UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

Departamento de Análisis Matemático

ON SMOOTH APPROXIMATION AND EXTENSION ON BANACH SPACES AND APPLICATIONS TO BANACH-FINSLER MANIFOLDS

MEMORIA PARA OPTAR AL GRADO DE DOCTOR PRESENTADA POR

Luis Sánchez González

Bajo la dirección de la doctora
Mar Jiménez Sevilla

Madrid, 2012

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Sobre aproximación y extensión suave en espacios de Banach y aplicaciones a variedades Banach-Finsler

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A mi padre

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Luis Sánchez

Index

Resumen

El marco de esta memoria es la teoría de diferenciabilidad en espacios de Banach y en variedades de tipo Banach-Finsler. En ella nos ocupamos de varias cuestiones diferentes que, como iremos viendo, están muy relacionadas entre sí.

El primer problema que abordamos trata de caracterizar los espacios de Banach separables donde existen funciones meseta diferenciables que localmente dependen de un número finito de coordenadas. Recordemos que una *función meseta* definida en un espacio de Banach X es una función real-valuada $f : X \to \mathbb{R}$ tal que su soporte es no vacío y acotado, es decir, existe un elemento $x \in X$ tal que $f(x) \neq 0$ y la clausura del conjunto $\{x \in X : f(x) \neq 0\}$ es acotada. La existencia de una función meseta con buenas condiciones nos proporciona una gran cantidad de propiedades geométricas y de herramientas que no disponemos en otros espacios de Banach. Un buen ejemplo es un espacio de Banach el cual admite una función meseta diferenciable, ya que ésta implica que el espacio sea Asplund y admita particiones de la unidad diferenciables. La propiedad en la que nosotros estamos interesados es la siguiente: una función $f : X \to \mathbb{R}$, definida en un espacio de Banach, *localmente depende de un número finito de coordenadas* (*LFC*, para acortar) siempre que localmente se pueda factorizar a través de espacios de dimensión finita, es decir, para todo $x \in X$, existan un entorno U de x, una cantidad finita de funcionales $\{f_1, \ldots, f_n\} \subset X^*$ y una función continua $g : \mathbb{R}^n \to \mathbb{R}$ tales que $f(y) = g(f_1(y), \ldots, f_n(y))$ para todo $y \in U$.

Fue Kuiper quien utilizó por primera vez el concepto de función LFC para construir renormamientos LFC y de clase C^∞ en c_0 (esta construcción apareció en [15]). Durante las últimas décadas, esta noción ha sido muy utilizada en teoría de renormamientos de espacios de Banach, de hecho, se ha convertido en una de las formas más comunes de tratar estos problemas. Probablemente, la aplicación más importante de las funciones LFC fue dada por H. Toruńczyk en [108], donde usó la existencia de funciones meseta LFC de clase C^∞ en $c_0(\Gamma)$ para construir particiones de la unidad de clase C^k en espacios reflexivos con funciones meseta C^k diferenciables. Recientemente, P. Hájek y M. Johanis en [55] y [56] han usado el concepto de funciones LFC para obtener sup-particiones de la unidad diferenciables y Lipschitz en cierta clase de espacios de Banach no separables.

La existencia de una función meseta continua y LFC proporciona propiedades extra en los espacios de Banach. En el artículo [101] (en el cual se define por primera vez la noción de una función LFC) J. Pechanec, J.H.M. Whitfield y V. Zizler demuestran que todo espacio de Banach que admite una función meseta LFC está saturado con copias de c_0. Algunos años más tarde, M. Fabian y V. Zizler [26] prueban que, además, estos espacios deben ser Asplund. Sin embargo, no todo espacio Asplund, c_0-saturado admite una función meseta LFC (ver [69]).

El problema que abordamos en el Capítulo 2 viene motivado por los artículos [51] y [54]. En el primero de ellos se demuestra que un espacio de Banach separable admite una norma LFC de clase C^∞ siempre que éste admita una norma LFC. Este resultado suscita varias cuestiones, entre ellas si ocurre lo mismo cuando sustituimos norma por función meseta. El segundo artículo mencionado supone un gran avance en esta dirección, demostrando que la respuesta es positiva siempre y cuando el espacio admita una base de Schauder. En este

capítulo presentamos una mejora de este resultado, caracterizando los espacios de Banach separables que admiten una función meseta LFC de clase C^∞ como aquellos que admiten una función meseta LFC continua, así, contestamos a un problema propuesto en [54], [57] y [26]. Concretamente, el resultado principal del Capítulo 2 es el siguiente:

Teorema 1. *Sea X un espacio de Banach separable. Entonces, X admite una función meseta continua LFC si y sólo si X admite una función meseta de clase C^∞ y LFC.*

