Journal of Physics: Conference Series **597** (2015) 012021 doi:10.1088/1742-6596/597/1/012021

Generalized harmonic functions and unitary representations of low dimensional Lie groups

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Abstract. The unitary representations of the three dimensional simple Lie groups are reconsidered from the perspective of harmonic functions acting on certain manifolds related to differential realisations of the groups themselves. By means of contractions of Lie groups, the procedure is also applied to the group E_2 of rotations-translations in two dimensions.

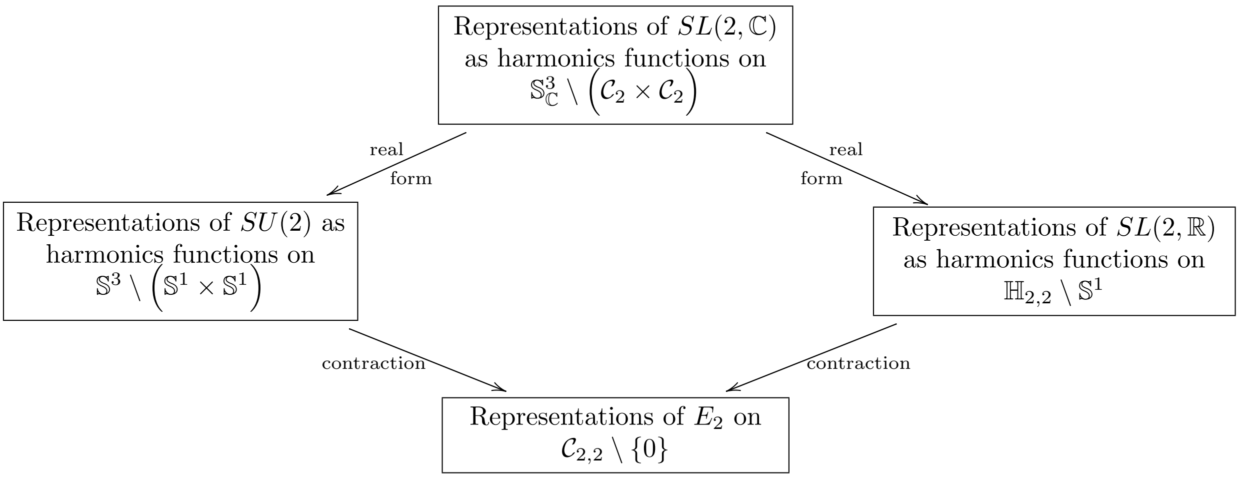
1. Introduction

Besides their fundamental importance in Analysis and Geometry, harmonic functions constitute an indispensable tool for many branches of Classical Physics, specially in connection with potential theory, providing an appropriate formalism to describe electrostatic, magnetic or gravitational potentials, as well as in Quantum physics, where harmonic functions play a relevant role in the operators techniques and the symmetry analysis of quantum systems, hence for the representation theory of these groups [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Their generalisation to higher dimensions is a natural consequence of these applications, particularly in the context of relativistic systems [2, 11, 12, 13, 14]. As representative examples of this connection we have the relationship between unitary representations of pseudo-orthogonal groups $SO(p,q)$ and harmonic functions, *i.e.* eigenfunctions of the Laplacian, on the $(n-1)-$sphere $\mathbb{S}^{n-1}$ or the hyperboloids (see [15] for references). Harmonic functions have also been considered for other types of simple Lie groups, as $SU(3)$ in the context of elementary particles [16, 17] or the group $Sp(6,\mathbb{R})$ in the quantum mechanics of three bodies [18].

In this work we propose an alternative construction of the representations of the three dimensional semisimple Lie groups (complex and real) from the point of view of harmonic functions. To proceed, we identify $SL(2,\mathbb{C})$ (resp. $SL(2,\mathbb{R}), SU(2)$) with the complex unit three-sphere $\mathbb{S}^3_{\mathbb{C}}$ (resp. with the hyperboloid of signature $(2,2)$ $\mathbb{H}_{2,2}$ or with three-sphere $\mathbb{S}^3$) that we describe by means of a specific system of coordinates. Removing appropriate regions from these manifolds (either cylinders or circles), these systems of coordinates give rise to a bijective correspondence. Using these systems of coordinates enables us to obtain differential realisations of the Lie algebras $\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ acting respectively on the spaces $\mathbb{S}^3_{\mathbb{C}} \setminus \Big(\mathcal{C}_2 \times \mathcal{C}_2\Big), \mathbb{H}_{2,2} \setminus \mathbb{S}^1$ or $\mathbb{S}^3 \setminus \Big(\mathbb{S}^1 \times \mathbb{S}^1\Big)$, thus allowing to construct all unitary representations in terms of harmonic functions.

