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in candidacy for the degree of

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To Ana María Maitín López,
with all my love

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Abstract

The main objective of this dissertation consists in analyzing the diffusive Lotka–Volterra competition model by studying the uniqueness, multiplicity and stability of its component-wise positive steady states, preferably, when the diffusion rates of the species in the inhabiting territory are sufficiently small, though some of our results do not require that. To accomplish this task, it is imperative to go down one step and analyze first the logistic equations obtained by uncoupling the system. As a consequence, this PhD Thesis has been distributed into two parts, the first one dealing with the equation and the second one with the diffusive competition system, which have been built as a selection of the most significant results found in the research papers [42, 39, 38, 41, 40], authored by the candidate together with his adviser.

Part I consists of Chapters 2 and 3 and covers both the analysis of the sublinear logistic equation and the associated superlinear indefinite problem. More precisely, Chapter 2, which polishes the theory developed in [41], begins providing us with a characterization of the regularity of $\partial\Omega$, for an open bounded set $\Omega \subset \mathbb{R}^N$, $N \geq 1$, through the regularity of the distance function along an outward vector field, or by means of the fact $\partial\Omega = \Psi^{-1}(0)$ for some smooth non-degenerate function, Ψ, defined on a neighborhood of $\partial\Omega$. This result is crucial to deal with non-classical mixed boundary conditions, as well as to adapt a technical device of López-Gómez [87], for constructing certain supersolutions, to our general setting here. Then, Chapter 2 establishes the existence, uniqueness and monotonicity properties of the sublinear diffusive logistic equation

$$\begin{cases} d\mathcal{L}u = u\,h(u,x) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a domain of class $\mathcal{C}^2$, $d > 0$ is the diffusion rate, $\mathcal{L}$ is a uniformly elliptic differential operator in divergence form, i.e.,

$$\mathcal{L} = -\mathrm{div}(A\nabla\cdot) + B\nabla + C,$$

for some $A \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^1(\bar\Omega))$, $B \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar\Omega))$ and $C \in \mathcal{C}(\bar\Omega)$, and $\mathcal{B}$ is a boundary operator of non-classical mixed type. It is mixed in the sense that $\mathcal{B}$ can be either of Dirichlet or Robin type at any component of $\partial\Omega$, and non-classical since the function $\beta \in \mathcal{C}(\partial\Omega)$ taking part in the Robin boundary operator,

$$\mathcal{R}u = \partial_\nu u + \beta u$$

can change sign. In the problem (1), $h(u,x)$ is a non-linear continuous function in $x \in \bar\Omega$, of class $\mathcal{C}^1$ in $u \in \mathbb{R}$, strictly decreasing in $u \geq 0$, among some other technical assumptions.

The main aim of this chapter is to ascertain the limiting profile of the positive solutions of (1) as $d > 0$ decays to zero, generalizing the already classical results of Cantrell and Cosner [18], Furter and López-Gómez [48], and Hutson, López-Gómez, Mischaikow and Vickers [63]. The singular perturbation result for the equation, established through Theorem 2.21, is necessary to obtain its counterpart for the diffusion-competition system, which is a central result of this dissertation.

Regarding Chapter 3, it consists of Sections 1–4 of [40]. Throughout this chapter, a new generalized Picone identity is used systematically to study the superlinear indefinite problem

$$\begin{cases} \mathcal{L}u = \lambda u - a(x)f(u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the function $f \in \mathcal{C}^r(\mathbb{R})$, $r \geq 2$, satisfies $f(0) = 0$ and $f \not\equiv 0$, and $a \in \mathcal{C}(\bar{\Omega})$ may change sign. As usual in this dissertation, $\lambda \in \mathbb{R}$ is regarded as a bifurcation parameter. In particular, Chapter 3 not only adapts the results of Gómez-Reñasco and López-Gómez [51, 52], and López-Gómez [89], obtained for the choice $f(u) = u^q$ for some $q \geq 2$, to cover the case of non-classical mixed boundary conditions, but it also provides us with their optimality, as we were able to complement the previous results by establishing that they cannot hold even when f is the sum of two monomials which is an arbitrarily small perturbation of u^q, $q \geq 2$.

Part II is made up of the remaining chapters, and deals with the diffusive Lotka–Volterra competition two species model

$$\begin{cases} \frac{\partial u}{\partial t} + d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial t} + d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \Omega \times (0, +\infty), \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{2}$$

with general uniformly elliptic operators and boundary operators of mixed non-classical type, much like in the equation. The associated non-diffusive model, obtained by switching off to zero the diffusion coefficients d_1 and d_2, goes back to the seminal works of Lotka [78] and Volterra [114, 115], and hence, the coefficient functions, $\lambda, \mu \in \mathcal{C}(\bar{\Omega})$ and $a, b, c, d \in \mathcal{C}(\bar{\Omega}; (0, +\infty))$, acquire a special meaning from the point of view of population dynamics: λ and μ are growth rates of the species (in the absence of competition), a and d measure the intraspecific competition among the individuals of the same species, and b and c measure the interspecific competition among the individuals of u and v.

Chapter 4 consists of Section 4 of [38], which generalizes the main result of [42] and those of Sections 3 and 5 of [39]. It studies the singular perturbation problem for the diffusive competition model, providing us with the limiting profile of the coexistence steady states in those regions of Ω where the non-diffusive counterpart of the model (2) exhibits global attractivity, i.e., the region where both species become extinct, Ω_{ext}, or both species are permanent, Ω_{per}, or one species dominates the other, Ω^u_{do} and Ω^v_{do}. According to this result, the limiting profile of the singular perturbation problem for the system may not be determined in the region Ω_{bi}, where the non-diffusive model exhibits founder control competition (two linearly stable semitrivial steady states). Thus, the singular perturbation result established in this dissertation improves substantially Theorem 4.1 of Hutson, López-Gómez, Mischaikow and Vickers [63], Theorem 5.2(iii) of He and Ni [53], and Lemma 3.3 and Theorem 1.2 of Hutson, Lou and Mischaikow [65], where $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, $\mathcal{B}_1 = \mathcal{B}_2$ are non-flux boundary operators, and either the competition coefficients are assumed to be constant or the behavior of the species is considered to be homogeneous in Ω. Although the main technical device to get these singular perturbation results is a monotone scheme introduced by López-Gómez and Sabina in Section 3.3 of [77] to estimate the region of extinction of the species, adapting it to a spatially heterogeneous setting is much more intricate and technically sophisticated.

Chapter 5 consists of Sections 5, 6 and 7 of [38] and provides us with the Induced Instability Principle, which establishes that the local instability of a steady state of the non-diffusive model in a region arbitrarily small of Ω is induced, globally, to any family of steady states of the diffusive model perturbing from it therein. This result is completely new and, among its main consequences, it provides us with a generalized version of Theorem 2.1(i) of Furter and López-Gómez [48]. Furthermore, the Induced Instability Principle allows us to obtain the first general multiplicity result for (2) based, exclusively, on the spatial heterogeneities of the domain, whatever its geometry is. Indeed, in the symmetric case, if there are regions of Ω where the underlying non-diffusive model exhibits permanence, i.e., $\Omega_{\mathrm{per}} \neq \emptyset$, and other regions where it exhibits founder control competition, i.e., $\Omega_{\mathrm{bi}} \neq \emptyset$, then the diffusive model admits at least three coexistence steady states, two of them stable and the other unstable, for sufficiently small diffusion rates, regardless the sizes of Ω_{per} and Ω_{bi}. As a by-product, this explains why the limiting profile of a family of coexistence steady states of the diffusive model is not determined by our singular perturbation result in Ω_{bi}.

To conclude, Chapter 6 tidies up, polishes and rearranges Sections 8 and 9 of [38] and Sections 5 and 6 of [40], providing us with two situations under which, for sufficiently small diffusion rates, the model (2) exhibits a unique coexistence steady state, which is actually globally asymptotically stable. First, this dissertation establishes that this phenomenology occurs if $\bar{\Omega} = \Omega_{\mathrm{per}}$. The result found here is a substantial generalization of Theorem 1.1 of Hutson, Lou and Mischaikow [65] that uses a completely different, much simpler and more versatile proof, which allows us to deal with uniformly elliptic differential operators other than $-\Delta$, and boundary conditions other than non-flux (Robin ones of non-classical type). Finally, uniqueness is also achieved in the heterogeneous diffusion-competition model when low competition occurs, i.e., $bc \lneq ad$ (and so $\Omega_{\mathrm{bi}} = \emptyset$), which provides us with a substantial extension of Theorem 3.4 of He and Ni [55], obtained for constant competition coefficients, with $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ under non-flux boundary conditions.

Resumen

El objetivo principal de esta tesis es analizar el modelo difusivo de tipo Lotka–Volterra competitivo mediante el estudio de la unicidad, multiplicidad y estabilidad de sus soluciones estacionarias no negativas, preferiblemente cuando la difusión de las especies en el medio es suficientemente pequeña, aunque algunos de los resultados presentados aquí no requieren esta hipótesis. Para llevar a cabo esta tarea, es imperativo descender un escalón y analizar primero las ecuaciones logísticas obtenidas al desacoplar el sistema. Como consecuencia, la tesis se ha dividido en dos partes, la primera dedicada a la ecuación y la segunda al sistema difusivo-competitivo, construidas a partir de una selección de los resultados más significativos de los artículos de investigación [42, 39, 38, 41, 40], que han sido realizados por el candidato a doctor junto con su tutor.

La Parte I consta de los Capítulos 2 y 3, y cubre tanto el análisis de la ecuación logística sublinear, como el problema superlinear indefinido asociado. De forma más precisa, el Capítulo 2, que refina la teoría desarrollada en [41], comienza caracterizando la regularidad de $\partial\Omega$, para un conjunto abierto y acotado $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a través de la regularidad de la función distancia en la dirección de un campo vectorial exterior a Ω, así como por medio del hecho de que $\partial\Omega = \Psi^{-1}(0)$ para alguna función suave no degenerada, Ψ, definida en un entorno de $\partial\Omega$. Este resultado es crucial para poder lidiar con condiciones de frontera mixtas no clásicas, así como para adaptar a este marco un recurso de carácter técnico introducido por López-Gómez [87], que permite construir ciertas supersoluciones. A raíz de este resultado, el Capítulo 2 establece la existencia, unicidad, y propiedades de monotonía, de soluciones positivas para la ecuación difusiva de tipo logístico sublineal

$$\begin{cases} d\mathcal{L}u = u\,h(u,x) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

donde $\Omega \subset \mathbb{R}^N$, $N \geq 1$, es un dominio de clase $\mathcal{C}^2$, $d > 0$ es la tasa de difusión, $\mathcal{L}$ es un operador diferencial uniformemente elíptico en forma de divergencia, esto es,

$$\mathcal{L} = -\mathrm{div}(A\nabla\cdot) + B\nabla + C,$$

con $A \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $B \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ y $C \in \mathcal{C}(\bar{\Omega})$, mientras que $\mathcal{B}$ es un operador de frontera de tipo mixto no clásico. $\mathcal{B}$ es mixto en tanto que, en cada componente de $\partial\Omega$, puede ser o bien de tipo Dirichlet o de tipo Robin, y es no clásico en el sentido de que la función $\beta \in \mathcal{C}(\partial\Omega)$ que aparece en el operador de frontera de tipo Robin,

$$\mathcal{R}u = \partial_{\boldsymbol{\nu}} u + \beta u$$

puede cambiar de signo. Por otra parte, en el problema (3), $h(u,x)$ es una función no lineal continua en $x \in \bar{\Omega}$, de clase $\mathcal{C}^1$ en $u \in \mathbb{R}$, y estrictamente decreciente en $u \geq 0$, entre otras hipótesis de carácter técnico.

El objetivo principal de este capítulo es proporcionar el perfil límite de las soluciones positivas de (3) cuando $d > 0$ converge a cero, lo cual generaliza los resultados más recientes desarrollados para este tipo de ecuaciones, esto es, los de Cantrell y Cosner [18], Furter y López-Gómez [48] y Hutson, López-Gómez, Mischaikow y Vickers [63]. El resultado de perturbación singular para la ecuación, establecido en el Teorema 2.21, es crucial para poder obtener su análogo para el sistema difusivo de tipo competitivo, uno de los resultados centrales de esta tesis.

En cuanto al Capítulo 3, este consiste en las Secciones 1–4 de [40]. A lo largo de este capítulo, se usa sistemáticamente una nueva identidad de Picone generalizada para estudiar el problema superlineal indefinido

$$\begin{cases} \mathcal{L}u = \lambda u - a(x)f(u) & \text{in } \ \Omega, \\ \mathcal{B}u = 0 & \text{on } \ \partial\Omega, \end{cases}$$

donde la función $f \in \mathcal{C}^r(\mathbb{R})$, $r \geq 2$, satisface $f(0) = 0$ y $f \not\equiv 0$, y $a \in \mathcal{C}(\bar{\Omega})$ puede cambiar de signo. Por otra parte, $\lambda \in \mathbb{R}$ es considerado como parámetro de bifurcación. En particular, el Capítulo 3 no solo adapta los resultados de Gómez-Reñasco y López-Gómez [51, 52], y López-Gómez [89], obtenidos para la elección $f(u) = u^q$ para algún $q \geq 2$, para que sigan siendo válidos en el caso de condiciones de frontera mixtas no clásicas, sino que también los complementa estableciendo su optimalidad, probando que dejan de ser ciertos incluso cuando la función f es una perturbación arbitrariamente pequeña de u^q, $q \geq 2$, definida como la suma de dos monomios.

La Parte II engloba los restantes capítulos, y centra su análisis en el modelo difusivo de tipo Lotka–Volterra competitivo para dos especies

$$\begin{cases} \frac{\partial u}{\partial t} + d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \ \Omega \times (0,+\infty), \\ \frac{\partial v}{\partial t} + d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \ \Omega \times (0,+\infty), \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \ \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x) & \text{in } \ \Omega, \end{cases} \tag{4}$$

con $\mathcal{L}_1$ y $\mathcal{L}_2$ operadores uniformemente elípticos, y $\mathcal{B}_1$ y $\mathcal{B}_2$ operadores de frontera mixtos no clásicos, como los de la ecuación. El modelo no difusivo asociado, esto es, el obtenido al hacer cero los coeficientes de difusión d_1 y d_2, se remonta a los trabajos de Lotka [78] y Volterra [114, 115], y por tanto las funciones $\lambda, \mu \in \mathcal{C}(\bar{\Omega})$ y $a, b, c, d \in \mathcal{C}(\bar{\Omega}; (0,+\infty))$, adquieren su propio significado desde el punto de vista de la dinámica de poblaciones: λ y μ son tasas de crecimiento de las especies (en ausencia de competición), a y d miden la competición intraespecífica entre los individuos de la misma especie, y b y c miden la competición interespecífica entre los individuos de u y v.

El Capítulo 4 recopila la teoría desarrollada en la Sección 4 de [38], que generaliza el resultado principal de [42] y los de las Secciones 3 y 5 de [39]. Este capítulo estudia el problema de perturbación singular en el modelo difusivo de competición, proporcionando el perfil límite de los estados estacionarios de coexistencia, esto es, aquellos con ambas componentes positivas, en las regiones de Ω donde el modelo no difusivo asociado a (4) exhibe un atractor global, esto es, la región donde ambas especies se extinguen, Ω_{ext}, la región donde ambas especies son permanentes, Ω_{per}, o las regiones donde una especie domina a la otra, Ω^{u}_{do} y Ω^{v}_{do}. De acuerdo con este resultado, el perfil límite para el problema de perturbación para el sistema puede no quedar determinado en la región Ω_{bi}, donde el modelo no difusivo exhibe 'founder control competition' (dos estados estacionarios semitriviales linealmente estables). Por lo tanto, el resultado de perturbación singular obtenido en esta tesis mejora sustancialmente el Teorema 4.1 de Hutson, López-Gómez, Mischaikow y Vickers [63], el Teorema 5.2 (iii) de He y Ni [53], y el Lema 3.3 y el Teorema 1.2 de Hutson, Lou y Mischaikow [65], en los que $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, $\mathcal{B}_1 = \mathcal{B}_2$ son operadores de frontera de tipo Neumann, y o bien los coeficientes de competición, a, b, c, y d, se suponen constantes, o el comportamiento de las especies se considera homogéneo en Ω. Aunque el principal recurso técnico usado para obtener estos resultados de perturbación singular es un esquema monótono introducido por López-Gómez y Sabina en la Sección 3.3 de [77] para estimar la región de extinción de las especies, adaptarlo para lidiar con un contexto espacialmente heterogéneo es mucho más intrincado y sofisticado técnicamente.

El Capítulo 5, por su parte, consiste en las Secciones 5, 6 y 7 of [38]. En él se establece el Principio de Inestabilidad Inducida, según el cual la inestabilidad local de un estado estacionario (u_*, v_*) del modelo no difusivo en una región arbitrariamente pequeña de Ω es inducida, globalmente, a una familia de estados estacionarios del modelo difusivo (4) que perturben desde (u_*, v_*) en dicha región. Este resultado es completamente nuevo y, entre sus principales consecuencias, está el hecho de que proporciona el Teorema 2.1(i) de Furter y López-Gómez [48]. Además, el Principio de Inestabilidad Inducida nos permite obtener el primer resultado general de multiplicidad para (4) basado exclusivamente en las heterogeneidades espaciales del dominio, sea cual sea su geometría. En efecto, en el caso simétrico, si hay regiones de Ω donde el modelo no difusivo asociado a (4) exhibe permanencia, es decir, $\Omega_{\text{per}} \neq \emptyset$, y regiones donde exhibe 'founder control competition', esto es, $\Omega_{\text{bi}} \neq \emptyset$, entonces el modelo difusivo admite al menos tres estados estacionarios de coexistencia, dos de ellos estables y el otro inestable, para coeficientes de difusión suficientemente pequeños, independientemente de los tamaños de Ω_{per} y Ω_{bi}. Como consecuencia, esto explica por qué el perfil límite de una familia arbitraria de estados estacionarios de coexistencia del modelo difusivo no queda determinado por nuestro resultado de perturbación singular en Ω_{bi}.

Para concluir, el Capítulo 6 limpia, pule y reorganiza las Secciones 8 y 9 de [38] y las Secciones 5 y 6 de [40], proporcionando dos situaciones en las que, si los coeficientes de difusión son suficientemente pequeños, el modelo (4) exhibe un único estado estacionario de coexistencia, que es, de hecho, un atractor global para las soluciones de (4) con ambas componentes positivas. En primer lugar, se prueba la unicidad en el caso $\bar{\Omega} = \Omega_{\text{per}}$. El resultado obtenido aquí supone una generalización sustancial del Teorema 1.1 de Hutson, Lou y Mischaikow [65], y usa una prueba completamente diferente, mucho más simple y más versátil, lo que nos permite tratar con operadores elípticos uniformes distintos de $-\Delta$, y condiciones de contorno distintas a las de tipo Neumann (Robin de tipo no clásico). Finalmente, la unicidad también se consigue en el modelo difusivo-competitivo heterogéneo cuando hay baja competición, esto es, $bc \lneq ad$ (luego $\Omega_{\text{bi}} = \emptyset$), lo que supone una extensión sustancial del Teorema 3.4 de He y Ni [55], obtenido para coeficientes de competición constantes, $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ y condiciones de frontera de tipo Neumann.

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Chapter 1

Introduction

The main objective of this dissertation consists in analyzing the diffusive Lotka–Volterra competition model by studying the uniqueness, multiplicity and stability of its component-wise positive solutions, as well as their limiting profiles, when the diffusions of the species in the inhabiting territory are sufficiently small, which is a situation rather realistic from the point of view of the applications in Population Dynamics. To accomplish this task it is necessary to go down one step and analyze first the logistic equation, because a sharp analysis of this equation is imperative to treat the system.

The mathematical theory of diffusive competing species, as we know it today, grew from the pioneering papers of Casten and Holland [21], Matano and Mimura [99], Kishimoto and Weinberger [69], Dancer [27, 28, 29], Blat and Brown [12, 13], Cosner and Lazer [23], Ahmad and Lazer [1], Hess and Lazer [57], López-Gómez and Pardo [93, 95], Eilbeck, Furter and López-Gómez [33], Furter and López-Gómez [47, 48], López-Gómez and Sabina [77], Hutson, López-Gómez, Mischaikow and Vickers [63], and Hsu, Smith and Waltman [62], where the foundations of the theory of competing species developed in this dissertation were settled. The pioneering work of Hsu, Hubbell and Waltman [61] had been formulated through a system of Ordinary Differential Equations.

Among the most astonishing effects of the spatial dispersion through random diffusion, Eilbeck, Furter and López-Gómez [33] were able to show, answering to a clever question of C. Cosner, that, under homogeneous Dirichlet boundary conditions, the species can be permanent even when the non-spatial associated model exhibits founder control competition. The related question of C. Cosner raised in a personal letter to J. C. Eilbeck short time after the Conference on Reaction-Diffusion Equations organized by K. J. Brown and A. Lacey in Heriot-Watt University (Edinburgh) was held on May 1988. Since then, it became apparent the crucial role played by the spatial dispersion in the biological problem of the permanence of competing species. The result of Eilbeck, Furter and López-Gómez [33] contrasts, very strongly, with a previous finding of Kishimoto and Weinberger [69], who established that, under non-flux boundary conditions, permanence cannot occur in the homogeneous competition model if its non-spatial counterpart exhibits founder control competition and the territory is convex. Some time later, Cano-Casanova and López-Gómez [17] established that permanence is possible, regardless of the level of the aggression between the competitors, as soon as each of them can refuge on some spatial protection zone, sufficiently large, where it can stay free from the aggression of the others.

This thesis studies, with much more generality, the role played by the spatial heterogeneities of the model on its dynamics. It has been built as a selection of the most significant results found in the next research papers:

[38] S. Fernández-Rincón and J. López-Gómez. Spatially heterogeneous Lotka–Volterra competition. *Nonlinear Analysis*, 165:33–79, 2017.

[39] S. Fernández-Rincón and J. López-Gómez. Spatial versus non-spatial dynamics for diffusive Lotka–Volterra competing species models. *Calculus of Variations and Partial Differential Equations*, 56(71):1–37, 2017.

[40] S. Fernández-Rincón and J. López-Gómez. The Picone identity: A device to get optimal uniqueness results and global dynamics in Population Dynamics. *arXiv:1911.05066*, pages 1–48, 2019.

[41] S. Fernández-Rincón and J. López-Gómez. The singular perturbation problem for a class of generalized logistic equations under non-classical mixed boundary conditions. *Advanced Nonlinear Studies*, 19:1–27, 2019.

[42] S. Fernández-Rincón and J. López-Gómez. A singular perturbation result in competition theory. *Journal of Mathematical Analysis and Applications*, 445:280–296, 2017.

This dissertation has been distributed in two parts: the first one dealing with the single equation, the second one with the competition model itself. Part I consists of Chapter 2, which polishes the theory developed in [41] for the logistic equation, and Chapter 3, which consists of Sections 1–4 of [40] and is focussed on the study of the superlinear indefinite problem. Part II consists of the remaining chapters and deals with the diffusive Lotka–Volterra competition system. In particular, Chapter 4 consists of Section 4 of [38], which generalizes the main result of [42] and those of Sections 3 and 5 of [39]. Precisely, it studies the singular perturbation problem in the competition model. Chapter 5 consists of Sections 5, 6 and 7 of [38] and establishes the Induced Instability Principle and some of its consequences. To conclude, Chapter 6 rearranges and adapts Sections 8 and 9 of [38] and Sections 5 and 6 of [40], providing two new situations where the diffusive Lotka–Volterra competition model exhibits a unique coexistence steady state. It should be emphasized that [40] also provides one of the first uniqueness results for diffusive Lotka–Volterra models of symbiotic type, though it has been left outside the scope of this dissertation, focused on the competition model.

Next, we provide an overview on the main results established in the thesis, highlighting their strengths while comparing them with those existing in the literature associated with problems of similar nature. As it will become apparent soon, they answer to many challenges of the research teams of W. M. Ni, one of the leading experts in PDE's.

1.1 Characterizing the regularity of the boundary

As soon as the diffusive competition problem studied in this thesis requires a place to be set, an habitat for the species to perform such a competition, the first result in this dissertation is focused on providing different ways to describe its regularity. More precisely, it provides us with a number of equivalent characterizations of the regularity of a subdomain, $\Omega \subset \mathbb{R}^N$, $N \geq 1$, whose boundary, the edges of the inhabiting area, is assumed to be a topological $(N-1)$-manifold. For the adequate statement of this result, it is appropriate to introduce notation related to what it is understood by *projection* onto the boundary of Ω, $\partial\Omega$, and *distance* with respect to $\partial\Omega$. Although these definitions are easily understood when the projection and the distance are taken along the normal vector field, and coincide with the usual ones in $\mathbb{R}^N$, we need to generalize these concepts to cover arbitrary vector fields as those considered in this dissertation.

According to Definition 2.2, given $\boldsymbol{\nu} : \partial\Omega \to \mathbb{R}^N$ a vector field on $\partial\Omega$, and $\mathcal{U} \subset \mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, a neighborhood of $\partial\Omega$, it is said that a function $\Pi_{\boldsymbol{\nu}} : \mathcal{U} \to \partial\Omega$ is a *projection onto* $\partial\Omega$ *along* $\boldsymbol{\nu}$ if it satisfies the next assumptions:

(i) *(Identity on $\partial\Omega$)* $\Pi_{\boldsymbol{\nu}}(x) = x$ for all $x \in \partial\Omega$. This implies that $\Pi_{\boldsymbol{\nu}}^2 = \Pi_{\boldsymbol{\nu}}$.

(ii) *(Constant along the vector field)* $\Pi_{\boldsymbol{\nu}}$ is constant along the ray $x + \lambda\boldsymbol{\nu}(x)$ for every $x \in \partial\Omega$ and $\lambda \in \mathbb{R}$ such that $x + \lambda\boldsymbol{\nu}(x) \in \mathcal{U}$. Analogously,

$$\frac{\partial \Pi_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}(x) = 0 \quad \text{for all} \;\; x \in \mathcal{U}.$$

It should be noted that every projection function onto $\partial\Omega$ admits an associated distance function. Indeed, the *distance to* $\partial\Omega$ *along* $\boldsymbol{\nu}$ can be defined through

$$\operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) := \frac{|x - \Pi_{\boldsymbol{\nu}}(x)|}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|} \quad \text{for every} \;\; x \in \mathcal{U},$$

where $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^N$. However, this distance function is not smooth on $\partial\Omega$, as it behaves much like the absolute value function, and hence, in practice, one needs to consider the *regularized distance function* $\mathfrak{d}_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ defined by

$$\mathfrak{d}_{\boldsymbol{\nu}}(x) := \begin{cases} \operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) & \text{if} \;\; x \in \mathcal{U} \cap \Omega, \\ -\operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) & \text{if} \;\; x \in \mathcal{U} \setminus \Omega. \end{cases}$$

As shown by Theorem 2.3 stated below, the regularity of the boundary can be characterized in terms of the regularity of the projection and regularized distance function (in (b)), or in terms of the existence and regularity of a function whose zeros describe $\partial\Omega$ (in (d) and (f)).

Excerpt from Theorem 2.3 (Characterization of the regularity of $\partial\Omega$).

Assume that Ω is an open subdomain of $\mathbb{R}^N$ such that $\partial\Omega$ is a topological $(N-1)$-manifold. Then, for every integer $r \geq 2$, the next assertions are equivalent:

(a) *$\partial\Omega$ is of class $\mathcal{C}^r$.*

(b) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$ and, for every outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$, there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a projection onto $\partial\Omega$ along $\boldsymbol{\nu}$, $\Pi_{\boldsymbol{\nu}} : \mathcal{U} \to \partial\Omega$, of class $\mathcal{C}^{r-1}$. Moreover, the associated regularized distance function $\mathfrak{d}_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ is of class $\mathcal{C}^r$.*

(d) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$ and, for every outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$, there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$ with $\partial\Omega \subset \mathcal{U}$, and a function $\psi \in \mathcal{C}^r(\mathcal{U};\mathbb{R})$ such that $\psi(x) < 0$ for all $x \in \Omega \cap \mathcal{U}$, $\psi(x) > 0$ for all $x \in \mathcal{U} \setminus \bar{\Omega}$, and*

$$\min_{x\in\partial\Omega} \frac{\partial\psi}{\partial\boldsymbol{\nu}}(x) > 0.$$

In particular, $\psi(x) = 0$ for all $x \in \partial\Omega$, by the continuity of ψ on $\mathcal{U}$.

(f) *There exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$ with $\partial\Omega \subset \mathcal{U}$ and a function $\Psi \in \mathcal{C}^r(\mathcal{U};\mathbb{R})$ such that $\Omega = \{x \in \mathcal{U} \,:\, \Psi(x) < 0\}$, $\partial\Omega = \Psi^{-1}(0)$, and $|\nabla\Psi(x)| = 1$ for all $x \in \partial\Omega$.*

Theorem 2.3 seems to be the first result in the literature establishing the equivalence between the regularity of $\partial\Omega$ by means of the definition using charts, and the regularity of the distance function, or the fact $\partial\Omega = \Psi^{-1}(0)$ for some smooth function, Ψ, defined on a neighborhood of $\partial\Omega$, such that $\Psi(x) = 0$ if and only if $x \in \partial\Omega$. However, the problem of determining the regularity of the *standard* distance function

$$\operatorname{dist}(x,\partial\Omega) := \min_{y\in\partial\Omega} |x-y|, \quad x \in \mathbb{R}^N,$$

from the regularity of Ω, is not new, as it received a great attention during the 20$^{\text{th}}$ century. Indeed, Serrin [110, Le. 1 in Sec. 3] showed that if S is an $N-1$ dimensional surface of class $\mathcal{C}^3$, then the distance function is of class $\mathcal{C}^2$ in a neighborhood of S controled by the normal curvature of S. So, a degree of regularity is lost. This result was later improved by Gilbarg and Trudinger [50], who proved that the regularity is preserved for classes greater or equal than $\mathcal{C}^2$. Precisely, their result reads as follows

Lemma 14.16 of [50]. *Let Ω be bounded and $\partial\Omega \in \mathcal{C}^k$ for $k \geq 2$. Then, there exists a positive constant μ depending on Ω such that $\operatorname{dist}(\cdot,\partial\Omega) \in \mathcal{C}^k(\Gamma_\mu)$, where*

$$\Gamma_\mu := \{x \in \bar{\Omega} \,|\, \operatorname{dist}(x,\partial\Omega) < \mu\}.$$

Some time later, Krantz and Parks [71, Ex. 4] showed that $\mathcal{C}^2$ is actually the minimal regularity condition on $\partial\Omega$ so that $\operatorname{dist}(x,\partial\Omega)$ is guaranteed to be well defined. Moreover, they proved in the same paper that the regularity result can be extended to the case $\mathcal{C}^1$ if there exits a neighborhood of $\partial\Omega$ where every point admits a unique *nearest point* in $\partial\Omega$.

Theorem 1 of [71]. *If $M \subset \mathbb{R}^N$, is a compact manifold of class $\mathcal{C}^1$ with dimension $N-1$, and there exists $\mu > 0$ such that every*

$$x \in U_\mu := \{x \in \mathbb{R}^N \,|\, \operatorname{dist}(x,\partial\Omega) < \mu\}$$

admits a unique 'nearest point' in M, then there exists an open neighborhood U

of M such that the function

$$\delta(x) := \begin{cases} \mathrm{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\mathrm{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

restricted to U is of class $\mathcal{C}^1$.

To conclude this discussion on the different approaches available to the problem of determining the regularity of the distance function, we should highlight the work of Foote [45], who provided free-coordinate proofs of Theorems 1 and 2 of Krantz and Parks [71], as well as the contributions of Li and Nirenberg [76], who established that if Ω is a domain in a smooth complete Finsler manifold with $\partial\Omega \in \mathcal{C}^{k,a}$, for some $0 < a \leq 1$ and $k \geq 2$, and G is the largest open subset of Ω such that, for every $x \in G$, there exists a unique 'closest point' from $\partial\Omega$ to x (measured in the Finsler metric), then the distance function from $\partial\Omega$ is in $\mathcal{C}^{k,a}_{\mathrm{loc}}(G \cup \partial\Omega)$.

Besides providing us with the first characterization of the regularity of $\partial\Omega$ through the regularity of the distance functions, Theorem 2.3 also seems to incorporate the *conormal vector field* into the regularity problem for the first time. But first, let us introduce the concept of *conormal vector field* associated to an operator in Ω. Assume that Ω is of class $\mathcal{C}^2$ and denote by $\mathbf{n}$ its normal vector field. Then, any uniformly elliptic operator in divergence form, namely

$$\mathcal{L} := -\mathrm{div}(A\nabla\cdot) + b\nabla + c$$

with $A \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $b \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $c \in \mathcal{C}(\bar{\Omega})$, admits an associated *conormal vector field*

$$\boldsymbol{\nu} := A\mathbf{n},$$

which is the most appropriate one to use when introducing the Robin boundary conditions. According to Theorem 2.3, since $\partial\Omega$ is of class $\mathcal{C}^2$, there exists an associated *conormal projection*, of class $\mathcal{C}^1$, and a *conormal distance*, of class $\mathcal{C}^2$, along the outward vector field $\boldsymbol{\nu}$. Despide the intuitiveness of these notations, a search in the *Web of Science Core Collection* shows no results for the terms 'conormal distance' and 'conormal projection', except for [41], though it shows 149 entries when combining 'conormal' and 'boundary'. Hence, there is no doubt that the concepts of conormal projection and conormal distance, as well as the problem of their regularities, are introduced and analyzed for the first time in Theorem 2.3, which was established in [41].

Before enunciating some of the most important implications of Theorem 2.3, we should remark that, as far as concerns the use of the different characterizations of the regularity of $\partial\Omega$ in the literature, although (a) is the most extended condition in differential geometry to discuss the regularity of a manifold, (f) is the condition used in some classical PDE papers, like in the works of Howes [58, 59, 60], De Santi [30], or Cantrell and Cosner [18]. Actually, condition (f), or equivalently (e), are crucial in the PDE framework when dealing with non-classical boundary conditions, as well as in constructing some special supersolutions, as it will become apparent below. However, it is quite surprising that its systematic use has not been yet popularized in the existing literature.

Besides the interest that Theorem 2.3 may deserve in differential geometry, as pointed out in the previous comments, it plays a crucial role when dealing with problems in partial differential equations. In particular, with those subject to non-classical Robin boundary conditions, i.e., those whose boundary operator has the form

$$\mathcal{R} := \frac{\partial}{\partial \boldsymbol{\nu}} + \beta = \langle A\nabla\cdot, \mathbf{n}\rangle + \beta = \langle \nabla\cdot, A\mathbf{n}\rangle + \beta \quad \text{on } \partial\Omega,$$

where the function $\beta \in \mathcal{C}(\bar{\Omega})$ can change sign (non-classical), and $\boldsymbol{\nu} := A\mathbf{n}$ is the conormal vector field associated to the operator $\mathcal{L}$. One may think of problems of the form

$$\begin{cases} (\mathcal{L}+\omega)\, u = f & \text{in } \Omega, \\ \mathcal{R}u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{or} \quad \begin{cases} \mathcal{L}u = u\, h(u,x) & \text{in } \Omega, \\ \mathcal{R}u = 0 & \text{on } \partial\Omega, \end{cases}$$

with appropriate functions $f \in L^2(\Omega)$ and $h \in \mathcal{C}^1(\mathbb{R}\times\Omega)$ such that h decreases in u uniformly in $\bar{\Omega}$. In these settings, the constant functions do not provide us with supersolutions anymore, as they used to do in the classical case when $\beta \geq 0$. This is a serious shortage, as they are very common to prove the existence of solutions through the method of sub and supersolutions. However, Lemma 2.7, by means of the characterization provided by Theorem 2.3, allows us to solve this problem by constructing the supersolutions as multiples of a function of the form

$$E_\mu(\cdot) := \exp(-\mu\, \mathfrak{d}_\nu(\cdot)),$$

for some $\mu \geq 0$ depending on the norm of the outward vector field $\boldsymbol{\nu}$ and the minimum of β. The function E_μ is of class $\mathcal{C}^2$ in a neighborhood of $\partial\Omega$ as soon as $\partial\Omega$ is of class $\mathcal{C}^2$, and it can be easily extended to the interior of Ω with the required regularity by means of cut-off functions.

Another application of Theorem 2.3 consists on transforming a non-classical problem, where β changes of sign, into a classical one, with the associated β non-negative, in such a way that the equation preserves its structure and main properties. As suggested by the previous example with the supersolution, and has already been established by Theorem 2.6, this can be achieved in a logistic type equation through a change of variables of the type

$$u = E_\mu w.$$

In the setting of this dissertation, such a change transforms the problem

$$\begin{cases} d\mathcal{L}u = u\, h(u,x) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{into} \quad \begin{cases} d\mathcal{L}_{E_\mu} w = w\, h_{E_\mu}(w,x) & \text{in } \Omega, \\ \mathcal{B}_{E_\mu} w = 0 & \text{on } \partial\Omega, \end{cases}$$

with both $\mathcal{L}$ and $\mathcal{L}_{E_\mu}$ being uniformly elliptic differential operators in divergence form, h and h_{E_μ} satisfying nearly the same substantial hypothesis, and $\mathcal{B}$ and $\mathcal{B}_{E_\mu}$ being boundary operators of non-classical and classical type, respectively.

The previous trick of considering a function like E_μ to prove the existence and uniqueness of weak solutions for linear BVP's goes back to Theorem 3.3 of López-Gómez [87], which reads as follows.

Adapted from Theorem 3.3 of [87]. *Suppose that Ω is a subdomain of $\mathbb{R}^N$, $N \geq 1$, of class $\mathcal{C}^1$, $\mathcal{L}$ is a uniformly elliptic operator in divergence form, and let $\mathcal{B}$ stand for a non-classical boundary operator of mixed type. Assume that a function $\psi \in \mathcal{C}^2(\bar{\Omega})$ exists such that*

$$\min_{x\in\partial\Omega} \frac{\partial \psi}{\partial \boldsymbol{\nu}}(x) > 0. \tag{1.1}$$

Then, then there exists $\omega_0 \in \mathbb{R}$ such that, for every $\omega > \omega_0$ and $f \in L^2(\Omega)$ the problem

$$\begin{cases} (\mathcal{L}+\omega)u = f & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

possesses a unique weak solution

$$u := (\mathcal{L}+\omega)^{-1} f \in W^{1,2}_{\Gamma_{\mathcal{D}}}(\Omega).$$

Here $\Gamma_{\mathcal{D}}$ stands for those components of $\partial\Omega$ where the boundary operator $\mathcal{B}$ is of Dirichlet type.

As a first step towards the proof of this result, Lemma 3.1 of [87] establishes the existence of the function ψ satisfying (1.1) if Ω is of class $\mathcal{C}^2$, which corresponds to the step (a) implies (d) of Theorem 2.3 for the $\mathcal{C}^2$-case. Immediately after Lemma 3.1 of [87], the change of variable

$$u = E_\mu v := \exp(\mu\psi)\, v,$$

is performed, for certain $\mu > 0$, so that problem (1.2), which was of non-classical type, can be transformed into the equivalent classical problem

$$\begin{cases} (\mathcal{L}_{E_\mu}+\omega)v = \frac{f}{E_\mu} & \text{in } \Omega, \\ \mathcal{B}_{E_\mu} v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\mathcal{B}_{E_\mu}$ has the (classical) coefficient $\beta_{E_\mu} \geq 0$. As a consequence, the celebrated theorem of Lax and Milgram [75], which lies on Theorem 2.1 of Gårding [49], can be applied to (1.3), hence providing us with the existence result for (1.2).

Although the original idea of considering a supersolution of exponential type, E, of the linear problem goes back to López-Gómez [87], using the change of variable $v = u/H$ for some positive function H such that $\mathcal{L}H \geq 0$ in Ω, for transforming an elliptic operator $\mathcal{L}$ with a coefficient c changing of sign into another one,

$$\mathcal{L}_H = -\mathrm{div}(A\nabla\cdot) + b_H\nabla + c_H$$

with $c_H = \mathcal{L}H/H \geq 0$, had been already used by Protter and Weinberger [107] to prove their generalized maximum principle from the Hopf's maximum principle; López-Gómez adapted that idea to transform, simultaneously, the boundary operator, and, at the end of the day, to characterize, whether or not the function H exists, which remained an open enigma in the celebrated book of Protter and Weinberger [107] (see López-Gómez [86]).

The last, but not less important, application of Theorem 2.3 consists in providing appropriate approximations of continuous functions through functions satisfying a certain boundary estimate related to the Robin boundary operator $\mathcal{R}$. Indeed, Lemma 2.9 establishes that, given a function $f \in \mathcal{C}(\bar{\Omega})$, for every $\varepsilon > 0$ there are functions $\Psi_1, \Psi_2 \in \mathcal{C}^2(\bar{\Omega})$ such that

$$f(x) - \varepsilon \leq \Psi_1(x), \Psi_2(x) \leq f(x) + \varepsilon \quad \text{for every} \;\; x \in \bar{\Omega},$$

with

$$\mathcal{R}\Psi_1(x) > 0 \quad \text{and} \quad \mathcal{R}\Psi_2(x) < 0 \quad \text{for all} \;\; x \in \partial\Omega.$$

This technical tool plays a pivotal role in the proof of the singular perturbation problem for the logistic equation which is introduced below.

1.2 Singular perturbations for sublinear logistic equations

Another main result of this thesis ascertains the limiting profile, as $d > 0$ decreases to zero, of the maximal non-negative solution of the diffusive logistic problem

$$\begin{cases} d\mathcal{L}u = u\,h(u,x) & \text{in} \;\; \Omega, \\ \mathcal{B}u = 0 & \text{on} \;\; \partial\Omega, \end{cases} \tag{1.4}$$

where $h(u,x)$ is a non-linear continuous function in $x \in \bar{\Omega}$, of class $\mathcal{C}^1$ in $u \in \mathbb{R}$, strictly decreasing in $u \geq 0$, among some other technical assumptions detailed at the beginning of Chapter 2. In (1.4), $\mathcal{L}$ is an uniformly elliptic differential operator in Ω in divergence form, i.e.,

$$\mathcal{L} := -\text{div}(A\nabla\cdot) + b\nabla + c \tag{1.5}$$

with $A \in \mathcal{M}_N^{\text{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $b \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $c \in \mathcal{C}(\bar{\Omega})$.

Throughout this dissertation, $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is assumed to be of class $\mathcal{C}^2$, and the boundary operator, $\mathcal{B}$, is allowed to be of non-classical mixed type, i.e., the components of $\partial\Omega$ are assumed to be distributed in two disjoint sets, $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{R}}$, so that $\mathcal{B}$ is the Dirichlet boundary operator on $\Gamma_{\mathcal{D}}$, i.e.,

$$\mathcal{B}u := \mathcal{D}u = u = 0 \quad \text{on} \;\; \Gamma_{\mathcal{D}} \;\; \text{for every} \;\; u \in W^{2,p}(\Omega), \;\; p > N, \tag{1.6}$$

whereas $\mathcal{B}$ as a non-classical Robin-type boundary operator on $\Gamma_{\mathcal{R}}$, i.e.,

$$\mathcal{B}u := \mathcal{R}u = \langle \nabla u, A\mathbf{n} \rangle + \beta u = 0 \quad \text{on} \;\; \Gamma_{\mathcal{R}}, \tag{1.7}$$

where $\beta \in \mathcal{C}(\Gamma_{\mathcal{R}})$ and $\mathbf{n}$ stands for the outward unit normal vector field along $\partial\Omega$. The boundary operator $\mathcal{B}$ is non-classical in the sense that β can change sign.

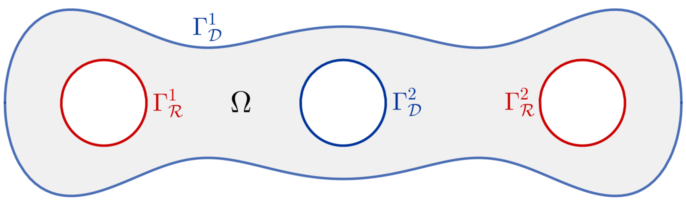


Figure 1.1: Plot of of the inhabiting territory distinguishing between the several components of $\partial\Omega$ where $\mathcal{B}$ acts as a Dirichlet or Robin boundary operator, i.e., $\Gamma_{\mathcal{D}}^1$ and $\Gamma_{\mathcal{D}}^2$, and $\Gamma_{\mathcal{R}}^1$ and $\Gamma_{\mathcal{R}}^2$, respectively. Here $\Gamma_{\mathcal{D}} = \Gamma_{\mathcal{D}}^1 \cup \Gamma_{\mathcal{D}}^2$ and $\Gamma_{\mathcal{R}} = \Gamma_{\mathcal{R}}^1 \cup \Gamma_{\mathcal{R}}^2$.

The main perturbation result for the diffusive logistic equation, stated in Theorem 2.21, establishes that the maximal non-negative solution of (1.4) converges, as d decays to zero, to the maximal non-negative solution of the equation obtained by switching off to zero the diffusion rate, d, i.e.,

$$u\,h(u,x)=0,\quad x\in\bar{\Omega}.$$

It is worth-mentioning that, for every $x\in\bar{\Omega}$, the maximal non-negative solution of this algebraic equation is the unique linearly stable, or linearly neutrally stable, non-negative steady state of the *kinetic* counterpart of (1.4), i.e.,

$$u'(t)=u(t)h(u(t),x),\quad t\in(0,+\infty),$$

which actually is a global attractor with respect to its component-wise positive solutions.

For the sake of simplicity in stating our theorem, in order to clarify as much as possible the challenges that we had to overcome to get it, as well as its main advantages and improvements with respect to the previous existing results, we will focus our attention into the simplest paradigmatic prototype equation

$$\begin{cases} d\mathcal{L}u=\lambda(x)u-a(x)u^q & \text{in } \Omega,\\ \mathcal{B}u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.8}$$

where $q\geq 2$, $d>0$, $\lambda\in\mathcal{C}(\bar{\Omega})$, and $a\in\mathcal{C}(\bar{\Omega},(0,+\infty))$, which holds for the special choice

$$h(u,x)=\lambda(x)-a(x)u^{q-1}.$$

Under certain circumstances, in particular when $\max_{\bar{\Omega}}\lambda>0$, the model admits a unique positive solution for sufficiently small $d>0$. Subsequently, we denote by $\theta_{\{d,\lambda,a\}}$ the maximal non-negative solution of (1.8). When switched off to zero the diffusion rate, d, in (1.8), we are lead to a *family* of ordinary differential equations parameterized by $x\in\bar{\Omega}$. These are the kinetic/non-diffusive problems associated to (1.8), which are explicitly given by

$$\begin{cases} u'(t)=\lambda(x)u(t)-a(x)u^q(t) & t\in(0,+\infty),\\ u(0)=u_0\geq 0,\end{cases}\tag{1.9}$$

whose steady states are $u=0$, and

$$u=\sqrt[q-1]{\frac{\lambda(x)}{a(x)}}\quad\text{as soon as}\quad \lambda(x)>0.$$

Thus, the maximal non-negative steady state of (1.9), which is actually the unique linearly stable one if $\lambda(x)\neq 0$, is provided by

$$u=\sqrt[q-1]{\frac{\lambda_+(x)}{a(x)}},$$

where λ_+ denotes the positive part of λ, i.e.,

$$\lambda_+:=\max\{\lambda,0\}.$$

Note that $u=0$ is linearly neutrally stable if $\lambda(x)=0$. In this setting, the next result follows as a very special case of Theorem 2.21, which is not only of interest in its own, but it plays a crucial role to prove the main singular perturbation result for the diffusive competition model.

Corollary of Theorem 2.21 (Singular perturbation for the equation).

Given a compact subset, K, of $\Omega \cup \Gamma_{\mathcal{R}}^{+} \cup \lambda_{+}^{-1}(0)$, we have that

$$\lim_{d\downarrow 0} \theta_{\{d,\lambda,a\}} = \sqrt[q-1]{\frac{\lambda_+}{a}} \quad \text{uniformly in } K.$$

Here, $\Gamma_{\mathcal{R}}^{+}$ denotes the union of the components of $\Gamma_{\mathcal{R}}$ where λ is everywhere positive.

Next, let us provide an overview on the perturbation problem for the diffusive logistic equation. To the best of our knowledge, the first singular perturbation result dealing with this kind of equations goes back to Result (A) of Berger and Fraenkel [10], which analyzes the problem

$$\begin{cases} \epsilon^2 \Delta u + u - g^2(x) u^3 = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$, $N \leq 3$, is of class $\mathcal{C}^\infty$, and $g \in \mathcal{C}^\infty$ is strictly positive in $\bar{\Omega}$. The precise result, which is proved in Sections 3 and 4 in [10] through the method of matched asymptotic expansions, reads as follows.

Result (A) of [10]. *For sufficiently small values of ϵ, the problem (1.10) has a unique smooth positive solution $u(x,\epsilon)$ which tends to $1/g(x)$ as $\epsilon \to 0$, outside a narrow boundary layer of width $O(\epsilon)$ concentrated near Ω.*

It should be noted that this result deals for the first time with the boundary layer that arises when considering the singular perturbation result under Dirichlet boundary conditions in the logistic equation, estimating the width of that layer in terms of the diffusion coefficient. However, it does not allows the kinetic counterpart of (1.10) to exhibit different linearly stable steady states in different regions of $\bar{\Omega}$, which would imply that there is a region in Ω where the kinetic model exhibits a linearly neutraly stable steady state. Actually, even though shortly later De Villiers [31], Fife [43], and Fife and Greenlee [44] sharpened the result of Berger and Fraenkel [10] by allowing more general non-linearities, the kinetic counterpart of the diffusion problem was still assumed to posses a unique steady state being linearly stable in the whole $\bar{\Omega}$. For instance, De Villiers [31] obtained the singular perturbation result for

$$\begin{cases} \epsilon^2 \Delta u + g(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $N \geq 1$ and $g \in \mathcal{C}^\infty(\bar{\Omega} \times \mathbb{R})$ admitting a strictly positive function $T \in \mathcal{C}(\bar{\Omega})$ (the steady state) such that

$$g(x, T(x)) = 0 \quad \text{and} \quad \partial_u g(x, T(x)) < 0 \quad \text{for all } x \in \bar{\Omega},$$

and, for every $s \in \partial\Omega$ and $\xi \in [0, T(s))$,

$$\int_{\xi}^{T(s)} g(s,y)\, dy > 0.$$

On the other hand, Fife [43] established the singular perturbation result for

$$\begin{cases} \epsilon^2 \mathrm{div}(A(x,\epsilon)\nabla u)u - g(x,u,\varepsilon) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where Ω, $A(x,\epsilon)$, φ and $g(x,u,\epsilon)$ are of class $\mathcal{C}^\infty$ and such that, for every $x \in \bar{\Omega}$, there exists $u_0(x)$ for which

$$g(x,u_0(x),0) = 0 \quad \text{and} \quad \partial_u g(x,u_0(x),0) > 0 \quad \text{for all } \; x \in \bar{\Omega},$$

and, for every $x \in \partial\Omega$ and $k \neq \emptyset$ in between $u_0(x)$ and $\varphi(x)$,

$$\int_{u_0(x)}^{k} g(x,u,0)\, du > 0. \tag{1.12}$$

Finally, Fife and Greenlee [44] established that an interior transition occurs for (1.11) if there are two functions, $u_{0,1}$ and $u_{0,2}$, of class $\mathcal{C}^\infty$ with $u_{0,1}(x) \neq u_{0,2}(x)$, such that

$$g(x,u_{0,i}(x),0) = 0 \quad \text{and} \quad \partial_u g(x,u_{0,i}(x),0) < 0 \quad \text{for all } \; x \in \bar{\Omega}, \quad i = 1,2,$$

and an integral condition of the type of (1.12) holds. In particular the transition between these solutions occurs in the curve originated by the zeros of the function

$$J(x) := \int_{u_{0,1}(x)}^{u_{0,2}(x)} g(x,u,0)\, du.$$

It should be remarked that the singular perturbation problem studied by Fife and Greenlee [44] is rather different from the one analyzed in this dissertation since the non-linearity is assumed to exhibit two separate linearly stable curves of solutions instead of two curves of steady states crossing each other and inter-exchanging their stability character. It deserves a special attention the fact that the non-linearity in the works of Fife [43] and Fife and Greenlee [44] depends smoothly on the diffusion parameter ϵ^2. Although Theorem 2.21 does not take into consideration this type of behavior, the main singular perturbation result for the diffusive competition Lotka–Volterra model established in this dissertation through Theorem 4.4 requires from that kind of results. Some additional light will be shed on this issue later in this introduction.

Continuing this review on the previous approaches to the singular perturbation problem for the diffusive logistic equation, we should mention the work of Howes [59, 58, 60] who studied the problem under Robin boundary conditions by using, among other techniques, the method of sub and supersolutions. Precisely, Howes considers, in the first part of [58], the problem

$$\begin{cases} \epsilon\Delta u = g(x,u) & \text{in } \Omega, \\ \nabla F(x)\nabla u(x) + \mu(x)u(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where

$$\Omega := \{x \in \mathbb{R}^N \;:\; F(x) < 0\}, \qquad N \geq 2,$$

and $\partial\Omega := F^{-1}(0)$ for some function $F \in \mathcal{C}^{2,\alpha}(\mathbb{R}^N)$, with $0 < \alpha < 1$, such that

$$\|\nabla F(x)\| = 1 \;\text{ for all }\; x \in \partial\Omega, \quad \text{and} \quad \max_{\bar{\Omega}} \|\nabla F(x)\|^2 = K \in \mathbb{R}.$$

Moreover, the author also assumes that $\mu, \varphi \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, with $\mu \geq 0$. Note that thanks to our characterization of the regularity of the boundary established in Theorem 2.3, the hypothesis on F can be equivalently stated by saying, simply, that $\partial\Omega$ is of $\mathcal{C}^2$; ∇F being the normal vector field on $\partial\Omega$.

In particular, Howes proves, through Theorem 2.1 of [58], that if g is smooth enough, say of class $\mathcal{C}^{0,\alpha}$ in x and of class $\mathcal{C}^1$ in u, and there exists $u_0 \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ such that

$$g(x, u_0(x)) = 0 \quad \text{for all} \ \ x \in \bar{\Omega} \quad \text{and} \quad \partial_u g(x, u_0(x)) > 0 \quad \text{for all} \ \ x \in \bar{\Omega},$$

i.e., u_0 is (I_0)-stable using the notation of the author, then (1.13) admits a solution $u(x,\epsilon)$ converging to $u_0(x)$ as $\epsilon \to 0$ for every $x \in \bar{\Omega}$. Hence, the perturbation result holds for the choice

$$g(x,u) := -\lambda(x)u + a(x)u^q, \quad \text{with} \ \ q \geq 2, \ \ \lambda, a \in \mathcal{C}^{2,\alpha}(\bar{\Omega}, (0,+\infty)),$$

providing uniform convergence to $\sqrt[q-1]{\frac{\lambda}{a}}$ in $\bar{\Omega}$, but also for the choice

$$g(x,u) := u^3 - u,$$

providing, in this case, the existence of two solutions converging to 1 and -1 uniformly in $\bar{\Omega}$, respectively, as $\epsilon \to 0$. This last result, stated in Example 2.2 of [58], lies closer to the type of results of Fife and Greenlee [44], and remains outside the general scope of this dissertation.

It should be noted that Howes [59, 58] also analyzes the singular perturbation problem when the perturbation parameter only affects the highest order derivatives of the differential operator, so that a partial differential equation is still obtained at $\epsilon = 0$. Such study is carried over both for Dirichlet [59, 60] and Robin [58] boundary conditions. However, this kind of results remain outside the general scope of this dissertation.

On the subsequent few years all these findings were improved by Angenent [6], De Santi [30], Clément and Sweers [22], and Kelley and Ko [68], among others, but either these authors still considered the situation of a *smooth* curve $u_*(x)$ that is a linearly stable solution of

$$u'(t) = g(x, u(t)), \ \ t \in (0, +\infty),$$

for every $x \in \bar{\Omega}$, or they studied interior layers (much like in the work of Fife and Greenlee [44]), or they assumed that the perturbation parameter was only affecting the highest order derivatives like in the work of Howes [59, 58]. In the later case, one might think about the concept of viscosity solution.

It was not until the work of Cantrell and Cosner [18] (see Theorems 4.7 and 4.8 therein, as well as the comments in between them), that a change of stability between intersecting solutions of $g(x,u) = 0$ happening inside Ω was taken in consideration. Shortly later, these results were improved substantially by Furter and López-Gómez [48]. Their main singular perturbation result, established for the equation

$$\begin{cases} -d\Delta u = \lambda(x)u - a(x)u^2 & \text{in} \ \ \Omega, \\ u = 0 & \text{on} \ \ \partial\Omega, \end{cases} \tag{1.14}$$

can be stated as follows:

Theorem 3.6 of [48]. *Assume that* $\lambda, a \in L^\infty(\bar\Omega)$ *with* $\sup_\Omega \lambda > 0$ *and* $a \geq k > 0$*. Let* $\Omega_c \subset \Omega$ *be an open set such that* λ *and* a *are continuous in* Ω_c*. Then*

$$\lim_{d\downarrow 0} \theta_d = \frac{\lambda_+}{a}$$

uniformly on compact subsets of Ω_c*.*

Finally, the singular perturbation result was proved, almost simultaneously, for (1.14) under non-flux boundary conditions, through Lemma 2.4 of Hutson, López-Gómez, Mischaikow and Vickers [63]. As under Neumann boundary conditions there is no boundary layer, the uniform convergence in $\bar\Omega$ is guaranteed.

As noticed from the previous general overview on the singular perturbation problem for the diffusive logistic equation, one of the main differences observed between the result provided herein and those established previously in the literature has to do with the generality of the differential operator and the non-classical mixed boundary conditions imposed through this dissertation. Considering a uniformly elliptic operator, $\mathcal{L}$, instead of $-\Delta$, makes more difficult to control the exact point where the maximum of the associated principal eigenfunctions is attained. This difficulty influences, strongly, the construction of appropriate subsolutions, and is solved herein by means of a technical device introduced in the proof of Theorem 4.1 of López-Gómez [83].

Moreover, this dissertation deals with non-classical mixed boundary conditions, where the coefficient β can take negative values. As already pointed out when presenting Theorem 2.3, this technical trouble can be overcome through a technical device introduced in Lemma 3.1 of López-Gómez [87]. In particular, the regularity of $\partial\Omega$ allows us to transform the problem (1.4), or (1.8), into another (classical) problem of identical nature with $\beta \geq 0$.

It should be emphasized that dealing with mixed non-classical boundary conditions introduces the combined difficulties inherent to both problems, Robin and Dirichlet, into our setting. So, the underlying problem becomes substantially more intricate. Not to mention the extremely delicate issue related to the greatest generality of the non-linearity, $uh(x,u)$. The fact that both curves of steady states of the associated kinetic problem $u' = uh(x,u)$ intersect each other, inter-exchanging their stabilities characters, increases substantially the difficulty of constructing a global supersolution satisfying the appropriate boundary conditions.

To conclude the analysis of the singular perturbation problem, like in the works of Fife [43], and Fife and Greenlee [44], this dissertation also deals with the problem of ascertaining the limiting profile of the diffusive logistic equation when the non-linearity depends appropriately on the diffusion parameter. In particular, the following results are obtained for the problem

$$\begin{cases} \delta\mathcal{L}u = \gamma_{(\delta,\eta)}(x)u - m_{(\delta,\eta)}(x)u^2 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

and play a crucial role in the proof of the singular perturbation problem for the diffusive Lotka–Volterra model. The first of them reads as follows.

Theorem 4.2 (Singular perturbation for the diffusive logistic equation with diffusion dependent coefficients I).

Consider $\gamma, m \in \mathcal{C}(\bar{\Omega})$, *and families* $\gamma_{(\delta,\eta)}, m_{(\delta,\eta)} \in \mathcal{C}(\bar{\Omega})$ *such that*

$$\lim_{(\delta,\eta)\to(0,0)} \gamma_{(\delta,\eta)} = \gamma \quad \text{and} \quad \lim_{(\delta,\eta)\to(0,0)} m_{(\delta,\eta)} = m \quad \text{uniformly in } \bar{\Omega}.$$

Then,

$$\lim_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} = \frac{\gamma_+}{m}$$

uniformly on compact subsets of $\Omega \cup \Gamma_{\mathcal{R}}^+ \cup (\gamma_+)^{-1}(0)$.

This result assumes the most natural, and actually most demanding, condition for the family of function coefficients. Namely, uniform convergence in $\bar{\Omega}$. However, such convergence cannot be returned in those components of $\Gamma_{\mathcal{D}}$ where $\mathcal{B}$ is a Dirichlet boundary operator, because of the formation of boundary layers in all those components of the boundary. Thus, one may want to get rid of this assumption.

The next result, whose proof is much more subtle, changes the uniform convergence in the closure of the whole domain, $\bar{\Omega}$, by the uniform convergence on compact subsets of an appropriate (not too restrictive) set

$$\mathcal{O} \subset \Omega \cup \Gamma_{\mathcal{R}}^+ \cup (\gamma_+)^{-1}(0)$$

obtaining identical convergence in return. However, it does require to impose uniform bounds for the coefficients in $\bar{\Omega}$.

Theorem 4.3 (Singular perturbation for the diffusive logistic equation with diffusion dependent coefficients II).

Consider $\gamma, m \in \mathcal{C}(\bar{\Omega})$, *two families* $\gamma_{(\delta,\eta)}, m_{(\delta,\eta)} \in \mathcal{C}(\bar{\Omega})$, *and* $\mathcal{O} \subset \Omega \cup \Gamma_{\mathcal{R}}^+ \cup \gamma_+^{-1}(0)$, *an open subset, with respect to the induced topology, such that either* $\bar{\mathcal{O}} \cap \Gamma_{\mathcal{R}}^+ = \emptyset$, *or* $\bar{\mathcal{O}} \cap \Gamma_{\mathcal{R}}^+$ *consists of components of* $\Gamma_{\mathcal{R}}^+$, *each one contained in either* $\mathcal{O}$ *or* $\mathbb{R}^N \setminus \mathcal{O}$. *Assume that*

$$\lim_{(\delta,\eta)\to(0,0)} \gamma_{(\delta,\eta)} = \gamma \quad \text{and} \quad \lim_{(\delta,\eta)\to(0,0)} m_{(\delta,\eta)} = m$$

uniformly on compact subsets of $\mathcal{O}$, *and there exist* $k > 0$ *and* $M > 0$ *such that, for every* $x \in \bar{\Omega}$,

$$m_{(\delta,\eta)}(x) \geq k \quad \text{and} \quad \frac{\gamma_{(\delta,\eta)}(x)}{m_{(\delta,\eta)}(x)} \leq M \quad \text{for sufficiently small } \delta, \eta > 0.$$

Then,

$$\lim_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} = \frac{\gamma_+}{m}$$

uniformly on compact subsets of $\mathcal{O}$.

1.3 The superlinear indefinite problem

As it has just been pointed out before the statement of the previous two singular perturbation results, the analysis on the perturbation problem for the sublinear equation, besides its evident interest on its own, it is imperative to deal with the diffusive spatially heterogeneous competing species model.

The analysis performed in Chapter 3 follows a quite different direction. Instead of providing us with the necessary intermediate results to be used in the context of the system, it focuses attention in establishing a series of non-trivial consequences from the Picone identity, a variational identity going back to Picone [106] which has shown to be a astonishingly powerful tool in analyzing the existence and stability of the positive solutions, not only for the equation, but, rather crucially, for competitive and cooperative systems.

The Picone identity has been extended to different settings by Kreith [72], Berestycki, Capuzzo-Dolcetta and Nirenberg [9], and López-Gómez [84], among other authors. In particular, a generalized version of the Picone identity is stated in this dissertation through Theorem 3.1, which establishes that if $u, v \in W^{2,p}(\Omega)$, $p > N$, with $\frac{v}{u} \in \mathcal{C}(\bar{\Omega})$, $\mathcal{L}$ is **self-adjoint**, and $g \in \mathcal{C}^1(\mathbb{R})$, then

$$\int_\Omega g\left(\frac{v}{u}\right)[u\mathcal{L}v - v\mathcal{L}u] = \int_\Omega u^2 g'\left(\frac{v}{u}\right)\langle\nabla\frac{v}{u}, A\nabla\frac{v}{u}\rangle - \int_{\partial\Omega} g\left(\frac{v}{u}\right)[\mathcal{D}u\mathcal{R}v - \mathcal{D}v\mathcal{R}u]. \quad (1.15)$$

In particular, (1.15) is used in Chapter 3 to study the existence and uniqueness of stable positive solutions for the superlinear indefinite problem

$$\begin{cases} \mathcal{L}u = \lambda u - a(x)f(u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.16)$$

where $f \in \mathcal{C}^r(\mathbb{R})$, $r \geq 2$, with $f(0) = 0$ and $f \not\equiv 0$, $a \in \mathcal{C}(\bar{\Omega})$ may change sign, and $\lambda \in \mathbb{R}$ is regarded as a bifurcation parameter. As our results invoke the Picone identity, $\mathcal{L}$ is required to be a uniformly elliptic *self-adjoint* operator in divergence form

$$\mathcal{L} := -\text{div}(A\nabla\cdot) + c,$$

i.e., $\mathcal{L}$ is defined as in (1.5) with $b = 0$. On the positive side, $\mathcal{B}$ is assumed to be a general non-classical boundary operator of mixed type, i.e., the boundary operator $\mathcal{B}$ and the associated components of $\partial\Omega$, $\Gamma_\mathcal{D}$ and $\Gamma_\mathcal{R}$, are defined much like in (1.6) and (1.7).

Theorem 3.6 is the first result of this dissertation using Picone identity to provide some necessary conditions for the existence of positive solutions for the superlinear indefinite problem (1.16). It extends Proposition 2.2 of Gómez-Reñasco and López-Gómez [51, 52], and Proposition 9.2 of López-Gómez [89] to cover the case of non-classical mixed boundary conditions. To formulate this result we need to introduce the notation

$$\sigma_0 := \sigma_1[\mathcal{L}; \mathcal{B}, \Omega]$$

for the principal eigenvalue of the triple $[\mathcal{L}; \mathcal{B}, \Omega]$, i.e., the lowest one associated to the linear eigenvalue problem

$$\begin{cases} \mathcal{L}w = \sigma w & \text{in } \Omega, \\ \mathcal{B}w = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 7.7 of López-Gómez [88], σ_0 is algebraically simple, strictly dominant, and it is the unique eigenvalue associated with it there is a positive eigenfunction, φ, often called principal. Actually, $\varphi \gg 0$ in the sense that

$$\varphi(x) > 0 \text{ for all } x \in \Omega \cup \Gamma_{\mathcal{R}} \text{ and } \frac{\partial \varphi}{\partial \mathbf{n}}(x) < 0 \text{ for all } x \in \Gamma_{\mathcal{D}}.$$

Throughout this dissertation, we denote by $\varphi_0 > 0$ the unique principal eigenfunction associated to σ_0 normalized so that $\|\varphi_0\|_2 = 1$. Using these notations, Theorem 3.6 can be stated as follows.

Theorem 3.6 (Non-existence for $\lambda \geq 0$ under subcritical bifurcation).

Assume that $q > 1$ exists such that $f(u) := u^q$ for every $u \geq 0$ and

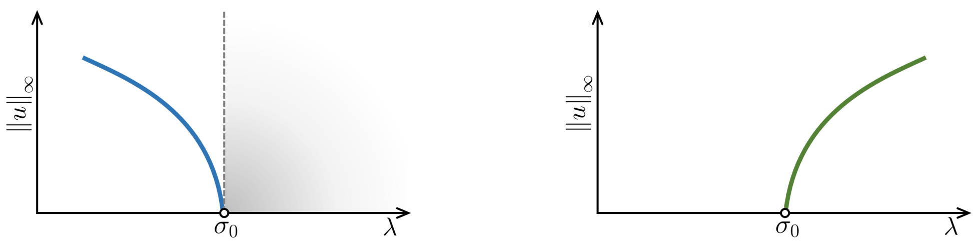
$$\mathcal{S}_q = \int_\Omega a(x)\varphi_0^{q+1}(x)\,dx \leq 0.$$

Then, $\lambda < \sigma_0$ if (1.16) admits a positive solution, (λ, u).

By Theorem 3.2 and Proposition 3.3, the hypothesis on the sign of $\mathcal{S}_q$ determines the nature of the local bifurcation of the curve of positive solutions from the trivial branch, $u \equiv 0$, at $\lambda = \sigma_0$, as a straightforward application of the main theorem of Crandall and Rabinowitz [24]. According to them, if $f \in \mathcal{C}^r(\mathbb{R})$, $r \geq 2$, satisfies $f(0) = f'(0) = 0$, and there exists $q \geq 1$ such that

$$\mathcal{S}_q := \lim_{s \to 0^+} \frac{f(s)}{s^q} \int_\Omega a(x)\varphi_0^{q+1}(x)\,dx$$

is finite, then a $\mathcal{C}^{r-1}$ curve of positive solutions of (1.16) bifurcates from the trivial branch $u \equiv 0$ at $\lambda = \sigma_0$. Moreover, the bifurcation is *supercritical* if $\mathcal{S}_q > 0$, and *subcritical* if $\mathcal{S}_q < 0$. Thus, Theorem 3.6 establishes that if the function f is of *monomial* type and the bifurcation at $\lambda = \sigma_0$ is *subcritical*, then (1.16) does not admits positive solutions for $\lambda \geq \sigma_0$, as illustrated by Figure 1.2(a) below.



(a) $f(u) = u^q$ for some $q \geq 2$ and $\mathcal{S}_q < 0$. (b) f arbitrary and $\mathcal{S}_q > 0$ for some $q > 1$.

Figure 1.2: Plot of the bifurcation diagram for (1.16), showing that if $f(u) = u^q$ for some $q \geq 2$, and the bifurcation is subcritical, then (1.16) does not admits positive solutions for $\lambda \geq \sigma_0$ as a consequence of Theorem 3.6.

The second important application of our generalized Picone identity is closely related to the problem of analyzing the shape of the curve of positive solutions and its relation with the linear stability character of the positive solutions filling it. The linear stability character of a solution (λ_0, u_0) of (1.16) is determined by the sign of the principal eigenvalue of the linearization of (1.16) at (λ_0, u_0), which is given through

$$\sigma_1[\mathcal{L} - \lambda_0 - a(x)f'(u_0)u_0; \mathcal{B}, \Omega], \tag{1.17}$$

with associated principal eigenfunction $\psi_0 \in W^{2,p}(\Omega)$, for $p > N$, normalized so that $\|\psi_0\|_2 = 1$. Indeed,

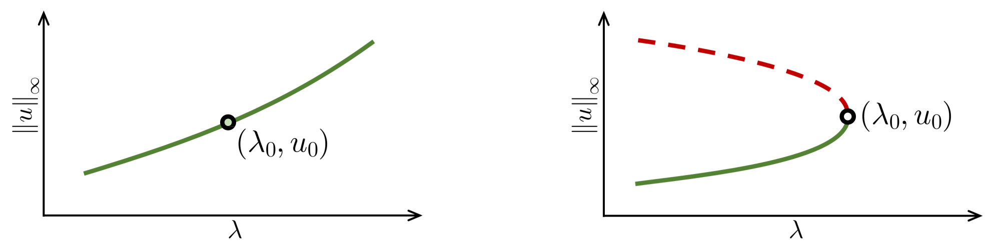
(i) (λ_0, u_0) is linearly neutrally stable if $\sigma_1[\mathcal{L} - \lambda_0 - a(x)f'(u_0)u_0; \mathcal{B}, \Omega] = 0$,

(ii) (λ_0, u_0) is linearly stable if $\sigma_1[\mathcal{L} - \lambda_0 - a(x)f'(u_0)u_0; \mathcal{B}, \Omega] > 0$,

(iii) (λ_0, u_0) is linearly unstable if $\sigma_1[\mathcal{L} - \lambda_0 - a(x)f'(u_0)u_0; \mathcal{B}, \Omega] < 0$.

Actually, by the principle of linearized stability of Lyapunov, (λ_0, u_0) is exponentially asymptotically stable if it is linearly stable, while it is unstable if it is linearly unstable. The local character of (λ_0, u_0) when it is linearly neutrally stable depends on the particular nature of the nonlinearities in the setting of the differential equation.

For an arbitrary function f, if a positive solution, (λ_0, u_0), is linearly stable, by the implicit function theorem, the curve of positive solutions through (λ_0, u_0) is always strictly increasing in u as λ increases (see Figure 1.3(a)). However, the behavior of the component on linearly unstable and linearly neutrally stable solutions is more subtle. In particular, Theorem 3.7, provides us with the local structure and the shape of the component of positive solutions through any linearly neutrally stable positive solution of (1.16), (λ_0, u_0), when f is of monomial type, establishing that (λ_0, u_0) is a quadratic subcritical turning point as illustrated in Figure 1.3(b).



(a) f arbitrary and (λ_0, u_0) linearly stable.

(b) $f(u) = u^q$ for some $q \geq 2$ and (λ_0, u_0) linearly neutrally stable.

Figure 1.3: Plot of the curve of positive solutions of (1.16) near a point (λ_0, u_0) depending on its linear stability character.

Theorem 3.7 extends Proposition 3.2 of Gómez-Reñasco and López-Gómez [51, 52], and Proposition 9.7 of López-Gómez [89], to cover the general case of non-classical mixed boundary conditions, and it can be stated as follows.

Excerpt from Theorem 3.7 (Subcritical turning point character).

Assume that $q \geq 2$ exists such that $f(u) = u^q$ for every $u \geq 0$. Let (λ_0, u_0) be a linearly neutrally stable positive solution of (1.16). Then, there exist $\varepsilon > 0$ and two functions of class $\mathcal{C}^2$,

$$\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad \text{and} \quad u : (-\varepsilon, \varepsilon) \to W^{2,p}_{\mathcal{B}}(\Omega)$$

such that

$$(\lambda(0), u(0)) = (\lambda_0, u_0), \quad (\lambda'(0), u'(0)) = (0, \psi_0), \quad \lambda''(0) < 0,$$

for which the curve $(\lambda(s), u(s))$ provides us with the set of solutions of (1.16) in a neighborhood of (λ_0, u_0). Moreover, shortening ε, if necessary, $u(s)$ is linearly stable if $s \in (-\varepsilon, 0)$ and linearly unstable if $s \in (0, \varepsilon)$.

For applying the Picone identity in the proof of Theorems 3.6 and 3.7 above, the hypothesis that the function f is a monomial is crucial. However, a priori, this should not entail that these results should not possess some variants for more general functions f. Surprisingly, one of the main novelties of this dissertation establishes that Theorems 3.6 and 3.7 are optimal as they are stated, in the sense that even in the very special case when f is a small perturbation of a monomial function by another monomial function, the indefinite problem (1.16) can admit, simultaneously, the following types of solutions:

(i) Positive solutions for $\lambda > \sigma_0$, even if the bifurcation from $u = 0$ is subcritical.

(ii) Neutrally stable solutions that are supercritical turning points.

The precise optimality result reads as follows.

Corollary of Theorem 3.8 (Optimality of Theorems 3.6 and 3.7).