Este resultado ha sido publicado en [65]. La idea clave de su demostración es un lema previo que, grosso modo, descompone una función LFC a través de c_0 en la unión finita de los abiertos donde finitamente está representada, es decir, dada una función $f : X \to \mathbb{R}$, si $f = g_1(f_1^1, \dots, f_{n_1}^1)$ en $B(x_1, 2r_1)$ y $f = g_2(f_1^2, \dots, f_{n_2}^2)$ en $B(x_2, 2r_2)$ con g_1, g_2 funciones continuas y $f_1^1, \dots, f_{n_1}^1, f_1^2, \dots, f_{n_2}^2 \in X^*$, entonces existen una aplicación lineal y continua $T : X \to c_0(\mathbb{N})$ y una función continua $g : c_0(\mathbb{N}) \to \mathbb{R}$ tales que $f(x) = g(T(x))$ en $B(x_1, r_1) \cup B(x_2, r_2)$.

Además, en la Sección 3 del Capítulo 2, generalizaremos la definición de funciones LFC a funciones que *localmente se factorizan a través de espacios de Banach* que pertenecen a una familia fijada de espacios de Banach $\mathcal{F}$ (*LF-$\mathcal{F}$*, para acortar), es decir, para todo $x \in X$, existen un entorno U de x en X, un espacio de Banach $E \in \mathcal{F}$, un operador lineal y continuo $T : U \to E$ y una función continua $g : E \to \mathbb{R}$ tales que $f(y) = g(T(y))$ en U. En este contexto obtenemos un teorema similiar al anterior:

Teorema 2. *Sea $\mathcal{F}$ una familia de espacios de Banach separables y sea X un espacio de Banach con dual separable que admite una función meseta continua LF-$\mathcal{F}$. Asumamos que todo espacio de Banach $E \in \mathcal{F}$ admite una función meseta b de clase C^k (de clase C^k y LFC). Entonces X admite una función meseta de clase C^k (de clase C^k y LFC, respectivamente).*

El Capítulo 3 es un breve resumen de los resultados de aproximación diferenciable y Lipschitz de funciones Lipschitz que se conocen hasta el momento. Un problema que surge de forma natural en cualquier contexto es el de intentar aproximar una función por otra con mejores propiedades. Es un hecho bien conocido que en el caso finito dimensional toda función Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ puede ser uniformemente aproximada por una función Lipschitz de clase C^∞ utilizando convoluciones integrales

$$f_n(x) = \int_{\mathbb{R}^n} f(y)\varphi_n(y - x)dy,$$

donde φ_n son funciones no negativas de clase C^∞ en $\mathbb{R}^n$ tales que $\int_{\mathbb{R}^n} \varphi_n = 1$ y $\operatorname{supp} \varphi_n \subset B(0, 1/n)$. Desafortunadamente, en espacios de Banach de dimensión infinita no es posible utilizar el método de convoluciones integrales al no existir una medida suficientemente buena como la medida de Lebesgue.

El problema de aproximar uniformemente una función continua por una función diferenciable o analítica en un espacio de Banach ha sido muy estudiado en el último siglo (ver [83], [15], [108], [48], [34], [17], [22], [40], [90], etc). En el Capítulo 3 nuestra atención se centra en los resultados existentes en aproximación uniforme de funciones Lipschitz por funciones Lipschitz diferenciables, es decir, dado $\varepsilon > 0$ y $f : X \to \mathbb{R}$ una función Lipschitz, cuándo existe una función $g : X \to \mathbb{R}$ de clase C^k y Lipschitz tal que $|f(x) - g(x)| < \varepsilon$ en X y

$\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$, donde $C_0 \geq 1$ es una constante que depende únicamente del espacio de Banach X. Los primeros resultados en esta dirección los obtuvieron J.M. Lasry y P.L. Lions [85]. En este artículo utilizan convoluciones infimales para demostrar que en todo espacio de Hilbert toda función Lipschitz puede ser uniformemente aproximada por una función de clase C^1 con la misma constante Lipschitz. Para más información y resultados sobre convoluciones infimales ver [4], [106], [107] y [16].

