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BAND-DIAGONAL OPERATORS ON BANACH LATTICES: MATRIX DYNAMICS AND INVARIANT SUBSPACES

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Abstract. We address the existence of non-trivial closed invariant ideals for positive operators defined on Banach lattices whose order is induced by an unconditional basis. In particular, for band-diagonal positive operators such existence is characterized whenever their matrix representations meet a positiveness criteria. For more general classes of positive operators, sufficient conditions are derived proving, particularly, the sharpness of such results from the standpoint of view of the matrix representations. The whole approach is based on studying the behavior of the dynamics of infinite matrices and the localization of the non-zero entries. Finally, we generalize a theorem of Grivaux regarding the existence of non-trivial closed invariant subspaces for positive tridiagonal operators to a more general class of band-diagonal operators showing, in particular, that a large subclass of them have non-trivial closed invariant subspaces but lack non-trivial closed invariant ideals.

1. Introduction

Understanding linear operators acting on finite-dimensional complex Banach spaces by means of their restrictions to their invariant subspaces may be considered as an initial approach to the more general framework of Banach spaces of infinite dimension. For a long time, it was an open problem whether the classical Perron-Frobenius Theorem for irreducible non-negative matrices had a natural extension of the form that irreducible positive compact operators on Banach lattices possess a positive spectral radius (i.e., possess a positive eigenvalue) and thus they are not quasinilpotent. In 1986, B. de Pagter [8] finally answered the question in the affirmative, where "irreducible" has to be taken in the sense of ideal irreducibility, namely, the operator does not leave any non-trivial ideal invariant.

Recall that if X is a Banach lattice, a closed subspace M of X is a sublattice if it is closed under the lattice operations. A closed subspace M of X is an ideal if $x \in M$ and $|y| \leq |x|$ imply $y \in M$. It is easy to see that every invariant ideal is an invariant sublattice, and that every invariant sublattice is an invariant subspace. Hence, de Pagter theorem establishes that every positive compact quasinilpotent operator on a Banach lattice has a non-trivial closed invariant ideal.

In fact, that every positive compact operator on a Banach lattice has a non-trivial closed invariant sublattice traces back to a pioneering work of Kreĭn and Rutman [11]. In 2006, A. K. Kitover and A. W. Wickstead [12] presented several examples of positive operators on Banach lattices with no invariant sublattices. In [14], the authors exhibit a few examples of positive operators on Banach lattices with no invariant sublattices. Moreover, the authors prove that there is a positive tridiagonal

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operator (even positive compact operators on ℓ^p, $1 \leq p < \infty$) with no invariant closed ideals and exactly one closed invariant sublattice.

In this framework, a complete understanding of the invariant subspaces of positive operators on Banach lattices seems to be far from reaching. Indeed, in the series of papers [2, 3, 4] Abramovich, Aliprantis and Burkinshaw undertook this problem establishing sufficient conditions which implied the existence of non-trivial closed invariant subspaces (indeed, non-trivial closed invariant ideals) for positive operators conjecturing that every positive operator on a Banach lattice has non-trivial closed invariant subspaces (see also the related works by Troitsky and coauthors [6], [13] or [15]).

On the other hand, a different perspective by means of moment sequences allowed Grivaux [10] to prove the existence of non-trivial closed invariant subspaces for positive tridiagonal operators on the classical Banach lattices ℓ^p, $1 \leq p < \infty$ or c_0. More recently, in [9] the authors studied positive operators on Banach lattices whose order is induced by unconditional basis showing, in particular, that every lattice homomorphism on such spaces has non-trivial closed invariant ideals (and hence invariant subspaces). Likewise, they characterized when positive tridiagonal operators have non-trivial closed invariant ideals.

At this regard, it is worthy to remark that even in the classical Banach lattices ℓ^p, $1 \leq p < \infty$, or c_0, it is still unknown whether every positive operator with a pentadiagonal matrix representation or, more generally, a band-diagonal matrix representation, has non-trivial closed invariant subspaces. This will be the driving motivation of this work.
In particular, we will study positive operators T acting on Banach lattices whose order is induced by an unconditional basis and such that the matrix representation of T respect to such a basis is band-diagonal. Our approach in order to provide non-trivial closed invariant subspaces (even ideals) will rely on studying the behavior of the dynamics of the associated infinite matrices.

In order to state some of the results obtained regarding band-diagonal matrices, it turns out simpler if we consider an alternative notation for matrices based on the sequences defining the corresponding non-zero diagonals instead of the classical one based on rows and columns. More precisely, let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be an infinite matrix and for every $i \in \mathbb{Z}$ and $n \in \mathbb{N}$ let us denote

$$\begin{cases} d_n^i = a_{n,n+i} & \text{if } i \geq 0 \\ d_n^i = a_{n-i,n} & \text{if } i < 0, \end{cases}$$

For each $i \in \mathbb{Z}$, let d^i be the sequence in the i-th diagonal of A counted from the diagonal, that is, $d^i = (d_n^i)_{n\geq 1}$. Note that d^0 denotes the sequence in the principal diagonal of A, while d^1 or d^{-1} denote the sequences defining the first superdiagonal and the first subdiagonal of A, respectively. Accordingly, we will say that the matrix A is *in diagonal notation* when it is expressed by $A = (d^i)_{i\in\mathbb{Z}}$. With this notation at hands, we introduce band-diagonal matrices as follows.

Definition. *An infinite matrix* in diagonal notation $A = (d^i)_{i\in\mathbb{Z}}$ *is* ***band-diagonal*** *if there exists* $k \in \mathbb{N} \cup \{0\}$ *such that* $d^i = 0$ *if* $|i| > k$. *The number* k *is called the* ***bandwidth*** *of the matrix. In this case, the matrix is said to be* ***(2k+1)-diagonal****.*

Observe that if $k = 0$ we obtain a diagonal matrix, if $k = 1$ we have a tridiagonal matrix while for $k = 2$ it is a pentadiagonal matrix. This definition leads naturally to the concept of band-diagonal operator. Particularly, if X is a Banach lattice whose order is induced by an unconditional basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$, a linear bounded operator T is a *band-diagonal operator* if its associated matrix with respect to $\mathcal{E}$ is band-diagonal. Likewise, we say T is a *(2k+1)-diagonal operator* for $k \geq 0$ if its associated matrix is *(2k+1)-diagonal*.

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Motivated by the study of positive operators T acting on Banach lattices whose order is induced by an unconditional basis, we will focus on studying the dynamics of infinite matrices with non-negative entries, in short, non-negative infinite matrices. For this purpose, we introduce the following class of matrices:

Definition (Class $\mathcal{D}$). *A non-negative infinite matrix A belongs to the class $\mathcal{D}$ if for every $N \in \mathbb{N}$ the power matrix A^N is well defined, that is, every entry of A^N is finite.*

For instance, every infinite band-diagonal matrix belongs to $\mathcal{D}$, since every row and column has a finite number of non-zero entries.

Note that not every matrix in the class $\mathcal{D}$ represents necessary the action of a positive linear operator on Banach lattices with an unconditional basis (it suffices to take an infinite diagonal matrix with unbounded diagonal positive sequence d^0 acting on ℓ_p, $1 \leq p < \infty$).

In order to relate the dynamics of non-negative infinite matrices to the existence of non-trivial closed invariant ideals of positive linear operators on Banach lattices, we introduce the following concept which will play a role throughout the rest of the manuscript.

Definition 1.1. *Let $A = (d^i)_{i\in\mathbb{Z}}$ be an infinite matrix in diagonal notation. Assume $A \in \mathcal{D}$ and write A^N as $A^N = (d(N)^i)_{i\in\mathbb{Z}}$ for each $N \in \mathbb{N}$. We say that A has dynamics with zeros if there exist $i_0 \in \mathbb{Z} \setminus \{0\}$ and $n_0 \in \mathbb{N}$ such that $d(N)^{i_0}_{n_0} = 0$ for every $N \in \mathbb{N}$. Otherwise, we will say that A has strictly positive dynamics.*

Non-negative matrices exhibiting dynamics with zeros are related to positive operator T acting on a Banach lattice whose order is induced by unconditional basis having non-trivial closed invariant ideals. Indeed, restating a result of Radjavi and Troitsky it follows that such operators T have non-trivial closed invariant ideals if and only if their matrix representations have dynamics with zeros (see [14, Proposition 1.2]). Though this latter result was originally proved for ℓ_p with $1 \leq p < \infty$ and c_0, it also holds in the setting of Banach lattices with unconditional basis.

Consequently, obtaining characterizations ensuring that such dynamics phenomenon happen provide results regarding the existence of non-trivial closed invariant ideals for positive operators whose order is induced by unconditional basis. In this setting, the authors of [9] characterized the existence of non-trivial closed invariant ideals for positive tridiagonal operators showing, in particular, that a non-negative infinite matrix A being tridiagonal has dynamics with zeros if and only if there are zeros on the subdiagonal or on the superdiagonal of A.

The rest of paper is organized as follows: in Section 2 we will characterize the existence of non-trivial closed invariant ideals for T whenever its matrix representation meets a positiveness criteria, generalizing the previous results in this context. For the sake of clarity, we will begin by providing the characterization for pentadiagonal operators. These results will follow directly upon applying the characterization of the dynamics of the associated matrices which turns out to be the core of our analysis. The main result in this direction reads as follows (see Theorem 2.4):

Theorem. *Let $A = (d^i)_{i\in\mathbb{Z}}$ be a $(2k+1)-$diagonal non-negative infinite matrix $(k \geq 2)$ belonging to the class $\mathcal{D}$. Assume that $d^i_n > 0$ for every $-k+1 \leq i \leq -1$ and $n \in \mathbb{N}$. Then, A has dynamics with zeros if and only if there exists $n_0 \in \mathbb{N}$ such that for every $i \in \{1, \cdots, k\}$ the element d^i_n is equal to zero for all $n \in \{n_0 + 1 - i, \cdots, n_0\} \cap \mathbb{N}$.*

It is worthy to remark that while in the statement of the previous theorem $k \geq 2$, the cases $k = 0, 1$ are also characterized. Indeed, if $k = 0$, the matrix A is diagonal and clearly it has dynamics with zeros since A^n is diagonal for every $n \in \mathbb{N}$. Likewise, if $k = 1$, A is a tridiagonal matrix and A has

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dynamics with zeros if and only if it has a zero entry either in the superdiagonal or in the subdiagonal (see [9, Corollary 4.3]).

Accordingly, if T is a band-diagonal positive operator acting on a Banach lattice X whose order is induced by an unconditional basis $\mathcal{E}$ and A denotes its associated matrix with respect to $\mathcal{E}$, T has non-trivial closed invariant ideals if and only if A has dynamics with zeros. So, the existence of non-trivial closed invariant ideals is characterized for $(2k+1)$-diagonal operators with matrix representation $A = (d^i)_{i\in\mathbb{Z}}$ satisfying $d^i_n > 0$ for every $-k+1 \leq i \leq -1$ and $n \in \mathbb{N}$ whenever $k \geq 2$ or having a zero entry either in the superdiagonal or in the subdiagonal when $k = 1$.

In Section 3 we deal with a subclass of positive operators that are not necessarily band-diagonal introducing the concept of *honeycomb matrices* and showing that positive operators whose matrix representation is honeycomb do have non-trivial closed invariant ideals (see Theorem 3.2). Likewise, we discuss the sharpness of our results and derive, within this class, a characterization of the existence of non-trivial closed invariant ideals for *pentadiagonal operators* and *heptadiagonal operators* whenever the outer subdiagonal and the outer superdiagonal of its matrix representation consist of strictly positive entries (see Theorem 3.8 and Theorem 3.13).