Assume that, for some $0 \leq q_1 < q_2$, the next estimate holds

$$\int_\Omega a(x)\varphi_0^{q_1+1}(x)\,dx < 0 < \int_\Omega a(x)\varphi_0^{q_2+1}(x)\,dx.$$

Then, there exists $\nu_0 > 0$ such that, for every $\nu \in (0, \nu_0)$, the problem

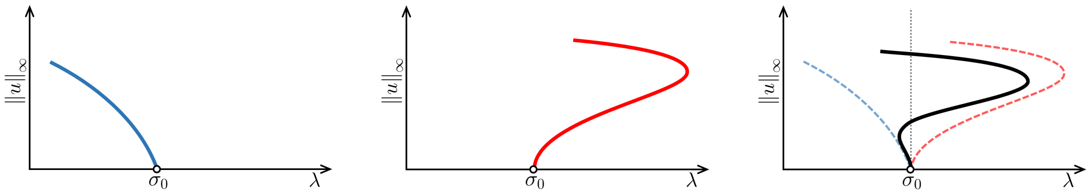
$$\begin{cases} \mathcal{L}u = \lambda u - a(x)(\nu u^{q_1} + u^{q_2}) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits positive solutions for values of the parameter λ at both sides of σ_0.

Figure 1.4 shows an sketch of the proof of Theorem 3.8. If equation (1.16) is considered only with $f_1(u) = \nu u^{q_1}$, then the bifurcation is subcritical, because

$$\int_\Omega a(x)\varphi_0^{q_1+1}(x)\,dx < 0,$$

and Theorem 3.6 applies, regardless of the value of $\nu > 0$!



(a) Case $f(x) = \nu u^{q_1}$ with subcritical bifurcation and nonexistence of positive solutions for $\lambda \geq \sigma_0$.

(b) Case $f(x) = u^{q_2}$ with supercritical bifurcation, where linearly neutrally stable positive solutions are quadratic subcritical folds.

(c) Case $f(x) = \nu u^{q_1} + u^{q_2}$ with subcritical bifurcation, positive solutions for values $\lambda > \sigma_0$, and linearly neutrally stable solutions that are supercritical folds.

Figure 1.4: Plot of bifurcation diagrams sketching the construction of the example of Theorem 3.8, which reveals the optimality of Theorems 3.6 and 3.7.

The bifurcation is still subcritical if we add a polinomial term of higher degree to f, i.e.,

$$f_{1,2}(u) = \nu u^{q_1} + u^{q_2}$$

with $q_2 > q_1$, for any value of $\nu > 0$. On the other hand, if (1.16) is considered only with $f_2(u) = u^{q_2}$, then the bifurcation is supercritical to a linearly stable curve, because

$$\int_\Omega a(x)\varphi_0^{q_2+1}(x)\,dx > 0.$$

As a consequence the bifurcation curve associated to $f_{1,2}$ bifurcates subcritically and, for sufficiently small values of $\nu > 0$, it perturbs from the curve associated to f_2 at the right side of σ_0, as illustrated by Figure 1.4(c).

As for as concerns the feasibility of the hypothesis of Theorem 3.8, it is easily seen that they are fulfilled even for the simplest choice $q_1 = 2$ and $q_2 = 3$, as shown by the next example. Let us consider the one-dimensional problem with

$$\Omega = \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big), \quad \mathcal{L} = -\frac{d^2}{dx^2} \quad \text{and} \quad \mathcal{B} = \mathcal{D} \text{ in } \partial\Omega \text{ (i.e., } \Gamma_{\mathcal{R}} = \emptyset),$$

and

$$a(x) := \cos x - 0.9 \quad \text{for every } x \in \bar{\Omega}.$$

Under these assumptions we have that

$$\sigma_0 = 1 \quad \text{and} \quad \varphi_0(x) = \sqrt{\tfrac{2}{\pi}}\cos x.$$

Thus,

$$\int_\Omega a(x)\varphi_0^3(x)\,dx = \tfrac{3\sqrt{2}}{32}\pi^{5/2} - 0.9\tfrac{\sqrt{2}}{3}\pi^{3/2} \simeq -0.04312 < 0$$

and

$$\int_\Omega a(x)\varphi_0^4(x)\,dx = \tfrac{4}{15}\pi^2 - 0.9\tfrac{3}{32}\pi^3 \simeq 0.0157399 > 0.$$

Hence, for every $\nu > 0$,

$$\mathcal{S}_2 = \lim_{s\to 0^+} \frac{\nu s^2 + s^3}{s^2} \int_\Omega a(x)\varphi_0^3(x)\,dx = \nu \int_\Omega a(x)\varphi_0^3(x)\,dx < 0.$$

Therefore, the curve of positive solutions bifurcates subcritically from the trivial branch at $\lambda = 1$ and, according to Theorem 3.8, it possesses some points (λ, u) with $\lambda > \sigma_0 = 1$. As a consequence, such a curve must exhibit a supercritical turning point on a linearly neutrally stable solution.

To conclude this study of the superlinear indefinite logistic equation (1.16) with $f(u) = u^q$, $u \geq 2$, we should remark that, as a consequence of Theorems 3.6 and 3.7, Theorem 3.11 establishes the existence and uniqueness of the linearly stable solution of (1.16). This result adapts for non-classical boundary conditions of mixed type Theorem 3.4, Corollary 3.5 and Theorem 3.6 of Gómez-Reñasco and López-Gómez [51], Corollary 3.2, Theorem 3.3 and Theorem 4.1 of [52], and Theorem 9.9, Corollary 9.10, Theorem 9.11, Theorem 9.12 and Proposition 9.14 of [89], which were originally established for homogeneous Dirichlet boundary conditions. In particular, our Theorem 3.11 establishes that the following assertions hold:

(i) Any positive solution, (λ_0, u_0), of (1.16) with $\lambda_0 \leq \sigma_0$ must be linearly unstable.

(ii) The problem (1.16) admits some linearly stable positive solution, (λ_0, u_0), if, and only if, $\mathcal{S}_q > 0$. Moreover, in such a case, $\lambda_0 > \sigma_0$.

(iii) If $\mathcal{S}_q > 0$ and $\lambda > \sigma_0$, then the unique positive linearly stable, or linearly neutrally stable solution, of (1.16) is the minimal one.

(iv) If (1.16) admits a positive solution (λ_0, u_0) for some $\lambda_0 > \sigma_0$, then it admits a minimal solution $(\lambda_0, u_{\min})$.

Theorem 3.11 ultimately provides us with the global structure of the stable positive solutions of (1.16) if $f(u) = u^q$, for some $q \geq 2$, through Theorem 3.12, which goes back to Theorem 3.7 of Gómez-Reñasco and López-Gómez [51], Theorem 5.1 of [52] and Theorem 9.16 of [89]. It can be stated as follows.

Theorem 3.12 (Global structure of stable positive solutions).

Suppose that $a(x)$ changes sign in Ω, $q \geq 2$ exists such that $f(u) = u^q$ for all $u \geq 0$, and $\mathcal{S}_q > 0$. Then, the supremum of the set of $\mu > \sigma_0$ for which (1.16) possesses a positive solution for each $\lambda \in (\sigma_0, \mu)$, λ_*, satisfies $\lambda_* \in (\sigma_0, +\infty)$. Moreover, the set of linearly stable positive solutions of (1.16) consists of a $\mathcal{C}^1$ strictly increasing curve

$$\mathfrak{C}_+ := \{(\lambda, u(\lambda)) \ : \ \lambda \in (\sigma_0, \lambda_*)\}.$$

Furthermore, some of the next excluding options occurs:

(i) *$\{u(\lambda)\}_{\lambda\in(\sigma_0,\lambda_*)}$ is bounded in $\mathcal{C}(\bar{\Omega})$, and then*

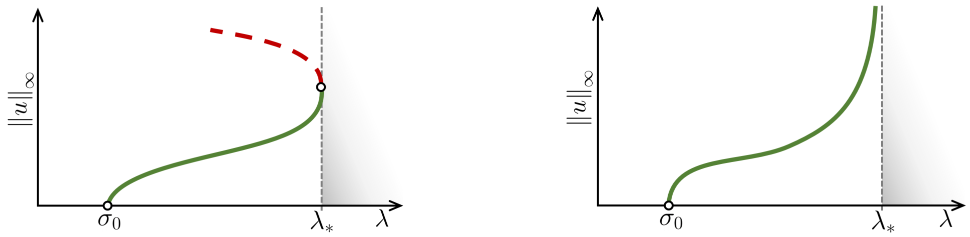
$$u_* := \lim_{\lambda\uparrow\lambda_*} u(\lambda)$$

is a linearly neutrally stable positive solution of (1.16) *at* $\lambda = \lambda_*$.

(ii) $\lim_{\lambda \uparrow \lambda_*} \|u(\lambda)\|_{\mathcal{C}(\bar{\Omega})} = +\infty$.

In both cases, (1.16) *cannot admit any further positive solution for* $\lambda > \lambda_*$.

Figure 1.5 shows an admissible plot of the two alternatives for the global structure of the stable positive solutions of (1.16) when f is of monomial type and the bifurcation from the trivial branch is supercritical, as a consequence of the previous result.



(a) Case (i): the problem (1.16) admits a positive solution at $\lambda = \lambda_*$, which is linearly neutrally stable.

(b) Case (ii): the problem (1.16) does not admit any positive solution at $\lambda = \lambda_*$.

Figure 1.5: Plot of the global structure of the curve of positive solutions of (1.16) if $f(u) = u^q$ for some $q \geq 2$.

It should be noted that the example provided by Theorem 3.8 also shows the optimality of Theorem 3.12, strengthening the value of the Picone identity as well as the accuracy of the estimates that it provides. Actually, not surprisingly by the quasi-cooperative internal structure of the model, in Chapter 6 the Picone identity also proves to be a useful technical tool to study the uniqueness of the coexistence state in the diffusive Lotka–Volterra competition model.

1.4 The singular perturbation problem for the competition system

Once the main singular perturbation result has been established for the single equation in Chapter 2, this dissertation focuses attention in ascertaining the behavior of the solutions of the Reaction-Diffusion model

$$\begin{cases} \frac{\partial u}{\partial t} + d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial v}{\partial t} + d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \Omega \times (0,+\infty), \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.18}$$

for sufficiently small diffusion rates, $d_1, d_2 > 0$. Note that, as soon as the model (1.18) deals with biological or chemical species, we are only interested in those solutions with $u \geq 0$ and $v \geq 0$.

The uniformly elliptic operators $\mathcal{L}_1$ and $\mathcal{L}_2$ taking part in the setting of the model (1.18) are assumed to be in divergence form, i.e., for $i = 1, 2$,

$$\mathcal{L}_i = -\mathrm{div}(A_i \nabla \cdot) + B_i \nabla + C_i,$$

where $A_i \in \mathcal{M}_N^{\text{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $B_i \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $C_i \in \mathcal{C}(\bar{\Omega})$. Moreover, $\mathcal{B}_1$ and $\mathcal{B}_2$ are allowed to be boundary operators of non-classical mixed type, i.e., for $i = 1, 2$,

$$\mathcal{B}_i := \begin{cases} \mathcal{D}_i u := u = 0 & \text{on } \Gamma^i_{\mathcal{D}}, \\ \mathcal{R}_i u := \langle \nabla u, A_i \mathbf{n} \rangle + \beta_i u = 0 & \text{on } \Gamma^i_{\mathcal{R}}, \end{cases} \quad \text{for every } \ u \in W^{2,p}(\Omega), \ \ p > N,$$

with $\beta_i \in \mathcal{C}(\Gamma_{\mathcal{R}})$. As illustrated by Figure 1.6, according to the type of the boundary operator acting on each component of $\partial\Omega$, one can split out $\partial\Omega$ into two ways:

$$\partial\Omega = \Gamma^1_{\mathcal{D}} \cup \Gamma^1_{\mathcal{R}} \quad \text{and} \quad \partial\Omega = \Gamma^2_{\mathcal{D}} \cup \Gamma^2_{\mathcal{R}},$$

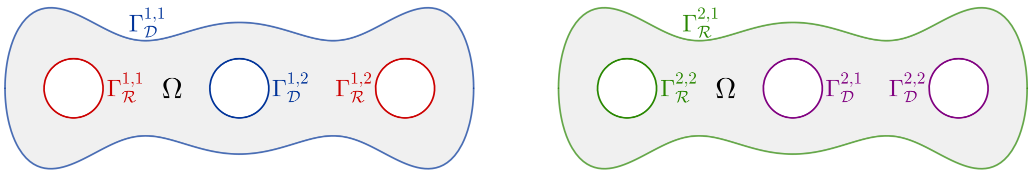
where, for every $i = 1, 2$, $\Gamma^i_{\mathcal{D}}$ consists of the components of $\partial\Omega$ where the species i is subject to homogeneous Dirichlet boundary conditions,

$$\Gamma^i_{\mathcal{D}} = \bigcup_{j=1}^{n^i_{\mathcal{D}}} \Gamma^{i,j}_{\mathcal{D}}, \qquad n^i_{\mathcal{D}} \geq 1,$$

while $\Gamma^i_{\mathcal{R}}$ is made of the components of $\partial\Omega$ where the species i is subject to non-flux boundary conditions,

$$\Gamma^i_{\mathcal{R}} = \bigcup_{j=1}^{n^i_{\mathcal{R}}} \Gamma^{i,j}_{\mathcal{R}}, \qquad n^i_{\mathcal{R}} \geq 1.$$

Note that, for every $i = 1, 2$, $\Gamma^i_{\mathcal{D}}$ and $\Gamma^i_{\mathcal{R}}$ are two disjoint subsets of $\partial\Omega$, simultaneously open and closed in the induced topology. Moreover, any of them could be empty. In such a case, we take $n^i_{\mathcal{D}} = 0$, or $n^i_{\mathcal{R}} = 0$, respectively.



(a) The several components of $\partial\Omega$ for the species u according to the behavior of the boundary operator $\mathcal{B}_1$

(b) The several components of $\partial\Omega$ for the species v according to the behavior of the boundary operator $\mathcal{B}_2$

Figure 1.6: An admissible $\partial\Omega$ where each of its components has been classified according to the behavior of the boundary operators associated to the species u and v.

When the diffusion coefficients, d_1 and d_2, are switched off to zero, we come up with a family (parameterized by $x \in \bar{\Omega}$) of classical competition models for two species

$$\begin{cases} \frac{\partial u}{\partial t} = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \ (0, +\infty), \\ \frac{\partial v}{\partial t} = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \ (0, +\infty), \\ u(0) = u_0, \ \ v(0) = v_0, \end{cases} \tag{1.19}$$

which where introduced by Lotka [78] and Volterra [114, 115] in their seminal works. Here, the parameters $\lambda, \mu \in \mathcal{C}(\bar{\Omega})$ play the role of the growth rates of the species u and v, respectively, if they are positive, whereas they stand for their death rates if $\lambda(x) < 0$ or $\mu(x) < 0$. Moreover, the function coefficients $a, d \in \mathcal{C}(\bar{\Omega}, (0, +\infty))$ measure the intra-specific competition of each of the species, i.e., the competition among the individuals of the same species. Finally, the coupling functions, $b, c \in \mathcal{C}(\Omega, (0, +\infty))$, stand for the inter-specific competition rates, as they measure the competition among the individuals of the species u and v.

At every location point in the inhabiting territory, $x \in \bar{\Omega}$, the *non-spatial*, or *non-diffusive*, or *kinetic*, model (1.19) may admit three different types of steady states:

(i) One *trivial* steady state with both components vanishing, $(u, v) = (0, 0)$,

(ii) Up to two *semitrivial* steady states, with only one positive component, which are given by

$$(u, v) = \left(\frac{\lambda(x)}{a(x)}, 0\right) \text{ if } \lambda(x) > 0, \quad \text{and} \quad (u, v) = \left(0, \frac{\mu(x)}{d(x)}\right) \text{ if } \mu(x) > 0.$$

(iii) *Coexistence* steady states, which are those steady-state solutions, (u, v), with $u > 0$ and $v > 0$. The model (1.19) admits only one coexistence steady state, given by

$$(u, v) = \left(\frac{\lambda(x)d(x) - \mu(x)b(x)}{a(x)d(x) - b(x)c(x)}, \frac{\mu(x)a(x) - \lambda(x)c(x)}{a(x)d(x) - b(x)c(x)}\right)$$

if the numerators and the denominator have the same sign, whereas (1.19) admits a continuum of coexistence steady states

$$(u_\eta, v_\eta) = \left(\eta\frac{\lambda(x)}{a(x)}, (1 - \eta)\frac{\mu(x)}{d(x)}\right) \quad \text{for } \eta \in (0, 1),$$

if the numerators and the denominator vanish. In any other case the model (1.19) cannot admit coexistence steady states.

The precise dynamics of the *kinetic* model (1.19) depends on the existence and linear stability character of the semitrivial steady states, and, thus, on the values of the coefficients $\lambda(x)$, $\mu(x)$, $a(x)$, $b(x)$, $c(x)$ and $d(x)$. For example, if the model (1.19) does not admits neither semitrivial nor coexistence steady states, i.e., if

$$\lambda(x) \leq 0 \quad \text{and} \quad \mu(x) \leq 0.$$

then $(0, 0)$ is a global attractor with respect to the component-wise non-negative solutions, and hence, both species become *extinct* as time goes to infinity. Figure 1.7(a) plots an admissible phase portrait for this case.

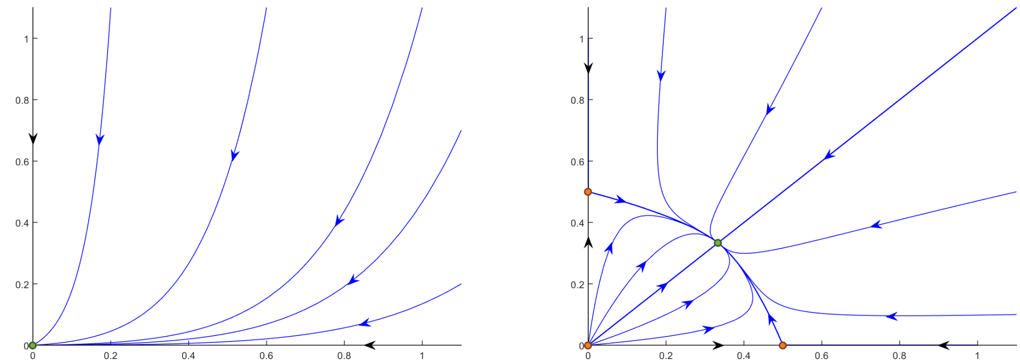
On the contrary, none of the species become extinct, and thus the model is said to exhibit *permanence* of both species, if both semitrivial steady states exist and they are linearly unstable. In such a case there exists a unique coexistence steady state which is a global attractor for the component-wise positive solutions of (1.19). This occurs when

$$\lambda(x) > 0, \quad \mu(x) > 0, \quad \lambda(x)d(x) > \mu(x)b(x) \quad \text{and} \quad \mu(x)a(x) > \lambda(x)c(x).$$

A paradigmatic plot of the phase portrait when permanence occurs is shown in Figure 1.7(b). Note that if the previous estimates hold, then

$$b(x)c(x) < a(x)d(x).$$

When this condition occurs, it is said that u and v compite with a *low competition* regime.



(a) Extinction of both species.
Choice: $\lambda = \mu = -1$, $a = b = 1$, $c = 5$, $d = 7$

(b) Permanence of both species.
Choice: $\lambda = \mu = 1$, $a = d = 2$, $c = d = 1$

Figure 1.7: Plot of the phase portraits for the Lotka–Volterra non-spatial model (1.19) for (a) $x \in \Omega_{\text{ext}}$ and (b) $x \in \Omega_{\text{per}}$.

When both semitrivial steady states are linearly stable, then the model exhibits a unique coexistence steady state, which is a saddle point. In such a case, it is said that the model exhibits *founder control competition*, because the result of the competition depends on the precise location of the initial values, (u_0, v_0). Indeed, the species u becomes extinct if (u_0, v_0) is placed within the basin of attraction of $(0, \mu(x)/d(x))$ (the upper half-quadrant with respect to the stable manifold, W^s, of the coexistence state), whereas the species v is driven to extinction by u if (u_0, v_0) lies within the basin of attraction of $(\lambda(x)/a(x), 0)$ (the lower half-quadrant with respect to W^s). In such a case, the coefficients satisfy

$$\lambda(x) > 0, \quad \mu(x) > 0, \quad \lambda(x)d(x) < \mu(x)b(x) \quad \text{and} \quad \mu(x)a(x) < \lambda(x)c(x).$$

Note that these estimates imply that

$$b(x)c(x) > a(x)d(x).$$

In this case, it is said that u and v compite with a *high competition* regime. Figure 1.8(a) shows a prototypical plot of the phase space under founder control competition.

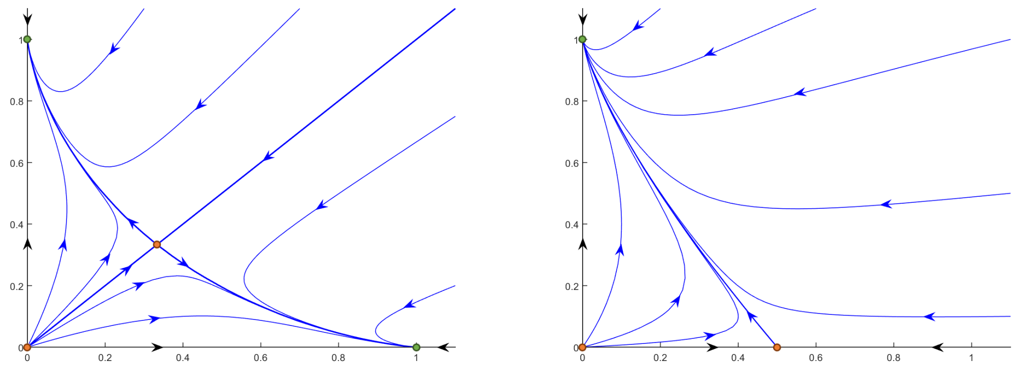
Finally, if one semitrivial steady state is linearly stable and either the other one does not exist, or it is linearly unstable, then the model does not admit any coexistence steady state, and the stable semitrivial steady state actually is a global attractor for the component-wise positive solutions. Thus, in this case one of the competitors is driven to extinction by the other. The one that prevails is said that *dominates* the dynamics. In particular, the species u prevails if

$$\lambda(x) > 0, \quad \lambda(x)d(x) > \mu(x)b(x) \quad \text{and} \quad \mu(x)a(x) < \lambda(x)c(x),$$

whereas v prevails and u becomes extinct if

$$\mu(x) > 0, \quad \lambda(x)d(x) < \mu(x)b(x) \quad \text{and} \quad \mu(x)a(x) > \lambda(x)c(x).$$

A genuine example where the later occurs has been plotted in Figure 1.8(b).



(a) Founder control competition.
Choice: $\lambda = \mu = 1$, $a = d = 1$, $b = c = 2$

(b) Dominance of the species v.
Choice: $\lambda = \mu = 1$, $a = b = 2$, $c = d = 1$

Figure 1.8: Plot of the phase spaces for the Lotka–Volterra non-spatial model (1.19) for (a) $x \in \Omega_{\text{bi}}$ and (b) $x \in \Omega^v_{\text{do}}$.

Adopting the methodology of Furter and López-Gomez [48], one can divide the inhabiting territory Ω into the following regions according to the dynamics of the associated non-spatial model (1.19):

$$\begin{aligned}
&\Omega_{\text{ext}} := \{x \in \bar{\Omega} \ : \ \lambda(x),\ \mu(x) \leq 0\},\\
&\Omega_{\text{per}} := \{x \in \bar{\Omega} \ : \ \lambda(x),\ \mu(x) > 0,\ \ \lambda(x)d(x) > \mu(x)b(x),\ \ \mu(x)a(x) > \lambda(x)c(x)\},\\
&\Omega_{\text{bi}} := \{x \in \bar{\Omega} \ : \ \lambda(x),\ \mu(x) > 0,\ \ \lambda(x)d(x) < \mu(x)b(x),\ \ \mu(x)a(x) < \lambda(x)c(x)\},\\
&\Omega^u_{\text{do}} := \{x \in \bar{\Omega} \ : \ \lambda(x) > 0,\ \ \lambda(x)d(x) > \mu(x)b(x),\ \ \mu(x)a(x) < \lambda(x)c(x)\},\\
&\Omega^v_{\text{do}} := \{x \in \bar{\Omega} \ : \ \mu(x) > 0,\ \ \lambda(x)d(x) < \mu(x)b(x),\ \ \mu(x)a(x) > \lambda(x)c(x)\},\\
&\Omega_{\text{junk}} := \bar{\Omega} \setminus (\Omega_{\text{ext}} \cup \Omega_{\text{per}} \cup \Omega_{\text{bi}} \cup \Omega^u_{\text{do}} \cup \Omega^v_{\text{do}}).
\end{aligned}$$

Note that the degenerate situation of having a continuum of coexistence steady states may occur in Ω_{junk}, which can be considered as a limiting area between the other regions. This dissertation does not focus any attention into this sort of marginal region of the inhabiting territory.

Going back to the diffusive model (1.18), it turns out that, similarly, its dynamics (in the first quadrant) are determined by the existence and linear stability character of its steady states, i.e, the component-wise non-negative solutions of the semilinear boundary value problem

$$\begin{cases} d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \ \Omega,\\ d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \ \Omega,\\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{1.20}$$

Precisely, (1.18) admits three types of steady states:

(i) The *trivial* steady state,

$$(u,v)=(0,0).$$

(ii) Up to two *semitrivial* steady states with only one positive component, given by

$$(u,v)=(\theta_{\{d_1,\lambda,a\}},0) \quad \text{and} \quad (u,v)=(0,\theta_{\{d_2,\mu,d\}}),$$

where $\theta_{\{d_1,\lambda,a\}}$ and $\theta_{\{d_2,\mu,d\}}$ are the respective positive solutions of the diffusive logistic elliptic problems

$$\begin{cases} d_1\mathcal{L}_1 u=\lambda(x)u-a(x)u^2 & \text{in } \Omega, \\ \mathcal{B}_1 u=0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} d_2\mathcal{L}_2 v=\mu(x)v-d(x)v^2 & \text{in } \Omega, \\ \mathcal{B}_2 v=0 & \text{on } \partial\Omega. \end{cases}$$

(iii) *Coexistence* steady states, which are those with both components positive.

Although the existence of *coexistence* steady states is guaranteed under certain circumstances, determining their uniqueness and providing exact multiplicity results is a far more intricate mathematical problem. In particular, if (1.18) admits both semitrivial steady states and they are linearly unstable, i.e.,

$$\sigma_1[d_1\mathcal{L}_1-\lambda+b\theta_{\{d_2,\mu,d\}};\mathcal{B}_1,\Omega]<0 \quad \text{and} \quad \sigma_1[d_2\mathcal{L}_2-\mu+c\theta_{\{d_1,\lambda,a\}};\mathcal{B}_2,\Omega]<0,$$

then (1.18) exhibits a stable coexistence steady state. Moreover, if it is unique, then it is a global attractor with respect to the component-wise positive solutions of (1.18). This result follows in our setting by adapting the proofs of Theorem 4.1 of Cosner and Lazer [23], Theorem 1 of Dancer [27], Remark 33.2 and Theorem 33.3 of Hess [56], Theorem 2.1 of Cantrell and Cosner [19], and Theorem 4.1 of López-Gómez and Sabina [77].

On the other hand, if (1.18) admits both semitrivial steady states and they are linearly stable, i.e.,

$$\sigma_1[d_1\mathcal{L}_1-\lambda+b\theta_{\{d_2,\mu,d\}};\mathcal{B}_1,\Omega]>0 \quad \text{and} \quad \sigma_1[d_2\mathcal{L}_2-\mu+c\theta_{\{d_1,\lambda,a\}};\mathcal{B}_2,\Omega]>0,$$

then (1.18) exhibits an unstable coexistence steady state. The proof of this result follows from Theorem 35.1 of Hess [56], Theorem 4.1 of López-Gómez [79], Theorem 5.3 of López-Gómez and Sabina [77] and Theorem 2.4 of Furter and López-Gómez [47].

The next result is the first one developed in this dissertation providing a relation between the dynamics of the diffusive and non-diffusive competition models. It establishes that the limiting profile of the coexistence steady states of the diffusive model (1.18) is provided by the global attractor of the non-diffusive model (1.19) in those regions of $\bar{\Omega}$ where it exists. More precisely, it can be stated as follows.

Excerpt from Theorem 4.4 (Singular perturbation for the system).

Let $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}$ *be a family of coexistence states of* (1.20). *Then,*

$$\lim_{(d_1,d_2)\to(0,0)} (u_{(d_1,d_2)}, v_{(d_1,d_2)}) = \begin{cases} (0,0) & \text{if } x \in \Omega_{\text{ext}}, \\ \left(\frac{\lambda d-\mu b}{ad-bc}(x), \frac{\mu a-\lambda c}{ad-bc}(x)\right) & \text{if } x \in \Omega_{\text{per}}, \\ \left(\frac{\lambda}{a}(x), 0\right) & \text{if } x \in {\Omega^u_{\text{do}}}^*, \\ \left(0, \frac{\mu}{d}(x)\right) & \text{if } x \in {\Omega^v_{\text{do}}}^*, \end{cases}$$

uniformly on compact subsets of

$$(\Omega_{\text{ext}} \cup \Omega_{\text{per}} \cup {\Omega^u_{\text{do}}}^* \cup {\Omega^v_{\text{do}}}^*) \cap (\Omega \cup \Gamma^{\text{per}}_{\mathcal{R}}),$$

where $\Gamma^{\text{per}}_{\mathcal{R}}$ *stand for the union of components of* $\Gamma^1_{\mathcal{R}} \cap \Gamma^2_{\mathcal{R}}$ *contained in* Ω_{per}.

The regions ${\Omega^u_{\text{do}}}^*$ and ${\Omega^v_{\text{do}}}^*$ in the statement of the previous result are extensions of Ω^u_{do} and Ω^v_{do} to cover the limiting regions in between Ω^u_{do} and Ω_{per}, and in between Ω^v_{do} and Ω_{per}. Thus,

$$\begin{aligned} {\Omega^u_{\text{do}}}^* &:= \{x \in \bar{\Omega} \; : \; \lambda(x) > 0, \;\; \lambda(x)d(x) > \mu(x)b(x), \;\; \mu(x)a(x) \leq \lambda(x)c(x)\}, \\ {\Omega^v_{\text{do}}}^* &:= \{x \in \bar{\Omega} \; : \; \mu(x) > 0, \;\; \lambda(x)d(x) \leq \mu(x)b(x), \;\; \mu(x)a(x) > \lambda(x)c(x)\}. \end{aligned}$$

Incorporating into the discussion these limiting regions, the global attractor of (1.19) is allowed to vary continuously between the semitrivial steady state and the coexistence steady state.

Regarding previous approaches to the singular perturbation problem in the context of competing species, to the best of our knowledge, the first perturbation result for a diffusive Lotka–Volterra competition model goes back to the 90's and, in particular, to Theorem 4.1 of Hutson, López-Gómez, Mischaikow and Vickers [63], where the next degenerate model was introduced for analyzing competition between mutant species

$$\begin{cases} \frac{\partial u}{\partial t} = \mu \Delta u + u(\alpha(x) - u - v) & \text{in } \; \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial t} = \mu \Delta v + v(\beta(x) - u - v) & \text{in } \; \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \; \partial\Omega \times (0, +\infty), \\ u(x,0) = u_0(x), \;\; v(x,0) = v_0(x) & \text{in } \; \Omega. \end{cases} \tag{1.21}$$

In this model the diffusion rates are assumed to be equal to each other, $\mu := d_1 = d_2$, whereas the growth rates, denoted by $\alpha := \lambda$ and $\beta := \mu$ are assumed to be 'sufficiently smooth'. Finally, the competition coefficients are assumed to be constant and equal to 1. The model is highly degenerate in the sense that we do not have neither high nor low competition. Precisely, the main singular perturbation result of [63] reads as follows.

Theorem 4.1 of [63]. *Let (u_μ, v_μ) be a family of coexistence states of the system* (1.21). *Set*

$$u_0(x) = \begin{cases} 0 & \text{if } \alpha(x) \le 0 \text{ or } \beta(x) > \alpha(x) > 0, \\ \alpha(x) & \text{if } \alpha(x) > \beta(x) \text{ and } \alpha(x) > 0, \end{cases}$$

and

$$v_0(x) = \begin{cases} 0 & \text{if } \beta(x) \le 0 \text{ or } \alpha(x) > \beta(x) > 0, \\ \beta(x) & \text{if } \beta(x) > \alpha(x) \text{ and } \beta(x) > 0. \end{cases}$$

Then,

$$\lim_{\mu \to 0} (u_\mu, v_\mu) = (u_0, v_0)$$

uniformly on compact subsets of

$$\bar{\Omega} \setminus \{x \in \bar{\Omega} \ : \ \alpha(x) = \beta(x)\}.$$

The proof of [63, Th. 4.1] follows by computing the limiting profile of a monotone scheme made of sub and supersolutions associated to (1.21). Such a monotone scheme had been introduced by López-Gómez and Sabina [77], as earlier as in 1992, to try to characterize the existence of coexistence states in competition systems, and, later, it has been intensively used when dealing with singular parturbation problems (see, e.g, Theorem 1.2 of [39], Theorem 1 of [42], Theorem 4.1 of [38], which is Theorem 4.4 in this dissertation, Theorem 1.2 and Lemma 3.3 of Hutson, Lou and Mischaikow [65], and Theorem 5.4(iii) of He and Ni [53], where the authors remarked that Theorem 4.1 of Hutson, López-Gómez, Mischaikow and Vickers [63] also holds for different diffusion rates).

Besides the fact that the model (1.21) only considers constant competition coefficients, one of the main differences between the models (1.18), the one considered in this dissertation, and (1.21) lies in the admissible spatial configurations of $\bar{\Omega}$ according to the associated non-spatial dynamics as pointed out in Figure 1.9. Indeed, the semitrivial steady states of the kinetic counterpart of (1.21),

$$\begin{cases} \frac{\partial u}{\partial t} = \lambda(x) u - u^2 - uv & \text{in } (0, +\infty), \\ \frac{\partial v}{\partial t} = \mu(x) v - v^2 - uv & \text{in } (0, +\infty), \\ u(0) = u_0, \quad v(0) = v_0, \end{cases}$$

are given by

$$(\alpha(x), 0) \quad \text{if } \alpha(x) > 0 \quad \text{and} \quad (0, \beta(x)) \quad \text{if } \beta(x) > 0.$$

Thus,

$$\Omega_{\text{per}} = \emptyset \quad \text{and} \quad \Omega_{\text{bi}} = \emptyset,$$

and the only admissible regions of $\bar{\Omega}$ in the setting of Theorem 4.1 of [63] are

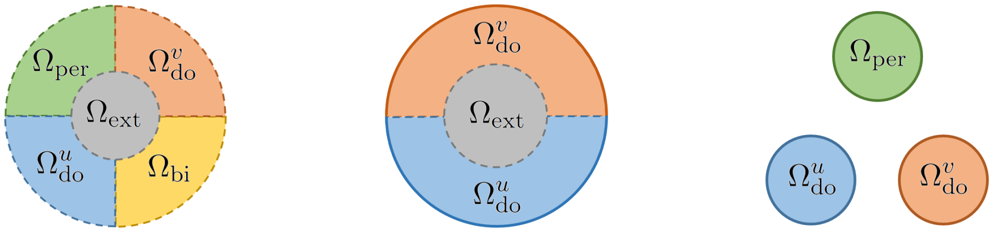
$$\Omega_{\text{ext}} := \{x \in \bar{\Omega} \ : \ \alpha(x), \beta(x) \le 0\},$$

$${\Omega^u_{\text{do}}}^* = \{x \in \bar{\Omega} \ : \ \alpha(x) > 0 \text{ and } \alpha(x) > \beta(x)\}$$

and

$${\Omega^v_{\text{do}}}^* = \{x \in \bar{\Omega} \ : \ \beta(x) > 0 \text{ and } \beta(x) > \alpha(x)\}$$

Note that, if $\alpha(x) = \beta(x) > 0$, then the kinetic model associated to (1.21) exhibits a continuum of coexistence steady states. Thus, $x \in \Omega_{\text{junk}}$.



(a) An admissible configuration of $\bar{\Omega}$ in the context of Theorem 4.4

(b) An admissible configuration of $\bar{\Omega}$ in the context of Theorem 4.1 of [63] and Theorem 5.2(iii) of [53]

(c) Three admissible configurations of $\bar{\Omega}$ in the context of Lemma 3.3 and Theorem 1.2 of [65]

Figure 1.9: Plot of the most general admissible configuration of $\bar{\Omega}$ in (a) this dissertation, (b) the works of Hutson, López-Gómez, Mischaikow and Vickers [63], and He and Ni [53], and (c) the work of Hutson, Lou and Mischaikow [65].

Therefore, the main singular perturbation theorem of this dissertation, Theorem 4.4, is the first existing result dealing with general spatially heterogeneous competing species systems.

The second difference between models (1.18) and (1.21) is related to the differential and boundary operators. Whereas our Theorem 4.4 allows $\mathcal{L}_1$ and $\mathcal{L}_2$ to be differential elliptic operators in divergence form, and $\mathcal{B}_1$ and $\mathcal{B}_2$ to be non-classical mixed boundary operators, [63, Th. 4.1] assumes that $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and that $\mathcal{B}_1 = \mathcal{B}_2$ are non-flux boundary conditions. Allowing $\mathcal{B}_1$ and $\mathcal{B}_2$ to be of Dirichlet type in some component of $\partial\Omega$ makes the family $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ generate a boundary layer in that component. Thus, the uniform convergence around that component is lost. As a consequence, it is difficult to control the singular limit. So, this dissertation deals with a truly *singular* perturbation problem.

The next significant contribution to the theory of singular perturbations in the context of competing species systems goes back to the beginning of the 21^{st} century, with the work of Hutson, Lou and Mischaikow [65], who considered the singular perturbation problem for the following *generalized* competition model of Lotka–Volterra type

$$\begin{cases} u_t = \mu\Delta u + uf(u,v,x) & \text{in } \ \Omega \times (0,+\infty), \\ v_t = \nu\Delta v + vg(u,v,x) & \text{in } \ \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \ \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \ \Omega, \end{cases} \tag{1.22}$$

under the following assumptions:

(H1) (Smoothness) $f, g : \mathcal{C}^1 \times \mathcal{C}^1 \times \mathcal{C}^1 \to \mathbb{R}$.

(H2) (Species limitation) There exists a positive constant M such that, for every

$x \in \bar{\Omega}$, $f(M,0,x) < 0$, $f(0,M,x) < 0$, $g(M,0,x) < 0$ and $g(0,M,x) < 0$.

(H3) (Inter and intra-specific competition) For every $x \in \bar{\Omega}$, $u \geq 0$, $v \geq 0$, $f_u(u,v,x) < 0$, $f_v(u,v,x) < 0$, $g_u(M,0,x) < 0$ and $g_v(u,v,x) < 0$.

Note that these initial hypothesis provide us with a more general reaction diffusion competition model than (1.18) from the point of view of the nonlinearities. Indeed, the genuine classical choice

$$f(u,v,x) := \lambda(x) - a(x)u - b(x)v \quad \text{and} \quad g(u,v,x) := \mu(x) - c(x)u - d(x)v,$$

going back to Lotka–Volterra, satisfies (H1)–(H3). Incidentally, these general competitive kinetics had been previously introduced, at least, by López-Gómez in [80] and in Chapter 7 of [85].

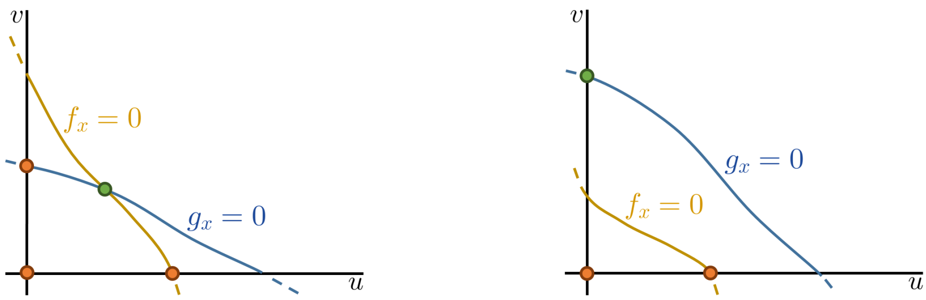
Despite their initial generality, the authors of [65] carry out the singular perturbation analysis in two specific situations where the associated kinetic model

$$\begin{cases} \frac{\partial u}{\partial t} = u\, f(u,v,x) & \text{in } (0,+\infty), \\ \frac{\partial v}{\partial t} = v\, g(u,v,x) & \text{in } (0,+\infty), \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.23}$$

is assumed to exhibit a global attractor, of the same type, for every $x \in \bar{\Omega}$. Depending on whether this attractor is a coexistence state, or a semitrivial steady state of (1.23), the authors differentiate between the *interior case* and the *boundary case*. Figure 1.10 shows an admissible plot of the isoclines $f_x = 0$ and $g_x = 0$, i.e., the curves,

$$\{(u,v) \in \mathbb{R}^2 \;:\; f(u,v,x) = 0\} \quad \text{and} \quad \{(u,v) \in \mathbb{R}^2 \;:\; g(u,v,x) = 0\}$$

for any $x \in \bar{\Omega}$ in each of these situations.



(a) Plot of the isoclines $f_x = 0$ and $g_x = 0$ in the *interior case*, i.e., under (H1)–(H5)

(b) Plot of the isoclines $f_x = 0$ and $g_x = 0$ in the *boundary case*, i.e., under (H1)–(H3) and (H6)–(H7)

Figure 1.10: Prototypical plot of the isoclines $f_x = 0$ and $g_x = 0$ for the generalized non-spatial model (1.23), i.e., for the choices $f_x := f(u,v,x)$ and $g_x := g(u,v,x)$, under different assumptions.

More precisely, the 'interior case' consists in assuming the following hypothesis, in addition to (H1)–(H3):

(H4) (Unique solution in non-negative cone) For every $x \in \bar{\Omega}$, $f(u,v,x) = g(u,v,x) = 0$ has a unique solution, denoted by $(u^*(x), v^*(x))$ in $\{(u,v) \in \mathbb{R}^2 : u \geq 0,\ v \geq 0\}$. Moreover $u^*(x) > 0$ and $v^*(x) > 0$ for every $x \in \bar{\Omega}$.

(H5) (Hyperbolicity and local attractivity) For every $x \in \bar{\Omega}$, the following holds:

$$(f_u g_v - f_v g_u)|_{(u,v,x)=(u^*(x),v^*(x),x)} > 0.$$

It easily follow from (H1)–(H5) that, for every $x \in \bar{\Omega}$, the associated kinetic model (1.23) exhibits both semitrivial steady states, which are linearly stable, and a unique coexistence steady state, which is linearly stable and actually it is a global attractor for the component-wise positive solutions of (1.23). Therefore,

$$\bar{\Omega} = \Omega_{\text{per}}.$$

So, essentially, the dynamic of the non-spatial model (1.23) is uniform in the entire inhabiting territory. Consequently, this model cannot be considered to be spatially heterogeneous in a wide sense.

For the sake of simplicity, Figure 1.11 shows a plot of the isoclines in the Lotka–Volterra model according to the kinetic dynamics at each location $x \in \bar{\Omega}$, helping us to compare the generalized competition model, whose isoclines have been plotted in Figure 1.10, and the Lotka–Volterra one.

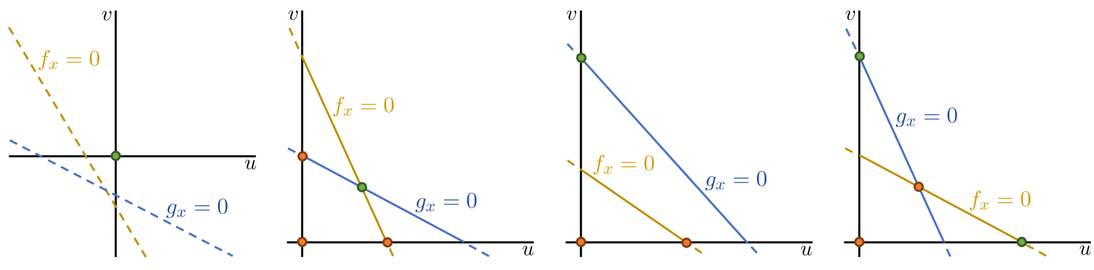


Figure 1.11: Prototypical plot of the isoclines $f_x = 0$ and $g_x = 0$ for the Lotka–Volterra non-spatial model (1.19), i.e., for the choices $f_x := \lambda(x) - a(x)u - b(x)v$ and $g_x := \mu(x) - c(x)u - d(x)v$, depending on the region of $\bar{\Omega}$.

The precise singular perturbation result delivered by Hutson, Lou and Mischaikow [65] in the *interior case* is stated as follows.

Lemma 3.3 of [65]. *Assume that (H1)–(H5) hold. Let (u,v) denote any coexistence state of (1.22). Then,* $\lim_{(\mu,\nu)\to(0,0)}(u,v) = (u^*,v^*)$ *uniformly in $\bar{\Omega}$.*

As expected, like in Theorem 4.4, in such a context the coexistence steady states of the diffusive model (1.22) converge as $(\mu,\nu) \to (0,0)$ to the unique coexistence steady state of the kinetic model (1.23). Under non-flux boundary conditions, this convergence is uniform even on the boundary of Ω. Under Dirichlet boundary conditions, the solutions must exhibit boundary layers. In such a case, one can only expect convergence on compact subsets of Ω, as stated in Theorem 4.4.

As far as concerns the *boundary case*, it holds by assuming the next two hypothesis, in addition to (H1)–(H3):

(H6) For every $x \in \bar{\Omega}$, $f(u,v,x) = g(u,v,x) = 0$ has no solution in $\{(u,v) \in \mathbb{R}^2 \;:\; u \geq 0,\; v \geq 0\}$.

(H7) For every $x \in \bar{\Omega}$, $f(0,0,x) > 0$ and $g(0,0,x) > 0$.

As a consequence of imposing (H1)–(H3) and (H6)–(H7), the kinetic model (1.23) admits both semitrivial steady states, and one of them is linearly stable whereas the other is linearly unstable. Moreover, (1.23) cannot admit coexistence steady states. Thus, the linearly stable semitrivial steady state is a global attractor for the component-wise positive solutions of (1.23). As a by-product, either

$$\bar{\Omega} = \Omega^u_{\text{do}}, \qquad \text{or} \qquad \bar{\Omega} = \Omega^v_{\text{do}}.$$

Again, much like in the *interior case*, the dynamics of the non-spatial model (1.23) are assumed to be uniform in $\bar{\Omega}$. Therefore, no spatial heterogeneities are really taken into account by Hutson, Lou and Mischaikow in [65] as all the situations dealt with are uniform all over $\bar{\Omega}$.

To conclude, in the *boundary case* the singular perturbation result lies hidden in the proof of Theorem 1.2 of Hutson, Lou and Mischaikow [65]. Indeed, the authors prove that under (H1)–(H3) and (H6)–(H7), if $(0, \bar{v}^*)$ is the global attracting equilibrium of the non-spatial system (1.23) for every $x \in \bar{\Omega}$, then

$$\lim_{(\mu,\nu)\to(0,0)} (u,v) = (0, \bar{v}^*)$$

uniformly in $\bar{\Omega}$. Moreover, they show that, in such a situation, with $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, under non-flux boundary conditions, the diffusive model (1.22) does not admit coexistence steady states through a contradiction argument involving the divergence theorem.

Summing up, at the light of the previous discussion on the findings of Hutson, Lou and Mischaikow [65], despite the fact that the admissible nonlinearities in the statement of Theorems 1.2 and Lemma 3.3 of [65] look more complex than the classical Lotka-Volterra kinetics of model (1.18), at the end of the day, the admissible configurations of the inhabiting territory, $\bar{\Omega}$, in the setting of Theorem 4.4 are far more rich (see Figure 1.9) than those treated by Hutson, Lou and Mischaikow [65].

Furthermore, our Theorem 4.4 not only provides us with the limiting profiles of the coexistence steady states of (1.18) in the regions Ω_{ext}, Ω_{per}, Ω^u_{do}, Ω^v_{do}, as well as in the regions between them, through the use of the monotone scheme introduced by Hutson, López-Gómez, Mischaikow and Vickers [63], but it also reveals that the limit may not be determined in the region Ω_{bi}. One may conjecture that this occurs because $\Omega_{\text{bi}} \neq \emptyset$ implies the existence of multiple coexistence steady states which might converge to different steady states of (1.19) in that region. In order to prove this conjecture, we need to establish the Induced Instability Principle, which is the next goal of this dissertation.

1.5 The Induced Instability Principle

The Induced Instability Principle is one of the main novelties, and most fruitful tools, developed in this dissertation, since it provides us with a connection between the linear stability character of the steady states of the spatial model (1.18) and that of the steady states of the non-spatial model (1.19).

Precisely, it establishes that if the non-spatial model (1.19) exhibits a linearly unstable steady state,

$$(u_*(x), v_*(x)) \quad \text{for every} \quad x \in \Omega_{\text{un}},$$

for some open subset $\Omega_{\text{un}} \subset \Omega$, which might be arbitrarily small, and the spatial model (1.18) exhibits a family of steady states, $\big(u_{(d_1,d_2)}, v_{(d_1,d_2)}\big)$ perturbing from (u_*, v_*) uniformly in Ω_{un} as $d_1, d_2 \downarrow 0$, then $\big(u_{(d_1,d_2)}, v_{(d_1,d_2)}\big)$ is linearly unstable for sufficiently small $d_1, d_2 > 0$. Therefore, the *local* instability of (u_*, v_*) in Ω_{un} turns into *global* instability of $\big(u_{(d_1,d_2)}, v_{(d_1,d_2)}\big)$ in Ω for sufficiently small d_1 and d_2. The Induced Instability Principle follows from Theorem 5.9 and Proposition 5.11 and can be stated as follows.

Induced Instability Principle (IIP).

Suppose that $\mathcal{L} := \mathcal{L}_1 = \mathcal{L}_2$ *and* $\mathcal{B} := \mathcal{B}_1 = \mathcal{B}_2$. *Let* $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}$ *be a family of steady states of* (1.18) *such that, for some open subset* $\Omega_{\text{un}} \subset \Omega$,

$$\lim_{(d_1,d_2)\to(0,0)} \big(u_{(d_1,d_2)}, v_{(d_1,d_2)}\big) = (u_*, v_*) \quad \textit{uniformly in} \;\; \Omega_{\text{un}}$$

with $(u_*(x), v_*(x))$ *linearly unstable for all* $x \in \Omega_{\text{un}}$, *as a steady-state solution of* (1.19). *Then,* $\delta > 0$ *exists such that* $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ *is linearly unstable for all* $d_1, d_2 < \delta$.

The proof of this principle when the family $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}$ consists of coexistence steady states (Theorem 5.9) is mainly based on the monotonicity, with respect to the domain, of the principal eigenvalue of the linearized system, which is of quasi-cooperative type,

$$\mathfrak{L}_{(d_1,d_2)} := \begin{pmatrix} d_1\mathcal{L} & 0 \\ 0 & d_2\mathcal{L} \end{pmatrix} + H \quad \text{with} \;\; H \in \mathcal{M}_2(\mathcal{C}(\bar{\Omega})), \;\; i,j = 1,2,$$

and the fact that, when H is made up of constant coefficients, the principal eigenvalue of $\mathfrak{L}_{(d_1,d_2)}$ converges to the lowest eigenvalue of H, as a 2×2 real valued matrix.

Proposition 5.11 establishes the previous principle when $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}$ consists of semitrivial steady states. In such a case, the proof follows different patterns where the assumptions $\mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{B}_1 = \mathcal{B}_2$ are unnecessary.

The first consequence of the Induced Instability Principle is Theorem 2.1(i) of Furter and López-Gómez [48]. Since it depends on Proposition 5.11, it allows $\mathcal{L}_1$ and $\mathcal{L}_2$, as well as $\mathcal{B}_1$ and $\mathcal{B}_2$, to be different. Such a result can be stated as follows.

Corollary 5.12 (Permanence when $\Omega_{\text{per}} \neq \emptyset$, or ${\Omega^u_{\text{do}}}^* \neq \emptyset$ and ${\Omega^v_{\text{do}}}^* \neq \emptyset$).

Suppose that either $\Omega_{\text{per}} \neq \emptyset$, or ${\Omega^u_{\text{do}}}^* \neq \emptyset$ and ${\Omega^v_{\text{do}}}^* \neq \emptyset$. Then, $\delta > 0$ exists such that the parabolic problem (1.18) is permanent for all $d_1, d_2 < \delta$ (the trivial and semitrivial steady states are linearly unstable). Therefore, it admits a stable coexistence steady state for these diffusion rates.

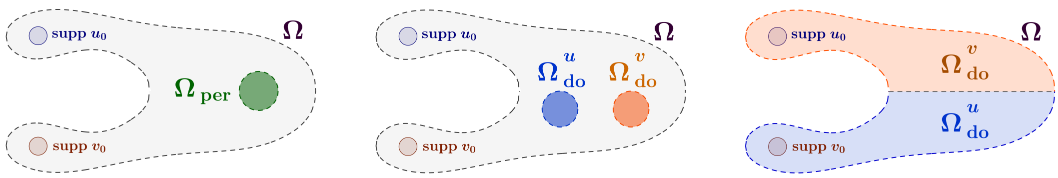
This result is an extension of Theorem 2.1(i) of [48] for arbitrary differential and boundary operators. As established by this result, and illustrated by Figure 1.12, the introduction of the diffusion through the elliptic operators in the Lotka–Volterra competition model facilitates the permanence of the species. For example, in each of the configurations of Ω considered in Figure 1.12 the species u and v are originally supported in hostile regions from the point of view of the kinetic model. Indeed, in cases (a) and (b) we have that

$$\text{supp } u_0 \subset \Omega_{\text{ext}} \quad \text{and} \quad \text{supp } v_0 \subset \Omega_{\text{ext}},$$

whereas in (c),

$$\text{supp } u_0 \subset \Omega^v_{\text{do}} \quad \text{and} \quad \text{supp } v_0 \subset \Omega^u_{\text{do}}.$$

Thus, according to the non-diffusive model (1.19), both species should become extinct. However, in the diffusive model (1.18), with small diffusion rates none of the species becomes extinct because the diffusion term allows the individuals of both species to reach a favorable area to maintain its population.



(a) $\Omega_{\text{per}} \neq \emptyset$ (b) $\Omega^u_{\text{do}} \neq \emptyset$ and $\Omega^v_{\text{do}} \neq \emptyset$ (c) Ω made of Ω^u_{do} and Ω^v_{do}

Figure 1.12: Different configurations of Ω for which the reaction-diffusion model (1.18) exhibits permanence for sufficiently small diffusion rates. Ω_{per}, Ω^u_{do} and Ω^v_{do} may be surrounded by Ω_{ext}.

The second consequence of the Induced Instability Principle, and one of the main novelties of this dissertation, provides us with an astonishing example of multiplicity of coexistence steady states in the heterogeneous Lotka–Volterra reaction–diffusion model. It turns out that the non-emptiness of the region Ω_{bi} gives rise to multiple coexistence steady states, at least, for the symmetric competition model.

$$\begin{cases} \frac{\partial u}{\partial t} + d_1 \mathcal{L} u = \lambda(x) u - a(x) u^2 - b(x) uv & \text{in } \ \Omega \times (0, +\infty), \\ \frac{\partial v}{\partial t} + d_1 \mathcal{L} v = \lambda(x) v - a(x) v^2 - b(x) uv & \text{in } \ \Omega \times (0, +\infty), \\ \mathcal{B} u = \mathcal{B} v = 0 & \text{on } \ \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x), \ \ v(x, 0) = v_0(x) & \text{in } \ \Omega. \end{cases} \tag{1.24}$$

Note that the special choice $\lambda = \mu$, $a = d$ and $b = c$, made in (1.24), produces

$${\Omega^u_{\text{do}}}^* = \emptyset \quad \text{and} \quad {\Omega^v_{\text{do}}}^* = \emptyset.$$

Thus, the inhabiting territory, Ω, is made, essentially, of $\Omega_{\rm ext}$, $\Omega_{\rm per}$ and $\Omega_{\rm bi}$, providing us with the perfect setting to study the effect of inserting a region $\Omega_{\rm bi}$ in an habitat consisting of regions where the limit of the coexistence steady states is determined by Theorem 4.4. The multiplicity result delivered in this dissertation reads as follows.

Theorem 5.16 (Multiplicity when $\Omega_{\rm bi} \neq \emptyset$).

Assume that $\Omega_{\rm per} \neq \emptyset$ and $\Omega_{\rm bi} \neq \emptyset$. Then $\delta_{\rm m} > 0$ exists such that, for every $\delta \in (0, \delta_{\rm m})$, (1.24) admits at least three coexistence steady states; two of them linearly stable and another one linearly unstable. Moreover, such a linearly unstable coexistence steady state perturbs from the coexistence steady state of the associated non-spatial problem in the region $\Omega_{\rm bi} \cup \Omega_{\rm per}$.

The fact that Theorem 5.16 only requires Ω to be of class $\mathcal{C}^2$, contrasts with the seminal work of Matano and Mimura [99], where the multiplicity result was a consequence of breaking down the convexity of the territory. Indeed, Theorem A of Matano and Mimura [99, Sec. 3] establishes that whenever $\bar{\Omega} = \Omega_{\rm bi}$, the spatially **homogeneous** diffusive competition model of Lotka-Volterra, with $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ and non-flux boundary conditions, exhibits multiple coexistence steady states if Ω is made of two convex domains connected by a sufficiently narrow corridor (dumbbell shape). Actually, this is a rather genuine phenomenology inherent to the superlinear internal character of the competition model. Astonishingly, our multiplicity result does not depend on the geometry of the inhabiting territory, but rather on the spatial heterogeneities themselves, regardless the size of the permanence and bistability zones.

In order to extract all the information provided by Theorem 5.16, and for being able to analyze the effect of inserting $\Omega_{\rm bi}$ in Ω, we should first go back to Theorem 1.1 of Hutson, Lou and Mischaikow [65]. According to it, in the *interior case* (see Figure 1.10(a)), i.e., under (H1)–(H5), the diffusive competition system (1.22) admits a unique coexistence steady state, which is a global attractor with respect to its component-wise positive solutions. In this dissertation, we provide a completely different, much simpler and more versatile proof of the result of Hutson, Lou and Mischaikow in the setting of (1.18), under non-classical Robin boundary conditions, which is valid for general second order elliptic operators, $\mathcal{L}_1$ and $\mathcal{L}_2$, not necessarily of Neumann type. The precise statement of our generalized version of Theorem 1.1 of [65] can be stated as follows.

Theorem 6.9 (Uniqueness when $\bar{\Omega} = \Omega_{\rm per}$).

Assume that $\bar{\Omega} = \Omega_{\rm per}$ and $\Gamma^1_{\mathcal{D}} = \Gamma^2_{\mathcal{D}} = \emptyset$. Then, $\delta > 0$ exists such that, for every $d_1, d_2 \in (0, \delta)$, the reaction-diffusion model (1.18) exhibits a unique coexistence steady state which is a global attractor with respect to the component-wise positive solutions.

Let us assume that we are in the diffusive symmetric context of the model (1.24) under Robin boundary conditions with $\bar{\Omega} = \Omega_{\rm per}$, i.e.,

$$\lambda(x) > 0 \quad \text{and} \quad b(x) < a(x) \quad \text{for all} \;\; x \in \bar{\Omega}.$$

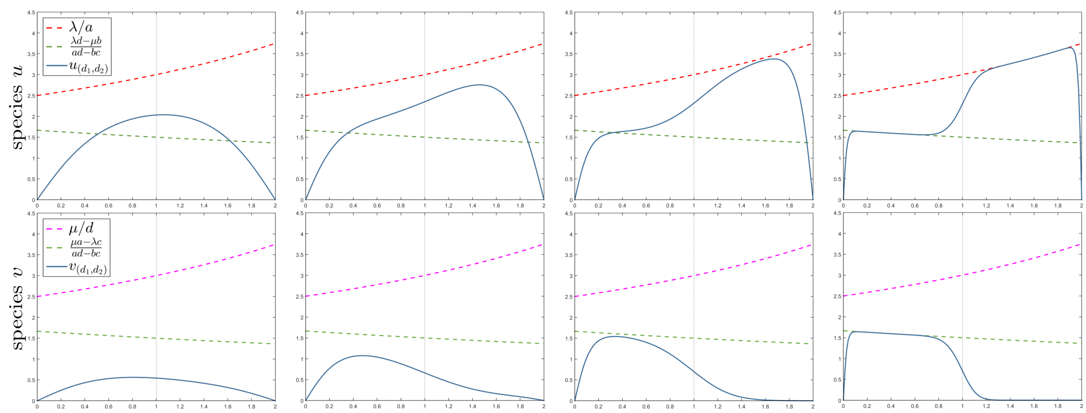
Then, thanks to Theorem 6.9, (1.24) admits a unique coexistence steady state, which is linearly stable and a global attractor with respect to the component-wise positive solutions, at least for sufficienly small diffusion rates. Moreover, it approximates the coexistence steady state of the associated kinetic model (1.19),

$$(u_*, v_*) = \left(\frac{\lambda}{a+b}, \frac{\lambda}{a+b} \right).$$

Therefore, the diffusive model mimics the dynamics of the its non-diffusive counterpart for sufficiently small diffusion rates.

Now, we can perturb the function $b(x)$ until there exists a unique $x_0 \in \Omega$ such that $b(x_0) = a(x_0)$. During such homotopy the diffusive model still exhibits uniqueness for sufficiently small diffusion rates. If we further increase b at x_0, so creating a bi-stability region, $\Omega_{\rm bi}$, surrounding x_0, then, by Theorem 5.16, the number of coexistence steady states of the diffusive model will increase, at least, to three. Therefore, the emergence of $\Omega_{\rm bi}$ generates multiplicity in the diffusive competition model, regardless of its geometry and size.

It is remarkable that the Induced Instability Principle also explains the reasons for which one may not be able to determine the limiting profile of the coexistence steady states in the region $\Omega_{\rm bi}$ in the context of the main singular perturbation result for the diffusive system. Indeed, in the previous example, one of the three coexistence steady states of (1.24), the unstable one, converges to the coexistence steady state of the associated kinetic model in $\Omega_{\rm bi}$, $(\frac{\lambda}{a+b}, \frac{\lambda}{a+b})$, which is linearly unstable therein. On the other hand, Theorem 5.16 ensures us the existence of two more coexistence steady states for the diffusive model (1.24), which are linearly stable. If the limits as $d_1, d_2 \downarrow 0$ of all these coexistence steady states of (1.24) were uniformly determined in any zone of $\Omega_{\rm bi}$, then they should equal $(\frac{\lambda}{a+b}, \frac{\lambda}{a+b})$, which is linearly unstable therein. Therefore, according to the Induced Instability Principle, these coexistence steady states should also be linearly unstable, leading to a contradiction.



(a) $d_1 = d_2 = 0.5$ (b) $d_1 = d_2 = 0.12662$ (c) $d_1 = d_2 = 0.027291$ (d) $d_1 = d_2 = 0.0013329$

Figure 1.13: Plot of the components $u_{(d_1,d_2)}$ (up) and $v_{(d_1,d_2)}$ (down) of a coexistence steady state of the diffusive competition system (1.18), for different diffusion rates, in the symmetric case with $\Omega = (0,2)$, $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, $a = d = 2 - \frac{1}{3}x$, $b = c = 1 + \frac{2}{3}x$, under Dirichlet boundary conditions. Note that $\Omega_{\rm per} = [0,1)$ and $\Omega_{\rm bi} = (1,2]$.

As a consequence of the previous analysis, one may conjecture that the two stable coexistence steady states exhibited by the symmetric reaction-diffusion model (1.24) converge to the *stable* semitrivial steady states of the non-diffusive/kinetic counterpart in $\Omega_{\rm bi}$. This conjecture is supported by Figure 1.13, which shows the behavior of one of those stable steady states when the diffusion rates go to zero.

1.6 Uniqueness under low competition

Once proven that perturbing an habitat made of $\Omega_{\rm per}$ by inserting on it a region $\Omega_{\rm bi}$ can provoke multiple coexistence steady states as $d_1, d_2 \downarrow 0$, so destroying the former uniqueness of the coexistence state, one may wonder if the same happens when, instead of $\Omega_{\rm bi}$, the regions $\Omega_{\rm ext}$, $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$ are non-empty. It should be remembered that, unlike $\Omega_{\rm bi}$, in these regions the limiting profile of the coexistence steady states of (1.18) is uniquely determined by Theorem 4.4, so one may expect that uniqueness still holds.

The uniqueness result established in this dissertation provides us with the first one in the literature for a Lotka–Volterra competition model with completely heterogeneous competition coefficients. It lies over three assumptions. First, it assumes that the model exhibits low competition in the habitat, i.e.,

$$bc \lneq ad \quad \text{in } \bar{\Omega} \tag{1.25}$$

Although the regions $\Omega_{\rm ext}$, $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$ can admit areas where high competition occurs, the condition (1.25) is imperative because the previous multiplicity result established that combining a high competition region, $\Omega_{\rm bi}$, with a low competition one, $\Omega_{\rm per}$, can provoke multiple coexistence steady states. Moreover, in the presence of low competition, the assumption $\bar{\Omega} = \Omega_{\rm per}$ produces uniqueness.

The remaining two assumptions come from the fact that our proof of the uniqueness result makes an intensive use of the Picone identity, which is needed to show that any coexistence steady state of (1.18) is linearly stable. By an additional argument involving the fixed point index in cones, this feature guarantees the uniqueness. Since the Picone identity needs the operator to be self-adjoint, $\mathcal{L}_1$ and $\mathcal{L}_2$ must be of the type

$$\mathcal{L}_i = -\text{div}(A_i \nabla \cdot) + C_i,$$

with $A_i \in \mathcal{M}_N^{\rm sym}(\mathcal{C}^1(\bar{\Omega}))$ and $C_i \in \mathcal{C}(\bar{\Omega})$. On the other hand, once Picone identity is applied, one needs to determine whether, or not, an integral is positive. The least constraining assumption to obtain such positivity consists on supposing that the next estimate holds:

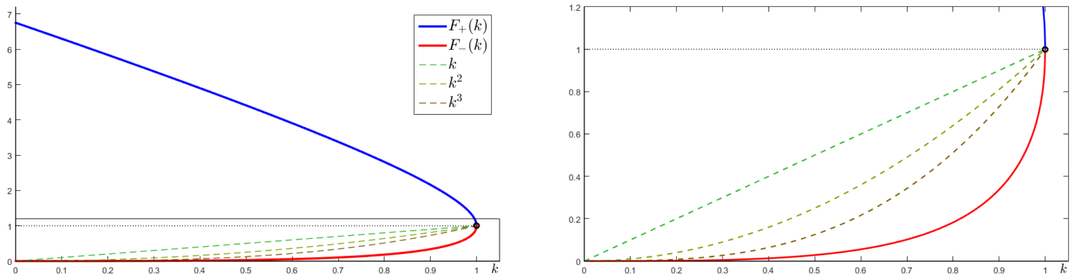
$$\max_{\bar{\Omega}} \left(\frac{ad^2}{c^3} F_- \left(\frac{bc}{ad} \right) \right) \leq \min_{\bar{\Omega}} \left(\frac{ad^2}{c^3} F_+ \left(\frac{bc}{ad} \right) \right), \tag{1.26}$$

with $F_\pm : [0,1] \to \mathbb{R}$ defined as

$$F_\pm(k) := \frac{1}{8} \left(27 - 18k - k^2 \pm (9-k)^{3/2}(1-k)^{1/2} \right), \qquad k \in [0,1].$$

Figure 1.14 plots $F_\pm$, as well as the functions $k \mapsto 1$, $k \mapsto k$, $k \mapsto k^2$ and $k \mapsto k^3$, to show that

$$F_-(k) \leq k^3 \leq k^2 \leq k \leq 1 \leq F_+(k) \quad \text{for all } \ k \in [0,1].$$



(a) Plot for $k \in [0,1]$ (b) Zoom in $[0,1] \times [0,1.2]$

Figure 1.14: Plot of the functions F_- and F_+ in $[0,1]$, and comparison with the functions $k \mapsto 1, k, k^2, k^3$.

Thus, particularizing at $k = \frac{bc}{ad}$ and multiplying by $\frac{ad^2}{c^3}$ we have that

$$\frac{ad^2}{c^3} F_-\left(\frac{bc}{ad}\right) \leq \frac{b^3}{a^2 d} \leq \frac{b^2}{ac} \leq \frac{bd}{c^2} \leq \frac{ad^2}{c^3} \leq \frac{ad^2}{c^3} F_+\left(\frac{bc}{ad}\right) \quad \text{in } \Omega.$$

Hence, the assumption (1.26) holds if

$$\frac{b^3}{a^2 d}, \quad \text{or} \quad \frac{b^2}{ac}, \quad \text{or} \quad \frac{bd}{c^2}, \quad \text{or} \quad \frac{ad^2}{c^3}, \quad \text{is constant all over } \Omega, \tag{1.27}$$

providing us with a far more friendly condition than (1.26).

The main uniqueness result established in this dissertation in the low competition regime reads as follows.

Theorem 6.13 (Uniqueness under low competition. Global dynamics).

Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are self-adjoint, $bc \lneq ad$ in Ω, and the estimate (1.26) holds. Then:

(a) *If both semitrivial solutions exist and are linearly unstable, (1.20) admits a unique coexistence state. Moreover, it is a global attractor with respect to the component-wise positive solutions of (1.18).*

(b) *In any other case, (1.20) cannot admit any coexistence state.*

(c) *Both semitrivial solutions of (1.20) cannot be linearly stable simultaneously.*

(d) *If a semitrivial solution of (1.20) is linearly stable, then it is a global attractor with respect to the component-wise positive solutions of (1.18).*

(e) *If the trivial solution of (1.20) is linearly stable, then it is a global attractor with respect to the component-wise positive solutions of (1.18).*

In particular, if either $\Omega_{\mathrm{per}} \neq \emptyset$, or $\Omega^u_{\mathrm{do}} \neq \emptyset$ and $\Omega^v_{\mathrm{do}} \neq \emptyset$, then (a) holds for sufficiently small $d_1, d_2 > 0$.

Let us discuss this result. First, note that there are no constrains neither on the size of the diffusion rates, nor on the growth rates of the species. Thus, as soon as a, b, c

and d satisfy (1.25) and (1.26) (or (1.27)), then the statements of Theorem 6.13 hold, regardless of the sizes of λ and μ. As a consequence, for any given *smooth* configuration of Ω consisting of patches of type Ω_{ext}, Ω_{per}, Ω^u_{do} and Ω^v_{do}, we can choose a, b, c, and d, to be under the assumptions of Theorem 6.13, and then choose λ and μ so that the spatial model induces that particular given configuration.

As far as concerns the condition (1.27), note that it allows us to choose, freely, three coefficients, among a, b, c and d, while the remaining one depends on them. For example, if we pick the condition of

$$\frac{b^3}{a^2 d} \quad \text{being constant in } \; \Omega,$$

then we are free to fix, arbitrarily, the values of the functions $a, b, c \in \mathcal{C}(\bar{\Omega}; (0, +\infty))$, whereas the function d needs to be of the form

$$d := \xi \frac{b^3}{a^2}, \quad \text{with } \; \xi \in \mathbb{R}, \;\; \xi > \max_{\bar{\Omega}} \frac{ca}{b^2}.$$

The latter estimate follows by imposing $bc < ad$ in Ω, so that (1.25) holds. Hence, assumption (1.27) provides us with a wide range of competition models for which Theorem 6.13 holds. It is unnecessary in the presence of constant coefficients, of course!

Furthermore, Theorem 6.13 is a result structurally stable in the sense that if the model (1.18) satisfies the conditions (1.25) and (1.26) with strict inequalities, so that it exhibits uniqueness for small diffusion rates by Theorem 6.13, then such uniqueness is still preserved under small perturbations of the coefficients λ, μ, a, b, c, and d.

Finally, Theorem 6.13 establishes that the curves of change of stability of the semi-trivial positive solutions divide the (λ, μ) plane into several regions according to the global dynamics that (1.18) can exhibit. Indeed, in the special case when λ and μ are assumed to be constant, then, as illustrated by Figure 1.15, the (λ, μ)-plane consists of the following complementary regions:

- The region Λ_{ext}, consisting of the pairs (λ, μ) such that

  $$\lambda < \sigma_1[d_1\mathcal{L}_1; \mathcal{B}_1, \Omega] \quad \text{and} \quad \mu < \sigma_1[d_2\mathcal{L}_2; \mathcal{B}_2, \Omega].$$

  These estimates corresponds to Theorem 6.13(e), i.e., $(0, 0)$ is a global attractor with respect to the component-wise positive solution of (1.18). Thus, both species become extinct in the diffusive model.

- The region Λ^v_{do}, integrated by the pairs (λ, μ) such that

  $$\mu > \sigma_1[d_2\mathcal{L}_2; \mathcal{B}_2, \Omega] \quad \text{and} \quad \lambda < \sigma_1[d_1\mathcal{L}_1 + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega].$$

  As these estimates fit the case (d) of Theorem 6.13, it becomes apparent that $(0, \theta_{\{d_2,\mu,d\}})$ is a global attractor with respect to the component-wise positive solution of (1.18). Therefore, the species v dominates the dynamics.

- The region Λ^u_{do}, consisting of the pairs (λ, μ) such that

  $$\lambda > \sigma_1[d_1\mathcal{L}_1; \mathcal{B}_1, \Omega] \quad \text{and} \quad \mu < \sigma_1[d_2\mathcal{L}_2 + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega].$$

Here, $(\theta_{\{d_1,\lambda,a\}},0)$ is a global attractor with respect to the component-wise positive solutions of (1.18). Thus, the species u dominates the dynamics, and hence, the species v is driven to extinction.

- The region Λ_{per}, made of the pairs (λ,μ) such that

$$\lambda > \sigma_1[d_1\mathcal{L}_1;\mathcal{B}_1,\Omega], \quad \mu > \sigma_1[d_2\mathcal{L}_2;\mathcal{B}_2,\Omega],$$

$$\lambda > \sigma_1[d_1\mathcal{L}_1 + b\theta_{\{d_2,\mu,d\}};\mathcal{B}_1,\Omega] \quad \text{and} \quad \mu > \sigma_1[d_2\mathcal{L}_2 + c\theta_{\{d_1,\lambda,a\}};\mathcal{B}_1,\Omega].$$

  As a consequence of Theorem 6.13(a), in this region (1.20) admits a unique coexistence steady state, which is a global attractor with respect to the component-wise positive solution of (1.18). Therefore, the species are permanent.

It should be noted that, as illustrated by Figure 1.15, the curves of change of stability of the semitrivial positive solutions cannot cross each other by Theorem 6.13(c), because in such a case they would enclose a region where both semitrivial steady states of (1.18) are linearly stable. Moreover, as pointed out at the end of Theorem 6.13, if either $\Omega_{\text{per}} \neq \emptyset$, or $\Omega^u_{\text{do}} \neq \emptyset$ and $\Omega^v_{\text{do}} \neq \emptyset$, then $(\lambda,\mu) \in \Lambda_{\text{per}}$ for sufficiently small $d_1, d_2 > 0$.