A diferencia de la aproximación diferenciable, en el caso de aproximación Lipschitz no podemos usar particiones de la unidad diferenciables, ya que, aunque todas las funciones involucradas sean Lipschitz, no tenemos ningún control en el tamaño de la suma de sus derivadas. Para solucionar este problema, R. Fry en el artículo [35] construye en todo espacio de Banach con dual separable, una familia de funciones diferenciables y Lipschitz con el supremo de las constantes Lipschitz acotado, cumpliendo las mismas propiedades que una partición de la unidad, excepto que en lugar de sumar uno en todo punto es el supremo de todas ellas lo que tiene que valer uno. Por esta razón, tiempo más tarde, son llamadas *sup-particiones de la unidad*. Utilizando las técnicas de sup-particiones de la unidad es posible aproximar en una gran clase de espacios de Banach cualquier función Lipschitz $f : X \to \mathbb{R}$ por funciones Lipschitz diferenciables $g : X \to \mathbb{R}$ con la constante Lipschitz contralada, es decir, $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$ donde $C_0 \geq 1$ es una constante que depende únicamente del espacio de Banach (ver [35], [10] y [56]). Recientemente se ha utilizado estas técnicas para aproximar uniformemente funciones Lipschitz por funciones analíticas y Lipschitz (ver [39], [8] y [9]).

En este capítulo también presentamos algunas mejoras en el problema de la aproximación Lipschitz y diferenciable. El resultado principal que obtenemos es que la constante $C_0 \geq 1$ es independiente del espacio de Banach siempre y cuando éste tenga un espacio dual separable, de hecho, puede ser tomada menor o igual a $4 + r$ para cada $r > 0$. Por último, indicar que la aproximación diferenciable y Lipschitz de funciones Lipschitz está muy relacionada con la aproximación C^1-fina, es decir, dada una función de clase C^1, aproximar al mismo tiempo la función y su derivada por otra de clase mayor (ver [92], [11] y [56]). Este capítulo y los resultados aquí enunciados serán claves para el desarrollo del resto de la memoria.

En el Capítulo 4 tratamos un problema de extensión diferenciable, el cual es uno de los principales objetivos de esta memoria. Es interesante preguntarse si una función con alguna buena propiedad definida en un subconjunto puede ser extendida a un espacio más grande conservando esa propiedad. Dos ejemplos clásicos son: el Teorema de extensión continua de Tietze, el cual asegura que toda función continua definida en un subespacio cerrado de un espacio topológico normal puede ser extendida a una función continua definida en todo el espacio; y el Teorema de Hahn-Banach, que extiende todo funcional lineal y continuo definido en un subespacio cerrado de un espacio de Banach a un funcional lineal y continuo definido en todo el espacio. Podemos preguntarnos si estos resultados siguen siendo válidos cuando queremos extender funciones diferenciables, es decir, cuándo una función definida en un subconjunto cerrado de un espacio de Banach es la restricción de una función diferenciable definida en todo el espacio, y cuándo una función diferenciable definida en un subespacio cerrado puede ser extendida a una función diferenciable en todo el espacio.

Un ejemplo trivial es cuando el subespacio es complementado, en este caso toda función de clase C^k definida en el subespacio se puede extender a una función de clase C^k definida en todo el espacio, sin más que componer con la proyección lineal y continua del subespacio. Usando un resultado clásico de J. Lindenstrauss y L Tzafriri [89], el cual asegura que si en un espacio de Banach todos los subespacios cerrados son complementados entonces este espacio es isomorfo a un espacio de Hilbert, el problema de extender funciones diferenciables desde subespacios cerrados únicamente será interesante en espacios de Banach de dimensión infinita que no sean Hilbert.

El problema de extensión desde subconjuntos cerrados en dimensión finita ha sido exhaustivamente estudiado. El primero en tratar tal problema fue H. Whitney en [111] y [112], quien caracterizó las funciones definidas en subconjuntos cerrados de $\mathbb{R}$ que pueden ser extendidas a funciones de clase C^k definidas en todo $\mathbb{R}$ en término de sus diferencias divididas. La resolución de este problema para mayores dimensiones fue dada por G. Glaeser [47] para el caso de extensión C^1 y, finalmente, C. Fefferman en una serie de artículos ([29] y [30]) completa la solución del problema y caracteriza las funciones definidas en subconjuntos compactos de $\mathbb{R}^n$ que pueden ser extendidas a funciones de clase C^k definidas en todo el espacio.

En espacios de Banach de dimensión infinita el problema ha sido bastante menos estudiado. Por un lado tenemos el trabajo de C.J. Atkin [3], que extiende funciones diferenciables definidas en una unión finita de abiertos convexos en un espacio de Banach separable sin función meseta diferenciable, siempre y cuando esta función admita una extensión diferenciable a todo el espacio cuando la restringimos a un cierto entorno de cada punto de su dominio. Por otra parte, D. Azagra, R. Fry y L. Keener [7] demuestran que toda función de clase C^1 definida en un subespacio cerrado de un espacio de Banach con dual separable puede ser extendida a una función de clase C^1 definida en todo el espacio de Banach. Éste es nuestro punto de partida.

