall \varphi \in W^{2,\infty} \cap W_0^{1,\infty}(\Omega). \end{cases} \tag{4.14}$$

In [DR10] the authors prove an existence result for the most general case. As a matter of fact, they also add a nonlinear term $\beta(u)$ to (4.13).

In this chapter $\Omega \subset \mathbb{R}^n$ and

$$p' = \frac{p}{p-1}. \tag{4.15}$$

Theorem 4.3 ([DR10])**.** *Let $V \in L^1_{loc}(\Omega)$ with $V \geq -\lambda \geq -\lambda_1$, where λ_1 is the first eigenvalue of $L = -\operatorname{div}(A\nabla u)$ where A is symmetric, uniformly elliptic and $C^{0,1}(\bar{\Omega})$. Then, there exists a unique solution $u \in L^{n',\infty}(\Omega) \cap W^{1,q}(\Omega, \delta)$ for every $1 \leq q < \frac{2n}{2n-1}$ satisfying*

$$\begin{cases} Vu \in L^1(\Omega, \delta), \\ \displaystyle\int_\Omega uL\varphi + \int_\Omega Vu\varphi = \int_\Omega f\varphi \quad \forall \varphi \in W^{2,\infty} \cap W_0^{1,\infty}(\Omega). \end{cases} \tag{4.16}$$

Furthermore

i) $\|Vu\|_{L^1(\Omega,\delta)} \leq C\|f\|_{L^1(\Omega,\delta)}$,

ii) $\|u\|_{L^{n',\infty}(\Omega)} C\|f\|_{L^1(\Omega,\delta)}$,

iii) The following also holds

$$\int_\Omega |\nabla u|^q \delta \leq C\|f\|^{\frac{q}{2}}_{L^1(\Omega,\delta)} \left(1 + \|f\|^{n'}_{L^1(\Omega,\delta)}\right)^{1-\frac{q}{2}}. \tag{4.17}$$

Nonetheless, since $\delta^{-r} \notin L^1(\Omega)$ for $r > 1$, $\delta^{-\alpha} \notin L^1(\Omega, \delta)$ for $\alpha > 2$. Therefore, we set out to see what could be done in this case. In [DGCRT17] the author, jointly with J. I. Díaz, J. M. Rakotoson and R. Temam, solved some cases left open in [DR10].

Theorem 4.4. *Let Ω be bounded and $V \geq cd(x, \partial\Omega)^{-s}, s > 2$. Then there exists a unique very weak solution $u \in L^1(\Omega)$ of the problem*

$$-\Delta u + Vu = f \qquad \text{in } \Omega$$

in the sense that

$$\begin{cases} Vu \in L^1(\Omega, \delta), \\ -\displaystyle\int_\Omega u \Delta\varphi + \int_\Omega V u \varphi = \int_\Omega f \varphi \quad \forall \varphi \in W_c^{2,\infty}(\Omega). \end{cases} \tag{4.18}$$

We will prove this result in Section 4.6.

Remark 4.1. Notice that the uniqueness theorem is stated without imposing any boundary conditions in a classical way (the test functions have compact support).

Later, Brezis proved the same result for $s = 2$, in personal communication to the author during his visit to Technion by an extension of the previous argument.

Theorem 4.5. *Let Ω be bounded and $V \geq cd(x, \partial\Omega)^{-2}$. Then there exists a unique $u \in L^1(\Omega)$ such that* (4.18)

We will include the details of the proof of this improvement[1].

4.3.1 Very weak solutions in problems with transport

In developing the theory, thanks to a very fruitful collaboration with J.M. Rakotoson (U. Poitiers, France) and R. Temam (U. Indiana, USA) we managed to extend the results to the problem with a transport term

$$\begin{cases} -\Delta u + \vec{b} \cdot \nabla u + Vu = f & \Omega, \\ u = 0 & \partial\Omega, \end{cases} \tag{4.19}$$

where

$$\begin{cases} \operatorname{div} \vec{b} = 0 & \Omega, \\ \vec{b} \cdot n = 0 & \partial\Omega. \end{cases} \tag{4.20}$$

[1] At the time of writing, a draft paper containing further improvements is in preparation jointly with J.I. Díaz and J.M. Rakotoson

This case is very relevant in incompressible flows. The very weak formulation of this problem can be written as

$$\int_\Omega u(-\Delta\varphi - \vec{b}\cdot\nabla\varphi + V\varphi) = \int_\Omega f\varphi \qquad \forall \varphi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega). \tag{4.21}$$

In order to be very clear about the definition of very weak solution, and the sense in which we define boundary conditions. We collect now some definitions:

Definition 4.3. Let $V, f \in L^1_{loc}$ and $\vec{b} \in L^n(\Omega)^n$, satisfy (4.20) in the sense that

$$\int_\Omega \varphi\nabla u\cdot\vec{b} = -\int_\Omega u\nabla\varphi\cdot\vec{b} \tag{4.22}$$

for all $\varphi \in W^{1,\infty}(\Omega)$ and $u \in W^{1,n'}(\Omega)$ (see an explanation of this definition in Remark 4.2).

Let us define the following types of very weak solutions

- **Local very weak solution of** (4.19) (i.e. without boundary condition). We say that u is v.w.s. without b.c. if (4.21) holds for every $\varphi \in \mathscr{C}_c^2(\Omega)$.
- **Very weak solution of** (4.19) **in the sense of weights** We say that u is v.w.s. with weight if it satisfies (4.21) holds for every $\varphi \in \mathscr{C}_c^2(\Omega)$ and $Vu \in L^1(\Omega,\delta)$.
- **Very weak solution of** (4.19) **in the sense of traces** We say that u is v.w.s. with Dirichlet homogeneous boundary conditions if (4.21) holds for every $\varphi \in \mathscr{C}^2(\overline{\Omega})$ such that $\varphi = 0$ on $\partial\Omega$ and $Vu \in L^1(\Omega,\delta)$.

We have the following result

Theorem 4.6. *Let $f \in L^1(\Omega,\delta), V \in L^1_{loc}(\Omega)$ and $\vec{b} \in L^{p,1}(\Omega)$ such that $\operatorname{div}\vec{b} = 0$ in Ω and $\vec{b}\cdot n = 0$ on $\partial\Omega$ where either*

i) *$p > n$ or*

ii) *$p = n$ and $\vec{b}$ is small in $L^{n,1}$ (in the sense that $\|\vec{b}\|_{n,1} \le K^{s0}_{n1}$ for a constant specified in [DGCRT17]).*

Then,

i) *there exists a very weak solution without boundary condition $u \in L^{n',\infty}(\Omega)$.*

ii) *If $V \in L^1(\Omega,\delta)$, then there exists a v.w.s. in the sense of traces in $u \in L^{n',\infty}(\Omega)$.*

iii) *If $V \in L^{p,1}(\Omega)$, then there exists a unique v.w.s. in the sense of traces $u \in L^{n',\infty}(\Omega)$.*

iv) If $V \geq c\delta^{-\alpha}$ for some $\alpha > 2$, then there exists a unique v.w.s. in the sense of weights $u \in L^{n',\infty}(\Omega)$.

We conclude this statement section by explaining our definition of (4.20).

Remark 4.2. Assume first that $\vec{b}, u, \varphi$ are smooth, and that $\vec{b}$ satisfies (4.20). Then

$$\begin{aligned} \operatorname{div}(u\varphi\vec{b}) &= \nabla(u\varphi)\cdot\vec{b} + u\varphi \operatorname{div}\vec{b} = \nabla(u\varphi)\cdot b \\ &= u\nabla\varphi\cdot\vec{b} + \varphi\nabla u\cdot\vec{b}. \end{aligned}$$

Integrating over Ω

$$\int_{\partial\Omega} u\varphi\vec{b}\cdot\vec{n} = \int_\Omega u\nabla\varphi\cdot\vec{b} + \int_\Omega \varphi\nabla u\cdot\vec{b}.$$

Since $\vec{b}\cdot\vec{n} = 0$ on $\partial\Omega$ we have that (4.22) holds. We can pass to the limit for less smooth $\vec{b}, u, \varphi$.

4.4 Existence and regularity

We will construct the solution as a limit of problems with cutoff. Let us define the cut-off operator, for $k > 0$

$$T_k(s) = \begin{cases} s & |s| \leq k, \\ k\operatorname{sign}(s) & |s| > k, \end{cases} \tag{4.23}$$

and let

$$V_k = T_k \circ V. \tag{4.24}$$

4.4.1 Regularity of the adjoint operator $-\Delta - \vec{u}\cdot\nabla$

Given $T \in H^{-1}(\Omega)$ we focus first on the regularity of the adjoint problem

$$\int_\Omega \nabla\phi\nabla\varphi - \int_\Omega \vec{b}\cdot\nabla\phi\varphi + \int_\Omega V\phi\varphi = \langle T, \varphi\rangle \tag{4.25}$$

$\forall\varphi \in H_0^1(\Omega)$.

By applying the Lax-Milgram theorem we can show that

Proposition 4.3. *Let $T \in H^{-1}(\Omega)$, $V \in L^0(\Omega)$ satisfying $V \geq -\lambda > -\lambda_1$ (where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition). Let*

$$\mathscr{W} = \{\varphi \in H_0^1(\Omega) : (V+\lambda)\phi^2 \in L^1(\Omega)\} \tag{4.26}$$

endowed with

$$[\varphi]_{\mathscr{W}}^2 = \|\varphi\|_{H_0^1(\Omega)}^2 + \int_\Omega (V+\lambda)\varphi^2. \tag{4.27}$$

Then there exists a unique element $\phi \in \mathscr{W}$ such that (4.25) *holds for every $\varphi \in \mathscr{W}$. Moreover*

$$\|\phi\|_{H_0^1(\Omega)} \leq \frac{\lambda_1}{\lambda_1 - \lambda}\|T\|_{H^{-1}}, \tag{4.28}$$

$$\left(\int_\Omega (V+\lambda)\varphi^2\right)^{\frac{1}{2}} \leq \left(\frac{\lambda_1}{\lambda_1-\lambda}\right)^{\frac{1}{2}} \|T\|_{H^{-1}}. \tag{4.29}$$

It is clear that if $T \in H^{-1}$ then there exists a unique solution $\phi_k \in H_0^1(\Omega)$ of

$$\int_\Omega \nabla\phi_k \nabla\varphi - \int_\Omega \vec{b}\nabla\phi_k\varphi + \int_\Omega V_k\phi_k\varphi = \langle T, \varphi\rangle, \qquad \forall \varphi \in H_0^1(\Omega). \tag{4.30}$$

It turns out that $\phi_k \to \phi$ strongly in $H_0^1(\Omega)$. We can show that the regularity can be improved

Proposition 4.4. *Let $T \in L^{\frac{n}{2},1}(\Omega) \subset H^{-1}(\Omega)$ and $V \geq 0$. Then $\phi \in L^\infty(\Omega)$ and there exists a constant $C = C(n,\Omega)$ such that*

$$\|\phi\|_{L^\infty(\Omega)} \leq C\|T\|_{L^{\frac{n}{2},1}(\Omega)}. \tag{4.31}$$

Proposition 4.5. *Let $V \in L^0(\Omega)$ and*

$$T = -\operatorname{div}\vec{F} \qquad \vec{F} \in L_F = \begin{cases} L^{n,1}(\Omega)^n & n \geq 3, \\ L^{2+\varepsilon}(\Omega)^2 & n = 2. \end{cases} \tag{4.32}$$

Then $\phi \in L^\infty(\Omega)$ and there exists a constant $C = C(n,\Omega)$ such that

$$\|\phi\|_{L^\infty(\Omega)} \leq C\|\vec{F}\|_{L_F}. \tag{4.33}$$

Proposition 4.6. *Let*

i) $\vec{b} \in L^{p,q}(\Omega)$ *with* $p > n$,

ii) $0 \leq V \in L^{r,q}(\Omega)$ *where* $r = \frac{np}{n+p}$, *iii)* $T = -\operatorname{div}\vec{F}$ *where* $\vec{F} \in L^{p,q}(\Omega)$ *for* $1 \le q \le +\infty$.

Then $\phi \in W^1L^{p,q}(\Omega)$*. Moreover, there exists* $K_{pq} = K(p,q,n,\Omega)$ *such that*

$$\|\nabla\phi\|_{L^{p,q}} \le K_{pq}(1 + \|\vec{b}\|_{L^{p,q}} + \|V\|_{L^{r,q}})\|F\|_{L^{p,q}(\Omega)^n}. \tag{4.34}$$

Proposition 4.7. *If* $\vec{b}, \vec{F} \in L^{p,\infty}(\Omega)^n$ *for* $p > n$ *then* $\phi \in \mathscr{C}^{0,\alpha}(\bar{\Omega})$ *for* $\alpha = 1 - \frac{n}{p}$.

As an auxiliary space we will use the spaces of bounded mean oscillation

Definition 4.4. A locally integrable function f on $\mathbb{R}^n$ is said to be in $\mathrm{bmo}(\mathbb{R}^N)$ if

$$\|f\|_{\mathrm{bmo}(\mathbb{R}^N)} = \sup_{0<\operatorname{diam}(Q)<1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx + \sup_{\operatorname{diam}(Q)\ge 1} \frac{1}{|Q|}\int_Q |f(x)|dx < +\infty$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^n$ the sides of which are parallel to the coordinates axes and

$$f_Q = \frac{1}{|Q|}\int_Q f(y)dy.$$

Definition 4.5. A locally integrable function f on a Lipschitz bounded domain Ω is said to be in $\mathrm{bmo}_r(\Omega)$ (r stands for restriction) if

$$\|f\|_{\mathrm{bmo}_r(\Omega)} = \sup_{0<\operatorname{diam}(Q)<1} \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,dx + \int_\Omega |f(x)|dx < +\infty, \tag{4.35}$$

where the supremum is taken over all cube $Q \subset \Omega$ the sides of which are parallel to the coordinates axes.

In this case, there exists a function $\widetilde{f} \in \mathrm{bmo}(\mathbb{R}^N)$ such that

$$\widetilde{f}\Big|_\Omega = f \text{ and } \|\widetilde{f}\|_{\mathrm{bmo}(\mathbb{R}^N)} \le c_\Omega \cdot \|f\|_{\mathrm{bmo}_r(\Omega)}. \tag{4.36}$$

Proposition 4.8. *Let* $\vec{b}, \vec{F} \in \mathrm{bmo}_r(\Omega)^n$ *and* $V \in \mathrm{bmo}_r(\Omega)$*. Then:*

i) $\vec{b}\phi \in \mathrm{bmo}_r(\Omega)^n$,

ii) $\nabla\phi \in \mathrm{bmo}_r(\Omega)^n$.

We can even estimate some second order derivatives

Proposition 4.9. *Let* $\vec{b} \in L^{p,q}$*,* $T, V \in L^{p,q}(\Omega)$ *for some* $p > n$ *and* $1 \le q \le +\infty$*. Then* $\phi \in W^2L^{p,q}(\Omega)$ *and there exists* $K = K(p,q,n,\Omega)$ *such that*

$$\|\phi\|_{W^2L^{p,q}} \le K\frac{1 + c_{\varepsilon_0}\|\vec{b}\|_{L^{p,q}} + \|V\|_{L^{p,q}}}{1 - \varepsilon_0\|\vec{b}\|_{L^{p,q}}}\|T\|_{L^{p,q}}, \tag{4.37}$$

where $\varepsilon_0 > 0$ *is such that* $\varepsilon_0 \|\vec{b}\|_{L^{p,q}} < 1$ *and* $c_{\varepsilon_0} \to +\infty$ *as* $\varepsilon \to 0$.

For the proof of the existence of solutions in Theorem 4.6 the idea is to consider u_k the solution of

$$-\Delta u_k + \vec{b}_k \cdot \nabla u_k + V_k u_k = f_k \tag{4.38}$$

where $f_k = T_k \circ f$ and $\vec{b}_k$ is an approximating sequence in

$$\mathscr{V} = \{\vec{b} \in \mathscr{C}_c^\infty(\Omega)^n : \operatorname{div}\vec{b} = 0 \text{ in } \Omega\}, \tag{4.39}$$

which has adherence in $L^{p,q}$ the set

V = { $\in L^{p,q}(\Omega)^n : \operatorname{div}\vec{b} = 0$ in $\Omega, \vec{b}\cdot\vec{n} = 0$ on $\partial\Omega\}$.(4.40)In order to get some uniform estimate (we want to apply the Dunford–Pettis compactness theorem) we consider the family of test functions

$$\begin{cases} -\Delta\phi_{k,E} - \vec{b}_j \cdot \nabla\phi_{k,E} = \chi_E & \Omega, \\ \phi_{k,E} = 0 & \partial\Omega. \end{cases} \tag{4.41}$$

The previously established regularity result assure that

$$\|\phi_{k,E}\|_{W^2L^{n,1}} \le C\|\chi_E\|_{L^{n,1}} \le C|E|^{\frac{1}{n}}. \tag{4.42}$$

From this reasoning we can extract some conclusions (see [DGCRT17] for the details)

$$\int_E u_j \le C|E|^{\frac{1}{n}} \int_\Omega f_k\delta, \tag{4.43}$$

$$\|u_j\|_{L^{n',\infty}} \le C\int_\Omega f\delta, \tag{4.44}$$

$$\int_\Omega V_k u_k \delta \le C(1+\|\vec{b}\|_{L^{n,1}}) \int_\Omega f\delta. \tag{4.45}$$

With some additional work we show that there exists $u \in L^1(\Omega)$ such that

i) $u_j \to u$ in $L^1(\Omega)$,

ii) $Vu \in L^1(\Omega)$ (by applying Fatou's lemma),

iii) $V_k u_k \delta \rightharpoonup Vu\delta$ in $L^1_{loc}(\Omega)$,

iv) if $V \in L^1(\Omega,\delta)$, then $V_k u_k \delta \rightharpoonup Vu\delta$ in $L^1(\Omega)$.

This is enough to show that u is a v.w.s. without boundary condition. If $V \in L^1(\Omega, \delta)$ then u is a v.w.s. in the sense of traces. If $V \geq C\delta^{-\alpha}$ with $\alpha > 2$ then

$$+\infty > \int_\Omega |u| V \delta \geq C \int_\Omega |u| \delta^{1-\alpha} \tag{4.46}$$

hence $u \in L^1(\Omega, \delta^{-r})$ for some $r > 1$. For the uniqueness of solutions in this last case we must set a suitable theory.

4.5 Maximum principles in some weighted spaces

The classical maximum principle states that, if $u \in \mathscr{C}^2(\Omega)$ and

$$\begin{cases} -\Delta u \leq 0 & \text{in } \Omega, \\ u \leq 0 & \text{on } \partial\Omega, \end{cases} \implies u \leq 0. \tag{4.47}$$

We will say that a space X satisfies a maximum principle if

$$\begin{cases} -\Delta u \leq 0 & \text{in } \mathscr{D}'(\Omega), \\ u \in X & \text{on } \partial\Omega, \end{cases} \implies u \leq 0. \tag{4.48}$$

Since it will be used very subtly in the following sections, we recall the following definition

Definition 4.6. Let u be an integrable function. We say that $-\Delta u = f$ in $\mathscr{D}'(\Omega)$ if

$$-\int_\Omega u \Delta \varphi = \int_\Omega f \varphi \qquad \forall \varphi \in \mathscr{C}_c^\infty(\Omega).$$

To show that some spaces satisfy the property above, let us state an approximation lemma for the space of test functions.

Remark 4.3. One of the useful properties of (4.47) is that it allows to prove uniqueness of solutions straightforwardly. Let $f \in \mathscr{C}(\overline{\Omega})$. Consider two solutions u_i such that

$$\begin{cases} -\Delta u_i = f & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.49}$$

Then, one immediately proves that $u_1 - u_2 \leq 0$ and $u_2 - u_1 \leq 0$. Hence $u_1 = u_2$.

In order to prove the relevant results in this Chapter we will use the following maximum principle:

Theorem 4.7. *Let $u \in L^1(\Omega)$ be such that*

$$-\int_\Omega u\Delta\varphi \le 0, \qquad \forall \varphi \ge 0,\ \varphi \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{4.50}$$

Then $u \le 0$.

A very useful result to be used in conjuction with this kind of maximum principle is known as Kato's inequality (which was originally published in [Kat72]). To present it we give the definition of the *positive sign function*: for $s \in \mathbb{R}$

$$\mathrm{sign}_+(s) = \begin{cases} 1 & s > 0, \\ 0 & s \le 0. \end{cases} \tag{4.51}$$

Theorem 4.8 (Kato's inequality as presented in [MV13])**.** *Assume that $u, f \in L^1_{loc}(\Omega)$ and $-\Delta u \le f$ in $\mathscr{D}'(\Omega)$. Then:*

i) $-\Delta|u| \le f \,\mathrm{sign}\, u$ *in* $\mathscr{D}'(\Omega)$.

ii) $-\Delta u_+ \le f \,\mathrm{sign}_+ u$ *in* $\mathscr{D}'(\Omega)$.

Remark 4.4. It is very important to compare the test functions in (4.50) with the ones of the definition of $-\Delta u \le 0$ in $\mathscr{D}'(\Omega)$.

4.5.1 Some approximation lemmas

Approximation in $W_0^{1,\infty}$ with weights In [DGCRT17] we proved the following result, which is stated for the spaces

$$W_c^{m,\infty}(\Omega, \delta^r) = \{ f \in W^{m,\infty}(\Omega, \delta^r) : \exists K \subset \Omega \text{ compact such that } f = 0 \text{ a.e. in } \Omega \setminus K \}.$$

Theorem 4.9. *The following density results hold:*

i) Let $r > m$. Then $W_c^{m,\infty}(\Omega, \delta^r)$ is dense in $W^{m,\infty}(\Omega, \delta^r)$

ii) Let $r > m - 1$. Then $W_c^{m,\infty}(\Omega, \delta^r)$ is dense in $W_0^{1,\infty}(\Omega) \cap W^{m,\infty}(\Omega, \delta^r)$.

Remark 4.5. Notice that, without the weights, the results do not hold. If a sequence $(f_n) \in W_c^{m,\infty}(\Omega)$ converges to a function f in the norm of this space, due to the continuity of trace $f \in W_0^{1,\infty}(\Omega)$. Hence, the adherence of $W_c^{m,\infty}(\Omega)$ with the $W^{m,\infty}(\Omega)$ norm can not be the whole space $W^{m,\infty}(\Omega)$.

We will prove that

Proposition 4.10. *Let $\varphi \in W^{m,\infty}$, then for $r > m$ there exists $(\varphi_n) \subset W_c^{m,\infty}$ such that*

$$\delta^r(\partial_\alpha \varphi_n) \xrightarrow{L^\infty} \delta^r(\partial_\alpha \varphi), \qquad |\alpha| < r, m.$$

If $\varphi \in W_0^{1,\infty} \cap W^{m,\infty}$ then

$$\delta^r(\partial_\alpha \varphi_n) \xrightarrow{L^\infty} \delta^r(\partial_\alpha \varphi), \qquad |\alpha| < r+1, m.$$

The proof is based on the existence and bounds of the cut-off function we will define now. Let $\psi \in \mathscr{C}^\infty(\mathbb{R})$ be a non decreasing function such that $0 \le \psi \le 1$ and

$$\psi(s) = \begin{cases} 1, & s \ge 1, \\ 0, & s \le 0. \end{cases}$$

Let, for $x \in \Omega$,

$$\eta_\varepsilon(x) = \psi\left(\frac{\delta(x) - \varepsilon}{\varepsilon}\right).$$

We have constructed a function η_ε which will be of relevance to us.

Lemma 4.5.1. *Let Ω be such that $\partial\Omega \in \mathscr{C}^2$. Then, there exists a sequence of function η_ε such that*

i) $\operatorname{supp} \eta_\varepsilon \subset \{\delta \ge \varepsilon\}$,

ii) $\operatorname{supp}(1 - \eta_\varepsilon) \subset \{\delta \le 2\varepsilon\}$,

iii) $|D^\alpha \eta_\varepsilon(x)| \le C\varepsilon^{-|\alpha|}$.

The approximating sequence that we construct to prove Proposition 4.10 is precisely, for $\varphi \in W^{m,\infty}(\Omega)$, given by $\varphi_n = \eta_{\frac{1}{n}} \varphi$. The details of the proof (which requires several sharp estimations) can be found [DGCRT17].

Approximation in $L^1(\Omega, \delta)'$ The mentioned improvement by Brezis is the following.

Theorem 4.10. *Let $u \in L^1(\Omega, \delta^{-1})$ and $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. Then*

$$\int_\Omega u \Delta(\varphi \eta_\varepsilon) \to \int_\Omega u \Delta \varphi. \tag{4.52}$$

Proof. Taking into account that

$$\Delta(\eta_\varepsilon \varphi) = \eta_\varepsilon \Delta\varphi + 2\nabla\eta_\varepsilon \cdot \nabla\varphi + \varphi\Delta\eta_\varepsilon, \tag{4.53}$$

we have that

$$-\int_\Omega u\Delta(\eta_\varepsilon\varphi) = -\int_\Omega u\eta_\varepsilon\Delta\varphi - 2\int_\Omega u\nabla\eta_\varepsilon\cdot\nabla\varphi - \int_\Omega u\varphi\Delta\eta_\varepsilon. \tag{4.54}$$

Using the fact that $\frac{u}{\delta} \in L^1(\Omega)$, $\delta\eta_\varepsilon \to \delta$ in $L^\infty(\Omega)$ and $\Delta\varphi \in L^\infty(\Omega)$:

$$-\int_\Omega u\eta_\varepsilon\Delta\varphi = -\int_\Omega \frac{u}{\delta}\delta\eta_\varepsilon\Delta\varphi \to -\int_\Omega \frac{u}{\delta}\delta\Delta\varphi = -\int_\Omega u\Delta\varphi.$$

On the other hand

$$\begin{aligned}
\left|\int_\Omega u\nabla\eta_\varepsilon\cdot\nabla\varphi\right| &\leq \left|\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{u}{\delta}\delta\nabla\eta_\varepsilon\cdot\nabla\varphi\right| \\
&\leq \int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{|u|}{\delta}\delta\|\nabla\eta_\varepsilon\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{L^\infty(\Omega)} \\
&\leq C\varepsilon\|\nabla\eta_\varepsilon\|_{L^\infty(\Omega)}\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{|u|}{\delta} \\
&\leq C\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{|u|}{\delta}.
\end{aligned}$$

Since $u/\delta \in L^1(\Omega)$ and the Lebesgue measure $m(\{\varepsilon < \delta < 2\varepsilon\}) \to 0$ we have

$$\int_\Omega u\nabla\eta_\varepsilon\cdot\nabla\varphi \to 0 \text{ as } \varepsilon \to 0.$$

Due the Hardy inequality in $W_0^{1,\infty}(\Omega)$ we have that $\frac{\varphi}{\delta} \in L^\infty(\Omega)$. Therefore

$$\begin{aligned}
\left|\int_\Omega u\varphi\Delta\eta_\varepsilon\right| &\leq \left|\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{u}{\delta}\frac{\varphi}{\delta}\delta^2\Delta\eta_\varepsilon\right| \\
&\leq \left\|\frac{\varphi}{\delta}\right\|_{L^\infty(\Omega)} \varepsilon^2\|\Delta\eta_\varepsilon\|_{L^\infty(\Omega)}\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{|u|}{\delta} \\
&\leq C\int_{\{\varepsilon<\delta<2\varepsilon\}} \frac{|u|}{\delta} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned}$$

This concludes the proof. □

4.5.2 Maximum principle of $-\Delta$ in L^1 with weights and without boundary condition

In [DGCRT17] we proved a first result in this direction, which we will write following the definitions in [MV13].

Theorem 4.11 ([DGCRT17])**.** *Let $u \in L^1(\Omega, \delta^{-r})$ for some $r > 1$ be such that $-\Delta u \leq 0$ in $\mathscr{D}'(\Omega)$, i.e.*

$$-\int_\Omega u\Delta\varphi \leq 0, \qquad \forall\varphi \in \mathscr{C}_c^\infty(\Omega), \quad \varphi \geq 0. \tag{4.55}$$

Then $u \leq 0$.

Proof. Assume first that $r > 1$. Let $\varphi \in W_0^{1,\infty} \cap W^{2,\infty}$ and let $\varphi_n \in W_c^{2,\infty}$ be the approximating sequence constructed in Proposition 4.10 (e.g. $\eta_{\frac{1}{n}}\varphi$ where η_ε is given by Lemma 4.5.1). Then

$$0 \geq -\int_\Omega u\Delta\varphi_n = -\int_\Omega u\delta^{-r}(\Delta\varphi_n)\delta^r.$$

Since $u\delta^{-r} \in L^1(\Omega)$ and $\delta^r\Delta\varphi_n \to \delta^r\Delta\varphi$ in L^∞ we can pass to the limit and obtain

$$0 \geq -\int_\Omega u\delta^{-r}\Delta\varphi\delta^r = -\int_\Omega u\Delta\varphi,$$

which proves the result. $\square$

Combining this fact with Theorem 4.8 we have the following result (without boundary condition)

Corollary 4.1. *Let $u \in L^1(\Omega, \delta^{-r})$ for some $r > 1$ be such that $-\Delta|u| \leq 0$ then $u = 0$.*

Prof. Häim Brezis improved Theorem 4.11 in a personal communication. The proof is a refinement of the one in [DGCRT17].

Theorem 4.12. *Let $u \in L^1(\Omega, \delta^{-1})$ be such that $-\Delta u \leq 0$. Then $u \leq 0$.*

Proof. Let $0 \leq \varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$. Since $0 \leq \eta_\varepsilon\varphi \in W_c^{2,\infty}(\Omega)$ we can use it as a test function. We have that

$$-\int_\Omega u\Delta(\varphi\eta_\varepsilon) \leq 0.$$

Therefore, due to Theorem 4.10 and the previous estimates,

$$0 \geq -\int_\Omega u\Delta(\varphi\eta_\varepsilon) \to -\int_\Omega u\Delta\varphi.$$

Finally, for any $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$, we have that

$$-\int_\Omega u\Delta\varphi \leq 0.$$

Due to Theorem 4.7, we have that $u \leq 0$. □

4.5.3 Maximum principle of $-\Delta + \vec{b}\cdot\nabla$ in L^1 with weights

Theorem 4.13 ([DGCRT17]). *Let $\bar{u} \in W^{1,1}_{loc}(\Omega)$ and $\vec{b}\bar{u} \in L^1_{loc}(\Omega)$ and*

$$L\bar{u} = -\Delta\bar{u} + \operatorname{div}(\vec{b}\bar{u}) \in L^1_{loc}(\Omega). \tag{4.56}$$

Define the dual operator

$$L^*\bar{\psi} = -\Delta\bar{\psi} - \vec{b}\cdot\nabla\bar{\psi}. \tag{4.57}$$

Then

i) For all $\psi \in \mathscr{D}(\Omega)$, $\psi \geq 0$ we have that

$$\int_\Omega \bar{u}_+ L^*\psi \leq \int_\Omega \psi \operatorname{sign}_+(\bar{u}) L\bar{u}. \tag{4.58}$$

That is, $L\bar{u}_+ \leq \operatorname{sign}_+(u_+)L\bar{u}$ in $\mathscr{D}'(\Omega)$.

ii) $L(|\bar{u}|) \leq \operatorname{sign}(\bar{u})L\bar{u}$ in $\mathscr{D}'(\Omega)$.

4.6 Uniqueness of very weak solutions of problem (4.18)

We provide here the proof of the extended result Theorem 4.5, which has not been published. For the proof of Theorem 4.4 can be found in [DGCRT17].

Proof of Theorem 4.5. The existence result was shown in [DGCRT17]. Since the problem is linear let us show uniqueness for $f = 0$. Since $Vu \in L^1(\Omega, \delta)$ we have that $u \in L^1(\Omega, \delta^{-1})$. We have that $-\Delta u = -Vu$ in the sense of distributions. Applying Theorem 4.8 we have that

$$-\Delta|u| \leq -(\operatorname{sign} u)Vu = -V|u| \leq 0 \tag{4.59}$$

Applying Theorem 4.12 we have $u = 0$. □

4.7 On weights and traces

It was noted on [Kuf85] that some power type weights in L^1 induce zero trace on continuous functions $L^1(\Omega,\delta^{-r}) \cap \mathscr{C}(\bar{\Omega}) \subset \mathscr{C}_0(\bar{\Omega})$. This naturally raises the question: Is $u\delta^{-\alpha} \in L^p$ for some α and p a sufficient condition to have uniqueness in an elliptic equation even when the solution does not neccesarily have a trace? Does the weight work as a trace, even when there is no trace? A number of results in this directions are provided in [Kuf85], for $p > 1$

$$W^{k,p}(\Omega,\delta^{\varepsilon}) = W_0^{k,p}(\Omega,\delta^{\varepsilon})(=\overline{C_0^\infty(\Omega)}^{W^{k,p}(\Omega,\delta^{\varepsilon})}) \tag{4.60}$$

and

$$u \in W^{1,p}(\Omega) \quad \text{and} \quad \frac{u}{\delta} \in L^p(\Omega) \iff u \in W_0^{1,p}(\Omega). \tag{4.61}$$

In this sense it is natural that something like this might be used as a boundary condition. In some cases, the fact that the solution is in such a weighted appears naturally.

In fact, due to Theorem 4.12, weights in the form of negative powers of the distance to the boundary can be used to define "Dirichlet boundary conditions" for elliptic equations and ensure uniqueness. In particular, our aim is to show that, if we assume

$$\begin{cases} -\Delta u = 0 \text{ in } \mathscr{D}'(\Omega), \\ u \in L^1(\Omega,\delta^{-1}), \end{cases} \qquad \text{or} \qquad \begin{cases} -\Delta u + Vu = 0 \text{ in } \mathscr{D}'(\Omega), \\ u \in L^1(\Omega,\delta^{-1}), \end{cases}$$

then $u = 0$ in Ω.

4.7.1 Weights and Hardy's inequalities in $L^1(\Omega)$

The equivalence (4.61), which holds for $p > 1$ and is useful throughout this Chapter, is heavily linked with Hardy's inequality for $W_0^{1,p}(\Omega)$ for $p > 1$:

$$\int_\Omega \left(\frac{|u|}{\delta}\right)^p \le C \int_\Omega |\nabla u|^p \qquad \forall u \in \mathscr{C}_c^\infty(\Omega). \tag{4.62}$$

(see [Har25; BM97]). Neither of these results is true for $p = 1$ (see, e.g., [Psa13]).

The following facts for the case $p = 1$ are known for a smooth bounded domain in $\mathbb{R}^n$:

i) If $u \in W_0^{1,1}(\Omega)$ and $-\Delta u = 0$ in $\mathscr{D}'(\Omega)$ then $u = 0$.

ii) If $u \in W^{1,1}(\Omega)$ and $\frac{u}{\delta} \in L^1(\Omega)$ then $u \in W_0^{1,1}(\Omega)$.

iii) $u \in W_0^{1,1}(\Omega)$ does not imply $\frac{u}{\delta} \in L^1(\Omega)$.

We proved that

iv) If $\frac{u}{\delta} \in L^1(\Omega)$ and $\Delta u = 0$ in $\mathscr{D}'(\Omega)$ then $u = 0$.

The following question[2] seems natural:

> Does it exist a weight function $p(x)$ such that:
>
> $$u \in W_0^{1,1}(\Omega) \implies \frac{u}{p} \in L^1(\Omega),$$
>
> and
>
> $$\left.\begin{array}{c} \Delta u = 0 \text{ in } \mathscr{D}'(\Omega) \\ \frac{u}{p} \in L^1(\Omega) \end{array}\right\} \implies u = 0, \tag{4.63}$$
>
> both hold?

If $p(x)$ satisfies (4.63), we will say that weight $\frac{1}{p}$ gives a Dirichlet boundary condition in a generalized sense.

We will focus in the case $\Omega = (0,1) \subset \mathbb{R}$. We consider the set of admissible weights:

$$\mathbb{X} = \{p \in \mathscr{C}([0,1]) : p(0) = 0, p(1) = 0, p > 0 \text{ in } (0,1)\}.$$

Naturally, the distance to the boundary is a function in this set.

The map $u \in W_0^{1,1}(0,1) \mapsto \frac{u}{p} \in L^1(0,1)$ is continuous if and only if there exists $C > 0$ such that the following Hardy-type inequality is satisfied:

$$\int_0^1 |u'| \geq C \int_0^1 \frac{|u|}{p}, \qquad \forall u \in \mathscr{C}_c^\infty(0,1). \tag{4.64}$$

Remark 4.6. In [Psa13] the author studies the possible nature of weights p such that (4.64) holds.

We can answer negatively to the question above.

---
[2] which was raised to the candidate by H. Brezis in May 2017

Theorem 4.14. *There exists no $p \in \mathbb{X}$ that satisfies both* (4.63) *and* (4.64) *for* $\Omega = (0,1)$.

For the proof we will state several intermediate results.

Lemma 4.7.1. *Let $p \in \mathbb{X}$ satisfy* (4.64). *Then* $\frac{1}{p} \in L^1(0,1)$.

Proof. For $0 < \varepsilon < 1$ define $u_\varepsilon = \chi_{[\varepsilon, 1-\varepsilon]} \in BV(0,1)$, the characteristic function of the interval $[\varepsilon, 1-\varepsilon]$. We have that $u_\varepsilon' = \delta_\varepsilon - \delta_{1-\varepsilon}$ and $|u'| = \delta_\varepsilon + \delta_{1-\varepsilon}$. By passing to the limit by an approximating sequence in $\mathscr{C}_c^\infty(0,1)$ and applying the coarea formula (see Section 2.4), we write (4.64) as

$$2 \geq C \int_\varepsilon^{1-\varepsilon} \frac{1}{p}.$$

As $\varepsilon \to 0$ we deduce that

$$\int_0^1 \frac{1}{p} \leq \frac{2}{C}.$$

This proves the lemma. □

Lemma 4.7.2. *If $p \in \mathbb{X}$ satisfies* (4.63) *then* $\frac{1}{p} \notin L^1$.

Proof. If $\frac{1}{p} \in L^1$ then we can take $u = 1$ and (4.63) is not satisfied. □

We have the following extra information:

Lemma 4.7.3. *If $\frac{1}{p} \notin L^1(0, \frac{1}{2})$ and $\frac{1}{p} \notin L^1(\frac{1}{2}, 1)$ then* (4.63) *holds.*

Proof. Let $u \in \mathscr{D}'(0,1)$ be such that $u'' = 0$. Then $u(x) = a + bx$ for some $a, b \in \mathbb{R}$. Assume, towards a contradiction that $u \not\equiv 0$. There exists at most one $c \in [0,1]$ such that $u(c) = 0$. We distinguish 4 cases. If no c exists then $|u(x)| \geq D > 0$. Then

$$+\infty > \frac{1}{D} \int_0^1 \frac{|u|}{p} \geq \int_0^1 \frac{1}{p}.$$

This is a contradiction. If $c = 0$ then $|u| \geq D > 0$ in $(\frac{1}{2}, 1)$. Then

$$+\infty > \frac{1}{D} \int_{\frac{1}{2}}^1 \frac{|u|}{p} \geq \int_{\frac{1}{2}}^1 \frac{1}{p}.$$

This is also a contradiction. The same happens if $c = 1$. If $c \in (0,1)$ then $|u| \geq D$ in $(0, \varepsilon) \cup (1-\varepsilon, 1)$. Then

$$+\infty > \frac{1}{D} \left(\int_0^\varepsilon \frac{|u|}{p} + \int_{1-\varepsilon}^1 \frac{1}{p} \right) \geq \int_0^\varepsilon \frac{1}{p} + \int_{1-\varepsilon}^1 \frac{|u|}{p}$$

This concludes the proof. □

4.7.2 A decomposition problem

The notions of trace and weighted boundary condition do not inter-relate. A question that emerges[3] is the follow:

> what happens if we know that a function is the sum of two parts, one satisfying a boundary condition in the sense of traces and the other one in the sense of weights (a generalized version of it).

We have the following result:

Proposition 4.11. *Let u satisfy the following:*

i) $\Delta u = 0$ *in* $\mathscr{D}'(\Omega)$.

ii) $u = u_1 + u_2$

iii) $u_1 \in W_0^{1,1}(\Omega)$

iv) $\frac{u_2}{\delta} \in L^1(\Omega)$.

Then $u = 0$.

Proof. Since $\frac{u_2}{\delta} \in L^1(\Omega)$, due to Theorem 4.10, it holds that

$$\int_\Omega u_2 \Delta(\eta_\varepsilon \varphi) \to \int_\Omega u_2 \Delta \varphi. \tag{4.65}$$

On the other hand, since $u_1 \in W_0^{1,1}(\Omega)$:

$$-\int_\Omega u_1 \Delta(\eta_\varepsilon \varphi) = \int_\Omega \nabla u_1 \nabla(\eta_\varepsilon \varphi) = \int_\Omega \eta_\varepsilon \nabla u_1 \nabla \varphi + \int_\Omega \varphi \nabla u_1 \nabla \eta_\varepsilon. \tag{4.66}$$

Therefore, applying the properties of η_ε we have that:

$$\left| \int_\Omega u_1 \Delta(\eta_\varepsilon \varphi) - \int_\Omega u_1 \Delta \varphi \right| \le \int_\Omega |1-\eta_\varepsilon| |\nabla u_1| |\nabla \varphi| + \int_\Omega |\varphi| |\nabla u_1| |\nabla \eta_\varepsilon| \tag{4.67}$$

$$\le \int_{\delta < 2\varepsilon} |\nabla u_1 \nabla \varphi| + C \int_{\varepsilon < \delta < 2\varepsilon} \delta |\nabla u_1| \varepsilon^{-1} \tag{4.68}$$

$$\le C \left(\int_{\delta < 2\varepsilon} |\nabla u_1| + \int_{\varepsilon < \delta < 2\varepsilon} |\nabla u_1| \right) \tag{4.69}$$

$$\to 0, \tag{4.70}$$

[3] Raised by H. Brezis to the candidate (Haifa, June 2017)

since $|\nabla u_1| \in L^1(\Omega)$. Therefore

$$\int_\Omega u_1 \Delta(\eta_\varepsilon \varphi) \to \int_\Omega u_1 \Delta \varphi. \tag{4.71}$$

Hence

$$0 = \int_\Omega u_1 \Delta(\eta_\varepsilon \varphi) + \int_\Omega u_2 \Delta(\eta_\varepsilon \varphi) \to \int_\Omega u_1 \Delta\varphi + \int_\Omega u_2 \Delta \varphi = \int_\Omega u \Delta \varphi. \tag{4.72}$$

We have that

$$\int_\Omega u \Delta \varphi = 0 \qquad \forall \varphi \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega). \tag{4.73}$$

Therefore $u = 0$. □

Remark 4.7. The conclusion of this result can be useful to prove the uniqueness of solutions of some suitable non-standard linear boundary value problems.

4.7.3 The L^1 weight as a trace operator in $W^{1,q}, q > 1$

Another approach to this problem is to study whether being in $L^1(\Omega, \delta^{-r})$ does imply having trace 0 at least for functions in $W^{1,p}(\Omega)$ for $p > 1$. In this direction, in a more functional presentation, we have proved the following new result[4]:

Theorem 4.15. *Let Ω be a bounded domain of class $\mathscr{C}^{0,1}$. Then, for all $r > 1$ and $q > 1$*

$$L^1(\Omega, \delta^{-r}) \cap W^{1,q}(\Omega) \hookrightarrow W_0^{1,q}(\Omega), \tag{4.74}$$

Lemma 4.7.4. *Let $1^* = \frac{n}{n-1}$, $n \geq 2$ and $\alpha > 1$. Then, there exists c_Ω such that for $u \in L^1(\Omega, \delta^{-\alpha}) \cap W^{1,1}(\Omega)$ one has*

$$\left(\int_\Omega \left| \frac{u}{\delta} \right|^p \right)^{\frac{1}{p}} \leq c_\Omega \|u\|_{L^{1^*}(\Omega)}^{1-\frac{1}{\alpha}} \left(\int_\Omega |u| \delta^{-\alpha} dy \right)^{\frac{1}{\alpha}}, \quad 1 < p < \min\left\{ \alpha, \frac{1^* \alpha}{\alpha - 1 + 1^*} \right\}. \tag{4.75}$$

Proof. By Hölder's inequality

$$\int_\Omega |u|^p \delta^{-p} = \int_\Omega |u|^{p(1-\frac{1}{\alpha})} \delta^{-p} |u|^{\frac{p}{\alpha}} \leq \left(\int_\Omega |u|^{\frac{p(\alpha-1)}{\alpha-p}} \right)^{1-\frac{p}{\alpha}} \left(\int_\Omega |u| \delta^{-\alpha} \right)^{\frac{p}{\alpha}}. \tag{4.76}$$

We impose that

$$\frac{p(\alpha-1)}{\alpha-p} \leq 1^* \tag{4.77}$$

[4]This candidate thanks J.M. Rakotoson and J.I. Díaz for their coversation on this topic.

which is exactly the condition on the statement. □

Proof of Theorem 4.15. Let $q < \min\left\{\alpha, \frac{1^*\alpha}{\alpha-1+1^*}\right\}$. Then $\frac{u}{\delta} \in L^q$ and therefore $u \in W_0^{1,q}$. If $q \geq \min\left\{\alpha, \frac{1^*\alpha}{\alpha-1+1^*}\right\}$ first we observe that the result holds for $\bar{q}$ in the previous case and hence we see that $u \in W_0^{1,\bar{q}}(\Omega)$. Therefore

$$u \in W^{1,q}(\Omega) \cap W_0^{1,\bar{q}}(\Omega) = W_0^{1,q}(\Omega). \tag{4.78}$$

This proves the result. □

Remark 4.8. Notice that we have substituted L^p to L^1 in the known result (4.61).

Part II

A problem in Fourier representation

Chapter 5

Optimal basis in Fourier representation

This chapter presents work developed while on a visit to Prof. Häim Brezis at Technion - Israel Institute of Technology in Haifa, Israel in April-July 2017. The candidate wishes to extend to Häim Brezis his warmest thanks for the hospitality and the mentoring. The visit and the work led to the publication of [BGC17].

5.1 A problem in image representation

While studying compression of meshes for 3D representation Ron Kimmel and his group stumbled upon the following question, of a strict mathematical nature:

> Which is the basis of $L^2(\Omega)$ that provides the best finite dimensional projections of functions in $H_0^1(\Omega)$?

First, we need to define the term "optimal basis". It is natural to define as optimal a basis $b = (b_i)$ of $L^2(\Omega)$ such that, for all $m \geq 1$,

$$\left\| f - \sum_{i=1}^{m} (f, b_i) b_i \right\|_{L^2}^2 \leq \alpha_m \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5.1}$$

with optimal constants α_m. This technique led the group of Ron Kimmel to the publication of several paper in this direction, in collaboration with Häim Brezis (see [ABK15; ABBKS16]).

5.2 The mathematical treatment

In the works above the authors had shown that, in a bounded smooth set $\Omega \subset \mathbb{R}^n$, an optimal basis for $H_0^1(\Omega)$-representation in the sense of (5.1) was formed by the eigenfunctions e_i of the Laplace operator

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ is the ordered sequence of eigenvalues repeated according to their multiplicity.

It is a classical result that

Theorem 5.1. *We have, for all $m \geq 1$,*

$$\left\| f - \sum_{i=1}^{m} (f, e_i) e_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \qquad \forall f \in H_0^1(\Omega). \tag{5.3}$$

The proof of this fact is tremendously simple, due to the orthogonality of the eigenfunctions. Indeed

$$\left\| f - \sum_{i=1}^{m} (f, e_i) e_i \right\|_{L^2}^2 = \left\| \sum_{i=m+1}^{+\infty} (f, e_i) e_i \right\|_{L^2}^2 = \sum_{i=m+1}^{+\infty} (f, e_i)^2$$

and

$$\|\nabla f\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i (f, e_i)^2 \geq \sum_{i=m+1}^{+\infty} \lambda_i (f, e_i)^2 \geq \lambda_{m+1} \sum_{i=m+1}^{+\infty} (f, e_i)^2.$$

Combining these expressions yields the result.

The authors of [ABK15] and [ABBKS16] have investigated the "optimality" in various directions of the basis (e_i), with respect to inequality (5.3). Here is one of their results restated in a slightly more general form:

Theorem 5.2 (Theorem 3.1 in [ABK15])**.** *There is no integer $m \geq 1$, no constant $0 \leq \alpha < 1$ and no sequence $(\psi_i)_{1 \leq i \leq m}$ in $L^2(\Omega)$ such that*

$$\left\| f - \sum_{i=1}^{m} (f, \psi_i) \psi_i \right\|_{L^2}^2 \leq \frac{\alpha}{\lambda_{m+1}} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5.4}$$

The proof in [ABK15] relies in the Fischer-Courant max-min principle (see, e.g., [Lax02] or [Wei74]). For the convenience of the reader we present a very elementary proof based on a simple and efficient device originally due to H. Poincaré [Poi90, p. 249-250] (and later rediscovered by many people, e.g. H. Weyl [Wey12, p. 445] and R. Courant [Cou20, p. 17-18]; see also H. Weinberger [Wei74, p. 56] and P. Lax [Lax02, p. 319]).

Suppose not, and set

$$f = c_1 e_1 + c_2 e_2 + \cdots + c_m e_m + c_{m+1} e_{m+1} \tag{5.5}$$

where $c = (c_1, c_2, \cdots, c_m, c_{m+1}) \in \mathbb{R}^{m+1}$. The under-determined linear system

$$(f, \psi_i) = 0, \qquad \forall i = 1, \cdots, m \tag{5.6}$$

of m equations with $m+1$ unknowns admits a non-trivial solution. Inserting f into (5.4) yields

$$\lambda_{m+1} \sum_{i=1}^{m+1} c_i^2 \leq \alpha \sum_{i=1}^{m} \lambda_i c_i^2 \leq \alpha \lambda_{m+1} \sum_{i=1}^{m+1} c_i^2. \tag{5.7}$$

Therefore $\sum_{i=1}^{m+1} c_i^2 = 0$ and thus $c = 0$. A contradiction. This proves Theorem 5.2. □

The authors of [ABBKS16] were thus led to investigate the question of whether inequality (5.3) holds *only* for the orthonormal bases consisting of eigenfunctions corresponding to ordered eigenvalues. They established that a "discrete", i.e. finite-dimensional, version does hold; see [ABBKS16, Theorem 2.1]. But their proof of "uniqueness" could not be adapted to the infinite-dimensional case (because it relied on a "descending" induction). It was raised there as an open problem (see [ABBKS16, p. 1166]). The following result solves this problem.

Theorem 5.3 ([BGC17]). *Let (b_i) be an orthonormal basis of $L^2(\Omega)$ such that, for all $m \geq 1$,*

$$\left\| f - \sum_{i=1}^{m} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{m+1}} \quad \forall f \in H_0^1(\Omega). \tag{5.8}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of $-\Delta$ with corresponding eigenvalues (λ_i).

In fact, a more general result, which was introduced in [BGC17] as a remark, also holds:

Theorem 5.4. *Let V and H be Hilbert spaces such that $V \subset H$ with compact and dense inclusion* ($\dim H \leq +\infty$)*. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form for which there exist constants $C, \alpha > 0$ such that, for all $v \in V$,*

$$a(v,v) \geq 0,$$
$$a(v,v) + C|v|_H^2 \geq \alpha \|v\|_V^2.$$

Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the sequence of eigenvalues associated with the orthonormal (in H) eigenfunctions $e_1, e_2, \cdots \in V$, i.e.,

$$a(e_i, v) = \lambda_i (e_i, v) \qquad \forall v \in V,$$

where $(\cdot,\cdot)$ denotes the scalar product[1] *in H. For every $m \geq 1$ and $f \in V$:*

$$\lambda_{m+1} \left| f - \sum_{i=1}^{m} (e_i, f) e_i \right|_H^2 \leq a(f,f). \tag{5.9}$$

Let (b_i) be an orthonormal basis of H such that for all $m \geq 1$ and $f \in V$

$$\lambda_{m+1} \left| f - \sum_{i=1}^{m} (b_i, f) b_i \right|_H^2 \leq a(f,f). \tag{5.10}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of a with corresponding eigenvalues (λ_i).

Remark 5.1. When $\dim H < +\infty$ and $V = H$ this result is originally due to [ABBKS16]. The proof of "rigidity" was quite different and could not be adapted to the infinite dimensional case. It was raised there as an open problem.

This more general formulation allows us to cover some of the most relevant situations in the applications:

- For the optimal representation of function in $H^1(\Omega)$ we must take

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad a(f,h) = \int_\Omega \nabla f \cdot \nabla h + \mu \int_\Omega fh \tag{5.11}$$

[1] We point that, in this general setting, it may happen that $\lambda_1 = 0$ (e.g. $-\Delta$ with Neumann boundary conditions); and λ_1 may have multiplicity greater than 1

where μ is a positive constant. Then, e_i are solutions of

$$\begin{cases} -\Delta e_i + \mu e_i = \lambda_i e_i \\ \frac{\partial e_i}{\partial n} = 0. \end{cases} \tag{5.12}$$

Notice that, depending of the choice of bilinear product, we have a different choice of eigenfunctions.

- Let $\mathscr{M}$ be a compact Riemmanian manifold without boundary. Then one can choose

$$H = L^2(\mathscr{M}), \qquad V = H^1(\mathscr{M}), \qquad a(f,h) = \int_{\mathscr{M}} \nabla_g f \cdot \nabla_g h \tag{5.13}$$

where g is the Riemmanian metric. Then, the basis are the solutions of

$$-\Delta_g e_i = \lambda_i e_i. \tag{5.14}$$

where $-\Delta_g$ is the Laplace-Beltrami operator. Since there is no boundary, there is no boundary condition.

The basic ingredient of our proof is the following lemma, the proof of which is based on Poincaré's *magic trick*:

Lemma 5.2.1. *Assume that* (5.8) *holds for all* $m \geq 1$ *and all* $f \in H_0^1(\Omega)$*, and that*

$$\lambda_i < \lambda_{i+1} \tag{5.15}$$

for some $i \geq 1$. *Then*

$$(b_j, e_k) = 0, \qquad \forall j,k \text{ such that } 1 \leq j \leq i < k. \tag{5.16}$$

Applying this lemma we can quickly complete the proof in the case of simple eigenvalues. Since $\lambda_1 < \lambda_2$ then, by the lemma,

$$(b_1, e_k) = 0 \qquad \forall k \geq 2. \tag{5.17}$$

Thus $b_1 = \pm e_1$. Next we apply the lemma with $\lambda_2 < \lambda_3$. We have that

$$(b_2, e_k) = 0 \qquad \forall k \geq 3. \tag{5.18}$$

Also, we have that

$$(b_2, e_1) = \pm(b_2, b_1) = 0. \tag{5.19}$$

Therefore $b_2 = \pm e_2$. Similarly, we have that $b_i = \pm e_i$ for $i \geq 3$.

5.3 Connection to the Fischer-Courant principles

It is a very relevant part of the proof in [BGC17] that (5.3) can be understood under the light of the Fischer-Courant principles. In particular, if one considers the functions

$$0 \neq f \in \operatorname{span}(e_1, \cdots, e_m)^{\perp}$$

then, automatically,

$$\lambda_{m+1} \leq \frac{\|\nabla f\|_{L^2}}{\|f\|_{L^2}} \qquad \forall f \in \operatorname{span}(e_1, \cdots, e_m)^{\perp}, f \neq 0, \quad \forall m \geq 1. \tag{5.20}$$

Recall that the usual Fischer-Courant max-min principle asserts that for every $m \geq 1$ we have

$$\lambda_{m+1} = \max_{\substack{M \subset L^2(\Omega) \\ M \text{ linear space} \\ \dim M = m}} \min_{\substack{0 \neq f \in H_0^1(\Omega) \\ f \in M^{\perp}}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2}, \tag{5.21}$$

(see, e.g., [Lax02] or [Wei74]). Therefore, in some sense our basis b must be a maximizer of (5.21) for every $m \geq 1$.

Applying the same technique as in the proof of our Theorem 5.3, we can prove the following:

Proposition 5.1. *Let (b_i) be an orthonormal sequence in $L^2(\Omega)$ such that, for every $m \geq 1$,*

$$\lambda_{m+1} = \min_{\substack{0 \neq f \in H_0^1(\Omega) \\ f \in M_m^{\perp}}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2} \qquad \text{where } M_m = \operatorname{span}(b_1, b_2, \cdots, b_m). \tag{5.22}$$

Then, each b_i is an eigenfunction associated to λ_i.

The natural way to establish eigen-decomposition is through a compact, symmetric operator $A : H \to H$. The resolvent operator of Dirichlet problem $A = (-\Delta)^{-1} : f \mapsto u$ where

u is given by the weak solution of

$$\begin{cases} -\Delta u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{5.23}$$

satisfies this properties with $H = L^2(\Omega)$, due to the compact embedding $H^1(\Omega) \to L^2(\Omega)$. For simplicity, we will consider μ_n its eigenvalues. Notice that

$$Ae_i = \lambda e_i \implies \frac{1}{\lambda_i} e_i = A^{-1} e_i. \tag{5.24}$$

Thus, we get that

$$\mu_m((-\Delta)^{-1}) = \frac{1}{\lambda_m(-\Delta)}. \tag{5.25}$$

The spectral theorem guaranties that $A = (-\Delta)^{-1}$ has a basis of eigenvalues that expand $L^2(\Omega)$, and the existence of a sequence of positive eigenvalues $\mu_m \to 0$. However, this guaranties the spectral decomposition for $-\Delta$.

The Courant-Fischer principles are usually written in the literature for Rayleigh quotient

$$R_A(x) = \frac{(Ax, x)}{\|x\|^2}. \tag{5.26}$$

of $A = (-\Delta)^{-1}$, rather than $(-\Delta)$. On the other hand, (5.21) is written in terms of $R_{-\Delta}$. Nonetheless, once the eigendecomposition of A is established, Theorems 5.2 and 5.3 and (5.20) gives us a direct proof of (5.21).

Remark 5.2. The Rayleigh quotients $R_{-\Delta}$ and $R_{(-\Delta)^{-1}}$ do not seem to be directly related. Notice that

$$R_{-\Delta}(u) = \frac{(-\Delta u, u)}{\|u\|_{L^2}^2} = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} \tag{5.27}$$

$$R_{(-\Delta)^{-1}}(f) = \frac{(Af, f)}{\|f\|_{L^2}^2} = \frac{(u, -\Delta u)}{\|\Delta u\|_{L^2}^2} \tag{5.28}$$

$$= \frac{\|\nabla u\|_{L^2}^2}{\|\Delta u\|_{L^2}^2}, \tag{5.29}$$

where (5.23). Nonetheless, notice that

$$R_{-\Delta}(e_i) = \lambda_i, \tag{5.30}$$

$$R_{(-\Delta)^{-1}}(e_i) = \frac{1}{\lambda_i}. \tag{5.31}$$

5.3.1 Some controversy about the Fischer-Courant principles

Principle (5.21) has several different presentations in the literature. Currently, there are two main presentations, which are due to Fischer in 1905 [Fis05] and Courant in 1920 [Cou20].

For the rest of the section we will focus on compact symmetric operators defined over the whole Hilbert space H. Many of the references provide sharper results, and only simplified versions are stated here.

Let us, first, state the principles as Lax does in [Lax02]. This appears to be the commonly accepted nomenclature.

Theorem 5.5. *Let A be a compact symmetric operator in a Hilbert space H and let μ_n be its eigenvalues. Then, the following statements hold:*

- *Fischer's principle:*
$$\mu_m = \max_{S_m} \min_{x\in S_m} R_A(x), \tag{5.32}$$
where S_m is any linear subspace of H of dimension m

- *Courant's principle:*
$$\mu_m = \min_{S_{m-1}} \max_{x\perp S_{m-1}} R_A(x). \tag{5.33}$$

Remark 5.3. It is important to notice that (5.33) with (5.21) are both the Courant principle even though max and min are in reverse order. This is due to (5.25). This relates strongly to Remark 5.2.

However, Weinberger in [Wei74] assigns the credit differently. Here, (5.32) is named Poincaré's principle (see [Wei74, Theorem 5.1]), due to Poincaré's seminal paper [Poi90] in 1890, in which he starts the theory of eigen-decomposition. Also, (5.33) is named Courant-Weyl's principle (see [Wei74, Theorem 5.2]) and it is written in a slightly more general way

$$\mu_m = \min_{\substack{l_1,\cdots,l_m \\ \text{linear functionals}}} \sup_{\substack{v\in H \\ l_1(v)=\cdots=l_m(v)=0}} R_A(x). \tag{5.34}$$

Notice that, in infinite dimensional spaces, a linear functional l_i need not be continuous, so it not be written $l_i(v) = (w, v)$ for some $w \in H$. Hence, there are many more functionals in this characterization. In Weinberger's text, the name of Fischer does not appear.

The inclusion of the name of Weyl is due to his paper [Wey12] in which he proves the asymptotic behaviour of eigenvalues (see also [Wey11]). Some books, e.g. [WS72], go as far as stating the following:

> An even more important variational characterization, the maximum- minimum principle, is claimed by Weyl, who used some of its consequences in his famous theory of asymptotic distribution of eigenvalues [W31, W32]. Later, Courant applied the principle contained in Weyl's fundamental inequality to a fairly general typical situation [C2].

In [WS72] (where the authors use A as the operator with increasing eigenvalues, and thus in direct conflict with [Lax02]) the following is stated.

Lemma 5.3.1 ([Wey12], as extracted from [WS72])**.** *Let A be a symmetric, compact operator. Let $p_1, \cdots, p_{m-1}$ be any arbitrary vectors in H. Then*

$$\max_{\substack{f \in H \\ (f,p_1)=\cdots=(f,p_{m-1})=0}} R_A(f) \geq \mu_m. \tag{5.35}$$

The proof again passes by the use of Poincaré's *magic trick*. This, which is presented in [WS72] as "Weyl's lemma" must not be confused with what is usually called Weyl's lemma, that is the regularity of functions such that $-\Delta u = 0$ in $\mathscr{D}'(\Omega)$. From this result the author extracts the proof of Courant's principle.

5.4 Some follow-up questions

In [WS72] the problem of whether the equality can hold in (5.35) for a finite orthonormal set $b = (b_1, \cdots, b_N)$ is studied. This question was also answered in [BGC17] in some of the relevant cases.

Remark 5.4. After the publication of the paper the authors were made aware of the interest of this question by many authors. The terminology employed by the specialists in this field is *n-widths*. See, e.g., [Pin85; EBBH09; FS17] and the references therein.

Let us present merely the case in which only $N = 2$.

Remark 5.5. Assume that $b = b_1 \in L^2(\Omega)$ is such that $\|b\|_{L^2} = 1$ and

$$\|f - (f,b)b\|_{L^2}^2 \leq \frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5.36}$$

Of course, (5.36) holds with $b = e_1$. From Lemma 5.2.1 we know that (5.36) implies that

$$(e_2, b) = 0. \tag{5.37}$$

Clearly, (5.37) is not sufficient. Indeed, take $b = e_3$. Then, (5.37) holds but (5.36) fails for $f = e_1$. We do not have a simple characterization of the functions b satisfying (5.36). But we can construct a large family of functions b (which need not be smooth) such that (5.36) holds. Assume that $0 < \lambda_1 \leq \lambda_2 < \lambda_3$. Let $\chi \in L^2(\Omega)$ be any function such that

$$(e_1, \chi) = 0, \tag{5.38}$$

$$(e_2, \chi) = 0, \tag{5.39}$$

$$\|\chi\|_{L^2}^2 = 1. \tag{5.40}$$

Set

$$b = \alpha e_1 + \varepsilon \chi \qquad \alpha^2 + \varepsilon^2 = 1, \text{ with } 0 < \varepsilon < 1. \tag{5.41}$$

Then, there exists $\varepsilon_0 > 0$, depending on $(\lambda_i)_{1 \leq i \leq 3}$, such that for every $0 < \varepsilon < \varepsilon_0$ (5.36) holds (see [BGC17]).

Remark 5.6. In the general setting of Theorem 5.4 it may happen that $0 = \lambda_1 < \lambda_2$. Suppose now that $b \in H$ is such that $\|b\|_H = 1$ and

$$\|f - (f,b)b\|_H^2 \leq \frac{1}{\lambda_2} a(f,f) \qquad \forall f \in V. \tag{5.42}$$

Claim: we have $b = \pm e_1$. Indeed, let $f = e_1$ in (5.42) we have that

$$\|e_1 - (e_1,b)b\|_H^2 \leq \frac{\lambda_1}{\lambda_2} = 0, \tag{5.43}$$

Therefore $b = \pm e_1$.

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Homogenization and Shape Differentiation of Quasilinear Elliptic Equations

Publications

David Gómez Castro

Advisor: Prof. Jesús Ildefonso Díaz Díaz

Dpto. de Matemática Aplicada &
Instituto de Matemática Interdisciplinar
Universidad Complutense de Madrid

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On the influence of pellet shape on the effectiveness factor of homogenized chemical reactions

J. I. Díaz[*], D. Gómez-Castro[*] and C. Timofte[†]

Abstract— One of the most popular principles of Nanotechnology, especially in the context of composite media, says, roughly speaking, that one of the reasons for the optimality of certain composite media comes from the fact that when the size of the small particles decreases (maintaining a prescribed total volume) then their total surface increases and this leads to peculiar properties which cannot be observed when the particles are big. What is really relevant in this context is a suitable balance between the total surface and the homogenized diffusion. In order to fix ideas, we consider the case of adsorption chemical reactions on the surface of a set of particles in a periodic composite structure (medium). We know that the solution of our problem converges to the solution of a related homogenized semilinear elliptic problem. Our main goal is to study the behaviour of the so-called *effectiveness factor* η_ε for the chemical reactions, defined at the microscale, and to establish the relation between this factor and the corresponding one η defined for the homogenized problem. Moreover, we shall study the effect of the shape of the pellets (in particular, their total surface $|\partial T|$ and the homogenized diffusion coefficient $a_0(T)$) in the homogenized effectiveness factor. We prove the existence of an optimal convex shape of the particles for the effectiveness functional.

Introduction

Let Ω be an open bounded connected set in $\mathbb{R}^N$ and let us insert in it a set of identical periodically distributed obstacles T^ε. Let us denote the resulting domain by Ω^ε ε being a small parameter related to the characteristic size of the obstacles. We assume that the size of the obstacles is of the order of $r(\varepsilon)$. In such a domain, we shall study a semilinear problem involving diffusion and suitable chemical reactions taking place on the boundary of the inclusions. There exists a critical size of the inclusions that separates different asymptotic behaviours of the solution of such a problem. We shall discuss here only the case of the so-called *big particles*. The case of *small particles* and, in particular, the interesting case of *critical particles* will be addressed in a forthcoming paper. Under suitable hypotheses, it is well-known that the solution of our problem converges, as ε goes to zero, to the solution of a new elliptic PDE, containing an extra-term generated by the chemical reactions taking place on the surface of the particles.

Our main goal is to study the behaviour of the so-called *effectiveness factor* η_ε and to establish the relation between this factor and the corresponding one defined for the homogenized problem. We shall be also interested in analyzing the effect of the shape of the particles (in particular, their total surface $|\partial T|$ and the homogenized diffusion coefficient $a_0(T)$) in both functionals. We shall prove the existence of convex shapes which maximize the effectiveness.

One of the most popular principles of Nanotechnology, especially in the context of composite media, says, roughly speaking, that one of the reasons for the optimality of certain composite media comes from the fact that when the size of the small particles decreases (maintaining a prescribed total volume) then their total surface increases and this leads to peculiar properties which cannot be observed when the particles are big (see e.g. [22], [18] and [5]).

We show some numerical experiments in which this ratio is not the only relevant parameter, but rather the one given by a balance between the measure of the surface of the pellets and their shape.

1 Problem setting

Let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be a bounded connected open set such that $|\partial\Omega| = 0$ and let $Y = (-\frac{1}{2}, \frac{1}{2})^N$ be the reference cell in $\mathbb{R}^N$. Let ε be a real parameter taking values in a sequence of positive numbers converging to zero. ε represents a small parameter related to the characteristic size of the particles. Let T be another open bounded subset of $\mathbb{R}^N$, with the boundary ∂T of class C^2. T will be called *the elementary particle* and

[*]Instituto de Matemática Interdisciplinar and Dpto. de Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid. Spain. Email: jidiaz@ucm.es, dgcastro@ucm.es
[†]Faculty of Physics, University of Bucharest, Romania. Email: claudia.timofte@g.unibuc.ro

we assume that 0 belongs to T and that T is star-shaped with respect to 0. Since T is bounded, without loss of generality, we can assume that $\overline{T} \subset Y$. Let $r : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous map, related to the size of the pellet. We shall assume that $r(\varepsilon) \sim \varepsilon$. These are known as *big particles*. The case of *small particles*, when $\lim_{\varepsilon\to 0} r(\varepsilon)/\varepsilon = 0$ and $r(\varepsilon) < \varepsilon/2$, will be treated in a forthcoming paper.

REMARK 1 Even though the usual term in homogenization theory for the inclusions is *holes* (in order to give the idea that something has been removed from the domain) here we will avoid this terminology. For us, these inclusions will be pellets, for example the ones that can be found in fixed bed reactors and towers. Therefore, we will refer to these *holes* as *pellets*, *particles* or even *inclusions* and *obstacles*.

For each ε and for any vector $\mathbf{i} \in \mathbb{Z}^N$, we shall denote by $T_{\mathbf{i}}^\varepsilon$ the translated image of $r(\varepsilon)T$ by the vector $\varepsilon\mathbf{i}$, $\mathbf{i} \in \mathbb{Z}^N$: $T_{\mathbf{i}}^\varepsilon = \varepsilon\mathbf{i} + r(\varepsilon)T$. Also, let us denote by T^ε the set of all the pellets contained in Ω, i.e.

$$T^\varepsilon = \bigcup \{T_{\mathbf{i}}^\varepsilon \mid \overline{T_{\mathbf{i}}^\varepsilon} \subset \Omega,\ \mathbf{i} \in \mathbb{Z}^N\}$$

and let the number of pellets be $n(\varepsilon) = \#\left\{\mathbf{i} \in \mathbb{Z}^N : \overline{T_{\mathbf{i}}^\varepsilon} \subset \Omega\right\}$. Set $\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}$. Therefore, Ω^ε is a periodically perforated structure with pellets of the size $r(\varepsilon)$. Let us notice that the inclusions do not intersect the fixed boundary $\partial\Omega$.

Let $S^\varepsilon = \cup\{\partial T_{\mathbf{i}}^\varepsilon \mid \overline{T_{\mathbf{i}}^\varepsilon} \subset \Omega, \mathbf{i} \in \mathbb{Z}^N\}$. So, $\partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon$. We shall consider the homogenization of problems in the form

$$(1) \qquad \begin{cases} -\Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} + \mu(\varepsilon) g(u^\varepsilon) = 0 & \text{on } S^\varepsilon, \\ u_\varepsilon = 1 & \text{on } \partial\Omega, \end{cases}$$

where ν is the exterior unit normal to S^ε,

$$(2) \qquad g \text{ is a maximal monotone graph such that } g(0) = 0,$$

(single-valued or even multivalued) and

$$(3) \qquad f \in L^2(\Omega), \quad f \geq 0.$$

A particular case we shall discuss is the Freundlich isotherm:

$$(4) \qquad g(u) = |u|^{p-1}u, \qquad p \in (0, 1].$$

Also, we can consider the limit case of zero order reactions:

$$(5) \qquad g(u) = \begin{cases} 0 & u < 0, \\ [0, 1] & u = 0, \\ 1 & u > 0. \end{cases}$$

We address here the interesting cases in which $\mu(\varepsilon)|S^\varepsilon| = O(1)$. In fact, we can consider that if $r(\varepsilon) = \varepsilon^\alpha$, then $\mu(\varepsilon) = \varepsilon^{-\gamma}$, $\gamma = \alpha(N-1) - N$. In our case, i.e. for big particles, $\alpha = 1$, and so $\gamma = -1$.

Through standard procedures in weak solution theory, one easily gets the following result (see, e.g., [6]).

PROPOSITION 2 (WELL-POSSEDNESS) *Under the assumptions* (2) *and* (3), *there exists a unique solution* $u \in H^2(\Omega)$ *of* (1).

PROPOSITION 3 (STRONG MAXIMUM PRINCIPLE) *Under the assumptions* (2) *and* (3), $u_\varepsilon > 0$ *in* Ω_ε.

Proof. . By the maximum principle, we have that $u_\varepsilon \geq 0$. Now, we can apply the comparison principle with $\underline{u}_\varepsilon$, the non-negative solution of

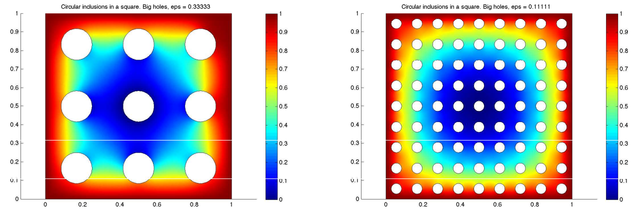
$$\begin{cases} -\Delta \underline{u}_\varepsilon = f & \text{in } \Omega^\varepsilon, \\ \underline{u}_\varepsilon = 0 & \text{on } \partial\Omega \cup S^\varepsilon, \end{cases}$$

to obtain $u_\varepsilon \geq \underline{u}_\varepsilon$ in Ω^ε. For $\underline{u}_\varepsilon$, we can apply the bound found in [15]

$$\underline{u}_\varepsilon(x) \geq c \left(\int_\Omega f(y)\, \mathrm{d}(y, \partial\Omega^\varepsilon)\, dy \right) \mathrm{d}(x, \partial\Omega^\varepsilon), \qquad x \in \Omega^\varepsilon,$$

which proves the result. □

We can see a couples of the steps of the homogenization process in the following COMSOL simulation.



(a) $\varepsilon = 0.33$ (b) $\varepsilon = 0.11$

Figure 1: Fixed bed reactors with big pellets ($r_\varepsilon = \varepsilon$) and the level set of the solution of problem (1) for $f = 0$, (4) where $p = \frac{1}{2}$, $\mu(\varepsilon) = \varepsilon$.

2 Homogenization of the state equation

Assume that $r(\varepsilon) = \varepsilon$ and either a smooth kinetic

$$(6) \qquad |g'(v)| \leq C(1 + |v|^q), \qquad 0 \leq q < \frac{N}{N-2}$$

or not necessarily a smooth one with bounded growth

$$(7) \qquad |g(v)| \leq C(1 + |v|^q), \qquad 0 \leq q < \frac{N}{N-2}.$$

Following the theory in [9] and [10], the solution u^ε of problem (1), properly extended to the whole of Ω, converges weakly in $H^1(\Omega)$, as $\varepsilon \to 0$, to $u \in H^1(\Omega)$, i.e. $u^\varepsilon \rightharpoonup u$, where u is the unique solution of the following homogenized problem

$$(8) \qquad \begin{cases} -\operatorname{div}(a_0(T)\nabla u) + \frac{|\partial T|}{|Y \setminus T|} g(u) = f & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega. \end{cases}$$

The proof of existence and uniqueness of a weak solution for this problem can be found, e.g., in [12]. Here, $a_0(T) \in \mathcal{M}_N(\mathbb{R})$ is the classical homogenized matrix (see, e.g., [9]). If we write $a_0(T) = (q_{ij})$, then

$$q_{ij} = \delta_{ij} + \frac{1}{|Y \setminus T|} \int_{Y \setminus T} \frac{\partial \chi_j}{\partial y_i} dy,$$

where χ_i are the solutions of the so-called *cell problems*:

$$(9) \qquad \begin{cases} -\Delta \chi_i = 0 & \text{in } Y \setminus T, \\ \frac{\partial(\chi_i + y_i)}{\partial \nu} = 0 & \text{on } \partial T, \\ \chi_i & Y\text{-periodic.} \end{cases}$$

For the corresponding result in the case of small particles, we refer to [16]. Let us mention that in the critical case, i.e. the case in which $r(\varepsilon) = \varepsilon^{N/(N-2)}$, the extra-term arising in the homogenized equation is defined in terms of the solution of a functional equation involving the nonlinear function g.

2.1 Effectiveness and homogenization

For the case of smooth kinetics, we shall assume that $g(0) = 0$ and we shall impose growth condition (6) on the nonlinearity g. Inspired by the definition given in the linear case $p = 1$ by the chemical engineer R. Aris (see [1] and [2]), we define the notion of **effectiveness** of the pellet in this more general setting as follows:

$$(10) \qquad \eta_\varepsilon(T) = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g(u_\varepsilon) d\sigma.$$

This is well defined since $g(u^\varepsilon) \in W_0^{1,\bar{q}}(\Omega)$, $\bar{q} = \frac{2N}{q(N-2)+N}$.

Definition (10) can be naturally extended to the homogenized case, as follows

$$(11) \qquad \eta(T) = \frac{1}{|\Omega|} \int_\Omega g(u) dx.$$

PROPOSITION 4 *For $\varepsilon \to 0$, it follows that $\eta_\varepsilon(T) \to \eta(T)$.*

Proof. From [8] (see also [9]), it holds that

$$\varepsilon \int_{S_\varepsilon} g(u^\varepsilon(x)) \mathrm{d}\sigma \to \frac{|\partial T|}{|Y|} \int_\Omega g(u(x)) \mathrm{d}x, \quad \text{as } \varepsilon \to 0.$$

Since, by explicit computation, $|S_\varepsilon| = n(\varepsilon)|\partial(\varepsilon T)| = n(\varepsilon)\varepsilon^{N-1}|\partial T|$, when the cells tend to cover the total volume,

$$n(\varepsilon)|Y|\varepsilon^N = n(\varepsilon)|\varepsilon Y| \to |\Omega|, \quad \text{as } \varepsilon \to 0,$$

we have that $|S_\varepsilon|\varepsilon \to |\Omega||\partial T|$, as $\varepsilon \to 0$. Hence, as $\varepsilon \to 0$,

$$\eta_\varepsilon(T) = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} g(u^\varepsilon(x)) \mathrm{d}\sigma \to \frac{1}{|\Omega|} \int_\Omega g(u) \mathrm{d}x = \eta(T),$$

which proves the result. □

REMARK 5 It is an open problem whether or not this convergence remains true under more general nonlinearities g. Our proof of the convergence relies on [8], in which one requires differentiability of $g(u^\varepsilon)$. We can define the effectiveness η_ε by means of $g(\mathrm{tr}_{S_\varepsilon}(u^\varepsilon))$. However, the proof, in essence, requires that we consider $\mathrm{tr}_{S_\varepsilon}(g(u^\varepsilon))$. It is our belief that a proof of the general case might need an extension of the results in [8] or a completely new approach.

REMARK 6 The convergence remains true for the kinetic (4) in the case of domains in which there exists $\delta > 0$ such that $u^\varepsilon \geq \delta$ uniformly on ε, that is, no dead core exists. For the solution u, the region where $u = 0$ (which might have positive measure) is known in the literature as a *dead core*. Conditions for the existence and location of a dead core in this and other kinds of equations can be found in [12], [4] and the references therein. In the case when a dead core exists, even though the limit theorem does not apply, the strong maximum principle (Proposition 3) suggests that the effectiveness is higher prior to the homogenization process.

3 Existence of optimal pellet shapes

Once we know the effect that a general obstacle T causes, it seems reasonable to perform domain optimization. First, we show an abstract result of existence of optimal hole shape. We will focus on the homogenized model (8). Our main result is the following one:

THEOREM 7 *Let $0 < \theta < |Y|$, C, D be fixed proper subsets of Y and $\tilde{\varepsilon} > 0$. Let us consider the hypothesis*

(12) *T satisfies the uniform $\tilde{\varepsilon}$-cone property.*

We define

$$U_{adm} = \{\overline{C} \subset T \subset \overline{D} : T \text{ satisfies }, (12) \text{ and } |T| = \theta\},$$
$$C_\theta(D) = \{T \subset D : T \text{ is open, convex and } |T| = \theta\}.$$

At fixed volume $\theta \in (0, |Y|)$, there exists a domain of maximal effectiveness in the class of $T \in U_{adm} \cap C_\theta(D)$.

REMARK 8 Optimization of the effectiveness considering the homogenized domain Ω (the chemical reactor) has also been studied (see [13], [14] and the references therein). In this situation, the existence of a *dead core* affects the effectiveness negatively.

REMARK 9 Dealing with the optimization of the domain Ω, **there exist no optimal shapes** considering a general framework (see [4], [14]). We conjecture that new results may be also obtained by applying methods analogous to the ones that follow.

REMARK 10 As in [9], the problem in which we consider reactions inside the pellets can also be addressed. Let us consider the system of equations

$$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_p \Delta v^\varepsilon + a g(v^\varepsilon) = 0 & \text{in } \Omega \setminus \overline{\Omega^\varepsilon}, \\ -D_f \frac{\partial u^\varepsilon}{\partial \nu} = D_p \frac{\partial v^\varepsilon}{\partial \nu} & \text{on } S^\varepsilon, \\ u^\varepsilon = v^\varepsilon & \text{on } S^\varepsilon, \\ u^\varepsilon = 1 & \text{on } \partial\Omega, \end{cases}$$

with $a, D_f, D_p > 0$ and $f \in L^2(\Omega)$. If we introduce the matrix $A = D_f \chi_{Y\setminus T} + D_p \chi_T$ where I is the identity matrix in $\mathcal{M}_N(\mathbb{R})$, then the homogenized problem for big pellets is (see [9])

$$\begin{cases} -\operatorname{div}(A^0 \nabla u) + a\frac{|T|}{|Y\setminus T|} g(u) = f & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega, \end{cases}$$

where $A^0 = (a^0_{ij})$ is the homogenized matrix, whose entries are defined as follows: $a^0_{ij} = \int_Y (a_{ij} + a_{ik}\frac{\partial \chi_j}{\partial y_k}) dy$, in terms of the functions χ_j, $i = 1, ..., N$, Y-periodic solutions of the cell problems $-\text{div}(A\nabla(y_j + \chi_j)) = 0$. In this context, the results would be analogous and the proofs perhaps even simpler.

We see in (8) that the effect of T is present in three terms: $a_0(T), |\partial T|$ and $|Y \setminus T|$. Therefore, any sensible choice of topology for the set of admissible domains T in a search for optimal obstacles must make this expressions continuous.

A logical choice of topology in the space of shape is the one given by the Hausdorff distance

$$d_H(\Omega_1, \Omega_2) = \sup\{\sup_{x\in\Omega_1} d(x, \Omega_2), \sup_{x\in\Omega_2} d(x, \Omega_1)\}.$$

For the optimization, we will restrict ourselves to a general enough family of domains, but in which we can define a topology which makes the family compact. It is well known (see, for example, [23]) that the following result holds true.

THEOREM 11 ([23]) *The class of closed subsets of a compact set D is compact for the Hausdorff convergence.*

A proof for the continuity of the effective diffusion under the Hausdorff distance in U_{adm} can be found in [17].