In analogous way, we consider an alternative construction of unitary representations of the group of rotations-translations in two dimensions E_2 on the cone with one point removed

$\mathcal{C}_{2,2} \setminus \{0\}$, which can be seen as a singular limit of $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$. It has to be emphasised that all these constructions are related by natural operations such as taking real forms or considering contractions of Lie algebras. We also observe that in the case of $SL(2,\mathbb{Z})$ and E_2, their first homotopy group being isomorphic to $\mathbb{Z}$, the representations are constructed in some special cases (more precisely, when the eigenvalues of the semisimple element or the spin do not belong to $\frac{1}{2}\mathbb{N}$) on a suitable covering space of the considered manifold. The proposed approach can be sumarised in the following diagram:



Alternative constructions of the representations considered in this paper can be found for example in [19, 20, 21]. We also observe that in [22], a unified construction of the Lie groups $SO(3), SO(1,2)$ and E_2, was developed, however basing on a different Ansatz and techniques as those used in this work.

The contents of the paper is the following. In Section 2 we explicitly parameterise the Lie groups $SL(2,\mathbb{C}), SL(2,\mathbb{R})$ and $SU(2)$. Considering appropriate systems of coordinates a bijective correspondence deduced from the paremeterisation is obtained after removal of some regions on the previous manifolds. Section 3 is devoted to an explicit construction of a differential realisation of the Lie algebras $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$. In Section 4 we construct explicitly unitary representations of $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ as harmonic functions on the corresponding manifold. A special emphasis on $\mathfrak{sl}(2,\mathbb{R})$ is given. Finally, Section 5 focuses on the extension of this approach for the description of unitary representations of E_2.

2. Parameterisation of the Lie groups

In this section we construct explicit systems of coordinates for the Lie groups $SL(2,\mathbb{C}), SU(2), SL(2,\mathbb{R})$. It is shown that removing appropriate regions of the corresponding manifolds allows us to obtain bijective parameterisations. These will be central for the construction of unitary representations of the corresponding Lie groups.

The Lie group $SL(2,\mathbb{C})$ is the set of two-by-two complex unimodular matrices

$$\begin{aligned} SL(2,\mathbb{C}) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \ \ \alpha,\beta,\gamma,\delta \in \mathbb{C} \ , \ \ \alpha\delta - \beta\gamma = 1 \right\} , \\ &\cong \ \mathbb{S}^3_{\mathbb{C}} = \left\{ z_1, z_2, z_3, z_4 \in \mathbb{C}, \ z_1^2 + z_2^2 + z_3^2 + z_4^2 = 1 \right\} \subset \mathbb{C}^4, \end{aligned}$$

with $\alpha = z_1 - iz_2, \delta = z_1 + iz_2, \beta = z_4 - iz_3, \gamma = -z_4 - iz_3$. An adapted system of coordinates on $\mathbb{S}^3_{\mathbb{C}}$ can be given as follows, writing $z_1^2 + z_2^2 + z_3^2 + z_4^2 = z_+ \bar{z}'_+ + z_- \bar{z}'_-$ We define $\Theta = \vartheta_0 + i\vartheta_1$, $\Phi_\pm = \varphi_{\pm 0} + i\varphi_{\pm 1} \in \mathbb{C}$

and set

$$\begin{aligned} z_+ &= \cos(\Theta)e^{i\Phi_+} \ , \\ z_- &= \sin(\Theta)e^{i\Phi_-} \ , \\ z'_+ &= \cos(\bar\Theta)e^{i\bar\Phi_+} \ , \\ z'_- &= \sin(\bar\Theta)e^{i\bar\Phi_-} \ , \end{aligned} \tag{1}$$

with

$$(\vartheta_0, \varphi_{+0}, \varphi_{-0}, \vartheta_1, \varphi_{+1}, \varphi_{-1}) \in [0, \frac{\pi}{2}] \times [0, 2\pi[\times[0, 2\pi[\times\mathbb{R}^3 \ .$$

It can be shown that although (1) covers $\mathbb{S}^3_\mathbb{C}$, the correspondence is not one-to-one. However, removing the two cylinders $\mathcal{C}_2$

$$\begin{aligned} \Theta &= 0 \ , \ \ (\varphi_{+0}, \varphi_{+1}) \in [0, 2\pi[\times\mathbb{R} \ , \\ \Theta &= \frac{\pi}{2} \ , \ \ (\varphi_{-0}, \varphi_{-1}) \in [0, 2\pi[\times\mathbb{R} \ . \end{aligned}$$

and denoting $\mathcal{I}^3_\mathbb{C} = \Big\{(\vartheta_0, \varphi_{+0}, \varphi_{-0}, \vartheta_1, \varphi_{+1}, \varphi_{-1}) \in [0, \frac{\pi}{2}] \times [0, 2\pi[\times[0, 2\pi[\times\mathbb{R}^3 \ , \text{s.t. } (\vartheta_0, \vartheta_1) \neq (0, 0), (\pi/2, 0)\Big\}$, we have a bijection between $\mathcal{I}^3_\mathbb{C}$ and $\mathbb{S}^3_\mathbb{C} \setminus \Big(\mathcal{C}_2 \times \mathcal{C}_2\Big)$ [23].