En este capítulo tratamos el problema de extensión diferenciable de funciones real-valuadas y vector-valuadas desde subconjuntos cerrados, para ello damos una condición necesaria para que una función se pueda extender de forma diferenciable y vemos en qué espacios esta condición es también suficiente. La condición que imponemos sobre un par de espacios de Banach (X, Z), donde X será el espacio de partida y Z el espacio de llegada, es una propiedad de aproximación diferenciable de funciones Lipschitz. Más concretamente, diremos que (X, Z) satisface *la propiedad (*)* si existe una constante $C_0 \geq 1$ que depende de X y Z, tal que para todo subconjunto A de X, toda aplicación Lipschitz $f : A \to Z$ y todo $\varepsilon > 0$, existe una aplicación $g : X \to Z$ Lipschitz y de clase C^1 tal que

$$||f(x) - g(x)|| < \varepsilon \qquad \text{para todo } x \in A, \text{ y} \qquad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

El principal resultado que presentamos en este capítulo es el siguiente:

Teorema 3. *Sea (X, Z) un par de espacios de Banach con la propiedad (*) y A un subconjunto cerrado de X. Una aplicación $f : A \to Z$ puede ser extendida a una aplicación de clase C^1 en todo el espacio X si y sólo si existe una aplicación continua $D : A \to \mathcal{L}(X, Z)$ tal que para todo $y \in A$ y todo $\varepsilon > 0$, existe $r > 0$ tal que*

$$||f(z) - f(w) - D(y)(z - w)|| \leq \varepsilon ||z - w||, \qquad \textit{para todo } z, w \in A \cap B(y, r). \tag{1}$$

Como podemos ver en el anterior teorema, la propiedad de extender funciones que admiten la desigualdad (1) para alguna aplicación continua D está muy cerca de la aproximación diferenciable de funciones Lipschitz, de hecho, en espacios separables obtenemos la siguiente caracterización:

Corolario 4. *Sea X un espacio de Banach separable. Son equivalentes:*

(i) *El par $(X, \mathbb{R})$ satisface la propiedad (*), es decir, existe una constante $C_0 \geq 1$ tal que toda función f Lipschitz definida en un subconjunto de X con valores reales se puede aproximar uniformemente por una función g de clase C^1 y Lipschitz con $\mathrm{Lip}(g) \leq C_0 \mathrm{Lip}(f)$.*

(ii) *Toda función real-valuada definida en un subconjunto cerrado $A \subset X$ que cumple la desigualdad (1) en A para una aplicación continua $D : A \to X^*$, se puede extender por una función de clase C^1 a todo el espacio de Banach X.*

(iii) *El espacio dual X^* es separable.*

En la Sección 3 del Capítulo 4 damos una extensa lista de pares de espacios de Banach que cumplen la propiedad (*) y, así, el Teorema 3. Es interesante notar que cuando $Z = \mathbb{R}$, la propiedad (*) se reduce a la aproximación diferenciable de funciones Lipschitz estudiada en el Capítulo 3.

En la Sección 4 de este capítulo tratamos la extensión de aplicaciones diferenciables definidas en subespacios cerrados. Para ello, necesitamos añadir una condición extra al par de espacios de Banach (X, Z). Diremos que (X, Z) satisface *la propiedad de extensión lineal* si existe una constante $\lambda \geq 1$, que depende de X y Z, tal que para todo subespacio cerrado $Y \subset X$ y todo operador lineal y continuo $T : Y \to Z$ existe un operador lineal y continuo $\widetilde{T} : X \to Z$ tal que $\widetilde{T}(y) = T(y)$ para todo $y \in Y$ y $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda ||T||_{\mathcal{L}(Y,Z)}$. En esta sección vemos que si un par de espacios de Banach satisface esta propiedad, entonces toda aplicación $f : Y \to Z$ de clase C^1 cumple la desigualdad (1) para una aplicación continua D, así podemos aplicar el Teorema 3 para obtener el siguiente corolario:

Corolario 5. *Sea (X, Z) un par de espacios de Banach el cual cumple la propiedad (*) y la propiedad de extensión lineal. Toda aplicación $f : Y \to Z$ de clase C^1 definida en un subespacio cerrado de X admite una extensión de clase C^1 a todo el espacio X.*

Es importante notar que cuando $Z = \mathbb{R}$ la propiedad de extensión lineal se reduce al conocido teorema de Hahn-Banach. Por lo que este resultado generaliza el obtenido en [7] a funciones vector-valuadas y a funciones definidas en cierta clase de espacios no separables. En esta sección también vemos una lista de pares de espacios de Banach que cumplen ambas propiedades.