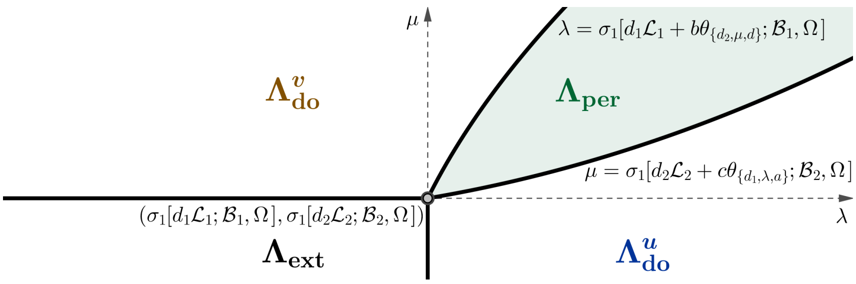


Figure 1.15: Plot of an admissible (λ,μ)-plane for (1.18), with λ and μ regarded as constant parameters varying in $\mathbb{R}$.

To conclude this general introduction, it should be remarked that Theorem 6.13 generalizes substantially Theorem 3.4(iii) of He and Ni [55], which dealt with (1.18) in the very special case when $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, $a = d = 1$, and b, c are positive constants satisfying $bc < 1$, under non-flux boundary conditions, i.e.,

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + u(m_1(x) - u - bv) & \text{in } \Omega\times(0,+\infty), \\ \frac{\partial v}{\partial t} = d_2\Delta v + v(m_2(x) - cu - v) & \text{in } \Omega\times(0,+\infty), \\ \partial_{\mathbf{n}} u = \partial_{\mathbf{n}} v = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.28}$$

with $d_1, d_2 > 0$, $b, c > 0$, and

$$m_i \in C^\alpha(\bar{\Omega}) \ \ (\alpha \in (0,1)), \quad \int_\Omega m_i \geq 0 \ \text{ and } \ m_i \not\equiv 0 \ \ (i = 1,2).$$

Note that the integral condition above implies that

$$\sigma_1[-d_1\Delta - m_1;\partial_{\mathbf{n}},\Omega] < 0, \quad \text{and} \quad \sigma_1[-d_2\Delta - m_2;\partial_{\mathbf{n}},\Omega] < 0,$$

(see He and Ni [55, Pr. 2.2(i)], and compare with the discussion of López-Gómez in Section 9.5 of [88]) and thus, (1.28) admits both semitrivial steady states,

$$(\theta_{\{d_1,m_1,1\}},0) \quad \text{and} \quad (0,\theta_{\{d_2,m_2,1\}}).$$

According to He and Ni [55], who clearly adopted the methodology of Furter and López-Gómez [48], we may distinguish between the following regions in the first quadrant of the (d_1,d_2)-plane, depending on the linear stability character of the semitrivial steady states of (1.28):

$$\Sigma_- := \left\{ (d_1,d_2)\in(0,+\infty)^2 \ : \ \begin{array}{l} (\theta_{\{d_1,m_1,1\}},0) \text{ and } (0,\theta_{\{d_2,m_2,1\}}) \\ \text{are linearly unstable} \end{array} \right\}$$

$$\Sigma_{U,*} := \left\{ (d_1,d_2)\in(0,+\infty)^2 \ : \ \begin{array}{l} (\theta_{\{d_1,m_1,1\}},0) \text{ is linearly stable} \\ \text{or linearly neutrally stable} \end{array} \right\}$$

$$\Sigma_{V,*} := \left\{ (d_1,d_2)\in(0,+\infty)^2 \ : \ \begin{array}{l} (0,\theta_{\{d_2,m_2,1\}}) \text{ is linearly stable} \\ \text{or linearly neutrally stable} \end{array} \right\}$$

$$\Pi := \left\{ (d_1,d_2)\in(0,+\infty)^2 \ : \ \begin{array}{l} (\theta_{\{d_1,m_1,1\}},0) \text{ and } (0,\theta_{\{d_2,m_2,1\}}) \\ \text{are linearly neutrally stable} \end{array} \right\}$$

Moreover, He and Ni focused their study on those b and c in the following region

$$\Xi = \{(b,c) : b,c>0,\ bc\leq 1\} \cup \{(b,c) : 0<c\leq 1/S_U\} \cup \{(b,c) : 0<b\leq 1/S_V\}$$

with

$$S_U := \sup_{d_1>0}\sup_{\bar\Omega} \frac{m_2}{\theta_{\{d_1,m_1,1\}}} \quad \text{and} \quad S_V := \sup_{d_2>0}\sup_{\bar\Omega} \frac{m_1}{\theta_{\{d_2,m_2,1\}}}.$$

Under all these assumptions, the main result of He and Ni [55], concerning uniqueness in the low competition regime, reads as follows.

Theorem 3.4 of [55]. *Assume that $(b,c)\in\Xi$. Then, the following mutually disjoint decomposition of $(0,+\infty)^2$ holds:*

$$(0,+\infty)^2 = (\Sigma_{U,*}\setminus\Pi)\cup(\Sigma_{V,*}\setminus\Pi)\cup\Sigma_-\cup\Pi.$$

Moreover, the following assertions are true for (1.28):

(i) *For every $(d_1,d_2)\in\Sigma_{U,*}\setminus\Pi$, $(\theta_{\{d_1,m_1,1\}},0)$ is globally asymptotically stable.*

(ii) *For every $(d_1,d_2)\in\Sigma_{V,*}\setminus\Pi$, $(0,\theta_{\{d_2,m_2,1\}})$ is globally asymptotically stable.*

(iii) *For every $(d_1,d_2)\in\Sigma_-$, (1.28) has a unique coexistence steady state which is globally asymptotically stable.*

(iv) *For every $(d_1,d_2)\in\Pi$, $\theta_{\{d_1,m_1,1\}}=c\theta_{\{d_2,m_2,1\}}$ and (1.28) has a compact global attractor consisting of a continuum of steady states*

$$\{(\xi\theta_{\{d_1,m_1,1\}},(1-\xi)\theta_{\{d_1,m_1,1\}}/c) \ : \ \xi\in[0,1]\}$$

connecting the two semitrivial steady states.

Let us compare Theorem 3.4 of He and Ni [55] with our Theorem 6.13. Although the structure of both results looks quite similar, one remarkable difference between them relies on the fact that He and Ni allow the model to exhibit high competition globally in Ω, by letting the pair (b, c) to be included in the sets

$$\Xi_U := \{(b, c) : 0 < c \leq 1/S_U \text{ and } b \geq S_U\}$$

and

$$\Xi_V := \{(b, c) : 0 < b \leq 1/S_V \text{ and } c \geq S_V\}.$$

However, as shown by Theorem 3.1 of He and Ni [55], these sets are not problematic since

$$\Sigma_{U,*} = (0, +\infty)^2 \quad \text{for all} \quad (b, c) \in \Xi_U$$

and

$$\Sigma_{V,*} = (0, +\infty)^2 \quad \text{for all} \quad (b, c) \in \Xi_V.$$

Thus, the model (1.28) cannot exhibit uniqueness if

$$(b, c) \in \Xi_U \cup \Xi_V.$$

Actually, the regions Ξ_U and Ξ_V receive a special treatment in the proof of Theorem 3.4 of [55] to show that this actually occurs.

Furthermore, one cannot expect a result of the type of part (iv) of Theorem 3.4 of [55] in the general setting covered by Theorem 6.13 of this dissertation. Firstly, because the boundary conditions for the species might be different. Secondly, because having a closer look at the proof of He and Ni reveals that b and c must be positive constants satisfying $bc = 1$ for the existence of a continuum of coexistence steady states (the coexistence steady states are degenerate).

Summarizing, this dissertation has succeeded in solving positively the singular perturbation problem and in deriving some very sharp uniqueness and multiplicity results for the diffusive competition model, as well as in incorporating to the usual scenario of the theory arbitrary second order elliptic operators, instead of $-\Delta$, general non-classical mixed boundary conditions, instead of homogeneous Neumann or Dirichlet boundary conditions, and truly spatially heterogeneous models, instead of models where Ω equals, simply, Ω_{per} or Ω_{bi}.

The techniques developed here can be adapted, almost *mutatis mutandis*, to cover a huge variety of competitive kinetics, like those introduced by López-Gómez in [80] and [85, Ch. 7], though we have refrained ourselves of delivering these novelties with the greatest generality possible by the sake of highlighting and emphasizing the true novelties of this dissertation, reducing the more sophisticated technicalities as much as possible. Even dealing with the most classical Lotka–Volterra kinetics, our findings enjoy a great interest in the field and have sharpened, very substantially, some very recent findings by some of the top leading experts in PDE's.

Part I

The Generalized Logistic Equation

Chapter 2

The singular perturbation problem for the generalized logistic equation

Introduction

This chapter analyzes the limiting behavior as $d \downarrow 0$ of the positive solutions of

$$\begin{cases} d\mathcal{L}u = uh(u,x) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, $d > 0$ is a positive constant, and the differential operator $\mathcal{L}$ is uniformly elliptic in Ω and has the form

$$\mathcal{L} = -\mathrm{div}(A\nabla\cdot) + b\nabla + c, \tag{2.2}$$

with $A \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^1(\bar{\Omega}))$, $b \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $c \in \mathcal{C}(\bar{\Omega})$. Given any Banach space, X, and two integers, $n, m \geq 1$, $\mathcal{M}_{n\times m}(X)$ stands for the vector space of the matrices with n rows and m columns with entries in X. Naturally, we set $\mathcal{M}_n(X) := \mathcal{M}_{n\times n}(X)$, and $\mathcal{M}_n^{\mathrm{sym}}(X)$ denotes the subset of the symmetric matrices.

Except in Section 2.1, $\partial\Omega$ is assumed to be an $(N-1)$-dimensional manifold of class $\mathcal{C}^2$ consisting of finitely many (connected) components

$$\Gamma_{\mathcal{D}}^j, \quad 1 \leq j \leq n_{\mathcal{D}}, \qquad \Gamma_{\mathcal{R}}^k, \quad 1 \leq k \leq n_{\mathcal{R}},$$

for some integers $n_{\mathcal{D}}, n_{\mathcal{R}} \geq 1$, and we denote by

$$\Gamma_{\mathcal{D}} := \bigcup_{j=1}^{n_{\mathcal{D}}} \Gamma_{\mathcal{D}}^j, \qquad \Gamma_{\mathcal{R}} := \bigcup_{j=1}^{n_{\mathcal{R}}} \Gamma_{\mathcal{R}}^j,$$

the Dirichlet and Robin portions of $\partial\Omega = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{R}}$. It should be noted that either $\Gamma_{\mathcal{D}}$, or $\Gamma_{\mathcal{R}}$, might be empty. Associated with this decomposition of $\partial\Omega$, arises in a rather natural way the boundary operator $\mathcal{B}$ defined by

$$\mathcal{B}u = \begin{cases} \mathcal{D}u := u & \text{on } \Gamma_{\mathcal{D}}, \\ \mathcal{R}u := \frac{\partial u}{\partial \nu} + \beta u & \text{on } \Gamma_{\mathcal{R}}, \end{cases} \qquad \text{for every } u \in W^{2,p}(\Omega),\ p > N, \tag{2.3}$$

where $\beta \in \mathcal{C}(\partial\Omega)$, $\mathbf{n}$ stands for the outward normal vector field along $\partial\Omega$, $\boldsymbol{\nu} := A\mathbf{n}$ is the conormal vector field associated to $\mathcal{L}$, and $\frac{\partial}{\partial \boldsymbol{\nu}}$ (which equals $\langle \nabla \cdot, \boldsymbol{\nu} \rangle$ under sufficient regularity) represents the derivative in the direction of the vector field $\boldsymbol{\nu}$.

As far as the function $h(u,x)$ concerns, it is assumed to satisfy some of the next hypothesis.

Hypothesis on the nonlinearity $h(u,x)$

(H1) $h : \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ is of class $\mathcal{C}^1$ in $u \geq 0$ and continuous in $x \in \bar{\Omega}$.

(H2) $\partial_u h(u,x) < 0$ for all $u > 0$ and $x \in \bar{\Omega}$.

In addition, through this chapter, it is imposed that, for some $d > 0$, the following hypothesis holds for a pair $(\mathcal{L}, h)$:

(H3) There exists a constant $M > 0$ such that $h(M,x) < dc(x)$ for all $x \in \bar{\Omega}$.

In particular, we eventually can assume (H3) with $d = 0$, i.e., that

(H4) There exists a constant $M > 0$ such that $h(M,x) < 0$ for all $x \in \bar{\Omega}$.

Note that (H4) implies (H3) for sufficiently small $d > 0$, regardless the sign of $c(x)$. A prototypical example of admissible h, for which (2.1) becomes closer to the classical diffusive logistic equation, is given by

$$h(u,x) = \ell(x) - a(x) f(u), \qquad u \in \mathbb{R}, \ \ x \in \bar{\Omega},$$

where $\ell \in \mathcal{C}(\bar{\Omega})$ can change of sign, $a \in \mathcal{C}(\bar{\Omega})$ and $f \in \mathcal{C}^1(\mathbb{R})$ satisfy $\min_{\bar{\Omega}} a > 0$, $f(0) = 0$, $f'(u) > 0$ for all $u > 0$, and $\lim_{u \uparrow \infty} f(u) = +\infty$. For this choice, it is easily seen that (H1) and (H2) hold. As far as concerns (H3), note that, for every $d > 0$,

$$h(M,x) = \ell(x) - a(x) f(M) \leq \max_{\bar{\Omega}} \ell - f(M) \min_{\bar{\Omega}} a < d \min_{\bar{\Omega}} c, \qquad x \in \bar{\Omega},$$

provided $M = M(d) > 0$ is sufficiently large, because $f(M) \uparrow \infty$ as $M \uparrow \infty$. Thus, (H3) holds for all $d > 0$. Moreover, by taking a sufficiently large $M > 0$ so that

$$f(M) > \frac{\max_{\bar{\Omega}} \ell}{\min_{\bar{\Omega}} a},$$

it is clear that (H4) also holds.

Under the general conditions (H1), (H2) and (H4), it is easily seen that the maximal non-negative solution of the non-spatial equation $uh(u,x) = 0$,

$$\Theta_h(x) := \begin{cases} 0 & \text{if } \ h(\xi, x) < 0 \ \text{ for all } \ \xi > 0, \\ \xi & \text{if } \ \xi > 0 \ \text{ exists such that } \ h(\xi, x) = 0, \end{cases}$$

is continuous in $\bar{\Omega}$. Actually, for every $x \in \bar{\Omega}$, $\Theta_h(x)$ is the unique non-negative linearly stable, or linearly neutrally stable, steady state of the *kinetic model* (2.23), associated to (2.1), i.e., the ordinary differential equation

$$u'(t) = u(t) h(u(t), x), \qquad t \geq 0.$$

According to Theorem 2.15 stated in Section 2.4, which is the main existence result of this chapter, for sufficiently small $d > 0$, (2.1) admits, at most, one positive solution. Let us denote by $\theta_{\{d,h\}}$ the maximal non-negative solution of (2.1), and by $\Gamma_{\mathcal{R}}^{+}$ the union of the components of $\Gamma_{\mathcal{R}}$ where Θ_h is everywhere positive. The main goal of this chapter is to provide with a singular perturbation result for (2.1), as stated in Theorem 2.21. Specifically, it establishes that the maximal non-negative solution of (2.1) approximates Θ_h as $d \downarrow 0$ uniformly on compact subsets of $\Omega \cup \Gamma_{\mathcal{R}}^{+} \cup \Theta_h^{-1}(0)$.

To the best of our knowledge, the most pioneering version of our convergence result, Theorem 2.21, goes back to [10], where the singular perturbation problem

$$\begin{cases} -d\Delta u = u(1 - a(x)u^2) & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases} \tag{2.4}$$

with Ω and $a(x)$ of class $\mathcal{C}^{\infty}$ and $\min_{\bar{\Omega}} a > 0$, was analyzed in dimension $N \leq 3$. Precisely, in [10], Berger and Fraenkel established that, for sufficiently small $d > 0$, problem (2.4) possesses a unique smooth positive solution, $u_d(x)$, which converges to $1/\sqrt{a(x)}$ as $d \downarrow 0$, outside a boundary layer of width $O(\sqrt{d})$. Moreover, a global continuation of u_d in d was performed up to the critical value of the diffusion where u_d bifurcates from $u = 0$. The main technical tool of [10] relies on the method of matched asymptotic expansions applied to approximate the positive solution. The global existence of the positive solution was derived from some classical results in critical point theory. An abstract version of this singular perturbation result for autonomous equations was given by the same authors in [11]. Two years later, De Villiers [31] sharpened these findings up to cover a general class of $\mathcal{C}^{\infty}$ functions, $g(u, x)$, instead of $u - a(x)u^3$. Almost simultaneously, Fife [43] and Fife and Greenlee [44] extended these results to a general class of nonhomogeneous Dirichlet boundary value problems including

$$\begin{cases} -d \operatorname{div}\left(A(x, d)\nabla u\right) = g(u, x, d) & \text{in } \ \Omega, \\ u = 0, & \text{on } \ \partial\Omega, \end{cases} \tag{2.5}$$

with Ω, $A(x, d)$ and $g(u, x, d)$ of class $\mathcal{C}^{\infty}$ and such that, for every $x \in \bar{\Omega}$, the equation $g(u, x, 0) = 0$ has a solution, $u_0(x)$, for which $\partial_u g(u_0(x), x, 0) < 0$. This negativity entails the linearized stability of the equilibrium solution $u_0(x)$ of the associated kinetic model

$$u'(t) = g(u(t), x, 0), \qquad t \geq 0, \tag{2.6}$$

for all $x \in \bar{\Omega}$. Much like in [10], the singular perturbations results of [43, 44] are based on a bound for the inverse of the linearization about the formal solution constructed with the matched asymptotic expansion. Fife and Greenlee [44] also analyzed the more general case when $g(u, x, 0) = 0$ possesses two $\mathcal{C}^{\infty}$-curves of solutions, $u_{0,1}(x)$ and $u_{0,2}(x)$, $x \in \bar{\Omega}$, which are linearly stable as steady-state solutions of (2.6) and separated away from each other.

Essentially, all these monographs adapted the former asymptotic expansion methods developed in the context of ODE's by the Russian School (e.g., see [15, 113]) to a PDE's framework. Naturally, working with ODE's many of the underlying technicalities can be easily overcome.

The first papers where some intrinsic techniques of the theory of PDE's, like the method of sub and supersolutions, were used to obtain singular perturbation results were those of

Howes [59, 58, 60]. As a result, the previous restrictive regularity assumptions were relaxed. Precisely, Howes [59] considered a general class of problems including (2.5) with $A = I$ and $g(u,x,d) = g(u,x)$ of class $\mathcal{C}^m$ for sufficiently large $m \geq 1$. Essentially, assuming that Ω is sufficiently smooth and that, for every $x \in \bar{\Omega}$, $g(u_0(x),x) = 0$ for some smooth $u_0(x)$ which is linearly stable as an equilibrium of (2.6), Howes found some sufficient conditions for the existence of a classical solution u_d of (2.5) such that

$$\lim_{d \downarrow 0} u_d = u_0 \quad \text{uniformly on compact subsets of } \Omega.$$

Almost simultaneously, Howes [58] extended these results to cover the following special class of Robin problems

$$\begin{cases} -d\Delta u = g(u,x) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(x) + \beta(x)u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $\beta \geq 0$ on $\partial\Omega$ and $\beta \in \mathcal{C}^{2,\mu}(\partial\Omega)$ for some $\mu \in (0,1)$. As a consequence, e.g., of [58, Th 2.1], Howes could infer in [58, Ex. 2.2] that in the special case when $g(u) = u - u^3$, $u_{0,\pm} \equiv \pm 1$ are I_0-stable zeroes of $g(u,x) = 0$, because $g'(\pm 1) = -2 < 0$, and therefore, (2.7) has two solutions, $u_{d,\pm}(x)$, such that

$$\lim_{d \downarrow 0} u_{d,\pm}(x) = \pm 1 \quad \text{uniformly in } \bar{\Omega}.$$

In these papers, the regularity of the support domain Ω is imposed through the existence of a function $F \in \mathcal{C}^{2,\mu}(\mathbb{R}^N;\mathbb{R})$ such that $|\nabla F(x)| = 1$ for all $x \in \partial\Omega$ and

$$\Omega = \{x \in \mathbb{R}^N \; : \; F(x) < 0\}, \qquad \partial\Omega = F^{-1}(0). \tag{2.8}$$

Incidentally, in the papers of Howes the problem of ascertaining whether, or not, a function F satisfying (2.8) exists, with the required regularity, remained open. Except for some pioneering results of Oleĭnik [103, 104, 105], for linear problems with transport terms, [58] seems to be the first paper dealing with the singular perturbation problem for a semilinear equation under Neumann or (classical) Robin boundary conditions with $\beta \geq 0$. The singular perturbation results of Howes for essential nonlinearities involving transport terms, like those of [58, Sec. 3 & 4] and [60], remain outside the general scope of this dissertation.

Some time later, these pioneering findings were slightly, and occasionally substantially, improved by Angenent [6], De Santi [30], Clément and Sweers [22] and Kelley and Ko [68], among many others, who dealt with the singular perturbation problem under Dirichlet boundary conditions through some comparison techniques based on the synthesis of Amann [2, 3], Sattinger [108] and Matano [98].

As shown by the simplest examples of truly spatially heterogeneous semilinear elliptic equations in the context of Population Dynamics, the most serious shortcoming of the classical singular perturbation theory is caused by the fact that the curves, $u_{0,j}(x)$, $1 \leq j \leq q$, $q = 1,2$, solving the equation $g(u,x) = 0$ must preserve their stability character for all $x \in \bar{\Omega}$, regarded as steady-state solutions of (2.6). For example, even in the simplest case situation when $g(u,x)$ inherits a logistic structure,

$$g(u,x) = \ell(x)u - a(x)u^2$$

for some functions $\ell, a \in \mathcal{C}(\bar{\Omega})$ such that $\ell(x)$ changes sign in Ω and $\min_{\bar{\Omega}} a > 0$, most of the assumptions imposed in the previous references fail to be true. Indeed, although $u_{0,1}(x) \equiv 0$ and $u_{0,2}(x) := \ell(x)/a(x)$, $x \in \bar{\Omega}$, might provide us with two smooth curves of $g(u, x) = 0$ for sufficiently smooth $\ell(x)$ and $a(x)$, it becomes apparent from $\partial_u g(u, x) = \ell(x) - 2a(x)u$ that

- $u_{0,1}(x) = 0$ is linearly stable, as an steady-state solution of (2.6), if, and only if, $\ell(x) < 0$,
- $u_{0,2}(x) = \ell(x)/a(x)$ is linearly stable if, and only if, $\ell(x) > 0$.

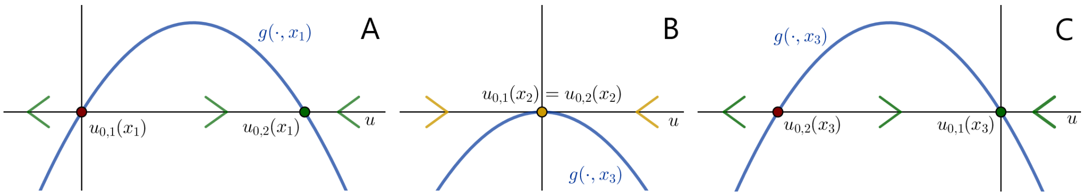


Figure 2.1: Plots of $u \mapsto g(u, x_i) := \ell(x_i)u - a(x_i)u^2$, $i \in \{1, 2, 3\}$, for a function $\ell \in \mathcal{C}(\bar{\Omega})$ that changes sign in Ω with $\ell(x_1) > 0$, $\ell(x_2) = 0$ and $\ell(x_3) < 0$. In the central case, (B), $u = 0$ must be a double zero of $g(\cdot, x_2)$. In each of these plots we have superimposed the 1-dimensional dynamics of (2.6) on the horizontal axis.

Therefore, the curves $u_{0,i}(x)$, $i = 1, 2$, cannot satisfy the requirements of the previous references, because they have a different stability character if $\ell(x) \neq 0$. Even considering the 'mixed interlaced branches' constructed from $u_{0,1}(x)$ and $u_{0,2}(x)$ through

$$\tilde{u}_{0,1}(x) := \max\{u_{0,1}(x), u_{0,2}(x)\}, \quad \tilde{u}_{0,2}(x) := \min\{u_{0,1}(x), u_{0,2}(x)\}, \quad x \in \bar{\Omega},$$

it is apparent that $\tilde{u}_{0,1}(x)$ is linearly stable if and only if $\ell(x) \neq 0$, and hence, the classical theory cannot be applied neither, because the linearized stability fails at $\ell^{-1}(0)$ and, in general, these curves are far from smooth. In these degenerate situations, not previously considered in the specialized literature, Furter and López-Gómez [48] established that the unique positive solution, u_d, of

$$\begin{cases} -d\Delta u = u(\ell(x) - a(x)u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies

$$\lim_{d \downarrow 0} u_d = \ell_+/a = \max\{0, \ell/a\} = \tilde{u}_{0,1} \quad \text{uniformly on compact subsets of } \Omega$$

(see [48, Th. 3.5]), which suggests the validity of the next general principle in the context of (2.1):

Singular Perturbation Principle (SPP). If for every $x \in \bar{\Omega}$ the associated kinetic problem possesses a unique linearly stable, or linearly neutrally stable, non-negative steady-state solution, $\Theta(x)$, which is somewhere positive in

Ω, then, for sufficiently small $d > 0$, the associated parabolic problem possesses a unique positive steady-state solution, θ_d. Moreover, $\lim_{d\downarrow 0} \theta_d = \Theta$ uniformly on any compact subset of Ω where $\Theta(x)$ is continuous.

This is widely confirmed by Theorems 2.15 and 2.21. Later, the same principle was shown to hold under homogeneous Neumann boundary conditions by Hutson, López-Gómez, Mischaikow and Vickers [63, Le. 2.4], as well as in the context of competitive systems (see [63, Th. 4.1] and [42, Th. 1], [39, Th. 1.2], [38, Th. 4.1] for some special cases when $\mathcal{L} = -\Delta$, or $b = 0$).

A problem of a different nature was studied by Nakashima, Ni and Su [102] in the special case when $\mathcal{L} = -\Delta$ and $g(u, x) = a(x)f(u)$, for the appropriate choices of the functions $a(x)$ and $f(u)$, under Neumann boundary conditions. In such case, the steady-state solutions of (2.6) are spatially homogeneous, though their linearized stabilities, viewed as equilibria of (2.6), vary with the location of x in $\bar{\Omega}$ according to the sign of $a(x)$. In spite of these differences, it turns out that this model also satisfies the *Singular Perturbation Principle* formulated above (see [102, Th. 1.3]).

Theorem 2.21 provides us with an extremely general version of all previous existing singular perturbation results for Kolmogorov nonlinearities of the form $g(u, x) = uh(u, x)$, where $h(u, x)$ satisfies (H1), (H2) and (H4). Actually, it is the first general result available for second order uniformly elliptic operators, $\mathcal{L}$, under general mixed boundary conditions of non-classical type. As the general linear existence theory developed in [88, Sect. 4.6] is only available for operators of the form (2.2), in this dissertation the principal part of $\mathcal{L}$ is required to be in divergence form. Nevertheless, even imposing this restriction, Theorem 2.21 is substantially sharper than most of the previous singular perturbation results for the generalized logistic equation.

The proof of Theorem 2.21 is based on the method of sub and supersolutions, which relies on the theorem of characterization of the Strong Maximum Principle of López-Gómez and Molina-Meyer [90, 82], and Amann and López-Gómez [5]. A comparison argument provides us with a global uniform supersolution of (2.1) on $\bar{\Omega}$, while the construction of the appropriate local subsolutions combined with a compactness argument, provides us with the necessary lower estimates to get Theorem 2.21. The main technical difficulties that must be overcome in the proof of Theorem 2.21 come from the following facts:

(I) The principal eigenfunctions associated to $\mathcal{L}$ in interior balls do not enjoy the nice symmetry properties of the principal eigenfunctions of $-\Delta$, which take the maximum on the center of these balls. This difficulty is overcome through a technical device introduced by López-Gómez in [83], which facilitates the construction of local subsolutions in the general non-autonomous case.

(II) A more subtle difficulty relies on the construction of a global supersolution of (2.1) sufficiently close to Θ_h, which is far from obvious when dealing with general mixed boundary conditions. As no previous singular perturbation result is available under mixed boundary conditions, these difficulties have been overcome for the first time in this dissertation.

(III) In our general setting, the coefficient function $\beta(x)$ can change sign. Thus, we must perform a preliminary change of variables for transforming (2.1) into an equivalent problem of the same nature with $\beta \geq 0$.

The resolution of the technical difficulties sketched in (II) and (III) relies on Theorem 2.3, which might be of independent interest in differential geometry. In particular, Theorem 2.3 establishes the equivalence between the following assertions, extracted from the statement of that result:

(a) $\partial\Omega$ is of class $\mathcal{C}^r$.

(b) $\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$ and, for every outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$, there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a projection onto $\partial\Omega$ along $\boldsymbol{\nu}$, $\Pi_{\boldsymbol{\nu}} : \mathcal{U} \to \partial\Omega$, of class $\mathcal{C}^{r-1}$ (*conormal projection*). Moreover, the function $\mathfrak{d}_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ defined by $\mathfrak{d}_{\boldsymbol{\nu}}(x) := \mathrm{dist}_{\boldsymbol{\nu}}(x,\partial\Omega)$ if $x \in \mathcal{U}\cap\Omega$, and by $\mathfrak{d}_{\boldsymbol{\nu}}(x) := -\mathrm{dist}_{\boldsymbol{\nu}}(x,\partial\Omega)$ if $x \in \mathcal{U}\setminus\Omega$, is of class $\mathcal{C}^r$ (*conormal distance*). Here, $\mathrm{dist}_{\boldsymbol{\nu}}(\cdot,\partial\Omega)$ stands for the distance to $\partial\Omega$ along $\boldsymbol{\nu}$.

(f) There exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$ with $\partial\Omega \subset \mathcal{U}$ and a function $\Psi \in \mathcal{C}^r(\mathcal{U};\mathbb{R})$ such that
$$\Omega = \{x \in \mathcal{U} \ : \ \Psi(x) < 0\}, \qquad \partial\Omega = \Psi^{-1}(0),$$
and $|\nabla\Psi(x)| = 1$ for all $x \in \partial\Omega$.

It should be noted that (f) is the condition used in some of the classical papers discussed above, with $F := \Psi$. It is astonishing that, in spite of the equivalence between (a) and (f), yet the existence of Ψ of class $\mathcal{C}^r$ satisfying (f) is far from adopted in the specialized literature as the most natural, and simple, definition for a bounded domain of class $\mathcal{C}^r$. Indeed, the usual definition in the most paradigmatic textbooks, like those of Gilbarg and Trudinger [50] and Evans [34], involves local charts at any point of the boundary, instead of the minimal requirements of (f).

On the other hand, to the best of our knowledge, the existence of the *conormal projection* and the *conormal distance* constructed in Theorem 2.3, as well as the proof of the fact that they inherit the regularity of $\partial\Omega$, seem completely new findings. Astonishingly, the Math. Sci. Net. of the Amer. Math. Soc. was unable to capture any entry with the words *conormal distance*, or *conormal projection*, though a huge list was given with *conormal*. Thus, Theorem 2.3 might be introducing these concepts into the debate of the characterization of the regularity of $\partial\Omega$ in terms of the regularity of the associated distance function, as pointed out by (b). Note that $\mathcal{C}^2$ is the minimal regularity of $\partial\Omega$ required to guarantee that the distance function through the 'nearest point' is well defined (see the paper of Krantz and Parks [71, Ex. 4]).

Actually, although Gilbarg and Trudinger [50, Le. 14.16] show that the distance function to the boundary, $\mathrm{dist}\,(x,\partial\Omega)$, is of class $\mathcal{C}^r$, $r \geq 2$, if $\partial\Omega$ is of class $\mathcal{C}^r$, and this result was later sharpened up to cover the case $r = 1$ by Krantz and Parks [71], even the problem of establishing the regularity of $\partial\Omega$ from the regularity of $\mathrm{dist}\,(x,\partial\Omega)$ remains open. These results actually sharpened a pioneering finding of Serrin [110] which established the $\mathcal{C}^{r-1}$-regularity of $\mathrm{dist}\,(x,\partial\Omega)$ from the $\mathcal{C}^r$-regularity of $\partial\Omega$. Some time later, Foote [45] generalized some of the results of [71] by establishing that, for every compact submanifold

M of class $\mathcal{C}^k$, $k \geq 2$, there exists a neighborhood, U, such that the distance function, $d(x, M)$, is $\mathcal{C}^k$ in $U \setminus M$. Under these assumptions, the fact that M has a neighborhood, U, with the unique nearest point property, as well as the fact that the projection map $\Pi : U \to M$ is $\mathcal{C}^{k-1}$, relies on the *tubular neighborhood theorem* with the added observation that Π factors through the map that creates the neighborhood. More recently, almost twenty years later, Li and Nirenberg [76] established that if Ω is a domain in a smooth complete Finsler manifold, and G stands for the largest open subset of Ω with the nearest point property in the Finsler metric, then the distance function from $\partial\Omega$ is in $\mathcal{C}^{k,a}_{\text{loc}}(G \cup \partial\Omega)$, $k \geq 2$ and $0 < a \leq 1$, if $\partial\Omega$ is of class $\mathcal{C}^{k,a}$. But no converse result, within the vain of the characterization provided by Theorem 2.3, seems to be available in the literature.

This chapter is distributed as follows. Section 2.1 states and proves Theorem 2.3, which, in particular, provides us with a characterization of the regularity of $\partial\Omega$ in terms of the regularity of distance functions. Section 2.2 uses Theorem 2.3 to reduce the general case (β changes sign) to the classical case ($\beta \geq 0$). This simplifies substantially the underlying analysis and the construction of the supersolutions. As a final consequence of Theorem 2.3, Subsection 2.2.3 provides with the approximation of continuous functions in $\bar{\Omega}$ by $\mathcal{C}^2(\bar{\Omega})$-functions with constant sign in $\Gamma_{\mathcal{R}}$, which is crutial in the proof the the perturbation result. Section 2.3 establishes some important monotonicity properties of the associated principal eigenvalues with respect to the domain and the potential. Section 2.4 proves Theorem 2.15, i.e., the main existence and uniqueness result, and derives from it some important monotonicity properties. Section 2.5 states the main perturbation result (see Theorem 2.21) and delivers its proof through a series of lemmas.

2.1 Regularity of the distance to the boundary function

Throughout this section we assume that Ω is an open subdomain of $\mathbb{R}^N$, $N \geq 1$, such that its boundary, $\partial\Omega$, is a topological $(N-1)$-manifold. Let us introduce some notation for the appropriate statement of Theorem 2.3.

Definition 2.1. Let $\boldsymbol{\nu} : \partial\Omega \to \mathbb{R}^N$ be a vector field on $\partial\Omega$. We will say that $\boldsymbol{\nu}$ is an *outward vector field* when the following conditions are satisfied:

(i) There exists $\varepsilon_0 > 0$ such that

$$x + \varepsilon\boldsymbol{\nu}(x) \in \mathbb{R}^N \setminus \bar{\Omega} \quad \text{and} \quad x - \varepsilon\boldsymbol{\nu}(x) \in \Omega$$

for all $x \in \partial\Omega$ and $0 < \varepsilon < \varepsilon_0$.

(ii) For every $x \in \partial\Omega$ there exist $\varepsilon_x > 0$ and $\alpha \in (0, 1)$ such that

$$x + \varepsilon\mathbf{v} \in \mathbb{R}^N \setminus \bar{\Omega} \quad \text{for all} \quad \varepsilon \in (0, \varepsilon_x) \quad \text{and} \quad \mathbf{v} \in \mathbb{S}^{N-1} \quad \text{with} \quad \langle \mathbf{v}, \boldsymbol{\nu} \rangle \in (0, \alpha|\boldsymbol{\nu}|).$$

Here, $\mathbb{S}^{N-1}$ stands for the $(N-1)$-dimensional sphere.

It should be noted that assumption (ii) imposes that for each $x \in \partial\Omega$ there is a small cone with vertex in x and axis in the direction of the vector $\boldsymbol{\nu}$ that lies outside $\bar{\Omega}$. It is easy to check that, under condition (i), when $\partial\Omega$ is of class $\mathcal{C}^1$, assumption (ii) is equivalent to the fact that, for every $x \in \partial\Omega$, $\boldsymbol{\nu}(x)$ is not tangent to $\partial\Omega$. Moreover, under such regularity, $\partial\Omega$ admits an outward normal vector field, $\mathbf{n}$, and every outward vector field, $\boldsymbol{\nu}$, satisfies $\langle \boldsymbol{\nu}, \mathbf{n} \rangle > 0$.

Definition 2.2. Let $\boldsymbol{\nu} : \partial\Omega \to \mathbb{R}^N$ be a vector field on $\partial\Omega$ and let $\mathcal{U} \subset \mathbb{R}^N$ be an open set such that $\partial\Omega \subset \mathcal{U}$. We will say that a function $\Pi_{\boldsymbol{\nu}} : \mathcal{U} \to \partial\Omega$ is the *projection onto* $\partial\Omega$ *along* $\boldsymbol{\nu}$ if

(i) $\Pi_{\boldsymbol{\nu}}(x) = x$ for all $x \in \partial\Omega$,

(ii) $\Pi_{\boldsymbol{\nu}}(x - \lambda\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))) = \Pi_{\boldsymbol{\nu}}(x)$ for every $x \in \mathcal{U}$ and $\lambda \in \mathbb{R}$ such that $x - \lambda\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x)) \in \mathcal{U}$. In particular,
$$\frac{\partial \Pi_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}(x) = 0 \quad \text{for all } \; x \in \mathcal{U}.$$

Naturally, given a projection onto $\partial\Omega$ along $\boldsymbol{\nu}$, the function *distance to the boundary along* $\boldsymbol{\nu}$ is defined through
$$\operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) := \frac{|x - \Pi_{\boldsymbol{\nu}}(x)|}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|}, \quad x \in \mathcal{U},$$
where $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^N$. In particular, when $\boldsymbol{\nu} = A\mathbf{n}$, i.e., when $\boldsymbol{\nu}$ is the conormal vector field, it is simply said that $\Pi_{\boldsymbol{\nu}}$ is a *conormal projection* and that $\operatorname{dist}_{\boldsymbol{\nu}}(\cdot, \partial\Omega)$ is the *conormal distance.*

The next result provides us with the characterization of the regularity of $\partial\Omega$ in terms of the regularity of the distance to the boundary function. Specifically, $\partial\Omega$ is of class $\mathcal{C}^r$ if, and only if, for some outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}$ the function $\operatorname{dist}_{\boldsymbol{\nu}}$ is of class $\mathcal{C}^r$ in $\mathcal{U} \setminus \partial\Omega$.

Theorem 2.3. *Assume that Ω is an open subdomain of $\mathbb{R}^N$ such that $\partial\Omega$ is a topological $(N-1)$-manifold. Then, for every integer $r \geq 2$, the next assertions are equivalent:*

(a) *$\partial\Omega$ is of class $\mathcal{C}^r$.*

(b) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$ and, for every outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$, there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a projection onto $\partial\Omega$ along $\boldsymbol{\nu}$, $\Pi_{\boldsymbol{\nu}} : \mathcal{U} \to \partial\Omega$, of class $\mathcal{C}^{r-1}$. Moreover, the function $\mathfrak{d}_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ defined by*
$$\mathfrak{d}_{\boldsymbol{\nu}}(x) := \begin{cases} \operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) & \text{if } \; x \in \mathcal{U} \cap \Omega, \\ -\operatorname{dist}_{\boldsymbol{\nu}}(x, \partial\Omega) & \text{if } \; x \in \mathcal{U} \setminus \Omega, \end{cases} \tag{2.9}$$
is of class $\mathcal{C}^r$.

(c) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega; \mathbb{R}^N)$ for which the property stated in (b) holds.*

(d) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$ and, for every outward vector field $\boldsymbol{\nu} \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$, there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$ with $\partial\Omega \subset \mathcal{U}$ and a function $\psi \in \mathcal{C}^r(\mathcal{U};\mathbb{R})$ such that $\psi(x) < 0$ for all $x \in \Omega \cap \mathcal{U}$, $\psi(x) > 0$ for all $x \in \mathcal{U} \setminus \bar{\Omega}$ and*

$$\min_{x\in\partial\Omega} \frac{\partial \psi}{\partial \boldsymbol{\nu}}(x) > 0.$$

In particular, $\psi(x) = 0$ for all $x \in \partial\Omega$ by the continuity of ψ on $\mathcal{U}$.

(e) *$\partial\Omega$ admits an outward vector field $\boldsymbol{\nu}_0 \in \mathcal{C}^{r-1}(\partial\Omega;\mathbb{R}^N)$ for which the property stated in (d) holds.*

(f) *There exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$ with $\partial\Omega \subset \mathcal{U}$ and a function $\Psi \in \mathcal{C}^r(\mathcal{U};\mathbb{R})$ such that*

$$\Omega = \{x \in \mathcal{U} \;:\; \Psi(x) < 0\}, \quad \partial\Omega = \Psi^{-1}(0),$$

and $|\nabla\Psi(x)| = 1$ for all $x \in \partial\Omega$.

Proof. It suffices to prove the following implications: (a) implies (b), (b) implies (c), (d) and (f), (c), or (d), or (f), implies (e), and (e) implies (a). First, we will prove that (a) implies (b). Note that the normal vector field is of class $\mathcal{C}^{r-1}$ as soon as $\partial\Omega$ is of class $\mathcal{C}^r$. Now, consider a field $\boldsymbol{\nu}$ satisfying the requirements of Part (b). For each $\varepsilon > 0$, let us denote by $Q_{\boldsymbol{\nu}} \in \mathcal{C}^{r-1}((-\varepsilon,\varepsilon)\times\partial\Omega;\mathbb{R}^N)$ the function defined by

$$\begin{aligned} Q_{\boldsymbol{\nu}} : (-\varepsilon,\varepsilon)\times\partial\Omega &\to \mathcal{U}_\varepsilon := \operatorname{Im} Q_{\boldsymbol{\nu}} \subset \mathbb{R}^N \\ (s,x) &\mapsto x - s\boldsymbol{\nu}(x) \end{aligned}$$

which establishes a bijection over its image for sufficiently small $\varepsilon > 0$. Moreover, shortening $\varepsilon > 0$, if necessary, $Q_{\boldsymbol{\nu}}^{-1}$ also is of class $\mathcal{C}^{r-1}$-regularity. Indeed, the proof of the injectivity proceeds by contradiction.

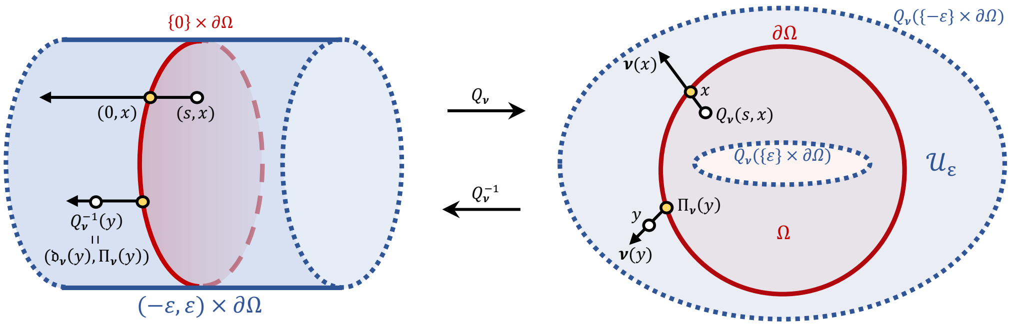


Figure 2.2: Scheme for the realization of $Q_{\boldsymbol{\nu}}$ and $Q_{\boldsymbol{\nu}}^{-1}$ and their relationships with the projection $\Pi_{\boldsymbol{\nu}}$ and the distance function $\mathfrak{d}_{\boldsymbol{\nu}}$.

Suppose that $Q_{\boldsymbol{\nu}}$ is not injective for sufficiently small $\varepsilon > 0$. Then, there exist $\{s_n^1\}_{n\geq1}, \{s_n^2\}_{n\geq1} \subset \mathbb{R}$, with $s_n^1 \to 0$ and $s_n^2 \to 0$, as $n \uparrow \infty$, and $\{x_n^1\}_{n\geq1}, \{x_n^2\}_{n\geq1} \subset \partial\Omega$ such that

$$(s_n^1, x_n^1) \neq (s_n^2, x_n^2) \quad \text{and} \quad Q_{\boldsymbol{\nu}}(s_n^1, x_n^1) = Q_{\boldsymbol{\nu}}(s_n^2, x_n^2) \quad \text{for all} \;\; n \geq 1.$$

In other words,

$$x_n^1 - s_n^1 \boldsymbol{\nu}(x_n^1) = x_n^2 - s_n^2 \boldsymbol{\nu}(x_n^2) \quad \text{for all} \;\; n \geq 1. \tag{2.10}$$

Moreover, without lost of generality, we can assume that $s_n^1 s_n^2 > 0$ for all $n \geq 1$. Otherwise, since $\boldsymbol{\nu}$ is an outward vector field, for sufficiently large $n \geq 1$, we have that

$$x_n^1 - s_n^1 \boldsymbol{\nu}(x_n^1) = x_n^2 - s_n^2 \boldsymbol{\nu}(x_n^2)$$

should lie, simultaneously, in Ω and in $\mathbb{R}^N \setminus \bar{\Omega}$, which is impossible.

Suppose $x_n^1 = x_n^2$ for some $n \geq 1$. Then, since $\boldsymbol{\nu}(x_n^1) \neq 0$, (2.10) implies that $s_n^1 = s_n^2$, which cannot hold. Hence, $x_n^1 \neq x_n^2$ for all $n \geq 1$. Since $\partial\Omega$ is compact, along some subsequences of $\{x_n^1\}$ and $\{x_n^2\}$, relabeled by n, we have that

$$\lim_{n\to\infty} x_n^j = x_\infty^j, \qquad j = 1, 2,$$

for some $x_\infty^1, x_\infty^2 \in \partial\Omega$. Subsequently, we are renaming by $\{s_n^1\}_{n\geq 1}$, $\{s_n^2\}_{n\geq 1}$, $\{x_n^1\}_{n\geq 1}$ and $\{x_n^2\}_{n\geq 1}$ the new subsequences. Letting $n \uparrow \infty$ in (2.10) yields

$$x_\infty^1 = x_\infty^2 =: x_\infty.$$

Now, for each $j = 1, 2$, we consider the sequence $\{\varsigma_n^j\}_{n\geq 1}$ defined through

$$\varsigma_n^j := s_n^i |\boldsymbol{\nu}(x_n^j)|, \qquad n \geq 1.$$

Then, by the continuity of $\boldsymbol{\nu}$, the new sequences still satisfy

$$\lim_{n\to\infty} \varsigma_n^1 = \lim_{n\to\infty} \varsigma_n^2 = 0, \tag{2.11}$$

and, setting $\xi := \boldsymbol{\nu}/|\boldsymbol{\nu}|$ for the unitary outward vector field, (2.10) can be equivalently expressed as

$$x_n^1 - x_n^2 = \varsigma_n^1 \xi(x_n^1) - \varsigma_n^2 \xi(x_n^2) = (\varsigma_n^1 - \varsigma_n^2)\xi(x_n^1) + \varsigma_n^2 \left(\xi(x_n^1) - \xi(x_n^2)\right) \tag{2.12}$$

for all $n \geq 1$. On the other hand, since $x_n^1 \neq x_n^2$, we have that

$$\frac{x_n^1 - x_n^2}{|x_n^1 - x_n^2|} \in \mathbb{S}^{N-1} \subset \mathbb{R}^N \quad \text{for all} \;\; n \geq 1,$$

where $\mathbb{S}^{N-1}$ stands for the $(N-1)$-dimensional sphere. As the sphere is compact, we can extract subsequences of $\{\varsigma_n^1\}_{n\geq 1}$, $\{\varsigma_n^2\}_{n\geq 1}$, $\{x_n^1\}_{n\geq 1}$ and $\{x_n^2\}_{n\geq 1}$, again labeled by n, such that

$$\tau_\infty := \lim_{n\to\infty} \frac{x_n^1 - x_n^2}{|x_n^1 - x_n^2|} \in T_{x_\infty}\partial\Omega,$$

where $T_{x_\infty}\partial\Omega$ stands for the tangent hyperplane of $\partial\Omega$ at x_∞. Note that $|\tau_\infty| = 1$. Moreover, by construction, we have that

$$|\varsigma_n^j| = |x_n^j - Q_{\boldsymbol{\nu}}(s_n^j, x_n^j)|, \quad Q_{\boldsymbol{\nu}}(s_n^1, x_n^1) = Q_{\boldsymbol{\nu}}(s_n^2, x_n^2), \quad j = 1, 2, \;\; n \geq 1.$$

Thus, since $s_n^1 s_n^2 > 0$ for all $n \geq 1$, the triangular inequality yields

$$|\varsigma_n^1 - \varsigma_n^2| = \big||\varsigma_n^1| - |\varsigma_n^2|\big| = \big||x_n^1 - Q_{\boldsymbol{\nu}}(s_n^1, x_n^1)| - |x_n^2 - Q_{\boldsymbol{\nu}}(s_n^2, x_n^2)|\big| \leq |x_n^1 - x_n^2|$$

for all $n \geq 1$. Consequently, by the Bolzano–Weierstrass theorem, there exist $\eta \in [-1, 1]$ and subsequences of $\{\varsigma_n^1\}_{n\geq 1}$, $\{\varsigma_n^2\}_{n\geq 1}$, $\{x_n^1\}_{n\geq 1}$ and $\{x_n^2\}_{n\geq 1}$, relabeled by n, such that

$$\lim_{n\to\infty} \frac{\varsigma_n^1 - \varsigma_n^2}{|x_n^1 - x_n^2|} = \eta. \tag{2.13}$$

Now we will show that, as a consequence of the regularity of ξ, the limit

$$\lim_{n\to\infty} \frac{\xi(x_n^1) - \xi(x_n^2)}{|x_n^1 - x_n^2|}$$

is well defined in $\mathbb{R}^N$. Indeed, since $\partial\Omega$ is a $\mathcal{C}^r$-manifold, there exist $\delta > 0$ and a local chart of $\partial\Omega$ on a neighborhood of x_∞, $\Phi \in \mathcal{C}^r(B_\delta(0); \mathbb{R}^N)$ with $\Phi(0) = x_\infty$. Subsequently, we set

$$y_n^j := \Phi^{-1}(x_n^j)$$

for $j = 1, 2$ and sufficiently large $n \geq 1$. By the continuity of Φ^{-1},

$$\lim_{n\to\infty} y_n^j = 0, \qquad j = 1, 2.$$

Since $x_n^1 \neq x_n^2$ and Φ is a local diffeomorphism, $y_n^1 \neq y_n^2$ and hence,

$$\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} \in \mathbb{S}^{N-2}, \qquad n \geq n_0.$$

Thus, by compactness, we can extract subsequences, relabeled by n, such that

$$\tilde{\tau}_\infty := \lim_{n\to\infty} \frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} \in \mathbb{S}^{N-2}. \tag{2.14}$$

Then, for every $\varphi \in \mathcal{C}^1(B_\delta(0); \mathbb{R}^N)$, we have that

$$\lim_{n\to\infty} \frac{\varphi(y_n^1) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} = \mathrm{D}\varphi(0)\tilde{\tau}_\infty = \frac{\partial\varphi}{\partial\tilde{\tau}_\infty}(0).$$

Indeed,

$$\begin{aligned}
\left|\frac{\varphi(y_n^1) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} - \mathrm{D}\varphi(0)\tilde{\tau}_\infty\right| &= \left|\frac{\varphi(y_n^2 + (y_n^1 - y_n^2)) - \varphi(y_n^2)}{|y_n^1 - y_n^2|} - \mathrm{D}\varphi(0)\tilde{\tau}_\infty\right| \\
&= \left|\frac{1}{|y_n^1 - y_n^2|}\int_0^1 \mathrm{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))(y_n^1 - y_n^2)dt - \int_0^1 \mathrm{D}\varphi(0)\tilde{\tau}_\infty dt\right| \\
&= \left|\int_0^1 \left(\mathrm{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} - \mathrm{D}\varphi(0)\tilde{\tau}_\infty\right) dt\right| \\
&\leq \int_0^1 \left|\mathrm{D}\varphi(y_n^2 + t(y_n^1 - y_n^2))\left(\frac{y_n^1 - y_n^2}{|y_n^1 - y_n^2|} - \tilde{\tau}_\infty\right)\right| dt \\
&+ \int_0^1 \left|\left(\mathrm{D}\varphi(y_n^2 + t(y_n^1 - y_n^2)) - \mathrm{D}\varphi(0)\right)\tilde{\tau}_\infty\right| dt
\end{aligned}$$

which, thanks to (2.14) and the uniform continuity of $\mathrm{D}\varphi$ in $B_{\delta/2}(0)$, converges to 0 as $n\uparrow\infty$. Hence, by the regularity of $\boldsymbol{\nu}$, and so of ξ, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{\xi(x_n^1)-\xi(x_n^2)}{|x_n^1-x_n^2|}&=\lim_{n\to\infty}\frac{(\xi\circ\Phi)(y_n^1)-(\xi\circ\Phi)(y_n^2)}{|\Phi(y_n^1)-\Phi(y_n^2)|}\\&=\lim_{n\to\infty}\frac{|y_n^1-y_n^2|}{|\Phi(y_n^1)-\Phi(y_n^2)|}\frac{(\xi\circ\Phi)(y_n^1)-(\xi\circ\Phi)(y_n^2)}{|y_n^1-y_n^2|}\\&=\frac{1}{|\mathrm{D}\Phi(0)\tilde{\tau}_\infty|}\mathrm{D}(\xi\circ\Phi)(0)\tilde{\tau}_\infty\in\mathbb{R}^N.\end{aligned}\tag{2.15}$$

Therefore, thanks to (2.11), (2.13) and (2.15), dividing by $|x_n^1-x_n^2|$ in (2.12) and letting $n\uparrow+\infty$ yields

$$\tau_\infty=\eta\xi(x_\infty)=\eta\frac{\boldsymbol{\nu}(x_\infty)}{|\boldsymbol{\nu}(x_\infty)|}.$$

Since $\tau_\infty\in\mathbb{S}^{N-1}$, taking norms in both sides provides us with $|\eta|=1$. However, since $\tau_\infty\in T_{x_\infty}\partial\Omega$ and ξ is an outward unit vector field along $\partial\Omega$, we have that

$$\langle\tau_\infty,\mathbf{n}(x_\infty)\rangle=0\quad\text{and}\quad\langle\xi(x_\infty),\mathbf{n}(x_\infty)\rangle>0,$$

respectively, which implies $\eta=0$, driving to a contradiction. Thus, there exists $\varepsilon>0$ such that $Q_{\boldsymbol{\nu}}:(-\varepsilon,\varepsilon)\to\mathcal{U}_\varepsilon$ is bijective. Note that $Q_{\boldsymbol{\nu}}$ inherits the regularity of $\boldsymbol{\nu}$. So, it is of class $\mathcal{C}^{r-1}((-\varepsilon,\varepsilon)\times\partial\Omega;\mathcal{U}_\varepsilon)$, and $Q_{\boldsymbol{\nu}}(0,x)=x$ for all $x\in\partial\Omega$.

It remains to show the regularity of

$$Q_{\boldsymbol{\nu}}^{-1}:\mathcal{U}_\varepsilon\to(-\varepsilon,\varepsilon)\times\partial\Omega$$

for sufficiently small $\varepsilon>0$. This is a consequence of the inverse function theorem. By continuity and compactness, it suffices to establish that $\mathrm{D}Q_{\boldsymbol{\nu}}$ is non-degenerate on $\{0\}\times\partial\Omega$. Indeed, since $\partial\Omega$ is a class $\mathcal{C}^r$ manifold, for each $x\in\partial\Omega$ there exist $\delta_x>0$ and a homeomorphism onto its image, $\Phi_x\in\mathcal{C}^r(B_{\delta_x}(0)\subset\mathbb{R}^{N-1};\mathbb{R}^N)$, with $\Phi_x(0)=x$ and $\Phi_x(B_{\delta_x}(0))\subset\partial\Omega$. Actually, Φ_x parameterizes $\partial\Omega$ in a neighborhood of x. Consider the function $\tilde{Q}_{\boldsymbol{\nu}}:(-\varepsilon,\varepsilon)\times B_{\delta_x}(0)\to\mathcal{U}_\varepsilon$ defined by

$$\tilde{Q}_{\boldsymbol{\nu}}(s,y):=Q_{\boldsymbol{\nu}}(s,\Phi_x(y))=\Phi_x(y)-s\boldsymbol{\nu}(\Phi_x(y)).$$

Then, for every $s\in(-\varepsilon,\varepsilon)$ and $y\in B_{\delta_x}(0)$, $\mathrm{D}Q_{\boldsymbol{\nu}}(s,\Phi_x(y))$ is represented by

$$\mathrm{D}\tilde{Q}_{\boldsymbol{\nu}}(s,y)=[-\boldsymbol{\nu}(\Phi_x(y)),\ \mathrm{D}\Phi_x(y)-s\mathrm{D}(\boldsymbol{\nu}\circ\Phi_x)(y)].$$

In particular,

$$\mathrm{D}\tilde{Q}_{\boldsymbol{\nu}}(0,y)=[-\boldsymbol{\nu}(\Phi_x(y)),\ \mathrm{D}\Phi_x(y)].$$

Since Φ_x is a local chart of a $\mathcal{C}^r$ $(N-1)$-dimensional manifold, $\operatorname{rank}\mathrm{D}\Phi_x(y)=N-1$ for all $y\in B_{\delta_x}(0)$ and hence, it generates the tangent space at $\Phi_x(y)$. Thus, since $\boldsymbol{\nu}(\Phi_x(y))$ is a non-tangential vector field, it becomes apparent that

$$\operatorname{rank}\mathrm{D}\tilde{Q}_{\boldsymbol{\nu}}(0,y)=N.$$

Consequently, $D\tilde{Q}_{\boldsymbol{\nu}}(0,y)$ is an isomorphism. Therefore, $Q_{\boldsymbol{\nu}}$ establishes a diffeomorphism of class $\mathcal{C}^{r-1}$ onto its image for sufficiently small $\varepsilon>0$. In order to complete the proof of (a) implies (b) it remains to construct the projection $\Pi_{\boldsymbol{\nu}}$ and show that the function $\mathfrak{d}_{\boldsymbol{\nu}}$ defined in (2.9) is of class $\mathcal{C}^r$. Let $P_1:\mathbb{R}\times\partial\Omega\to\mathbb{R}$ and $P_2:\mathbb{R}\times\partial\Omega\to\partial\Omega$ denote the projections on the first and the second component, respectively, i.e.,

$$\begin{aligned} P_1:\mathbb{R}\times\partial\Omega &\to \mathbb{R}, \\ (s,x) &\mapsto s \end{aligned} \qquad \begin{aligned} P_2:\mathbb{R}\times\partial\Omega &\to \partial\Omega, \\ (s,x) &\mapsto x \end{aligned}$$

respectively. Obviously, P_1 and P_2 are of class $\mathcal{C}^\infty$ and, by construction, it is easily seen that the map

$$\Pi_{\boldsymbol{\nu}}:=P_2\circ Q_{\boldsymbol{\nu}}^{-1}:\mathcal{U}_\varepsilon\to\partial\Omega\subset\mathbb{R}^N$$

satisfies all the requirements of Part (b). Indeed, $\Pi_{\boldsymbol{\nu}}$ also is of class $\mathcal{C}^{r-1}$, as $Q_{\boldsymbol{\nu}}^{-1}$ and P_2. Moreover, for every $x\in\partial\Omega$, we have that

$$\Pi_{\boldsymbol{\nu}}(x)=P_2\circ Q_{\boldsymbol{\nu}}^{-1}(x)=P_2(0,x)=x.$$

Since $Q_{\boldsymbol{\nu}}$ is a diffeomorphism, for every $x\in\mathcal{U}_\varepsilon$ there exists $s\in(-\varepsilon,\varepsilon)$ such that

$$x=Q_{\boldsymbol{\nu}}(s,\Pi_{\boldsymbol{\nu}}(x))=\Pi_{\boldsymbol{\nu}}(x)-s\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x)).$$

Hence, if $\lambda\in\mathbb{R}$ satisfies $x-\lambda\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\in\mathcal{U}_\varepsilon$, we find that

$$\begin{aligned}\Pi_{\boldsymbol{\nu}}(x-\lambda\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))) &= \Pi_{\boldsymbol{\nu}}\Big(\Pi_{\boldsymbol{\nu}}(x)-s\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))-\lambda\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\Big)\\ &= P_2\circ Q_{\boldsymbol{\nu}}^{-1}(Q_{\boldsymbol{\nu}}(s+\lambda,\Pi_{\boldsymbol{\nu}}(x)))=\Pi_{\boldsymbol{\nu}}(x).\end{aligned}$$

In particular, this entails that

$$\frac{\partial\Pi_{\boldsymbol{\nu}}}{\partial\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}(x)=0\quad\text{for all}\quad x\in\mathcal{U}_\varepsilon.$$

By the definition of $Q_{\boldsymbol{\nu}}$, $\mathfrak{d}_{\boldsymbol{\nu}}=P_1\circ Q_{\boldsymbol{\nu}}^{-1}$, and so it is of class $\mathcal{C}^{r-1}(\mathcal{U}_\varepsilon)$. Moreover, for every $x\in\mathcal{U}_\varepsilon$,

$$x=\Pi_{\boldsymbol{\nu}}(x)-\mathfrak{d}_{\boldsymbol{\nu}}(x)\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x)).$$

Thus,

$$\mathfrak{d}_{\boldsymbol{\nu}}(x)=\frac{1}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^2}\langle\Pi_{\boldsymbol{\nu}}(x)-x,\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\rangle$$

and hence, combining the Leibniz rule with the properties of the projection $\Pi_{\boldsymbol{\nu}}$, we find that, for every $x \in \mathcal{U}_\varepsilon$,

$$\begin{aligned}
\mathrm{D}\mathfrak{d}_{\boldsymbol{\nu}}(x) &= -\frac{2\langle \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x)), \mathrm{D}(\boldsymbol{\nu}\circ\Pi_{\boldsymbol{\nu}})(x)\rangle}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^4}\langle \Pi_{\boldsymbol{\nu}}(x)-x, \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\rangle \\
&\quad + \frac{1}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^2}\Big(\langle \mathrm{D}\Pi_{\boldsymbol{\nu}}(x) - Id, \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\rangle + \langle \Pi_{\boldsymbol{\nu}}(x)-x, \mathrm{D}(\boldsymbol{\nu}\circ\Pi_{\boldsymbol{\nu}})(x)\rangle\Big) \\
&= \frac{1}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^2}\Big(\langle \mathrm{D}\Pi_{\boldsymbol{\nu}}(x), \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\rangle - \mathfrak{d}_{\boldsymbol{\nu}}(x)\langle \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x)), \mathrm{D}(\boldsymbol{\nu}\circ\Pi_{\boldsymbol{\nu}})(x)\rangle \\
&\quad - \langle Id, \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\rangle\Big) \\
&= \frac{1}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^2}\Big(\frac{\partial \Pi_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}(x) - \mathfrak{d}_{\boldsymbol{\nu}}(x)\frac{\partial(\boldsymbol{\nu}\circ\Pi_{\boldsymbol{\nu}})}{\partial \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}(x) - \boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))\Big) \\
&= -\frac{\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))}{|\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))|^2}.
\end{aligned}$$

because $\Pi_{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}\circ\Pi_{\boldsymbol{\nu}}$ are constant along each direction $\boldsymbol{\nu}(\Pi_{\boldsymbol{\nu}}(x))$. Therefore, $\mathrm{D}\mathfrak{d}_{\boldsymbol{\nu}} \in \mathcal{C}^{r-1}$, which entails $\mathfrak{d}_{\boldsymbol{\nu}} \in \mathcal{C}^r$ and ends the proof of (a) implies (b).

The fact that Part (b) implies Part (c) is immediate. Next, we will prove that (b) implies (d) and (f). Suppose (b) and consider any outward vector field, $\boldsymbol{\nu} \in \mathcal{C}^{r-1}$. Then,

$$\tilde{\boldsymbol{\nu}} := \boldsymbol{\nu}/|\boldsymbol{\nu}| \in \mathcal{C}^{r-1}.$$

Let $\mathcal{U}$, $\Pi_{\tilde{\boldsymbol{\nu}}}$ and $\mathfrak{d}_{\tilde{\boldsymbol{\nu}}}$ denote, respectively, the open set, the projection and the 'regularized distance' (2.9) provided by Part (b). Then, the function $\psi_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ defined by $\psi_{\boldsymbol{\nu}} := -\mathfrak{d}_{\tilde{\boldsymbol{\nu}}}$ satisfies

$$\nabla\psi_{\boldsymbol{\nu}}(x) = \mathrm{D}\psi_{\boldsymbol{\nu}}(x) = -\mathrm{D}\mathfrak{d}_{\tilde{\boldsymbol{\nu}}} = \tilde{\boldsymbol{\nu}}(\Pi_{\tilde{\boldsymbol{\nu}}}(x))$$

for all $x \in \mathcal{U}$. In particular, $\nabla\psi_{\boldsymbol{\nu}}(x) = \tilde{\boldsymbol{\nu}}(x)$ for every $x \in \partial\Omega$, and hence,

$$\frac{\partial \psi_{\boldsymbol{\nu}}}{\partial \boldsymbol{\nu}}(x) = \langle \nabla\psi_{\boldsymbol{\nu}}(x), \boldsymbol{\nu}(x)\rangle = \langle \tilde{\boldsymbol{\nu}}(x), \boldsymbol{\nu}(x)\rangle = |\boldsymbol{\nu}(x)| > 0,$$

which ends the proof of (b) implies (d). Actually, since

$$|\nabla\psi_{\boldsymbol{\nu}}(x)| = |\tilde{\boldsymbol{\nu}}(x)| = 1$$

for all $x \in \partial\Omega$, $\Psi := \psi_{\boldsymbol{\nu}}$ satisfies the requirements of Part (f).

The fact that (d) implies (e) is trivial, and the proof of (c) implies (e) follows the same patterns as the proof of (b) implies (d). The fact that (f) implies (e) follows from the fact that $\boldsymbol{\nu}(x) := \nabla\Psi(x)$ is an outward vector field of class $\mathcal{C}^{r-1}$ satisfying

$$\frac{\partial \Psi}{\partial \boldsymbol{\nu}}(x) = |\nabla\Psi(x)|^2 = 1 > 0$$

for all $x \in \partial\Omega$. Thus, Part (e) holds by choosing $\psi := \Psi$.

It remains to prove that (e) implies (a). By the properties of the function ψ guaranteed by Part (e), it is apparent that $\partial\Omega := \psi^{-1}(0)$. Let us consider $x_0 \in \partial\Omega$ and $\boldsymbol{\nu}(x_0)$, and let $\{\mathbf{e}_j\}_{j=1}^{N-1}$ be an orthonormal basis of $\operatorname{span}[\boldsymbol{\nu}(x_0)]^\perp$ in $\mathbb{R}^N$. Subsequently, for every $\delta > 0$, we denote by

$$F_\delta : (-\delta, \delta) \times (-\delta, \delta)^{N-1} \to \mathbb{R}^N$$

the $\mathcal{C}^\infty$ map defined through

$$F_\delta(z,\mathbf{y}) := x_0 + z\boldsymbol{\nu}(x_0) + \sum_{j=1}^{N-1} y_j \mathbf{e}_j, \quad \mathbf{y} = (y_1, ..., y_{N-1}).$$

This map establishes a diffeomorphism onto its image, which is an open neighborhood of x_0 denoted by $\mathcal{W}_\delta$. Note that $F_\delta(0,0) = x_0$. Choose $\delta > 0$ such that $\mathcal{W}_\delta \subset \mathcal{U}$, where $\mathcal{U}$ is the open neighborhood of $\partial\Omega$ guaranteed by Part (c). Lastly, consider the function

$$G_\delta := \psi \circ F_\delta \in \mathcal{C}^r((-\delta,\delta)^N;\mathbb{R}).$$

Obviously, $G_\delta(0,0) = 0$. Moreover,

$$\frac{\partial G_\delta}{\partial z}(0,0) = [\mathrm{D}\psi(x_0)]\Big(\frac{\partial F_\delta}{\partial z}(0,0)\Big) = \mathrm{D}\psi(x_0)(\boldsymbol{\nu}(x_0)) = \frac{\partial \psi}{\partial \boldsymbol{\nu}}(x_0) > 0.$$

Thus, according to the implicit function theorem, there exists $\delta_0 > 0$ and a function $\zeta \in \mathcal{C}^r((-\delta_0,\delta_0)^{N-1};\mathbb{R})$ such that

$$G_{\delta_0}^{-1}(0) = \{(\zeta(\mathbf{y}),\mathbf{y}) \in \mathbb{R}^N \,:\, \mathbf{y} \in (-\delta_0,\delta_0)^{N-1}\}.$$

In particular, the function

$$(-\delta_0,\delta_0)^{N-1} \ni \mathbf{y} \mapsto F_{\delta_0}(\zeta(\mathbf{y}),\mathbf{y})$$

provides us with a class $\mathcal{C}^r$ parametrization of $\partial\Omega \cap \mathcal{W}_{\delta_0}$. Since x_0 was arbitrary, $\partial\Omega$ is an $(N-1)$−manifold of class $\mathcal{C}^r$. This ends the proof of Theorem 2.3 □

Remark 2.4. According to López-Gómez [88, Lem. 2.1], using a partition of the unity of class $\mathcal{C}^r$, or a cut-off function, the function $\psi(x)$ in Part (d), as well as Ψ in Part (f), can be assumed to be globally defined in a neighborhood of $\bar{\Omega}$, or even in $\mathbb{R}^N$, and in such case $\psi(x) < 0$ (resp. $\Psi(x) < 0$) for all $x \in \Omega$ and $\psi(x) > 0$ (resp. $\Psi(x) > 0$) for all $x \in \mathbb{R}^N \setminus \bar{\Omega}$.

A further (deeper) analysis of the role played by the regularity of the outward vector field reveals the validity of the next result.

Corollary 2.5. *If $\partial\Omega$ is an $(N-1)$-dimensional manifold of class $\mathcal{C}^r$, $r \geq 1$, and $\boldsymbol{\nu} \in \mathcal{C}^k(\partial\Omega;\mathbb{R}^N)$, $k \geq 1$, is an outward vector field, then there exist an open subset $\mathcal{U}$ of $\mathbb{R}^N$, with $\partial\Omega \subset \mathcal{U}$, and a function $\Pi_{\boldsymbol{\nu}} \in \mathcal{C}^{\min\{r,k\}}(\mathcal{U};\partial\Omega)$ which is a projection onto $\partial\Omega$ along $\boldsymbol{\nu}$. In particular, the function $\mathfrak{d}_{\boldsymbol{\nu}} : \mathcal{U} \to \mathbb{R}$ defined in (2.9) is of class $\mathcal{C}^{\min\{r,k+1\}}$.*

2.2 Consequences of the theorem of characterization of the regularity of the boundary

This section collects a series of consequences of Theorem 2.3 if $\partial\Omega$ is assumed to be of class $\mathcal{C}^2$, when applying such regularity characterization in the context of problem (2.1). It should be remembered that this assumption will be maintained in the forthcoming sections.

2.2.1 A canonical transformation

The next result allows to transform the original problem into a problem with $\beta \geq 0$ preserving the properties (H1), (H2), and (H4) of the nonlinearity. Hence, in general, one can assume that $\beta \geq 0$.

Theorem 2.6. *Assume that $\partial\Omega$ is of class $\mathcal{C}^2$. Then, there exits $E \in \mathcal{C}^2(\bar{\Omega})$, with $E(x) > 0$ for all $x \in \bar{\Omega}$, such that (2.1) can be equivalently expressed as*

$$\begin{cases} d\mathcal{L}_E w = w h_E(w,x) & \text{in } \Omega, \\ \mathcal{B}_E w = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.16}$$

where

(i) $h_E(w,x) = h(E(x)w, x)$ *for all* $w \geq 0$ *and* $x \in \bar{\Omega}$.

(ii) $\mathcal{L}_E = -\operatorname{div}(A\nabla\cdot) + b_E\nabla + c_E$, *with*

$$b_E := b - 2A\frac{\nabla E}{E} \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega})), \qquad c_E := \frac{\mathcal{L}E}{E} \in \mathcal{C}(\bar{\Omega}).$$

(iii) $\mathcal{B}_E = \mathcal{D}$ *on* $\Gamma_{\mathcal{D}}$ *and* $\mathcal{B}_E = \frac{\partial}{\partial\nu} + \beta_E$ *on* $\Gamma_{\mathcal{R}}$, *with* $\beta_E(x) := \frac{\mathcal{B}E}{E}(x) > 0$ *for all* $x \in \Gamma_{\mathcal{R}}$.

Moreover, h_E satisfies (H1), (H2) and (H4) if h does too.

Proof. First, let us consider an arbitrary $E \in \mathcal{C}^2(\bar{\Omega})$ such that $E(x) > 0$ for all $x \in \bar{\Omega}$. Suppose that u is a non-negative solution of (2.1). Then, $w := u/E$ satisfies

$$\begin{aligned} \mathcal{L}u = \mathcal{L}(Ew) &= -\operatorname{div}(A\nabla(Ew)) + b\nabla(Ew) + cEw \\ &= -\operatorname{div}(EA\nabla w) - \operatorname{div}(wA\nabla E) + Eb\nabla w + wb\nabla E + wcE \\ &= -\nabla EA\nabla w - E\operatorname{div}(A\nabla w) - \nabla wA\nabla E - w\operatorname{div}(A\nabla E) \\ &\quad + Eb\nabla w + wb\nabla E + wcE \\ &= -E\operatorname{div}(A\nabla w) + Eb\nabla w - \nabla EA\nabla w - \nabla wA\nabla E \\ &\quad + w(-\operatorname{div}(A\nabla E) + b\nabla E + cE). \end{aligned}$$

By the symmetry of A, we have that

$$\nabla wA\nabla E = \nabla EA\nabla w,$$

and thus

$$\mathcal{L}u = E\left(-\operatorname{div}(A\nabla w) + \left(b - 2A\frac{\nabla E}{E}\right)\nabla w + \frac{\mathcal{L}E}{E}w\right) = E\mathcal{L}_E w, \quad \text{in } \Omega.$$

Hence,

$$d\mathcal{L}_E w = \frac{1}{E}d\mathcal{L}u = \frac{1}{E}uh(u,\cdot) = \frac{1}{E}Ewh(Ew,\cdot) = wh_E(w,\cdot) \qquad \text{in } \Omega.$$

As for the boundary, we find that

$$\mathcal{B}_E w(x) = w(x) = u(x)/E(x) = 0$$

for all $x \in \Gamma_{\mathcal{D}}$, whereas

$$
\begin{aligned}
0 = \mathcal{B}u(x) = \mathcal{B}(Ew)(x) &= \frac{\partial (Ew)}{\partial \boldsymbol{\nu}}(x) + \beta(x)E(x)w(x) \\
&= E(x)\frac{\partial w}{\partial \boldsymbol{\nu}}(x) + \left(\frac{\partial E}{\partial \boldsymbol{\nu}}(x) + \beta(x)E(x)\right) w(x) = E(x)\mathcal{B}_E w(x)
\end{aligned}
$$

for all $x \in \Gamma_{\mathcal{R}}$. In order to choose E such that $\beta_E > 0$, note that, according to Theorem 2.3 and Remark 2.1, there exist an open set $\mathcal{U}$, $\bar{\Omega} \subset \mathcal{U} \subset \mathbb{R}^N$, and a function $\psi \in \mathcal{C}^2(\mathcal{U})$ such that $\psi(x) < 0$ for all $x \in \Omega$, $\psi(x) = 0$ for all $x \in \partial\Omega$ and $\min_{\Gamma_{\mathcal{R}}} \frac{\partial \psi}{\partial \boldsymbol{\nu}} > 0$. Consider

$$E := \exp(\mu\psi)$$

with $\mu > 0$ to be determined. Then, for each $x \in \Gamma_{\mathcal{R}}$, $E(x) = 1$ and, hence,

$$\beta_E(x) = \frac{\mathcal{B}E(x)}{E(x)} = \beta(x) + \frac{1}{E(x)}\frac{\partial E}{\partial \boldsymbol{\nu}}(x) = \beta(x) + \mu\frac{\partial E}{\partial \boldsymbol{\nu}}(x).$$

Thus, since $\min_{\Gamma_{\mathcal{R}}} \frac{\partial \psi}{\partial \boldsymbol{\nu}} > 0$, it becomes apparent that $\beta_E(x) > 0$ for all $x \in \Gamma_{\mathcal{R}}$ for sufficiently large $\mu > 0$.