LEMMA 12 ([17]) *If U_{adm} is compact with respect to the Hausdorff metric and if $T_n \to T$, $(T_n) \subset U_a$ as $n \to \infty$, $T \in U_{adm}$, then $a_0(T_n) \to a_0(T)$ in $\mathcal{M}_N(\mathbb{R})$.*

The behaviour of the measure $|Y - T|$ is slightly more delicate (we include a commentary even though, in our family, this will be constant). For this, a distance with a definition similar to Hausdorff metric, the Hausdorff complementary distance

$$d_{H^c}(\Omega_1, \Omega_2) = \sup_{x\in\mathbb{R}^n} |d(x, \Omega_1^c) - d(x, \Omega_2^c)|,$$

has the following property: for open domains, $d_{H^c}(\Omega_n, \Omega) \to 0$ as $n \to \infty$ implies $\liminf_n |\Omega_n| \geq |\Omega|$. However, lower semicontinuity of the measure of the boundary ($|\partial T|$) is, in general, false (see [17] for some counterexamples). Nevertheless, the set of **convex** domains has a number of very interesting properties (see [24]).

LEMMA 13 ([24]) *The topological spaces $(C_\theta(D), d_H)$ and $(C_\theta(D), d_{H^c})$ are equivalent.*

The continuity of the boundary measure is provided by the following theorem, proved in [7].

LEMMA 14 ([7]) *Let $(\Omega_n), \Omega \in C_\theta(D)$. If $\Omega_1 \subset \Omega_2$, then $|\partial\Omega_1| \leq |\partial\Omega_2|$. Moreover, if $\Omega_n \overset{d_H}{\to} \Omega$, then $|\Omega_n| \to |\Omega|$ and $|\partial\Omega_n| \to |\partial\Omega|$, as $n \to \infty$.*

For the continuity of solutions with respect to T, we need the following theorem of continuity of Nemitskij operators (see, for example, [19], [11] and [21]).

LEMMA 15 ([21]) *Let $G : \Omega \to \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that (7) with $q = \frac{r}{t}$ with $r \geq 1$ and $t < \infty$. Then, the map*

$$L^r(\Omega) \to L^t(\Omega) \qquad v \mapsto G(x, v(x))$$

is continuous in the strong topologies.

LEMMA 16 *Let $\mathcal{A}$ be the set of elliptic matrices and let g satisfy (7). Then, the application*

$$\mathcal{A} \times \mathbb{R}_+ \to H^1(\Omega) \qquad (A, \lambda) \mapsto u,$$

where u is the unique solution of

$$\begin{cases} -div(A\nabla u) + \lambda g(u) = f, & \text{in } \Omega, \\ u = 1, & \text{on } \partial\Omega, \end{cases}$$

is continuous in the weak topology.

Proof. . Let $G(u) = \int_0^u g(s)ds$ and

$$J_{A,\lambda}(v) = \frac{1}{2}\int_\Omega (A\nabla v)\cdot \nabla v + \int_\Omega \lambda G(v) - \int_\Omega fv.$$

We know that $u(A, \lambda)$ is the unique minimizer of this functional. Let $A_n \to A$ and $\lambda_n \to \lambda$ be two converging sequences. It is easy to prove that $u_n = u(A_n, \lambda)$ is bounded in $H^1(\Omega)$ and, up to a subsequence, $u_n \rightharpoonup u$ in H^1 as $n \to \infty$. Therefore, $\int_\Omega (A\nabla u)\nabla u \leq \liminf_n \int_\Omega (A_n \nabla u_n)\nabla u_n$. We can apply Theorem 15 to show that $G(u_n) \to G(u)$ in L^1 as $n \to \infty$ (see details for a similar proof, for example, in [9]) and we have that $u = u(A, \lambda)$. □

COROLLARY 17 *In the hypotheses of the previous lemma, the map $(A, \lambda) \mapsto \int_\Omega g(u(A, \lambda))$ is continuous.*

With these tools, we can prove now our main result.

Proof. of Theorem 7 First, we have that Lemmas 12 and 14 imply that the application $T \mapsto (a_0(T), \lambda(T))$ is continuous. Then, Corollary 17 implies that $T \mapsto \eta(T)$ is continuous. Therefore, since $C_\theta(D)$ is closed and U_{adm} is compact, by Lemma 12 we have the compactness of $U_{adm} \cap C_\theta(D)$ and the existence of maximizers. $\square$

4 Effectiveness for obstacles with some symmetries. Numerical experiments

There exists a large literature on the computation and behaviour of the homogenized coefficient $a_0(T)$, both from the mathematics and the engineering part (see, e.g., [3], [17], [20]). In these papers, one can find power series techniques and numerical analysis, generally for spherical obstacles. As it is common in the literature (e.g. [3]), we use the commercial software COMSOL. As said on the introduction, in nanotechnology, however, it is a common misconception that the measure of the surface alone, $|\partial T|$, is a good indicator of the effectiveness of the obstacle.

Considering obstacles with some symmetries (for $N = 2$ it is sufficient that they are invariant under a 90^0 rotation) in general, it is well known that $a_0(T) = \alpha(T)I$, where $\alpha(T)$ is a scalar (see, for example, [3], [20]) and I is the identity matrix in $\mathcal{M}_N(\mathbb{R})$. In this case, it can be easily proved that the effectiveness is an decreasing function of $\lambda(T) = |\partial T|/(\alpha(T)|Y \setminus T|)$ (it is a direct consequence of the comparison principle, see [12]). In fact, this is the only relevant parameter (once $g(u)$ is fixed) of the equation (8). The behaviour of the effectiveness with respect to the coefficient λ can also be numerically computed:

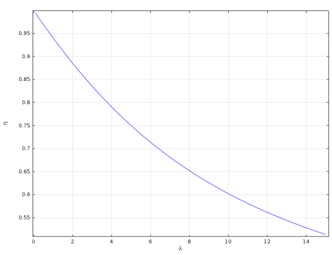
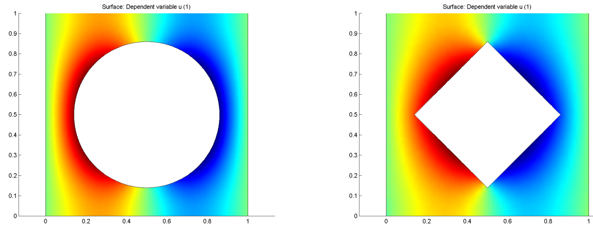


Figure 2: Plot of η as a function of λ when Ω is a 2D circle.

Let us consider, in two dimensions for simplicity, the following obstacles:



(a) Circular inclusions (b) Square inclusions

Figure 3: Two obstacles T, and the level sets of the solution of the cell problem (9)

We can numerically compute the homogenized diffusion coefficient $a_0(T)$ via a parametric sweep on the size of the obstacle.

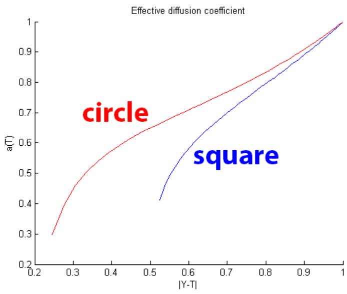
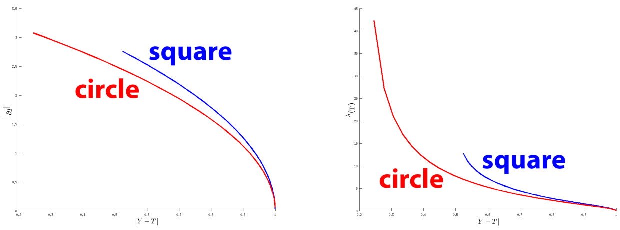


Figure 4: The effective diffusion coefficient $\alpha(T)$. Circular particles in red and square particles in blue

Now, we can couple this with direct computations of $|\partial T|$ and compare the behaviour of both indicators.



(a) Classical coefficient $|\partial T|$. (b) New coefficient $\lambda(T)$ as a function of the concentration $|Y \setminus T|$

Figure 5: Coefficients $|\partial T|$ and $\lambda(T)$. Circular particles in red and square particles in blue

Since $|Y \setminus T| = 1 - \theta$ and since η is monotone decreasing with respect to λ, what Figure 5 represents is a comparison between the effectiveness of circular and square pellets for different θ. We can conclude that, in the computed cases, circular pellets are more efficient. This could have also been deduced solely from the consideration of $|\partial T|$. However, even though the relative order is not affected, what we see in Figure 5 is that the behaviour close to minimum and maximum admissible θ (which correspond with $\theta = 0$ and the pellet touching the boundary of the cell) on each is radically different (notice the steepness). The fact that the circle appears to be more effective contrast with the fact that in the homogenized reactor Ω a sphere is worst (see [4], [13], [14]).

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Orthogonal $\perp$

THE EFFECTIVENESS FACTOR OF REACTION-DIFFUSION EQUATIONS: HOMOGENIZATION AND EXISTENCE OF OPTIMAL PELLET SHAPES

JESÚS ILDEFONSO DÍAZ, DAVID GÓMEZ-CASTRO, AND CLAUDIA TIMOFTE

Dedicated to an exceptional mathematician, David Kinderlehrer, with admiration.

Abstract. We study the asymptotic behaviour of the so-called *effectiveness factor* η_ε of a nonlinear diffusion equation that occurs on the boundary of periodically distributed inclusions (or particles) in an ε-periodic medium. Here, ε is a small parameter related to the characteristic size of the inclusions, which, in the homogenization process, will tend to 0. The inclusions are modeled as homothecy of a fixed pellet T, rescaled by a factor $r(\varepsilon)$. We study the cases in which $r(\varepsilon) = O(\varepsilon^\alpha)$, known as *big holes*, for $\alpha = 1$, as well as *non-critical small holes*, for $1 < \alpha < \frac{n}{n-2}$. We will prove the existence of some convex shapes which maximize the effectiveness of the homogenized problem. In particular, we deduce that for small holes the sphere is the domain of highest effectiveness.

1. Introduction

We study the asymptotic behaviour of the so-called *effectiveness factor* η_ε of nonlinear diffusion equations for which a reaction occurs on the boundary of periodically distributed inclusions (or particles) in an ε-periodic medium.

To be more precise, let $\Omega \subset \mathbb{R}^N$, with $N \geq 3$, be a bounded connected open set such that $|\partial\Omega| = 0$ and let $Y = (-\frac{1}{2}, \frac{1}{2})^N$ be the reference cell in $\mathbb{R}^N$. Let T be another open bounded subset of $\mathbb{R}^N$, with the boundary ∂T of class C^2. T will be called *the elementary particle*. We assume that 0 belongs to T and that T is star-shaped with respect to 0. Since T is bounded, without loss of generality, we can assume that $\overline{T} \subset Y$. We point out that, even though the usual term in homogenization theory for inclusions is *holes* (in order to give the idea that something has been removed from the domain), here we will avoid this terminology. For us, these inclusions will be pellets, for example the ones that can be found in fixed bed chemical reactors and towers. Therefore, we will refer to these *holes* as *pellets*, *particles* or even *inclusions* and *obstacles*.

2010 *Mathematics Subject Classification.* homogenization, effectiveness factor, semilinear elliptic equations.

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MATHEMATICS

Homogenization of the p-Laplacian with Nonlinear Boundary Condition on Critical Size Particles: Identifying the Strange Term for the Some non Smooth and Multivalued Operators[1]

J. I. Diaz[a]*, D. Gómez-Castro[a], A. V. Podol'skii[b]***, and T. A. Shaposhnikova[b]******

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Abstract—We extend previous papers in the literature concerning the homogenization of Robin type boundary conditions for quasilinear equations, in the case of microscopic obstacles of critical size: here we consider nonlinear boundary conditions involving some maximal monotone graphs which may correspond to discontinuous or non-Lipschitz functions arising in some catalysis problems.

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Papers [2–10] were devoted to the study of asymptotic behavior of the solution to the boundary value problem for the p-Laplacian ($p \in [2, n)$) in ε-periodically perforated domain with nonlinear Robin-type boundary condition that contains function $\sigma(x, u)$. It was supposed there that $\sigma(x, u)$ is a smooth function of it's arguments, monotone by variable u. In this paper we extend the method introduced in [3, 4, 7–10] to deal with the problems with more general conditions on the function $\sigma(x, u)$. As in all papers in which the holes are of critical size and the adsorption parameter has a critical power of ε (we will precise this later) we observe a change in the nature of the nonlinearity. Our aim is to present this change in the case $\sigma(u) = C|u|^{q-1}u$, $0 < q < 1$, which is not differentiable at 0, and in the case when σ is the maximal monotone operator for the Heaviside function, which is a multivalued operator, and $p \in [2, n)$. In a further paper [12] we extend the arguments to the case of general maximal monotone graphs σ and $p \in (1, n)$.

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[a] *Complutense de Madrid. Inst. Mat. Interdisciplinar & Dep. Mat. Aplicada. Fac. Mat. Plaza de las Ciencias n3 28040, Madrid, Spain*

[b] *Moscow State University, Faculty of Mechanics and Mathematics, Moscow, 19992 Russia*

*e-mail: *jidiaz@ucm.es, **dgcastro@ucm.es, ***originalea@ya.ru, ****shaposh.tan@mail.ru*

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$. Denote by G_0 the unit ball centered at the origin. For $\delta > 0$ and $0 < \varepsilon \ll 1$ we define sets $\delta B = \{x | \delta^{-1}x \in B\}$ and $\tilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x, \partial\Omega) > 2\varepsilon\}$. Let $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$ and C_0 is positive number. Define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n | (a_\varepsilon G_0 + \varepsilon j) \cap \bar{\tilde{\Omega}}_\varepsilon \neq \varnothing\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$, Z^n is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$ and note that $\bar{G}_\varepsilon^j \subset Y_\varepsilon^j$ and center of the ball G_ε^j coincides with the center of the cube Y_ε^j. Define

$$\Omega_\varepsilon = \Omega \setminus \bar{G}_\varepsilon, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma_q(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \in [2, n)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε, $\gamma = \alpha(p-1)$. We suppose that $f \in L^{p'}(\Omega)$, $p' = \dfrac{p}{p-1}$.

Function $\sigma_q(\lambda)$, $q \in [0,1]$, is the maximal monotone continuous mapping [11], that depends on parameter q

$$\sigma_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda^q, & \lambda \in (0,1), \\ 1, & \lambda > 1. \end{cases} \tag{2}$$

We note that $\sigma_0(\lambda)$ is the maximal monotone mapping for the Heaviside function, i.e. multivalued function

$$\sigma_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ [0,1], & \lambda = 0, \\ 1, & \lambda > 0. \end{cases} \tag{3}$$

Boundary conditions of this type correspond to the presence of so called chemical reaction of order q on the boundary of cavities [5, 6]. The motivation to truncate the powers comes from the chemical modeling, in which concentrations impose range in [0, 1], but it also corresponds to the case $f \in L^\infty(\Omega)$, for which the solution is bounded.

Applying monotonicity tools (see, e.g., [11]) it is easy to see that problem (1) is equivalent to ask for $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, satisfying the integral inequality

$$\begin{aligned} &\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla(\phi - u_\varepsilon) dx \\ &+ \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} (\psi_q(\phi) - \psi_q(u_\varepsilon)) ds \geq \int\limits_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \end{aligned} \tag{4}$$

for any arbitrary function $\phi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, where $\psi_q(\lambda)$ for $q \in (0,1]$ is the primitive of σ_q. For $q \in (0,1]$ we have that

$$\psi_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \dfrac{\lambda^{q+1}}{q+1}, & \lambda \in (0,1), \\ \lambda - \dfrac{q}{q+1}, & \lambda > 1 \end{cases} \tag{5}$$

and if $q = 0$ then

$$\psi_0(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ \lambda, & \lambda > 0. \end{cases} \tag{6}$$

Space $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the closure in $W^{1,p}(\Omega_\varepsilon)$ of the set of infinitely differentiate functions in $\bar{\Omega}_\varepsilon$, that vanish near the boundary $\partial\Omega$.

It is well known that problem (1) has unique weak solution (see., e.g. [1, Theorem 8.5]). The following estimation is valid

$$\|u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|u_\varepsilon\|_{L^1(S_\varepsilon \cap \{x | u_\varepsilon > 1\})} \leq K, \tag{7}$$

where constant K here and below is independent of ε.

Let $H_q(\lambda)$ be the function given through functional equation

$$B_0 |H_q|^{p-2} H_q \in \sigma_q(\lambda - H_q), \tag{8}$$

where $B_0 = \text{const} > 0$. Note that, for any prescribed λ equation (8) has unique solution. In the case if $q = 0$

$$H_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda, & 0 < \lambda < B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > B_0^{-\frac{1}{p-1}} \end{cases} \tag{9}$$

and if $q \in (0,1]$ then

$$H_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ (b_q)^{-1}(\lambda), & 0 < \lambda < 1 + B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > 1 + B_0^{-\frac{1}{p-1}}, \end{cases} \tag{10}$$

where $b_q(s)$ is the strictly monotone function given for $s \geq 0$ by

$$b_q(s) \equiv s + B_0^{\frac{1}{q}} s^{\frac{p-1}{q}} = \lambda. \tag{11}$$

Note that, in both cases, $H_q(\lambda)$ is bounded and Lipschitz continuous.

Denote by $\tilde{u}_\varepsilon$ a $W^{1,p}$-extension of function u_ε (see [10]). Using estimate (7) we get following inequality

$$\|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K. \tag{12}$$

Therefore, there exists a subsequence (denoted as the original sequence), such that $\varepsilon \to 0$:

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega). \tag{13}$$

The following theorem gives a description of the limit function u.

Theorem 1. *Let* $n \geq 3$, $p \in [2, n)$, $q \in [0,1]$, $\alpha = \dfrac{n}{n-p}$, $\gamma = \alpha(p-1)$ *and* u_ε *is the weak solution of the problem* (1). *Suppose that* $H_q(\lambda)$ *is the function given by equation* (8), *in which* $B_0 = \left(\dfrac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$. *Then the limit function* u, *introduced in* (13), *is the weak solution of the problem*

$$\begin{cases} -\Delta_p u + A(n,p) |H_q(u)|^{p-2} H_q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{14}$$

which is understood as a function $u \in W_0^{1,p}(\Omega)$ *satisfying the variational inequality*

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v-u)dx + $$
$$+ A(n,p)\int_\Omega |H_q(v)|^{p-2}H_q(v)(v-u)dx \geq$$
$$\geq \int_\Omega f(v-u)dx,$$

for an arbitrary function $v \in W_0^{1,p}(\Omega)$. *Here,* $A(n,p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$, *with* ω_n *the surface area of the unit sphere in* $\mathbb{R}^n$.

Proof. (a) Consider the case $q = 0$. Denote $B_1 = B_0^{-\frac{1}{p-1}}$. Note that the integral inequality in the case when $q = 0$ has the form

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla(\phi-u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+-u_\varepsilon^+)ds \geq \int_{\Omega_\varepsilon} f(\phi-u_\varepsilon)dx, \tag{15}$$

where ϕ^+ is the positive part of function ϕ, $\phi = \phi^+ - \phi^-$. From (15) we conclude

$$\varepsilon^{-\gamma}\left\|u_\varepsilon^+\right\|_{L^1(S_\varepsilon)} \leq K. \tag{16}$$

Using the monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$ we derive inequality for u_ε

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2}\nabla \phi\nabla(\phi-u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+-u_\varepsilon^+)ds \geq \int_{\Omega_\varepsilon} f(\phi-u_\varepsilon)dx, \tag{17}$$

that is valid for an arbitrary function $\varphi \in W_0^{1,p}(\Omega)$.

Let us take a test function in inequality (17)

$$\phi(x) = v(x) - W_\varepsilon(x)H_0(v(x)),$$

where $v \in C_0^\infty(\Omega)$, $H_0(\lambda)$ is given by the formula (9), function $W_\varepsilon(x)$ is defined as follows

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \quad j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup\limits_{j\in\Upsilon_\varepsilon} T_\varepsilon^j. \end{cases} \tag{18}$$

Here, T_ε^j is the ball of radius $\varepsilon/4$ center of which coincides with the center P_ε^j of G_ε^j,

$$w_\varepsilon^j(x) = \frac{|x-P_\varepsilon^j|^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}{a_\varepsilon^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}.$$

Note that

$$W_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad W^{1,p}(\Omega) \tag{19}$$

as $\varepsilon \to 0$. Using that

$$v = H_0(v) \Leftrightarrow v \in [0, B_1],$$
$$v < H_0(v) \Leftrightarrow v < 0,$$
$$v > H_0(v) \Leftrightarrow v > B_1$$

and $\phi^+|_{S_\varepsilon} = (v - H_0(v))^+ = v - H_0(v)$ if $v > H_0(v)$ we get

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+-u_\varepsilon^+)ds \leq \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds - \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ ds. \tag{20}$$

Substituting introduced test function in inequality (17) and using (19) and (20), we get that the limit as $\varepsilon \to 0$ of the left-hand side of the inequality (17) doesn't exceed the limit of

$$\begin{aligned}&\int_{\Omega_\varepsilon}|\nabla v|^{p-2}\nabla v\nabla(v-u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds \\ &- \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ds - \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2} \\ &\times H_0(v-H_0(v)-u_\varepsilon)ds \\ &- \sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2}H_0(v-u_\varepsilon)ds.\end{aligned} \tag{21}$$

The limit of the right-hand side of inequality (17) is equal to

$$\int_\Omega f(v-u)dx. \tag{22}$$

Consider the integrals over S_ε, included in the expression (21):

$$\begin{aligned}&- \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2} H_0(v-H_0(v)-u_\varepsilon)ds \\ &+ \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds - \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ds \\ &= \varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0|^{p-2}H_0(v-H_0(v) \\ &- u_\varepsilon)ds + \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds \\ &- \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ds\end{aligned}$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-\varepsilon^{-\gamma}B_0\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds \quad (23)$$

$$-\frac{B_0\varepsilon^{-\gamma}\alpha_\varepsilon}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+ ds+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-B_0\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}B_0^{-1}(v-B_1-u_\varepsilon^+)ds$$

$$+\varepsilon^{-\gamma}B_0\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}v^{p-1}u_\varepsilon^+ ds$$

$$-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+ ds$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+(1-B_0v^{p-1}(x))ds$$

$$-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$= J^\varepsilon-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds,$$

where

$$J^\varepsilon \equiv -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$
$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+(1-B_0v^{p-1})ds\le 0, \quad (24)$$

and $\alpha_\varepsilon\to 0$ if $\varepsilon\to 0$.

Using that $\varepsilon^{-\gamma}\|u_\varepsilon^+\|_{L^1(S_\varepsilon)}\le K$, we get that the limit of the expression (21) doesn't exceed the limit of the following expression

$$\int_\Omega|\nabla v|^{p-2}\nabla v\nabla(v-u)dx$$
$$-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_{\nu_p}w_\varepsilon^j|H_0(v)|^{p-2}H_0(v)(v-u_\varepsilon)ds. \quad (25)$$

Using an equality proved in [3] we get

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0(v)|^{p-2}H_0(v-u_\varepsilon)ds$$
$$= -A(n,p)\int_\Omega|H_0(v)|^{p-2}H_0(v)(v-u)dx. \quad (26)$$

It follows from (20)–(26) that u satisfies inequality, for any $v\in W_0^{1,p}(\Omega)$,

$$\int_\Omega|\nabla v|^{p-2}\nabla v\nabla(v-u)dx+A(n,p)$$
$$\times\int_\Omega|H_0(v)|^{p-2}H_0(v)(v-u)dx\ge\int_\Omega f(v-u)dx. \quad (27)$$

Taking $v=u+\lambda w$, with $w\in W_0^{1,p}(\Omega)$ arbitrary, and making $\lambda\to 0$, since $H_0$is Lipschits continuous and bounded, we get that u is a weak solution of problem (14) for $q=0$ in the usual sense.

(b) Now we consider the case $q\in(0,1]$. In this case we make similar reasoning. We set $\phi=v-W_\varepsilon H_q(v)$ in inequality (17) as a test function, where $H_q(\lambda)$ is defined by (10). Further, we only need to explain the method of the comparison of the integrals over S_ε, included in the obtained variational inequality. Note that in this case variational inequality has the form

$$\int_{\Omega_\varepsilon}|\nabla\phi|^{p-2}\nabla\phi(\nabla\phi-\nabla u_\varepsilon)dx+\varepsilon^{-\gamma}$$
$$\times\int_{S_\varepsilon}(\psi_q(\phi)-\psi_q(u_\varepsilon))ds\ge\int_{\Omega_\varepsilon}f(\phi-u_\varepsilon)dx, \quad (28)$$

where

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(\psi_q(\varphi)-\psi_q(u_\varepsilon))ds$$

$$\le\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\in(0,1)\}}\left(\frac{(v-H_q(v))^{q+1}}{q+1}-\frac{u_\varepsilon^{q+1}}{q+1}\right)ds$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\le 0\}}\frac{(v-H_q(v))^{q+1}}{q+1}ds$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon>1\}}\left(\frac{(v-H_q)^{q+1}}{q+1}-u_\varepsilon+\frac{q}{q+1}\right)ds \quad (29)$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon\in(0,1)\}}\left(v-H_q-\frac{q}{q+1}-\frac{u_\varepsilon^{q+1}}{q+1}\right)ds$$

$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon\le 0\}}\left(v-H_q-\frac{q}{q+1}\right)ds$$

$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon>1\}}\left(v-H_q-\frac{q}{q+1}-u_\varepsilon+\frac{q}{q+1}\right)ds.$$

We substitute the test function in (17) and we consider the remaining integrals over S_ε in the left-hand side of variational inequality (17):

$$\begin{aligned}
&-\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_q|^{p-2}H_q(v-H_q(v)-u_\varepsilon)ds\\
&=-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}B_0H_q^{p-1}(v)(v-H_q(v)-u_\varepsilon^+)ds\\
&-\frac{B_0\varepsilon^{-\gamma}\alpha_\varepsilon}{1-\alpha_\varepsilon}\int\limits_{S_\varepsilon}H_q^{p-1}(v)(v-H_q(v)-u_\varepsilon^+)ds\\
&-\varepsilon^{-\gamma}\frac{B_0}{1-\alpha_\varepsilon}\int\limits_{S_\varepsilon}H_q{}^{p-1}u_\varepsilon^-ds.
\end{aligned}\tag{30}$$

Note that

$$\begin{aligned}
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}B_0H_q^{p-1}(v)(v-H_q(v)-u_\varepsilon^+)ds\\
&=-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\in(0,1)\}}(v-H_q(v))^q\\
&\quad\times(v-H_q(v)-u_\varepsilon)ds\\
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon>1\}}(v-H_q(v))^q(v-H_q(v)-u_\varepsilon)ds\\
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\le 0\}}(v-H_q(v))^{q+1}ds\\
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon\le 0\}}(v-B_1)ds\\
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon\in(0,1)\}}(v-B_1-u_\varepsilon)ds\\
&-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon>1\}}(v-B_1-u_\varepsilon)ds.
\end{aligned}\tag{31}$$

Next we compare integrals over the same subsets of S_ε in the left-hand side of inequality (17). We have

$$\begin{aligned}
I_\varepsilon&\equiv\varepsilon^{-\gamma}\int\limits_{M_\varepsilon}\left\{\frac{(v-H_q(v))^{q+1}}{q+1}-\frac{u_\varepsilon^{q+1}}{q+1}\right.\\
&\left.-(v-H_q(v))^{q+1}+(v-H_q(v))^qu_\varepsilon\right\}ds\\
&=-\varepsilon^{-\gamma}\int\limits_{M_\varepsilon}\left(\frac{q(v-H_q(v))^{q+1}}{q+1}+\frac{u_\varepsilon^{q+1}}{q+1}\right.\\
&\left.-u_\varepsilon(v-H_q(v))^q\right)ds,
\end{aligned}\tag{32}$$

where $M_\varepsilon=S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\in(0,1)\}$. Using Young's inequality we get

$$u_\varepsilon(v-H_q(v))^q\le\frac{q(v-H_q(v))^{q+1}}{q+1}+\frac{u_\varepsilon^{q+1}}{q+1}.\tag{33}$$

Therefore,

$$I_\varepsilon\le 0.\tag{34}$$

The remaining integrals over subsets of S_ε are considered in the similar way and we establish that the sum of all integrals over the corresponding subsets of S_ε is non-positive. Therefore, the limit function u satisfy variational inequality

$$\begin{aligned}
&\int\limits_\Omega|\nabla v|^{p-2}\nabla v\nabla(v-u)dx+A(n,p)\\
&\times\int\limits_\Omega H_q^{p-1}(v)(v-u)dx\ge\int\limits_\Omega f(v-u),
\end{aligned}$$

for an arbitrary function $v\in W_0^{1,p}(\Omega)$. Again taking $v=u+\lambda w$, with $w\in W_0^{1,p}(\Omega)$arbitrary, and making $\lambda\to 0$ we obtain that u is a weak solution the usual sense.

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HOMOGENIZATION OF VARIATIONAL INEQUALITIES OF SIGNORINI TYPE FOR THE p-LAPLACIAN IN PERFORATED DOMAINS WHEN $p \in (1, 2)$

J.I. DÍAZ, D. GÓMEZ-CASTRO, A.V. PODOLSKIY, AND T.A. SHAPOSHNIKOVA

Works [6, 4] are concerned with the investigation of the asymptotic behavior of the solution of the variational inequality for the p - Laplace operator, where $p \in [2, n)$ and ε-periodically perforated domain with nonlinear Robin type boundary condition. In the present work we investigate a similar homogenization problem for the p - Laplacian in the case when $p \in (1, 2)$. It is known (see [2]) that for this values of p the considered problems describe the motion of non-Newtonian fluids. This type of diffusion is also used to describe certain problems of Newtonian fluids in turbulent regime (see, e.g., [3]). The operator also has some interest in the context on non-linear elasticity.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$. Denote $Y = (-1/2, 1/2)^n$ and let G_0 be the unit ball centered at the origin. For $\delta > 0$ we define and a given set $B \subset \mathbb{R}^n$ we define $\delta B = \{x | \delta^{-1} x \in B\}$. We also define, for $j \in \mathbb{Z}^n$, $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$,

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x, \partial\Omega) > 2\varepsilon\}, \qquad G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

(where $0 < \varepsilon << 1$), $a_\varepsilon = C_0 \varepsilon^\alpha$, $\alpha = \frac{n}{n-p}$ and

$$\Upsilon_\varepsilon = \left\{ j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}_\varepsilon} \neq \emptyset \right\}.$$

It is easy to check that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ where $d > 0$ is a constant. Finally, let us define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $j \in \Upsilon_\varepsilon$ (where we point out that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and that the center of the ball G_ε^j coincides with the center of Y_ε^j) and

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \qquad S_\varepsilon = \partial G_\varepsilon, \qquad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

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In this setting we consider the following nonlinear diffusion problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $p \in (1, 2)$, $\Delta_p u \equiv div(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$ and with ν the outward unit normal to S_ε and $\gamma = \alpha(p-1)$, $f \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and σ the following maximal monotone graph

$$\sigma(\lambda) = \begin{cases} \sigma_0(\lambda), & \lambda > 0, \\ (-\infty, 0], & \lambda = 0, \\ \emptyset, & \lambda < 0, \end{cases} \tag{2}$$

where $\sigma_0 \in C^1(\mathbb{R})$, $\sigma_0(0) = 0$, $\sigma_0'(\lambda) \geq k_1 > 0$ and k_1 is a constant.

We note that boundary value problem (1) with a function such as $\sigma(\lambda)$ in the boundary condition corresponds to the problem with the one-sided restrictions, i.e. Signorini type problem

$$\begin{cases} u_\varepsilon \geq 0, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon) \geq 0 \text{ and} \\ u_\varepsilon(\partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon)) = 0, \quad \text{on } S_\varepsilon. \end{cases}$$

Let us define the following functions

$$\hat{\psi}(\lambda) = \int_0^\lambda \sigma_0(\tau)d\tau, \tag{3}$$

$$\psi(\lambda) = \begin{cases} \hat{\psi}(\lambda), & \lambda \geq 0, \\ +\infty, & \lambda < 0. \end{cases} \tag{4}$$

This convex l.s.c. function ψ has σ as its subdifferential, in the sense that

$$\psi(\lambda) - \psi(\mu) \leq \xi(\lambda - \mu), \quad \forall \lambda, \mu \in \mathbb{R}, \xi \in \sigma(\lambda) \tag{5}$$

This is typically denoted $\sigma = \partial\psi$. The weak solution of the problem (1) is defined as a function

$$u_\varepsilon \in K_\varepsilon = \{g \in W^{1,p}(\Omega_\varepsilon, \partial\Omega) : g \geq 0 \text{ a.e. on } S_\varepsilon\},$$

satisfying the integral inequality

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\hat{\psi}(\phi) - \hat{\psi}(u_\varepsilon))ds$$

$$\geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx \tag{6}$$

for any arbitrary function $\phi \in K_\varepsilon$.

Let $H(\lambda)$ be the solution of the functional inclusion

$$B_0|H|^{p-2}H \in \sigma(\lambda - H), \tag{7}$$

where $B_0 > 0$ is a constant. In the case of σ as in (2), inclusion (7) has a unique solution of the form

$$H(\lambda) = \begin{cases} H_0(\lambda), & \lambda > 0, \\ \lambda, & \lambda \leq 0, \end{cases} \tag{8}$$

where $H_0(\lambda)$ is the solution of the functional equation

$$B_0|H_0|^{p-2}H_0 = \sigma_0(\lambda - H_0). \tag{9}$$

Note that $H_0(0) = 0$. If we decompose $u = u^+ - u^-$ where $u^+, u^- \geq 0$ are the positive and negative parts of u then we have

$$H(u) = H_0(u^+) - u^-, \qquad |H(u)|^{p-2}H(u) = |H_0(u^+)|^{p-2}H_0(u^+) - |u^-|^{p-2}u^-.$$

Also,

Lemma 1. *For every $s \neq 0$, $0 < H'(s) \leq 1$. In particular, H is a Lipschitz continuous function.*

Proof. If $H_0(s) \leq 0$, since $\sigma_0(0) = 0$, $\sigma_0'(s) \geq k_1 > 0$

$$0 \geq B_0|H_0|^{p-2}H_0 = \sigma_0(s - H_0(s)) \geq k_1(s - H_0(s))$$

then $s \leq 0$. So, for $s > 0$, $H(s) = H_0(s) > 0$. Hence, for $s > 0$ $B_0H_0(s)^{p-1} = \sigma_0(s - H_0(s))$. Differentiating with respect to s, for $s > 0$

$$B_0(p-1)H_0(s)^{p-2} = \sigma_0'(s - H_0(s))(1 - H_0'(s)),$$

$$H_0'(s) = \frac{\sigma_0'(s - H_0(s))}{B_0(p-1)H_0(s)^{p-2} + \sigma_0'(s - H_0(s))},$$

which is in $(0,1)$. Since, for $s<0$, $H(s)=s$ we finish the proof. $\square$

Remark 1. *In if σ is given by (2), $H(s)\le s$ for all $s\in\mathbb{R}$. For $s\le 0$ this is obvious and for $s>0$ we point out that $H(0)=0$ and $H'(s)\le 1$.*

Let $\widetilde{u_\varepsilon}\in W_0^{1,p}(\Omega)$ be a $W^{1,p}$ – extension of u_ε, that satisfies the following inequalities

$$\|\widetilde{u_\varepsilon}\|_{W^{1,p}(\Omega)}\le K\|u_\varepsilon\|_{W^{1,p}(\Omega_\varepsilon)},\qquad \|\nabla\widetilde{u_\varepsilon}\|_{L^p(\Omega)}\le K\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)},\tag{10}$$

Considering (6) it is easy to check that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)}\le K.$$

Hence, using this inequality and estimations (10) we conclude that there exists a subsequence (denote as the original sequence), such that as $\varepsilon\to 0$

$$\widetilde{u_\varepsilon}\rightharpoonup u\qquad\text{weakly in } W_0^{1,p}(\Omega).\tag{11}$$

We will use systematically that the function

$$\Phi_p: L^p(\Omega)^N\to L^{p'}(\Omega)^N,\quad \xi\mapsto|\xi|^{p-2}\xi\tag{12}$$

is continuous in the strong topology (see [8]).

The following theorem gives us the description of function u. What is remarkable in it is that a sequence of variational inequalities converges to the solution of a single-valued quasilinear equation with a Lipschitz absorption term.

Theorem 1. *Let $\alpha=\frac{n}{n-p}$, $\gamma=\alpha(p-1)$, $p\in(1,2)$, $n\ge 3$. Suppose that $u_\varepsilon\in W^{1,p}(\Omega_\varepsilon,\partial\Omega)$ is the weak solution of the problem (1), where $\sigma(\lambda)$ is given by formula (2) and $\widetilde{u_\varepsilon}\in W_0^{1,p}(\Omega)$ is a $W^{1,p}$-extension of u_ε satisfying (10). Then, the function u defined in (11) is a weak solution of the following problem*

$$\begin{cases}-\Delta_p u+\mathcal{A}(n,p)|H(u)|^{p-2}H(u)=f, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{13}$$

where function $H(\lambda)$ is given by formula (8), $H_0(\lambda)$ is a solution of the equation (9) for $B_0=\left(\frac{n-p}{p-1}\right)^{p-1}C_0^{1-p}$, $\mathcal{A}(n,p)=\left(\frac{n-p}{p-1}\right)^{p-1}C_0^{n-p}\omega_n$ and ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

We will use the following auxiliary function W_ε defined as follows

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} T_\varepsilon^j, \end{cases}$$

where w_ε^j is the solution of the following value problem

$$\begin{cases} \Delta_p w_\varepsilon^j = 0, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 1, & x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0, & x \in \partial T_\varepsilon^j, \end{cases}$$

and T_ε^j denotes the ball of radius $\varepsilon/4$ which center coincides with the center of cube Y_ε^j. It is easy to show that

$$\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q dx \leq K\varepsilon^{n(p-q)/(n-p)}, \tag{14}$$

where $1 \leq q \leq p$. $W_\varepsilon \to 0$ in $W_0^{1,q}(\Omega)$ $\varepsilon \to 0$, for $q < p$. Also, the $W_0^{1,p}$ norm is bounded, so it has a weakly convergent subsequence. The limit of that sequence must be its $W_0^{1,q}$ limit, hence $W_\varepsilon \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$.

Proof of Theorem 1. Taking into account (3) and using the monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$, from inequality (6) we derive that u_ε satisfies the following inequality

$$\int\limits_{\Omega_\varepsilon} |\nabla \phi|^{p-2}\nabla\phi\nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma_0(\phi)(\phi - u_\varepsilon)ds \geq \int\limits_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx, \tag{15}$$

for any function $\phi \in K_\varepsilon$.

Let $v \in W^{1,\infty}(\Omega)$ and let us consider $\phi = v - W_\varepsilon H(v)$ as a test function, where $H(\lambda)$ is defined by (8). Notice that $\phi|_{S_\varepsilon} = v - H(v) \geq 0$ due to Remark 1, and hence $\phi \in K_\varepsilon$. Let us define $\psi_\varepsilon = \phi - \widetilde{u_\varepsilon}$, and rewrite (15) as $I_\varepsilon^1 + I_\varepsilon^2 \geq I_\varepsilon^3$ where

$$I_\varepsilon^1 = \int\limits_{\Omega_\varepsilon} |\nabla(v - W_\varepsilon H(v))|^{p-2} \nabla \psi_\varepsilon dx$$

$$I_\varepsilon^2 = \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma(v - H(v))\psi_\varepsilon ds \qquad I_\varepsilon^3 = \int\limits_{\Omega_\varepsilon} f\psi_\varepsilon$$

Let us define

$$\xi_1 = \Phi_p(\nabla(v - W_\varepsilon H(v))), \quad \xi_2 = \Phi_p(\nabla v) \quad \xi_3 = \Phi_p(\nabla(W_\varepsilon H(v))),$$

We write $I_\varepsilon^1 = J_\varepsilon^1 + J_\varepsilon^2 + J_\varepsilon^3$ where

$$J_\varepsilon^1 = \int\limits_{\Omega_\varepsilon} (\xi_1 - (\xi_2 - \xi_3)) \cdot \nabla \psi_\varepsilon, \quad J_\varepsilon^2 = \int\limits_{\Omega_\varepsilon} \xi_2 \cdot \nabla \psi_\varepsilon, \quad J_\varepsilon^3 = -\int\limits_{\Omega_\varepsilon} \xi_3 \cdot \nabla \psi_\varepsilon.$$

Lemma 3 below implies the inequality $|\xi_1 - (\xi_2 - \xi_3)| \leq C(|\xi_2||\xi_3|)^{\frac{p-1}{2}}$. Hence, we can write

$$\begin{aligned}
|J_\varepsilon^1| &\leq K \int\limits_{\Omega_\varepsilon} |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} \Big(|\nabla(W_\varepsilon H(v))| + |\nabla v| + |\nabla u_\varepsilon| \Big) dx \\
&\leq K \int\limits_{\Omega_\varepsilon} \Big(|\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p+1}{2}} + |\nabla v|^{\frac{p+1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} \\
&\qquad + |\nabla v|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(v))|^{\frac{p-1}{2}} |\nabla u_\varepsilon| \Big) dx \\
&\leq K \int\limits_{\Omega_\varepsilon} \Big(|\nabla W_\varepsilon|^{\frac{p+1}{2}} + |\nabla u_\varepsilon| |\nabla W_\varepsilon|^{\frac{p-1}{2}} \Big) dx
\end{aligned}$$

Applying Hölder's inequality for p on the second term

$$|J_\varepsilon^1| \leq K \left\{ \|\nabla W_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega)}^{\frac{2}{p+1}} + \|\nabla u_\varepsilon\|_{L^p(\Omega)} \left(\int_\Omega |\nabla W_\varepsilon|^{\frac{p}{2}} \right)^{\frac{p-1}{p}} \right\} \to 0$$

as $\varepsilon \to 0$ by taking into account that $\frac{p}{2}, \frac{p+1}{2} < p$ and estimate (14). Moreover, convergence (11) implies

$$\lim_{\varepsilon \to 0} J_\varepsilon^2 = \int\limits_\Omega |\nabla v|^{p-2} \nabla v \nabla (v - u) dx. \tag{16}$$

Consider J_ε^3. Splitting $\nabla(W_\varepsilon H(v)) = W_\varepsilon \nabla H(v) + H(v)\nabla W_\varepsilon$, since $W_\varepsilon \nabla H(v) \to 0$ in $L^p(\Omega)^N$, Φ_p is continuous and ψ_ε is bounded in $W^{1,p}$ we have that

$$\lim_{\varepsilon\to 0} J_\varepsilon^3 = -\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} \Phi_p(H(v)\nabla W_\varepsilon)\cdot\nabla\psi_\varepsilon.$$

On the other hand, it is easy to check that

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\cdot\nabla(|H(v)|^{p-2}H(v)\psi_\varepsilon) = \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon} \Phi_p(H(v)\nabla W_\varepsilon)\cdot\nabla\psi_\varepsilon.$$

Hence

$$J_\varepsilon^3 = -\int\limits_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\cdot\nabla[|H(v)|^{p-2}H(v)(v - W_\varepsilon H(v) - u_\varepsilon)]dx + \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. From Green's formula we derive that $J_\varepsilon^3 = K_\varepsilon^1 + K_\varepsilon^2$

$$K_\varepsilon^1 = -\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H(v)|^{p-2}H(v)(v - H(v) - u_\varepsilon)ds$$

$$K_\varepsilon^2 = -\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H(v)|^{p-2}H(v)(v - u_\varepsilon)ds + \alpha_\varepsilon$$

Taking into account that

$$\partial_{\nu_p} w_\varepsilon^j\Big|_{\partial G_\varepsilon^j} = \frac{(n-p)\varepsilon^{-\frac{n}{n-p}}}{(p-1)C_0(1-\kappa_\varepsilon)}, \tag{17}$$

$$\partial_{\nu_p} w_\varepsilon^j\Big|_{\partial T_\varepsilon^j} = -\frac{(n-p)2^{2(n-1)/(p-1)}C_0^{(n-p)/(p-1)}\varepsilon^{1/(p-1)}}{(p-1)(1-\kappa_\varepsilon)}, \tag{18}$$

where $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$ and that $\gamma = \alpha(p-1)$ we obtain, taking into account (9) that

$$\begin{aligned} K_\varepsilon^1 + I_\varepsilon^2 &= \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\Big[\sigma(v - H(v)) - B_0|H(v)|^{p-2}H(v)\Big](v - H(v) - u_\varepsilon)ds + \beta_\varepsilon \\ &= \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\Big[|v_-|^{p-2}v_-\Big](v_+ - H(v_+) - u_\varepsilon)ds + \beta_\varepsilon \\ &= \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\Big[|v_-|^{p-2}v_-\Big](-u_\varepsilon)ds + \kappa_\varepsilon \le \beta_\varepsilon \end{aligned}$$

where $\beta_\varepsilon \to 0$ as $\varepsilon \to 0$ since $u_\varepsilon \geq 0$ on S_ε.

We will use the next lemma to pass to the limit in K_ε^1 (see [10]).

Lemma 2. *Let $u_\varepsilon \in H_0^1(\Omega)$ and $u_\varepsilon \rightharpoonup u_0$ as $\varepsilon \to 0$ in $H_0^1(\Omega)$, then*

$$\left| 2^{2(n-1)}\varepsilon \sum_{j=1}^{N_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} u_\varepsilon dS - \omega_n \int\limits_\Omega u_0 dx \right| \to 0,\ \varepsilon \to 0,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Due Lemma 2 we deduce that

$$K_\varepsilon^1 \to \mathcal{A}(n,p) \int\limits_\Omega |H(v)|^{p-2} H(v)(v-u)dx, \qquad \text{as } \varepsilon \to 0 \tag{19}$$

where $\mathcal{A}(n,p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$. From (15)-(19) we derive that u satisfies following inequality

$$\int\limits_\Omega |\nabla u|^{p-2}\nabla v \nabla(v-u)dx + \mathcal{A}(n,p)\int\limits_\Omega |H(v)|^{p-2}H(v)|^{p-2}(v-u)dx$$
$$\geq \int\limits_\Omega f(v-u)dx. \tag{20}$$

This inequality implies that u is a weak solution of the problem (13). $\square$

In the next theorem we will prove convergence in the norm of space $W_0^{1,p}(\Omega_\varepsilon)$ of the solution of the problem (1) with a corrector to the solution of the homogenized problem.

Theorem 2. *Let $\alpha = \frac{n}{n-p}$, $\gamma = \alpha(p-1)$, $p \in (1,2)$, $n \geq 3$. Suppose that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon)$ is a weak solution of the problem (1) and u is a weak solution of the problem (13) possessing the additional smoothness $u \in W^{1,\infty}(\Omega)$. Then*

$$\|\nabla(u_\varepsilon + W_\varepsilon H(u) - u)\|_{L^p(\Omega^\varepsilon)} \to 0, \qquad \text{as } \varepsilon \to 0 \tag{21}$$

In particular, since $W_\varepsilon \to 0$ in $W^{1,q}(\Omega)$ for $q < p$, we have, for all $q < p$

$$\|\nabla(u_\varepsilon - u)\|_{L^q(\Omega_\varepsilon)} \to 0, \qquad \varepsilon \to 0$$

Remark 2. *Under some smoothness hypothesis of* σ_0 *and* f, $u \in W^{1,\infty}(\Omega)$ *is often achieve. See* [5, 1, 9, 7].

Proof of Theorem 2. Inequality (6) implies that

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma_0(\phi)(\phi - u_\varepsilon)ds$$
$$\geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx. \qquad (22)$$

In inequality (22) we substitute $\phi = u - W_\varepsilon H(u)$ and in the weak formulation of problem (13), namely,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v + \mathcal{A}(n,p)\int_\Omega |H(u)|^{p-2}H(u)v = \int_\Omega fv$$

we take, as a test function, $v = -\Psi_\varepsilon$, where $\Psi_\varepsilon = u - W_\varepsilon H(u) - \widetilde{u_\varepsilon}$ and $\widetilde{u_\varepsilon}$ is a $W^{1,p}$-extension u_ε on Ω. Let us define,

$$\xi_1^\varepsilon = \Phi_p(\nabla u_\varepsilon), \qquad \xi_2 = \Phi_p(\nabla u)$$

By adding the two expressions we obtain $I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon \geq I_4^\varepsilon$ where

$$I_1^\varepsilon = \int_{\Omega_\varepsilon} (\xi_1^\varepsilon - \xi_2)\cdot\nabla\Psi_\varepsilon, \qquad I_2^\varepsilon \int_{G_\varepsilon} \xi_2 \cdot \nabla\Psi_\varepsilon$$
$$I_3^\varepsilon = \varepsilon^{-\gamma}\int_{S_\varepsilon} \sigma(u - H(u))\Psi_\varepsilon - \mathcal{A}(n,p)\int_\Omega |H(u)|^{p-2}H(u)\Psi_\varepsilon$$
$$I_4^\varepsilon = \int_{G_\varepsilon} f\Psi_\varepsilon$$

It is clear that $I_2^\varepsilon, I_4^\varepsilon \to 0$ as $\varepsilon \to 0$ due to weak convergence and the fact that $|G_\varepsilon| \to 0$. We define

$$\xi_3^\varepsilon = \Phi_p(\nabla(W_\varepsilon H(u))), \qquad \xi_4^\varepsilon = \Phi_p(\nabla(u - W_\varepsilon H(u)))$$

We decompose $I_1^\varepsilon = J_1^\varepsilon + J_2^\varepsilon + J_3^\varepsilon$

$$J_1^\varepsilon = \int_{\Omega_\varepsilon} (\xi_1^\varepsilon - \xi_4^\varepsilon)\cdot\nabla\Psi_\varepsilon \qquad J_2^\varepsilon = \int_{\Omega_\varepsilon} (\xi_4^\varepsilon - \xi_2 + \xi_3^\varepsilon)\cdot\nabla\Psi_\varepsilon \qquad J_3^\varepsilon = -\int_{\Omega_\varepsilon} \xi_3^\varepsilon \cdot \nabla\Psi_\varepsilon$$

Applying Lemma 3 we have that

$$|J_2^\varepsilon| \leq C \int\limits_{\Omega_\varepsilon} |\nabla u|^{\frac{p-1}{2}} |\nabla(W_\varepsilon H(u))|^{\frac{p-1}{2}} |\nabla(u - W_\varepsilon H(u) - u_\varepsilon)| dx \to 0, \text{ as } \varepsilon \to 0.$$

On the other hand, we can write

$$\begin{aligned} J_3^\varepsilon &= -\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^{p-2} \nabla W_\varepsilon \nabla \left(|H(u)|^{p-2} H(u) \Psi_\varepsilon \right) + \delta_\varepsilon \\ &= -\sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H(u)|^{p-2} H(u) \Psi_\varepsilon ds - \\ &\quad - \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial T_\varepsilon^j} \partial_{\nu_p} w_\varepsilon^j |H(u)|^{p-2} H(u) \Psi_\varepsilon ds + \delta_\varepsilon, \end{aligned}$$

where $\delta_\varepsilon \to 0$. Therefore, $J_3^\varepsilon + I_3^\varepsilon \to 0$ as $\varepsilon \to 0$, due to the explicit expression of $\partial_{\nu_p} w_\varepsilon^j$ and H. So, finally, $J_1^\varepsilon \to 0$. We will use the following inequality (see [2]). For all $1 < p < 2$ and $\xi, \eta \in \mathbb{R}^n$

$$C \frac{|\xi - \eta|^2}{|\xi|^{2-p} + |\eta|^{2-p}} \leq (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta). \tag{23}$$

Hence, for $\xi = \xi_1^\varepsilon$ and $\eta = \xi_4^\varepsilon$, we deduce that

$$C \int\limits_{\Omega_\varepsilon} \frac{|\nabla(u_\varepsilon - u + W_\varepsilon H(u))|^2}{|\nabla u_\varepsilon|^{2-p} + |\nabla(u - W_\varepsilon H(u))|^{2-p}} \leq \int\limits_{\Omega_\varepsilon} (\xi_1^\varepsilon - \xi_4^\varepsilon) \cdot \nabla \Psi_\varepsilon = J_1^\varepsilon \to 0$$

as $\varepsilon \to 0$. Using Holder's inequality (21), which concludes the proof. $\square$

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Appendix A. An auxiliary lemma

Lemma 3. *Let $p \in (1,2)$, $n \geq 2$. Then there exists constant $C = C(n,p)$ such that for all $a, b \in \mathbb{R}^n$ following inequality is valid*

$$\Big| |a-b|^{p-2}(a-b) - (|a|^{p-2}a - |b|^{p-2}b) \Big| \leq C \Big(|a||b| \Big)^{\frac{p-1}{2}}.$$

Proof. Without loss of generality we can assume that $|a| \geq |b| > 0$. Let $u = \frac{a}{|a|}$, $v = \frac{b}{|b|}$, $|u| = |v| = 1$, $\xi = u \cdot v$, $\xi \in [-1, 1]$, $k = \frac{|a|}{|b|} \geq 1$. The desired inequality in written in these new variables takes the following form

$$||ku - v|^{p-2}(ku - v) - (k^{p-1}u - v)| \leq Ck^{(p-1)/2}.$$

By squaring this inequality we get

$$\begin{aligned} \mathfrak{K}(k, \xi) = (k^2 - 2k\xi + 1)^{p-1} + k^{2(p-1)} + 1 - 2k^{p-1}\xi \\ - 2(k^2 - 2k\xi + 1)^{(p-2)/2}(k^p + 1 - k\xi - k^{p-1}\xi) \leq C^2 k^{p-1}. \end{aligned}$$

Consider function

$$\begin{aligned} f(k, \xi) = \frac{\mathfrak{K}(k, \xi)}{k^{p-1}} = k^{p-1}\Big(1 - \frac{2\xi}{k} + \frac{1}{k^2}\Big)^{p-1} + k^{p-1} + k^{1-p} - 2\xi - \\ - 2\Big(1 - \frac{2\xi}{k} + \frac{1}{k^2}\Big)^{(p-2)/2}(k^{p-1} - \xi - k^{p-2}\xi + k^{-1}). \end{aligned}$$

Decomposing functions $(1 - 2\xi/k + 1/k^2)^\beta$ for $\beta = p - 1, (p-2)/2$ in Taylor series as $k \to \infty$, $k > 1 + \sqrt{2}$, and equating the coefficients of corresponding degrees, we obtain

$$f(k, \xi) = \alpha k^{1-p} + \beta k^{p-2} + o\Big(\frac{1}{k}\Big),$$

where α and β depend only on p and ξ. Hence, $f(k, \xi) \to 0$ as $k \to \infty$. Thus there exists $k_1 > 1 + \sqrt{2}$ such that $f(k, \xi) < 1$ for all $k > k_1$, $|\xi| \leq 1$. It's easy to show that function $f(k, \xi)$ is continuous on the set $D = \{(k, \xi) | 1 \leq k \leq k_1, |\xi| \leq 1\}$. So there exists a positive constant M that depends on p such that $\max\limits_{(k.\xi) \in D} |f(k, \xi)| \leq M$. Hence, function $|f|$ is bounded by $\max(M, 1)$ for all permissible k and ξ. □

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Dpto. Matemática Aplicada & Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid
E-mail address: jidiaz@ucm.es

E-mail address: dgcastro@ucm.es

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On the asymptotic limit of the effectiveness of reaction–diffusion equations in periodically structured media

J.I. Díaz [a,b,*], D. Gómez-Castro [a,b], A.V. Podolskii [c], T.A. Shaposhnikova [c]

[a] *Dpto. de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain*
[b] *Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Spain*
[c] *Moscow State University, Faculty of Mechanics and Mathematics, Moscow, 19992 Russia*

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ABSTRACT

This paper addresses an investigation of the asymptotic behaviour as $\varepsilon \to 0$ of the solution to the boundary value problem associated with the p-Laplace operator in an ε-periodically structured domain with a nonlinear Robin-type condition specified on the boundary of the periodic subdomains. This kind of domains include the so called perforated media as well as the case of isolated particles distributed in a periodic way. This second case arises quite often in Chemical Engineering. Here we consider a non-critical size of the particles. The objective of this paper is twofold. First we study the homogenization of solutions in the case of a continuous nonlinear reaction term on the boundary of the periodic structure. Then, we move to studying the homogenization of the effectiveness factor of the reactor, which is of relevance in Chemical Engineering.

0. Introduction

We will study asymptotic behaviour as $\varepsilon \to 0$ of the solution to the boundary value problem associated with the p-Laplace operator in an ε-periodically structured domain with a nonlinear Robin-type condition specified on the boundary of the periodic subdomains. This kind of domains include the so called perforated media as well as the case of isolated particles distributed in a periodic way. This second case arise quite often in Chemical Engineering. Here we consider a non-critical size of the perforations or particles. The objective of this paper is twofold. First, a homogenized problem is constructed and a theorem is proved stating weak convergence as $\varepsilon \to 0$ of the solution of the original problem to the solution of the homogenized one. The closest papers in the literature are [31,32] where the case $p = 2$ was considered, [17–19,26] dedicated to the

* Corresponding author at: Dpto. de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain.

E-mail addresses: jidiaz@ucm.es (J.I. Díaz), dgcastro@ucm.es (D. Gómez-Castro), originalea@ya.ru (A.V. Podolskii), shaposh.tan@mail.ru (T.A. Shaposhnikova).

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case $2 < p < n$ and [11] where the case $p > n$ was investigated. In contrast to the mentioned papers we consider here that the reaction function σ needs not be smooth. In order to achieve this result we introduce uniform approximation arguments, that allow us to deal with such reaction functions.

The case when the size of particles is non critical is characterized by the fact that the homogenized problem contains the same nonlinearity as the nonhomogeneous problem. As we will see, if the condition is $\partial_n u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = 0$ on the boundary of the inclusions (see (3)), then the homogenized interior equation contains a reaction term of the form $\lambda\sigma(u)$ (see (19)). However, there are critical cases in which the nature of the nonlinearity changes. More precisely, the limit equation for the interior contains a reaction term of the form $\lambda H(u)$, where H is different from σ, instead of the term $\lambda\sigma(u)$ in (19) (see [9,10,12,17–20,25,29,32]). For further reference see [18,17,19]. In the critical case, due to the technique used, the considered inclusions are balls whereas in this noncritical case a general shape is considered.

The second main result of the paper is the analysis of the asymptotic limit of the *effectiveness* functional (as introduced by Aris, see [2,3]), which extends results in [13,14] to the cases $p \neq 2$ and σ Hölder continuous.

1. Statement of results

1.1. Problem setting

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = (-\frac{1}{2}, \frac{1}{2})^n$. Let G_0 be a smooth open set such that $\overline{G}_0 \subset Y$. For $\delta > 0$ and $B \subset \mathbb{R}^n$ let $\delta B = \{x \in \mathbb{R}^n : \delta^{-1}x \in B\}$. For $\varepsilon > 0$ we define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$, where ρ is the distance function. Let $a_\varepsilon > 0$, define the set of inclusions

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, $\mathbb{Z}^n$ is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$, and note that $\overline{G}_\varepsilon^j \subset \overline{Y}_\varepsilon^j$. Finally, we define

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Notice that the number of inclusions is of the order of ε^{-n}, in the sense that

$$\lim_{\varepsilon \to 0} \frac{|\Upsilon_\varepsilon|}{\varepsilon^{-n}} = |\Omega|. \tag{1}$$

Throughout this paper we will write

$$a_\varepsilon \ll b_\varepsilon \iff \lim_{\varepsilon \to 0} a_\varepsilon b_\varepsilon^{-1} = 0$$
$$a_\varepsilon \sim b_\varepsilon \iff \lim_{\varepsilon \to 0} a_\varepsilon b_\varepsilon^{-1} \in (0, +\infty).$$

We will consider that the sizes of the particles are smaller than their repetition, in the sense that

$$a_\varepsilon \ll \varepsilon. \tag{2}$$

Sometimes, this case is known as *tiny holes* (in our case they can be thought of as tiny particles). We consider the problem of nonlinear diffusion in Ω_ε with a nonlinear reaction taking place on S_ε:

$$\begin{cases} -\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) = \beta(\varepsilon) g, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε , σ is a continuous nondecreasing function such that $\sigma(0) = 0$, $f \in L^{p'}(\Omega)$ and $g \in W^{1,\infty}(\Omega)$ and, for every $\varepsilon > 0$, $\beta(\varepsilon)$ is a nonnegative constant.

This problem can be obtained as a change of variable $u = 1 - w$, $\sigma(u) = \tilde{\sigma}(1) - \tilde{\sigma}(w)$ of the following problem, which appears in Chemical Engineering in the design of fixed-bed reactors (see, for example, [30])

$$\begin{cases} -\Delta_p w_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} w_\varepsilon + \beta(\varepsilon)\tilde{\sigma}(w_\varepsilon) = 0, & x \in S_\varepsilon, \\ w_\varepsilon = 1, & x \in \partial\Omega. \end{cases} \tag{4}$$

In this setting, a typical nonlinearity in the applications is $\tilde{\sigma}(w) = |w|^{q-1}w$ with $0 < q < 1$. This kinetics is not locally Lipschitz and therefore the approaches in the literature do not apply directly to this case.

A quantity of great interest in the applications is the *effectiveness*, which can be expressed as

$$\mathcal{E}_\varepsilon(\Omega, G_0) = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \tilde{\sigma}(w_\varepsilon)\, \mathrm{d}S, \tag{5}$$

in the nonhomogeneous case and as

$$\mathcal{E}(\Omega, G_0) = \frac{1}{|\Omega|} \int_{\Omega} \tilde{\sigma}(w)\, \mathrm{d}x, \tag{6}$$

in the homogenized case. It represents the ratio of the actual amount of reactant consumed per unit time in Ω to the amount that would be consumed if the interior concentration were everywhere equal to the ambient concentration. A high effectiveness is desirable in most applications. For isothermal and endothermic reactions, we see that $0 \le \mathcal{E}_\varepsilon, \mathcal{E} < 1$. This definition was introduced by Aris in the linear case ($p = 2$ and $\sigma = \lambda u$, see [1,21,3]). The study of this functional is equivalent to the study of the *ineffectiveness*

$$\eta_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u_\varepsilon)\, \mathrm{d}S, \qquad \eta = \frac{1}{|\Omega|} \int_{\Omega} \sigma(u)\, \mathrm{d}x. \tag{7}$$

The mathematical properties have widely been studied, see [4–8]. The aim of this paper is to prove that $\eta_\varepsilon \to \eta$ as $\varepsilon \to 0$.

1.2. Weak formulations

Let us define the energy functional

$$J_\varepsilon(v) = \frac{1}{p} \int_{\Omega_\varepsilon} |\nabla v|^p\, \mathrm{d}x + \beta(\varepsilon) \int_{S_\varepsilon} \Phi(v)\, \mathrm{d}S - \int_{\Omega_\varepsilon} f v\, \mathrm{d}x - \beta(\varepsilon) \int_{S_\varepsilon} g v\, \mathrm{d}S \tag{8}$$

where $\Phi(s) = \int_0^s \sigma(\tau)d\tau$. Its subdifferential $A_\varepsilon = \partial J_\varepsilon$ is given by

$$\langle A_\varepsilon v, w\rangle = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla w \,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(v)w \,\mathrm{d}S - \int_{\Omega_\varepsilon} fw \,\mathrm{d}x - \beta(\varepsilon)\int_{S_\varepsilon} gw \,\mathrm{d}S. \tag{9}$$

We say that u_ε is a weak solution of (3) if $A_\varepsilon u_\varepsilon = 0$. However, $\sigma(u_\varepsilon)$ is usually not a well behaved sequence (in the sense that it may not converge in $H^1(\Omega)$). We would rather work with an equivalent formulation that does not include it. In this direction, we have the following characterization of minimizers:

Lemma 1 *(Chapter 1 in [16]). Let X be a reflexive Banach space, $J : X \to (-\infty, +\infty]$ be a convex functional $A = \partial J : X \to \mathcal{P}(X')$ be its subdifferential. Then the following are equivalent:*

i) *u is a minimizer of J,*
ii) *$u \in D(A)$ and $0 \in Au$.*

If either hold, then

iii) *For every $v \in D(A)$ and $\xi \in Av$*

$$\langle \xi, v-u\rangle \geq 0. \tag{10}$$

Furthermore, assume that J is Gâteaux-differentiable on X and A is continuous on X then iii) *is also equivalent to* i)*.*

Remark 1. Naturally, if there is uniqueness of iii) then the i)–iii) are also equivalent.

Remark 2. One should not confuse condition iii) with the Stampacchia formulation (see e.g. [6]). For a bilinear form a and a linear function F this function is

$$a(u, v-u) \geq G(v-u) \tag{11}$$

for all v in the correspondent space, whereas with this formulation we have $a(v, v-u)$. The advantage of the representation we consider is that one of the elements can be taken constant as $\varepsilon \to 0$.

We will say that u_ε is a weak solution of (3) if it is a minimizer of J_ε in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

Proposition 1 *([26]). Let $p > 1$. Then there exists an extension operator*

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W_0^{1,p}(\Omega). \tag{12}$$

Furthermore, there exists a constant C independent of ε such that

$$\|\nabla P_\varepsilon u_\varepsilon\|_{L^p(\Omega)} \leq C\|\nabla u_\varepsilon\|_{L^p(\Omega)}. \tag{13}$$

Hence, there exists a subsequence of the original sequence $P_\varepsilon u_\varepsilon$ that admits a weak $W_0^{1,p}(\Omega)$ limit, which we will define as u. The aim of this paper is to characterize u.

1.3. Homogenization of solutions for $1 < p < n$

We state two approximation lemmas, which are key to our arguments.

Lemma 2. *Let $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing such that $\sigma(0) = 0$. Then there exists $\sigma_\varepsilon \in C^1(\mathbb{R})$ non decreasing such that $\sigma_\varepsilon(0) = 0$ and $\|\sigma - \sigma_\varepsilon\|_\infty \leq \varepsilon$.*

Let us define the critical values of a_ε and β, for $1 < p < n$

$$a_\varepsilon^* = \varepsilon^{\frac{n}{n-p}}, \qquad \beta^*(\varepsilon) = a_\varepsilon^{-(n-1)}\varepsilon^n, \tag{14}$$

which separates different asymptotic behaviours of the solution. We focus on the cases $a_\varepsilon \gg a_\varepsilon^*$, since the critical case is $a_\varepsilon \sim a_\varepsilon^*$. Notice that

$$|S_\varepsilon| = |\Upsilon_\varepsilon||a_\varepsilon \partial G_0| = a_\varepsilon^{n-1}|\Upsilon_\varepsilon||\partial G_0|. \tag{15}$$

Taking into account (1) we get

$$\frac{|S_\varepsilon|}{\beta^*(\varepsilon)|\Omega||\partial G_0|} \to 1. \tag{16}$$

The value β^* separates the behaviours as shown by the following theorem. In fact, let us define

$$\beta_0 = |\partial G_0| \lim_{\varepsilon \to 0} \beta(\varepsilon)\beta^*(\varepsilon)^{-1}. \tag{17}$$

Theorem 1. *Let $1 < p < n$, $g \in W^{1,\infty}(\Omega)$, $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, $\sigma \in \mathcal{C}(\mathbb{R})$ nondecreasing such that $\sigma(0) = 0$ and*

$$|\sigma(v)| \leq C(1 + |v|^{p-1}). \tag{18}$$

Then the following results hold:

i) *Let $\beta_0 < +\infty$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is the unique solution of*

$$\begin{cases} -\Delta_p u + \beta_0 \sigma(u) = f + \beta_0 g & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{19}$$

ii) *Let $\beta_0 = +\infty$, $g = 0$ and $\sigma \in \mathcal{C}^1$. Then, up to a subsequence $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and u satisfies*

$$u(x) \in \sigma^{-1}(0) \tag{20}$$

a.e. in Ω. In other words, $\sigma(u(x)) = 0$ for a.e. $x \in \Omega$. In particular, if σ is strictly increasing then $u = 0$.

Remark 3. In particular, if $\beta_0 = 0$ then the limit problem does not contain any reaction term. If $a_\varepsilon = C_0\varepsilon^\alpha$ and $\beta(\varepsilon) = \varepsilon^{-\gamma}$ we have

$$\alpha \in \left(1, \frac{n}{n-p}\right)$$

$$\beta_0 = \begin{cases} 0 & \gamma < \alpha(n-p) - n, \\ C_0^{n-1}|\partial G_0| & \gamma = \alpha(n-p) - n, \\ +\infty & \gamma > \alpha(n-p) - n. \end{cases}$$

Remark 4. The same result holds for $p = n$, where the condition on the size a_ε is

$$\varepsilon^{\frac{n}{n-1}} \ln(a_\varepsilon^{-1}\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0 \tag{21}$$

(see [27]) and for $p > n$, where a critical size of inclusions doesn't exist so there is no condition on a_ε^* (see [11]). We can write the critical size for any $p > 1$ as:

$$a_\varepsilon^* = \begin{cases} \varepsilon^{\frac{n}{n-p}} & \text{if } 1 < p < n, \\ \varepsilon e^{-\left(\frac{1}{\varepsilon}\right)^{1-\frac{1}{n}}} & \text{if } p = n, \\ 0 & \text{if } p > n. \end{cases} \tag{22}$$

The value of β^* is still

$$\beta^*(\varepsilon) = a_\varepsilon^{-(n-1)} \varepsilon^n. \tag{23}$$

1.4. Homogenization of the effectiveness factor when $p > 1$

We conclude by stating a theorem on homogenization of the *effectiveness* functional that improves previous results by the authors (see [13,14]). We give conditions so that

$$\frac{1}{|S_\varepsilon|} \int\limits_{S_\varepsilon} \sigma(u_\varepsilon)\, \mathrm{d}S \to \frac{1}{|\Omega|} \int\limits_{\Omega} \sigma(u)\, \mathrm{d}x \qquad \text{as } \varepsilon \to 0. \tag{24}$$

To achieve this we need a stronger approximation result for the family of Hölder-continuous functions.

Remark 5. If I is a bounded interval then $\mathcal{C}^1(I) \subset C^{0,\alpha}(I)$. This is not true if I is unbounded. For example, all functions in $\mathcal{C}^{0,\alpha}(\mathbb{R})$ are sublinear. We introduce the following condition

$$|\sigma(t) - \sigma(s)| \le C(|t-s|^\alpha + |t-s|^p) \qquad \forall t, s \in \mathbb{R}, \tag{25}$$

that represents "local Hölder" continuity, in the sense that there is no need for the function to be differentiable. On the other hand, as $|s-t| \to +\infty$, the function σ behaves like a power, and then σ can be non sublinear.

Lemma 3. *Let $\sigma \in \mathcal{C}(\mathbb{R})$, nondecreasing and there exists $0 < \alpha \le 1$, $p > 1$ such that (25) holds. Then, for every $0 < \varepsilon < \frac{1}{4C}$ there exists $\sigma_\varepsilon \in \mathcal{C}(\mathbb{R})$ (piecewise linear) such that*

$$\|\sigma_\varepsilon - \sigma\|_{\mathcal{C}(\mathbb{R})} \le \varepsilon, \tag{26}$$

$$0 \le \sigma_\varepsilon' \le D\varepsilon^{1-\frac{1}{\alpha}}, \tag{27}$$

where D depends only on C, α, p.

Theorem 2. *Let $p > 1$, $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, $\beta \sim \beta^*$ and σ be continuous such that $\sigma(0) = 0$. Let u_ε and u be the solutions of (3) and (19). Lastly, assume either:*

i) *σ is uniformly Lipschitz continuous ($\sigma' \in L^\infty$), or*
ii) *$\sigma \in \mathcal{C}(\mathbb{R})$ and there exists $0 < \alpha \le 1$ and $q > 1$ such that we have (25) and*

$$(\sigma(t) - \sigma(s))(t - s) \ge C|t - s|^q, \qquad \forall t, s \in \mathbb{R}. \tag{28}$$

Then (24) holds.

Remark 6. If $0 < q < 1$ root functions: $\sigma(s) = |s|^{q-1}s$ do not satisfy (28). However, the following cut-off to a linear function does satisfy (28):

$$\sigma(s) = \begin{cases} |s|^{q-1}s & |s| \le s_0, \\ \sigma_0 + \lambda s & |s| > s_0, \end{cases} \qquad \text{where } s_0^q = \sigma_0 + \lambda s_0. \tag{29}$$

Hence, the results in this paper hold for $\sigma(s) = |s|^{q-1}s$ where $0 < q < 1$ holds, at least for bounded solutions. We point out that in many context in the applications u is bounded (e.g. if f is a bounded function). For example, in Chemical Engineering u typically represents a concentration, so $0 \le u \le 1$.

2. Auxiliary results and estimates

2.1. Estimates on the boundary integrals

First let us introduce a uniform trace information in S_ε.

Proposition 2 *([26]). Let $p > 1$ and assume (2). Then*

i) *There exists C, independent of ε, such that, for $u \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, it holds that*

$$\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^p \, \mathrm{d}S \le C \int_{\Omega_\varepsilon} |\nabla u|^p \, \mathrm{d}x. \tag{30}$$

ii) *If $v_\varepsilon \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ and $a_\varepsilon^* \ll a_\varepsilon \ll \varepsilon$, then*

$$\beta^*(\varepsilon) \int_{S_\varepsilon} v_\varepsilon \, \mathrm{d}S \to |\partial G_0| \int_\Omega v \, \mathrm{d}x. \tag{31}$$

Remark 7. Notice that the natural trace on S_ε, $\int_{S_\varepsilon} \cdot \, \mathrm{d}S$, is not well behaved in order to pass to the limit (in the sense if $u_0 \in H_0^1(\Omega)$ and $u_0(x) \ge C_0 > 0$ in some set $A \subset \Omega$ of positive measure, then $\int_{S_\varepsilon} u_0 \, \mathrm{d}S \to +\infty$ as $\varepsilon \to 0$). However, the average over S_ε, $\frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} \cdot \, \mathrm{d}S$, behaves much better, as it is shown by Proposition 2.

Lemma 4. *Let $0 < r < s$. Then, there exists C, independent of ε, such that*

$$\left(\beta^*(\varepsilon) \int_{S_\varepsilon} |u|^r \, \mathrm{d}S \right)^{\frac{1}{r}} \le C \left(\beta^* \int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{1}{s}}. \tag{32}$$

Proof. Let $q = \frac{s}{r} > 1$. Then $q' = \frac{s}{s-r}$. Applying Hölder's inequality we find that

$$\int_{S_\varepsilon} |u|^r \, \mathrm{d}S \le C \left(\int_{S_\varepsilon} |u|^s \, \mathrm{d}S \right)^{\frac{r}{s}} \left(\int_{S_\varepsilon} 1^{\frac{s}{s-r}} \, \mathrm{d}S \right)^{\frac{s-r}{s}}$$

$$\begin{aligned}\beta^*(\varepsilon)\int_{S_\varepsilon}|u|^r\,\mathrm{d}S &\le C\beta^*(\varepsilon)^{\frac{1}{q}+\frac{1}{q'}}\left(\int_{S_\varepsilon}|u|^s\,\mathrm{d}S\right)^{\frac{r}{s}}|S_\varepsilon|^{\frac{s-r}{s}}\\ &\le C\left(\beta^*(\varepsilon)\int_{S_\varepsilon}|u|^s\,\mathrm{d}S\right)^{\frac{r}{s}}(\beta^*(\varepsilon)|S_\varepsilon|)^{\frac{s-r}{s}}\\ &\le C\left(\beta^*(\varepsilon)\int_{S_\varepsilon}|u|^s\,\mathrm{d}S\right)^{\frac{r}{s}},\end{aligned}$$

which concludes the result. □

With these results we can prove that

Proposition 3. *Let $p > 1$. Then, for every $\varepsilon > 0$ there exists a unique weak solution of (3) $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Furthermore, there exists a constant C independent of ε such that*

$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \le C(\|f\|_{L^{p'}(\Omega_\varepsilon)} + \beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|g\|_{L^\infty(\mathbb{R})}). \tag{33}$$

Proof. Considering u_ε as a test function in the weak formulation of (3)

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^p \le \int_{\Omega_\varepsilon} f u_\varepsilon + \beta(\varepsilon)\int_{S_\varepsilon} g u_\varepsilon \tag{34}$$

$$\le \int_{\Omega_\varepsilon} f u_\varepsilon + \beta(\varepsilon)\beta^*_\varepsilon(\varepsilon)^{-1}\beta^*_\varepsilon(\varepsilon)\int_{S_\varepsilon} g u_\varepsilon \tag{35}$$

$$\le C(\|f\|_{L^{p'}} + \beta(\varepsilon)\beta^*_\varepsilon(\varepsilon)^{-1}\|g\|_{L^\infty})\|\nabla u_\varepsilon\|_{L^p}. \tag{36}$$

This proves the result. □

2.2. Characterization of solutions

The proof of the *furthermore* statement in Lemma 1 can be found in [16]. In fact we state the following characterization, which could improve the regularity required, but that we do not apply due to the applied homogenization techniques.

Lemma 5 *(Proposition 2.2 of Chapter II in [16]). Let us assume that $J = J_1 + J_2$ and J_1 and J_2 being l.s.c. convex functions on a convex set $\mathcal{C}$ into $\mathbb{R}$, J_1 being Gâteaux-differentiable with differential J_1'. Then if $u \in \mathcal{C}$, the following three conditions are equivalent:*

i) *u is a minimizer of J;*
ii) *For every $v \in \mathcal{C}$*

$$\langle J_1'(u), v-u\rangle + J_2(v) - J_2(u) \ge 0; \tag{37}$$

iii) *For every $v \in \mathcal{C}$*

$$\langle J_1'(v), v-u\rangle + J_2(v) - J_2(u) \ge 0. \tag{38}$$

We have the following lemma, which is classical (see, e.g. [6]):

Lemma 6. *Let $1 < p < +\infty$ and σ be a nondecreasing function. Then if*

$$
\begin{aligned}
X &= W^{1,p}(\Omega_\varepsilon, \partial\Omega) \qquad \mathcal{C} = \{v \in X : \Phi(v) \in L^1(S_\varepsilon)\} \\
J(v) &= E_\varepsilon(v) \qquad Av = A_\varepsilon v \\
J_1(v) &= \frac{1}{p}\int\limits_{\Omega_\varepsilon} |\nabla v|^p \, \mathrm{d}x - \int\limits_{\Omega_\varepsilon} fv \, \mathrm{d}x - \beta(\varepsilon)\int\limits_{S_\varepsilon} gv \, \mathrm{d}S \\
J_1'(v)(w) &= \int\limits_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla w \, \mathrm{d}x - \int\limits_{\Omega_\varepsilon} fw \, \mathrm{d}x - \beta(\varepsilon)\int\limits_{S_\varepsilon} gw \, \mathrm{d}S \\
J_2(v) &= \beta(\varepsilon)\int\limits_{S_\varepsilon} \Phi(v) \, \mathrm{d}S \\
\langle J_2'(v), w\rangle &= \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma(v) w \, \mathrm{d}S,
\end{aligned}
$$

or

$$
\begin{aligned}
X &= W^{1,p}(\Omega), \qquad \mathcal{C} = \{v \in X : \Phi(v) \in L^1(\Omega)\} \\
J &= J_1 + J_2 \\
J_1(v) &= \frac{1}{p}\int\limits_{\Omega} |\nabla v|^p \, \mathrm{d}x - \int\limits_{\Omega} fv \, \mathrm{d}x - \beta_0\int\limits_{\Omega} gv \, \mathrm{d}x \\
\langle J_1'(v), w\rangle &= \frac{1}{p}\int\limits_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla w \, \mathrm{d}x - \int\limits_{\Omega} fw \, \mathrm{d}x - \beta_0\int\limits_{\Omega} gw \, \mathrm{d}x \\
J_2(v) &= \beta_0\int\limits_{\Omega} \Phi(v) \, \mathrm{d}x \\
J_2'(v)(w) &= \beta_0\int\limits_{\Omega} \sigma(v) w \, \mathrm{d}x,
\end{aligned}
$$

we have that $J_1, J_2 : \mathcal{C} \to \mathbb{R}$, J_1 is convex, J_1 is Gâteaux differentiable. Furthermore, if (18) holds then $\mathcal{C} = X$, J_2 is Gâteaux-differentiable in X and J' is continuous on X.

Remark 8. The *furthermore* statement in Lemma 1 was first proved in [15]. Condition (18) is given by the fact that, $v \mapsto G(v)$ is $L^r(\Omega) \to L^t(\Omega)$ is continuous if $|G| \le C(1 + |v|^{\frac{r}{t}})$. Notice that, for $r = p$ and $t = p'$ we have $\frac{r}{t} = p - 1$. In this case, J satisfies the continuity condition for $L^p \to L^1$, which is enough to make J continuous. It is likely that (18) is purely a technical requirement so that iii) implies i).