Por último, es importante indicar que a lo largo de este capítulo trabajamos tanto con extensión diferenciable como con extensión diferenciable Lipschitz. De hecho, obtenemos que la propiedad (*) es equivalente a que toda aplicación Lipschitz que cumpla la desigualdad (1) para una aplicación continua y acotada, pueda extenderse a una aplicación Lipschitz y de clase C^1 con la constante Lipschitz controlada. En esta línea, la propiedad de extensión lineal es también necesaria para que toda aplicación Lipschitz y de clase C^1 definida en un subespacio cerrado pueda ser extendida por una aplicación Lipschitz y de clase C^1 con la constante Lipschitz controlada. Los resultados presentados en este capítulo han sido recogidos en [67] y [68].

Aparte de los problemas de diferenciación en espacios de Banach, el estudio de las variedades Banach-Finsler es uno de los principales objetivos de esta memoria. Así, en los Capítulos 5, 6 y 7 se presentan varias propiedades de esta clase de variedades.

Una variedad Banach-Finsler (en adelante, variedad Finsler) es una variedad diferenciable modelada en un espacio de Banach tal que admite una aplicación continua que a cada punto de la variedad le asigna una norma definida en el espacio tangente (isomorfo al espacio de Banach donde la variedad está modelada). Gracias a esta estructura podemos definir la longitud de caminos sobre la variedad de forma habitual y, de este modo, introducir la distancia Finsler, la cual será una métrica equivalente con la topología de la variedad. R.S. Palais en [100] introdujo una propiedad extra, concretamente estudió las variedades Finsler cuyos espacios tangentes tienen normas localmente $(1+\varepsilon)$-equivalentes para todo $\varepsilon > 0$; estas variedades son conocidas como *variedades Finsler en el sentido de Palais* y generalizan a las variedades de Riemann. Por otra parte, H. Upmeier en [110] y K.H. Neeb en [98] trabajaron con una propiedad algo más débil, pidiendo, únicamente, que para cada punto de la variedad exista un entorno y una constante $K \geq 1$ tal que las normas sean K-equivalentes en ese entorno; estas variedades son conocidas como *variedades Finsler en el sentido de Neeb-Upmeier* y forman una clase de variedades más amplia que la de variedades Finsler en el sentido de Palais.

En el Capítulo 5 introducimos dos clases de variedades Finsler, *variedad Finsler en el sentido de Neeb-Upmeier débil-uniforme* y *en el sentido de Neeb-Upmeier uniforme*, tales que generalizan las variedades Finsler en el sentido de Palais y las variedades de Riemann, respectivamente, pero con mejores propiedades que las variedades Finsler en el sentido de Neeb-Upmeier. Es interesante hacer notar que las cuatro clases de variedades Finsler son equivalentes en variedades finito dimensionales. En este capítulo estudiamos varias propiedades de estas clases de variedades y las relaciones existentes entre ellas, también obtenemos algunas desigualdades del valor medio, y un resultado sobre existencia de cartas bi-Lipschitz, las cuales serán herramientas de gran utilidad en los siguientes capítulos.

En el Capítulo 6 se obtienen diferentes resultados sobre aproximación diferenciable de funciones Lipschitz, extensión diferenciable y extensión diferenciable Lipschitz en el contexto de variedades Finsler. El estudio de la aproximación diferenciable de funciones Lipschitz en variedades viene motivado por el trabajo de D. Azagra, J. Ferrera, F. López-Mesas y Y.C. Rangel [6], en el que demuestran que toda función Lipschitz definida en una variedad Riemanniana separable puede ser aproximada por una función Lipschitz de clase C^∞ con casi la misma constante Lipschitz. Usando este hecho, no es difícil ver que toda variedad Riemanniana separable es C^∞*-uniformemente mesetable* (concepto introducido en [5]), es decir, existen dos números positivos $r > 0$ y $R > 1$ tales que, para cada punto $p \in M$ y cada $\delta \in (0, r)$, existe una función $b : M \to [0, 1]$ de clase C^∞, tal que $b(p) = 1$, $b(x) = 0$ siempre que $d_M(x, p) \geq \delta$, y $\sup_{x \in M} ||db(x)||_x \leq R/\delta$. La importancia de que una variedad Riemanniana sea uniformemente mesetable se puede ver en los artículos [5] y [42]. En el primero de ellos se demuestra el principio variacional de Deville-Godefroy-Zizler para toda variedad Riemanniana completa y C^1-uniformemente mesetable; este principio variacional es una herramienta esencial en el análisis no-suave y en la resolución de las ecuaciones de Hamilton-Jacobi. En el segundo artículo se obtiene una versión infinito dimensional del teorema de Myers-Nakai para toda variedad Riemanniana completa y C^1-uniformemente mesetable. Sobre estas dos cuestiones hablaremos más adelante.