Now, let us analyze the properties of h_E. The regularity required for (H1) is a byproduct of the regularity of both h and E. On the other hand, for every $u > 0$ and $x \in \bar{\Omega}$ we have that

$$\frac{\partial h_E}{\partial w} = \frac{\partial}{\partial w}\left(h(E(x)w, x)\right) = E(x)\frac{\partial h}{\partial u}(E(x)w, x) < 0.$$

Hence h_E satisfies (H2). To conclude, since h satisfies (H4) there exists $M > 0$ such that $\max_{\bar{\Omega}} h(M, \cdot) < 0$. Therefore, setting

$$M_E := \frac{M}{\min_{\bar{\Omega}} E} > 0$$

and taking into account that h is decreasing in u by (H2), we conclude that, for every $x \in \bar{\Omega}$,

$$h_E(M_E, x) = h(M_E E(x), x) = h\big(E(x)\tfrac{M}{\min_{\bar{\Omega}} E}, x\big) \leq h(M, x) < 0,$$

which ends the proof. $\square$

2.2.2 Existence of sufficiently large supersolutions

The next results will allow us to prove the existence of solutions of (2.1) in Section 2.4.

Lemma 2.7. *Assume that problem* (2.16) *derived in Theorem 2.6 satisfies (H2) and (H3) for some $d > 0$. Then, for such $d > 0$ and for every $M > 0$, the problem* (2.1) *possesses a supersolution greater than M in $\bar{\Omega}$.*

Proof. Consider the function $E \in \mathcal{C}^2(\bar{\Omega})$ as derived in Theorem 2.6, and denote $\bar{u} := \kappa E$ for some constant $\kappa > 0$ to be determined. Applying the boundary operator to $\bar{u}$ we obtain

$$\mathcal{B}\bar{u} = \kappa > 0 \quad \text{in } \Gamma_{\mathcal{D}}$$

and, by the analysis carried out in Theorem 2.6,

$$\mathcal{B}\bar{u} = \frac{\partial}{\partial \boldsymbol{\nu}}\kappa + \beta_E \kappa \geq 0, \quad \text{in } \Gamma_{\mathcal{R}}.$$

Moreover,

$$\bar{u}h(\bar{u}, \cdot) = \kappa E h(\kappa E, \cdot) = \kappa E h_E(\kappa, \cdot)$$

and

$$d\mathcal{L}\bar{u} = dE\mathcal{L}_E\kappa = dEc_E\kappa.$$

Hence, $\bar{u}$ is a strict supersolution of (2.1) if

$$dc_E(x) \geq h_E(\kappa, x) \quad \text{for all } \ x \in \bar{\Omega}.$$

Therefore, because problem (2.16) is assumed to satisfy (H2) and (H3), it suffices to choose $\kappa > \frac{M}{\min_{\bar{\Omega}}}$ greater than the value provided by hypothesis (H3). □

As an immediate consequence of Lemma 2.7 and the fact that properties (H2) and (H4) are preserved by the transformation carried out in Theorem 2.6, the next result holds.

Lemma 2.8. *Assume that* (2.1) *satisfies (H2) and (H4). Then, $d_0 > 0$ exists such that for every $M > 0$ and $d \in (0, d_0)$, the problem* (2.1) *possesses a supersolution greater than M in $\bar{\Omega}$.*

2.2.3 Approximation by functions of class $\mathcal{C}^2$ with a fixed sign on $\Gamma_{\mathcal{R}}$

The next result will play a crucial role in the proof of Theorem 2.21.

Lemma 2.9. *Let $\xi_1, \xi_2 \in \mathcal{C}(\bar{\Omega})$ be such that $\xi_1(x) < \xi_2(x)$ for all $x \in \bar{\Omega}$. Then the following hold:*

(a) *There exists $\Phi \in \mathcal{C}^2(\bar{\Omega})$ such that $\xi_1 \leq \Phi \leq \xi_2$ in $\bar{\Omega}$ and $\mathcal{R}\Phi(x) > 0$ for all $x \in \Gamma_{\mathcal{R}}$.*

(b) *There exists $\Phi \in \mathcal{C}^2(\bar{\Omega})$ such that $\xi_1 \leq \Phi \leq \xi_2$ in $\bar{\Omega}$ and $\mathcal{R}\Phi(x) < 0$ for all $x \in \Gamma_{\mathcal{R}}$.*

Proof. By Theorem 2.3 applied to the conormal vector field, there exist an open neighborhood $\mathcal{U} \subset \mathbb{R}^N$ of $\partial\Omega$, a function $\psi \in \mathcal{C}^2(\mathcal{U}; \mathbb{R})$ and a constant $\tau > 0$ such that $\psi(x) < 0$ for all $x \in \mathcal{U} \cap \Omega$, $\psi(x) = 0$ for each $x \in \partial\Omega$ and

$$\frac{\partial \psi}{\partial \boldsymbol{\nu}}(x) \geq \tau \quad \text{for all } \ x \in \partial\Omega. \tag{2.17}$$

Let $\varepsilon > 0$ be such that

$$\varepsilon < \min_{\bar{\Omega}} (\xi_2 - \xi_1).$$

Then,

$$\xi_1(x) + \frac{\varepsilon}{2} < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } \ x \in \bar{\Omega},$$

and, hence, there exists $\phi \in \mathcal{C}^\infty(\bar{\Omega})$ such that

$$\xi_1(x) + \frac{\varepsilon}{2} < \phi(x) < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } \ x \in \bar{\Omega}.$$

Consider, for each $M \in \mathbb{R}$, the map $\phi_M \in \mathcal{C}^2(\mathcal{U} \cap \bar{\Omega})$ defined by

$$\phi_M(x) := \phi(x) - 1 + e^{M\psi(x)}, \qquad x \in \mathcal{U} \cap \bar{\Omega}.$$

By the continuity of ϕ_M, and the fact that $\phi_M(x) = \phi(x)$ for all $x \in \partial\Omega$, we can reduce $\mathcal{U}$ to some open set $\mathcal{U}_M$, with $\partial\Omega \subset \mathcal{U}_M \subset \mathcal{U}$, so that

$$\xi_1(x) + \frac{\varepsilon}{2} < \phi_M(x) < \xi_2(x) - \frac{\varepsilon}{2} \quad \text{for all } \ x \in \mathcal{U}_M \cap \bar{\Omega}.$$

On the other hand, since $\psi(x) = 0$ for all $x \in \partial\Omega$, it becomes apparent that, for every $x \in \Gamma_{\mathcal{R}}$,

$$\begin{aligned}\mathcal{R}\phi_M(x) &= \mathcal{R}\phi(x) + \mathcal{R}(e^{M\psi(x)} - 1) \\ &= \mathcal{R}\phi(x) + Me^{M\psi(x)}\frac{\partial\psi}{\partial\boldsymbol{\nu}}(x) + \beta(x)(e^{M\psi(x)} - 1) \\ &= \mathcal{R}\phi(x) + M\frac{\partial\psi}{\partial\boldsymbol{\nu}}(x).\end{aligned}$$

According to (2.17), for sufficiently large $M > 0$, one can get $\mathcal{R}\phi_M(x) > 0$ for all $x \in \Gamma_{\mathcal{R}}$. So, in order to get Part (a), it suffices to choose Φ equal to ϕ_M in a neighborhood of $\partial\Omega$. Similarly, by choosing $M < 0$ sufficiently large, Part (b) can be easily accomplished.

In each of these cases, once we have fixed the appropriate M, it remains to take Φ as any smooth extension of ϕ_M from a neighborhood $\mathcal{V}$ of $\partial\Omega$, with $\mathcal{V} \subset \mathcal{U}_M$, to $\bar{\Omega}$ in such a way that

$$\xi_1(x) < \Phi(x) < \xi_2(x) \quad \text{for all } \ x \in \bar{\Omega}.$$

This can be accomplished through an appropriate cutoff function of class $\mathcal{C}^\infty$. □

Remark 2.10. Note that if $\xi_1 \geq 0$ in $\bar{\Omega}$, then the function Φ provided by Lemma 2.9(a) satisfies $\mathcal{B}\Phi \geq 0$ on $\partial\Omega$, whereas if $\xi_2 \leq 0$ in $\bar{\Omega}$ then the function Φ provided by Lemma 2.9(b) verifies $\mathcal{B}\Phi \leq 0$ on $\partial\Omega$.

As an immediate consequence of Lemma 2.9, continuous function in $\bar{\Omega}$ can be approximated by class $\mathcal{C}^2(\bar{\Omega})$-functions, Ψ, with either $\mathcal{R}\Psi > 0$, or $\mathcal{R}\Psi < 0$, in $\Gamma_{\mathcal{R}}$.

Corollary 2.11. *Consider $\xi \in \mathcal{C}(\bar{\Omega})$. Then, for every $\varepsilon > 0$ there exist $\Psi_1, \Psi_2 \in \mathcal{C}^2(\bar{\Omega})$ such that*

(a) *$\Psi_i(x) \in (\xi(x) - \varepsilon, \xi(x) + \varepsilon)$ for all $x \in \bar{\Omega}$ and $i = 1, 2$.*

(b) *$\mathcal{R}\Psi_1(x) > 0 > \mathcal{R}\Psi_2(x)$ for all $x \in \Gamma_{\mathcal{R}}$.*

2.3 Monotonicity properties of the principal eigenvalue

Throughout the remaining of this dissertation, for any given $V \in \mathcal{C}(\bar{\Omega})$, we will denote by $\sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega]$ the *principal eigenvalue*, i.e., the *lowest real eigenvalue*, of the linear eigenvalue problem

$$\begin{cases} d\mathcal{L}\varphi + V(x)\varphi = \sigma\varphi & \text{in } \ \Omega, \\ \mathcal{B}\varphi = 0 & \text{on } \ \partial\Omega. \end{cases}$$

According to López-Gómez [88, Th. 7.7], the principal eigenvalue is algebraically simple and strictly dominant and it provides us with the unique eigenvalue that is associated with a positive (principal) eigenfunction, say φ_0. In such a case, $\varphi_0 \gg 0$ in the sense that

$$\varphi_0(x) > 0 \;\text{ for all }\; x \in \Omega \cup \Gamma_{\mathcal{R}} \;\text{ and }\; \frac{\partial \varphi_0}{\partial \mathbf{n}}(x) < 0 \;\text{ for all }\; x \in \Gamma_{\mathcal{D}}. \tag{2.18}$$

Moreover, according to [88, Ch. 5], $\varphi_0 \in W^{2,\infty}_{\mathcal{B}}(\Omega)$, where

$$W^{2,\infty}_{\mathcal{B}}(\Omega) := \bigcap_{p>N} W^{2,p}_{\mathcal{B}}(\Omega), \quad W^{2,p}_{\mathcal{B}}(\Omega) := \{u \in W^{2,p}(\Omega) \;:\; \mathcal{B}u = 0 \;\text{ on }\; \partial\Omega\}. \tag{2.19}$$

Thus, $\varphi_0 \in \mathcal{C}^1_{\mathcal{B}}(\bar{\Omega}) \cap \mathcal{C}^{1,\nu}(\Omega)$ for all $\nu < 1$ and it is almost everywhere twice differentiable in Ω, much like the weak positive solutions of (2.1).

The next result, which slightly extends [41, Th. 4.1], collects the main properties of the function $\Sigma : (0, +\infty) \times \mathcal{C}(\bar{\Omega}) \to \mathbb{R}$ defined by

$$\Sigma(d, V) := \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega].$$

It extends [38, Th. 2.1] to deal with general differential operators, $\mathcal{L}$, not necessarily self-adjoint. In particular, Parts (a) and (b) provide us with the monotonicity and continuity of the principal eigenvalue with respect to the potential, respectively.

Theorem 2.12. *$\Sigma(d, V)$ has the following properties:*

(a) *For every $d > 0$, the map $\Sigma(d, \cdot) : C(\bar{\Omega}) \to \mathbb{R}$ is strictly increasing, i.e., $\Sigma(d, V_1) < \Sigma(d, V_2)$ if $V_1, V_2 \in \mathcal{C}(\bar{\Omega})$ with $V_1 \lneq V_2$.*

(b) *For every $d > 0$, the map $\Sigma(d, \cdot) : C(\bar{\Omega}) \to \mathbb{R}$ is Lipschitz continuous.*

(c) *For every $V \in \mathcal{C}(\bar{\Omega})$*
$$\Sigma(0, V) := \lim_{d \to 0} \Sigma(d, V) = \min_{\bar{\Omega}} V.$$

Proof. Let $\varphi_1 \gg 0$ denote the (unique) principal eigenfunction associated to $\sigma_1[d\mathcal{L} + V_1; \mathcal{B}, \Omega]$ such that $\|\varphi_1\|_\infty = 1$. Then,

$$\begin{cases} (d\mathcal{L}+V_2-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega])\varphi_1 \gneq (d\mathcal{L}+V_1-\sigma_1[d\mathcal{L}+V_1;\mathcal{B},\Omega])\varphi_1 = 0 & \text{in } \Omega, \\ \mathcal{B}\varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, the function φ_1 provides us with a positive strict supersolution of the differential operator

$$d\mathcal{L} + V_2 - \sigma_1[d\mathcal{L} + V_1; \mathcal{B}, \Omega]$$

subject to the boundary operator $\mathcal{B}$ on $\partial\Omega$ and hence, thanks to the theorem of characterization of López-Gómez [88, Th. 7.10], its principal eigenvalue must be positive. Thus,

$$\sigma_1[d\mathcal{L} + V_2; \mathcal{B}, \Omega] - \sigma_1[d\mathcal{L} + V_1; \mathcal{B}, \Omega] = \sigma_1[d\mathcal{L} + V_2 - \sigma_1[d\mathcal{L} + V_1; \mathcal{B}, \Omega]; \mathcal{B}, \Omega] > 0,$$

which ends the proof of Part (a).

The Lipschitz continuity stated in Part (b) is an immediate consequence of the monotonicity. Indeed, by Part (a), for any given $V_1, V_2 \in \mathcal{C}(\bar{\Omega})$, we have that

$$\Sigma(d, V_1) = \Sigma(d, V_1 - V_2 + V_2) \leq \Sigma(d, \|V_1 - V_2\|_\infty + V_2) = \|V_1 - V_2\|_\infty + \Sigma(d, V_2),$$

and, thus,

$$\Sigma(d, V_1) - \Sigma(d, V_2) \leq \|V_1 - V_2\|_\infty.$$

Changing V_1 by V_2 provides us with

$$|\Sigma(d, V_1) - \Sigma(d, V_2)| \leq \|V_1 - V_2\|_\infty.$$

For the convergence in Part (c), we first note that, thanks to Part (a),

$$\sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega] \geq d\sigma_1[\mathcal{L}; \mathcal{B}, \Omega] + \min_{\bar{\Omega}} V.$$

Thus,

$$\liminf_{d \to 0} \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega] \geq \min_{\bar{\Omega}} V.$$

Now, arguing by contradiction, suppose that

$$\limsup_{d \to 0} \sigma_1[d\mathcal{L} + V; \mathcal{B}, \Omega] > \min_{\bar{\Omega}} V.$$

Then, there exist $\varepsilon > 0$ and a sequence $\{d_n\}_{n \geq 1} \subset (0, +\infty)$ with

$$\lim_{n \to \infty} d_n = 0,$$

such that, for every $n \geq 1$,

$$\sigma_1[d_n\mathcal{L} + V; \mathcal{B}, \Omega] > \min_{\bar{\Omega}} V + \varepsilon.$$

Equivalently,

$$\sigma_1[d_n\mathcal{L} + V - \min_{\bar{\Omega}} V - \varepsilon; \mathcal{B}, \Omega] > 0,$$

and hence, by López-Gómez [88, Th. 7.10], for every $n \geq 1$ the problem

$$\begin{cases} (d_n\mathcal{L} + V - \min_{\bar{\Omega}} V - \varepsilon)u = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a strict supersolution, $\varphi_n \gg 0$. So, φ_n satisfies

$$\begin{cases} (d_n\mathcal{L} + V - \min_{\bar{\Omega}} V - \varepsilon)\varphi_n \geq 0 & \text{in } \Omega, \\ \mathcal{B}\varphi_n \geq 0 & \text{on } \partial\Omega, \end{cases}$$

with some of these inequalities strict. Let $x_0 \in \bar{\Omega}$ be such that

$$V(x_0) = \min_{\bar{\Omega}} V.$$

By continuity, there exists $\rho > 0$ such that

$$V(x) < \min_{\bar{\Omega}} V + \frac{\varepsilon}{2}$$

for all $x \in B_\rho(x_0) \cap \bar{\Omega}$. In particular, this estimate holds in an open ball $B \subset B_\rho(x_0) \cap \Omega$. Thus, for every $n \geq 1$ we have that

$$\begin{cases} (d_n\mathcal{L} - \frac{\varepsilon}{2})\varphi_n \gneq 0 & \text{in } B, \\ \varphi_n > 0 & \text{on } \partial B. \end{cases}$$

Consequently, thanks again to [88, Th. 7.10], we find that

$$\sigma_1[d_n\mathcal{L} - \tfrac{\varepsilon}{2}; \mathcal{D}, B] > 0,$$

which contradicts the fact that

$$\lim_{n\to\infty} \sigma_1[d_n\mathcal{L} - \tfrac{\varepsilon}{2}; \mathcal{D}, B] = \lim_{n\to\infty} d_n\sigma_1[\mathcal{L}; \mathcal{D}, \Omega] - \tfrac{\varepsilon}{2} = -\tfrac{\varepsilon}{2}.$$

This contradiction ends the proof. □

For establishing the monotonicity of the principal eigenvalue with respect to the underlying domain, we need to introduce some notations.

Definition 2.13. Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω and $\mathcal{B}_0$ a boundary operator on $\partial\Omega_0$. We will say that $(\mathcal{B}_0, \Omega_0)$ *is comparable with* $(\mathcal{B}, \Omega)$, and write

$$(\mathcal{B}_0, \Omega_0) \preceq (\mathcal{B}, \Omega),$$

when the following conditions are satisfied:

(i) Each component, Γ, of $\partial\Omega_0$ is either a component of $\partial\Omega$, or $\Gamma \subset \Omega$.

(ii) The boundary operator $\mathcal{B}_0$ satisfies

$$\mathcal{B}_0 := \begin{cases} \mathcal{D} & \text{on } \partial\Omega_0 \cap \Omega, \\ \tilde{\mathcal{B}} & \text{on } \partial\Omega_0 \cap \partial\Omega, \end{cases}$$

where, for every component, Γ, of $\partial\Omega_0 \cap \partial\Omega$, either $\tilde{\mathcal{B}} = \mathcal{D}$ on Γ, or $\Gamma \subset \Gamma_{\mathcal{R}}$ and there is $\beta_0 \in \mathcal{C}(\partial\Omega_0)$ with $\beta_0 \geq \beta$ such that

$$\tilde{\mathcal{B}} = \frac{\partial}{\partial \boldsymbol{\nu}} + \beta_0 \quad \text{on } \Gamma.$$

We will write

$$(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$$

if, in addition, $(\mathcal{B}_0, \Omega_0) \neq (\mathcal{B}, \Omega)$.

It should be noted that, according to [16, Th. 9.1], the Dirichlet boundary operator on each component of $\partial\Omega$ can be approximated by letting $\min_\Gamma \beta \uparrow \infty$. Thus, the larger β_0, the closer are $\mathcal{B}_0$ and $\mathcal{D}$.

Lemma 2.14. *Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω and $\mathcal{B}_0$ a boundary operator on $\partial\Omega_0$. If $(\mathcal{B}_0,\Omega_0)\prec(\mathcal{B},\Omega)$, then*

$$\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega]<\sigma_1[d\mathcal{L}+V;\mathcal{B}_0,\Omega_0],\quad \textit{for every } d>0 \textit{ and } V\in\mathcal{C}(\bar{\Omega}).$$

Proof. Let $\varphi\gg 0$ be the principal eigenfunction associated to $\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega]$ normalized so that $\|\varphi\|_\infty=1$. Then, according to Definition 2.13, as long as $(\mathcal{B}_0,\Omega_0)\prec(\mathcal{B},\Omega)$, there exist a component, $\Gamma\neq\emptyset$, of $\partial\Omega$ for which some of the following alternatives hold

- $\Gamma\subset\Omega$ and $\mathcal{B}_0\varphi=\varphi>0$ on Γ. Actually, this occurs if Ω_0 is a proper subdomain of Ω.
- $\Gamma\subset\Gamma_{\mathcal{R}}$ and $\mathcal{B}_0\varphi=\varphi$ on Γ. Then, since $\varphi(x)>0$ for all $x\in\Gamma_{\mathcal{R}}$, we have that $\mathcal{B}_0\varphi>0$ on Γ.
- $\Gamma\subset\Gamma_{\mathcal{R}}$ and $\mathcal{B}_0=\frac{\partial}{\partial\boldsymbol{\nu}}+\beta_0$ with $\beta_0\gneq\beta$ on Γ. Then, since $\varphi(x)>0$ for all $x\in\Gamma_{\mathcal{R}}$, we find that
$$\mathcal{B}_0\varphi=\frac{\partial\varphi}{\partial\boldsymbol{\nu}}+\beta_0\varphi\gneq\frac{\partial\varphi}{\partial\boldsymbol{\nu}}+\beta\varphi=\mathcal{B}\varphi=0\quad\text{on }\Gamma.$$

Hence, φ satisfies

$$\begin{cases}(d\mathcal{L}+V-\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega])\varphi=0 & \text{in }\Omega_0,\\ \mathcal{B}_0\varphi\gneq 0 & \text{on }\partial\Omega_0.\end{cases}$$

In particular, φ is a positive strict supersolution of

$$\begin{cases}(d\mathcal{L}+V-\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega])u=0 & \text{in }\Omega_0,\\ \mathcal{B}_0u=0 & \text{on }\partial\Omega_0.\end{cases}$$

Therefore, we can conclude from [88, Th 7.10] that

$$\sigma_1[d\mathcal{L}+V;\mathcal{B}_0,\Omega_0]-\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega]=\sigma_1[d\mathcal{L}+V-\sigma_1[d\mathcal{L}+V;\mathcal{B},\Omega];\mathcal{B}_0,\Omega_0]>0,$$

which ends the proof. □

2.4 Existence and uniqueness of positive solutions

The main result of this section, Theorem 2.15, characterizes the existence and establishes the uniqueness of the positive solution of (2.1) in terms of the linearized instability of $u=0$ as a steady-state solution of its parabolic counterpart. Note that this result complements Lemma 3.4 of Fraile, Koch, López-Gómez and Merino [46].

Theorem 2.15. *Assume that problem* (2.16) *derived in Theorem 2.6 satisfies (H1), (H2) and (H3) for some $d>0$. Then, for such $d>0$, problem* (2.1) *has a positive solution $u\in\bigcap_{p>N}W^{2,p}(\Omega)$ if and only if*

$$\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]<0.$$

Moreover, it is unique if it exists.

Proof. As a consequence of Lemma 2.7, $\bar{u} := \kappa E > 0$ is a supersolution of (2.1) for any sufficiently large $\kappa > 0$, with $E \in \mathcal{C}^2(\bar{\Omega})$ being the function constructed in the proof of Theorem 2.6. Now, suppose that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0$$

and let $\phi \gg 0$ be any associated eigenfunction. We claim that $\underline{u} := \varepsilon\phi$ is a subsolution of (2.1) for sufficiently small $\varepsilon > 0$. Since

$$\mathcal{B}(\varepsilon\phi) = \varepsilon\mathcal{B}\phi = 0 \quad \text{on} \;\; \partial\Omega,$$

it suffices to show that

$$d\mathcal{L}(\varepsilon\phi) \leq \varepsilon\phi h(\varepsilon\phi, \cdot) \quad \text{in} \;\; \Omega.$$

By the choice of ϕ, we have that

$$d\mathcal{L}(\varepsilon\phi) = \varepsilon\Big(\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega]\phi + h(0,\cdot)\phi\Big) \quad \text{in} \;\; \Omega.$$

Hence, dividing by $\varepsilon\phi$ we should make sure that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] \leq h(\varepsilon\phi, \cdot) - h(0,\cdot) \quad \text{in} \;\; \Omega. \tag{2.20}$$

Since h is uniformly continuous on $[0,1] \times \bar{\Omega}$ and $\varepsilon\phi$ converges to 0 uniformly in $\bar{\Omega}$ as $\varepsilon \downarrow 0$, we find that

$$\lim_{\varepsilon \to 0} \|h(\varepsilon\phi, \cdot) - h(0,\cdot)\|_\infty = 0.$$

Thus, condition (2.20) holds for sufficiently small ε and, hence, $\underline{u} := \varepsilon\phi$ is a subsolution of (2.1). Since ε can be shortened up to get $\varepsilon\phi \leq \kappa$, (2.1) possesses a (strong) positive solution, u, such that $\varepsilon\phi \leq u \leq \kappa$.

Next, we will show that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < 0$$

is necessary for the existence of a positive solutions. Indeed, if (2.1) admits a positive solution, u, then

$$\sigma_1[d\mathcal{L} - h(u,\cdot); \mathcal{B}, \Omega] = 0,$$

by the uniqueness of the principal eigenvalue. Thus, by (H2), it follows from Theorem 2.12(a) that

$$\sigma_1[d\mathcal{L} - h(0,\cdot); \mathcal{B}, \Omega] < \sigma_1[d\mathcal{L} - h(u,\cdot); \mathcal{B}, \Omega] = 0.$$

As for establishing the uniqueness, assume that

$$u_1, u_2 \in \bigcap_{p>N} W^{2,p}(\Omega)$$

are two positive solutions of (2.1). In particular, $u_1, u_2 \gg 0$. Thanks to the first part of the proof, we already know that (2.1) admits a subsolution, $\underline{u} = \varepsilon\phi$, and a supersolution, $\bar{u} = \kappa > M$, such that

$$\underline{u} \leq u_1, u_2 \leq \bar{u}.$$

This can be easily obtained by shortening $\varepsilon > 0$ and enlarging κ as much as necessary. For these choices, thanks to Theorem 3 of Amann [2], problem (2.1) admits two strong solutions,

$$u_*, u^* \in \bigcap_{p>N} W^{2,p}(\Omega),$$

which are the minimal and maximal solutions of (2.1), respectively, in the order interval $[\underline{u}, \bar{u}]$. In particular, we have that

$$\underline{u} \le u_* \le u_1, u_2 \le u^* \le \bar{u}$$

and, since $u_1 \neq u_2$, necessarily $u_* < u^*$. Since they are solutions of (2.1), we already know that

$$\sigma_1[d\mathcal{L} - h(u_*, \cdot); \mathcal{B}, \Omega] = \sigma_1[d\mathcal{L} - h(u^*, \cdot); \mathcal{B}, \Omega] = 0. \tag{2.21}$$

and, thanks to (H2),

$$h(u_*, \cdot) \gneq h(u^*, \cdot) \quad \text{in} \quad \Omega.$$

Thus, by Theorem 2.12(a),

$$\sigma_1[d\mathcal{L} - h(u_*, \cdot); \mathcal{B}, \Omega] < \sigma_1[d\mathcal{L} - h(u^*, \cdot); \mathcal{B}, \Omega],$$

which contradicts (2.21). Therefore, $u_1 = u_2$. This ends the proof. □

As a byproduct of Lemma 2.8 and Theorem 2.15 the next result holds.

Corollary 2.16. *Assume that $h(u, v)$ satisfies (H1), (H2) and (H4). Then, for every sufficiently small $d > 0$, problem (2.1) has a positive solution*

$$u \in \bigcap_{p>N} W^{2,p}(\Omega)$$

if and only if

$$\sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0.$$

Moreover, it is unique if it exists.

By linearizing (2.1) at $u = 0$ it is easily seen that $u = 0$ is linearly unstable if, and only if,

$$\sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0,$$

while it is linearly stable, or linearly neutrally stable, in any other case.

Throughout the rest of this dissertation, we will denote by $\theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega}$ the maximal non-negative solution of (2.1). By Theorem 2.15,

$$\theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega} = 0 \quad \text{if} \quad \sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] \ge 0,$$

while

$$\theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega} \gg 0 \quad \text{if} \quad \sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0.$$

Should not exist any ambiguity, we will simply set

$$\theta_{\{d,h\}} := \theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega},$$

or, alternatively, omit some of these indexes. As a byproduct of Theorems 2.12(c) and 2.15, the positiveness of $\theta_{\{d,h\}}$ can be characterized for small $d>0$ in terms of the sign of $\max_{\bar\Omega} h(0,\cdot)$ as established by the next result.

Corollary 2.17. *Assume that problem* (2.16) *derived in Theorem 2.6 satisfies (H1), (H2) and (H3) for sufficiently small $d>0$. Then, the following properties hold:*

(a) *If $\max_{\bar\Omega} h(0,\cdot)<0$, then a maximal $d_0\in(0,+\infty]$ exists such that $\theta_{\{d,h\}}=0$ for $d\in(0,d_0)$.*

(b) *If $\max_{\bar\Omega} h(0,\cdot)>0$, then a maximal $d_0\in(0,+\infty]$ exists such that $\theta_{\{d,h\}}\gg 0$ for $d\in(0,d_0)$.*

In the intermediate case when

$$\max_{\bar\Omega} h(0,\cdot)=0,$$

Theorem 2.12(c) implies that

$$\lim_{d\downarrow 0}\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]=\min_{\bar\Omega}(-h(0,\cdot))=-\max_{\bar\Omega} h(0,\cdot)=0.$$

Thus, the sign of the principal eigenvalue $\sigma_1[d\mathcal{L}-h(0,\cdot);\mathcal{B},\Omega]$ for sufficiently small $d>0$ might depend on the nature of the coefficients of $\mathcal{L}$ as well as on the boundary operator $\mathcal{B}$, or even the geometry and the size of Ω. Indeed, if $\mathcal{L}=-\Delta$ is the Laplace operator and we assume that $\Gamma_{\mathcal{R}}=\emptyset$, i.e., $\mathcal{B}$ is the Dirichlet operator, $\mathcal{D}$, and $h(0,\cdot)=0$, then

$$\sigma_1[-d\Delta;\mathcal{D},\Omega]=d\sigma_1[-\Delta;\mathcal{D},\Omega]>0$$

for all $d>0$ and hence, by Theorem 2.15, $\theta_{\{d,h\}}=0$ for all $d>0$. But if we assume that $\mathcal{L}=-\Delta-1$, $h(0,\cdot)=0$, $\Gamma_{\mathcal{D}}=\emptyset$ and $\beta\equiv 0$ on $\Gamma_{\mathcal{R}}=\partial\Omega$, i.e., $\mathcal{B}$ is the Neumann operator, $\mathcal{R}_0$, then

$$\sigma_1[d(-\Delta-1);\mathcal{R}_0,\Omega]=d\sigma_1[-\Delta;\mathcal{R}_0,\Omega]-d=-d<0$$

for all $d>0$. Therefore, due to Theorem 2.15, $\theta_{\{d,h\}}\gg 0$ for all $d>0$. Finally, note that, according to a celebrated variational inequality of Faber [35] and Krahn [70] (see, e.g., [88, Prop. 8.6]), the sign of

$$\sigma_1[d(-\Delta-1);\mathcal{D},\Omega]=d\,(\sigma_1[-\Delta;\mathcal{D},\Omega]-1)$$

depends on the Lebesgue measure of Ω. Indeed, for sufficiently small $|\Omega|$,

$$\sigma_1[-\Delta;\mathcal{D},\Omega]>1,$$

and hence, $\theta_{\{d,h\}}=0$ for all $d>0$, while, for sufficiently large $|\Omega|$,

$$\sigma_1[-\Delta;\mathcal{D},\Omega]<1$$

and therefore, $\theta_{\{d,h\}}\gg 0$ for all $d>0$.

The following result provides us with the monotonicity of the maximal non-negative solution of (2.1) with respect to the non-linearity, domain and boundary operator.

Lemma 2.18. *Assume that problem (2.16) derived in Theorem 2.6 satisfies (H1), (H2) and (H3) for some* $d > 0$*. Let* Ω_0 *be a subdomain of class* $\mathcal{C}^2$ *of* Ω*,* $\mathcal{B}_0$ *a boundary operator on* $\partial\Omega_0$ *such that, according to Definition 2.13,* $(\mathcal{B}_0, \Omega_0) \preceq (\mathcal{B}, \Omega)$*, and suppose* $h_0 \in \mathcal{C}(\mathbb{R} \times \bar{\Omega}_0)$ *satisfies (H1), (H2) and* $h_0 \leq h$ *in* $[0, +\infty) \times \bar{\Omega}_0$*. Then,*

$$\theta_{\{d,h_0\}}^{\mathcal{L},\mathcal{B}_0,\Omega_0} \leq \theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega} \quad \text{in } \Omega_0.$$

If, in addition, $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$*, or* $h_0(u, \cdot) \neq h(u, \cdot)$ *in* Ω_0 *for all* $u \geq 0$*, then*

$$\theta_{\{d,h_0\}}^{\mathcal{L},\mathcal{B}_0,\Omega_0} \ll \theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega} \quad \text{in } \Omega_0$$

provided $\theta_{\{d,h\}}^{\mathcal{L},\mathcal{B},\Omega} > 0$*.*

Proof. For the sake of simplicity, throughout this proof we will denote

$$\theta := \theta_{\{\nu,h\}}^{\mathcal{L},\mathcal{B},\Omega}, \qquad \theta_0 := \theta_{\{\nu,h_0\}}^{\mathcal{L},\mathcal{B}_0,\Omega_0}.$$

By Theorem 2.12(a) and Lemma 2.14 we have that

$$\sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] \leq \sigma_1[d\mathcal{L} - h_0(0, \cdot); \mathcal{B}_0, \Omega_0].$$

Thus, due to Theorem 2.15,

$$\begin{array}{ll} \theta = \theta_0 = 0 & \text{if } \sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] \geq 0, \\ \theta \gg \theta_0 = 0 & \text{if } \sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0 \leq \sigma_1[d\mathcal{L} - h_0(0, \cdot); \mathcal{B}_0, \Omega_0]. \end{array}$$

Hence, it remains to study the case when

$$\sigma_1[d\mathcal{L} - h_0(0, \cdot); \mathcal{B}_0, \Omega_0] \leq \sigma_1[d\mathcal{L} - h(0, \cdot); \mathcal{B}, \Omega] < 0.$$

Then, by Theorem 2.15, $\theta, \theta_0 \gg 0$. Subsequently, we will consider the function $f \in \mathcal{C}(\bar{\Omega}_0)$ defined, for each $x \in \bar{\Omega}_0$, by

$$f(x) := \begin{cases} \frac{\theta(x) h_0(\theta(x), x) - \theta_0(x) h_0(\theta_0(x), x)}{\theta(x) - \theta_0(x)} & \text{if } \theta(x) \neq \theta_0(x), \\ h_0(\theta_0(x), x) + \theta_0(x) \frac{\partial}{\partial u} h_0(\theta_0(x), x) & \text{if } \theta(x) = \theta_0(x). \end{cases}$$

By definition, $\theta - \theta_0$ satisfies

$$d\mathcal{L}(\theta - \theta_0) = \theta h(\theta, \cdot) - \theta_0 h_0(\theta_0, \cdot) \geq \theta h_0(\theta, \cdot) - \theta_0 h_0(\theta_0, \cdot) = (\theta - \theta_0) f \quad \text{in } \Omega_0,$$

with strict inequality if $h(u, \cdot) \gneq h_0(u, \cdot)$ in Ω_0 for every $u > 0$. Moreover, since $(\mathcal{B}_0, \Omega_0) \preceq (\mathcal{B}, \Omega)$, we have that

$$\mathcal{B}_0(\theta - \theta_0) = \mathcal{B}_0 \theta \geq 0 \qquad \text{on } \partial\Omega_0,$$

with strict inequality if $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$. Thus, $\theta - \theta_0$ is a supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } \Omega_0, \\ \mathcal{B}_0 u = 0 & \text{on } \partial\Omega_0, \end{cases}$$

and, actually, it is a strict supersolution if $(\mathcal{B}_0, \Omega_0) \prec (\mathcal{B}, \Omega)$, or $h_0(u, \cdot) \neq h(u, \cdot)$ in Ω_0 for all $u \geq 0$. We claim that $\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > 0$. Thanks to Theorem 7.10 of López-Gómez

[88], this entails that $\theta - \theta_0 \geq 0$ in Ω_0 and that, actually, $\theta \gg \theta_0$ if it is strict, and so concluding the proof.

To prove

$$\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > 0,$$

we can argue as follows. Let $x \in \bar{\Omega}_0$ be such that $\theta(x) = \theta_0(x)$. Then, by definition, and thanks to (H2),

$$f(x) = h_0(\theta_0(x), x) + \theta_0(x)\frac{\partial h_0}{\partial u}(\theta_0(x), x) \leq h_0(\theta_0(x), x),$$

with strict inequality if $\theta_0(x) > 0$, while if $x \in \bar{\Omega}_0$ with $\theta(x) \neq \theta_0(x)$, then

$$\begin{aligned} f(x) &= \frac{\theta(x)h_0(\theta(x), x) - \theta_0(x)h_0(\theta_0(x), x)}{\theta(x) - \theta_0(x)} \\ &= h_0(\theta_0(x), x) + \theta(x)\frac{h_0(\theta(x), x) - h_0(\theta_0(x), x)}{\theta(x) - \theta_0(x)} \leq h_0(\theta_0(x), x) \end{aligned}$$

with strict inequality if $\theta(x) > 0$. Note that $\theta(x) > 0$ and $\theta_0(x) > 0$ for all $x \in \Omega_0$ and hence, both inequalities are strict for all $x \in \Omega_0$. Therefore,

$$f \lneq h_0(\theta_0, \cdot) \quad \text{in} \quad \bar{\Omega}_0$$

and, hence, owing to Theorem 2.12(a),

$$\sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_0] > \sigma_1[d\mathcal{L} - h_0(\theta_0, \cdot); \mathcal{B}_0, \Omega_0] = 0,$$

which ends the proof. □

2.5 The singular perturbation problem

Throughout this section, we assume that h satisfies (H1), (H2), and (H4). Thus, by Theorem 2.6, the same hypothesis hold for problem (2.16). In particular, (2.16) satisfies (H3) for sufficiently small $d > 0$ and hence, the results of Section 2.4 can be applied. The precise range of d where this occurs is unimportant for the proof of the perturbation results and, so, it is not specified. It should be remembered that under the previous hypothesis the function

$$\Theta_h(x) := \begin{cases} 0 & \text{if } \ h(\xi, x) < 0 \ \text{ for all } \ \xi > 0, \\ \xi & \text{if there exists } \ \xi > 0 \ \text{ such that } \ h(\xi, x) = 0, \end{cases} \tag{2.22}$$

is well defined for all $x \in \bar{\Omega}$ and is continuous in $\bar{\Omega}$. Let $\Gamma_{\mathcal{R}}^+$ denote the union of the components of $\Gamma_{\mathcal{R}}$ where Θ_h is everywhere positive.

Remark 2.19. For every $x \in \bar{\Omega}$, $\Theta_h(x)$ provides us with the unique non-negative linearly stable, or linearly neutrally stable, steady-state solution of the associated kinetic model

$$\begin{cases} u'(t) = u(t)h(u(t), x) & t \in [0, +\infty), \\ u(0) = u_0 \geq 0. \end{cases} \tag{2.23}$$

Note that $\Theta_h(x) = 0$ if $h^{-1}(\cdot, x)(0) = \emptyset$ and that

$$\Theta_h(x) = \max\{0, h^{-1}(\cdot, x)(0)\} \quad \text{if } \; h^{-1}(\cdot, x)(0) \neq \emptyset.$$

Moreover, Θ_h is monotonically increasing with respect to h, i.e., if h_1 and h_2 are two functions satisfying (H1), (H2), and (H4), with $h_1(u, x) \leq h_2(u, x)$ for all $u \geq 0$ and $x \in \bar{\Omega}$, then $\Theta_{h_1} \leq \Theta_{h_2}$ in $\bar{\Omega}$. Indeed, if, for some $x \in \bar{\Omega}$, $\Theta_{h_1}(x) = 0$ then the result is trivial. If $\Theta_{h_1}(x) > 0$ then

$$h_2(\Theta_{h_1}(x), x) \geq h_1(\Theta_{h_1}(x), x) = 0.$$

By (H4),

$$h_2(\Theta_{h_2}(x), x) = 0.$$

Moreover, thanks to (H2), h_2 is strictly decreasing in the first variable. Therefore, $\Theta_{h_1} \leq \Theta_{h_2}$. In particular, if $h_1 \lneq h_2$ in $\bar{\Omega}$, then $\Theta_{h_1} \lneq \Theta_{h_2}$ in $\bar{\Omega}$.

Remark 2.20. The condition (H4) is necessary for the continuity of Θ_h on $\bar{\Omega}$, as the following simple example shows

$$\begin{cases} d(-\Delta u + u) = u(-x^2 + e^{-u}) & \text{in } \Omega = (-1, 1), \\ \mathcal{B}u = 0 & \text{on } \partial\Omega = \{-1, 1\}, \end{cases}$$

where $h(u, x) = -x^2 + e^{-u}$ for all $x \in (-1, 1)$ and $u \in \mathbb{R}$. According to (2.22), it becomes apparent that

$$\Theta_h(x) = -\log x^2, \qquad x \in [-1, 1] \setminus \{0\},$$

which is discontinuous, and unbounded, at $x = 0$. It turns out that in this example the function $h(u, x)$ satisfies (H1), (H2) and (H3) for sufficiently small $d > 0$, however it does not satisfies (H4). Therefore, condition (H4) is the minimal necessary condition required to guarantee the continuity of $\Theta_h(x)$.

Next theorem is the main perturbation result in Chapter 2. It provides us with the limiting profile of the maximal non negative solution of (2.1) when the diffusion rate goes to zero.

Theorem 2.21. *Assume that h satisfies (H1), (H2) and (H4), and let $\Gamma_{\mathcal{R}}^+$ denote the union of the components of $\Gamma_{\mathcal{R}}$ where Θ_h is everywhere positive. Then, for any compact subset, K, of $\Omega \cup \Gamma_{\mathcal{R}}^+ \cup \Theta_h^{-1}(0)$,*

$$\lim_{d \downarrow 0} \theta_{\{d,h\}} = \Theta_h \quad \textit{uniformly in} \;\; K.$$

The proof of Theorem 2.21 follows after a series of technical results. The next one provides us with a global uniform estimate in $\bar{\Omega}$, when $d \sim 0$, for the non-negative solutions of (2.1).

Lemma 2.22. *For every $\varepsilon > 0$, there exists $d_0 = d_0(\varepsilon) > 0$ such that*

$$\theta_{\{d,h\}} \leq \Theta_h + \varepsilon \quad \textit{in} \;\; \bar{\Omega}$$

for all $d \in (0, d_0)$.

Proof. Subsequently, we suppose that d has been chosen sufficiently small so that (H3) holds for (2.16). For a given $\varepsilon > 0$, set

$$\xi_1 := \Theta_h + \frac{\varepsilon}{2} > 0, \qquad \xi_2 := \Theta_h + \varepsilon.$$

By Lemma 2.9(a) and Remark 2.10, there exists $\Phi \in C^2(\bar{\Omega})$ such that

$$0 < \Theta_h + \frac{\varepsilon}{2} \leq \Phi \leq \Theta_h + \varepsilon \quad \text{in } \bar{\Omega}, \quad \text{and} \quad \mathcal{B}\Phi \geq 0 \quad \text{on } \partial\Omega.$$

In particular, $\Phi(x) > \Theta_h(x)$ for all $x \in \bar{\Omega}$. Thus, since $h(\Theta_h(x), x) \leq 0$ for all $x \in \bar{\Omega}$ and, owing to (H2), it is strictly decreasing in the first variable, we find that

$$h(\Phi(x), x) < 0 \quad \text{for all } x \in \bar{\Omega}.$$

Hence, setting

$$d_0 := \frac{\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{\min\{0, \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)\}} \in (0, +\infty],$$

it becomes apparent that, for every $d < d_0$,

$$\Phi(x)h(\Phi(x), x) \leq \max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x)) \leq d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \leq d\mathcal{L}\Phi(x) \quad \text{in } \bar{\Omega}.$$

Note that this estimate holds true for all $d > 0$ if

$$\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \geq 0,$$

because, by construction,

$$\Phi h(\Phi, \cdot) < 0 \qquad \text{in } \bar{\Omega}.$$

This explains why we are setting $d_0 = +\infty$ when $\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) \geq 0$. On the other hand, when

$$\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) < 0,$$

then the value of d_0 becomes

$$d_0 := \frac{\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)} = \frac{-\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))}{-\min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x)} > 0.$$

Thus,

$$-d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x) < -\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x))$$

for all $d < d_0$, or, equivalently,

$$\max_{x\in\bar{\Omega}}(\Phi(x)h(\Phi(x), x)) < d \min_{x\in\bar{\Omega}} \mathcal{L}\Phi(x),$$

which also shows the previous estimate in this case.

Consequently, Φ provides us with a positive supersolution of (2.1). Consider the function $f \in \mathcal{C}(\bar{\Omega})$ defined, for each $x \in \bar{\Omega}$, by

$$f(x) := \begin{cases} \dfrac{\Phi(x)h(\Phi(x),x) - \theta_{\{d,h\}}(x)h(\theta_{\{d,h\}}(x),x)}{\Phi(x) - \theta_{\{d,h\}}(x)} & \text{if } \Phi(x) \neq \theta_{\{d,h\}}(x), \\ h(\Phi(x),x) + \Phi(x)\frac{\partial h}{\partial u}(\Phi(x),x) & \text{if } \Phi(x) = \theta_{\{d,h\}}(x). \end{cases}$$

Therefore, the function $\Phi - \theta_{\{d,h\}}$ is a supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, either $\theta_{\{d,h\}} \equiv 0$, which ends the proof, or

$$\theta_{\{d,h\}} \gg 0, \qquad \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}},\cdot);\mathcal{B},\Omega] = 0.$$

In the latter case, it is easily seen that (H2) implies

$$f \lneq h(\theta_{\{d,h\}},\cdot) \quad \text{in } \bar{\Omega}.$$

Thus, for every $d \in (0,d_0)$, it follows from Theorem 2.12(a) that

$$\sigma_1[d\mathcal{L} - f;\mathcal{B},\Omega] > \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}},\cdot);\mathcal{B},\Omega] = 0.$$

By Theorem 7.10 of López-Gómez [88], we may infer that, for every $0 < d < d_0$,

$$\theta_{\{d,h\}}(x) \leq \Phi(x) \leq \Theta_h(x) + \varepsilon \quad \text{for all } x \in \bar{\Omega}.$$

The proof is complete. □

The following result provides us with Theorem 2.21 in the special case when $\Gamma_{\mathcal{R}} = \emptyset$.

Proposition 2.23. *For any compact subset, K, of $\Omega \cup \Theta_h^{-1}(0)$, we have that*

$$\lim_{d \downarrow 0} \theta^{\Omega}_{\{d,h\}} = \Theta_h \quad \text{uniformly in } K.$$

Proof. Fix $\varepsilon > 0$. By Lemma 2.22, there exists $d_0 = d_0(\varepsilon) > 0$ such that

$$\theta_{\{d,h\}} \leq \Theta_h + \varepsilon \quad \text{for all } x \in K \subset \bar{\Omega}, \ d \in (0,d_0).$$

In order to get a lower estimate, we will first assume $h(u,x)$ to be autonomous, i.e.,

$$h(u,x) = h(u) \quad \text{for all } (u,x) \in \mathbb{R} \times \bar{\Omega}.$$

In such a case, Θ_h is a non-negative constant. Since $\theta_{\{d,h\}}$ is non negative, it is obvious that

$$\theta_{\{d,h\}} > \Theta_h - \varepsilon \quad \text{in } \bar{\Omega}$$

for all $d > 0$ if $\Theta_h = 0$. Thus, the following estimate holds:

$$\Theta_h - \varepsilon \leq \theta_{\{d,h\}} \leq \Theta_h + \varepsilon \quad \text{for all } x \in K \subset \bar{\Omega}, \ d \in (0,d_0).$$

In order to get the lower estimate when Θ_h is a positive constant (necessarily, $h(0) > 0$, $h(\Theta_h) = 0$ and $K \subset \Omega$, because $\Theta_h^{-1}(0) = \emptyset$), we consider $\tilde{\varepsilon} \in (0, \min\{2\Theta_h, \varepsilon\})$, $x_0 \in K$, and $\rho > 0$ be such that $\rho < \mathrm{dist}(K, \partial\Omega)$. For these choices, $\bar{B}_\rho(x_0) \subset \Omega$. Let $\varphi \gg 0$ be the principal eigenfunction associated to $\sigma_1[\mathcal{L}; \mathcal{D}, B_\rho(x_0)]$ normalized so that $\|\varphi\|_\infty = 1/2$, and define the function $\phi \in \cap_{p>N} W^{2,p}(B_\rho(x_0))$ through

$$\phi := \begin{cases} \varphi & \text{in } \bar{B}_\rho(x_0) \setminus \bar{B}_{\rho/2}(x_0), \\ \tilde{\varphi} & \text{in } \bar{B}_{\rho/2}(x_0), \end{cases}$$

where $\tilde{\varphi}$ is any sufficiently smooth function chosen so that $\phi(x) > 0$ for all $x \in B_\rho(x_0)$, $\phi(x_0) = 1$ and $\|\phi\|_\infty = 1$. Set

$$\Phi := \left(\Theta_h - \frac{\tilde{\varepsilon}}{2}\right)\phi \quad \text{in } \bar{B}_\rho(x_0).$$

Then, by construction, $\mathcal{D}\Phi = 0$ on $\partial B_\rho(x_0)$ and

$$0 < \Phi(x) \le \Theta_h - \frac{\tilde{\varepsilon}}{2} \quad \text{for all } x \in B_\rho(x_0),$$

since Θ_h is a constant greater than $\tilde{\varepsilon}/2$ and $\|\phi\|_\infty = 1$. Thus, taking into account that, owing to (H2), $h(u)$ is strictly decreasing in $u > 0$, it is apparent that

$$h(\Phi(x)) \ge h(\Theta_h - \tfrac{\tilde{\varepsilon}}{2}) > h(\Theta_h) = 0 \quad \text{for all } x \in \bar{B}_\rho(x_0),$$

and hence,

$$\min_{x \in \bar{B}_\rho(x_0)} h(\Phi(x)) \ge h(\Theta_h - \tfrac{\tilde{\varepsilon}}{2}) > 0.$$

On the other hand, the function $\mathcal{L}\Phi/\Phi$ is continuous in $\bar{B}_\rho(x_0)$ because $\Phi(x) > 0$ for all $x \in B_\rho(x_0)$ and

$$\mathcal{L}\Phi(x)/\Phi(x) = \sigma_1[\mathcal{L}; \mathcal{D}, B_\rho(x_0)] \in \mathbb{R} \quad \text{for all } x \in \partial B_\rho(x_0).$$

Although unnecessary, $\rho(x_0)$ can be shortened so that

$$\sigma_1[\mathcal{L}; \mathcal{D}, B_\rho(x_0)] > 0,$$

because, due to the Faber–Krahn inequality [35, 70],

$$\lim_{\rho \to 0} \sigma_1[\mathcal{L}; \mathcal{D}, B_\rho(x_0)] = +\infty$$

(see, e.g., Proposition 8.6 of López-Gómez [88]). Thus, setting

$$0 < d_{x_0} < \frac{\min_{\bar{B}_\rho(x_0)} h(\Phi)}{\max_{\bar{B}_\rho(x_0)} |\mathcal{L}\Phi/\Phi|},$$

we have that, for every $d \in (0, d_{x_0})$,

$$d \max_{\bar{B}_\rho(x_0)} |\mathcal{L}\Phi/\Phi| \lneq h(\Phi) \quad \text{in } \bar{B}_\rho(x_0)$$

and so

$$d\mathcal{L}\Phi = d\Phi\mathcal{L}\Phi/\Phi \le d\Phi \max_{\bar{B}_\rho(x_0)} |\mathcal{L}\Phi/\Phi| \lneq \Phi h(\Phi) \quad \text{in } \bar{B}_\rho(x_0).$$

Therefore, Φ provides us with a strict subsolution of

$$\begin{cases} d\mathcal{L}u = uh(u) & \text{in } B_\rho(x_0), \\ u = 0 & \text{on } \partial B_\rho(x_0). \end{cases}$$

Equivalently, $\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)} - \Phi$ is a strict supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } B_\rho(x_0), \\ u = 0 & \text{on } \partial B_\rho(x_0), \end{cases}$$

where $f \in \mathcal{C}(\bar{B}_\rho(x_0))$ stands for the function defined, for every $x \in \bar{B}_\rho(x_0)$, by

$$f(x) := \begin{cases} \dfrac{\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x) h\big(\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x)\big) - \Phi(x)h(\Phi(x))}{\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x) - \Phi(x)} & \text{if } \Phi(x) \neq \theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x), \\ h(\Phi(x)) + \Phi(x)h'(\Phi(x)) & \text{if } \Phi(x) = \theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x). \end{cases}$$

Moreover, thanks to (H2),

$$f \lneq h\big(\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}\big) \quad \text{in } \bar{B}_\rho(x_0)$$

and thus, by the monotonicity of the principal eigenvalue with respect to the potential established by Theorem 2.12(a), it becomes apparent that

$$\sigma_1[d\mathcal{L} - f; \mathcal{D}, B_\rho(x_0)] > \sigma_1\big[d\mathcal{L} - h\big(\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}\big); \mathcal{D}, B_\rho(x_0)\big] = 0.$$

Note that $h(0) > 0$ and hence, owing to Corollary 2.17(b), $\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)} \gg 0$ for sufficiently small $d > 0$. Consequently, by [88, Th. 7.10], we find that, for every $d \in (0, d_{x_0})$,

$$\Phi(x) < \theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)}(x) \quad \text{for all } x \in B_\rho(x_0). \tag{2.24}$$

Moreover, by Lemma 2.18,

$$\theta_{\{d,h\}}^{\mathcal{D},B_\rho(x_0)} \le \theta_{\{d,h\}}^{\mathcal{B},\Omega} \quad \text{in } B_\rho(x_0). \tag{2.25}$$

On the other hand, since $\Phi \in \mathcal{C}(\bar{B}_\rho(x_0))$ and

$$\Phi(x_0) = \Theta_h - \frac{\tilde{\varepsilon}}{2},$$

there exist $\rho_{x_0} \in (0, \rho)$ such that

$$\Phi(x) \ge \Theta_h - \tilde{\varepsilon} > \Theta_h - \varepsilon \quad \text{for all } x \in B_{\rho_{x_0}}(x_0). \tag{2.26}$$

According to (2.24), (2.25), and (2.26), we find that

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega} > \Theta_h - \varepsilon \quad \text{in } B_{\rho_{x_0}}(x_0) \ \text{ for all } d \in (0, d_{x_0}).$$

As K is compact, we can extract $x_1, ..., x_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} B_{\rho_{x_i}}(x_i)$$

and, hence, for every $d < \min_i d_{x_i}$, $d > 0$, the estimate

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega} \geq \Theta_h - \varepsilon$$

holds in K. This ends the proof when h is independent of x.

Subsequently, we will assume that $h(u,x)$ is a general function satisfying (H1), (H2) and (H4). Then, for sufficiently small $d > 0$, it is obvious that

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega}(x) \geq 0 \geq \Theta_h(x) - \varepsilon \quad \text{for all} \;\; x \in \Theta_h^{-1}([0,\varepsilon]).$$

As this provides us with a satisfactory lower estimate in $K \cap \Theta_h^{-1}([0,\varepsilon])$, in order to extend it to K, it remains to show that there exists $d_1 > 0$ such that, for every $d \in (0, d_1)$,

$$\theta_{\{d,h\}}^{\mathcal{B},\Omega}(x) \geq \Theta_h(x) - \varepsilon \quad \text{for all} \;\; x \in K_0 := K \cap \Theta_h^{-1}([\varepsilon, \max_{\bar{\Omega}} \Theta_h]) \subset \Omega.$$

Should K_0 be empty, the proof is complete. So, suppose that K_0 is nonempty and pick $x_0 \in K_0$ and $\rho > 0$ such that

$$\bar{B}_\rho(x_0) \subset \Omega \cap \Theta_h^{-1}(\tfrac{\varepsilon}{2}, +\infty).$$

By construction,

$$\Theta_h(x) > \frac{\varepsilon}{2} > 0 \quad \text{for all} \;\; x \in \bar{B}_\rho(x_0)$$

and hence, owing to (H2),

$$\min_{\bar{B}_\rho(x_0)} h(0,\cdot) > 0 \quad \text{and} \quad \min_{\bar{B}_\rho(x_0)} \Theta_h > \frac{\varepsilon}{2} > 0.$$

Actually, by continuity, $\rho > 0$ can be shortened, if necessary, so that

$$\min_{\bar{B}_\rho(x_0)} \Theta_h \geq \Theta_h(x) - \frac{\varepsilon}{2} \quad \text{for all} \;\; x \in \bar{B}_\rho(x_0). \tag{2.27}$$

The rest of the proof consists in reducing the problem to the previous case, by establishing the existence of an autonomous function, $H(u)$, satisfying (H1), (H2), (H4), and such that

$$H(u) \leq h(u,x) \quad \text{for all} \;\; u \geq 0, \;\; x \in \bar{B}_\rho(x_0), \tag{2.28}$$

and

$$\min_{x \in \bar{B}_\rho(x_0)} \Theta_h(x) - \frac{\varepsilon}{4} \leq \Theta_H \leq \min_{x \in \bar{B}_\rho(x_0)} \Theta_h(x). \tag{2.29}$$

The most natural candidate function for a (globally defined in $\mathbb{R}$) $H(u)$ is

$$h_{\min}(u) := \begin{cases} \min_{x \in \bar{B}_\rho(x_0)} h(u,x) & \text{if} \;\; u \geq 0, \\ \min_{x \in \bar{B}_\rho(x_0)} h(0,x) - u & \text{if} \;\; u < 0. \end{cases}$$

Obviously, $h_{\min} \in \mathcal{C}(\mathbb{R})$ and it is strictly decreasing, though, in general, it is not of class $\mathcal{C}^1(\mathbb{R})$. Thus, in order to construct $H(u)$ satisfying (2.28), (2.29), (H1), (H2) and (H4) we begin by considering the function

$$G(u) := \min\left\{-\delta, \frac{4}{\varepsilon} h_{\min}\Big(u+\frac{\varepsilon}{4}\Big)\right\} < 0, \qquad u \in \mathbb{R},$$

with sufficiently small $\delta > 0$, to be chosen later, and then we take, for every $u \in \mathbb{R}$,

$$H(u) := \int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u} G(s)\,ds, \quad \text{where } \Theta_{\min} \equiv \min_{x\in \bar{B}_\rho(x_0)} \Theta_h(x).$$

Since G is a continuous function, H is a function of class $\mathcal{C}^1(\mathbb{R})$ and hence, (H1) holds. Moreover, by definition,

$$H'(u) = G(u) < 0 \quad \text{for all} \quad u \in \mathbb{R}.$$

Thus, (H2) holds. Furthermore, since

$$H\Big(\Theta_{\min} - \frac{\varepsilon}{4}\Big) = 0,$$

(H4) also holds, because $H(u) < 0$ for all $u > \Theta_{\min} - \frac{\varepsilon}{4}$. Actually, (2.29) holds too, since $\Theta_H = \Theta_{\min} - \frac{\varepsilon}{4}$, by definition (see (2.22) if necessary). It remains to shorten δ, if necessary, to get (2.28). Suppose $u \leq \Theta_{\min} - \frac{\varepsilon}{4}$. Then, $u + \frac{\varepsilon}{4} \leq \Theta_{\min}$ and hence, $h_{\min}\left(u+\frac{\varepsilon}{4}\right) \geq 0$ and $G(u) = -\delta$, which implies

$$H(u) = -\delta\left(u - \Theta_{\min} + \frac{\varepsilon}{4}\right).$$

Thus, for sufficiently small δ,

$$H(0) = \delta\left(\Theta_{\min} - \frac{\varepsilon}{4}\right) < h_{\min}\left(\Theta_{\min} - \frac{\varepsilon}{4}\right)$$

and therefore,

$$H(u) \leq H(0) < h_{\min}\left(\Theta_{\min} - \frac{\varepsilon}{4}\right) \leq h_{\min}(u)$$

for all $u \in [0, \Theta_{\min} - \frac{\varepsilon}{4}]$. So, (2.28) holds in this interval. When

$$u \in \Big(\Theta_{\min} - \frac{\varepsilon}{4}, \Theta_{\min}\Big),$$

by construction,

$$H(u) = \int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u} G(s)\,ds < 0 < h_{\min}(u)$$

and hence, (2.28) holds in $[0, \Theta_{\min})$. Finally, when $u \geq \Theta_{\min}$, we find that

$$G(u) = \min\left\{-\delta, \frac{4}{\varepsilon} h_{\min}\Big(u+\frac{\varepsilon}{4}\Big)\right\} \leq \frac{4}{\varepsilon} h_{\min}\Big(u+\frac{\varepsilon}{4}\Big) < 0$$

and, consequently,

$$H(u)=\int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u}G(s)\,ds\leq\frac{4}{\varepsilon}\int_{\Theta_{\min}-\frac{\varepsilon}{4}}^{u}h_{\min}\Big(s+\frac{\varepsilon}{4}\Big)\,ds=\frac{4}{\varepsilon}\int_{\Theta_{\min}}^{u+\frac{\varepsilon}{4}}h_{\min}(t)\,dt$$
$$\leq\frac{4}{\varepsilon}\int_{u}^{u+\frac{\varepsilon}{4}}h_{\min}(t)\,dt<\frac{4}{\varepsilon}\int_{u}^{u+\frac{\varepsilon}{4}}h_{\min}(u)\,dt=h_{\min}(u),$$

which shows (2.28).

By Lemma 2.18, for sufficiently small $d>0$, the following estimate holds:

$$\theta_{\{d,H\}}^{\mathcal{D},B_\rho(x_0)}\leq\theta_{\{d,h\}}^{\mathcal{B},\Omega}\quad\text{in}\ \ B_\rho(x_0). \tag{2.30}$$

By the first part of the proof, since $H(u)$ does not depend on $x\in\Omega$, there exists $d_{x_0,\varepsilon}>0$ such that

$$\Theta_H-\frac{\varepsilon}{4}\leq\theta_{\{d,H\}}^{\mathcal{D},B_\rho(x_0)}\quad\text{in}\ \ \bar{B}_{\rho/2}(x_0)\ \ \text{for all}\ \ d\in(0,d_{x_0,\varepsilon}), \tag{2.31}$$

Combining (2.27), (2.29), (2.30) and (2.31) yields

$$\Theta_h-\varepsilon\leq\min_{\bar{B}_\rho(x_0)}\Theta_h-\frac{\varepsilon}{2}\leq\Theta_H-\frac{\varepsilon}{4}\leq\theta_{\{d,H\}}^{\mathcal{D},B_\rho(x_0)}\leq\theta_{\{d,h\}}^{\mathcal{B},\Omega}\quad\text{in}\ \ B_{\rho/2}(x_0)$$

for all $d\in(0,d_{x_0,\varepsilon})$. Lastly, since K_0 is compact, there exist $x_1,...,x_n\in K_0$ such that

$$K_0\subset\bigcup_{i=1}^{n}B_{\rho_i/2}(x_i).$$

Therefore,

$$\Theta_h-\varepsilon\leq\theta_{\{d,h\}}^{\mathcal{B},\Omega}\quad\text{in}\ \ K_0\ \ \text{for all}\ \ d<d_0:=\min_{1\leq i\leq n}d_{x_i,\varepsilon},$$

which ends the proof. □

We already have all the necessary tools to complete the proof of Theorem 2.21.

Proof of Theorem 2.21. Since h satisfies (H2),

$$\Theta_h\equiv0\quad\text{if}\ \ \max_{\bar{\Omega}}h(0,\cdot)\leq0.$$

Should it be the case, the result is a direct consequence from Proposition 2.23. So, subsequently, we assume that

$$\max_{\bar{\Omega}}h(0,\cdot)>0.$$

Then, by Corollary 2.17(b), $\theta_{\{d,h\}}\gg0$ for sufficiently small $d>0$.

Thanks to Proposition 2.23, Theorem 2.21 holds on any compact subset of $\Omega\cup\Theta_h^{-1}(0)$. Hence, it remains to prove the theorem on a neighborhood of $\Gamma_{\mathcal{R}}^+$. Let γ be a component of $\Gamma_{\mathcal{R}}^+$. By the definition of $\Gamma_{\mathcal{R}}^+$, we have that

$$\Theta_h(x)>0\quad\text{for all}\ \ x\in\gamma.$$

By the continuity of Θ_h, there exists $\rho > 0$ such that

$$\varepsilon_0 := \min_{\bar{\Omega}_{\gamma,\rho}} \Theta_h > 0, \quad \text{where } \Omega_{\gamma,\rho} \equiv \{x \in \Omega \ : \ \mathrm{dist}\,(x,\gamma) < \rho\}.$$

Pick $\varepsilon \in (0, \varepsilon_0)$. By the proof of Theorem 2.3, we can shorten ρ, if necessary, so that

$$\{x \in \Omega \ : \ \mathrm{dist}\,(x,\gamma) = \rho\} = \partial\Omega_{\gamma,\rho} \cap \Omega$$

is diffeomorphic to γ, and so of class $\mathcal{C}^2$. Hence, $\Omega_{\gamma,\rho}$ is an open subdomain of Ω with boundary of class $\mathcal{C}^2$, consisting of two components, $\partial\Omega_{\gamma,\rho} \cap \Omega$ and γ, for sufficiently small $\rho > 0$.

Subsequently, we consider the compact subset of Ω,

$$K_{\gamma,\rho} := \{x \in \Omega \ : \ \rho/2 \leq \mathrm{dist}\,(x,\gamma) \leq \rho\}.$$

By Proposition 2.23, there exists $d_\rho > 0$ such that

$$\Theta_h - \frac{\varepsilon}{2} \leq \theta_{\{d,h\}} \quad \text{in } K_{\gamma,\rho} \text{ for all } d < d_\rho. \tag{2.32}$$

By applying Lemma 2.9(a) and Remark 2.10 with the choices

$$\xi_1(x) := \Theta_h(x) - \varepsilon \ (\geq \varepsilon_0 - \varepsilon > 0)$$

and

$$\xi_2(x) := \Theta_h(x) - \frac{3\varepsilon}{4} < \Theta_h(x), \quad x \in \bar{\Omega}_{\gamma,\rho/2},$$

there exists $\Phi \in \mathcal{C}^2(\bar{\Omega}_{\gamma,\rho/2})$ such that

$$\Theta_h - \varepsilon \leq \Phi \leq \Theta_h - \frac{3\varepsilon}{4} \quad \text{in } \Omega_{\gamma,\rho/2} \text{ and } \mathcal{R}\Phi \leq 0 \text{ on } \gamma. \tag{2.33}$$

In particular, since $\partial\Omega_{\gamma,\rho/2} \cap \Omega \subset K_{\gamma,\rho}$, we may infer from (2.32) and (2.33) that

$$\theta_{\{d,h\}} \geq \Theta_h - \frac{\varepsilon}{2} = \Theta_h - \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} \geq \Phi + \frac{\varepsilon}{4} \quad \text{on } \partial\Omega_{\gamma,\rho/2} \cap \Omega \text{ for all } d < d_\rho. \tag{2.34}$$

Moreover, by (H2), since

$$\Phi(x) < \Theta_h(x) \quad \text{for all } x \in \bar{\Omega}_{\gamma,\rho/2},$$

we have that

$$\min_{x \in \bar{\Omega}_{\gamma,\rho/2}} h(\Phi(x), x) > \min_{x \in \bar{\Omega}_{\gamma,\rho/2}} h(\Theta_h(x), x) = 0.$$

Thus, shortening d_ρ, if necessary, so that

$$d_\rho < \frac{\min_{x\in\bar{\Omega}_{\gamma,\rho/2}} \Phi(x)h(\Phi(x),x)}{\max\{0, \max_{\bar{\Omega}_{\gamma,\rho/2}} \mathcal{L}\Phi\}},$$

we are driven to

$$d\mathcal{L}\Phi \leq \Phi h(\Phi,\cdot) \quad \text{in } \Omega_{\gamma,\rho/2} \text{ for all } d < d_\rho.$$

Let denote by $f \in \mathcal{C}(\bar{\Omega}_{\gamma,\rho/2})$ the function defined, for every $x \in \bar{\Omega}_{\gamma,\rho/2}$, through

$$f(x) := \begin{cases} \dfrac{\theta_{\{d,h\}}(x)h(\theta_{\{d,h\}}(x),x) - \Phi(x)h(\Phi(x),x)}{\theta_{\{d,h\}}(x) - \Phi(x)} & \text{if } \ \theta_{\{d,h\}}(x) \neq \Phi(x), \\ h(\Phi(x),x) + \Phi(x)\frac{\partial h}{\partial u}(\Phi(x),x) & \text{if } \ \theta_{\{d,h\}}(x) \neq \Phi(x). \end{cases}$$

Then, for every $d < d_\rho$, taking into account (2.34), the function

$$w := \theta_{\{d,h\}} - \Phi$$

satisfies

$$\begin{cases} (d\mathcal{L} - f)w \geq 0 & \text{in } \ \Omega_{\gamma,\rho/2}, \\ \mathcal{B}w = \mathcal{R}w > 0 & \text{on } \ \gamma, \\ w \geq \frac{\varepsilon}{4} > 0 & \text{on } \ \partial\Omega_{\gamma,\rho/2} \cap \Omega. \end{cases}$$

Therefore, w provides us with a strict supersolution of

$$\begin{cases} (d\mathcal{L} - f)u = 0 & \text{in } \ \Omega_{\gamma,\rho/2}, \\ \mathcal{B}_0 u = 0 & \text{on } \ \partial\Omega_{\gamma,\rho/2}, \end{cases}$$

where

$$\mathcal{B}_0 := \begin{cases} \mathcal{B} & \text{on } \ \gamma, \\ \mathcal{D} & \text{on } \ \partial\Omega_{\gamma,\rho/2} \setminus \gamma. \end{cases}$$

Since, owing to (H2), h is strictly decreasing in the first variable,

$$f \lneq h(\theta_{\{d,h\}}, \cdot) \quad \text{in } \ \Omega_{\gamma,\rho/2}.$$

Moreover, we already know that $\theta_{\{d,h\}} \gg 0$ for sufficiently small $d > 0$. Thus, it follows from Theorem 2.12(a) and Lemma 2.14 that

$$\begin{aligned} \sigma_1[d\mathcal{L} - f; \mathcal{B}_0, \Omega_{\gamma,\rho/2}] &> \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}_0, \Omega_{\gamma,\rho/2}] \\ &> \sigma_1[d\mathcal{L} - h(\theta_{\{d,h\}}, \cdot); \mathcal{B}, \Omega] = 0 \end{aligned}$$

for sufficiently small $d > 0$. Therefore, due to [88, Th. 7.10], and taking into account (2.33), we conclude that

$$\theta_{\{d,h\}} \gg \Phi \geq \Theta_h - \varepsilon \quad \text{in } \ \Omega_{\gamma,\rho/2} \quad \text{for sufficiently small } \ d > 0.$$

The proof is complete. □

Chapter 3

A general class of superlinear indefinite problems

Introduction

In this chapter, the generalized version of Picone's identity, as stated in Theorem 3.1, is used to study the existence and global dynamics of the positive solutions of the superlinear indefinite problem

$$\begin{cases} \mathcal{L}u = \lambda u - a(x)f(u) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with boundary, $\partial\Omega$, of class $\mathcal{C}^2$, $a \in \mathcal{C}(\bar{\Omega})$ is allowed to change sign, $\lambda \in \mathbb{R}$ is a parameter, $f \in \mathcal{C}(\mathbb{R}) \setminus \{0\}$ with $f(0) = 0$, and $\mathcal{L}$ is the operator defined in (2.2) with $b = 0$, i.e., $\mathcal{L}$ is the uniformly elliptic self-adjoint operator in divergence form

$$\mathcal{L} := -\operatorname{div}(A\nabla\cdot) + c \tag{3.2}$$

with $A \in \mathcal{M}_N^{\text{sym}}(\mathcal{C}^1(\bar{\Omega}))$ and $c \in \mathcal{C}(\bar{\Omega})$. As far as concerns the boundary of Ω, much like in Chapter 2, it is assumed that

$$\partial\Omega = \Gamma_{\mathcal{D}} \cup \Gamma_{\mathcal{R}},$$

where $\Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{R}}$ are two disjoint closed and open subsets of $\partial\Omega$ associated with the mixed boundary operator defined in (2.3), which can equivalently written as

$$\mathcal{B} := \begin{cases} \mathcal{D} = \text{Id} & \text{on } \Gamma_{\mathcal{D}}, \\ \mathcal{R} = \langle \nabla\cdot, A\mathbf{n} \rangle + \beta & \text{on } \Gamma_{\mathcal{R}}, \end{cases}$$

for some $\beta \in \mathcal{C}(\Gamma_{\mathcal{R}})$. Here, $\mathbf{n}$ stands for the outward unit normal vector field along $\partial\Omega$. It should be noted that, as β might change of sign, this boundary operator is of general mixed non-classical type.