2.3. On the coercivity of the p-Laplacian, when $1 < p < 2$

We will need the following auxiliary lemma, that deals with the coercivity of the p-Laplace operator:

Lemma 7. *Let* $1 < p < 2$ *and* $u, v \in W^{1,p}(\Omega)$. *Then*

$$\int_\Omega |\nabla(u-v)|^p \, \mathrm{d}x \le C \left(\int_\Omega \frac{|\nabla(u-v)|^2}{|\nabla u|^{2-p} + |\nabla v|^{2-p}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}}$$
$$\le C \left(\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v) \, \mathrm{d}x \right)^{\frac{p}{2}}$$
$$\times \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}}. \tag{39}$$

Proof. The first inequality is a direct consequence of the Hölder inequality

$$\left(\int_\Omega \frac{|\nabla(u-v)|^p}{(|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2}}} \left(|\nabla u|^{2-p} + |\nabla v|^{2-p}\right)^{\frac{p}{2}} \, \mathrm{d}x \right)$$
$$\le \left(\int_\Omega \frac{|\nabla(u-v)|^2}{|\nabla u|^{2-p} + |\nabla v|^{2-p}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u|^{2-p} + |\nabla v|^{2-p})^{\frac{p}{2-p}} \, \mathrm{d}x \right)^{\frac{2-p}{p}},$$

and the second one is due to the estimate for vectors, $\xi, \eta \in \mathbb{R}^n$, not both zero:

$$\frac{|\xi - \eta|^2}{|\xi|^{2-p} + |\eta|^{2-p}} \le C \left(|\eta|^{p-2}\eta - |\xi|^{p-2}\xi \right) \cdot (\eta - \xi),$$

this concludes the proof. $\square$

2.4. Comparison of solutions with different kinetics

We have the following comparison lemma for the solutions:

Lemma 8. *Let* $\sigma, \hat{\sigma}$ *be continuous functions,* σ *satisfies* (28) *for some* $q > 1$ *and let* u_ε *and* $\hat{u}_\varepsilon$ *be the corresponding solutions of* (3) *with* $\beta \sim \beta^*$. *Then*

$$\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - \hat{u}_\varepsilon|^q \, \mathrm{d}s \le C \|\sigma - \hat{\sigma}\|_{C(\mathbb{R})}^{\frac{q}{q-1}}. \tag{40}$$

Proof. We use $u - \hat{u}$ as a test function, and via the monotonicity of σ we have

$$\beta(\varepsilon) \int_{S_\varepsilon} (\sigma(u) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S \le \int_{\Omega_\varepsilon} |\nabla(u - \hat{u})|^p \, \mathrm{d}S + \beta(\varepsilon) \int_{S_\varepsilon} (\sigma(u) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S$$
$$\le \beta(\varepsilon) \int_{S_\varepsilon} (\hat{\sigma}(\hat{u}) - \sigma(\hat{u}))(u - \hat{u}) \, \mathrm{d}S$$
$$\le \|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}| \, \mathrm{d}S$$
$$\le C \|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})} \left(\beta(\varepsilon) \int_{S_\varepsilon} |u - \hat{u}|^q \, \mathrm{d}S \right)^{\frac{1}{q}}.$$

Due to (28) we have that

$$\beta(\varepsilon)\int_{S_\varepsilon} |u-\hat{u}|^q \,\mathrm{d}S \leq C\|\sigma-\hat{\sigma}\|_{\mathcal{C}(\mathbb{R})}\left(\beta(\varepsilon)\int_{S_\varepsilon} |u-\hat{u}|^q \,\mathrm{d}S\right)^{\frac{1}{q}}$$

$$\left(\beta(\varepsilon)\int_{S_\varepsilon} |u-\hat{u}|^q \,\mathrm{d}S\right)^{1-\frac{1}{q}} \leq C\|\sigma-\hat{\sigma}\|_{\mathcal{C}(\mathbb{R})},$$

which concludes the result. □

Lemma 9. *Let $\sigma, \hat{\sigma}$ be continuous nondecreasing functions such that $\sigma(0)=0$ and $u, \hat{u}$ be their respective solutions of (3). Then, there exist constants C depending on p, but independent of ε, such that*

i) *If $1<p<2$*

$$\|\nabla(u_\varepsilon-\hat{u}_\varepsilon)\|_{L^p(\Omega_\varepsilon)} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma-\hat{\sigma}\|_{\mathcal{C}(\mathbb{R})}\left(\|\nabla u_\varepsilon\|^{2-p}_{L^p(\Omega_\varepsilon)}+\|\nabla \hat{u}_\varepsilon\|^{2-p}_{L^p(\Omega_\varepsilon)}\right)^{\frac{2}{p}}. \tag{41}$$

ii) *If $p \geq 2$ then*

$$\|\nabla(u_\varepsilon-\hat{u}_\varepsilon)\|^{p-1}_{L^p(\Omega_\varepsilon)} \leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma-\hat{\sigma}\|_{\mathcal{C}(\mathbb{R})}. \tag{42}$$

Proof. By considering the difference of weak formulations we can write, for the test function u_2-u_1,

$$\int_\Omega (|\nabla u_2|^{p-2}\nabla u_2-|\nabla u_1|^{p-2}\nabla u_1)\cdot\nabla(u_2-u_1)\,\mathrm{d}x+\beta(\varepsilon)\int_{S_\varepsilon}(\sigma_2(u_2)-\sigma_2(u_1))(u_2-u_1)\,\mathrm{d}S$$

$$=\beta(\varepsilon)\int_{S_\varepsilon}(\sigma_1(u_1)-\sigma_2(u_1))(u_2-u_1)\,\mathrm{d}S.$$

Applying monotonicity, Proposition 2 and Lemma 4

$$\int_\Omega (|\nabla u_2|^{p-2}\nabla u_2-|\nabla u_1|^{p-2}\nabla u_1)\cdot\nabla(u_2-u_1)\,\mathrm{d}x$$

$$\leq \beta(\varepsilon)\|\sigma_2-\sigma_1\|_\infty\beta^*(\varepsilon)^{-1}\left(\beta^*(\varepsilon)\int_{S_\varepsilon}|u_1-u_2|^p\,\mathrm{d}S\right)^{\frac{1}{p}}$$

$$\leq C\beta(\varepsilon)\|\sigma_2-\sigma_1\|_\infty\beta^*(\varepsilon)^{-1}\|\nabla(u_1-u_2)\|_{L^p(\Omega_\varepsilon)}.$$

Part ii) follows directly. Let us prove part i). Applying Lemma 7 we have that

$$\|\nabla(u_1-u_2)\|^p_{L^p} \leq C\left(\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2-\sigma_1\|_\infty\|\nabla(u_1-u_2)\|_{L^p}\right)^{\frac{p}{2}}$$

$$\times\left(\int_\Omega (|\nabla u_1|^{2-p}+|\nabla u_2|^{2-p})^{\frac{p}{2-p}}\,\mathrm{d}x\right)^{\frac{2-p}{p}}$$

$$
\begin{aligned}
\|\nabla(u_1-u_2)\|_{L^p} &\le C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2-\sigma_1\|_\infty \left(\int_\Omega (|\nabla u_1|^{2-p}+|\nabla u_2|^{2-p})^{\frac{p}{2-p}}\,\mathrm{d}x\right)^{\frac{2}{p}\frac{2-p}{p}} \\
&\le C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_2-\sigma_1\|_\infty \left(\|\nabla u_1\|_{L^p}^{2-p}+\|\nabla u_2\|_{L^p}^{2-p}\right)^{\frac{2}{p}},
\end{aligned}
$$

which proves the result. □

2.5. Proof of the approximation lemmas

There is extensive literature on the approximation of functions in bounded intervals, in particular approximation that preserve the monotonicity. For example, it is known that Bernstein polynomials of a monotone function are also monotone, and the convolution with a positive kernel also preserves global monotonicity. Finer results are known as to the approximation of functions which are piecewise monotone by functions that share their monotonicity (i.e. *comonotone* functions. In this direction see, e.g. [22–24,28]).

One of the canonical options in this direction is the Yosida approximation, but, in general this only converges pointwise. This is natural, since one can approximate a discontinuous function by Lipschitz continuous ones, and therefore the limit cannot be uniform. We choose, locally, a convolution with mollifiers.

Proof of Lemma 2. Let $\sigma_{\varepsilon,0} \in \mathcal{C}^1([-1,1])$ be an approximation of σ such that

$$
\begin{cases}
\sigma_{\varepsilon,0} = \sigma \text{ in } \{-1,0,1\}, \\
\|\sigma_{\varepsilon,0}-\sigma\|_{\mathcal{C}([-1,1])} \le \varepsilon \\
\sigma_{\varepsilon,0} \text{ is nondecreasing.}
\end{cases}
$$

This can be done, since, for example, the convolution of σ with nonnegative mollifiers are nondecreasing. Let $\sigma_{\varepsilon,1} \in \mathcal{C}^1([1,2])$ be an approximation of σ in $[1,2]$ such that

$$
\begin{cases}
\sigma_{\varepsilon,1} = \sigma \text{ in } \{1,2\}, \\
\sigma'_{\varepsilon,1}(1) = \sigma'_{\varepsilon,0}(1), \\
\|\sigma_{\varepsilon,1}-\sigma\|_{\mathcal{C}([1,2])} \le \varepsilon, \\
\sigma_{\varepsilon,1} \text{ is nondecreasing.}
\end{cases}
$$

We proceed analogously in $[n,n+1], [-(n+1),-n]$ for $n \in \mathbb{N}$. We finally construct $\sigma_\varepsilon \in \mathcal{C}^1(\mathbb{R})$ by matching the pieces. □

Proof of Lemma 3. Let $\varepsilon < \frac{1}{4C}$ and $\delta = \left(\frac{\varepsilon}{4C}\right)^{\frac{1}{\alpha}} < 1$. If $|x-y| \le \delta$ then

$$
\begin{aligned}
|\sigma(x)-\sigma(y)| &\le D(|x-y|^\alpha + |x-y|^p) \le D(\delta^\alpha+\delta^p) \\
&\le 2D\delta^\alpha = \varepsilon.
\end{aligned}
$$

We define

$$
\sigma_\varepsilon(n\delta) = \sigma(n\delta), \qquad n \in \mathbb{Z} \tag{43}
$$

and linear in $(n, n+1)$. Since σ is nondecreasing so is σ_ε. For $x \in [\delta(n-1), \delta n]$ we have

$$
\begin{aligned}
|\sigma(x) - \sigma_\varepsilon(x)| &\le |\sigma(x) - \sigma(\delta n)| + |\sigma(\delta n) - \sigma_\varepsilon(x)| \\
&\le \frac{\varepsilon}{2} + |\sigma_\varepsilon(\delta n) - \sigma_\varepsilon(x)| \\
&\le \frac{\varepsilon}{2} + \big(\sigma_\varepsilon(\delta n) - \sigma_\varepsilon(\delta(n-1))\big) \\
&= \frac{\varepsilon}{2} + \big(\sigma(\delta n) - \sigma(\delta(n-1))\big) \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}
$$

On the other hand, for $x \in (\delta(n-1), \delta n)$ we have

$$
0 \le \sigma_\varepsilon'(x) = \frac{\sigma(\delta n) - \sigma(\delta(n-1))}{\delta} \le C(\delta^{\alpha-1} + \delta^{p-1}) \le D\varepsilon^{1-\frac{1}{\alpha}},
$$

which concludes the result. $\square$

3. Proof of Theorem 1

Proof of Theorem 1. We rewrite the problem, due to Lemma 1 as

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(v)(v-u_\varepsilon)\,\mathrm{d}S \\
&\qquad \ge \int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x + \beta(\varepsilon)\int_{S_\varepsilon} g(v-u_\varepsilon)\,\mathrm{d}S \qquad \forall v \in W_0^{1,p}(\Omega).
\end{aligned}
\tag{44}
$$

Let us start by considering $\sigma \in \mathcal{C}^1(\mathbb{R})$. If either $\beta_0 = +\infty$ and $g = 0$ or $\beta_0 < +\infty$, we can apply Proposition 3 to show that $P_\varepsilon u_\varepsilon$ are uniformly bounded in $W_0^{1,p}(\Omega)$, and therefore there exists $u \in W_0^{1,p}(\Omega)$ and a subsequence of $P_\varepsilon u_\varepsilon$ (denoted as the original sequence) such that

$$
P_\varepsilon u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{45}
$$

Then it is known that (see [26,29,32]), for $v \in W_0^{1,\infty}(\Omega)$ we have

$$
\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u_\varepsilon)\,\mathrm{d}x &\to \int_\Omega |\nabla v|^{p-2} v \cdot \nabla(v-u)\,\mathrm{d}x \\
\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x &\to \int_\Omega f(v-u)\,\mathrm{d}x \\
\beta^*(\varepsilon)\int_{S_\varepsilon} \sigma(v)(v-u_\varepsilon)\,\mathrm{d}S &\to |\partial G_0| \int_\Omega \sigma(v)(v-u)\,\mathrm{d}x \\
\beta^*(\varepsilon)\int_{S_\varepsilon} g(v-u_\varepsilon)\,\mathrm{d}S &\to |\partial G_0| \int_\Omega g(v-u_\varepsilon)\,\mathrm{d}x,
\end{aligned}
$$

as $\varepsilon \to 0$. If $\beta_0 < +\infty$ we can pass to the limit in (44) as $\varepsilon \to 0$ and obtain

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u)\,\mathrm{d}x + \beta_0 \int_\Omega \sigma(v)(v-u)\,\mathrm{d}x \geq \int_\Omega f(v-u)\,\mathrm{d}x + \beta_0 \int_\Omega g(v-u)\,\mathrm{d}x \qquad \forall v \in W_0^{1,\infty}(\Omega). \tag{46}$$

Applying density, Lemma 6 and Lemma 1 this is equivalent to u being a solution of (19).

If $\beta_0 = +\infty$ and $g = 0$ then we write (44) as

$$\beta^*(\varepsilon)\beta(\varepsilon)^{-1} \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u_\varepsilon)\,\mathrm{d}x + \beta^*(\varepsilon) \int_{S_\varepsilon} \sigma(v)(v-u_\varepsilon)\,\mathrm{d}S \geq \beta^*(\varepsilon)\beta(\varepsilon)^{-1} \int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,\mathrm{d}x.$$

By passing to the limit we obtain

$$\int_\Omega \sigma(v)(v-u_\varepsilon)\,\mathrm{d}x \geq 0, \quad v \in W_0^{1,\infty}(\Omega).$$

Again, applying Lemma 1 we deduce the result.

Let $\sigma \in \mathcal{C}(\mathbb{R})$ and $\beta_0 < +\infty$. By Lemma 2 there exist nondecreasing functions $(\sigma_m) \subset \mathcal{C}^1(\mathbb{R})$ such that $\sigma_m(0) = 0$, $\sigma_m \to \sigma$ in $\mathcal{C}(\mathbb{R})$. Let $u_{\varepsilon,m}$ and u_m be the solutions of (3) and (19) with kinetics σ_m, which by the previous proof satisfy

$$P_\varepsilon u_{\varepsilon,m} \rightharpoonup u_m \qquad \text{in } W_0^{1,p}(\Omega) \text{ as } \varepsilon \to 0. \tag{47}$$

Applying Lemma 9 we have that

$$\begin{aligned} \|\nabla(u_\varepsilon - u_{m,\varepsilon})\|_{L^p(\Omega)} &\leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} && \text{if } 1 < p < 2, \\ \|\nabla(u_\varepsilon - u_{m,\varepsilon})\|_{L^p(\Omega)}^{p-1} &\leq C\beta(\varepsilon)\beta^*(\varepsilon)^{-1}\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} && \text{if } 2 \leq p < n. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ in these estimates we get

$$\begin{aligned} \|\nabla(u - u_m)\|_{L^p(\Omega)} &\leq C\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} && \text{if } 1 < p < 2, \\ \|\nabla(u - u_m)\|_{L^p(\Omega)}^{p-1} &\leq C\|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} && \text{if } 2 \leq p < n. \end{aligned}$$

By uniform boundedness there exists $\hat{u} \in W_0^{1,p}(\Omega)$ such that $u_m \rightharpoonup \hat{u}$ in $W^{1,p}(\Omega)$ as $m \to +\infty$. By continuity of the equation with respect to the kinetics we know that $\hat{u}$ is the solution of (19). From the previous estimate we have that $u = \hat{u}$, which concludes the proof. □

Remark 9. Notice that condition (18) is only used to show that (46) implies that u is a solution of (19). However, if we show that (46) has a unique solution then condition (18) can be removed. Also, if u is bounded then this condition can also be removed.

4. Proof of Theorem 2

Proof of Theorem 2. Applying the results of this paper for the case $1 < p < n$, which extend naturally to $p = n$ (see Remark 4) and [11] for the case $p > n$ we have that $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$.

First, let us suppose that $\sigma' \in L^\infty$. Then $\sigma(u_\varepsilon)$ is bounded in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Hence, it is easy to show that $P_\varepsilon \sigma(u_\varepsilon) \rightharpoonup \sigma(u)$ in $W_0^{1,p}(\Omega)$. Hence, applying Proposition 2 we have the result for σ uniformly Lipschitz.

Let $\sigma \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ such that (25), (28) are satisfied. According to Lemma 3 there exist a sequence of nondecreasing functions $(\sigma_m) \subset \mathcal{C}(\mathbb{R})$ such that $\sigma_m' \in L^\infty$ and $\sigma_m \to \sigma$ in $\mathcal{C}(\mathbb{R})$.

Let $u_{\varepsilon,m}$ be the corresponding solution of (3) with kinetics σ_m. Then we have

$$\begin{aligned}
\left| \beta(\varepsilon) \int_{S_\varepsilon} \sigma(u) \,\mathrm{d}S - \beta(\varepsilon) \int_{S_\varepsilon} \sigma_m(u_{\varepsilon,m}) \,\mathrm{d}S \right| &\le \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_\varepsilon) - \sigma_m(u_\varepsilon)| \,\mathrm{d}S \\
&\le \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_\varepsilon) - \sigma(u_{\varepsilon,m})| \,\mathrm{d}S + \beta(\varepsilon) \int_{S_\varepsilon} |\sigma(u_{\varepsilon,m}) - \sigma_m(u_{\varepsilon,m})| \,\mathrm{d}S \\
&\le C\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - u_{\varepsilon,m}|^\alpha \,\mathrm{d}S + \beta(\varepsilon)|S_\varepsilon| \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \\
&\le C \left(\beta(\varepsilon) \int_{S_\varepsilon} |u_\varepsilon - u_{\varepsilon,m}|^q \,\mathrm{d}S \right)^{\frac{\alpha}{q}} + \beta(\varepsilon)|S_\varepsilon| \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \\
&\le C \left(\|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}^{\frac{\alpha}{q-1}} + \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \right).
\end{aligned}$$

In particular, taking any $m \in \mathbb{Z}$ we show that up to a subsequence the following convergence holds

$$\eta_\varepsilon = \frac{1}{|S_\varepsilon|} \int_{S_\varepsilon} \sigma(u) \,\mathrm{d}S \to \eta_0 \qquad \text{as } \varepsilon \to 0.$$

Applying the first part of the proof, we have that

$$\left| \eta_0 - \frac{1}{|\Omega|} \int_\Omega \sigma_m(u_m) \,\mathrm{d}x \right| \le C \left(\|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})}^{\frac{\alpha}{q-1}} + \|\sigma - \sigma_m\|_{\mathcal{C}(\mathbb{R})} \right).$$

Due to Lemma 9 we have that, as $m \to +\infty$, $u_m \to u$ in $L^p(\Omega)$. Also, due (25) we have that $\sigma(u_m) \to \sigma(u)$ in $L^1(\Omega)$. Hence

$$\begin{aligned}
\|\sigma_m(u_m) - \sigma(u)\|_{L^1(\Omega)} &\le \|\sigma_m(u_m) - \sigma(u_m)\|_{L^1(\Omega)} + \|\sigma(u_m) - \sigma(u)\|_{L^1(\Omega)} \\
&\le \|\sigma_m - \sigma\|_{\mathcal{C}(\mathbb{R})} + \|\sigma(u_m) - \sigma(u)\|_{L^1(\Omega)}.
\end{aligned}$$

Therefore, $\sigma_m(u_m) \to \sigma(u)$ in $L^1(\Omega)$. Hence,

$$\eta_0 = \frac{1}{|\Omega|} \int_\Omega \sigma(u) \,\mathrm{d}x.$$

Since every convergent subsequence of (η_ε) has the same limit η_0 we conclude the proof. □

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CHANGE OF HOMOGENIZED ABSORPTION TERM IN DIFFUSION PROCESSES WITH REACTION ON THE BOUNDARY OF PERIODICALLY DISTRIBUTED ASYMMETRIC PARTICLES OF CRITICAL SIZE

JESÚS ILDEFONSO DÍAZ, DAVID GÓMEZ-CASTRO,
TATIANA A. SHAPOSHNIKOVA, MARIA N. ZUBOVA

Abstract. The main objective of this article is to get a complete characterization of the homogenized global absorption term, and to give a rigorous proof of the convergence, in a class of diffusion processes with a reaction on the boundary of periodically "microscopic" distributed particles (or holes) given through a nonlinear microscopic reaction (i.e. under nonlinear Robin microscopic boundary conditions). We introduce new techniques to deal with the case of non necessarily symmetric particles (or holes) of critical size which leads to important changes in the qualitative global homogenized reaction (such as it happens in many problems of the Nanotechnology). Here we shall merely assume that the particles (or holes) G_ε^j, in the n-dimensional space, are diffeomorphic to a ball (of diameter $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$ for some $C_0 > 0$). To define the corresponding "new strange term" we introduce a one-parametric family of auxiliary external problems associated to canonical cellular problem associated to the prescribed asymmetric geometry G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$ (which is assumed to be merely a Hölder continuous function). We construct the limit homogenized problem and prove that it is a well-posed global problem, showing also the rigorous convergence of solutions, as $\varepsilon \to 0$, in suitable functional spaces. This improves many previous papers in the literature dealing with symmetric particles of critical size.

1. Introduction

It is well-known that the asymptotic behaviour of the solution of many relevant diffusion processes with reaction on the boundary of periodically "microscopic" distributed particles (or holes) is described through the solution of a global reaction-diffusion problem in which the global reaction term (usually an absorption term if the microscopic reactions are given by monotone non-decreasing functions) maintains the same structural properties as the microscopic reaction (see, for instance, [3]) and its many references to previous results in the literature).

A certain critical size of the "microscopic particles" may be responsible of a change in the nature of the homogenized global absorption term, with respect to

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the structural assumptions on the microscopic boundary reaction kinetic. It seems that the first result in that direction was presented in the pioneering paper by V. Marchenko and E. Hruslov [15] dealing with microscopic non-homogeneous Neumann boundary condition (see also the study made by E. Hruslov concerning linear microscopic Robin boundary conditions in [12, 11]). A –perhaps– more popular presentation of the appearance of some "strange term" was due to D. Cioranescu and F.Murat [2] dealing with microscopic Dirichlet boundary conditions (see also [13]).

This change of behavior from the microscopic reaction to the global homogeneized reaction term is one of the characteristics of the nanotechnological effects (see, e.g., [20]) and it does not appear for particles of bigger size (relative to their repetition) than the critical scale (see, e.g., [6] and the references therein). The total identification of the new or "strange" reaction term is an important task which was considered by many authors under different technical assumptions. In the case of nonlinear microscopic boundary reactions the first result in the literature was the 1997 paper by Goncharenko [10] (see also the precedent paper [13]). The identification (and the rigorous proof of the convergence in the homogenization process) requires to assume that the particles (or holes) are symmetric balls of diameter $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$, for some $C_0 > 0$. Many other researches were developed for different problems concerning critical sized balls (see [22, 18, 5] and the references therein). Recently, a unifying study concerning the homogenization for particles (or holes) given by symmetric balls of critical order was presented in [7]: the treatment was extended to a microscopic reaction given by a general maximal monotone graph which allows to include, as special problems, the cases of Dirichlet or nonlinear Robin microscopic boundary conditions. The case of particles of general shape when $n = 2$ was studied in [19], with the limit behaviour being similar to the case of spherical inclusions and $n \geq 2$.

The main task of this paper is to get a complete characterization of the homogenized global absorption term in the class of problems given through a nonlinear microscopic reaction (i.e. under nonlinear Robin microscopic boundary conditions) and for non necessarily symmetric particles (or holes). Here we will merely assume that the particles (or holes) G_ε^j are a rescaled version of a set G_0, diffeomorphic to a ball (where the scaling factor is $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$ for some $C_0 > 0$). To define the corresponding new "strange term" we introduce a one-parametric family of auxiliary external problems associated to canonic cellular problem, which play the role of a "nonlinear capacity" of G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$ (which is assumed to be merely a Hölder continuous function). We construct the limit homogenized problem and prove that it is well-posed global problem, showing also the rigorous convergence of solutions, as $\varepsilon \to 0$, in suitable functional spaces.

2. Statement of main results

Let Ω be a bounded domain in $\mathbb{R}^n$ $n \geq 3$ with a piecewise smooth boundary $\partial\Omega$. The case $n = 2$ requires some technical modifications which will not be presented here. Let G_0 be a domain in $Y = (-\frac{1}{2}, \frac{1}{2})^n$, and $\overline{G_0}$ be a compact set diffeomorphic to a ball. Let $C_0, \varepsilon > 0$ and set

$$a_\varepsilon = C_0\varepsilon^\alpha \quad \text{for } \alpha = \frac{n}{n-2}. \tag{2.1}$$

For $\delta > 0$ and B a set let $\delta B = \{x \,|\, \delta^{-1}x \in B\}$. Assume that ε is small enough so that $a_\varepsilon G_0 \subset \varepsilon Y$. We define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$. For $j \in \mathbb{Z}^n$ we define

$$P_\varepsilon^j = \varepsilon j, \quad Y_\varepsilon^j = P_\varepsilon^j + \varepsilon Y, \quad G_\varepsilon^j = P_\varepsilon^j + a_\varepsilon G_0.$$

We define the set of admissible indexes:

$$\Upsilon_\varepsilon = \big\{ j \in \mathbb{Z}^n : G_\varepsilon^j \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset \big\}.$$

Notice that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ where $d > 0$ is a constant. Our problem will be set in the following domain:

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j, \quad \Omega_\varepsilon = \Omega \setminus \overline{G}_\varepsilon$$

Finally, let

$$\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega, \quad S_\varepsilon = \partial G_\varepsilon.$$

We consider the following boundary value problem in the domain Ω_ε

$$\begin{gathered} -\Delta u_\varepsilon = f, \quad x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = 0, \quad x \in S_\varepsilon, \\ u_\varepsilon = 0, \quad x \in \partial\Omega, \end{gathered} \tag{2.2}$$

where $\gamma = \alpha = \frac{n}{n-2}$, $f \in L^2(\Omega)$, ν is the unit outward normal vector to the boundary S_ε, $\partial_\nu u$ is the normal derivative of u. Furthermore, we suppose that the function $\sigma : \mathbb{R} \to \mathbb{R}$, describing the microscopic nonlinear Neumann boundary condition, is nondecreasing, $\sigma(0) = 0$, and there exist constants k_1, k_2 such that

$$|\sigma(s) - \sigma(t)| \le k_1|s - t|^\alpha + k_2|s - t| \quad \forall s, t \in \mathbb{R}, \quad \text{for some } 0 < \alpha \le 1. \tag{2.3}$$

Remark 2.1. Condition (2.3) means that σ is locally Hölder continuous, but it is only sublinear towards infinity. This condition is weaker than $u \in \mathcal{C}^{0,\alpha}(\mathbb{R})$ or σ Lipschitz, that correspond, respectively, to $k_2 = 0$ and $k_1 = 0$.

Remark 2.2. Condition (2.3) on σ is a purely technical requirement. This kind of regularity can probably be improved. In particular, as shown in [4, 7] the kind of homogenization techniques and result that will be presented later can be expect for any maximal monotone graph σ.

For any prescribed set G_0, as before, and for any given $u \in \mathbb{R}$, we define $\widehat{w}(y; G_0, u)$, for $y \in \mathbb{R}^n \setminus G_0$, as the solution of the following one-parametric family of auxiliary external problems associated to the prescribed asymmetric geometry G_0 and the nonlinear microscopic boundary reaction $\sigma(s)$:

$$\begin{gathered} -\Delta_y \widehat{w} = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y} \widehat{w} - C_0\sigma(u - \widehat{w}) = 0, \quad \text{if } y \in \partial G_0, \\ \widehat{w} \to 0 \quad \text{as } |y| \to \infty. \end{gathered} \tag{2.4}$$

We will prove in Section 4 that the above auxiliary external problems are well defined and, in particular, there exists a unique solution $\widehat{w}(y; G_0, u) \in H^1(\mathbb{R}^n \setminus \overline{G_0})$, for any $u \in \mathbb{R}$. Concerning the corresponding "new strange term", for any prescribed asymmetric set G_0, as before, and for any given $u \in \mathbb{R}$ we introduce the following definition.

Definition 2.3. Given G_0 we define $H_{G_0} : \mathbb{R} \to \mathbb{R}$ by means of the identity

$$\begin{aligned} H_{G_0}(u) &:= \int_{\partial G_0} \partial_{\nu_y} \widehat{w}(y; G_0, u)\, \mathrm{d}S_y \\ &= C_0 \int_{\partial G_0} \sigma(u - \widehat{w}(y; G_0, u))\, \mathrm{d}S_y, \quad \text{for any } u \in \mathbb{R}. \end{aligned} \tag{2.5}$$

Remark 2.4. Let $G_0 = B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. We can find the solution of problem (2.4) in the form $\widehat{w}(y; G_0, u) = \frac{\mathcal{H}(u)}{|y|^{n-2}}$, where, in this case, $\mathcal{H}(u)$ is proportional to $H_{B_1(0)}(u)$. We can compute that

$$\begin{aligned} H_{G_0}(u) &= \int_{\partial G_0} \partial_\nu \widehat{w}(u, y)\, \mathrm{d}S_y \\ &= \int_{\partial G_0} (n-2) H_{G_0}(u)\, \mathrm{d}S_y \\ &= (n-2)\mathcal{H}(u)\omega(n), \end{aligned}$$

where $\omega(n)$ is the area of the unit sphere. Hence, due to (2.5), $\mathcal{H}(u)$ is the unique solution of the following functional equation

$$(n-2)\mathcal{H}(u) = C_0 \sigma(u - \mathcal{H}(u)). \tag{2.6}$$

In this case, it is easy to prove that H is nonexpansive (Lipschitz continuous with constant 1). This equation has been considered in many papers (see [7] and the references therein).

We shall prove several results on the regularity and monotonicity of the homogenized reaction $H_{G_0}(u)$ in the next section. Concerning the convergence as $\varepsilon \to 0$ the following statement collects some of the more relevant aspects of this process:

Theorem 2.5. *Let $n \geq 3$, $a_\varepsilon = C_0 \varepsilon^{-\gamma}$, $\gamma = \frac{n}{n-2}$, σ a nondecreasing function such that $\sigma(0) = 0$ and that satisfies (2.3). Let u_ε be the weak solution of (2.2). Then there exists an extension to $H_0^1(\Omega)$, still denoted by u_ε, such that $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$ as $\varepsilon \to 0$, where $u_0 \in H_0^1(\Omega)$ is the unique weak solution of*

$$\begin{gathered} -\Delta u_0 + C_0^{n-2} H_{G_0}(u_0) = f \quad \Omega, \\ u_0 = 0 \quad \partial\Omega. \end{gathered} \tag{2.7}$$

Remark 2.6. Since $|H_{G_0}(u)| \leq C(1 + |u|)$ it is clear that $H_{G_0}(u_0) \in L^2(\Omega)$.

3. On the ε-global problem

Some comments on the well-posednes and some a priori estimates concerning the ε-global problem (2.2), when the nondecreasing function $\sigma \in \mathcal{C}(\mathbb{R})$, $\sigma(0) = 0$ satisfies (2.3), are collected in this section. We start by introducing some notations:

Definition 3.1. Let U be an open set and $\Gamma \subset \partial\Omega$. We define the functional space

$$H^1(U, \Gamma) = \overline{\{f \in \mathcal{C}^\infty(U) : f|_\Gamma = 0\}}^{H^1(U)}.$$

Thanks to well-known results (see, e.g. the references given in [7]) there exists a unique weak solution of problem (2.2): i.e. $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ is the unique function such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi \, \mathrm{d}x + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(u_\varepsilon) \varphi \, \mathrm{d}S = \int_{\Omega_\varepsilon} f \varphi \, \mathrm{d}x, \tag{3.1}$$

for every $\varphi \in H^1(\Omega_\varepsilon, \partial\Omega)$. As a matter of fact, in order to get a proof of the convergence of u_ε as $\varepsilon \to 0$, under the general assumption (2.3), it is useful to recall that, thanks to the monotonicity of $\sigma(u)$, we can write the weak formulation of (2.2) in the following equivalent way (for details see [6]):

$$\int_{\Omega_\varepsilon} \nabla\varphi \cdot \nabla(\varphi - u_\varepsilon)\,\mathrm{d}x + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(\varphi)(\varphi - u_\varepsilon) ds \geq \int_{\Omega_\varepsilon} f(\varphi - u_\varepsilon)\,\mathrm{d}x, \tag{3.2}$$

for every $\varphi \in H_0^1(\Omega)$.

Concerning some initial apriori estimates, we recall that we can work with $\widetilde{u}_\varepsilon \in H_0^1(\Omega)$ given as an extension of u_ε to Ω such that

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega)} \leq K\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}, \quad \|\nabla\widetilde{u}_\varepsilon\|_{L^2(\Omega)} \leq K\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \tag{3.3}$$

where K does not depend on ε. The construction of such an extension is given, e.g., in [17] (the $W^{1,p}$ equivalent, for $p \neq 2$, can be found in [18]).

Now, considering in the weak formulation (3.1) the test function $\varphi = u_\varepsilon$, and using the monotonicity of σ, we obtain

$$\|\nabla u_\varepsilon\|^2_{L_2(\Omega_\varepsilon)} \leq K. \tag{3.4}$$

where K does not depend on ε. From (3.4) we derive that there are a subsequence of $\widetilde{u}_\varepsilon$ (still denote by $\widetilde{u}_\varepsilon$) and $u_0 \in H_0^1(\Omega)$ such that, as $\varepsilon \to 0$, we have

$$\widetilde{u}_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \tag{3.5}$$

$$\widetilde{u}_\varepsilon \to u_0 \quad \text{strongly in } L^2(\Omega). \tag{3.6}$$

In Section 4 we characterize the limit function $u_0 \in H_0^1(\Omega)$.

4. On the regularity of the strange term

4.1. **Auxiliary function $\widehat{w}$.** The existence and regularity of solution in domains

$$\mathcal{O} = \mathbb{R}^n \setminus \overline{G_0} \tag{4.1}$$

which are commonly known as exterior domains, has been extensively studied (see, e.g., [9] and the references therein).

Based on the rate of convergence to 0 as $|y| \to +\infty$ we consider the space

$$\mathbb{X} = \big\{w \in L^1_{loc}(\mathcal{O}) : \nabla w \in L^2(\mathcal{O}),\ w|_{\partial G_0} \in L^2(\partial G_0),\ |w| \leq \frac{K}{|y|^{n-2}}\big\} \tag{4.2}$$

It is a standard result, known as Weyl's lemma, that any harmonic function is smooth (of class $\mathcal{C}^\infty$) in the interior of the domain. It was first proved for the whole space by Hermann Weyl [21], and later extended by others to any open set in $\mathbb{R}^n$ (see, e.g., [14]).

Remark 4.1. Notice that $\widetilde{w}(y; G_0, u) = -\widehat{w}(y; G_0, -u)$ is a solution of (2.4) that corresponds to $\widetilde{\sigma}(s) = -\sigma(-s)$. Hence, any comparison we prove for $u \geq 0$ we automatically prove for $u \leq 0$.

4.1.1. *A priori estimates.*

Lemma 4.2 (Weak maximum principle in exterior domains)**.** *Assume that $w \in \mathbb{X}$ is such that*

$$\begin{gathered}-\Delta w \leq 0 \quad \mathcal{D}'(\mathcal{O}), \\ w \leq 0 \quad \partial G_0.\end{gathered}$$

Then $w \leq 0$ in $\overline{\mathcal{O}}$.

Proof. Let $R > 0$. Consider $\mathcal{O}_R = \mathcal{O} \cap B_R$. Since $w \in \mathbb{X}$ then $w \leq \frac{K}{|y|^{n-2}}$. Using the hypothesis $w \leq 0$ on ∂G_0 and this fact, $\frac{K}{R^{n-2}}$ on $\partial \mathcal{O}_R$. We can apply the standard weak maximum principle for weak solutions in $\mathcal{O}_R$ to show that $w \leq \frac{K}{R^{n-2}}$ on $\overline{\mathcal{O}}_R$. As $R \to +\infty$ we prove the result. □

Analogously, we have the strong maximum principle.

Lemma 4.3. *Let σ nondecreasing, $u \in \mathbb{R}$, $\widehat{w} \in \mathbb{X}$ be a weak solution of* (2.4)*. Then*

$$\min\{0, u\} \leq \widehat{w} \leq \max\{0, u\} \tag{4.3}$$

Proof. For $u = 0$, $w = 0$ follows from a monotonicity argument. Assume $u > 0$. Let $\psi \in W^{1,\infty}(\mathbb{R})$ non-increasing such that

$$\psi(s) = \begin{cases} 1 & s < \frac{1}{2} \\ 0 & s > 1 \end{cases}$$

and consider the test function $\varphi = (w-u)_+ \psi\big(\frac{d(\cdot, \partial G_0)}{R}\big)$. Then

$$\begin{aligned}&\int_{\mathcal{O}} |\nabla (w-u)_+|^2 \psi\Big(\frac{d(x,\partial G_0)}{R}\Big)\, \mathrm{d}x + \int_{\mathcal{O}} (w-u)_+ \frac{\psi'\big(\frac{d(x,\partial G_0)}{R}\big)}{R} \nabla w \cdot \nabla d \, \mathrm{d}x \\ &= C_0 \int_{\partial G_0} \sigma(u-w)(w-u)_+ \, \mathrm{d}S \leq 0\end{aligned}$$

and

$$\begin{aligned}&\Big| \int_{\mathcal{O}} (w-u)_+ \frac{\psi'\big(\frac{d(x,\partial G_0)}{R}\big)}{R} \nabla w \cdot \nabla d \, \mathrm{d}x \Big| \, \mathrm{d}x \\ &\leq C \int_{\{\frac{R}{2} < d < R\}} \frac{w}{R} |\nabla w| \, \mathrm{d}x \\ &\leq C \Big(\int_{\{\frac{R}{2} < d < R\}} \frac{|w|^2}{R^2} \, \mathrm{d}x \Big)^{1/2} \Big(\int_{\mathcal{O}} |\nabla w|^2 \, \mathrm{d}x \Big)^{1/2} \\ &\leq \frac{C}{R^{\frac{n-2}{2}}} \Big(\int_{\mathcal{O}} |\nabla w|^2 \, \mathrm{d}x \Big)^{1/2} \to 0,\end{aligned}$$

as $R \to \infty$. Therefore,

$$\begin{aligned}0 &\leq \int_{0 \leq d < \frac{R}{2}} |\nabla (w-u)_+|^2 \, \mathrm{d}x \leq \int_{\mathcal{O}} |\nabla (w-u)_+|^2 \psi\Big(\frac{d(x,\partial G_0)}{R}\Big) \, \mathrm{d}x \\ &\leq - \int_{\mathcal{O}} (w-u)_+ \frac{\psi'\Big(\frac{d(x,\partial G_0)}{R}\Big)}{R} \nabla w \cdot \nabla d \, \mathrm{d}x.\end{aligned}$$

As $R \to +\infty$ we obtain that

$$\int_{\mathcal{O}} |\nabla (w-u)_+|^2 \,\mathrm{d}x = 0. \tag{4.4}$$

In particular $(w-u)_+ \geq 0$ is a constant. Since, as $|y| \to +\infty$ we show that the constant must be $(-u)_+ = 0$ we deduce that $w - u \leq 0$.

If $u < 0$ we apply the previous argument with $\tilde{\sigma}(s) = -\sigma(-s)$. □

Lemma 4.4. *Let $u \in \mathbb{R}$, $w \in \mathbb{X}$ such that $w \leq u$ in ∂G_0 and $-\Delta w \leq 0$ and*

$$K_0 = \max_{z \in \partial G_0} |z|^{n-2}.$$

Then

$$w \leq \frac{K_0 u}{|y|^{n-2}} \quad \forall y \in \overline{\mathcal{O}}.$$

Proof. Notice that

$$\max_{z \in \partial G_0} w(z)|z|^{n-2} \leq u \max_{z \in \partial G_0} |z|^{n-2} = K_0 u.$$

Then

$$w \leq \frac{K_0 u}{|y|^{n-2}} \quad y \in \partial G_0.$$

Since $w - \frac{K_0 u}{|y|^{n-2}}$ is subharmonic and tends to 0 as $|y| \to +\infty$, we can apply the weak maximum principle to deduce that

$$w(y) \leq \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathbb{R}^n \setminus G_0.$$

This proves the result. □

By the same argument it is easy to prove that any classical solution $\widehat{w} \in \mathcal{C}^2(\mathcal{O}) \cap \mathcal{C}(\overline{\mathcal{O}})$ is, in fact, in $\mathbb{X}$. Furthermore, we have an explicit expression of the K in the definition of $\mathbb{X}$ for the solutions of (2.4):

Lemma 4.5. *Let $\widehat{w} \in \mathbb{X}$ be a solution of* (2.4)*. Then*

$$|\widehat{w}(y; G_0, u)| \leq \frac{K_0 |u|}{|y|^{n-2}} \quad \forall y \in \overline{\mathcal{O}}. \tag{4.5}$$

Lemma 4.6. *Let $R_0 = \max_{\partial G_0} |y|$, $\widehat{w} \in \mathbb{X}$ be a weak solution of* (2.4)*. Then*

$$\max_{|y|=R} |\nabla \widehat{w}(y; G_0, u)| \leq \frac{K|u|}{(R-R_0)^{n-1}} \quad \forall R > R_0 \tag{4.6}$$

where K does not depend on u or R.

Proof. Let $|y_0| = R$. Let B be a ball centered at y_0 of radius $\frac{R-R_0}{2}$. In B we have $|y| \geq \frac{R-R_0}{2}$. Since $\frac{\partial \widehat{w}}{\partial x_i}$ is a harmonic function, and applying Lemma 4.5, we have

$$\frac{\partial \widehat{w}}{\partial x_i}(y_0) = \frac{1}{|B|}\int_B \frac{\partial \widehat{w}}{\partial x_i}\,\mathrm{d}y = \frac{1}{|B|}\int_{\partial B} \widehat{w}\nu_i \,\mathrm{d}S,$$

$$\Big|\frac{\partial \widehat{w}}{\partial x_i}(y_0)\Big| \leq \frac{|\partial B|}{|B|}\frac{K|u|}{(R-R_0)^{n-2}} \leq \frac{K|u|}{(R-R_0)^{n-1}}.$$

This completes the proof. □

4.1.2. *Uniqueness, comparison and approximation of solutions.*

Lemma 4.7. *Let $u \in \mathbb{R}$, σ_1, σ_2 be two nondecreasing functions such that $\sigma_1 \le \sigma_2$ in $[0, +\infty)$ and let $w_1, w_2 \in \mathbb{X}$ satisfy (2.4). Then $w_1 \le w_2$.*

Proof. We subtract the two weak formulations, and consider $\varphi = (w_1 - w_2)_+$ as a test function. We obtain that

$$\int_{\mathbb{R}^n \setminus G_0} |\nabla (w_1 - w_2)_+|^2 \,\mathrm{d}x = \int_{\partial G_0} (\sigma_1(u - w_1) - \sigma_2(u - w_2))(w_1 - w_2)_+ \,\mathrm{d}S$$

Thus, in the set $\{w_2 \le w_1\}$ we have that $u - w_2 \ge u - w_1$ and, hence,

$$\sigma_2(u - w_2) \ge \sigma_2(u - w_1) \ge \sigma_1(u - w_1),$$

so

$$\sigma_1(u - w_1) - \sigma_2(u - w_2) \le 0.$$

Thus, since $(w_1 - w_2)_+ \ge 0$ a.e. in ∂G_0, we have that

$$\int_{\mathbb{R}^n \setminus G_0} |\nabla (w_1 - w_2)_+|^2 \,\mathrm{d}x \le 0. \tag{4.7}$$

Hence $(w_1 - w_2)_+ = c$ constant. Since $(w_1 - w_2)_+ \to 0$ as $|y| \to +\infty$, we have that $c = 0$ and thus $w_1 \le w_2$. □

Corollary 4.8. *There exists, at most, one solution $w \in \mathbb{X}$ of (2.4).*

Lemma 4.9. *Let $\sigma_1, \sigma_2 \in \mathcal{C}(\mathbb{R})$ be two nondecreasing function. Let $\widehat{w}_i(\cdot; G_0, u) \in \mathbb{X}$ be a solution of*

$$\begin{gathered} -\Delta_y \widehat{w}_i = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \partial_{\nu_y} \widehat{w}_i - C_0 \sigma_i(u - \widehat{w}_i) = 0, \quad \text{if } y \in \partial G_0, \\ \widehat{w_i} \to 0 \quad \text{as } |y| \to \infty. \end{gathered} \tag{4.8}$$

Then

$$\|\nabla(\widehat{w}_1 - \widehat{w}_2)\|^2_{L^2(\mathcal{O})} \le C|u| \|\sigma_1 - \sigma_2\|_{L^\infty(I)}, \tag{4.9}$$

where

$$I = \{u - \widehat{w}_1(y; G_0, u) : y \in \mathbb{R}^n \setminus \overline{G}_0\} \subset \mathbb{R}, \tag{4.10}$$

and C is independent of u.

Proof. By taking as test function $\varphi = \widehat{w}_1 - \widehat{w}_2$ in the weak formulation of these equations we have that

$$\begin{aligned} \|\nabla(\widehat{w}_1 - \widehat{w}_2)\|^2_{L^2(\mathcal{O})} &\le \int_{\mathcal{O}} |\nabla(\widehat{w}_1 - \widehat{w}_2)|^2 \,\mathrm{d}x \\ &\quad + \int_{\partial G_0} (\sigma_2(u - \widehat{w}_2) - \sigma_2(u - \widehat{w}_1))(\widehat{w}_1 - \widehat{w}_2) \,\mathrm{d}S \\ &= \int_{\partial G_0} (\sigma_1(u - \widehat{w}_1) - \sigma_2(u - \widehat{w}_1))(\widehat{w}_1 - \widehat{w}_2) \,\mathrm{d}S \\ &\le \|\sigma_1 - \sigma_2\|_\infty \int_{\partial G_0} |\widehat{w}_1 - \widehat{w}_2| \,\mathrm{d}S \\ &\le C|u| \|\sigma_1 - \sigma_2\|_\infty. \end{aligned}$$

This completes the proof. □

4.1.3. *Existence and regularity.*

Lemma 4.10. *Let $u \in \mathbb{R}$ and σ uniformly Lipschitz. Then, there exists $\widehat{w} \in \mathbb{X}$ a weak solution of* (2.4)*. Furthermore, $\widehat{w}$ satisfies* (4.5)*.*

Proof. Let us assume that $u > 0$. Let $\lambda > 0$, and consider $\mu > 0$ such that

$$F : z \mapsto C_0\sigma(u - z) + \mu z$$

is nondecreasing. Let $w_0 = 0$. We define the sequence $w_k \in H^1(\mathcal{O})$ as the solutions of

$$\begin{gathered} -\Delta w_{k+1} + \lambda w_{k+1} = \lambda w_k \quad \mathcal{O}, \\ \partial_\nu w_{k+1} + \mu w_{k+1} = F(w_k) \quad \partial G_0, \\ w_{k+1} \to 0 \quad |y| \to +\infty. \end{gathered}$$

This sequence is well defined, since $\lambda > 0$ applying the Lax-Milgram theorem. Indeed, if $w_k \in H^1(\mathcal{O})$ then $F(w_k) \in H^{1/2}(\partial G_0)$ so that $w_{k+1} \in H^1(\mathcal{O})$.

Let us show that $0 \le w_k \le w_{k+1} \le u$ a.e. in $\mathcal{O}$ and ∂G_0 for every $n \ge 1$. We start by showing that $0 \le w_1$. This is immediate because $F(0) = C_0\sigma(u) \ge 0$. Let us now show that, if $w_{k-1} \le w_k$ then $w_k \le w_{k+1}$. Considering the weak formulations:

$$\begin{aligned} &\int_{\mathcal{O}} \nabla w_{k+1} \nabla v \,\mathrm{d}x + \lambda \int_{\mathcal{O}} w_{k+1} v \,\mathrm{d}x + \mu \int_{\partial G_0} w_k v \,\mathrm{d}S \\ &= \lambda \int_{\mathcal{O}} w_{k+1} v \,\mathrm{d}x + \int_{\partial G_0} F(w_k) v \,\mathrm{d}x \end{aligned} \tag{4.11}$$

we have that

$$\begin{aligned} &\int_{\mathcal{O}} \nabla (w_k - w_{k+1}) \nabla v \,\mathrm{d}x + \lambda \int_{\mathcal{O}} (w_k - w_{k+1})\varphi \,\mathrm{d}x + \mu \int_{\partial G_0} (w_k - w_{k+1}) v \,\mathrm{d}S \\ &= \lambda \int_{\mathcal{O}} (w_{k-1} - w_k) v \,\mathrm{d}x + \int_{\partial G_0} (F(w_{k-1}) - F(w_k)) v \,\mathrm{d}S \end{aligned}$$

Consider $v = (w_k - w_{k+1})_+ \ge 0$. We have that $w_{k-1} \le w_k$ therefore $w_{k-1} - w_k, F(w_{k-1}) - F(w_k) \le 0$. Hence

$$\begin{aligned} &\int_{\mathcal{O}} |\nabla (w_k - w_{k+1})_+|^2 \,\mathrm{d}x + \lambda \int_{\mathcal{O}} |(w_k - w_{k+1})_+|^2 \,\mathrm{d}x + \mu \int_{\partial G_0} |(w_k - w_{k+1})_+|^2 \,\mathrm{d}S \\ &= \lambda \int_{\mathcal{O}} (w_k - w_{k+1}) v \,\mathrm{d}x + \int_{\partial G_0} (F(w_{k-1}) - F(w_k)) v \,\mathrm{d}S \le 0 \end{aligned}$$

so that $(w_k - w_{k+1})_+ = 0$. Hence $w_k \le w_{k+1}$ a.e. in $\mathcal{O}$ and in ∂G_0. With an argument similar to the one in Lemma 4.3, one proves that $w_{k+1} \le u$ a.e. in $\mathcal{O}$ and ∂G_0.

The sequence w_k is pointwise increasing a.e. in $\mathcal{O}$. Therefore, there exists a function w such that

$$w_k(y) \nearrow w(y) \quad \text{a.e. } \mathcal{O}. \tag{4.12}$$

Taking traces, the same happens in ∂G_0. Hence

$$w_k(y) \nearrow w(y) \quad \text{a.e. } \partial G_0. \tag{4.13}$$

Thus $F(w_k) \nearrow F(w)$ a.e. in $L^2(\partial G_0)$. Since $F(w) \le F(u)$ and ∂G_0 has bounded measure, we have that

$$F(w_k) \to F(w) \quad \text{in } L^2(\partial G_0). \tag{4.14}$$

We have that

$$-\Delta w_{k+1} = \lambda(w_k - w_{k+1}) \le 0 \quad \mathcal{O}.$$

Hence, w_k are all subharmonic. Then, since $w_k \to 0$ as $|y| \to 0$ and $w_k \in \mathbb{X}$ and $w_k \le u$ on ∂G_0, by Lemma 4.4 we have that

$$0 \le w_k \le \frac{K_0 u}{|y|^{n-2}}.$$

in particular $w_n \in \mathbb{X}$. Passing to the limit we deduce that

$$0 \le w \le \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathcal{O}$$

Hence $w \to 0$ as $|y| \to +\infty$. Applying an equivalent argument to the one in Lemma 4.9 we have that ∇w_k is a Cauchy sequence in $L^2(\mathcal{O})^n$. In particular, there exists $\xi \in L^2(\mathcal{O})^n$ such that

$$\nabla w_k \to \xi \quad \text{in } L^2(\mathcal{O})^n.$$

Consider $\mathcal{O}' \subset \mathcal{O}$ open and bounded. Then we have that

$$\int_{\mathcal{O}'} |\nabla w_k|^2 \, \mathrm{d}y \le \int_{\mathcal{O}} |\nabla w_n|^2 \, \mathrm{d}y$$

is bounded, and

$$\int_{\mathcal{O}'} |w_k|^2 \, \mathrm{d}y \le |u|^2 |\mathcal{O}'|.$$

Hence, there is convergent subsequence in $H^1(\mathcal{O}')$. Any convergent subsequence must have the same limit, so $w_k \rightharpoonup w$ $H^1(\mathcal{O}')$. In particular

$$\xi = \nabla w \quad \text{a.e. } \mathcal{O}'.$$

Since this works for every $\mathcal{O}'$ bounded we have that $\nabla w \in L^2(\mathcal{O})^n$, hence $w \in \mathbb{X}$, and

$$\nabla w_n \to \nabla w \quad \text{in } L^2(\mathcal{O})^n.$$

Using this fact and (4.14), we can pass to the limit in the weak formulation to deduce that

$$\begin{gathered} -\Delta w = 0 \quad \mathcal{O}, \\ \frac{\partial w}{\partial n} = C_0 \sigma(u - w) \quad \partial G_0. \end{gathered}$$

In particular, a solution of (2.4). The same reasoning applies to case $u < 0$. □

Corollary 4.11. *Let $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing be such that*

$$|\sigma(u)| \le C(1 + |u|). \tag{4.15}$$

Then, there exists a unique solution of (2.4).

Proof. Let us assume first that $u > 0$. Let $\sigma_m \in C^1([0, |u|])$ be a pointwise increasing sequence that approximates σ uniformly in $[0, |u|]$. Since σ_m is Lipschitz, then $\widehat{w}_m$ exists by the previous part. Because of Lemma 4.7, the sequence $\widehat{w}_m$ of solutions of (2.4) is pointwise increasing. Since we know that, we have that $\widehat{w}_m \le u$ then, for a.e. $y \in \mathcal{O}$, $\widehat{w}_m(y)$ is a bounded and increasing sequence

$$\widehat{w}_m(y) \nearrow w(y).$$

for some $w(y)$. In particular

$$0 \le w(y) \le \frac{K_0 u}{|y|^{n-2}} \quad y \in \mathcal{O}.$$

Applying as in the proof of Lemma 4.10 we deduce that $w \in \mathbb{X}$ and it is a solution of (2.4). The proof for $u < 0$ follows in the same way, by taking a pointwise decreasing sequence σ_m. □

With the same techniques we can prove the following (applying that $u - w \ge 0$ for $u \ge 0$ and $u - w \le 0$ for $u \le 0$):

Lemma 4.12. *Let $u \in \mathbb{R}$, $\mathcal{O}' \subset \mathcal{O}$ bounded, σ, σ_m be nondecreasing continuous functions such that $\sigma(0) = \sigma_m(0) = 0$ and $|\sigma_m| \le |\sigma|$ and $\sigma_m \to \sigma$ in $\mathcal{C}([-2|u|, 2|u|])$. Then:*

$$\widehat{w}_m(\cdot; G_0, u) \to \widehat{w}(\cdot; G_0, u) \quad \text{strongly in } H^1(\mathcal{O}'). \tag{4.16}$$

Furthermore,

(1) *If $u \ge 0$ then $\widehat{w}_m \nearrow \widehat{w}$ a.e. $y \in \mathcal{O}$ and $y \in \partial G_0$.*
(2) *If $u \le 0$ then $\widehat{w}_m \searrow \widehat{w}$ a.e. $y \in \mathcal{O}$ and $y \in \partial G_0$.*

4.1.4. *Lipschitz continuity with respect to u.*

Lemma 4.13. *For every $y \in \mathbb{R}^n \setminus G_0$, $\widehat{w}(y; G_0, u)$ is a nondecreasing Lipschitz-continuous function with respect to u. In fact,*

$$|\widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2)| \le |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R},\ \forall y \in \mathbb{R}^n \setminus G_0. \tag{4.17}$$

Furthermore, for every $y \in \partial G_0$, $\partial_\nu \widehat{w}(y; G_0, u)$ is also nondecreasing in u.

Proof. Let us consider first that $\sigma \in \mathcal{C}^1(\mathbb{R})$. We have that $\widehat{w}(\cdot; G_0, u) \in \mathcal{C}(\overline{\mathcal{O}}) \cap \mathcal{C}^2(\mathcal{O})$ for every $u \in \mathbb{R}$ and the equation is satisfied pointwise (see [14]).

Let us first consider $u_1 > u_2$. We want to prove the following

$$0 \le \widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2) \le u_1 - u_2 \tag{4.18}$$

$$\partial_\nu \widehat{w}(u_1; G_0, y) \ge \partial_\nu \widehat{w}(y; G_0, u_2). \tag{4.19}$$

That

$$\widehat{w}(u_1; G_0, y) \ge \widehat{w}(y; G_0, u_2). \tag{4.20}$$

follows from the comparison principle. Indeed, let us plug $\widehat{w}(u_1; G_0, y)$ in the equation for $\widehat{w}(y; G_0, u_2)$:

$$\begin{gathered} -\Delta \widehat{w}(u_1; G_0, y) = 0 \quad \mathbb{R}^n \setminus G_0, \\ \partial_{\nu_y} \widehat{w}(u_1; G_0, y) - C_0 \sigma(u_2 - \widehat{w}(u_1; G_0, y)) \\ = C_0 \left(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2))\right) \ge 0 \quad \partial G_0, \\ \widehat{w}(u_1; G_0, y) \to 0 \quad |y| \to +\infty. \end{gathered} \tag{4.21}$$

Therefore, $\widehat{w}(u_1; G_0, y)$ is a supersolution of the problem for $\widehat{w}(y; G_0, u_2)$. Applying the comparison principle we deduce (4.20).

We define

$$g(u_1, u_2, y) = \widehat{w}(u_1; G_0, y) - \widehat{w}(y; G_0, u_2) \ge 0. \tag{4.22}$$

The function g is the solution of the following elliptic problem:

$$\begin{gathered}\Delta_y g = 0 \quad \text{if } y \in \mathbb{R}^n \setminus \overline{G_0},\\ \partial_{\nu_y} g - C_0\Big(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2))\Big) = 0 \quad \text{if } y \in \partial G_0, \\ g \to 0 \quad \text{as } |y| \to \infty.\end{gathered} \tag{4.23}$$

Let us consider the boundary condition for $y \in \partial G_0$:

$$\partial_{\nu_y} g(y) = C_0(\sigma(u_1 - \widehat{w}(u_1; G_0, y)) - \sigma(u_2 - \widehat{w}(y; G_0, u_2)))$$

multiplying by $u_1 - u_2 - g(u_1, u_2, y)$, and applying the monotonicity of σ, we have

$$(\partial_\nu g(y))(u_1 - u_2 - g(y)) \geq 0 \quad \forall y \in \partial G_0. \tag{4.24}$$

Let $g(y_0) = \max_{\partial G_0} g$ for some $y_0 \in \partial G_0$. By the strong maximum principle $g(y_0) = \max_{\mathbb{R}^n \setminus G_0} g$. Hence $g(y) \leq g(y_0)$ for $y \in \mathbb{R}^n \setminus G_0$ and we have

$$\partial_{\nu_y} g(y_0) \geq 0.$$

Assume, first, that σ is strictly increasing. We study two cases. If $\partial_{\nu_y} g(y_0) > 0$ then, by (4.24),

$$u_1 - u_2 \geq g(y_0) \geq g(y) \quad \forall y \in \mathbb{R}^n \setminus G_0.$$

If $\partial_\nu g(y_0) = 0$ then, by (4.23),

$$\begin{gathered}\sigma(u_1 - \widehat{w}(y_0; G_0, u_1)) = \sigma(u_2 - \widehat{w}(y_0; G_0, u_2))\\ u_1 - \widehat{w}(y_0; G_0, u_1) = u_2 - \widehat{w}(y_0; G_0, u_2)\\ u_1 - u_2 = g(y_0) \geq g(y) \quad \forall y \in \mathbb{R}^n \setminus G_0.\end{gathered}$$

Either way, we deduce that (4.18) holds. Hence,

$$\sigma(u_1 - \widehat{w}(u_1; G_0, y)) \geq \sigma(u_2 - \widehat{w}(y; G_0, u_2)) \quad \forall y \in \partial G_0$$

so (4.19) holds. This concludes the proof when σ is strictly increasing.

Let σ be a nondecreasing function and $U = \max\{|u_1|, |u_2|\}$. We consider an approximation sequence σ_m of σ in $[-2U, 2U]$ by strictly increasing smooth functions such that $|\sigma_m| \leq |\sigma|$. Consider $\widehat{w}_m$ as defined in Lemma 5.7. We have that

$$u_i - \widehat{w}(y; G_0, u_i) \in [-2U, 2U] \quad \forall i = 1, 2, \forall y \in \mathbb{R}^n \setminus \overline{G}_0.$$

By the previous part $\widehat{w}_m$ satisfies (4.18) and (4.19). Applying Lemma 4.12 we have a.e.-pointwise convergence $\widehat{w}_m(u_i, y) \to \widehat{w}(u_i, y)$ for $i = 1, 2$, up to a subsequence, as $m \to +\infty$. Therefore (4.18) and (4.19) hold almost everywhere in y. Since $\widehat{w}$ is continuous, (4.18) and (4.19) hold everywhere. This concludes the proof in the case $u_1 > u_2$.

If $u_1 < u_2$ we can exchange the roles of u_1 and u_2 in (4.18) to deduce (4.17). This concludes the proof. □

4.1.5. *Auxiliary function $\widehat{w}_\varepsilon^j$.* We conclude this section by introducing the following function:

Definition 4.14. Let $u \in \mathbb{R}$, $j \in \Upsilon_\varepsilon$ and $\varepsilon > 0$. We define

$$\widehat{w}_\varepsilon^j(x; G_0, u) = \widehat{w}\Big(\frac{x - P_\varepsilon^j}{a_\varepsilon}; G_0, u\Big). \tag{4.25}$$

It is clear that this function is the solution of the problem

$$\begin{aligned} -\Delta \widehat{w}_\varepsilon^j &= 0 \quad \mathbb{R}^n \setminus G_\varepsilon^j, \\ \partial_n \widehat{w}_\varepsilon^j - \varepsilon^{-\gamma}\sigma(u - \widehat{w}_\varepsilon^j) &= 0 \quad \partial G_\varepsilon^j \\ \widehat{w}_\varepsilon^j &\to 0 \quad |x| \to +\infty. \end{aligned} \tag{4.26}$$

We have the following estimates:

Lemma 4.15. *Let $\varepsilon, r > 0$ and $x \in \partial T_{r\varepsilon}^j$. Then*

$$|\widehat{w}_\varepsilon^j(x; G_0, u)| \le \frac{K|u|}{\left|\frac{x - P_\varepsilon^j}{a_\varepsilon}\right|^{n-2}} \le \frac{K|u| a_\varepsilon^{n-2}}{r^{n-2}\varepsilon^{n-2}} \le \frac{K|u|}{r^{n-2}}\varepsilon^2 \tag{4.27}$$

where K does not depend on $r, |u|$ or ε.

Lemma 4.16. *For $\varepsilon, r > 0$ be such that $a_\varepsilon < \frac{r\varepsilon}{2R_0}$. Let $x \in \partial T_{r\varepsilon}^j$. Then*

$$|\nabla \widehat{w}_\varepsilon^j(x; G_0, u)| \le \frac{K|u|}{r^{n-1}}\varepsilon, \tag{4.28}$$

where K does not depend on r, ε or j.

Proof. By the definition of $\widehat{w}_\varepsilon^j$ we have

$$\nabla \widehat{w}_\varepsilon^j(x; G_0, u) = a_\varepsilon^{-1}(\nabla \widehat{w})\Big(\frac{x - P_\varepsilon^j}{a_\varepsilon}; G_0, u\Big)$$

Therefore, for $x \in \partial T_{r\varepsilon}^j$,

$$\begin{aligned} |\nabla \widehat{w}_\varepsilon^j| = a_\varepsilon^{-1}\big|\nabla \widehat{w}\Big(\frac{x - P_\varepsilon^j}{a_\varepsilon}\Big)\big| &\le \frac{K|u| a_\varepsilon^{-1}}{\Big(\big|\frac{x - P_\varepsilon^j}{a_\varepsilon}\big| - R_0\Big)^{n-1}} \\ &\le \frac{K|u| a_\varepsilon^{n-2}}{(r\varepsilon - a_\varepsilon R_0)^{n-1}} \le \frac{K|u| a_\varepsilon^{n-2}}{\big(\frac{r\varepsilon}{2}\big)^{n-1}} \\ &\le \frac{K|u|}{r^{n-1}}\varepsilon. \end{aligned}$$

This completes the proof. □

4.2. **Properties of H_{G_0}.**

Lemma 4.17. *H_{G_0} is a nondecreasing function. Furthermore:*

1. *If σ satisfies (2.3), then so does H_{G_0}.*
2. *If $\sigma \in \mathcal{C}^{0,\alpha}(\mathbb{R})$, then so is H_{G_0}.*
3. *If $\sigma \in \mathcal{C}^1(\mathbb{R})$, then H_{G_0} is locally Lipschitz continuous.*
4. *If $\sigma \in W^{1,\infty}(\mathbb{R})$, then so is H_{G_0}.*

Proof. Let us prove the monotonicity of $H_{G_0}(u)$ given by (2.5). Let $u_1 > u_2$. By applying (4.19) we deduce that $H_{G_0}(u_1) \ge H_{G_0}(u_2)$.

Assume (2.3). Indeed, taking into account (4.17) we deduce

$$\begin{aligned}
&|H_{G_0}(u) - H_{G_0}(v)| \\
&\le C_0 \int_{\partial G_0} |\sigma(u - \widehat{w}(y;G_0,u)) - \sigma(v - \widehat{w}(y;G_0,v))| \,\mathrm{d}S_y \\
&\le C_0 k_1 \int_{\partial G_0} \Big(|u-v| + |\widehat{w}(y;G_0,u) - \widehat{w}(y;G_0,v)|\Big)^\alpha \mathrm{d}S_y \\
&\quad + C_0 k_2 \int_{\partial G_0} \Big(|u-v| + |\widehat{w}(y;G_0,u) - \widehat{w}(y;G_0,v)|\Big) \mathrm{d}S_y \\
&\le K_1|u-v|^\alpha + K_2|u-v|
\end{aligned} \tag{4.29}$$

In particular, if $u \in \mathcal{C}^{0,\alpha}(\mathbb{R})$, then $k_2 = 0$ and $K_2 = 0$.

Assume now that $\sigma \in \mathcal{C}^1(\mathbb{R})$. Let $u_1, u_2 \in \mathbb{R}$. We have that, for $y \in \partial G_0$

$$\begin{aligned}
&|\partial_{\nu_y}\widehat{w}(u_1;G_0,y) - \partial_{\nu_y}\widehat{w}(u_2,y)| \\
&= C_0|\sigma(u_1 - \widehat{w}(u_1;G_0,y)) - \sigma(u_2 - \widehat{w}(y;G_0,u_2))| \\
&\le C|\sigma'(\xi)|\Big(|u_1-u_2| + |\widehat{w}(u_1;G_0,y) - \widehat{w}(y;G_0,u_2)|\Big) \\
&\le C|\sigma'(\xi)||u_1-u_2|.
\end{aligned}$$

for some ξ between $u_1 - \widehat{w}(y;G_0,u_1)$ and $u_2 - \widehat{w}(y;G_0,u_2)$. Since $|\widehat{w}(u,y)| \le |u|$, for every $K \subset \mathbb{R}$ compact there exists a constant C_K such that

$$|\partial_{\nu_y}\widehat{w}(y;G_0,u_1) - \partial_{\nu_y}\widehat{w}(y;G_0,u_2)| \le C_K|u_1-u_2| \quad \forall u_1,u_2 \in K.$$

Therefore,

$$|H_{G_0}(u) - H_{G_0}(v)| \le \widetilde{C}_K|u_1-u_2| \quad \forall u_1,u_2 \in K.$$

Let $\sigma \in W^{1,\infty}(\mathbb{R})$. By approximation by nondecreasing functions $\sigma_n \in W^{1,\infty} \cap \mathcal{C}^1$, we obtain that

$$|\partial_{\nu_y}\widehat{w}(y;G_0,u_1) - \partial_{\nu_y}\widehat{w}(y;G_0,u_2)| \le 2\|\sigma'\|_\infty|u_1-u_2|. \tag{4.30}$$

Therefore,

$$|H_{G_0}(u) - H_{G_0}(v)| \le 2\|\sigma'\|_\infty|\partial G_0||u_1-u_2| \quad \forall u_1,u_2 \in \mathbb{R}. \tag{4.31}$$

This completes the proof. □

Lemma 4.18. *Let $u \in \mathbb{R}$. Let σ, σ_m be nondecreasing continuous functions such that $\sigma(0) = \sigma_m(0) = 0$ satisfy* (2.3) *with the same constants k_1, k_2 and α, $|\sigma_m| \le |\sigma|$ and $\sigma_m \to \sigma$ in $\mathcal{C}([-2U,2U])$ for some $U > 0$. Then*

$$H_{G_0,m} \to H_{G_0} \quad \text{in } \mathcal{C}([-U,U]). \tag{4.32}$$

Proof. Let $u \in [0,U]$. By Lemma 4.12 we know that

$$u - \widehat{w}_m(y;G_0,u) \searrow u - \widehat{w}(y;G_0,u) \text{ for a.e. } y \in \partial G_0.$$

In particular, due to the dominated convergence theorem, $H_{G_0,m}(u) \to H_{G_0}(u)$. An equivalent argument applies to $u \in [-U,0]$. Hence

$$H_{G_0,m} \to H_{G_0} \quad \text{pointwise in } [-U,U].$$

Since all σ_m satisfy (2.3) with the same k_1, k_2, α, we know that H_m satisfies (4.29) with the same K_1, K_2 and α. Hence, $H_{G_0,m}$ is an equicontinuous sequence. Applying the Ascoli-Arzela theorem we know that the sequence is relatively compact in $\mathcal{C}([-U,U])$ with the supremum norm. It has, at least, a uniformly convergent subsequence. Since every convergent subsequence has to converge to H_{G_0}, we know that the the whole sequence $H_{G_0,m}$ converges to H_{G_0} uniformly in $[-U,U]$. $\square$

Remark 4.19. When ∂G_0 is assumed C^2 it is possible to develop other type of techniques (which we shall not present in detail here) showing the existence and uniqueness of solution $\widehat{w}(y;G_0,u)$. Indeed, the existence of $\widehat{w}(y;G_0,u)$ can be built through passing to the limit after a truncation of the domain process (with the artificial boundary condition $\widehat{w}(y;G_0,u) = 0$ on the new truncated boundary). The maximum principle for classical solutions (see, e.g. [14], [8, p.206] or [1]) allows to get universal a priori estimates which justify the weak convergence and thanks to the monotonicity of the nonlinear term in the interior boundary condition the passing to the limit can well-justified. In addition, it can be proved (see the indicated references) that the limit is also a classical solution on the whole exterior domain. Moreover, the same technique (i.e. the maximum principle for classical solutions) implies the comparison, uniqueness and continuous dependence of the solution $\widehat{w}(y;G_0,u)$.

5. Proof in the smooth case $\sigma \in \mathcal{C}^1(\mathbb{R})$

5.1. **Auxiliary function w_ε^j.** To pass to the limit as $\varepsilon \to 0$ in (3.2) we need some auxiliary functions.

Definition 5.1. Let $u \in \mathbb{R}$, $\varepsilon > 0$ and $j \in \Upsilon_\varepsilon$. We define the function $w_\varepsilon^j(\cdot;G_0,u)$ as the solution of the problem

$$\begin{gathered} \Delta w_\varepsilon^j = 0 \quad \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ \partial_{\nu_x} w_\varepsilon^j - \varepsilon^{-\gamma}\sigma(u - w_\varepsilon^j) = 0 \quad \text{if } x \in \partial G_\varepsilon^j, \\ w_\varepsilon^j = 0 \quad \text{if } x \in \partial T^j_{\varepsilon/4}, \end{gathered} \tag{5.1}$$

where

$$T_r^j = \{x \in \mathbb{R}^n : |x - P_\varepsilon^j| \le r\}, \tag{5.2}$$

P_ε^j is the center of Y_ε^j. Finally, we define

$$W_\varepsilon(x;G_0,u) = \begin{cases} w_\varepsilon^j(x;G_0,u) & \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, j \in \Upsilon_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \cup_{j\in\Upsilon_\varepsilon} \overline{T^j_{\varepsilon/4}}. \end{cases} \tag{5.3}$$

Applying the comparison principle we obtain the following result.

Lemma 5.2. *Let $u \ge 0$. Then $0 \le w_\varepsilon^j(\cdot;G_0,u) \le \widehat{w}_\varepsilon^j(\cdot;G_0,u)$. If $u \le 0$ then $\widehat{w}_\varepsilon^j(\cdot;G_0,u) \le w_\varepsilon^j(\cdot;G_0,u) \le 0$.*

Remark 5.3. Note that, if $u = 0$, then $w_\varepsilon^j(\cdot;G_0,0) \equiv 0$.

Let us prove some properties of $W_\varepsilon(x;G_0,u)$. First, we introduce the following lemma.

Lemma 5.4 (Uniform trace constant)**.** *There exists a constant $C_T > 0$ such that, for all $\varepsilon > 0$*

$$\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |f|^2 \, \mathrm{d}S \le C_T \int_{T^j_{\varepsilon/4} \setminus G^j_\varepsilon} |\nabla f|^2 \, \mathrm{d}x \quad \forall f \in H^1\big(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\frac{\varepsilon}{4}}\big) \tag{5.4}$$

Proof. First, we extend f to $H_0^1(Y_\varepsilon^j)$ where $Y_\varepsilon^j = \varepsilon j + \varepsilon Y$. In [17] we find that

$$\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |f|^2 \le C\Big(\int_{Y_\varepsilon^j} |f|^2 + \int_{Y_\varepsilon^j} |\nabla f|^2 \Big). \tag{5.5}$$

Since $f = 0$ on ∂Y_ε^j, taking $\tilde f(y) = f(P_\varepsilon^j + \varepsilon y)$, we have

$$\int_Y |\tilde f|^2 \,\mathrm{d}y \le C \int_Y |\nabla \tilde f|^2 \,\mathrm{d}y. \tag{5.6}$$

Since $\nabla_x f = \varepsilon \nabla_y \tilde f$ we have

$$\int_{Y_\varepsilon^j} |f|^2 \le \varepsilon^2 \int_{Y_\varepsilon^j} |\nabla f|^2. \tag{5.7}$$

Hence, the result is proved. □

We have some precise estimates on the norm of W_ε:

Lemma 5.5. *For all $u \in \mathbb{R}$, we have*

$$\|\nabla W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K(|u| + |u|^2), \tag{5.8}$$

$$\|W_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^2. \tag{5.9}$$

Proof. Let $u \in \mathbb{R}$ be fixed. If we take w_ε^j as a test function in weak formulation of problem (5.1) we obtain

$$\int_{T^j_{\varepsilon/4} \setminus G_\varepsilon^j} |\nabla w_\varepsilon^j|^2 \,\mathrm{d}x - \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j) w_\varepsilon^j \,\mathrm{d}S = 0.$$

We rewrite this as follows:

$$\int_{T^j_{\varepsilon/4} \setminus G_\varepsilon^j} |\nabla w_\varepsilon^j|^2 \,\mathrm{d}x + \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) \,\mathrm{d}S = \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} \sigma(u - w_\varepsilon^j) u \,\mathrm{d}S.$$

Since σ is nondecreasing we have that

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)} \le \varepsilon^{-\gamma} |u| \int_{\partial G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| \,\mathrm{d}S.$$

Because of (2.3) and that $|s|^\alpha \le 1 + |s|$ for every $s \in \mathbb{R}$, we have

$$\begin{aligned}
\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| \,\mathrm{d}S &\le k_1 \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j|^\alpha \,\mathrm{d}S + k_2 \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \,\mathrm{d}S \\
&\le k_1 \varepsilon^{-\gamma} |\partial G_\varepsilon^j| + (k_1 + k_2)\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \,\mathrm{d}S.
\end{aligned}$$

Applying Lemma 5.4 and that, for every $a, b, C \in \mathbb{R}$ it holds that $ab \le \frac{C^2}{2} a^2 + \frac{1}{2C^2} b^2$, we obtain

$$\begin{aligned}
(k_1 + k_2)\varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j| \,\mathrm{d}S &\le \varepsilon^{-\gamma} C |\partial G_\varepsilon^j| + \frac{1}{2C_T|u|} \varepsilon^{-\gamma} \int_{\partial G_\varepsilon^j} |u - w_\varepsilon^j|^2 \,\mathrm{d}S \\
&\le C|u| \varepsilon^{-\gamma} |\partial G_\varepsilon^j| + \frac{1}{2C_T|u|} \|u - w_\varepsilon^j\|^2_{L^2(\partial G_\varepsilon^j)} \\
&\le C|u| \varepsilon^n + \frac{1}{2|u|} \|\nabla(u - w_\varepsilon^j)\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)} \\
&= C|u| \varepsilon^n + \frac{1}{2|u|} \|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4} \setminus G_\varepsilon^j)}.
\end{aligned}$$

Therefore,

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^n + \frac{1}{2}\|\nabla w_\varepsilon^j\|^2_{L^2(\partial G^j_\varepsilon)}.$$

Thus, we have

$$\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^n.$$

Adding over $j \in \Upsilon_\varepsilon$, and taking into account that $\#\Upsilon_\varepsilon \le d\varepsilon^{-n}$, we deduce that (5.8) holds. Using Friedrich's inequality we obtain

$$\|w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le \varepsilon^2 K\|\nabla w_\varepsilon^j\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)},$$

so (5.9) holds. This completes the proof. □

5.2. **Auxiliary function $v_\varepsilon^j = w_\varepsilon^j - \widehat{w}_\varepsilon^j$.** Let us define:

$$v_\varepsilon^j = w_\varepsilon^j - \widehat{w}_\varepsilon^j. \tag{5.10}$$

This functions is the solution of the problem

$$\begin{gathered} \Delta v_\varepsilon^j = 0 \quad \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ \partial_\nu v_\varepsilon^j - \varepsilon^{-\gamma}(\sigma(u-w_\varepsilon^j) - \sigma(u-\widehat{w}_\varepsilon^j)) = 0, \quad \text{if } x \in \partial G^j_\varepsilon, \\ v_\varepsilon^j = -\widehat{w}_\varepsilon^j(x;G_0,u), \quad \text{if } x \in \partial T^j_{\varepsilon/4}. \end{gathered} \tag{5.11}$$

Lemma 5.6. *The following estimates hold*

$$\sum_{j\in\Upsilon_\varepsilon} \|\nabla(w_\varepsilon^j(x;G_0,u) - \widehat{w}_\varepsilon^j(x;G_0,u))\|^2_{L^2(T^j_{\frac{\varepsilon}{4}}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^2, \tag{5.12}$$

$$\sum_{j\in\Upsilon_\varepsilon} \|w_\varepsilon^j(x;G_0,u) - \widehat{w}_\varepsilon^j(x;G_0,u)\|^2_{L^2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u|+|u|^2)\varepsilon^4. \tag{5.13}$$

Proof. From Lemma 5.2 it is clear that

$$|v_\varepsilon^j(x;G_0,u)| \le |\widehat{w}_\varepsilon^j(x;G_0,u)| \quad \forall x \in \overline{T^j_{\varepsilon/4}} \setminus G^j_\varepsilon. \tag{5.14}$$

Integrating by parts $v_\varepsilon^j(\Delta v_\varepsilon^j)$ and using (5.11) we deduce that

$$\begin{aligned} &\int_{T^j_{\varepsilon/4}\setminus G^j_\varepsilon} |\nabla v_\varepsilon^j|^2\,\mathrm{d}x - \varepsilon^{-\gamma}\int_{\partial G^j_\varepsilon} (\sigma(u-w_\varepsilon^j) - \sigma(u-\widehat{w}))v_\varepsilon^j\,\mathrm{d}S \\ &= -\int_{\partial T^j_{\varepsilon/4}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j(x;G_0,u)\,\mathrm{d}S \end{aligned}$$

By the monotonicity of σ and applying Green's first identity, we have

$$\begin{aligned} \|\nabla v_\varepsilon^j\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} &\le -\int_{\partial T^j_{\varepsilon/4}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j(x;G_0,u)\,\mathrm{d}S \\ &= -\int_{T^j_{\varepsilon/4}\setminus T^j_{\varepsilon/8}} \nabla v_\varepsilon^j \nabla \widehat{w}_\varepsilon^j\,\mathrm{d}x + \int_{\partial T^j_{\varepsilon/8}} (\partial_\nu v_\varepsilon^j)\widehat{w}_\varepsilon^j\,\mathrm{d}S. \end{aligned}$$

Applying Lemmas 4.15 and 4.16 we have

$$\begin{gathered} |v_\varepsilon^j(x;G_0,u)| \le |\widehat{w}_\varepsilon^j(x;G_0,u)| \le K|u|\varepsilon^2. \\ |\nabla \widehat{w}_\varepsilon^j(x;G_0,u)| \le K|u|\varepsilon \end{gathered}$$

for all $x \in T^j_{\varepsilon/8}$, where K does not depend on ε. Since v^j_ε is harmonic, denoting $T^x_r = \{z \in \mathbb{R}^n : |x - z| < r\}$ we have

$$|\frac{\partial v^j_\varepsilon}{\partial x_i}(x)| = \frac{1}{|T^x_{\varepsilon/16}|}\Big|\int_{T^x_{\varepsilon/16}} \frac{\partial v^j_\varepsilon}{\partial x_i}\,\mathrm{d}x\Big| = \frac{K}{\varepsilon^n}\Big|\int_{\partial T^x_{\varepsilon/16}} v^j_\varepsilon \nu_i\,\mathrm{d}S\Big| \le K|u|\varepsilon.$$

for all $x \in T^j_{\varepsilon/4} \setminus T^j_{\frac{\varepsilon}{8}}$, since $T^x_{\varepsilon/16} \subset T^j_{\varepsilon/4} \setminus T^j_{\varepsilon/16}$. Hence, we have

$$\begin{gathered}\Big|\int_{T^j_{\varepsilon/4}\setminus T^j_{\varepsilon/8}} \nabla v^j_\varepsilon \nabla \widehat{w}^j_\varepsilon\,\mathrm{d}x\Big| \le K(|u| + |u|^2)\varepsilon^{n+2},\\ \Big|\int_{\partial T^j_{\frac{\varepsilon}{8}}} (\partial_\nu v^j_\varepsilon)\widehat{w}^j_\varepsilon\,\mathrm{d}S\Big| \le K(|u| + |u|^2)\varepsilon^{n+2}.\end{gathered}$$

From this we deduce that

$$\|\nabla v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^{n+2}.$$

From Friedrich's inequality,

$$\|v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^{n+4}.$$

Then, adding over $j \in \Upsilon_\varepsilon$ we obtain

$$\begin{gathered}\sum_{j\in\Upsilon_\varepsilon} \|\nabla v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^2,\\ \sum_{j\in\Upsilon_\varepsilon} \|v^j_\varepsilon\|^2_{L_2(T^j_{\varepsilon/4}\setminus G^j_\varepsilon)} \le K(|u| + |u|^2)\varepsilon^4.\end{gathered}$$

This estimates completes the proof. □

5.3. Convergence of integrals over $\cup_{j\in\Upsilon_\varepsilon}\partial T^j_{\varepsilon/4}$.

Lemma 5.7. *Let $H_{G_0}(u)$ be defined by formula (2.5), $\phi \in C^\infty_0(\Omega)$ and $h_\varepsilon, h \in H^1_0(\Omega)$ be such that $h_\varepsilon \rightharpoonup h$ in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Then, we have that*

$$-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}} \Big(\partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon))\Big)h_\varepsilon(x)\,\mathrm{d}S = C^{n-2}_0\int_\Omega H_{G_0}(\phi(x))\,h(x)\,\mathrm{d}x \tag{5.15}$$

where ν is an unit outward normal vector to $T^j_{\varepsilon/4}$.