The group $SL(2, \mathbb{R})$ is the set of two-by-two real unimodular matrices

$$\begin{aligned} SL(2,\mathbb{R}) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \ , \ \ \alpha, \beta, \gamma, \delta \in \mathbb{R} \ , \ \ \alpha\delta - \beta\gamma = 1 \right\} \ , \\ &\cong \ \mathbb{H}_{2,2} = \left\{ x_1, x_2, x_3, x_4 \in \mathbb{R}, \ x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1 \right\} \subset \mathbb{R}^4. \end{aligned}$$

Since $SL(2, \mathbb{R})$ is a real form on $SL(2, \mathbb{C})$, we consider the real form of $\mathbb{S}^3_\mathbb{C}$ parameterised using

$$\Theta = i\vartheta_1 = i\rho \ , \ \ \Phi_+ = \varphi_{+0} = \varphi_+ \ , \Phi_- = \phi_{-0} = \varphi_- \ , \tag{2}$$

such that (1) reduces to

$$\zeta_+ = \cosh \rho e^{i\varphi_+} \ , \ \ \zeta_- = \sinh \rho e^{i\varphi_-} \ , \tag{3}$$

with $(\rho, \varphi_+, \varphi_-) \in \mathbb{R}_+ \times [0, 2\pi[\times[0, 2\pi[$. We obviously have

$$\left|\zeta_+\right|^2 - \left|\zeta_-\right|^2 = 1 \ .$$

The variables $\zeta_\pm$ clearly parameterise the hyperboloid $\mathbb{H}_{2,2}$. If we remove the circle

$$\rho = 0 \ , \ \ \varphi_+ \in [0, 2\pi[\ ,$$

that is $|\zeta_+|^2 = 1$, from $\mathbb{H}_{2,2}$, we have a bijection between $\mathcal{I}_{2,2} = \mathbb{R}^*_+ \times [0, 2\pi[\times[0, 2\pi[$ and $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$. Further it can be seen that the first homotopy group of $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ [23]. Several coverings of $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$ that will be relevant in the sequel can be defined (for other coverings see [23]):

- the $(p, 1)-$sheeted covering $\widetilde{\mathbb{H}_{2,2} \setminus \mathbb{S}^1}^{(p,1)}$ with $\rho \in \mathbb{R}^*_+, \ 0 \le \varphi_+ < 2p\pi, \ 0 \le \varphi_- < 2\pi$,
- the $(\infty, 1)-$sheeted covering $\widetilde{\mathbb{H}_{2,2} \setminus \mathbb{S}^1}^{(\infty,1)}$ with $\rho \in \mathbb{R}^*_+, \varphi_+ \in \mathbb{R}, \ 0 \le \varphi_- < 2\pi$.

The Lie group $SU(2)$ is the set of two-by-to unitary complex matrices of determinant one

$$\begin{aligned} SU(2) &= \left\{ U = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \ , \alpha, \beta \in \mathbb{C} \ , \left|\alpha\right|^2 + \left|\beta\right|^2 = 1 \right\} \\ &\cong \mathbb{S}^3 = \left\{ x_1, x_2, x_3, x_4 \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\} \subset \mathbb{R}^4 \ . \end{aligned}$$

Since $SU(2)$ is a real form of $SL(2, \mathbb{C})$ we consider the real form of $\mathbb{S}^3_{\mathbb{C}}$ parametrised by

$$\Theta = \vartheta_0 = \theta \ , \Phi_+ = \varphi_{+0} = \varphi_+ \ , \Phi_- = \varphi_{-0} = \varphi_- \ , \tag{4}$$

which gives in (1)

$$w_+ = \cos\theta e^{i\varphi_+} \ , \ w_- = \sin\theta e^{i\varphi_-} \ ,$$

leading to

$$\left|w_+\right|^2 + \left|w_-\right|^2 = 1 \ ,$$

with $(\theta, \varphi_+, \varphi_-) \in [0, \pi/2] \times [0, 2\pi[\times[0, 2\pi[$. The variables $w_\pm$ clearly parameterise the sphere $\mathbb{S}^3$. Now, removing two circles from $\mathbb{S}^3$

$$\begin{aligned} \theta &= 0 \ , \ \ \varphi_+ \in [0, 2\pi[\ , \\ \theta &= \frac{\pi}{2} \ , \ \ \varphi_- \in [0, 2\pi[\ , \end{aligned}$$

that is, the circles $|w_+|^2 = 1$ and $|w_-|^2 = 1$, we have a bijection between $\mathcal{I}_3 =]0, \pi/2[\times[0, 2\pi[\times[0, 2\pi[$ and $\mathbb{S}^3 \setminus \left(\mathbb{S}^1 \times \mathbb{S}^1\right)$ [23].