The problem (3.1) is a generalized version of the simple prototype analyzed by Gómez-Reñasco and López-Gómez [51, 52], where $\mathcal{L} = -\Delta$, $\mathcal{B} = \mathcal{D}$ is the Dirichlet operator on $\partial\Omega$, and

$$f(u) = u^q \quad \text{for some} \quad q \geq 2.$$

As a by-product of our generalized Picone identity, in the special case when $f(u) = u^q$, we can extend the results of [51, 52] characterizing whether, or not, (3.1) admits a linearly stable positive solution, as well as establishing its uniqueness if it exists. This is a rather intriguing uniqueness result as it is folklore that some simple prototypes of (3.1) possess an arbitrarily large number of positive solutions for the appropriate parameter ranges (see Gómez-Reñasco and López-Gómez [51], López-Gómez, Molina-Meyer and Tellini [92], López-Gómez, Tellini and Zanolin [97], López-Gómez and Tellini [96], Sovrano and Zanolin [112], Boscaggin, Feltrin and Zanolin [14], Sovrano [111], Feltrin and Sovrano [37], as well as the recent monograph of Feltrin [36]). This striking uniqueness theorem relies on the fact that, for the special choice $f(u) = u^q$, $q \geq 2$, any linearly neutrally stable positive solution of (3.1) must be a quadratic subcritical turning point in the entire set of positive solutions, (λ, u), of the problem (3.1). Two of the main novelties of this chapter, Theorems 3.8 and 3.11, establish that there are arbitrarily small perturbations of the function $f(u) = u^q$ for which the previous uniqueness result fails to be true. Therefore, our extension of the pioneering findings of [51, 52] seems optimal.

Chapter 3 is distributed as follows. Section 3.1 delivers, through Theorem 3.1, a generalized identity of Picone type valid for arbitrary boundary conditions of mixed type, classical and non-classical, which generalizes, substantially, the previous ones of Picone [106], Kreith [72], Berestycki, Capuzzo-Dolcetta and Nirenberg [9], López-Gómez [84], and [38], as it works out under general boundary conditions of non-classical mixed type. In Section 3.2 several bifurcation results are collected. The main goal of this chapter is to infer, from this generalized Picone identity, the nonexistence of solutions at the right side of the principal eigenvalue of $\mathcal{L}$ in Ω under $\mathcal{B}$ in Section 3.3, and the fact that neutrally stable solutions are quadratic subcritical turning points in Section 3.4. These results hold, exclusively, in the case $f(u) = u^q$, $q \geq 2$, as shown in Section 3.5. In particular, Theorem 3.8 states that polynomial perturbations on f makes the previous properties disappear. In Section 3.6 the existence and uniqueness of the stable positive solution of (3.1) when $f(u) = u^q$, $q \geq 2$, is proved. Section 3.7 and, in particular, Theorem 3.12, provides us the global structure of the set of stable positive solutions of (3.1) if $f(u) = u^q$, $q \geq 2$, and if an appropriate inequality holds.

3.1 A generalized Picone identity

The next result provides us with a generalized version of a celebrated identity of Picone [106]. In this result, the symmetric matrix $A(x)$ is not required to be positive definite, i.e., the second order differential operator $\mathcal{L}$ defined in (3.2) might not be of elliptic type.

This identity will play a crucial role in the forthcoming sections of this chapter, but also in Chapter 6, where the dynamics of the competition Lotka–Volterra diffusive model are analyzed.

Theorem 3.1. *Suppose that Ω is a bounded open subdomain of $\mathbb{R}^N$, $N \geq 1$, of class $\mathcal{C}^2$,* n *stands for the outward unit normal vector field along $\partial\Omega$, and let $u, v \in W^{2,p}(\Omega)$, $p > N$, be such that $\frac{v}{u} \in \mathcal{C}^1(\bar{\Omega})$ and $\mathcal{L}u, \mathcal{L}v \in \mathcal{C}(\bar{\Omega})$, with $\mathcal{L}$ the self-adjoint operator in divergence* *form* (3.2). *Consider* $\beta \in \mathcal{C}(\partial\Omega)$ *and let* $\mathcal{D}$, $\mathcal{R}$ *be boundary operators on* $\partial\Omega$ *defined by*

$$\begin{cases} \mathcal{D}w = w, \\ \mathcal{R}w = \langle \nabla w, A\mathrm{n}\rangle + \beta w, \end{cases} \qquad w \in W^{2,p}(\Omega).$$

Then, for every $g \in \mathcal{C}^1(\mathbb{R})$ *the next identity holds*

$$\int_\Omega g\left(\frac{v}{u}\right)[u\mathcal{L}v - v\mathcal{L}u] = \int_\Omega u^2 g'\left(\frac{v}{u}\right)\langle \nabla\frac{v}{u}, A\nabla\frac{v}{u}\rangle - \int_{\partial\Omega} g\left(\frac{v}{u}\right)[\mathcal{D}u\mathcal{R}v - \mathcal{D}v\mathcal{R}u]. \quad (3.3)$$

Proof. Expanding the integrand on the left hand side and using the symmetry of A yields

$$\begin{aligned} g\left(\frac{v}{u}\right)[u\mathcal{L}v - v\mathcal{L}u] &= g\left(\frac{v}{u}\right)[v\,\mathrm{div}(A\nabla u) - u\,\mathrm{div}(A\nabla v)] \\ &= g\left(\frac{v}{u}\right)\mathrm{div}(vA\nabla u - uA\nabla v) \\ &= \mathrm{div}\left[g\left(\frac{v}{u}\right)(vA\nabla u - uA\nabla v)\right] - \langle \nabla g\left(\frac{v}{u}\right), vA\nabla u - uA\nabla v\rangle \\ &= \mathrm{div}\left[g\left(\frac{v}{u}\right)(vA\nabla u - uA\nabla v)\right] - g'\left(\frac{v}{u}\right)\langle \nabla\frac{v}{u}, A(v\nabla u - u\nabla v)\rangle \\ &= \mathrm{div}\left[g\left(\frac{v}{u}\right)(vA\nabla u - uA\nabla v)\right] + u^2 g'\left(\frac{v}{u}\right)\langle \nabla\frac{v}{u}, A\nabla\frac{v}{u}\rangle. \end{aligned}$$

Thus, integrating in Ω, we find that

$$\int_\Omega g\left(\frac{v}{u}\right)[u\mathcal{L}v - v\mathcal{L}u] = \int_\Omega \mathrm{div}\left[g\left(\frac{v}{u}\right)(vA\nabla u - uA\nabla v)\right] + \int_\Omega u^2 g'\left(\frac{v}{u}\right)\langle \nabla\frac{v}{u}, A\nabla\frac{v}{u}\rangle.$$

As integrating by parts shows that

$$\begin{aligned} \int_\Omega \mathrm{div}\left[g\left(\frac{v}{u}\right)(vA\nabla u - uA\nabla v)\right] &= \int_{\partial\Omega} g\left(\frac{v}{u}\right)\langle vA\nabla u - uA\nabla v, \mathrm{n}\rangle \\ &= \int_{\partial\Omega} g\left(\frac{v}{u}\right)[v(\langle A\nabla u, \mathrm{n}\rangle + \beta u) - u(\langle A\nabla v, \mathrm{n}\rangle + \beta v)] \\ &= -\int_{\partial\Omega} g\left(\frac{v}{u}\right)[\mathcal{D}u\mathcal{R}v - \mathcal{D}v\mathcal{R}u], \end{aligned}$$

the identity (3.3) holds. □

3.2 Positive solutions bifurcating from the trivial branch

Throughout this chapter, and according to Theorem 7.7 of López-Gómez [88, Th. 7.7], we will denote by σ_0 the (unique) *principal eigenvalue* associated to the eigenvalue problem

$$\begin{cases} \mathcal{L}\varphi = \sigma\varphi & \text{in } \Omega, \\ \mathcal{B}\varphi = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.4)$$

which is algebraically simple and strictly dominant. Moreover, we will keep the notations stated in Chapter 2, Section 2.3, for the principal eigevalues. Hence, we set

$$\sigma_0 := \sigma_1[\mathcal{L}; \mathcal{B}, \Omega].$$

It should be remembered that the principal eigenvalue is algebraically simple and it is the unique eigenvalue associated with a positive eigenfunction, φ_0. Actually, $\varphi_0 \gg 0$ in the sense of (2.18) and $\varphi_0 \in W^{2,\infty}_{\mathcal{B}}(\Omega)$, as defined in (2.19). Hence,

$$\varphi_0 \in \mathcal{C}^1_{\mathcal{B}}(\bar{\Omega}) \cap \mathcal{C}^{1,\nu}(\Omega) \quad \text{for all } \ \nu < 1$$

and it is almost everywhere twice differentiable in Ω, like the weak positive solutions of (3.1).

The following result establishes the existence of a curve of positive solutions, (λ, u), of (3.1) emanating from $u = 0$ as λ crosses σ_0. It is a straightforward application of the main theorem of Crandall and Rabinowitz [24] based on the fact that σ_0 is algebraically simple.

Theorem 3.2. *Assume that f is of class $\mathcal{C}^r$, $r \geq 2$, in a neighborhood of zero and $f(0) = f'(0) = 0$. Let $\varphi_0 \in W^{2,p}_{\mathcal{B}}(\Omega)$, $p > N$, be the principal eigenfunction associated with σ_0 normalized so that*

$$\int_\Omega \varphi_0^2(x)\, dx = 1.$$

Then, there exist $\varepsilon > 0$ and two maps of class $\mathcal{C}^{r-1}$,

$$\mu : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad y : (-\varepsilon, \varepsilon) \to W^{2,p}_{\mathcal{B}}(\Omega),$$

such that $\mu(0) = 0$, $y(0) = 0$, $\int_\Omega y(s)\varphi_0 = 0$ for all $s \in (-\varepsilon, \varepsilon)$, and

$$(\lambda(s), u(s)) := (\sigma_0 + \mu(s), s(\varphi_0 + y(s))) \tag{3.5}$$

solves (3.1) for every $s \in (-\varepsilon, \varepsilon)$. Moreover, there exists a neighborhood of $(\sigma_0, 0)$ in $\mathbb{R} \times W^{2,p}_{\mathcal{B}}(\Omega)$, $\mathcal{U}$, such that, for any solution $(\lambda, u) \in \mathcal{U}$ of (3.1), either $u = 0$, or there exists $s \in (-\varepsilon, \varepsilon)$ such that $(\lambda, u) = (\lambda(s), u(s))$.

Proof. Let $\omega > 0$ be such that $\omega > -\sigma_0$. Then,

$$\sigma_1[\mathcal{L} + \omega; \mathcal{B}, \Omega] = \sigma_0 + \omega > 0.$$

Hence, the solutions of (3.1) are given by the zeroes of the operator

$$\mathfrak{F}(\lambda, u) := u - (\mathcal{L} + \omega)^{-1}[(\lambda + \omega)u - a(x)f(u)], \qquad (\lambda, u) \in \mathbb{R} \times L^p(\Omega),$$

which is a compact perturbation of the identity map of class $\mathcal{C}^r$; in particular, it is Fredholm of index zero. We have that $\mathfrak{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Moreover, the Fréchèt differential $\mathfrak{L}(\lambda) := D_u\mathfrak{F}(\lambda, 0)$ is given by

$$D_u\mathfrak{F}(\lambda, 0)u = u - (\mathcal{L} + \omega)^{-1}[(\lambda + \omega)u].$$

Thus, it is apparent that $\mathfrak{L}(\lambda)$ is an isomorphism if λ is not an eigenvalue of (3.4). Furthermore,

$$\text{Ker}\, D_u\mathfrak{F}(\sigma_0, 0) = \text{span}\,[\varphi_0]$$

and the next transversality condition holds

$$\mathfrak{L}'(\sigma_0)\varphi_0 = -(\mathcal{L} + \omega)^{-1}\varphi_0 \notin \text{Im}\, \mathfrak{L}(\sigma_0).$$

On the contrary, assume that, for some $u \in W^{2,p}_{\mathcal{B}}(\Omega)$,

$$u - (\mathcal{L}+\omega)^{-1}[(\sigma_0+\omega)u] = -(\mathcal{L}+\omega)^{-1}\varphi_0.$$

Then,

$$\begin{cases} (\mathcal{L}-\sigma_0)u = -\varphi_0 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases}$$

which contradicts the simplicity established by Theorem 7.8 of López-Gómez [88]. Therefore, the desired result follows by applying the main theorem of Crandall and Rabinowitz [24], with

$$Y := \left\{ y \in W^{2,p}_{\mathcal{B}}(\Omega) \ : \ \int_\Omega \varphi_0(x)y(x)\,dx = 0 \right\}.$$

The proof is complete. □

As a consequence of the definition of $u(s)$, we have that $u'(0) = \varphi_0 \gg 0$. Hence, ε can be shortened, if necessary, so that

$$u'(s) := \frac{du}{ds}(s) \gg 0 \quad \text{for all } \ s \in (-\varepsilon, \varepsilon).$$

Moreover, $u(s) \gg 0$ if $s \in (0,\varepsilon)$, while $u(s) \ll 0$ if $s \in (-\varepsilon, 0)$, and the next result holds.

Proposition 3.3. *Under the same assumptions of Theorem 3.2, the following assertions are true:*

(i) *For every $q \geq 1$,*

$$\lim_{s\to 0^\pm} \frac{\lambda(s)-\sigma_0}{|s|^{q-1}} = \lim_{s\to 0^\pm} \frac{f(s)}{s|s|^{q-1}} \int_\Omega a(x)\varphi_0^{q+1}(x)\,dx \tag{3.6}$$

if the limit on the right hand side exists.

(ii) *If $\lambda'(s)u(s) > 0$ for some $s \in (-\varepsilon,\varepsilon)$, then $u(s)$ is linearly stable as a steady-state solution of the parabolic problem*

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u = \lambda u - a(x)f(u) & (x,t) \in \Omega \times (0,+\infty), \\ \mathcal{B}u = 0 & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) \geq 0 & x \in \Omega. \end{cases} \tag{3.7}$$

In other words,

$$\sigma_1[\mathcal{L} + a(x)f'(u(s)) - \lambda(s); \mathcal{B}, \Omega] > 0. \tag{3.8}$$

Proof. Substituting (3.5) in (3.1) we are driven to

$$s\mathcal{L}(\varphi_0 + y(s)) = s(\sigma_0 + \mu(s))(\varphi_0 + y(s)) - af(s(\varphi_0 + y(s))).$$

Thus,

$$s(\mathcal{L}-\sigma_0)y(s) = s\mu(s)(\varphi_0 + y(s)) - af(s(\varphi_0 + y(s))).$$

Hence, for any given $q \geq 1$, multiplying both sides of this identity by $\frac{\varphi_0}{s|s|^{q-1}}$, $s \neq 0$, it is apparent that

$$\frac{1}{|s|^{q-1}}\varphi_0(\mathcal{L}-\sigma_0)y(s) = \frac{\mu(s)}{|s|^{q-1}}\varphi_0(\varphi_0+y(s)) - a\varphi_0\frac{f(s(\varphi_0+y(s)))}{s|s|^{q-1}}.$$

Therefore, since

$$\int_\Omega \varphi_0(\mathcal{L}-\sigma_0)y(s) = \int_\Omega y(s)(\mathcal{L}-\sigma_0)\varphi_0 = 0,$$

we find that

$$0 = \frac{\mu(s)}{|s|^{q-1}}\int_\Omega \varphi_0(\varphi_0+y(s)) - \int_\Omega a\varphi_0(\varphi_0+y(s))^q\frac{f(s(\varphi_0+y(s)))}{s|s|^{q-1}(\varphi_0+y(s))^q}. \tag{3.9}$$

The identity (3.6) follows from Lebesgue's dominated convergence theorem by letting $s \to 0$ in (3.9) provided $q \geq 1$ satisfy

$$\lim_{s\to 0^\pm}\frac{f(s)}{s|s|^{q-1}} \in \mathbb{R}.$$

Finally, differentiating with respect to s the identity

$$\mathfrak{F}(\lambda(s), u(s)) = 0, \qquad s \in (-\varepsilon, \varepsilon),$$

inverting $(\mathcal{L}+\omega)^{-1}$ and rearranging terms, it becomes apparent that

$$(\mathcal{L}-\lambda(s)+af'(u(s)))u'(s) = \lambda'(s)u(s), \quad s \in (-\varepsilon, \varepsilon).$$

Since shortening ε, we can assume that $u'(s) \gg 0$ for all $s \in (-\varepsilon, \varepsilon)$, it follows from Theorem 7.10 of López-Gómez [88] that $\lambda'(s)u(s) > 0$ implies (3.8), ending the proof. □

It should be noted that (3.6) provides us with the sign of $\mu(s) = \lambda(s) - \sigma_0$ and hence, the bifurcation direction of the curve of positive solutions, $(\lambda(s), u(s))$, $s > 0$, in terms of the behavior of $f(u)$ at $u = 0$ and the sign of the integral

$$\int_\Omega a(x)\varphi_0^{q+1}(x)\,dx.$$

However, as we are applying Theorem 3.2, f is required to be of class $\mathcal{C}^2$ regularity. In particular, the next result holds.

Corollary 3.4. *Under the same assumptions of Theorem 3.2, suppose that in addition* $f(u) := u|u|^{q-1}$ *for some* $q \geq 2$. *Then,*

$$\lim_{s\to 0^\pm}\frac{\lambda(s)-\sigma_0}{|s|^{r-1}} = 0 \;\; \text{for all} \;\; r \in [1, q) \;\; \text{and} \;\; \lim_{s\to 0^\pm}\frac{\lambda(s)-\sigma_0}{|s|^{q-1}} = \int_\Omega a(x)\varphi_0^{q+1}(x)\,dx.$$

Thus, the bifurcation to positive solutions is supercritical if

$$\mathcal{S} := \int_\Omega a(x)\varphi_0^{q+1}(x)\,dx > 0,$$

while it is subcritical, if $\mathcal{S} < 0$.

In the general case when f is merely continuous, one can still use the unilateral global bifurcation theorem of López-Gómez [88, Th. 6.2.4] to infer that the set of solutions of (3.1) possesses a (connected) component, $\mathfrak{C}^+$, of positive solutions which is unbounded in $\mathbb{R} \times \mathcal{C}(\bar{\Omega})$ and satisfies $(\sigma_0, 0) \in \bar{\mathfrak{C}}^+$. But, in this general case, the sharp information provided by Theorem 3.2 in a neighborhood of $(\sigma_0, 0)$ is lost.

3.3 Nonexistence of small positive solutions for $\lambda \geq \sigma_0$.

In this section we provide the first consequences of Picone's identity delivered in Theorem 3.1: the nonexistence of positive solutions at the right side of σ_0. In particular, the next result provides us with a sufficient condition so that (3.1) cannot admit positive small solutions for $\lambda \geq \sigma_0$ even in the general case when f is continuous.

Remember that $\sigma_0 = \sigma_1[\mathcal{L}; \mathcal{B}, \Omega]$ is associated with the L^2-normalized positive principal eigenfunction φ_0.

Theorem 3.5. *Assume that, for some $q \geq 1$,*

$$\lim_{s \downarrow 0} \frac{f(s)}{s^q} \int_\Omega a(x)\varphi_0^{q+1}(x)\, dx < 0. \tag{3.10}$$

Then, there exists $\varepsilon > 0$ such that $\lambda < \sigma_0$ if (3.1) admits a solution, (λ, u), with $u \gneq 0$ and $\|u\|_\infty < \varepsilon$. In other words, (3.1) cannot admit small positive solutions if $\lambda \geq \sigma_0$.

Proof. Let (λ, u) be a positive solution of (3.1). Then, since $q \geq 1$, it is easily seen that $u \gg 0$ in Ω, in the sense of (2.18). Thus, since $\varphi_0 \gg 0$, the quotient $\frac{\varphi_0}{u}$ preserves its regularity even on the Dirichlet components of $\partial\Omega$. Thus, applying Theorem 3.1 with $g(t) = t|t|^{q-1}$, $t \in \mathbb{R}$, to the functions u and φ_0 and taking into account that $A(x)$ is positive definite yields the estimate

$$\int_\Omega \left(\frac{\varphi_0}{u}\right)^q (u\mathcal{L}\varphi_0 - \varphi_0\mathcal{L}u) = p\int_\Omega \frac{\varphi_0^{q-1}}{u^{q-3}}\langle \nabla\frac{\varphi_0}{u}, A\nabla\frac{\varphi_0}{u}\rangle \geq 0,$$

since $\mathcal{B}u = \mathcal{B}\varphi_0 = 0$ on $\partial\Omega$ and, so, either $\mathcal{D}u = \mathcal{D}\varphi_0 = 0$, or $\mathcal{R}u = \mathcal{R}\varphi_0 = 0$, on each component of $\partial\Omega$. On the other hand, using the fact that u solves (3.1) it follows from the definition of φ_0 that

$$u\mathcal{L}\varphi_0 - \varphi_0\mathcal{L}u = (\sigma_0 - \lambda)u\varphi_0 + a(x)f(u)\varphi_0.$$

Hence, multiplying this identity by $\frac{\varphi_0^q}{u^q}$ and integrating in Ω we obtain that

$$(\sigma_0 - \lambda)\int_\Omega \frac{\varphi_0^{q+1}}{u^{q-1}} + \int_\Omega a\varphi_0^{q+1}\frac{f(u)}{u^q} = \int_\Omega \left(\frac{\varphi_0}{u}\right)^q (u\mathcal{L}\varphi_0 - \varphi_0\mathcal{L}u) \geq 0. \tag{3.11}$$

Therefore, by the Lebesgue's dominated convergence theorem, it follows from (3.10) that

$$(\sigma_0 - \lambda)\int_\Omega \frac{\varphi_0^{q+1}}{u^{q-1}} \geq -\int_\Omega a\varphi_0^{q+1}\frac{f(u)}{u^q} \xrightarrow{\|u\|_\infty \to 0,\ u \geq 0} -\lim_{s \to 0^+}\frac{f(s)}{s^q}\int_\Omega a\varphi_0^{q+1} > 0.$$

Consequently, since

$$\int_\Omega \frac{\varphi_0^{q+1}}{u^{q-1}} > 0,$$

it is apparent that $\sigma_0 > \lambda$. This ends the proof. ∎

Note that, in Theorem 3.5, the size of $\varepsilon > 0$ only depends on how

$$\int_\Omega a\varphi_0^{q+1}\frac{f(u)}{u^q} \ \text{approximates} \ \lim_{s\downarrow 0}\frac{f(s)}{s^q}\int_\Omega a\varphi_0^{q+1} \ \text{as} \ \|u\|_\infty \to 0. \tag{3.12}$$

In particular, the next result shows that when the approximation (3.12) occurs suddenly, i.e., when $f(u) = u|u|^{q-1}$, $u \in \mathbb{R}$, for some $q > 1$, then $\varepsilon = +\infty$, i.e., (3.1) cannot admit a positive solution if $\lambda \geq \sigma_0$. It is a substantial extension of Theorem 2 of Berestycki, Capuzzo-Dolcetta and Nirenberg [9] and Theorem 4.2 of López-Gómez [84]. Later, in Theorem 3.8, the optimality of this result will be established.

Theorem 3.6. *Assume that $q > 1$ exists such that $f(s) := s^q$ for every $s \geq 0$ and*

$$\int_\Omega a(x)\varphi_0^{q+1}(x)\,dx \leq 0. \tag{3.13}$$

Then, $\lambda < \sigma_0$ if (3.1) admits a positive solution, (λ, u).

Proof. The proof can be easily adapted from the proof of Theorem 3.5. First, assume that $\frac{\varphi_0}{u}$ is not constant. Then, by (3.11) and Theorem 3.1,

$$(\sigma_0 - \lambda)\int_\Omega \frac{\varphi_0^{q+1}}{u^{q-1}} + \int_\Omega a\varphi_0^{q+1} = p\int_\Omega \frac{\varphi_0^{q-1}}{u^{q-3}}\langle\nabla\frac{\varphi_0}{u}, A\nabla\frac{\varphi_0}{u}\rangle > 0.$$

Hence,

$$(\sigma_0 - \lambda)\int_\Omega \frac{\varphi_0^{q+1}}{u^{q-1}} > -\int_\Omega a\varphi_0^{q+1} \geq 0,$$

and so $\lambda < \sigma_0$. On the other hand, if $\frac{\varphi_0}{u}$ is a (positive) constant, then u satisfies

$$\lambda u - a(x)u^q = \mathcal{L}u = \sigma_0 u$$

and hence,

$$a(x)u^{q-1}(x) = \lambda - \sigma_0 \quad \text{for all} \ \ x \in \Omega.$$

Thus, a cannot change sign, which contradicts (3.13). Therefore, $\lambda < \sigma_0$. This ends the proof. □

3.4 Quadratic subcritical turning point character of neutrally stable solutions

The main result of this section, which is based on Theorem 3.1, provides us with the local structure of the set of solutions of (3.1) around any *neutrally stable* positive solution, (λ_0, u_0), i.e., any positive solution of (3.1), such that

$$\sigma_1[\mathcal{L} - \lambda_0 + a(x)f'(u_0); \mathcal{B}, \Omega] = 0. \tag{3.14}$$

Theorem 3.7 is a significative generalization of Proposition 3.2 of Gómez-Reñasco and López-Gómez [51]. Based on this result, we will establish in Section 3.6 the uniqueness of the linearly stable positive solution of (3.1) if it exists. This uniqueness result generalize, very substantially, the corresponding uniqueness theorems of Gómez-Reñasco and López-Gómez [51, 52], slightly polished by López-Gómez [89, Ch. 9].

Theorem 3.7. *Assume that one of the following three condition holds:*

(i) $\Gamma_{\mathcal{D}} = \emptyset$ *and* $f(u) = u \log u$ *for every* $u \geq 0$,

(ii) $\Gamma_{\mathcal{D}} = \emptyset$ *and* $f(u) = u^q$ *for every* $u \geq 0$ *with* $q \in (0,1) \cup (1,2)$,

(iii) $f(u) = u^q$ *for every* $u \geq 0$ *with* $q \geq 2$.

Let (λ_0, u_0) *be a neutrally stable positive solution of* (3.1) *such that* $u_0 \geq 1$ *in case* (i) *and* $u_0 \geq \tau > 0$ *in case* (ii). *Let* $\psi_0 \in W^{2,p}(\Omega)$, $p > N$, *denote the principal eigenfunction associated with* (3.14) *normalized so that* $\int_\Omega \psi_0^2 = 1$. *Then, there exist* $\varepsilon > 0$ *and two functions of class* $\mathcal{C}^2$,

$$\lambda : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad \text{and} \quad u : (-\varepsilon, \varepsilon) \to W^{2,p}_{\mathcal{B}}(\Omega)$$

such that

$$(\lambda(0), u(0)) = (\lambda_0, u_0), \quad (\lambda'(0), u'(0)) = (0, \psi_0), \quad \lambda''(0) < 0,$$

for which the curve $(\lambda(s), u(s))$ *provides us with the set of solutions of* (3.1) *in a neighborhood of* (λ_0, u_0). *Moreover, shortening* ε, *if necessary,* $u(s)$ *is linearly stable if* $s \in (-\varepsilon, 0)$ *and linearly unstable if* $s \in (0, \varepsilon)$.

Proof. The existence of a real analytic curve of solutions is an immediate consequence of Proposition 20.7 of Amann [3]. A more recent approach, where the underlying analysis has been considerably tidied up, can be found in Proposition 9.7 of López-Gómez [89]. In these references the existence of the curve follows from the implicit function theorem after a Lyapunov–Schmidt decomposition. The fact that

$$(\lambda(0), u(0)) = (\lambda_0, u_0), \quad u'(0) = \psi_0,$$

follows easily from these previous constructions. Differentiating with respect to s in

$$\mathcal{L}u(s) = \lambda(s)u(s) - a(x)f(u(s))$$

yields

$$\mathcal{L}u'(s) = \lambda'(s)u(s) + \lambda(s)u'(s) - a(x)f'(u(s))u'(s). \tag{3.15}$$

Thus, particularizing at $s = 0$, multiplying the resulting identity by ψ_0 and integrating by parts in Ω yields

$$\lambda'(0) = \frac{\int_\Omega \psi_0[\mathcal{L} - \lambda_0 + a(x)f'(u_0)]\psi_0}{\int_\Omega u_0\psi_0} = 0.$$

Similarly, by differentiating (3.15) with respect to s, it follows that

$$\begin{aligned}\mathcal{L}u''(s) = \lambda''(s)u(s) + 2\lambda'(s)u'(s) + \lambda(s)u''(s) \\ - a(x)f''(u(s))(u'(s))^2 - a(x)f'(u(s))u''(s).\end{aligned}$$

Thus, since $\lambda'(0) = 0$, particularizing at $s = 0$ shows that

$$\mathcal{L}u''(0) = \lambda''(0)u_0 + \lambda_0 u''(0) - a(x)f''(u_0)\psi_0^2 - a(x)f'(u_0)u''(0).$$

Hence, multiplying by ψ_0 and integrating by parts in Ω yields

$$\lambda''(0) = \frac{\int_\Omega u''(0)[\mathcal{L} - \lambda_0 + af'(u_0)]\psi_0 + \int_\Omega a\psi_0^3 f''(u_0)}{\int_\Omega u_0\psi_0} = \frac{\int_\Omega a\psi_0^3 f''(u_0)}{\int_\Omega u_0\varphi_0}.$$

To prove that $\lambda''(0) < 0$ one can argue as follows. By the definition of u_0 and ψ_0, we have that

$$\begin{aligned}\left(\frac{\psi_0}{u_0}\right)^2 (u_0\mathcal{L}\psi_0 - \psi_0\mathcal{L}u_0) &= \frac{\psi_0^2}{u_0}\mathcal{L}\psi_0 - \frac{\psi_0^3}{u_0^2}\mathcal{L}u_0 \\ &= \frac{\psi_0^2}{u_0}\left(\lambda_0\psi_0 - af'(u_0)\psi_0\right) - \frac{\psi_0^3}{u_0^2}\left(\lambda_0 u_0 - af(u_0)\right) \\ &= -a\psi_0^3\frac{f'(u_0)u_0 - f(u_0)}{u_0^2} = -a\psi_0^3\left(\frac{f(u)}{u}\right)'\Big|_{u=u_0}.\end{aligned} \tag{3.16}$$

Thus, integrating in Ω, we find that

$$\int_\Omega \left(\frac{\psi_0}{u_0}\right)^2 (u_0\mathcal{L}\psi_0 - \psi_0\mathcal{L}u_0) = -\int_\Omega a\psi_0^3\left(\frac{f(u)}{u}\right)'\Big|_{u=u_0}.$$

As it turns out that the functions $f(u) = u^q$, $q \in (0, +\infty) \setminus \{1\}$, and $f(u) = u\log u$, are the unique ones satisfying

$$\left(\frac{f(u)}{u}\right)' = \frac{1}{q}f''(u) \quad \text{and} \quad \left(\frac{f(u)}{u}\right)' = f''(u),$$

respectively, it becomes apparent that, for these functions,

$$\operatorname{sgn}\lambda''(0) = \operatorname{sgn}\int_\Omega a\psi_0^3 f''(u_0) = -\operatorname{sgn}\int_\Omega \left(\frac{\psi_0}{u_0}\right)^2 (u_0\mathcal{L}\psi_0 - \psi_0\mathcal{L}u_0).$$

Consequently, the fact that $\lambda''(0) < 0$ follows easily from Theorem 3.1, which provides us with the identity

$$\int_\Omega \left(\frac{\psi_0}{u_0}\right)^2 (u_0\mathcal{L}\psi_0 - \psi_0\mathcal{L}u_0) = 2\int_\Omega \psi_0 u_0\langle\nabla\frac{\psi_0}{u_0}, \nabla\frac{\psi_0}{u_0}\rangle > 0,$$

because u_0 cannot be a multiple of ψ_0. Indeed, on the contrary case, $\psi_0 = \kappa u_0$ for some $\kappa > 0$ and hence, it follows from (3.16) that

$$a\left(f(u_0) - u_0 f'(u_0)\right) = 0 \;\text{ in }\; \Omega.$$

Thus, since we are assuming that $a \in \mathcal{C}(\bar{\Omega})$ satisfies $a \neq 0$, there exists $x_0 \in \Omega$ such that

$$f(u_0(x_0)) = u_0(x_0)f'(u_0(x_0)).$$

However, for the special choices $f(u) = u\log u$, and $f(u) = u^q$, $q \in (0, +\infty) \setminus \{1\}$, this identity entails $u_0(x_0) = 0$. Therefore, we are in case (iii) and necessarily $x_0 \in \partial\Omega$, which contradicts $x_0 \in \Omega$.

For the stability of the positive solution $(\lambda(s), u(s))$ for sufficiently small $s \sim 0$, we have to ascertain the sign of the principal eigenvalue

$$\Sigma(s) := \sigma_1[\mathcal{L} - \lambda(s) + a(x)f'(u(s)); \mathcal{B}, \Omega].$$

In particular, since $\Sigma(0) = 0$, it suffices to show that $\Sigma'(0) < 0$. As $\Sigma(s)$ is a simple eigenvalue of class $\mathcal{C}^1$ in s, it follows from the abstract theory of Kato [67] that ψ_0 admits a $\mathcal{C}^1$ perturbation, $\psi(s)$, $s \sim 0$, such that $\psi(0) = \psi_0$ and $\int_\Omega \psi^2(s) = 1$ for sufficiently small s (see also Lemma 2.2.1 of López-Gómez [85]). Thus, differentiating with respect to s the identity

$$\mathcal{L}\psi(s) - \lambda(s)\psi(s) + af'(u(s))\psi(s) = \Sigma(s)\psi(s),$$

we are driven to the identity

$$[\mathcal{L} - \lambda(s) + af'(u(s))]\psi'(s) - \lambda'(s)\psi(s) + af''(u(s))u'(s)\psi(s) = \Sigma(s)\psi'(s) + \Sigma'(s)\psi(s).$$

So, particularizing at $s = 0$, we have that

$$[\mathcal{L} - \lambda_0 + af'(u_0)]\psi'(0) + af''(u_0)\psi_0^2 = \Sigma'(0)\psi_0.$$

Therefore, multiplying by ψ_0 this identity and integrating by parts in Ω the next identity holds

$$\Sigma'(0) = \int_\Omega af''(u_0)\psi_0^3 = \lambda''(0)\int_\Omega u_0\psi_0 < 0,$$

which ends the proof. □

Figure 3.1 represents a genuine quadratic subcritical turning point. Theorem 3.7 establishes that this is the bifurcation diagram of (3.1) in a neighborhood of any linearly neutrally stable positive solution, (λ_0, u_0). The half low branch, plotted with a continuous line, is filled in by linearly stable positive solutions, while the upper one, plotted with a discontinuous line, consists of linearly unstable positive solutions with one-dimensional unstable manifold.

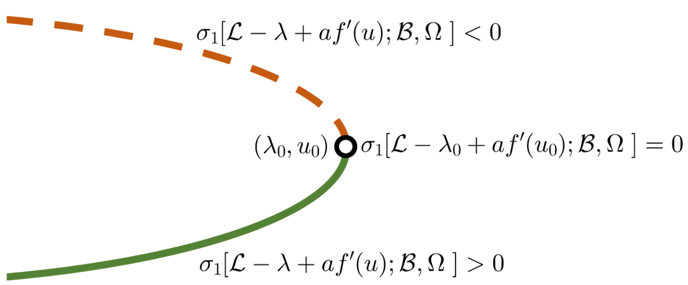


Figure 3.1: Local bifurcation diagram at a linearly neutrally stable positive solution, (λ_0, u_0).

3.5 Optimality of Theorems 3.6 and 3.7

This section is devoted to show the optimality of the results stated in Sections 3.3 and 3.4. In particular, the next result shows the optimality of Theorem 3.6, in the sense that it fails to be true when $f(u)$ does not have the required form. Again, remember that $\varphi_0 \gg 0$ stands for the principal eigenfunction associated with $\sigma_0 := \sigma_1[\mathcal{L};\mathcal{B},\Omega]$ normalized in $L^2(\Omega)$.

Theorem 3.8. *Assume that there exist* $0 < q_1 < q_2$ *and* F, $G \in \mathcal{C}^r([0,+\infty);\mathbb{R})$, $r \geq 2$, *such that*

$$F(0) = G(0) = 0, \quad F'(0) = G'(0) = 0$$

and

$$\lim_{s\to 0} \frac{F(s)}{s|s|^{q_1-1}} \int_\Omega a\varphi_0^{q_1+1} < 0 < \lim_{s\to 0} \frac{G(s)}{s|s|^{q_2-1}} \int_\Omega a\varphi_0^{q_2+1}.$$

Then, there exists $\nu_0 > 0$ *such that, for every* $\nu \in (0,\nu_0)$, *the problem*

$$\begin{cases} \mathcal{L}u = \lambda u - a(x)(\nu F(u) + G(u)) & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

admits positive solutions for values of the parameter λ *at both sides of* σ_0.

Proof. Suppose $\nu \geq 0$ and $p > N$. Then, owing to Theorem 3.2, there exist $s_0 = s_0(\nu) > 0$ and two maps of class $\mathcal{C}^{r-1}$

$$\lambda(s): [0,s_0) \to \mathbb{R}, \qquad u(s): [0,s_0) \to W^{2,p}_{\mathcal{B}}(\Omega),$$

such that

$$(\lambda(0),u(0)) = (\sigma_0,0), \quad u'(0) = \varphi_0,$$

and $(\lambda(s),u(s))$ solves (3.1) for all $s \in (-s_0,s_0)$. Moreover, by Proposition 3.3, for every $\nu > 0$,

$$\begin{aligned} \lim_{s\downarrow 0} \frac{\lambda(s)-\sigma_0}{s^{q_1-1}} &= \lim_{s\downarrow 0} \frac{\nu F(s)+G(s)}{s^{q_1}} \int_\Omega a\varphi_0^{q_1+1} \\ &= \nu \lim_{s\downarrow 0} \frac{F(s)}{s_1^q} \int_\Omega a\varphi_0^{q_1+1} + \lim_{s\downarrow 0}\left(s^{q_2-q_1}\frac{G(s)}{s^{q_2}}\right)\int_\Omega a\varphi_0^{q_1+1} \\ &= \nu \lim_{s\downarrow 0} \frac{F(s)}{s^{q_1}} \int_\Omega a\varphi_0^{q_1+1} < 0, \end{aligned}$$

whereas, for $\nu = 0$,

$$\lim_{s\downarrow 0} \frac{\lambda(s)-\sigma_0}{s^{q_2-1}} = \lim_{s\downarrow 0} \frac{G(s)}{s_2^q} \int_\Omega a\varphi_0^{q_2+1} > 0.$$

Thus, when $\nu = 0$, the bifurcation of the curve of positive solutions $(\lambda(s),u(s))$ from the trivial branch is supercritical. In such case, by Proposition 3.3, we also have that $u(s)$, as an steady-state solution of (3.7), is linearly stable for sufficiently small $s \in (0,s_0)$. Subsequently, we shorten s_0, if necessary, so that $u(s)$ is linearly stable for all $s \in (0,s_0)$. Then, $\lambda'(s) > 0$ for each $s \in (0,s_0)$. Thus, there exists $\varepsilon > 0$ such that the curve of positive solutions $(\lambda(s), u(s))$, $s \sim 0$, can be parameterized by λ, $(\lambda, u(\lambda))$, with $\lambda \in (\sigma_0, \sigma_0 + \varepsilon)$, in a neighborhood of the bifurcation point $(\sigma_0, 0)$. In particular,

$$\sigma_1[\mathcal{L} - \lambda + aG'(u(\lambda)); \mathcal{B}, \Omega] > 0, \qquad \lambda \in (\sigma_0, \sigma_0 + \varepsilon). \tag{3.18}$$

Pick $\omega > -\sigma_0$ arbitrary and two values $\lambda_1, \lambda_2 \in (\sigma_0, \sigma_0 + \varepsilon)$, with $\lambda_1 < \lambda_2$. Then, setting

$$\mathfrak{F}(\nu, \lambda, u) := u - (\mathcal{L} + \omega)^{-1}[(\lambda + \omega)u - a(\nu F(u) + G(u))],$$

we have that, for every $\lambda \in [\lambda_1, \lambda_2]$,

$$D_u\mathfrak{F}(0, \lambda, u(\lambda))u = u - (\mathcal{L} + \omega)^{-1}\left[(\lambda + \omega)u + aG'(u(\lambda))u\right].$$

Since it is a compact perturbation of the identity map, $D_u\mathfrak{F}(0, \lambda, u(\lambda))$ is Fredholm of index zero in $L^p(\Omega)$ for all $p > N$. Moreover, owing to (3.18), it is a topological isomorphism. Therefore, by the implicit function theorem and the compactness of $[\lambda_1, \lambda_2]$, $u(\lambda)$ can be regarded as a function of class $\mathcal{C}^r$-regularity of λ and ν, $u(\lambda, \nu)$, in $[\lambda_1, \lambda_2] \times [0, \nu_0]$ for sufficiently small $\nu_0 > 0$. Furthermore, by (3.18), it becomes apparent that $(\lambda, u(\lambda, \nu))$ is linearly asymptotically stable for all $\lambda \in [\lambda_1, \lambda_2]$ and $\nu \in [0, \nu_0]$.

On the other hand, as soon as $\nu > 0$, we have that

$$\lim_{s \downarrow 0} \frac{\lambda(s) - \sigma_0}{s^{q_1 - 1}} = \lim_{s \downarrow 0} \frac{\nu F(s) + G(s)}{s^{q_1}} \int_\Omega a\varphi_0^{q_1+1} = \nu \lim_{s \downarrow 0} \frac{F(s)}{s^{q_1}} \int_\Omega a\varphi_0^{q_1+1} < 0.$$

Thus, the bifurcation of $(\lambda(s), u(s))$ from $(\lambda, 0)$ is subcritical. Consequently, for every $\nu \in (0, \nu_0)$, (3.1) admits positive solutions at both sides of σ_0, which ends the proof of the theorem. □

Remark 3.9. It should be noted that the hypothesis of Theorem 3.8 can be fulfilled even with polynomials. For example, the choices

$$\Omega = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \qquad \mathcal{L} = -\frac{d^2}{dx^2}, \qquad \Gamma_{\mathcal{R}} = \emptyset,$$

and

$$a(x) = \cos x - 0.9, \qquad F(u) = u^2, \qquad G(u) = u^3,$$

satisfy all its assumptions with $q_1 = 2$ and $q_2 = 3$. Indeed, in this case

$$\sigma_0 = 1, \qquad \varphi_0(x) = \sqrt{\tfrac{2}{\pi}} \cos x,$$

and

$$\lim_{s \to 0} \frac{F(s)}{s|s|} \int_\Omega (\cos x - 0.9) \cos^3 x \, dx = -0.0219028 < 0,$$

$$\lim_{s \to 0} \frac{G(s)}{s|s|^2} \int_\Omega (\cos x - 0.9) \cos^4 x \, dx = 0.00637915 > 0.$$

Therefore, all the requirements of Theorem 3.8 hold.

Remark 3.10. With a little bit more of effort, much like in the proof of Theorem 1.7 of Crandall and Rabinowitz [24], one can also define the auxiliary operator

$$\mathfrak{G}(s,\nu,\lambda,y) := \begin{cases} s^{-1}\mathfrak{F}(\nu,\lambda,s(\varphi_0+y)), & \text{if } s\neq 0,\\ D_u\mathfrak{F}(\nu,\lambda,0)(\varphi_0+y), & \text{if } s=0,\end{cases}$$

and apply the implicit function theorem to it at $(0,0,\sigma_0,0)$ to infer that, actually, the local bifurcation diagram of positive solutions of (3.17) for sufficiently small $\nu>0$ looks like shows Figure 3.2. This is a direct consequence from the uniqueness, uniform in ν, of the smooth curve of positive solutions of (3.17) bifurcating from $(\sigma_0,0)$. Therefore, the example after the statement of Theorem 3.8 also shows the optimality of Theorem 3.7 in the sense that if condition (iii) of Theorem 3.7 fails, then the problem can admit supercritical turning points at linearly neutrally stable positive solutions, like the one shown on the right plot of Figure 3.2.

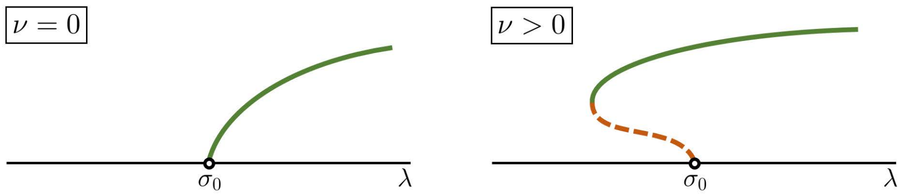


Figure 3.2: Bifurcation diagrams of (3.17) when $\nu=0$ and $\nu>0$.

3.6 Uniqueness of the stable positive solution when $f(u)=u^q$, $q\geq 2$

The next result relies on Theorem 3.7. It is a substantial extension of the previous results of Gómez-Reñasco and López-Gómez [51, 52], for as here we are dealing with general boundary operators of mixed type.

Theorem 3.11. *Suppose that $f(u)=u^q$ for all $u\geq 0$ with $q\geq 2$. Then:*

(i) *Any positive solution, (λ_0,u_0) of (3.1) with $\lambda_0\leq\sigma_0$ must be linearly unstable, as an steady state of (3.7), i.e.,*

$$\sigma_1[\mathcal{L}-\lambda_0+af'(u_0);\mathcal{B},\Omega]=\sigma_1[\mathcal{L}-\lambda_0+qau_0^{q-1};\mathcal{B},\Omega]<0.$$

(ii) *The problem (3.1) admits some linearly stable positive solution, (λ_0,u_0), if, and only if,*

$$\mathcal{S}:=\int_\Omega a(x)\varphi_0^{q+1}(x)\,dx>0. \tag{3.19}$$

Moreover, in such case, $\lambda_0>\sigma_0$.

(iii) *Suppose (3.19) and $\lambda>\sigma_0$. Then, the unique positive linearly stable or linearly neutrally stable solution of (3.1) is the minimal one.*

(iv) *Suppose that* (3.1) *admits a positive solution* (λ_0, u_0) *for some* $\lambda_0 > \sigma_0$. *Then, it admits a minimal solution* $(\lambda_0, u_{\min})$.

Proof. First we will prove Part (i). Let (λ_0, u_0) be a positive solution of (3.1) with $\lambda_0 \leq \sigma_0$. Then, by the uniqueness of the principal eigenvalue, it follows from (3.1) that

$$\lambda_0 = \sigma_1[\mathcal{L} + au_0^{q-1}; \mathcal{B}, \Omega]. \tag{3.20}$$

Thus, when $a \lneq 0$, it follows from the monotonicity of the principal eigenvalue with respect to the potential that

$$\sigma_1[\mathcal{L} + qau_0^{q-1} - \lambda_0; \mathcal{B}, \Omega] < \sigma_1[\mathcal{L} + au_0^{q-1} - \lambda_0; \mathcal{B}, \Omega] = 0.$$

This monotonicity, in the general setting covered in this chapter, was established by Cano-Casanova and López-Gómez [16, Pr. 3.3]. Moreover, in this case, it also follows from (3.20) that

$$\lambda_0 = \sigma_1[\mathcal{L} + au_0^{q-1}; \mathcal{B}, \Omega] < \sigma_1[\mathcal{L}; \mathcal{B}, \Omega] = \sigma_0.$$

Thus, (3.1) cannot admit a positive solution if $\lambda \geq \sigma_0$.

The proof of Part (i) in the general case when $a(x)$ changes of sign is far more subtle than in the special case when $a \lneq 0$. Our proof here is an adaptation of the proof of Theorem 9.9 of López-Gómez [89]. It proceeds by contradiction. Suppose that (3.1) possesses a positive solution, (λ_0, u_0), such that

$$\lambda_0 \leq \sigma_0, \qquad \sigma_1[\mathcal{L} - \lambda_0 + qau_0^{q-1}; \mathcal{B}, \Omega] \geq 0.$$

According to Theorem 3.7, this entails the existence of some positive solution, (λ_1, u_1), of (3.1) such that

$$\lambda_1 \leq \lambda_0 \leq \sigma_0, \qquad \sigma_1[\mathcal{L} - \lambda_1 + qau_1^{q-1}; \mathcal{B}, \Omega] > 0.$$

By the implicit function theorem applied to the operator $\mathfrak{F}(\lambda, u)$ defined in the proof of Theorem 3.2, it becomes apparent that (λ_1, u_1) lies on a smooth curve of positive solutions, $(\lambda, u(\lambda))$, $\lambda \sim \lambda_1$, such that

$$\sigma_1[\mathcal{L} - \lambda + qau^{q-1}(\lambda); \mathcal{B}, \Omega] > 0 \quad \text{for} \ \ \lambda \sim \lambda_1.$$

By global continuation of the curve $(\lambda, u(\lambda))$ for $\lambda < \lambda_1$, one of the following options occurs:

(a) $u(\lambda) \gg 0$ and

$$\sigma_1[\mathcal{L} - \lambda + qau^{q-1}(\lambda); \mathcal{B}, \Omega] > 0 \quad \text{for all} \ \ \lambda < \lambda_1.$$

(b) There exists $\lambda_2 < \lambda_1$ such that $u(\lambda) \gg 0$ and

$$\sigma_1[\mathcal{L} - \lambda + qau^{q-1}(\lambda); \mathcal{B}, \Omega] > 0 \quad \text{for all} \ \ \lambda < \lambda_2,$$

though $u(\lambda_2) = 0$.

(c) There exists $\lambda_3 < \lambda_1$ such that $u(\lambda) \gg 0$ and

$$\sigma_1[\mathcal{L} - \lambda + qau^{q-1}(\lambda); \mathcal{B}, \Omega] > 0 \quad \text{for all} \ \ \lambda < \lambda_3,$$

though $u(\lambda_3) \gg 0$ and

$$\sigma_1[\mathcal{L} - \lambda_3 + qau_1^{q-1}(\lambda_3); \mathcal{B}, \Omega] = 0.$$

The option (b) cannot occur, because $\lambda_2 < \lambda_0$ and $(\lambda_0, 0)$ is the unique bifurcation point from $(\lambda, 0)$ to positive solutions of (3.1). By Theorem 3.7, the option (c) cannot occur neither. Therefore, (a) occurs. By differentiating

$$\mathfrak{F}(\lambda, u(\lambda)) = 0$$

with respect to λ it becomes apparent that

$$\left(\mathcal{L} - \lambda + qau^{q-1}(\lambda)\right) u'(\lambda) = u(\lambda)$$

for all $\lambda \leq \lambda_1$ and hence,

$$u'(\lambda) = \left(\mathcal{L} - \lambda + qau^{q-1}(\lambda)\right)^{-1} u(\lambda) \gg 0.$$

Therefore, the map $\lambda \mapsto u(\lambda)$ is point-wise increasing. In particular, there exists a constant $\tau > 0$ such that

$$\|u(\lambda)\|_{\mathcal{C}(\bar{\Omega})} \leq \tau \quad \text{for all} \ \ \lambda \leq \lambda_1. \tag{3.21}$$

Now, consider the change of variables given by

$$u(\lambda) = |\lambda|^{\frac{1}{q-1}} v(\lambda), \quad \text{for all} \ \ \lambda \leq \lambda_2 := \min\{0, \lambda_1\}.$$

Then, owing to (3.21), we have that

$$(v(\lambda))^{q-1} \leq \frac{\tau^{q-1}}{|\lambda|} \quad \text{for all} \ \ \lambda \leq \lambda_2. \tag{3.22}$$

Moreover, for every $\lambda < \lambda_2$, $v(\lambda)$ is a positive solution of

$$\frac{1}{|\lambda|}\mathcal{L}v = -v - a(x)v|v|^{q-1} \quad \text{in} \ \ \Omega. \tag{3.23}$$

Let us denote by $\varphi_0 \gg 0$ the principal eigenfunction associated with $\sigma_0 = \sigma_1[\mathcal{L}; \mathcal{B}, \Omega]$ normalized so that $\int_\Omega \varphi_0^2 = 1$. Then, multiplying (3.23) by φ_0 and integrating in Ω yields

$$\frac{\sigma_0}{|\lambda|} \int_\Omega \varphi_0 v(\lambda) = -\int_\Omega v(\lambda)\varphi_0 - \int_\Omega a\varphi_0 (v(\lambda))^q.$$

Thus, thanks to (3.22), we find that

$$\left(\frac{\sigma_0}{|\lambda|} + 1\right) \int_\Omega \varphi_0 v(\lambda) \leq \frac{1}{|\lambda|} \|a\|_\infty \tau^{q-1} \int_\Omega \varphi_0 v(\lambda).$$

Hence, for every $\lambda<\lambda_2$,

$$\frac{\sigma_0}{|\lambda|}+1\leq \frac{1}{|\lambda|}\|a(x)\|_\infty \tau^{q-1},$$

which is impossible. This contradiction ends the proof of Part (i).

Part (ii) is an immediate consequence of Proposition 3.3, Theorem 3.6 and Part (i). Indeed, if $\mathcal{S}>0$ then Proposition 3.3 provides us with a supercritical bifurcation, from the trivial branch, of a curve of linearly stable positive solutions, whereas, if $\mathcal{S}\leq 0$, then Theorem 3.6 restricts to $\lambda\in(-\infty,\sigma_0)$ the values of the parameter for which the problem (3.1) admits positive solutions. By Part (i), these solutions are linearly unstable. Therefore, (3.1) cannot admit a linearly stable positive solution if $\mathcal{S}\leq 0$.

To prove the uniqueness in Part (iii), let $\lambda_0>\sigma_0$ be the value of λ for which (3.1) has two linearly stable positive solutions of (3.1), u_0 and v_0, $u_0\neq v_0$. As a consequence of the implicit function theorem, much like in the proof of Part (i), we can get two different curves of solutions, $u_0(\lambda)$ and $v_0(\lambda)$, for every λ in a neighborhood of λ_0. By a rather standard global continuation argument, each of these curves should satisfy some of the alternatives, (a), (b) or (c), as in the proof of Part (i). Moreover, as all these solutions are non-degenerate, $u_0(\lambda)\neq v_0(\lambda)$, as soon as some of these solutions is linearly stable. As a consequence of Part (i), option (a) cannot occur. Similarly, by Theorem 3.7, these curves cannot satisfy the option (c) neither. Therefore, $u_0(\lambda)$ and $v_0(\lambda)$ should bifurcate supercritically from the trivial branch at $(\lambda,u)=(\sigma_0,0)$, which contradicts the local uniqueness at $(\sigma_0,0)$ obtained as an application of Theorem 3.2.

The fact that the minimal positive solution of (3.1) is linearly stable, or linearly neutrally stable, for any $\lambda>\sigma_0$ where it admits a positive solution can be easily inferred by adapting the argument given in the proof of Theorem 9.12 of López-Gómez [89], which was adapted from López-Gómez, Molina-Meyer and Tellini [92] and Amann [3].

The proof of Part (iv) follows similar patterns as the proof of [89, Th. 9.13]. Fix $\lambda=\lambda_0>\sigma_0$. Under this assumption, $\underline{u}_\varepsilon:=\varepsilon\varphi_0$, where $\varphi_0\gg 0$ stands for the normalized principal eigenfunction associated to σ_0, is a subsolution of (3.1) for sufficiently small $\varepsilon>0$. Let w_ε denote the unique solution of

$$\begin{cases} \frac{\partial w}{\partial t}+\mathcal{L}w=\lambda_0 w-a(x)w^q & \text{in } \Omega\times(0,+\infty),\\ \mathcal{B}w=0 & \text{on } \partial\Omega\times(0+\infty),\\ w(\cdot,0)=\underline{u}_\varepsilon & \text{in } \Omega.\end{cases}$$

Since this equation preserves the ordering, we find that

$$w_\varepsilon(t)\leq u_0 \quad \text{in } \Omega \text{ for all } t>0.$$

Moreover, thanks to the abstract theory of Sattinger [109], $w_\varepsilon(t)$ is increasing and globally defined in time. Actually, owing to the main theorem of Langlais and Phillips [74], the limit

$$w_\varepsilon^*:=\lim_{t\to+\infty} w_\varepsilon(t)$$

is well defined and provides us with a positive solution of (3.1) for $\lambda=\lambda_0$. Moreover, by construction,

$$w_{\varepsilon_1}^*\leq w_{\varepsilon_2}^*\leq u_0 \quad \text{if } 0<\varepsilon_1<\varepsilon_2.$$

Thus, since ε can be chosen sufficiently small so that $\underline{u}_\varepsilon$ lies below any given positive solution of (3.1), the limit

$$w^*_{\min} := \min_{\varepsilon>0} w^*_\varepsilon$$

is the minimal positive solution of (3.1), because we are assuming that $\lambda_0 > \sigma_0$ and σ_0 is the unique bifurcation point to positive solutions from $u = 0$. Note that $w^*_{\min}$ stays below any positive solution of (3.1), by construction.

It remains to prove that the minimal positive solution, $w^*_{\min}$, is either linearly stable or neutrally stable. To show it we will argue by contradiction assuming that

$$\sigma_{\min} := \sigma_1[\mathcal{L} - \lambda_0 + a(x)q(w^*_{\min})^{q-1}; \mathcal{B}, \Omega] < 0.$$

Let $\varphi_{\min} \gg 0$ be a positive eigenfunction associated with $\sigma_{\min}$. Then, for sufficiently small $\varepsilon > 0$,

$$\overline{u}_\varepsilon := w^*_{\min} - \varepsilon\varphi_{\min}$$

provides us with a supersolution of (3.1) such that

$$\overline{u}_\varepsilon \ll w^*_{\min}.$$

Indeed, as $\varepsilon \downarrow 0$, we find that

$$\begin{aligned}
\mathcal{L}\overline{u}_\varepsilon &= \lambda_0 w^*_{\min} - a(x)(w^*_{\min})^q - \varepsilon\mathcal{L}\varphi_{\min} \\
&= \lambda_0 w^*_{\min} - a(x)(w^*_{\min})^q - \varepsilon\left(\sigma_{\min}\varphi_{\min} + \lambda_0\varphi_{\min} - a(x)q(w^*_{\min})^{q-1}\varphi_{\min}\right) \\
&= \lambda_0\overline{u}_\varepsilon - a(x)\left((w^*_{\min})^q + \varepsilon q(w^*_{\min})^{q-1}\varphi_{\min}\right) - \varepsilon\sigma_{\min}\varphi_{\min} \\
&= \lambda_0\overline{u}_\varepsilon - a(x)\overline{u}_\varepsilon^q - a(x)\left((w^*_{\min})^q + \varepsilon q(w^*_{\min})^{q-1}\varphi_{\min} - (w^*_{\min} - \varepsilon\varphi_{\min})^q\right) - \varepsilon\sigma_{\min}\varphi_{\min} \\
&= \lambda_0\overline{u}_\varepsilon - a(x)\overline{u}_\varepsilon^q - a(x)o(\varepsilon) - \varepsilon\sigma_{\min}\varphi_{\min}.
\end{aligned}$$

Therefore, since $\sigma_{\min} < 0$, for sufficiently small $\varepsilon > 0$,

$$\mathcal{L}\overline{u}_\varepsilon > \lambda_0\overline{u}_\varepsilon - a(x)\overline{u}_\varepsilon^q \quad \text{in} \;\; \Omega.$$

Since (3.1) admits arbitrarily small positive subsolutions, it becomes apparent that (3.1) possesses a positive solution below $\overline{u}_\varepsilon$ and, hence, below $w^*_{\min}$, which is impossible. This ends the proof. □

3.7 Global structure of the set of stable positive solutions when $f(u) = u^q$, $q \geq 2$

The next result provides us with the global structure of the set of linearly stable and linearly neutrally stable positive solutions of (3.1) in the special case when $f(u) = u^q$ for some $q \geq 2$.

Theorem 3.12. *Suppose that $a(x)$ changes sign in Ω, $f(u) = u^q$, $u \geq 0$, with $q \geq 2$, and that (3.19) holds. Then, the supremum of the set of $\mu > \sigma_0$ for which (3.1) possesses a* *positive solution for each* $\lambda \in (\sigma_0, \mu)$*,* λ_**, satisfies* $\lambda_* \in (\sigma_0, +\infty)$*. Moreover, the set of linearly stable positive solutions of* (3.1) *consists of a* $\mathcal{C}^1$ *strictly increasing curve*

$$\mathfrak{C}_+ := \{(\lambda, u(\lambda)) \; : \; \lambda \in (\sigma_0, \lambda_*)\}.$$

Furthermore, some of the next excluding options occurs:

(i) $\{u(\lambda)\}_{\lambda\in(\sigma_0,\lambda_*)}$ *is bounded in* $\mathcal{C}(\bar{\Omega})$*, and then*

$$u_* := \lim_{\lambda\uparrow\lambda_*} u(\lambda)$$

is a linearly neutrally stable positive solution of (3.1) *at* $\lambda = \lambda_*$*.*

(ii) $\lim_{\lambda\uparrow\lambda_*} \|u(\lambda)\|_{\mathcal{C}(\bar{\Omega})} = +\infty$*.*

In both cases, (3.1) *cannot admit any further positive solution for* $\lambda > \lambda_*$*.*

Proof. The existence of λ_*, as well as the fact that $\lambda_* \in (\sigma_0, +\infty]$, is a direct consequence of Theorem 3.2 and Proposition 3.3. The fact that λ_* is finite follows with the next argument. Let (λ, u) be a positive solution of (3.1). Then,

$$\lambda = \sigma_1[\mathcal{L} + a(x)u^{q-1}; \mathcal{B}, \Omega].$$

Moreover, since $a(x)$ changes sign, there exists a ball, $B \subset \Omega$, such that $a(x) < 0$ for all $x \in B$. Thus, thanks to Corollary 3.6 of Cano-Casanova and López-Gómez [16],

$$\lambda = \sigma_1[\mathcal{L} + a(x)u^{q-1}; \mathcal{B}, \Omega] < \sigma_1[\mathcal{L} + a(x)u^{q-1}; \mathcal{D}, B] < \sigma_1[\mathcal{L}; \mathcal{D}, B].$$

Therefore, $\lambda_* \leq \sigma_1[\mathcal{L}; \mathcal{D}, B]$. In particular, $\lambda_* \in (\sigma_0, \infty)$.

Thanks to Proposition 3.3, the solutions bifurcating from $u = 0$ at σ_0 are linearly stable. Therefore, thanks to implicit function theorem applied to the integral equation associated to (3.1), they consist of a $\mathcal{C}^1$ curve parameterized by λ which is strictly increasing. By a global continuation argument involving the implicit function theorem, this curve can be globally parameterized by λ in the form $(\lambda, u(\lambda))$, with $\lambda \in (\sigma_0, \lambda_{\max})$, for some maximal $\lambda_{\max} \in (\sigma_0, +\infty)$. Necessarily, $\lambda_{\max} \leq \lambda_*$. Moreover, thanks to their linearized stability, $u'(\lambda) \gg 0$ for all $\lambda \in (\sigma_0, \lambda_{\max})$. By construction, the curve

$$\mathfrak{C}_+ := \{(\lambda, u(\lambda)) \; : \; \lambda \in (\sigma_0, \lambda_{\max})\},$$

provides us with the maximal set of linearly stable positive solutions of (3.1) that bifurcates from $u = 0$. By the monotonicity of the solution on this curve, either

$$\lim_{\lambda\uparrow\lambda_{\max}} \|u(\lambda)\|_{\mathcal{C}(\bar{\Omega})} = +\infty,$$

much like illustrated in the right picture of Figure 3.3, or $\{u(\lambda)\}_{\lambda\in(\sigma_0,\lambda_{\max})}$ stays bounded. In the latest case, by a rather standard compactness argument, it is easily seen that

$$u_{\max} := \lim_{\lambda\uparrow\lambda_{\max}} u(\lambda)$$

is a solution of (3.1) for $\lambda = \lambda_{\max}$. By the continuity of the principal eigenvalue with respect to the potential, $u_{\max}$ is either linearly stable or neutrally stable. In the former case, the implicit function theorem would allow us to continue the curve $\mathfrak{C}_+$ beyond $\lambda_{\max}$, which contradicts the maximality of $\lambda_{\max}$. Thus, $u_{\max}$ is neutrally stable and, due to Theorem 3.7, the set of solutions surrounding it consists of a subcritical quadratic turning point, as illustrated by the left picture of Figure 3.3.

Lastly, the proof that $\lambda_{\max} = \lambda_*$ proceeds by contradiction. Suppose $\lambda_{\max} < \lambda_*$. Then, (3.1) admits a positive solution for some $\lambda_1 > \lambda_{\max}$. By Theorem 3.11(iv), (3.1) also admits a minimal positive solution, which is either linearly stable, and hence part of an increasing curve of solutions, or neutrally stable, and hence a subcritical quadratic turning point. By a backwards global continuation argument in λ starting at λ_1, one can construct an analytic curve of linearly stable positive solutions up to reach $(\lambda, u) = (\sigma_0, 0)$, which contradicts the definition of $\lambda_{\max}$ and ends the proof. □

Figure 3.3 shows two admissible global bifurcation diagrams for each of the cases (i) and (ii) discussed by Theorem 3.12. Some general conditions ensuring that the option (i) of Theorem 3.12 occurs can be formulated from the a priori bounds of Amann and López-Gómez [5]. The problem of ascertaining whether or not each of these options can occur has not been solved yet.

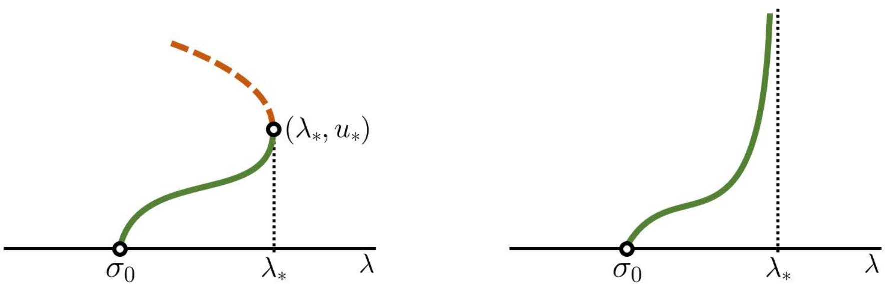


Figure 3.3: Two admissible global bifurcation diagrams of linearly stable positive solutions.

Part II

The Reaction-Diffusion Competition Lotka–Volterra System

Chapter 4

The singular perturbation problem

Introduction

In the second part of this dissertation we consider the Lotka–Volterra competition reaction-diffusion heterogeneous system

$$\begin{cases} \frac{\partial u}{\partial t} + d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega\times(0,+\infty), \\ \frac{\partial v}{\partial t} + d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ u(\cdot,0)=u_0>0, \quad v(\cdot,0)=v_0>0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

as well as its associated elliptic counterpart

$$\begin{cases} d_1\mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega, \\ d_2\mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

whose solutions are the steady states of the evolutionary model (4.1). In this model, Ω is a bounded domain of $\mathbb{R}^N$ with boundary, $\partial\Omega$, of class $\mathcal{C}^2$, and $\mathcal{L}_i$, $i=1,2$, are two uniformly elliptic operators in Ω of the type

$$\mathcal{L}_i = -\mathrm{div}(\mathrm{A_i}\nabla\cdot) + \mathrm{B_i}\nabla + \mathrm{C_i}, \quad \mathrm{i}=1,2, \tag{4.3}$$

with $A_i \in \mathcal{M}_N^{\mathrm{sym}}(\mathcal{C}^2(\bar\Omega))$, $B_i \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar\Omega))$ and $C_i \in \mathcal{C}(\bar\Omega)$. As far as concerns $\partial\Omega$, it is throughout assumed to be a $(N-1)$-dimensional manifold of class $\mathcal{C}^2$ consisting, for each $i\in\{1,2\}$, of finitely many connected components of class $\mathcal{C}^2$

$$\Gamma_{\mathcal{D}}^{i,j}, \quad \Gamma_{\mathcal{R}}^{i,k}, \quad 1\le j\le n_{\mathcal{D}}^i, \quad 1\le k\le n_{\mathcal{R}}^i,$$

for some integers $n_{\mathcal{D}}^i, n_{\mathcal{R}}^i \ge 0$. By the definition of component, they must be disjoint (see, e.g., Munkres [101]) and each of them must be, simultaneously, a relatively open and closed subset of $\partial\Omega$, because $\partial\Omega$ is a compact manifold without boundary. Some, or several, of these components might be empty, of course. We denote by

$$\Gamma_{\mathcal{D}}^i = \bigcup_{j=1}^{n_{\mathrm{D}}^i}\Gamma_{\mathcal{D}}^{i,j}, \quad \Gamma_{\mathcal{R}}^i = \bigcup_{j=1}^{n_{\mathrm{R}}^i}\Gamma_{\mathcal{R}}^{i,j}, \quad i=1,2,$$

the Dirichlet and Robin portions of

$$\partial\Omega = \Gamma^i_{\mathcal{D}} \cup \Gamma^i_{\mathcal{R}}, \quad i = 1, 2.$$

Associated with these decompositions of $\partial\Omega$, there are two boundary operators $\mathcal{B}_i$, $i = 1, 2$, defined by

$$\mathcal{B}_i h = \begin{cases} \mathcal{D}_i h := h & \text{in } \Gamma^i_{\mathcal{D}}, \\ \mathcal{R}_i h := \langle \mathbf{n}, A_i \nabla h \rangle + \beta_i h & \text{in } \Gamma^i_{\mathcal{R}}, \end{cases} \quad \text{for every } h \in W^{2,p}(\Omega), \;\; p > N, \tag{4.4}$$

where $\beta_i \in \mathcal{C}(\partial\Omega)$ and $\mathbf{n}$ stands for the outward normal vector field along $\partial\Omega$. Thus, for each $i = 1, 2$, $\Gamma^i_{\mathcal{D}}$ and $\Gamma^i_{\mathcal{R}}$ are the portions of the edges of the inhabiting territory, $\partial\Omega$, where the corresponding species, u, or v, obeys a boundary condition of Dirichlet ($\mathcal{D}$) or Robin ($\mathcal{R}$) type, respectively. In particular, we may denote $\mathcal{B}_i = \mathcal{D}$ when $\Gamma^i_{\mathcal{D}} = \partial\Omega$.

For the coefficients in the setting of (4.1), d_1, $d_2 > 0$ measure the strength of the diffusivities of the species u and v, λ, $\mu \in \mathcal{C}(\bar{\Omega})$ stand for the growth, or decay, rates of the species, $a, d \in \mathcal{C}(\bar{\Omega}; (0, +\infty))$ are the intra-specific competition rates of u and v, respectively, and $b, c \in \mathcal{C}(\bar{\Omega}; (0, +\infty))$ represent the competition effects between both populations. Subsequently, we are assuming that $\lambda, \mu, a, b, c, d \in \mathcal{C}(\bar{\Omega})$ satisfy

$$b(x) > 0 \;\text{ and }\; c(x) > 0 \;\text{ for all }\; x \in \Omega, \quad \min_{\bar{\Omega}} a > 0, \quad \min_{\bar{\Omega}} d > 0,$$

though in this chapter the hypothesis on b and c can be relaxed to $b, c \geq 0$ in $\bar{\Omega}$.

Throughout this chapter, for any given function $h \in \mathcal{C}(\Omega)$, we denote

$$h_+ := \max\{h, 0\}.$$

It is said that h is positive, $h > 0$ or $h \gneq 0$ (in Ω), if $h \geq 0$ with $h \neq 0$. Also, for any given $h \in \mathcal{C}^1(\bar{\Omega})$, it is said that h is strongly positive (in Ω), $h \gg 0$, if it satisfies

$$h(x) > 0 \quad \text{for all } \; x \in \Omega \quad \text{and} \quad \frac{\partial h}{\partial \mathbf{n}}(x) := \langle \mathbf{n}(x), \nabla h(x) \rangle < 0 \quad \text{for all } \; x \in h^{-1}(0) \cap \partial\Omega.$$

Except for the general existence results of Chapter 7 of López-Gómez [85], most of the available literature on Lotka–Volterra competing species models dealt with the very special cases when either $\Gamma^1_{\mathcal{R}} = \Gamma^2_{\mathcal{R}} = \emptyset$, or the species are subject to non-flux boundary conditions, where $\Gamma^1_{\mathcal{D}} = \Gamma^2_{\mathcal{D}} = \emptyset$ and $\beta_1 = \beta_2 = 0$; in particular, those of Blat and Brown [12, 13], Dancer [29], Eilbeck, Furter and López-Gómez [33], López-Gómez [80, 81], López-Gómez and Sabina [77], Hutson, López-Gómez, Mischaikow and Vickers [63], Furter and López-Gómez [48], Dockery, Hutson, Mischaikow and Pernarowski [32], Hutson, Lou and Mischaikow [65], Cantrell and Cosner [20], He and Ni [55, 54], as well as [42, 39] and most of the references therein. Consequently, as the results of this thesis are valid for general boundary conditions of mixed type, our findings are substantially more general than all previous existing results.

As in most of the applications to Ecology, Environmental Sciences, Biology and Medical Sciences, the diffusion rates of the species, measured by d_1 and d_2, are very small in comparison with the relative size of the remaining coefficients involved in the setting of the model, our attention in the second part of the thesis is mainly focused into the problem of characterizing the dynamics of (4.1) for sufficiently small d_1 and d_2, which is an important mathematical challenge, as we are dealing with a singular perturbation problem for a parabolic system in the presence of spatial heterogeneities. The reader should be aware that no previous singular perturbation result under mixed boundary conditions is available, even for the single diffusive logistic equation!

Much like in [48, 42, 39], also under general mixed boundary conditions the dynamics of (4.1) for small diffusion rates is based on the nature of the dynamics of the associated non-spatial model

$$\begin{cases} u'(t) = \lambda(x)u(t) - a(x)u^2(t) - b(x)u(t)v(t) \\ v'(t) = \mu(x)v(t) - d(x)v^2(t) - c(x)u(t)v(t) \end{cases} \quad t > 0, \tag{4.5}$$

where $x \in \Omega$ is regarded as a parameter, though our new results here provide with some new findings that are extremely important from the point of view of the applications, as it will become apparent soon. Adopting the methodology of Furter and López-Gómez [48] and, according to the nature of the dynamics of (4.5), the inhabiting territory, $\bar{\Omega}$, consists of the following regions:

$$\begin{aligned}
&\Omega_{\rm ext} := \{x \in \bar{\Omega} \;:\; \lambda(x),\ \mu(x) \le 0\},\\
&\Omega_{\rm per} := \{x \in \bar{\Omega} \;:\; \lambda(x),\ \mu(x) > 0,\ \ \lambda(x)d(x) > \mu(x)b(x),\ \ \mu(x)a(x) > \lambda(x)c(x)\},\\
&\Omega_{\rm bi} := \{x \in \bar{\Omega} \;:\; \lambda(x),\ \mu(x) > 0,\ \ \lambda(x)d(x) < \mu(x)b(x),\ \ \mu(x)a(x) < \lambda(x)c(x)\},\\
&\Omega^u_{\rm do} := \{x \in \bar{\Omega} \;:\; \lambda(x) > 0,\ \ \lambda(x)d(x) > \mu(x)b(x),\ \ \mu(x)a(x) < \lambda(x)c(x)\},\\
&\Omega^v_{\rm do} := \{x \in \bar{\Omega} \;:\; \mu(x) > 0,\ \ \lambda(x)d(x) < \mu(x)b(x),\ \ \mu(x)a(x) > \lambda(x)c(x)\},\\
&\Omega_{\rm junk} := \bar{\Omega} \setminus (\Omega_{\rm ext} \cup \Omega_{\rm per} \cup \Omega_{\rm bi} \cup \Omega^u_{\rm do} \cup \Omega^v_{\rm do}).
\end{aligned} \tag{4.6}$$

As already suggested by the names given to each of these zones, $\Omega_{\rm ext}$ consists of the set of $x \in \bar{\Omega}$ where $(0, 0)$ is a global attractor with respect to the positive solutions of (4.5); $\Omega_{\rm per}$ stands for the set of $x \in \bar{\Omega}$ where the semi-trivial positive steady-state solutions, $(\frac{\lambda(x)}{a(x)}, 0)$ and $(0, \frac{\mu(x)}{d(x)})$, are linearly unstable — and so, the model (4.5) is permanent —, $\Omega_{\rm bi}$ consists of the set of $x \in \bar{\Omega}$ where $(\frac{\lambda(x)}{a(x)}, 0)$ and $(0, \frac{\mu(x)}{d(x)})$ are linearly stable, where (4.5) exhibits a genuine founder control competition, and the portions $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$ stand for the zones of $\bar{\Omega}$ where one of the semi-trivial steady states is positive and linearly stable, while the other one is non-positive, or it is positive but linearly unstable. Should it be the case, the linearly stable positive semi-trivial solution is a global attractor for the component-wise positive solutions of (4.5). Finally, we are denoting by $\Omega_{\rm junk}$ the supplement in $\bar{\Omega}$ of the union of the previous regions. It is folklore that in $\Omega_{\rm per}$ the non-spatial model possesses a unique coexistence steady state which is a global attractor for the component-wise positive solutions of (4.5), whereas in $\Omega_{\rm bi}$ there is a unique coexistence state which is a saddle point, whose stable manifold, linking $(0, 0)$ to the coexistence state, divides the first quadrant, $u > 0$, $v > 0$, in two regions, each of them being the attraction source of one of the semi-trivial positive solutions.