Finally, in Section 4, we extend a theorem of Grivaux regarding the existence of non-trivial closed invariant subspaces for positive tridiagonal operators to a more general class of band-diagonal operators exhibiting, in particular, a large class of positive band-diagonal operators having non-trivial closed invariant subspaces but lacking non-trivial closed invariant ideals.

We close this introductory section with some preliminaries which will be of interest throughout the rest of the manuscript.

1.1. **Preliminaries.** Let X be an infinite dimensional (real or complex) Banach space and denote by $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$ an unconditional basis in X. Clearly, for every $x \in X$ there exists a unique sequence of scalars $(x_n)_{n\in\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} x_n e_n.$$

Every unconditional basis gives rise to a natural closed cone

$$\mathcal{C}_{\mathcal{E}} := \left\{ \sum_{n=1}^{\infty} x_n e_n : x_n \geq 0 \text{ for all } n \in \mathbb{N} \right\},$$

which defines a partial order in X as follows: $x \leq y$ if and only if $y - x \in \mathcal{C}_{\mathcal{E}}$. In addition, it is well known that every cone $\mathcal{C}$ in a Banach space X determines a partial order $\leq$ by letting $x \leq y$ whenever $y - x \in \mathcal{C}$. The elements of $\mathcal{C}$ are known as positive vectors and the pair $(X, \mathcal{C})$ is an ordered Banach space.

Following the standard lattice notation, the supremum (least upper bound) and the infimum (greatest lower bound) of a pair of vectors x, $y \in X$ will be denoted by $x \wedge y$ and $x \vee y$ respectively, namely

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\}.$$

For x in a vector lattice, its *positive part*, its *negative part* and its *absolute value* are defined by:

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad \text{and } |x| = x \vee (-x),$$

respectively.

A sublattice of a vector lattice is a subspace which is also closed under the lattice operations (that is, for each x, y in a sublattice, $x \vee y$ and $x \wedge y$ also belong to the sublattice). An ideal M in a vector lattice is a subspace such that for every x, y such that if $|x| \leq |y|$ and $y \in M$ then $x \in M$.

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Recall that a norm in a vector lattice is a lattice norm if it satisfies $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$. Thus, a Banach lattice is a complete vector lattice equipped with a lattice norm. Throughout the rest of the manuscript, we restrict ourselves to Banach lattices whose orders have been induced by an unconditional basis.

It is worthy to point out that for such Banach lattices, every closed ideal is of the form

$$\left\{ \sum_{n=1}^{\infty} x_n e_n \in X : x_n = 0 \text{ for every } n \in N \right\},$$

where $N \subset \mathbb{N}$.

As it is standard, $\mathcal{L}(X)$ will denote the Banach algebra consisting of bounded linear operators defined on X. We say that $T \in \mathcal{L}(X)$ is **positive** respect to the basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$ if $T(\mathcal{C}_{\mathcal{E}}) \subset \mathcal{C}_{\mathcal{E}}$, or equivalently, $Tx \geq 0$ for each $x \geq 0$. Observe that if $\mathcal{E}$ is unconditional, the linear functionals e_n^* defined on X as

$$e_n^* \left(\sum_{m=1}^{\infty} x_m e_m \right) = x_n$$

satisfy that $(e_n^*)_{n\in\mathbb{N}}$ is an unconditional sequence in the dual space X^* but not necessarily a basis.

Note that every operator $T \in \mathcal{L}(X)$ has an associated matrix given by

$$A = \left(e_i^*(Te_j)\right)_{i,j\in\mathbb{N}}.$$

In particular, identifying T with the infinite matrix A, it holds that T is a positive operator (respect to $\mathcal{E}$) if and only if $e_i^*(Te_j) \geq 0$ for every pair (i, j). Moreover, since $\mathcal{E}$ is unconditional, every positive operator is automatically continuous; see [1, Theorem 1.31]. Clearly, if $T \in \mathcal{L}(X)$ is a positive operator, then its associated matrix belongs to the class $\mathcal{D}$.

2. Invariant closed ideals for band-diagonal operators

As we have just pointed out, every positive linear operator on a Banach lattice whose order is induced by an unconditional basis is automatically bounded. Next lemma establishes that positive band-diagonal operators acting on c_0 and ℓ^p spaces, $1 \leq p < \infty$, have indeed bounded *upper and lower diagonals.*

Lemma 2.1. *Let $A = (d^i)_{-k\leq i\leq k}$ be a non-negative band-diagonal matrix in diagonal notation and T the positive linear operator induced by A acting or ℓ^p, $1 \leq p < \infty$, or c_0. Then, each sequence d^i is bounded.*

Proof. We argue by contradiction. Assume there exists $0 \leq i \leq k$ such that d^i is not a bounded sequence. The case $-k \leq i \leq 0$ is analogous. Then, there exists a subsequence such that $d^i_{n_k} \to \infty$. For the sake of simplicity, let us suppose that $d^i_n \to \infty$ when $n \to \infty$. Consider the sequence of basis vectors $(e_{n+i})_{n\geq 1}$. Observe that

$$Te_{n+i} \geq d^i_n e_n,$$

so $\|Te_{n+i}\| \geq d^i_n$. Having in mind that T is bounded and letting $n \to \infty$, we deduce that $\|Te_{n+i}\| \to \infty$, which yield a contradiction. $\square$

Note that the converse of Lemma 2.1 holds in a straightforward manner in c_0 or ℓ^p spaces. Likewise, it extends naturally to more general Banach spaces, the next remark being the key to such an extension.

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Remark 2.2. *If X is a Banach space with an unconditional basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$, S and B denote the shift and the backward shift operator on X, respectively*

$$Se_n = e_{n+1} \qquad n \geq 1$$

$$Be_n = \begin{cases} 0 & n = 1, \\ e_{n-1} & n \geq 2, \end{cases}$$

and for $-k \leq i \leq k$, D_i is the diagonal operator with coefficients $(d_n^i)_n$

$$D_i e_n = d_n^i e_n,$$

then

$$T = \sum_{i=-k}^{-1} S^i D_i + \sum_{i=0}^{k} D_i B^i \tag{1}$$

is a linear operator whose matrix representation respect to $\mathcal{E}$ is band-diagonal. The converse is also clear. Needless to say that if S, B and each D_i act boundedly on X, so does T.

In order to study the dynamics of non-negative infinite matrices, we first observe that the zero entries in the main diagonal do not affect the behavior of its dynamics in the sense of Definition 1.1.

Lemma 2.3. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$ and $d = (d_n)_{n\in\mathbb{N}}$ a bounded sequence of strictly positive numbers. Let $B = (b_{i,j})_{i,j\in\mathbb{N}}$ defined by*

$$b_{i,j} = \begin{cases} d_i & \text{if } i = j \\ a_{i,j} & \text{if } i \neq j. \end{cases}$$

Then, A has dynamics with zeros if and only if B has.

Proof. For each $n \in \mathbb{N}$, let us denote by $B^n = (b_{i,j}^{(n)})_{i,j\in\mathbb{N}}$ and $A^n = (a_{i,j}^{(n)})_{i,j\in\mathbb{N}}$. First, let us assume that A has dynamics with zeros. Our goal is proving that for each $i, j \in \mathbb{N}$ with $i \neq j$ such that $a_{i,j}^{(n)} = 0$ for every $n \in \mathbb{N}$, then $b_{i,j}^{(n)} = 0$ for every $n \in \mathbb{N}$. Since A has dynamics with zeros, this argument will imply that B has dynamics with zeros as well.

We argue by induction in $n \in \mathbb{N}$. If $n = 1$, it is obvious that $b_{i,j} = 0$ whenever $a_{i,j} = 0$ with $i, j \in \mathbb{N}$ and $i \neq j$, by the construction of B.

Now, let $n_0 \in \mathbb{N}$ and assume that for $1 \leq k \leq n_0$ and each $i, j \in \mathbb{N}$ such that $i \neq j$ and $a_{i,j}^{(k)} = 0$ we have $b_{i,j}^{(k)} = 0$. Arguing by contradiction, suppose that there exist $i_0, j_0 \in \mathbb{N}$ with $i_0 \neq j_0$ such that $a_{i_0,j_0}^{(k)} > 0$ for every $1 \leq k \leq n_0 + 1$ and $b_{i_0,j_0}^{(n_0+1)} > 0$. We have

$$b_{i_0,j_0}^{(n_0+1)} = \sum_{m=1}^{\infty} b_{i_0,m}^{(n_0)} b_{m,j_0}.$$

Then, there exists $m_0 \in \mathbb{N}$ such that $b_{i_0,m_0}^{(n_0)} b_{m_0,j_0} > 0$. Let us show that $m_0 \neq i_0, j_0$. If $m_0 = i_0$, observe that $b_{i_0,j_0} = a_{i_0,j_0} = 0$, which would be a contradiction. On the other hand, if $m = j_0$, then $b_{i_0,j_0}^{(n_0)} = 0$ by the induction hypothesis, since $a_{i_0,j_0}^{(n_0)} = 0$. Now, for each $1 \leq k \leq n_0$, the equality

$$0 = a_{i_0,j_0}^{(k+1)} = \sum_{m=1}^{\infty} a_{i_0,m}^{(k)} a_{m,j_0}$$

yields to deduce that $a_{i_0,m_0}^{(k)} = 0$ for each $1 \leq k \leq n_0$, since $a_{m_0,j_0} = b_{m_0,j_0} > 0$. Then, the induction hypothesis implies that $b_{i_0,m_0}^{(n_0)} = 0$, which is also a contradiction.

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Accordingly, the induction argument is proved and B has dynamics with zeros.

For the converse, assume that B has dynamics with zeros and let us show that A has as well. For such a task, observe that if $b_{i,j} = 0$ for some $i, j \in \mathbb{N}$, then $a_{i,j} = 0$. In particular, this implies that if $b_{i,j}^{(n)} = 0$ for some $i, j \in \mathbb{N}$ and every $n \in \mathbb{N}$, then $a_{i,j}^{(n)} = 0$ as well. Consequently, A has dynamics with zeros, as claimed. This concludes the proof of the lemma. $\square$

As announced in the Introduction, our main result in this section reads as follows:

Theorem 2.4. *Let $A = (d^i)_{i\in\mathbb{Z}}$ be a $(2k+1)-$diagonal non-negative infinite matrix ($k \geq 2$) belonging to the class $\mathcal{D}$. Assume that $d_n^i > 0$ for every $-k+1 \leq i \leq -1$ and $n \in \mathbb{N}$. The matrix A has dynamics with zeros if and only if the following condition*

($*$) *there exists $n_0 \in \mathbb{N}$ such that for every $i \in \{1, \cdots, k\}$, $d_n^i = 0$ for all $n \in \{n_0+1-i, \cdots, n_0\} \cap \mathbb{N}$*

holds. Consequently, if T is a band-diagonal positive operator acting on a Banach lattice X whose order is induced by an unconditional basis $\mathcal{E}$ being its associated matrix A, T has non-trivial closed invariant ideals if and only if A satisfies ($*$).

It may be illustrative to show the expression of a band-diagonal matrix satisfying the hypothesis of Theorem 2.4.

$$A = \begin{pmatrix} * & * & * & 0 & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & 0 & \cdots \\ * & * & * & 0 & 0 & 0 & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes a strictly positive real number. In such a case, if each diagonal is a bounded sequence, the matrix A defines a linear bounded positive operator in ℓ^p spaces, for instance.

Observe that A is a heptadiagonal matrix satisfying the hypotheses of Theorem 2.4 with $n_0 = 3$. Roughly speaking, we are asking the matrix to have a rectangle of zeros with infinite length and width n_0, starting in the $n_0 + 1$-th column.

A word about notation. In order to simplify notation throughout this section, given a matrix *in diagonal notation* $A = (d^i)_{i\in\mathbb{Z}}$, we will adopt that $d_n^i = 0$ whenever $n \leq 0$.