Proof. Let us consider the auxiliary problem

$$\begin{gathered}\Delta\theta^j_\varepsilon = \mu^j_\varepsilon \quad x \in Y^j_\varepsilon \setminus \overline{T}^j_{\varepsilon/4},\ j \in \Upsilon_\varepsilon,\\ -\partial_\nu\theta^j_\varepsilon = \partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon)) \quad x \in \partial T^j_\varepsilon,\\ -\partial_\nu\theta^j_\varepsilon = 0 \quad x \in \partial Y^j_\varepsilon,\\ \langle\theta^j_\varepsilon\rangle_{Y^j_\varepsilon\setminus\overline{T}^j_{\varepsilon/4}} = 0,\end{gathered} \tag{5.16}$$

where ν is a unit inwards normal vector of the boundary of $Y^j_\varepsilon \setminus T^j_{\varepsilon/4}$. We choose, against the convention, the inward normal vector so that it coincides with the unit outward normal vector of $T^j_{\varepsilon/4} \setminus G^j_\varepsilon$ in their shared boundary. We changed the sign accordingly. The constant μ_ε^j is given by the compatibility condition of the problem (5.16):

$$\begin{aligned}\mu_\varepsilon^j\varepsilon^n\big|Y\setminus T_{1/4}^0\big| &= \int_{\partial T_{\varepsilon/4}^j}\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\,\mathrm{d}S\\ &= -\int_{\partial G_\varepsilon^j}\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\,\mathrm{d}S\\ &= -a_\varepsilon^{n-2}\int_{\partial G_0}\partial_{\nu_y}\widehat{w}(\phi(P_\varepsilon^j),y)\,\mathrm{d}S_y,\end{aligned}$$

Therefore,

$$\mu_\varepsilon^j=\frac{-a_\varepsilon^{n-2}H_{G_0}(\phi(P_\varepsilon^j))}{|Y\setminus T_{1/4}^0|\varepsilon^n}=\frac{-C_0^{n-2}H_{G_0}(\phi(P_\varepsilon^j))}{|Y\setminus T_{1/4}^0|},$$

From the integral identity for the problem (5.16) we obtain

$$-\int_{Y_\varepsilon^j\setminus T_\varepsilon^j}|\nabla\theta_\varepsilon^j|^2\,\mathrm{d}x=\mu_\varepsilon^j\int_{Y_\varepsilon^j\setminus T_{\varepsilon/4}^j}\theta_\varepsilon^j dx-\int_{\partial T_{\varepsilon/4}^j}\Big(\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\Big)\,\theta_\varepsilon^j\,\mathrm{d}S. \quad (5.17)$$

Applying Lemma 4.16 and using the estimates from [17], we deduce

$$\begin{aligned}&\int_{\partial T_{\varepsilon/4}^j}\big|\big(\partial_{\nu_x}\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\big)\theta_\varepsilon^j\big|\,\mathrm{d}S\\ &\le K|\phi(P_\varepsilon^j)|\varepsilon\int_{\partial T_{\varepsilon/4}^j}|\theta_\varepsilon|\,\mathrm{d}S\\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n-1}{2}+1}\|\theta_\varepsilon^j\|_{L_2(\partial T_{\varepsilon/4}^j)}\\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n+1}{2}}\big\{\varepsilon^{-\frac12}\|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j\setminus\overline{T}_{\varepsilon/4}^j)}+\sqrt{\varepsilon}\|\nabla\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j\setminus\overline{T}_{\varepsilon/4}^j)}\big\}\\ &\le K|\phi(P_\varepsilon^j)|\varepsilon^{\frac{n+2}{2}}\|\nabla\theta_\varepsilon^j\|_{L_2(Y_\varepsilon\setminus\overline{T}_{\varepsilon/4}^j)}.\end{aligned}$$

In particular, since $|\phi(P_\varepsilon^j)|\le\|\phi\|_\infty$ we can make a uniform bound, independent of j and ε. Thus, we have

$$\|\nabla\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j\setminus\overline{T}_\varepsilon^j)}^2\le K\varepsilon^{n+2}. \quad (5.18)$$

Adding over $j\in\Upsilon_\varepsilon$ we have

$$\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T}_{\varepsilon/4}^j}|\nabla\theta_\varepsilon^j|^2\,\mathrm{d}x\le K\varepsilon^2. \quad (5.19)$$

Hence, by the definition of θ_ε^j, we obtain

$$\begin{aligned}&\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\Big(\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\Big)h_\varepsilon\,\mathrm{d}S-\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x\Big|\\ &=\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\nabla\theta_\varepsilon^j\nabla h_\varepsilon\,\mathrm{d}x\Big|\le K\varepsilon\|h_\varepsilon\|_{H_1(\Omega,\partial\Omega)}.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\Big(\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\Big)\,h_\varepsilon\,\mathrm{d}S=\lim_{\varepsilon\to0}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x.$$

From the definition of μ_ε^j we deduce

$$\begin{aligned}
&\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x+\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))\,h_\varepsilon\,\mathrm{d}x\\
&=-\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\Big(H_{G_0}(\phi(P_\varepsilon^j))-H_{G_0}(\phi(x))\Big)h_\varepsilon\,\mathrm{d}x.
\end{aligned}$$

Using (4.17) we obtain

$$\begin{aligned}
&\Big|\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}(H_{G_0}(\phi(P_\varepsilon^j))-H_{G_0}(\phi(x)))h_\varepsilon\,\mathrm{d}x\Big|\\
&\le K\|h_\varepsilon\|_{L_2(\Omega)}\max_j\Big|\int_{\partial G_0}\partial_{\nu_y}\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\partial_{\nu_y}\widehat{w}(y;G_0,\phi(x))\,\mathrm{d}S_y\Big|\\
&=K\|h_\varepsilon\|_{L_2(\Omega)}\max_j\Big|\int_{\partial G_0}\sigma\Big(\phi(P_\varepsilon^j)-\widehat{w}(y;G_0,\phi(x))\Big)\\
&\quad-\sigma\Big(\phi(P_\varepsilon^j)-\widehat{w}(y;G_0,\phi(P_\varepsilon^j))\Big)\,\mathrm{d}S_y\Big|\\
&\le K\max_j\Big(\Big|\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\widehat{w}(y;G_0,\phi(x))\Big|\\
&\quad+\Big|\widehat{w}(y;G_0,\phi(P_\varepsilon^j))-\widehat{w}(y;G_0,\phi(x))\Big|^\alpha\Big)\\
&\le K\max_j\Big(|\phi(P_\varepsilon^j)-\phi(x)|+|\phi(P_\varepsilon^j)-\phi(x)|^\alpha\Big)\\
&\le K(a_\varepsilon+a_\varepsilon^\alpha)\to 0\quad\text{as }\varepsilon\to 0.
\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}\mu_\varepsilon^j h_\varepsilon\,\mathrm{d}x=-\lim_{\varepsilon\to 0}\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))h_\varepsilon\,\mathrm{d}x.$$

From [16, Corollary 1.7] we derive

$$\lim_{\varepsilon\to 0}\frac{C_0^{n-2}}{|Y\setminus T_{1/4}^0|}\sum_{j\in\Upsilon_\varepsilon}\int_{Y_\varepsilon^j\setminus\overline{T_{\varepsilon/4}^j}}H_{G_0}(\phi(x))h_\varepsilon\,\mathrm{d}x=C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))h\,\mathrm{d}x.$$

This completes the proof. □

Lemma 5.8. *Let $H_{G_0}(u)$ be defined by formula (2.5), $\phi\in C_0^\infty(\Omega)$ and $h_\varepsilon,h\in H_0^1(\Omega)$ be such that $h_\varepsilon\rightharpoonup h$ in $H_0^1(\Omega)$ as $\varepsilon\to 0$. Then, we have*

$$-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon}\int_{\partial T_{\frac{\varepsilon}{4}}^j}\big(\partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\big)\,h_\varepsilon\,\mathrm{d}S=C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))h\,\mathrm{d}x. \tag{5.20}$$

Proof. Using Lemma 5.6 and applying Green's identity we obtain

$$\begin{aligned}
&\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\Big(\partial_\nu\widehat{w}_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))-\partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))\Big)h_\varepsilon\,\mathrm{d}S\\
&=-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_{\varepsilon/4}^j}\partial_\nu v_\varepsilon^j h_\varepsilon\,\mathrm{d}S\\
&=-\sum_{j\in\Upsilon_\varepsilon}\int_{T_{\varepsilon/4}^j\setminus G_\varepsilon^j}\nabla v_\varepsilon^j\nabla h_\varepsilon\,\mathrm{d}x+\int_{\partial G_\varepsilon^j}\partial_\nu v_\varepsilon^j h_\varepsilon\,\mathrm{d}S
\end{aligned}$$

$$
\begin{aligned}
&= - \sum_{j\in\Upsilon_\varepsilon} \int_{T^j_{\varepsilon/4}\setminus G^j_\varepsilon} \nabla v^j_\varepsilon \nabla h_\varepsilon \,\mathrm{d}x \\
&\quad + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} \big(\sigma(\phi(P^j_\varepsilon) - w^j_\varepsilon) - \sigma(\phi(P^j_\varepsilon) - \widehat{w})\big)\, h_\varepsilon \,\mathrm{d}S.
\end{aligned}
$$

From Cauchy's inequality and the properties of v^j_ε we have

$$
\begin{aligned}
\Big| \sum_{j\in\Upsilon_\varepsilon} \int_{T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}} \nabla v^j_\varepsilon \nabla h_\varepsilon \,\mathrm{d}x \Big| &\le \varepsilon^{-1} \sum_{j\in\Upsilon_\varepsilon} \|\nabla v^j_\varepsilon\|^2_{L^2(T^j_{\frac{\varepsilon}{4}})} + \varepsilon \|\nabla h_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \\
&\le K\varepsilon.
\end{aligned}
$$

Using the estimates from Lemma 5.6 we deduce

$$
\begin{aligned}
&\varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \Big| \int_{\partial G^j_\varepsilon} \big(\sigma(\phi(P^j_\varepsilon) - w^j_\varepsilon) - \sigma(\phi(P^j_\varepsilon) - \widehat{w}^j_\varepsilon)\big)\, h_\varepsilon \,\mathrm{d}S \Big| \\
&\le \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} \|\sigma'\|_{L^\infty([-2\|\phi\|_\infty, 2\|\phi\|_\infty])} |v^j_\varepsilon| |h_\varepsilon| \,\mathrm{d}S \\
&\le K\varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G^j_\varepsilon} |v^j_\varepsilon| |h_\varepsilon| \,\mathrm{d}S \\
&\le K\varepsilon\varepsilon^{-\gamma/2} \|h_\varepsilon\|_{L_2(S_\varepsilon)} \\
&\le K\varepsilon \|\nabla h_\varepsilon\|_{L^2(\Omega)},
\end{aligned}
$$

where K depends on $\|\phi\|_\infty$. Therefore,

$$
\Big| \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \big(\partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon)) - \partial_\nu w^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon))\big)\, h_\varepsilon \,\mathrm{d}S \Big| \le K\varepsilon. \tag{5.21}
$$

From this inequality and Lemma 5.7 we deduce that

$$
\begin{aligned}
&- \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon} \int_{\partial T^j_{\varepsilon/4}} \Big(\partial_\nu w^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon))\Big) h_\varepsilon \,\mathrm{d}S \\
&= - \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \Big(\partial_\nu \widehat{w}^j_\varepsilon(x; G_0, \phi(P^j_\varepsilon))\Big)\, h_\varepsilon \,\mathrm{d}S \\
&= C_0^{n-2} \int_\Omega H_{G_0}(\phi(x))\, h \,\mathrm{d}x.
\end{aligned}
$$

This completes the proof. □

5.4. **Proof of Theorem 2.5 for $\sigma \in \mathcal{C}^1(\mathbb{R})$.** Let $\phi \in C_0^\infty(\Omega)$. We define

$$
\widetilde{W}_\varepsilon(x;\phi) = \begin{cases} W_\varepsilon(x; G_0, \phi(P^j_\varepsilon)) & Y^j_\varepsilon \setminus \overline{G^j_\varepsilon}, j \in \Upsilon_\varepsilon \\ 0 & \Omega \setminus \cup_{j\in\Upsilon_\varepsilon} \overline{Y}^j_\varepsilon, j \in \Upsilon_\varepsilon. \end{cases} \tag{5.22}
$$

We have that $\widetilde{W}_\varepsilon(\cdot;\phi) \in H_0^1(\Omega)$ and $\widetilde{W}_\varepsilon(\cdot;\phi) \rightharpoonup 0$ in $H^1(\Omega)$ as $\varepsilon \to 0$. Using $\varphi = \phi - \widetilde{W}_\varepsilon(x;\phi)$ as a test function in inequality (3.2) we obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(\phi - \widetilde{W}_\varepsilon(x;\phi))\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi)) - u_\varepsilon)\,\mathrm{d}x \\
&+ \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S \qquad (5.23)\\
&\ge \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Taking into account that $w_\varepsilon^j(x;G_0,u)$ is a solution of the problem (5.1), we can rewrite this in the form

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x \\
&- \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))(\phi - u_\varepsilon)\,\mathrm{d}S \\
&- \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi(P_\varepsilon^j) - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S \\
&\qquad (5.24)\\
&+ \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon)\,\mathrm{d}S \qquad (5.25)\\
&\ge \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

We choose the boundary condition for w_ε^j so that (5.24) cancels (5.25) out in the limit. We observe that

$$\begin{aligned}
\rho_\varepsilon &= \Big|\varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \Big(\sigma(\phi(P_\varepsilon^j) - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))) - \sigma(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)))\Big) \\
&\quad \times \Big(\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon\Big)\,\mathrm{d}S\Big| \\
&\le \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \|\sigma'\|_{L^\infty([-U,U])} \|\nabla\phi\|_{L^\infty(\Omega)} a_\varepsilon |\phi - w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j)) - u_\varepsilon|\,\mathrm{d}S \\
&\le K a_\varepsilon \to 0,
\end{aligned}$$

where $U = 2\|\phi\|_\infty$ and K depends of $\|\phi\|_\infty$. Taking this into account we have

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\frac{\varepsilon}{4}}} \partial_\nu w_\varepsilon^j(x;G_0,\phi(P_\varepsilon^j))(\phi - u_\varepsilon)\,\mathrm{d}S \qquad (5.26)\\
&\ge \int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x - \rho_\varepsilon.
\end{aligned}$$

From Lemma 5.5 we have

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x = \int_\Omega \nabla\phi\nabla(\phi - u_0)\,\mathrm{d}x, \qquad (5.27)$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} f(\phi - \widetilde{W}_\varepsilon(x;\phi) - u_\varepsilon)\,\mathrm{d}x = \int_\Omega f(\phi - u_0)\,\mathrm{d}x. \tag{5.28}$$

Applying Lemma 5.8 for $h_\varepsilon = \phi - u_\varepsilon$ we have

$$-\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T^j_{\frac{\varepsilon}{4}}}\Big(\partial_\nu w^j_\varepsilon(x;G_0,\phi(P^j_\varepsilon))\Big)(\phi-u_\varepsilon)\,\mathrm{d}S = C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))(\phi-u_0)dx.$$

Therefore u_0 satisfies the inequality

$$\int_\Omega \nabla\phi\nabla(\phi-u_0)\,\mathrm{d}x + C_0^{n-2}\int_\Omega H_{G_0}(\phi(x))(\phi-u_0)\,\mathrm{d}x \ge \int_\Omega f(\phi-u_0)\,\mathrm{d}x.$$

for any $\phi \in H^1_0(\Omega)$. Therefore, $u \in H^1_0(\Omega)$ satisfies the identity

$$\int_\Omega \nabla u_0\nabla\phi\,\mathrm{d}x + C_0^{n-2}\int_\Omega H_{G_0}(u_0)\phi\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x,$$

where $\phi \in H^1_0(\Omega)$. Thus, u is a weak solution of (2.7). This completes the proof of the Theorem 2.5 when σ is $\mathcal{C}^1(\mathbb{R})$. □

6. Proof in the Hölder-continuous case

Let $\sigma \in \mathcal{C}(\Omega)$ be satisfying (2.3). Applying [6, Lemma 2] we deduce there a sequence of nondecreasing functions $\sigma_\delta \in C^1(\mathbb{R})$ such that $\sigma_\delta(0) = 0$, $|\sigma_\delta| \le |\sigma|$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}(\mathbb{R})$. Therefore σ_δ satisfies (2.3). Applying the result in the previous section, we have that

$$P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta \quad \text{in } H^1(\Omega). \tag{6.1}$$

where u_δ is the solution of (2.7) with H_δ instead of H_{G_0}.

By the approximation lemmas in [6] we have

$$\|\nabla(u_\varepsilon - u_{\varepsilon,\delta})\|_{L^2(\Omega_\varepsilon)} \le C\|\sigma_\delta - \sigma\|_\infty \tag{6.2}$$

Therefore,

$$\|\nabla(u - u_\delta)\|_{L^2(\Omega)} \le C\|\sigma_\delta - \sigma\|_\infty \tag{6.3}$$

Since, by Lemma 4.18, H_{δ,G_0} converges uniformly over compacts to HG, applying standard methods (see Lemma 7.1) we deduce that $u_\delta \to \widehat{u}_0$, where $\widehat{u}_0$ is the solution of (2.7). Notice that, due to Lemma 4.17, we have that, if $u_0 \in L^2(\Omega)$ then $H_{G_0}(u_0) \in L^2(\Omega)$.

By uniqueness of the limit $u_0 = \widehat{u}$ and it is the solution of (2.7). This completes the proof of Theorem 2.5 in the general case. □

7. Appendix: A convergence lemma

Lemma 7.1. *Let $H_m, H : \mathbb{R} \to \mathbb{R}$ be nondecreasing functions that satisfy* (2.3) *with the same constants k_1, k_2, and such that $H_m \to H$ uniformly over compacts. Let u_m, u be the corresponding solutions of* (2.7) *with H_m and H respectively. Then*

$$u_m \rightharpoonup u \text{ in } H^1_0(\Omega). \tag{7.1}$$

Proof. We have

$$\int_\Omega |\nabla u_m|^2\,\mathrm{d}x \le C\int_\Omega |f|^2\,\mathrm{d}x \tag{7.2}$$

Therefore, up to a subsequence, there is a weak limit in $H^1_0(\Omega)$, let this be $\widetilde{u}$. A further subsequence guaranties that

$$u_m \to \widetilde{u} \quad \text{in } L^2(\Omega),$$

$$u_m \to \widetilde{u} \quad \text{a.e. } \Omega.$$

Let $x \in \Omega$ such that $u_m(x) \to u(x)$ in $\mathbb{R}$. In particular the sequence is bounded so $H_m(u_m(u(x))) \to H(u(x))$ because of the uniform convergence over compact sets. Hence

$$H_m(u_m) \to H(\widetilde{u}) \quad \text{a.e. in } \Omega. \tag{7.3}$$

On the other hand, we have

$$\begin{gathered}
|H_m(u_m)| \le k_1|u_m|^\alpha + k_2|u_m| \le k_1 + (k_1+k_2)|u_m|, \\
\int_\Omega |H(u_m)|^2 \,\mathrm{d}x \le C\Big(|\Omega| + \int_\Omega |u_m|^2 \,\mathrm{d}x\Big) \\
\le C\Big(|\Omega| + \int_\Omega |f|^2 \,\mathrm{d}x\Big)
\end{gathered}$$

Hence, up to a subsequence, there exists $\widetilde{H} \in L^2(\Omega)$ such that

$$H_m(u_m) \rightharpoonup \widetilde{H} \text{ in } L^2(\Omega).$$

By Egorov's theorem, we have that, for every $\delta > 0$ there exists A_δ measurable such that $|A_\delta| < \delta$ and $H_m(u_m) \to H(\widetilde{u})$ uniformly $\Omega \setminus A_\delta$. Since $H_m(u_m) \rightharpoonup \widetilde{H}$ in $L^2(\Omega \setminus A_\delta)$ we have that $H(\widetilde{u}) = \widetilde{H}$ a.e. in $\Omega \setminus A_\delta$. Hence $H(\widetilde{u}) = \widetilde{H}$ in a.e. Ω, so

$$H_m(u_m) \rightharpoonup H(\widetilde{u}) \text{ in } L^2(\Omega).$$

By passing to the limit in the weak formulation we deduce that $\widetilde{u} = u$. □

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Jesús Ildefonso Díaz
Instituto de Matematica Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain
E-mail address: `ildefonso.diaz@mat.ucm.es`

David Gómez-Castro
Instituto de Matematica Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain
E-mail address: `dgcastro@ucm.es`

Tatiana A. Shaposhnikova
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
E-mail address: `shaposh.tan@mail.ru`

Maria N. Zubova
Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia
E-mail address: `zubovnv@mail.ru`

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ORIGINAL PAPER

Non existence of critical scales in the homogenization of the problem with p-Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n-dimensional domains when $p > n$

J. I. Díaz[1] · **D. Gómez-Castro**[1] · **A. V. Podolskii**[2] · **T. A. Shaposhnikova**[2]

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Abstract In previous works, the homogenization of the problem with p-Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n-dimensional domains has been studied in the cases where $p \leq n$. The main trait of the cases $p \leq n$ is the existence of a critical size of the particles, for which the nonlinearity arising of the limit problem does not coincide with the non linear term of the microscopic reaction. The main result of this paper proves that in the case $p > n$ there exists no critical size.

Keywords Homogenization · p-Laplace diffusion · Non-linear boundary reaction · Non-critical sizes

Mathematics Subject Classification 35B27 · 35J66 · 35J60 · 35J92 · 35J62

1 Introduction

The main goal of this paper is to study the behaviour arising in the homogenization process applied to chemical reactions taking place on fixed-bed nanoreactors, at the microscopic level, on the boundary of the particles

✉ D. Gómez-Castro
dgcastro@ucm.es

J. I. Díaz
jidiaz@ucm.es

A. V. Podolskii
originalea@ya.ru

T. A. Shaposhnikova
shaposh.tan@mail.ru

1 Dpto. Matemática Aplicada & I.M.I., Universidad Complutense de Madrid, Madrid, Spain

2 Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon) & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases} \tag{1}$$

for a very general type of chemical kinetics (here given by the maximal monotone graph σ of $\mathbb{R}^2$). Here the diffusion is modeled by the quasilinear operator $\Delta_p u_\varepsilon \equiv \mathrm{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$ with $p > 1$. Notice that $p = 2$ corresponds to the linear diffusion operator, and that $p \neq 2$ appears in turbulent regime flows or non-Newtonian flows (see [2]). The "normal derivative" must be then understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nu$, where ν is outward unit normal vector on the boundary of the particles $S_\varepsilon \subset \partial\Omega_\varepsilon$. In fact we shall consider the structural assumption

$$n < p < +\infty \text{ and } n \geq 3. \tag{2}$$

In previous works, the cases where $p \leq n$ have been studied (see [3–6,9,10,12] for the details). The main trait of this cases is the existence of a critical size of the particles, for which the non linear term arising of the limit problem does not coincide with the non linear term of the microscopic reaction. If the size of the particles is larger than this critical size then the limit problem is of the form

$$\begin{cases} -\Delta_p u + \mathcal{A}\sigma(u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{3}$$

where $\mathcal{A} > 0$. If the size of the particles is critical, the limit problem becomes

$$\begin{cases} -\Delta_p u + \mathcal{B}|H(u)|^{p-2}H(u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{4}$$

where $\mathcal{B} > 0$ and H is the solution of functional equation depending only on σ, n and the shape of the particle.

The main result of this paper proves that for $p > n$ there exists no critical size. That is, the solution u_ε converges to the homogenized solution u of problem (3) where $\mathcal{A}$ is a constant that will be specified later.

The plan of the rest of the paper is the following: Sect. 2 will be devoted to the statement of the main results, whilst Sects. 3 and 4 are devoted to the proofs.

2 Statement of results

Definition 1 (*Perforated domain* Ω_ε) Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = (-1/2, 1/2)^n$. Denote by G_0 a smooth open set such that $\overline{G_0} \subset Y$. For $\delta > 0$ and B an open set we define $\delta B = \{x \mid \delta^{-1}x \in B\}$. For $\varepsilon > 0$ we define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \mid \rho(x, \partial\Omega) > 2\varepsilon\}$. Let $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$ and C_0 is positive number. Define

$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j, \tag{5}$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}_\varepsilon} \neq \emptyset\}$, $\mathbb{Z}^n$ is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$. It is clear that $\overline{G_j^\varepsilon} \subset Y_j^\varepsilon$. Define

$$\Omega_\varepsilon = \Omega \backslash \overline{G_\varepsilon}, \ S_\varepsilon = \partial G_\varepsilon, \ \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

It can be checked that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, for some constant $d > 0$, in the sense that $|\Upsilon_\varepsilon|/\varepsilon^{-n} \to d$ as $\varepsilon \to 0$.

In this geometry we consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f(x) & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma} \sigma(u_\varepsilon) = 0 & x \in S_\varepsilon, \\ u_\varepsilon = 0 & x \in \partial\Omega, \end{cases} \tag{6}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε and σ is a nondecreasing function such that $\sigma(0) = 0$ and $f \in L^{p'}(\Omega)$. In this paper we will be interested in the case $p > n$ and $\alpha > 1$.

We define $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as the closure in $W^{1,p}$ of $\{f \in \mathcal{C}^\infty(\overline{\Omega}_\varepsilon) : f|_{\partial\Omega} = 0\}$.

Definition 2 Let (Ω_ε) be a sequence of domains $\Omega_\varepsilon \subset \Omega \subset \mathbb{R}^n$ and $\partial\Omega \subset \partial\Omega_\varepsilon$ where Ω is bounded. We say that the sequence has a uniformly bounded sequence of extension operators in $W^{1,p}$ if there exists a sequence (P_ε) where:

$$P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega) \tag{7}$$

where $P_\varepsilon u|_{\Omega_\varepsilon} = u_\varepsilon$ for every $u \in W^{1,p}(\Omega_\varepsilon)$ and there exists $K_p > 0$ independent of ε such that

$$\|\nabla P_\varepsilon u\|_{L^p(\Omega)} \le K_p \|\nabla u\|_{L^p(\Omega_\varepsilon)}, \quad \text{for every} \quad \varepsilon > 0. \tag{8}$$

Applying the techniques in [8] we can prove that

Lemma 1 *The sequence (Ω_ε) has an uniformly bounded sequence of extension operators.*

We will use the existence of a Poincaré constant for $W_0^{1,p}(\Omega)$, $C_{p,\Omega}$, such that

$$\|v\|_{L^p(\Omega)} \le C_{p,\Omega} \|\nabla v\|_{L^p(\Omega)}, \quad v \in W_0^{1,p}(\Omega). \tag{9}$$

In fact we can also show the following, which is seldom stated

Theorem 2 *Let $p > 1$. If there exists a sequence of uniformly bounded extension operators in $W_0^{1,p}$ then there exists a uniform Poincaré constant for $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$. In particular, if (8) holds and $C_{p,\Omega}$ is a Poincaré constant for $W_0^{1,p}(\Omega)$, then, $K_p C_{p,\Omega}$ is a Poincaré constant for $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.*

Proof We simply indicate that

$$\|v\|_{L^p(\Omega_\varepsilon)} \le \|P_\varepsilon v\|_{L^p(\Omega)} \le C_{p,\Omega} \|\nabla P_\varepsilon v\|_{L^p(\Omega)} \le C_{p,\Omega} K_p \|\nabla v\|_{L^p(\Omega_\varepsilon)} \tag{10}$$

which concludes the proof. □

Our aim is to prove the following results

Theorem 3 *Let $n < p < +\infty$, $\alpha > 1$, σ be a continuous nondecreasing function such that $\sigma(0) = 0$, u_ε be the solution of (6) and let*

$$\gamma^* = \alpha(n-1) - n. \tag{11}$$

Then, $P_\varepsilon u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ where $u \in W_0^{1,p}(\Omega)$ is the unique weak solution of one of the following problem

1. *If* $\gamma = \gamma^*$ *then*

$$\begin{cases} -\Delta_p u + \mathcal{A}\sigma(u) = f, & \Omega, \\ u = 0 & \partial\Omega \end{cases} \tag{12}$$

where $\mathcal{A} = C_0^{n-1}|\partial G_0|$.

2. *If* $\gamma < \gamma^*$ *then*

$$\begin{cases} -\Delta_p u = f & \Omega, \\ u = 0 & \partial\Omega. \end{cases} \tag{13}$$

Lemma 4 *Let* $n < p < +\infty, \alpha > 1$ *and* $\sigma \equiv 0$. *Then* $P_\varepsilon u_\varepsilon \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *where* u *is the unique solution of* (13) (*equivalently* (12) *for* $\sigma \equiv 0$).

Theorem 5 *Let* $n < p < +\infty, \alpha > 1, \gamma > \gamma^*$ *and* $\sigma \in \mathcal{C}^1(\mathbb{R})$ *nondecreasing function such that* $\sigma(0) = 0$. *Then, there exists* $u \in W_0^{1,p}(\Omega)$ *such that, up to a subsequence,* $P_\varepsilon u_\varepsilon \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\sigma(u(x)) = 0, \ a.e. \ x \in \Omega. \tag{14}$$

In other words, $u(x) \in \sigma^{-1}(0)$ *for a.e.* $x \in \Omega$.

Remark 1 In this setting ($p > n$) there exists no critical exponent α^*. This is quite natural since, for $p < n$ the critical exponent results $\alpha^* = \frac{n}{n-p}$. The case $p = n$ was done in [9].

We will use the following comparison result, which will be proved later

Lemma 6 *Let* $p > 2$ *and let* $u_\varepsilon, \hat{u}_\varepsilon$ *be the solutions of* (6) *with* σ *and* $\hat{\sigma}$ *continuous functions. Then,*

$$\|\nabla(u_\varepsilon - \hat{u}_\varepsilon)\|_{L^p(\Omega_\varepsilon)}^{p-1} \le C\varepsilon^{\frac{\gamma^*-\gamma}{p}} \|\sigma - \hat{\sigma}\|_{\mathcal{C}(\mathbb{R})}. \tag{15}$$

Remark 2 Since any function $v \in W^{1,p}(\Omega)$, $p > n$ is Hölder with the estimate

$$|v(x) - v(y)| \le C|x - y|^{1-\frac{n}{p}} \|\nabla v\|_{L^p(\Omega)}, \quad \text{if } [x, y] \subset \Omega \tag{16}$$

where $[x, y] = \{\lambda x + (1 - \lambda y) : \lambda \in [0, 1]\}$, we have that $(P_\varepsilon u_\varepsilon)$ is uniformly Hölder continuous, and therefore (u_ε) is also uniformly Hölder continuous.

We need some information on the traces on S_ε. We can compute the following lemma, analogous to results in [8] which, for the proof, points to [7].

Lemma 7 *Let* $p > n$ *and* $u \in W^{1,p}(Y_\varepsilon)$ *where* $Y_\varepsilon = \varepsilon Y \backslash \overline{a_\varepsilon G_0}$. *Then,*

$$\int_{a_\varepsilon S_0} |u|^p dS \le K \left(a_\varepsilon^{n-1} \varepsilon^{-n} \int_{Y_\varepsilon} |u|^p dx + a_\varepsilon^{n-1} \varepsilon^{p-n} \int_{Y_\varepsilon} |\nabla u|^p dx \right) \tag{17}$$

where K *is independent of* ε.

Remark 3 In particular, if $a_\varepsilon = C_0 \varepsilon^\alpha$ we have

$$\int_{a_\varepsilon S_0} |u|^p dS \le K \left(\varepsilon^{\gamma^*} \int_{Y_\varepsilon} |u|^p dx + a_\varepsilon^{n-1} \varepsilon^{p-n} \int_{Y_\varepsilon} |\nabla u|^p dx \right) \tag{18}$$

This explains the choice of γ^*. If $p < n$ then $a_\varepsilon^{n-1}\varepsilon^{p-n}$ is replaced by a_ε^{p-1}. In that case

$$\frac{a_\varepsilon^{p-1}}{\varepsilon^{\gamma^*}} = C_0^{p-1} \varepsilon^{\alpha(p-n)+n}. \tag{19}$$

For the cases $p < n$ this exponent is the one that produces the appearance of a critical case, which corresponds to $\alpha = \frac{n}{n-p}$. In the case $p = n$ a similar expression exists, but is more self-involved (see [9]).

The following result will be instrumental in the proof. Nonetheless it has a great intrinsic mathematical value.

Proposition 1 *Let $p > n$, $\alpha > 1$, $\gamma^* = \alpha(n-1) - n$ and $v_\varepsilon \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Then*

$$\varepsilon^{-\gamma^*} \int_{S_\varepsilon} v_\varepsilon dS \to \mathcal{A} \int_{\Omega} v dS \tag{20}$$

where

$$\mathcal{A} = C_0^{n-1} |\partial G_0|. \tag{21}$$

This result does not hold if $p < n$, and this causes the appearance of a term known as *strange term*, first noticed by Cioranescu and Murat for the linear problem [1], and which has been well documented also in the nonlinear case (see, e.g., [6,12]).

The technique for the proof of this result uses the following auxiliary result. Define function $M_\varepsilon(x)$ as Y_ε—periodic solution of the boundary value problem

$$\begin{cases} \Delta_p m_\varepsilon = \mu_\varepsilon, \;\; x \in Y_\varepsilon = \varepsilon Y \backslash \overline{a_\varepsilon G_0}; \\ \partial_{\nu_p} m_\varepsilon = 1, \;\; x \in \partial(a_\varepsilon G_0) = S_\varepsilon^0; \\ \partial_{\nu_p} m_\varepsilon = 0, \;\; x \in \partial Y_\varepsilon \backslash S_\varepsilon^0; \end{cases} \quad , \quad \mu_\varepsilon = \frac{C_0^{n-1} \varepsilon^{\alpha(n-1)-n} |\partial G_0|}{1 - (a_\varepsilon \varepsilon^{-1})^n |G_0|},$$
$$\text{and} \int_{Y_\varepsilon} m_\varepsilon(x) dx = 0. \tag{22}$$

This has the nice property of allowing us to write, for any test function $\varphi \in W^{1,p}(Y_\varepsilon)$

$$- \int_{Y_\varepsilon} |\nabla m_\varepsilon|^{p-2} \nabla m_\varepsilon \nabla \varphi dx + \int_{S_\varepsilon^0} \varphi dS = \mu_\varepsilon \int_{Y_\varepsilon} \varphi dx. \tag{23}$$

Denote by P_ε^j the center of the ball $G_\varepsilon^j = P_\varepsilon^j + a_\varepsilon G_0$. Let T_ε^j denote the ball of radius $\varepsilon/4$ centered at the point P_ε^j. Let $M_\varepsilon^j = m_\varepsilon(x - P_j^\varepsilon)$ be the solution of the boundary value problem. We will use the following fact, which we will prove later

Lemma 8 *The following estimate holds*

$$\|\nabla M_\varepsilon\|_{L^p(\cup_j Y_\varepsilon^j)} \le C(a_\varepsilon \varepsilon^{-1})^{\frac{n-1}{p-1}} \tag{24}$$

3 Proof of Proposition 1

Proof of Lemma 8 Setting in (23) $\varphi = m_\varepsilon$ and applying Theorem 2, Lemma 7 and the definition of $m_\varepsilon(x)$, we obtain

$$
\begin{aligned}
\|\nabla m_\varepsilon\|^{p^2}_{L^p(Y_\varepsilon)} &\le \left(\left| \int_{S^0_\varepsilon} m_\varepsilon dS \right| + \mu_\varepsilon \left| \int_{Y_\varepsilon} m_\varepsilon dx \right| \right)^p \\
&\le \left(\int_{S^0_\varepsilon} 1 dS \right)^{p-1} \|m_\varepsilon\|^p_{L^p(S^0_\varepsilon)} \le C_1 a_\varepsilon^{(n-1)(p-1)} \|m_\varepsilon\|^p_{L^p(S^0_\varepsilon)} \\
&\le C_2 a_\varepsilon^{(n-1)(p-1)} \left(a_\varepsilon^{n-1} \varepsilon^{-n} \|m_\varepsilon\|^p_{L^p(Y_\varepsilon)} + a_\varepsilon^{n-1} \varepsilon^{p-n} \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)} \right) \\
&\le C_3 \left(a_\varepsilon^{p(n-1)} \varepsilon^{p-n} + a_\varepsilon^{p(n-1)} \varepsilon^{p-n} \right) \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)} \qquad (25) \\
&\le C_4 a_\varepsilon^{p(n-1)} \varepsilon^{p-n} \|\nabla m_\varepsilon\|^p_{L^p(Y_\varepsilon)}, \qquad (26)
\end{aligned}
$$

Finally, we have the following inequality

$$
\|\nabla m_\varepsilon\|_{L^p(Y_\varepsilon)} \le K a_\varepsilon^{\frac{n-1}{p-1}} \varepsilon^{\frac{p-n}{p(p-1)}}. \qquad (27)
$$

Hence, since $\#\Upsilon_\varepsilon \le C\varepsilon^{-n}$ we get the estimate

$$
\|\nabla M_\varepsilon\|_{L^p(\cup_j Y^j_\varepsilon)} \le C(a_\varepsilon \varepsilon^{-1})^{\frac{n-1}{p-1}}, \qquad (28)
$$

which concludes the proof. □

Remark 4 Notice that from (25) to (26) we apply that $p > n$. In the case $p < n$ the other term is dominant, and hence the comparison is $\|\nabla M_\varepsilon\|_{L^p} \le C(a_\varepsilon \varepsilon^{-1})^{\frac{n}{p}}$ (see [8]).

Let $M^j_\varepsilon(x)$ be a restriction of function $M_\varepsilon(x)$ on Y^j_ε. Using the definition of $M^j_\varepsilon(x)$, we can make the following transformations

$$
\begin{aligned}
\varepsilon^{-\gamma} \int_{S_\varepsilon} v_\varepsilon dS = \; & \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{Y^j_\varepsilon} div(|\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon v_\varepsilon) dx \\
= \; & \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{Y^j_\varepsilon} |\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon \nabla v_\varepsilon dx \\
& + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{Y^j_\varepsilon} (\Delta_p M^j_\varepsilon) v_\varepsilon dx \\
= \; & \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \int_{Y^j_\varepsilon} |\nabla M^j_\varepsilon|^{p-2} \nabla M^j_\varepsilon \nabla v_\varepsilon dx \\
& + \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon} \mu_\varepsilon \int_{Y^j_\varepsilon} v_\varepsilon dx \qquad (29)
\end{aligned}
$$

Using (28), we get

$$\varepsilon^{-\gamma}\int\limits_{\Omega_\varepsilon}|\nabla M_\varepsilon|^{p-1}|\nabla v_\varepsilon|dx \le C\varepsilon^{-\gamma}\left(\int\limits_{\Omega_\varepsilon}|\nabla M_\varepsilon|^p dx\right)^{\frac{p-1}{p}}$$
$$\le C\varepsilon^{-\alpha(n-1)+n}a_\varepsilon^{n-1}\varepsilon^{1-n} = C\varepsilon. \tag{30}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{Y_\varepsilon^j}|\nabla M_\varepsilon^j|^{p-2}\nabla M_\varepsilon^j\cdot\nabla v_\varepsilon dx = 0 \tag{31}$$

and, finally, we use the fact (see [13]) that, since $v_\varepsilon \rightharpoonup v$ in $W^{1,2}(\Omega)$ we have

$$\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\mu_\varepsilon\int\limits_{Y_\varepsilon^j}v_\varepsilon dx \to C_0^{n-1}|\partial G_0|\int\limits_\Omega v dx. \tag{32}$$

Remark 5 Notice that, for $p<n$ estimate (30) transform into $C\varepsilon^{\frac{1}{p}(n-\alpha(n-p))}$ producing the appearance of a critical α (see [8]).

4 Proof of Theorem 3

First, let us prove the auxiliary lemma

Proof of Lemma 6 By considering the difference of weak formulations we can write, for the test function u_1-u_2

$$\int\limits_\Omega(|\nabla u_2|^{p-2}\nabla u_2-|\nabla u_1|^{p-2}\nabla u_1)\cdot\nabla(u_2-u_1)dx$$
$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma_2(u_2)-\sigma_2(u_1))(u_2-u_1)dS$$
$$=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma_1(u_1)-\sigma_2(u_1))(u_2-u_1)dS. \tag{33}$$

For $p\ge 2$ it is true that (see [11] or [2, Lemma 4.10])

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le \left|\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma_2(u_1)-\sigma_1(u_1))(u_2-u_1)dS\right| \tag{34}$$
$$\le \varepsilon^{-\gamma}|S_\varepsilon|^{\frac{1}{p'}}\|\sigma_2-\sigma_1\|_\infty\|u_1-u_2\|_{L^p(S_\varepsilon)} \tag{35}$$
$$\le C\varepsilon^{-\frac{\gamma}{p}}\|\sigma_2-\sigma_1\|_\infty\|u_1-u_2\|_{L^p(S_\varepsilon)}, \tag{36}$$

since $|S_\varepsilon|\le C\varepsilon^{-\gamma}$. By applying Lemma 7 we deduce that

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le K\varepsilon^{-\frac{\gamma}{p}}\|\sigma_1-\sigma_2\|_\infty\varepsilon^{\frac{\gamma^*}{p}}$$
$$\times\left(\|u_1-u_2\|_{L^p(\Omega_\varepsilon)}+\|\nabla(u_1-u_2)\|_{L^p(\Omega_\varepsilon)}\right). \tag{37}$$

Applying the uniform Poincaré inequality we deduce

$$\|\nabla(u_1-u_2)\|^p_{L^p(\Omega_\varepsilon)} \le K\varepsilon^{\frac{\gamma^*-\gamma}{p}}\|\sigma_1-\sigma_2\|_\infty\|\nabla(u_1-u_2)\|_{L^p(\Omega_\varepsilon)}. \tag{38}$$

which concludes the proof.

We consider the weak formulation

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla v dx+\varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(u_\varepsilon)v dS=\int_{\Omega_\varepsilon} f v dx, \quad \forall v\in W_0^{1,p}(\Omega). \tag{39}$$

Since u_ε is a weak solution and $p>n$ we have that

$$\|\nabla u_\varepsilon\|^{p-1}_{L^p(\Omega_\varepsilon)} \le \|f\|_{L^p(\Omega_\varepsilon)} \tag{40}$$

Therefore (u_ε) is a bounded sequence in $W^{1,p}(\Omega_\varepsilon)$. Hence $(P_\varepsilon u_\varepsilon)$ is a uniformly Hölder sequence in Ω, and therefore uniformly bounded

$$\|u_\varepsilon\|_{\mathcal{C}(\Omega)} \le \|P_\varepsilon u_\varepsilon\|_{\mathcal{C}(\Omega)} \le C, \quad \text{for some } C>0. \tag{41}$$

Hence we have that

$$\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\cdot\nabla v dx \to \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx, \tag{42}$$

$$\int_{\Omega_\varepsilon} f v dx \to \int_\Omega f v dx. \tag{43}$$

Proof of Theorem 3 First let us assume that $\gamma<\gamma^*$. Let $u_{\varepsilon,0}$ be the solution corresponding to $\sigma\equiv 0$. Then

$$\|u_\varepsilon-u_{\varepsilon,0}\|_{W^{1,p}(\Omega_\varepsilon)} \le \varepsilon^{\frac{\gamma-\gamma^*}{p(p-1)}}\|\sigma\|^{\frac{1}{p-1}}_{\mathcal{C}(K)} \tag{44}$$

where K is a compact such that $\|P_\varepsilon u_\varepsilon\|_{L^\infty}, \|P_\varepsilon u_{\varepsilon,0}\|_{L^\infty}\in K\subset\mathbb{R}$. Then $P_\varepsilon u_\varepsilon \rightharpoonup u_0$ the solution of (13) by applying Lemma 4.

Assume that $\gamma=\gamma^*$. We start by considering $\sigma\in\mathcal{C}^1(\mathbb{R})$. Since the solutions are uniformly bounded and continuous, we have that

$$\|\sigma'(u_\varepsilon)\|_{\mathcal{C}(S_\varepsilon)} \le \|\sigma'(u_\varepsilon)\|_{\mathcal{C}(\Omega)} \le C \tag{45}$$

since σ' is continuous. Notice that $\sigma(P_\varepsilon u_\varepsilon)=P_\varepsilon(\sigma(u_\varepsilon))$ on $\overline{\Omega}_\varepsilon$. Hence

$$\|\nabla(\sigma(u_\varepsilon))\|_{L^p(\Omega_\varepsilon)} \le \|\sigma'(u_\varepsilon)\|_{\mathcal{C}(\Omega)}\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C. \tag{46}$$

Therefore there exists $\hat\sigma\in W^{1,p}(\Omega)$ such that $P_\varepsilon\sigma(u_\varepsilon)\rightharpoonup\hat\sigma$. Since $p>n$ the convergence is also in the sense of $\mathcal{C}(\Omega)$, and therefore $\hat\sigma=\sigma(u)$. Hence, we conclude that for $v\in W^{1,p}$ we have

$$\varepsilon^{-\gamma^*}\int_{S_\varepsilon}\sigma(u_\varepsilon)v dS \to \mathcal{A}\int_\Omega \sigma(u)v dx. \tag{47}$$

Then, limit becomes

$$\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v dx+\mathcal{A}\int_\Omega\sigma(u)v dx=\int_\Omega f v dx, \quad \forall v\in W_0^{1,p}(\Omega). \tag{48}$$

Let $\sigma \in \mathcal{C}(\Omega)$. Let us consider an approximating sequence $\sigma \in \mathcal{C}^1$, $\sigma^{-1}(0) = 0$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M, M])$ as $\delta \to 0$ where $\|P_\varepsilon u_\varepsilon\|_{\mathcal{C}(\overline{\Omega})} < M$ for all $\varepsilon > 0$. We have that

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{W^{1,p}}^{p-1} \leq C\|\sigma_\delta - \sigma\|_{\mathcal{C}([-M,M])}. \tag{49}$$

Passing to the limit we have that

$$\|u - u_\delta\|_{W^{1,p}}^{p-1} \leq C\|\sigma_\delta - \sigma\|_{\mathcal{C}([-M,M])}, \tag{50}$$

where u_δ satisfies (12), with σ_δ instead of σ. As $\delta \to 0$ the sequence $u_\delta \to w$ where w is the solution of (12). Therefore, due to (50) we have that $u = w$, which concludes the proof. □

Proof of Theorem 5 If $\gamma > \gamma^*$ we write

$$\varepsilon^{\gamma-\gamma^*} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla v dx + \varepsilon^{-\gamma^*} \int_{S_\varepsilon} \sigma(u_\varepsilon) v dS = \varepsilon^{\gamma-\gamma^*} \int_{\Omega_\varepsilon} f v dx, \tag{51}$$

for all $v \in W_0^{1,p}(\Omega)$. Hence, in the limit

$$\mathcal{A} \int_\Omega \sigma(u) v \, dx = 0, \quad \forall v \in W_0^{1,p}(\Omega). \tag{52}$$

That is $\sigma(u(x)) = 0$ for a.e. $x \in \Omega$. □

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Research Article

Jesus Ildefonso Díaz*, David Gómez-Castro, Alexander V. Podol'skii and Tatiana A. Shaposhnikova

Characterizing the strange term in critical size homogenization: Quasilinear equations with a general microscopic boundary condition

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Abstract: The aim of this paper is to consider the asymptotic behavior of boundary value problems in n-dimensional domains with periodically placed particles, with a general microscopic boundary condition on the particles and a p-Laplace diffusion operator on the interior, in the case in which the particles are of critical size. We consider the cases in which $1 < p < n$, $n \geq 3$. In fact, in contrast to previous results in the literature, we formulate the microscopic boundary condition in terms of a Robin type condition, involving a general maximal monotone graph, which also includes the case of microscopic Dirichlet boundary conditions. In this way we unify the treatment of apparently different formulations, which before were considered separately. We characterize the so called "strange term" in the homogenized problem for the case in which the particles are balls of critical size. Moreover, by studying an application in Chemical Engineering, we show that the critically sized particles lead to a more effective homogeneous reaction than noncritically sized particles.

Keywords: Homogenization, p-Laplace diffusion, nonlinear boundary reaction, noncritical sizes, maximal monotone graphs

MSC 2010: 35B25, 35B40, 35J05, 35J20

1 Introduction

A well-known effect in homogenization theory is the appearance of some changes in the structural modelling of the homogenized problem for suitable critical size of the elements configuring the "micro-structured" material. It seems that the first result in this direction was presented in the pioneering paper by Marchenko and Hruslov [27]. A more popular presentation of the appearance of some "strange terms" was due to Cioranescu and Murat [4]. Both articles dealt with linear equations with Neumann and Dirichlet boundary conditions, respectively. Since then many papers were devoted to different formulations, e.g., more general elliptic partial differential equations (possibly of quasilinear type), Robin type and other boundary conditions of different nature, etc. It is impossible to mention all of them here (a few of them will be mentioned in the rest of the introduction) but the reader may imagine that the nature of this "strange term" may be completely different according to the peculiarities of the formulation in consideration (something that was already indicated at the end of the introduction of the paper by Cioranescu and Murat [4]).

***Corresponding author: Jesus Ildefonso Díaz:** Departamento de Matemática Aplicada e I.M.I., Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain, e-mail: jidiaz@ucm.es. http://orcid.org/0000-0003-1730-9509
David Gómez-Castro: Departamento de Matemática Aplicada e I.M.I., Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain, e-mail: dgcastro@ucm.es
Alexander V. Podol'skii, Tatiana A. Shaposhnikova: Faculty of Mechanics and Mathematics, Moscow State University, Moscow 199992, Russia, e-mail: originalea@ya.ru, shaposh.tan@mail.ru

The main goal of this paper is to characterize the change of structural behavior arising in the homogenization process when applied to chemical reactions taking place on fixed-bed nanoreactors, at the microscopic level, on the boundary of the particles

$$\begin{cases} -\Delta_p u_\varepsilon = f(x), & x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma(u_\varepsilon), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

for a very general type of chemical kinetics (here given by the maximal monotone graph σ of $\mathbb{R}^2$). Thanks to this generality on the maximal monotone graph σ, our treatment also includes the case of microscopic Dirichlet boundary conditions. In this way we unify the treatment of apparently different formulations, which before were considered separately.

The diffusion is modeled by the quasilinear operator $\Delta_p u_\varepsilon \equiv \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon)$, with $p > 1$. Notice that $p = 2$ corresponds to the linear diffusion operator, and that $p \neq 2$ appears in turbulent regime flows or non-Newtonian flows (see [8]). As it is well known, this operator appears in many other contexts and is one of the best examples of quasilinear operators leading to a formulation in terms of nonlinear monotone operators (see, e.g., [1, 7, 26]).

The "normal derivative" must be then understood as $\partial_{\nu_p} u_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \boldsymbol{\nu}$, where $\boldsymbol{\nu}$ is the outward unit normal vector on the boundary of the particles $S_\varepsilon \subset \partial\Omega_\varepsilon$. In fact, we will consider the structural assumption

$$1 < p < n, \quad n \geq 3.$$

The cases $p \geq n$ are completely different, see [31, 32] (see also, for instance, the study made for a general monotone quasilinear equation with Dirichlet boundary conditions in [7]).

As mentioned before, the generality assumed on the maximal monotone graph σ of $\mathbb{R}^2$ allows to treat, in a unified way, different cases as the case of Dirichlet boundary conditions, which corresponds to the choice of σ given by

$$D(\sigma) = \{0\} \quad \text{and} \quad \sigma(0) = (-\infty, +\infty) \tag{1.2}$$

(see, e.g. [1]), and the case of nonlinear Robin type boundary conditions, which corresponds (see, e.g. [24]) to the case in which $D(\sigma) = \mathbb{R}$ and σ is a continuous nondecreasing function.

The domain $\Omega_\varepsilon \subset \mathbb{R}^n$ is assumed to have an ε-periodical structure. Since our main goal is to get a very precise description of the so-called "strange term" in the homogenized problem, we shall assume that the particles are balls of radius $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$. One of the interesting properties that arise from our precise characterization is that there is uniqueness of solutions of the homogenized problem. This was not always proved in previous results (cf. the general framework considered in [7], and how their characterization, given in their Lemma 5.1, is not enough to get the uniqueness of solution of their homogenized problem). The consideration of particles of a general shape is a difficult task, especially the exact identification of the "strange terms". A similar formulation to the one considered in this paper for that case can be obtained, at least for continuous σ, and has been the subject of a different paper (see [15]).

The problem has two different parameters: α, the size of the particles, and γ, the normalization factor of the boundary condition on S_ε. When they have critical values

$$\alpha = \frac{n}{n-p}, \quad \gamma = \alpha(n-1) - n = \alpha(p-1),$$

then our main result in this paper shows that the homogenized problem involves a different distributed chemical kinetics nonlinearity:

$$\begin{cases} -\Delta_p u + \mathcal{A}|H(u)|^{p-2}H(u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where

$$\mathcal{A} = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n \tag{1.4}$$

and $H\colon \mathbb{R} \to \mathbb{R}$ is given by

$$H(r) = (I + \sigma^{-1} \circ \Theta_{n,p})^{-1}(r), \tag{1.5}$$

with

$$\Theta_{n,p}(s) = \mathcal{B}_0 |s|^{p-2} s \quad \text{for } s \in \mathbb{R}$$

and

$$\mathcal{B}_0 = \Big(\frac{n-p}{C_0(p-1)} \Big)^{p-1}, \tag{1.6}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. We show that, for any maximal monotone graph σ, H is a nondecreasing contraction, and thus the existence, uniqueness and continuous dependence of solutions of the homogenized problem is consequence of well-known results on monotone operator theory.

The change of behavior from the nonlinearity of type σ in the nonhomogeneous problem to the nonlinearity H in the homogeneous problem is one of the characteristics of the nanotechnological effects (see, e.g., [33]) and does not appear if $1 \le \alpha < \frac{n}{n-p}$ (see [5, 34]).

Before presenting the details of the notation used above, let us mention that our main aim is to provide a common roof and extend (under different points of view) some previous results in the literature concerning different structural assumptions (i.e., the functions σ and H) after the homogenization process.

The case of Robin boundary conditions $\partial_n u + \beta(\varepsilon)\sigma(u) = 0$ on S_ε was first studied by Marchenko and Hruslov in a series of papers dealing mainly with the linear case $\sigma(u) = \lambda u$, see [19–21, 27]. Some references on different choices of smooth functions σ can be found in [6, 18, 22, 24, 25, 29, 37] and the references therein. For further references, see [13, 16, 17, 23]. Some previous results by the authors [11], formulated there for some not necessarily Lipschitz functions σ and $p \in [2, n)$, will be here extended to the case a general maximal monotone graph σ (which includes the case of Dirichlet boundary conditions) and $p \in (1, n)$.

The special case of Dirichlet boundary condition $u_\varepsilon = 0$ on S_ε, covered by (1.2), gives $\sigma^{-1}(s) = 0$ for any $s \in \mathbb{R}$, and so $H(r) = r$ for any $r \in \mathbb{R}$. Therefore, the "strange term" arising in the homogenized equation becomes $\mathcal{A}|u|^{p-2}u$. This was shown for $p = 2$ in the pioneering paper by Cioranescu and Murat [4]. However, even in this simple case, the treatment in [7] for the case $p \neq 2$ is not as sharp as in our case. In [7], Dal Maso and Skrypnik do not provide an explicit expression for this strange term. In fact, their characterization (see [7, Lemma 5.1]) does not guaranty uniqueness of solutions of the homogenized problem.

The case of the boundary condition

$$u_\varepsilon \ge 0, \quad \partial_n u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon) \ge 0, \quad u_\varepsilon(\partial_n u_\varepsilon + \varepsilon^{-\gamma}\sigma_0(u_\varepsilon)) = 0 \quad \text{on } S_\varepsilon,$$

which was studied for smooth σ_0 in [22] by ad hoc techniques, is also covered by the common proof provided in this paper, by taking

$$D(\sigma) = [0, +\infty), \qquad \sigma(u) = \begin{cases} (-\infty, 0] & \text{if } u = 0, \\ \sigma_0(u) & \text{if } u > 0. \end{cases}$$

See also [12, 28].

The choice of the critical values of α and γ might appear arbitrary. Let us give some reasons why this is a good choice. First, if $N(\varepsilon)$ is the number of particles, then $N(\varepsilon) \sim \varepsilon^{-n}$. It is easy to see that $|S_\varepsilon| = N(\varepsilon)|\partial(a_\varepsilon G_0)| \sim \varepsilon^{\alpha(n-1)-n}$, where G_0 is the unit ball centered at 0. Let us analyze the choice of γ. If we consider the reaction term on the weak formulation, with $\sigma(u_\varepsilon)$ a bounded sequence in L^∞ and v a bounded test function, then

$$\frac{1}{|S_\varepsilon|} \int\limits_{S_\varepsilon} \sigma(u_\varepsilon) v \, dS \sim \varepsilon^{-(\alpha(n-1)-n)} \int\limits_{S_\varepsilon} \sigma(u_\varepsilon) v \, dS$$

is a bounded sequence. Hence, if the sequence u_ε is bounded in L^∞ and v is a bounded test function, then

$$\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma(u_\varepsilon) v \, dS$$

can only be expect to tend to either 0 or $+\infty$ if $\gamma \neq \alpha(n-1) - n$, and hence we will lose the reaction term on the equation on the homogenized equation or we lose the equation altogether. If the macroscopic behavior is given by a reaction diffusion equation (with nontrivial reaction), then the choice scaling γ as $\varepsilon \to 0$ can be no other.

The appearance of the critical value of α has to do with a property of traces. It is known (see [30]) that

$$\int_{a_\varepsilon S_0} |u|^p \, dS \le K\Big(a_\varepsilon^{n-1}\varepsilon^{-n} \int_{Y_\varepsilon} |u|^p \, dx + a_\varepsilon^{p-1} \int_{Y_\varepsilon} |\nabla u|^p \, dx\Big).$$

As it turns out, the critical scale is the one in which both terms in the right-hand side have the same order of convergence. Notice that, in the critical case $\alpha = \frac{n}{n-p}$, we have $\gamma = \alpha(p-1)$.

Notice that for a Newtonian fluid in $\mathbb{R}^3$ ($n = 3, p = 2$), the critical size corresponds to $\alpha = 3$. Obviously the critical value of α is an increasing function of p. Therefore, for non-Newtonian dilatant fluids or a Newtonian flow in turbulent regime ($p > 2$), our assumption means $\alpha > 3$, the particles are tiny with respect to their repetition, whereas for pseudoplastic fluids ($p < 2$), the critical particles satisfy $\alpha < 3$, and hence are not so tiny with respect to their repetition.

A relevant application of our results is the following. Let us consider the usual formulation in Chemical Engineering (see [9, 35]) with a constant external supply

$$\begin{cases} -\Delta w_\varepsilon = 0, & x \in \Omega_\varepsilon, \\ \partial_\nu w_\varepsilon + \varepsilon^{-\gamma} g(w_\varepsilon) = 0, & x \in S_\varepsilon, \\ w_\varepsilon = 1, & x \in \partial\Omega, \end{cases}$$

where g is a nondecreasing real function such that $g(0) = 0$. In order to adapt our results, we introduce the change in variable $u = 1 - w$ and $\sigma(u) = g(1) - g(1-u)$, and the problem becomes

$$\begin{cases} -\Delta u_\varepsilon = 0, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma} g(1), & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega. \end{cases}$$

Notice that the presence of $w_\varepsilon = 1$ on $\partial\Omega$ is translated to a source in S_ε for u_ε. We will see later (Theorem 6.2) that the new equation for H, when $\alpha = \frac{n}{n-2}$, is

$$\frac{n-2}{C_0} H(s) = \sigma(s - H(s)) - g(1), \tag{1.7}$$

that is,

$$H(u) = -\Big(g^{-1}\Big(\frac{n-2}{C_0}\,\cdot\Big) + \mathrm{Id}\Big)^{-1}(1-u),$$

so that an extension of w_ε converges weakly in $H^1(\Omega)$ to w_{crit}, the solution of

$$\begin{cases} -\Delta w_{\text{crit}} + \mathcal{A} h(w_{\text{crit}}) = 0 & \text{in } \Omega, \\ w_{\text{crit}} = 1 & \text{on } \partial\Omega, \end{cases}$$

and h is given by

$$h(w) = \Big(g^{-1}\Big(\frac{n-2}{n}\,\cdot\Big) + \mathrm{Id}\Big)^{-1}(w).$$

Notice that in the case of Neumann problems, $\sigma(s) \equiv 0$ for any $s \in R$, and although σ^{-1} is a well-known maximal monotone graph, the more direct identification of the "strange term" $H(u)$ is obtained trough the implicit equation (1.7), since in this case we get that

$$H(s) = -\frac{C_0 g(1)}{n-2} \quad \text{for any } s \in \mathbb{R}.$$

In the noncritical cases, $1 < \alpha < \frac{n}{n-2}$, we will show that an extension of w_ε converges weakly in $H^1(\Omega)$ to $w_{\text{non-crit}}$, the solution of

$$\begin{cases} -\Delta w_{\text{non-crit}} + \hat{\mathcal{A}} g(w_{\text{non-crit}}) = 0 & \text{in } \Omega, \\ w_{\text{non-crit}} = 1 & \text{on } \partial\Omega, \end{cases}$$

with $\hat{\mathcal{A}} = C_0^{n-1}|\partial G_0|$. Finally, we will show in Theorem 6.3 that

$$w_{\text{crit}} \geq w_{\text{non-crit}},$$

so we have a pointwise “better” reaction in the critical case [10]. We point out that a different criterion to establish the optimality of the reaction in terms of the so-called “chemical effectiveness” was considered by the authors in [14].

The plan of the rest of the paper is the following: Section 2 will be devoted to the statement of the main results, Section 3 contains the proof of the existence results for equation (1.1) and the characterization of H, Section 4 is devoted to the proof of the main result, Theorem 2.4, and Section 5 contains the proof of the auxiliary Theorem 2.9, which studies the limit of the diffusion. We conclude the paper with Section 6, where we study the noncritical case and the pointwise comparison of its homogenized solution with the critical case.

2 Statement of the main results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$, and let $Y = (-\frac{1}{2}, \frac{1}{2})^n$. Denote by $G_0 = B_1(0)$ the unit ball centered at the origin. This plays a crucial role in the proof. As far as we known, no results are known in the critical cases if G_0 is not a ball. For $\delta > 0$ and $\varepsilon > 0$, we define sets $\delta B = \{x : \delta^{-1}x \in B\}$ and $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}$. Let

$$a_\varepsilon = C_0\varepsilon^\alpha,$$

where $\alpha > 1$ and C_0 is a given positive number. Define

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, $N(\varepsilon) = |\Upsilon_\varepsilon| \cong \varepsilon^{-n}$, and $\mathbb{Z}^n$ denotes the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$ and note that $\overline{G}_\varepsilon^j \subset \overline{Y}_\varepsilon^j$ and center of the ball G_ε^j coincides with the center of the cube Y_ε^j. Our “microscopic domain” is defined as

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

We define the space $W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ as the completion, with respect to the norm of $W^{1,p}(\Omega_\varepsilon)$, of the set of infinitely differentiable functions in $\overline{\Omega}_\varepsilon$ equal to zero in a neighborhood of $\partial\Omega$, that is,

$$W_0^{1,p}(\Omega_\varepsilon, \partial\Omega) = \{u \in W^{1,p}(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\}.$$

Concerning the solvability of problem (1.1), we start by introducing the notion of weak solution. Since we assume that $\sigma\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ denotes the set of subsets of $\mathbb{R}$, we recall, by well-known results (see, e.g., [2]), that

$$\sigma \text{ is a maximal monotone graph of } \mathbb{R}^2,\ 0 \in \sigma(0), \tag{2.1}$$

and that there exists a convex lower semicontinuous function $\Psi\colon \mathbb{R} \to (-\infty, +\infty]$, with $\Psi(0) = 0$, such that $\sigma = \partial\Psi$ is its subdifferential. We also know that if we define

$$D(\sigma) = \{r \in \mathbb{R} \text{ such that } \sigma(r) \neq \emptyset\},$$

where $\emptyset$ denotes the empty set, and

$$D(\Psi) = \{r \in \mathbb{R} \text{ such that } \Psi(r) < +\infty\},$$

then $D(\sigma) \subset D(\Psi) \subset \overline{D(\Psi)} = \overline{D(\sigma)}$.

In the rest of the paper we will always assume that $f \in L^{p'}(\Omega)$, where, as usual, $p' = \frac{p}{p-1}$.

Since u_ε is the minimizer of the following energy functional in $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ (see [1, 26]):

$$E(u) = \int_{\Omega_\varepsilon} |\nabla u|^p \, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \Psi(u) \, dS - \int_{\Omega_\varepsilon} fu \, dx,$$

we consider the following definition of weak solution.

Definition 2.1. We will say that $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is a weak solution of problem (1.1) if $u_\varepsilon(x) \in D(\Psi)$ for a.e. $x \in S_\varepsilon$, and for all $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla(v - u_\varepsilon) \, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon)) \, dS \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon) \, dx. \tag{2.2}$$

The existence and uniqueness of a weak solution to problem (2.2) is an easy consequence of well-known results:

Proposition 2.2. *There exists a unique $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ weak solution of* (2.2). *Besides, there exists $K > 0$ independent of ε such that*

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|\Psi(u_\varepsilon)\|_{L^1(S_\varepsilon)} \leq K. \tag{2.3}$$

The homogenized problem will involve the function $H\colon \mathbb{R} \to \mathbb{R}$ given by (1.5). Let us present some of the properties satisfied by H.

Lemma 2.3. *If σ satisfies* (2.1), *then the function H defined by* (1.5) *is a nondecreasing nonexpansion on $\mathbb{R}$ (i.e., a nondecreasing Lipschitz continuous function of Lipschitz constant $L \leq 1$). Moreover, this function H is the unique function $H\colon \mathbb{R} \to \mathbb{R}$ satisfying the relation*

$$\mathcal{B}_0 |H(r)|^{p-2} H(r) \in \sigma(r - H(r)) \quad \textit{for any } r \in \mathbb{R}. \tag{2.4}$$

Concerning the homogenized problem (1.3), we point out that since H is a nondecreasing nonexpansion on $\mathbb{R}$, for the parameters $\mathcal{A}$ and $\mathcal{B}_0$ given by (1.4) and (1.6), and for $f \in L^{p'}(\Omega)$, there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ of problem (1.3). Moreover, $|H(u)|^{p-2} H(u) \in L^{p'}(\Omega)$. For the proof it is enough to set $V = W_0^{1,p}(\Omega)$ and define the operator $A\colon V \to V'$ by

$$\langle Av, w \rangle = \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w \, dx + \int_\Omega \mathcal{A} |H(v)|^{p-2} H(v) w \, dx \quad \text{for any } w \in V. \tag{2.5}$$

Notice that, since H is Lipschitz, $H(v) \in L^p(\Omega)$ for any $v \in L^p(\Omega)$. Then A is a hemicontinuous strictly monotone coercive operator, and the existence and uniqueness of a weak solution u is standard (see, e.g., [26]).

We will make fundamental use of the following reformulation of a weak solution. Since the *limit operator* $A\colon V \to V'$, with $V = W_0^{1,p}(\Omega)$, given by (2.5), is hemicontinuous and monotone, we can use the Brezis–Sibony characterization (see [3, Lemma 1.1] or [26, Chapter 2, Theorem 2.2]), that is, $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.3) if and only if

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u) \, dx + \int_\Omega \mathcal{B}_0 |H(v)|^{p-2} H(v)(v - u) \, dx \geq \int_\Omega f(v - u) \, dx \quad \text{for any } v \in W_0^{1,p}(\Omega). \tag{2.6}$$

The main result of this paper is the following convergence result.

Theorem 2.4. *Let $n \geq 3$, $1 < p < n$, $\alpha = \frac{n}{n-p}$ and $\gamma = \alpha(p - 1)$. Let σ be any maximal monotone graph of $\mathbb{R}^2$, with $0 \in \sigma(0)$, and let $f \in L^{p'}(\Omega)$. Let $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be the (unique) weak solution of problem* (1.1). *Then there exists an extension $\tilde{u}_\varepsilon$ of u_ε such that $\tilde{u}_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$, where $u \in W_0^{1,p}(\Omega)$ is the (unique) weak solution of problem* (1.3) *associated to the function H, defined by* (1.5).

Remark 2.5. The case $n = 2$ can be studied by similar techniques, although some of the computations vary. In particular, the critical value of α does not verify the same formula.

The other key result we will prove in this paper is Theorem 2.9 below, the statement of which requires some preliminary lemmas. The extension $\tilde{u}_\varepsilon$ of solutions u_ε can be obtained by applying the methods of [30].

Lemma 2.6. *Let Ω_ε be the domain defined above and let $1 < p < n$, $n \geq 3$. Then there exists an extension operator $P_\varepsilon : W^{1,p}(\Omega_\varepsilon) \to W^{1,p}(\Omega)$ such that*

$$\|P_\varepsilon u\|_{W^{1,p}(\Omega)} \leq C_1 \|u\|_{W^{1,p}(\Omega_\varepsilon)},$$
$$\|\nabla(P_\varepsilon u)\|_{L_p(\Omega)} \leq C_2 \|\nabla u\|_{L_p(\Omega_\varepsilon)}.$$

Moreover, by applying this extension theorem and the methods introduced in [30], we can prove the following useful estimates.

Lemma 2.7. (i) *Let $u \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$, $p > 1$ and $n \geq 3$. Then there exists positive constant C such that*

$$\|u\|_{L^p(\Omega_\varepsilon)} \leq C \|\nabla u\|_{L^p(\Omega_\varepsilon)}.$$

(ii) *Let $u \in W^{1,p}(Y_\varepsilon)$ be such that $\int_{Y_\varepsilon} u = 0$. Then*

$$\|u\|_{L^p(Y_\varepsilon)} \leq K_1 \varepsilon \|\nabla u\|_{L^p(Y_\varepsilon)},$$

where the constant K_1 is independent of ε.

Thanks to the a priori estimate (2.3) and the properties of the extension operator $P_\varepsilon : W_0^{1,p}(\Omega_\varepsilon, \partial\Omega) \to W^{1,p}(\Omega)$, we know that and there exists $u \in W_0^{1,p}(\Omega)$ such that

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega).$$

The difficult task is to show that $u \in W_0^{1,p}(\Omega)$ is the weak solution of problem (1.3) such as it is ensured in Theorem 2.4.

Motivated by this and (2.6), we will also use the fact that if $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the weak solution of problem (1.1), then

$$\int_{\Omega_\varepsilon} |\nabla v|^{p-2} \nabla v \cdot \nabla(v - u_\varepsilon)\, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v) - \Psi(u_\varepsilon))\, dS \geq \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\, dx \tag{2.7}$$

for any test function $v \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$.

The problematic term, in order to pass to the limit, is the boundary integrals over S_ε. Here we will follow a technique of proof introduced by the last author (Shaposhnikova) in collaboration with different co-authors (see, e.g., [28, 34, 37]), which can be applied in different frameworks.

Lemma 2.8. *Let $z_\varepsilon \in W_0^{1,p}(\Omega)$ for some $p > 1$, and assume that $z_\varepsilon \rightharpoonup z_0$ in $W_0^{1,p}(\Omega)$ as $\varepsilon \to 0$. Then*

$$\Big| 2^{2(n-1)} \varepsilon \sum_{j \in \Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} z_\varepsilon\, dS - \omega_n \int_\Omega z_0\, dx \Big| \to 0 \quad \text{as } \varepsilon \to 0,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$.

This lemma (which we remark is independent of α and γ, see the proof in [37]) is the key point of the homogenization technique in the critical case. It is based in the general idea that if P^j_ε is the center of the ball $G^j_\varepsilon = \{x \in Y^j_\varepsilon : |x - P^j_\varepsilon| < a_\varepsilon\}$ and if T^j_ε denotes the ball of radius $\varepsilon/4$ centered at the point P^j_ε, then we can get several explicit estimates on the solution $w^j_\varepsilon(x)$ for $j = 1, \dots, N(\varepsilon)$ of the auxiliary cellular boundary value problem

$$\begin{cases} \Delta_p w^j_\varepsilon = 0, & x \in T^j_\varepsilon \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon = 1, & x \in \partial G^j_\varepsilon, \\ w^j_\varepsilon = 0, & x \in \partial T^j_\varepsilon. \end{cases} \tag{2.8}$$

One of the many remarkable properties of this cellular problem is that its (unique) weak solution, w^j_ε, is radially symmetric (recall that G_0 is a ball) and satisfies that $\partial_{\nu_p} w^j_\varepsilon$ is constant on ∂T^j_ε and on ∂G^j_ε. Due to the divergence theorem,

$$\int_{G^j_\varepsilon} |\nabla w^j_\varepsilon|^{p-2} \nabla w^j_\varepsilon \cdot \nabla z\, dx = \int_{\partial T^j_\varepsilon} z \partial_{\nu_p} w^j_\varepsilon\, dS + \int_{\partial G^j_\varepsilon} z \partial_{\nu_p} w^j_\varepsilon\, dS \quad \text{for any } z \in W^{1,p}(T^j_\varepsilon \setminus \overline{G^j_\varepsilon}).$$

Furthermore, we can make explicitly several computations. Hence, we have an explicit way to compare the reaction term on S_ε with an auxiliary term on balls with radius $C\varepsilon$, and Lemma 2.8 becomes very useful.

Another key idea of our proof is to relate a general test function $v \in W_0^{1,p}(\Omega)$, used to check the limit characterization (2.6), with some suitable *correction* v_ε, which is a better fitted test function in the microscopic weak formulation (2.7). In fact, by density, it will be enough to do that with a smooth test function $v \in \mathcal{C}_c^\infty(\Omega)$. We will construct such adaptation among test functions in the form $v_\varepsilon = v - hW_\varepsilon$, where, for the moment, $h \in W^{1,\infty}(\Omega)$ without any other property, and, which is crucial, $W_\varepsilon \in W_0^{1,\infty}(\Omega)$ defined as

$$W_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j},\ j = 1, \dots, N(\varepsilon) = |\Upsilon_\varepsilon|, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j=1}^{N(\varepsilon)} T_\varepsilon^j, \end{cases} \tag{2.9}$$

with w_ε^j the solution of the auxiliary cellular boundary value problem (2.8). The following technical result will explain why the function H, arising in the limit problem (1.3), was taken in this concrete form (more precisely, so that (2.4) holds), different from the boundary kinetics σ.

Theorem 2.9. *Let $u_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$, $1 < p < n$, be a sequence of uniformly bounded norm, and let $v \in \mathcal{C}_c^\infty(\Omega)$, $h \in W^{1,\infty}(\Omega)$ and $v_\varepsilon = v - hW_\varepsilon$. Then*

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\,dx\Big) = \lim_{\varepsilon\to 0}(I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon}),$$

where

$$I_{1,\varepsilon} = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u_\varepsilon)\,dx, \tag{2.10}$$

$$I_{2,\varepsilon} = -\varepsilon^{-\gamma}\mathcal{B}_0 \int_{S_\varepsilon} |h|^{p-2}h(v - h - u_\varepsilon)\,dS,$$

$$I_{3,\varepsilon} = -A_\varepsilon \varepsilon \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T_\varepsilon^j} |h|^{p-2}h(v - u_\varepsilon)\,dS, \tag{2.11}$$

with A_ε being a bounded sequence, see (5.1). *Besides, if $\tilde u_\varepsilon$ is an extension of u_ε and $\tilde u_\varepsilon \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then, for any $v \in W_0^{1,p}(\Omega)$,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - hW_\varepsilon - u_\varepsilon)\,dx = \int_{\Omega} |\nabla v|^{p-2}\nabla v \cdot \nabla(v - u)\,dx.$$

The aforementioned corrector term in the form hW_ε, where $h \in W^{1,\infty}(\Omega_\varepsilon, \partial\Omega)$ will be taken to satisfy the condition $h(x) = H(v(x))$ for a.e. $x \in \Omega$, with H given by (2.4). These conditions rise naturally so that the term $I_{2,\varepsilon}$ above cancels out with the reaction term.

Remark 2.10. In general, it is expected that the convergence $\tilde u_\varepsilon \to u$ can be improved to strong convergence by adding a corrector term. In fact, if σ is smooth, it is known that $u_\varepsilon - H(u_\varepsilon)W_\varepsilon \to u$ strongly in $W_0^{1,p}(\Omega)$ (see, e.g., [37]). It is possible to adapt these arguments to the case of some maximal monotone graphs as, for instance, the one given by the Signorini boundary condition (see [12]).

3 Existence of u_ε and characterization of the function H

Proof of Proposition 2.2. Consider the Banach space $V = W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Let $A_\varepsilon\colon V \to V'$ be the operator defined by

$$\langle A_\varepsilon v, w\rangle = \int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v \cdot \nabla w\,dx \quad \text{for any } w \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega).$$

Then A is a hemicontinuous strictly monotone coercive operator [26]. Define $\varphi^\varepsilon: W_0^{1,p}(\Omega_\varepsilon, \partial\Omega) \to (-\infty, +\infty]$ by

$$\varphi^\varepsilon(u) = \begin{cases} \varepsilon^{-\gamma} \int_{S_\varepsilon} \Psi(\mathrm{tr}_{S_\varepsilon}(u))\, dS & \text{if } \mathrm{tr}_{S_\varepsilon}(u(x)) \in D(\Psi) \text{ for a.e. } x \in S_\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

It is clear that φ^ε is a convex lower semicontinuous function with $\varphi^\varepsilon \not\equiv +\infty$. Since $f \in V'$, we have that u_ε is a weak solution of problem (1.1) if and only if

$$\langle A_\varepsilon(u_\varepsilon) - f, v - u_\varepsilon \rangle + \varphi^\varepsilon(v) - \varphi^\varepsilon(u_\varepsilon) \geq 0 \quad \text{for all } v \in V.$$

Thus, the existence and uniqueness of a weak solution u_ε of problem (1.1) is consequence of [26, Chapter 2, Theorem 8.5].

In order to prove the a priori bound (2.3), let $v \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$. Then, we have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p\, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \Psi(u_\varepsilon)\, dS \leq \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla v\, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \Psi(v)\, dS - \int_{\Omega_\varepsilon} f(v - u_\varepsilon)\, dx.$$

Given $\delta \in (0, 1)$, we apply Young's inequality, $ab \leq \delta |a|^{p'} + C_\delta |b|^p$, to get

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \cdot \nabla v\, dx \leq \delta \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^p\, dx + C_\delta \int_{\Omega_\varepsilon} |\nabla v|^p\, dx.$$

Therefore, since $\Psi \geq 0$, taking $v = 0$ and applying Hölder's and Poincaré's inequalities, we have

$$(1 - \delta) \|\nabla u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|\Psi(u_\varepsilon)\|_{L^1(S_\varepsilon)} \leq \int_{\Omega_\varepsilon} f u_\varepsilon\, dx \leq C \|f\|_{L^{p'}(\Omega)} \|\nabla u_\varepsilon\|_{L^p(\Omega_\varepsilon)},$$

which leads to the result. □

Proof of Lemma 2.3. Let $\Theta_{n,p}(s) = \mathcal{B}_0 |s|^{p-2} s$ for $s \in \mathbb{R}$. Since σ^{-1} is also a maximal monotone graph of $\mathbb{R}^2$, for any $p > 1$ and $\mathcal{B}_0 > 0$, the graph $\sigma^{-1} \circ \Theta_{n,p}$ is also a maximal monotone graph of $\mathbb{R}^2$. Indeed, let $D(\sigma^{-1}) = [a, b]$ for some $-\infty \leq a < b \leq +\infty$, and let $(\sigma^{-1})^0$ be the principal section (i.e., the nondecreasing function) of the graph σ^{-1}. This means that

$$(\sigma^{-1})^0(r) = \inf \sigma^{-1}(r), \quad r \in [a, b].$$

Then, since $\Theta_{n,p}$ is strictly increasing, $\sigma^{-1} \circ \Theta_{n,p}$ is a monotone graph,

$$D(\sigma^{-1} \circ \Theta_{n,p}) = [\Theta_{n,p}^{-1}(a), \Theta_{n,p}^{-1}(b)] \quad \text{and} \quad (\sigma^{-1} \circ \Theta_{n,p})^0 = (\sigma^{-1})^0 \circ \Theta_{n,p}.$$

In particular, if σ^{-1} is multivalued in some point $c \in (a, b)$, then $\sigma^{-1} \circ \Theta_{n,p}(c)$ is the full interval

$$\sigma^{-1} \circ \Theta_{n,p}(c) = [(\sigma^{-1})^0(\Theta_{n,p}(c)^-), (\sigma^{-1})^0(\Theta_{n,p}(c)^+)],$$

and this implies that $\sigma^{-1} \circ \Theta_{n,p}$ is a maximal monotone graph of $\mathbb{R}^2$ (see [2, Example 2.8.1]).

Now, since $\sigma^{-1} \circ \Theta_{n,p}$ is also a maximal monotone graph of $\mathbb{R}^2$, we know that $(I + \sigma^{-1} \circ \Theta_{n,p})$ is an injective application such that $R(I + \sigma^{-1} \circ \Theta_{n,p}) = \mathbb{R}$ (see [2]). Thus, if H is defined by (1.5), then H is a nonexpansion on $\mathbb{R}$ (see [2, Proposition 2.2]). Hence,

$$(I + \sigma^{-1} \circ \Theta_{n,p})(H(r)) = r$$

for any $r \in \mathbb{R}$ and, in consequence,

$$H(r) + \sigma^{-1} \circ \Theta_{n,p}(H(r)) = r.$$

In other words,

$$\sigma^{-1} \circ \Theta_{n,p}(H(r)) = r - H(r).$$

This implies that $r - H(r) \in D(\sigma)$ for any $r \in \mathbb{R}$ and that $\Theta_{n,p}(H(r)) \in \sigma(r - H(r))$ for any $r \in \mathbb{R}$, which proves that $H(r)$ satisfies relation (2.4). Moreover, from the definition of H, it is obvious that H is nondecreasing (in fact if σ is strictly increasing, then H is also a strictly increasing function).

On the other hand, such function $H(r)$ is the unique function satisfying relation (2.4), since applying the inverse graph

$$\sigma^{-1} \circ \Theta_{n,p} \circ H \supset (I - H)$$

implies that $(I + \sigma^{-1} \circ \Theta_{n,p}) \circ H = I$, and so, necessarily, $H = (I + \sigma^{-1} \circ \Theta_{n,p})^{-1}$. Of course, from the implicit formula, H is strictly increasing. □

4 Proof of Theorem 2.4

Since G_0 is ball, it is easy to see that

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}{(C_0\varepsilon^\alpha)^{-\frac{n-p}{p-1}} - (\varepsilon/4)^{-\frac{n-p}{p-1}}}, \quad x \in T_\varepsilon^j, \setminus \overline{G_\varepsilon^j}, \tag{4.1}$$

is the unique solution of (2.8).