The main result of this chapter is a substantial generalization of Theorem 4.1 of Hutson, López-Gómez, Mischaikow and Vickers [63] that provides us with a sharp relation between the dynamics of (4.1) and the dynamics of the associated kinetic problem (4.5). As [63, Th. 4.1] had shown to be a milestone for the generation of new results in the theory of competing species in the presence of spatial heterogeneities, our result should deserve a huge amount of attention over the next years. In terms of attractivity our main result, precisely stated in Theorem 4.4, establishes that the coexistence steady-state solutions of (4.1) (component-wise positive steady states) approximate, for small diffusion rates, $d_1 > 0$ and $d_2 > 0$, the global attractor of the non-spatial model (4.5) in the regions of Ω where such global attractor exists, i.e., in particular, Ω_{ext}, Ω_{per}, Ω^u_{do} and Ω^v_{do}.

Our main result in this chapter, Theorem 4.4, also sharpens the main theorem of Hutson, Lou and Mischaikow [64] as well as Theorem 4.2 (iii) of He and Ni [53], where the following (very degenerate) problem, introduced by Hutson, López-Gómez, Mischaikow and Vickers in [63], which later generated a huge literature in the field, was analyzed

$$\begin{cases} \frac{\partial u}{\partial t} - d_1\Delta u = \lambda(x)u - u^2 - uv & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial v}{\partial t} - d_2\Delta v = \mu(x)v - uv - v^2 & \text{in } \Omega \times (0,+\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial\Omega \times (0,+\infty), \\ u(\cdot,0) = u_0 > 0, \quad v(\cdot,0) = v_0 > 0, & \text{in } \Omega. \end{cases}$$

All these precursors of our main theorem, imposed some very strong structural conditions on each of the regions, Ω_{per}, Ω^u_{do}, Ω^v_{do}, Ω_{bi}, Ω_{ext} and Ω_{junk}. Indeed, while Hutson, López-Gómez, Mischaikow and Vickers [63] and He and Ni [53] required that

$$\Omega_{\text{per}} = \emptyset = \Omega_{\text{bi}},$$

the singular perturbation theorem of Hutson, Lou and Mischaikow [64] imposed the very strong restriction that either $\bar{\Omega} = \Omega_{\text{per}}$, or $\bar{\Omega} = \Omega^u_{\text{do}}$, or $\bar{\Omega} = \Omega^v_{\text{do}}$, all of them extremely severe, as they do not allow us to deal with truly spatially heterogeneous landscapes as those considered by Theorem 4.4, where each of the (dynamical) patches Ω_{per}, Ω^u_{do}, Ω^v_{do}, Ω_{bi}, Ω_{ext} and Ω_{junk} can exhibit an arbitrary structure and, in particular, can be either empty, or non-empty; not requiring any additional restriction for applying our result.

Theorem 4.4 cannot be improved up to provide us with the asymptotic limit of the coexistence states as diffusion rates tend to zero in the bi-stability region Ω_{bi}, because (4.1) may possess three coexistence states for sufficiently small diffusion rates if $\Omega_{\text{bi}} \neq \emptyset$: two of them might perturb from the stable semitrivial solutions and the third one from the unstable coexistence state of the non-spatial model, as it is shown in the next chapter, in Section 5.4. As a bi-product, the limit of an arbitrary family of coexistence steady states of (4.1) when the diffusion rates, $d_1, d_2 > 0$, go to zero cannot be uniquely determined in the bi-stability region Ω_{bi}. Therefore, Theorem 4.1 is optimal, in the sense that it cannot admit any further substantial generalization, though it might be generalized to consider arbitrary kinetics of competitive type, of course!

Although the proof of Theorem 4.4 relies upon the monotone scheme introduced by Hutson, López-Gómez, Mischaikow and Vickers [63], which later inspired most of the existing singular perturbation results in competing species models (see, e.g., Hutson, Lou and Mischaikow [64, 65], Hutson, Lou, Mischaikow and Polácik [66], and He and Ni [53]), the proof given here is substantially sharper than all previous existing ones. It requires an extremely deep analysis of the monotone scheme introduced in [63], and a sharp study of the limiting profile of the solution of the logistic equation to deal with the mixed boundary conditions.

Monotone scheme techniques in the context of reaction diffusion equations go back, at least, to the influential works of Amann [2] and Sattinger [108] and have shown to be extremely useful in studying a huge variety of diffusive Lotka–Volterra systems of predator-prey type (see, e.g., López-Gómez and Pardo [94]), or competition type (see, e.g., López-Gómez and Sabina [77], Hutson, López-Gómez, Mischaikow and Vickers [63], as well as the references therein), or symbiotic type (see, e.g., Lam and Lou [73]), as well as the seminal work of Molina-Meyer [100], where some rather pioneering results within the same vain as those of Lam and Lou [73] where found through the theorem of characterization of López-Gómez and Molina-Meyer [90], later sharpened by Amann and López-Gómez [5] and Amann [4]).

This chapter is structured as follows. Section 4.1 is devoted to ascertain the limiting profile of a family of the solution of the diffusive logistic equation, under general mixed boundary conditions of non-classical type, when the coefficients also depend on the diffusion rate. The result provided herein relies on the theory developed in Chapter 2. Section 4.2 uses the singular perturbation result of Section 4.1 to derive a general singular perturbation theorem for the elliptic system (4.2) as $(d_1, d_2) \to (0,0)$ from the monotone scheme introduced by Hutson, López-Gómez, Mischaikow and Vickers in [63], later refined by the author in [42, 39]. Our singular perturbation result is substantially sharper than the previous ones because it is valid for general mixed boundary conditions and general differential operators in divergence form. All the previous ones were given for the $-\Delta$ in both equations under Dirichlet or Neumann boundary conditions, and some of them, like those of Hutson, López-Gómez, Mischaikow and Vickers [63, Th. 4.1], or He and Ni [53, Th. 4.2(iii)], with constant competition rates.

4.1 Singular perturbation results for a family of logistic equations

This section focuses attention in the problem of ascertaining the limiting profile (as $\delta \downarrow 0$) of the maximal non-negative solution of the semilinear elliptic boundary value problem

$$\begin{cases} \delta \mathcal{L} u = \gamma(x) u - m(x) u^2 & \text{in } \Omega, \\ \mathcal{B} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

where $\gamma, m \in \mathcal{C}(\bar{\Omega})$, $\min_{\bar{\Omega}} m > 0$, $\mathcal{L}$ is a uniformly elliptic differential operator of the type (4.3), and $\mathcal{B}$ is a boundary operator of mixed non-classical type, much like the one defined in (4.4). Note that (4.7) is equation (2.1) for the choice

$$h(u, x) := \gamma(x) - m(x) u$$

and $d := \delta$. Moreover, this nonlinearity satisfies hypothesis (H3) for all $d > 0$ and thus, the results developed in Chapter 2 hold for all $\delta > 0$. Note that according to (2.22), and Remark 2.19,

$$\Theta_h = \frac{\gamma_+}{m}$$

is the unique non-negative linearly stable, or neutrally stable, steady state of the associated kinetic model

$$\begin{cases} u'(t) = \gamma(x)u(t) - m(x)u^2(t) & t \in [0, +\infty), \\ u(0) = u_0 > 0. \end{cases}$$

On the other hand, following Theorem 2.15, we denote by $\theta_{\{\delta,\gamma,m\}}$ the maximal non-negative solution of (4.7). Note that

$$\theta_{\{\delta,\gamma,m\}} = 0 \quad \text{if} \;\; \sigma_1[\delta\mathcal{L} - \gamma; \mathcal{B}, \Omega] \geq 0,$$

and that

$$\theta_{\{\delta,\gamma,m\}} \gg 0 \quad \text{if} \;\; \sigma_1[\delta\mathcal{L} - \gamma; \mathcal{B}, \Omega] < 0.$$

Concerning the perturbation problem, the next result is a immediate consequence of Theorem 2.21. Here, much like in Section 2.5, we will denote by $\Gamma_{\mathcal{R}}^{+}$ the union of the set of components of $\Gamma_{\mathcal{R}}$ where the function γ is everywhere positive. Note that

$$\gamma_+^{-1}(0) = \{x \in \bar{\Omega} : \gamma_+(x) = 0\}.$$

Theorem 4.1. *Let K be a compact subset of $\Omega \cup \Gamma_{\mathcal{R}}^{+} \cup \gamma_+^{-1}(0)$. Then,*

$$\lim_{\delta \downarrow 0} \theta_{\{\delta,\gamma,m\}} = \frac{\gamma_+}{m} \quad \textit{uniformly in } \; K,$$

where $\theta_{\{\delta,\gamma,m\}}$ stands for the maximal non-negative solution of (4.7).

Note that this theorem is a substantial generalization of Theorem 3.5 of Furter and López-Gómez [48] and of Theorem 3.3 of [39], which were established for the very special case when $\mathcal{L} = -\eta$ and $\Gamma_{\mathcal{R}} = \emptyset$, as well as of Lemma 2.5 of Hutson, López-Gómez, Mischaikow and Vickers [63], which was found for the very special case when $\mathcal{L} = -\Delta$, $\Gamma_{\mathcal{D}} = \emptyset$ and $\beta \equiv 0$ on $\partial\Omega$. Astonishingly, Theorem 4.1 seems to be the first singular perturbation result for semilinear elliptic equations under mixed boundary conditions. It is optimal from two different points of view. First, because the boundary conditions are completely general; in particular, substantially more general than the ones of Nakashima, Ni and Su [102]. Secondly, because in general the convergence cannot be expected to be uniform on $\bar{\Omega}$, as $\mathcal{B}$ might be of Dirichlet type on some, or several, of the components of $\partial\Omega$, where the positive solution must develop boundary layers. The available techniques do not work out to deal with our more general setting. Actually, our proofs are based on some rather sophisticated technical devices developed from Lemma 2.1 and Theorem 1.9 of López-Gómez [88], though the overall proof relies on a clever use of the method of sub and supersolutions, like in the available, less general, results. Naturally, from a technical point of view, it is much more intricate constructing these sub and supersolutions under arbitrary mixed boundary conditions.

The main aim of this section is to provide with the limiting profile of the maximal non-negative solution of (4.7) as $\delta \downarrow 0$ in the general case when the coefficients of the model also depend on the diffusion, δ, in a controlled way. In particular, we consider the problem

$$\begin{cases} \delta\mathcal{L}u = \gamma_{(\delta,\eta)}(x)u - m_{(\delta,\eta)}(x)u^2 & \text{in } \; \Omega, \\ \mathcal{B}u = 0 & \text{on } \; \partial\Omega, \end{cases}$$

for $\delta, \eta > 0$ decreasing to zero. The next results provide us with some variants of Theorem 4.1 for 'floating' coefficients, depending on δ. The second one, is a key ingredient in the proof of the singular perturbation result for the system.

Theorem 4.2. *Consider $\gamma, m \in \mathcal{C}(\bar{\Omega})$, $J \subset (0,+\infty)^2$ with $(0,0) \in \bar{J}$, and families $\{\gamma_{(\delta,\eta)}\}_{(\delta,\eta)\in J}$, $\{m_{(\delta,\eta)}\}_{(\delta,\eta)\in J} \subset \mathcal{C}(\bar{\Omega})$ such that*

$$\lim_{(\delta,\eta)\to(0,0)} \gamma_{(\delta,\eta)} = \gamma \quad \text{and} \quad \lim_{(\delta,\eta)\to(0,0)} m_{(\delta,\eta)} = m \quad \text{uniformly in } \bar{\Omega}. \tag{4.8}$$

Then,

$$\lim_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} = \frac{\gamma_+}{m}$$

uniformly on compact subsets of $\Omega \cup \Gamma^+_{\mathcal{R}} \cup (\gamma_+)^{-1}(0)$.

Proof. Let $\varepsilon > 0$ be such that $\varepsilon < \min_{\bar{\Omega}} m$. Then, by (4.8), $\delta_\varepsilon > 0$ exists such that

$$\gamma - \varepsilon \leq \gamma_{(\delta,\eta)} \leq \gamma + \varepsilon \quad \text{and} \quad 0 < m - \varepsilon \leq m_{(\delta,\eta)} \leq m + \varepsilon \quad \text{in } \Omega$$

for all $\delta, \eta < \delta_\varepsilon$, $(\delta,\eta) \in J$. Thanks to Lemma 2.18, for such range of δ and η we have that

$$\theta_{\{\delta,\gamma-\varepsilon,m+\varepsilon\}} \leq \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \theta_{\{\delta,\gamma+\varepsilon,m-\varepsilon\}} \quad \text{in } \Omega. \tag{4.9}$$

Let K be a compact subset of $\Omega \cup \Gamma^+_{\mathcal{R}} \cup \gamma_+^{-1}(0)$. By Theorem 4.1, letting $(\delta,\eta) \in J$ approximate $(0,0)$ in the restriction of the estimate (4.9) to K yields

$$\frac{(\gamma-\varepsilon)_+}{m+\varepsilon} \leq \varliminf_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \varlimsup_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} = \frac{(\gamma+\varepsilon)_+}{m-\varepsilon}$$

uniformly in K. Letting $\varepsilon \to 0$ ends the proof. □

Essentially, the next result sharpens Theorem 4.2 by relaxing the uniform convergence in $\bar{\Omega}$ to a uniform convergence on compact subsets of $\Omega \cup \Gamma^+_{\mathcal{R}} \cup \gamma_+^{-1}(0)$.

Theorem 4.3. *Consider $\gamma, m \in \mathcal{C}(\bar{\Omega})$, $J \subset (0,+\infty)^2$ with $(0,0) \in \bar{J}$, and*

$$\mathcal{O} \subset \Omega \cup \Gamma^+_{\mathcal{R}} \cup \gamma_+^{-1}(0),$$

an open subset, with respect to the induced topology, such that either $\bar{\mathcal{O}} \cap \Gamma^+_{\mathcal{R}} = \emptyset$, or $\bar{\mathcal{O}} \cap \Gamma^+_{\mathcal{R}}$ consists of components of $\Gamma^+_{\mathcal{R}}$, each one contained in either $\mathcal{O}$ or $\mathbb{R}^N \setminus \mathcal{O}$. Let

$$\{\gamma_{(\delta,\eta)}\}_{(\delta,\eta)\in J} \subset \mathcal{C}(\bar{\Omega}) \quad \text{and} \quad \{m_{(\delta,\eta)}\}_{(\delta,\eta)\in J} \subset \mathcal{C}(\bar{\Omega})$$

be such that

$$\lim_{(\delta,\eta)\to(0,0)} \gamma_{(\delta,\eta)} = \gamma \quad \text{and} \quad \lim_{(\delta,\eta)\to(0,0)} m_{(\delta,\eta)} = m \tag{4.10}$$

uniformly on compact subsets of $\mathcal{O}$. Assume that there exists $k > 0$ and $M > 0$ such that

$$m_{(\delta,\eta)}(x) \geq k \quad \text{and} \quad \frac{\gamma_{(\delta,\eta)}(x)}{m_{(\delta,\eta)}(x)} \leq M \quad \text{for all } (\delta,\eta) \in J, \;\; x \in \bar{\Omega}. \tag{4.11}$$

Then,

$$\lim_{(\delta,\eta)\to(0,0)} \theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} = \frac{\gamma_+}{m}$$

uniformly on compact subsets of $\mathcal{O}$*. Moreover, for every* $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that, for every* $(\delta,\eta) \in J$ *with* $\delta,\eta < \delta_\varepsilon$*,*

$$\theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq M + \varepsilon \quad \textit{in} \quad \Omega.$$

The assumption (4.11) was unnecessary in the statement of Theorem 4.2 as it is a direct consequence of (4.8).

Proof. To prove the convergence, we will obtain first the upper limit. Fix $\varepsilon > 0$ and consider a compact subset, K, of $\mathcal{O}$. Subsequently, for each $r > 0$, we will denote by

$$K_r := \{x \in \bar{\Omega} \ : \ \mathrm{dist}(x, K) \leq r\}$$

the compact r-neighborhood of K. By construction, for sufficiently small $r > 0$,

$$K \subset K_r \subset \mathcal{O}.$$

According to (4.10), the quotients $\frac{\gamma_{(\delta,\eta)}}{m_{(\delta,\eta)}}$ converge uniformly to $\frac{\gamma}{m}$ in K_r as $(\delta,\eta) \to (0,0)$, $(\delta,\eta) \in J$. Thus, there exists $\delta_{\varepsilon,1} > 0$ such that

$$\frac{\gamma_{(\delta,\eta)}}{m_{(\delta,\eta)}} \leq \frac{\gamma}{m} + \frac{\varepsilon}{2} \leq \frac{\gamma_+}{m} + \frac{\varepsilon}{2} \quad \text{in} \;\; K_r \;\; \text{for} \;\; \delta,\eta < \delta_{\varepsilon,1}, \; (\delta,\eta) \in J. \tag{4.12}$$

On the other hand, by Urysohn's Lemma, a function $\xi \in \mathcal{C}(\bar{\Omega})$ exists such that $\xi(x) \in [0,1]$ for all $x \in \bar{\Omega}$, $\xi = 0$ in K, and $\xi = 1$ on $\bar{\Omega} \setminus K_r$. Thanks to (4.10) and (4.11), we have that $\frac{\gamma}{m} \leq M$ in K_r. Thus,

$$\frac{\gamma_+}{m} \leq M \quad \text{in} \;\; K_r,$$

because $M > 0$ and so,

$$\frac{\gamma_+}{m} = 0 \leq M \quad \text{if} \;\; \gamma < 0.$$

Hence, for the previous choice of ξ, the auxiliary non-negative function

$$\psi := \frac{\gamma_+}{m}(1-\xi) + M\xi + \frac{\varepsilon}{2} \in \mathcal{C}(\bar{\Omega})$$

satisfies

$$\psi \leq M(1-\xi) + M\xi + \frac{\varepsilon}{2} = M + \frac{\varepsilon}{2} \quad \text{in} \;\; \bar{\Omega} \quad \text{and} \quad \psi \begin{cases} = \frac{\gamma_+}{m} + \frac{\varepsilon}{2} & \text{in} \;\; K, \\ \geq \frac{\gamma_+}{m} + \frac{\varepsilon}{2} & \text{in} \;\; K_r \setminus K, \\ = M + \frac{\varepsilon}{2} & \text{in} \;\; \bar{\Omega} \setminus K_r. \end{cases}$$

Consequently, by (4.11) and (4.12), we find that

$$\psi \geq \frac{\gamma_{(\delta,\eta)}}{m_{(\delta,\eta)}} \quad \text{in} \;\; \Omega \;\; \text{for} \;\; \delta,\eta < \delta_{\varepsilon,1}, \; (\delta,\eta) \in J. \tag{4.13}$$

By Lemma 2.9(a) and Remark 2.10, there exists $\Psi \in \mathcal{C}^2(\bar{\Omega})$ such that $\mathcal{B}\Psi \geq 0$ on $\partial\Omega$ and

$$\psi + \frac{\varepsilon}{4} \leq \Psi \leq \psi + \frac{\varepsilon}{2} \quad \text{in} \;\; \bar{\Omega}.$$

Since $\psi > 0$, we have that $\Psi \geq \frac{\varepsilon}{4} > 0$, and, owing to (4.13),

$$\Psi \leq \frac{\gamma_+}{m} + \varepsilon \;\; \text{in} \;\; K, \qquad \text{and} \qquad \Psi \geq \frac{\gamma_{(\delta,\eta)}}{m_{(\delta,\eta)}} + \frac{\varepsilon}{4} \;\; \text{in} \;\; \Omega, \tag{4.14}$$

for every $\delta, \eta < \delta_{\varepsilon,1}$ with $(\delta,\eta) \in J$. If necessary, $\delta_{\varepsilon,1} > 0$ can be shortened so that

$$\delta_{\varepsilon,1} < \left(\frac{\varepsilon}{4}\right)^2 \frac{k}{\|\mathcal{L}\Psi\|_\infty}.$$

Then, for every $\delta, \eta < \delta_{\varepsilon,1}$, $(\delta,\eta) \in J$, taking into account (4.14) and the hypothesis $m_{(\delta,\eta)} \geq k$ in $\bar{\Omega}$, we obtain that

$$\begin{aligned} \gamma_{(\delta,\eta)}\Psi - m_{(\delta,\eta)}\Psi^2 = m_{(\delta,\eta)}\Psi \left(\frac{\gamma_{(\delta,\eta)}}{m_{(\delta,\eta)}} - \Psi\right) &\leq -k\Psi\frac{\varepsilon}{4} \\ &\leq -k\left(\frac{\varepsilon}{4}\right)^2 < -\delta\|\mathcal{L}\Psi\|_\infty \leq \delta\mathcal{L}\Psi \quad \text{in} \;\; \Omega. \end{aligned}$$

Therefore, for such range of values of the parameters, Ψ is a strict supersolution of

$$\begin{cases} \delta\mathcal{L}u = \gamma_{(\delta,\eta)}u - m_{(\delta,\eta)}u^2 & \text{in} \;\; \Omega, \\ \mathcal{B}u = 0 & \text{on} \;\; \partial\Omega, \end{cases}$$

and hence, by the maximum principle,

$$\theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \Psi \quad \text{in} \;\; \bar{\Omega}$$

for all $\delta, \eta < \delta_{\varepsilon,1}$ with $(\delta,\eta) \in J$. Consequently, thanks to (4.14), we also have that

$$\theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \frac{\gamma_+}{m} + \varepsilon \quad \text{in} \;\; K$$

and

$$\theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \psi + \frac{\varepsilon}{2} \leq M + \varepsilon \quad \text{in} \;\; \bar{\Omega},$$

for these values of the parameters, which provides us with the upper estimate and the global bound of the last assertion of the theorem. Note that, since

$$\theta_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \geq 0 \geq \frac{\gamma_+}{m} - \varepsilon \quad \text{in} \;\; K \cap \left(\frac{\gamma_+}{m}\right)^{-1}[0,\varepsilon],$$

the lower estimate of the theorem holds in $K \cap (\frac{\gamma_+}{m})^{-1}[0,\varepsilon]$ for every $\delta, \eta < \delta_{\varepsilon,1}$ with $(\delta,\eta) \in \cap J$. So, it remains to get this estimate on the compact set

$$K_0 := K \cap (\frac{\gamma_+}{m})^{-1}\Big[\frac{\varepsilon}{2}, +\infty\Big).$$

To apply the available comparison results in a regular open neighborhood of K_0, $\mathcal{O}_0$, with $\bar{\mathcal{O}}_0 \subset \mathcal{O}$, a little bit more of technical work is needed. Since

$$K \subset \mathcal{O} \subset \Omega \cup \Gamma^+_{\mathcal{R}} \cup \gamma_+^{-1}(0),$$

it becomes apparent that

$$K_0 \subset \mathcal{O} \setminus \gamma_+^{-1}(0) \subset \Omega \cup \Gamma^+_{\mathcal{R}}$$

and hence, for sufficiently small $r > 0$, the open set

$$\mathcal{O}_r := \{x \in \mathcal{O} \,:\, \mathrm{dist}\left(x, \partial\mathcal{O} \setminus (\mathcal{O} \cap \Gamma^+_{\mathcal{R}})\right) > r\} \setminus (\mathcal{O} \cap \Gamma^+_{\mathcal{R}})$$

satisfies

$$K_0 \subset \mathcal{O}_r \cup (\mathcal{O} \cap \Gamma^+_{\mathcal{R}})$$

and $\partial\mathcal{O}_r$ consists of two types of components, $\mathcal{O} \cap \Gamma^+_{\mathcal{R}}$ and $\partial\mathcal{O}_r \cap \Omega$. Note that if $\partial\mathcal{O} \subset \mathcal{O} \cap \Gamma^+_{\mathcal{R}}$, then $\partial\mathcal{O} \setminus (\mathcal{O} \cap \Gamma^+_{\mathcal{R}}) = \emptyset$ and hence, $\mathcal{O}_r = \mathcal{O} \setminus \partial\mathcal{O}$ is an open set of class $\mathcal{C}^2$. Should this be the case, then we can define $\mathcal{O}_0 := \mathcal{O} \setminus \partial\mathcal{O}$. Otherwise, let $n \geq 1$ denote the number of components of $\partial\mathcal{O}_r \setminus (\mathcal{O} \cap \Gamma^+_{\mathcal{R}})$. Note that, for sufficiently small $r > 0$, $\partial\mathcal{O}_r$ and $\partial\mathcal{O}$ have the same number of components. In such case, $\mathcal{O}_{r/2} \setminus \mathcal{O}_r$ consists of n (open connected) components, $\mathfrak{O}_j$, $1 \leq j \leq n$. Next, for every $j \in \{1, ..., n\}$, let $\mathfrak{M}_j$ be any $(N-1)$-dimensional compact manifold without boundary of class $\mathcal{C}^\infty$ such that

$$\mathfrak{M}_j \subset \mathfrak{O}_j, \qquad \mathcal{O}_r \subset \mathrm{int}\,\mathfrak{M}_j, \qquad \mathbb{R}^N \setminus \mathcal{O}_{r/2} \subset \mathrm{ext}\,\mathfrak{M}_j,$$

where $\mathrm{int}\,\mathfrak{M}_j$ and $\mathrm{ext}\,\mathfrak{M}_j$ stand for the two components of $\mathbb{R}^N \setminus \mathfrak{M}_j$. Lastly, in this case, we consider

$$\mathcal{O}_0 := \cap_{j=1}^n \mathrm{int}\,\mathfrak{M}_j \cap \mathcal{O}_{r/2}.$$

Then, $\mathcal{O}_0$ is an open subset of $\mathbb{R}^N$ of class $\mathcal{C}^2$ with

$$K_0 \subset \mathcal{O}_0 \cup (\mathcal{O} \cap \Gamma^+_{\mathcal{R}})$$

and $\partial\mathcal{O}_0$ consists of two types of components. Namely, those of $\mathcal{O} \cap \Gamma^+_{\mathcal{R}}$ and those of

$$\partial\mathcal{O}_0 \cap \Omega = \cup_{j=1}^n \mathfrak{M}_j \subset \mathcal{O}_{r/2}.$$

Thus, $\bar{\mathcal{O}}_0$ is a compact subset of $\mathcal{O}$ and hence, due to (4.10), $\gamma_{(\delta,\eta)}$ and $m_{(\delta,\eta)}$ converge uniformly to γ and m, respectively, in $\bar{\mathcal{O}}_0$ as $(\delta, \eta) \to (0, 0)$ in J. Consequently, applying Theorem 4.2 in $\mathcal{O}_0$ provides us with a $\delta_{\varepsilon,2} > 0$, $\delta_{\varepsilon,2} \leq \delta_{\varepsilon,1}$, such that, for every $\delta, \eta < \delta_{\varepsilon,2}$ with $(\delta, \eta) \in J$,

$$\theta^{\mathcal{B}_0,\mathcal{O}_0}_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \geq \frac{\gamma_+}{m} - \varepsilon \qquad \text{in } \; K_0,$$

where $\mathcal{B}_0$ stands for the boundary operator on $\partial\mathcal{O}_0$ defined by $\mathcal{B}_0 := \mathcal{B}$ on $\mathcal{O} \cap \Gamma^+_{\mathcal{R}}$ and by $\mathcal{B}_0 := \mathcal{D}$ on $\partial\mathcal{O}_0 \cap \Omega$. On the other hand, owing to Lemma 2.18,

$$\theta^{\mathcal{B}_0,\mathcal{O}_0}_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \leq \theta^{\mathcal{B},\Omega}_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \qquad \text{in } \; \mathcal{O}_0 \; \text{ for all } \; (\delta, \eta) \in J.$$

Therefore, for this range of values of the parameters,

$$\theta^{\mathcal{B},\Omega}_{\{\delta,\gamma_{(\delta,\eta)},m_{(\delta,\eta)}\}} \geq \frac{\gamma_+}{m} - \varepsilon \qquad \text{in } \; K_0,$$

which ends the proof. □

4.2 A singular perturbation theorem for competing species models

The main result of this section establishes that, given any family of coexistence states of (4.2), they must approximate as $d_1, d_2 \to 0$ the global hyperbolic attractor of the non-spatial model on every patch of the inhabiting territory where it exists. As in this section we are dealing with general boundary conditions of mixed type, its main result is completely new in its greatest generality. Since the proof uses the singular perturbation results established in Section 4.1 for the scalar equation, which go back to Section 2.5, it is far from immediate. Actually, it is rather elaborated.

Besides the regions Ω_{ext}, Ω_{per}, Ω_{bi}, Ω^u_{do}, Ω^v_{do} and Ω_{junk} already defined in (4.6), in order to state the main result of this section it is imperative to differentiate some important areas within Ω_{junk}. Precisely, we will denote by $\Omega^{\text{per},u}_{\text{junk}}$ the set of points for which the non-spatial model can be perturbed to exhibit either permanence or dominance of the species u, i.e.,

$$\Omega^{\text{per},u}_{\text{junk}} := \{x \in \bar{\Omega} \ : \ \lambda(x) > 0, \ \ \lambda(x)d(x) > \mu(x)b(x), \ \ \mu(x)a(x) = \lambda(x)c(x)\}.$$

By symmetry, we also define

$$\Omega^{\text{per},v}_{\text{junk}} := \{x \in \bar{\Omega} \ : \ \mu(x) > 0, \ \ \lambda(x)d(x) = \mu(x)b(x), \ \ \mu(x)a(x) > \lambda(x)c(x)\}.$$

Finally, the remaining part of Ω_{junk} is added to Ω_{bi} by considering

$$\begin{aligned}\Omega^{\star}_{\text{bi}} &:= \Omega_{\text{bi}} \cup \Omega_{\text{junk}} \setminus \left(\Omega^{\text{per},u}_{\text{junk}} \cup \Omega^{\text{per},v}_{\text{junk}}\right) = \bar{\Omega} \setminus \left(\Omega_{\text{ext}} \cup \Omega_{\text{per}} \cup \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}} \cup \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}}\right) \\ &= \{x \in \bar{\Omega} \ : \ \lambda(x), \mu(x) > 0, \ \ \lambda(x)d(x) \leq \mu(x)b(x), \ \ \mu(x)a(x) \leq \lambda(x)c(x)\},\end{aligned}$$

which consists of the set of values of $\bar{\Omega} \setminus \Omega_{\text{ext}}$ such that the non-spatial model can be perturbed to exhibit founder control competition. Using these notations, it is easily seen that, for every

$$x \in \Omega_{\max} := \bar{\Omega} \setminus \Omega^{\star}_{\text{bi}} = \Omega_{\text{ext}} \cup \Omega_{\text{per}} \cup \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}} \cup \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}},$$

the non-spatial model possesses a steady state which is a global attractor with respect to the component-wise positive solutions. It is worth-emphasizing that such a steady state might not be of hyperbolic type, and that, for every $x \in \Omega_{\max}$, is given by

$$(u_*(x), v_*(x)) = \begin{cases} (0,0) & \text{if } \ x \in \Omega_{\text{ext}}, \\ \left(\frac{\lambda(x)d(x)-\mu(x)b(x)}{a(x)d(x)-b(x)c(x)}, \frac{\mu(x)a(x)-\lambda(x)c(x)}{a(x)d(x)-b(x)c(x)}\right) & \text{if } \ x \in \Omega_{\text{per}}, \\ \left(\frac{\lambda(x)}{a(x)}, 0\right) & \text{if } \ x \in \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}}, \\ \left(0, \frac{\mu(x)}{d(x)}\right) & \text{if } \ x \in \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}}. \end{cases}$$

In agreement with the notations introduced in Section 2.5 and Section 4.1, we will denote by $\Gamma^{1,+}_{\mathcal{R}}$ the union of the components of $\Gamma^1_{\mathcal{R}}$ where $\lambda > 0$ everywhere, while $\Gamma^{2,+}_{\mathcal{R}}$ stands for the union of the components of $\Gamma^2_{\mathcal{R}}$ such that $\mu > 0$ in the whole component. Similarly, we will denote by $\Gamma_{\mathcal{R}}^{\mathrm{per}}$ the union of the components of $\Gamma_{\mathcal{R}}^1 \cap \Gamma_{\mathcal{R}}^2$ for which the non-spatial model exhibits permanence everywhere. In particular,

$$\Gamma_{\mathcal{R}}^{\mathrm{per}} \subset \Gamma_{\mathcal{R}}^{1,+} \cap \Gamma_{\mathcal{R}}^{2,+} \cap \Omega_{\mathrm{per}}.$$

We are ready to state the main result of this section.

Theorem 4.4. *Consider a family of coexistence states of* (4.2), $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}_{(d_1,d_2)\in J}$, *with* $J \subset (0,+\infty)^2$ *such that* $(0,0) \in \bar{J}$. *Then,*

$$\lim_{(d_1,d_2)\to(0,0)} (u_{(d_1,d_2)}, v_{(d_1,d_2)}) = (u_*, v_*)$$

uniformly on compact subsets of

$$\Omega_{\max} \cap (\Omega \cup \Gamma_{\mathcal{R}}^{\mathrm{per}}) = (\Omega \setminus \Omega_{\mathrm{bi}}^{\star}) \cup \Gamma_{\mathcal{R}}^{\mathrm{per}}.$$

In particular, if $\Omega_{\mathrm{per}} = \bar{\Omega}$ *and* $\Gamma_{\mathcal{D}}^1 = \Gamma_{\mathcal{D}}^2 = \emptyset$, *i.e.,* $\Gamma_{\mathcal{R}}^{\mathrm{per}} = \partial\Omega$, *then*

$$\lim_{(d_1,d_2)\to(0,0)} (u_{(d_1,d_2)}, v_{(d_1,d_2)}) = (u_*, v_*) \quad \textit{uniformly in } \bar{\Omega}.$$

The proof of this result follows from a series of technical lemmas. The monotone scheme introduced by the next result goes back to Hutson, López-Gómez, Mischaikow and Vickers [63], thought it had been previously introduced by López-Gómez and Sabina [77] in another context.

Lemma 4.5. *Fix* $d_1, d_2 > 0$ *and consider the families*

$$\{\bar{u}_{(d_1,d_2,n)}\}_{n\geq 0}, \quad \{\underline{u}_{(d_1,d_2,n)}\}_{n\geq 0}, \quad \{\bar{v}_{(d_1,d_2,n)}\}_{n\geq 0} \quad \textit{and} \quad \{\underline{v}_{(d_1,d_2,n)}\}_{n\geq 0},$$

defined recursively by

$$\begin{cases} \underline{v}_{(d_1,d_2,0)} := 0, & \underline{v}_{(d_1,d_2,n)} := \theta_{\{d_2,\mu-c\bar{u}_{(d_1,d_2,n-1)},d\}}, \\ \bar{u}_{(d_1,d_2,0)} := \theta_{\{d_1,\lambda,a\}}, & \bar{u}_{(d_1,d_2,n)} := \theta_{\{d_1,\lambda-b\underline{v}_{(d_1,d_2,n)},a\}}, \end{cases} \quad n \geq 1,$$

and

$$\begin{cases} \underline{u}_{(d_1,d_2,0)} := 0, & \underline{u}_{(d_1,d_2,n)} := \theta_{\{d_1,\lambda-b\bar{v}_{(d_1,d_2,n-1)},a\}}, \\ \bar{v}_{(d_1,d_2,0)} := \theta_{\{d_2,\mu,d\}}, & \bar{v}_{(d_1,d_2,n)} := \theta_{\{d_2,\mu-c\underline{u}_{(d_1,d_2,n)},d\}}, \end{cases} \quad n \geq 1.$$

Then,

$$\begin{matrix} \underline{u}_{(d_1,d_2,n)} \leq \underline{u}_{(d_1,d_2,n+1)} \leq \bar{u}_{(d_1,d_2,n+1)} \leq \bar{u}_{(d_1,d_2,n)} \\ \underline{v}_{(d_1,d_2,n)} \leq \underline{v}_{(d_1,d_2,n+1)} \leq \bar{v}_{(d_1,d_2,n+1)} \leq \bar{v}_{(d_1,d_2,n)} \end{matrix} \quad \textit{for every } n \geq 0.$$

Moreover, if the elliptic model (4.2) *admits a coexistence state,* $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$, *then*

$$\underline{u}_{(d_1,d_2,n)} \leq u_{(d_1,d_2)} \leq \bar{u}_{(d_1,d_2,n)} \quad \textit{and} \quad \underline{v}_{(d_1,d_2,n)} \leq v_{(d_1,d_2)} \leq \bar{v}_{(d_1,d_2,n)} \quad \textit{for all } n \geq 0.$$

Proof. We will only prove the estimates for the first component, u, as those of v follow the same patterns. By Lemma 2.18, the maps

$$\mathcal{C}(\bar{\Omega}) \ni h \mapsto \theta_{\{d_1,\lambda-bh,a\}} \quad \text{and} \quad \mathcal{C}(\bar{\Omega}) \ni h \mapsto \theta_{\{d_2,\mu-ch,d\}}$$

are non-increasing. Thus,

$$\begin{aligned}\mathcal{F}_{(d_1,d_2)} : \mathcal{C}(\bar{\Omega}) &\to \mathcal{C}(\bar{\Omega})\\ h &\mapsto \theta_{\{d_1,\lambda-b\theta_{\{d_2,\mu-ch,d\}},a\}}\end{aligned}$$

is a non-decreasing, or order preserving, operator such that

$$\underline{u}_{(d_1,d_2,0)} = 0 \leq \mathcal{F}_{(d_1,d_2)}[h] \leq \theta_{\{d_1,\lambda,a\}} = \bar{u}_{(d_1,d_2,0)} \quad \text{for all} \;\; h \in \mathcal{C}(\bar{\Omega}). \tag{4.15}$$

The last estimate follows from Lemma 2.18, while the first one holds by definition. Hence, for any $n \geq 1$, applying n times the operator $\mathcal{F}_{(d_1,d_2)}$ produces

$$\begin{aligned}0 &\leq \mathcal{F}^n_{(d_1,d_2)}[\underline{u}_{(d_1,d_2,0)}] \leq \mathcal{F}^{n+1}_{(d_1,d_2)}[\underline{u}_{(d_1,d_2,0)}]\\ &\leq \mathcal{F}^{n+1}_{(d_1,d_2)}[\bar{u}_{(d_1,d_2,0)}] \leq \mathcal{F}^{n}_{(d_1,d_2)}[\bar{u}_{(d_1,d_2,0)}] \leq \bar{u}_{(d_1,d_2,0)}.\end{aligned}$$

On the other hand, the iterates $\{\bar{u}_{(d_1,d_2,n)}\}_{n\geq 0}$ and $\{\underline{u}_{(d_1,d_2,n)}\}_{n\geq 0}$ can be recursively defined in terms of $\mathcal{F}_{(d_1,d_2)}$ by

$$\begin{aligned}\underline{u}_{(d_1,d_2,0)} &= 0, & \underline{u}_{(d_1,d_2,n)} &= \mathcal{F}_{(d_1,d_2)}[\underline{u}_{(d_1,d_2,n-1)}], \quad n \geq 1,\\ \bar{u}_{(d_1,d_2,0)} &= \theta_{\{d_1,\lambda,a\}}, & \bar{u}_{(d_1,d_2,n)} &= \mathcal{F}_{(d_1,d_2)}[\bar{u}_{(d_1,d_2,n-1)}], \quad n \geq 1.\end{aligned}$$

The last assertion follows from (4.15) taking into account that $u_{(d_1,d_2)}$ is a fixed point of $\mathcal{F}_{(d_1,d_2)}$. This ends the proof. □

Lemma 4.6. *Fix $J \subset (0,+\infty)^2$ with $(0,0) \in \bar{J}$. For every $(d_1,d_2) \in J$ let*

$$\{\bar{u}_{(d_1,d_2,n)}\}_{n\geq 1}, \quad \{\underline{u}_{(d_1,d_2,n)}\}_{n\geq 1}, \quad \{\bar{v}_{(d_1,d_2,n)}\}_{n\geq 1} \quad \text{and} \quad \{\underline{v}_{(d_1,d_2,n)}\}_{n\geq 1}$$

be the sequences introduced in Lemma 4.5. Then, the sequences

$$\{\bar{U}_n\}_{n\geq 1}, \quad \{\underline{U}_n\}_{n\geq 1}, \quad \{\bar{V}_n\}_{n\geq 1}, \quad \{\underline{V}_n\}_{n\geq 1} \subset \mathcal{C}(\bar{\Omega}),$$

defined by

$$\begin{aligned}\underline{U}_0 &:= 0, \quad \bar{U}_0 := \frac{\lambda_+}{a}, \quad \underline{U}_n := \frac{1}{a}\left(\lambda - \frac{b}{d}(\mu - c\bar{U}_{n-1})_+\right)_+, \quad n \geq 1,\\ \underline{V}_0 &:= 0, \quad \bar{V}_0 := \frac{\mu_+}{d}, \quad \underline{V}_n := \frac{1}{d}\left(\mu - \frac{c}{a}(\lambda - b\bar{V}_{n-1})_+\right)_+, \quad n \geq 1,\end{aligned}$$

satisfy

$$\lim_{(d_1,d_2)\to(0,0)} (\bar{u}_{(d_1,d_2,n)}, \underline{u}_{(d_1,d_2,n)}, \bar{v}_{(d_1,d_2,n)}, \underline{v}_{(d_1,d_2,n)}) = (\bar{U}_n, \underline{U}_n, \bar{V}_n, \underline{V}_n)$$

uniformly on compact subsets of $\Omega \cup \Gamma^{\text{per}}_{\mathcal{R}}$.

Proof. We will restrict ourselves to prove the assertions for $\bar{u}_{(d_1,d_2,n)}$ by induction. The limiting behaviors of $\underline{u}_{(d_1,d_2,n)}$, $\bar{v}_{(d_1,d_2,n)}$ and $\underline{v}_{(d_1,d_2,n)}$ can be obtained similarly. By definition,

$$\Gamma^{\text{per}}_{\mathcal{R}} \subset \Gamma^{1,+}_{\mathcal{R}} \cap \Gamma^{2,+}_{\mathcal{R}}$$

and

$$\lambda(x)d(x)-\mu(x)b(x)>0 \quad \text{and} \quad \mu(x)a(x)-\lambda(x)c(x)>0 \quad \text{for all } \ x\in\Gamma_{\mathcal{R}}^{\mathrm{per}}.$$

Moreover,

$$0\le \bar{U}_n\le \frac{\lambda_+}{a} \quad \text{in } \ \bar{\Omega} \ \text{ for all } \ n\ge 0.$$

Thus, since $\lambda(x)>0$ for each $x\in\Gamma_{\mathcal{R}}^{\mathrm{per}}$, we find that, for every $n\ge 0$,

$$\mu(x)-c(x)\bar{U}_n(x)\ge \mu(x)-c(x)\frac{\lambda(x)}{a(x)}>0 \quad \text{for all } \ x\in\Gamma_{\mathcal{R}}^{\mathrm{per}}, \tag{4.16}$$

and hence,

$$\lambda(x)-\frac{b(x)}{d(x)}\big(\mu(x)-c(x)\bar{U}_n(x)\big)_+\ge \lambda(x)-\frac{b(x)}{d(x)}\mu(x)>0 \quad \text{for all } \ x\in\Gamma_{\mathcal{R}}^{\mathrm{per}}. \tag{4.17}$$

In the case $n=0$, according to Theorem 4.1,

$$\lim_{(d_1,d_2)\to(0,0)}\bar{u}_{(d_1,d_2,0)}=\lim_{(d_1,d_2)\to(0,0)}\theta_{\{d_1,\lambda,a\}}=\frac{\lambda_+}{a}=\bar{U}_0$$

uniformly on compact subsets of $\Omega\cup\Gamma_{\mathcal{R}}^{1,+}\cup\lambda_+^{-1}(0)$. In particular, this limit holds on compact subsets of $\Omega\cup\Gamma_{\mathcal{R}}^{\mathrm{per}}$. So, we are done in this case. As induction hypothesis, assume that, for some $n\ge 0$,

$$\lim_{(d_1,d_2)\to(0,0)}\bar{u}_{(d_1,d_2,n)}=\bar{U}_n \quad \text{uniformly on compact subsets of } \ \Omega\cup\Gamma_{\mathcal{R}}^{\mathrm{per}}. \tag{4.18}$$

Then, by (4.16), $\Omega\cup\Gamma_{\mathcal{R}}^{\mathrm{per}}$ is an open subset of

$$\Omega\cup\Gamma_{\mathcal{R}}^{2,*}\cup(\mu-c\bar{U}_n)_+^{-1}(0),$$

where $\Gamma_{\mathcal{R}}^{2,*}$ stands for the union of the components of $\Gamma_{\mathcal{R}}^2$ such that $\mu-c\bar{U}_n>0$ everywhere. Note that $\Gamma_{\mathcal{R}}^{\mathrm{per}}$ consists of finitely many components of $\Gamma_{\mathcal{R}}^{2,*}$. Moreover, by definition,

$$\bar{u}_{(d_1,d_2,n)}\ge 0 \quad \text{for all } \ (d_1,d_2)\in J.$$

Thus,

$$\mu-c\bar{u}_{(d_1,d_2,n)}\le \max_{\bar{\Omega}}\mu \quad \text{in } \ \bar{\Omega}$$

and hence, it follows from (4.18) and Theorem 4.3 that

$$\lim_{(d_1,d_2)\to(0,0)}\underline{v}_{(d_1,d_2,n+1)}=\lim_{(d_1,d_2)\to(0,0)}\theta_{\{d_2,\mu-c\bar{u}_{(d_1,d_2,n)},d\}}=\frac{1}{d}(\mu-c\bar{U}_n)_+$$

uniformly on compact subsets of $\Omega\cup\Gamma_{\mathcal{R}}^{\mathrm{per}}$. Similarly, thanks to (4.17), $\Omega\cup\Gamma_{\mathcal{R}}^{\mathrm{per}}$ is an open subset of

$$\Omega\cup\Gamma_{\mathcal{R}}^{1,*}\cup\Big(\lambda-\frac{b}{d}(\mu-c\bar{U}_n)_+\Big)_+^{-1}(0),$$

where $\Gamma_{\mathcal{R}}^{1,*}$ stands for the union of the components of $\Gamma_{\mathcal{R}}^{1}$ such that

$$\lambda - \frac{b}{d}\left(\mu - c\bar{U}_n\right)_+ > 0$$

everywhere. As above, $\Gamma_{\mathcal{R}}^{\mathrm{per}}$ consists of finitely many components of $\Gamma_{\mathcal{R}}^{1,*}$. Since

$$\lambda - b\underline{v}_{(d_1,d_2,n+1)} \le \max_{\bar{\Omega}} \lambda \quad \text{in} \quad \bar{\Omega}, \qquad (d_1,d_2) \in J,$$

it follows from Theorem 4.3 that

$$\begin{aligned}\lim_{(d_1,d_2)\to(0,0)} \bar{u}_{(d_1,d_2,n+1)} &= \lim_{(d_1,d_2)\to(0,0)} \theta_{\{d_1,\lambda-b\underline{v}_{(d_1,d_2,n+1)},a\}} \\ &= \frac{1}{a}\left(\lambda - \frac{b}{d}(\mu - c\bar{U}_n)_+\right)_+ = \bar{U}_{n+1}\end{aligned}$$

uniformly on compact subsets of $\Omega \cup \Gamma_{\mathcal{R}}^{\mathrm{per}}$. This ends the proof. □

The next result is Lemma 11 of [42].

Lemma 4.7. *Setting*

$$S_n := \sum_{j=0}^{n} \left(\frac{bc}{ad}\right)^j \quad \textit{in} \quad \Omega, \quad \textit{for every} \quad n \ge 0,$$

the continuous functions defined in Lemma 4.6 are given by

$$\bar{U}_n(x) = \begin{cases} 0 & \textit{if} \;\; \lambda(x) \le 0, \\ \frac{\lambda(x)}{a(x)} & \textit{if} \;\; \lambda(x) > 0 \;\; \textit{and} \;\; \mu(x) \le \frac{c(x)}{a(x)}\lambda(x), \\ \frac{1}{a(x)}\left(\lambda(x)S_n(x) - \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x)\right) & \textit{if} \;\; \lambda(x) > 0, \;\; \mu(x) > \frac{c(x)}{a(x)}\lambda(x) \;\; \textit{and} \\ & \quad \lambda(x)S_n(x) > \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x), \\ 0 & \textit{if} \;\; \lambda(x) > 0, \;\; \mu(x) > \frac{c(x)}{a(x)}\lambda(x) \;\; \textit{and} \\ & \quad \lambda(x)S_n(x) \le \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x), \end{cases}$$

$$\underline{U}_{n+1}(x) = \begin{cases} \frac{\lambda_+(x)}{a(x)} & \textit{if} \;\; \mu(x) \le 0, \\ 0 & \textit{if} \;\; \mu(x) > 0 \;\; \textit{and} \;\; \lambda(x) \le \frac{b(x)}{d(x)}\mu(x), \\ \frac{1}{a(x)}\left(\lambda(x) - \frac{b(x)}{d(x)}\mu(x)\right)S_n(x) & \textit{if} \;\; \mu(x) > 0, \;\; \lambda(x) > \frac{b(x)}{d(x)}\mu(x) \;\; \textit{and} \\ & \quad \mu(x)S_n(x) > \frac{c(x)}{a(x)}\lambda(x)S_{n-1}(x), \\ \frac{\lambda(x)}{a(x)} & \textit{if} \;\; \mu(x) > 0, \;\; \lambda(x) > \frac{b(x)}{d(x)}\mu(x) \;\; \textit{and} \\ & \quad \mu(x)S_n(x) \le \frac{c(x)}{a(x)}\lambda(x)S_{n-1}(x), \end{cases}$$

$$\bar{V}_n(x) = \begin{cases} 0 & \text{if } \mu(x) \le 0, \\ \frac{\mu(x)}{d(x)} & \text{if } \mu(x) > 0 \text{ and } \lambda(x) \le \frac{b(x)}{d(x)}\mu(x), \\ \frac{1}{d(x)}\Big(\mu(x)S_n(x) - \frac{c(x)}{a(x)}\lambda(x)S_{n-1}(x)\Big) & \text{if } \mu(x) > 0,\ \lambda(x) > \frac{b(x)}{d(x)}\mu(x) \text{ and} \\ & \quad \mu(x)S_n(x) > \frac{c(x)}{a(x)}\lambda(x)S_{n-1}(x), \\ 0 & \text{if } \mu(x) > 0,\ \lambda(x) > \frac{b(x)}{d(x)}\mu(x) \text{ and} \\ & \quad \mu(x)S_n(x) \le \frac{c(x)}{a(x)}\lambda(x)S_{n-1}(x), \end{cases}$$

$$\underline{V}_{n+1}(x) = \begin{cases} \frac{\mu_+(x)}{d(x)} & \text{if } \lambda(x) \le 0, \\ 0 & \text{if } \lambda(x) > 0 \text{ and } \mu \le \frac{c(x)}{a(x)}\lambda(x), \\ \frac{1}{d(x)}\Big(\mu(x) - \frac{c(x)}{a(x)}\lambda(x)\Big)S_n(x) & \text{if } \lambda(x) > 0,\ \mu(x) > \frac{c(x)}{a(x)}\lambda(x) \text{ and} \\ & \quad \lambda(x)S_n(x) > \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x), \\ \frac{\mu(x)}{d(x)} & \text{if } \lambda(x) > 0,\ \mu(x) > \frac{c(x)}{a(x)}\lambda(x) \text{ and} \\ & \quad \lambda(x)S_n(x) \le \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x). \end{cases}$$

for every $x \in \bar{\Omega}$ and $n \ge 1$.

By the monotone character of these sequences of iterates, they must have a point-wise limit. The next result characterizes it.

Lemma 4.8. *For every $x \in \bar{\Omega}$,the sequence $(\bar{U}_n(x), \underline{V}_n(x))$ converges to*

$$(\bar{U}_*(x), \underline{V}_*(x)) := \begin{cases} (0,0) & \text{if } x \in \Omega_{\text{ext}}, \\ \Big(0, \frac{\mu(x)}{d(x)}\Big) & \text{if } x \in \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}}, \\ \Big(\frac{\lambda(x)}{a(x)}, 0\Big) & \text{if } x \in \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}} \cup \Omega^\star_{\text{bi}}, \\ \Big(\frac{\lambda(x)d(x)-\mu(x)b(x)}{a(x)d(x)-b(x)c(x)}, \frac{\mu(x)a(x)-\lambda(x)c(x)}{a(x)d(x)-b(x)c(x)}\Big) & \text{if } x \in \Omega_{\text{per}}, \end{cases}$$

whereas the sequence $(\underline{U}_n(x), \bar{V}_n(x))$ converges to

$$(\underline{U}_*(x), \bar{V}_*(x)) := \begin{cases} (0,0) & \text{if } x \in \Omega_{\text{ext}} \\ \Big(0, \frac{\mu(x)}{d(x)}\Big) & \text{if } x \in \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}} \cup \Omega^\star_{\text{bi}}, \\ \Big(\frac{\lambda(x)}{a(x)}, 0\Big) & \text{if } x \in \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}}, \\ \Big(\frac{\lambda(x)d(x)-\mu(x)b(x)}{a(x)d(x)-b(x)c(x)}, \frac{\mu(x)a(x)-\lambda(x)c(x)}{a(x)d(x)-b(x)c(x)}\Big) & \text{if } x \in \Omega_{\text{per}}, \end{cases}$$

Moreover, these limits are uniform on any compact subset of $\Omega_{\max} = \bar{\Omega} \setminus \Omega^\star_{\text{bi}}$.

Proof. Fix $x \in \bar{\Omega}$. Note that $S_n(x) = 1$ for all $n \ge 1$ if $b(x)c(x) = 0$. If $b(x)c(x) > 0$, then $S_n(x)$ is increasing and

$$\lim_{n\to\infty} S_n(x) = \begin{cases} +\infty & \text{if } a(x)d(x) \le b(x)c(x), \\ \frac{a(x)d(x)}{a(x)d(x)-b(x)c(x)} & \text{if } a(x)d(x) > b(x)c(x). \end{cases}$$

Moreover,

$$S_n = \frac{bc}{ad} S_{n-1} + 1, \quad \text{for all } \ n \geq 1,$$

and hence, $\frac{S_n(x)}{S_{n-1}(x)}$ is decreasing, with

$$\lim_{n\to\infty} \frac{S_n(x)}{S_{n-1}(x)} = \max\left\{1, \frac{b(x)c(x)}{a(x)d(x)}\right\} = \begin{cases} \frac{b(x)c(x)}{a(x)d(x)} & \text{if } \ a(x)d(x) \leq b(x)c(x), \\ 1 & \text{if } \ a(x)d(x) > b(x)c(x). \end{cases} \tag{4.19}$$

To ascertain the limiting profile of $(\bar{U}_n, \underline{V}_n)$ as $n \to \infty$, we will differentiate several different cases. First, suppose that $x \in \Omega_{\text{ext}}$. Then, $\lambda(x), \mu(x) \leq 0$ and hence, by Lemma 4.7,

$$(\bar{U}_n(x), \underline{V}_n(x)) = (0,0) \quad \text{for every } \ n \geq 2.$$

Now, suppose that $x \in \Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}}$. Then,

$$\mu(x) > 0, \quad \lambda(x)d(x) \leq \mu(x)b(x), \quad \text{and} \quad \mu(x)a(x) > \lambda(x)c(x).$$

When, in addition, $\lambda(x) \leq 0$, then the first rows of the developments given by Lemma 4.7 provide us with

$$(\bar{U}_n(x), \underline{V}_n(x)) = \left(0, \frac{\mu}{d}\right), \quad \text{for every } \ n \geq 2.$$

So, assume that $\lambda(x) > 0$. If

$$\lambda(x)d(x) = \mu(x)b(x),$$

then

$$a(x)d(x) = a(x)\frac{b(x)\mu(x)}{\lambda(x)} > b(x)c(x)$$

and hence, S_n/S_{n-1} decreases towards 1, or equals 1, and consequently,

$$b(x)\frac{\mu(x)}{d(x)} = \lambda(x) \leq \frac{S_n(x)}{S_{n-1}(x)}\lambda(x) \quad \text{for all } \ n \geq 1.$$

Thus, by Lemma 4.7,

$$\bar{U}_n(x) = \frac{1}{a(x)}\left(\lambda(x)S_n(x) - \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x)\right)$$

and

$$\underline{V}_n(x) = \frac{1}{d(x)}\left(\mu(x) - \frac{c(x)}{a(x)}\lambda(x)\right) S_{n-1}(x)$$

for all $n \geq 2$ and therefore,

$$\lim_{n\to\infty} (\bar{U}_n(x), \underline{V}_n(x)) = \left(\frac{\lambda(x)d(x) - \mu(x)b(x)}{a(x)d(x) - b(x)c(x)}, \frac{\mu(x)a(x) - \lambda(x)c(x)}{a(x)d(x) - b(x)c(x)}\right) = \left(0, \frac{\mu(x)}{d(x)}\right).$$

In the case when

$$\lambda(x)d(x) < \mu(x)b(x),$$

since

$$\lambda(x)c(x) < \mu(x)a(x),$$

we have that

$$\lambda(x) < \frac{b(x)}{d(x)}\mu(x) \quad \text{and} \quad \frac{b(x)c(x)}{a(x)d(x)}\lambda(x) < \frac{b(x)a(x)}{a(x)d(x)}\mu(x) = \frac{b(x)}{d(x)}\mu(x).$$

Thus,

$$\max\left\{1, \frac{b(x)c(x)}{a(x)d(x)}\right\}\lambda(x) < \frac{b(x)}{d(x)}\mu(x).$$

Therefore, thanks to (4.19), there exists $n_0 \geq 1$ such that

$$\lambda(x) \leq \max\left\{1, \frac{b(x)c(x)}{a(x)d(x)}\right\}\lambda(x) \leq \lambda(x)\frac{S_n(x)}{S_{n-1}(x)} < \frac{b(x)}{d(x)}\mu(x) \quad \text{for all } \; n \geq n_0.$$

Consequently, owing to Lemma 4.7, it becomes apparent that

$$(\bar{U}_n(x), \underline{V}_n(x)) = \left(0, \frac{\mu(x)}{d(x)}\right) \quad \text{for all } \; n > n_0.$$

If $x \in \Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}} \cup \Omega^\star_{\text{bi}}$, then

$$\lambda(x) > 0 \quad \text{and} \quad \mu(x)a(x) \leq \lambda(x)c(x),$$

and so, thanks to the second line of the developments in the statement of Lemma 4.7,

$$(\bar{U}_n(x), \underline{V}_n(x)) = \left(\frac{\lambda(x)}{a(x)}, 0\right) \quad \text{for all } \; n \geq 2.$$

Finally, suppose that $x \in \Omega_{\text{per}}$. Then,

$$a(x)d(x) > b(x)c(x)$$

and

$$\lambda(x),\ \mu(x) > 0, \quad \lambda(x)d(x) > \mu(x)b(x), \quad \mu(x)a(x) > \lambda(x)c(x).$$

Thus,

$$b(x)\frac{\mu(x)}{d(x)} < \lambda(x) \leq \frac{S_n(x)}{S_{n-1}(x)}\lambda(x) \quad \text{for all } \; n \geq 1.$$

Therefore, by the third row of the developments of Lemma 4.7, it is apparent that

$$\bar{U}_n(x) = \frac{1}{a(x)}\left(\lambda(x)S_n(x) - \frac{b(x)}{d(x)}\mu(x)S_{n-1}(x)\right)$$

and

$$\underline{V}_n(x) = \frac{1}{d(x)}\left(\mu(x) - \frac{c(x)}{a(x)}\lambda(x)\right)S_{n-1}(x).$$

for all $n \geq 2$. Letting $n \to \infty$ in this identity, provides us with $(\bar{U}_*, \underline{V}_*)$. The uniform convergence is an easy consequence from Dini's criterion, because the point-wise limits, $\bar{U}_*, \underline{V}_*$ are continuous in $\Omega_{\max}$. The convergence of $(\underline{U}_n, \bar{V}_n)$ follows similarly. The proof is complete. □

We are ready to prove the singular perturbation result for the system.

Proof of Theorem 4.4. We will only obtain the limit of the first component, as the limit of the second one follows similarly. Thanks to Lemmas 4.5 and 4.6, for every $n \geq 0$, we have that

$$\begin{aligned} \underline{U}_n = \lim_{(d_1,d_2)\to(0,0)} \underline{u}_{(d_1,d_2,n)} &\leq \liminf_{(d_1,d_2)\to(0,0)} u_{(d_1,d_2)} \\ &\leq \limsup_{(d_1,d_2)\to(0,0)} u_{(d_1,d_2)} \leq \lim_{(d_1,d_2)\to(0,0)} \bar{u}_{(d_1,d_2,n)} = \bar{U}_n \end{aligned}$$

uniformly on compact subsets $\Omega \cup \Gamma_{\mathcal{R}}^{\mathrm{per}}$. Hence, by Lemma 4.8, letting $n \to \infty$ yields

$$\underline{U}_* \leq \liminf_{(d_1,d_2)\to(0,0)} u_{(d_1,d_2)} \leq \limsup_{(d_1,d_2)\to(0,0)} u_{(d_1,d_2)} \leq \bar{U}_*$$

uniformly on compact subsets of $\Omega_{\max} \cap \big(\Omega \cup \Gamma_{\mathcal{R}}^{\mathrm{per}}\big)$. Since $\underline{U}_* = \bar{U}_*$, this ends the proof. □

Chapter 5

The Induced Instability Principle. Some consequences

Introduction

The main goal of this chapter is establishing a new principle in the theory of competing species models, which may be formulated as follows:

> Induced Instability Principle (IIP): As soon as any steady-state solution of the non-spatial model (4.5) is linearly unstable somewhere in Ω, any steady state of the spatial model (4.1) perturbing uniformly from it therein — as (d_1, d_2) moves away from $(0,0)$ — must be linearly unstable with respect to the associated parabolic model (4.1).

Therefore, any localized instability of a non-spatial steady-state solution has a global effect on the dynamics of the spatial parabolic model (4.1) for sufficiently small $d_1 > 0$ and $d_2 > 0$. This result has some astonishing consequences. For example, when $\Omega_{\mathrm{per}} \neq \emptyset$, then the non-spatial semi-trivial solutions $(\frac{\lambda(x)}{a(x)}, 0)$ and $(0, \frac{\mu(x)}{d(x)})$ are linearly unstable for all $x \in \Omega_{\mathrm{per}}$. Thus, according to the IIP, rigorously established by Theorem 5.9 and Proposition 5.11, each of the semi-trivial positive solutions of the spatial model, which perturb from those of (4.5) by Theorem 4.1, must be linearly unstable with respect to (4.1) for sufficiently small d_1 and d_2. Therefore, (4.1) must be permanent, which provides us with Theorem 2.1 of Furter and López-Gómez [48] and Corollary 4.6 of [39]. The most intriguing feature of this fact is that Ω_{per} might be $B_\varepsilon(x_0)$, for some $x_0 \in \Omega$, ε being the inverse of the universe radius measured in Angstroms, while simultaneously $\Omega_{\mathrm{bi}} = \bar{\Omega} \setminus \bar{\Omega}_{\mathrm{per}}$! Under this special patch configuration, the smaller ε the smaller the values of d_1 and d_2 are so that (4.1) can be permanent.

Naturally, the same conclusion holds as soon as Ω^u_{do} and Ω^v_{do} are non-empty. Indeed, if $\Omega^u_{\mathrm{do}} \neq \emptyset$, then, for every $x \in \Omega^u_{\mathrm{do}}$ with $\mu(x) > 0$, the semi-trivial positive solution $(0, \frac{\mu(x)}{d(x)})$ must be unstable for all $x \in \Omega^u_{\mathrm{do}}$. Thus, according to Proposition 5.11, the semi-trivial positive steady-state solution of (4.1) perturbing from it, $(0, v)$, must be linearly unstable. By symmetry, since $\Omega^v_{\mathrm{do}} \neq \emptyset$, also the semi-trivial positive steady state of the form $(u, 0)$ must be linearly unstable for sufficiently small diffusion rates. Therefore, as in the previous case when $\Omega_{\rm per} \neq \emptyset$, also when $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$ are non-empty, the problem (4.1) is permanent for sufficiently small d_1 and d_2. The striking fact that these permanence results do not depend on the sizes of the patches $\Omega_{\rm per}$, $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$ reveals the strength of our Induced Instability Principle. Actually, this might explain why in most of empirical studies on competing species permanence is much more usual than expected in heterogeneous environments (see, e.g., López-Gómez and Molina-Meyer [91], Belovsky, Mellison, Larson and Van Zandt [8], and the references there in). Going beyond it was shown in López-Gómez and Molina-Meyer [91] how "Most of field experiments and paleontology data corroborate that in the presence of refuge areas, the species persist during long periods of time, even under drastic changes in competition patterns as a result of sudden environmental 'disasters', so confirming that in many circumstances the competitive exclusion principle is false."

The second goal of this chapter consists in establishing, as an important consequence of the Induced Instability Principle, the multiplicity of coexistence steady states for sufficiently small diffusion rates when $\Omega_{\rm bi} \neq \emptyset$ in the symmetric model where

$$d_1 = d_2, \quad \mathcal{L}_1 = \mathcal{L}_2, \quad \mathcal{B}_1 = \mathcal{B}_2, \quad \lambda = \mu, \quad a = d \quad \text{and} \quad b = c.$$

For the validity of this result we need to assume (4.1) to be permanent for small diffusion rates in order to guarantee the existence of a stable coexistence steady state. We already know that the permanence for sufficiently small d_1 and d_2 can be reached by simply imposing that $\Omega_{\rm per} \neq \emptyset$, or $\Omega^u_{\rm do} \neq \emptyset$ and $\Omega^v_{\rm do} \neq \emptyset$. Although we were unable to establish the multiplicity result in the general non-symmetric case, we do make the following conjecture:

> CONJECTURE: If (4.1) is permanent for sufficiently small diffusion rates and $\Omega_{\rm bi} \neq \emptyset$, then (4.1) possesses at least three coexistence steady-state solutions for sufficiently small d_1 and d_2. Two among them linearly stable and perturbing from each of the semi-trivial steady states in $\Omega_{\rm bi}$ and another one linearly unstable perturbing from the coexistence steady state of the associated non-spatial model in $\Omega_{\rm bi}$ as d_1 and d_2 move away from 0.

Besides this result should not depend on the size of $\Omega_{\rm bi}$, the number of coexistence steady-state solutions for sufficiently small d_1 and d_2 might depend on the number of components of $\Omega_{\rm bi}$.

This chapter is distributed as follows. Section 5.1 contains a version of the theorem of characterization of the maximum principle for quasi-cooperative two species systems, as well as some important monotonicity properties of the underlying principal eigenvalues. Our results are refinements of some previous findings of [77], based on [90]. Section 5.2 uses most of the previous results to prove the Induced Instability Principle stated above. Section 5.3 provides us with Theorem 2.1 of Furter and López-Gómez [48] as a byproduct of the IIP. Section 5.4 establishes the multiplicity of coexistence steady-state solutions when $\Omega_{\rm bi} \neq \emptyset$ provided (4.1) is permanent for sufficiently small d_1 and d_2. So, corroborating the validity of the Conjecture above.

5.1 The strong maximum principle for quasi-cooperative systems

Throughout this section, for every $d_1, d_2 > 0$, we denote

$$\mathcal{L}^{1,2}_{(d_1,d_2)} := \begin{pmatrix} d_1\mathcal{L}_1 & 0 \\ 0 & d_2\mathcal{L}_2 \end{pmatrix} \quad \text{and} \quad \mathcal{L}_{(d_1,d_2)} := \begin{pmatrix} d_1\mathcal{L} & 0 \\ 0 & d_2\mathcal{L} \end{pmatrix} \quad \text{if } \mathcal{L} := \mathcal{L}_1 = \mathcal{L}_2.$$

The next result establishes the existence of the principal eigenvalue for any operator of quasi-cooperative type. The proof of Theorem 6.5 of López-Gómez and Sabina [77] can be easily adapted to get the first part. The strict dominance can be obtained arguing as in the proof of Theorem 7.9 of López-Gómez [88]. So, we will omit the technical details here.

Theorem 5.1. *Let $M \in \mathcal{M}_2(\mathcal{C}(\bar{\Omega}))$ be such that $m_{12}(x) > 0$ and $m_{21}(x) > 0$ for all $x \in \Omega$. Then, the boundary value problem*

$$\begin{cases} \left(\mathcal{L}^{1,2}_{(d_1,d_2)} + M\right)\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sigma \begin{pmatrix} \phi \\ \psi \end{pmatrix} & \text{in } \Omega, \\ \mathcal{B}_1\phi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique eigenvalue, denoted by

$$\sigma_0 := \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right],$$

associated to an eigenfunction, (ϕ, ψ), such that $\phi \gg 0$ and $\psi \ll 0$. Moreover, it is real and algebraically simple. Furthermore, it is strictly dominant in the sense that any other eigenvalue σ satisfies $\operatorname{Re}\sigma > \sigma_0$.

In order to state the remaining results of this section, the next concepts are needed. They slightly generalize Definitions 6.1 and 6.2 of López-Gómez and Sabina [77].

Definition 5.2. Let $M \in \mathcal{M}_2(\mathcal{C}(\bar{\Omega}))$ be such that $m_{12}(x) > 0$ and $m_{21}(x) > 0$ for all $x \in \Omega$. A pair $(\phi, \psi) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega)$, $p > N$, is said to be a *supersolution* of $[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ if

$$\begin{cases} \left(\mathcal{L}^{1,2}_{(d_1,d_2)} + M\right)\begin{pmatrix} \phi \\ \psi \end{pmatrix} \begin{matrix} \geq 0 \\ \leq 0 \end{matrix} & \text{in } \Omega, \\ \mathcal{B}_1\phi \geq 0, \quad \mathcal{B}_2\psi \leq 0 & \text{on } \partial\Omega. \end{cases}$$

If any of these inequalities is strict, then (ϕ, ψ) is said to be a *strict supersolution*.

Definition 5.3. Let $M \in \mathcal{M}_2(\mathcal{C}(\bar{\Omega}))$ be such that $m_{12}(x) > 0$ and $m_{21}(x) > 0$ for all $x \in \Omega$. Then:

(i) The tern $[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ is said to satisfy the *maximum principle* if $\phi \geq 0$ and $\psi \leq 0$ for every supersolution, (ϕ, ψ).

(ii) The tern $[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ is said to satisfy the *strong maximum principle* if $\phi \gg 0$ and $\psi \ll 0$ for every supersolution, $(\phi, \psi) \neq (0, 0)$, and, in particular, for any strict supersolution.

The next result slightly sharpens Theorem 6.3 of López-Gómez and Sabina [77]; it goes back to López-Gómez and Molina-Meyer [90] for general cooperative systems. It is necessary for the proof of Lemma 5.7.

Theorem 5.4. *Let $M \in \mathcal{M}_2(\mathcal{C}(\bar\Omega))$ be such that $m_{12}(x) > 0$ and $m_{21}(x) > 0$ for all $x \in \Omega$. Then, the following conditions are equivalent:*

(i) *The principal eigenvalue $\sigma_0 := \sigma_1[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ is positive,*

(ii) *$[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ admits a strict supersolution, (ϕ, ψ), with $\phi \gneq 0$ and $\psi \lneq 0$,*

(iii) *$[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ satisfies the strong maximum principle,*

(iv) *$[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega]$ satisfies the maximum principle.*

Proof. For (i) implies (ii), let (ϕ, ψ) be a principal eigenfunction associated to $\sigma_0 > 0$. Then $\phi \gg 0$, $\psi \ll 0$ and, since $\sigma_0 > 0$, we have that

$$\begin{cases} \begin{pmatrix} d_1\mathcal{L}_1 & 0 \\ 0 & d_2\mathcal{L}_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + M \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sigma_1 \begin{pmatrix} \phi \\ \psi \end{pmatrix} \begin{matrix} > 0 \\ < 0 \end{matrix} & \text{in } \Omega, \\ \mathcal{B}_1\phi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, (ϕ, ψ) is a strict supersolution with $\phi \gg 0$ and $\psi \ll 0$.