2.1. **A first step: the pentadiagonal case.** Before proceeding with the proof of Theorem 2.4, and for the sake of clarity, we will first show the particular instance of $k = 2$, namely, the pentadiagonal case. We state it for the sake of completeness.

Theorem 2.5. *Let $A = (d_i)_{i\in\mathbb{Z}}$ be a pentadiagonal non-negative infinite matrix such that $d_n^{-1} > 0$ for every $n \in \mathbb{N}$. The matrix A has dynamics with zeros if and only if there exists $n_0 \in \mathbb{N}$ such that $d_{n_0}^1 = d_{n_0-1}^2 = d_{n_0}^2 = 0$.*

Observe that, because of the convention, if $n_0 = 1$ then $d_0^2 = 0$.

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Proof. As a consequence of Lemma 2.3, we may assume without loss of generality that $d_n^0 = 0$ for every $n \in \mathbb{N}$. Assume A has dynamics with zeros, and let us show the existence of $n_0 \in \mathbb{N}$ such that $d_{n_0}^1 = d_{n_0-1}^2 = d_{n_0}^2 = 0$.

Denote A^2 by $A^2 = (b^i)_{i\in\mathbb{Z}}$. Observe that for every $n \in \mathbb{N}$

$$b_n^1 \geq d_{n-1}^{-1} d_{n-1}^2 + d_n^2 d_{n+1}^{-1}. \tag{2}$$

Obviously, A^2 has dynamics with zeros as well. Define $\tilde{B} = (\tilde{b}^i)_{i\in\mathbb{Z}}$ the matrix in diagonal notation given by

$$\begin{cases} \tilde{b}_n^i = b_n^i & \text{if } |i| < 2, \\ \tilde{b}_n^i = 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. Obviously, $\tilde{B}$ has dynamics with zeros as well and is a tridiagonal non-negative matrix, so by [9, Corollary 4.3], there exists $n_0 \in \mathbb{N}$ such that $\tilde{b}_{n_0}^1 = 0$ or $\tilde{b}_{n_0}^{-1} = 0$. Recall that we are assuming $d_n^{-1} > 0$ for every $n \in \mathbb{N}$, so $b_{n_0}^1 > 0$ as well. It yields that $\tilde{b}_{n_0}^1 = 0$.

Now, since $d_n^0 = 1$ for every $n \in \mathbb{N}$, it follows that $\tilde{b}_n^i \geq d_n^i$ for every $|i| < 2$ and $n \in \mathbb{N}$. Hence, $d_{n_0}^1 = 0$.

Let us show that, in addition, $d_{n_0-1}^2 = d_{n_0}^2 = 0$. Using (2), we deduce that

$$d_{n_0-1}^{-1} d_{n_0-1}^2 + d_{n_0}^2 d_{n_0+1}^{-1} = 0.$$

Observe that if $n_0 \geq 2$, both $d_{n_0+1}^{-1}$ and $d_{n_0-1}^{-1}$ are strictly positive by hypothesis. Moreover, since $d_{n_0-1}^2$ and $d_{n_0}^2$ are non-negative, we deduce that $d_{n_0-1}^2 = d_{n_0}^2 = 0$.

Likewise, if $n_0 = 1$ we automatically have $d_0^2 = 0$ and hence, the same argument works in order to show that $d_1^2 = 0$, as wished.

For the reverse implication, we will make use of the standard matrix notation $A = (a_{i,j})_{i,j\in\mathbb{N}}$, since it simplifies the presentation of the rest of the proof. Accordingly, reformulating the statement, we will show that if there exists $n_0 \in \mathbb{N}$ such that $a_{n_0,n_0+1} = a_{n_0-1,n_0+1} = a_{n_0,n_0+2} = 0$ then A has dynamics with zeros.

Assume then the existence of such $n_0 \in \mathbb{N}$ and denote $A^n = (a_{i,j}^{(n)})_{i,j\in\mathbb{N}}$ for every $n \in \mathbb{N}$. By induction on $k \in \mathbb{N}$, we are showing that for every pair $(i,j) \in \{1,\cdots,n_0\} \times \{n_0+1, n_0+2, \cdots\}$ it holds

$$a_{i,j}^{(k)} = 0.$$

Observe that by hypothesis, the induction case for $k = 1$ holds. Assume the induction hypothesis for $k \in \mathbb{N}$ and consider A^{k+1}. Let $i \in \{1,...,n_0\}$ and $j \geq n_0 + 1$.

Clearly $A^{k+1} = A\,A^k$ and the (i,j) element of the matrix A^{k+1} is the product of the i-th row of A by the j-th column of A^k. Observe that all the positive elements of the i-th row of A are in the first n_0 coordinates. Moreover, by induction, the first n_0 coordinates of the j-th column of A^k are null. Hence, the corresponding product of the i-th row of A and the j-th column of A^k is zero, and therefore the element (i,j) of A^{k+1} is zero, as wished to show. This completes the proof. $\square$

2.2. **The General Case.** Now we proceed with the proof of Theorem 2.4 and derive some consequences.

Proof of Theorem 2.4. As in the proof of Theorem 2.5, we can assume without loss of generality that $d_n^0 = 1$ for every $n \in \mathbb{N}$.

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We argue by induction in $k \geq 2$ showing that every $(2k+1)-$diagonal non-negative infinite matrix satisfying condition $(*)$ has dynamics with zeros. If $k = 2$, the conclusion follows from Theorem 2.5.

For the general induction step, assume that every $(2k-1)-$diagonal non-negative infinite matrix satisfies the statement of the Theorem 2.4. We will show it holds for every $(2k+1)-$diagonal operator.

Let $A = (d^i)_{i\in\mathbb{Z}}$ be a $(2k+1)-$diagonal non-negative infinite matrix in diagonal notation, and denote A^2 by $A^2 = (b^i)_{i\in\mathbb{Z}}$. Observe that for every $n \in \mathbb{N}$ and $l \in \{1, \cdots, k-1\}$

$$b_n^{k-l} \geq d_{n-l}^{-l} d_{n-l}^{k} + d_n^k d_{n+k-l}^{-l}. \tag{3}$$

Analogously as in the proof of Theorem 2.5, let $\tilde{B} = (\tilde{b}^i)_{i\in\mathbb{Z}}$ be the matrix in diagonal notation given by

$$\begin{cases} \tilde{b}_n^i = b_n^i & \text{if } |i| < k, \\ \tilde{b}_n^i = 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbb{N}$. Once again, $\tilde{B}$ has dynamics with zeros, but $\tilde{B}$ is a $(2k-1)-$diagonal non-negative infinite matrix which satisfies by contruction that $\tilde{b}_n^i \geq d_n^i > 0$ for every $n \in \mathbb{N}$ and $-k+2 \leq i \leq -1$. Thus, there exists $N \in \mathbb{N}$ such that for every $i \in \{1, \cdots, k\}$

$$\tilde{b}_n^i = 0 \qquad \text{for every } n \in \{N+1-i, \cdots N\}.$$

Again, since $d_n^0 = 1$ we have that $\tilde{b}_n^i \geq d_n^i$ for every $|i| < k$ and $n \in \mathbb{N}$. So, for each $i \in \{1, \cdots, k-1\}$

$$d_n^i = 0 \qquad \text{for every } n \in \{N+1-i, \cdots N\}.$$

Let us show that $d_n^k = 0$ for every $n \in \{N+1-k, \cdots, N\}$, in order to prove that the implication is necessary.

If $n \geq k$, observe that, by hypothesis, $d_{n-l}^{-l} > 0$ for every $l \in \{1, \cdots, k-1\}$. This property, with the inequality (3) and along with the fact that $b_n^{k-l} = 0$ for every $l \in \{1, \cdots, k-1\}$, imply

$$d_n^k = 0 \qquad \text{for every } n \in \{N+1-k, \cdots, N\},$$

as claimed.

If $n \in \{1, \cdots, k-1\}$, the desired statement follows by recalling that $d_n^i = 0$ for every $i \in \mathbb{Z}$ and $n \leq 0$.

For the converse implication, by means of standard matrix notation $A = (a_{i,j})_{i,j\in\mathbb{N}}$ as in Theorem 2.5, we are required to show that A has dynamics with zeros whether there exists $n_0 \in \mathbb{N}$ such that $a_{i,j} = 0$ for every $(i,j) \in \{1, \cdots, n_0\} \times \{n_0+1, n_0+2, \cdots\}$.

The proof runs as that of Theorem 2.5 showing, by induction on $k \in \mathbb{N}$, that for every $(i,j) \in \{1, \cdots, n_0\} \times \{n_0+1, n_0+2, \cdots\}$

$$a_{i,j}^{(k)} = 0,$$

where $A^n = (a_{i,j}^{(n)})_{i,j\in\mathbb{N}}$ for every $n \in \mathbb{N}$. This completes the proof of Theorem 2.4.

□

As an immediate corollary, we observe that the statement of Theorem 2.4 follows if we assume that the upper diagonals consist of strictly positive entries instead of the lower ones:

Corollary 2.6. *Let $A = (d_i)_{i\in\mathbb{Z}}$ be a $(2k+1)-$diagonal non-negative infinite matrix in diagonal notation $(k \geq 2)$ and assume that $d_n^i > 0$ for every $1 \leq i \leq k-1$ and $n \in \mathbb{N}$. The matrix A has dynamics with zeros if and only if the following condition*

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($*'$) *there exists* $n_0 \in \mathbb{N}$ *such that for every* $i \in \{-k, \cdots, -1\}$, $d_n^i = 0$ *for all* $n \in \{n_0 + 1 + i, \cdots, n_0\} \cap \mathbb{N}$

holds. Consequently, if T *is a band-diagonal positive operator acting on a Banach lattice* X *whose order is induced by an unconditional basis* $\mathcal{E}$ *being its associated matrix* A, T *has non-trivial closed invariant ideals if and only if* A *verifies* ($*'$).

Remark 2.7. It its worth pointing out that Theorem 2.4 provides a sufficient condition for a positive operator T to have a non-trivial closed invariant ideal, namely the existence of $n_0 \in \mathbb{N}$ such that $d_n^i = 0$ for every $i \in \{1, \cdots, k\}$ and every $n \in \{n_0 + 1 - i, \cdots, n_0\} \cap \mathbb{N}$. Indeed, the assumption that T is band-diagonal is not required in this implication.

2.3. **A class of positive operators not meeting Theorem 2.4.** Having in mind Theorem 2.4, the existence of zero entries in the upper or lower diagonals of a band-diagonal non-negative infinite matrix characterizes the dynamics of the matrix in the sense of Definition 1.1, which turns out to be equivalent to the existence of non-trivial closed invariant ideals for positive operators induced by such a matrix. In this sense, Theorem 2.4 leads to the following natural question: if $A = (d^i)_{i\in\mathbb{Z}}$ is a $(2k+1)-$diagonal non-negative infinite matrix in diagonal notation and the sequences d^i for $-k+1 \leq i \leq -1$ contain null elements, does the matrix A have dynamics with zeros?

At this regard, we introduce a class of non-negative matrices which will not meet Theorem 2.4 but have dynamics with zeros.