Lemma 4.1. *If W_ε is defined by (2.9), then the following estimate holds:*

$$\int_{\Omega_\varepsilon} |\nabla W_\varepsilon|^q \, dx \le K\varepsilon^{\frac{n(p-q)}{n-p}} \quad \textit{for any } 1 \le q \le p. \tag{4.2}$$

In particular,

$$W_\varepsilon \rightharpoonup 0 \quad \textit{in } W_0^{1,p}(\Omega) \textit{ as } \varepsilon \to 0.$$

Proof. Estimate (4.2) is an explicit computation. For $q = p$, we obtain from it that, up to a subsequence, there exists $W_0 \in W_0^{1,p}(\Omega)$ such that $W_\varepsilon \rightharpoonup W_0$ in $W_0^{1,p}(\Omega)$. For $q < p$, we have that $W_\varepsilon \to 0$ in $W_0^{1,q}(\Omega)$, hence $W_0 = 0$. □

Proof of Theorem 2.4. Let $v \in \mathcal{C}_c^\infty(\Omega)$ and $h = H(v)$, with $H\colon \mathbb{R} \to \mathbb{R}$ given by (1.5). Then $h \in W^{1,\infty}(\Omega)$. Let $v_\varepsilon = v - hW_\varepsilon \in W_0^{1,p}(\Omega_\varepsilon, \partial\Omega)$, with $W_\varepsilon \in W_0^{1,\infty}(\Omega)$ defined by (2.9). Due to (2.7), we know that u_ε satisfies the inequality

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \cdot \nabla(v_\varepsilon - u_\varepsilon)\, dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v_\varepsilon) - \Psi(u_\varepsilon))\, dS \ge \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\, dx.$$

Since $W_\varepsilon \to 0$ in $L^p(\Omega)$ (due to the compact inclusion), by Theorem 2.9, we can deduce that

$$\lim_{\varepsilon\to 0} \Big[I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v_\varepsilon) - \Psi(u_\varepsilon))\, dS \Big] \ge \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} f(v_\varepsilon - u_\varepsilon)\, dx = \int_\Omega f(v - u)\, dx.$$

Since $H\colon \mathbb{R} \to \mathbb{R}$ satisfies (2.4), by applying that if $\xi \in \partial\Psi(s_0) = \sigma(s_0)$, then $\Psi(s) - \Psi(s_0) \ge \xi(s - s_0)$, we can write

$$I_{2,\varepsilon} + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\Psi(v_\varepsilon) - \Psi(u_\varepsilon))\, dS = \varepsilon^{-\gamma} \int_{S_\varepsilon} [\Psi(v - H(v)) - \Psi(u_\varepsilon) - \mathcal{B}_0|H(v)|^{p-2}H(v)(v - H(v) - u_\varepsilon)]\, dS \le 0,$$

since $\mathcal{B}_0|H(v(x))|^{p-2}H(v(x)) \in \sigma(v(x) - H(v(x)))$ for any $x \in \overline{\Omega}$. We can pass also to the limit in (2.10) and (2.11) to get that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla(v - u)\, dx + \int_\Omega \mathcal{B}_0|H(v)|^{p-2}H(v)(v - u)\, dx \ge \int_\Omega f(v - u)\, dx,$$

and since $v \in \mathcal{C}_c^\infty(\Omega)$ is arbitrary, by density, this also holds for every $v \in W_0^{1,p}(\Omega)$. Hence, we get that u is the unique weak solution of (1.3). □

5 Proof of Theorem 2.9

The proof of Theorem 2.9 for $p = 2$ can be found in [36], and for $2 < p < n$ in [34]. Here we will complete the proof for $1 < p < 2$. We need some auxiliary results.

Lemma 5.1 ([12]). *Let $1 < p < 2$. Then there exists positive constant $C = C(p)$ such that the inequality*

$$\big||\mathbf{a}-\mathbf{b}|^{p-2}(\mathbf{a}-\mathbf{b}) - (|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b})\big| \le C(|\mathbf{a}||\mathbf{b}|)^{\frac{p-1}{2}}$$

is valid for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$.

By using this result, we prove the following lemma.

Lemma 5.2. *Let $1 < p < 2$, $n \ge 3$, $v \in W_0^{1,\infty}(\Omega)$ and $\varphi \in W_0^{1,p}(\Omega)$. Let $\eta_\varepsilon \in W^{1,p}(\Omega)$ be such that $\|\nabla\eta_\varepsilon\|_{L^q(\Omega)} \to 0$ for some $q \in [1, p)$ as $\varepsilon \to 0$. Then*

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla(v-\eta_\varepsilon)|^{p-2}\nabla(v-\eta_\varepsilon)\cdot\nabla\varphi\,dx\Big) = \lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon} |\nabla v|^{p-2}\nabla v\cdot\nabla\varphi\,dx - \int_{\Omega_\varepsilon}|\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon\cdot\nabla\varphi\,dx\Big).$$

Proof. By Lemma 5.1, by applying Hölder's inequality, we have

$$\begin{aligned}\Big|\int_{\Omega_\varepsilon} &|\nabla(v-\eta_\varepsilon)|^{p-2}\nabla(v-\eta_\varepsilon)\cdot\nabla\varphi\,dx - (\nabla v|^{p-2}\nabla v - |\nabla\eta_\varepsilon|^{p-2}\nabla\eta_\varepsilon)\cdot\nabla\varphi\,dx\Big|\\ &\le C\int_{\Omega_\varepsilon}|\nabla v|^{\frac{p-1}{2}}|\nabla\eta_\varepsilon|^{\frac{p-1}{2}}|\nabla\varphi|\,dx\\ &\le K\|\nabla v\|_\infty^{\frac{p-1}{2}}\|\nabla\eta_\varepsilon\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)}^{\frac{p-1}{2}}\|\nabla\varphi\|_{L^{\frac{p+1}{2}}(\Omega_\varepsilon)},\end{aligned}$$

since $1 < \frac{p+1}{2} < p$. This proves the result. □

We have all the tools we need for the proof of Theorem 2.9.

Proof of Theorem 2.9. As said before, it is enough to consider the case $p \in (1, 2)$. Applying Lemma 5.2, we obtain

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon}|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon\cdot\nabla(v_\varepsilon - u_\varepsilon)\,dx\Big) = \lim_{\varepsilon\to 0}(J_{1,\varepsilon} + J_{2,\varepsilon}),$$

where

$$\begin{aligned}J_{1,\varepsilon} &= \int_{\Omega_\varepsilon}|\nabla v|^{p-2}\nabla v\cdot\nabla(v - hW_\varepsilon - u_\varepsilon)\,dx,\\ J_{2,\varepsilon} &= \int_{\Omega_\varepsilon}|\nabla(hW_\varepsilon)|^{p-2}\nabla(hW_\varepsilon)\cdot\nabla(v - hW_\varepsilon - u_\varepsilon)\,dx.\end{aligned}$$

Moreover,

$$\lim_{\varepsilon\to 0}J_{1,\varepsilon} = \lim_{\varepsilon\to 0}\Big(I_{1,\varepsilon} + \int_{\Omega_\varepsilon}|\nabla v|^{p-2}\nabla v\cdot\nabla(hW_\varepsilon)\,dx\Big) = \lim_{\varepsilon\to 0}I_{1,\varepsilon}.$$

On the other hand,

$$\begin{aligned}\lim_{\varepsilon\to 0}J_{2,\varepsilon} &= \lim_{\varepsilon\to 0}\Big(\int_{\Omega_\varepsilon}|\nabla W_\varepsilon|^{p-2}\nabla W_\varepsilon\cdot\nabla(v - hW_\varepsilon - u_\varepsilon)\,dx\Big)\\ &= \lim_{\varepsilon\to 0}\Big(\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|h|^{p-2}h(v-u_\varepsilon)\,dS + \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|h|^{p-2}h(v-h-u_\varepsilon)\,dS\Big),\end{aligned}$$

where $\partial_\nu g$ is the usual normal derivative of g. Using (4.1), we get

$$\begin{aligned}\partial_\nu w_\varepsilon^j\big|_{\partial T_\varepsilon^j} &= \frac{d}{dr}w_\varepsilon^j\Big|_{r=\varepsilon/4} = -\frac{(n-p)2^{\frac{2n-2}{p-1}}C_0^{\frac{n-p}{p-1}}\varepsilon^{\frac{1}{p-1}}}{(p-1)(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}}2^{\frac{2n-2p}{p-1}})},\\ \partial_\nu w_\varepsilon^j\big|_{\partial G_\varepsilon^j} &= -\frac{d}{dr}w_\varepsilon^j\Big|_{r=a_\varepsilon} = \frac{(n-p)\varepsilon^{\frac{-n}{n-p}}}{(p-1)C_0(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}}2^{\frac{2n-2p}{p-1}})}.\end{aligned}$$

Therefore,

$$\lim_{\varepsilon\to 0} J_{2,\varepsilon} = \lim_{\varepsilon\to 0}\Big(A_\varepsilon \varepsilon \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T_\varepsilon^j} |h|^{p-2}h(v-u_\varepsilon)\,ds - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\Big(\Big(\frac{n-p}{p-1}\Big)^{p-1} C_0^{1-p}|h|^{p-2}h\Big)(v-h-u_\varepsilon)\,ds - Q_\varepsilon\Big),$$

where

$$A_\varepsilon = \Big(\frac{n-p}{p-1}\Big)^{p-1} \frac{2^{2n-2}C_0^{n-p}}{(1-(C_0\varepsilon^\alpha)^{\frac{n-p}{p-1}}\varepsilon^{-\frac{n-p}{p-1}}2^{\frac{2n-2p}{p-1}})^{p-1}} \tag{5.1}$$

and

$$Q_\varepsilon = \frac{1-(1-a_\varepsilon^{\frac{n-p}{p-1}}\varepsilon^{\frac{p-n}{p-1}}2^{\frac{2n-2p}{p-1}})^{p-1}}{C_0^{p-1}(1-a_\varepsilon^{\frac{n-p}{p-1}}\varepsilon^{\frac{p-n}{p-1}}2^{\frac{2n-2p}{p-1}})^{p-1}}\Big(\frac{n-p}{p-1}\Big)^{p-1}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}|h|^{p-2}h(v-h-u_\varepsilon)\,dS.$$

It is an easy (but tedious) task to check that

$$\lim_{\varepsilon\to 0} Q_\varepsilon = 0,$$

which concludes the proof. □

6 Noncritical case and pointwise comparison of homogenized solutions with the critical case

For $A \subset \mathbb{R}^m$, let $\mathcal{C}(A)$ denote the space of continuous functions on A.

Theorem 6.1. *Let $n \geq 3$, $p \in [2,n)$, $1 < \alpha < \frac{n}{n-p}$, $f \in L^\infty(\Omega)$ and $r \in \mathcal{C}(\bar\Omega)$. Let also $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing such that $\sigma(0) = 0$ and let u_ε be the solution of*

$$\begin{cases} -\Delta_p u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ \partial_{\nu_p} u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma} r, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

Then $\tilde u_\varepsilon \rightharpoonup u_{\text{non-crit}}$ in $W_0^{1,p}(\Omega)$, where $u_{\text{non-crit}}$ is the solution of

$$\begin{cases} -\Delta_p u + \hat{\mathcal{A}}\sigma(u) = f + \hat{\mathcal{A}} r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

with $\hat{\mathcal{A}} = C_0^{n-1}|\partial G_0|$.

Proof. Assume first that

$$0 < k_1 \leq \sigma' \leq k_2.$$

Then the result holds by [34, Theorem 3].

Applying the estimates in [29], we check that $(P_\varepsilon u_\varepsilon)$ is bounded in $W_0^{1,p}(\Omega)$, hence there exists a limit $\hat u$ such that, up to a subsequence, $P_\varepsilon u_\varepsilon \to \hat u$ strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$.

Let M be such that $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leq M$ (see [8]). Let σ_δ be a sequence such that $0 < k_{1,\delta} \leq \sigma_\delta' \leq k_{2,\delta}$ and $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M,M])$ as $\delta \to 0$. Let $u_{\varepsilon,\delta}$ be the solution of (6.1) with σ_δ. We can check, again from estimates in [29], that

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{L^p(\Omega_\varepsilon)} \leq C\|\sigma - \sigma_\delta\|_{\mathcal{C}([-M,M])}.$$

Passing to the limit as $\varepsilon \to 0$, indicating that $P_\varepsilon u_{\varepsilon,\delta} \rightharpoonup u_\delta$ in $W_0^{1,p}(\Omega)$, where u_δ is the solution of (6.2) with σ_δ, we have that

$$\|\hat u - u_\delta\|_{L^p(\Omega)} \leq C\|\sigma - \sigma_\delta\|_{\mathcal{C}([-M,M])}.$$

It is easy to check that $u_\delta \to u$ in $L^p(\Omega)$, where u is the solution of the problem with σ. Therefore, $P_\varepsilon u_\varepsilon \to u$ in $L^p(\Omega)$ as $\varepsilon \to 0$ and $u = \hat u$. □

Theorem 6.2. *Let $n \geq 3$, $p \in [2, n)$, $\alpha = \frac{n}{n-p}$, $f \in L^\infty(\Omega)$ and $r \in \mathcal{C}(\bar{\Omega})$. Let also $\sigma \in \mathcal{C}(\mathbb{R})$ be nondecreasing such that $\sigma(0) = 0$ and let u_ε be the solution of (6.1). Then $\tilde{u}_\varepsilon \rightharpoonup u_{\mathrm{crit}}$ in $W_0^{1,p}(\Omega)$, where u_{crit} is the solution of*

$$\begin{cases} -\Delta_p u + \mathcal{A}|H(x,u)|^{p-2}H(x,u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and H is the solution of

$$\mathcal{B}_0|H(x,s)|^{p-2}H(x,s) = \sigma(s - H(x,s)) - r(x) \quad \text{a.e. in } \Omega.$$

Sketch of proof. We can apply the same reasoning as before and the fact that $H_\delta \to H$, in the sense of maximal monotone graphs, as $\sigma_\delta \to \sigma$ in $\mathcal{C}([-M, M])$. □

Theorem 6.3. *Assume the conditions of the two previous theorems, $f = 0$ and $r(x) \equiv g(1) = 1$. Then, we have that $u_{\mathrm{crit}} \leq u_{\text{non-crit}}$.*

Proof. The condition on f and r guarantee that $0 \leq u \leq 1$ in both cases. It is easy to check that H is increasing, and $H(s) \leq 0$ for $s \in [0, 1]$. It is easy to establish the following inequality on the zero order terms:

$$\mathcal{B}_0|H(s)|^{p-2}H(s) \geq \hat{\mathcal{A}}(\sigma(s) - g(1)).$$

Therefore, applying the comparison principle (see, e.g., [8]), we have the result. □

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A Mathematical Proof in Nanocatalysis: Better Homogenized Results in the Diffusion of a Chemical Reactant Through Critically Small Reactive Particles

Jesús Ildefonso Díaz and David Gómez-Castro

Abstract We consider a reaction-diffusion where the reaction takes place on the boundary of the reactive particles. In this sense the particles can be though as a catalysts, that produce as change in the ambient concentration w_ε of a reactive element. It is known that depending on the size of the particles with respect to their periodic repetition there are different homogeneous behaviors. In particular, there is a case in which the kind of nonlinear dynamics changes, and becomes more smooth. This case can be linked with the strange behaviors that arise with the use of nanoparticles. In this paper we show that that concentrations of a catalyst are always higher when nanoparticles are applied.

1 Introduction

We consider a reaction-diffusion problem in which the reaction takes place on the boundary of the inclusions. In this sense the inclusions can be though as a catalysts, that produce as change in the ambient concentration w_ε of a reactive element. This is standardly modeled as

$$\begin{cases} -\Delta w_\varepsilon = 0 & \Omega_\varepsilon, \\ \partial_\nu w_\varepsilon + \varepsilon^{-\gamma} g(w_\varepsilon) = 0 & S_\varepsilon, \\ w_\varepsilon = 1 & \partial\Omega, \end{cases} \tag{1}$$

J.I. Díaz
Instituto de Matemática Aplicada & Dpto. de Matemática Aplicada, Plaza de las Ciencias 3, 28040 Madrid (Spain), e-mail: jidiaz@ucm.es

D. Gómez Castro
Instituto de Matemática Aplicada & Dpto. de Matemática Aplicada, Plaza de las Ciencias 3, 28040 Madrid (Spain), e-mail: dgcastro@ucm.es

where g is a nondecreasing function such that $g(0) = 0$, Ω_ε is a perforated domain, $\partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega$, R. Aris defined (see, e.g., [1]) the effectiveness of a reactor Ω as

$$\eta_\varepsilon = \frac{1}{|S_\varepsilon|}\int_{S_\varepsilon} g(w_\varepsilon). \tag{2}$$

In the case where the particles are large (in a sense that would be precised later), the the problem can be though homogenous, as $\Omega_\varepsilon \to \Omega$ then $w_\varepsilon \to w$ (in a sense that would be precised lated), where the effective problem results

$$\begin{cases} -\Delta w + Ag(w) = 0 & \Omega, \\ w = 1 & \partial\Omega, \end{cases} \tag{3}$$

for a certain constant A. In this setting Aris defined the effectiveness for the effective problem as

$$\eta = \frac{1}{|\Omega|}\int_\Omega g(w)dx. \tag{4}$$

This kind of problems, when g is not Lipschitz, has been shown to develop, in some cases, a region of positive measure $\{x \in \Omega : u(x) = 0\}$. This region, which is sometimes known as a dead core, has been studied in [2, 5].

Nonetheless, when the holes are of a sufficiently small size with respect to their repetition, the behaviour of the limit changes and becomes

$$\begin{cases} -\Delta w + Bh(w) = 0 & \Omega, \\ w = 1 & \partial\Omega, \end{cases} \tag{5}$$

and h is a new nonlinearity, which we will introduce later, and $B > 0$ is a constant.

This change in behaviour, which is related to the pioneering paper [3], will be linked to new surprising properties that arise with the use of nanoparticles (see [11]). In this setting, the good definition for the effectiveness of the limit problem is unclear.

The aim of this paper is to show that homogenized problem is more effective in the case associated with nanoparticles than the other cases. It represents a mathematical proof of some experimental facts in the literature.

2 Statement of results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = (-1/2, 1/2)^n$. Denote by $G_0 = B_1(0)$ the unit ball centered at the origin. For $\delta > 0$ and $\varepsilon > 0$ we define $\delta B = \{x \,|\, \delta^{-1}x \in B\}$ and $\widetilde{\Omega}_\varepsilon = \{x \in \Omega \,|\, \rho(x, \partial\Omega) > 2\varepsilon\}$.

Let
$$a_\varepsilon = C_0 \varepsilon^\alpha, \tag{6}$$
where $\alpha > 1$ and C_0 is a given positive number. Define
$$G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j,$$
where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \overline{\widetilde{\Omega}}_\varepsilon \neq \emptyset\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = const > 0$, $\mathbb{Z}^n$ is the set of vectors z with integer coordinates. The reference cell is represented by Figure 1.

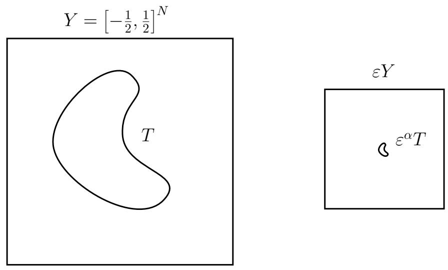

Fig. 1 The reference cell Y and the scalings by ε and ε^α, for $\alpha > 1$. Notice that, for $\alpha > 1$, $\varepsilon^\alpha T$ (for a general particle shaped as T) becomes smaller relative to εY, which scales as the repetition. In our case T will be a ball $B_1(0)$.

Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$ and note that $\overline{G}_\varepsilon^j \subset \overline{Y}_\varepsilon^j$ and center of the ball G_ε^j coincides with the center of the cube Y_ε^j. Our "microscopic domain" is defined as
$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \qquad S_\varepsilon = \partial G_\varepsilon, \qquad \partial \Omega_\varepsilon = \partial \Omega \cup S_\varepsilon,$$
which can be represented as in Figure 2.

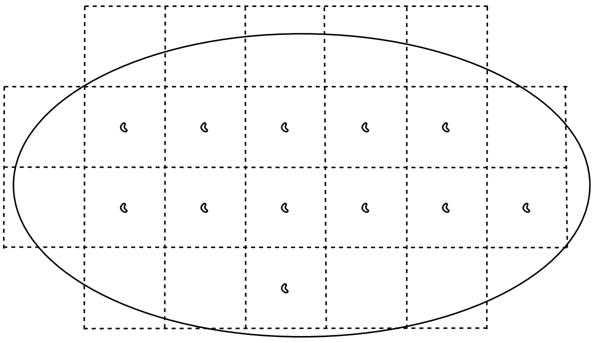

Fig. 2 The fixed bed reactor, i.e., the domain Ω_ε.

We define the space $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ be the completion, with respect to the norm of $W^{1,p}(\Omega_\varepsilon)$, of the set of infinitely differentable functions in $\overline{\Omega}_\varepsilon$ equal to zero in a neighborhood of $\partial\Omega$.

We are interest in understating the comparison of the limits of (1) when $\alpha \in (1, \frac{n}{n-2})$ and $\alpha = \frac{n}{n-2}$, which are known as the subcritical and critical cases in homogenization. The case $\alpha = 1$ was studied in [4]. In order to do this, we consider the change in variable $u = 1 - w$, $\sigma(u) = g(1) - g(1-u)$ we have

$$\begin{cases} -\Delta u_\varepsilon = 0 & \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(u_\varepsilon) = \varepsilon^{-\gamma} g(1) & S_\varepsilon, \\ u_\varepsilon = 0 & \partial\Omega. \end{cases} \tag{7}$$

Studying the family of solution $(u_\varepsilon)_{\varepsilon>0}$ is difficult, since they are not defined in the same domain. We consider a family of linear extension operators (see [10])

$$P_\varepsilon : \{u \in H^1(\Omega_\varepsilon) : u = 0, \partial\Omega\} \to H_0^1(\Omega) \tag{8}$$

such that

$$\|\nabla P_\varepsilon u\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega_\varepsilon)}. \tag{9}$$

We define the different possible limits:

- If $\alpha \in \left(1, \frac{n}{n-2}\right)$ the $u = u_{\text{non-crit}}$, which for $A = C_0^{n-1}\omega_n$ satisfies

$$\begin{cases} -\Delta u_{\text{non-crit}} + A\sigma(u_{\text{non-crit}}) = Ag(1) & \Omega, \\ u_{\text{non-crit}} = 0 & \partial\Omega. \end{cases} \tag{10}$$

- If $\alpha = \frac{n}{n-2}$ then u_{crit}, which for $B = (n-2)C_0^{n-2}\omega_n = \frac{n-2}{C_0}A$ satisfies

$$\begin{cases} -\Delta u_{\text{crit}} + BH(u_{\text{crit}}) = 0 & \Omega, \\ u_{\text{crit}} = 0 & \partial\Omega, \end{cases} \tag{11}$$

where H is the solution of the functional equality

$$\frac{n-2}{C_0} H(s) = \sigma(s - H(s)) - g(1). \tag{12}$$

We will start by indicating that, in the sense of maximal monotone graphs, in the particular case of $\sigma(u) = g(1) - g(1-u)$ one has

Lemma 1. *Let σ be a maximal monotone graph, then the solution H of* (12) *is given by*

$$H(u) = -\left(g^{-1}\left(\frac{n-2}{C_0}\cdot\right) + Id\right)^{-1}(1-u). \tag{13}$$

Hence $H(u) \le 0$ for every $u \in [0,1]$.

Remark 1. Notice that, in particular, in equation (5) we have

$$h(w) = \left(g^{-1}\left(\frac{n-2}{C_0}\cdot\right) + Id\right)^{-1}(w) \tag{14}$$

which is a nondecreasing function such that $h(0) = 0$.

Lemma 2. *Let σ be a bounded maximal monotone graph of $[0,1]\times\mathbb{R}$, then H is non-expansive in $[0,1]$ (and hence Lipschitz continuous).*

Proof. If $\sigma \in \mathscr{C}^1([0,1])$, differentiating (12) with respect to s we derive

$$H'(s) = \frac{\sigma'(s - H(s))}{\frac{n-2}{C_0} + \sigma'(s - H(s))} \in (0,1). \tag{15}$$

Hence,

$$|H(t) - H(s)| \le |t - s| \tag{16}$$

for all $t,s \in [0,1]$. If σ is a maximal monotone graph, let $\sigma_\delta \in \mathscr{C}^1([0,1])$ be an approximation in the sense of maximal monotone graphs $\sigma_\delta \to \sigma$. In particular, $H_\delta \to H$ pointwise, and hence

$$|H(t) - H(s)| \le |t - s| \tag{17}$$

which concludes the proof. □

We have the following homogenization result.

Theorem 1 ([12]). *Let $\alpha > 1$, $\gamma = \alpha(n-1) - n$ and $\sigma \in \mathscr{C}^1(\mathbb{R})$ be such that $\sigma(0) = 0$ and*

$$0 < k_1 \le \sigma'(s) \le k_2 \tag{18}$$

and let u_ε be the weak solution of (7). *Then, the extension $P_\varepsilon u_\varepsilon$ converge as $\varepsilon \to 0$*

$$P_\varepsilon u_\varepsilon \to \begin{cases} u_{non\text{-}crit} & \text{if } \alpha \in \left(1, \frac{n}{n-2}\right), \\ u_{crit} & \text{if } \alpha = \frac{n}{n-2}, \end{cases} \tag{19}$$

strongly in $W_0^{1,p}(\Omega)$ for $1 \le p < 2$ and weakly in $H_0^1(\Omega)$.

Since, in our case $0 \le u_\varepsilon \le 1$ then we can have a simple corollary:

Corollary 1. *Let $\sigma \in \mathscr{C}([0,1])$, nondecreasing and such that $\sigma(0) = 0$, then* (19) *weakly in $H_0^1(\Omega)$.*

Proof. Applying the estimates in [9] we check that $(P_\varepsilon u_\varepsilon)$ is bounded in $H_0^1(\Omega)$, hence there exists a limit $\hat{u}$ such that, up to a subsequence, $P_\varepsilon u_\varepsilon \to \hat{u}$ strongly in L^2.

Let σ_δ be such that it satisfies Theorem (1) and $\sigma_\delta \to \sigma$ in $\mathscr{C}([0,1])$ as $\delta \to 0$. Let $u_{\varepsilon,\delta}$ the solution of (7) with σ_δ. We can check, again with estimates in [9] that

$$\|u_\varepsilon - u_{\varepsilon,\delta}\|_{L^2(\Omega^\varepsilon)} \le C\|\sigma - \sigma_\delta\|_{\mathscr{C}([0,1])}. \tag{20}$$

Passing to the limit as $\varepsilon \to 0$

$$\|\hat{u} - u_\delta\|_{L^2(\Omega)} \le C\|\sigma - \sigma_\delta\|_{\mathscr{C}([0,1])}. \tag{21}$$

On the other hand, applying the theory of maximal monotones graphs, it is easy to check that H_δ the solution of (12) with σ_δ satisfies $H_\delta \to H$ in the sense of maximal monotone graphs. In both cases, then, it is easy to check that $u_\delta \to u$ in L^2 where u is the solution of the problem with H or σ. Therefore $u_\varepsilon \to u$ as $\varepsilon \to 0$. $\square$

It is known already (see [8]) that, for the non critical cases the effectiveness behaves as expected

$$\eta_\varepsilon \to \eta, \qquad \text{as } \varepsilon \to 0. \tag{22}$$

However, the noncritical case the dynamics changes. Therefore it is not clear whether it is natural to define the effectiveness in the usual way. Nonetheless, we can give a pointwise inequality. Let u_{crit} and $u_{\text{non-crit}}$ be the solutions of (11) and (10)

Theorem 2. *Let $\sigma \in \mathscr{C}([0,1])$ be such that $\sigma(0) = 0$. Then*

$$u_{crit} \le u_{non\text{-}crit}. \tag{23}$$

Proof. Since $H(s) \le 0$ we have that

$$BH(s) = (n-2)C_0^{n-2}\omega_n H(s) = C_0^{n-1}\omega_n \frac{n-2}{C_0} H(u) \tag{24}$$

$$= C_0^{n-1}\omega_n\left(\sigma(s - H(s)) - g(1)\right) = A\left(\sigma(s - H(s)) - g(1)\right) \tag{25}$$

$$\ge A\left(\sigma(s) - g(1)\right). \tag{26}$$

Therefore, applying the comparison principle, $u_{\text{crit}} \le u_{\text{non-crit}}$. $\square$

Remark 2. Therefore, if $g \in \mathscr{C}([0,1])$, the concentration w in the critical case is always larger than in the non critical cases

$$w_{\text{crit}} \geq w_{\text{non-crit}}. \tag{27}$$

We have a pointwise better reaction.

Remark 3. This kind of result has been proved in many different cases. In particular, for non smooth σ in the form of a root or a Heaviside function and nonlinear diffusion in the form of a p-Laplacian see [6]. The case of Signorini type boundary conditions can be found in [7].

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Pure and Applied Geophysics

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On the Effectiveness of Wastewater Cylindrical Reactors: an Analysis Through Steiner Symmetrization

J. I. Díaz[1] and D. Gómez-Castro[1]

Abstract—The mathematical analysis of the shape of chemical reactors is studied in this paper through the research of the optimization of its effectiveness η such as introduced by R. Aris around 1960. Although our main motivation is the consideration of reactors specially designed for the treatment of wastewaters our results are relevant also in more general frameworks. We simplify the modeling by assuming a single chemical reaction with a monotone kinetics leading to a parabolic equation with a non-necessarily differentiable function. In fact we consider here the case of a single, non-reversible catalysis reaction of chemical order $q, 0 < q < 1$ (i.e., the kinetics is given by $\beta(w) = \lambda w^q$ for some $\lambda > 0$). We assume the chemical reactor of cylindrical shape $\Omega = G \times (0, H)$ with G and open regular set of $\mathbb{R}^2$ not necessarily symmetric. We show that among all the sections G with prescribed area the ball is the set of lowest effectiveness $\eta(t, G)$. The proof uses the notions of Steiner rearrangement. Finally, we show that if the height H is small enough then the effectiveness can be made as close to 1 as desired.

Key words: Wastewater treatment, chemical reactor tanks, effectiveness, Steiner symmetrization.

1. Introduction

One of the most important problems on environment in Geosciences is the treatment of wastewater flows. Most industrial wastewater treatments are carried out in a series of cylindrical-type tanks. In some of them a diffusion-reaction process takes place specially in the trickling filter phase in which wastewater flows downward through a bed of rocks, gravel, slag, peat moss, or plastic media reacting on a layer (or film) of microbial slime covering the bed media. The process (see , e.g., Rodriguez *et al.* 2012; Vicente *et al.* 2011; Rosas *et al.* 2014 and its references) involves adsorption of organic compounds in the wastewater by the microbial slime layer, diffusion of air into the slime layer to provide the oxygen required for the biochemical oxidation of the organic compounds. In this paper, we shall assume that an ideal homogenization process was applied (by passing to the limit $\varepsilon \to 0$ on the porosity of the solid bed) so that the chemical reaction can be assumed as distributed over all the reactor cylinder (see, e.g., Conca *et al.* 2003, 2004 and their references). Simplifying the modeling process we arrive to the consideration of a single, non-reversible catalysis reaction of q-order on a chemical reactor Ω of cylindrical shape

$$\Omega = G \times (0, H),$$

with G an open regular set of $\mathbb{R}^2$ (or more in general $\mathbb{R}^N$) not necessarily symmetric. We point out that, in spite of the abovementioned motivation, our mathematical results can be applied to a larger framework (for instance the own structure of the set Ω can be taken much more in general (see Sect. 3). It is useful to separate the boundary of Ω in its lateral parts $\partial_l \Omega$ and its horizontal parts $\partial_h\Omega$, so that $\partial_l\Omega = \partial G \times (0, H)$ and $\partial_h\Omega$ consists in the union of the top and bottom boundaries: $\partial_h\Omega = (\partial_h\Omega)^H \cup (\partial_h\Omega)_0$ with $(\partial_h\Omega)^H = \Omega \times \{H\}$ and $(\partial_h\Omega)_0 = G \times \{0\}$. We shall use also the notation $\mathbf{x} = (x, y)$ with $x = (x_1, x_2) \in G$ and $y \in (0, H)$. A similar notation can be introduced if $\mathbb{R}^2$ is replaced by $\mathbb{R}^N$ and (0, H) by a set in $\mathbb{R}^m$.

In order to fix ideas we shall consider here the following parabolic model

[1] Dpto. de Matemática Aplicada, Facultad de Ciencias Matemáticas and Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Madrid, Spain. E-mail: ildefonso.diaz@mat.ucm.es; dgomez91@gmail.com

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$$\begin{cases} \frac{\partial w}{\partial t} - \Delta w + \lambda\beta(w) = 0 & \text{in } (0,+\infty)\times\Omega, \\ w = 1 & \text{on } (0,+\infty)\times\partial_l\Omega, \\ \frac{\partial w}{\partial n} = \mu(1-w) & \text{on } (0,+\infty)\times\partial_h\Omega, \\ w(0,\mathbf{x}) = w_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1}$$

where

$$\beta(w) = w^q, \quad 0 < q \le 1$$

(q is called reaction order), $\lambda > 0$,

$$w_0 \in L^\infty(\Omega), \quad 0 \le w_0 \le 1, \tag{2}$$

$\mathbf{n}$ denotes the unit normal exterior vector to $\partial_h\Omega$ and the Robin coefficient μ is taken in a generalized way as $\mu \in [0,+\infty]$. In fact, we assume that the value of μ can be different for the top or the bottom surfaces , i.e.,

$$\mu = \begin{cases} \mu_H & \text{on } (\partial_h\Omega)^H = G\times\{H\}, \\ \mu_0 & \text{on } (\partial_h\Omega)_0 = G\times\{0\}. \end{cases}$$

So, very often $\mu_H = 0$ (which corresponds to the case of an open tank) and/or $\mu_0 = +\infty$ (which must be understood as a Dirichlet type boundary condition $w = 1$ on $(0,+\infty)\times(\partial_h\Omega)_0$ and that corresponds to a tank alimented also from the bottom).

The limit case, the case of 0-order reactions, $q = 0$, can also be considered (see Remark 5) with the help of some special multivalued maximal monotone graph of $\mathbb{R}^2$. We also mention that some larger generality can be considered also concerning the differential operator (see Remark 5).

As mentioned before, as proved in Conca *et al.* (2004), this is the limit as $\varepsilon \to 0$ of the following models

$$\begin{cases} -\Delta w^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ \frac{\partial w_\varepsilon}{\partial \nu} + \mu(\varepsilon)\beta(w^\varepsilon) = 0 & \text{on } S^\varepsilon, \\ w_\varepsilon = 1 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω^ε is a domain with fixed obstacles (which due to the chemical implications we will call pellets), where $\varepsilon > 0$ has to do with the size of each obstacle and S^ε represents the boundary of the pellets and $\partial\Omega$ the boundary of the reactor. This kind of problem is an intuitive model of fixed bed reactors. In this sense problem (1) can be seen as a homogenized problem for fixed bed reactors. This can model a large number of process in Geosciences.

Due to one of those "gifts" of interdisciplinarity, problem (1) also models the heat energy stored by the Earth. The so-called Sellers model considers the averaged surface atmospheric climate proposing the equation:

$$\frac{\partial w}{\partial t} - \Delta w + \beta(u) = QH(u).$$

In the above model, the functions H and β are assumed Lipschitz continuous. There is a different model proposed by Budyko in which the function H is assumed to be discontinuous but we shall not pay attention here to this case (see Díaz 1996). The techniques presented in this paper (mainly the application of the Trotter–Kato formula and its consequences) are extensible to the Sellers model, since the operator is omega-accretive (see Benilan *et al.*, unfinished manuscript).

We shall also consider, as by-product of our results concerning the parabolic problem, the associated stationary problem (formally obtained when making $t \to +\infty$)

$$\begin{cases} -\Delta w + \lambda\beta(w) = 0 & \text{in } \Omega, \\ w = 1 & \text{on } \partial_l\Omega, \\ \frac{\partial w}{\partial n} = \mu(1-w) & \text{on } \partial_h\Omega. \end{cases} \tag{4}$$

The main optimality element in the study of the shape of such chemical reactors is given in terms of a notion introduced in 1957 by R. Aris (see references in Strieder and Aris 1973): the so called *effectiveness* factor which is defined as

$$\eta(t : G, H) := \frac{1}{H|G|}\int_\Omega \beta(w(t,\mathbf{x}))d\mathbf{x}.$$

In a pioneering work, R. Aris presented, in his book (Strieder and Aris 1973), in collaboration with W. Strieder, the study of a linear model ($q = 1$) for a finite number of catalyst particles, which they always consider spherical. Here we will consider cylinders of arbitrary basis and reactions of order less or equal than one, which are much more frequent in practice, but which result in non-linear models requiring delicate mathematical tools. We recall that when $0 < q < 1$ the solutions may give rise to a *dead core*, an interior region where no reaction is taking place. This dead core, which can be defined, for a given $t \ge 0$, as

$$N_w(t) = \{\mathbf{x} \in \Omega : w(t,\mathbf{x}) = 0\}.$$

We shall not give here estimates on the size and location of the dead core regions (see Sect. 4, Remark 4). Obviously, the presence of dead cores affects negatively the global effectiveness, and is to be avoided in the shape optimization process. Intuitively, it represents volume where no catalyst is present, and thus no reaction is taking place.

Although more realistic models may incorporate more complex and sophisticated aspects that the ones here presented, our main goal is to give a conceptual justification of why these reactors are wide and low. In fact, we shall prove here that among all the sections G, with prescribed area, the ball is the set of lowest effectiveness $\eta(t : G, H)$ (Theorem 2.1). Our proof uses the notions of Steiner rearrangement. In contrast to that, we shall also show that if the height of the tank H is small enough then the effectiveness can be made as close to 1 as desired (Theorem 2.2).

The organization of this paper is the following: the above main results are stated in Sect. 2 where some numerical experiences are commented. Section 3 is devoted to the proof of Theorem 2.1. The notion of Steiner rearrangement of a function is introduced and several properties showing the comparison in mass of the Steiner rearrangement of the solution of problem (1) and the solution of the "symmetrized problem" are given. In particular, we show how the so-called Trotter–Kato formula can be applied even under non-autonomous formulation. Finally, Sect. 4 contains the proof of Theorem 2.2 as well as a series of remarks on more general frameworks in which our main results remain valid.

2. *Main Results and Some Numerical Experiences*

Thanks to the maximum principle, it is clear that the solution w of (1) must satisfy that $0 \le w(t, \mathbf{x}) \le 1$ for a.e. $\mathbf{x} \in \Omega$ and for any $t \ge 0$. Then, in which follows, it will be useful to introduce the change of unknown $u = 1 - w$ for which the problem may be rewritten as

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \lambda g(u) = \lambda \beta(1) & \text{in } (0, +\infty) \times \Omega, \\ u = 0 & \text{on } (0, +\infty) \times \partial_l \Omega, \\ -\frac{\partial u}{\partial n} = \mu u & \text{on } (0, +\infty) \times \partial_h \Omega, \\ u(0, x) = u_0(x) & \text{on } \Omega, \end{cases} \tag{5}$$

where

$$g(u) = \beta(1) - \beta(1 - u). \tag{6}$$

Thus, we can assume that g is a continuous increasing function with $g(0) = 0$. We recall that the existence and uniqueness of a weak solution $u \in C([0, +\infty) : L^1(\Omega)) \cap L^\infty((0, +\infty) \times \Omega)$ is today a well-known result. Moreover, it is also known that when $t \to +\infty$ then $u(t, \cdot) \to u_\infty(\cdot)$ in $L^2(\Omega)$ (see e.g., Díaz 1994 and its references).

We shall start by giving a rigorous proof of the well-known principle (from an experimental point of view) that among all cylindrical reactors with prescribed volume the one with a circular section is the least effective:

Theorem 2.1 *For fixed basis volume $|G|$effectiveness is least on an circle. That is, let $A > 0$ and let B the ball centered at the origin and let G be any other n-dimensional open regular set such that $|G| = |B| = A$. Then*

$$\eta(t : B, H) \le \eta(t : G, H).$$

Moreover, the same inequality holds for the associated stationary problems.

Remark 1 In contrast to the case in which *the effectiveness is compared with the one on a ball of* $\mathbb{R}^3$ having the same volume than Ω, the proof of the above theorem for the stationary case seems quite complicated without proving first the analogous result for the associated parabolic problem. That was one of our motivations not to simplify our formulation to the easier case of the stationary problem.

In order to illustrate the conclusion of Theorem 2.1 we produced a numerical experience concerning a particular (one-parametric) family of elliptic cylinders $G_a \times (0, H)$. The elliptic cylinders are assumed with a prescribed volume V. So, given the lower semiaxis a, the greater semiaxis b_a is given by the identity $\pi a b_a = \frac{V}{H}$. In other words, the ellipse family is defined by the parameter a through the expression

$$\begin{aligned} G_a &= \left\{ (x_1, x_2) \in \mathbb{R}^2 : \left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b_a}\right)^2 = 1 \right\}, \\ b_a &= \frac{V}{H\pi a}. \end{aligned} \tag{7}$$

The image below shows a minimum of the effectiveness over this one-parametric family of elliptic cylinders $\Omega_a = E_a \times (0,1)$, in which if we choose $V = \pi H$ and so the value $a = 1$ corresponds to the case of a circular section (Fig. 1).

Our second main result deals with the pure Dirichlet problem ($\mu = +\infty$) and gives a detailed statement of the well-known principle (from an experimental point of view) that among all cylindrical reactors with the prescribed volume low reactors are very effective. We introduce the auxiliary function $\psi \in C^2(\Omega)$ given as the unique solution of

$$\begin{cases} -\Delta\psi = 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

Theorem 2.2 *Assume $\mu = +\infty$. Let $V = |\Omega| = |G|H = AH > 0$ be a fixed volume and let B_H be the ball of $\mathbb{R}^N$ centered at the origin such that $|B_H| = A$. Assume also*

$$1 - w_0(\mathbf{x}) \le \lambda\psi(\mathbf{x}) \quad \text{a.e. } \mathbf{x} \in \Omega.$$

Then

$$\eta(t : G, H) \to 1 \quad \text{as } H \to 0. \tag{9}$$

More precisely, for any $t > 0$ and a.e $\mathbf{x} \in \Omega$

$$1 \ge \beta(w(t,\mathbf{x})) \ge 1 - \left(\frac{V(4 + 2(N+1))(2N+1)^{-\frac{N+1}{2}}}{\pi^2 \omega_{N+1}} H^2 \right)^{2/(N+3)}. \tag{10}$$

The above estimate holds also for the solution of the associated stationary problem (4).

In order to illustrate quantitatively conclusion 2 we produced a numerical experience concerning the family of symmetric cylinder reactors $B_r \times (0, H)$. Motivated by the special case considered in Aris (1975) (see its Figure 4.5.1) when computing curves for this phenomenon for the linear case $q = 1$, we have taken $H = \gamma^{-2}\left(\frac{16}{3}\right)^{\frac{1}{3}}$ and $r = \gamma\left(\frac{2}{3}\right)^{\frac{1}{3}}$ with γ a variable parameter. In the next figure we can see how $H \to 0$ implies $\eta \to 1$ (Fig. 2). We can also see how, in this case, $\eta \to 1$ as $q \to 0$ (this is because, for this volume, no dead core exists even in the worst case scenario).

Remark 2 The numerical experiences were produced by using a semi-implicit iterative algorithm [see Spigler and Vianello 1995 for a proof of the convergence in an abstract framework which includes, as a special case, problem (1) under the conditions assumed in this paper). The chosen scheme applies finite differences in time and finite elements in space. The time discretization for time step h is

$$u_{n+1} - h\Delta u_{n+1} = u_n - hg(u_n).$$

The scheme is chosen implicit in time on the diffusion so that the operator in u_{n+1} is coercive, and thus the sequence is uniquely determined in $H_0^1(\Omega)$. However, the method is explicit in the non-linearity,

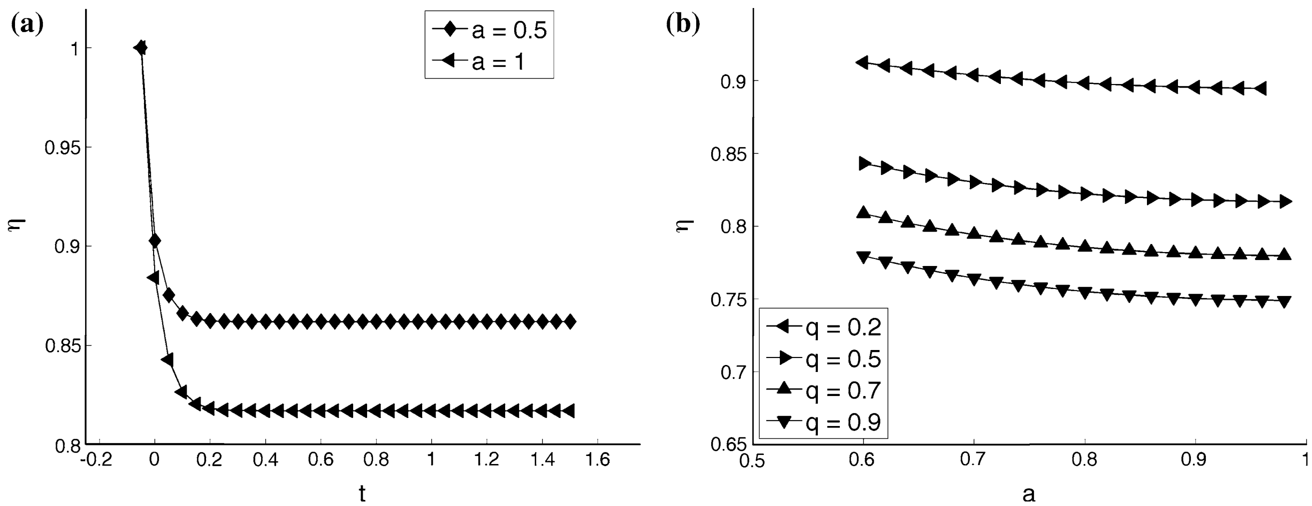

Figure 1
Effectiveness factor for a family of ellipses with the same area. **a** Time evolution of the effectiveness for two cylinders, one circular $a = 1$ and one elliptical $a = 0.5$ both of the same volume, with initial condition $w_0 = 1$ on Ω. **b** Effectiveness for the elliptic problem for different values of G_a and q

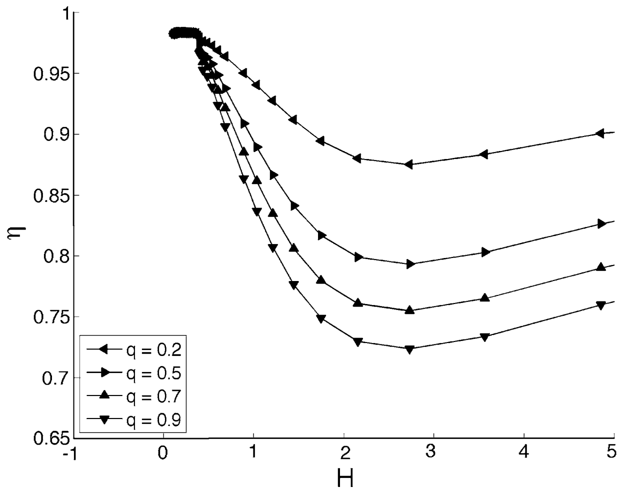

Figure 2
Effectiveness for the elliptic problem on cylinder with varying aspect ratio. Simulation

which makes the problem linear in u_{n+1}, thus allowing for faster simulations. The implementation of the finite element method was performed through the automated library FEniCS, which meshes simple domains in two and three dimensions, constructs the continuous Galerkin finite elements necessary and solves the linear systems.

3. *The Circular Section is the Least Effective: Steiner Symmetrization. Proof of Theorem 2.1*

The proof of Theorem 2.1 will use some inequalities on Steiner symmetrization obtained in Alvino *et al.* (1996). As a matter of fact, we shall improve also a previous result by the authors (Díaz and Gómez-Castro 2014a) corresponding, essentially, to the case $q \geq 1$. It turns out that our result remains true under a more general setting by replacing the *vertical space* $\mathbb{R}$ by $\mathbb{R}^m$. We start by recalling that given a general measurable function $h : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}$, with $N, m \geq 1$, for a fixed $y \in \mathbb{R}^m$ we can define the Steiner distribution function $\mu_h : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ by means of

$$\mu_h(t, y) = |\{x \in \mathbb{R}^N : |h(x, y)| > t\}|.$$

The Hardy–Littlewood–Polya decreasing rearrangement $h^* : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}$ is given as

$$\begin{aligned} h^*(s, y) &= \sup\{t > 0 : \mu_h(t, y) > s\} \\ &= \inf\{t > 0 : \mu_h(t, y) \leq s\}. \end{aligned}$$

It is well known that if ω represents a generic measurable subset of $\mathbb{R}^N \times \mathbb{R}^m$ then

$$\int_0^s h^*(\sigma, y)\, \mathrm{d}\sigma = \sup_{|\omega| = s} \int_\omega h(\mathbf{x}, y)\, \mathrm{d}\mathbf{x}. \tag{11}$$

Finally, for $y \in \mathbb{R}^m$ prescribed, we define the Steiner symmetrization of h with respect to $\mathbf{x}$ as

$$h^\#(\mathbf{x}, y) = h^*(\omega_N |\mathbf{x}|^N, y),$$

where ω_N is the measure of the N-dimensional ball. The basic idea underlying Steiner symmetrization is to consider the integral of the function over *slices*. Given $s > 0$ and $y \in \mathbb{R}^m$ we take very particular *slices* of the form

$$G(y) = \{\mathbf{x} \in \mathbb{R}^N : u(\mathbf{x}, y) > u^*(s, y)\},$$

where $|G(y)| = s$ (by construction of u^*). Variable s should formally be included in the definition but this will not lead to confusion.

Explicit calculations can be performed in simple cases. The following figure provides and example of the exact distribution function and Steiner rearrangement for the function

$$u(x, y) = \begin{cases} 0, & |(x, y)| > 1, \\ 2(1 - x^2 - y^2), & \frac{1}{2} \leq |(x, y)| \leq 1, \\ 1, & |(x, y)| > 1. \end{cases}$$

In this section we shall use a more general framework. We introduce the following notations (Fig. 3):

$$\Omega = \Omega' \times \Omega''$$

and $(x, y) \in \Omega' \times \Omega''$ for an arbitrary point (note that in our initial framework $\Omega' = G$ and $\Omega'' = (0, H)$). We shall denote by B a ball such that $|B| = |\Omega'|$ and then we introduce

$$\Omega^\# = B \times \Omega''.$$

Remark 3 In the case where we rearrange with respect to all variable, i.e., no y is presented (and, in an abuse of notation $m = 0$) the symmetrization is know as Schwarz symmetrization. Since it will be

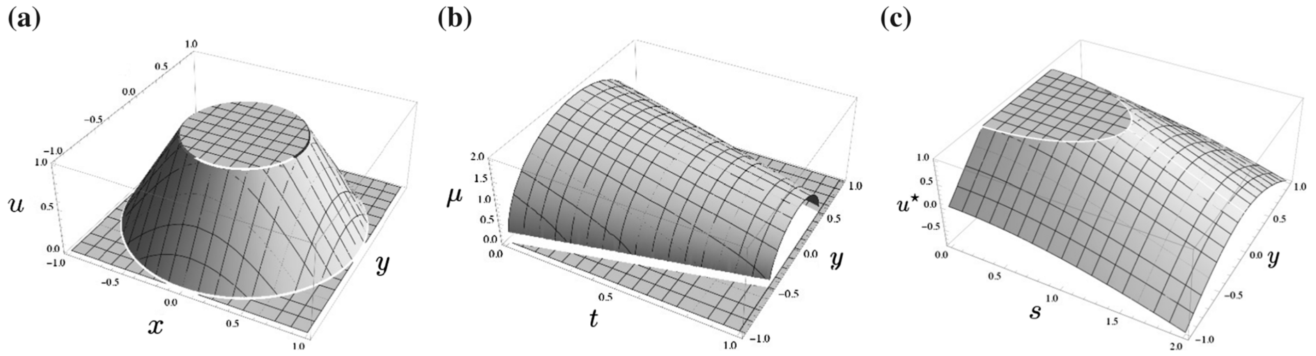

Figure 3
Computation of Steiner symmetrization. **a** Function u. **b** Distribution function μ. **c** Steiner rearrangement u^*

useful to use both symmetrizations, for Schwarz rearrangement we will use the notation $\tilde{u}$. We also introduce the truncation at level $y \in \Omega''$ as

$$u_y(x) = u(x,y), \quad (x,y) \in \Omega' \times \Omega''.$$

Is clear from the definition that,

$$\widetilde{u_y}(s) = u^*(s,y).$$

For the case where time is introduced, even though the application is written $u(t, x, y)$ we will never rearrange with respect to t.

The image below (Fig. 4) shows an artistic comparison between Steiner and Schwarz symmetrizations for the function

$$u(x,y,z) = e^{-10x^2-5y^2-10z^2}(1-x)x(1-y)y(1-z)z, \quad (x,y,z) \in [0,1]^3.$$

This function has a single maximum point, and we show cross cuts of symmetrizations.

Our main result leading to the conclusion of Theorem 2.1 is the following:

Theorem 3.1 *Let β be a concave continuous non-decreasing function such that $\beta(0) = 0$. Give $T > 0$arbitrary and let $f \in L^2(0, T : L^2(\Omega))$with $f \geq 0$ in (0, T) and let $w_0 \in L^2(\Omega)$ be such that $0 \leq w_0 \leq 1$. Let $\delta > 0$ be fixed and $w \in C([0,T] : L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$and $z \in C([0,T] : L^2(\Omega^\#)) \cap L^2(\delta, T : H_0^1(\Omega^\#))$ be the unique solutions of*

$$(P)\begin{cases} \frac{\partial w}{\partial t} - \Delta w + \lambda\beta(w) = f(t) & \text{in } \Omega \times (0,T), \\ w = 1 & \text{on } \partial\Omega \times (0,T), \\ w(0) = w_0 & \text{on } \Omega, \end{cases}$$

$$(P^\#)\begin{cases} \frac{\partial z}{\partial t} - \Delta z + \lambda\beta(z) = f^\#(t), & \text{in } \Omega^\# \times (0,T), \\ z = 1, & \text{on } \partial\Omega^\# \times (0,T), \\ z(0) = z_0, & \text{on } \Omega^\#, \end{cases}$$

where $z_0 \in L^2(\Omega^\#)$, $0 \leq z_0 \leq 1$ is such that

$$\int_s^{|\Omega'|} z_0^*(\sigma,y)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} w_0^*(\sigma,y)\mathrm{d}\sigma, \quad \forall s \in [0, |\Omega'|] \text{ and a.e. } y \in \Omega''.$$

Then, for any $t \in [0,T]$, $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$

$$\int_s^{|\Omega'|} z^*(t,\sigma,y)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} \mathrm{w}^*(\mathrm{t},\sigma,\mathrm{y})\mathrm{d}\sigma. \tag{12}$$

In terms of the comparison of the effectiveness we have the following consequence (which will be proved in Sect. 3) leading to the proof of Theorem 2.1:

Corollary 3.2 *In the assumptions of Theorem 3.1, for any $t \in [0, +\infty)$ we have*

$$\int_{\Omega^\#} \beta(z(t,\mathbf{x}))\mathrm{d}\mathbf{x} \leq \int_\Omega \beta(\mathrm{w}(\mathrm{t},\mathbf{x}))\mathrm{d}\mathbf{x}. \tag{13}$$

The interest on the above two results is that the conclusions remains true for the associated stationary problems.

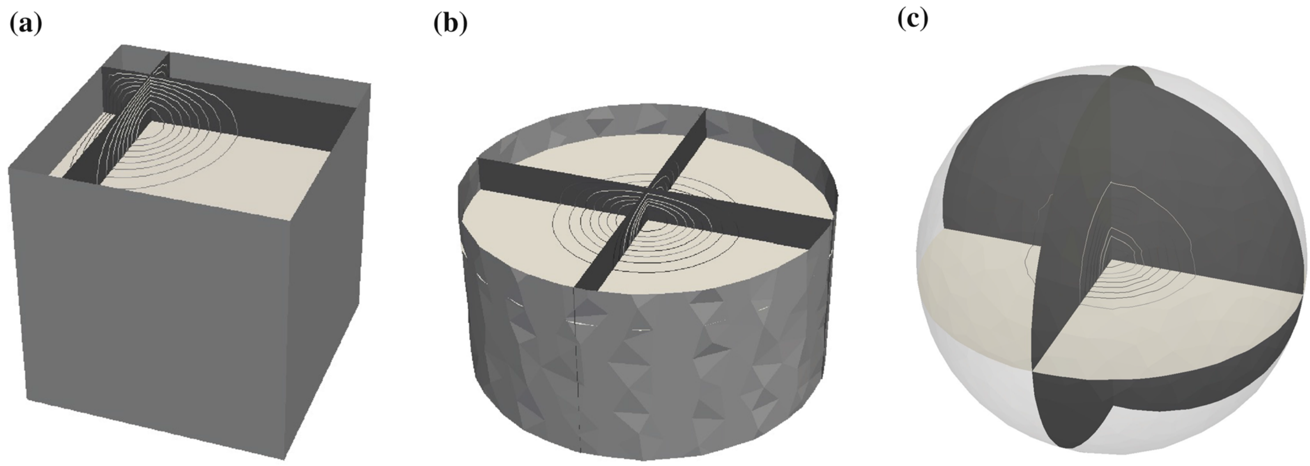

Figure 4
Comparison of Steiner and Schwarz rearrangements of a given function. **a** A given measurable function on $\Omega = [0, 1]^3$, which we choose constant on the boundary. **b** Steiner symmetrization with respect to (x, y). **c** Schwarz symmetrization

Corollary 3.3 *The mass and effectiveness comparison given by* (12) and (13), *respectively*, *remain valid for the solutions of the corresponding stationary problems*.

As mentioned before, Theorem 3.1 extends previous result by the authors (Díaz and Gómez-Castro 2014a). For the proof of this result we apply, essentially, the same techniques as in the cited article, but with some refinements concerning the nature of the non-linear term $\beta(w)$ (i.e., $g(u)$ in the equivalent formulation (5)). In contrast to our work (Díaz and Gómez-Castro 2014a) we shall work with the increasing rearrangement. We start by recalling the following simple property: if $f : [0, |\Omega'|] \to \mathbb{R}$ is a real function such that $0 \le f \le L$ then $(L-f)^*(s) = L - f^*(|\Omega'| - s)$ and in particular

$$\int_0^s (L - f(t))^* \mathrm{d}t = L - \int_{|\Omega'|-s}^{|\Omega'|} f^*(t) \mathrm{d}t$$

(the proof can be found, for instance, in Mossino 1984).

As in Díaz and Gómez-Castro (2014a), we shall prove the above theorem by means of the Trotter–Kato formula. So we shall need to consider previously two auxiliary problems. The first problem corresponds to the associated linear diffusion problem:

Proposition 3.4 *Let* $0 \le w_0, z_0 \le 1$

$$(A)\begin{cases} \frac{\partial w}{\partial t} - \Delta w = 0, & (0,T) \times \Omega \\ w = 1, & (0,T) \times \partial\Omega \\ w = w_0, & \{0\} \times \Omega \end{cases}$$

$$(A^\#)\begin{cases} \frac{\partial z}{\partial t} - \Delta z = 0, & (0,T) \times \Omega^\# \\ z = 1, & (0,T) \times \partial\Omega^\# \\ z = z_0, & \{0\} \times \Omega^\# \end{cases}$$

and

$$\int_s^{|\Omega'|} z_0^*(\sigma, y) \, \mathrm{d}\sigma \le \int_s^{|\Omega'|} \mathrm{w}_0^*(\sigma) \, \mathrm{d}\sigma, \quad \mathrm{s} \in [0, |\Omega|].$$

Then

$$\int_s^{|\Omega'|} z^*(t, \sigma, y) \, \mathrm{d}\sigma \le \int_s^{|\Omega'|} \mathrm{w}^*(\mathrm{t}, \sigma, \mathrm{y}) \, \mathrm{d}\sigma,$$
$$s \in [0, |\Omega|].$$

Proof Let us consider $u = 1 - w$ and $v = 1 - z$. Then u and v are solutions of the problems

$$(B)\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & (0,T) \times \Omega \\ u = 0, & (0,T) \times \partial\Omega \\ u = u_0, & \{0\} \times \Omega \end{cases}$$

$$(B^\#)\begin{cases} \frac{\partial v}{\partial t} - \Delta v = 0, & (0,T) \times \Omega^\# \\ v = 0, & (0,T) \times \partial\Omega^\# \\ v = v_0, & \{0\} \times \Omega^\# \end{cases}$$

where now $u_0, v_0 \ge 0$ are given as $u_0 = 1 - w_0$ and $v_0 = 1 - z_0$. Since, for any $\tau \in [0, |\Omega'|]$, we have that

$$\int_0^\tau u_0^*(\sigma)\ \mathrm{d}\sigma = L - \int_{|\Omega'|-\tau}^{|\Omega'|} w_0^*(\sigma)\ \mathrm{d}\sigma$$
$$\leq L - \int_{|\Omega-\tau}^{|\Omega'|} z_0^*(\sigma) z_0 \leq \int_0^\tau v_0^*(\sigma)\ \mathrm{d}\sigma$$

then

$$\int_0^\tau u_0^*(\sigma, y)\ \mathrm{d}\sigma \leq \int_0^\tau v_0^*(\sigma, y)\ \mathrm{d}\sigma.$$

Under this conditions, it is proven in Chiacchio (2004) that, for any $t \geq 0$ and for any $\tau \in [0, |\Omega'|]$, we have the comparison

$$\int_0^\tau u^*(t,\sigma,y)\mathrm{d}\sigma \leq \int_0^\tau v^*(t,\sigma,y)\mathrm{d}\ \sigma. \tag{14}$$

The key idea the proof of the result in Chiacchio (2004) is to integrate each term of the equation of problem (B) over the sets $\Omega_y(s) = \{x \in \mathbb{R}^N : (x,y) \in \Omega\, \text{and}\, u(t,x,y) > u^*(t,s,y)\}$ for each $t > 0$ and to use the differentiation formula

$$\left(\frac{\partial^2 F}{\partial y_i \partial y_j}\right)_{i,j} \geq \int_{\Omega_y(s)} \left(\frac{\partial^2 u}{\partial y_i \partial y_j}\right)_{i,j}, \tag{15}$$

where

$$F(t,s,y) = \int_0^s u^*(t,\sigma,y)\mathrm{d}\sigma.$$

Inequality (15) was proved for the first time in the literature in the paper (Alvino *et al.* 1996) (see also an alternative proof in Ferone and Mercaldo 1998). In Chiacchio (2004) we find the application of this formula to the parabolic problem (with the additional proof of the comparison with respect the formula obtained for the case of radially symmetric sections).

Applying (14), finally we arrive to the conclusion since

$$\int_{|\Omega'|-\tau}^{|\Omega'|} z^* = L - \int_0^\tau v^* \leq L - \int_0^\tau u^* = \int_{|\Omega'|-\tau}^{|\Omega'|} w^*.$$

which concludes the proof. □

The second auxiliary problem corresponds to a distributed non-linear ordinary differential equation.

Proposition 3.5 *Let β be a concave continuous non-decreasing function such that $\beta(0) = 0$. Let u, v satisfy*

$$(B)\ \begin{cases} w_t + \lambda\beta(w) = 0, & \Omega \times (0,T), \\ w = w_0, & \Omega \times \{0\}, \end{cases}$$
$$(B^\#)\ \begin{cases} z_t + \lambda\beta(z) = 0, & \Omega^\# \times (0,T), \\ z = z_0, & \Omega^\# \times \{0\}. \end{cases}$$

Assume

$$\int_s^{|\Omega'|} z_0^*(\sigma,y)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} w_0^*(\sigma,y)\mathrm{d}\sigma,$$
$$\forall s \in [0,|\Omega'|], \quad \text{a.e.} y \in \Omega''.$$

Then we have

$$\int_s^{|\Omega'|} z^*(t,\sigma,y)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} \mathrm{w}^*(\mathrm{t},\sigma,\mathrm{y})\mathrm{d}\sigma$$
$$\forall t > 0, s \in [0,|\Omega'|],\ \text{a.e. } y \in \Omega''.$$

Proof For any $\varepsilon > 0$ and $y \in \Omega''$ prescribed, let $w_{\varepsilon,y}(t,x)$, $z_{\varepsilon,y}(t,x)$ be the solutions of the (ε,y)-parametric family of semilinear parabolic problems

$$(P(\varepsilon,y))\begin{cases} \frac{\partial w}{\partial t} - \varepsilon\Delta_x w + \lambda\beta(w) = f_y(t) & \text{in } \Omega' \times (0,T), \\ w = 1 & \text{on } \partial\Omega' \times (0,T), \\ w(0) = (w_0)_y & \text{on } \Omega', \end{cases}$$
$$(P^\#(\varepsilon,y))\begin{cases} \frac{\partial z}{\partial t} - \varepsilon\Delta z + \lambda\beta(z) = f_y^\#(t) & \text{in } B \times (0,T), \\ z = 1 & \text{on } \partial B \times (0,T), \\ z(0) = (z_0)_y & \text{on } B. \end{cases}$$

Notice that the diffusion operator is only dependent of the x-variables. Then, by Theorem 1 of Díaz (1991) we know that, for any $\varepsilon > 0$ and $y \in \Omega''$ prescribed,

$$\int_s^{|\Omega'|} \widetilde{z_{\varepsilon,y}}(t,\sigma)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} \widetilde{w_{\varepsilon,y}}(t,\sigma)\mathrm{d}\sigma \qquad \forall t > 0, s \in [0,|\Omega'|]. \tag{16}$$

Moreover, we can know apply Theorem 3.16 on Brezis (1973)

$$z_{\varepsilon,y} \to z_y \quad \text{as } \varepsilon \to 0 \quad \text{in } C([0,T] : L^2(B)),$$
$$w_{\varepsilon,y} \to w_y \quad \text{as } \varepsilon \to 0 \quad \text{in } \mathrm{C}([0,\mathrm{T}] : \mathrm{L}^2(\mathrm{G})).$$

Then, passing to the limit in (16) we get

$$\int_s^{|\Omega'|} \widetilde{z_y}(t,\sigma)\mathrm{d}\sigma \leq \int_s^{|\Omega'|} \widetilde{w_y}(t,\sigma)\mathrm{d}\sigma$$
$$\forall t > 0, s \in [0,|\Omega'|].$$

Finally, it is enough to observe that since $y \in \Omega''$ is prescribed then the Schwarz rearrangement $\widetilde{w_y}(t, \sigma)$ coincides with the Steiner rearrangement $w^*(t, \sigma, y)$ (see Remark 3) and the result holds. □

3.1. Proof of Theorem 3.1.

Proof of Theorem 3.1 The special case $f = 0$ is easier. Since we know

$$\int_{\tau}^{|\Omega'|} z_0^*(\sigma, y)\ \mathrm{d}\sigma \leq \int_{\tau}^{|\Omega'|} w_0^*(\sigma, y)\ \mathrm{d}\sigma, \quad \forall s, \forall y$$

applying Propositions 3.4 and 3.5 inductively we get

$$\int_{\tau}^{|\Omega'|} \left[\left(S_A\left(\frac{t}{n}\right)S_B\left(\frac{t}{n}\right)\right)^n z_0\right]^*(\sigma, y)\ \mathrm{d}\sigma \leq \int_{\tau}^{|\Omega'|} \left[\left(S_{A^\#}\left(\frac{t}{n}\right)S_{B^\#}\left(\frac{t}{n}\right)\right)^n w_0\right]^*(\sigma, y)\ \mathrm{d}\sigma$$

where S_A is the semigroup associated to problem (A) and analogously for S_B, S_P, $S_{A^\#}$, $S_{B^\#}$, and $S_{P^\#}$.Taking limits, applying the Trotter–Kato formula (see Proposition 4.3 Brezis 1973) and applying convergence under the integral sign we get

$$\int_0^s [S_P(t)z_0]^*(\sigma, y)\mathrm{d}\sigma \leq \int_0^s [S_{P^\#}(t)w_0]^*(\sigma, y)\mathrm{d}\sigma$$

for any $t \in [0, T]$, for any $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$.

For the case $f \neq 0$ and time dependent, the Trotter–Kato formula can be also applied (see, e.g., Vuillermot *et al.* 2008). In fact, to deal with the affine case $f(t) \neq 0$ we shall use a "reduction of order technique" argument which can be found on Benilan *et al.* (unfinished manuscript). We point out that by an approximation argument and then passing to the limit process we can assume, without loss of generality, that in fact $f \in H^1(0, T; L^2(\Omega))$. We shall argue by using the formulation of the problem with homogeneous Dirichlet condition, that is $u = 1 - w$ as unknown, for the case of the general set Ω and with $v = 1 - z$ as unknown for the ball $\Omega^\#$. We also introduce the following notations:

$$\hat{f}(t) = \lambda\beta(1) - f(t).$$

and given any function $\theta \in H^1(0, T; L^2(\Omega))$, for a.e. $t \in (0, T)$ we define the function $\theta(t + \cdot) \in H^1(0, T; L^2(\Omega))$ by the application $s \mapsto \theta(t + s)$. We also introduce the vectorial function $U(t) = (u(t), f(t + \cdot)) \in L^2(\Omega) \times H^1(0, T; L^2(\Omega))$. We proceed in a similar way for the case of the domain $\Omega^\#$: we define $V(t) = (v(t), f^\#(t + \cdot)) \in L^2(\Omega^\#) \times H^1(0, T; L^2(\Omega^\#))$. Then, it is easy to see that U, V are the respective unique solutions of the "autonomous vectorial problems"

$$\begin{cases} \frac{\partial U}{\partial t} + \hat{L}U = 0, & t \in (0, T) \\ U(0) = (u_0, \hat{f}) \end{cases}$$
$$\begin{cases} \frac{\partial V}{\partial t} + \hat{L}V = 0, & t \in (0, T) \\ V(0) = (v_0, \hat{f}^\#) \end{cases}$$

where

$$\hat{L}(u, \xi) = (-\Delta u + g(u) - \xi(0 + \cdot), \xi').$$

Here ξ' represents simply the derivative of ξ. We can use a decomposition $\hat{L} = \hat{L}_1 + \hat{L}_2$ in the following way:

$$\hat{L}_1(u, \xi) = (-\Delta u + h(t)g(u), 0),$$
$$\hat{L}_2(u, \xi) = (-\xi(0 + \cdot), \xi').$$

Let us define the problems

$$(C)\begin{cases} \frac{\partial U}{\partial t} + \hat{L}_1 U = 0, \\ U(0) = (u_0, \hat{f}), \end{cases} \quad (C^\#)\begin{cases} \frac{\partial V}{\partial t} + \hat{L}_1 V = 0, \\ V(0) = (v_0, \hat{f}^\#), \end{cases}$$
$$(D)\begin{cases} \frac{\partial U}{\partial t} + \hat{L}_2 U = 0, \\ U(0) = (u_0, \hat{f}), \end{cases} \quad (D^\#)\begin{cases} \frac{\partial V}{\partial t} + \hat{L}_2 V = 0, \\ V(0) = (v_0, \hat{f}^\#), \end{cases}$$

and the correspondent solution operators

$$S_C(t)(u_0, \hat{f}) = (S_P(t)u_0, \hat{f}),$$
$$S_{C^\#}(t)(v_0, \hat{f}^\#) = (S_P(t)u_0, \hat{f}^\#),$$

$$S_D(t)(u_0, \hat{f}) = \left(u_0 + \int_0^t \hat{f}(s)ds, \hat{f}\right),$$
$$S_{D^\#}(t)(v_0, f^\#) = \left(v_0 + \int_0^t \hat{f}^\#(s)ds, \hat{f}^\#\right).$$

Let Q be the projection operator such that $u(t) = QU(t)$. Let us study QS_C and QS_D. Since, for any $t \in [0, T]$, for any $s \in [0, |\Omega'|]$ and a.e. $y \in \Omega''$,

$$\int_0^s u_0^*(\sigma, y)\ \mathrm{d}\sigma \leq \int_0^s v_0^*(\sigma, y)\ \mathrm{d}\sigma,$$

we have, by the above explicit formulas (for the first component we apply the similar proof as in the case $f = 0$)

$$\int_0^s [Q\ S_C(t)(u_0,f)]^*(\sigma,y)\mathrm{d}\sigma$$
$$\leq \int_0^s [Q\ S_{C^\#}(t)(v_0,f^\#)]^*(\sigma,y)\mathrm{d}\sigma,$$
$$\int_0^s [Q\ S_D(t)(u_0,f)]^*(\sigma,y)\mathrm{d}\sigma$$
$$\leq \int_0^s [Q\ S_{D^\#}(t)(v_0,f^\#)]^*(\sigma,y)\mathrm{d}\sigma.$$

By applying an induction argument again we get

$$\int_0^s \left[Q\left(S_C\left(\frac{t}{n}\right)S_D\left(\frac{t}{n}\right)\right)^n(u_0,f)\right]^*(\sigma,y)\mathrm{d}\sigma$$
$$\leq \int_0^s \left[Q\left(S_{C^\#}\left(\frac{t}{n}\right)S_{D^\#}\left(\frac{t}{n}\right)\right)^n(v_0,f^\#)\right]^*(\sigma,y)\mathrm{d}\sigma.$$

Finally, since all the operators are maximal monotone operators on their respective Hilbert spaces, we can take limits by applying the Trotter–Kato formula (which justify the convergence of the limits) and the result holds. □

3.2. *Proof of Corollary 3.2: End of the Proof of Theorem 2.1*

For the proof we shall need a classical result.

Lemma 3.6 (Hardy *et al.* 1929) *Let* $\tilde{y},\tilde{z}\in L^1(0,M)$, $\tilde{y},\tilde{z}\geq 0$ *a.e.. Suppose y is non-increasing and*

$$\int_0^s \tilde{y}(\sigma)\ \mathrm{d}\sigma \leq \int_0^s \tilde{z}(\sigma)\ \mathrm{d}\sigma, \quad \forall s\in[0,M].$$

Then, for every continuous non-decreasing convex function Φ *we have*

$$\int_0^s \Phi(\tilde{y}(\sigma))\ \mathrm{d}\sigma \leq \int_0^s \Phi(\tilde{z}(\sigma))\ \mathrm{d}\sigma \quad \forall s\in[0,M].$$

Proof of Corollary 3.2 Applying the theorem and Lemma 3.6 to

$$\Phi(s) = \beta(1)-\beta(1-s),$$
$$\tilde{y}(\sigma) = 1 - z^*(|\Omega'|-\sigma,y),$$
$$\tilde{z}(\sigma) = 1 - w^*(|\Omega'|-\sigma,y)$$

we get that

$$\int_s^{|\Omega'|} \beta(z^*(t,\sigma,y))\mathrm{d}\sigma \leq \int_s^{|\Omega'|} \beta(w^*(t,\sigma,y))\mathrm{d}\sigma.$$

It is a classical result (see Mossino 1984) that for F Borel and u measurable it holds that

$$\int_{\Omega'} F(u) = \int_0^{|\Omega'|} F(u^*)$$

In particular, the comparison holds between w and z. All that remains is to integrate on Ω'', apply Fubini's theorem and the result follows. □

3.3. *The Elliptic Case*

Proof of Corollary 3.3 Since there is uniqueness of solutions for the stationary problem (4) then, by applying Corollary 3 of Díaz (1994) we get that $w(t)\to w$ in $H^1(\Omega)$, as $t\to+\infty$ (with w the unique solution of problem (4) with $\mu=+\infty$, i.e., the Dirichlet problem $w=1$ on $\partial\Omega$). Moreover, since the application $u\mapsto u^*$ is continuous with respect to the convergence in L^1 (see e.g., Mossino 1984) we get that the mass comparison is stable by passing to the limit as $t\to+\infty$ and the result holds.

4. *Proof of Theorem 2.2 and Further Remarks*

We shall use the function $\overline{u}=\lambda\psi$ is a supersolution (we recall that ψ is given by (8)). We shall apply the following previous result in the literature due to Bandle (1985):

Theorem 4.1 *Let* $\Omega\subset\mathbb{R}^n$ *be an open bounded set of measure* $V=|\Omega|$ *such that* Ω *is contained between two parallel* $(n-1)$*-dimensional hyperplanes at distance* 2ρ *and let* ψ *be the solution of problem* (8). *Then*

$$\|\psi\|_\infty^{1+\frac{n}{2}}\leq CV\rho^2$$

with

$$C=\frac{(4+2n)(2n)^{-\frac{n}{2}}}{\pi^2\omega_n}. \tag{17}$$

Proof of Theorem 2.2 Thanks to the assumption on the initial datum, since we are dealing with the Dirichlet problem $[\mu=+\infty$ in (5)] and $0\leq u=1-w\leq 1$, $0\leq g(u)\leq 1$, we get that $\overline{u}=\lambda\psi$ is a supersolution of problem (5). Then, applying

Theorem 4.1 to $\Omega = G \times (0, H)$, i.e., with $n = N + 1$ and $2\rho = H$, we get that

$$\|u\|_{L^\infty(0,T;L^\infty(\Omega))} \to 0, \quad \text{as H} \to 0,$$

and, in particular

$$\operatorname*{essinf}_{(0,T)\times\Omega} \beta(w) \to 1, \quad \text{as } H \to 0.$$

More precisely, for any $t \geq 0$ and a.e $\mathbf{x} \in \Omega$

$$1 \geq \beta(w(\mathbf{x}, t)) \geq 1 - \left(\frac{V(2N+6)(2N+1)^{-\frac{N+1}{2}}}{\pi^2 \omega_{N+1}} H^2 \right)^{2/(N+3)}, \tag{18}$$

which proves the assertion for the case of the parabolic problem (even if $V = |\Omega| = |G|H$ is prescribed). In the case of the associated stationary problem, since we know that $w(t) \to w$ in $H^1(\Omega)$, as $t \to +\infty$ (see the proof of Corollary 3.3) then, by the dominated Lebesgue theorem we know that $\beta(w(t)) \to \beta(w)$ in $L^\infty(\Omega)$, as $t \to +\infty$ and thus the estimate (18) remains valid replacing $\beta(w(t))$ by $\beta(w)$ (since the bounds are independent of t). □

Remark 4 We shall not enter in this paper in the study of the free boundary (the boundary of the dead core) associated to the solutions $w(t)$ and w of the parabolic and elliptic problems (1) and (4), respectively. We recall that the key assumption for the formation of such free boundary is the condition $0 < q < 1$. We send the reader to the monographs Díaz (1985) and Antontsev *et al.* (2001) for an extensive treatment with numerous references.

Remark 5 All the results of this paper can be extended to more general frameworks according different point of views. For instance, with respect to the diffusion operator it is possible to replace the Laplacian operator $-\Delta w$ by a general second-order elliptic operator of the type

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x,y)\frac{\partial u}{\partial x_j}\right) - \sum_{h,k=1}^{m} \frac{\partial}{\partial y_k}\left(b_{hk}(y)\frac{\partial u}{\partial y_h}\right) - \sum_{i=1}^{N}\sum_{h=1}^{m} \frac{\partial}{\partial y_h}\left(c_{ih}(y)\frac{\partial u}{\partial x_i}\right) - \sum_{i=1}^{N}\sum_{h=1}^{m} \frac{\partial}{\partial x_i}\left(d_{hi}(y)\frac{\partial u}{\partial y_h}\right) \tag{19}$$

with bounded coefficients (here we followed the notation of Sect. 3). In that case, the comparison via Steiner symmetrization is made with respect the solution (on a cylinder of symmetric section) associated to the operator

$$L^\# v = -\Delta_x v - \sum_{h,k=1}^{m} \frac{\partial}{\partial y_k}\left(b_{hk}(y)\frac{\partial v}{\partial y_h}\right).$$

No special change in the statements arises if the operator Lu involves transport first-order terms of the type

$$\sum_{k=1}^{m} b_k(y)\frac{\partial u}{\partial y_k}.$$

Quasilinear terms with respect to the x-variable (that is, those in which Steiner symmetrization is performed) can be allowed, too. The presence of transport terms in the x-variable can also be considered, but then the expression of the rearranged operator $L^\# v$ must be modified (see,e.g., Chiacchio and Monetti 2001 and its references). On the other hand, it is still an open problem how to deal with quasilinear terms in the y-variables. We point out that Theorem 4.1 (which play a fundamental role in the proof of Theorem 2.2) was obtained in Bandle (1985) for the case of a general second-order elliptic operator of the type (19). Concerning the reaction term $\beta(w) = w^q$, the results of this paper can be extended also to the case $q = 0$ by means of the consideration of the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(w) = 0 \text{ if } w < 0,\ \beta(w) = 1 \text{ if } w > 0 \quad \text{and} \quad \beta(0) = [0,1]. \tag{20}$$

(see, e.g., Díaz 1985, Chapter 2). As a matter of fact, the proof of Proposition 3.5 (an thus Theorem 3.1) remains valid under the same assumptions on β that Theorem 1 on Díaz (1991), i.e., β non-decreasing function with $\beta(0) = 0$ and such that

$$\beta = \beta_1 + \beta_2 \tag{21}$$

where β_1 is concave and β_2 is convex. The results can be extended also to the "enthalpy formulation" of some porous media type equations (associated to a linear operator Lu) in the spirit of the framework presented in Díaz (1991, 1992, 2001). It is also possible to extend the results to the more realistic case of suitable coupled systems of the type

$$\begin{cases} \frac{\partial w}{\partial t} - d_w \Delta w + R_1(w,u) = 0 & \text{in } (0,+\infty)\times\Omega, \\ \frac{\partial u}{\partial t} - d_u \Delta u + R_2(w,u) = 0 & \text{in } (0,+\infty)\times\Omega, \end{cases}$$

under suitable structural assumptions on the coupling reaction terms $R_1(w,u)$ and $R_2(w,u)$ (see Theorem 3 of Díaz 1991 for $d_w, d_u > 0$ and Díaz and Stakgold 1994 for $d_w > 0$ and $d_u = 0$). Some results on the Steiner rearrangement for the case of Neumann boundary conditions can be found in Ferone and Mercaldo (2005) and Chiacchio (2004).

Remark 6 It can be shown (see Bandle and Vernier-Piro 2003) that, in spite of Theorem 2.2, domains Ω of optimal effectiveness do not exist for reactions $\beta(w) = w^q$ with $0<q<1$. Nevertheless, for the limit case of zero order reactions [with $\beta(w)$ given by (20)] any result proving that there is no dead core for a concrete Ω shows that the effectiveness attains its maximum value for this domain Ω (several criteria for the non-formation of the dead core were given in Chapter 2 of Díaz 1985).

Remark 7 The study of the optimality of the effectiveness factor in terms of shape differentiation on Ω is the main object of the paper (Díaz and Gómez-Castro 2014b).

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STEINER SYMMETRIZATION FOR CONCAVE SEMILINEAR ELLIPTIC AND PARABOLIC EQUATIONS AND THE OBSTACLE PROBLEM

J.I. Díaz and D. Gómez-Castro

Instituto de Matemática Interdisciplinar and Dpto. de Matemática Aplicada
Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
Plaza de las Ciencias, 3, 28040 Madrid, Spain

Abstract. We extend some previous results in the literature on the Steiner rearrangement of linear second order elliptic equations to the semilinear concave parabolic problems and the obstacle problem.

1. **Introduction.** In this paper we extend some previous results in the literature on the Steiner rearrangement of second order semilinear parabolic problems of the type

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + h(t)g(u) = f, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $h \in W^{1,\infty}(0,T)$ is such that $h(t) \geq 0$ for all $t \in (0,T)$ and g is a concave continuous nondecreasing function such that $g(0) = 0$ satisfying that

$$\int_0^\tau \frac{d\sigma}{g(\sigma)} < \infty, \qquad \forall \tau > 0. \tag{H}$$

We recall that the existence and uniqueness of a weak solution $u \in C([0,T] : L^2(\Omega)) \cap L^2(\delta, T : H_0^1(\Omega))$ for any $\delta \in (0,T)$ can be obtained, for instance, by the application of the theory of maximal monotone operators in $L^2(\Omega)$ (see [6], [2] and [4]).