Note finally that the manifolds $\mathbb{S}^3 \setminus \left(\mathbb{S}^1 \times \mathbb{S}^1\right)$ and $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$ are real forms of the manifold $\mathbb{S}^3_{\mathbb{C}} \setminus \left(\mathcal{C}_2 \times \mathcal{C}_2\right)$ [23]. The real manifolds can be endowed with a scalar product and a Laplacian related to a scalar product and Laplacian on $\mathbb{S}^3 \setminus \left(\mathbb{S}^1 \times \mathbb{S}^1\right)$. In this note we only consider the case of $SL(2, \mathbb{R})$. The Laplacian is given by [23]

$$\Delta = -\frac{1}{\cosh\rho \sinh\rho} \partial_\rho (\cosh\rho \sinh\rho \partial_\rho) + \frac{1}{\cosh^2\rho} \partial^2_{\varphi_+} - \frac{1}{\sinh^2\rho} \partial^2_{\varphi_-} \ . \tag{5}$$

The scalar product firstly defined on $\mathbb{H}_{2,2} \setminus \mathbb{S}^1$ extends naturally to the covering considered above. On $\widetilde{\mathbb{H}_{2,2} \setminus \mathbb{S}^1}^{(p,1)}$ we have

$$(f, g)_{(p,1)} = \frac{1}{p} \frac{1}{(2\pi)^2} \int\limits_0^{+\infty} \cosh\rho \sinh\rho \ \mathrm{d}\rho \int\limits_0^{2p\pi} \mathrm{d}\varphi_+ \int\limits_0^{2\pi} \mathrm{d}\varphi_- \bar f(\rho, \varphi_+, \varphi_-) g(\rho, \varphi_+, \varphi_-) \ , \tag{6}$$

and on $\widetilde{\mathbb{H}_{2,2} \setminus \mathbb{S}^1}^{(\infty,1)}$ it reduces to

$$(f, g)_{(\infty,1)} = \frac{1}{(2\pi)^2} \int\limits_0^{+\infty} \cosh\rho \sinh\rho \ \mathrm{d}\rho \int\limits_{-\infty}^{+\infty} \mathrm{d}\varphi_+ \int\limits_0^{2\pi} \mathrm{d}\varphi_- \bar f(\rho, \varphi_+, \varphi_-) g(\rho, \varphi_+, \varphi_-) \ . \tag{7}$$

3. Differential realisations of the Lie groups

The parameterisations of Section 2 enable us to obtain explicit differential realisations of the Lie algebras $\mathfrak{sl}(2,\mathbb{C}), \mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ which define a left action respectively on $\mathbb{S}^3_{\mathbb{C}} \setminus \Big(\mathcal{C}_2 \times \mathcal{C}_2\Big), \mathbb{H}_{2,2} \setminus \mathbb{S}^1$ and $\mathbb{S}^3 \setminus \Big(\mathbb{S}^1 \times \mathbb{S}^1\Big)$.

For the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ the realisation is explicitly given by