Vista la importancia de poder aproximar uniformemente una función Lipschitz por una función Lipschitz y diferenciable en una variedad Riemanniana separable, es natural preguntarse si es posible obtener el mismo resultado en el caso no separable, y si también ocurre cuando la variedad es Finsler. Ambos problemas los resolvemos simultáneamente en la Sección 1 del Capítulo 6. Uno de los resultados principales que obtenemos es el siguiente:

Teorema 6. *Sea M una variedad Finsler de clase C^k en el sentido de Neeb-Upmeier débil-uniforme modelada en un espacio de Banach separable con función meseta C^k y Lipschitz (M está modelada en un espacio de Hilbert no separable). Entonces, para toda función continua $\varepsilon : M \to (0,\infty)$ y toda función Lipschitz $f : M \to \mathbb{R}$, existe una función Lipschitz $g : M \to \mathbb{R}$ de clase C^k (de clase C^1, respectivamente) tal que*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \text{para todo } x \in M, \text{ y} \qquad \operatorname{Lip}(g) \leq C \operatorname{Lip}(f),$$

donde $C \geq 1$ es una constante que depende únicamente de M.

Como corolario inmediato de este teorema obtenemos la siguiente generalización del resultado dado en [6] a toda variedad Riemanniana, pero en este caso obtenemos únicamente aproximación por funciones de clase C^1 y no de clase C^∞.

Corolario 7. *Sea M una variedad Riemanniana. Para todo $r > 0$, toda función continua $\varepsilon : M \to (0,\infty)$ y toda función Lipschitz $f : M \to \mathbb{R}$, existe una función Lipschitz $g : M \to \mathbb{R}$ de clase C^1 tal que*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \text{para todo } x \in M, \text{ y} \qquad \operatorname{Lip}(g) \leq \operatorname{Lip}(f) + r.$$

En la Sección 2 del Capítulo 6 se introduce el concepto de *variedad Finsler C^k-uniformemente mesetable* de forma análoga a la definición en variedades Riemannianas dada en [5]. No es difícil ver que toda variedad Finsler donde se puede aproximar uniformemente funciones Lipschitz por funciones Lipschitz y diferenciables es uniformemente mesetable, por lo que el teorema anterior nos da una gran cantidad de ejemplos de variedades Finsler uniformemente mesetables, entre ellas toda variedad Riemanniana. Además, utilizando las técnicas desarrolladas en [35], [10] y [56] construimos sup-particiones de la unidad en variedades Finsler separables uniformemente mesetables y demostramos la siguiente caracterización:

Teorema 8. *Sea M una variedad Finsler en el sentido de Neeb-Upmeier de clase C^k y separable. Entonces, M es C^k uniformemente mesetable si y sólo si toda función Lipschitz puede ser aproximada por una función de clase C^k con la constante Lipschitz controlada.*

En la última sección de este capítulo se prueban algunos resultados sobre extensión diferenciable en variedades de Banach y extensión diferenciable y Lipschitz en variedades Finsler, que se obtienen utilizando los resultados vistos en el Capítulo 4.

En el Capítulo 7 presentamos algunas aplicaciones de los anteriores resultados. Como ya hemos comentado anteriormente, toda variedad Riemanniana donde se pueden aproximar funciones Lipschitz por funciones Lipschitz y diferenciables es C^1-uniformemente mesetable. Por lo que obtenemos, como consecuencia inmediata del Corolario 7, que toda variedad Riemanniana es C^1-uniformemente mesetable, así contestamos a un problema propuesto en [5], [6] y [42], y concluimos que el principio variacional de Deville-Godefroy-Zizler y el teorema de Myers-Nakai se cumplen para toda variedad Riemanniana completa.

También aplicamos estos resultados, concretamente el teorema del valor medio y el carácter bi-Lipschitz de las cartas, a un problema de derivadas escalares. Recordar que si M y N son espacios métricos y $f : M \to N$ es una aplicación continua, dado $x \in M$ podemos definir la *derivada escalar superior* de f en x como

$$D_x^+ f = \limsup_{\substack{z \to x \\ z \neq x}} \frac{d_N(f(z), f(x))}{d_M(z, x)}.$$

Cuando M y N son espacios de Banach o variedades Riemannianas conexas y completas, y la aplicación es diferenciable en x, la derivada escalar superior en x coincide con la norma de la diferencial de la función en el punto x (ver [70] y [50], respectivamente). O. Gutú y J.A. Jaramillo en [50] prueban que si M y N son variedades Finsler en el sentido de Neeb-Upmeier conexas y completas, y $f : M \to N$ es una aplicación diferenciable en M, entonces $D_x^+ f \leq ||df(x)||_x$ para todo x de M. En la Sección 2 probamos que ambas expresiones son iguales para todo par de variedades Finsler en el sentido de Palais que sean conexas y completas, de este modo contestamos a una cuestión planteada en [50].