Let us start by recalling that given a general measurable function $v : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, with $n, m \geq 1$ and $n + m = N$, for a fixed $y \in \mathbb{R}^m$ we can define the function $\mu_v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by means of

$$\mu_v(t,y) = |\{x \in \mathbb{R}^n : |v(x,y)| > t\}|.$$

The Hardy-Littlewood-Polya decreasing rearrangement $v^* : [0,+\infty) \times \mathbb{R}^m \to \mathbb{R}$ is given as

$$v^*(s,y) = \sup\{t > 0 : \mu_v(t,y) > s\} = \inf\{t > 0 : \mu_v(t,y) \leq s\}.$$

It can be shown that, if ω represents a generic measurable subset of $\mathbb{R}^n \times \mathbb{R}^m$

$$\int_0^s v^*(\sigma, y)\, \mathrm{d}\sigma = \sup_{|\omega|=s} \int_\omega v(x,y)\, \mathrm{d}x, \quad \text{a.e. } y \in \mathbb{R}^m. \tag{1}$$

Finally we define the Steiner symmetrization of v with respect to x as

$$v^\#(x,y) = v^*(\omega_n |x|^n, y), \quad \text{a.e. } y \in \mathbb{R}^m,$$

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where ω_n is the measure of the n-dimensional ball (see details, for instance, in [10], [11]).

The basic idea underlying Steiner symmetrization is to consider the integral of the function over *slices*. We take very particular *slices* of the form

$$G(y) = \{x \in \mathbb{R}^m : u(x,y) > u^*(s,y)\}$$

where $|G(y)| = s$ (by construction of u^*). Variable s should formally be included in the definition but this will not lead to confusion.

We shall use the following notations:

$$\Omega = \Omega' \times \Omega''$$

is a product domain, where $(x,y) \in \Omega' \times \Omega''$. We shall denote by B a ball such that $|B| = |\Omega'|$ and then we introduce

$$\Omega^\# = B \times \Omega'' \qquad \Omega^* = (0, |\Omega'|) \times \Omega''.$$

Our main result is the following:

Theorem 1.1. *Let g be concave, verifying (H). Let $h \in W^{1,\infty}(0,T)$, such that $h(t) \geq 0$ for all $t \in (0,T)$, $f \in L^2(0,T : L^2(\Omega))$ with $f \geq 0$ in $(0,T)$ and let $u_0 \in L^2(\Omega)$ be such that $u_0 \geq 0$. Let $u \in C([0,T], L^2(\Omega)) \cap L^2(\delta, T : H^1_0(\Omega))$ and $v \in C([0,T] : L^2(\Omega^\#)) \cap L^2(\delta, T : H^1_0(\Omega^\#))$ be the unique solutions of*

$$(P) \begin{cases} \frac{\partial u}{\partial t} - \Delta u + h(t) g(u) = f(t), & \textit{in } \Omega \times (0,T), \\ u = 0, & \textit{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \textit{on } \Omega, \end{cases}$$

$$(P^\#) \begin{cases} \frac{\partial v}{\partial t} - \Delta v + h(t) g(v) = f^\#(t), & \textit{in } \Omega^\# \times (0,T), \\ v = 0, & \textit{on } \partial\Omega^\# \times (0,T), \\ v(0) = v_0, & \textit{on } \Omega^\#, \end{cases}$$

where $v_0 \in L^2(\Omega^\#)$, $v_0 \geq 0$ is such that

$$\int_0^s u_0^*(\sigma, y) d\sigma \leq \int_0^s v_0^*(\sigma, y) d\sigma, \qquad \forall s \in [0, |\Omega'|] \textit{ and a.e. } y \in \Omega''.$$

Then, for any $t \in [0,T]$ and $s \in [0, |\Omega'|]$

$$\int_0^s u^*(t, \sigma, y) d\sigma \leq \int_0^s v^*(t, \sigma, y) d\sigma \qquad \textit{a.e. } y \in \Omega''.$$

The main idea of the proof is to use a generalization of the Trotter-Kato formula and to decompose the process in two different steps: the parabolic case without any absorption term ($g \equiv 0$) and the consideration of the auxiliary distributed ODE

$$\begin{cases} \xi_t + h(t) g(\xi) = 0, \\ \xi(0) = \xi_0. \end{cases}$$

Theorem 1 extends previous results in the literature on the comparison of Steiner rearrangements which until now were merely related to linear problems (see [1], [13], [7], [8], [9] and their references). The case in which g is convex is considered in [12].

2. **Some definitions on the Steiner symmetrization.** We recall that Hardy's inequality and (1) provides us with the estimate

$$\int_{\Omega(y)} f(x,y)\ \mathrm{d}x \leq \int_0^s f^*(\sigma, y)\ \mathrm{d}\sigma, \qquad \text{a.e. } y \in \mathbb{R}^m.$$

Now, let u be a measurable function. We define the auxiliary function

$$F(s,y) = \int_0^s u^*(\sigma, y)\ \mathrm{d}\sigma, \qquad \text{a.e. } y \in \mathbb{R}^m.$$

From the definition of the rearrangement we have that

$$F(s,y)=\int_{\Omega(y)} u(x,y)\ \mathrm{d}x, \qquad \text{a.e. } y\in\mathbb{R}^m.$$

In [1] it was shown that:

Lemma 2.1. *Let F be defined as before and let u be regular enough. Then,*

$$\frac{\partial F}{\partial y_i}=\int_{\Omega(y)}\frac{\partial u}{\partial y_i}$$

$$\left(\frac{\partial^2 F}{\partial y_i\partial y_j}\right)\geq\left(\int_{\Omega(y)}\frac{\partial^2 u}{\partial y_i\partial y_j}\right)$$

in the sense of matrices.

The results in [1] where presented on the stationary case and without non linear perturbation (the integro-differential equation which results is very difficult to treat by maximum principle arguments). Now, we may consider t, the time variable as a first y component, we may extend all of the above to the evolutionary case. It, then, holds that

$$\frac{\partial F}{\partial t}=\int_{\Omega(t,y)}\frac{\partial u}{\partial t}$$

and the analogous for the second derivative, which we will not need. This is also a consequence of other results ([3], [14], [15]).

To conclude the definitions we define the concentration relation as

$$u\preccurlyeq v\equiv\int_0^s u^*(\sigma,y)\ \mathrm{d}\sigma\leq\int_0^s v^*(\sigma,y)\ \mathrm{d}\sigma, \qquad \text{a.e. } y\in\Omega'', \quad \text{for any } s\in[0,|\Omega'|].$$

Although it is not strictly a result on symmetrization the following lemma (see, e.g., [10]) is a very useful tool for what follows.

Lemma 2.2. *Let $y,z\in L^1(0,M)$, $y,z\geq 0$ a.e., suppose y is non-increasing and*

$$\int_0^s y(\sigma)\ d\sigma\leq\int_0^s z(\sigma)\ d\sigma, \qquad \forall s\in[0,M].$$

Then, for every continuous non-decreasing function Φ we have

$$\int_0^s \Phi(y(\sigma))\ d\sigma\leq\int_0^s \Phi(z(\sigma))\ d\sigma \qquad \forall s\in[0,M].$$

Written in terms of the concentration relation the above property can be read as

$$y\preccurlyeq z\implies\Phi(y)\preccurlyeq\Phi(z)$$

for any function Φ convex and increasing.

Extending the above concentration relation we can define

Definition 2.3. Let $\Omega_1\equiv\Omega$ and $\Omega_2\equiv\Omega^{\#}$. Let

$$S_i: L^2(\Omega_i)\to\mathcal{C}([0,T]:L^2(\Omega_i))$$

We say that the pair (S_1,S_2) is *Steiner concentration monotone* if given $u_i\in L^2(\Omega_i)$ we have that

$$u_1\preccurlyeq u_2\implies S_1(t)u_1\preccurlyeq S_2(t)u_2, \qquad \text{for any } t\in[0,T].$$

It will be useful to recall that if $(u_n^i)\in L^2(\Omega_i)$ are two L^2- convergent sequences such that $u_n^i\to u^i$ and $u_n^1\preccurlyeq u_n^2$ then $u^1\preccurlyeq u^2$.

3. **Steiner comparison for linear parabolic equations and for a distributed nonlinear ODE.** We first compare the semigroup of a linear equation an its Steiner symmetrization, to show they are Steiner concentration monotone pairs. The following result can be proven by using as fundamental ingredient the proof for the elliptic case: see [8] for a detailed proof.

Proposition 1. *Let*

$$(A) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, , \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

$$(A^{\#}) \qquad \begin{cases} \frac{\partial v}{\partial t} - \Delta v = 0, & \text{in } (0,T)\times\Omega^{\#}, \\ v = 0, & \text{on } (0,T)\times\partial\Omega^{\#}, \\ v(0) = v_0, & \text{on } \Omega^{\#}, \end{cases}$$

and let S_Δ and $S_{\Delta^\#}$ be their associated L^2 semigroups on Ω and $\Omega^{\#}$ respectively. Then $(S_\Delta, S_{\Delta^\#})$ is a Steiner concentration monotone pair. That is, if

$$\int_0^s u_0^*(\sigma,y)d\sigma \le \int_0^s v_0^*(\sigma,y)d\sigma, \qquad \forall s\in[0,|\Omega'|], \ a.e.\ y\in\Omega'',$$

then we have

$$\int_0^s u^*(t,\sigma,y)\sigma \le \int_0^s v^*(t,\sigma,y)\sigma \qquad \forall t\in[0,T], \forall s\in[0,|\Omega'|], a.e.\ y\in\Omega'',$$

where $u = S_\Delta(\cdot)u_0$ and $v = S_{\Delta^\#}(\cdot)v_0$.

Concerning nonlinear distributed ODEs we have:

Proposition 2. *Let g be concave verifying (H) and let $h\in L^\infty(0,T)$, $h\ge 0$. Let u, v satisfy*

$$(B) \qquad \begin{cases} \frac{\partial u}{\partial t} + h(t)g(u) = 0, & \text{in } (0,T)\times\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases},$$

$$(B^{\#}) \qquad \begin{cases} \frac{\partial v}{\partial t} + h(t)g(v) = 0, & \text{in } (0,T)\times\Omega^{\#}, \\ v(0) = v_0, & \text{on } \Omega^{\#}. \end{cases}$$

Finally, let S_B and $S_{B^\#}$ be their associated evolution Green operators (i.e. the associated semigroups if $h(t)$ is constant). Then $(S_B, S_{B^\#})$ is a Steiner concentration monotone pair. That is, if

$$\int_0^s u_0^*(\sigma,y)d\sigma \le \int_0^s v_0^*(\sigma,y)d\sigma, \qquad \forall s\in[0,|\Omega'|], \ a.e.\ y\in\Omega'',$$

then we have

$$\int_0^s u^*(t,\sigma,y)d\sigma \le \int_0^s v^*(t,\sigma,y)d\sigma \qquad \forall t>0, s\in[0,|\Omega'|], \ a.e.\ y\in\Omega'',$$

for the solutions $u = S_B(\cdot)u_0$, $v = S_{B^\#}(\cdot)v_0$.

Proof. In a first step we assume, in addition that g is Lipschitz continuous and $g(0)=\varepsilon>0$. Let

$$\Phi(\xi) = \int_0^\xi \frac{d\sigma}{g(\sigma)}, \quad \Psi = \Phi^{-1}, \qquad H(t) = \int_0^t h(\sigma)d\sigma.$$

It is easy to check that

$$(S_B(t)u)(x,y) = \Psi(\Phi(u_0(x,y) - H(t)), \qquad (S_{B^\#}(t)v)(x,y) = \Psi(\Phi(v_0(x,y) - H(t)).$$

For these solutions

$$\begin{aligned}\mu_{S_B(t)u_0}(\tau,y) =&|\{x\in\mathbb{R}^n : |u(t,x,y)|>\tau\}|\\ =&|\{x\in\mathbb{R}^n : u_0(x,y)>\Phi(\Psi(\tau+H(t)))\}|\\ =&\mu_{u_0}(\Phi(\Psi(\tau)+H(t)),y).\end{aligned}$$

Since Φ, Ψ are monotone increasing then

$$\begin{aligned}\Big(S_B(t)u_0\Big)^*(s,y) =&\inf\{\tau>0 : \mu_{u_0}(\Phi(\Psi(\tau)+t),y)\le s\}\\ =&\inf\{\Phi(\Psi(\sigma)-H(t)) : \mu_{u_0}(\sigma,y)\le s\}\\ =&\Phi\Big(\Psi\big(\inf\{\sigma>0 : \mu_{u_0}(\sigma,y)\ge s\}\big)-H(t)\Big)\\ =&\Phi\Big(\Psi\big(u_0^*(s,y)\big)-H(t)\Big)=u^*(t,s,y).\end{aligned}$$

Therefore, u^* satisfies

$$\frac{\partial u^*}{\partial t}+h(t)g(u^*)=0.$$

Now let $w=e^{\lambda t}u$, then we have by the lemma $w^*=e^{\lambda t}u^*$, and so we have that w^* satifies

$$\frac{\partial w^*}{\partial t}+e^{\lambda t}h(t)g(e^{-\lambda t}w^*)-\lambda w^*=0.$$

We choose λ large enough so that $e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w$ be nonincreasing function on u for every $t\in(0,T)$. Analogous calculations provided information on $z=e^{\lambda t}v$. Let

$$\tilde{T}=\sup\left\{t : \int_0^s u^*(t,\sigma,y)\le\int_0^s v^*(t,\sigma,y)\sigma, \forall s\in[0,|\Omega'|]\right\}\ge 0.$$

Since $e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w$ is concave, for $t<\tilde{T}$, we apply lemma 2.2 and get

$$\begin{aligned}&\frac{d}{dt}\int_0^s(w^*(t,\sigma,y)-z^*(t,\sigma,y))d\sigma\\ &=\int_0^s h(t)(e^{\lambda t}h(t)g(e^{-\lambda t}z)-\lambda z-(e^{\lambda t}h(t)g(e^{-\lambda t}w)-\lambda w))d\sigma\le 0.\end{aligned}$$

So, we get

$$e^{\lambda t}\int_0^s(u^*(t,\sigma,y)-v^*(t,\sigma,y))d\sigma=\int_0^s(w^*(t,\sigma,y)-z^*(t,\sigma,y))d\sigma\le 0$$

and the result follows once g is Lipschitz continuous and $g(0)=\varepsilon$.

In the general case since g is associated to a maximal monotone graph of $\mathbb{R}^2$ we can approximate it by its Yosida approximation (which is still concave and satisfies (H)) and we get the result by passing to the limit. Finally we make $\varepsilon\downarrow 0$ and use the continuity of solutions with respect to g (see [5] and [4]). □

4. **Proof of the main theorem.** The special case $f=0$ and $h(t)\equiv h$ independent on t is easier. Since we know

$$\int_0^s u_0^*(\sigma,y)\,d\sigma\le\int_0^s v_0^*(\sigma,y)\,d\sigma,\qquad\forall s,\forall y$$

applying Proposition 1 and Proposition 2 inductively we get

$$\begin{aligned}&\int_0^s\left[\left(S_A\left(\frac{t}{n}\right)S_B\left(\frac{t}{n}\right)\right)^n u_0\right]^*(\sigma,y)\,d\sigma\\ &\le\int_0^s\left[\left(S_{A^\#}\left(\frac{t}{n}\right)S_{B^\#}\left(\frac{t}{n}\right)\right)^n v_0\right]^*(\sigma,y)\,d\sigma\end{aligned}$$

where S_A is the semigroup associated to problem (A) and S_B is the semigroup associated to problem (B) and analogously for $S_{A\#}$ and $S_{B\#}$.

Taking limits, applying the Trotter-Kato formula (see [5]) and applying convergence under the integral sign we get

$$\int_0^s [S_P(t)u_0]^*(\sigma,y)d\sigma \le \int_0^s [S_{P\#}(t)v_0]^*(\sigma,y)d\sigma$$

for any $t\in[0,T]$, for any $s\in[0,|\Omega'|]$ and a.e. $y\in\Omega''$. For the case $f\neq 0$ and $h(t)$ time dependent the Trotter-Kato formula can be also applied (see, e.g. [16]). In fact, to deal with the affine case $f(t)\neq 0$ we shall use a "reduction of order technique" argument which can be found on [4]. We point out that by an approximation argument and posterior passing to the limit process we can assume, without loss of generality, that in fact $f\in H^1(0,T;L^2(\Omega))$. Now, let us define $f(t+\cdot)\in H^1(0,T;L^2(\Omega)) : s\mapsto f(t+s)$ and $U(t)=(u(t),f(t+\cdot))\in L^2(\Omega)\times H^1(0,T;L^2(\Omega))$, $V(t)=(v(t),f^\#(t+\cdot))\in L^2(\Omega^\#)\times H^1(0,T;L^2(\Omega^\#))$. Let us note that U is the unique solution of

$$\begin{cases}\frac{\partial U}{\partial t}+\hat{L}U=0, & t\in(0,T)\\ U(0)=(u_0,f)\end{cases}\qquad\qquad \begin{cases}\frac{\partial V}{\partial t}+\hat{L}V=0, & t\in(0,T)\\ V(0)=(v_0,f^\#)\end{cases}$$

where

$$\hat{L}(u,\xi)=(-\Delta u+h(t)g(u)-\xi(0),\xi').$$

We can use a decomposition $\hat{L}=\hat{L}_1+\hat{L}_2$ in the following way:

$$\hat{L}_1(u,\xi)=(-\Delta u+h(t)g(u),0),\qquad \hat{L}_2(u,\xi)=(-\xi(0),\xi').$$

Let us define the problems

$$(C)\begin{cases}\frac{\partial U}{\partial t}+\hat{L}_1U=0,\\ U(0)=(u_0,f),\end{cases},\qquad (C^\#)\begin{cases}\frac{\partial V}{\partial t}+\hat{L}_1V=0,\\ V(0)=(v_0,f^\#),\end{cases}$$

$$(D)\begin{cases}\frac{\partial U}{\partial t}+\hat{L}_2U=0,\\ U(0)=(u_0,f),\end{cases},\qquad (D^\#)\begin{cases}\frac{\partial V}{\partial t}+\hat{L}_2V=0,\\ V(0)=(v_0,f^\#),\end{cases}$$

and the correspondent solution operators

$$S_C(t)(u_0,f)=(S_P(t)u_0,f),\quad S_{C\#}(t)(v_0,f^\#)=(S_P(t)u_0,f),$$

$$S_D(t)(u_0,f)=\left(u_0+\int_0^t f(s)ds,f\right),\quad S_{D\#}(t)(v_0,f^\#)=\left(v_0+\int_0^t f^\#(s)ds,f^\#\right).$$

Let Q be the projection operator such that $u(t)=QU(t)$. Let us study QS_C and QS_D. Since

$$\int_0^s u_0^*(\sigma,y)\,d\sigma\le\int_0^s v_0^*(\sigma,y)\,d\sigma,\qquad \text{for all } s\in[0,|\Omega'|] \text{ and a.e. } y\in\Omega''$$

we have, by the above explicit formulas (for the first component we apply the similar proof as in the case $f=0$)

$$\int_0^s [Q\ S_C(t)u_0]^*(\sigma,y)d\sigma\le\int_0^s [Q\ S_{C\#}(t)v_0]^*(\sigma,y)d\sigma,$$

$$\int_0^s [Q\ S_D(t)u_0]^*(\sigma,y)d\sigma\le\int_0^s [Q\ S_{D\#}(t)v_0]^*(\sigma,y)d\sigma.$$

By applying an induction argument again we get

$$\int_0^s\left[Q\left(S_C\left(\frac{t}{n}\right)S_D\left(\frac{t}{n}\right)\right)^n u_0\right]^*(\sigma,y)d\sigma$$
$$\le\int_0^s\left[Q\left(S_{C\#}\left(\frac{t}{n}\right)S_{D\#}\left(\frac{t}{n}\right)\right)^n v_0\right]^*(\sigma,y)d\sigma.$$

Finally, since all the operators are maximal monotone on their respective Hilbert spaces, we can take limits, apply the Trotter-Kato formula to justify the convergence of the limits and the result holds.

5. **Remarks and applications.** We point out that the main result applies to the parabolic obstacle problem:

$$\begin{cases} \left.\begin{array}{c} \frac{\partial u}{\partial t} - \Delta u - f(t,x) \geq 0, \ \ u \geq 0 \\ (\frac{\partial u}{\partial t} - \Delta u - f(t,x))u = 0 \end{array}\right\} & \text{in } (0,T)\times\Omega, \\ \qquad\qquad u = 0, & \text{on } (0,T)\times\partial\Omega, \\ \qquad\qquad u(0) = u_0, & \text{on } \Omega, \end{cases}$$

assumed $u_0 \in L^2(\Omega)$, $u_0 \geq 0$ and $f \in L^2(0,T : L^2(\Omega))$. The main argument is it can be reformulated in terms of

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \beta(u) \ni f(t,x) + 1, & \text{in } (0,T)\times\Omega, \\ u = 0, & \text{on } (0,T)\times\partial\Omega, \\ u(0) = u_0, & \text{on } \Omega, \end{cases}$$

where β is the maximal monotone graph of $\mathbb{R}^2$ given by

$$\beta(r) = \begin{cases} \emptyset, & r < 0, \\ (-\infty, 1], & r = 0, \\ \{1\}, & r > 0, \end{cases}$$

and that $\beta(u)$ can be approximated by a sequence $\beta_\lambda(u)$ of non decreasing concave functions satisfying (H) (take, for instance, $\beta_\lambda(u)$ such that $\beta_\lambda(u) = u^{\frac{1}{\lambda}}$ if $u \geq 0$). It is well known that the correspondent solutions u_λ converge strongly in $C([0,T] : L^2(\Omega))$ to the solution u of the obstacle problem and so the comparison of the associated Steiner rearrangements is mantained after passing to the limit.

Finally, we also mention that the associated nonlinear elliptic equation

$$\begin{cases} -\Delta u + g(u) = f(x), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

can be considered in this framework (since we can write $u(x) = \lim_{t\to\infty} u(t,x)$ for some suitable $u(t,x)$ solution of a nonlinear parabolic problem for which we can apply Theorem 1.1).

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Received September 2014; revised January 2015.

E-mail address: `jidiaz@ucm.es`
E-mail address: `dgcastro@ucm.es`

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AN APPLICATION OF SHAPE DIFFERENTIATION TO THE EFFECTIVENESS OF A STEADY STATE REACTION-DIFFUSION PROBLEM ARISING IN CHEMICAL ENGINEERING

JESÚS ILDEFONSO DÍAZ, DAVID GÓMEZ-CASTRO

Dedicated to our colleague and good friend Alfonso Casal on his 70th birthday

Abstract. In applications it is common to arrive at a problem where the choice of an optimal domain is considered. One such problem is the one associated with the steady state reaction diffusion equation given by a semilinear elliptic equation with a monotone nonlinearity g. In some contexts, in particular in chemical engineering, it is common to consider the functional given by the integral of this nonlinear term of the solution dived by the measure of the domain Ω in which the pde takes place. This is often related with the effectiveness of the reaction. In this paper our aim is to study the differentiability of such functional as study connected to the optimality of the best chemical reactor.

1. Introduction and statement of results

The main goal of this article is to analyze the differentiability, with respect to the domain Ω, of the *effectiveness factor*

$$\mathcal{E}(\Omega) = \frac{1}{|\Omega|}\int_\Omega \beta(w_\Omega)dx$$

where w_Ω is the solution of the problem arising in chemical catalysis [2, 3]

$$\begin{gathered} -\Delta w + \beta(w) = \hat{f}, \quad \text{in } \Omega, \\ w = 1, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

The model can be obtained in different ways, including homogenization techniques: see, e.g. [6] and [5]. By introducing the change in variable $u = 1 - w$ the problem can be reformulated as

$$\begin{gathered} -\Delta u + g(u) = f, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

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where $g(u) = \beta(1) - \beta(1-u)$ and $f = \beta(1) - \hat{f}$. In this case instead of the effectiveness factor we can study $\eta(\Omega) = 1 - \mathcal{E}(\Omega)$

$$\eta(\Omega) = \frac{1}{|\Omega|} \int_\Omega g(u_\Omega) dx \,, \tag{1.3}$$

where u_Ω is the solution of (1.2). In the chemical context this factor represents the amount of reaction taking place.

This kind of problems fall with the family of problems studied by several authors in the literature (see, e.g. [18, 19, 20] and the references therein). In the most general case this family of problems may be described by:

$$\begin{aligned} A(u(D)) &= f, \quad \text{in } D, \\ B(u(D)) &= g, \quad \text{on } \partial D \end{aligned} \tag{1.4}$$

and the functional can by given generally as

$$J(D) = \int_D C(u_D) dx,$$

where A, B, C may contain also some derivatives of u_D. In this paper we shall concentrate our attention in problem (1.2) and we shall provide elementary and direct proofs of results which could be obtained from the general theory but under stronger assumptions (see, for instance, the statement taken from [20] which is reproduced here in Section 2).

As mentioned before, our aim is to study the differentiability of functional (1.3). We consider a fixed domain open bounded regular set of $\mathbb{R}^n$, Ω_0, and study its deformations given by a function $\theta : \mathbb{R}^n \to \mathbb{R}^n$, so that the new domain is $\Omega = (Id+\theta)\Omega_0$. We consider, as it is the case in chemistry catalysis, g and f such that $0 \le u \le 1$. We also mention that this kind of differentiation result also appears in many other contexts. Besides the above mentioned references we recall here the articles [7] for a linear problem with a Dirichlet constant boundary condition and [17] were a semilinear equation arising in combustion was considered (corresponding, in our formulation to take $g(u) = -e^u$).

To obtain this properties in the sense of derivatives, we consider two approaches, mimicking the approach in differential geometry. We first consider the global differentiability of solutions (as it was done in the linear cases in [15, 1] and the most general case in [20]), which unfortunately requires derivatives in spaces of too regular functions, and then we take advantage of the differentiation along curves (the approach followed in [21]).

Let us call, for simplicity, u_Ω the solution of (1.2). This corresponds to the Lagrangian understanding of the problem in the sense that the functional under study is study in terms of the direct domain Ω. However, we can consider the Eulerian understanding of the problem by recalling that in this family of domains, $\Omega = (Id + \theta)\Omega_0$, we can introduce a new function $v_\theta : \Omega_0 \to \mathbb{R}$ defined by

$$v_\theta = (I+\theta)^* u_{(I+\theta)\Omega_0} = u_{(I+\theta)\Omega_0} \circ (I+\theta), \tag{1.5}$$

simplifying the study of the differentiability of u_Ω and the functional $\eta(\Omega)$ with respect to Ω.

Our proof relies heavily on the Implicit Function Theorem. The application of this theorem requires an uniform choice of functional space, which would require some additional information on u. This kind of problems in the functional setting is well portrayed in [4].

For the nonlinearity g we shall consider the following assumptions:

Hypothesis 1.1. g is nondecreasing

Hypothesis 1.2. The Nemitskij operator for g (which we will denote again by g in some circumstances, as a widely accepted abuse of notation) $G : H^1(\Omega) \to L^2(\Omega)$ defined by

$$G(u) = g \circ u \tag{1.6}$$

is of class C^m for some $m \geq 1$.

We recall that Hypothesis 1.2 immediately implies that $[DG](v)\varphi = g'(v)\varphi$ for $\varphi, v \in H^1(\Omega)$ and that if G is of class $\mathcal{C}^k$ with $k > 1$ then necessarily $g(s) = as + b$ for some $a, b \in \mathbb{R}$.

Our first result collects some general results on the differentiability of the solution u_Ω with respect to Ω:

Theorem 1.3. *Let g satisfy Hypothesis 1.1 and 1.2. Then, the map $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to H^1_0(\Omega_0)$,*

$$\theta \mapsto v_\theta$$

(where v_θ is defined by (1.5)) is of class $\mathcal{C}^l$ in a neighbourhood of 0 if $f \in H^k(\mathbb{R}^n)$ where $l = \min\{k, l\}$. Furthermore, the application $u : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$$\theta \mapsto u_{(I+\theta(\Omega_0))}$$

(where u_θ is extended by zero outside $(I + \theta)(\Omega_0)$) is differentiable at 0. In fact $u' : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to H^1(\Omega)$ and

$$u'(0)\theta + \nabla u_{\Omega_0} \cdot \theta \in H^1_0(\Omega).$$

As in differential geometry, to compute a derivative we can take two routes. The first one is to show the existence of a global derivative, and this allows to compute some properties of our functions. The other one, is to compute the derivative along curves.

Definition 1.4. We say that Φ is a curve of deformations if $\Phi : [0, T) \to W^{1,\infty}(\Omega_0)$ with $\det \Phi(\tau) > 0$.

Hypothesis 1.5. We will say that θ is a curve of small perturbations of the identity (with direction V) if $\Phi(\tau) = I + \theta(\tau)$ is a curve of deformations and

(1) $\theta : [0, T) \to W^{1,\infty}(\mathbb{R}^n)$ is differentiable at 0,
(2) $\theta(0) = 0$,
(3) $\theta'(0) = V$.

Sometimes we consider higher order derivatives too. We will refer to θ or Φ indistinctively, since they relate by $\Phi(\tau) = I + \theta(\tau)$. Thus, the above theorem leads to:

Corollary 1.6. *Let Φ be a a curve of deformations of class $\mathcal{C}^k$. Then $\tau \mapsto v_{\theta(\tau)}$ is of class $\mathcal{C}^l$ with $l = \min\{m, k\}$.*

Our second result concerns the characterization of u'.

Theorem 1.7. *Let g satisfy Hypothesis 1.1 and 1.2. Let θ be a curve satisfying assumptions 1.5. Then u is differentiable along Φ at least at 0. That is, the directional derivative $\frac{d}{d\tau}(u \circ \Phi)$ exists, and it is the solution u' of*

$$\begin{aligned} -\Delta u' + \lambda g'(u_{\Omega_0})u' = 0 \quad \text{in } \Omega_0, \\ u' = -\nabla u_{\Omega_0} \cdot V, \quad \text{on } \partial\Omega_0. \end{aligned} \tag{1.7}$$

We point out that the above result shows, in other terms, that $u'(0)\theta$ is the unique weak solution of

$$\begin{aligned} -\Delta u' + \lambda g'(u_{\Omega_0})u' = 0, \quad \text{in } \Omega_0, \\ u' = -\nabla u_{\Omega_0} \cdot \theta, \quad \text{on } \partial\Omega_0. \end{aligned} \tag{1.8}$$

As consequence we have the following result.

Corollary 1.8. *The function $u' : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to H^1(\Omega)$ is continuous. In fact, since the solution u of (1.2) $u \in W^{2,p}(\Omega)$ for any $p \in [1, +\infty)$ then for any $q \in [1, p]$,*

$$\begin{aligned} |u'(0)(\theta)|_q &\le c|\nabla u \cdot \theta|_{L^p(\partial\Omega_0)} \\ &\le c|\theta|_\infty |\nabla u_{\Omega_0}|_{L^p(\partial\Omega_0)} \\ &\le c(p)|\theta|_\infty |u_{\Omega_0}|_{W^{2,p}(\Omega_0)}. \end{aligned}$$

Concerning the differentiability of the effectiveness factor functional we have the following theorem.

Theorem 1.9. *Under the assumptions of Theorem 1.3, let*

$$\hat{\eta}(\theta) = \int_{(I+\theta)\Omega_0} g(u_{(I+\theta)\Omega_0}) dx. \tag{1.9}$$

Then η is of class $\mathcal{C}^m$ in a neighbourhood of 0. It holds that

$$\hat{\eta}^{(m)}(0)(\theta_1, \cdots, \theta_m) = \int_{\Omega_0} \frac{d^n}{d\theta_n \cdots d\theta_1} (g(v_\theta) J_\theta)\, dx. \tag{1.10}$$

Its first derivative can be expressed in terms of u,

$$\hat{\eta}'(0)(\theta) = \int_{\Omega_0} (g'(u_{\Omega_0})u' \ + \operatorname{div}(g(u_{\Omega_0})\theta))\, dx, \tag{1.11}$$

and if ∂G is Lipschitz

$$\hat{\eta}'(0)(\theta) = \int_{\Omega_0} g'(u_{\Omega_0})u'\, dx + g(0) \int_{\partial\Omega_0} \theta \cdot n\, dS, \tag{1.12}$$

where $u' = u'(0)(\theta)$.

Corollary 1.10. *Under the assumptions of Theorem 1.3 it holds that*

$$\eta'(\theta) = \frac{1}{|\Omega_0|} \Big(\int_{\Omega_0} g'(u_{\Omega_0})u'\, dx - \eta(0) \int_{\partial\Omega_0} \theta \cdot n\, dS \Big).$$

Corollary 1.11. *Under the assumptions of Theorem 1.3 if Φ is a volume preserving curve then*

$$\eta'(\theta) = \frac{1}{|\Omega_0|} \int_{\Omega_0} g'(u_{\Omega_0})u'\, dx\,.$$

We point out that if g is Lipschitz (i.e. $g \in W^{1,\infty}(\mathbb{R})$) then we obtain

$$|\eta(\theta) - \eta(0)| = |\eta'(0)(\lambda\theta)| \leq c|g'|_\infty |u|_{W^{2,p}} |\theta|_\infty.$$

This allows to get some generalizations of the last result in cases in which the absorption term g is not so regular, as for instance when $\beta(w) = w^q$ and $q \in (0,1)$. Nevertheless, if there is a non-empty dead core (in the literature the dead core is defined as $\{x \in \Omega : w_\Omega(x) = 0\}$ where w_Ω is the solution of (1.1)) some additional arguments must be developed, in the line of [14], where some unbounded potentials are considered. This will the subject of a separated paper by the authors [12].

We end this paper by presenting, in Section 5, some applications of the above results in terms of the Schwarz and Steiner symmetrization as well as by illustrating them for some special families of domains by means of some numerical experiences.

2. Functional setting: Nemitskij operators and the implicit function theorem

Let us formalize what we mean by a shape functional. At the most fundamental level it should be a function defined over a set of domain, that is defined over $\mathfrak{C} \subset \mathcal{P}(\mathbb{R}^n)$. Since we want to differentiate we, at the very least, need to define proximity, that is a way to define *neighbourhood of a set*. As it is usual in the literature of shape optimization we work over the set of weakly differentiable bounded deformations with bounded derivative, the Sobolev space $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 2.1. We say that J is defined on a neighbourhood of $\Omega_0 \subset \mathbb{R}^n$ if there exists U a neighbourhood of 0 on $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ such that J is defined over $\{(Id + \theta)(\Omega_0) : \theta \in U\}$. We say that J is differentiable at Ω_0 if the application $W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n) \to \mathbb{R}$,

$$\theta \mapsto J((Id + \theta)(\Omega_0))$$

is differentiable at 0.

We present a sufficient condition so that Hypothesis 1.2 holds. This is widely used in the context of partial differential equations, but as far as we know no reference is known besides it being an exercise in [16]. That being the case we provide the usual proof. Other conditions, mainly on the growth of g can be considered so that assumptions 1.1.1.2 holds.

Lemma 2.2. *Let $g \in W^{2,\infty}(\mathbb{R})$. Then the Nemitskij operator* (1.6) *in the sense $L^p(\Omega) \to L^2(\Omega)$ is of class $\mathcal{C}^1$ for all $p > 2$. In particular, Hypothesis 1.2 holds.*

Proof. Let us define G the Nemitskij operator defined in (1.6). Consider it $G : L^p(\Omega) \to L^2(\Omega)$ for $p \geq 2$. We first have that, for $L = \max\{\|g\|_\infty, \|g'\|_\infty, \|g''\|_\infty\}$

$$\|G(u) - G(v)\|_{L^2}^2 = \int_\Omega |g(u) - g(v)|^2 dx \leq L \int_\Omega |u - v|^2 dx2$$

so that F is continuous. For $p > 2$ let $\varphi \in \mathcal{C}^\infty(\Omega)$ we compute

$$\|g(u + \varphi) - g(u) - g'(u)\varphi\|_{L^2}^2 = \int_\Omega |g'(\xi(x)) - g'(u(x))|^2 |\varphi(x)|^2 dx$$

for some function ξ by the intermediate value theorem. We, of course, have that

$$|g'(\xi(x)) - g'(u((x))| \leq L|\xi(x) - u(x)| \leq L|\varphi(x)|$$
$$|g'(\xi(x)) - g'(u(x))| \leq 2L$$

$$|g'(\xi(x)) - g'(u(x))| \leq L2^{1-\alpha}|\varphi(x)|^{\alpha}, \quad \forall \alpha \in (0,1)\,.$$

Therefore,

$$\|g(u+\varphi) - g(u) - g'(u)\varphi\|_{L^2}^2 \leq L^2 2^{2-2\alpha} \int_\Omega |\varphi(x)|^{2+2\alpha} dx\,.$$

Let $2 < p < 4$ then we have that $p = 2 + 2\alpha$ with $0 < \alpha < 1$. We then have that

$$\|g(u+\varphi) - g(u) - g'(u)\varphi\|_{L^2} \leq L2^{1-\alpha}\|\varphi(x)\|_{L^p}^{1+\alpha}$$

which proves the Frechet differentiability. Of course for $p > 4$ we have that $L^p(\Omega) \hookrightarrow L^3(\Omega)$. Furthermore, for any given dimension n we have Sobolev inclusions $H^1(\Omega) \hookrightarrow L^p(\Omega)$ with $p > 2$, proving the differentiability. □

Some other well-known results are quoted now.

Theorem 2.3. *Let $g \in W^{1,p}(\mathbb{R}^n)$. Then the map $\mathfrak{G} : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n)$ given by $\theta \mapsto g \circ (I + \theta)$ is differentiable in a neighbourhood of 0 and*

$$\mathfrak{G}'(0) = (\nabla g) \circ (I + \theta)\,.$$

Theorem 2.4 ([15, Lemme 5.3.3.])**.** *Let $g \in W^{1,p}(\mathbb{R}^n)$,*

$$\Psi : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$$

continuous at 0 with $\Psi(0) = I$, $W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n) \times L^\infty(\Omega)$,

$$\theta \mapsto (g(\theta), \Psi(\theta))$$

differentiable at 0, with $g(0) \in W^{1,p}(\mathbb{R}^n)$ and

$$g'(0) : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$$

continuous. Then the application $\mathfrak{G} : W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n)$,

$$\mathfrak{G}(\theta) = g(\theta) \circ \Psi(\theta)$$

is differentiable at 0 and

$$\mathfrak{G}'(0) = g'(0) + \nabla g(0) \cdot \Psi'(0).$$

To conclude this section we state a classical result.

Theorem 2.5 (Implicit Function Theorem)**.** *Let X, Y and Z be Banach spaces and let U, V be neighbourhoods on X and Y, respectively. Let $F : U \times V \to Z$ be continuous and differentiable, and assume that $D_y F(0,0) \in \mathcal{L}(Y, Z)$ is bijective. Let assume, further, that $F(0,0) = 0$. Then there exists W neighbourhood of 0 on X and a differentiable map $f : W \to Y$ such that $F(x, f(x)) = 0$. Furthermore, for x and y small, $f(x)$ is the only solution y of the equation $F(x,y) = 0$. If F is of class $\mathcal{C}^m$ then so is f.*

3. Differentiation of solutions. Proof of Theorems 1.3 and 1.7

For the reader convenience we repeat here the general result in [20]:

Theorem 3.1. *Let D be a bounded domain such that ∂D be a piecewise $\mathcal{C}^1$ and assume that D is locally on one side of ∂D. Let u_0 be the solution of (1.4). Let us use the notation $\mathcal{C}^k = \mathcal{C}^k(\mathbb{R}^n, \mathbb{R}^n)$ and $k \geq 1$. Assume that*

$$u(\theta) \in W^{m,p}((I + \theta)D) \tag{3.1}$$

and that for every open set D' close to D (for example $D' = (I+\theta)D$ for small θ' in the norm of $\mathcal{C}^k$),

$$\begin{gathered} A : W^{m,p}(D') \to \mathcal{D}'(D') \\ B : W^{m,p}(D'^{1,1}(D') \\ C : W^{m,p}(D'^{1}(D') \\ A, B, C : W^{m-1,p}(D') \to \mathcal{D}'(D) \ \textit{differentiable} \end{gathered} \tag{3.2}$$

and $\mathcal{C}^k \to W^{m,p}$: $\theta \mapsto u(\theta) \circ (I+\theta)$ is differentiable at 0. Then:

(1) *The solution is differentiable in the sense that $u : \mathcal{C}^k \to W^{m-1,p}_{\rm loc}(D)$ is differentiable and the derivative the local derivative u' in the direction of τ satisfies*
$$\frac{\partial A}{\partial u}(u_0)u' = 0, \quad \textit{in } D. \tag{3.3}$$
(2) *If $\theta \mapsto B(u(\theta)) \circ (I+\theta)$ is differentiable at 0, into $W^{1,1}(D)$, $B(u_0) \in W^{2,1}(D)$ and $g \in W^{2,1}(\mathbb{R}^n)$, then u' satisfies*
$$\frac{\partial B}{\partial u}(u_0)u' = -\tau \cdot n \frac{\partial}{\partial n}(B(u_0) - g). \tag{3.4}$$
(3) *If $\theta \mapsto C(u(\theta)) \circ (I+\theta)$ is differentiable at 0 into $L^1(D)$, and $C(u_0) \in W^{1,1}(D)$, then $\theta \mapsto J(\theta)$ is differentiable and its directional derivative in the direction of τ is:*
$$\frac{\partial J}{\partial \theta}(0)\tau = \int_D \frac{\partial C}{\partial u} u' \, dx + \int_{\partial D} \tau \cdot n C(u_0) \, dS. \tag{3.5}$$

Let us prove now our first contribution.

Proof of Theorem 1.3. We take several steps. For simplicity, allow the notation
$$\Omega_\theta = (I+\theta)(\Omega_0).$$
We first check that v_θ satisfies
$$-\operatorname{div}(A(\theta)\nabla v) + \lambda J_\theta g(v_\theta) = (f \circ (I+\theta))J_\theta$$
in $H^{-1}(\Omega)$, where
$$A(\theta) = J_\theta (I + D\theta)^{-1}(I +^t D\theta)^{-1}, \quad J_\theta = \det J(I+\theta)\,.$$
For that, consider for a given $\varphi \in H^1_0(\Omega_0)$ the auxiliar function $\varphi_\theta = \varphi \circ (I+\theta)^{-1} \in H^1_0(\Omega_\theta)$ by definition of u_θ we have
$$\int_{\Omega_\theta} (\nabla u_\theta \nabla \varphi_\theta + \lambda g(u_\theta)\varphi_\theta) \, dx = \int_{\Omega_\theta} f \varphi_\theta \, dS \quad \forall \varphi \in H^1_0(\Omega_0).$$
Then by a change of variable, the result follows.

Let us define the operator $F : W^{1,\infty} \times H^1_0(\Omega_0) \to H^{-1}(\Omega_0)$, by
$$F(\theta, v) = \operatorname{div}(A(\theta)\nabla v) + \lambda J_\theta g(v) - (f \circ (I+\theta))J_\theta$$
of class $\mathcal{C}^1$ (or $\mathcal{C}^m$) in a neighbourhood of $\theta = 0$. On that direction we check

- $\theta \in W^{1,\infty} \mapsto J_\theta = \det(I + D\theta) \in L^\infty$ of class $\mathcal{C}^\infty$ since $\theta \in W^{1,\infty} \to I + D\theta \in L^\infty(\mathbb{R}^n, \mathcal{M}_n)$ and det is a polynomic operator.
- $\theta \in W^{1,\infty} \mapsto (I+D\theta)^{-1} = \sum_{q\ge 0} (-1)^q D\theta^q \in L^\infty(\mathbb{R}^n, \mathcal{M}_n)$ is $\mathcal{C}^\infty$,
- $(A, v) \in L^\infty(\mathbb{R}^n, \mathcal{M}_n) \times H^1_0(G) \mapsto -\operatorname{div}(A\nabla v) \in H^{-1}(G)$ is $\mathcal{C}^\infty$ since it is bilinear and continuous.

- Through the lemma $\theta \mapsto k(\theta) = (f \circ (I+\theta))J_\theta \in L^2(\mathbb{R}^n) \subset H^{-1}(\Omega_0)$ is $\mathcal{C}^1$ so $F \in \mathcal{C}^1$. Note that, if $f = 0$ then $F \in \mathcal{C}^m$.

It holds that

$$D_v F(0,0)\varphi = -\Delta\varphi + \lambda g'(u(\cdot : 0))\varphi$$

and, since $g' \geq 0$, we have that $D_v(0,v) : H_0^1(G) \to H^{-1}(G)$ is a isomorphism by Lax-Milgram's theorem. Through the implicit function theorem (theorem 2.5) there exists a map $\theta \in W^{1,\infty} \to v(\theta) \in H_0^1(\Omega_0)$ of class $\mathcal{C}^1$ is $f \in H^1(\mathbb{R}^n)$ and $\mathcal{C}^m$ if $f = 0$ such that

$$F(\theta, v(\theta)) = 0\,.$$

If we we consider uniqueness for the elliptic problem we find that

$$v(\theta) = v_\theta.$$

Simple substitution returns u_θ. By Theorem 2.4 we have the differentiability of u. □

Once this is done we can make explicit calculations for the directional derivative.

Proof of Theorem 1.7. Let us now characterize the directional derivative. Let $\theta \in W^{1,\infty}$ be fixed, let us call $u' = u'(0)(\theta)$ and let Φ a curve of perturbations of the identity with $V = \theta$. We differenciate on the variational formulation

$$\int_{\mathbb{R}^n} f\varphi \,\mathrm{d}x = \int_{\mathbb{R}^n} \left(-u_\tau \Delta\varphi + \lambda g(u_\tau)\varphi\right)\, dx \quad \varphi \in \mathcal{C}_c^\infty(\Omega)$$

to obtain

$$0 = \int_{\Omega_0} (-u'\Delta\varphi + \lambda g'(u_0)u'\varphi)\, dx, \quad \varphi \in \mathcal{C}_c^\infty(\Omega) \tag{3.6}$$

(observe that $h(x) = \lambda g'(u_0(x))$ is a known function). This means that u' is a very weak solution of the aforementioned equation (1.8). Since we know that $u' \in L^2(\mathbb{R}^n)$ we can apply regularity theory for this equation.

For the boundary condition $v_\theta = 0$ on $\partial\Omega_0$, for all θ and therefore $v' = 0, \partial\Omega_0$. Since $v_\tau = u_\tau \circ \Phi(\tau)$ we have

$$u' + \nabla u_{\Omega_0} \cdot \theta = v' \in H_0^1(\Omega_0)$$

which provides the boundary condition. Therefore, we have

$$\int_{\Omega_0} \left(-u'\Delta\varphi + \lambda g'(u_0)u'\varphi\right)\, dx = \int_{\partial\Omega_0} \left((\nabla u_{\Omega_0} \cdot \theta)\partial_{\mathbf{n}}\varphi\right)\, dS, \quad \varphi \in \mathcal{C}_0^2(\Omega) \tag{3.7}$$

we can obtain a Kato type inequality to shows uniqueness of very weak solutions (see [13]). For the regularity we apply the following classical trick. Since u' is know we can take $\tilde{f} = -\lambda g'(u_0)u' \in L^2(\Omega)$ and $\tilde{\eta} = -\nabla u \cdot \theta \in L^2(\partial\Omega)$ and find z the unique solution in $H^1(\Omega_0)$ of

$$\begin{gathered} -\Delta z = \tilde{f}, \quad \text{in } \Omega \\ z = \tilde{\eta}, \quad \text{on } \partial\Omega \end{gathered}$$

classical theory. Then z is a very weak solution of (3.7) and, by uniqueness, $u'(0) = z \in H^1(\Omega)$. □

Remark 3.2. In the case that further regularity is necessary $v \in H_0^1 \cap H^m$ then deformation must taken in $W^{m,\infty}$. A theory analogous to that on [15] for higher differentiability can be obtained for the non-linear case.

4. Differentiation the functional. Proof of Theorem 1.9 and its corollaries

We shall follow a reasoning similar to the one presented in [15]. Let us define $G_t = \Phi(t, G)$ and consider a function f such that $f(\tau) \in L^1(G_t)$. We take interest on the map $I : \mathbb{R} \to \mathbb{R}$,

$$I(\tau) = \int_{G_\tau} f(\tau, x)\, dx = \int_G f(\tau, \Phi(\tau, y)) J(\tau, y)\, dy \tag{4.1}$$

where $f(\tau, x) = f(\tau)(x)$,

$$J(\tau, y) = \det(D_y \Phi(\tau, y)).$$

Theorem 4.1. *Let Φ very assumptions 1.5, f such that $f : [0, T) \to L^1(\mathbb{R}^n)$ is differentiable at 0 and*

$$f(0) \in W^{1,1}(\mathbb{R}^N)\,.$$

Then, $\tau \mapsto I(\tau) = \int_{G_\tau} f(\tau)$ is differentiable at 0 and

$$I'(0) = \int_G f'(0) + \operatorname{div}(f(0)V)\,.$$

If G is an open set with Lipschitz boundary then

$$I'(0) = \int_G f'(0) + \int_{\partial G} f(0) n \cdot V.$$

In simpler terms, under regularity it holds that

$$\frac{\partial}{\partial \tau}\Big|_{\tau=0} \Big(\int_{G_\tau} f(\tau, x) dx \Big) = \int_{\Omega_0} \Big\{ \frac{\partial f}{\partial \tau}(0, x) + \operatorname{div} \Big(f(0, x) \frac{\partial \Phi}{\partial \tau}(0, x) \Big) \Big\} dx. \tag{4.2}$$

We have some immediate consequences of Theorem 4.1

Lemma 4.2. *Let $g \in W^{1,1}(\mathbb{R}^N)$ and $\Psi : [0, T) \to W^{1,\infty}$ be continuous at 0 such that $\Psi : [0, T) \to L^\infty$ is differentiable at 0, and let Z be its derivative. Then the mapping $[0, T) \to L^1(\mathbb{R}^n)$,*

$$\tau \mapsto g \circ \Psi(\tau)$$

is differentiable at 0 and $G'(0) = \nabla g \cdot Z$.

Lemma 4.3 (Differentiation under the integral sign)**.** *Let E be a Banach space and $f : E \times \Omega \to \mathbb{R}$ be such that $\tilde{f} : E \to L^1(\Omega)$*

$$\tilde{f}(x) = f(x, \cdot)$$

is differentiable at x_0. Let $F : E \to \mathbb{R}$,

$$F(x) = \int_\Omega f(x, y) dy\,.$$

Then F is differentiable at x_0 and

$$DF(x) = \int_\Omega (D_x \tilde{f})(x)(y).$$

Now we can prove the third of our main results.

Proof of Theorem 1.9. It is classical that we can differentiate under the integral sign

$$\int_{\Omega} f(t,x)dx$$

with respect to t as many times as f is differentiable, and that the integral commutates with the derivative. This shows the derivability with vJ_θ under the integral sign. For the remaining equations we have to be a little more subtle and apply the previous theorem. Let $f(\tau) = g \circ u_\tau$. From the know formulas we must compute

$$f'(\tau) = (g' \circ u_0)u'$$

Thus

$$\frac{\partial}{\partial \tau}\Big|_{\tau=0}\Big(\int_{G_\tau} g(u_\tau)\,dx\Big) = \int_{\Omega_0} \{g'(u_0)u' + \operatorname{div}(g(u_0)\Phi'(0))\}\,dx.$$

If Ω_0 is Lipschitz then

$$\frac{\partial}{\partial \tau}\Big|_{\tau=0}\Big(\int_{G_\tau} g(u_\tau)\,dx\Big) = \int_{\Omega_0} g'(u_0)u'x + g(0)\int_{\partial\Omega_0} \Phi'(0)\cdot n\,dS. \tag{4.3}$$

Equation (1.10) is guaranteed since $g(v) : W^{1,\infty} \to H_0^1(\Omega) \to L^1(\Omega)$ is $\mathcal{C}^1$, and so we can differentiate under the integral sign. □

To show equation (1.10) we need a formula of differentiation under the integral sign

Proof of Corollary 1.10. Given the functional

$$I(\Omega) = \frac{1}{|\Omega|}\int_{\Omega} g \circ u_\Omega dx$$

If we do not impose constant volume we have also to differentiate the volume measure

$$I(\Phi) = \frac{\int_{\Phi(G)} g \circ u_{\Phi(\Omega_0)}dx}{\int_{\Phi(G)} dx}$$

over a curve of deformations $\Phi(\tau)$ we have, applying the formula of differentiation of fractions

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\tau}\Big|_{\tau=0} = \frac{1}{|\Omega_0|^2}\Big(&|\Omega_0|\frac{d}{d\tau}\Big(\int_{\Phi(G)} g \circ u_{\Phi(\Omega_0)}dx\Big)\\ &- \Big(\int_{\Omega_0} \operatorname{div}\Phi'(0)\cdot n\,dx\Big)\Big(\int_{\Phi(G)} g \circ u_{\Phi(\Omega_0)}dx\Big)\Big),\end{aligned}$$

which, once simplified, gives the result. □

The proof of Corollary 1.11 relies on the following Proposition.

Proposition 4.4. *Let $\Phi(\tau)$ be a volume preserving family of deformations of Ω_0 in the sense of Hypothesis 1.5. Then*

$$\int_{\Omega_0} \operatorname{div}\Phi'(0)\,dx = 0.$$

If G is Lipschitz then

$$\int_{\partial\Omega_0} \Phi'(0)\cdot n\,dS = 0.$$

Proof. Define $G_\tau = \Phi(\Omega_0, \tau)$; then
$$c = \int_{G_\tau} 1 \, \mathrm{d}x \, .$$
From this and theorem 4.1 we obtain
$$0 = \int_{\Omega_0} \frac{\partial 1}{\partial \tau} + \operatorname{div}\left(1 \Phi'(0)\right) \, \mathrm{d}x,$$
which proves the first part of the result. The second is an immediate consequence of the divergence theorem. ☐

Remark 4.5. Note that the condition $\Phi(0) = I$ is paramount. For example consider the family of deformations
$$\Phi(\tau)(x, y) = \Big((1+\tau)x, \frac{1}{1+\tau} y \Big) \, .$$
These are isovolumetric deformations of any circle centered at 0, and of course $\Phi(0) = 0$. We can compute
$$\operatorname{div} \Phi'(\tau) = 1 - \frac{1}{(1+\tau)^2} \, .$$
This is only zero at $\tau = 0$ (that is where $\Phi(\tau) = I$) even though the transformations are isovolumetric at any given τ.

Remark 4.6. *For generalizing to the case $g = g(x, u)$, we need to assume that the Nemitskij operator $G : W^{1,\infty}(\Omega) \times H^1(\Omega) \to L^2(\Omega)$,*
$$G(\Phi, v) = g(\Phi(x), v(x))$$
is C^m and that
$$\frac{\partial g}{\partial v}(x, v) \geq 0.$$
In this case the operator on the implicit function theorem will be
$$F(\theta, v) = -\operatorname{div}(A(\theta) v) + g((I+\theta)^{-1}, v) J_\theta = f J_\theta$$
with derivative
$$D_v F(0, v)\varphi = -(\Delta \varphi)(x) + \frac{\partial g}{\partial v}(x, v(x))\varphi(x).$$

5. Applications

Rearrangement techniques: Schwarz and Steiner symmetrization. From Schwarz symmetrization we know (see e.g. [8], [9]) that, if g is either concave or convex and θ is volume preserving then $\eta(\theta) \leq \eta(0)$ (that is: the sphere is the least effective reactor). Therefore
$$\int_G g'(u_0) u' = \tilde{\eta}'(0) = 0.$$

For the Steiner symmetrization we know that, as we have proven in [10], for concave g, and in [11], for convex g (note that this is equivalent to concave β), the following holds:

Theorem 5.1. *Let g be a concave or convex continuous nondecreasing function such that $g(0) = 0$. Let $f \in L^2(\Omega)$ be nonnegative, i.e. $f \geq 0$, and $|B| = |\Omega''|$ with B a ball. Then*

$$\eta(\Omega' \times \Omega'') \leq \eta(\Omega' \times B). \tag{5.1}$$

So, for $G = B \times G_2 \ni (x, y)$, we have for all deformations $\theta = (\theta_1, 0)$ with θ_1 volume preserving and g convex or concave,

$$\begin{aligned}
\int_G g'(u_0)u' &= \int_{G_2} \int_B (g'(u_0)u' + \operatorname{div}(g(u_0)\theta) \\
&= \int_{G_2} \int_B (g'(u_0)u' + \operatorname{div}_x(g(u_0)\theta_1) \\
&= \int_{G_2} \Big\{ \int_B g'(u_0)u' + g(0) \int_{\partial B} \theta_1 \cdot n \Big\} \\
&= \int_{G_2} \int_B g'(u_0)u' \\
&= \int_G g'(u_0)u' \,.
\end{aligned}$$

Whenever the Nemitskij operator for g is of class $\mathcal{C}^2$ we get

$$\eta'(0)(\theta) = 0, \quad \eta''(0)(\theta, \theta) \leq 0\,.$$

Applying the bounds for $\eta'(0)$ we have as consequence an a priori estimate of the effectiveness factor in terms of the value of the functional for a circular cylinder:

Proposition 5.2. *If B is a ball such that $|B| = |\Omega'|$ then*

$$\eta(B \times \Omega'') - c(p)|g'|_\infty |u|_{W^{2,p}} |\theta|_\infty \leq \eta(\Omega' \times \Omega'') \leq \eta(B \times \Omega'').$$

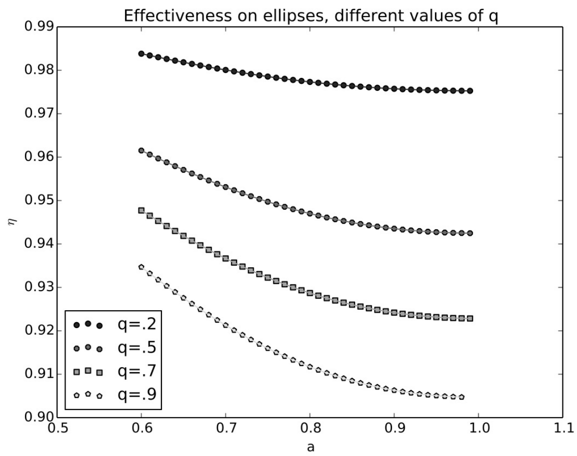
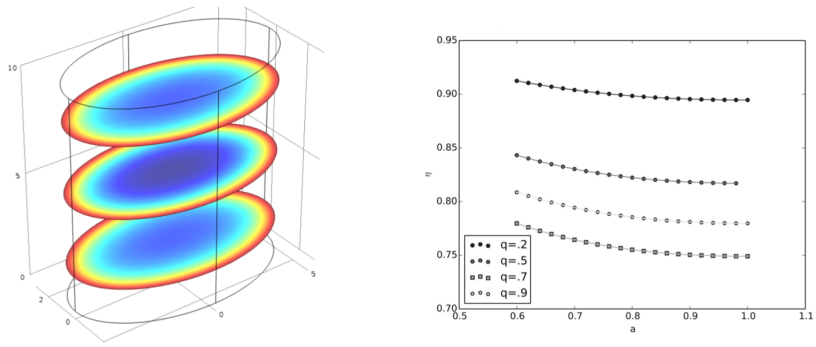


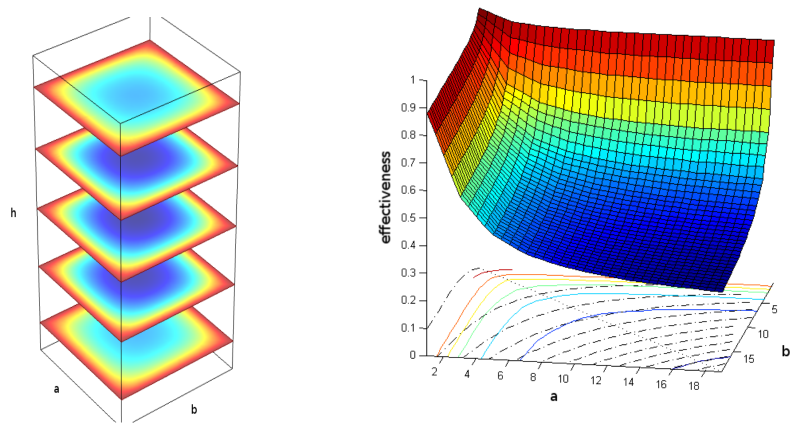
FIGURE 1. Effectiveness on isovolumetric ellipses with smaller semiaxes a, for the kinetic $g(u) = 1 - (1 - u)^{1/q}$.

Numerical experiments. The following numerical experiments were performed with COMSOL Multiphysics.



Solution in elliptic cylinder — Curve of the effectiveness

FIGURE 2. Effectiveness on elliptic cylinders with smaller semiaxes a, for the kinetic $g(u) = 1 - (1-u)^{1/q}$, $0 < q < 1$ (this kinetic corresponds to $\beta(w) = w^q$, which is known in chemistry as the Freundlich isotherm).



Solution in rectangular cylinder — Surface of the effectiveness for $h = 10$ and a, b as parameters. Dotted lines represent curves of equal area

FIGURE 3. Effectiveness on rectangular cylinders $[0, a] \times [0, b] \times [0, h]$, for the kinetic $g(u) = 1 - (1-u)^2$ and $h = 10$.

Example 5.3 (Schwarz symmetrization)**.** Let $g = g_1 + g_2$ where g_1 is convex and g_2 is concave. It is well known, see [8] and [9], that a sphere is the least effective reactor for our problem in each isoperimetric family (to be more precise, isovolumetric families). We can see this in terms of derivatives through a family of ellipses

$$\Phi(x, y, \tau) = (a(\tau)x, a(-\tau)y)$$

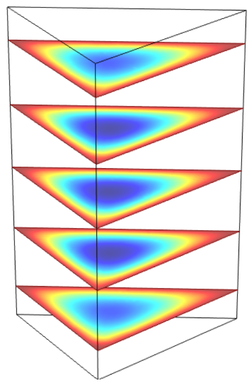


FIGURE 4. A triangular cylinder.

for a regular such that $a(0) = 1$, even when we have no volume conservation. It turns out that since this is a symmetric curve of linear transformations we have that

$$I(\tau) = I(-\tau)\,.$$

Since we have differentiability it must hold that $I'(0) = 0$. Since we have that this is a minimum and we are able to differentiate twice $I''(0) = 0$.

Example 5.4 (Steiner symmetrization)**.** The same computations hold for transformations

$$\Phi(x, y, z, \tau) = (a(\tau)x, a(-\tau)y, z)$$

This is a particular case of the results in [10] and [11]. If we consider a (uniparametric) family of elliptic cylinders of fixed height then we have the analogous result .

We can even do this analysis on two parametric families, for example in square or triangular cylinder were we consider both dimensions on the basis.

This analysis can be repeated over other families, like triangular cylinders with results of the same exact nature.

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Jesús Ildefonso Díaz
Instituto de Matemática Interdisciplinar and Dpto. de Matemá tica Aplicada, Facultad de Ciencias Matemáticas.Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Spain
E-mail address: `jidiaz@ucm.es`

David Gómez-Castro
Instituto de Matemática Interdisciplinar and Dpto. de Matemá tica Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Spain
E-mail address: `dgcastro@ucm.es`

SHAPE DIFFERENTIATION OF A STEADY-STATE REACTION-DIFFUSION PROBLEM ARISING IN CHEMICAL ENGINEERING: THE CASE OF NON-SMOOTH KINETIC WITH DEAD CORE

DAVID GÓMEZ-CASTRO

ABSTRACT. In this paper we consider an extension of the results in shape differentiation of semilinear equations with smooth nonlinearity presented in Díaz, J.I., Gómez-Castro, D.: An Application of Shape Differentiation to the Effectiveness of a Steady State Reaction-Diffusion Problem Arising in Chemical Engineering. Electron. J. Differ. Equations. 22, 31—45 (2015) to the case in which the nonlinearities might be less smooth. Namely we will show that Gateaux shape derivatives exists when the nonlinearity is only Lipschitz continuous, and we will give a definition of the derivative when the nonlinearity has a blow up. In this direction, we will study the case of root-type nonlinearities.

1. INTRODUCTION

In this paper we consider the shape differentiation of a family of diffusion-reaction problems introduced by Aris in the context of optimization of chemical reactors depending on the spatial domain (see [1]). It was later shown that the model can be rigorously deduced as a limit of different nonhomogeneous microscopic models (see [3, 4]). In particular we will be interested in the solutions of problem

$$\begin{cases} -\Delta w + \beta(w) = f, & \text{in } \Omega, \\ w = 1, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and their behaviour as we deform the domain Ω.

It will be sometimes useful to consider the change in variable $u = 1 - w, g(u) = \beta(1) - \beta(1-u)$ and $\widehat{f} = \beta(1) - f$, so that we have $u = 0$ on the boundary. After this change in variable we have that u is the solution of

$$\begin{cases} -\Delta u + g(u) = \widehat{f}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

These functions will be sometimes denoted u_Ω, w_Ω when different domains are considered.

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In [8] (see also [15, 13, 14]) the authors showed that, if $\beta \in W^{2,\infty}(\mathbb{R})$ and $f \in L^2(\Omega)$ then the maps

$$\begin{aligned} W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) &\to H_0^1(\Omega) \\ \theta &\mapsto u_{(I+\theta)\Omega} \circ (I+\theta) \\ W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) &\to L^2(\mathbb{R}^n) \\ \theta &\mapsto u_{(I+\theta)\Omega}, \end{aligned}$$

where the extension by 0 is considered in $\mathbb{R}^n \setminus \Omega$, are Fréchet differentiable at 0. Fixing $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ it was shown in [8] that the directional derivative (the derivative of $u_\tau = u_{(I+\tau\theta)\Omega}$ with respect to τ, $\frac{du_\tau}{d\tau} = \frac{du_\tau}{d\tau}|_{\tau=0}$) is the solution of the following problem

$$\begin{cases} -\Delta \frac{du_\tau}{d\tau} + g'(u_\Omega)\frac{du_\tau}{d\tau} = 0, & \text{in } \Omega, \\ \frac{du_\tau}{d\tau} = -\nabla u_\Omega \cdot \theta, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Notice that, since $u = 1 - w$, we have that $\frac{du_\tau}{d\tau} = -\frac{dw_\tau}{d\tau}$. Hence, taking into account that $g'(u) = -\beta'(w)$, we have that

$$\begin{cases} -\Delta \frac{dw_\tau}{d\tau} + \beta'(w_\Omega)\frac{dw_\tau}{d\tau} = 0, & \text{in } \Omega, \\ \frac{dw_\tau}{d\tau} = -\nabla w_\Omega \cdot \theta, & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The aim of this paper is to extend this kind of results to the case when $\beta \notin W^{2,\infty}$. First, we will show that, when $\beta \in W^{1,\infty}$ then the Gateaux shape derivative exists. However, if β is not locally Lipschitz continuous, the solution of (1.1) might develop a region of positive measure

$$N_\Omega = \{x \in \Omega : w_\Omega(x) = 0\}. \tag{1.5}$$

This region, known as *dead core*, was studied at length in [5, 2]. It is a necessary condition for the existence of this region that $\beta'(w_\Omega) = +\infty$. Hence, equation (1.4) cannot be understood immediately in a standard way. In this setting, we will show that there exists a limit of the previous theory.

2. Statement of results

For the rest of the paper $\Omega \subset \mathbb{R}^n$ will be a fixed domain, of class $\mathcal{C}^2$, and $n \geq 2$.

2.1. **Existence and estimates of shape derivatives.**

2.1.1. *Existence of Gateaux derivative when $\beta \in W^{1,\infty}$.* In [8] the authors prove the existence of a shape derivative in the Fréchet sense when $\beta \in W^{2,\infty}(\mathbb{R})$. Nonetheless, as is it usually the case, the equation for the derivative is well defined in a straightforward way when $\beta \in W^{1,\infty}(\mathbb{R})$. In fact, the following result shows that, if $\beta \in W^{1,\infty}(\mathbb{R})$ rather than $W^{2,\infty}(\mathbb{R})$, then the shape derivative exists only in the Gateaux sense, which is weaker than the Fréchet sense.

Theorem 2.1. *Let $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$, $\beta \in W^{1,\infty}(\mathbb{R})$ be nondecreasing such that $\beta(0) = 0$ and $f \in H^1(\mathbb{R}^n)$. Then, the applications*

$$\begin{aligned} \mathbb{R} &\to L^2(\Omega) \\ \tau &\mapsto u_{(I+\tau\theta)\Omega} \circ (I+\tau\theta), \end{aligned}$$

and

$$\begin{array}{rcl} \mathbb{R} & \to & L^2(\mathbb{R}^n) \\ \tau & \mapsto & u_{(I+\tau\theta)\Omega} \end{array}$$

are differentiable at 0. *Furthermore,* $\frac{du_\tau}{d\tau}|_{\tau=0}$ *is the unique solution of* (1.3).

Remark 2.2. *In most case, the process of homogenization mentioned in the introduction gives an homogeneous equation* (1.1) *in which* β *is the same as in the microscopic limit, and thus it is natural that* β *be singular. However, it sometimes happens that the limit kinetic is different. In the homogenization of problems with particles of critical size (see* [9]*) it turns out that the resulting kinetic in the macroscopic homogeneous equation* (1.1) *satisfies* $\beta \in W^{1,\infty}$, *even when the original kinetic of the microscopic problem was a general maximal monotone graph.*

2.1.2. *From* $W^{2,\infty}$ *to* $W^{1,\infty} \cap \mathcal{C}^1$. Let us show that the shape derivative is continuously dependent on the nonlinearity, and thus that we can make a smooth transition from the Fréchet scenario presented in [8] to our current case. For the rest of the paper we will use the notation:

$$v = \frac{dw_\tau}{d\tau}\Big|_{\tau=0} \tag{2.1}$$

Lemma 2.3. *Let* $f \in L^2(\mathbb{R}^n)$, $\beta \in W^{1,\infty}(\mathbb{R})$ *be nondecreasing functions such that* $\beta(0) = 0$ *and let* $\beta_n \in W^{2,\infty}(\mathbb{R})$ *nondecreasing such that* $\beta_n(0) = 0$. *Let* w_n *be the unique solution of*

$$\begin{cases} -\Delta w_n + \beta_n(w_n) = f & \Omega, \\ w_n = 1 & \partial\Omega. \end{cases} \tag{2.2}$$

Then

$$\|w_n - w\|_{H^1(\Omega)} \le C\|\beta_n - \beta\|_{L^\infty} \tag{2.3}$$

$$\|w_n - w\|_{H^2(\Omega)} \le C(1 + \|\beta'\|_{L^\infty})\|\beta_n - \beta\|_{L^\infty}. \tag{2.4}$$

Furthermore, let $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ *and* v_n *be the unique solution of*

$$\begin{cases} -\Delta v_n + \beta_n'(w_n)v_n = 0 & \Omega, \\ v_n + \nabla w_n \cdot \theta = 0 & \partial\Omega. \end{cases} \tag{2.5}$$

Then

$$v_n \rightharpoonup v \text{ in } H^1(\Omega). \tag{2.6}$$

Remark 2.4. *In* (2.3) *the notation*

$$\|\beta_n - \beta\|_{L^\infty} = \sup_{x\in\mathbb{R}} |\beta_n(x) - \beta(x)|$$

does mean that either β_n *or* β *are* $L^\infty(\mathbb{R})$ *functions themselves, but rather that their difference is pointwise bounded, and, in fact, this bound is destined to go* 0 *as* $n \to +\infty$. *We will use this notation throughout the paper.*

2.1.3. *Shape derivative with a dead core.* We can prove that the shape derivative in the smooth case has, under some assumptions, a natural limit when β not smooth.

In some cases in the applications (see [5]) we can take β so that $\beta'(w_\Omega)$ has a blow up. It is common, specially in Chemical Engineering, that $\beta'(0) = +\infty$ and N_Ω exists (see [5]). In this case $\beta'(w_\Omega) = +\infty$ in N_Ω. Due to this fact, the natural behaviour of the weak solutions of (1.4) is $v = 0$ in N_Ω. We have the following result

Theorem 2.5. *Let β be nondecreasing, $\beta(0) = 0$, $\beta'(0) = +\infty$,*

$$\beta \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R} \setminus \{0\}),$$

and assume that $|N_\Omega| > 0$, $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ and $0 \le f \le \beta(1)$. Then, there exists v a solution of

$$\begin{cases} -\Delta v + \beta'(w_\Omega) v = 0 & \Omega \setminus N_\Omega, \\ v = 0 & \partial N_\Omega, \\ v = -\nabla w_\Omega \cdot \theta & \partial\Omega, \end{cases} \tag{2.7}$$

in the sense that $v \in H^1(\Omega)$, $v = 0$ in N_Ω, $v = -\nabla w_\Omega \cdot \theta$ in $L^2(\partial\Omega)$, $\beta'(w_\Omega)v^2 \in L^1(\Omega)$ and

$$\int_{\Omega\setminus N_\Omega} \nabla v \nabla \varphi + \int_{\Omega \setminus N_\Omega} \beta'(w) v \varphi = 0 \tag{2.8}$$

for every $\varphi \in W_c^{1,\infty}(\Omega \setminus N_\Omega)$. Furthermore, for $m \in \mathbb{N}$, consider β_m defined by

$$\beta_m'(s) = \min\{m, \beta'(s)\}, \qquad \beta_m(0) = \beta(0) = 0,$$

and let w_m, v_m be the unique solutions of (2.2) *and* (2.5)*. Then,*

$$v_m \rightharpoonup v, \quad \text{in } H^1(\Omega), \tag{2.9}$$

where v is a solution of (2.7)*.*

The uniqueness of solutions of (2.7) when $\beta'(w_\Omega)$ blows up is by no means trivial. Problem (2.7) can be written in the following way:

$$-\Delta v + V v = f \tag{2.10}$$

where $V = \beta'(w_\Omega)$ may blow up as a power of the distance to a piece of the boundary. This kind of problems are common in Quantum Physics, although their mathematical treatment is not always rigorous (cf. [6, 7]).

In the next section we will show estimates on $\beta'(w_\Omega)$. Let us state here some uniqueness results depending on the different blow-up rates.

When the blows is subquadratic (i.e. not *too* rapid), by applying Hardy's inequality and the Lax-Migram theorem, we have the following result (see [6, 7]).

Corollary 2.6. *Let N_Ω have positive measure and $\beta'(u(x)) \le C d(x, N_\Omega)^{-2}$ for a.e. $x \in \Omega \setminus N_\Omega$. Then the solution v is unique.*

The study of solutions of problem (2.10) in Ω when $V \in L^1_{loc}(\Omega)$ by many authors (see [11, 10] and the references therein). Existence and uniqueness of this problem in the case $V(x) \geq Cd(x, \partial\Omega)^{-r}$ with $r > 2$ was proved in [10]. Applying these techniques one can show that

Corollary 2.7. *Let N_Ω have positive measure and $\beta'(u(x)) \geq Cd(x, N_\Omega)^{-r}, r > 2$ for a.e. $x \in \Omega \setminus N_\Omega$. Then the solution v is unique.*

Similar techniques can be applied to the case $\beta'(u(x)) \geq Cd(x, N_\Omega)^{-2}$. This will be the subject of a further paper.

2.2. **Estimates of w_Ω close to N_Ω.** Let us study the solution w_Ω on the proximity of the dead core and the blow up behaviour of $\beta'(w_\Omega)$. First, we present a known example

Example 2.8. *Explicit radial solutions with dead core are known when $\beta(w) = |w|^{q-1}w$ ($0 < q < 1$), Ω is a ball of large enough radius and f is radially symmetric. In this case it is known that N_Ω exists, has positive measure and*

$$\frac{1}{C} d(x, N_\Omega)^{-2} \leq \beta'(w_\Omega) \leq Cd(x, N_\Omega)^{-2}.$$

For the details see [5].

In fact, we present here a more general result to study the behaviour in the proximity of the dead core, based on estimates from [5].

Proposition 2.9. *Let $f = 0$, β be continuous, monotone increasing such that $\beta(0) = 0$, w be a solution of* (1.1) *that develops a dead core N_Ω of positive measure and $\partial N_\Omega \in \mathcal{C}^1$. Assume that*

$$G(t) = \sqrt{2} \left(\int_0^t \beta(\tau) d\tau + \alpha t \right)^{\frac{1}{2}}, \quad \text{where } \alpha = \max \left\{ 0, \min_{x \in \partial\Omega} H(x) \frac{\partial w}{\partial n}(x) \right\}, \tag{2.11}$$

is such that $\frac{1}{G} \in L^1(\mathbb{R})$. Then

$$w_\Omega(x) \leq \Psi^{-1}(d(x, N_\Omega)), \quad \text{where } \Psi(s) = \int_0^s \frac{dt}{G(t)}, \tag{2.12}$$

in a neighbournood of N_Ω.

Example 2.10 (Root type reactions)**.** *Let $f = 0$, $\beta(s) = \lambda |s|^{q-1} s$ with $0 < q < 1$ and Ω be convex such that N_Ω exists and $\partial N_\Omega \in \mathcal{C}^1$. Then*

$$w_\Omega(x) \leq Cd(x, N_\Omega)^{\frac{2}{1-q}}. \tag{2.13}$$

Furthermore

$$\beta'(w_\Omega(x)) \geq Cd(x, N_\Omega)^{-2}. \tag{2.14}$$

3. Proof of Theorem 2.1

For the rest of the paper let us note

$$u_\tau = u_{(I+\tau\theta)\Omega}. \tag{3.1}$$

Notice that $u_0 = u_\Omega$.