Definition 2.8. *For* $k \in \mathbb{N}$, *let* $\mathcal{C}_k$ *be the class consisting of non-negative infinite matrices in diagonal notation* $A = (d^i)_{i\in\mathbb{Z}}$ *belonging to* $\mathcal{D}$ *such that*

$$\mathcal{C}_k = \{A = (d^i)_{i\in\mathbb{Z}} \in \mathcal{D} : d_n^i = 0 \text{ for every } i \in \mathbb{Z} \setminus k\mathbb{Z} \text{ and } n \in \mathbb{N}\}.$$

For instance, the class $\mathcal{C}_3$ consists of matrices of the form

$$\begin{pmatrix} * & 0 & 0 & * & 0 & 0 & \cdots \\ 0 & * & 0 & 0 & * & 0 & \cdots \\ 0 & 0 & * & 0 & 0 & * & \cdots \\ * & 0 & 0 & * & 0 & 0 & \cdots \\ 0 & * & 0 & 0 & * & 0 & \cdots \\ 0 & 0 & * & 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes any non-negative real number.

Next lemma establishes that each class $\mathcal{C}_k$ is closed under matrix multiplication.

Lemma 2.9. *Let* $k \in \mathbb{N}$ *and* $A, B \in \mathcal{C}_k$. *Then,* $AB \in \mathcal{C}_k$.

Proof. We make use of the standard matrix notation. Let $A = (a_{n,m})_{n,m\in\mathbb{N}}$ and $B = (b_{n,m})_{n,m\in\mathbb{N}}$ be matrices in $\mathcal{C}_k$. The assumption on A and B implies that

$$a_{n,n+j} = b_{n,n+j} = 0 \qquad (j \in \mathbb{Z} \setminus k\mathbb{Z}, n \in \mathbb{N}, n + j \in \mathbb{N})$$

and

$$a_{n+j,n} = b_{n+j,n} = 0 \qquad (j \in \mathbb{Z} \setminus k\mathbb{Z}, n \in \mathbb{N}, n + j \in \mathbb{N}).$$

Denote $AB = (t_{n,m})_{n,m\in\mathbb{N}}$. In order to show that $AB \in \mathcal{C}_k$, let $n \in \mathbb{N}$ and $j \in \mathbb{Z} \setminus k\mathbb{Z}$ such that $n + j \in \mathbb{N}$. Let us prove that $t_{n,n+j} = 0$ (the proof for $t_{n+j,n} = 0$ is analogous).

Journal Pre-proof

Observe that

$$t_{n,n+j} = \sum_{m=1}^{\infty} a_{n,m} b_{m,n+j},$$

so the sum is zero if every summand is zero. Let $m \in \mathbb{N}$ and let us prove that $a_{n,m} b_{m,n+j} = 0$, which will yield the proof.

Observe that if $a_{n,m} \neq 0$, then $n - m \in k\mathbb{Z}$. Hence, $n - m + j \notin k\mathbb{Z}$ since $j \notin k\mathbb{Z}$. Accordingly, $b_{m,n+j} = 0$ and therefore $a_{n,m} b_{m,n+j} = 0$ as claimed. □

As a consequence, we are in position to state a result regarding operators with matrix representation in $\mathcal{C}_k$.

Theorem 2.10. *For $k \geq 2$, every infinite matrix belonging to $\mathcal{C}_k$ has dynamics with zeros. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis such that its associated matrix belongs to $\mathcal{C}_k$ for some $k \geq 2$ has non-trivial closed invariant ideals.*

A direct consequence of Theorem 2.10 allows us to ensure the existence of non-trivial closed invariant ideals for combinations of powers of shifts, backward shifts and diagonal operators in the classical Banach spaces ℓ^p,$1 \leq p < \infty$, or c_0.

Corollary 2.11. *Let $X = c_0$ or $X = \ell^p$ for some $1 \leq p < \infty$. Let $S, B \in \mathcal{L}(X)$ denote the shift operator and backward shift operator, respectively, and $(D_n)_{n\in\mathbb{Z}} \subset \mathcal{L}(X)$ a sequence of positive diagonal operators. Assume that there exists $k \geq 2$ such that*

$$T = D_0 + \sum_{n=1}^{\infty} S^{kn} D_n + \sum_{n=1}^{\infty} B^{kn} D_{-n}$$

is bounded. Then, $T \in \mathcal{L}(X)$ has non-trivial closed invariant ideals.

Observe that the associated matrix of T is a non-negative matrix that belongs to $\mathcal{C}_k$, so Theorem 2.10 guarantees the existence of a non-trivial invariant closed ideal for T.

As particular instance of Corollary 2.11, taking $D_n = a_n I$ with $a_n \geq 0$ for every $n \in \mathbb{Z}$, every linear combination of powers of S and B of the kind

$$T = a_0 I + \sum_{n=1}^{\infty} a_n S^{kn} + \sum_{n=1}^{\infty} a_{-n} B^{kn},$$

where $k \geq 2$ and T is bounded in X has a non-trivial closed invariant ideal.

A remark. We close Section 2 with a brief discussion regarding both Theorem 2.4 and Theorem 2.10 in the context of positive band-diagonal operators.

Assume that $T \in \mathcal{L}(X)$ is a positive band-diagonal operator and that there exists $k \geq 2$ such that the matrix representation of T belongs to the class $\mathcal{C}_k$. Further, suppose that the upper superdiagonal consists of strictly positive entries. Clearly, while Theorem 2.10 assures the existence of non-trivial closed invariant ideals for T, Theorem 2.4 does not apply. Moreover, T no longer satisfies the condition stating the existence of $n_0 \in \mathbb{N}$ such that for every $i \in \{1, \cdots, k\}$ the entries d_n^i are equal to zero for every $n \in \{n_0 + 1 - i, \cdots, n_0\} \cap \mathbb{N}$.

Accordingly, a characterization based just on this latter property does not hold for every positive band-diagonal operator and therefore, the assumption $d_n^i > 0$ for $n \in \mathbb{N}$ and $i \in \{-k+1, \cdots, -1\}$ is not superfluous in Theorem 2.4.

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As an example, consider

$$A = \begin{pmatrix} * & 0 & * & 0 & * & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & 0 & \cdots \\ * & 0 & * & 0 & * & 0 & * & \cdots \\ 0 & * & 0 & * & 0 & * & 0 & \cdots \\ * & 0 & * & 0 & * & 0 & * & \cdots \\ 0 & * & 0 & * & 0 & * & 0 & \cdots \\ 0 & 0 & * & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes any strictly positive real number. If the diagonals are bounded, this hepta-diagonal matrix induces a positive, bounded operator T on Banach lattices like ℓ^p with $1 \leq p < \infty$ or c_0 and has non-trivial closed invariant ideals.

3. Honeycomb matrices

Some of the previous ideas may be further generalize in order to study how zero entries in some of the upper or lower diagonals may provide the existence of non-trivial closed invariant ideals for operators not covered by Theorem 2.4.

In this context, we introduce the concept of k-honeycomb matrices. For the sake of clarity, let us first introduce honeycomb matrices (which will correspond to $k = 2$) and discuss the general case later on.

Definition 3.1. *An infinite matrix $A = (a_{n,m})_{n,m\in\mathbb{N}}$ is said to be a* ***honeycomb matrix*** *if one of the following conditions is satisfied:*

(i) $a_{2n,2m-1} = 0$ *for every* $n, m \in \mathbb{N}$.
(ii) $a_{2n-1,2m} = 0$ *for every* $n, m \in \mathbb{N}$.

An example of a honeycomb matrix is the following:

$$A = \begin{pmatrix} * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes a non-zero real number. Observe that a honeycomb matrix with positive entries can be understood as a matrix of the class $\mathcal{C}_2$ where non-zero elements have been "added" in the alternate entries of each zero upper and lower diagonals.

Next result shows that every non-negative honeycomb infinite matrix behaves well with respect the dynamics:

Theorem 3.2. *Every non-negative honeycomb infinite matrix $A \in \mathcal{D}$ has dynamics with zeros. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis such that its matrix representation is honeycomb has non-trivial closed invariant subspaces.*

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Proof. Let $A = (a_{n,m})_{n,m\in\mathbb{N}}$ be a non-negative honeycomb matrix in $A = (a_{n,m})_{n,m\in\mathbb{N}} \in \mathcal{D}$. Let us assume A satisfies condition (i) of Definition 3.1 (if A satisfies (ii), the argument is similar).

For each $k \in \mathbb{N}$, denote by $A^k = (a^{(k)}_{n,m})_{n,m\in\mathbb{N}}$. Let us show that A^k also satisfies condition (i) of Definition 3.1, which will show that A has dynamics with zeros. We argue by induction.

By hypothesis, $a_{2n,2m-1} = 0$ for every $n, m \in \mathbb{N}$. Assume this is true for $k-1 \in \mathbb{N}$, and let $n, m \in \mathbb{N}$. Note that

$$a^{(k)}_{2n,2m-1} = \sum_{j=1}^{\infty} a^{(k-1)}_{2n,j} a_{j,2m-1}.$$

We are showing that every summand of the previous series is zero.

Let $j \in \mathbb{N}$. If $a^{(k-1)}_{2n,j} = 0$, then clearly $a^{(k-1)}_{2n,j} a_{j,2m-1} = 0$. Assume, hence, that $a^{(k-1)}_{2n,j} \neq 0$. Then j must be an even number, so $a^{(k-1)}_{j,2m-1} = 0$. Accordingly, $a_{2n,2m-1} = 0$ for every $n, m \in \mathbb{N}$ since A is honeycomb. This completes the proof. $\square$

Next result illustrates that modifying a honeycomb matrix changing just one zero entry by a strictly positive number yields non-negative infinite matrices with strictly positive dynamics. In this sense, the concept of honeycomb matrix seems to be a threshold from the standpoint of the existence of non-trivial closed invariant ideals for those positive operators with honeycomb matrix representation.

Theorem 3.3. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Assume*

(1) $a_{1,2}\, a_{2,3} > 0$, *and*
(2) $a_{i,i+2}\, a_{i+2,i} > 0$ *for every* $i \in \mathbb{N}$.

Then, A has strictly positive dynamics. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis whose matrix representation satisfies both (1) and (2) is irreducible, namely, it lacks non-trivial closed invariant ideals.

Proof. Denote by $A^n = (a^{(n)}_{i,j})_{i,j\in\mathbb{N}}$ for each $n \in \mathbb{N}$. We may assume without loss of generality that $a_{i,i} = 1$ by Lemma 2.3. As a consequence, if $a^{(k)}_{i,j} > 0$ for some $i, j \in \mathbb{N}$ and $k \in \mathbb{N}$, then $a^{(n)}_{i,j} > 0$ for every $n \geq k$.

By contradiction, assume that A has dynamics with zeros. In particular, if we denote by $\tilde{A}$ the tridiagonal non-negative matrix that coincides with A in the three main diagonals, then $\tilde{A} = (\tilde{a}_{i,j})_{i,j\in\mathbb{N}}$ has dynamics with zeros as well. This follows since $\tilde{a}_{i,j} \leq a_{i,j}$ for every $i, j \in \mathbb{N}$: note that if $\tilde{A}^n = (\tilde{a}^{(n)}_{i,j})_{i,j\in\mathbb{N}}$ for each $n \in \mathbb{N}$, then $\tilde{a}^{(n)}_{i,j} \leq a^{(n)}_{i,j}$ for every $i, j \in \mathbb{N}$ and $n \in \mathbb{N}$.

Hence, by Theorem 2.4 there exists $N_0 \in \mathbb{N}$ such that $a^{(n)}_{N_0,N_0+1} = 0$ for every $n \in \mathbb{N}$ or $a^{(n)}_{N_0,N_0-1} = 0$ for every $n \in \mathbb{N}$. Without loss of generality, we may assume that $a^{(n)}_{N_0,N_0+1} = 0$ for every $n \in \mathbb{N}$, the second case is equivalent.