$$\begin{aligned}
L_+ &= \frac{1}{4} e^{i(-\varphi_{-0} - i\varphi_{-1} + \varphi_{+0} + i\varphi_{+1})} \Big(\tan(\vartheta_0 + i\vartheta_1)\big(-i\partial_{\varphi_{+0}} - \partial_{\varphi_{+1}}\big) + \\
&\quad \partial_{\vartheta_0} - i\partial_{\vartheta_1} + \cot(\vartheta_0 + i\vartheta_1)\big(-i\partial_{\varphi_{-0}} - \partial_{\varphi_{-1}}\big)\Big) \\
&= \frac{1}{2} e^{i(\Phi_+ - \Phi_-)} \Big(-i \tan\Theta \partial_{\Phi_+} + \partial_\Theta - i\cot\Theta \partial_{\Phi_-}\Big) , \\
L_- &= \frac{1}{4} e^{i(\varphi_{-0} + i\varphi_{-1} - \varphi_{+0} - i\varphi_{+1})} \Big(\tan(\vartheta_0 + i\vartheta_1)\big(-i\partial_{\varphi_{+0}} - \partial_{\varphi_{+1}}\big) + \\
&\quad -\partial_{\vartheta_0} + i\partial_{\vartheta_1} + \cot(\vartheta_0 + i\vartheta_1)\big(-i\partial_{\varphi_{-0}} - \partial_{\varphi_{-1}}\big)\Big) \\
&= \frac{1}{2} e^{i(\Phi_- - \Phi_+)} \Big(-i \tan\Theta \partial_{\Phi_+} - \partial_\Theta - i\cot\Theta \partial_{\Phi_-}\Big) , \\
L_0 &= -\frac{i}{4}\Big(\partial_{\phi_{+0}} - i\partial_{\varphi_{+1}} - \partial_{\varphi_{-0}} + i\partial_{\varphi_{-1}}\Big) = -\frac{i}{2}\Big(\partial_{\Phi_+} - \partial_{\Phi_-}\Big) ,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\bar{L}_+ &= \frac{1}{4} e^{i(-\varphi_{-0} + i\varphi_{-1} + \varphi_{+0} - i\varphi_{+1})} \Big(\tan(\vartheta_0 - i\vartheta_1)\big(-i\partial_{\varphi_{+0}} + \partial_{\varphi_{+1}}\big) + \\
&\quad \partial_{\vartheta_0} + i\partial_{\vartheta_1} + \cot(\vartheta_0 - i\vartheta_1)\big(-i\partial_{\varphi_{-0}} + \partial_{\varphi_{-1}}\big)\Big) \\
&= \frac{1}{2} e^{i(\bar\Phi_+ - \bar\Phi_-)} \Big(-i \tan\bar\Theta \partial_{\bar\Phi_+} + \partial_{\bar\Theta} - i\cot\bar\Theta \partial_{\bar\Phi_-}\Big) , \\
\bar{L}_- &= \frac{1}{4} e^{i(\varphi_{-0} - i\varphi_{-1} - \varphi_{+0} + i\varphi_{+1})} \Big(\tan(\vartheta_0 - i\vartheta_1)\big(-i\partial_{\varphi_{+0}} + \partial_{\varphi_{+1}}\big) + \\
&\quad -\partial_{\vartheta_0} - i\partial_{\vartheta_1} + \cot(\vartheta_0 - i\vartheta_1)\big(-i\partial_{\varphi_{-0}} + \partial_{\varphi_{-1}}\big)\Big) \\
&= \frac{1}{2} e^{i(\bar\Phi_- - \bar\Phi_+)} \Big(-i \tan\bar\Theta \partial_{\bar\Phi_+} - \partial_{\bar\Theta} - i\cot\bar\Theta \partial_{\bar\Phi_-}\Big) , \\
\bar{L}_0 &= -\frac{i}{4}\Big(\partial_{\varphi_{+0}} + i\partial_{\varphi_{+1}} - \partial_{\varphi_{-0}} - i\partial_{\varphi_{-1}}\Big) = -\frac{i}{2}\Big(\partial_{\bar\Phi_+} - \partial_{\bar\Phi_-}\Big) ,
\end{aligned} \tag{9}$$

such that we have the $\mathfrak{sl}(2,\mathbb{C})$ commutation relations

$$\begin{aligned}
\big[L_0, L_\pm\big] = \pm L_\pm \ , \quad \big[L_+, L_-\big] = 2L_0 \ , \\
\big[\bar{L}_0, \bar{L}_\pm\big] = \pm \bar{L}_\pm \ , \quad \big[\bar{L}_+, \bar{L}_-\big] = 2\bar{L}_0 \ , \\
\big[L_a, \bar{L}_b\big] = 0 \ .
\end{aligned}$$

The differential realisation of $\mathfrak{sl}(2,\mathbb{R})$ is obtained using (8) and (9) through the real form (2)

$$\begin{aligned}
J_+ &= \frac{1}{2} e^{i(\varphi_+ - \varphi_-)} \Big(-i\tanh(\rho)\partial_{\varphi_+} - \partial_\rho + i\coth\rho \partial_{\varphi_-}\Big) , \\
J_- &= \frac{1}{2} e^{i(\varphi_- - \varphi_+)} \Big(-i\tanh(\rho)\partial_{\varphi_+} + \partial_\rho + i\coth\rho \partial_{\varphi_-}\Big) , \\
J_0 &= -\frac{i}{2}\Big(\partial_{\varphi_+} - \partial_{\varphi_-}\Big) ,
\end{aligned} \tag{10}$$

and satisfies the $\mathfrak{sl}(2,\mathbb{R})$ commutation relations

$$\Big[J_0, J_\pm\Big] = \pm J_\pm \ , \quad \Big[J_+, J_-\Big] = -2J_0 \ .$$

Journal of Physics: Conference Series **597** (2015) 012021 doi:10.1088/1742-6596/597/1/012021