Let us define $U_\tau = u_{(I+\tau\theta)\Omega} \circ (I+\tau\theta) \in H_0^1(\Omega)$. Again $U_0 = u_0 = u_\Omega$. We have that

$$\int_\Omega A_\tau \nabla U_\tau \nabla \varphi + \int_\Omega g(U_\tau)\varphi J_\tau = \int_\Omega f_\tau \varphi J_\tau, \tag{3.2}$$

where J_τ is the Jacobian of the transformation. $f_\tau = f \circ (I+\tau\theta)$ and A_τ is the corresponding diffusion matrix (see [8] for the explicit expression). Fortunately, $J_\tau \geq 0$ and, for τ small, we have that $\xi \cdot A_\tau \xi \geq A_0 |\xi|^2$ for some $A_0 > 0$ constant. Considering the difference of the weak formulations of U_τ and $U_0 = u_\Omega$ we have that

$$\begin{aligned}\int_\Omega A_\tau \nabla (U_\tau - u_0)\nabla\varphi + \int_\Omega (g(U_\tau) - g(u_0))J_\tau \varphi = \int_\Omega (f_\tau J_\tau - f)\varphi + \\ + \int_\Omega (I - A_\tau)\nabla u_0 \nabla \varphi \\ + \int_\Omega (J_\tau - 1) g(u_0)\varphi.\end{aligned}$$

Hence, due to the monotonicity of g, we have that

$$\left\| \nabla \left(\frac{U_\tau - u}{\tau} \right) \right\|_{L^2} \leq C \left(\left\| \frac{f_\tau - f}{\tau} \right\|_{L^2} + \left\| \frac{A_\tau - I}{\tau} \right\|_{L^\infty} \|\nabla u_0\|_{L^2} + \left\| \frac{J_\tau - 1}{\tau} \right\|_{L^\infty} \|g(u_0)\|_{L^2} \right)$$

Since f_τ, A_τ and J_τ are differentiable at 0, there is weak $H_0^1(\Omega)$ limit. Hence, the limit is strong in $L^2(\Omega)$. Therefore, the function

$$u_\tau = U_\tau \circ (I + \tau\theta)^{-1} \tag{3.3}$$

is differentiable with respect to $\tau \in \mathbb{R}$ with images in $L^2(\Omega)$ at $\tau = 0$. Besides,

$$H_0^1(\Omega) \ni \frac{dU_\tau}{d\tau}\Big|_{\tau=0} = \frac{du_\tau}{d\tau}\Big|_{\tau=0} + \nabla u_0 \cdot \theta. \tag{3.4}$$

To characterize the derivative, we differenciate on the variational formulation

$$\int_{\mathbb{R}^n} f\varphi = \int_{\mathbb{R}^n} (-u_\tau \Delta\varphi + g(u_\tau)\varphi) \qquad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

Considering the difference of the equations for u_τ and u_0 and diving by τ

$$0 = \int_{\mathbb{R}^n} \left(-\frac{u_\tau - u_0}{\tau}\Delta\varphi + \frac{g(u_\tau) - g(u_0)}{\tau}\varphi \right) \tag{3.5}$$

$$= \int_{\mathbb{R}^n} \frac{u_\tau - u_0}{\tau} \left(-\Delta\varphi + \frac{g(u_\tau) - g(u_0)}{u_\tau - u_0}\varphi \right). \tag{3.6}$$

Notice that

$$\left| \frac{g(u_\tau) - g(u_0)}{u_\tau - u_0} \right| \leq \|g'\|_{L^\infty}.$$

Therefore, up to a subsequence, $\frac{g(u_\tau)-g(u_0)}{u_\tau - u_0}$ converges weakly in $L^2(\Omega)$. On the other hand since $u_\tau \to u_0$ pointwise, again up to a subsequence, so

$$\frac{g(u_\tau) - g(u_0)}{u_\tau - u_0} \to g'(u_0) \quad \text{a.e. in } \Omega. \tag{3.7}$$

Via a Césaro mean argument we have that the weak L^2 limit and pointwise limit coincide. Hence, passing to the limit in $L^2(\Omega)$

$$0 = \int_\Omega \frac{du_\tau}{d\tau}\Big|_{\tau=0} \left(-\Delta\varphi + g'(u_0)\varphi\right), \qquad \varphi \in \mathcal{C}_c^\infty(\Omega). \tag{3.8}$$

Therefore $\frac{du_\tau}{d\tau}$ is the unique solution of (1.3). □

4. Proof of Lemma 2.3

By considering the difference of the weak formulations we have that

$$\int_\Omega \nabla(w_m - w)\nabla\varphi + \int_\Omega (\beta_m(w_m) - \beta_m(w))\varphi = \int_\Omega (\beta(w) - \beta_m(w))\varphi.$$

Taking $\varphi = w_m - w$, and using the monotonicity of β_m we have that

$$\|\nabla(w_m - w)\|_{L^2}^2 \le \|\beta_m - \beta\|_{L^\infty}\|w_m - w\|_{L^1(\Omega)}.$$

Using Poincaré inequality and the embedding $L^1 \hookrightarrow L^2$ we have that

$$\|w_m - w\|_{L^2} \le C\|\beta_m - \beta\|_{L^\infty}.$$

By considering the equation

$$\begin{aligned}\|\Delta(w_m - w)\|_{L^2} &= \|\beta(w) - \beta_m(w_m)\|_{L^2} \\ &\le \|\beta(w) - \beta(w_m)\|_{L^2} + \|\beta(w_m) - \beta_m(w_m)\|_{L^2} \\ &\le \|\beta'\|_{L^\infty}\|w_m - w\|_{L^2} + \|\beta_m - \beta\|_{L^\infty}.\end{aligned}$$

Hence, to deduce (2.4) we apply that

$$\|w_m - w\|_{H^2} \le C(\|\Delta(w_m - w)\|_{L^2} + \|w_m - w\|_{L^2}).$$

Considering the difference of the weak formulations of the problems for v_m and v we have that

$$\begin{aligned}\int_\Omega \nabla(v_m - v)\nabla\varphi &= \int_\Omega (\beta'(w)v - \beta_m'(w_m)v_m)\varphi \\ &= \int_\Omega (\beta'(w) - \beta_m'(w_m))v_m\varphi + \int_\Omega \beta'(w)(v - v_m)\varphi \\ &= \int_\Omega (\beta'(w) - \beta'(w_m))v_m\varphi + \int_\Omega (\beta'(w_m) - \beta_m'(w_m))v_m\varphi \\ &\quad + \int_\Omega \beta'(w)(v - v_m)\varphi \end{aligned} \tag{4.1}$$

for all $\varphi \in H_0^1(\Omega)$. Considering the test function $\varphi = v_m - v + \nabla(w_m - w)\cdot\theta \in H_0^1(\Omega)$ we have, applying (2.4)

$$\begin{aligned}\int_\Omega |\nabla(v_m - v)|^2 \le C(1 + \|w_m - w\|_{H^2}) \\ \times \Big((1 + \|\beta'(w)\|_{L^\infty})\|w_m - w\|_{H^2} \\ + \|v_m\|_{L^2}(\|\beta_m' + \beta'\|_{L^\infty} + \|\beta'(w_m) - \beta'(w)\|_{L^\infty})\Big).\end{aligned}$$

We cannot guaranty that $\|\beta'(w_m)-\beta'(w)\|_\infty$ goes to zero. However it is, indeed, bounded by $2\|\beta'\|_{L^\infty}$. On the other hand, taking into account the boundary condition

$$\|v_m - v\|_{L^2(\partial\Omega)} \le C\|\nabla(w_m - w)\|_{L^2(\partial\Omega)} \le C\|w_m - w\|_{H^2(\Omega)} \le C\|\beta_m - \beta\|_{L^2} \to 0. \quad (4.2)$$

Hence, there is a weak limit $\widehat{v} \in H^1(\Omega)$

$$v_m - v \rightharpoonup \widehat{v} \text{ in } H^1(\Omega). \quad (4.3)$$

Due to (4.2) we have that $\widehat{v} \in H_0^1(\Omega)$. Taking into account (4.1) and the fact that $\beta'(w_m) \to \beta'(w)$ a.e. in Ω, have that

$$\int_\Omega \nabla\widehat{v}\nabla\varphi + \int_\Omega \beta'(w)\widehat{v}\varphi = 0 \qquad \forall\varphi \in H_0^1(\Omega). \quad (4.4)$$

Taking $\varphi = \widehat{v} \in H_0^1(\Omega)$ as a test function we deduce that $\widehat{v} = 0$. □

5. Proof of Theorem 2.5

We start by pointing out that, due to the condition on f we have that $0 \le w_m \le 1$. Since $\beta_m \nearrow \beta$ in $[0,1]$ we have w_m is pointwise decreasing (see [12]). Hence, there exists a pointwise limit w such that $w_m \searrow w$ a.e. in Ω. In particular $0 \le w \le 1$. Due to the Dominated Convergence Theorem we have that

$$w_m \to w \text{ in } L^p(\Omega) \quad \forall 1 \le p < +\infty. \quad (5.1)$$

Let $U \subset \Omega$ be an open neighbourhood of $\partial\Omega$ such that $\overline{U} \cap N_\Omega = \emptyset$ and $\partial U \in \mathcal{C}^2$. Then

$$\underline{w}_U = \inf_U w > 0. \quad (5.2)$$

We have that $w_m \ge w \ge \underline{w}_U$. We have that $\beta \in \mathcal{C}^1([\underline{w}_U, 1])$ and, hence, $\beta_m \to \beta$ in $\mathcal{C}^1([\underline{w}_U, 1])$. Therefore

$$\beta_m(w_m) \to \beta(w) \text{ in } L^p(\Omega \setminus \overline{U}) \quad \forall 1 \le p < +\infty, \quad (5.3)$$

Since $\|w_m\|_{H^1} \le C(1 + \|\beta_m(w_m)\|_{L^2} + \|f\|_{L^2})$ we have that $w_m \rightharpoonup w$ in $H^1(\Omega)$, and thus that w is the unique solution of (1.1). Applying this

$$\Delta w_m = \beta_m(w_m) - f \to \beta(w) - f = \Delta w \text{ in } L^p(\Omega \setminus \overline{U}). \quad (5.4)$$

Thus

$$\|w_m - w\|_{H^2(\Omega\setminus\overline{U})} \le C(\|\Delta(w_m - w)\|_{L^2(\Omega\setminus\overline{U})} + \|w_m - w\|_{L^2(\Omega\setminus\overline{U})}) \to 0. \quad (5.5)$$

Hence

$$w_m \to w \text{ in } H^2(\Omega \setminus \overline{U}).$$

In particular

$$\nabla w_m \to \nabla w \text{ in } H^{\frac{1}{2}}(\partial\Omega)^n.$$

Since $\beta_m' \in L^\infty(\mathbb{R})$ we take the "shape derivative" v_m solution of (2.5), which is well defined. Let us find their limit.

Let us show we show that

$$\beta_m'(w_m) \to \beta'(w) \text{ a.e. in } \Omega. \quad (5.6)$$

First, let $x \notin N_\Omega$. Then β is C^1 in $w(x)$. Therefore $\beta'(w_m(x)) \to \beta'(w(x))$. Hence, the sequence $\beta'(w_m(x))$ is bounded, so $\beta'(w_m(x)) \le m_0$ for some m_0 large. Thus $\beta_m'(w_m(x)) = \beta'(w_m(x))$ for $m \ge m_0$. Hence the convergence is proved for $x \notin N_\Omega$.

Let $x \in N_\Omega$. Then $\beta'(w(x)) = +\infty$. Since $w_m(x) \to w(x)$ then $\beta'(w_m(x)) \to +\infty$. In that case, we have that

$$\beta'_m(w_m(x)) = \beta(w_m(x)) \wedge m \to +\infty = \beta(w(x)).$$

This completes the proof of (5.6).
Let us show that sequence (v_m) is bounded in $H^1(\Omega)$. There exist two open sets $U_0, U_1 \subset \Omega$ such that $\partial\Omega \subset U_1, N_\Omega \subset U_0$, $U_0 \cap U_1 = \emptyset$. There also exists a smooth transition function Ψ such that $\Psi = 0$ in U_0 and $\Psi = 1$ in U_1. Let us define $g_m = \Psi \nabla w_m \cdot \theta \in H^1(\Omega)$. Then $\varphi = v_m + g_m \in H_0^1(\Omega)$ and it can be used as a test function in the weak formulation. Hence

$$\int_\Omega \nabla v_m \nabla (v_m + g_m) + \int_\Omega \beta'_m(w_m) v_m (v_m + g_m) = 0.$$

Therefore, through standard arguments

$$\begin{aligned}
\int_\Omega |\nabla v_m|^2 + \int_\Omega \beta'_m(w_m) v_m^2 &= -\int_\Omega \nabla v_m \nabla g_m - \int_\Omega \beta'_m(w_m) v_m g_m \\
&\leq \left(\int_\Omega |\nabla v_m|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla g_m|^2\right)^{\frac{1}{2}} \\
&\quad + \left(\int_\Omega \beta'_m(w_m) v_m^2\right)^{\frac{1}{2}} \left(\int_\Omega \beta'_m(w_m) g_m^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\left(\int_\Omega |\nabla v_m|^2 + \int_\Omega \beta'_m(w_m) v_m^2\right) \\
&\quad + C\left(\int_\Omega |\nabla g_m|^2 + \int_\Omega \beta'_m(w_m) g_m^2\right).
\end{aligned}$$

Since $\beta'_m(w_m)$ is uniformly bounded in $L^\infty(\Omega \backslash \overline{U_0})$ we have that the sequence is bounded:

$$\left(\int_\Omega |\nabla v_m|^2 + \int_\Omega \beta'_m(w_m) v_m^2\right) \leq C\left(\int_\Omega |\nabla g_m|^2 + \int_\Omega \beta'_m(w_m) g_m^2\right) \leq C.$$

In particular, there exists $v \in H^1(\Omega)$ such that, up to a subsequence,

$$v_m \rightharpoonup v \text{ in } H^1(\Omega).$$

Also, due to Fatou's lemma

$$\int_\Omega \beta'(w) v^2 \leq C. \tag{5.7}$$

Since $\beta'(w) = +\infty$ in N_Ω we have that $v = 0$ a.e. in N_Ω. For $\varphi \in W_c^{1,\infty}(\Omega \setminus N_\Omega)$ we have that

$$\int_{\Omega \backslash N_\Omega} \nabla v_m \nabla \varphi + \int_{\Omega \backslash N_\Omega} \beta'_m(w_m) v_m \varphi = 0. \tag{5.8}$$

Let us consider the compact subset $K = \text{supp}\varphi \subset \Omega \setminus N_\Omega$. Let us show that $\beta'(w_m) \to \beta'(w)$ in $L^2(K)$. We have $0 < \underline{w}_K \leq w \leq w_m$ in K. Due to the Dominated Convergence Theorem we have that $\beta'_m(w_m) \to \beta'(w)$ strongly in $L^p(K)$ for $1 \leq p < +\infty$.
Hence, by passing to the limit we deduce that

$$\int_{\Omega \backslash N_\Omega} \nabla v \nabla \varphi + \int_{\Omega \backslash N_\Omega} \beta'(w) v \varphi = 0. \tag{5.9}$$

This completes the proof. $\square$

6. Proof of Proposition 2.9

Let us consider $x_0 \in \partial N_\Omega$ and

$$W(t) = w_\Omega(x_0 + tn(x_0)) \tag{6.1}$$

where $n(x_0)$ represents the normal vector to ∂N_Ω at x_0. Due to Theorem 1.24 in [5], we have that

$$\frac{1}{2}|\nabla w_\Omega(x)|^2 \le \int_0^{w_\Omega(x)} \beta(s)ds + \alpha w_\Omega(x) \tag{6.2}$$

for all $x \in \overline{\Omega}$. Hence

$$\begin{aligned}\frac{dW}{dt} \le \left|\frac{dW}{dt}\right| &= |\nabla w_\Omega(x_0 + tn(x_0)) \cdot n(x_0)| \\ &\le |\nabla w_\Omega(x_0 + tn(x_0))| \le G(w_\Omega(x_0 + tn(x_0))) \\ &= G(W(t)).\end{aligned}$$

Thus, W is a solution of the following Ordinary Differential Inequality

$$\begin{cases}\frac{dW}{dt}(t) \le G(W(t)),\\ W(0) = 0.\end{cases} \tag{6.3}$$

Let us consider W_ε the solution of

$$\begin{cases}\frac{dW_\varepsilon}{dt}(t) = G(W_\varepsilon(t)),\\ v_\varepsilon(0) = \varepsilon.\end{cases} \tag{6.4}$$

This problem has a unique smooth solution, since $G \in \mathcal{C}^1(\mathbb{R} \setminus \{0\}) \cap \mathcal{C}(\mathbb{R})$ is strictly increasing and $G(0) = 0$. In fact, solving this simply separable O.D.E., we obtain that

$$W_\varepsilon(t) = \Psi^{-1}(t + \Psi(\varepsilon)). \tag{6.5}$$

Due to the monotonicity of G we have that

$$W(t) \le W_\varepsilon(t) \quad \forall t \ge 0. \tag{6.6}$$

Passing to the limit as $\varepsilon \to 0$ in (6.5) we have that

$$W(t) \le \Psi^{-1}(t). \tag{6.7}$$

Hence, since we can parametrize a neighbourhood of ∂N_Ω by $(x,t) \in \partial N_\Omega \times (-\lambda_0, \lambda_0) \mapsto x + tn(x)$, we deduce that

$$w(x) \le \Psi^{-1}(d(x, N_\Omega)) \tag{6.8}$$

at least in a neighbournood of ∂N_Ω. This proves the result. □

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David Gómez-Castro

Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Plaza de las Ciencias 3, 28040 Madrid, Spain

E-mail address: `dgcastro@ucm.es`

Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach

J.I. Díaz and D. Gómez-Castro

Instituto de Matematica Interdisciplinar & Dpto. de Matematica Aplicada,
Universidad Complutense de Madrid
Plaza de las Ciencias, 3, 28040 Madrid, Spain
jidiaz@ucm.es, dgcastro@ucm.es

Jean Michel Rakotoson[1]

Université de Poitiers
Laboratoire de Mathématiques et Applications - UMR CNRS 7348 - SP2MI, France
Bd Marie et Pierre Curie, Téléport 2, F-86962 Chasseneuil Futuroscope Cedex, France
Jean-Michel.Rakotoson@math.univ-poitiers.fr

Roger Temam

Institute for Scientific Computation and Applied Mathematics
Indiana University
Bloomington, Indiana 47405, U.S.A.
temam@indiana.edu

Abstract

In this paper we prove the existence and uniqueness of very weak solutions to linear diffusion equations involving a singular absorption potential and/or an unbounded convective flow on a bounded open set of $\mathbb{R}^N$. In most of the paper we consider homogeneous Dirichlet boundary conditions but we prove that when the potential function grows faster than the distance to the boundary to the power -2 then no boundary condition is required to get the uniqueness of very weak solutions. This result is new in the literature and must be distinguished from other previous results in which such uniqueness of solutions without any boundary condition was proved for degenerate diffusion operators (which is not our case). Our approach, based on the treatment on some distance to the boundary weighted spaces, uses a suitable regularity of the solution of the associated dual problem which is here established. We also consider the delicate question of the differentiability of the very weak solution and prove that some suitable additional hypothesis on the data is required since otherwise the gradient of the solution may not be integrable on the domain.

Keywords linear diffusion equations, singular absorption potential, unbounded convective flow, no boundary conditions, dual problem, local Kato inequality, distance to the boundary weighted spaces.

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[1]corresponding author

1 Introduction

In this paper we want to develop a weighted space approach to study the existence, uniqueness and regularity of linear diffusion equations involving singular and unbounded coefficients of the type

$$-\Delta\omega + \vec{u}\cdot\nabla\omega + V\omega = f \text{ on } \Omega, \tag{1}$$

where V is a very singular potential being in general non negative and locally integrable. To fix ideas, we shall consider mainly the case of Dirichlet boundary conditions

$$\omega = 0 \text{ on } \partial\Omega, \tag{2}$$

but our weighted space approach can also be adapted to the case of Neumann boundary conditions and, what is more remarkable, to the case of **no boundary conditions on** $\partial\Omega$ (but still getting the uniqueness of solutions) for some specially singular potentials (see the subsection 4.2 in section 4). Here Ω is an open bounded smooth (for instance with $\partial\Omega$ of class $C^{2,1}$) of $\mathbb{R}^N$, $N \geqslant 2$, (the case $N=1$ and u =constant is considerably simpler) . The external forcing term $f(x)$ will be assumed such that

$$f \in L^1(\Omega;\delta) \tag{3}$$

where the weight in this space is given by

$$\delta(x) = d(x,\partial\Omega) \tag{4}$$

(sharper results will require some slight restrictions to (3) (see for instance section 4.3). We recall that (3) is optimal in the cases $V \equiv 0$ and $\vec{u} = \vec{0}$ as it can be shown by explicitly computing the Green kernel for special domains.
Although we shall indicate later the detailed assumptions on the data, we anticipate now that we shall always assume that the convective flow vector $\vec{u}$ satisfies

$$\begin{cases} \vec{u} \in L^N(\Omega)^N, \ \operatorname{div}\vec{u} = 0 & \text{in} \quad \mathcal{D}'(\Omega) \text{ and} \\ \vec{u}\cdot\vec{n} = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

where $\vec{n}$ denotes the unit exterior normal vector to $\partial\Omega$. Notice that, due to (5), the weak solution notion adapted to equation (1) is equivalent to the one defined for the treatment of the equation in divergent form that is

$$-\Delta\omega + \operatorname{div}(\vec{u}\,\omega) + V\omega = f \quad \text{ in } \ \Omega. \tag{6}$$

It is well-known that the mathematical treatment of diffusion equations such as (1) $\big($or (6)$\big)$ leads to quite satisfactory results (in view of some applications) when the data f, $\vec{u}$ and V are assumed to be bounded. Nevertheless, the main interest of this work concerns the limit cases in which $V(x)$ is assumed to be a singular function (mainly with its singularity located on $\partial\Omega$) and/or when $\vec{u}$ is an unbounded vector (satisfying (5)). Let us indicate some relevant applications leading to the consideration of such limit cases :

1. *The vorticity equation in fluid mechanics.* Equation (1) can be derived from the stationary Navier-Stokes in 2D
$$-\Delta\vec{u} + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{F} \tag{7}$$
taking the curl of the equation and setting
$$f = \vec{F}\cdot\vec{k}, \qquad \omega = \operatorname{curl}\vec{u}\cdot\vec{k}, \tag{8}$$
where $\vec{k}$ is the last element of the canonical basis in $\mathbb{R}^3$ (see e.g. [46]). Nevertheless, as far as we know no satisfactory theory is available in the literature under the general condition that $\vec{F}\cdot\vec{k} \in L^1(\Omega;\delta)$.

2. *Schrödinger equation with singular potentials.* It is well-known that the consideration of the bound states $\psi(x,t) = e^{-iEt}\omega(x)$ leads to the stationary Schrödinger equation

$$-\Delta\omega + V(x)\omega = E\omega \qquad \text{in } \mathbb{R}^N. \tag{9}$$

The Heisenberg uncertainty principle makes specially interesting the consideration of potentials which are critically singular on $\partial\Omega$ more precisely, such that

$$V(x) \geqslant \frac{c}{\delta(x)^2}, \qquad a.e\ x \in \Omega, \tag{10}$$

for some $c > 0$, which implies that $\omega = \dfrac{\partial\omega}{\partial\vec{n}} = 0$ on $\partial\Omega$, so that we can assume that $\omega \equiv 0$ on $\mathbb{R}^N - \Omega$ (see [15, 16]). Here we shall not consider any eigenvalue problem like (9) but the study of (1) for potentials $V(x)$ satisfying (10) will be very useful for later works in this direction.

3. *Linearization of singular and/or degenerate nonlinear equations.* For many different purposes, it is very convenient to "approximate" the solutions of quasilinear diffusion equations of the type

$$-\Delta\varphi(w) + \operatorname{div}\big(\vec{\phi}(w)\big) + g(w) = f(x) \qquad \text{in } \Omega \tag{11}$$

by the solutions of the associated linearized equation. This is what appears, for instance, in the study of the stability of the associated parabolic or hyperbolic equations and also in some control problems associated with (11). Usually, it is assumed that φ is a strictly increasing function. So by considering $\theta := \varphi(w)$ we get

$$-\Delta\theta + \operatorname{div}\big(\vec{\psi}(\theta)\big) + h(\theta) = f(x) \qquad \text{in } \Omega, \tag{12}$$

with

$$\begin{cases} \vec{\psi} : \mathbb{R} \to \mathbb{R}^N, & \vec{\psi} = \vec{\phi} \circ \varphi^{-1}, \\ h = g \circ \varphi^{-1}. \end{cases} \tag{13}$$

Now, assume that $\theta_\infty(x)$ is a given solution of (12), satisfying, for instance, $\theta_\infty = 0$ on $\partial\Omega$. Then the "formal linearization" of equation (13) around the solution $\theta_\infty(x)$ coincides with equation (1) when we take

$$\vec{u}(x) := \vec{\psi}\big(\theta_\infty(x)\big)$$

and

$$V(x) = h'\big(\theta_\infty(x)\big).$$

What makes difficult the study of the corresponding problem (1) is the fact that in many cases relevant in the reaction-diffusion theory (see e.g. [26]) functions $\vec{\psi}'(r)$ and $h'(r)$ present a singularity at $r = 0$ and so, at least on $\partial\Omega$, the coefficients $\vec{u}$ and $\vec{V}$ are singular. A qualitative information on the behavior of $\theta_\infty(x)$ near $\partial\Omega$ allows us to get the precise information about the singularities of $\vec{u}$ and/or V near $\partial\Omega$ (which, for instance, is of the type (10)).

4. *Shape optimization in Chemical Engineering.* When dealing with the problem of shape optimization for chemical reactors and applying technics of shape differentiation, it was shown that if $g \in W^{2,\infty}(\mathbb{R})$, then the solutions u_0 of the problem

$$\begin{cases} -\Delta u + g(u) = f, & \Omega, \\ u = 1, & \partial\Omega, \end{cases} \tag{14}$$

are differentiable with respect to the domain in the sense of Hadamard [25] and after developed in Murat and Simon [32, 43] and the derivative u' in the direction of a deformation $\theta \in W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n)$ is the solution of the problem

$$\begin{cases} -\Delta u' + g'(u_0)u' = 0, \\ u' + \theta \cdot \nabla u \in H_0^1(\Omega). \end{cases} \tag{15}$$

Applying the theory developed for the general case (1), we can give a meaning to the shape derivative if the domain is not smooth as, for example, for root type kinetics (see [17, 24]). These nonlinear terms $g(u)$ are known in chemistry as Freundlich kinetics and have signifiant importance. Once again, taking $V(x) \equiv g'\big(u_0(x)\big)$ we arrive to problem (1).

Some previous papers dealing with data in $L^1(\Omega; \delta)$ and/or singular potentials (with usually $\vec{u} = \vec{0}$) are [20, 18, 37, 1, 29, 40, 6] (see also the references therein).

We also mention that sometimes it is possible to get conclusions for the stationary problem (1) (with $\vec{u} = \vec{0}$) through the consideration of the associated evolution equations (see e.g. [7], [8] and its references).

In this paper we shall work with the notion of "very weak solutions" (v.w.s.) of problem (1).

Definition 1.1. (Very weak solutions of problem (1)).
Let f be in $L^1(\Omega; \delta)$ and $\vec{u} \in L^{N,1}(\Omega)^N$ with $\operatorname{div}(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$, $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega$, V measurable and non negative function. A very weak solution ω of (1) is a function $\omega \in L^{N',\infty}(\Omega)$ satisfying

$$V\omega \in L^1(\Omega; \delta) \text{ and } \int_\Omega \omega\big[-\Delta\phi - \vec{u} \cdot \nabla\phi + V\phi\big]\,dx = \int_\Omega f\phi\,dx, \tag{16}$$

for all $\phi \in C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$, if $V \in L^1(\Omega; \delta)$, or for all $\phi \in C_c^2(\Omega)$ if $V \in L^1_{loc}(\Omega)$.

Notice that we look for a function in the space $L^{N',\infty}(\Omega)$ where $N' = \frac{N}{N-1}$ instead of $\omega \in L^1(\Omega)$ as usual, in order to get more general assumptions on $\vec{u}$ and V.

We also also point out that our study will be concentrated in the case of "absorption" potentials $V(x) \geqslant 0$ a.e. $x \in \Omega$. In fact, as we shall see later, the study is also applicable to some general potentials such that e.g. $V(x) \geqslant -\lambda$ with $0 < \lambda < \lambda_1$ (λ_1 being the first eigenvalue of the Laplacian on Ω with zero Dirichlet boundary condition). As we shall show, this does not induce a restriction on the growth of the singularity of such absorption potentials near $\partial\Omega$ (in contrast with the well-known results for **negative** potentials, see e.g. [7]).

The detailed definition of the Lorentz spaces $L^{p,q}(\Omega)$ and some other spaces which we shall use in our study will be the object of Section 2 of this paper. Other preliminary results and the statement of some of our main conclusions will be also presented there.

The proof of the existence and uniqueness of a very weak solution (v.w.s.) for (1) needs a deep study of the dual problem associated with (1)

$$\begin{cases} -\Delta\phi - \vec{u} \cdot \nabla\phi + V\phi = T & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

Notice the change of sign in the convection term. We anticipate that in some cases no boundary condition will be assumed on ϕ.

In Section 3, we discuss, depending on V and $\vec{u}$, the existence and the regularity of the solution of the dual problem. After this, we shall be concerned with the existence of the very weak solution in $L^{N',\infty}(\Omega)\cap L^1(\Omega;V\delta)$, when $V\geqslant 0$ is locally integrable. We will show that the very weak solution ω of equation (1) under zero Dirichlet or Neumann boundary condition has its gradient in the Sobolev-Lorentz weighted space $W^1L^{1+\frac{1}{N},\infty}(\Omega;\delta)$ in particular we shall get the estimate

$$\int_{\{x:|\nabla\omega|(x)<\lambda\}}\delta(x)dx\leqslant\frac{constant}{\lambda^{1+\frac{1}{N}}}\text{ for all }\lambda>0, \tag{18}$$

under the mere assumption $\vec{u}\in L^{N,1}(\Omega)^N$. Thus, we can conclude that $\nabla\omega\in L^1_{loc}(\Omega)$.

The question of uniqueness of v.w.s. given by (16), when V is only in $L^1_{loc}(\Omega)$ is one of the major difficulties in this general framework. When V is sufficiently integrable, say $V\in L^{N,1}(\Omega)$, then we derive the uniqueness thanks to the regularity of the dual problem. If V is only locally integrable, but V is bounded from below by $c\delta^{-r}$, $r>2$ near the boundary, then the v.w.s. is unique even when no boundary condition is specified on $\partial\Omega$ (but we additionally know that $V\omega\in L^1(\Omega;\delta)$).
The uniqueness proof relies on the $L^1(\Omega;\delta)$-accretiveness property of the operator (see [36]) $T\overline{\omega}=-\Delta\overline{\omega}+\operatorname{div}(\vec{u}\,\overline{\omega})$ when $\overline{\omega}\in L^1(\Omega;\delta^{-r})\cap W^{1,1}_{loc}(\Omega)$. This is given through the following local version of the Kato's type inequality

$$\int_\Omega\overline{\omega}_+T^*\psi\,dx\leqslant\int_\Omega\psi\operatorname{sign}_+(\overline{\omega})T\overline{\omega}\,dx,\text{ whenever }T\overline{\omega}\in L^1_{loc}(\Omega),\ \psi\in\mathcal{D}(\Omega), \tag{19}$$

and a special approximation of test function φ in $C^2(\overline{\Omega})$ by a sequence of functions of the type $\varphi_n(x)=\delta(x)^rh_n(x)$ with $h\in C^2_c(\Omega)$ and $r>0$ (see Lemma 4.4). We point out that, besides the concrete interest of (19) in itself; such an inequality has many consequences since it allows to apply the semigroup operators theory on suitable functional spaces.
Concerning very weak solutions (where no differentiability is asked to the function ω), a natural question (originally set by H. Brézis in 1972 when $\vec{u}=0$) is then : when should we have $|\nabla\omega|$ in $L^1(\Omega)$? The answer to this question will require some suitable additional integrability conditions on f and $\vec{u}$.

Note that for proving some additional integrability for the very weak solutions ω is a delicate task. Indeed, we shall show that for some special cases of $\vec{u}\in C^{0,\alpha}(\overline{\Omega})$, $\alpha>0$, there exists $f\in L^1_+(\Omega;\delta)$ such that $||\omega||_{L^{N'}}=+\infty$ when $N\geqslant 3$.This leads to an additional question : under what conditions could we improve the integrability of ω, to say $\omega\in L^{N'}(\Omega)$? The answer to this question is also one of the main results of this paper.

Before stating the study of the main equation (1), we shall recall some notations and functional spaces that we shall use.

2 Notations, preliminary definitions and results

Before stating our main results concerning equation (1) we need to recall some notations and some functional spaces which are relevant for the study of the "dual problem" (17) under very general regularity assumptions on the coefficients $\vec{u}$ and T.

Definition 2.1. ($\mathrm{bmo}(\mathbb{R}^N)$) [23].
A locally integrable function f on $\mathbb{R}^N$ is said to be in $\mathrm{bmo}(\mathbb{R}^N)$ if

$$\sup_{0<\mathrm{diam}\,(Q)<1} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \sup_{\mathrm{diam}\,(Q)\geqslant 1} \frac{1}{|Q|}\int_Q |f(x)|dx$$
$$\equiv ||f||_{\mathrm{bmo}(\mathbb{R}^N)} < +\infty,$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^N$ the sides of which are parallel to the coordinates axes.
Here $f_Q = \dfrac{1}{|Q|}\displaystyle\int_Q f(y)dy.$

Definition 2.2. ($\mathrm{bmo}_r(\Omega)$) [11, 12].
A locally integrable function f on a Lipschitz bounded domain Ω is said to be in $\mathrm{bmo}_r(\Omega)$ (r stands for restriction) if

$$\sup_{0<\mathrm{diam}\,(Q)<1} \frac{1}{|Q|}\int_Q |f(x)-f_Q|\,dx + \int_\Omega |f(x)|dx \equiv ||f||_{\mathrm{bmo}_r(\Omega)} < +\infty, \tag{20}$$

where the supremum is taken over all cube $Q \subset \Omega$ the sides of which are parallel to the coordinates axes.
In this case, there exists a function $\widetilde{f} \in \mathrm{bmo}(\mathbb{R}^N)$ such that

$$\widetilde{f}\Big|_\Omega = f \text{ and } ||\widetilde{f}||_{\mathrm{bmo}(\mathbb{R}^N)} \leqslant c_\Omega \cdot ||f||_{\mathrm{bmo}_r(\Omega)}. \tag{21}$$

Remark 1.
The above definition adapted to the case where the domain Ω is bounded, is equivalent to the definition given in [12, 11]. The main property (21) is due to P.W Jones [27].
This extension result implies that $\mathrm{bmo}_r(\Omega)$ embeds continuously into $L_{exp}(\Omega)$ (a space which we shall introduce below in Definition 2.5.)

Definition 2.3. (Campanato space $\mathcal{L}^{2,N}(\Omega)$.)
A function $u \in \mathcal{L}^{2,N}(\Omega)$ if

$$||u||_{L^2(\Omega)} + \sup_{x_0\in\Omega, r>0}\left[r^{-N}\int_{Q(x_0,r)\cap\Omega} |u-u_r|^2\,dx\right]^{\frac{1}{2}} := ||u||_{\mathcal{L}^{2,N}(\Omega)} < +\infty.$$

Here
$$u_r := \frac{1}{|Q(x_0;r)\cap\Omega|}\int_{Q(x_0;r)\cap\Omega} u(x)\,dx.$$

In fact the two above definitions are equivalent :

Theorem 2.1. [40]
For a Lipschitz bounded domain Ω one has

$$\mathcal{L}^{2,N}(\Omega) = \mathrm{bmo}_r(\Omega), \text{ with equivalent norms.}$$

We set
$$L^0(\Omega) = \Big\{v : \Omega \to \mathbb{R} \text{ Lebesgue measurable}\Big\}$$

and we denote by $L^p(\Omega)$ the usual Lebesgue space $1 \leqslant p \leqslant +\infty$. Although it is not too standard, we shall use the notation $W^{1,p}(\Omega) = W^1L^p(\Omega)$ for the associate Sobolev space. We shall need the following definitions:

Definition 2.4. (of the distribution function and monotone rearrangement.)
Let $u \in L^0(\Omega)$. The distribution function of u is the decreasing function

$$m = m_u : \mathbb{R} \mapsto [0, |\Omega|]$$

$$m_u = m_u(t) = \text{measure}\{x : u(x) > t\} = |\{u > t\}|.$$

The generalized inverse u_* of m is defined by

$$u_*(s) = \inf\Big\{t : |\{u > t\}| \leqslant s\Big\}, \quad s \in [0, |\Omega|[$$

and is called the decreasing rearrangement of u. We shall set $\Omega_* =]0, |\Omega|\,[$.

We recall now the following definitions :

Definition 2.5.
Let $1 \leqslant p \leqslant +\infty,\ 0 < q \leqslant +\infty$:
• If $q < +\infty$, one defines the following norm for $u \in L^0(\Omega)$

$$||u||_{p,q} = ||u||_{L^{p,q}} := \left[\int_{\Omega_*} \left[t^{\frac{1}{p}}|u|_{**}(t)\right]^q \frac{dt}{t}\right]^{\frac{1}{q}} \text{ where } |u|_{**}(t) = \frac{1}{t}\int_0^t |u|_*(\sigma)d\sigma.$$

• If $q = +\infty$,

$$||u||_{p,\infty} = \sup_{0<t\leqslant|\Omega|} t^{\frac{1}{p}}|u|_{**}(t).$$

The space $L^{p,q}(\Omega) = \Big\{u \in L^0(\Omega) : ||u||_{p,q} < +\infty\Big\}$ is called a **Lorentz space**.
• If $p = q = +\infty$, $\qquad L^{\infty,\infty}(\Omega) = L^\infty(\Omega)$.
The dual of $L^{1,1}(\Omega)$ is called $L_{exp}(\Omega)$

Remark 2. We recall that $L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega)$ for any $p > 1,\ q \geqslant 1$.

For $\alpha > 0$, we define

$$L^\alpha_{exp}(\Omega) = \left\{ v : \Omega \to \mathbb{R},\ \sup_{0<s<|\Omega|} \frac{|v|_*(s)}{\left(1 - \mathrm{Log}\,\dfrac{s}{|\Omega|}\right)^\alpha} < +\infty \right\},$$

$$L^p(\mathrm{Log}\,L)^\alpha = \left\{ f : \Omega \to \mathbb{R},\ \int_{\Omega_*} \left[\left(1 - \mathrm{Log}\,\frac{s}{|\Omega|}\right)^\alpha |f|_*(s)\right]^p ds < +\infty \right\}.$$

When there is no possible confusion, we denote by the same symbol the space product V^N and V.
We recall also that if $v,\ u \in L^1(\Omega)$, then

$$v_{*u} \doteq \lim_{\lambda \searrow 0} \frac{(u + \lambda v)_* - u_*}{\lambda}$$

exists in a weak sense and it is called the relative rearrangement of v with respect to u. More precisely, we have the following result (see [31, 35]).

Theorem 2.2.
Let Ω be a bounded measurable set in $\mathbb{R}^N$, u and v two functions in $L^1(\Omega)$ and let $w : \overline{\Omega}_* \to \mathbb{R}$ be defined by:

$$w(s) = \int_{\{u>u_*(s)\}} v(x)dx + \int_0^{s-|u>u_*(s)|} \left(v\Big|_{\{u=u_*(s)\}} \right)_* (\sigma)d\sigma,$$

where $v\Big|_{\{u=u_*(s)\}}$ is the restriction of v to $\{u = u_*(s)\}$.
Then

$$\frac{(u+\lambda v)_* - u_*}{\lambda} \underset{\lambda \to 0}{\rightharpoonup} \frac{dw}{ds} \text{ in } \begin{cases} L^p(\Omega_*)\text{-weak} & if\ v \in L^p(\Omega),\ 1 \leqslant p < +\infty \\ L^\infty(\Omega_*)\text{-weak-star} & if\ v \in L^\infty(\Omega) \end{cases}.$$

Moreover, $\left|\dfrac{dw}{ds}\right|_{L^p(\Omega_*)} \leqslant |v|_{L^p(\Omega)}$.

One property that we shall use for the relative rearrangement is the following one:

Proposition 1.
Let $v \geqslant 0$, and u be two functions in $L^1(\Omega)$. Then

$$(v_{*u})_{**} \leqslant v_{**}.$$

There is a link between the derivative of u_* and the relative rearrangement of the gradient of u as it was proved in [35, 41]. We will use only the following result (see [35])

Theorem 2.3.

(a) Let $u \in W_0^{1,1}(\Omega), u \geqslant 0$. Then

$$-u_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} |\nabla u|_{*u}(s) \quad \text{a.e in } \Omega_*,$$

and

$$-u_{**}'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}} (|\nabla u|_{*u})_{**}(s) \ a.e. \text{ in } \ \Omega_*.$$

(b) Let $u \in W^{1,1}(\Omega)$. Then if Ω is a Lipschitz connected open set of $\mathbb{R}^n$

$$-u_*'(s) \leqslant \frac{\min(s, |\Omega| - s)^{\frac{1}{N}-1}}{Q(\Omega)} |\nabla u|_{*u}(s),$$

where $Q(\Omega)$ is a suitable constant depending only on Ω.

Note that u_* is in $W_{loc}^{1,1}(\Omega_*)$ under statements (a) and (b) (see [35, 41]).

Let V be a Banach space contained in $L_{loc}^1(\Omega)$.The norm on V is denoted by $||\cdot||_V$ (or simply $||\cdot||$). We define the Sobolev space over V, for $m \in \mathbb{N}$ by

$$W^m V = \Big\{ v \in L_{loc}^1(\Omega) : D^\alpha v \in V \text{ for any } |\alpha| = \alpha_1 + \ldots + \alpha_N \leqslant m \Big\}.$$

In particular, $W_0^1 V = W^1 V \cap W_0^{1,1}(\Omega)$.

The following density result can be found in [22, 38, 40]:

Theorem 2.4. (Density)
Let Ω be a bounded set of class $C^{1,1}$. Then, the set $\{\varphi \in C^2(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\Big\}$ is dense in $\Big\{\varphi \in W^2L^{p,q}(\Omega) : \varphi = 0 \text{ on } \partial\Omega\Big\}$, $1 < p < +\infty$, $1 \leqslant q \leqslant +\infty$.

Remark 3.
Here and along the paper $\vec{u}$ is at least in $L^N(\Omega)^N$, $\operatorname{div}(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$ and $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega$, if $N \geqslant 3$ and $\vec{u} \in L^{2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$ if $N = 2$. The value of $\vec{u} \cdot \vec{n}$ on $\partial\Omega$ is defined through the Green's formula (see [46]).

The following density result can be proved using the same argument as for the L^p-case (see [46, 13])

Proposition 2. (Density of smooth functions).
Let $1 < p < +\infty$ and $1 \leqslant q \leqslant \infty$. Then the closure of the set

$$\mathcal{V} = \Big\{\vec{u} \in C_c^\infty(\Omega)^N : \operatorname{div}(\vec{u}) = 0 \text{ in } \Omega\Big\}$$

in $L^{p,q}(\Omega)^N$ (resp. $(L^N(\operatorname{Log} L)^\alpha)^N$, $\alpha > 0$) is the space

$$\overline{\mathcal{V}} := \Big\{\vec{u} \in L^{p,q}(\Omega)^N \text{ (resp. } (L^N(\operatorname{Log} L)^\alpha)^N,\ \alpha > 0\text{) : } \operatorname{div}(\vec{u}) = 0 \text{ and } \vec{u} \cdot \vec{n} = 0 \text{ on } \partial\Omega\Big\}.$$

Due to Proposition 2, a standard approximation argument leads to :

Lemma 2.6.
For all Lipschitz mappings $G : \mathbb{R} \to \mathbb{R}$, and for all $\phi \in W_0^1L^{N'}(\Omega)$ with $N' = \dfrac{N}{N-1}$, one has

$$\int_\Omega (\vec{u} \cdot \nabla\phi)\, G(\phi)\, dx = 0.$$

Lemma 2.7.
For all $\varpi \in H_0^1(\Omega)$, and for all $\phi \in H_0^1(\Omega)$

$$\int_\Omega (\vec{u} \cdot \nabla\varpi)\, \phi\, dx = -\int_\Omega \vec{u} \cdot \nabla\phi\, \varpi\, dx.$$

Let us remark that,
- if $N \geqslant 3$
$$\left|\int_\Omega \vec{u} \cdot \nabla\varpi\, \phi\, dx\right| \leqslant ||\vec{u}||_{L^N} ||\nabla\varpi||_{L^2} ||\phi||_{L^{2^*}} \text{ where } \frac{1}{2^*} + \frac{1}{2} + \frac{1}{N} = 1, \tag{22}$$
- if $N = 2$ the above inequality holds true after replacing N by $2+\varepsilon$ and 2^* by $\dfrac{2(2+\varepsilon)}{\varepsilon}$.

We shall need the following classical result (see [28]) :

Lemma 2.8.
Let $X \hookrightarrow_c Y \hookrightarrow Z$ be three Banach spaces each continuously embedded in the next one, the first inclusion is supposed to be compact. Then, for all $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $\forall\, \phi \in X$
$$||\phi||_Y \leqslant \varepsilon ||\phi||_X + c_\varepsilon ||\phi||_Z.$$

3 Existence, uniqueness, regularity and results for the dual problem

3.1 Case where the potential V is only measurable and bounded from below

We first study the solvability of the dual problem (17) (equivalent to (23) below and the regularity of its solutions.
The following result, consequence of the Lax-Milgram theorem, is a remarkable fact due to the low regularity assumed on the data $\vec{u}$ and V:

Proposition 3.
Let $T \in H^{-1}(\Omega)$ (dual space of $H_0^1(\Omega)$), $\vec{u}$ satisfying (5) and let $V \in L^0(\Omega)$ satisfying $V \geqslant -\lambda$ for some $\lambda \in [0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta$ under the zero Dirichlet boundary condition. Define $W = \Big\{\varphi \in H_0^1(\Omega) : (V+\lambda)\varphi^2 \in L^1(\Omega)\Big\}$, and let W' denotes its dual.
Then, there exists a unique $\phi \in H_0^1(\Omega)$, with $(V+\lambda)\phi^2 \in L^1(\Omega)$, such that

$$(\mathcal{P})_{V,T} \qquad -\Delta\phi - \vec{u}\cdot\nabla\phi + V\phi = T \text{ in } W'. \tag{23}$$

Moreover,

$$||\phi||_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla\phi|^2 dx\right)^{\frac{1}{2}} \leqslant \frac{\lambda_1}{\lambda_1-\lambda}||T||_{H^{-1}(\Omega)},$$

$$\left(\int_\Omega (V+\lambda)\phi^2 dx\right)^{\frac{1}{2}} \leqslant \left(\frac{\lambda_1}{\lambda_1-\lambda}\right)^{\frac{1}{2}} ||T||_{H^{-1}(\Omega)},$$

$$V\phi \in L^1_{loc}(\Omega).$$

If furthermore $V \in L^1_{loc}(\Omega)$, then the equation (23) holds in the sense of distributions in $\mathcal{D}'(\Omega)$

Proof. We endow W with the following norm

$$[\varphi]_W^2 = ||\varphi||^2_{H_0^1(\Omega)} + \int_\Omega (V+\lambda)\varphi^2 dx.$$

Let us consider the bilinear form on W given by

$$\begin{aligned} a(\psi,\varphi) &= \int_\Omega \nabla\psi\cdot\nabla\varphi dx - \int_\Omega \vec{u}\cdot\nabla\psi\varphi dx + \int_\Omega (V+\lambda)\psi\varphi dx \\ &\quad -\lambda\int_\Omega \psi\varphi dx, \quad (\psi,\varphi) \in W^2. \end{aligned}$$

Then, by Lemmas 2.6 and 2.7

$$a(\psi,\psi) = \int_\Omega |\nabla\psi|^2 - \lambda\int_\Omega \psi^2 dx + \int_\Omega (V+\lambda)\psi^2 dx \geqslant \alpha_0\left[\int_\Omega (V+\lambda)\psi^2 + \int_\Omega |\nabla\psi|^2\right], \tag{24}$$

with $\alpha_0 > 0$.

According to the above remark (22), since $\vec{u} \in L^N(\Omega)^N$, the bilinear form is continuous on W and we have

$$|a(\psi,\varphi)| \leqslant M[\psi]_W[\varphi]_W \ ,$$

with $M = 3(1 + ||\vec{u}||_{L^N})$. Moreover, since $W \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow W'$ we have

$$\langle T, \psi \rangle_{H^{-1}H_0^1} \leqslant ||T||_{H^{-1}}[\psi]_W, \ \forall \psi \in W.$$

Thus we may apply the Lax-Milgram theorem to derive the existence of a unique $\phi \in W$, such $a(\phi, \psi) = \langle T, \psi \rangle_{H^{-1}H_0^1} \ \forall \psi \in W$. The estimate on ϕ follows from (24).
If $V \in L^1_{loc}(\Omega)$ then one has

$$\mathcal{D}(\Omega) \subset W.$$

Moreover, since $\displaystyle\int_\Omega (V + \lambda)\phi^2 dx$ is finite, the Cauchy-Schwarz inequality yields

$$0 \leqslant \int_{\Omega'} (V + \lambda)|\phi| dx \leqslant \left(\int_\Omega (V + \lambda)\phi^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega'} (V + \lambda) dx \right)^{\frac{1}{2}} < +\infty \tag{25}$$

for any open set Ω' relatively compact in Ω.
Writing

$$\int_{\Omega'} |V\phi| dx \leqslant \int_{\Omega'} (V + \lambda)|\phi| dx + \lambda \int_\Omega |\phi| dx,$$

the right hand is finite taking into account (25) and the fact that $\phi \in L^2(\Omega)$. Thus, we have $\forall \Omega' \subset\subset \Omega, \ V\phi \in L^1(\Omega')$. We conclude that $V\phi \in L^1_{loc}(\Omega)$. □

As usual in some problems of Quantum Mechanics (see e.g. Lemma 2.1 of [15]) it is very useful to approximate the solution $\phi \in H_0^1(\Omega)$ of the dual problem (23) found in Proposition 3 by a sequence of solutions ϕ_k corresponding to a sequence of bounded potentials V_k approximating V. Let us define V_k by

$$V_k = \min(V, k).$$

Proposition 4. (**Approximation by bounded potentials).**
Let $T \in H^{-1}(\Omega)$, $\vec{u}$ and V as in Proposition 3. Then, the sequence $\phi_k \in H_0^1(\Omega)$ of solutions of the problems

$$(\mathcal{P})_{V_k,T} : \int_\Omega \nabla\phi_k \cdot \nabla\psi dx - \int_\Omega \vec{u}\nabla\phi_k \phi dx + \int_\Omega V_k \phi_k \psi dx = \langle T, \psi \rangle, \quad \forall \psi \in H_0^1(\Omega),$$

converges to ϕ strongly in $H_0^1(\Omega)$, where ϕ is the unique solution of $(\mathcal{P})_{V,T}$ found in Proposition 3.

Sketch of the proof of Proposition 4. One has, following the arguments of the Proposition 3, that

$$||\phi_k||_{H_0^1} + \left(\int_\Omega (V_k + \lambda)\phi_k^2 dx \right)^{\frac{1}{2}} \leqslant 2 \left(\frac{\lambda_1}{\lambda_1 - \lambda} \right) ||T||_{H^{-1}(\Omega)}. \tag{26}$$

Thus, ϕ_k remains in a bounded set of $H_0^1(\Omega)$. So we may assume that it converges to a function φ weakly in $H_0^1(\Omega)$ and a.e. in Ω. The above relation (26) implies that:

$$\left(\int_\Omega (V + \lambda)\varphi^2 dx \right)^{\frac{1}{2}} + ||\varphi||_{H_0^1} \leqslant 2 \left(\frac{\lambda_1}{\lambda_1 - \lambda} \right) ||T||_{H^{-1}(\Omega)}. \tag{27}$$

This shows that $\varphi \in W$ (where W is the space defined in the proof of Proposition 3). Moreover, since for all $\psi \in W$ we have $\vec{u}\psi \in L^{2^{*'}}(\Omega)$ (see the above remark), we deduce

$$\lim_{k \to +\infty} \int_\Omega \vec{u} \cdot \nabla\phi_k \psi dx = \int_\Omega \vec{u} \cdot \nabla\varphi\psi dx. \tag{28}$$

The sequence $(V_k+\lambda)\phi_k\psi$ satisfies Vitali's condition, since for any measurable subset $B\subset\Omega$, we have

$$\left|\int_B (V_k+\lambda)\phi_k\psi dx\right| \leqslant 2\left(\frac{\lambda_1}{\lambda_1-\lambda}\right)||T||_{H^{-1}(\Omega)}\left(\int_B (V+\lambda)\psi^2 dx\right)^{\frac{1}{2}} \tag{29}$$

and

$$\lim_{k\to+\infty}(V_k+\lambda)(x)\phi_k(x)\psi(x)=(V+\lambda)(x)\varphi(x)\psi(x). \tag{30}$$

Thus

$$\lim_{k\to+\infty}\int_\Omega (V_k+\lambda)\phi_k\psi dx=\int_\Omega (V+\lambda)\varphi\,\psi dx. \tag{31}$$

We then deduce that φ is solution of the problem $(\mathcal{P})_{V,T}$ and by uniqueness $\varphi=\phi$. Therefore, the whole sequence ϕ_k converges to ϕ weakly in W and strongly in $L^2(\Omega)$.

To prove the strong convergence in $H^1_0(\Omega)$, let us note, using the equations $(\mathcal{P})_{V_k,T}$ and $(\mathcal{P})_{V,T}$, that

$$\lim_{k\to+\infty}\int_\Omega |\nabla\phi_k|^2 dx+\int_\Omega (V_k+\lambda)\phi_k^2 dx=\lambda\int_\Omega \phi^2 dx+\langle T,\phi\rangle=\int_\Omega (V+\lambda)\phi^2+\int_\Omega |\nabla\phi|^2 dx.$$

Therefore, if we introduce $U_k=(\nabla\phi_k;\phi_k\sqrt{V_k+\lambda})\in L^2(\Omega)^{N+1}$, $U_\infty=(\nabla\phi;\phi\sqrt{V+\lambda})$ we have

- $\lim_{k\to+\infty}|U_k|^2_{L^2(\Omega)^{N+1}}=|U_\infty|^2_{L^2(\Omega)^{N+1}}$,
- U_k converges to U_∞ weakly in $L^2(\Omega)^{N+1}$.

Thus U_k converges to U_∞ strongly in $L^2(\Omega)^{N+1}$. □

Remark 4.
Let us notice that for $\phi\in L^2(\Omega)$ the conditions $(V+\lambda)\phi^2\in L^1(\Omega)$ and $|V|\phi^2\in L^1(\Omega)$, $\phi\in L^2(\Omega)$ are equivalent. Indeed, since $V+\lambda=|V+\lambda|$,

$$\int_\Omega |V|\phi^2 dx\leqslant\int_\Omega (V+\lambda)\phi^2 dx+\lambda\int_\Omega\phi^2\leqslant\int_\Omega |V|\phi^2 dx+2\lambda\int_\Omega \phi^2 dx.$$

For this reason, from now, **we will assume that** $\lambda=0$.

Proposition 5.
Under the same assumptions as for Proposition 3 (with $\lambda=0$), if $T\geqslant 0,\ T\in L^1(\Omega)\cap H^{-1}(\Omega)$ then $\phi\geqslant 0$.

Proof. We have $\phi_-\in W$ and

$$0\geqslant-\int_\Omega |\nabla\phi_-|dx-\int_\Omega V\phi_- dx=\int_\Omega T\phi_- dx\geqslant 0.$$

Thus

$$\phi_-=0.$$

□

For the treatment of (1) we shall need some additional regularity for the solutions of the dual problem (23) independent of $\vec{u}$ or V. We start by proving the boundedness of ϕ by means of some rearrangement technics ([35] p.126 of Th 5.5.1, see also [45]).

We point out that $L^{\frac{N}{2},1}(\Omega)\hookrightarrow H^{-1}(\Omega)$.

Proposition 6. **(L^∞-estimates).**
Let ϕ be the solution of (23) when $T \in L^{\frac{N}{2},1}(\Omega)$, $V \geqslant 0$. Then $\phi \in L^\infty(\Omega)$ and there exists a constant $K_N(\Omega)$ independent of $\vec{u}$ and V such that

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||T||_{L^{\frac{N}{2},1}(\Omega)}.$$

Proof. We shall argue in a way similar to the proof of Theorem 5.3.1 in [35]. According to Proposition 4 , it is enough to prove the proposition for $V \in L^\infty_+(\Omega)$, and and for $T \geqslant 0$, since the equation (23) is linear. Thus $\phi \geqslant 0$, therefore, in this proof $v = |\phi| = \phi$, but we shall keep the notation v because in the general case we cannot use anymore this maximum principle. Let $v = |\phi|$, $G_s(\sigma) = (\sigma - v_*(s))_+ \operatorname{sign}(\sigma)$, $\sigma \in \mathbb{R}$, $s \in \Omega_*$. The mapping $\sigma \mapsto G_s(\sigma)$ is Lipschitz. Then following Lemma 2.6

$$\int_\Omega (\vec{u} \cdot \nabla\phi) G_s(\phi)\, dx = 0.$$

Therefore, we derive

$$\int_\Omega \nabla\phi \cdot \nabla G_s(\phi) = \int_{v>v_*(s)} |\nabla\phi|^2 dx = \int_\Omega T(x)G_s(\phi)(x)dx - \int_\Omega V(x)G_s(\phi)dx.$$

Differentiating this relation with respect to s, we find

$$\frac{d}{ds}\int_{v>v_*(s)} |\nabla\phi|^2 dx = -v'_*(s)\int_{v>v_*(s)} \big(T(x) - V(x)\big)dx \leqslant -v'_*(s)\int_0^s T_*(\sigma)d\sigma$$

where T_* is the monotone rearrangement of T (we use the fact that $V \geqslant 0$).
Therefore, we arrive at

$$\big[\,|\nabla\phi|^2\big]_{*v}(s) \leqslant -v'_*(s)\int_0^s T_*(\sigma)d\sigma. \tag{32}$$

Since

$$|\nabla\phi| = |\nabla v|, \text{ and } \quad -v'_*(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}|\nabla v|_{*s}(s)$$

$\big($the PSR property (see Theorem 3 of [35])$\big)$ and $|\nabla v|_{*v} \leqslant \big[\,|\nabla v|^2\big]_{*v}^{\frac{1}{2}}$, we infer from (32)

$$-v'_*(s) \leqslant \frac{s^{\frac{2}{N}-2}}{(N\alpha_N^{\frac{1}{N}})^2}\int_0^s T_*(\sigma)d\sigma. \tag{33}$$

Thus, integrating (33) between 0 to $|\Omega|$, we find

$$||\phi||_{L^\infty} \leqslant c_N \int_0^{|\Omega|} s^{\frac{2}{N}} T_{**}(s)\frac{ds}{s} \equiv c_N||T||_{L^{\frac{N}{2},1}(\Omega)}.$$

□

An analogous result can be obtained when $T = -\operatorname{div}(\vec{F})$, with $\vec{F} \in L^{N,1}(\Omega)^N$.

Proposition 7.
Let $N \geqslant 2$, and let ϕ be a solution of (23) when $T = -\operatorname{div}(\vec{F})$, $\vec{F} \in L^{N,1}(\Omega)^N$ if $N \geqslant 3$, $\vec{F} \in L^{2+\varepsilon}(\Omega)^2$ if $N = 2$.
Then $\phi \in L^\infty(\Omega)$ and there exists a constant $K_N(\Omega) > 0$ independent of $\vec{u}$ and V such that

$$||\phi||_{L^\infty(\Omega)} \leqslant K_N(\Omega)||\vec{F}||_{L_V} \text{ with } L_V = L^{N,1}(\Omega)^N \text{ if } N \geqslant 3,\ L^{2+\varepsilon}(\Omega)^2 \text{ if } N = 2.$$

Proof. For convenience, we write F for $\vec{F}$. Thanks to Proposition 4, we can use the same test function $G_s(\phi)$ as in the proof of Proposition 6. Then

$$\int_\Omega \nabla\phi \cdot \nabla G_s(\phi)dx + \int_\Omega V(x)G_s(\phi)dx = \int_\Omega F \cdot \nabla G_s(\phi)dx.$$

We differentiate this equation with respect to s as before, for a.e. $s \in \Omega_*$, and find

$$\big[\,|\nabla v|^2\big]_{*v}(s) - v_*'(s)\int_{v>v_*(s)} V(x)dx = \big[\,F \cdot \nabla\phi\,\big]_{*v}(s). \tag{34}$$

Since, $V \geqslant 0$ and $v_*'(s) \leqslant 0$, we obtain

$$\big[\,|\nabla v|^2\big]_{*v}(s) \leqslant \big[\,|\,F\,|^2\big]_{*v}^{\frac{1}{2}}\big[\,|\nabla v|_{*v}^2\big]^{\frac{1}{2}}(s), \tag{35}$$

$$\big[\,|\nabla v|^2\big]_{*v}^{\frac{1}{2}}(s) \leqslant \big[\,|\,F\,|^2\big]_{*v}^{\frac{1}{2}}(s). \tag{36}$$

We have as before:

$$-\,v_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}\big[\,|\nabla v\,|^2\big]_{*v}^{\frac{1}{2}}(s). \tag{37}$$

We infer that for a.e. s

$$-\,v_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}\big[\,|\,F\,|^2\big]_{*v}^{\frac{1}{2}}. \tag{38}$$

Integrating this relation between 0 and $|\Omega|$ and using the Hardy-Littlewood inequality (see [35] p.118-121) we obtain

$$||\phi||_{L^\infty} \leqslant \begin{cases} c_N \displaystyle\int_{\Omega_*} \sigma^{\frac{1}{N}-1}\big(\,|\,F\,|^2\big)_{**}^{\frac{1}{2}}(\sigma)d\sigma, & \text{if } N \geqslant 3,\\ c_{2,\varepsilon}\|F\,\|_{L^{2+\varepsilon}(\Omega)^2}, & \text{if } N = 2.\end{cases}$$

We conclude as in [35] p. 118-120, Proposition 5.2.2. □

Remark 5. The problem considered in this Section 3.1 was previously considered by other authors in the special case of $\overrightarrow{u} \equiv \overrightarrow{0}$ (see, e.g. [14] and its references), nevertheless we emphasize that the results of this section must be understood as preliminary results with respect the study we shall present in the following sections of this paper. In particular, what is specially important for us is to obtain a continuous dependence estimate with respect to the data (namely the velocity $\overrightarrow{u}$, the potential V, and the right hand side f) since we need to carry out several perturbations of those data in the next sections. As far as we know, such estimates are new in the literature (and, of course, they were not given in the above mentioned reference).

3.2 Some regularity results with an integrable potential V and bounded from below

As a first consequence of Proposition 3 and Proposition 7 we can deduce Meyer's type regularity giving a better information on the gradient of the solution of (23).

Proposition 8. **($W^1L^{p,q}$-estimate)**
Let $N \geqslant 2$. Assume that there exists $p > N$ and $q \in [1, +\infty]$, such that

$$\begin{cases} \vec{u} \in L^{p,q}(\Omega)^N & V \geqslant 0,\ V \in L^{r,q}(\Omega),\ r = \dfrac{Np}{N+p}, \\ T = -\mathrm{div}\,(\vec{F}) & \text{with } \vec{F} \in L^{p,q}(\Omega)^N. \end{cases}$$

Then, the unique solution ϕ of the equation (23) belongs to $W^1L^{p,q}(\Omega)$. Moreover, there exists a constant $K_{pq} > 0$ independent of $\vec{u}$ such that :

$$||\nabla\phi||_{L^{p,q}(\Omega)} \leqslant K_{pq}\,(1 + ||\vec{u}||_{L^{p,q}} + ||V||_{L^{r,q}})\,||F||_{L^{p,q}(\Omega)^N}.$$

Proof. (We shall simply write $F,\ F_0,\ F_1$ for $\vec{F},\ \vec{F}_0,\ \vec{F}_1$). We first assume that $\vec{u} \in \mathcal{V}$. We know from Proposition 7 that $\phi \in L^\infty(\Omega)$ and that there exists a constant independent of $\vec{u}$, V and $\vec{F}$ and V such that

$$||\phi||_\infty \leqslant K_N(\Omega)||F||_{L^{p,q}(\Omega)}. \tag{39}$$

Therefore, there exists a vector field $F_0 \in L^{p,q}(\Omega)^N$ such that

$$V\phi = -\mathrm{div}\,(F_0) \text{ and } ||F_0||_{L^{p,q}} \leqslant K_{1,N}(\Omega)||V||_{L^{r,q}}||\phi||_\infty,$$

that is

$$||F_0||_{L^{p,q}} \leqslant K_{1N}(\Omega)||V||_{L^{r,q}}||F||_{L^{p,q}(\Omega)}.$$

Setting $F_1 = F - F_0$, we can write (23) as

$$-\Delta\phi = -\mathrm{div}\,(F_1 - \vec{u}\phi). \tag{40}$$

But, we have $\vec{u}\phi \in L^{p,q}(\Omega)^N$ since $\phi \in L^\infty(\Omega)$ according to the above Proposition 7. Hence

$$||\vec{u}\phi||_{L^{p,q}(\Omega)^N} \leqslant ||\vec{u}||_{L^{p,q}}||\phi||_{L^\infty} \leqslant K_N||F||_{L^{p,q}}||\vec{u}||_{L^{p,q}}.$$

We may apply the $W^1L^{p,q}$ result to (40) (see [42, 9, 2, 36]) to deduce that

$$||\nabla\phi||_{L^{p,q}} \leqslant K_p||F_1 - \vec{u}\phi||_{L^{p,q}} \leqslant K_{pNq}(1 + ||\vec{u}||_{L^{p,q}} + ||V||_{L^{r,q}})||F||_{L^{p,q}}. \tag{41}$$

For the general case, we consider $u_k \in \mathcal{V}$ such that $u_k \to u$ strongly in $L^{p,q}(\Omega)^N$. Let ϕ_k be the solution of equation (23) where ϕ is replaced by ϕ_k

$$-\Delta\phi_k - \vec{u}_k\nabla\phi_k + V\phi_k = T = -\mathrm{div}\,(F).$$

The sequence $(\phi_k)_k$ satisfies

$$||\phi_k||_{L^\infty} \leqslant K_N||F||_{L^{r,q}} \text{ and } ||\phi_k||_{H_0^1} \leqslant ||T||_{H^{-1}},$$

and then $(\phi_k)_k$ converges weakly in $H_0^1(\Omega)$ to ϕ the solution of (23). Since ϕ_k satisfies (41), we deduce that ϕ also satisfies (41) and (23). □

As an immediate consequence of the above result.

Proposition 9.
Let $\vec{u}$ and $\vec{F}$ be in $L^{p,\infty}(\Omega)^N$ for some $p > N$. Then, the solution of (23) satisfies

$$\phi \in C^{0,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{p}.$$

Proof. According to the Sobolev embedding (see [35]), we have

$$W^1L^{p,\infty}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}), \text{ with } \alpha = 1 - \frac{N}{p}.$$

□

Now we shall consider the case of more general data $\vec{u}$ and V.

Proposition 10.
Assume that $\vec{u}$ and $\vec{F}$ are in $\mathrm{bmo}_r(\Omega)^N$ and V is in $\mathrm{bmo}_r(V)$. Then the solution ϕ of the equation (23) satisfies

1. $\vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$
2. $\nabla\phi \in \mathrm{bmo}_r(\Omega)^N$.

Proof. Since $\mathrm{bmo}_r(\Omega) \hookrightarrow L^{p,q}(\Omega)$ for all $p > N$ and $q \in [1, +\infty]$, we deduce from Proposition 8 and Proposition 9 that :

$$\phi \in C^{0,\alpha}(\overline{\Omega}) \quad \forall\, \alpha \in\ [0, 1\,[\text{ and } -\Delta\phi = -\mathrm{div}\,(\vec{F_1} - \vec{u}\phi),$$

where $\vec{F_1}$ was defined in the proof of Proposition 8 (see equation (40)). From Stegenga multiplier's result, $\vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$ whenever $\vec{u}$ is in $\mathrm{bmo}_r(\Omega)^N$ [44, 47]. Therefore $\vec{F_1} - \vec{u}\phi \in \mathrm{bmo}_r(\Omega)^N$. We may appeal to Campanato's result [10] to derive then that $\nabla\phi \in \mathrm{bmo}_r(\Omega)^N$ and

$$||\nabla\phi||_{\mathrm{bmo}_r} \leqslant K\Big(||F||_{\mathrm{bmo}_r} + ||\vec{u}\,\phi||_{\mathrm{bmo}_r} + ||F_0||_{\mathrm{bmo}_r}\Big).$$

□

We shall end this paragraph by proving a $W^2L^{p,q}(\Omega)$-regularity result for the solutions of the dual problem (23) which will lead to interesting conclusions for the direct problem (1).
For this, we shall use the following ADN constant

$$K^s_{pq} = \sup_{v \in H^1_0(\Omega)\cap W^2L^{p,q}(\Omega)} \frac{||v||_{W^2L^{p,q}(\Omega)}}{||v||_{L^{p,q}(\Omega)} + ||\Delta v||_{L^{p,q}(\Omega)}}, \tag{42}$$

which is finite due to the well-known Agmon-Douglis-Nirenberg's regularity result combined with the Marcinkiewicz interpolation Theorem.
We shall improve now the regularity obtained in Proposition 10. We consider $\varepsilon_0 > 0$ (fixed) so that $K^s_{pq}\varepsilon_0||\vec{u}||_{L^{p,q}(\Omega)} \leqslant \frac{1}{2}$.

Proposition 11. ($W^2L^{p,q}(\Omega)$ regularity for $p > N$)
Let ϕ be the solution of (23) when $T \in L^{p,q}(\Omega),\ p > N,\ q \in [1, +\infty]$. Assume, furthermore, that $\vec{u} \in L^{p,q}(\Omega)^N$ and $V \in L^{p,q}(\Omega)$. Then

$$\phi \in W^2L^{p,q}(\Omega).$$

Moreover, there exist constants $c_{\varepsilon_0},\ K_{pqN} > 0$ such that

$$||\phi||_{W^2L^{p,q}(\Omega)} \leqslant \frac{K_{pqN}c_{\varepsilon_0}(1 + ||V||_{L^{p,q}} + ||\vec{u}||_{L^{p,q}(\Omega)})}{1 - K^s_{pq}\varepsilon_0||\vec{u}||_{L^{p,q}(\Omega)}}||T||_{L^{p,q}(\Omega)}.$$

Proof. We assume first that $\vec{u} \in \mathcal{V}$. Arguing as in Proposition 8, since we can assume that $T = \operatorname{div} \vec{F}$ for suitable $\vec{F}$ we get that the solution ϕ of (23) is in $W^1L^{p,q}(\Omega)$ and then

$$-\Delta\phi = \vec{u}\,\nabla\phi + T - V\phi \in L^{p,q}(\Omega).$$

By the Agmon-Douglis-Nirenberg regularity results and the Marcinkiewicz interpolation theorem we deduce that $\phi \in W^2L^{p,q}(\Omega)$. Moreover, since $p > N$ and $q \in [1, +\infty]$, we have the following continuous embeddings :

$$W^2L^{p,q}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \hookrightarrow L^{p,q}(\Omega).$$

The first inclusion is compact so we may appeal to Lemma 2.8 to derive that $\forall\, \varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$||\nabla\,\phi||_\infty \leqslant \varepsilon||\phi||_{W^2L^{p,q}(\Omega)} + c_\varepsilon||\phi||_{L^{p,q}(\Omega)}. \tag{43}$$

From the equation satisfied by ϕ, we have

$$||\Delta\phi||_{L^{p,q}(\Omega)} \leqslant ||\vec{u}||_{L^{p,q}(\Omega)}||\nabla\phi||_\infty + ||T||_{L^{p,q}(\Omega)} + ||V||_{L^{p,q}}||\phi||_\infty, \tag{44}$$

and using the ADN constant

$$||\phi||_{W^2L^{p,q}(\Omega)} \leqslant K^s_{pq}\Big(||\phi||_{L^{p,q}(\Omega)} + ||\Delta\phi||_{L^{p,q}(\Omega)}\Big). \tag{45}$$

We combine those last three equations and derive that for any $\varepsilon > 0$

$$\begin{aligned} ||\phi||_{W^2L^{p,q}(\Omega)}(1 - \varepsilon K^s_{pq}||\vec{u}||_{L^{p,q}(\Omega)}) \quad &\leqslant \quad K^s_{pq}||\phi||_{L^{p,q}(\Omega)}\Big(1 + c_\varepsilon||\vec{u}||_{L^{p,q}(\Omega)}\Big) \\ &\quad + K^s_{pq}||T||_{L^{p,q}(\Omega)}(1 + ||V||_{L^{p,q}})K_{2N}. \end{aligned} \tag{46}$$

Next, we consider $\vec{u}_k \in \mathcal{V}$ such that $\vec{u_k} \to \vec{u} \in \overline{\mathcal{V}}$. Then, choosing $\varepsilon = \varepsilon_0 > 0$ such that $\varepsilon_0 K^s_{pq} \sup\limits_k ||\vec{u_k}||_{L^{p,q}(\Omega)} \leqslant \dfrac{1}{2}$, we deduce from relation (46) that ϕ_k corresponding to the solution of (23) , that is $-\Delta\phi_k - \vec{u_k}\cdot\nabla\phi_k + V\phi_k = T \in L^{p,q}(\Omega)$, belongs to a bounded set of $W^2L^{p,q}(\Omega)$ when k varies. Therefore, the strong limit ϕ in $C^1(\overline{\Omega})$ is the solution of (23) and it satisfies also the relation (46) for all $\varepsilon \in]0, \varepsilon_0\,]$. From Proposition 6, we have

$$||\phi||_{L^{p,q}(\Omega)} \leqslant K_N(\Omega)||T||_{L^{p,q}(\Omega)}. \tag{47}$$

Combining relations (46) and (47) with $\varepsilon = \varepsilon_0$, we derive the result. □

The case where $p = N$ can also be treated in the same way provided that the norm of $\vec{u}$ in $L^{N,1}(\Omega)$ is small enough in the sense that

$$||\vec{u}||_{L^{N,1}(\Omega)} \quad \leqslant \quad \theta K^{s0}_{N1} \text{ for some } \theta \in [\,0, 1\,[, \tag{48}$$

$$K^{s0}_{N1} \quad = \quad K^s_{N1} \sup_{\phi\in H^1_0(\Omega)\cap W^2L^{N,1}(\Omega)} \frac{||\nabla\phi||_\infty}{||\phi||_{W^2L^{N,1}}}. \tag{49}$$

Proposition 12. (Regularity in $W^2L^{N,1}(\Omega)$).
Let ϕ be the solution of (23) when $T \in L^{N,1}(\Omega)$, $V \in L^{N,1}(\Omega)$. Assume that $\vec{u}$ satisfies relation (48). Then $\phi \in W^2L^{N,1}(\Omega)$. Moreover, there exists a constant $K'_N(\Omega)$ (independent of $\vec{u}$) such that

$$||\phi||_{W^2L^{N,1}(\Omega)} \leqslant \frac{K'_N(\Omega)(1 + ||V||_{L^{N,1}})}{1 - K^{s0}_{N1}||\vec{u}||_{L^{N,1}}}||T||_{L^{N,1}(\Omega)}.$$

Proof. The proof follows the same argument as for the proof of Proposition 11. Nevertheless, the embedding $W^1L^{N,1} \subset C(\overline{\Omega})$ is not compact and this explains the condition (48). $\square$

There are many other spaces between the space $L^{p,1}(\Omega)$ and $L^{N,1}(\Omega)$ for which we can obtain a regularity result for the second derivatives of ϕ.

Here we want only to consider the space $\Lambda = (L^N(\mathrm{Log}\, L)^{\frac{\beta}{N}})^N$ for $\beta > N-1$.

Indeed this space is included in $L^{N,1}(\Omega)$ and contains $L^p(\Omega)$ for all $p > N$.

Theorem 3.1. (Regularity in $W^2L^N(\Omega)$).
Let T and V be in $L^N(\Omega)$, $\vec{u} \in \Lambda$, $\mathrm{div}\,(\vec{u}) = 0$ and $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Then the unique solution ϕ of (23) belongs to $W^2L^N(\Omega)$ and choosing $\varepsilon > 0$ such that $\varepsilon||\vec{u}||_\Lambda \leqslant \dfrac{1}{2}$, there exists a constant $K_\varepsilon > 0$ such that

$$||\phi||_{W^2L^N(\Omega)} \leqslant \frac{K_\varepsilon(1 + ||\vec{u}||_\Lambda + ||V||_{L^N})}{1-\varepsilon||\vec{u}||_\Lambda}||T||_{L^N(\Omega)}.$$

The proof firstly depends on the following Trudinger's type embedding :

Lemma 3.1. (Trudinger's embedding)
We have

$$W_0^1L^N(\Omega) \hookrightarrow L_{exp}^{\frac{1}{N'}}(\Omega).$$

Moreover, for all $v \in W_0^1L^N(\Omega)$

$$\sup_{t\leqslant|\Omega|} \frac{|v|_*(t)}{\left(1 + \mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\frac{1}{N'}}} \leqslant K_0||\nabla v||_{L^N(\Omega)}, \text{ with } K_0 = \frac{1}{N\alpha_N^{\frac{1}{N}}}.$$

Proof. According to the pointwise Sobolev inequality for the relative rearrangement, we have for $u = |v|$ (see Theorem 2.3)

$$-u_*'(s) \leqslant \frac{s^{\frac{1}{N}-1}}{N\alpha_N^{\frac{1}{N}}}|\nabla u|_{*u}(s). \tag{50}$$

We integrate this formula from t to $|\Omega|$ knowing that $u_*(|\Omega|) = 0$, and using the Hölder inequality, we get

$$u_*(t) \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\int_t^{|\Omega|} s^{\frac{1}{N}-1}|\nabla u|_{*u}(s)ds \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}\left(\mathrm{Log}\,\frac{|\Omega|}{t}\right)^{\frac{1}{N'}} ||\,|\nabla u|_{*u}||_{L^N}. \tag{51}$$

Therefore from (51), implies using Theorem 2.2

$$\sup_{t\leqslant|\Omega|} \frac{u_*(t)}{\left(1 + \mathrm{Log}\,\dfrac{|\Omega|}{t}\right)^{\frac{1}{N'}}} \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}||\,|\nabla u|_{*u}||_{L^N} \leqslant \frac{1}{N\alpha_N^{\frac{1}{N}}}||\nabla u||_{L^N}.$$

$\square$

The key result for the proof of Theorem 3.1 is the following compactness inclusion :

Theorem 3.2. (Compact inclusion for $W_0^1L^N(\Omega)$).
$W_0^1L^N(\Omega)$ is compactly embedded in $L_{exp}^\alpha(\Omega)$ for $\alpha > \dfrac{1}{N'}$.

Proof. Let $(u_n)_n$ be a bounded sequence in $W_0^1L^N(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $W_0^1L^N(\Omega)$-weakly and almost everywhere in Ω. Let $c = \operatorname*{Max}_n ||u_n - u||_{L_{exp}^{\frac{1}{N'}}} < +\infty$.
For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{c}{\left(1 + \operatorname{Log}\dfrac{|\Omega|}{t}\right)^{\alpha - \frac{1}{N'}}} \leqslant \varepsilon \text{ for all } t \leqslant \delta.$$

Therefore, we have : if $t \leqslant \delta$

$$\frac{|u_n - u|_*(t)}{\left(1 + \operatorname{Log}\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \frac{c}{\left(1 + \operatorname{Log}\dfrac{|\Omega|}{t}\right)^{\alpha - \frac{1}{N'}}} \leqslant \varepsilon;$$

if $t > \delta$ then, since $|u_n - u|_*$ is nonincreasing

$$|u_n - u|_*(t) \leqslant \frac{1}{\delta}\int_0^{\delta} |u_n - u|_*(s)ds,$$

so that

$$\sup_{t \geqslant \delta} \frac{|u_n - u|_*(t)}{\left(1 + \operatorname{Log}\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \frac{1}{\delta}\int_0^{\delta} |u_n - u|_*(s)ds.$$

The right hand side of this inequality tends to zero as n goes to infinity. Hence, for $n \geqslant n_\varepsilon$ with n_ε large enough

$$\sup_{0<t<|\Omega|} \frac{|u_n - u|_*(t)}{\left(1 + \operatorname{Log}\dfrac{|\Omega|}{t}\right)^{\alpha}} \leqslant \varepsilon.$$

□

As a corollary of the above theorem, since $W^2L^N \cap W_0^1L^N \hookrightarrow W_0^1L_{exp}^{\alpha} \hookrightarrow L^N$, we have:

Corollary 1. (of Theorem 3.2)
Let $\alpha > \frac{1}{N'}$. Then, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that $\forall\, v \in W^2L^N(\Omega) \cap H_0^1(\Omega)$

$$||\nabla v||_{L_{exp}^{\alpha}} \leqslant \varepsilon ||\Delta v||_{L^N} + c_\varepsilon ||v||_{L^N}.$$

Proof. We use the equivalence of norms $||v||_{W^2L^N(\Omega)\cap H_0^1} \equiv ||\Delta v||_{L^N} + ||v||_{L^N}$ and apply Lemma 2.8 with

$$Y = W_0^1L_{exp}^{\alpha}(\Omega), \quad X = W^2L^N(\Omega) \cap H_0^1(\Omega), \quad Z = L^N(\Omega).$$

□

Proof of Theorem 3.1. We first assume that $\vec{u} \in \mathcal{V}$, and $T \in L^\infty(\Omega)$. Then, the unique solution ϕ of (23) satisfies

$$\begin{aligned} ||\Delta\phi||_{L^N} &\leqslant ||T||_{L^N} + ||\vec{u}\cdot\nabla\phi||_{L^N} + ||V||_{L^N}||\phi||_\infty \\ &\leqslant K_N(1 + ||V||_N)||T||_N + ||\vec{u}\cdot\nabla\phi||_{L^N}. \end{aligned} \tag{52}$$

We have

$$||\vec{u}\cdot\nabla\phi||_{L^N}^N \leqslant \int_{\Omega_*}|\vec{u}|_*^N|\nabla\phi|_*^N dt \leqslant \sup_{t\in\Omega_*}\frac{|\nabla\phi|_*^N(t)}{\left(1+\operatorname{Log}\dfrac{|\Omega|}{t}\right)^\beta}\int_{\Omega_*}|\vec{u}|_*^N(t)\left(1+\operatorname{Log}\frac{|\Omega|}{t}\right)^\beta dt,$$

which implies

$$||\vec{u}\nabla\phi||_{L^N} \leqslant ||\nabla\phi||_{L^\alpha_{exp}}||\vec{u}||_\Lambda \text{ with } \alpha=\frac{\beta}{N}>\frac{1}{N'}. \tag{53}$$

Let $\varepsilon>0$ be fixed. There exists $c_\varepsilon>0$ such that

$$||\vec{u}\cdot\nabla\phi||_{L^N} \leqslant (\varepsilon||\Delta\phi||_{L^N}+c_\varepsilon||\phi||_{L^N})||\vec{u}||_\Lambda$$

(see Corollary 1 of Theorem 3.2). Combining this with relation (52), we have $\forall\,\varepsilon>0,\ \exists\, c_\varepsilon^1>0$

$$||\Delta\phi||_{L^N}(1-\varepsilon||\vec{u}||_\Lambda) \leqslant c_\varepsilon^1(1+||\vec{u}||_\Lambda+||V||_{L^N})||T||_{L^N}. \tag{54}$$

Secondly, we consider $T\in L^N(\Omega)$ and $\vec{u}\in\overline{\mathcal{V}}$. There exist $\vec{u_k}\in\mathcal{V}$ such that $\vec{u}_k\to\vec{u}$ strongly in Λ and $T_k\in L^\infty(\Omega)$ with

$$||T_k||_{L^N} \leqslant ||T||_{L^N}.$$

Then from relation (54), the solution ϕ_k of (23) satisfies

$$||\Delta\phi_k||_{L^N}(1-\varepsilon||\vec{u_k}||_\Lambda) \leqslant c_\varepsilon^1(1+||\vec{u}_k||_\Lambda+||V||_{L^N})||T||_{L^N}. \tag{55}$$

We choose $\varepsilon_0>0$ such that

$$\varepsilon_0\sup_k||u_k||_\Lambda \leqslant \frac{1}{2}.$$

Then ϕ_k remains in a bounded set of $W^2L^N(\Omega)\cap H_0^1(\Omega)$. So it converges to ϕ weakly in $W^2L^N(\Omega)\cap H_0^1(\Omega)$ and we have

$$||\Delta\phi||_{L^N}(1-\varepsilon_0||\vec{u}||_\Lambda) \leqslant c_{\varepsilon_0}^1(1+||\vec{u}||_\Lambda+||V||_{L^N})||T||_{L^N}, \tag{56}$$

and

$$||\phi||_{L^N} \leqslant |\Omega|^{\frac{1}{N}}||\phi||_\infty \leqslant K_N(\Omega)||T||_{L^N(\Omega)}$$

(according to Proposition 6). This gives the results. ☐

4 Very weak solutions of problem (1) with and without the Dirichlet boundary condition.

We now want to apply all those regularity results to the study of equation (1). We first start with some definitions of the weak solution associated with (1).

4.1 Existence and regularity of the very weak solution for a locally integrable potential $V \geqslant 0$

We start by considering the existence of very weak solutions of equation (1) with the Dirichlet boundary condition (23) when the potential V is a nonnegative locally integrable function.
We can use the definition of very weak solution (see Definition 1.1).

Theorem 4.1.
Let $f \in L^1(\Omega;\delta)$. Let $\vec{u}$ be in $L^{p,1}(\Omega)^N$ with $\operatorname{div}(\vec{u}) = 0$ in $\mathcal{D}'(\Omega)$, $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Furthermore, assume that either $p > N$ or $p = N$ and $||\vec{u}||_{L^{N,1}} < K_{N1}^{s0}$ (see (48)). Then, there exists a very weak solution ω in the sense of (16), which is unique, if $V \in L^{p,1}(\Omega)$.

Remark 6.
In section 4.2, we shall discuss the uniqueness of the v.w.s when $V \notin L^{N,1}(\Omega)$.