Now, we claim the following

Claim. For each $n \in \mathbb{N}$

$$a^{(2n-1)}_{1+2(n-1),2+2(n-1)}\, a^{(2n-1)}_{2+2(n-1),3+2(n-1)} > 0, \tag{4}$$

and

$$a^{(2n)}_{1+2(n-1),2+2n}\, a^{(2n)}_{2+2(n-1),3+2n} > 0. \tag{5}$$

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Observe that (4) together with the fact that $a_{i,i} > 0$ implies that there exists $n_0 \in \mathbb{N}$ such that $a^{(n_0)}_{N_0,N_0+1} > 0$, which will yield the contradiction.

Proof of Claim. We proceed with an induction argument.
For $n = 1$, we have $a_{1,2}\, a_{2,3} > 0$ by assumption (1). Hence, we are just required to show that

$$a^{(2)}_{1,4}\, a^{(2)}_{2,5} > 0.$$

Observe that, for each $j \in \{1,2\}$

$$a^{(2)}_{j,(j+1)+2} = \sum_{m=1}^{\infty} a_{j,m} a_{m,(j+1)+2} \geq a_{j,j+1} a_{(j+1),(j+1)+2} > 0,$$

since $a_{i,i+2} > 0$ for every $i \in \mathbb{N}$. This proves the case $n = 1$.

Now, assume the induction hypothesis for $n-1 \in \mathbb{N}$ and let us show that both (4) and (5) hold for n.

A little computation shows that for every $j \in \{1,2\}$
(6)

$$a^{(2n-1)}_{j+2(n-1),(j+1)+2(n-1)} = \sum_{m=1}^{\infty} a_{j+2(n-1),m} a^{(2n-2)}_{m,(j+1)+2(n-1)} \geq a_{j+2(n-1),j+2(n-2)} a^{(2n-2)}_{j+2(n-2),(j+1)+2(n-1)}.$$

By induction hypothesis

$$a^{(2n-2)}_{j+2(n-2),(j+1)+2(n-1)} > 0.$$

Likewise, assumption (2) gives that

$$a_{j+2(n-1),j+2(n-2)} > 0.$$

So, replacing in (6),

$$a^{(2n-1)}_{j+2(n-1),(j+1)+2(n-1)} > 0$$

which shows that (4) holds for n.

In order to show that (5) also holds for n, observe that

$$a^{(2n)}_{j+2(n-1),(j+1)+2n} = \sum_{m=1}^{\infty} a^{(2n-1)}_{j+2(n-1),m} a_{m,(j+1)+2n} \geq a^{(2n-1)}_{j+2(n-1),(j+1)+2(n-1)} a_{(j+1)+2(n-1),(j+1)+2n}. \tag{7}$$

Since we have just shown that

$$a^{(2n-1)}_{j+2(n-1),(j+1)+2(n-1)} > 0,$$

having in mind (2)and (7), it follows that

$$a_{(j+1)+2(n-1),(j+1)+2n} > 0.$$

Accordingly, (5) also holds for n. This completes the induction argument and so, the proof of the Claim.

□

As a consequence, as we previously pointed out, we obtain a contradiction and deduce that A has strictly positive dynamics, as we wanted to show.

□

Indeed, the conclusion of Theorem 3.3 also follows if assumption (1) is replaced by a more general one, namely, the existence of $m > 1$ such that $a_{m,m+1}\, a_{m+1,m+2} > 0$.

Corollary 3.4. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Assume that*

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(1) *there exists* $m \in \mathbb{N}$ *such that* $a_{m,m+1}\, a_{m+1,m+2} > 0$, *and*
(2) $a_{i,i+2}a_{i+2,i} > 0$ *for every* $i \in \mathbb{N}$.

Then, A has strictly positive dynamics. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis whose matrix representation satisfies both (1) and (2) is irreducible.

In order to prove Corollary 3.4, we show a lemma that describes the dynamics of certain positive elements in an infinite matrix.

Lemma 3.5. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$ and assume that there exists $k \geq 2$ such that $a_{i,i+k}a_{i+k,i} > 0$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$ denote $A^n = (a^{(n)}_{i,j})_{i,j\in\mathbb{N}}$. If $a_{i_0,i_0+j_0} > 0$ for $i_0 \geq k+1$ and $j_0 \in \{0, \cdots, k\}$, then*

$$a^{(3)}_{i_0-k,i_0+j_0-k} > 0.$$

Proof. First, we show that $a^{(2)}_{i_0-k,i_0+j_0} > 0$. Note that

$$a^{(2)}_{i_0-k,i_0+j_0} = \sum_{m=1}^{\infty} a_{i_0-k,m}a_{m,i_0+j_0} \geq a_{i_0-k,i_0}a_{i_0,i_0+j_0} > 0 \tag{8}$$

since a_{i_0,i_0+j_0} and a_{i_0-k,i_0} are strictly positive by hypotheses (note that in this latter case $i_0-k \geq 1$). Finally, we have

$$a^{(3)}_{i_0-k,i_0+j_0-k} = \sum_{m=1}^{\infty} a^{(2)}_{i_0-k,m}a_{m,i_0+j_0-k} \geq a^{(2)}_{i_0-k,i_0+j_0}a_{i_0+j_0,i_0+j_0-k} > 0,$$

since $a^{(2)}_{i_0-k,i_0+j_0} > 0$ by (8) and $a_{i_0+j_0,i_0+j_0-k} > 0$ because $a_{i+k,i} > 0$ for every $i \in \mathbb{N}$ by hypotheses (note that $i_0 + j_0 \geq k+1$). □

Proof of Corollary 3.4. We apply Theorem 3.3 to deduce the result. As usual, we may assume without loss of generality, that $a_{i,i} = 1$.

Let us show that there exists $n \in \mathbb{N}$ such that A^n satisfies the hypotheses of Theorem 3.3 what would yield that A^n and, in particular, A, has strictly positive dynamics.

Since $a_{i,i} > 0$ for every $i \in \mathbb{N}$, it follows that $a^{(n)}_{i,i+2}a^{(n)}_{i+2,i} > 0$ for every $n, i \in \mathbb{N}$ so condition (2) in Theorem 3.3 is fulfilled for any power of A. Likewise, upon applying Lemma 3.5 reiteratively, it follows that there exists $N \in \mathbb{N}$ such that

$$a^{(N)}_{1,2}a^{(N)}_{2,3} > 0.$$

Thus, condition (1) in Theorem 3.3 is also satisfied and the proof is complete.

□

Observe that assumption (1) in Corollary 3.4 might have been also stated for two consecutive positive entries in the corresponding lower diagonal of the matrix A.

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Example 3.6. *Let A be the matrix*

$$A = \begin{pmatrix} * & 0 & * & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & 0 & \cdots \\ * & 0 & * & * & * & 0 & \cdots \\ 0 & * & 0 & * & * & * & \cdots \\ 0 & 0 & * & 0 & * & 0 & \cdots \\ 0 & 0 & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes a strictly positive real number. If the diagonals are bounded, this matrix induces a positive, bounded operator T on ℓ^p for $1 \leq p < \infty$ or c_0.

Observe that A is a pentadiagonal matrix that fails to be honeycomb and verifies the hypotheses of Corollary 3.4, since $a_{3,4}a_{4,5} > 0$. Then, A has strictly positive dynamics and therefore T is irreducible. Nevertheless, the matrix defined $\tilde{A} = (\tilde{a}_{i,j})_{i,j\in\mathbb{N}}$ defined as:

$$\begin{cases} \tilde{a}_{i,j} = a_{i,j} & \text{if } (i,j) \neq (3,4) \\ \tilde{a}_{i,j} = 0 & \text{if } (i,j) = (3,4) \end{cases}$$

does have dynamics with zeros by Theorem 3.2, since $\tilde{A}$ is a honeycomb matrix. As a consequence, the induced operator $\tilde{T}$ has non-trivial closed invariant ideals.

Remark 3.7. *Note that Example 3.6 shows the sharpness of Theorem 3.2 regarding the existence of zeros of matrices in $\mathcal{D}$ in order to characterize the dynamics of such matrices and the existence of non-trivial closed invariant ideals for the induced operators.*

Finally, for pentadiagonal matrices, while Theorem 2.5 provided a characterization about the existence of non-trivial closed invariant ideals assuming the positiveness of the elements of one subdiagonal, next theorem considers honeycomb matrices.

Theorem 3.8. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Assume A is pentadiagonal and $a_{i,i+2}\, a_{i+2,i} > 0$ for every $i \in \mathbb{N}$. Then, A has dynamics with zeros if and only if is honeycomb. Consequently, every operator T acting on a Banach lattice X whose order is induced by an unconditional basis such that its matrix representation A is pentadiagonal and satisfies $a_{i,i+2}\, a_{i+2,i} > 0$ for every $i \in \mathbb{N}$ has non-trivial closed invariant ideals if and only if A is honeycomb.*

Proof. By means of Theorem 3.2, it suffices to prove that if A has dynamics with zeros, A is a honeycomb matrix.

We may assume, without loss of generality, that $a_{i,i} > 0$ for every $i \in \mathbb{N}$. We argue by contradiction assuming that A is not a honeycomb matrix. Accordingly, there exists $n, m \in \mathbb{N}$ such that $a_{2n,2m-1} > 0$ and $N, M \in \mathbb{N}$ with $a_{2N-1,2M} > 0$.

Now, A is a pentadiagonal matrix, so writing $A = (d^i)_{-2\leq i\leq 2}$ in diagonal notation, we note that both entries must belong to $d^1 = (d^1_\ell)_{\ell\in\mathbb{N}}$ or $d^{-1} = (d^{-1}_\ell)_{\ell\in\mathbb{N}}$. We distinguish two possibilities.

(i) Suppose that both entries $a_{2n,2m-1}$ and $a_{2N-1,2M}$ belong to d^1 or both belong to d^{-1}. Hence, upon applying Lemma 3.5, it follows that there exists $k \in \mathbb{N}$ such that A^k satisfies the hypotheses of Corollary 3.4. Thus, A has strictly positive dynamics, which yields to a contradiction.

(ii) Assume that one of $\{a_{2n,2m-1},\, a_{2N-1,2M}\}$ belongs to d^1 and the other one to d^{-1}.

Without loss of generality, we may suppose that the entries $a_{2n,2n+1}$ and $a_{2N-1,2N-2}$ are strictly positive (if, instead, $a_{2n,2n-1} > 0$ and $a_{2N-1,2N} > 0$, the argument is analogous).

Journal Pre-proof

We claim that, in such a case, $a^{(2)}_{2n,2n+1}a^{(2)}_{2N-3,2N-2} > 0$. Observe that $a^{(2)}_{2n,2n+1} > 0$ follows by the hypotheses on the main diagonal of A. Likewise,

$$a^{(2)}_{2N-3,2N-2} \geq a_{2N-3,2N-1}a_{2N-1,2N-2} > 0,$$

so the claim follows. Now, arguing as in case (i), it follows that A has strictly positive dynamics, a contradiction.

Hence, in either case we arrive a contradiction, and therefore, the proof of Theorem 3.8 is complete. □

3.1. **k-honeycomb matrices.** As mentioned at the beginning of this section, the concept of honeycomb matrix is a particular instance of k-honeycomb matrices defined below. This latter concept will allow to draw results for positive operators $T \in \mathcal{L}(X)$ whose associated matrices are k-honeycomb regarding the absence of non-trivial closed invariant ideals (see Theorem 3.11).

Definition 3.9. *Let $k \geq 2$ be an integer. An infinite matrix $A = (a_{n,m})_{n,m\in\mathbb{N}}$ is said to be **k-honeycomb** if there exists $j \in \{0, \cdots, k-1\}$ such that one of the following conditions is satisfied:*

(i) $a_{n,m} = 0$ *for every $n \in k\mathbb{N} - j$ and $m \in \mathbb{N} \setminus (k\mathbb{N} - j)$.*
(ii) $a_{n,m} = 0$ *for every $n \in \mathbb{N} \setminus (k\mathbb{N} - j)$ and $m \in k\mathbb{N} - j$.*

Note that $k = 2$ corresponds to a honeycomb matrix. Next result is the analogous of Theorem 3.2 when $k > 2$.