Finally, the differential realisation of $\mathfrak{su}(2)$ is obtained using (8) and (9) through the real form (4) and is given by

$$\begin{aligned} R_+ &= \frac{1}{2}e^{i(\varphi_+-\varphi_-)}\Big(-i\tan\theta\partial_{\varphi_+}+\partial_\theta-i\cot\theta\partial_{\varphi_-}\Big) \ , \\ R_- &= \frac{1}{2}e^{i(\varphi_--\varphi_+)}\Big(-i\tan\theta\partial_{\varphi_+}-\partial_\theta-i\cot\theta\partial_{\varphi_-}\Big) \ , \\ R_0 &= -\frac{i}{2}\Big(\partial_{\varphi_+}-\partial_{\varphi_-}\Big) \ , \end{aligned}$$

and satisfies the $\mathfrak{su}(2)$ commutations relations

$$\big[R_0,R_\pm\big]=\pm R_\pm \ \ , \big[R_+,R_-\big]=2R_0 \ .$$

4. Representations as harmonic functions for the three dimensional simple Lie groups

In this section we show that all unitary representations of the Lie groups $SL(2,\mathbb{C}), SL(2,\mathbb{R})$ and $SU(2)$ can be obtained from their spinor(s) representation(s). The case of $SU(2)$ is trivial and it is not difficult to see that all unitary representations can be defined as harmonic functions on $\mathbb{S}^3\setminus\Big(\mathbb{S}^1\times\mathbb{S}^1\Big)$. This case will not be considered here (see [23] for the details).

Irreducible representations of $SL(2,\mathbb{C})$ have been studied by Gel'fand and can be given in terms of homogenous functions in $\mathbb{C}^2$ [2, 24, 25, 26, 27] (see [28] for an English translation of [24]). Unitary representations are characterised by two numbers ℓ_0 and ℓ_1 and correspond either to $\ell_0\in\frac{1}{2}\mathbb{N}$ and $\ell_1=i\sigma,\sigma\in\mathbb{R}$ for the principal series or to $\ell_0=0$ and $0<\ell_1\le 1$ for the complementary series. We have shown that we can extend the Gel'fand formulæ for all unitary representations such that they are living on $\mathbb{S}^3_{\mathbb{C}}\setminus\Big(\mathcal{C}_2\times\mathcal{C}_2\Big)$ and are harmonic. Since the expressions are complicated and not very enlightening we refer to [23] for explicit expressions.

Unitary representations of $SL(2,\mathbb{R})$ have been studied by various authors [6, 8, 9, 29, 30] and divide into two types: the discrete series (which is bounded either from below or above) and the continuous (principal and supplementary) series, which are unbounded. We have shown that all unitary representations of $SL(2,\mathbb{R})$ can be defined as harmonic functions of $\mathbb{H}_{2,2}\setminus\mathbb{S}^1$ (or one of his coverings), but in this note we only consider the case of discrete series. The discrete series is characterised by $s>0$. For the discrete series bounded from below we have

$$\begin{aligned} \mathcal{D}^+_s=\Bigg\{\Psi^+_{s,n} &= \sqrt{\frac{2\Gamma(n+2s)}{\Gamma(-1+2s)\Gamma(n+1)}}\bar{\zeta}_+{}^{-2s-n}\bar{\zeta}_-{}^{n} \\ &= \sqrt{\frac{2\Gamma(n+2s)}{\Gamma(-1+2s)\Gamma(n+1)}}e^{-in\varphi_-+i(2s+n)\varphi_+}\cosh^{-2s-n}\rho\sinh^n\rho \ , \quad n\in\mathbb{N}\Bigg\} \ , \end{aligned}$$

while for discrete series bounded from above

$$\begin{aligned} \mathcal{D}^-_s=\Bigg\{\Psi^-_{s,n} &= \sqrt{\frac{2\Gamma(n+2s)}{\Gamma(-1+2s)\Gamma(n+1)}}\zeta_+^{-2s-n}\zeta_-^{n} \\ &= \sqrt{\frac{2\Gamma(n+2s)}{\Gamma(-1+2s)\Gamma(n+1)}}e^{-i(2s+n)\varphi_++in\varphi_-}\cosh^{-2s-n}\rho\sinh^n\rho \ , \quad n\in\mathbb{N}\Bigg\} \ , \end{aligned}$$

which satisfy

$$\left\{\begin{aligned} J_+\Psi^+_{s,n} &= \sqrt{(n+1)(n+2s)}\Psi^+_{s,n+1} \ , \\ J_-\Psi^+_{s,n} &= \sqrt{n(n+2s-1)}\Psi^+_{s,n-1} \ , \\ J_0\Psi^+_{s,n} &= (n+s)\Psi^+_{s,n} \ , \end{aligned}\right. \qquad \left\{\begin{aligned} J_+\Psi^-_{s,n} &= -\sqrt{n(n+2s-1)}\Psi^-_{s,n-1} \\ J_-\Psi^-_{s,n} &= -\sqrt{(n+1)(n+2s)}\Psi^-_{s,n+1} \\ J_0\Psi^-_{s,n} &= -(n+s)\Psi^-_{s,n} \ . \end{aligned}\right. \ ,$$