En la Sección 3 se generaliza el principio variacional de Deville-Godefroy-Zizler a variedades Finsler uniformemente mesetables. En concreto, obtenemos que si M es una variedad Finsler en el sentido de Neeb-Upmeier débil-uniforme C^1-uniformemente mesetable y completa, dada una función $f : M \to \mathbb{R} \cup \{\infty\}$ inferiormente semicontinua, acotada inferiormente y $f \not\equiv +\infty$, entonces para todo $\varepsilon > 0$ existe una función $\varphi : X \to \mathbb{R}$ Lipschitz de clase C^1 y acotada tal que $f - \varphi$ alcanza su mínimo fuerte en M, $||\varphi||_\infty = \sup_{x \in M} |\varphi(x)| < \varepsilon$ y $||d\varphi||_\infty = \sup_{x \in M} ||d\varphi(x)||_x < \varepsilon$. Este principio variacional es una herramienta muy útil para el desarrollo del análisis no-suave y para encontrar teoremas de existencia y unicidad de soluciones de viscosidad de las ecuaciones de Hamilton-Jacobi (ver [21], [22] y [5]). Por ello, pensamos que éste es un buen punto de partida para comenzar a desarrollar el análisis no-suave y las ecuaciones de Hamilton-Jacobi en variedades Finsler, lo que puede ser una de las futuras aplicaciones de estos resultados.

Otra cuestión que tratamos en el Capítulo 7 es un problema del tipo Banach-Stone, en este caso tratamos de caracterizar la estructura métrica y diferenciable de las variedades Finsler por sus álgebras de funciones diferenciables y Lipschitz. El clásico teorema de Myers-Nakai [94] y [96] asegura que la estructura de una variedad Riemanniana M de dimensión finita está caracterizada por el álgebra de Banach de todas las funciones real-valuadas, acotadas, de clase C^1 y con derivada acotada, dotada de la norma del supremo de la función y de su derivada. Este resultado ha sido extendido por I. Garrido, J.A. Jaramillo y Y.C. Rangel en [42] a variedades Riemannianas infinito dimensionales y completas, siempre que ellas sean C^1-uniformemente mesetables (como anteriormente hemos visto, podemos eliminar esta última condición ya que toda variedad Riemanniana es C^1-uniformemente mesetable). En la última sección de este capítulo, extendemos este resultado a variedades Finsler en el sentido de Palais completas y C^k-uniformemente mesetables. Desafortunadamente, no sabemos si, al igual que en el caso Riemann, la estructura métrica determina la estructura diferenciable, es decir, no sabemos si una isometría métrica entre dos variedades Finsler es un difeomorfismo, lo que en variedades Riemannianas se conoce como teorema de Myers-Steenrod [95]. Por esta razón los resultados que obtenemos son algo más débiles que en el caso Riemanniano. Para intentar solventar este problema trabajamos con las álgebras de funciones $C_b^k(M)$ con $k \in \mathbb{N}$, definidas como las álgebras de funciones real-valuadas, acotadas, de clase C^k y con primera derivada acotada, dotadas de la norma del supremo de la función y de su primera derivada. Así, uno de los resultados principales que obtenemos es el siguiente:

Teorema 9. *Sean M y N dos variedades Finsler en el sentido de Palais de clase C^k, completas y modeladas en espacios de Banach separables con funciones meseta de clase C^k y Lipschitz (espacios de Banach WCG con funciones meseta de clase C^1). Entonces, las álgebras $C_b^k(M)$ y $C_b^k(N)$ ($C_b^1(M)$ y $C_b^1(N)$, respectivamente) son equivalentes como álgebras normadas si y sólo si M y N son C^k-débil equivalentes (C^1-débil equivalentes, respectivamente) como variedades Finsler, es decir, existe una isometría $h : M \to N$ tal que tanto h como h^{-1} son C^k-débil diferenciables. En particular, h es una isometría y un C^{k-1} difeomorfismo.*

Usando las propiedades básicas de las funciones débil diferenciables podemos obtener el teorema de Myers-Nakai, en toda su potencia, para variedades Finsler finito dimensionales.

Corolario 10. *Sean M y N dos variedades Finsler de clase C^k, completas y finito dimensionales. Entonces, las álgebras $C_b^k(M)$ y $C_b^k(N)$ son equivalentes como álgebras normadas si y sólo si M y N son C^k equivalentes como variedades Finsler, es decir, existe un isometría $h : M \to N$ el cual es un difeomorfismo de clase C^k.*

Para terminar, obtenemos una interesante aplicación de las variedades Finsler a los espacios de Banach.