Theorem 3.10. *For every $k \geq 2$, every positive k-honeycomb matrix $A \in \mathcal{D}$ has dynamics with zeros. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis such that its matrix representation is $k-$honeycomb has non-trivial closed invariant ideals.*

The proof runs as the one of Theorem 3.2 and we omit it.

Next result generalizes Theorem 3.3 to the context of positive operators with matrix representation being $k-$honeycomb.

Theorem 3.11. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Assume that there exists $k \in \mathbb{N}$ such that*

(1) $a_{j,j+1} > 0$ *for every $j \in \{1, \cdots, k\}$ and*
(2) $a_{i,i+k}a_{i+k,i} > 0$ *for every $i \in \mathbb{N}$.*

Then A has strictly positive dynamics. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis whose matrix representation satisfies both (1) and (2) is irreducible.

Though the proof follows the same ideas of that of Theorem 3.3, it requires a bit of care with the indexes due to the assumptions. We sketch it, for the sake of completeness.

Proof. Denote by $A^n = (a^{(n)}_{i,j})_{i,j\in\mathbb{N}}$ for each $n \in \mathbb{N}$. As in Theorem 3.3, we assume that $a_{i,i} = 1$ for every $i \in \mathbb{N}$ and assume that A has dynamics with zeros. Thus, there exists $N_0 \in \mathbb{N}$ such that $a^{(n)}_{N_0,N_0+1} = 0$ for every $n \in \mathbb{N}$.

We claim the following

Claim. For each $n \in \mathbb{N}$ and $j \in \{1, \cdots, k\}$

$$a^{(2n-1)}_{j+k(n-1),(j+1)+k(n-1)} > 0, \tag{9}$$

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and

$$a^{(2n)}_{j+k(n-1),(j+1)+kn} > 0. \tag{10}$$

Observe that (9) yields, together with the fact that $a_{i,i} = 0$ for every $i \in \mathbb{N}$, that there exists $n_0 \in \mathbb{N}$ such that $a^{(n_0)}_{N_0,N_0+1} > 0$, which yields the desired contradiction. Finally, as the corresponding claim in the proof of Theorem 3.3, the Claim is proved by an equivalent induction argument in n and we omit it, so th proof is done.

□

Note that Theorem 3.11 establishes a threshold for k-honeycomb matrices in order to have dynamics with zeros. Namely, if we ask that any of the coefficients involve in assumption (1) of Theorem 3.11 is zero, it is always possible to construct a $k-$honeycomb matrix A, that will have dynamics with zeros, satisfying assumption (2).

Roughly speaking, replacing one of the strictly positive entries in assumption (1) of Theorem 3.11 by zero sets up the limit in order to have dynamics with zeros, and therefore, also provides a threshold of existence of non-trivial closed invariant ideals for the induced operators.

Analogously to Corollary 3.4, assumption (1) of Theorem 3.11 may be replaced by a more general one as the next result states.

Corollary 3.12. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Assume that there exists $k \geq 2$ such that*

1. *there exists $m \in \mathbb{N}$ such that $a_{m+j,m+j+1} > 0$ for every $j \in \{1, \cdots, k\}$ and*
2. *$a_{i,i+k}a_{i+k,i} > 0$ for every $i \in \mathbb{N}$.*

Then, A has strictly positive dynamics. As a consequence, every positive operator acting on a Banach lattice X whose order is induced by an unconditional basis whose matrix representation satisfies both (1) and (2) is irreducible.

The proof of Corollary 3.12 runs as that of Corollary 3.4, so we omit it.

We conclude this subsection by showing a result for heptadiagonal operators ($k = 3$) in the spirit of Theorem 3.8 and exhibiting an example that illustrates that an analogous result does not hold for $k \geq 4$.

Theorem 3.13. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Suppose A is heptadiagonal and $a_{i,i+3}a_{i+3,i} > 0$ for every $i \in \mathbb{N}$. Then, A has dynamics with zeros if and only if A is 3-honeycomb matrix. Consequently, every operator T acting on a Banach lattice X whose order is induced by an unconditional basis such that its matrix representation A is heptadiagonal and $a_{i,i+3}a_{i+3,i} > 0$ for every $i \in \mathbb{N}$ has non-trivial closed invariant ideals if and only if A is 3-honeycomb.*

In order to prove Theorem 3.13, we require the following lemma in the sense of Lemma 3.5.

Lemma 3.14. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$ and assume that there exists $k \geq 2$ such that $a_{i,i+k}a_{i+k,i} > 0$ for every $i \in \mathbb{N}$. For each $n \in \mathbb{N}$ denote $A^n = (a^{(n)}_{i,j})_{i,j\in\mathbb{N}}$.*

1. *If $a_{i_0,i_0+j_0} > 0$ for $i_0 \in \mathbb{N}$ and $j_0 \in \{0, \cdots, k\}$, then $a^{(2)}_{i_0+k,i_0+j_0} > 0$.*
2. *If $a_{i_0+j_0,i_0} > 0$ for $i_0 \in \mathbb{N}$ and $j_0 \in \{0, \cdots, k\}$ then $a^{(2)}_{i_0+j_0,i_0+k} > 0$.*

Journal Pre-proof

For the proof of Lemma 3.14 note that, in particular,

$$a^{(2)}_{i_0+k,i_0+j_0} = \sum_{m=1}^{\infty} a_{i_0+k,m}\, a_{m,i_0+j_0} \geq a_{i_0+k,i_0}\, a_{i_0,i_0+j_0},$$

and $a_{i_0+k,i_0} > 0$ by hypotheses. This along with assumption of (1) yields $a^{(2)}_{i_0+k,i_0+j_0} > 0$ as stated. The assertion in (2) is similar.

Proof of Theorem 3.13. Without loss of generality, we may assume that $a_{i,i} > 0$ for every $i \in \mathbb{N}$. Note that this along with the fact that A has non-negative entries implies, in particular, that if $a_{i,j} > 0$ for some $i, j \in \mathbb{N}$, then $a^{(n)}_{i,j} > 0$ for every $n \in \mathbb{N}$.

Because of Theorem 3.10, it suffices to prove that A is 3-honeycomb whenever A has dynamics with zeros. So, let us suppose that A has dynamics with zeros and argue by contradiction assuming A is not a 3-honeycomb matrix.

Hence, for each $j \in \{0, 1, 2\}$ there exist $n_j \in 3\mathbb{N} - j$ and $m_j \in \mathbb{N} \setminus (3\mathbb{N} - j)$ such that

$$a_{n_j,m_j} > 0, \tag{11}$$

and $N_j \in \mathbb{N} \setminus (3\mathbb{N} - j)$ and $M_j \in 3\mathbb{N} - j$ such that

$$a_{N_j,M_j} > 0. \tag{12}$$

Since A is a heptadiagonal matrix it follows that $1 \leq |n_j - m_j| \leq 2$ and $1 \leq |N_j - M_j| \leq 2$. Equivalently, if $A = (d^i)_{i\in\mathbb{Z}}$ *in diagonal notation*, the entries in (11) and (12) belong to the sequence $\Delta = d^{-2} \cup d^{-1} \cup d^1 \cup d^2$. Denote by

$$S := \{(1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (3, 5)\}.$$

We claim the following

Claim 1. There exists $k_0 \in \mathbb{N}$ such that for each $j \in \{0, 1, 2\}$ there exist $\ell_j \in 3\mathbb{N} - j$ and $\lambda_j \in \mathbb{N} \setminus (3\mathbb{N} - j)$ with $(\ell_j, \lambda_j) \in S$ such that

$$a^{(k_0)}_{\ell_j,\lambda_j} > 0, \tag{13}$$

and, similarly, there exist $L_j \in \mathbb{N} \setminus (3\mathbb{N} - j)$ and $\Lambda_j \in 3\mathbb{N} - j$ with $(L_j, \Lambda_j) \in S$ such that

$$a^{(k_0)}_{L_j,\Lambda_j} > 0. \tag{14}$$

Roughly speaking, the claim states that there exists a power of the matrix A^{k_0} such that the corresponding conditions (11) and (12) are satisfied with entries whose indexes belong to the set S.

Proof of Claim 1. For $j \in \{0, 1, 2\}$ let $a_{n_j,m_j} > 0$ be given by (11). For each of the $j \in \{0, 1, 2\}$, we show the existence of a positive integer $k_{0,j}$ and a pair $(\ell_j, \lambda_j) \in S$ such that

$$a^{(k_{0,j})}_{\ell_j,\lambda_j} > 0. \tag{15}$$

Let $j \in \{0, 1, 2\}$ be fixed. We distinguish two possibilities:

(i) The entry $a_{n_j,m_j} \in d^1 \cup d^2$. If $(n_j, m_j) \in S$, take $\ell_j = n_j \in 3\mathbb{N}$, $\lambda_j = m_j \in \mathbb{N} \setminus (3\mathbb{N} - j)$ and $k_{0,j} = 1$.
   On the contrary, if $(n_j, m_j) \notin S$, Lemma 3.5 yields that

$$a^{(3)}_{n_j-3,m_j-3} > 0.$$

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If $(n_j-3, m_j-3) \in S$, take $\ell_j = n_j - 3 \in 3\mathbb{N}$, $\lambda_j = m_j - 3\mathbb{N} \setminus (3\mathbb{N}-j)$ and $k_{0,j} = 3$. If not, upon applying Lemma 3.5 a finite number of times, there exists $i \in \mathbb{N}$ such that $(n_j - 3i, m_j - 3i) \in S$ and

$$a^{(3i)}_{n_j-3i,m_j-3i} > 0.$$

Thus, consider $\ell_j = n_j - 3i \in 3\mathbb{N}$, $\lambda_j = m_j - 3i \in \mathbb{N} \setminus (3\mathbb{N}-j)$ and $k_{0,j} = 3i$ and (15) follows.

(ii) The entry $a_{n_j,m_j} \in d^{-1} \cup d^{-2}$. Upon applying Lemma 3.14 we have $a^{(2)}_{n_j+3,m_j} > 0$. Observe that $n_j + 3 \in 3\mathbb{N} - j$. Accordingly, case (i) applies to A^2 and hence, there exist $k_{0,j}$ and a pair (ℓ_j, λ_j) such that (15) holds.

A similar reasoning yields the existence of $\tilde{k}_{0,j}$ and a pair $(L_j, \Lambda_j) \in S$ such that

$$a^{(\tilde{k}_{0,j})}_{L_j,\Lambda_j} > 0. \tag{16}$$

Now, taking $k_0 = \max\{k_{0,j}, \tilde{k}_{0,j} : \ j = 0,1,2\}$ and having in mind that $a_{i,i} > 0$ for every $i \in \mathbb{N}$, the entries of the power of the matrix A^{k_0} whose indexes belong to the set S satisfy (13) and (14). Accordingly, the statement of Claim 1 is proved.

□

Now, simplifying the notation, let us denote by $B = (b_{i,j})_{i,j\in\mathbb{N}}$ the matrix A^{k_0}, namely, $b_{i,j} = a^{(k_0)}_{i,j}$ for $i, j \in \mathbb{N}$. Likewise, let $B = (\delta^i)_{i\in\mathbb{Z}}$ be *in diagonal notation*. In order to finish the proof, we claim the following

Claim 2. There exists $N \in \mathbb{N}$ such that either

$$b^{(N)}_{1,2} b^{(N)}_{2,3} b^{(N)}_{3,4} > 0$$

or

$$b^{(N)}_{2,1} b^{(N)}_{3,2} b^{(N)}_{4,3} > 0.$$

In order to prove Claim 2, we distinguish four cases depending on the number of positive entries $b_{i,j}$ satisfying (13) and (14) and lying on δ^1.