Note that $\mathcal{D}^+_s$ and $\mathcal{D}^-_s$ are isomorphic representations. In order that the expressions above are well defined they have to be defined on some covering of $\mathbb{H}_{2,2}\backslash\mathbb{S}^1$. In particular, when $2s = p/q \in \mathbb{Q}$, the representation $\mathcal{D}^\pm_s$ is defined on $\widetilde{\mathbb{H}_{2,2}\setminus\mathbb{S}^1}^{(q,1)}$ and corresponds to a representation of the $q-$sheeted covering of $SL(2,\mathbb{R})$ and when $s\in\mathbb{R}\backslash\mathbb{Q}$ the representation $\mathcal{D}^\pm_s$ is defined on $\widetilde{\mathbb{H}_{2,2}\setminus\mathbb{S}^1}^{(\infty,1)}$ and corresponds to a representation of the universal covering of $SL(2,\mathbb{R})$. Using (5), a direct computation gives

$$\Delta\Psi^\pm_{s,m} = -4s(s-1)\Psi^\pm_{s,m} = -4Q\Psi^\pm_{s,m} \ ,$$

where $Q = J_0^2 - 1/2(J_+J_- + J_-J_+)$ is the Casimir operator of $\mathfrak{sl}(2,\mathbb{R})$ and the functions $\Psi^\pm_{s,m}$ are harmonic functions defined on $\mathbb{H}_{2,2}\setminus\mathbb{S}^1$ (or one of his coverings).

Now we check unitarity of the representations using the scalar product (6) or (7). In order to proceed we have to check the convergence of the various integrals, since we are integrating on non-compact manifolds. Using hypergeometric functions, a routine computation shows that in all cases, after integration upon the angles and performing a change of variables, the integrals involving ρ reduce to

$$I_{s,n} = \int\limits_1^{+\infty} r^{-4s-2n+1}(r^2-1)^n \mathrm{d}r \ ,$$

with $n \geq 0$. These integrals converge if $s > 1/2$ and give

$$I_{s,n} = \frac{1}{2}\frac{\Gamma(n+1)\Gamma(2s-1)}{\Gamma(2s+n)} \ .$$

In particular, this means that when $2s = p/q, 2s' = p'/q'$, introducing q'' as the least common multiple of q and q' (6), provides the following identity when $s, s' > 1/2$

$$(\Psi^{\varepsilon'}_{s',n'}, \Psi^{\varepsilon}_{s,n})_{q''} = \delta^{\varepsilon'\varepsilon}\delta_{s's}\delta_{n'n} \ ,$$

and when either s or s' is an irrational number, when $s, s' > 1/2$ holds we have from (7) that

$$(\Psi^{\varepsilon'}_{s',n'}, \Psi^{\varepsilon}_{s,n})_{q''} = \delta^{\varepsilon'\varepsilon}\delta_{n'n}\delta(s-s') \ .$$

Observe that when $s \in \mathbb{R}$, the eigenvalues of the Casimir operator Q are continuous hence the eigenfunctions $\Psi^\pm_{s,n}$ cannot be normalised.

5. Extension of the formalism for the group of rotations-translations in two dimensions

Interestingly all the formalism we have applied for simple Lie groups extends for the group E_2 of rotation translations in two-dimensions. This group can be, for instance, obtained by an Inönü-Wigner contraction of the group $SU(2)$ or $SL(2,\mathbb{R})$. Studying a singular limit of the hyperboloid $\mathbb{H}_{2,2} \cong SL(2,\mathbb{R})$ enables us to describe the cone $\mathcal{C}_{2,2}$ of equation

$$x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0 \ ,$$

which is strongly related to E_2 [23]. To have an explicit realisation of unitary representations of E_2 we have to proceed in several steps. Firstly, considering a singular limit of (3) leads to a bijective parameterisation of the cone with one point removed $\mathcal{C}\setminus\{0\}$. Secondly, considering the same singular limit for (10) leads to a differential realisation of $\mathfrak{e}_2$ (the Lie algebra of E_2) and an explicit realisation of all unitary representations of $\mathfrak{e}_2$ on $\mathcal{C}_{2,2}\setminus\{0\}$. Next, to define an invariant scalar product we observe that $\mathcal{C}_{2,2}\setminus\{0\}$ can be continuously deformed (is homeomorphic) to $\mathbb{R}\times\mathbb{S}^1\times\mathbb{S}^1$ and since $\mathbb{S}^1$ is the one-point compactification of $\mathbb{R}$, we have explicitly constructed unitary representations of $\mathfrak{e}_2$ on $\mathbb{S}^1\times\mathbb{S}^1\times\mathbb{S}^1$, parameterised by $\Psi, \varphi_+, \varphi_- \in [0, 2\pi[$. With this realisation the generators of $\mathfrak{e}_2$ are given by