Next, we will show that (ii) implies (iii). Let (ϕ, ψ) be a strict supersolution with $\phi \gneq 0$ and $\psi \lneq 0$. Then,

$$\begin{cases} (d_1\mathcal{L}_1 + m_{11})\phi \geq -m_{12}\psi \gneq 0 & \text{in } \Omega, \\ \mathcal{B}_1\phi \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} (d_2\mathcal{L}_2 + m_{22})(-\psi) \geq m_{21}\phi \gneq 0 & \text{in } \Omega, \\ \mathcal{B}_2(-\psi) \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, ϕ and $-\psi$ are positive strict supersolutions of the terns

$$[d_1\mathcal{L}_1 + m_{11}; \mathcal{B}_1, \Omega] \quad \text{and} \quad [d_2\mathcal{L}_2 + m_{22}; \mathcal{B}_2, \Omega],$$

respectively, according to Definition 7.3 of López-Gómez [88]. Thus, by [88, Th. 7.10],

$$\sigma_1[d_1\mathcal{L}_1 + m_{11}; \mathcal{B}_1, \Omega] > 0 \quad \text{and} \quad \sigma_1[d_2\mathcal{L}_1 + m_{22}; \mathcal{B}_2, \Omega] > 0,$$

$\phi \gg 0$, $\psi \ll 0$, and the resolvents, $(d_1\mathcal{L}_1 + m_{11})^{-1}$ and $(d_2\mathcal{L}_2 + m_{22})^{-1}$, subject to the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$, are strongly positive and compact. Subsequently, we consider

$$\tilde\phi := (d_1\mathcal{L}_1 + m_{11})^{-1}(-m_{12}\psi) \gg 0 \quad \text{and} \quad \tilde\psi := -(d_2\mathcal{L}_2 + m_{22})^{-1}(m_{21}\phi) \gg 0.$$

Then, $\phi - \tilde\phi$ and $(-\psi) - (-\tilde\psi)$ are supersolutions of

$$\begin{cases} (d_1\mathcal{L}_1 + m_{11})w = 0 & \text{in } \Omega, \\ \mathcal{B}_1 w = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} (d_2\mathcal{L}_2 + m_{22})w = 0 & \text{in } \Omega, \\ \mathcal{B}_2 w = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

respectively, one of them being strict. Thus, Theorem 7.10 of López-Gómez [88] yields $\phi \geq \tilde{\phi}$, $\psi \leq \tilde{\psi}$. Moreover, since one of them is strict, either $\phi \gg \tilde{\phi}$ or $\psi \ll \tilde{\psi}$. Hence,

$$\begin{aligned} \tilde{\phi} &= (d_1\mathcal{L}_1 + m_{11})^{-1}(-m_{12}\psi) \geq (d_1\mathcal{L}_1 + m_{11})^{-1}(-m_{12}\tilde{\psi}) \\ &= (d_1\mathcal{L}_1 + m_{11})^{-1}\left(m_{12}\,(d_2\mathcal{L}_2 + m_{22})^{-1}\,(m_{21}\phi)\right) \\ &\geq (d_1\mathcal{L}_1 + m_{11})^{-1}\left(m_{12}\,(d_2\mathcal{L}_2 + m_{22})^{-1}\,(m_{21}\tilde{\phi})\right), \end{aligned} \tag{5.2}$$

with one of these inequalities being strict. Similarly,

$$\begin{aligned} -\tilde{\psi} &= (d_2\mathcal{L}_2 + m_{22})^{-1}(m_{21}\phi) \geq (d_2\mathcal{L}_2 + m_{22})^{-1}(m_{21}\tilde{\phi}) \\ &= (d_2\mathcal{L}_2 + m_{22})^{-1}\left(m_{21}\,(d_1\mathcal{L}_1 + m_{11})^{-1}\,(-m_{12}\psi)\right) \\ &\geq (d_2\mathcal{L}_2 + m_{22})^{-1}\left(m_{21}\,(d_1\mathcal{L}_1 + m_{11})^{-1}\,(-m_{12}\tilde{\psi})\right), \end{aligned} \tag{5.3}$$

with one of these inequalities strict. Now, we introduce the strongly positive compact operators defined by

$$\begin{aligned} T_1 &:= (d_1\mathcal{L}_1 + m_{11})^{-1}\left[m_{12}\,(d_2\mathcal{L}_2 + m_{22})^{-1}\,(m_{21}\,\cdot)\right], \\ T_2 &:= (d_2\mathcal{L}_2 + m_{22})^{-1}\left[m_{21}\,(d_1\mathcal{L}_1 + m_{11})^{-1}\,(m_{12}\,\cdot)\right]. \end{aligned}$$

subject to the boundary operators $\mathcal{B}_1$ and $\mathcal{B}_2$. We already know that $\tilde{\phi} > 0$, $-\tilde{\psi} > 0$,

$$\tilde{\phi} - T_1(\tilde{\phi}) > 0 \quad \text{and} \quad (-\tilde{\psi}) - T_2(-\tilde{\psi}) > 0 \quad \text{in } \Omega,$$

and

$$\mathcal{B}_1(\tilde{\phi}) = \mathcal{B}_2(-\tilde{\psi}) = 0 \quad \text{on } \partial\Omega.$$

Hence, by Theorem 3.2(iv) of Amann [3],

$$\operatorname{spr} T_1 < 1 \quad \text{and} \quad \operatorname{spr} T_2 < 1.$$

Now, consider a supersolution, $(u, v) \neq (0, 0)$. Then,

$$\begin{cases} (d_1\mathcal{L}_1 + m_{11})u \geq -m_{12}v & \text{in } \Omega, \\ \mathcal{B}_1 u \geq 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} (d_2\mathcal{L}_2 + m_{22})(-v) \geq m_{21}u & \text{in } \Omega, \\ \mathcal{B}_2(-v) \geq 0 & \text{on } \partial\Omega, \end{cases}$$

and setting

$$\tilde{u} := (d_1\mathcal{L}_1 + m_{11})^{-1}(-m_{12}v), \qquad \tilde{v} := -(d_2\mathcal{L}_2 + m_{22})^{-1}(m_{21}u),$$

we have that $u - \tilde{u}$ and $-v - (-\tilde{v})$ are respective supersolutions of (5.1). Thus, since the respective principal eigenvalues are positive, it follows from Theorem 7.10 of López-Gómez [88] that either $u \gg \tilde{u}$ or $u = \tilde{u}$, and either $v \ll \tilde{v}$ or $v = \tilde{v}$ in Ω. Reasoning as in (5.2) and (5.3), the estimates $\tilde{u} \geq T_1(\tilde{u})$ and $(-\tilde{v}) \geq T_2(-\tilde{v})$ hold in Ω. Since $\operatorname{spr} T_1, \operatorname{spr} T_2 < 1$, by Theorem 6.3(d) of López-Gómez [88], the resolvent operators $(I - T_i)^{-1}$, $i = 1, 2$, are strongly positive. Hence, either $\tilde{u} \gg 0$ or $\tilde{u} = 0$, and either $\tilde{v} \ll 0$ or $\tilde{v} = 0$. Consequently, either $u \gg 0$ or $u = 0$, and either $v \ll 0$ or $v = 0$. It remains to show that neither $u = 0$ nor $v = 0$. If, for example, $u = 0$, then $0 \geq -m_{12}v$ in Ω and so $v \geq 0$. Hence $v = 0$, which contradicts $(u, v) \neq (0, 0)$. This shows that (ii) implies (iii).

The fact that (iii) implies (iv) is immediate. For the proof of (iv) implies (i), let (ϕ, ψ) be an eigenfunction associated to the principal eigenvalue σ_0 and suppose that $\sigma_0 \leq 0$. Then, since $\phi \gg 0$ and $\psi \ll 0$, we obtain that

$$\begin{cases} \begin{pmatrix} d_1\mathcal{L}_1 & 0 \\ 0 & d_2\mathcal{L}_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} + M \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sigma_0 \begin{pmatrix} \phi \\ \psi \end{pmatrix} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix} & \text{in } \Omega, \\ \mathcal{B}_1\phi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying by -1, we have that $(-\phi, -\psi)$ is a supersolution of the tern

$$\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right].$$

Since the tern

$$\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right]$$

satisfies the maximum principle, $-\phi \geq 0$ and $-\psi \leq 0$, which contradicts the assumption. Therefore $\sigma_0 > 0$. □

Next, we will derive from Theorem 5.4 some of the main monotonicity properties of the principal eigenvalue

$$\sigma_1 \left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right].$$

These results extend the monotonicity properties established in Theorem 2.12 and Lemma 2.14 to cover the case of quasi-cooperative systems. To state them, it is convenient to introduce the following ordering.

Definition 5.5. Let $F, G \in \mathcal{M}_2(C(\bar{\Omega}))$ be two matrices with continuous coefficients. Then, F is said to be *greater* than G in Ω, $F \succ G$ in Ω, if $F \neq G$ and

$$f_{ii} \geq g_{ii} \quad \text{in } \Omega \text{ for all } i \in \{1, 2\} \quad \text{and} \quad f_{ij} \leq g_{ij} \quad \text{in } \Omega \text{ for all } i, j \in \{1, 2\}, \quad i \neq j.$$

Then, the next comparison result holds.

Lemma 5.6. *Let $F, G \in \mathcal{M}_2(C(\bar{\Omega}))$ be such that $F \succ G$ in Ω and $f_{12}(x), f_{21}(x) > 0$ for all $x \in \Omega$. Then,*

$$\sigma_1 \left[\mathcal{L}^{1,2}_{(d_1,d_2)} + F; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right] > \sigma_1 \left[\mathcal{L}^{1,2}_{(d_1,d_2)} + G; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right].$$

Proof. Since $F \succ G$ in Ω we have that

$$0 < f_{12}(x) \leq g_{12}(x) \quad \text{and} \quad 0 < f_{21}(x) \leq g_{21}(x) \quad \text{for all } x \in \Omega.$$

Thus, thanks to Theorem 5.1, the principal eigenvalues

$$\sigma_{0,F} := \sigma_1 \left[\mathcal{L}^{1,2}_{(d_1,d_2)} + F; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right] \quad \text{and} \quad \sigma_{0,G} := \sigma_1 \left[\mathcal{L}^{1,2}_{(d_1,d_2)} + G; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right]$$

are well defined. Then, any principal eigenfunction, (ϕ,ψ), associated to $\sigma_{0,G}$, satisfies $\phi \gg 0$, $\psi \ll 0$, and

$$\begin{cases} d_1\mathcal{L}_1\phi + g_{11}\phi + g_{12}\psi - \sigma_{0,G}\phi = 0 & \\ d_2\mathcal{L}_2\psi + g_{21}\phi + g_{22}\psi - \sigma_{0,G}\psi = 0 & \text{in } \Omega, \\ \mathcal{B}_1\phi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $F \succ G$ in Ω, we find from $\phi \gg 0$ and $\psi \ll 0$ that

$$0 = d_1\mathcal{L}_1\phi + g_{11}\phi + g_{12}\psi - \sigma_{0,G}\phi \leq d_1\mathcal{L}_1\phi + f_{11}\phi + f_{12}\psi - \sigma_{0,G}\phi,$$

and

$$0 = d_2\mathcal{L}_2\psi + g_{21}\phi + g_{22}\psi - \sigma_{0,G}\psi \geq d_2\mathcal{L}_2\psi + f_{21}\phi + f_{22}\psi - \sigma_{0,G}\psi,$$

with some of these inequalities strict. Thus,

$$\begin{cases} d_1\mathcal{L}_1\phi + f_{11}\phi + f_{12}\psi - \sigma_{0,G}\phi \geq 0 & \\ d_2\mathcal{L}_2\psi + f_{21}\phi + f_{22}\psi - \sigma_{0,G}\psi \leq 0 & \text{in } \Omega, \\ \mathcal{B}_1\phi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega, \end{cases}$$

with some of these inequalities strict. Therefore, by Theorem 5.4, we conclude that

$$0 < \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + F - \sigma_{0,G}; (\mathcal{B}_1,\mathcal{B}_2), \Omega\right] = \sigma_{0,F} - \sigma_{0,G},$$

which ends the proof. □

Lemma 5.7. *Let Ω_0 be a subdomain of class $\mathcal{C}^2$ of Ω such that $\partial\Omega_0\cap\partial\Omega$ consists of finitely many components of $\partial\Omega$, if it is non-empty. For each $i\in\{1,2\}$, let $\mathcal{B}_{i,0}$ be any boundary operator of the type*

$$\mathcal{B}_{i,0}h := \begin{cases} h & \text{on } \partial\Omega_0\cap\Omega, \\ \tilde{\mathcal{B}}_i h & \text{on } \partial\Omega_0\cap\partial\Omega, \end{cases} \qquad \text{for every } h\in W^{2,p}(\Omega), \;\; p>N,$$

where, on each component of $\partial\Omega_0\cap\partial\Omega$, either $\tilde{\mathcal{B}}_i h = h$, or $\tilde{\mathcal{B}}_i h = \mathcal{B}_i h$. If

$$(\mathcal{B}_1,\mathcal{B}_2,\Omega) \neq (\mathcal{B}_{1,0},\mathcal{B}_{2,0},\Omega_0),$$

then, for every $d_1, d_2 > 0$ and $M\in\mathcal{M}_2(\mathcal{C}(\bar{\Omega}))$, such that

$$m_{12}(x), m_{21}(x) > 0 \quad \text{for all } x\in\Omega,$$

we have that

$$\sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1,\mathcal{B}_2), \Omega\right] < \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_{1,0},\mathcal{B}_{2,0}), \Omega_0\right].$$

Proof. Let (ϕ,ψ) be a principal eigenfunction associated to

$$\sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1,\mathcal{B}_2), \Omega\right]$$

such that $\phi \gg 0$, $\psi \ll 0$. Then,

$$\mathcal{B}_{1,0}\phi = \begin{cases} \phi > 0 & \text{on } \partial\Omega_0 \cap \Omega, \\ \tilde{\mathcal{B}}_1\phi = \phi > 0 & \text{on } \partial\Omega_0 \cap \partial\Omega \text{ if } \tilde{\mathcal{B}}_1 \neq \mathcal{B}_1, \\ \tilde{\mathcal{B}}_1\phi = \mathcal{B}_1\phi = 0 & \text{on } \partial\Omega_0 \cap \partial\Omega \text{ if } \tilde{\mathcal{B}}_1 = \mathcal{B}_1, \end{cases}$$

and

$$\mathcal{B}_{2,0}\psi = \begin{cases} \psi < 0 & \text{on } \partial\Omega_0 \cap \Omega, \\ \tilde{\mathcal{B}}_2\psi = \psi < 0 & \text{on } \partial\Omega_0 \cap \partial\Omega \text{ if } \tilde{\mathcal{B}}_2 \neq \mathcal{B}_2, \\ \tilde{\mathcal{B}}_2\psi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega_0 \cap \partial\Omega \text{ if } \tilde{\mathcal{B}}_2 = \mathcal{B}_2. \end{cases}$$

Since

$$(\mathcal{B}_1, \mathcal{B}_2, \Omega) \neq (\mathcal{B}_{1,0}, \mathcal{B}_{2,0}, \Omega_0),$$

either $\mathcal{B}_{1,0}\phi \geq 0$ or $\mathcal{B}_{2,0}\psi \leq 0$ on $\partial\Omega_0$, with some of these inequalities strict. Thus,

$$\begin{cases} \left(\mathcal{L}^{1,2}_{(d_1,d_2)} + M - \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right]\right)\begin{pmatrix} \phi_{|\Omega_0} \\ \psi_{|\Omega_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega_0, \\ \mathcal{B}_{1,0}\phi_{|\Omega_0} \geq 0, \quad \mathcal{B}_{2,0}\psi_{|\Omega_0} = 0 & \text{on } \partial\Omega_0, \end{cases}$$

with some of the boundary inequalities strict. Thus, $(\phi_{|\Omega_0}, \psi_{|\Omega_0})$ is a strict supersolution of the tern

$$\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M - \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right]; (\mathcal{B}_{1,0}, \mathcal{B}_{2,0}), \Omega_0\right]$$

with $\phi_{|\Omega_0} > 0$ and $\psi_{|\Omega_0} < 0$. Therefore, by Theorem 5.4,

$$\sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M - \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right]; (\mathcal{B}_{1,0}, \mathcal{B}_{2,0}), \Omega_0\right] > 0,$$

and so,

$$\sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_{1,0}, \mathcal{B}_{2,0}), \Omega_0\right] - \sigma_1\left[\mathcal{L}^{1,2}_{(d_1,d_2)} + M; (\mathcal{B}_1, \mathcal{B}_2), \Omega\right] > 0.$$

□

5.2 Perturbation from a kinetic equilibrium which is somewhere unstable in Ω

The main result of this section establishes the Induced Instability Principle, according with it any family of non-trivial states of (4.2) perturbing (uniformly) from any steady state, $(u_*(x), v_*(x))$, of the non-spatial model (4.5), with x varying on some open subset, Ω_{un}, of Ω, must be linearly unstable provided $(u_*(x), v_*(x))$ is linearly unstable for all $x \in \Omega_{\text{un}}$. Note that for every $x \in \Omega_{\text{un}}$, such equilibria, $(u_*(x), v_*(x))$, satisfy

$$\begin{cases} [\lambda(x) - a(x)u_*(x) - b(x)v_*(x)]u_*(x) = 0, \\ [\mu(x) - d(x)v_*(x) - c(x)u_*(x)]v_*(x) = 0, \end{cases}$$

and hence, one of the following alternatives hold:

(i) $(u_*(x), v_*(x)) = (0,0)$,

(ii) $(u_*(x), v_*(x)) = \left(\frac{\lambda(x)}{a(x)}, 0\right)$ if $\lambda(x) > 0$,

(iii) $(u_*(x), v_*(x)) = \left(0, \frac{\mu(x)}{d(x)}\right)$ if $\mu(x) > 0$,

(iv) $(u_*(x), v_*(x)) = \left(\frac{\lambda(x)d(x)-\mu(x)b(x)}{a(x)d(x)-b(x)c(x)}, \frac{\mu(x)a(x)-\lambda(x)c(x)}{a(x)d(x)-b(x)c(x)}\right)$ if both components are well defined and positive.

The fact that this 'general principle' holds independently of the size of the instability region Ω_{un} is a rather astonishing feature. The next lemma is the main technical tool to prove such result, together with the monotonicity of the principal eigenvalue with respect to the domain. Remember that we are assuming that $b(x) > 0$ and $c(x) > 0$ for all $x \in \Omega$.

Lemma 5.8. *Assume that $\mathcal{L} := \mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{B} := \mathcal{B}_1 = \mathcal{B}_2$. Let $J \in (0,+\infty)^2$ be with $(0,0) \in \bar{J}$, and consider a family of matrices, $\{H_{(d_1,d_2)}\}_{(d_1,d_2)\in J} \subset \mathcal{M}_2(\mathbb{R})$, such that*

(i) *the off-diagonal entries of $H_{(d_1,d_2)}$ are positive for every $(d_1,d_2) \in J$,*

(ii) *$H_{(d_1,d_2)}$ converges to some matrix $H_* \in \mathcal{M}_2(\mathbb{R})$ as $J \ni (d_1,d_2) \to (0,0)$.*

If $\sigma_{\text{low}}[H_*]$ stands for the lower eigenvalue of H_*, then

$$\lim_{(d_1,d_2)\to(0,0)} \sigma_1[\mathcal{L}_{(d_1,d_2)} + H_{(d_1,d_2)}; (\mathcal{B},\mathcal{B}), \Omega] = \sigma_{\text{low}}[H_*].$$

Proof. Note that any matrix

$$H = (h_{ij}) \in \mathcal{M}_2(\mathbb{R})$$

with $h_{12}h_{21} \geq 0$ has real eigenvalues. Indeed, they are given by

$$\begin{aligned}\sigma_\pm[H] &= \frac{h_{11}+h_{22} \pm \sqrt{(h_{11}+h_{22})^2 - 4(h_{11}h_{22}-h_{12}h_{21})}}{2} \\ &= \frac{h_{11}+h_{22} \pm \sqrt{(h_{11}-h_{22})^2 + 4h_{12}h_{21}}}{2}.\end{aligned}$$

Hence, such a property holds for every $H_{(d_1,d_2)}$, $(d_1,d_2) \in J$, and, by the assumptions, also for H_*. On the other hand, by (i), Theorem 5.1 provides us with the existence and uniqueness of the principal eigenvalue

$$\sigma_{(d_1,d_2)} := \sigma_1[\mathcal{L}_{(d_1,d_2)} + H_{(d_1,d_2)}; (\mathcal{B},\mathcal{B}), \Omega],$$

associated with it there is an eigenfunction, $(\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)})$, with components $\phi_{(d_1,d_2)} \gg 0$ and $\psi_{(d_1,d_2)} \ll 0$, unique up to a multiplicative nontrivial constant. Let σ_0 and $\varphi_0 \gg 0$ denote the principal eigenpair of $\mathcal{L}$ in Ω, with φ_0 normalized so that $\|\varphi_0\|_\infty = 1$. Now, let us show that $(\xi_{(d_1,d_2)}\varphi_0, \zeta_{(d_1,d_2)}\varphi_0)$ provides us with a principal eigenfunction associated to $\sigma_{(d_1,d_2)}$ for appropriate values of $\xi_{(d_1,d_2)} > 0$ and $\zeta_{(d_1,d_2)} < 0$. By definition,

$$\begin{pmatrix} d_1\xi_{(d_1,d_2)}\mathcal{L}\varphi_0 \\ d_2\zeta_{(d_1,d_2)}\mathcal{L}\varphi_0 \end{pmatrix} + H_{(d_1,d_2)} \begin{pmatrix} \xi_{(d_1,d_2)}\varphi_0 \\ \zeta_{(d_1,d_2)}\varphi_0 \end{pmatrix} = \sigma_{(d_1,d_2)} \begin{pmatrix} \xi_{(d_1,d_2)}\varphi_0 \\ \zeta_{(d_1,d_2)}\varphi_0 \end{pmatrix},$$

which can be equivalently expressed as

$$\phi_0 \left[\begin{pmatrix} d_1 \xi_{(d_1,d_2)} \sigma_0 \\ d_2 \zeta_{(d_1,d_2)} \sigma_0 \end{pmatrix} + H_{(d_1,d_2)} \begin{pmatrix} \xi_{(d_1,d_2)} \\ \zeta_{(d_1,d_2)} \end{pmatrix} \right] = \varphi_0 \sigma_{(d_1,d_2)} \begin{pmatrix} \xi_{(d_1,d_2)} \\ \zeta_{(d_1,d_2)} \end{pmatrix}.$$

Thus, dividing by $\varphi_0 \gg 0$, it becomes apparent that $(\xi_{(d_1,d_2)}, \zeta_{(d_1,d_2)})$ must satisfy

$$\left[\sigma_0 \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} + H_{(d_1,d_2)} \right] \begin{pmatrix} \xi_{(d_1,d_2)} \\ \zeta_{(d_1,d_2)} \end{pmatrix} = \sigma_{(d_1,d_2)} \begin{pmatrix} \xi_{(d_1,d_2)} \\ \zeta_{(d_1,d_2)} \end{pmatrix},$$

i.e., $(\xi_{(d_1,d_2)}, \zeta_{(d_1,d_2)})$ is an eigenvector of the matrix

$$\tilde{H}_{(d_1,d_2)} := \sigma_0 \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} + H_{(d_1,d_2)}$$

associated with the eigenvalue $\sigma_{(d_1,d_2)}$. It remains to show that the lower eigenvalue of $\tilde{H}_{(d_1,d_2)}$ admits an eigenvector with components of opposite sign. But this is also a consequence from the fact that the off-diagonal entries of $H_{(d_1,d_2)}$, and so of $\tilde{H}_{(d_1,d_2)}$, are positive. Indeed, for any matrix $H \in \mathcal{M}_2(\mathbb{R})$ with $h_{12}, h_{21} > 0$

$$\sigma_{\text{low}}[H] = \frac{h_{11} + h_{22} - \sqrt{(h_{11} - h_{22})^2 + 4h_{12}h_{21}}}{2}$$

and any associated eigenvector, (ξ, ζ), satisfies

$$(h_{11} - \sigma_{\text{low}}[H])\,\xi = -h_{12}\zeta \quad \text{and} \quad (h_{22} - \sigma_{\text{low}}[H])\,\zeta = -h_{21}\xi.$$

Thus, $\xi\zeta < 0$ if and only if $h_{11}, h_{22} > \sigma_{\text{low}}[H]$. But, since $h_{12}h_{21} > 0$, we have that

$$\sigma_{\text{low}}[H] < \frac{h_{11} + h_{22} - |h_{11} - h_{22}|}{2} = \min\{h_{11}, h_{22}\}.$$

Therefore, $\tilde{H}_{(d_1,d_2)}$ admits an eigenfunction with components of opposite sign associated to its lower eigenvalue. By the uniqueness of the principal eigenvalue, $\sigma_{(d_1,d_2)}$, necessarily

$$\sigma_1[\mathcal{L}_{(d_1,d_2)} + H_{(d_1,d_2)}; (\mathcal{B}, \mathcal{B}), \Omega] = \sigma_{(d_1,d_2)} = \sigma_{\text{low}}[\tilde{H}_{(d_1,d_2)}].$$

Finally, thanks to (ii), the entries of $\tilde{H}_{(d_1,d_2)}$ converge to those of H_*. So, it does the lower eigenvalue of these matrices. □

The next result establishes the Induced Instability Principle when the perturbed steady states are coexistence states.

Theorem 5.9. *Suppose that $\mathcal{L} := \mathcal{L}_1 = \mathcal{L}_2$ and $\mathcal{B} := \mathcal{B}_1 = \mathcal{B}_2$. Consider a sequence of coexistence states of (4.2), $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}_{(d_1,d_2) \in J}$, with $J \subset (0, +\infty)^2$ and $(0,0) \in \bar{J}$, such that, for some open subset $\Omega_{\text{un}} \subset \Omega$,*

$$\lim_{(d_1,d_2) \to (0,0)} \left(u_{(d_1,d_2)}, v_{(d_1,d_2)}\right) = (u_*, v_*) \quad \textit{uniformly in } \Omega_{\text{un}}$$

with $(u_*(x), v_*(x))$ linearly unstable for all $x \in \Omega_{\text{un}}$, as a nontrivial steady-state solution of (4.5). Then, $\delta > 0$ exists such that $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ is linearly unstable for all $d_1, d_2 < \delta$, $(d_1, d_2) \in J$.

Proof. The linear instability follows from the negativity of the principal eigenvalue of the linearization of (4.2) at the coexistence state, i.e., $\sigma_1[\mathfrak{L}_{(d_1,d_2)};(\mathcal{B},\mathcal{B}),\Omega]<0$, where

$$\mathfrak{L}_{(d_1,d_2)} := \mathcal{L}_{(d_1,d_2)} + \begin{pmatrix} -\lambda+2au_{(d_1,d_2)}+bv_{(d_1,d_2)} & bu_{(d_1,d_2)} \\ cv_{(d_1,d_2)} & -\mu+2dv_{(d_1,d_2)}+cu_{(d_1,d_2)} \end{pmatrix}. \tag{5.4}$$

Since the off-diagonal entries of $\mathfrak{L}_{(d_1,d_2)}$, $bu_{(d_1,d_2)}$ and $cv_{(d_1,d_2)}$, are positive, the existence of $\sigma_1[\mathfrak{L}_{(d_1,d_2)};(\mathcal{B},\mathcal{B}),\Omega]$ follows from Theorem 5.1. Consider $x_0\in\Omega_{\mathrm{un}}$ and $\varepsilon>0$ such that $\bar{B}_\varepsilon(x_0)\subsetneq\Omega_{\mathrm{un}}$. Then, by Lemma 5.7,

$$\sigma_1[\mathfrak{L}_{(d_1,d_2)};(\mathcal{B},\mathcal{B}),\Omega] < \sigma_1[\mathfrak{L}_{(d_1,d_2)};(\mathcal{D},\mathcal{D}),B_\varepsilon(x_0)]. \tag{5.5}$$

Subsequently, we set

$$\begin{aligned}
\alpha_{(\varepsilon,d_1,d_2)} &:= \max_{\bar{B}_\varepsilon(x_0)} \left\{-\lambda+2au_{(d_1,d_2)}+bv_{(d_1,d_2)}\right\},\\
\beta_{(\varepsilon,d_1,d_2)} &:= \min_{\bar{B}_\varepsilon(x_0)} \left\{bu_{(d_1,d_2)}\right\},\\
\gamma_{(\varepsilon,d_1,d_2)} &:= \min_{\bar{B}_\varepsilon(x_0)} \left\{cv_{(d_1,d_2)}\right\},\\
\rho_{(\varepsilon,d_1,d_2)} &:= \max_{\bar{B}_\varepsilon(x_0)} \left\{-\mu+2dv_{(d_1,d_2)}+cu_{(d_1,d_2)}\right\}.
\end{aligned}$$

Then, according to Lemma 5.6, the next estimate holds

$$\sigma_1[\mathfrak{L}_{(d_1,d_2)};(\mathcal{D},\mathcal{D}),B_\varepsilon(x_0)] \le \sigma_1\left[\mathcal{L}_{(d_1,d_2)} + \begin{pmatrix}\alpha_{(\varepsilon,d_1,d_2)} & \beta_{(\varepsilon,d_1,d_2)}\\ \gamma_{(\varepsilon,d_1,d_2)} & \rho_{(\varepsilon,d_1,d_2)}\end{pmatrix};(\mathcal{D},\mathcal{D}),B_\varepsilon(x_0)\right] \tag{5.6}$$

for every $d_1,d_2>0$, $(d_1,d_2)\in J$. Since $u_{(d_1,d_2)}$ and $v_{(d_1,d_2)}$ converge to u_* and v_*, respectively, uniformly in Ω_{un} as $(d_1,d_2)\to(0,0)$, the following limits are well defined

$$\begin{aligned}
\alpha_{(\varepsilon,*)} &:= \lim_{(d_1,d_2)\to(0,0)} \alpha_{(\varepsilon,d_1,d_2)}, & \beta_{(\varepsilon,*)} &:= \lim_{(d_1,d_2)\to(0,0)} \beta_{(\varepsilon,d_1,d_2)},\\
\gamma_{(\varepsilon,*)} &:= \lim_{(d_1,d_2)\to(0,0)} \gamma_{(\varepsilon,d_1,d_2)}, & \rho_{(\varepsilon,*)} &:= \lim_{(d_1,d_2)\to(0,0)} \rho_{(\varepsilon,d_1,d_2)}.
\end{aligned}$$

Moreover, letting $\varepsilon\downarrow 0$ yields

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\alpha_{(\varepsilon,*)} &= -\lambda(x_0)+2a(x_0)u_*(x_0)+b(x_0)v_*(x_0),\\
\lim_{\varepsilon\downarrow 0}\beta_{(\varepsilon,*)} &= b(x_0)u_*(x_0),\\
\lim_{\varepsilon\downarrow 0}\gamma_{(\varepsilon,*)} &= c(x_0)v_*(x_0),\\
\lim_{\varepsilon\downarrow 0}\rho_{(\varepsilon,*)} &= -\mu(x_0)+c(x_0)u_*(x_0)+2d(x_0)v_*(x_0).
\end{aligned}$$

Note that

$$-\begin{pmatrix} -\lambda(x_0)+2a(x_0)u_*(x_0)+b(x_0)v_*(x_0) & b(x_0)u_*(x_0)\\ c(x_0)v_*(x_0) & -\mu(x_0)+c(x_0)u_*(x_0)+2d(x_0)v_*(x_0)\end{pmatrix}$$

provides us with the linearization of the non-spatial model at $(u_*(x_0), v_*(x_0))$, which is linearly unstable because $x_0 \in \Omega_{\rm un}$. Thus, this matrix has a positive eigenvalue, and hence, for sufficiently small $\varepsilon > 0$, the matrix

$$\begin{pmatrix} \alpha_{(\varepsilon,*)} & \beta_{(\varepsilon,*)} \\ \gamma_{(\varepsilon,*)} & \rho_{(\varepsilon,*)} \end{pmatrix}$$

possesses a negative eigenvalue. Therefore, owing to Lemma 5.8, for sufficiently small $\varepsilon > 0$ and $d_1, d_2 > 0$, with $(d_1, d_2) \in J$, we obtain that

$$\sigma_1 \left[\mathcal{L}_{(d_1,d_2)} + \begin{pmatrix} \alpha_{(\varepsilon,d_1,d_2)} & \beta_{(\varepsilon,d_1,d_2)} \\ \gamma_{(\varepsilon,d_1,d_2)} & \rho_{(\varepsilon,d_1,d_2)} \end{pmatrix}; (\mathcal{D}, \mathcal{D}), B_\varepsilon(x_0) \right] < 0.$$

Consequently, according to (5.5) and (5.6), we find that $\sigma_1[\mathfrak{L}_{(d_1,d_2)}; (\mathcal{B}, \mathcal{B}), \Omega] < 0$ for every sufficiently small $d_1, d_2 > 0$, with $(d_1, d_2) \in J$. This ends the proof. □

Although in Theorem 5.9 the steady state (u_*, v_*) might have some component vanishing, or both, the next result shows that actually the coexistence steady states of (4.1) cannot perturb from $(0, 0)$ uniformly in some open subset $\Omega_0 \subset \Omega$ as $d_1, d_2 \to 0$ if $(0, 0)$ is linearly unstable in Ω_0 as a steady state of (4.5).

Proposition 5.10. *Assume that* $\{(u_{(d_1,d_2)}, v_{(d_1,d_2)})\}_{(d_1,d_2)\in J}$*, with* $J \subset (0, +\infty)^2$ *and* $(0, 0) \in \bar{J}$*, is a sequence of coexistence states of* (4.2) *such that*

$$\lim_{(d_1,d_2)\to(0,0)} \left(u_{(d_1,d_2)}, v_{(d_1,d_2)}\right) = (0, 0) \quad \textit{uniformly in } \Omega_0$$

for some subdomain $\Omega_0 \Subset \Omega$ *of class* $\mathcal{C}^2$*. Then,* $(0, 0)$ *cannot be linearly unstable at any* $x \in \Omega_0$ *as a steady state of* (4.5)*.*

Proof. On the contrary, suppose that $(0, 0)$ is linearly unstable with respect to (4.5) at some $x_0 \in \Omega_0$. Then, the linearization at $(0, 0)$ of (4.5) for $x = x_0$, which is given by

$$\begin{pmatrix} \lambda(x_0) & 0 \\ 0 & \mu(x_0) \end{pmatrix},$$

has a positive eigenvalue. Thus, either

$$\max_{\bar{\Omega}_0} \lambda \geq \lambda(x_0) > 0$$

or

$$\max_{\bar{\Omega}_0} \mu \geq \mu(x_0) > 0.$$

Suppose $\max_{\bar{\Omega}_0} \lambda > 0$. Then, the monotonicity with respect to the domain established in Lemma 2.14 yields

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + au_{(d_1,d_2)} + bv_{(d_1,d_2)}; \mathcal{B}_1, \Omega] < \sigma_1[d_1\mathcal{L}_1 - \lambda + au_{(d_1,d_2)} + bv_{(d_1,d_2)}; \mathcal{D}, \Omega_0] \quad (5.7)$$

for all $(d_1, d_2) \in J$, while, thanks to Theorem 2.12, the uniform convergence in Ω_0 provides us with

$$\lim_{(d_1,d_2)\to(0,0)} \sigma_1[d_1\mathcal{L}_1 - \lambda + au_{(d_1,d_2)} + bv_{(d_1,d_2)}; \mathcal{D}, \Omega_0] = \min_{\bar{\Omega}_0} (-\lambda) = -\max_{\bar{\Omega}_0} \lambda < 0. \quad (5.8)$$

But, since $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ is a coexistence state,

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + au_{(d_1,d_2)} + bv_{(d_1,d_2)}; \mathcal{B}_1, \Omega] = 0 \quad \text{for all} \quad (d_1, d_2) \in J,$$

which contradicts (5.7) y (5.8) and ends the proof. $\square$

Theorem 5.9 admits the next counterpart for semitrivial solutions of (4.2). So, the Induced Instability Principle holds.

It should be remembered that, thanks to Theorem 4.1, which goes back to Theorem 2.21, i.e., the singular perturbation result for the single equation, the family of semitrivial solutions of (4.2), $(\theta_{\{d_1,\lambda,a\}}, 0)$, with $d_1 > 0$, converges to $(\frac{\lambda_+}{a}, 0)$ uniformly in compact subsets of Ω as $d_1 \downarrow 0$. Similarly, $(0, \theta_{\{d_2,\mu,d\}})$ converges to $(0, \frac{\mu_+}{d})$ uniformly in compact subsets of Ω as $d_2 \downarrow 0$.

Proposition 5.11. *The following assertions hold:*

(i) *Suppose that there exists an open subset* $\Omega_{\text{un}} \Subset \Omega$ *of class* $\mathcal{C}^2$ *such that, for every* $x \in \bar{\Omega}_{\text{un}}$, $(\frac{\lambda_+(x)}{a(x)}, 0)$ *is linearly unstable as a steady-state solution of* (4.5). *Then,* $\delta > 0$ *exists such that* $(\theta_{\{d_1,\lambda,a\}}, 0)$ *is linearly unstable for every* $d_1, d_2 < \delta$.

(ii) *Assume that there exists an open subset* $\Omega_{\text{un}} \Subset \Omega$ *of class* $\mathcal{C}^2$ *such that, for every* $x \in \bar{\Omega}_{\text{un}}$, $(0, \frac{\mu_+(x)}{d(x)})$ *is linearly unstable as a steady-state solution of* (4.5). *Then,* $\delta > 0$ *exists such that* $(0, \theta_{\{d_2,\mu,d\}})$ *is linearly unstable for every* $d_1, d_2 < \delta$.

Proof. As (ii) follows by symmetry, we will only prove (i). The linearization of (4.2) at $(\theta_{\{d_1,\lambda,a\}}, 0)$ can be easily determined from (5.4) and provides us with the eigenvalue problem

$$\begin{cases} (d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}})\phi_{(d_1,d_2)} + b\theta_{\{d_1,\lambda,a\}}\psi_{(d_1,d_2)} = \sigma\phi_{(d_1,d_2)} & \text{in } \Omega, \\ (d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}})\psi_{(d_1,d_2)} = \sigma\psi_{(d_1,d_2)} & \text{in } \Omega, \\ \mathcal{B}_1\phi_{(d_1,d_2)} = \mathcal{B}_2\psi_{(d_1,d_2)} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

It suffices to establish the existence and negativity of one eigenvalue, $\sigma_{(d_1,d_2)}$, associated to an eigenfunction $(\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)})$, with $\phi_{(d_1,d_2)} \gg 0$ and $\psi_{(d_1,d_2)} \ll 0$. Actually, this eigenvalue must be the principal one of $d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}$. Let $\psi_{(d_1,d_2)} \ll 0$ be a principal eigenfunction associated to

$$\Sigma_{(d_1,d_2)} := \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega].$$

Then, the monotonicity with respect to the domain established by Lemma 2.14 provides us with the estimate

$$\Sigma_{(d_1,d_2)} = \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] < \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{D}, \Omega_{\text{un}}].$$

By the uniform convergence of $\theta_{\{d_1,\lambda,a\}}$ to $\frac{\lambda_+}{a}$ in $\bar{\Omega}_{\text{un}}$ as $d_1 \to 0$, it follows from Theorem 2.12 that

$$\lim_{(d_1,d_2)\to(0,0)} \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{D}, \Omega_{\text{un}}] = \min_{\bar{\Omega}_{\text{un}}} \left(-\mu + c\frac{\lambda_+}{a}\right) < 0,$$

because $\left(\frac{\lambda_+}{a}, 0\right)$ is linearly unstable in $\bar{\Omega}_{\rm un}$. Therefore, $\delta > 0$ exists such that

$$\Sigma_{(d_1,d_2)} < 0 \quad \text{for all} \;\; d_1, d_2 < \delta.$$

It remains to solve the first equation of (5.9), i.e.,

$$(d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}} - \Sigma_{(d_1,d_2)})\phi_{(d_1,d_2)} = -b\theta_{\{d_1,\lambda,a\}}\psi_{(d_1,d_2)} \quad \text{in} \;\; \Omega,$$

subject to

$$\mathcal{B}_1\phi_{(d_1,d_2)} = 0 \quad \text{on} \;\; \partial\Omega.$$

By the monotonicity with respect to the potential established by Theorem 2.12,

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}} - \Sigma_{(d_1,d_2)}; \mathcal{B}_1, \Omega] > \sigma_1[d_1\mathcal{L}_1 - \lambda + a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega] = 0.$$

Thus, the operator

$$\left(d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{\delta_1,\lambda,a\}} - \Sigma_{(d_1,d_2)}\right)^{-1}$$

is strongly positive. Therefore, the previous equation has a unique solution, $\phi_{(d_1,d_2)} \gg 0$, because $-b\theta_{\{d_1,\lambda,a\}}\psi_{(d_1,d_2)} > 0$. This ends the proof. □

5.3 Permanence for small diffusion rates when $\Omega_{\rm per} \neq \emptyset$, or $\Omega^u_{\rm do} \neq \emptyset$ and $\Omega^v_{\rm do} \neq \emptyset$

As a direct consequence from Proposition 5.11, the following substantial extension of Theorem 2.1 of Furter and López-Gómez [48] holds. Note that it also refines [39, Co. 4.6] as we are dealing with a general class of linear second order self-adjoint elliptic operators under general non-classical mixed boundary conditions, where the weight functions β_i of the boundary operators $\mathcal{B}_i$ are allowed to change sign. Therefore, the result should be considered new in its greatest generality.

Subsequently, the model (4.1), or (4.5), is said to be *permanent* if its trivial and semitrivial steady states are linearly unstable. In this case the theory of Hess [56] and López-Gómez [79] and, in particular, Theorems 3.1 and 4.1 of López-Gómez and Sabina [77], show that the model possesses a stable coexistence steady state, which is a global attractor with respect to the component-wise positive solutions of the model if it is unique. Note that the results in all those references can be easily adapted to cover our general framework here in.

Corollary 5.12. *Suppose that either $\Omega_{\rm per} \neq \emptyset$, or $\Omega^u_{\rm do} \cup \Omega^{{\rm per},u}_{\rm junk} \neq \emptyset$ and $\Omega^v_{\rm do} \cup \Omega^{{\rm per},v}_{\rm junk} \neq \emptyset$. Then, $\delta > 0$ exists such that the parabolic problem (4.1) is permanent for all $d_1, d_2 < \delta$. Therefore, it admits a stable coexistence steady state for these diffusion rates.*

Proof. Suppose $\Omega_{\rm per} \neq \emptyset$. Then, the semitrivial solutions of (4.5),

$$\left(\frac{\lambda(x)}{a(x)}, 0\right) \quad \text{and} \quad \left(0, \frac{\mu(x)}{d(x)}\right),$$

are linearly unstable for all $x \in \Omega_{\rm per}$. Thus, thanks to Proposition 5.11, $\delta > 0$ exists such that $(\theta_{\{d_1,\lambda,a\}}, 0)$ and $(0, \theta_{\{d_2,\mu,d\}})$ are linearly unstable for all $d_1 < \delta$ and $d_2 < \delta$, respectively. Therefore, (4.1) is permanent.

Now, suppose

$$\Omega^u_{\rm do} \cup \Omega^{{\rm per},u}_{\rm junk} \neq \emptyset \quad \text{and} \quad \Omega^v_{\rm do} \cup \Omega^{{\rm per},v}_{\rm junk} \neq \emptyset.$$

Then, $\left(\frac{\lambda_+(x)}{a(x)}, 0\right)$ is linearly unstable for all $x \in \Omega^v_{\rm do} \cup \Omega^{{\rm per},v}_{\rm junk}$, while $\left(0, \frac{\mu_+(x)}{d(x)}\right)$ is linearly unstable for all $x \in \Omega^u_{\rm do} \cup \Omega^{{\rm per},u}_{\rm junk}$. Proposition 5.11 ends the proof as in the previous case. □

Note that Corollary 5.12 holds independently of the size and the shape of $\Omega_{\rm per}$, $\Omega^u_{\rm do} \cup \Omega^{{\rm per},u}_{\rm junk}$ and $\Omega^v_{\rm do} \cup \Omega^{{\rm per},v}_{\rm junk}$. Essentially, this entails that (4.1) is permanent for sufficiently small diffusivities for many non-spatial kinetic patterns. This is an extremely surprising feature at the light shared by the following simple examples. Suppose $\Omega = \Omega_{\rm ext}$, and, in particular, that $\lambda(x) < 0$ and $\mu(x) < 0$ for all $x \in \Omega$. Then, not only the non-spatial model (4.5) exhibits extinction, but also the spatial model (4.1) for small diffusion rates. Suppose, in addition, that we perturb λ and mu until $\lambda(x_0) = 0$, $\mu(y_0) = 0$, $\lambda(y_0) < 0$ and $\mu(x_0) < 0$ for some $x_0, y_0 \in \Omega$, $x_0 \neq y_0$. Then, by slightly perturbing λ and μ, for instance taking $\lambda + \varepsilon$ and $\mu + \varepsilon$, for sufficiently small $\varepsilon > 0$, we can get $\lambda > 0$ and $\mu < 0$ on some small ball around x_0, and $\mu > 0$ and $\lambda < 0$ on some small ball around y_0. In these balls we have that $\Omega^u_{\rm do} \neq \emptyset$, respectively $\Omega^v_{\rm do} \neq \emptyset$. Therefore, by Corollary 5.12, (4.1) is permanent for sufficiently small d_1 and d_2. Similarly, if $\lambda(x_0) = \mu(x_0) = 0$ and $a(x_0)d(x_0) > b(x_0)c(x_0)$ for some $x_0 \in \Omega$, we can perturb λ and μ nearby x_0 in such a way that $\Omega_{\rm per} \neq \emptyset$ around x_0. This simple example tells us how a very small perturbation of the coefficients can provoke dramatic changes on the dynamics of the diffusive model, at least for small diffusion rates. The independence of these permanence results on the size of the regions where the non-spatial model (4.5) is permanent, and on the sizes of the regions where there is dominance of u and v, is utterly attributable to the Induced Instability Principle established by Proposition 5.11, according with it the semitrivial solutions of the spatial model are linearly unstable for small diffusion rates as soon as the semitrivial steady-state solutions of the non-spatial model are linearly unstable somewhere in Ω.

5.4 Multiplicity for small diffusion rates when $\Omega_{\rm bi} \neq \emptyset$

This section is devoted to establish the existence of multiple coexistence steady states of (4.1) when a region of bi-stability, $\Omega_{\rm bi}$, arises in Ω. In particular, the next multiplicity result follows as an immediate consequence of the Induced Instability Principle derived in Theorem 5.9.

Corollary 5.13. *Suppose (4.1) is permanent for sufficiently small diffusion rates, $d_1, d_2 > 0$, $\Omega_{\rm bi} \neq \emptyset$ and the non-spatial coexistence steady-state solution of (4.5), $(u^*(x), v^*(x))$, $x \in \Omega_{\rm bi}$, admits a perturbed coexistence steady state of (4.1), $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$, for sufficiently small $d_1, d_2 > 0$, in the sense that*

$$\lim_{(d_1,d_2)\to(0,0)} (u_{(d_1,d_2)}, v_{(d_1,d_2)}) = (u^*, v^*) \quad \textit{uniformly on compact subsets of } \Omega_{\rm bi}.$$

Then, $\delta > 0$ exists such that (4.1) possesses at least two coexistence states for each $d_1, d_2 < \delta$.

Proof. By Theorem 5.9, the perturbation $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ must be linearly unstable for small diffusion rates. According to Theorem 3.1 of López-Gómez and Sabina [77], (4.1) possesses a stable coexistence state for sufficiently small $d_1, d_2 > 0$, because it is permanent. This ends the proof. □

It remains an open problem to ascertain whether or not such a perturbation from (u^*, v^*) exists. However, the next section provides us with an example exhibiting this behavior.

5.4.1 The symmetric model

Subsequently, we consider the symmetric Lotka–Volterra reaction-diffusion symmetric competition model, i.e., (4.1) under the assumptions

$$\mathcal{L} := \mathcal{L}_1 = \mathcal{L}_2, \quad \mathcal{B} := \mathcal{B}_1 = \mathcal{B}_2, \quad \delta := d_1 = d_2 > 0, \quad \lambda = \mu,$$

with $\max_{\bar\Omega} \lambda > 0$,

$$a = d \quad \text{with} \quad \min_{\bar\Omega} a > 0,$$

and $b = c$ with $b(x) > 0$ for all $x \in \Omega$. Hence, its elliptic counterpart is given by

$$\begin{cases} \delta\mathcal{L}u = \lambda(x)u - a(x)u^2 - b(x)uv & \text{in } \Omega, \\ \delta\mathcal{L}v = \lambda(x)v - a(x)v^2 - b(x)uv & \text{in } \Omega, \\ \mathcal{B}u = \mathcal{B}v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

Note that, under the previous assumptions, $\Omega^u_{\rm do} = \Omega^v_{\rm do} = \emptyset$ and hence,

$$\bar\Omega = \Omega_{\rm per} \cup \Omega_{\rm bi} \cup \Omega_{\rm ext} \cup \Omega_{\rm junk},$$

which allows $\Omega_{\rm per}$ and $\Omega_{\rm bi}$ to be nonempty. Moreover, thanks to the symmetry of the problem, for every solution of (5.10), (u, v), with $u \neq v$, we have that (v, u) also is a solution. Furthermore, (5.10) admits a solution with $u = v$, as shown by the next result.

The main aim of this section is to derive Theorem 5.16, which establishes the existence of at least three coexistence states of (5.10) for small diffusion rates, two of them stable and the remaining one linearly unstable. In particular, Theorem 5.16 follows from Corollary 5.12 and a series of lemmas.

Lemma 5.14. *Assume that $\max_{\bar\Omega} \lambda > 0$. Then, there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ the problem (5.10) admits a unique coexistence state, (u, v), with $u = v$, given by (w_δ, w_δ) with $w_\delta := \theta_{\{\delta,\lambda,a+b\}}$. Moreover, it converges to $\left(\frac{\lambda_+}{a+b}, \frac{\lambda_+}{a+b}\right)$ uniformly on compact subsets of $\Omega \cup \Gamma^+_{\mathcal{R}} \cup (\lambda_+)^{-1}(0)$ as $\delta \downarrow 0$.*

Proof. The pair (w,w) is a component-wise positive solution of (5.10) if and only if w satisfies

$$\begin{cases} \delta\mathcal{L}w = \lambda(x)w - (a(x)+b(x))w^2 & \text{in } \Omega, \\ \mathcal{B}w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

By Corollary 2.17(b), $\delta_0 > 0$ exists such that, for every $\delta \in (0,\delta_0)$, (5.11) admits a unique positive solution, $w_\delta := \theta_{\{\delta,\lambda,a+b\}}$. Its limiting behavior as $\delta \downarrow 0$ follows from Theorem 2.21. □

Note that the coexistence state whose existence has been established by Lemma 5.14 actually exists if and only if $\sigma_1[\delta\mathcal{L}-\lambda;\mathcal{B},\Omega] < 0$, which is the same condition guaranteeing the existence of the semitrivial states. By a rather standard comparison argument, it readily follows that, in general, the existence of the semitrivial states is necessary for the existence of coexistence states.

On the other hand, Theorem 5.16 requires the analysis of the attractivity properties of the coexistence states. The next result provides us with the instability of (w_δ,w_δ) for sufficiently small $\delta > 0$ when $\Omega_{\rm bi} \neq \emptyset$.

Lemma 5.15. *If $\Omega_{\rm bi} \neq \emptyset$, then there exists $\delta_{\rm un} > 0$ such that (w_δ,w_δ) is linearly unstable for all $\delta \in (0,\delta_{\rm un})$.*

Proof. Since $\Omega_{\rm bi} \neq \emptyset$, $\max_{\bar\Omega}\lambda > 0$. Thus, Lemma 5.14 can be applied to infer that (w_δ,w_δ) is a coexistence state for $\delta < \delta_0$ that converges uniformly on compact subsets of Ω to $(\frac{\lambda_+}{a+b},\frac{\lambda_+}{a+b})$ as $\delta \to 0$. As there is a smooth open subset of $\Omega\cap\Omega_{\rm bi}$, D, with $\bar D \subset \Omega$, such that, for every $x \in \bar D$, the coexistence state

$$\left(\frac{\lambda_+(x)}{a(x)+b(x)},\frac{\lambda_+(x)}{a(x)+b(x)}\right)$$

is linearly unstable as a coexistence state of the non-spatial model, it follows from Theorem 5.9 that (w_δ,w_δ) must be linearly unstable for sufficiently small $\delta > 0$. This ends the proof. □

Combining this result with Corollary 5.12 provides us with the multiplicity result.

Theorem 5.16. *Assume that*

$$\Omega_{\rm per} \neq \emptyset \quad \textit{and} \quad \Omega_{\rm bi} \neq \emptyset.$$

Then $\delta_{\rm m} > 0$ exists such that, for every $\delta \in (0,\delta_{\rm m})$, (5.10) admits at least three coexistence states; two of them linearly stable and another one linearly unstable. Moreover, such linearly unstable coexistence state perturbs from the coexistence steady state of the non-spatial problem in the region $\Omega_{\rm bi}\cup\Omega_{\rm per}$.

Proof. Since $\Omega_{\rm per} \neq \emptyset$, by Corollary 5.12, the problem (5.10) is permanent for sufficiently small $\delta > 0$. Thus, it admits a linearly stable coexistence state, (u_δ,v_δ). Moreover, since $\Omega_{\rm bi} \neq \emptyset$, according to Lemma 5.15, for sufficiently small δ, the coexistence state (w_δ,w_δ) is linearly unstable. Hence, $(u_\delta,v_\delta) \neq (w_\delta,w_\delta)$. Since (w_δ,w_δ) provides us with the unique coexistence state, (u,v), such that $u = v$, we find that $u_\delta \neq v_\delta$ and therefore, (v_δ,u_δ)

provides us with a third coexistence state with the same stability character. Note that, by similar reasons, the remaining coexistence states must appear by pairs. Thus, either the spatial model (5.10) exhibits a continuum of coexistence states, or it admits an odd number of them. □

In the context of Theorem 5.16, Theorem 4.4 provides us with the limiting profiles of all the coexistence steady states of (5.10) in

$$\Omega_{\max} \cap (\Omega \cup \Gamma_{\mathcal{R}}^{\text{per}}) = (\Omega_{\text{per}} \cup \Omega_{\text{ext}}) \cap (\Omega \cup \Gamma_{\mathcal{R}}^{\text{per}})$$

as $\delta \downarrow 0$, where the three coexistence states constructed in Theorem 5.16 approximate $\left(\frac{\lambda_+}{a+b}, \frac{\lambda_+}{a+b}\right)$ as $\delta \downarrow 0$. Note that $\left(\frac{\lambda_+}{a+b}, \frac{\lambda_+}{a+b}\right)$ is a coexistence state (if $\lambda(x) > 0$), or the trivial solution (if $\lambda(x) \leq 0$), of the non-spatial model. The analysis of the precise behavior of the stable coexistence states constructed in Theorem 5.16 in the region Ω_{bi} remains open, though it seems apparent that they should perturb from each of the semi-trivial steady state solutions in the region Ω_{bi} as $\delta > 0$ perturbs from zero. Nevertheless, according to Theorem 5.9 and Proposition 5.10, they cannot perturb uniformly from the trivial solution or the coexistence steady state of the non-spatial model.

Chapter 6

Uniqueness of the coexistence state. Global dynamics

Introduction

Our multiplicity result in Section 5.4 allows us to establish the optimality of a substantial extension of the main uniqueness theorem of Hutson, Lou and Mischaikow [65] (see Theorem 1.1 and Proposition 3.5 therein) that establishes the uniqueness of a coexistence steady-state solution of (4.1) for sufficiently small d_1 and d_2 under the rather natural assumption that

$$\Omega_{\text{per}} = \bar{\Omega}.$$

In such case, the unique coexistence steady state must be a global attractor with respect to the component-wise positive solutions of (4.1). Particularly, the multiplicity result when Ω_{bi} is non-empty shows that $\Omega_{\text{per}} = \bar{\Omega}$ is optimal for the uniqueness in the following sense. If we replace $b(x)$ and $c(x)$ by $\rho b(x)$ and $\rho c(x)$, where $\rho > 0$ is regarded as a real parameter, then there are choices of $b(x)$ and $c(x)$ for which condition $\Omega_{\text{per}} = \bar{\Omega}$ holds for all $\rho \in (0,1)$, but it fails at the single point $x_0 \in \Omega$ when $\rho = 1$. So, $\Omega_{\text{per}} = \bar{\Omega} \setminus \{x_0\}$ if $\rho = 1$. As there are examples of $b(x)$ and $c(x)$ with non-empty Ω_{bi} for $\rho > 1$ sufficiently close to 1, such that Ω_{bi} shrinks to x_0 as $\rho \downarrow 1$, our multiplicity theorem shows that if $\Omega_{\text{per}} = \bar{\Omega}$ fails at a single point $x_0 \in \bar{\Omega}$, then the problem (4.1) might exhibit a bifurcation to multiple coexistence steady-steady solutions.

In addition to the fact that our extension of the uniqueness theorem of Hutson, Lou and Mischaikow [65], collected in Theorems 6.9 and 6.10 of Section 6.2, is new in its greatest generality, because it is valid for a general class of differential operators subject to general mixed boundary conditions, the proof given in this chapter overcomes the highly sophisticated technicalities of the proof of Proposition 3.5 of Hutson, Lou and Mischaikow [65] by means of a quasi-cooperative version for mixed boundary conditions of Theorem 2.1 of López-Gómez and Molina-Meyer [90], Theorem 6.3 of López-Gómez and Sabina [77] and Theorem 2.4 of Amann and López-Gómez [5]. Indeed, our proof is a rather direct, very elegant, application of the theorem of characterization of the strong maximum principle through the construction of an appropriate supersolution. It should be noted that Proposition 3.5 of Hutson, Lou and Mischaikow [65] was found for Neumann boundary conditions.

The last goal of this thesis is establishing a general, rather astonishing, uniqueness result covering the general case when $\Omega_{\rm per}$ is a proper subset of $\bar{\Omega}$. Naturally, according to our main multiplicity result, established in Chapter 5, in order to get uniqueness one should assume that $\Omega_{\rm bi} = \emptyset$. The simplest way to get it is imposing that

$$bc \lneq ad \quad \text{in } \Omega. \tag{6.1}$$

Under (6.1), our general uniqueness theorem precisely stated in Theorem 6.13, establishes that, if

$$\max_{\bar{\Omega}} \left(\frac{ad^2}{c^3} F_-(\kappa) \right) \leq \min_{\bar{\Omega}} \left(\frac{ad^2}{c^3} F_+(\kappa) \right), \tag{6.2}$$

with

$$F_\pm(k) := \frac{1}{8} \left[27 - 18k - k^2 \pm (9-k)^{3/2}(1-k)^{1/2} \right], \qquad k \in [0,1],$$

then any coexistence state of (4.2) must be linearly stable and hence, unique, if it exists. In particular, we obtain (6.2) if either $\frac{b^2}{ac}$, or $\frac{b^3}{a^2 d}$, or $\frac{c^2}{db}$, or $\frac{c^3}{ad^2}$, is a positive constant in Ω (see Corollary 6.12). Naturally, these conditions hold when $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are positive constants such that $a = d = 1$ and $bc < 1$, as it was recently imposed by He and Ni [55]. Consequently, our result, Theorem 6.13, provides us with an extremely sharp and substantial extension of Theorem 3.4(iii) of He and Ni [55], because it is valid for general spatially heterogeneous systems subject to mixed boundary conditions.

As a byproduct of Theorem 6.13, under the previous assumptions, as soon as the model possesses two non-degenerate semi-trivial positive steady states, (4.1) exhibits three different types of behavior. Namely, either both semi-trivial positive solutions are linearly unstable, and then the problem has a unique coexistence steady state which is a global attractor with respect to the component-wise positive solutions of (4.1), or one of the semi-trivial positive solutions is linearly stable, while the other one is linearly unstable, and in such case the stable one must be a global attractor, much like in the non-spatial model.

As far as concerns the restrictions imposed on the function coefficients $a(x)$, $b(x)$, $c(x)$ and $d(x)$ in Theorem 6.13, and more specifically in Corollary 6.12, it should be noted that each of them involves three of these four coefficients: either a, b, c in the assumption that $\frac{b^2}{ac}$ is constant in Ω, or a, b, d if we impose that $\frac{b^3}{a^2 d}$ is constant, or a, c, d when $\frac{c^3}{ad^2}$ is constant, or b, c, d if, instead, $\frac{c^2}{db}$ is constant in Ω. Thus, in either cases we have complete freedom to chose, arbitrarily, three of the coefficients, while the fourth one is uniquely determined by the remaining ones, up to a positive multiplicative constant chosen to satisfy (6.1). Rather astonishingly, in this uniqueness theorem $\lambda(x)$ and $\mu(x)$ can be chosen arbitrarily.

This chapter has been organized as follows. Section 6.1 is devoted to the calculation of the fixed point index of the steady states of (4.1) in order to derive the uniqueness of coexistence steady states when all of them are linearly stable. Section 6.2 proves the uniqueness result in the special case when $\Omega_{\rm per} = \bar{\Omega}$. Finally, Section 6.3 derives the general uniqueness result when $\Omega_{\rm per}$ is a proper subset of $\bar{\Omega}$ through Picone's identity by assuming that the differential operators are selfadjoint.

6.1 Towards a characterization of the global dynamics

The uniqueness results to be delivered in this chapter will follow from the fact that if all coexistence steady states in the competition model (4.1) are linearly stable, then there exists, at most, only one of them. Much like in López-Gómez [79] and López-Gómez and Sabina [77], this section is devoted to provide such auxiliary result by applying the fixed point index to a certain integral operator associated to the problem

$$\begin{cases} d_1\mathcal{L}_1 u = \lambda u - au^2 - \gamma buv & \text{in } \Omega, \\ d_2\mathcal{L}_2 v = \mu v - dv^2 - \gamma cuv & \text{in } \Omega, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

where $\gamma \in [0,1]$ is regarded as an homotopy parameter to uncouple (4.2). Subsequently, we will set

$$W^{2,\infty}_{\mathcal{B}_i} := \bigcap_{p>N} W^{2,p}_{\mathcal{B}_i}(\Omega) \quad \text{and} \quad P_{W^{2,\infty}_{\mathcal{B}_i}} = \{w \in W^{2,\infty}_{\mathcal{B}_i} \; : \; w \geq 0 \; \text{ in } \; \Omega\}, \quad i = 1, 2,$$

which shares the notation introduced in (2.19). Then, $w \in \operatorname{int} P_{W^{2,\infty}_{\mathcal{B}_i}}$, for some $i = 1, 2$, if $w \in W^{2,\infty}_{\mathcal{B}_i}$ and satisfies (2.18) with $(\Gamma_{\mathcal{R}}, \Gamma_{\mathcal{D}}) = (\Gamma^i_{\mathcal{R}}, \Gamma^i_{\mathcal{D}})$.

The next result provides us with a bounded open set independent of $b, c > 0$ containing all the non-negative solutions of (4.2). In particular, it contains all the non-negative solutions of (6.3) uniformly in $\gamma \in [0,1]$. This is crucial in order to apply the fixed point index in cones.

Lemma 6.1. *There exits a bounded set*

$$\mathcal{U} \times \mathcal{V} \subset W^{2,\infty}_{\mathcal{B}_1} \times W^{2,\infty}_{\mathcal{B}_2},$$

independent of b and c, such that $(u, v) \in \operatorname{int}(\mathcal{U} \times \mathcal{V})$ if (u, v) is a solution of (4.2) with

$$(u, v) \in P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}. \tag{6.4}$$

Proof. It suffices to note that, if (u, v) is a solution of (4.2) satisfying (6.4), then

$$\begin{cases} d_1\mathcal{L}_1 u = \lambda u - au^2 - buv \leq \lambda u - au^2 & \text{in } \Omega, \\ \mathcal{B}_1 u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} d_2\mathcal{L}_2 v = \mu v - dv^2 - cuv \leq \mu v - dv^2 & \text{in } \Omega, \\ \mathcal{B}_2 v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, u and v are subsolutions of the associated logistic boundary value problems. To construct the appropriate supersolutions to these problems we will proceed much like in Section 2.2.2. Consider E defined by

$$E(x) := \exp\left(-M \operatorname{dist}_{\partial\Omega}(x)\right) \quad \text{with } \; M > \max_{\partial\Omega} \left\{0, \frac{-\beta_1}{\langle \mathbf{n}, A_1\mathbf{n}\rangle}, \frac{-\beta_2}{\langle \mathbf{n}, A_2\mathbf{n}\rangle}\right\},$$

on a sufficiently narrow neighborhood of $\partial\Omega$. According to Theorem 2.3, this function is of class $\mathcal{C}^2$, and can be extended to the entire $\bar{\Omega}$ with smoothness and positiveness by mean of cut-off functions, as sugested in Remark 2.4. Furthermore, E satisfies

$$\mathcal{B}_i E > 0 \quad \text{on} \;\; \Gamma^i_{\mathcal{D}},$$

whereas, on $\Gamma^i_{\mathcal{R}}$,

$$\mathcal{B}_i E = \langle \mathbf{n}, A_i \nabla E\rangle + \beta_i E = -EM\langle \mathbf{n}, A_i \nabla \mathrm{dist}_{\partial\Omega}(\cdot)\rangle + \beta_i E = E\left(M\langle \mathbf{n}, A_i \mathbf{n}\rangle + \beta_i\right) > 0$$

Note that, in the definition of $E(x)$, the function $-\mathrm{dist}_{\partial\Omega}(\cdot)$ can be changed by any function, ψ, of class $\mathcal{C}^2$, like those derived in Theorem 2.3(d), i.e., such that $\psi(x) < 0$ for all $x \in \Omega$, $\psi(x) = 0$ for all $x \in \partial\Omega$ and

$$\min_{\Gamma_{\mathcal{R}}} \langle \mathbf{n}, A_i \nabla \psi\rangle > 0, i = 1, 2.$$

Hence, if $\kappa > 0$ is a constant such that

$$\kappa > \max\left\{ \max_{\bar{\Omega}} \frac{\lambda - d_1 \frac{\mathcal{L}_1 E}{E}}{aE}, \max_{\bar{\Omega}} \frac{\mu - d_2 \frac{\mathcal{L}_2 E}{E}}{dE} \right\},$$

then κE provides us with a supersolution for both problems. Therefore, by the uniqueness of solution to these problems, it follows from Theorem 7.10 of López-Gómez [88], or Lemma 3.4 of Fraile, Koch, López-Gómez and Merino [46], that

$$0 \le u \le \theta_{\{d_1,\lambda,a\}} \le \kappa E \quad \text{and} \quad 0 \le v \le \theta_{\{d_2,\mu,d\}} \le \kappa E \quad \text{in} \;\; \Omega.$$

This ends the proof. □

Remark 6.2. As an immediate, but important, by-product of Lemma 6.1, we have that the existence of semitrivial solutions is a necessary condition for the existence of coexistence solutions of (4.2). Moreover, according to Theorem 2.15, model 4.2 admits semitrivial solutions, given by $(\theta_{\{d_1,\lambda,a\}}, 0)$ and $(0, \theta_{\{d_2,\mu,d\}})$, if, and only if, $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$ and $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$, respectively.

6.1.1 Fixed point index calculations

Subsequently, let us consider $\mathcal{U} \times \mathcal{V}$, the bounded set provided by Lemma 6.1. Let $m > 0$ be large enough so that

$$\sigma_1[d_1\mathcal{L}_1 + m; \mathcal{B}_1, \Omega] > 1, \qquad \sigma_1[d_2\mathcal{L}_2 + m; \mathcal{B}_2, \Omega] > 1, \tag{6.5}$$

and

$$\lambda - au - \gamma bv + m > 0 \quad \text{and} \quad \mu - dv - \gamma cu + m > 0 \quad \text{in} \;\; \bar{\Omega}$$

for all $(\gamma, u, v) \in [0, 1] \times \mathcal{U} \times \mathcal{V}$. Note that, in particular,

$$\lambda + m > 0 \quad \text{and} \quad \mu + m > 0 \quad \text{in} \;\; \Omega, \tag{6.6}$$

because $(0,0)\in\mathcal{U}\times\mathcal{V}$, by construction.

Consider the family of operators $\mathcal{I}:[0,1]\times\mathcal{U}\times\mathcal{V}\to W^{2,\infty}_{\mathcal{B}_1}\times W^{2,\infty}_{\mathcal{B}_2}$ defined through

$$\mathcal{I}(\gamma,u,v):=\begin{pmatrix}(d_1\mathcal{L}_1+m)^{-1}[(\lambda-au-\gamma bv+m)u]\\(d_2\mathcal{L}_2+m)^{-1}[(\mu-dv-\gamma cu+m)v]\end{pmatrix},$$

which are compact order preserving operators, by our assumptions on m. Moreover, their fixed points are the solutions of (6.3) in $P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}$, for each $\gamma\in[0,1]$, respectively. The following results provide us with the fixed point indices of the trivial, semitrivial and coexistence states of (4.2) as fixed points of $\mathcal{I}$ for $\gamma=1$.

Lemma 6.3. *The total fixed point index of $\mathcal{I}(1,\cdot,\cdot)$ in $\operatorname{int}(\mathcal{U}\times\mathcal{V})$ is given by*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V}))=1.$$

Proof. By the homotopy invariance property of the fixed point index (see Theorem 1.11(iii) of Amann [3]),

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V}))=i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(0,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V})).$$

Thus, owing to the product formula, we find that

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V}))=i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{I}_1(\cdot),\operatorname{int}\mathcal{U})\cdot i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{I}_2(\cdot),\operatorname{int}\mathcal{V}),$$

where $\mathcal{I}_1:\mathcal{U}\to W^{2,\infty}_{\mathcal{B}_1}$ and $\mathcal{I}_2:\mathcal{V}\to W^{2,\infty}_{\mathcal{B}_2}$ are the operators defined by

$$\mathcal{I}_1(u):=(d_1\mathcal{L}_1+m)^{-1}[(\lambda-au+m)u]\quad\text{and}\quad\mathcal{I}_2(v):=(d_2\mathcal{L}_2+m)^{-1}[(\mu-dv+m)v].$$

Now, consider the homotopies

$$\mathcal{G}_1(\gamma,u):=(d_1\mathcal{L}_1+m)^{-1}\left[(m+\sigma_1[d_1\mathcal{L}_1;\mathcal{B}_1,\Omega]-1+\gamma(\lambda-\sigma_1[d_1\mathcal{L}_1;\mathcal{B}_1,\Omega]+1-au))\,u\right],$$
$$\mathcal{G}_2(\gamma,v):=(d_2\mathcal{L}_2+m)^{-1}\left[(m+\sigma_1[d_2\mathcal{L}_2;\mathcal{B}_2,\Omega]-1+\gamma(\mu-\sigma_1[d_2\mathcal{L}_2;\mathcal{B}_2,\Omega]+1-dv))\,v\right].$$

By the homotopy invariance property,

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{I}_1(\cdot),\operatorname{int}\mathcal{U})=i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{G}_1(1,\cdot),\operatorname{int}\mathcal{U})=i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{G}_1(0,\cdot),\operatorname{int}\mathcal{U}),$$
$$i_{P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}_2(\cdot),\operatorname{int}\mathcal{V})=i_{P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{G}_2(1,\cdot),\operatorname{int}\mathcal{V})=i_{P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{G}_2(0,\cdot),\operatorname{int}\mathcal{V}).$$

On the other hand, 0 is the unique fixed point of $\mathcal{G}_1(0,\cdot)$ in $\operatorname{int}\mathcal{U}$ and $\mathcal{G}_2(0,\cdot)$ in $\operatorname{int}\mathcal{V}$. Moreover, the spectral radio of $\mathcal{G}_j(0,\cdot)$, $j=1,2$, is

$$\varrho(\mathcal{G}_j(0,\cdot))=\frac{m+\sigma_1[d_j\mathcal{L}_j;\mathcal{B}_j,\Omega]-1}{m+\sigma_1[d_j\mathcal{L}_j;\mathcal{B}_j,\Omega]}<1.$$

Thus, thanks to Theorem 13.1 of Amann [3], we can infer that

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}}(\mathcal{G}_1(0,\cdot),\operatorname{int}\mathcal{U})=i_{P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{G}_2(0,\cdot),\operatorname{int}\mathcal{V})=1.$$

This ends the proof. □

The next result provides us with the fixed point index of $(0,0)$ when it is non-degenerate.

Lemma 6.4. *Assume that*

$$\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]\cdot\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]\neq 0.$$

Then, the following statements hold:

(a) *If* $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]>0$ *and* $\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]>0$, *then*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0))=1.$$

(b) *If* $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]<0$, *or* $\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]<0$, *then*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0))=0.$$

Proof. Differentiating $\mathcal{I}(1,\cdot,\cdot)$ with respect to (u,v) and particularizing at $(0,0)$ yields

$$D_{(u,v)}\mathcal{I}(1,0,0)(u,v)=\begin{pmatrix}(d_1\mathcal{L}_1+m)^{-1}[(\lambda+m)u]\\(d_2\mathcal{L}_2+m)^{-1}[(\mu+m)v]\end{pmatrix}.$$

Suppose that

$$\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]>0\quad\text{and}\quad\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]>0, \tag{6.7}$$

which entails the linearized stability of $(0,0)$ and the non-existence of semitrivial solutions of (6.3).

Let $r_0\in\mathbb{R}$ be an eigenvalue of $D_{(u,v)}\mathcal{I}(1,0,0)$ to a component-wise non-negative eigenvector, $(\varphi,\psi)\neq(0,0)$. Without loss of generality, we can assume that $\varphi>0$. Then,

$$\sigma_1\left[d_1\mathcal{L}_1+m-\frac{m+\lambda}{r_0};\mathcal{B}_1,\Omega\right]=0.$$

Moreover, thanks to (6.6), by the strict monotonicity and continuity of the principal eigenvalue with respect to the potential delivered in Theorem 2.12, it becomes apparent that the map

$$r\mapsto\sigma_1\left[d_1\mathcal{L}_1+m-\frac{m+\lambda}{r};\mathcal{B}_1,\Omega\right]$$

is strictly increasing and, in addition, is continuous in $(0,+\infty)$. Taking into account (6.5),

$$\lim_{r\to\xi}\sigma_1\left[d_1\mathcal{L}_1+m-\frac{m+\lambda}{r};\mathcal{B}_1,\Omega\right]=\begin{cases}\sigma_1[d_1\mathcal{L}_1+m;\mathcal{B}_1,\Omega]>1 & \text{if }\ \xi=+\infty,\\ \sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega] & \text{if }\ \xi=1,\\ -\infty & \text{if }\ \xi=0.\end{cases} \tag{6.8}$$

By (6.7) and (6.8), we have that $r_0<1$. Thus, $D_{(u,v)}\mathcal{H}(1,0,0)$ cannot admit a positive eigenvector to an eigenvalue greater or equal than one. Therefore, owing to Theorem 13.1 of Amann [3], we find that

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0))=1.$$

Now, assume that some of the principal eigenvalues in (6.7) is negative, instead of positive. Without loss of generality, suppose that

$$\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]<0.$$

Then, by (6.8), there is a unique $r_0>1$ such that

$$\sigma_1\left[d_1\mathcal{L}_1+m-\frac{m+\lambda}{r_0};\mathcal{B}_1,\Omega\right]=0.$$

Let $\varphi>0$ be any principal eigenfunction associated to this eigenvalue. Then, $(\varphi,0)$ provides us with a positive eigenvector of $D_{(u,v)}\mathcal{I}(1,0,0)$ to an eigenvalue greater than one. Therefore, thanks to Theorem 13.1 of Amann [3],

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0))=0.$$

This ends the proof. □

To calculate the indices of the semitrivial solutions we will make an intensive use of Lemma 4.1 of López-Gómez [79], which goes back to Lemmas 2 and 4 of Dancer [26]. In our setting, it can be stated as follows. An analogous version holds true for $(0,\theta_{\{d_2,\mu,d\}})$. Subsequently, we will denote by

$$\mathrm{Proj}_2: W^{2,\infty}_{\mathcal{B}_1}\times W^{2,\infty}_{\mathcal{B}_2}\to\{0\}\times W^{2,\infty}_{\mathcal{B}_2}$$

the projection on the second component, i.e., $\mathrm{Proj}_2(u,v):=(0,v)$.

Lemma 6.5. *Assume that $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]<0$. So, $\theta_{\{d_1,\lambda,a\}}\neq 0$. Then, the following assertions hold:*

(a) *If the operator $I-D_{(u,v)}\mathcal{I}(1,\theta_{\{d_1,\lambda,a\}},0)$ is injective in $W^{2,\infty}_{\mathcal{B}_1}\times W^{2,\infty}_{\mathcal{B}_2}$ and the spectral radius of the operator*

$$\mathrm{Proj}_2 D_{(u,v)}\mathcal{I}(1,\theta_{\{d_1,\lambda,a\}},0)_{|\{0\}\times W^{2,\infty}_{\mathcal{B}_2}}$$

is greater than one, then

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(\theta_{\{d_1,\lambda,a\}},0))=0.$$

(b) *If the operator $I-D_{(u,v)}\mathcal{I}(1,\theta_{\{d_1,\lambda,a\}},0)$ is injective in $W^{2,\infty}_{\mathcal{B}_1}\times W^{2,\infty}_{\mathcal{B}_2}$ and the spectral radius of the operator*

$$\mathrm{Proj}_2 D_{(u,v)}\mathcal{I}(1,\theta_{\{d_1,\lambda,a\}},0)_{|\{0\}\times W^{2,\infty}_{\mathcal{B}_2}}$$

is less than one, then

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(\theta_{\{d_1,\lambda,a\}},0))=(-1)^{\chi},$$

where χ stands for the sum of the algebraic multiplicities of the eigenvalues of the operator $D_{(u,v)}\mathcal{I}(1,\theta_{\{d_1,\lambda,a\}},0)$ greater than one.