Proof of Claim 2.

(a) Assume that there exist three positive entries $b_{i,j}$ satisfying (13) and (14) lying on δ^1. Then $b_{1,2}, b_{2,3}, b_{3,4} > 0$, and Claim 2 holds for $N = 1$.

(b) Assume that there exist only two positive entries $b_{i,j}$ satisfying (13) and (14) lying on δ^1. In other words, there are only two positive elements belonging to $\{b_{1,2}, b_{2,3}, b_{3,4}\}$. We may assume that $b_{1,2}b_{2,3} > 0$ and $b_{3,4} = 0$, since an analogous argument works for the rest of the cases.

Observe that, in such a case, $b_{3,5} > 0$ (since (13) and (14) holds). Moreover, there must exist $i \in \{2, 3\}$, such that the entry $b_{i,4}$ is strictly positive. Since $b_{3,4} = 0$ by assumption, we deduce that $b_{2,4} > 0$.

Upon applying Lemma 3.14, we deduce that

$$b^{(2)}_{6,5} > 0$$

since $b_{3,5} > 0$. Now, by Lemma 3.5, it follows that

$$b^{(6)}_{3,2} > 0.$$

Journal Pre-proof

Hence,

$$b_{3,4}^{(7)} \geq b_{3,2}^{(6)} b_{2,4} > 0.$$

Finally, having in mind that $a_{i,i} > 0$ for every $i \in \mathbb{N}$ which, in particular, implies that $b_{i,i} > 0$ for every $i \in \mathbb{N}$, yields that

$$b_{1,2}^{(7)} b_{2,3}^{(7)} > 0.$$

Accordingly,

$$b_{1,2}^{(7)} b_{2,3}^{(7)} b_{3,4}^{(7)} > 0,$$

and Claim 2 holds.

(c) Assume that there exist only one positive entry $b_{i,j}$ satisfying (13) and (14) lying on δ^1. In other words, there is only one positive element in the set $\{b_{1,2}, b_{2,3}, b_{3,4}\}$. Then, there must be at least two positive entries belonging to $\{b_{1,3}, b_{2,4}, b_{3,5}\}$. Note that, upon applying Lemmas 3.5 and 3.14 this case is reduced to have two positive elements on δ^{-1}. Then, considering the transpose matrix of B and arguing as in item (b), the statement of Claim 2 also follows.

(d) Last, assume that there do not exist positive entries $b_{i,j}$ satisfying (13) and (14) lying on δ^1. Then $b_{1,3} b_{2,4} b_{3,5} > 0$, and again, as a byproduct of Lemmas 3.5 and 3.14, there exists $N \in \mathbb{N}$ such that

$$b_{2,1}^{(N)} b_{3,2}^{(N)} b_{4,3}^{(N)} > 0,$$

which ends up the proof of Claim 2.

□

Note that Claim 2 completes the proof of Theorem 3.13: indeed, by Proposition 3.11 the matrix $A^{k_0 N}$ would have strictly positive dynamics, which would contradict that A has dynamics with zeros. Accordingly, the matrix A is 3-honeycomb and the theorem is proved.

□

Finally, we state an example that illustrates that a similar statement to that of Theorem 3.13 does not hold for $k \geq 4$.

Example 3.15. *Let $A = (a_{i,j})_{i,j \in \mathbb{N}} \in \mathcal{D}$ such that:*

(i) $a_{i,i} > 0$ *for every* $i \in \mathbb{N}$.
(ii) $a_{i,i+4} a_{i+4,i} > 0$ *for every* $i \in \mathbb{N}$.
(iii) $a_{1,3} a_{1,4} a_{2,3} a_{3,2} a_{4,5} > 0$.
(iv) *Every other entry is zero.*

Observe that A is a band-diagonal matrix with bandwidth $k = 4$:

$$A = \begin{pmatrix} * & 0 & * & * & * & 0 & \cdots \\ 0 & * & * & 0 & 0 & * & \cdots \\ 0 & * & * & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & * & * & 0 & \cdots \\ * & 0 & 0 & 0 & * & 0 & \cdots \\ 0 & * & 0 & 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that A is not a 2-honeycomb matrix because $a_{2,3} a_{1,4} > 0$. Likewise, A is neither a 4-honeycomb matrix because of the election of the positive entries in (iii). Accordingly, A is not k-honeycomb for any k.

Nevertheless, let us prove that if A has dynamics with zeros.

Journal Pre-proof

If $A^n = (a^{(n)}_{i,j})_{i,j\in\mathbb{N}}$ we assert that

$$a^{(n)}_{2+4i,4j} = a^{(n)}_{2+4i,1+4j} = 0 \qquad i,j \geq 0,$$

and

$$a^{(n)}_{3+4i,4j} = a^{(n)}_{3+4i,1+4j} = 0, \qquad i,j \geq 0$$

for every $n \in \mathbb{N}$.

Note that, indeed, this is the case for $n = 1$. To prove the assertion for $n > 1$, let $B = (b_{i,j})_{i,j\in\mathbb{N}}$ and $C = (c_{i,j})_{i,j\in\mathbb{N}}$ be two non-negative matrices satisfying

$$b_{2+4i,4j} = c_{2+4i,1+4j} = 0 \qquad i,j \geq 0,$$
$$b_{2+4i,4j} = c_{2+4i,1+4j} = 0 \qquad i,j \geq 0,$$
$$b_{3+4i,4j} = c_{3+4i,1+4j} = 0 \qquad i,j \geq 0,$$

and

$$b_{3+4i,4j} = c_{3+4i,1+4j} = 0 \qquad i,j \geq 0.$$

Then, the matrix product $D = BC = (d_{i,j})_{i,j\in\mathbb{N}}$ also satisfies all the previous identities. Let us show the first one, since the other three are analogous.

Let $i,j \geq 0$. Clearly,

$$d_{2+4i,4j} = \sum_{m=1}^{\infty} b_{2+4i,m}\, c_{m,4j}.$$

The goal will be to show that every summand in the previous series is zero. Let $m \in \mathbb{N}$ and assume $b_{2+4i,m} > 0$ (otherwise there is nothing to show). Then, $m \neq 4l, 1+4l$ for any $l \in \mathbb{N}$. Accordingly, there exists l_0 such that $m = 2 + 4l_0$ or $m = 3 + 4l_0$. Observe that, in both cases, $c_{m,1+4j} = 0$, so $b_{2+4i,m}\, c_{m,4j} = 0$, as claimed.

Observe that Example 3.15 shows a distribution of zeros in the matrix A that makes A have dynamics with zeros though A is not k-honeycomb for any k. Nevertheless, in a broader sense, the "geometric distribution" of the zeros is similar: the matrix A has equal spaced 2×2 squares of zeros while honeycomb matrices have equal spaced rows or columns of zeros. At this regard, in the spirit of Theorems 3.8 and 3.13, we ask the following

Question. *Let $A = (a_{i,j})_{i,j\in\mathbb{N}} \in \mathcal{D}$. Suppose that A is a $(2k+1)-$diagonal matrix for $k \geq 4$ and $a_{i,i+k}a_{i+k,i} > 0$ for every $i \in \mathbb{N}$. Does there exist a characterization involving a description of the zero entries of A ensuring that A has dynamics with zeros?*

4. A generalization of a theorem of Grivaux for band-diagonal operators

In this last section, we obtain a generalization of a theorem of Grivaux [10] regarding the existence of non-trivial closed invariant subspaces for positive tridiagonal operators.

Recall that if X is a separable Banach space and $T \in \mathcal{L}(X)$, it is said that T has a **moment sequence** if there exists $x \in X \setminus \{0\}$, $x^* \in X^* \setminus \{0\}$ and a non-negative Borel measure μ on $\mathbb{R}$ such that

$$x^*\left(T^n x\right) = \int_{\mathbb{R}} t^n d\mu(t)$$

for every $n \geq 0$. The pair (x, x^*) is usually called a **moment pair** for T. Moment sequences for operators play a role regarding the existence of non-trivial closed invariant subspaces (see [7, Chapter 9] for instance). In particular, the following proposition holds.

Journal Pre-proof

Proposition 4.1. *Let X be a real Banach lattice whose order is induced by an unconditional basis and denote by $X_{\mathbb{C}} = X + iX$ its complexification. Assume $T \in \mathcal{L}(X_{\mathbb{C}})$ is an operator that leaves X invariant. The following conditions are equivalent:*

(i) *T has a non-trivial closed invariant subspace on $X_{\mathbb{C}}$.*
(ii) *$T\,|_X$ has a non-trivial closed invariant subspace on X.*
(iii) *$T\,|_X$ has a moment sequence.*

In the context of this work, if X is a complex Banach lattice whose order is induced by an unconditional basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$, then X is the complexification of the real Banach space

$$\left\{ \sum_{n=1}^{\infty} x_n e_n \in X : x_n \in \mathbb{R} \text{ for every } n \in \mathbb{N} \right\}.$$

In what follows, let us denote by X_0 the (non-closed) subspace consisting of vectors with finite support respect to $\mathcal{E}$ on X. Observe that X_0 is invariant under every band-diagonal operator. The following proposition stated in [10] for Banach spaces of real or complex sequences indexed by $\mathbb{N}$ will be relevant later on.

Proposition 4.2 ([10])**.** *Let X be a Banach lattice whose order is induced by an unconditional basis $\mathcal{E} = (e_n)_{n\in\mathbb{N}}$, and let $T : X \to X$ be a bounded operator that leaves invariant X_0. Assume that there exists*

(i) *A symmetric matrix S with $S(X_0) \subseteq X_0$.*
(ii) *A matrix L such that $L(X_0) \subseteq X_0$, L is invertible on X_0, and its adjoint L^t also satisfies $L^t(X_0) \subset X_0$.*
(iii) *Either $LT = SL$ or $TL = LS$ on X_0.*

Then, there exists a moment pair for T and hence, T has non-trivial closed invariant subspaces.

It is important to note that both matrices S and L are not required to induce bounded operators on X.

In order to extend the theorem of Grivaux [10, Theorem 3.2] to a larger class of positive operators, we introduce the following definition.

Definition 4.3. *An infinite matrix in diagonal notation $A = (d^i)_{i\in\mathbb{Z}}$ satisfies condition **(G)** if for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that*

$$d_n^{k_n}\, d_n^{-k_n} > 0$$

and

$$d_n^m = 0 \textit{ for every } m \in \mathbb{N} \setminus \{-k_n, 0, k_n\}.$$

In particular, if A is a positive band-diagonal matrix satisfying **(G)**, there exists $k \in \mathbb{N}$ such that $k_n \leq k$ for every $n \in \mathbb{N}$. We present some examples within this class of matrices.

Examples 4.4.

(i) *Every tridiagonal matrix with strictly positive entries in the subdiagonal and the superdiagonal satisfies condition **(G)**.*

Journal Pre-proof

(ii) *Every pentadiagonal matrix of the form*

$$A = \begin{pmatrix} * & 0 & * & 0 & 0 & 0 & \cdots \\ 0 & * & 0 & * & 0 & 0 & \cdots \\ * & 0 & * & 0 & * & 0 & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ 0 & 0 & * & 0 & * & 0 & \cdots \\ 0 & 0 & 0 & * & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes a strictly positive real number satisfies condition ***(G)****.*

(iii) *More general band-diagonal matrices can be constructed to satisfy condition* ***(G)****:*

$$A = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & 0 & 0 & \cdots \\ 0 & 0 & * & 0 & 0 & * & \cdots \\ 0 & * & 0 & * & 0 & * & \cdots \\ 0 & 0 & 0 & 0 & * & * & \cdots \\ 0 & 0 & * & * & * & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each $*$ denotes a strictly positive real number. In this example, we have $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 2$, $k_5 = 1, \cdots$

Note that each matrix $A = (d^i)_{i\in\mathbb{Z}}$ above induces a bounded operator on Banach lattices like ℓ^p $(1 \le p < \infty)$ or c_0 whenever every diagonal sequence d^i is bounded for every $i \in \mathbb{Z}$.