$$\begin{aligned} P_+ &= -\frac{1}{2}e^{i(\varphi_+-\varphi_-)}\partial_\Psi \ , \\ P_- &= \frac{1}{2}e^{i(\varphi_--\varphi_+)}\partial_\Psi \ , \\ J &= -\frac{i}{2}(\partial_{\varphi_+} - \partial_{\varphi_-}) \ , \end{aligned}$$

with commutations relations

$$[J, P_\pm] = \pm P_\pm \ , [P_+, P_-] = 0 \ .$$

Unitary representations of $\mathfrak{e}_2$ are characterised by two numbers $p \in \mathbb{R}$ and $-1/2 < s \leq 1/2$ and are given by

$$\mathcal{D}_{p,s} = \left\{ \Lambda_{p,s,n} = e^{i2p\Psi} e^{i(n+s)\varphi_+ - in\varphi_-} \ , n \in \mathbb{Z} \right\},$$

such that

$$\begin{aligned} P_+ \Lambda_{p,s,n} &= -ip\Lambda_{p,s,n+1} \ , \\ P_- \Lambda_{p,s,n} &= ip\Lambda_{p,s,n-1} \ , \\ J\Lambda_{p,s,n} &= (s+n)\Lambda_{p,s,n} \ . \end{aligned}$$

As for the $SL(2, \mathbb{R})$ case, the functions Λ are defined on appropriate coverings. These coverings will be given by their parameterisation. Two cases are considered. The first one corresponds to $2s = r/q$ with $\Psi \in \mathbb{R}, \varphi_+ \in [0, 2q\pi[, \varphi_- \in [0, 2\pi[$, and second case to $s \in \mathbb{R} \setminus \mathbb{Q}$ with $\Psi, \varphi_+ \in \mathbb{R}, \varphi_- \in [0, 2\pi[$.

Now, if f and g are two functions defined on an appropriate covering of $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, we define the scalar product

$$(f, g)_q = \frac{1}{q} \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \mathrm{d}\Psi \int_0^{2q\pi} \mathrm{d}\varphi_+ \int_0^{2\pi} \mathrm{d}\varphi_- \bar{f}(\Psi, \varphi_+, \varphi_-) g(\Psi, \varphi_+, \varphi_-) \ ,$$

in the first case and

$$(f, g)_\infty = \frac{2}{(2\pi)^2} \int_{-\infty}^{+\infty} \mathrm{d}\Psi \int_{-\infty}^{+\infty} \mathrm{d}\varphi_+ \int_0^{2\pi} \mathrm{d}\varphi_- \bar{f}(\Psi, \varphi_+, \varphi_-) g(\Psi, \varphi_+, \varphi_-) \ ,$$

in the second case. It is then straightforward to check that for $s, s' \in \mathbb{Q}$, *i.e.*, $2s = r/q, 2s' = r'/q'$ (q'' being the least common multiple of q, q'), we have

$$(\Lambda_{p', \frac{r'}{2q'}, n'}, \Lambda_{p, \frac{r}{2q}, n})_{q''} = \delta_{nn'} \delta_{\frac{r}{q} \frac{r'}{q'}} \delta(p - p') \ ,$$

while for s or s' a real number (but not a rational), we get

$$(\Lambda_{p', s', n'}, \Lambda_{p, s, n})_\infty = \delta_{nn'} \delta(s' - s) \delta(p - p') \ .$$

The functions $\Lambda_{p,s,n}$ are hence orthonormal, and the representations $\mathcal{D}_{p,s}$ are unitary.

6. Final remarks

Unitary representations of the three dimensional simple Lie groups have been constructed in terms of generalised harmonic functions acting on appropriate smooth manifolds related to differential realisations. Using the well known relation of these groups with the group of rotations-translations in two dimensions via Inönü-Wigner contractions, the argument is extended to the construction of unitary multiplets of the latter. This procedure provides an alternative perspective concerning the central role of spinor representations. In principle, up to the restrictions expected in the integration on non-compact manifolds, the method can be formally proposed for higher rank simple Lie groups. The corresponding analysis for SU(3) in the context of the Elliott model or the pseudo-orthogonal groups $\mathrm{SO}(5-p, p)$ in the frame of kinematical groups constitute physically motivating situations to further enlarge the Ansatz to the case of subduced representations or its application in reductions chains of symmetry groups.

Acknowledgments
The authors express their gratitude to Marcus Slupinski for fruitful discussions and valuable suggestions that improved the manuscript. This work was partially supported by the research project MTM2010-18556 of the MICINN (Spain).

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