(c) *If* $I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$ *is injective in* $W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$*, instead of in* $W^{2,\infty}_{\mathcal{B}_1} \times W^{2,\infty}_{\mathcal{B}_2}$*, and there exists* $w \in W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$ *such that the equation*

$$(I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0))y = w$$

has no solution $y \in W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$*, then*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1, \cdot, \cdot), (\theta_{\{d_1,\lambda,a\}}, 0)) = 0.$$

As a direct consequence of Lemma 6.5, the next result establishes that the fixed point index of each semitrivial solution is determined by its linear stability as a steady-state solution of(4.1), as soon as it is a non-degenerate steady state.

Lemma 6.6. *The following statements hold:*

(a) *Assume that* $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$*. Then* $\theta_{\{d_1,\lambda,a\}} \gg 0$ *and*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1, \cdot, \cdot), (\theta_{\{d_1,\lambda,a\}}, 0)) = \begin{cases} 0 & \text{if } \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] < 0, \\ 1 & \text{if } \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0, \end{cases} \tag{6.9}$$

(b) *Assume that* $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$*. Then* $\theta_{\{d_2,\mu,d\}} \gg 0$ *and*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1, \cdot, \cdot), (0, \theta_{\{d_2,\mu,d\}})) = \begin{cases} 0 & \text{if } \sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] < 0, \\ 1 & \text{if } \sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] > 0. \end{cases} \tag{6.10}$$

Proof. We will only prove (6.9), because (6.10) follows by symmetry. Since

$$\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0,$$

by Theorem 2.15 and Corollary 2.16, $\theta_{\{d_1,\lambda,a\}} \gg 0$. On the other hand, differentiating $\mathcal{I}(1, \cdot, \cdot)$ at this semitrivial solution yields

$$D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)(u, v) = \begin{pmatrix} (d_1\mathcal{L}_1 + m)^{-1}\left[(\lambda - 2a\theta_{\{d_1,\lambda,a\}} + m)u - b\theta_{\{d_1,\lambda,a\}}v\right] \\ (d_2\mathcal{L}_2 + m)^{-1}[(\mu - c\theta_{\{d_1,\lambda,a\}} + m)v] \end{pmatrix}.$$

Moreover, thanks to (6.5) and (6.6), $D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$ maps $W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$ into itself. Assume that

$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0. \tag{6.11}$$

Then, we claim that $I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$ is injective on $W^{2,\infty}_{\mathcal{B}_1} \times W^{2,\infty}_{\mathcal{B}_2}$. Indeed, if there exists $(u, v) \in W^{2,\infty}_{\mathcal{B}_1} \times W^{2,\infty}_{\mathcal{B}_2}$ such that

$$D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)(u, v) = (u, v),$$

then

$$\left(d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}}\right)u = -b\theta_{\{d_1,\lambda,a\}}v \tag{6.12}$$

and

$$\left(d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}\right) v = 0. \tag{6.13}$$

If $v \neq 0$, then 0 is an eigenvalue of the tern

$$\left[d_2\mathcal{L}_2 - \mu - c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega\right].$$

Thus,

$$\sigma_1[d_2\mathcal{L}_2 - \mu - c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] \leq 0,$$

which contradicts (6.11). Hence, $v = 0$ and (6.12) becomes

$$\left(d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}}\right) u = 0.$$

If $u \neq 0$, then 0 is an eigenvalue of $d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}}$ in Ω under $\mathcal{B}_1$. Thus,

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega] \leq 0. \tag{6.14}$$

Consequently, by the monotonicity of the principal eigenvalue with respect to the potential established in Theorem 2.12, we have that

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega] < \sigma_1[d_1\mathcal{L}_1 - \lambda + 2a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega] \leq 0. \tag{6.15}$$

This contradicts the fact that

$$(d_1\mathcal{L}_1 - \lambda + a\theta_{\{d_1,\lambda,a\}})\theta_{\{d_1,\lambda,a\}} = 0$$

because this entails that

$$\sigma_1[d_1\mathcal{L}_1 - \lambda + a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega] = 0. \tag{6.16}$$

Therefore, $(u, v) = (0, 0)$ and, hence,

$$I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$$

is injective. For applying Lemma 6.5, it remains to estimate the spectral radio of the operator

$$\mathrm{Proj}_2 D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)v := (d_2\mathcal{L}_2 + m)^{-1}[(\mu + c\theta_{\{d_1,\lambda,a\}} + m)v], \quad v \in W^{2,\infty}_{\mathcal{B}_2}.$$

A direct calculation shows that the spectral radius, r_0, of this operator satisfies

$$\sigma_1\left[d_2\mathcal{L}_2 + m - \frac{\mu - c\theta_{\{d_1,\lambda,a\}} + m}{r_0}; \mathcal{B}_2, \Omega\right] = 0.$$

Arguing as in the proof of Lemma 6.4, since

$$\mu - c\theta_{\{d_1,\lambda,a\}} + m > 0 \quad \text{in } \bar{\Omega}$$

by the hypothesis on m, it is apparent that the map $S : (0, +\infty) \to \mathbb{R}$ defined through

$$S(r) := \sigma_1\left[d_2\mathcal{L}_2 + m - \frac{\mu - c\theta_{\{d_1,\lambda,a\}} + m}{r}; \mathcal{B}_2, \Omega\right]$$

is continuous, strictly increasing and, owing to (6.11), satisfies

$$\lim_{r\to\xi} S(r) = \begin{cases} \sigma_1[d_2\mathcal{L}_2 + m; \mathcal{B}_2, \Omega] > 1 & \text{if } \xi = +\infty, \\ \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0 & \text{if } \xi = 1, \\ -\infty & \text{if } \xi = 0. \end{cases}$$

Hence, $r_0 < 1$ because $S(r_0) = 0$. Therefore, due to Lemma 6.5(b),

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot), (\theta_{\{d_1,\lambda,a\}}, 0)) = (-1)^{\chi},$$

where χ is the sum of the multiplicities of the eigenvalues of $D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$ greater than one. Now, assume that $\tau > 1$ is an eigenvalue of $D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$ with associated eigenvector $(u, v) \neq (0, 0)$, i.e.,

$$D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)(u, v) = \tau(u, v). \tag{6.17}$$

If $v \neq 0$, then, using the v-equation of (6.17), we have that 0 is an eigenvalue of the tern

$$\left[d_2\mathcal{L}_2 + m - \frac{\mu - c\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_2, \Omega\right].$$

Hence, by the dominance of the principal eigenvalue,

$$\sigma_1\left[d_2\mathcal{L}_2 + m - \frac{\mu - c\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_2, \Omega\right] \leq 0, \tag{6.18}$$

However, by the strict monotonicity of the principal eigenvalue with respect to the potential delivered in Theorem 2.12, it follows from (6.11) that

$$\sigma_1\left[d_2\mathcal{L}_2 + m - \frac{\mu - c\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_2, \Omega\right] > \sigma_1\left[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega\right] > 0,$$

which contradicts (6.18). Consequently, $v = 0$. Hence, $u \neq 0$. Thus, it follows from the u-equation of (6.17) that

$$\sigma_1\left[d_1\mathcal{L}_1 + m - \frac{\lambda - 2a\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_1, \Omega\right] \leq 0.$$

Moreover, by Theorem 2.12 and (6.16),

$$\begin{aligned} \sigma_1\left[d_1\mathcal{L}_1 + m - \frac{\lambda - 2a\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_1, \Omega\right] &> \sigma_1\left[d_1\mathcal{L}_1 + m - \frac{\lambda - a\theta_{\{d_1,\lambda,a\}} + m}{\tau}; \mathcal{B}_1, \Omega\right] \\ &> \sigma_1\left[d_1\mathcal{L}_1 - \lambda + a\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_1, \Omega\right] = 0, \end{aligned}$$

which again leads to a contradiction. Therefore, $\chi = 0$ and

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}} \times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot), (\theta_{\{d_1,\lambda,a\}}, 0)) = 1.$$

This ends the proof of the second identity of (6.9).

Next, suppose that

$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] < 0, \tag{6.19}$$

instead of (6.11). We claim that

$$I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$$

is an injective operator on $W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$. Indeed, if there is a

$$(u, v) \in W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$$

such that

$$D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)(u, v) = (u, v),$$

then the identities (6.12) and (6.13) hold. Since $v \in P_{W^{2,\infty}_{\mathcal{B}_2}}$, if $v \neq 0$, then

$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] = 0,$$

which contradicts (6.19). Thus, $v = 0$. Similarly, arguing as in the previous case, by (6.14) and (6.15), one can easily infer that $u = 0$. Hence,

$$I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$$

is injective on $W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$. According to Lemma 6.5(c), to complete the proof of (6.9), it suffices to show that

$$I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)$$

is not surjective on $W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$. To prove this, we proceed by contradiction. Assume that, for every $(w_1, w_2) \in W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$ there exists $(u, v) \in W^{2,\infty}_{\mathcal{B}_1} \times P_{W^{2,\infty}_{\mathcal{B}_2}}$ such that

$$[I - D_{(u,v)}\mathcal{I}(1, \theta_{\{d_1,\lambda,a\}}, 0)](u, v) = (w_1, w_2),$$

i.e.,

$$u - (d_1\mathcal{L}_1 + m)^{-1}\left[(\lambda - 2a\theta_{\{d_1,\lambda,a\}} + m)u - b\theta_{\{d_1,\lambda,a\}}v\right] = w_1$$

and

$$v - (d_2\mathcal{L}_2 + m)^{-1}[(\mu - c\theta_{\{d_1,\lambda,a\}} + m)v] = w_2. \tag{6.20}$$

In particular, since we are assuming that

$$\sigma_1[d_2\mathcal{L}_2 + m; \mathcal{B}_2, \Omega] > 1 > 0,$$

for every $w \in P_{W^{2,\infty}_{\mathcal{B}_2}} \setminus \{0\}$, the function w_2 defined by

$$w_2 := (d_2\mathcal{L}_2 + m)^{-1}\omega$$

is strongly positive. For this choice, (6.20) becomes

$$(d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}})v = \omega > 0.$$

This implies that $v > 0$ and hence, thanks to [88, Th.7.10],

$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0,$$

which contradicts (6.19). This ends the proof. $\square$

Finally, the next result provides us with the fixed point index of the linearly stable coexistence steady states.

Lemma 6.7. *Assume that* (u, v) *is a linearly stable coexistence steady state of* (4.1). *Then*

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(u,v)) = 1$$

Proof. This is an immediate consequence of the dominance of the principal eigenvalue of the linearization of (4.2) at (u, v), and the definition of the fixed point index (see Theorem 11.4 of Amann [3]). □

6.1.2 Linear stability of coexistence steady states induces uniqueness

The following result characterizes the existence and uniqueness of coexistence solutions in terms of the existence and linearized stability of the semitrivial solutions, $(\theta_{\{d_1,\lambda,a\}}, 0)$ and $(0, \theta_{\{d_2,\mu,d\}})$ when every coexistence steady states of (4.1) is linearly stable. It constitutes one of the the main tools of this chapter.

Theorem 6.8. *Assume that* $\Omega_{\text{per}} \neq \emptyset$ *or* $\Omega^u_{\text{do}} \cup \Omega^{\text{per},u}_{\text{junk}} \neq \emptyset$ *and* $\Omega^v_{\text{do}} \cup \Omega^{\text{per},v}_{\text{junk}} \neq \emptyset$. *Suppose that every coexistence steady state of* (4.1) *is linearly stable. Then, there exists* $\delta > 0$ *such that for every* $d_1, d_2 \in (0, \delta)$ *the model* (4.1) *exhibits a unique coexistence steady state. Moreover, it is a global attractor for the component-wise positive solutions of* (4.1).

Proof. By Corollary 5.12 we have that (4.1) exhibits stable coexistence steady states for small diffusion rates because both semitrivial steady states exist and they are linearly unstable. On the other hand, let us denote by Υ the set of coexistence states of (4.2). By construction, we have that Υ lies in the interior of $\mathcal{U} \times \mathcal{V}$. Then, by the additivity property of the fixed point index (see Theorem 11.1 of Amann [3]), we obtain that

$$\begin{aligned} i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V})) &= i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0)) \\ &\quad + i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(\theta_{\{d_1,\lambda,a\}},0)) \\ &\quad + i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,\theta_{\{d_2,\mu,d\}})) \\ &\quad + \sum_{(u,v)\in\Upsilon} i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(u,v)) \end{aligned} \tag{6.21}$$

Moreover, by Lemmas 6.3, 6.4, 6.6, and 6.7, we have that

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),\operatorname{int}(\mathcal{U}\times\mathcal{V})) = 1, \qquad i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,0)) = 0,$$

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(\theta_{\{d_1,\lambda,a\}},0)) = 0, \qquad i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(0,\theta_{\{d_2,\mu,d\}})) = 0,$$

and

$$i_{P_{W^{2,\infty}_{\mathcal{B}_1}}\times P_{W^{2,\infty}_{\mathcal{B}_2}}}(\mathcal{I}(1,\cdot,\cdot),(u,v)) = 1 \quad \text{for every} \quad (u,v)\in\Upsilon.$$

Hence, for that range of diffusion rates, the system (4.2) admits a unique coexistence state (linearly stable). The fact that it is a global attractor for the component-wise positive solutions of (4.1) is an immediate consequence of the uniqueness and the fact that the system is compressive as pointed out in Remark 33.2 and Theorem 33.3 of Hess [56]. □

6.2 Uniqueness under permanence

This section shows that under Robin boundary conditions the problem (4.2) admits a unique coexistence state for sufficiently small diffusion coefficients, $d_1 > 0$ and $d_2 > 0$, if $\bar{\Omega} = \Omega_{\text{per}}$. In particular, this uniqueness result applies to the next competition Lotka–Volterra reaction-diffusion prototype model under Robin boundary conditions

$$\begin{cases} d_1\mathcal{L}_1 u = \lambda u - au^2 - buv & \text{in } \ \Omega, \\ d_2\mathcal{L}_2 v = \mu v - dv^2 - cuv & \text{in } \ \Omega, \\ \mathcal{R}_1 u = \mathcal{R}_2 v = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{6.22}$$

Here, $d_1, d_2 > 0$ and $\lambda, \mu, a, b, c, d \in \ \mathcal{C}(\bar{\Omega})$ with

$$ad - bc > 0, \quad \lambda d - \mu b > 0 \quad \text{and} \quad \mu a - \lambda c > 0 \quad \text{in}\bar{\Omega},$$

which entails $\bar{\Omega} = \Omega_{\text{per}}$. Moreover, in contrast to Section 6.3, in this section, for every $i = 1, 2$,

$$\mathcal{L}_i = -\text{div}(\mathrm{A_i}\nabla\cdot) + \mathrm{B_i}\nabla + \mathrm{C_i}, \qquad \mathrm{i} = 1, 2,$$

with $A_i \in \mathcal{M}_N^{\text{sym}}(\mathcal{C}^2(\bar{\Omega}))$, $B_i \in \mathcal{M}_{1\times N}(\mathcal{C}(\bar{\Omega}))$ and $C_i \in \mathcal{C}(\bar{\Omega})$, and $\mathcal{R}_i := \frac{\partial}{\partial A_i \mathbf{n}} + \beta_i$ is the robin boundary operator, where $\beta_i \in \mathcal{C}(\Gamma_{\mathcal{R}}^i)$ and $\frac{\partial}{\partial A_i \mathbf{n}}$ stands for the directional derivative with respect to the conormal vector field $\boldsymbol{\nu}_i := A_i\mathbf{n}$.

Although the first version of Theorems 6.9 and 6.10 below was given in Hutson, Lou and Mischaikow [65], the proof here is substantially simpler than the original one of [65], which established for the $-\Delta$ operator under non-flux boundary conditions. Furthermore, as a result of its simplicity, it can be easily adapted to get the result for wider classes of differential operators in divergence form subject to Robin boundary conditions. Once established one of the main uniqueness results of this thesis, its optimality is discussed in Section 6.2.1 at the light of the multiplicity result established in Chapter 5.

Note that the condition $\Gamma_{\mathcal{D}}^1 = \Gamma_{\mathcal{D}}^2 = \emptyset$ must be imposed since in the proof of Theorems 6.9 and 6.10 we are using the uniform convergence of the coexistence states as $(d_1, d_2) \to (0, 0)$ established by Theorem 4.4.

Theorem 6.9. *Assume that $\bar{\Omega} = \Omega_{\text{per}}$ and $\Gamma_{\mathcal{D}}^1 = \Gamma_{\mathcal{D}}^2 = \emptyset$. Then, $\delta > 0$ exists such that for all $d_1, d_2 \in (0, \delta)$ the diffusive model (6.22) exhibits a unique coexistence state which is a global attractor with respect to the component-wise positive solutions.*

Proof. By Corollary 5.12, $\delta > 0$ exists such that (4.1) is permanent for $d_1, d_2 \in (0, \delta)$. As it was mentioned in Section 5.3, in such a case, the existence of at least a stable coexistence state is ensured by previous results (see, e.g, Hess [56], López-Gómez [79], and López-Gómez and Sabina [77]), which are easily adaptable to cover the case of general operators subject to general boundary conditions of mixed type. Furthermore, such a coexistence state is a global attractor if it is unique.

Thus, it suffices to establish the uniqueness of the coexistence state for small diffusion rates. According to the theory developed in the previous section (see Theorem 6.8), which goes back to [79, 77], this is a consequence from the fact that all the coexistence states are linearly stable.

Let $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$ be a coexistence steady state of (6.22) for diffusion rates $d_1, d_2 > 0$. Then, its linear stability follows from the positivity of the principal eigenvalue, σ_0, of the associated eigenvalue problem

$$\begin{cases} \left[d_1\mathcal{L}_1 - \lambda + 2au_{(d_1,d_2)} + bv_{(d_1,d_2)}\right]\phi + bu_{(d_1,d_2)}\psi = \sigma\phi & \text{in } \Omega, \\ \left[d_2\mathcal{L}_2 - \mu + 2dv_{(d_1,d_2)} + cu_{(d_1,d_2)}\right]\psi + cv_{(d_1,d_2)}\phi = \sigma\psi & \\ \mathcal{R}_1 u = \mathcal{R}_2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.23}$$

obtained by linearizing (6.22) at the given coexistence state. Since $b(x)u_{(d_1,d_2)}(x) > 0$ and $c(x)v_{(d_1,d_2)}(x) > 0$ for all $x \in \Omega$, (6.23) is a problem of quasi-cooperative type as those analyzed in Section 5.1. Thus, the existence of the principal eigenvalue, σ_0, follows from Theorem 5.1. As a consequence of Theorem 5.4, $\sigma_0 > 0$ if, and only if,

$$\mathfrak{L}_{(d_1,d_2)} := \begin{pmatrix} d_1\mathcal{L}_1 - \lambda + 2au_{(d_1,d_2)} + bv_{(d_1,d_2)} & bu_{(d_1,d_2)} \\ cv_{(d_1,d_2)} & d_2\mathcal{L}_2 - \mu + 2dv_{(d_1,d_2)} + cu_{(d_1,d_2)} \end{pmatrix},$$

subject to the boundary operator $(\mathcal{R}_1, \mathcal{R}_2)$, admits a strict supersolution, $(\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)})$, with $\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)} \in W^{2,p}(\Omega)$, for all $p > N$, and $\phi_{(d_1,d_2)} \gneq 0$, $\psi_{(d_1,d_2)} \lneq 0$ in Ω.

Since $\bar{\Omega} = \Omega_{\text{per}}$, we have that

$$b(x)c(x) < a(x)d(x) \quad \text{for all} \;\; x \in \bar{\Omega}.$$

Thus, $\tau \in \mathcal{C}(\bar{\Omega})$ exists such that

$$\frac{b(x)}{a(x)} < \tau(x) < \frac{d(x)}{c(x)} \quad \text{for all} \;\; x \in \bar{\Omega}.$$

Indeed, we can take

$$\tau := \frac{b+\xi}{a}$$

for some $\xi > 0$ small enough. Now, for sufficiently small $\eta > 0$, according to Corollary 2.17, let us denote by $\theta^1_{\eta,\tau}$ and $\theta^2_{\eta,1}$ the (unique) solutions of

$$\begin{cases} \eta\mathcal{L}_1\theta = \tau\theta - \theta^2 & \text{in } \Omega, \\ \mathcal{R}_1\theta = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \eta\mathcal{L}_2\theta = \theta - \theta^2 & \text{in } \Omega, \\ \mathcal{R}_2\theta = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively, which, according to Theorem 4.1, which goes back to Theorem 2.21, converge to τ and 1 uniformly in $\bar{\Omega}$ as $\eta \downarrow 0$. Hence, $\eta_0 > 0$ exists such that

$$\frac{b(x)}{a(x)} < \frac{\theta^1_{\eta_0,\tau}(x)}{\theta^2_{\eta_0,1}(x)} < \frac{d(x)}{c(x)} \quad \text{for all} \;\; x \in \bar{\Omega},$$

and so, $\varepsilon > 0$ exists such that

$$a(x)\theta^1_{\eta_0,\tau}(x) - b(x)\theta^2_{\eta_0,1}(x) > \varepsilon > 0$$

and

$$d(x)\theta^2_{\eta_0,1}(x) - c(x)\theta^1_{\eta_0,\tau}(x) > \varepsilon > 0$$

for all $x \in \bar{\Omega}$. Next, we will show that, for sufficiently small $d_1, d_2 > 0$, the pair

$$(\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)}) := (\theta^1_{\eta_0,\tau}, -\theta^2_{\eta_0,1})$$

provides us with the desired supersolution of $[\mathfrak{L}_{(d_1,d_2)}; (\mathcal{R}_1, \mathcal{R}_2), \Omega]$. Indeed, thanks to Theorem 4.4,

$$\lim_{(d_1,d_2)\to(0,0)} (u_{(d_1,d_2)}, v_{(d_1,d_2)}) = (u_*, v_*) = \left(\tfrac{\lambda d-\mu b}{ad-bc}, \tfrac{\mu a-\lambda c}{ad-bc}\right) \qquad \text{uniformly in } \bar{\Omega},$$

and hence, applying $\mathfrak{L}_{(d_1,d_2)}$ yields

$$\begin{aligned}
&(d_1\mathcal{L}_1 - \lambda + 2au_{(d_1,d_2)} + bv_{(d_1,d_2)})\theta^1_{\eta_0,\tau} - bu_{(d_1,d_2)}\theta^2_{\eta_0,1}\\
&\quad = d_1\mathcal{L}_1\theta^1_{\eta_0,\tau} - (\lambda - au_{(d_1,d_2)} - bv_{(d_1,d_2)})\theta^1_{\eta_0,\tau} + (a\theta^1_{\eta_0,\tau} - b\theta^2_{\eta_0,1})u_{(d_1,d_2)}\\
&\quad > d_1\mathcal{L}_1\theta^1_{\eta_0,\tau} - (\lambda - au_{(d_1,d_2)} - bv_{(d_1,d_2)})\theta^1_{\eta_0,\tau} + \varepsilon u_{(d_1,d_2)}\\
&\quad \xrightarrow{d_1,d_2\to 0} -(\lambda - au_* - bv_*)\theta^1_{\eta_0,\tau} + \varepsilon u_* = \varepsilon u_* > 0
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&cv_{(d_1,d_2)}\theta^1_{\eta_0,\tau} - (d_2\mathcal{L}_2 - \mu + 2dv_{(d_1,d_2)} + cu_{(d_1,d_2)})\theta^2_{\eta_0,1}\\
&\quad = -d_2\mathcal{L}_2\theta^2_{\eta_0,1} + (\mu - dv_{(d_1,d_2)} - cu_{(d_1,d_2)})\theta^2_{\eta_0,1} - (d\theta^2_{\eta_0,1} - c\theta^1_{\eta_0,\tau,})v_{(d_1,d_2)}\\
&\quad < -d_2\mathcal{L}_2\theta^2_{\eta_0,1} + (\mu - dv_{(d_1,d_2)} - cu_{(d_1,d_2)})\theta^2_{\eta_0,1} - \varepsilon v_{(d_1,d_2)}\\
&\quad \xrightarrow{d_1,d_2\to 0} (\mu - dv_* - cu_*)\theta^2_{\eta_0,1} - \varepsilon v_* = -\varepsilon v_* < 0,
\end{aligned}$$

with uniform convergence in $\bar{\Omega}$. By the choice of $\phi_{(d_1,d_2)}$ and $\psi_{(d_1,d_2)}$, they are independent of (d_1, d_2) and satisfy

$$\mathcal{R}_1\phi_{(d_1,d_2)} = \mathcal{R}_2\psi_{(d_1,d_2)} = 0 \quad \text{on } \partial\Omega.$$

This ends the proof. □

Naturally, the technical device introduced in the proof of Theorem 6.9 can be also adapted to derive a substantial generalization of Theorem 1.1 of Hutson, Lou and Mischaikow [65], which was established for the $-\Delta$ operator under non-flux boundary conditions. Note that here we are dealing with a more general class of differential operators under mixed boundary conditions of Robin type. Precisely, Theorem 6.9 can be extended to cover the next class of reaction-diffusion systems

$$\begin{cases} \frac{\partial u}{\partial t} + d_1\mathcal{L}_1 u = uf(u,v,x) & \text{in } \Omega\times(0,+\infty),\\ \frac{\partial v}{\partial t} + d_2\mathcal{L}_2 v = vg(u,v,x) & \\ \mathcal{R}_1 u = \mathcal{R}_2 v = 0 & \text{on } \partial\Omega\times(0,+\infty),\\ u(\cdot,0) = u_0 > 0,\;\; v(\cdot,0) = v_0 > 0, & \text{in } \Omega, \end{cases} \tag{6.24}$$

under the following general assumptions on f and g emphasizing the fact that the system must be of competitive type:

(H1) $f, g : \mathbb{R}\times\mathbb{R}\times\bar{\Omega} \to \mathbb{R}$ are of class C^1 in each variable.

(H2) For every $u, v \geq 0$ and $x \in \bar{\Omega}$, $\partial_w f(u,v,x) < 0$ and $\partial_w g(u,v,x) < 0$ where $w \in \{u,v\}$.

(H3) There exists a positive constant M such that $f(M,0,x) < 0$, $f(0,M,x) < 0$, $g(M,0,x) < 0$ and $g(0,M,x) < 0$ for every $x \in \bar{\Omega}$.

(H4) For every $x \in \bar{\Omega}$, there exists a unique $(u_*(x), v_*(x))$ in the non-negative cone $\{(u,v) \in \mathbb{R}^2 \, : \, u,v \geq 0\}$ such that $f(u,v,x) = 0$ and $g(u,v,x) = 0$. Moreover, $u_*(x) > 0$ and $v_*(x) > 0$ for all $x \in \bar{\Omega}$.

(H5) For every $x \in \bar{\Omega}$, $(\partial_u f \partial_v g - \partial_v f \partial_u g)_{|(u_*(x), v_*(x), x)} > 0$.

According to (H2) and (H5), for every $x \in \bar{\Omega}$, the linearization of the non-spatial model at $(u_*(x), v_*(x))$,

$$\begin{pmatrix} u_*(x)\partial_u f(u_*(x), v_*(x), x) & u_*(x)\partial_v f(u_*(x), v_*(x), x) \\ v_*(x)\partial_u g(u_*(x), v_*(x), x) & v_*(x)\partial_v g(u_*(x), v_*(x), x) \end{pmatrix},$$

has two negative eigenvalues. In addition, by (H3), the remaining steady states of the non-spatial model are linearly unstable. This entails $(u_*(x), v_*(x))$ to be a global hyperbolic attractor for the non-spatial model with respect to the positive cone. Consequently, the previous conditions are actually imposing that $\Omega_{\text{per}} = \bar{\Omega}$.

Theorem 6.10. *Suppose (H1)-(H5). Then, $\delta > 0$ exists such that, for every $d_1, d_2 < \delta$, the reaction-diffusion system (6.24) possesses a unique coexistence steady state, which is a global attractor with respect to the positive solutions.*

Proof. The proof follows the same patterns as the proof of Theorem 6.9. Let us consider $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$, a coexistence steady state of (6.24). Its linear stability is a consequence of the positivity of the principal eigenvalue of the linearization of (6.24) at $(u_{(d_1,d_2)}, v_{(d_1,d_2)})$, i.e., the principal eigenvalue of the operator $\mathfrak{L}_{(d_1,d_2)}$ defined through

$$\mathfrak{L}_{(d_1,d_2)} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Big[d_1\mathcal{L}_1 - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - u_{(d_1,d_2)}\partial_u f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\Big]\phi \\ -u_{(d_1,d_2)}\partial_v f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\,\psi \\ \\ \Big[d_2\mathcal{L}_2 - g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - v_{(d_1,d_2)}\partial_v g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\Big]\psi \\ -v_{(d_1,d_2)}\partial_u g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\,\phi \end{pmatrix}$$

subject to the boundary conditions $(\mathcal{R}_1, \mathcal{R}_2)$. Since $\partial_v f$ and $\partial_u g$ are negative in $[0,+\infty)^2 \times \bar{\Omega}$ (see (H2)), the off-diagonal entries of $\mathfrak{L}_{(d_1,d_2)}$,

$$-u_{(d_1,d_2)}\partial_v f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) \qquad \text{and} \qquad -v_{(d_1,d_2)}\partial_u g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)$$

are positive functions in $\bar{\Omega}$. Thus, $\mathfrak{L}_{(d_1,d_2)}$ is of quasi-cooperative type and hence satisfies the hypothesis of Theorems 5.1 and 5.4. Therefore, its principal eigenvalue is positive if it admits a strict supersolution, (ϕ, ψ), with $\phi > 0$ and $\psi < 0$. In particular, it suffices that these functions satisfy $\mathcal{R}_1\phi = \mathcal{R}_2\psi = 0$ on $\partial\Omega$, whereas in Ω

$$\begin{aligned} &[d_1\mathcal{L}_1 - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - u_{(d_1,d_2)}\partial_u f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)]\phi \\ &\qquad\qquad - u_{(d_1,d_2)}\partial_v f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\psi > 0 \end{aligned} \tag{6.25}$$

and

$$\begin{aligned}[d_2\mathcal{L}_2 - g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - v_{(d_1,d_2)}\partial_v g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)]\psi \\ - v_{(d_1,d_2)}\partial_u g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\phi < 0.\end{aligned} \tag{6.26}$$

In the special case covered by this theorem, i.e., when $\Omega_{\rm per} = \bar\Omega$ and $\partial\Omega = \Gamma^1_{\mathcal{R}} = \Gamma^2_{\mathcal{R}}$, the singular perturbation results of Section 2.5 and Chapter 4 can be adapted to cover the slightly more general Kolmogorov system of competitive type (6.24) in order to obtain that

$$\lim_{(d_1,d_2)\to(0,0)} \big(u_{(d_1,d_2)}, v_{(d_1,d_2)}\big) = (u_*, v_*) \qquad \text{uniformly in } \bar\Omega.$$

As the corresponding proofs follow *mutatis mutandis* the general patterns of the proofs of Section 2.5 and Chapter 4, and a version is provided in Lemma 3.3 of Hutson, Lou and Mischaikow [65], its technical details are omitted here.

Thus, thanks to (H5), it follows that

$$\Big(\partial_u f \partial_v g - \partial_v f \partial_u g\Big)(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x) > 0$$

for every $x \in \bar\Omega$ and sufficiently small $d_1, d_2 > 0$. Hence, by (H2), there exists a function $\tau \in \mathcal{C}(\bar\Omega)$ such that

$$\left(\frac{\partial_v f}{\partial_u f}\right)(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x) < \tau(x) < \left(\frac{\partial_v g}{\partial_u g}\right)(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x).$$

Subsequently, we consider the same positive solutions, $\theta^1_{\eta_0,\tau}$ and $\theta^2_{\eta_0,1}$, as in the proof of Theorem 6.9, with η_0 small enough so that

$$\left(\frac{\partial_v f}{\partial_u f}\right)(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x) < \frac{\theta^1_{\eta_0,\tau}}{\theta^2_{\eta_0,1}}(x) < \left(\frac{\partial_v g}{\partial_u g}\right)(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x)$$

for all $x \in \bar\Omega$ and sufficiently small $d_1, d_2 > 0$. Therefore, owing to (H2), $\varepsilon > 0$ exists such that

$$\partial_v f(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x)\theta^2_{\eta_0,1}(x) - \partial_u f(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x)\theta^1_{\eta_0,\tau}(x) > \varepsilon > 0$$

and

$$\partial_u g(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x)\theta^1_{\eta_0,\tau}(x) - \partial_v g(u_{(d_1,d_2)}(x), v_{(d_1,d_2)}(x), x)\theta^2_{\eta_0,1}(x) > \varepsilon > 0.$$

It remains to show that, for sufficiently small $d_1, d_2 > 0$, the vectorial function

$$\Big(\phi_{(d_1,d_2)}, \psi_{(d_1,d_2)}\Big) := \Big(\theta^1_{\eta_0,\tau}, -\theta^2_{\eta_0,1}\Big),$$

satisfies (6.25) and (6.26). Indeed, by substituting, we find that

$$\begin{aligned}&\Big[d_1\mathcal{L}_1 - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - u_{(d_1,d_2)}\partial_u f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\Big]\theta^1_{\eta_0,\tau} \\ &\qquad + u_{(d_1,d_2)}\partial_v f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1} \\ &\quad = d_1\mathcal{L}_1\theta^1_{\eta_0,\tau} - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau} \\ &\qquad + u_{(d_1,d_2)}\Big[\partial_v f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1} - \partial_u f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau}\Big] \\ &\quad > d_1\mathcal{L}_1\theta^1_{\eta_0,\tau} - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau} + \varepsilon u_{(d_1,d_2)}\end{aligned}$$

and

$$\lim_{d_1,d_2\to 0} d_1\mathcal{L}_1\theta^1_{\eta_0,\tau} - f(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau} + \varepsilon u_{(d_1,d_2)} \\ = -f(u_*, v_*, \cdot)\theta^1_{\eta_0,\tau} + \varepsilon u_* = \varepsilon u_* > 0$$

uniformly in $\bar{\Omega}$. Similarly,

$$\begin{aligned} -\Big[d_2\mathcal{L}_2 - g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot) - v_{(d_1,d_2)}\partial_v g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\Big]\theta^2_{\eta_0,1} \\ - v_{(d_1,d_2)}\partial_u g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau} \\ = -d_2\mathcal{L}_2\theta^2_{\eta_0,1} + g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1} \\ - v_{(d_1,d_2)}\Big[\partial_u g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^1_{\eta_0,\tau} - \partial_v g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1}\Big] \\ < -d_2\mathcal{L}_2\theta^2_{\eta_0,1} + g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1} - \varepsilon v_{(d_1,d_2)} \end{aligned}$$

and

$$\lim_{d_1,d_2\to 0} -d_2\mathcal{L}_2\theta^2_{\eta_0,1} + g(u_{(d_1,d_2)}, v_{(d_1,d_2)}, \cdot)\theta^2_{\eta_0,1} - \varepsilon v_{(d_1,d_2)} \\ = g(u_*, v_*, \cdot)\theta^2_{\eta_0,1} - \varepsilon v_* = -\varepsilon v_* < 0$$

uniformly in $\bar{\Omega}$. Note that, by (H4), u_* and v_* are positive and separated away from zero on $\bar{\Omega}$. This ends the proof. □

6.2.1 Optimality of the uniqueness result

Theorem 6.9 is optimal in the sense that if condition $\Omega_{\rm per} = \bar{\Omega}$ fails to be true in an arbitrarily small ball, $B_\varepsilon(x_0)$, centered at some $x_0 \in \Omega$ with radius $\varepsilon > 0$, where the non-spatial model exhibits a founder control competition (in other words, $B_\varepsilon(x_0) \subset \Omega_{\rm bi}$), then, owing to Theorem 5.16, (6.22) might admit at least three coexistence states for sufficiently small diffusion rates. A possible strategy to realize what is going on consists in modeling this change of behavior through some additional parameter incorporated to the setting of the model, in order to mimic such transition in a continuous way. Let $a(x)$, $d(x)$, $\mathfrak{b}(x)$ and $\mathfrak{c}(x)$ be four positive continuous functions on $\bar{\Omega}$ such that for some $x_0 \in \Omega$

$$\frac{a(x_0)}{\mathfrak{c}(x_0)} = \frac{\lambda(x_0)}{\mu(x_0)} = \frac{\mathfrak{b}(x_0)}{d(x_0)}, \tag{6.27}$$

while

$$\frac{a(x)}{\mathfrak{c}(x)} > \frac{\lambda(x)}{\mu(x)} > \frac{\mathfrak{b}(x)}{d(x)} \quad \text{for all } \; x \in \bar{\Omega} \setminus \{x_0\}. \tag{6.28}$$

For instance, the one-dimensional choices in $\Omega = (-1, 1)$ given by

$$a(x) = d(x) = 2, \;\; \lambda(x) = \mu(x) = 1, \;\; \mathfrak{b}(x) = \mathfrak{c}(x) = 2 - x^2, \quad x \in [-1, 1], \tag{6.29}$$

satisfy (6.27) and (6.28) with $x_0 = 0$. Next, we consider (6.22) for these choices of $\lambda(x)$, $\mu(x)$, $a(x)$, $d(x)$, and $(b, c) := (b_\rho, c_\rho)$, where

$$b_\rho(x) := \rho\mathfrak{b}(x), \quad c_\rho(x) := \rho\mathfrak{c}(x), \qquad x \in \bar{\Omega}, \;\; \rho > 0,$$

where ρ is regarded as a parameter measuring the intensity of the aggressions between the antagonist species, u and v. According to (6.27) and (6.28), it becomes apparent that, for every $\rho \in (0,1)$,

$$\frac{a(x)}{\rho\mathfrak{c}(x)} = \frac{a(x)}{c(x)} > \frac{\lambda(x)}{\mu(x)} > \frac{b(x)}{d(x)} = \frac{\rho\mathfrak{b}(x)}{d(x)} \qquad \text{for all } \ x \in \bar{\Omega}.$$

Consequently, in such range of values of ρ, $\Omega_{\text{per}} = \bar{\Omega}$ and, owing to Theorem 6.9, (6.22) possesses a unique (linearly stable) coexistence state, which is a global attractor with respect to the positive solutions of (4.1) for sufficiently small d_1 and d_2. By construction, the condition $\Omega_{\text{per}} = \bar{\Omega}$ fails to be true at $\rho = 1$, where $\Omega_{\text{per}} = \bar{\Omega} \setminus \{x_0\}$, as well as for $\rho > 1$ sufficiently close to 1, where there are a maximal $\varepsilon_1 := \varepsilon_1(\rho) > 0$ and a minimal $\varepsilon_2 := \varepsilon_2(\rho) > \varepsilon_1$ such that

$$B_{\varepsilon_1}(x_0) \subset \Omega_{\text{bi}} \quad \text{and} \quad \bar{\Omega} \setminus B_{\varepsilon_2}(x_0) \subset \Omega_{\text{per}}.$$

Actually, $\varepsilon_2(\rho)$ can be taken arbitrarily small by choosing $\rho > 1$ sufficiently close to 1, i.e., $\lim_{\rho\downarrow 1} \varepsilon_2(\rho) = 0$. In other words, for $\rho > 1$ sufficiently close to 1, the main assumption of Theorem 6.9 is 'almost' satisfied, except for a small ball centered at x_0, $B_{\varepsilon_2}(x_0)$, where $\Omega_{\text{bi}} \neq \emptyset$. Thus, thanks to the multiplicity result delivered in Theorem 5.16, at least for the symmetric choice (6.29) with $d_1 = d_2$, the problem (6.22) might exhibit, in general, at least three coexistence states for sufficiently small diffusion rates. Figure 6.1 shows an admissible bifurcation diagram of coexistence states in terms of the parameter ρ.

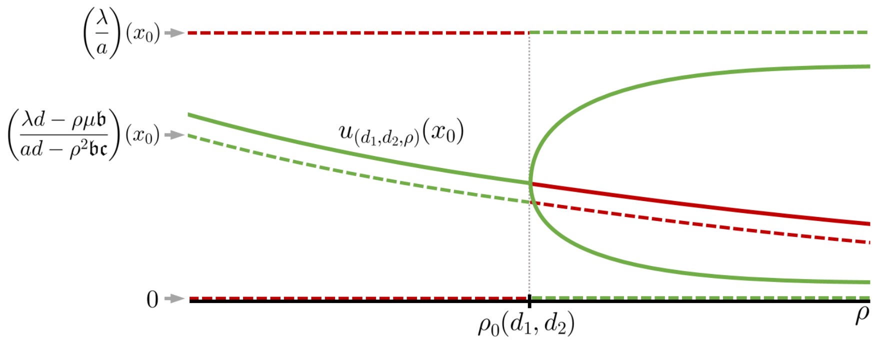


Figure 6.1: An admissible bifurcation diagram of $u_{(d_1,d_2,\rho)}$ versus ρ.

As one cannot represent the entire functions on the ordinate axis, Figure 6.1 plots, for small fixed values of d_1 and d_2, the components $u_{(d_1,d_2,\rho)}$ of the coexistence states $(u_{(d_1,d_2,\rho)}, v_{(d_1,d_2,\rho)})$ of (6.22) versus the continuation parameter ρ using a continuous line, together with the u-components of the unique coexistence state and the semitrivial positive solutions of the associated non-spatial model with dashed lines, i.e.,

$$\left(\frac{\lambda d - \mu b}{ad - bc}, \frac{\mu a - \lambda c}{ad - bc}\right) = \left(\frac{\lambda d - \mu\rho\mathfrak{b}}{ad - \rho^2\mathfrak{bc}}, \frac{\mu a - \lambda\rho\mathfrak{c}}{ad - \rho^2\mathfrak{bc}}\right), \quad \left(\frac{\lambda}{a}, 0\right) \quad \text{and} \quad \left(0, \frac{\mu}{d}\right).$$

As $\rho < 1$, the assumptions of Theorem 6.9 hold and the unique coexistence state of (6.22) is a global attractor for the component-wise positive solutions of (4.1). Similarly, the coexistence steady state of the non-spatial model is a global attractor for its component-wise positive solutions, and each of the semitrivial positive steady states is linearly unstable. Note that, according to Theorem 4.4, $u_{(d_1,d_2,\rho)}$ can be taken as close as we wish to $\frac{\lambda d-\mu b}{ad-bc}$ by choosing d_1 and d_2 sufficiently small. According to Theorem 5.16, as ρ crosses some critical value, $\rho_0(d_1,d_2)\sim 1$, the principal eigenvalue, say $\sigma_1(d_1,d_2,\rho)$, of the linearized system at the coexistence state that perturbs from the coexistence steady state of the non-spatial model crosses zero and becomes negative, provoking a pitchfork bifurcation to, at least, two additional coexistence states of (6.22), which, according to the linearized stability principle, [25], should be linearly stable. As $d_1, d_2\to 0$ we conjecture that these stable coexistence states approximate each of the (linearly stable) semitrivial positive steady states of the non-spatial model, and that

$$\lim_{d_1,d_2\to 0}\rho_0(d_1,d_2)=1.$$

From a technical point of view, the fact that $\sigma_1(d_1,d_2,\rho)$ changes sign as ρ crosses $\rho_0(d_1,d_2)$ for sufficiently small d_1 and d_2 shows how the proof of Theorem 6.9 works out exclusively when $\Omega_{\rm per}=\bar\Omega$.

Summarizing, when the coefficients of the model move away from their original values where $\Omega_{\rm per}=\bar\Omega$ to any other situation case such that $\Omega_{\rm per}$ is a proper subset of $\bar\Omega$ and $\Omega_{\rm bi}$ is non-empty, the principal eigenvalue of the global attractor looses positivity crossing zero just when $\Omega_{\rm bi}$ becomes non-empty. Since the fact that

$$\lim_{\substack{d_1,d_2\to 0\\ \rho\uparrow 1}}\sigma_1(d_1,d_2,\rho)=0$$

is exclusively based on the values of the coefficients of the model for $\rho\le 1$, it becomes apparent that actually this is the main technical difficulty for getting general uniqueness results in truly spatially heterogeneous landscapes, where permanence and dominance regions, i.e., $\Omega_{\rm per}$, $\Omega^u_{\rm do}$ and $\Omega^v_{\rm do}$, can coexist within the same habitat, in the sense that the proof Theorem 6.9 cannot be adapted to treat these general situations. However, this difficulties are overcome for a wide family of models in the next section through the use of Picone's Identity.

6.3 Global uniqueness under low competition

This section derives from Picone's identity, as stated in Theorem 3.1, a general sufficient condition so that every coexistence state of a general class of diffusive Lotka–Volterra low competition systems is linearly stable, which entails their uniqueness. Note that it is said that *low competition* occur whenever

$$bc\lneq ad \quad \text{in } \Omega.$$

Unlike previous sections, here we assume that, for every $i=1,2$, $\mathcal{L}_i$ is an uniformly elliptic second order *self-adjoint* operator of the form

$$\mathcal{L}_i:=-\mathrm{div}(A_i\nabla\cdot)+C_i,$$

where $A_i\in\mathcal{M}_N^{\rm sym}(\mathcal{C}^1(\bar\Omega))$ is definite positive, and $C_i\in\mathcal{C}(\bar\Omega)$.

More precisely, throughout this section the linear stability of the coexistence steady states of (4.1) is established under low competition if the estimate (6.31), involving the coefficients a, b, c, and d, and the functions

$$F_{\pm}(k) := \frac{1}{8}\left(27 - 18k - k^2 \pm (9-k)^{3/2}(1-k)^{1/2}\right), \qquad k \in [0,1], \tag{6.30}$$

holds. The precise profile of both F_+ and F_- is plotted in Figure 6.2. In particular, it shows that strict low competition in the whole $\bar{\Omega}$ facilitates the linear stability of all coexistence steady states, with independence of the values of the diffusion rates, d_1 and d_2, and the growth rates, λ and μ.

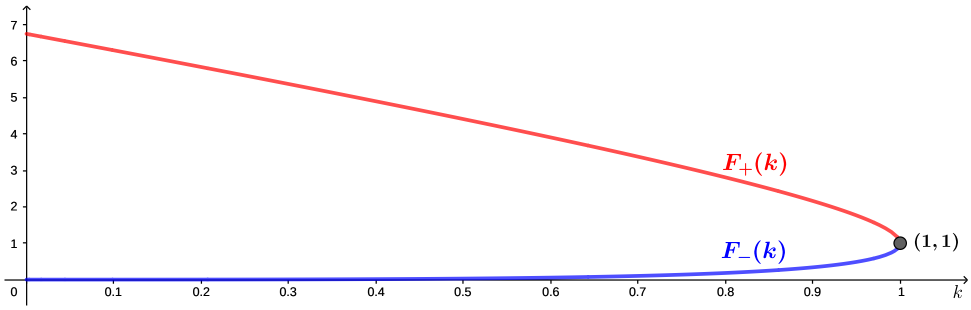


Figure 6.2: Plots of the functions F_- and F_+ with respect to k.

The main result of this section can be stated as follows.

Theorem 6.11. *Suppose that* $\kappa := \frac{bc}{ad} \lneq 1$ *in* Ω, *and that*

$$\max_{\bar{\Omega}}\left(\frac{ad^2}{c^3}F_-(\kappa)\right) \leq \min_{\bar{\Omega}}\left(\frac{ad^2}{c^3}F_+(\kappa)\right), \tag{6.31}$$

with $F_{\pm}$ defined as in (6.30)*. Then, every coexistence state of* (4.2) *is linearly stable.*

Proof. Let (u,v) be a coexistence state of (4.2). Then, the linearized stability of (u,v) as an steady-state solution of (4.1) is given by the signs of the eigenvalues of the linear eigenvalue problem

$$\begin{cases} [d_1\mathcal{L}_1 - \lambda + 2au + bv]\varphi + bu\psi = \sigma\varphi & \text{in } \Omega, \\ [d_2\mathcal{L}_2 - \mu + 2dv + cu]\psi + cv\varphi = \sigma\psi & \text{in } \Omega, \\ \mathcal{B}_1\varphi = \mathcal{B}_2\psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.32}$$

Since $b(x)u(x) > 0$ and $c(x)v(x) > 0$ for all $x \in \Omega$, it follows from Theorem 5.1, which goes back to [7, Th. 1.3], among other references, that (6.32) possesses a unique principal eigenvalue, σ_0, associated with it there is an eigenfunction (φ, ψ) with $\varphi \gg 0$ and $\psi \ll 0$. Note that [7, Th. 1.3] extends the findings of [90] to cover our general setting even in the context of periodic-parabolic problems.

Particularizing (6.32) at $\sigma = \sigma_0$, multiplying the first equation of (6.32) by u and using the u-equation of (4.2) yields

$$\begin{aligned} \sigma_0 u\varphi &= ud_1\mathcal{L}_1\varphi - \varphi u(\lambda - au - bv) + u^2(a\varphi + b\psi) \\ &= d_1(u\mathcal{L}_1\varphi - \varphi\mathcal{L}_1 u) + u^2(a\varphi + b\psi). \end{aligned} \tag{6.33}$$

Similarly, multiplying the second equation of (6.32) by v and using the v-equation of (4.2), it is easily seen that

$$\begin{aligned}\sigma_0 v\psi &= v d_2 \mathcal{L}_2 \psi - \psi v(\mu - dv - cu) + v^2(d\psi + c\varphi)\\ &= d_2(v\mathcal{L}_2\psi - \psi\mathcal{L}_2 v) + v^2(d\psi + c\varphi).\end{aligned} \tag{6.34}$$

Multiplying (6.33) and (6.34) by $\frac{\varphi^2}{u^2}$ and $\frac{\psi^2}{v^2}$, respectively, and integrating in Ω it becomes apparent that

$$\sigma_0 \int_\Omega \frac{\varphi^3}{u} = d_1 \int_\Omega \left(\frac{\varphi}{u}\right)^2 (u\mathcal{L}_1\varphi - \varphi\mathcal{L}_1 u) + \int_\Omega \varphi^2(a\varphi + b\psi),$$

$$\sigma_0 \int_\Omega \frac{\psi^3}{v} = d_2 \int_\Omega \left(\frac{\psi}{v}\right)^2 (v\mathcal{L}_2\psi - \psi\mathcal{L}_2 v) + \int_\Omega \psi^2(d\psi + c\varphi).$$

On the other hand, applying Theorem 3.1 with $g(t) = t^2$ and using the uniform ellipticity of $\mathcal{L}_1$ and $\mathcal{L}_2$ provides us with the estimates

$$\begin{aligned}\int_\Omega \left(\frac{\varphi}{u}\right)^2 (u\mathcal{L}_1\varphi - \varphi\mathcal{L}_1 u) &= \int_\Omega 2u\varphi\langle\nabla\frac{\varphi}{u}, A_1\nabla\frac{\varphi}{u}\rangle - \int_{\partial\Omega}\left(\frac{\varphi}{u}\right)^2 [\mathcal{D}u\mathcal{R}\varphi - \mathcal{D}\varphi\mathcal{R}u]\\ &= \int_\Omega 2u\varphi\langle\nabla\frac{\varphi}{u}, A_1\nabla\frac{\varphi}{u}\rangle \geq 0,\end{aligned} \tag{6.35}$$

and

$$\begin{aligned}\int_\Omega \left(\frac{\psi}{v}\right)^2 (v\mathcal{L}_2\psi - \psi\mathcal{L}_2 v) &= \int_\Omega 2v\psi\langle\nabla\frac{\psi}{v}, A_2\nabla\frac{\psi}{v}\rangle - \int_{\partial\Omega} [\mathcal{D}v\mathcal{R}\psi - \mathcal{D}\psi\mathcal{R}v]\\ &= \int_\Omega 2v\psi\langle\nabla\frac{\psi}{v}, A_2\nabla\frac{\psi}{v}\rangle \leq 0,\end{aligned} \tag{6.36}$$

where we have used that

$$\begin{aligned}&\mathcal{D}u = \mathcal{D}\varphi = 0 \quad \text{on } \Gamma^1_\mathcal{D}, &&\mathcal{R}u = \mathcal{R}\varphi = 0 \quad \text{on } \Gamma^1_\mathcal{R},\\ &\mathcal{D}v = \mathcal{D}\psi = 0 \quad \text{on } \Gamma^2_\mathcal{D}, &&\mathcal{R}v = \mathcal{R}\psi = 0 \quad \text{on } \Gamma^2_\mathcal{R},\end{aligned}$$

as well as the fact that $u, v, \phi \gg 0$ and $\psi \ll 0$ in Ω. Hence, the inequalities derived in (6.35) and (6.36) yield the estimates

$$\sigma_0 \int_\Omega \frac{\varphi^3}{u} \geq \int_\Omega \varphi^2(a\varphi + b\psi) \quad \text{and} \quad -\sigma_0 \int_\Omega \frac{\psi^3}{v} \geq -\int_\Omega \psi^2(c\varphi + d\psi).$$

Therefore, for every positive constant $\xi > 0$, we find that

$$\sigma_0\left(\int_\Omega \frac{\varphi^3}{u} - \xi\int_\Omega \frac{\psi^3}{v}\right) \geq \int_\Omega \left[\varphi^2(a\varphi + b\psi) - \xi\psi^2(c\varphi + d\psi)\right]. \tag{6.37}$$

Next, we will ascertain the values of $\xi > 0$ for which

$$\varphi^2(a\varphi + b\psi) - \xi\psi^2(c\varphi + d\psi) \geq 0 \quad \text{in } \Omega.$$

Dividing by $-\psi^3$ and setting $y := -\varphi/\psi$, it suffices to show that, for every $y \geq 0$, ξ satisfies

$$y^2(ay - b) - \xi(cy - d) \geq 0 \quad \text{in } \Omega.$$

Further, setting $z = \frac{c}{d}y \geq 0$ and dividing by d yields

$$\frac{ad^2}{c^3} z^2 \left(z - \frac{bc}{ad} \right) \geq \xi(z-1) \quad \text{in } \Omega. \tag{6.38}$$

Note that, since $\kappa = \frac{bc}{ad} \lneq 1$ in $\bar{\Omega}$, (6.38) holds if $z \in \{0, 1\}$ for every $\xi > 0$. Hence, the inequality (6.38) can be split into

$$\frac{ad^2}{c^3} z^2 \frac{z-\kappa}{z-1} \geq \xi \quad \text{in } \Omega \text{ for all } z > 1,$$

and

$$\frac{ad^2}{c^3} z^2 \frac{z-\kappa}{z-1} \leq \xi \quad \text{in } \Omega \text{ for all } z \in (0,1).$$

Therefore, in order to get (6.38) for all $z \geq 0$ and $x \in \Omega$ it suffices to make sure that the constant ξ satisfies

$$\frac{ad^2}{c^3} \sup_{0<z<1} \left(z^2 \frac{z-\kappa}{z-1} \right) \leq \xi \leq \frac{ad^2}{c^3} \inf_{z>1} \left(z^2 \frac{z-\kappa}{z-1} \right) \quad \text{in } \Omega. \tag{6.39}$$

Subsequently we consider the function

$$F(z;k) := z^2 \frac{z-k}{z-1}, \qquad z > 0, \;\; z \neq 1, \quad k \in (0,1].$$

Note that here k is a constant, not a function, like $\kappa := \kappa(x)$. By differentiating with respect to z yields

$$\frac{dF}{dz}(z;k) = \frac{(3z^2 - 2kz)(z-1) - (z^3 - kz^2)}{(z-1)^2} = \frac{2z^3 - (3+k)z^2 + 2kz}{(z-1)^2}.$$

Thus, the critical points of $F(\cdot, k)$ are given by the roots of

$$(2z^2 - (3+k)z + 2k)z = 0,$$

which are $z = 0$ plus

$$z_\pm(k) := \frac{3 + k \pm \sqrt{(3+k)^2 - 16k}}{4} = \frac{3 + k \pm \sqrt{(9-k)(1-k)}}{4}.$$

It is straightforward to check that if $k < 1$, then $F(\cdot, k)$ has local minimum at $z_+(k) \in (1, +\infty)$, which is global in $z \in (1 + \infty)$, and it has a local maximum at $z_-(k) \in (0,1)$, which is global in $z \in [0,1)$. Moreover,

$$\begin{aligned} F_\pm(k) &:= F(z_\pm(k);k) = z_\pm^2(k) \frac{z_\pm(k) - k}{z_\pm(k) - 1} \\ &= \frac{\left((3+k)^2 - 8k \pm (3+k)\sqrt{(9-k)(1-k)}\right)\left(3 - 3k \pm \sqrt{(9-k)(1-k)}\right)}{8\left(-1 + k \pm \sqrt{(9-k)(1-k)}\right)} \\ &= \frac{\left((3+k)^2 - 8k \pm (3+k)\sqrt{(9-k)(1-k)}\right)\left(3\sqrt{1-k} \pm \sqrt{9-k}\right)}{8\left(-\sqrt{1-k} \pm \sqrt{9-k}\right)} \end{aligned}$$

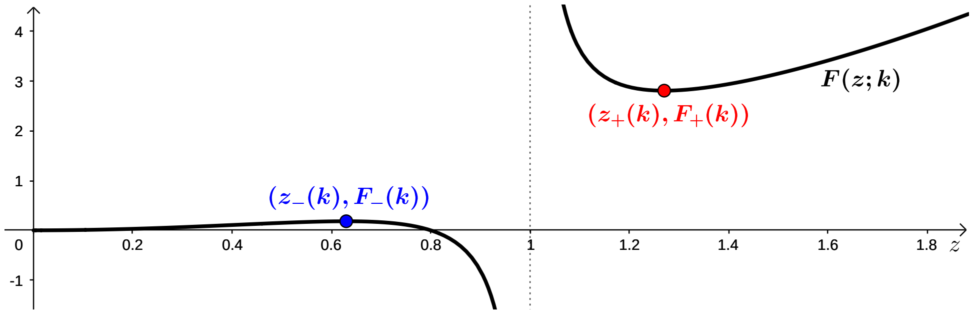


Figure 6.3: Plot of the function F for $k = 0.8$, together with its relative minimum and maximum.

and, rationalizing and simplifying, we find that

$$F_{\pm}(k) = \frac{\left((3+k)^2 - 8k \pm (3+k)\sqrt{(9-k)(1-k)}\right)\left(3 - k \pm \sqrt{(9-k)(1-k)}\right)}{16}$$
$$= \frac{1}{8}\left[-k^2 - 18k + 27 \pm (9-k)\sqrt{(9-k)(1-k)}\right].$$

Hence, the condition (6.39) can be rewritten, equivalently, as

$$\frac{ad^2}{c^3}F_-(\kappa) \le \xi \le \frac{ad^2}{c^3}F_+(\kappa) \quad \text{in } \Omega. \tag{6.40}$$

Therefore, if there exists a constant $\xi > 0$ satisfying (6.40), then (6.37) yields $\sigma_0 \ge 0$. Naturally, the condition (6.31) guarantees the existence of $\xi > 0$ such that (6.40) holds. Let us check that actually $\sigma_0 > 0$. Indeed, arguing by contradiction, assume that $\sigma_0 = 0$. We claim that, in such case,

$$\int_\Omega 2u\varphi\langle\nabla\frac{\varphi}{u}, A_1\nabla\frac{\varphi}{u}\rangle = \int_\Omega 2v\psi\langle\nabla\frac{\psi}{v}, A_2\nabla\frac{\psi}{v}\rangle = 0.$$

Indeed, if either

$$\int_\Omega 2u\varphi\langle\nabla\frac{\varphi}{u}, A_1\nabla\frac{\varphi}{u}\rangle > 0, \quad \text{or} \quad \int_\Omega 2v\psi\langle\nabla\frac{\psi}{v}, A_2\nabla\frac{\psi}{v}\rangle < 0,$$

then,

$$\sigma_0\int_\Omega \frac{\varphi^3}{u} > \int_\Omega \varphi^2(a\varphi + b\psi), \quad \text{or} \quad -\sigma_0\int_\Omega \frac{\psi^3}{v} > -\int_\Omega \psi^2(c\varphi + d\psi),$$

respectively and, hence, by the choice of ξ, we find that

$$0 = \sigma_0\left(\int_\Omega \frac{\varphi^3}{u} - \xi\int_\Omega \frac{\psi^3}{v}\right) > \int_\Omega \left[\varphi^2(a\varphi + b\psi) - \xi\psi^2(c\varphi + d\psi)\right] \ge 0,$$

which is a contradiction. Thus, since A_1 and A_2 are definite positive, it becomes apparent that φ and ψ are proportional to u and v, respectively. Consequently, going back to (6.33) and (6.34), we find that

$$0 = \varphi^2(a\varphi + b\psi) \quad \text{and} \quad 0 = \psi^2(d\psi + c\varphi) \quad \text{in } \Omega,$$

or, equivalently,

$$0 = a\varphi + b\psi = d\psi + c\varphi \quad \text{in} \;\; \Omega,$$

which implies $ad = bc$ in Ω and contradicts our assumption that

$$\kappa = \frac{bc}{ad} \lneq 1 \quad \text{in} \;\; \Omega.$$

This contradiction shows that $\sigma_0 > 0$ and ends the proof. □

The next result provides us with easy-to-check conditions on the coefficients a, b, c and d, so that Theorem 6.11 holds. It is [38, Th. 9.1].

Corollary 6.12. *Assume that $bc \lneq ad$ in Ω and that $\frac{ad^2}{c^3}$, or $\frac{bd}{c^2}$, or $\frac{b^2}{ac}$, or $\frac{b^3}{a^2 d}$, is constant on Ω. Then, every coexistence state of (4.2) is linearly stable.*

Proof. It suffices to show that, under the assumptions, (6.31) holds. On the one hand, setting $\kappa := \frac{bc}{ad}$, we have that

$$\frac{bd}{c^2} = \kappa \frac{ad^2}{c^3}, \qquad \frac{b^2}{ac} = \kappa^2 \frac{ad^2}{c^3}, \qquad \frac{b^3}{a^2 d} = \kappa^3 \frac{ad^2}{c^3},$$

On the other hand, for every $k \in [0,1]$,

$$\begin{aligned} F_+(k) - 1 &= \frac{1}{8}\left[-k^2 - 18k + 19 + (9-k)\sqrt{(9-k)(1-k)}\right] \\ &= \frac{1}{8}\left[(19+k)(1-k) + (9-k)\sqrt{(9-k)(1-k)}\right] \geq 0 \end{aligned}$$

and

$$\begin{aligned} k^3 - F_-(k) &= \frac{1}{8}\left[8k^3 + k^2 + 18k - 27 + (9-k)\sqrt{(9-k)(1-k)}\right] \\ &= \frac{1}{8}\left[-(8k^2 + 9k + 27)(1-k) + (9-k)\sqrt{(9-k)(1-k)}\right] \\ &= \frac{1}{8}\,\frac{16k^3(1-k)(4k^2+5k+23)}{(8k^2+9k+27)(1-k) + (9-k)\sqrt{(9-k)(1-k)}} \geq 0. \end{aligned}$$

Thus,

$$F_-(k) \leq k^3 \leq k^2 \leq k \leq 1 \leq F_+(k) \quad \text{for all} \;\; k \in [0,1],$$

which can be appreciated on Figure 6.2. Hence,

$$\frac{ad^2}{c^3} F_-\left(\frac{bc}{ad}\right) \leq \frac{b^3}{a^2 d} \leq \frac{b^2}{ac} \leq \frac{bd}{c^2} \leq \frac{ad^2}{c^3} \leq \frac{ad^2}{c^3} F_+\left(\frac{bc}{ad}\right) \quad \text{in} \;\; \bar{\Omega}.$$

□

Note that, in particular, Corollary 6.12, and thus Theorem 6.11, always applies when a, b, c and d are positive constants such that $\kappa = bc/ad < 1$, situation studied by He and Ni [55, Th. 3.4 (iii)] for the choice

$$\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$$

under non-flux boundary conditions. However, Theorem 6.11 provides us with the linear stability of the coexistence states of (4.2) not only in the case of constant coefficients, but also in a wide range of situations where two among the coefficients a, b, c and d are arbitrary while the remaining ones are chosen so that all assumptions of Theorem 6.11 are fulfilled. For example, choose c and d arbitrary and pick $\eta > 0$, take

$$a := \frac{\eta c^3}{d^2},$$

and finally choose any function b such that

$$b \lneq \frac{ad}{c} = \frac{\eta c^2}{d}.$$

Note that b can be arbitrary by choosing η sufficiently large. Another advantage of Theorem 6.11 is that it provides us with a method that guarantees the linearized stability of any coexistence state though an easily computable condition.

It is worth-emphasizing that the growth rates of the species, λ and μ, do not play any role in Theorem 6.11. However, they are ultimately responsible of the dynamics of the associated non-spatial model ($d_1 = d_2 = 0$). Thus, for any given domain Ω and functions a, b, c and d satisfying the hypothesis of Theorem 6.11, λ and μ can be chosen so that, for every $x \in \Omega$, the non-spatial model (4.5) can exhibit any desired low-competition dynamics, as soon as it respects the continuity of λ and μ. This feature reveals the huge versatility of Theorem 6.11, for as it can be applied independently of the underlying non-spatial dynamics of (4.5).

Furthermore, Theorem 6.11 is optimal in the sense that it is not true in its greatest generality (non-dependence on d_1, d_2), if $\kappa(x) \geq 1$ for some $x \in \Omega$, as shown by the multiplicity result for the symmetric model delivered in Theorem 5.16.

To conclude, the next result characterizes the existence and uniqueness of coexistence solutions in terms of the existence and linearized stability of the semitrivial solutions, $(\theta_{\{d_1,\lambda,a\}}, 0)$ and $(0, \theta_{\{d_2,\mu,d\}})$. Such characterization constitute the main result of this chapter, which can be stated as follows.

Theorem 6.13. *Assume that* $\kappa = \frac{bc}{ad} \lneq 1$ *in* Ω*. If*

$$\max_{\bar\Omega} \left(\frac{ad^2}{c^3} F_-(\kappa) \right) \leq \min_{\bar\Omega} \left(\frac{ad^2}{c^3} F_+(\kappa) \right),$$

with $F_\pm$ *defined as in* (6.30)*, then:*

(a) *If both semitrivial solutions exist and are linearly unstable, then* (4.2) *admits a unique coexistence state. Moreover, it is a global attractor for the component-wise positive solutions of* (4.1).

(b) *In any other case the system* (4.2) *does not admit any coexistence state.*

(c) *Both semitrivial solutions of* (4.2) *cannot be linearly stable simultaneously.*

(d) *If a semitrivial solution of* (4.2) *is linearly stable, then it is a global attractor for the component-wise positive solutions of* (4.1).

(e) *If the trivial solution of* (4.2) *is linearly stable, then it is a global attractor for the component-wise positive solutions of* (4.1).

In particular, if $\Omega_{\mathrm{per}} \neq \emptyset$*, or* $\Omega^u_{\mathrm{do}} \neq \emptyset$ *and* $\Omega^v_{\mathrm{do}} \neq \emptyset$*, then* (a) *holds for sufficiently small* $d_1, d_2 > 0$ *by Corollary 5.12.*

Proof. Much like in the proof of Theorem 6.8, the equality (6.21) holds by the additivity property of the fixed point index (see Theorem 11.1 of Amann [3]), provided both semitrivial states exist. Thus, as a straightforward consequence of Lemmas 6.3, 6.4, 6.6, and 6.7, the system (4.2) admits a unique coexistence state in the next case

(A) If $\sigma_1[d_1\mathcal{L}_1 - \lambda, ; \mathcal{B}_1, \Omega] < 0$, $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$, and
$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] < 0, \qquad \sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] < 0,$$
i.e., if both semitrivial states do exist and are linearly unstable.

On the other hand, (4.2) cannot admit a coexistence state in each of the following cases:

(B) If $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$, $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$, and
$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] < 0, \qquad \sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] > 0.$$

(C) If $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] > 0$ and $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$, which implies $\sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] > 0$.

(D) If $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] > 0$, $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] > 0$, i.e., there are no semitrivial states.

(E) If $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$ and $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] > 0$, which implies $\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0$.

(F) If $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$, $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$, and
$$\sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0, \qquad \sigma_1[d_1\mathcal{L}_1 - \lambda + b\theta_{\{d_2,\mu,d\}}; \mathcal{B}_1, \Omega] < 0.$$

By the additivity property of the fixed point index, should they exist, the semitrivial solutions cannot be simultaneously linearly stable. All these regions have been represented in Figure 6.4 in the special case when λ and μ are positive constants.

It remains to make sure that there are no coexistence states on each of the following limiting cases:

(I) $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0$ and $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] = 0$, or $\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] = 0$ and $\sigma_1[d_2\mathcal{L}_2 - \mu; \mathcal{B}_2, \Omega] < 0$.

(II) $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]\geq 0$ and $\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]=0$, or $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]=0$ and $\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]\geq 0$.

(III) $\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]<0$ and $\sigma_1[d_2\mathcal{L}_2-\mu+c\theta_{\{d_1,\lambda,a\}};\mathcal{B}_2,\Omega]=0$, or $\sigma_1[d_2\mathcal{L}_2-\mu;\mathcal{B}_2,\Omega]<0$ and $\sigma_1[d_1\mathcal{L}_1-\lambda+b\theta_{\{d_2,\mu,d\}};\mathcal{B}_1,\Omega]=0$.

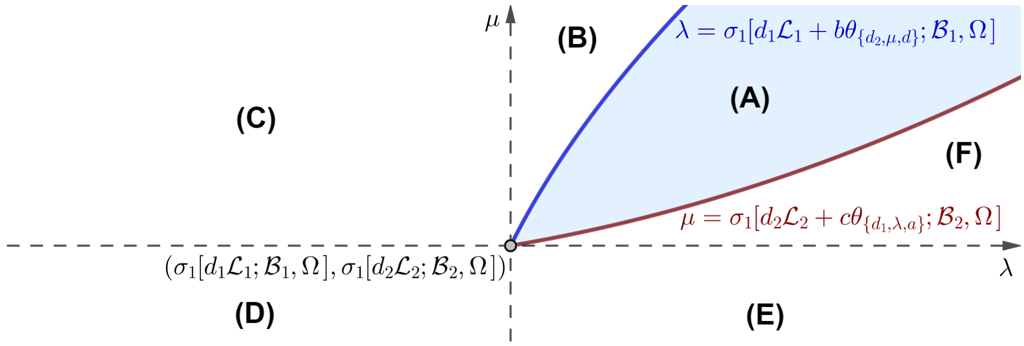


Figure 6.4: Plot of an admissible (λ,μ)-plane for a problem whose coefficients satisfy the hypothesis of Theorem 6.13.

The non-existence in Cases (I) and (II) follows from the fact that the existence of both semitrivial steady states is necessary for the existence of coexistence steady states. Indeed, if there exists a coexistence steady state, (u_0,v_0), then, by the monotonicity of the principal eigenvalue with respect to the potential established in Theorem 2.12(a), we have that

$$0=\sigma_1[d_1\mathcal{L}_1-\lambda+au_0+bv_0;\mathcal{B}_1,\Omega]>\sigma_1[d_1\mathcal{L}_1-\lambda;\mathcal{B}_1,\Omega]$$

and

$$0=\sigma_1[d_2\mathcal{L}_2-\mu+cu_0+dv_0;\mathcal{B}_2,\Omega]>\sigma_1[d_1\mathcal{L}_1-\mu;\mathcal{B}_2,\Omega].$$

It should be noted that this argument also provides us with the non-existence in situations (C), (D) and (E)

On the other hand, the non-existence in Case (III) will be proved through an argument involving the implicit function theorem. Such argument reads as follows. Suppose that (4.2) admits a coexistence state, (u_0,v_0), and, for every $\varepsilon_1,\varepsilon_2\in\mathbb{R}$, consider the problem

$$\begin{cases} d_1\mathcal{L}_1u=(\lambda+\varepsilon_1)u-au^2-buv & \text{in } \Omega,\\ d_2\mathcal{L}_2v=(\mu+\varepsilon_2)v-dv^2-cuv & \text{in } \Omega,\\ \mathcal{B}_1u=\mathcal{B}_2v=0 & \text{on } \partial\Omega. \end{cases} \tag{6.41}$$

Note that (6.41) satisfies the hypothesis of Theorem 6.13 as soon as (4.2) does. Now, the coexistence state (u_0,v_0) solves (6.41) for the choice $(\varepsilon_1,\varepsilon_2)=(0,0)$. Moreover, the linearization of (6.41) with respect to (u,v), particularized at $(\varepsilon_1,\varepsilon_2)=(0,0)$, evaluated at (u_0,v_0), is invertible, because, due to Theorem 6.11, its principal eigenvalue is positive. Then, the implicit function theorem provides us with a smooth surface of coexistence states for sufficiently small $\varepsilon_1,\varepsilon_2$. In particular, there exists ε_0 such that (6.41) admits a coexistence state for all $\varepsilon_1,\varepsilon_2\in[-\varepsilon_0,\varepsilon_0]$. Furthermore, due to Theorem 6.11, these coexistence states are linearly stable.

Now, suppose that we are under the assumptions of the first part of Case (III), i.e.,

$$\sigma_1[d_1\mathcal{L}_1 - \lambda; \mathcal{B}_1, \Omega] < 0 \quad \text{and} \quad \sigma_1[d_2\mathcal{L}_2 - \mu + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] = 0,$$

and that (4.2) admits a coexistence state. Then, by the implicit function theorem, (6.41) also admits a coexistence state with $\varepsilon_1 = 0$ and $\varepsilon_2 = -\varepsilon_0$ for sufficiently small $\varepsilon_0 > 0$. Moreover,

$$\sigma_1[d_2\mathcal{L}_2 - (\mu - \varepsilon_0) + c\theta_{\{d_1,\lambda,a\}}; \mathcal{B}_2, \Omega] > 0,$$

which fits the situation (F) provided ε_0 is chosen so that

$$\sigma_1[d_2\mathcal{L}_2 - (\mu - \varepsilon_0); \mathcal{B}_2, \Omega] < 0.$$

This is impossible, as we already know that the problem cannot admit a coexistence state in that situation.

To conclude the proof of Theorem 6.13, it suffices to note that if (4.1) admits a unique linearly stable steady state, and the remaining steady states are linearly unstable, or linearly neutrally stable, then the model (4.1) is compressive. Therefore, by Remark 33.2, Theorem 33.3 and Theorem 34.1 of Hess [56], such a linearly stable steady state is a global attractor for the component-wise positive solutions of (4.1). □

Going back to Figure 6.4, and according to Theorem 6.13, we have that when $(\lambda, \mu) \in \mathbb{R}^2$ is located in the shaded region of Figure 6.4, i.e., region (A), then (4.2) admits a unique coexistence state which is a global attractor for the component-wise positive solutions. Moreover, if (λ, μ) belongs to either regions (B), (C), (E) or (F), and the semi-axis between them, then the stable semitrivial solution ($(\theta_{\{d_2,\mu,d\}})$ in (B) and (C), and $(\theta_{\{d_1,\lambda,a\}}, 0)$ in (E) and (F)) is actually a global attractor for the component-wise positive solutions because in those situations there are no coexistence states and the system is compressive towards the appropriate semitrivial steady state (see Theorem 34.1 of Hess [56]). Finally, in region (D) the trivial solution must be a global attractor with respect to the positive solutions of (4.1).

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