Our main result in this section provides non-trivial closed invariant subspaces (non necessarily ideals) for positive band diagonal operators whose associated matrix satisfies condition **(G)**.

Theorem 4.5. *Let X be a Banach lattice whose order is induced by an unconditional basis $\mathcal{E}$ and let $T \in \mathcal{L}(X)$ be a positive band-diagonal operator such that its associated matrix $A = (d^i_n)_{i\in\mathbb{Z}}$ respect to $\mathcal{E}$ in diagonal notation satisfies condition* **(G)**. Then, T has a non-trivial closed invariant subspace.

Proof. Since T is a positive operator, it leaves invariant the real part of X. Hence, by Proposition 4.1, we may assume that X is a real Banach lattice and prove that T has non-trivial closed invariant subspaces in X. Moreover, it suffices to construct a moment pair for T.

We proceed by constructing two matrices L, S satisfying Proposition 4.2. Let us define the sequence of positive numbers $(\alpha_n)_{n\ge1}$ as follows:

<u>Step 1</u> Set $\alpha_1 = 1$. Since A satisfies condition **(G)**, let $k_1 \in \mathbb{N}$ be the first non-negative integer given by such a condition and write $m_1 = 1$ and $m_2 = 1 + k_1$. We define

$$\alpha_{m_2} = \alpha_1 \sqrt{\frac{d_1^{k_1}}{d_1^{-k_1}}}.$$

A recursive argument allows us to construct a strictly increasing subsequence of natural numbers $(m_n)_{n\ge1}$ and a sequence of positive real numbers $(\alpha_{m_n})_{n\ge1}$ by defining

$$m_n = m_{n-1} + k_{m_{n-1}} \qquad (n \ge 2),$$

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and

$$\alpha_{m_n} = \alpha_{m_{n-1}} \sqrt{\frac{d_{m_{n-1}}^{k_{m_{n-1}}}}{d_{m_{n-1}}^{-k_{m_{n-1}}}}}.$$

Note that if the sequence of natural numbers $(m_n)_{n\geq 1}$ coincides with the set of natural numbers $\mathbb{N}$, Step 1 provides the desired sequence $(\alpha_n)_{n\geq 1}$. Otherwise, we proceed to Step 2.

Step 2 Assume $(m_n)_{n\in\mathbb{N}} \subsetneq \mathbb{N}$ and let N be the smallest positive integer not belonging to $(m_n)_{n\in\mathbb{N}}$. Let $M_1 = N$ and define

$$M_n = M_{n-1} + k_{M_{n-1}} \qquad (n \geq 2).$$

There are two possibilities:

(a) Either the sequence $(M_n)_{n\geq 1}$ does not intersect $(m_n)_{n\geq 1}$: in such a case we define α_{M_n} as in Step 1, namely,

$$\alpha_{M_1} = 1$$

and

$$\alpha_{M_n} = \alpha_{M_{n-1}} \cdot \sqrt{\frac{d_{M_{n-1}}^{k_{M_{n-1}}}}{d_{M_{n-1}}^{-k_{M_{n-1}}}}}, \qquad (n \geq 2).$$

(b) Or, there exist $n_1, n_2 \in \mathbb{N}$ such that $M_{n_1} = m_{n_2}$. In such a case, we define

$$\alpha_{M_{n_1+j}} = \alpha_{m_{n_2+j}} \text{ for } j \geq 0,$$

and for those $j \in \{-n_1+1, \cdots, -1\}$ we define the *backward steps* until we reach M_1:

$$\alpha_{M_{n_1+j}} = \alpha_{M_{n_1+j+1}} \sqrt{\frac{d_{M_{n_1+j}}^{-k_{M_{n_1+j}}}}{d_{M_{n_1+j}}^{k_{M_{n_1+j}}}}}.$$

If the sequence of natural numbers $(M_n)_{n\geq N} \cup (m_n)_{n\in\mathbb{N}} = \mathbb{N}$, steps 1 and 2 provide the desired sequence $(\alpha_n)_{n\geq 1}$. Otherwise, we consider Step 2 with the sequence $(M_n)_{n\in\mathbb{N}} \cup (m_n)_{n\in\mathbb{N}}$.

Note that for each $N \in \mathbb{N}$, the element α_N is well defined after reiterating previous steps a finite number of times (which generally depends on N). Indeed, since in each reiteration it is chosen the smallest positive integer not previously covered by Step 2, the element α_N is well defined after at most N steps.

Accordingly, the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is defined.

Now, let L be the diagonal matrix with diagonal sequence $(\alpha_n)_{n\in\mathbb{N}}$. Observe that $L^t = L$. If we compute $S := LAL^{-1}$ and $S = (s^i)_{i\in\mathbb{Z}}$ in diagonal notation we obtain that for every $n \in \mathbb{N}$ we have

$$s_n^m = \frac{\alpha_n}{\alpha_{n+m}} d_n^m \qquad m \geq 0$$

and

$$s_n^m = \frac{\alpha_{n-m}}{\alpha_n} d_n^m \qquad m < 0.$$

As a consequence:

(i) The main diagonals of S and A coincide, that is, $s^0 = d^0$.
(ii) For every $n \in \mathbb{N}$ we have $s_n^m = 0$ for every $m \notin \{-k_n, 0, k_n\}$.

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(iii) For every $n \in \mathbb{N}$ we have
$$s_n^{k_n} = \frac{\alpha_n}{\alpha_{n+k_n}} d_n^{k_n}$$
and
$$s_n^{-k_n} = \frac{\alpha_{n+k_n}}{\alpha_n} d_n^{-k_n}.$$

Observe that S is a band-diagonal matrix because $|k_n|$ is bounded since A is band-diagonal. In particular, S leaves invariant X_0. Moreover, by the construction of the sequence $(\alpha_n)_{n\in\mathbb{N}}$, S is a symmetric matrix: we have $s_n^{k_n} = s_n^{-k_n}$ for every $n \in \mathbb{N}$ and $s_n^m = 0$ for every $m \notin \{-k_n, 0, k_n\}$ by (ii). Note that this latter property follows since $d_n^m = 0$ for $m \notin \{-k_n, 0, k_n\}$. Observe that the required symmetry would not longer hold in this construction if such assumption is dropped.

Finally, note that both matrices L and S satisfy the hypotheses of Proposition 4.2 and indeed $LT = SL$. Accordingly, there exists a moment pair for T, and hence, non-trivial closed invariant subspaces. □

Remark 4.6. *Theorem 4.5 answers, partially, to the question posed by Grivaux in* [10] *if every positive, pentadiagonal operator has a finitely supported moment pair. We have shown that every positive, pentadiagonal operator satisfying condition **(G)** does have a finitely supported moment pair but the general question remains still open.*

We close this section by exhibiting an example of a positive band-diagonal operator T such that its associated matrix satisfies condition **(G)**, T lacks non-trivial closed invariant ideals and does have non-trivial invariant subspaces. This concrete example illustrates that Theorem 4.5 provides non-trivial closed invariant subspaces for operators not necessarily covered by those results in Section 2.

Example 4.7. *Let $A = (d^i)_{i\in\mathbb{Z}}$ be a matrix in diagonal notation where*
(i) *$d_n^0 > 0$ for every $n \in \mathbb{N}$.*
(ii) *$d_{2n-1}^1 d_{2n-1}^{-1} > 0$ for every $n \in \mathbb{N}$.*
(iii) *$d_{2_n}^2 d_{2_n}^{-2} > 0$ for every $n \in \mathbb{N}$.*
(iv) *Every other entry is zero.*

In particular, A is of the form
$$A = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 & 0 & \cdots \\ * & * & 0 & * & 0 & 0 & 0 & \cdots \\ 0 & 0 & * & * & 0 & 0 & 0 & \cdots \\ 0 & * & * & * & 0 & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & * & * & 0 & \cdots \\ 0 & 0 & 0 & * & * & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$
where each $*$ denotes a strictly positive real number.

*Note that the matrix A clearly satisfies condition **(G)** with $k_{2n-1} = 1$ and $k_{2_n} = 2$ for every $n \in \mathbb{N}$.*

Let X be a Banach lattice whose order is induced by an unconditional basis such that A induces a bounded operator T on X. For instance, $X = c_0$ or $X = \ell^p$ with $1 \le p < \infty$ whenever the sequences d^i are bounded. By Theorem 4.5, it follows that T has non-trivial closed invariant subspaces.

Indeed, it is possible to show explicitly the construction of the sequence $(\alpha_n)_{n\in\mathbb{N}}$ addressed in the proof of Theorem 4.5. Start with Step 1 by setting $\alpha_1 = 1$. Hence, by definition, $k_1 = 1$, and write

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$m_1 = 1$ and $m_2 = 1 + 1 = 2$. Thereby, define

$$\alpha_2 = \alpha_1 \sqrt{\frac{d_1^1}{d_1^{-1}}}.$$

Now, since $k_{2n} = 2$ for every $n \in \mathbb{N}$, we obtain

$$m_{n+1} = 2n$$

for $n \in \mathbb{N}$. Accordingly, define

$$\alpha_{2n} = \alpha_{2n-2} \sqrt{\frac{d_{2n-2}^2}{d_{2n-2}^{-2}}}$$

for every $n \geq 2$. This ends this first iteration of Step 1. At this point, we have defined α_1 and α_{2n} for every $n \in \mathbb{N}$. It remains to define α_{2n+1} for $n \in \mathbb{N}$ applying the Step 2.

We choose the smallest integer not covered yet in the indexed sequence, namely 3. By writing $M_1 = 3$, we obtain $M_2 = 4$, a number that is already covered, so case (b) of Step 2 applies. Set

$$\alpha_3 = \alpha_4 \sqrt{\frac{d_3^{-1}}{d_3^1}}.$$

Hence, we have defined so far α_1, α_3 and α_{2n} for every $n \in \mathbb{N}$.

In order to define α_n for the rest of uncovered indexes n, we are required to apply Step 2 again.

Observe that in the next iteration of the process α_5 is defined applying Step 2 once again. In general, in each iteration of the process that follows the entry α_{2n+1} for $n \geq 2$ is obtained where the entry α_{2n-1} was defined in the previous step. Each α_{2n+1} is defined as follows:

$$\alpha_{2n+1} = \alpha_{2n+2} \sqrt{\frac{d_{2n+1}^{-1}}{d_{2n+1}^1}}.$$

Note that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is defined for every $n \in \mathbb{N}$ and, as we observed, defining each α_n requires only a finite number of steps which, in turn, depends on $n \in \mathbb{N}$.

Finally, in order to show that T lacks non-trivial closed invariant ideals, let us prove that if $A^2 = (b^i)_{i\in\mathbb{Z}}$ in diagonal notation, then

$$b_n^1 \, b_n^{-1} > 0 \tag{17}$$

for every $n \in \mathbb{N}$. Hence, by Theorem 2.4, T^2 does not have non-trivial closed invariant ideals and neither does T.

Since $d_n^0 > 0$ for every $n \in \mathbb{N}$, we deduce that $b_{2n-1}^1 b_{2n-1}^{-1} > 0$ for every $n \in \mathbb{N}$. Now, observe that

$$b_{2n}^1 \geq d_{2n}^2 d_{2n+1}^{-1} > 0.$$

Equivalently, it follows that $b_{2n}^{-1} > 0$. Hence, (17) holds as we wished to show.

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