Proof. First, we assume that $f \geqslant 0$. Let $u_j \in \mathcal{V}$ be such that $\vec{u}_j \to \vec{u}$ strongly in $L^{p,1}(\Omega)^N$ and $f_j \in L^\infty(\Omega)$ such that $0 \leqslant f_j(x) \leqslant f(x)$ a.e and $f_j(x) \to f(x)$ a.e. According to Proposition 4, Proposition 11 or Proposition 12, there exists a unique function $\omega_j \geqslant 0$ such that

$$\begin{cases} -\Delta\omega_j + \vec{u}_j\cdot\nabla\omega_j + V_j\omega_j = f_j, \\ \omega_j \in H_0^1(\Omega)\cap W^2L^{p,1}(\Omega), \end{cases} \tag{57}$$

which is equivalent to saying that

$$\begin{cases} \displaystyle\int_\Omega \omega_j\Big[-\Delta\phi - \vec{u}_j\cdot\nabla\phi\Big]dx = \int_\Omega f_j\phi\,dx - \int_\Omega V_j\omega_j\phi dx, \\ \forall\,\phi \in W^2L^{p,1}(\Omega)\cap H_0^1(\Omega). \end{cases} \tag{58}$$

We argue as in [20, 18, 36]. Let E be a measurable subset of Ω and χ_E its characteristic function. Then, there exists a non negative function $\phi_j \in W^2L^m(\Omega)$, $\forall\, m < +\infty$, satisfying

$$\begin{cases} -\Delta\phi_j - \vec{u}_j\nabla\phi_j = \chi_E \text{ in } \Omega, \\ \phi_j = 0 \text{ on } \partial\Omega. \end{cases} \tag{59}$$

We consider a small number $\varepsilon > 0$ such $\varepsilon\sup_j ||\vec{u}_j||_{L^{N,1}} \leqslant \dfrac{1}{2}$. Therefore, we have

$$||\phi_j||_{W^2L^{N,1}} \leqslant K_0||\chi_E||_{L^{N,1}} \leqslant K_1|E|^{\frac{1}{N}}.$$

Thus

$$\begin{aligned} \int_E \omega_j dx = \int_\Omega \omega_j\big[-\Delta\phi_j - \vec{u_j}\nabla\phi_j\big]dx &\leqslant \int_\Omega f_j\phi_j \leqslant K_1\left(\int_\Omega |f_j|\delta\right)||\phi_j||_{W^2L^{N,1}} \\ &\leqslant K_0|E|^{\frac{1}{N}}\int_\Omega |f_j|\delta dx. \end{aligned} \tag{60}$$

By the Hardy-Littlewood property we conclude that

$$\sup_{t\leqslant|\Omega|} t^{\frac{1}{N'}}|\omega_j|_{**}(t) \leqslant K_0\int_\Omega |f_j|\delta dx \leqslant K_0\int_\Omega |f|\delta dx. \tag{61}$$

Moreover, choosing $\phi = \varphi_1$ as the test function with $-\Delta\varphi_1 = \lambda_1\varphi_1$, and $\varphi_1 = 0$ on $\partial\Omega$, we have

$$\begin{aligned} \lambda_1\int_\Omega \omega_j\varphi_1 dx + \int_\Omega V_j\omega_j\varphi_1 dx &\leqslant ||\nabla\varphi_1||_\infty\,||\omega_j||_{L^{N',\infty}}||\vec{u}_j||_{L^{N,1}} + c\int_\Omega |f_j|\delta dx \\ &\leqslant c\big(1+||\vec{u}_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx, \end{aligned}$$

for a suitable constant $c > 0$. Thus $V_j\omega_j$ remains in a bounded set of $L^1(\Omega;\delta)$ and

$$\int_\Omega V_j\omega_j\delta dx \leqslant c\big(1 + ||\vec{u}_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx. \tag{62}$$

If f has a constant sign, we write $f_j = f_{j+} - f_{j-}$ with $f_{j+} = \max(f_j, 0) \geqslant 0$.
Denoting by ω_j^+ the v.w.s. associated to f_{j+} and by ω_j^- the one associated to f_{j-},we see that $\omega_j = \omega_j^+ - \omega_j^-$ satisfies (58) and we have also the estimates (61) and (62).
In particular, since $|\omega_j| \leqslant \omega_j^+ + \omega_j^-$

$$\int_\Omega V_j|\omega_j|\delta dx \leqslant c\big(1 + ||u_j||_{L^{N,1}}\big)\int_\Omega |f_j|\delta dx. \tag{63}$$

We conclude that $(\omega_j)_j$ converges weak-* to ω in $L^{N',\infty}(\Omega) = \big(L^{N,1}(\Omega)\big)^*$. To obtain a strong convergence, we need a local estimate of the gradient. For that purpose, we shall prove the boundedness of ω_j in the Lorentz-Sobolev weighted space $W^1L^{1+\frac{1}{N},\infty}(\Omega;\delta)$. For this, we shall need the following result due to Philippe Bénilan and co-authors whose proof can be found in [5] Lemma 4.2, with generalization in [40].

Proposition 13.
Let $v \in L^1(\Omega, \delta^\alpha)$, and $\alpha \in [\,0, 1\,]$. Assume that there exists a constant $c_0 > 0$ such that for all $k > 0$

$$T_k(v) := \min(|v|; k)\ \mathrm{sign}\,(v) \in W^1L^2(\Omega, \delta^\alpha),$$

and

$$\int_\Omega |\nabla T_k(v)|^2\delta^\alpha dx + \int_\Omega |T_k(v)|^2\delta^\alpha dx \leqslant c_0 k. \tag{64}$$

Then, there exists a constant c, depending continuously on $c_0 > 0$, such that for all $\lambda > 0$

$$\int_{\{x:|\nabla v|(x)>\lambda\}} \delta^\alpha(x)dx \leqslant \frac{c}{\lambda^{1+\frac{1}{N+\alpha-1}}}.$$

In particular, if v_j is a sequence converging weakly in $L^1(\Omega)$ to a function v, satisfying the inequality (64)

$$\int_\Omega |\nabla T_k(v_j)|^2\delta^\alpha dx \leqslant c_0 k \qquad\qquad \forall j,\ \forall k,$$

then v_j converges to v weakly in $W^{1,q}(\Omega')$ for all $q \in \left[1, \dfrac{N+\alpha}{N+\alpha-1}\right[$ and all $\Omega' \subset\subset \Omega$, with a subsequence, $v_j(x) \to v(x)$ a.e. in Ω.

We first need to prove the following a priori estimate :

Proposition 14.
Let ω_j be the solution of (57), ω its weak limit in $L^{N',\infty}(\Omega)$. Under the same assumptions as for Theorem 3.1, there exists a constant $c_0 > 0$ such that:

$$\int_\Omega \big|\nabla T_k(\omega_j)\big|^2\delta dx + \int_\Omega \big|\nabla T_k(\omega)\big|^2\delta dx \leqslant c_0 k \quad \forall\, k > 0,\ \forall\, j.$$

Proof. Let φ_1 be the first eigenvalue of the Dirichlet problem $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$. Then, there exist constants such that $c_1\delta(x) \leqslant \varphi_1(x) \leqslant c_2\delta(x) \quad \forall x \in \Omega$. We consider the approximate problem given in equation (57) say

$$\begin{cases} -\Delta\omega_j + \vec{u}_j \cdot \nabla\omega_j + V_j\omega_j = f_j, \\ \omega_j \in W_0^{1,1}(\Omega) \cap W^2L^{p,1}(\Omega), \end{cases}$$

with $|f_j(x)| \leqslant |f(x)|$, $f_j \to f$ a.e, $\vec{u}_j \to \vec{u}$ in $L^{p,1}(\Omega)^N-$strongly and $\omega_j \to \omega$ weakly-* in $L^{N',\infty}(\Omega)$.
For $k > 1$, we choose $T_k(\omega_j)\varphi_1$ as a test function; then $V_j\omega_jT_k(\omega_j)\varphi_1 \geqslant 0$ and we derive after some integrations by parts :

$$\int_\Omega |\nabla T_k(\omega_j)|^2\varphi_1 dx + \lambda_1\int_\Omega \varphi_1\left(\int_0^{\omega_j} T_k(\sigma)d\sigma\right) dx - \int_\Omega \vec{u}_j \cdot \nabla\varphi_1 \int_0^{\omega_j} T_k(\sigma)d\sigma dx \leqslant c_2k\int_\Omega |f|\delta dx. \quad (65)$$

This relation implies:

$$\int_\Omega |\nabla T_k(\omega_j)|^2\delta(x) \leqslant c_3k\int_\Omega |\omega_j|\delta dx + c_2k\int_\Omega |f|\delta dx + c_3k\int_\Omega |\vec{u}_j|\,|\omega_j|dx. \quad (66)$$

By the Hölder inequality

$$\int_\Omega |\vec{u}_j|\,|\omega_j|dx \leqslant c_4||\vec{u}_j||_{L^{N,1}} \cdot ||\omega_j||_{L^{N',\infty}} \leqslant c_4||\vec{u}_j||\int_\Omega |f|\delta dx. \quad (67)$$

From relation (66) and (67), we then have :

$$\int_\Omega |\nabla T_k(\omega_j)|^2\delta(x)dx \leqslant c_5(1 + ||\vec{u}_j||_{L^{N,1}})\left(\int_\Omega |f|\delta dx\right) k. \quad (68)$$

Letting $j \to +\infty$, we deduce from (68) and Proposition 13 :

$$\int_\Omega |\nabla T_k(\omega)|^2\delta(x)dx \leqslant c_0k \text{ with } c_0 = c_5(1 + ||\vec{u}||_{L^{N,1}})\int_\Omega |f|\delta dx.$$

Then the $L^{N',\infty}$-regularity of ω implies

$$\int_\Omega |T_k(\omega)|^2\delta dx \leqslant c_0k\int_\Omega |\omega|dx.$$

□

Corollary 2 (of Propositions 13 and 14)**.**
Let ω be as in the proof of the previous proposition. Then, there exists a constant $c_6 > 0$ such that

$$||\nabla\omega||_{L^{1+\frac{1}{N},\infty}(\Omega;\delta)} \leqslant c_6\int_\Omega |f(x)|\delta(x)dx.$$

In particular, we have, for all $q < 1 + \dfrac{1}{N}$,

$$\int_\Omega |\nabla\omega|^q\delta(x)dx \leqslant c_q\int_\Omega |f(x)|\delta(x)dx.$$

To pass to the limit in (57), we argue as in [19] p. 1041. We emphasize the main differences due to the additional term $\vec{u}\cdot\nabla\omega$.

Let us note that by the above Proposition 11, we have (for a subsequence still denoted as $(\omega_j)_j$) that

1. $\omega_j(x)\to\omega(x)$ a.e. (and thus $V_j\omega_j\to V\omega$ a.e. in Ω).
2. $\omega_j\rightharpoonup\omega$ weakly in $W^{1,q}(\Omega;\delta),\ \forall\, q<1+\dfrac{1}{N}$.
3. $\omega_j\to\omega$ strongly in $L^r(\Omega)$, for any $r<N'$.

In particular, we deduce from the above statement 1., relation (63) and Fatou's lemma

Lemma 4.1.
Under the assumptions of Theorem 4.1 and Proposition 14 one has

$$\int_\Omega V|\omega|\delta dx\leqslant c\Big(1+||u||_{L^{N,1}}\Big)\int_\Omega|f|\delta dx.$$

Lemma 4.2.
Under the assumptions of Theorem 4.1 and Proposition 14 one has

$$\lim_{j\to+\infty}\int_\Omega|\vec{u}_j\omega_j-\vec{u}\omega|dx=0.$$

Proof. Since $\vec{u}_j\to\vec{u}$ in $L^{N,1}(\Omega)$, and a.e. in Ω, we have

$$\lim_{j\to+\infty}\vec{u}_j(x)\omega(x)=\vec{u}(x)\omega(x)\ a.e.$$

It is enough to show that $(\vec{u}_j\omega_j)_j$ satisfies Vitali's condition : $\forall\,\varepsilon>0\ \exists\eta>0$ such that if $E\subset\Omega$ is measurable with $|E|\leqslant\eta$ then

$$\limsup_{j\to+\infty}\int_E|\vec{u}_j\omega_j|dx\leqslant\varepsilon.$$

But from Hölder's inequality we have

$$\int_E|\vec{u}_j\omega_j|dx\leqslant||\vec{u}_j||_{L^{N,1}(E)}||\omega_j||_{L^{N',\infty}(\Omega)}\leqslant c||\vec{u}_j||_{L^{N,1}(E)},$$

so that

$$\limsup_{j\to+\infty}\int_E|\vec{u}_j\omega_j|dx\leqslant c||\vec{u}||_{L^{N,1}(E)}.$$

Since

$$||\vec{u}||_{L^{N,1}(E)}\xrightarrow[|E|\to 0]{}0,$$

we derive that it satisfies the Vitali condition. Therefore, we have proved the lemma. □

Then we have the following result analogous to Lemma 2.3 of [19].

Lemma 4.3.
We assume that $V \in L^1_{loc}(\Omega)$, and $V \geqslant 0$. Then

$$V_j\omega_j\delta \rightharpoonup V\omega\delta \ \text{ weakly in } L^1_{loc}(\Omega).$$

Furthermore, if $V \in L^1(\Omega;\delta)$, then

$$V_j\omega_j\delta \rightharpoonup V\omega\delta \text{ weakly in } L^1(\Omega).$$

Proof. Let $t \in \mathbb{R}_+$. Consider a sequence of functions γ_m in $C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that

$$\begin{array}{rcl} \gamma_m' \geqslant 0 & & \forall\, s \in \mathbb{R}, \\ \gamma_m(s) & \to & -1 \text{ for } s < -t \text{ as } m \to +\infty, \\ \gamma_m(s) & \to & 1 \text{ for } s > t \text{ as } m \to +\infty, \\ \gamma_m(s) & = & 0 \text{ on } -t \leqslant s \leqslant t, \end{array}$$

and let $\varphi_1 \in C^2(\overline{\Omega})$ with $-\Delta\varphi_1 = \lambda_1\varphi_1$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, $\lambda_1 > 0$.
Taking $\varphi_1\gamma_m(\omega_j)$ as a test function in relation (57) we get

$$\begin{aligned} \int_\Omega \nabla\omega_j \cdot \nabla\big(\varphi_1\gamma_m(\omega_j)\big) + \int_\Omega V_j\omega_j\varphi_1\big(\gamma_m(\omega_j)\big)dx \quad &+ \quad \int_\Omega \vec{u}_j \cdot \nabla\omega_j\gamma_m(\omega_j)\varphi_1 dx \\ &= \quad \int_\Omega f_j\gamma_m(\omega_j)\varphi_1 dx. \end{aligned} \tag{69}$$

We write $\nabla\omega_j\gamma_m(\omega_j) = \nabla\left[\int_0^{\omega_j}\gamma_m(\sigma)dx\right]$ so that

$$\begin{aligned} \int_\Omega (\vec{u}_j \cdot \nabla\omega_j)\gamma_m(\omega_j)\varphi_1 dx &= -\int_\Omega \operatorname{div}(u_j\varphi_1)\int_0^{\omega_j}\gamma_m(\sigma)d\sigma dx \\ &= -\int_\Omega \vec{u}_j\nabla\varphi_1\left(\int_0^{\omega_j}\gamma_m(\sigma)d\sigma\right)dx. \end{aligned}$$

As $m \to +\infty$, treating the remaining terms in (69) as in [19], we derive

$$\int_{|\omega_j|>t} V_j|\omega_j|\delta dx \leqslant c\left[\int_{|\omega_j|\geqslant t}|f|\delta dx + \int_{|\omega_j|\geqslant t}|\omega_j|\delta dx + \int_{|\omega_j|\geqslant t}|\vec{u}_j|\,|\omega_j|dx\right]. \tag{70}$$

This relation proves that $V_j\omega_j\delta$ remains in a bounded set of $L^1(\Omega)$ but also that the set $\Big\{V_j|\omega_j|\delta,\ j \in \mathbb{N}\Big\}$ is x compact for the $\sigma(L^1;L^\infty)$-topology, so we may appeal to the Dunford-Pettis to conclude. Indeed, let us set

$$\Gamma_j(t) := \int_{|\omega_j|\geqslant t}|f(x)|\delta(x)dx + \int_{|\omega_j|\geqslant t}|\omega_j|\delta dx + \int_{|\omega_j|\geqslant t}|\vec{u}_j\omega_j|dx.$$

For a.e. $t > 0$,

$$\lim_{j\to+\infty}\Gamma_j(t) = \Gamma(t) = \int_{|\omega|>t}|f(x)|\delta(x)dx + \int_{|\omega|>t}|\omega|\delta dx + \int_{|\omega|>t}|\vec{u}\omega|dx,$$

and

$$\Big|\big\{|\omega| > t\big\}\Big| + \sup_j\Big|\big\{|\omega_j| > t\big\}\Big| \leqslant \frac{constant}{t} \xrightarrow[t\to+\infty]{} 0,$$

we deduce that for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that, for all $j \in \mathbb{N}$,

$$\Gamma_j(t_\varepsilon) \leqslant \varepsilon.$$

Let $\Omega_0 \subset \Omega$ such that $V\delta \in L^1(\Omega_0)$ (thus $\Omega_0 \neq \Omega$ if V is only locally integrable). Then by the Lebesgue convergence dominate theorem for a.e. t,

$$\lim_{j\to+\infty} \int_{\Omega_0} \Big| \chi_{|\omega_j|\leqslant t}(x) V_j \omega_j(x) - \chi_{\{|\omega|\leqslant t\}}(x) V(x)\omega(x) \Big| \delta(x) dx = 0,$$

since

$$\lim_{|A|\to 0} \int_A V|\omega|\delta dx = 0 \quad \Big(V\omega\delta \in L^1(\Omega)\Big).$$

Therefore there exists $\eta > 0$ such that if $A \subset \Omega_0,\ |A| \leqslant \eta$, then for all $j \in \mathbb{N}$,

$$\int_{A\cap\{|\omega_j|\leqslant t_\varepsilon\}} V_j|\omega_j|\delta dx \leqslant \varepsilon.$$

Hence, for all $j \in \mathbb{N}$, all $A \subset \Omega_0$, with $|A| \leqslant \eta$

$$\int_A V_j|\omega_j|\varphi dx \leqslant \Gamma_j(t_\varepsilon) + \int_A V_j|\omega_j|\delta dx \leqslant 2\varepsilon.$$

This conclude the proof of Lemma 7. $\square$

The passage to the limit, we will distinguish two different cases :

1. Case $V \in L^1(\Omega;\delta)$ For all $\phi \in C^2(\overline{\Omega}),\ \phi = 0$, we have

$$\lim_j \int_\Omega V_j\omega_j\phi dx = \int_\Omega V\omega\phi dx \tag{71}$$

(since $\dfrac{\phi}{\delta} \in L^\infty(\Omega)$ and $V_j\omega_j\delta$ converges to $V\omega\delta$ for $\sigma(L^1;L^\infty)$ topology). Therefore, since

$$-\int_\Omega \omega_j\Delta\phi dx - \int_\Omega \vec{u}_j\omega_j\nabla\phi dx + \int_\Omega V_j\omega_j\phi dx = \int_\Omega f_j\phi dx, \tag{72}$$

we let $j \to +\infty$ to deduce that ω is a v.w.s. using Lemma 4.2 and the convergences of ω_j.

2. Case $V \in L^1_{loc}(\Omega)$ We consider $\phi \in W^2L^{N,1}(\Omega)$ with support ϕ be a compact in Ω. Then the same argument holds since $V_j\omega_j\delta$ tends to $V\omega\delta$ weakly in $L^1_{loc}(\Omega)$. Then (71) and (72) hold true

$$\begin{cases} \displaystyle\int_\Omega \omega\big[-\Delta\phi - \vec{u}\nabla\phi + V\phi\big]dx = \int_\Omega f\phi dx, \\ \forall\,\phi \in W^2L^{N,1}(\Omega),\ \text{support}(\phi) \text{ compact in } \Omega. \end{cases} \tag{73}$$

If $V \in L^{p,1}(\Omega)$, the solution is unique. Indeed, if we denote by ω the difference of two solutions then

$$\int_\Omega \Big[-\Delta\varphi - \vec{u}\nabla\varphi + V\varphi\Big]\omega dx = 0 \quad \forall\,\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \delta\Omega.$$

Let us consider the function ϕ solution of

$$\begin{cases} -\Delta\phi - \vec{u}\nabla\phi + V\phi = \operatorname{sign}(\omega), \\ \phi \in H^1_0(\Omega). \end{cases} \tag{74}$$

Then $\phi \in W^2L^{N,1}\Omega \hookrightarrow C^1(\overline{\Omega})$ for $V \in L^{N,1}(\Omega)$. Thus

$$\int_\Omega \omega\big[-\Delta\phi - \vec{u}\nabla\phi + V\phi\big]dx = 0, \tag{75}$$

since $\Big\{\varphi \in C^2(\overline{\Omega}) : \varphi = 0 \text{ on } \partial\Omega\Big\}$ is dense in $W^2L^{N,1}(\Omega)\cap H_0^1(\Omega)$. Combining the relations (74) and (75) we find :

$$\int_\Omega |\omega|dx = 0 \quad i.e.\ \omega \equiv 0.$$

□

4.2 A result of uniqueness of solution when the potential is bounded from below by $c\,\delta^{-r}, r > 2$

The purpose of this section is to show the following uniqueness result.

Theorem 4.2.
Assume that V is locally integrable $V \geqslant 0$, and such that

$$\exists\, c > 0,\ V(x) \geqslant c\delta(x)^{-r},\ \text{in a neighborhood } U \text{ of the boundary, with } \ r > 2.$$

Then, the v.w.s. ω found in Theorem 4.1 is unique.

This theorem relies on the following general result which does not require any information about the boundary condition, since the required additional information is written in another way :

Theorem 4.3. (Comparison principle)
Let ϖ be in $L^1(\Omega;\delta^{-r})\cap W^{1,1}_{\text{loc}}(\Omega)$, $r > 1$. Let $\varpi \in L^{N',\infty}(\Omega)$ and $\vec{u} \in L^{p,1}(\Omega)$ with $p > N$ or $p = N$ with a small norm. Assume that

$$\mathrm{L}\varpi \doteq -\Delta\varpi + \operatorname{div}(\vec{u}\,\varpi) \leqslant 0 \text{ in } \mathcal{D}'(\Omega).$$

Then

$$\varpi \leqslant 0 \text{ in } \Omega.$$

As an immediate corollary of the above theorem we have

Corollary 3. of Theorem 4.3
Assume the hypotheses of Theorem 4.3 hold and let $f \in L^1_{loc}(\Omega)$. Then there exists at most one function $\varpi \in L^1(\Omega;\delta^{-r})\cap W^{1,1}_{\text{loc}}(\Omega)$, $r > 1$ solution of $\mathrm{L}\varpi = f$ in $\mathcal{D}'(\Omega)$.

For the proof of Theorem 4.3, we need the following extension of the Kato's inequality whose proof is similar to the one given in [30] :

Theorem 4.4. (Local Kato's inequality)
Let $\varpi \in W^{1,1}_{loc}(\Omega)$ with $\vec{u}\,\varpi \in L^1_{loc}(\Omega)$. Assume that $\mathrm{L}\varpi = -\,\Delta\varpi + \operatorname{div}(\vec{u}\,\varpi)$ belongs to $L^1_{loc}(\Omega)$. Then

1. $\forall\,\psi \in \mathcal{D}(\Omega),\ \psi \geqslant 0,\ \displaystyle\int_\Omega \varpi_+\mathrm{L}^*\psi dx \leqslant \int_\Omega \psi \operatorname{sign}_+(\varpi)\mathrm{L}(\varpi)dx,$

$$i.e.\ \mathrm{L}(\varpi_+) \leqslant \operatorname{sign}_+(\varpi)\mathrm{L}(\varpi) \ \text{ in } \ \mathcal{D}'(\Omega).$$

2. $\mathrm{L}(|\overline{\omega}|) \leqslant \mathrm{sign}\,(\overline{\omega})\mathrm{L}(\overline{\omega})$ in $\mathcal{D}'(\Omega)$.
   Here

$$\mathrm{sign}_+(\sigma) = \begin{cases} 1 & if\ \sigma > 0, \\ 0 & if\ \sigma \leqslant 0, \end{cases} \qquad \mathrm{sign}\,(\sigma) = \begin{cases} 1 & if \sigma > 0, \\ -1 & if\ \sigma < 0, \end{cases}$$

$$\mathrm{L}^*\psi = -\Delta\psi - \vec{u}\cdot\nabla\psi,\ for\ \psi \in C_c^\infty(\Omega).$$

Proof. Following [30], we first remark that for any $\alpha \in C_c^\infty(\Omega)$, $\mathrm{L}(\alpha\overline{\omega}) \in L^1(\Omega)$ since, one has, in $\mathcal{D}'(\Omega)$,

$$\mathrm{L}(\alpha\overline{\omega}) = \alpha\mathrm{L}\overline{\omega} - \overline{\omega}\Delta\alpha - 2\nabla\overline{\omega}\cdot\nabla\alpha + (\vec{u}\,\overline{\omega})\cdot\nabla\alpha \in L^1(\Omega).$$

Thus, the conclusion 1. will be proved if we show that

$$\mathrm{L}(\alpha\overline{\omega})_+ \leqslant \mathrm{sign}_+(\alpha\omega)\mathrm{L}(\alpha\overline{\omega}) \ \text{ in } \ \mathcal{D}'(\Omega).$$

For this purpose, we may assume that $\overline{\omega} \in W^{1,1}(\Omega)$ with compact support and $\mathrm{L}\overline{\omega} \in L^1(\Omega)$. Moreover, if $\rho_j \in C_c^\infty(\mathrm{I\!R}^N)$ is a sequence of mollifiers, and $\overline{\omega}\star\rho_j \in C_c^\infty(\Omega)$ we have

$$\mathrm{L}(\overline{\omega}\star\rho_j) = \mathrm{L}\overline{\omega}\star\rho_j \to \mathrm{L}\overline{\omega} \ \text{ in } \ L^1(\Omega).$$

So, it is sufficient to show the inequality number for $\overline{\omega} \in C_c^\infty(\Omega)$. From here, we argue as for the case where L is replaced by the Laplacian operator (see Proposition 1.5.4 p.21 in [30] for more details). We approximate the functions sign_+ by a sequence of convex, non-decreasing functions h_ε such that

$$\lim_{\varepsilon\to 0} h_\varepsilon'(t) = \mathrm{sign}_+(t); \quad \lim_{\varepsilon\to 0} h_\varepsilon(t) = t_+$$

$$\sup_{\varepsilon>0} |h_\varepsilon'|(t) \text{ is independent of } \varepsilon.$$

Thus, for all $\psi \in C_c^\infty(\Omega)$, $\psi \geqslant 0$, we have

$$\int_\Omega h_\varepsilon(\overline{\omega})\mathrm{L}^*\psi dx \leqslant \int_\Omega \psi h_\varepsilon'(\overline{\omega})\mathrm{L}\overline{\omega}dx, \tag{76}$$

where $\mathrm{L}^*\psi = -\Delta\psi - \vec{u}\cdot\nabla\psi$.

Indeed, $\psi h_\varepsilon'(\overline{\omega})$ is in $C_c^\infty(\Omega)$ and then the convexity of h_ε implies

$$\int_\Omega \psi h_\varepsilon'(\overline{\omega})\mathrm{L}\overline{\omega}dx \geqslant -\int_\Omega h_\varepsilon(\overline{\omega})\Delta\psi dx + \int_\Omega \vec{u}\psi h_\varepsilon'(\overline{\omega})\cdot\nabla\overline{\omega}dx.$$

Since $\mathrm{div}\,(\vec{u}) = 0$, and $h_\varepsilon'(\overline{\omega})\nabla\overline{\omega} = \nabla h_\varepsilon(\overline{\omega})$ we have

$$\int_\Omega \vec{u}\psi h_\varepsilon'(\overline{\omega})\cdot\nabla\overline{\omega}dx = \int_\Omega \vec{u}\psi\cdot\nabla h_\varepsilon(\overline{\omega})dx = -\int_\Omega \vec{u}\cdot\nabla\psi h_\varepsilon(\overline{\omega})dx.$$

Thus we get (76).

As in [30], letting $\varepsilon \to 0$, we have

$$\int_\Omega \overline{\omega}_+\mathrm{L}^*\psi dx \leqslant \int_\Omega \psi\mathrm{sign}_+(\overline{\omega})\mathrm{L}\overline{\omega}dx \quad \forall\,\psi \in \mathcal{D}(\Omega),\ \psi \geqslant 0.$$

We derive conclusion1., as in [30], for $\overline{\omega} \in W_c^{1,1}(\Omega)$ and the same for conclusion 2. □

To extend the set of test functions from $\mathcal{D}(\Omega)$ to other sets of functions we need the following approximation result.

Lemma 4.4. (Approximation of functions in $W^{m,\infty}(\Omega)$ by a sequence in $W_c^{m,\infty}(\Omega)$) Let $W_c^{m,\infty}(\Omega) = \Big\{\varphi \in W^{m,\infty}(\Omega)$ with compact support$\Big\}$, $1 < m < +\infty$ and assume that $\partial\Omega$ is of class C^m, $r > 0$. Then, for $\varphi \in W^{m,\infty}(\Omega)$ there exists a sequence $(\varphi_n)_n$, $\varphi_n \in W_c^{m,\infty}(\Omega)$, such that

1. $\delta^r(D^\alpha\varphi_n) \to \delta^r(D^\alpha\varphi)$ strongly in $L^\infty(\Omega)$, for all α such that $|\alpha| < r$.
2. Moreover, if $\varphi \in W_0^{1,\infty}(\Omega)$ then
$$\sup_n ||\nabla\varphi_n||_\infty \leqslant c_\Omega ||\nabla\varphi||_\infty,\ (c_\Omega \text{ with independent of } \varphi),$$
$$\delta^r(D^\alpha\varphi_n) \to \delta^r(D^\alpha\varphi) \text{ strongly in } L^\infty(\Omega) \text{ for } |\alpha| < r+1.$$
3. If $\varphi \geqslant 0$ then one can take $\varphi_n \geqslant 0$.
4. If $\varphi \in C^m(\overline{\Omega})$ then $\varphi_n \in C_c^m(\Omega)$. By the density of $C_c^\infty(\Omega)$ in $C_c^m(\Omega)$, φ_n in this case can be taken in $C_c^\infty(\Omega)$.

Proof. Let $h \in C^\infty(\mathbb{R})$ be such that $0 \leqslant h \leqslant 1$, $h(\sigma) = \begin{cases} 1 & \text{if } \sigma \geqslant 1, \\ 0 & \text{if } \sigma \leqslant 0 \end{cases}$
Since $\partial\Omega \in C^m$, δ is of class C^m in a neighborhood U of $\partial\Omega$ (see [22]). Let $0 < \varepsilon < 1$ be such that
$$\Big\{x \in \Omega : \delta(x) \leqslant \varepsilon\Big\} \subset U$$
and define, for $x \in \Omega$,
$$h_\varepsilon(x) = h\left(\frac{2\delta(x) - \varepsilon}{\varepsilon}\right), \tag{77}$$
so that $h_\varepsilon(x) = 1$ if $\delta(x) > \varepsilon$, $h_\varepsilon(x) \to 1$ as $\varepsilon \to 0$, and $h_\varepsilon(x) = 0$ if $\delta(x) < \varepsilon/2$.
One has
$$|D^\alpha h_\varepsilon(x)| \leqslant c\,\varepsilon^{-|\alpha|},\ \text{for a constant } c > 0 \text{ independent of } x \text{ and } \varepsilon.$$
Since we have, by Leibniz's formula
$$D^\alpha\big(\varphi(1-h_\varepsilon)\big)(x) = \sum_{\beta+\gamma=\alpha} c_{\gamma\beta} D^\beta\varphi(x) D^\gamma(1-h_\varepsilon)(x), \tag{78}$$
($c_{\gamma\beta}$ are constant depending only on γ, β) and for $\gamma \neq 0$.
$$\delta^r(x)\big|D^\gamma h_\varepsilon(x)\big| \leqslant c\,\varepsilon^{-|\gamma|+r}, \tag{79}$$
we then deduce, that
$$\delta^r(x)\big|D^\alpha\big(\varphi(1-h_\varepsilon)\big)(x)\big| \leqslant c\left[\sum_{\beta+\gamma=\alpha,\,\gamma\neq 0} |D^\beta\varphi(x)|\varepsilon^{-|\gamma|+r} + \delta^r|D^\alpha\varphi|(1-h_\varepsilon)\right].$$
Therefore
$$\sup_{x\in\Omega} \delta^r(x)\big|D^\alpha\varphi(1-h_\varepsilon)(x)\big| \leqslant c\,\varepsilon^{-|\alpha|+r}. \tag{80}$$

Taking $\varepsilon = \dfrac{1}{n}$, and $\varphi_n = h_{\frac{1}{n}}\varphi$ is convenient for large $n \geqslant n_0$. If furthermore $\varphi \in W_0^{1,\infty}(\Omega)$ then

$$|\varphi(x)| \leqslant \delta(x)||\nabla\varphi||_\infty.$$

Hence,

$$\delta^r\big|D^\alpha\big(\varphi(1-h_\varepsilon)\big)(x)\big| \leqslant c\,\delta^{r+1}(x)\varepsilon^{-|\alpha|} + c\sum_{\beta\neq 0,\beta+\gamma=\alpha}\big|D^\beta\varphi|(x)\big|D^\gamma(1-h_\varepsilon)|\delta^r(x) \leqslant c\,\varepsilon^{-|\alpha|+r+1}.$$

On the other hand

$$\begin{cases} \text{on } \delta(x) \leqslant \varepsilon & \big|\nabla(\varphi h_\varepsilon)(x)\big| \leqslant |\varphi(x)|\nabla h_\varepsilon(x)| + 2||\nabla\varphi||_\infty \leqslant c\,||\nabla\varphi||_\infty\left[1 + \dfrac{\delta(x)}{\varepsilon}\right] \\ & \leqslant c\,||\nabla\varphi||_\infty, \\ \text{on } \delta(x) > \varepsilon & \big|\nabla\varphi_n(x)\big| \leqslant 2||\nabla\varphi||_\infty. \end{cases}$$

Moreover, one has

$$\delta^r(x)\Big|\nabla\big(\varphi(1-h_\varepsilon)\big)\Big|(x) \leqslant \delta^r|D\varphi|(x)\big(1-h_\varepsilon(x)\big) + c\,\delta^{r+1}(x)||\nabla\varphi||_\infty|\nabla h_\varepsilon| \leqslant c\,\varepsilon^r.$$

□

Thanks to the above approximation lemma we can modify the set of the test functions in the Kato's inequality as follows

Corollary 4. (of Theorem 4.4 : Variant of Kato's inequality)
Let $\overline{\omega}$ be in $W^{1,1}_{\text{loc}}(\Omega) \cap L^{N',\infty}(\Omega)$, $\overline{\omega} \in L^1(\Omega;\delta^{-r})$ for $r > 1$ and $\vec{u} \in L^{N,1}(\Omega)^N$ with $\operatorname{div}(\vec{u}) = 0$, $\vec{u}\cdot\vec{n} = 0$. Assume furthermore that $\mathrm{L}\overline{\omega} = -\Delta\overline{\omega} + \operatorname{div}(\vec{u}\overline{\omega})$ is in $L^1(\Omega;\delta)$.
Then for all $\phi \in C^2(\overline{\Omega})$, $\phi = 0$ on $\partial\Omega$, $\phi \geqslant 0$ one has

1. $\displaystyle\int_\Omega \overline{\omega}_+\mathrm{L}^*\phi dx \leqslant \int_\Omega \phi\,\operatorname{sign}_+(\overline{\omega})\mathrm{L}(\overline{\omega})dx,$
2. $\displaystyle\int_\Omega |\omega|\mathrm{L}^*\phi dx \leqslant \int_\Omega \phi\,\operatorname{sign}(\overline{\omega})\mathrm{L}(\overline{\omega})dx,$

where $\mathrm{L}^*\phi = -\Delta\phi - \vec{u}\cdot\nabla\phi = -\Delta\phi - \operatorname{div}(\vec{u}\phi)$.

Proof. Let $\phi \geqslant 0$ be in $C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$. Then according to Lemma 4.4, we have a sequence $\phi_n \in C^2_c(\Omega)$, $\phi \geqslant 0$, such that

$$\begin{cases} \delta^r\Delta\phi_n \to \delta^r\Delta\phi & \text{in} \quad C(\overline{\Omega}) \text{ for } r > 1, \\ \delta^r\nabla\phi_n \to \delta^r\nabla\phi & \text{in} \quad C(\overline{\Omega})^N,\ ||\nabla\phi_n||_\infty \leqslant c||\nabla\phi||_\infty. \end{cases}$$

Therefore

$$\lim_{n\to+\infty}\int_\Omega \overline{\omega}_+\Delta\phi_n dx = \lim_{n\to+\infty}\int_\Omega \overline{\omega}_+\delta^{-r}\cdot\delta^r\Delta\phi_n dx = \int_\Omega \overline{\omega}_+\Delta\phi dx,$$

since $\overline{\omega}_+ \in L^1(\Omega;\delta^{-r})$ and $r > 1$.
By the Lebesgue dominated convergence theorem, one has

$$\lim_{n\to+\infty}\int_\Omega \vec{u}\cdot\nabla\phi_n\overline{\omega}_+dx = \int_\Omega \vec{u}\cdot\nabla\phi\overline{\omega}_+dx, \text{ since } \vec{u}\cdot\overline{\omega}_+ \in L^1(\Omega)^N.$$

Therefore

$$\begin{aligned}\int_\Omega \overline{\omega}_+ \mathrm{L}^*\phi dx &= \lim_{n\to\infty}\int_\Omega \overline{\omega}_+\mathrm{L}^*\phi_n dx \leqslant \lim_{n\to+\infty}\int_\Omega \phi_n \mathrm{sign}_+\overline{\omega}\,\mathrm{sign}_+\mathrm{L}\overline{\omega}dx\\ &= \int_\Omega \phi\,\mathrm{sign}_+\overline{\omega}\mathrm{L}\overline{\omega}\ (\text{since } \frac{|\phi_n|}{\delta}\leqslant ||\nabla\phi_n||_\infty \leqslant c||\nabla\phi||_\infty).\end{aligned}$$

Now we come to the proof of the uniqueness result stated in Theorem 4.2.

Proof of Theorem 4.2. Since the v.w.s. ω satisfies $V\omega\in L^1(\Omega;\delta)$, so if $V\geqslant c\,\delta^{-r}$, for $r>2$, we have in a neighborhood U of $\partial\Omega$

$$\int_\Omega |\omega|\delta^{-(r-1)}dx \leqslant c\int_U V|\omega|\delta dx + c_1\int_\Omega|\omega|dx < +\infty.$$

Thus $\omega\in L^1(\Omega;\delta^{-\widetilde{r}})$ with $\widetilde{r}=r-1>1$ for $r>2$.
If $\omega_1,\ \omega_2$ are two v.w.s. then $\omega=\omega_1-\omega_2$

$$\mathrm{L}\omega = \mathrm{L}(\omega_1-\omega_2) = -\Delta\omega + \mathrm{div}\,(\vec{u}\omega) = -V\omega\in L^1(\Omega;\delta).$$

We deduce from the Corollary 4 of Theorem 4.4 that $\forall\,\phi\geqslant 0,\ \phi\in C^2(\overline{\Omega}),\ \phi=0$ on $\partial\Omega$

$$\int_\Omega |\omega|\mathrm{L}^*\phi dx \leqslant -\int_\Omega \phi\,\mathrm{sign}\,(\omega)V\omega dx = -\int_\Omega \phi\, V|\omega|dx \leqslant 0.$$

For $\vec{u}\in L^{p,1}(\Omega)^N$, ($p\geqslant N$ as in the statement of Theorem 4.2) let us consider $\phi_0\in H^1_0(\Omega)$ solution of

$$\mathrm{L}^*\phi_0 = -\Delta\phi_0 - \vec{u}\,\nabla\phi_0 = 1.$$

Then $\phi_0\geqslant 0,\ \phi_0\in W^2L^{p,1}(\Omega)$ according to the above regularity result, (see Propositions 11 or 12) and ϕ_0 can be approximated by a sequence $\phi_{0j}\in C^2(\overline{\Omega}),\ \phi_{0j}\geqslant 0$, $\phi_{0j}=0$ on $\partial\Omega$ satisfying

$$\mathrm{L}^*_j\phi_{0j} = -\Delta\phi_{0j} - \vec{u_j}\cdot\nabla\phi_{0j} = 1,\ \vec{u}_j\to\vec{u} \text{ in } L^{p,1},\ \vec{u}_j\in\mathcal{V},$$

so that

$$||\phi_{0j}||_{W^2L^{p,1}}\leqslant c.$$

Indeed, we may assume that ϕ_{0j} converges weakly to a function $\overline{\phi}_0$ in $W^2L^{p,1}(\Omega)$,

$$\nabla\phi_{0j}(x)\to\nabla\overline{\phi}_0(x) \text{ and } \phi_{0j}(x)\to\overline{\phi}_0(x)\ a.e.\ x\in\Omega.$$

Since

$$\int_\Omega |\omega|\,|\vec{u_j}-\vec{u}| \leqslant ||\vec{u_j}-\vec{u}||_{L^{N,1}}|\omega|_{L^{N',\infty}},$$

and

$$||\nabla\phi_{0j}||_\infty\leqslant c,$$

we deduce that

$$\lim_{j\to+\infty}\int_\Omega |\omega|\vec{u}_j\cdot\nabla\phi_{0j} = \int_\Omega|\omega|\vec{u}\cdot\nabla\overline{\phi}_0 dx.$$

Thus

$$\mathrm{L}^*\overline{\phi}_0 = 1,\quad \overline{\phi}_0\in W^2L^{N,1}(\Omega)\cap H^1_0(\Omega).$$

By uniqueness $\overline{\phi}_0 = \phi_0$ and then $\mathrm{L}_j^*\phi_{0j} \rightharpoonup \mathrm{L}^*\phi_0$ weakly in $L^{N,1}$. Since, we have

$$\int_\Omega |\omega|\mathrm{L}^*\phi_{0j}dx \leqslant 0.$$

$$0 \leqslant \int_\Omega |\omega|dx = \int_\Omega |\omega|\mathrm{L}_j^*\phi_{0j}dx \leqslant \int_\Omega |\omega|(\mathrm{L}_j^*\phi_{0j} - \mathrm{L}^*\phi_{0j})dx \xrightarrow[j\to+\infty]{} 0,$$

we arrive to $\omega = 0$. □

Remark 7.
In Theorem 4.3 and Theorem 4.4, if $\vec{u} \equiv 0$ (or $\vec{u} \in C^1(\overline{\Omega})^N$) then we can weaken the conditions on $\overline{\omega}$ reducing it to $\overline{\omega}$ belongs to $L^1(\Omega;\delta^{-r}),\ r > 1$. Then the above conclusions hold true.

Remark 8.
In fact, in Corollary 3, we can state that the unique solution of (1) (without any indication of the boundary condition) **must** satisfy that $\omega = 0$ on $\partial\Omega$ at least if ω is differentiable. Indeed, a consequence of Lemma 7 we have

$$L^1(\Omega,\delta^{-r}) \cap W^1L^{p,q}(\Omega) = W_0^1L^{p,q}(\Omega) \text{ if } r > 1\ (1 \leqslant p, q \leqslant +\infty).$$

Remark 9. There is a large amount of works in the literature in which the uniqueness of solutions of suitable elliptic problems is established without indicating any boundary condition but these previous papers deal with degenerate elliptic operators (see, e.g. [3], [4], [21] and the references therein). We point out that the main reason to get this type of results in our case (in which the diffusion operator is the simplest one and is not degenerate) is the presence of a very singular coefficient of the zero order term (the potential $V(x)$) which is "pathological" since it is more singular on the boundary of the domain than what the Hardy inequality may allow.

4.3 Boundedness in $L^{N'}(\Omega)$ of the v.w.s., regularity and blow-up in absence of any potential ($V = 0$)

Since the very weak solutions found in Theorem 4.1 needs not be in $L^1(\Omega)$ our main goal now (assuming $V \equiv 0$) is to analyze under which conditions ω is globally integrable. We have

Theorem 4.5. (Integrability in $L^{N'}(\Omega)$.)
Let f be in $L^1\Big(\Omega;\delta\big(1+|\mathrm{Log}\,\delta|\big)^{\frac{1}{N'}}\Big)$, $\dfrac{1}{N}+\dfrac{1}{N'}=1$, $V=0$, $\vec{u}\in(L^N(\mathrm{Log}\,L)^{\frac{\beta}{N}})^N$, with $\beta > N-1$, $\mathrm{div}\,(\vec{u}) = 0$ in Ω and $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Then the unique very weak solution ω of equation (1) belongs to $L^{N'}(\Omega)$.

We recall the

Lemma 4.5. (see [37])
Let Ω be a bounded open Lipschitz set and $\alpha > 0$. Then, there exists a constant $c_\alpha(\Omega) > 0$ such that $\forall\,\phi \in W_0^1L^\alpha_{exp}(\Omega)$

$$|\phi(x)| \leqslant c_\alpha(\Omega)\delta(x)(1 + |\mathrm{Log}\,\delta(x)|)^\alpha||\nabla\phi||_{L^\alpha_{exp}(\Omega)}.$$

Proof of Theorem 4.5 (boundedness in $L^{N'}(\Omega)$). Let ω be the very weak solution found in Theorem 4.1 and assume that

$$f \in L^1\big(\Omega;\delta(1 + |\mathrm{Log}\,\delta|)^{\frac{1}{N'}}\big).$$

We know that there exists a sequence $\vec{u}_j \in \mathcal{V}$ such that the corresponding sequence $(\omega_j)_j$ satisfying relation (58) verifies $\omega_j \rightharpoonup \omega$ weak-* in $L^{N',\infty}$ and that $\forall\,\phi \in H_0^1 \cap W^2L^N(\Omega)$

$$\int_\Omega \omega_j\big[-\Delta\phi - \vec{u}_j\nabla\phi\big]dx = \int_\Omega f\phi dx. \tag{81}$$

Here $\vec{u}_j$ converges in $(L^N(\operatorname{Log} L)^{\frac{\beta}{N}})^N = \Lambda$ to $\vec{u}$ strongly where $\beta > N-1$. Let $g \in L^N(\Omega)$and let ϕ_j be the solution of

$$\phi_j \in W^2L^N(\Omega) \text{ such that } -\Delta\phi_j - \vec{u}_j\nabla\phi_j = g \text{ in } \Omega,\ \phi_j = 0 \text{ on } \partial\Omega.$$

Then according to Theorem 3.1, we have

$$||\phi_j||_{W^2L^N(\Omega)} \leqslant K_\varepsilon \frac{1+||\vec{u}_j||_\Lambda}{1-\varepsilon||\vec{u}_j||_\Lambda}||g||_{L^N(\Omega)},$$

with

$$\varepsilon \sup_j ||\vec{u}_j||_\Lambda \leqslant \frac{1}{2}, \text{ for some } \varepsilon > 0.$$

Thus

$$||\phi_j||_{W^2L^N(\Omega)} \leqslant K(\Omega)||g||_{L^N(\Omega)}. \tag{82}$$

By the Trudinger's type inclusion (see Lemma 3.1)

$$||\nabla\phi_j||_{L_{exp}^{\frac{1}{N'}}} \leqslant K_{10}||\phi_j||_{W^2L^N(\Omega)} \leqslant K_{11}||g||_{L^N(\Omega)}. \tag{83}$$

Therefore, considering equation (81), we have

$$\int_\Omega \omega_j g dx = \int_\Omega f\phi_j dx, \tag{84}$$

with the help of Lemma 4.5 with $\alpha = \dfrac{1}{N'}$ and estimate (83), this relation gives:

$$\int_\Omega \omega_j g dx \leqslant K_{12}||g||_{L^N}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx. \tag{85}$$

Hence

$$\sup_{||g||_{L^N}=1}\int_\Omega \omega_j g dx \leqslant K_{12}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx, \tag{86}$$

which shows that :

$$||\omega||_{L^{N'}(\Omega)} \leqslant K_{12}\int_\Omega |f|\delta(x)(1+|\operatorname{Log}\delta(x)|)^{\frac{1}{N'}}dx, \tag{87}$$

proving the result. □

For the case $V \equiv 0$, we can always obtain the $W^{1,q}(\Omega)$-regularity, for $q \geqslant 1$, provided some integrability on f but also on $\vec{u}$. Here is a first result in that direction :

Theorem 4.6.
Let f be in $L^1(\Omega;\delta(1+|\operatorname{Log}\delta|))$, $V = 0$, and $\vec{u}$ in $\mathrm{bmo}_r(\Omega)^N$. Then, the very weak solution found in Theorem 4.1 belongs to $W_0^{1,1}(\Omega)$.

Proof. As before we consider the approximating problem (57) with $\vec{u}_j = \vec{u}$, say

$$\begin{cases} -\Delta\omega_j + \vec{u}\cdot\nabla\omega_j = f_j & \text{in} \quad \Omega, \\ \omega_j \in H_0^1(\Omega)\cap W^2L^{p,1}(\Omega) & \forall\, p < +\infty. \end{cases}$$

Thus, taking $\phi \in W_0^1\mathrm{bmo}_r(\Omega)$ we have

$$\int_\Omega \nabla\omega_j\cdot\nabla\phi\,dx + \int_\Omega \vec{u}\cdot\nabla\omega_j\phi\,dx = \int_\Omega f_j\phi dx \Longleftrightarrow \int_\Omega \big[\nabla\omega_j\cdot\nabla\phi - \vec{u}\cdot\nabla\phi\,\omega_j\big]dx = \int_\Omega f_j\phi\,dx.$$

Let $F_j = \dfrac{\nabla\omega_j}{|\nabla\omega_j|}$ if $\nabla\omega_j \neq 0$, and 0 otherwise, $F_j \in L^\infty(\Omega)^N$, $||F_j||_\infty \leqslant 1$. According to Proposition 10, there exists a function $\phi_j \in W_0^1\mathrm{bmo}_r(\Omega)$ such that

$$-\Delta\phi_j - \vec{u}\nabla\phi_j = -\mathrm{div}\,(F_j), \text{ and } ||\phi_j||_{W_0^1L^q} \leqslant c_9||F_j||_{L^q} \leqslant c_q < +\infty\ \forall\, q > 1,$$

$$\Longleftrightarrow \int_\Omega \nabla\phi_j\nabla\varphi\,dx - \int_\Omega \vec{u}\nabla\phi_j\varphi\,dx = \int_\Omega F_j\nabla\varphi\,dx \quad \forall\,\varphi\in H_0^1(\Omega).$$

Choosing $\varphi = \omega_j$, we have

$$\int_\Omega |\nabla\omega_j|dx = \int_\Omega \nabla\phi_j\cdot\nabla\omega_j\,dx - \int_\Omega \vec{u}\cdot\nabla\phi_j\omega_j\,dx = \int_\Omega f_j\phi_j\,dx. \tag{88}$$

From Lemma 4.5, and by the John-Nirenberg inequality (see [47]) we have :

$$|\phi_j(x)| \leqslant c(\Omega)\delta(x)(1+|\mathrm{Log}\,\delta(x)|)||\nabla\phi_j||_{L_{exp}} \leqslant c(\Omega)\delta(x)(1+|\mathrm{Log}\,\delta(x)|)||\nabla\phi_j||_{\mathrm{bmo}_r(\Omega)}. \tag{89}$$

We recall that

$$||\nabla\phi_j||_{\mathrm{bmo}_r} \leqslant K(||F_j||_\infty + ||\vec{u}\phi_j||_{\mathrm{bmo}_r}) \leqslant c, \tag{90}$$

since $\phi_j \to \phi$ strongly in $C^{0,\alpha}(\overline{\Omega})$ (see Proposition 10).
Combining (88) to (90), we have

$$\int_\Omega |\nabla\omega_j|dx \leqslant c\int_\Omega |f_j|\delta(x)(1+|\mathrm{Log}\,\delta|)dx \leqslant K\int_\Omega |f|\delta(1+\mathrm{Log}\,\delta|)dx; \tag{91}$$

using also the fact that

$$\begin{cases} \omega_j \to \omega \text{ strongly in } L^q(\Omega)\ q < N', \\ \omega_j \rightharpoonup \omega \text{ weakly in } W^{1,q}_{\mathrm{loc}}(\Omega)\ 1 < q < 1 + \dfrac{1}{N}, \end{cases}$$

we deduce that :

$$\int_\Omega |\nabla\omega|dx \leqslant c\int_\Omega |f|\delta(1+|\mathrm{Log}\,\delta|)dx.$$

$\square$

Let us prove that if we enhance the integrability condition on f to $f \in L^1(\Omega,\delta^\alpha)$ for some $\alpha \in]\,0,1\,[$ then we can weaken the condition on $\vec{u}$ to $\vec{u} \in L^{\frac{N}{1-\alpha}}(\Omega)^N$ and in that case we have

Theorem 4.7.
Let f be in $L^1(\Omega, \delta^\alpha)$ for some $\alpha \in]0,1[$, $V = 0$, $\vec{u} \in L^{\frac{N}{1-\alpha}}(\Omega)$ with $\mathrm{div}\,(\vec{u}) = 0$, $\vec{u}\cdot\vec{n} = 0$ on $\partial\Omega$. Then, the very weak solution ω found in Theorem 4.1 belongs to $W_0^1 L^{\frac{N}{N-1+\alpha}}(\Omega)$. Moreover, there exists a constant $K(\alpha;\Omega) > 0$ such that

$$||\omega||_{W_0^1 L^{\frac{N}{N-1+\alpha}}(\Omega)} \leqslant K(\alpha;\Omega)\big(1 + ||\vec{u}||_{L^{\frac{N}{1-\alpha}}}\big)||f||_{L^1(\Omega,\delta^\alpha)}.$$

The proof of Theorem 4.7 relies on the following result, dual of Proposition 8.

Proposition 15.
Let $\vec{u} \in L^{p,q}(\Omega)$, $p > N, q \in [1, +\infty]$, $V = 0$, and $F \in L^{p',q'}(\Omega)^N$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1 = \dfrac{1}{q} + \dfrac{1}{q'}$. Then there exists $\overline{\omega} \in W_0^1 L^{p',q'}(\Omega)$ such that

$$-\Delta\overline{\omega} + \vec{u}\cdot\nabla\overline{\omega} = -\mathrm{div}\,(F), \tag{92}$$

which is equivalent to

$$a(\overline{\omega};\phi) = \int_\Omega \nabla\overline{\omega}\cdot\nabla\phi\,dx + \int_\Omega \vec{u}\cdot\nabla\overline{\omega}\phi\,dx = \int_\Omega F\cdot\nabla\phi\,dx \tag{93}$$

$\forall\,\phi \in W_0^1 L^{p,q}(\Omega)$. Moreover

$$||\nabla\omega||_{L^{p',q'}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}}$$

Proof. Let G be in $L^{p,q}(\Omega)^N$, $p > N$. Following Proposition 8, there exists a function $\phi_0 \in W_0^1 L^{p,q}(\Omega)$ such that

$$\int_\Omega \nabla\phi_0\cdot\nabla\varphi\,fx - \int_\Omega \vec{u}\cdot\nabla\phi_0\varphi\,dx = \int_\Omega G\cdot\nabla\varphi\,dx \qquad \forall\,\varphi \in C_c^\infty(\Omega).$$

Since

$$-\int_\Omega \vec{u}\cdot\nabla\phi_0\varphi\,dx = \int_\Omega \vec{u}\cdot\nabla\varphi\phi_0\,dx,$$

by using a density argument over the set of test functions there exists

$$a(\varphi,\phi_0) = \int_\Omega \nabla\phi_0\cdot\nabla\varphi + \int_\Omega \vec{u}\cdot\nabla\varphi\phi_0 = \int_\Omega G\cdot\nabla\varphi\,dx \qquad \forall\,\varphi \in W_0^1 L^{p',q'}(\Omega). \tag{94}$$

Let $F_k \in L^\infty(\Omega)^N$, with $|F_k(x)| \leqslant |F(x)|$ in Ω. Then we have that $\overline{\omega}_k \in W_0^1 L^{p',q'}(\Omega) \cap H^1(\Omega)$ such that

$$a(\overline{\omega}_k,\phi) = \int_\Omega F_k\cdot\nabla\phi\,dx \qquad \forall\phi \in W_0^1 L^{p,q}(\Omega). \tag{95}$$

Choosing $\phi = \phi_0$ in this last equation, we find that

$$\int_\Omega G\cdot\nabla\overline{\omega}_k\,dx = a(\overline{\omega}_k,\phi_0) = \int_\Omega F_k\cdot\nabla\phi_0\,dx. \tag{96}$$

Following Proposition 8, we have

$$||\nabla\phi_0||_{L^{p,q}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||G||_{L^{p,q}}. \tag{97}$$

From relation (96) and (97), we have

$$\int_\Omega G \cdot \nabla \overline{\omega}_k \, dx \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F_k||_{L^{p',q'}}||G||_{L^{p,q}}. \tag{98}$$

So that we have

$$\sup_{||G||_{L^{p,q}}=1} \int_\Omega G \cdot \nabla \overline{\omega}_k dx \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}} \tag{99}$$

$$||\nabla \overline{\omega_k}||_{L^{p',q'}} \leqslant K_{pq}(1 + ||\vec{u}||_{L^{p,q}})||F||_{L^{p',q'}}. \tag{100}$$

By standard argument, we derive the existence of $\overline{\omega}$ satisfying (92) as a weak limit of $\overline{\omega}_k$ in $W^1L^{p',q'}(\Omega)$. □

Proof of Theorem 4.7. Since $f \in L^1(\Omega; \delta^\alpha)$, according to [18], there exists $F = \nabla v \in L^{\frac{N}{N-1+\alpha}}(\Omega)^N$, $f = -\text{div}\,(F)$. Moreover, the function $F_k = \nabla v_k$ satisfying $-\Delta v_k = T_k(f)$ converge to F strongly in $L^{\frac{N}{N-1+\alpha}}(\Omega)^N$ (v_k and v are in $W_0^1 L^{\frac{N}{N-1+\alpha}}(\Omega)$).

Since the very weak solution ω found in Theorem 4.1 is the weak-* limit of the solutions of the regularized problem

$$\begin{cases} -\Delta\omega_k + \vec{u} \cdot \nabla\omega_k = f_k = T_k(f) = -\text{div}\,(F_k), \\ \omega_k \in W^2L^q(\Omega) \cap H_0^1(\Omega) \text{ with } q = \dfrac{N}{1-\alpha} > N, \end{cases}$$

and

$$||\nabla\omega_k||_{L^{q'}(\Omega)} \leqslant K_q(1 + ||\vec{u}||_{L^q})||F_k||_{L^{q'}(\Omega)}, \quad q' = \frac{N}{N-1+\alpha},$$

letting $k \to +\infty$, we derive the result once we know that $||F||_{L^{q'}} \leqslant c||f||_{L^1(\Omega,\delta^\alpha)}$. □

When $\alpha = 0$, that is $f \in L^1(\Omega)$, we can weaken the integrability assumption on $\vec{u}$ as we state in the following result :

Theorem 4.8.
Let f be in $L^1(\Omega)$, $V = 0$, $\vec{u} \in L^N(\Omega)^N$ with $\text{div}\,(\vec{u}) = 0$ on $\partial\Omega$, $\vec{u} \cdot \vec{n} = 0$ on $\partial\Omega$. Then, the very weak solution ω found in Theorem 4.1 belongs to $W_0^1 L^{N',\infty}(\Omega)$.
Moreover, there exists a constant $c(\Omega) > 0$, independent of $\vec{u}$, such that

$$||\nabla\omega||_{L^{N',\infty}(\Omega)} \leqslant c(\Omega)||f||_{L^1(\Omega)}.$$

Proof. Let $\vec{u}_j \in \mathcal{V}$ be such that $\vec{u}_j \to \vec{u}$ in $L^N(\Omega)^N$, and let $f_j \in L^\infty(\Omega)$ be such that $|f_j(x) \leqslant |f(x)|$ and $f_j(x) \to f(x)$ a.e, $x \in \Omega$.
Let us consider the functions $\omega_j \in W^2L^m(\Omega) \cap H_0^1(\Omega)\ \forall\, m < +\infty$ satisfying

$$-\Delta\omega_j + \vec{u}_j \cdot \nabla\omega_j = f_j.$$

Then

$$\int_\Omega |\nabla T_k(\omega_j)|^2 dx + \int_\Omega \vec{u}_j \cdot \nabla \int_0^{\omega_j} T_k(\sigma)d\sigma = \int_\Omega T_k(\omega_j) f_j(x) dx,$$

and since by integration by parts we have $\displaystyle\int_\Omega \vec{u}_j \cdot \nabla \int_0^{\omega_j} T_k(\sigma)d\sigma = 0$ we get

$$\int_\Omega |\nabla T_k(\omega_j)|^2 dx \leqslant k \int_\Omega |f(x)| dx. \tag{101}$$

By the Poincaré-Sobolev inequality

$$\int_\Omega |T_k(\omega_j)|^2 dx \leqslant c_\Omega k \int_\Omega |f(x)|dx.$$

By Proposition 13, we deduce that

$$\|\nabla\omega_j||_{L^{N',\infty}(\Omega)} \leqslant c_\Omega \int_\Omega |f(x)|dx.$$

Since $\vec{u}_j \to \vec{u}$ in $L^N(\Omega)^N$ and by compactness $\omega_j \to \omega$ in $L^{N'}(\Omega)$ $\Big($note that $W^1L^{N',\infty}(\Omega) \hookrightarrow L^{\frac{N}{N-2},\infty}(\Omega)$ for $N \geqslant 3$ (see [35])$\Big)$, we then have for all $\phi \in C^2(\overline{\Omega})$ with $\phi = 0$ on $\partial\Omega$,

$$\int_\Omega \omega_j \vec{u}_j \nabla\phi dx \xrightarrow[j\to+\infty]{} \int_\Omega \omega\vec{u}\cdot\nabla\phi dx,$$

so that ω solves (16) for $V \equiv 0$. □

As for the case $\vec{u} = 0$, the additional regularity questions are numerous; for instance, does there exists a datum $f \in L^1(\Omega;\delta)$ for which we have

$$\int_\Omega |\nabla\omega|dx = +\infty \text{ or } \int_\Omega |\omega|^{N'}dx = +\infty?$$

For the explosion of the norm of ω in $L^{N'}$, we can adopt the same proof as for the explosion of the gradient in $L^1(\Omega)$. We have

Theorem 4.9. (blow-up in $L^{N'}(\Omega)$)
Assume that $N \geqslant 3$, $\vec{u} \in C^{0,\alpha}(\overline{\Omega})^N$, $\alpha > 0$, $V = 0$. Then there exists a function f in $L^1_+(\Omega;\delta)\backslash L^1(\Omega,\delta(1+|\mathrm{Log}\,\delta|)^{\frac{1}{N'}})$ such that the very weak solution ω found in Theorem 4.1 satisfies that ω does not belong to $L^{N'}(\Omega)$).

First we recall the following result that can be proved as in [39] (see also [40]).

Lemma 4.6. Let $N \geqslant 3$. There exists a function $g \in L^N_+(\Omega)$ such that the unique solution $\psi \in W^2L^N(\Omega) \cap H^1_0(\Omega)$ of $-\Delta\psi - \vec{u}\cdot\nabla\psi = g$ satisfies :

1. $\psi(x) \geqslant c_1\delta(x),\ \forall\, x,$
2. $\sup\left\{\dfrac{\psi(x)}{\delta(x)},\ x \in \Omega\right\} = +\infty,$
3. $L^1_+(\Omega;\delta)\backslash L^1(\Omega,\psi)$ is non empty.

Arguing as in [39], [1], we consider $g_k = T_k(g)$, g given by Lemma 4.6 such that

$$\psi_k \in W^2L^q(\Omega) \cap H^1_0(\Omega) \text{ for all } q < +\infty,\ -\Delta\psi_k - \vec{u}\nabla\phi_k = T_k(g).$$

Now assume that for all $f \in L^1(\Omega;\delta)$, we have for the v.w.s. $||\omega||_{L^{N'}} < +\infty$. Then by the Banach-Steinhauss uniform boundedness theorem as in [1, 39], we derive the existence of a constant $c_0 > 0$ such that

$$||\omega||_{L^{N'}} \leqslant c_0 \int_\Omega |f|\delta dx \quad \forall\, f \in L^1(\Omega;\delta),$$

and

$$\int_\Omega \omega\big[-\Delta\phi - \vec{u}\cdot\nabla\phi\big]\,dx = \int_\Omega f\phi\,dx \qquad \forall\,\phi\in W^2L^{N,1}(\Omega)\cap H_0^1(\Omega).$$

Taking $\phi=\psi_k$, and $f\in L^1_+(\Omega;\delta)\backslash L^1_+(\Omega,\psi)$ we see that

$$0\leqslant \int_\Omega f\psi_k = \int_\Omega \omega g_k\,dx \leqslant ||\omega||_{L^{N'}}||g||_{L^N} < +\infty. \tag{102}$$

Letting $k\to+\infty$, we have a contradiction since

$$\underline{\lim}_{k\to+\infty}\int_\Omega f\psi_k \geqslant \int_\Omega f\psi\,dx = +\infty,$$

which concludes the proof Theorem 4.9. □

Remark 10.
We can give the more precise information that the function f in Theorem 4.9 is not in $L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^{\frac{1}{N'}})$ (due to Theorem 4.5).

4.4 Some final conclusion

In the opinion of the authors, the results of this paper open many different further applications in different directions. Besides the consideration of the list of concrete problems mentioned in the Introduction other studies can be carried out. For instance, following the arguments of [19], it is not complicated to extend many of the results of this paper to the study of semilinear problems for which equation (1) is replaced by the equation

$$-\Delta\omega + \vec{u}\cdot\nabla\omega + V\omega + \beta(x,u,\nabla u) = f(x) \text{ on } \Omega,$$

when β is nondecreasing in u. Moreover the consideration of parabolic problems of the type

$$\omega_t - \Delta\omega + \vec{u}\cdot\nabla\omega + V\omega + \beta(x,u,\nabla u) = f(t,x) \textit{ on } \Omega\times(0,T),$$

can be carried out with the help of the results of this paper (mainly the $L^1(\Omega;\delta)$-accretiveness property of the associates operator). The details will be given in some separate work by the authors.

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After this article was completed, we learned, during a presentation at a conference (March 29-30, 2017) in Poitiers, France, that L. Orsina and A. Ponce have obtained related results in the references [33, 34]. Their results deal essentially with the existence and the use of the normal derivative for any function in $W_0^{1,1}(\Omega)$. *In the improved version [34] that they sent to us by the authors after the conference, they add a new proposition (Proposition 2.7) which provides a complement to our results since it gives a qualitative property for* ω *solution of our problem (16) if the velocity* $\vec{u}$ *is zero when the solution is integrable on the whole domain (for a right hand side* f *in* $L^1(\Omega,\delta(1+|Log\delta|))$. *We note also that J.I.Díaz has already derived results similar to their Proposition 2.7 in [15, 16].*

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Partial differential equations/Theory of signals

Rigidity of optimal bases for signal spaces

Rigidité des bases optimales pour les espaces de signaux

Haïm Brezis [a,b,c], David Gómez-Castro [d,e]

[a] *Department of Mathematics, Hill Center, Busch Campus, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA*
[b] *Departments of Mathematics and Computer Science, Technion, Israel Institute of Technology, 32000 Haifa, Israel*
[c] *Laboratoire Jacques-Louis-Lions, Université Pierre-et-Marie-Curie, 4, place Jussieu, 75252 Paris cedex 05, France*
[d] *Dpto. de Matemática Aplicada, Universidad Complutense de Madrid, Spain*
[e] *Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, Spain*

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ABSTRACT

We discuss optimal L^2-approximations of functions controlled in the H^1-norm. We prove that the basis of eigenfunctions of the Laplace operator with Dirichlet boundary condition is the only orthonormal basis (b_i) of L^2 that provides an optimal approximation in the sense of

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \le \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \ge 1.$$

This solves an open problem raised by Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel, and N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

RÉSUMÉ

On s'intéresse à l'approximation optimale pour la norme L^2 de fonctions contrôlées en norme H^1. On prouve que la base des fonctions propres du laplacien avec condition de Dirichlet au bord est l'unique base orthonormale (b_i) de L^2 qui réalise une approximation optimale au sens de

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \le \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \ge 1.$$

Ceci résout un problème ouvert posé par Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel et N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

E-mail addresses: brezis@math.rutgers.edu (H. Brezis), dgcastro@ucm.es (D. Gómez-Castro).

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1. Introduction and main result

This note is a follow-up of the papers by Y. Aflalo, H. Brezis and R. Kimmel [2] and Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel and N. Sochen [1].

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $e = (e_i)$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ is the ordered sequence of eigenvalues repeated according to their multiplicity.

We first recall a very standard result:

Theorem 1.1. *We have, for all $n \geq 1$,*

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 \leq \frac{\| \nabla f \|_{L^2}^2}{\lambda_{n+1}} \qquad \forall f \in H_0^1(\Omega). \tag{2}$$

Here and throughout the rest of this paper $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.

Indeed, we may write

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 = \left\| \sum_{i=n+1}^{+\infty} (f, e_i) e_i \right\|_{L^2}^2 = \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{3}$$

On the other hand,

$$\|\nabla f\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i (f, e_i)^2 \geq \sum_{i=n+1}^{+\infty} \lambda_i (f, e_i)^2 \geq \lambda_{n+1} \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{4}$$

Combining (3) and (4) yields (2). □

The authors of [2] and [1] have investigated the "optimality" in various directions of the basis (e_i), with respect to inequality (2). Here is one of their results restated in a slightly more general form.

Theorem 1.2 *(Theorem 3.1 in [2]). There is no integer $n \geq 1$, no constant $0 \leq \alpha < 1$ and no sequence $(\psi_i)_{1 \leq i \leq n}$ in $L^2(\Omega)$ such that*

$$\left\| f - \sum_{i=1}^{n} (f, \psi_i) \psi_i \right\|_{L^2}^2 \leq \frac{\alpha}{\lambda_{n+1}} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5}$$

The proof in [2] relies on the Fischer–Courant max–min principle (see Remark 3.3 below). For the convenience of the reader, we present a very elementary proof based on a simple and efficient device originally due to H. Poincaré [5, pp. 249–250] (and later rediscovered by many people, e.g., H. Weyl [7, p. 445] and R. Courant [3, pp. 17–18]; see also H. Weinberger [6, p. 56] and P. Lax [4, p. 319]).

Suppose not, and set

$$f = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n + c_{n+1} e_{n+1} \tag{6}$$

where $c = (c_1, c_2, \cdots, c_n, c_{n+1}) \in \mathbb{R}^{n+1}$. The under-determined linear system

$$(f, \psi_i) = 0, \qquad \forall i = 1, \cdots, n \tag{7}$$

of n equations with $n+1$ unknowns admits a non-trivial solution. Inserting f into (5) yields

$$\lambda_{n+1} \sum_{i=1}^{n+1} c_i^2 \leq \alpha \sum_{i=1}^{n+1} \lambda_i c_i^2 \leq \alpha \lambda_{n+1} \sum_{i=1}^{n+1} c_i^2. \tag{8}$$

Therefore $\sum_{i=1}^{n+1} c_i^2 = 0$ and thus $c = 0$. A contradiction. This proves Theorem 1.2. □

The authors of [1] were thus led to investigate the question of whether inequality (2) holds *only* for the orthonormal bases consisting of eigenfunctions corresponding to ordered eigenvalues. They established that a "discrete", i.e., finite-dimensional, version does hold; see [1, Theorem 2.1] and Remark 3.2 below. But their proof of "uniqueness" could not be adapted to the infinite-dimensional case (because it relied on a "descending" induction). It was raised there as an open problem (see [1, p. 1166]). Our next result solves this problem.

Theorem 1.3. *Let* (b_i) *be an orthonormal basis of* $L^2(\Omega)$ *such that, for all* $n \geq 1$,

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega). \tag{9}$$

Then, (b_i) *consists of an orthonormal basis of eigenfunctions of* $-\Delta$ *with corresponding eigenvalues* (λ_i).

2. Proof of Theorem 1.3

A basic ingredient of the argument is the following lemma:

Lemma 2.1. *Assume that (9) holds for all* $n \geq 1$ *and all* $f \in H_0^1(\Omega)$, *and that*

$$\lambda_i < \lambda_{i+1} \tag{10}$$

for some $i \geq 1$. *Then*

$$(b_j, e_k) = 0, \qquad \forall j, k \text{ such that } 1 \leq j \leq i < k. \tag{11}$$

Proof. Fix $k > i$. Let l be the largest integer $l \leq k-1$ such that

$$\lambda_l < \lambda_{l+1}. \tag{12}$$

Clearly

$$i \leq l \tag{13}$$

and

$$\lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_k. \tag{14}$$

Applying (9) for $n = l$, we get

$$\left\| f - \sum_{i=1}^{l} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{l+1}} \quad \forall f \in H_0^1(\Omega). \tag{15}$$

We use again Poincaré's "magic trick". Take f of the form

$$f = c_1 e_1 + \cdots + c_l e_l + c e_k \tag{16}$$

such that

$$(f, b_j) = 0 \qquad \forall j = 1, \cdots, l. \tag{17}$$

This is a system of l linear equations with $l+1$ unknowns, so that there are nontrivial solutions. We may as well assume that

$$c_1^2 + \cdots + c_l^2 + c^2 = 1. \tag{18}$$

By (15) and (14), we have

$$\lambda_{l+1} \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2 + \lambda_k c^2 = \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2 + \lambda_{l+1} c^2. \tag{19}$$

From (18) we get

$$\lambda_{l+1}(c_1^2 + \cdots + c_l^2) \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2. \tag{20}$$

Thus

$$(\lambda_{l+1} - \lambda_1) c_1^2 + \cdots + (\lambda_{l+1} - \lambda_l) c_l^2 \leq 0. \tag{21}$$

By (12) the coefficients $\lambda_{l+1} - \lambda_i$ are positive for every $i = 1, \cdots, l$. Therefore

$$c_1 = \cdots = c_l = 0. \tag{22}$$

Hence $c = \pm 1$ so that $f = \pm e_k$ and by (17)

$$(b_j, e_k) = 0 \qquad \forall j = 1, \cdots, l. \tag{23}$$

The conclusion follows from (23) and (13). □

Before we present the proof in the general case, for the convenience of the reader, we start with the case of simple eigenvalues. Since $\lambda_1 < \lambda_2$ then, by the lemma,

$$(b_1, e_k) = 0 \qquad \forall k \geq 2. \tag{24}$$

Thus $b_1 = \pm e_1$. Next we apply the lemma with $\lambda_2 < \lambda_3$. We have that

$$(b_2, e_k) = 0 \qquad \forall k \geq 3. \tag{25}$$

Also, we have that

$$(b_2, e_1) = \pm(b_2, b_1) = 0. \tag{26}$$

Therefore $b_2 = \pm e_2$. Similarly, we have that $b_i = \pm e_i$ for $i \geq 3$.

We now turn to the general case:

Proof of Theorem 1.3. As above we have $b_1 = \pm e_1$. Consider the first index $i \geq 2$ such that $\lambda_i < \lambda_{i+1}$. Call it i_1. From the lemma we have that

$$(b_j, e_k) = 0 \qquad \forall j, k \text{ such that } 1 \leq j \leq i_1 < k. \tag{27}$$

Therefore $b_2, \cdots, b_{i_1} \in \operatorname{span}(e_2, \cdots, e_{i_1})$. Hence, each b_j with $2 \leq j \leq i_1$ is an eigenfunction of $-\Delta$ with corresponding eigenvalue $\lambda = \lambda_2 = \cdots = \lambda_{i_1}$. Therefore, due to dimensions, $b_2, \cdots, b_{i_1}$ is an orthonormal basis of

$$\operatorname{span}(b_2, \cdots, b_{i_1}) = \operatorname{span}(e_2, \cdots, e_{i_1}) = \ker(-\Delta - \lambda_{i_1} I); \tag{28}$$

in particular each

$$e_k \in \operatorname{span}(b_1, \cdots, b_{i_1}) \qquad k = 1, \cdots, i_1. \tag{29}$$

Consider the next block

$$\lambda = \lambda_{i_1+1} = \cdots = \lambda_{i_2} < \lambda_{i_2+1}. \tag{30}$$

From the lemma we have that

$$(b_j, e_k) = 0 \qquad \forall j, k \text{ such that } 1 \leq j \leq i_2 < k. \tag{31}$$

We also know that for $j \geq i_1 + 1$,

$$(b_j, e_k) = 0 \qquad k = 1, \cdots, i_1 \tag{32}$$

because of (29). Combining (31) and (32) yields

$$(b_j)_{i_1+1 \leq j \leq i_2} \in \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2}. \tag{33}$$

As above, we conclude, using (30), that $b_{i_1+1}, \cdots, b_{i_2}$ is an orthonormal basis of

$$\operatorname{span}(b_j)_{i_1+1 \leq j \leq i_2} = \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2} = \ker(-\Delta - \lambda_{i_2} I). \tag{34}$$

Similarly for the next blocks. □

3. Final remarks

Remark 3.1. We call the attention of the reader to the fact that the functions b_i are only assumed to be in $L^2(\Omega)$ and we deduce from Theorem 1.3 that (surprisingly) they belong to $H_0^1(\Omega) \cap \mathcal{C}^\infty(\Omega)$.

Remark 3.2. Theorem 1.3 holds in a more general setting. Let V and H be Hilbert spaces such that $V \subset H$ with compact and dense inclusion ($\dim H \leq +\infty$). Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form for which there exist constants $C, \alpha > 0$ such that, for all $v \in V$,

$$\begin{aligned} a(v, v) &\geq 0, \\ a(v, v) + C|\, v|_H^2 &\geq \alpha \|v\|_V^2. \end{aligned}$$

Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the sequence of eigenvalues associated with the orthonormal (in H) eigenfunctions $e_1, e_2, \cdots \in V$, i.e.,

$$a(e_i, v) = \lambda_i (e_i, v) \qquad \forall v \in V,$$

where $(\cdot, \cdot)$ denotes the scalar product in H. We point out that, in this general setting, it may happen that $\lambda_1 = 0$ (e.g., $-\Delta$ with Neumann boundary conditions); and λ_1 may have multiplicity > 1. Recall that, for every $n \geq 1$ and $f \in V$:

$$\lambda_{n+1} \left| f - \sum_{i=1}^{n} (e_i, f) e_i \right|_H^2 \le a(f, f). \tag{35}$$

Let (b_i) be an orthonormal basis of H such that for all $n \ge 1$ and $f \in V$

$$\lambda_{n+1} \left| f - \sum_{i=1}^{n} (b_i, f) b_i \right|_H^2 \le a(f, f). \tag{36}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of a with corresponding eigenvalues (λ_i). The proof is identical to the one above.

When $\dim H < +\infty$ and $V = H$, this result is originally due to [1]. The proof of rigidity was quite different and could not be adapted to the infinite-dimensional case. It was raised there as an open problem.

Remark 3.3. Recall that the usual Fischer–Courant max–min principle asserts that for every $n \ge 1$, we have

$$\lambda_{n+1} = \max_{\substack{M \subset L^2(\Omega) \\ M \text{ linear space} \\ \dim M = n}} \ \min_{\substack{0 \ne f \in H_0^1(\Omega) \\ f \in M^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2}, \tag{37}$$

(see, e.g., [4] or [6]). Our technique sheds some light about the structure of the maximizers in (37). Let (b_i) be an orthonormal sequence in $L^2(\Omega)$ such that, for every $n \ge 1$,

$$\lambda_{n+1} = \min_{\substack{0 \ne f \in H_0^1(\Omega) \\ f \in M_n^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2} \qquad \text{where } M_n = \operatorname{span}(b_1, b_2, \cdots, b_n). \tag{38}$$

Then, each b_i is an eigenfunction associated with λ_i. This is an easy consequence of the proof of Theorem 1.3.

Remark 3.4 *(rigidity of the tail).* Assume that (9) holds only for $n = k, k+1, \cdots$. Let the eigenvalues be simple. Applying the same reasoning as in our proof gives

$$\operatorname{span}(b_1, \cdots, b_n) = \operatorname{span}(e_1, \cdots, e_n) \qquad n = k, k+1, \cdots \tag{39}$$

The same argument as before yields $b_i = \pm e_i$ for $i = k+1, k+2, \cdots$. Concerning the b_i's for $i \le k$, we only know that $b_1, \cdots, b_k \in \operatorname{span}(e_1, \cdots, e_k)$ and therefore they are smooth. A similar result holds if the eigenvalues are not simple.

Remark 3.5. We now turn to the reverse situation, i.e., we assume that (9) holds only for $1 \le n \le k$. In this case (9) yields very little information on the b_i's. Consider for example the case $n = k = 1$. In other words, assume that $b = b_1 \in L^2(\Omega)$ is such that $\|b\|_{L^2} = 1$ and

$$\| f - (f, b) b\|_{L^2}^2 \le \frac{1}{\lambda_2} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{40}$$

Of course, (40) holds with $b = e_1$. From Lemma 2.1, we know that (40) implies that

$$(e_2, b) = 0. \tag{41}$$

Clearly, (41) is not sufficient. Indeed, take $b = e_3$. Then, (41) holds but (40) fails for $f = e_1$. We do not have a simple characterization of the functions b satisfying (40). But we can construct a large family of functions b (which need not be smooth) such that (40) holds. Assume that $0 < \lambda_1 \le \lambda_2 < \lambda_3$. Let $\chi \in L^2(\Omega)$ be any function such that

$$(e_1, \chi) = 0, \tag{42}$$

$$(e_2, \chi) = 0, \tag{43}$$

$$\|\chi\|_{L^2}^2 = 1. \tag{44}$$

Set

$$b = \alpha e_1 + \varepsilon \chi \qquad \alpha^2 + \varepsilon^2 = 1, \text{ with } 0 < \varepsilon < 1. \tag{45}$$

Claim: there exists $\varepsilon_0 > 0$, depending on $(\lambda_i)_{1\le i\le 3}$, such that for every $0 < \varepsilon < \varepsilon_0$ (40) holds. We have, for $f \in H_0^1(\Omega)$, and with $c_i = (f, e_i)$,

$$\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f-(f,b)b\|_{L^2}^2 = \frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \left(\|f\|_{L^2}^2 - (f,b)^2\right) \tag{46}$$

$$= \sum_{i=1}^{+\infty}\frac{\lambda_i}{\lambda_2}c_i^2 - \sum_{i=1}^{+\infty}c_i^2 + (f,b)^2. \tag{47}$$

On the other hand

$$(f,b)^2 = (\alpha(f,e_1)+\varepsilon(f,\chi))^2 \tag{48}$$

$$= \alpha^2c_1^2 + 2\alpha\varepsilon(f,e_1)(f,\chi) + \varepsilon^2(f,\chi)^2 \tag{49}$$

$$= \alpha^2c_1^2 + 2\alpha\varepsilon(f-c_2e_2,e_1)(f-c_2e_2,\chi) + \varepsilon^2(f,\chi)^2 \tag{50}$$

$$\ge \alpha^2c_1^2 - 2\varepsilon\|f-c_2e_2\|_{L^2}^2 \tag{51}$$

$$= \alpha^2c_1^2 - 2\varepsilon\sum_{i\neq 2}c_i^2. \tag{52}$$

Going back to (47), using (45) and choosing $\varepsilon < \varepsilon_0$ small enough, yields

$$\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f-(f,b)b\|_{L^2}^2$$

$$\ge \left(\frac{\lambda_1}{\lambda_2} - 2\varepsilon - \varepsilon^2\right)c_1^2 + \sum_{i=3}^{+\infty}\left(\frac{\lambda_i}{\lambda_2} - 1 - 2\varepsilon\right)c_i^2 \tag{53}$$

$$\ge 0. \tag{54}$$

Remark 3.6. In the general setting of Remark 3.2, it may happen that $0 = \lambda_1 < \lambda_2$. Suppose now that $b \in H$ is such that $\|b\|_H = 1$ and

$$\|\, f - \ (f,b)b\|_H^2 \le \frac{1}{\lambda_2}a(f,f) \qquad \forall f \in V. \tag{55}$$

Claim: we have $b = \pm e_1$. Indeed, let $f = e_1$ in (55) we have that

$$\|\, e_1 - \ (e_1,b)b\|_H^2 \le \frac{\lambda_1}{\lambda_2} = 0. \tag{56}$$

Therefore $b = \pm e_1$.

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