Corolario 11. *Sean X e Y dos espacios de Banach WCG con funciones meseta de clase C^1. Entonces, X e Y son isométricos si y sólo si $C_b^1(X)$ y $C_b^1(Y)$ son equivalentes como álgebras normadas.*

Los resultados del Capítulo 5, Capítulo 6 y de la primera parte del Capítulo 7 han sido publicados en [66]. La caracterización de las variedades Finsler en términos de sus álgebras de funciones ha sido recogida en el trabajo [63]. Por último, indicar que algunas de las herramientas y técnicas desarrolladas en estos capítulos ya han sido utilizadas en [64] para obtener resultados de inversión global de aplicaciones no-suaves en variedades Finsler de dimensión finita.

Al final de cada capítulo incluimos una breve sección de problemas abiertos, la cual contiene algunas cuestiones en el área que siguen - a nuestro saber - abiertas.

Chapter 1

Introduction

1.1 Introduction

The main topic of this thesis is the differential theory on Banach spaces and on Banach-Finsler manifolds. We deal here with several different questions which are interrelated, as we will see.

The first problem we address is to obtain a characterization of separable Banach spaces that admit a smooth bump function which locally depends on finitely many coordinates. Let us recall that a *bump function* on a Banach space X is a real-valued function $f : X \to \mathbb{R}$ such that it has non-empty and bounded support, i.e. there exists an element $x \in X$ such that $f(x) \neq 0$ and the closure of $\{x \in X : f(x) \neq 0\}$ is bounded. It is well known that the existence of a bump function with good properties implies many geometrical benefits on the Banach space, and it provides several tools that we do not have in other Banach spaces. A good example of this fact is the existence of a smooth bump function on a Banach space, which implies that the Banach space is Asplund and admits smooth partitions of unity. We are interested in the following property: a function $f : X \to \mathbb{R}$ defined on a Banach space, *locally depends on finitely many coordinates* (*LFC*, for short) whenever it can locally be factorized through finite-dimensional Banach spaces, i.e. if for every $x \in X$ there are a neighborhood U of x, a finite subset $\{f_1, \ldots, f_n\} \subset X^*$ and a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f(y) = g(f_1(y), \ldots, f_n(y))$ for every $y \in U$.

Kuiper was the first to use the concept of LFC function in order to obtain a C^∞-smooth and LFC equivalent norm on c_0 (the Kuiper's construction appeared in [15]). Over the last several decades, this notion has been used successfully in renorming of Banach spaces, in fact, it is one of the most common tools to deal with this class of problems. One of the most important applications of LFC functions is the use of C^∞-smooth, LFC bump functions on $c_0(\Gamma)$ in the construction of C^k-smooth partitions of unity in reflexive Banach spaces admitting a C^k-smooth bump, due to H. Toruńczyk [108]. Recently, P. Hájek and M. Johanis in [55] and [56] have used the concept of LFC function to obtain Lipschitz and smooth sup-partitions of unity on a certain class of non-separable Banach spaces.

The existence of a continuous, LFC bump function provides several extra-properties in the Banach spaces. In the paper [101] J. Pechanec, J.H.M. Whitfield and V. Zizler explicitly introduce the LFC notion and show that every Banach space admitting an LFC bump function is saturated with copies of c_0. A few years later, M. Fabian and V. Zizler [26] prove that these spaces must also be Asplund. However, not every Asplund, c_0-saturated Banach space admits an LFC bump function (see [69]).

The question we address in Chapter 2 is motivated by the papers [51] and [54]. In the first one, the author shows that a separable Banach space admits a C^∞-smooth and LFC norm provided that it admits an LFC norm. We can therefore wonder whether the same property holds when LFC norm is replaced with LFC bump. An important step in this direction is given in the second paper, in which a positive answer is provided as long as the Banach space admits a Schauder basis. In this chapter we present an extension of this result and establish a characterization of the class of separable Banach spaces which admit a C^∞-smooth and LFC bump function as those that admit a continuous and LFC bump function. This result answers a problem posed in [54], [57] and [26]. Precisely, the main result in Chapter 2 is the following:

Theorem 1.1.1. *Let X be a separable Banach space. Then, X admits a continuous and LFC bump function if and only if X admits a C^∞-smooth and LFC bump function.*

This work has been published in [65]. The proof of this result relies on a previous lemma, which, roughly speaking, obtains a factorization of an LFC function in the finite union of some neighborhoods where the LFC function is locally factorized, by a suitable composition through the space c_0. In other words, given a function $f : X \to \mathbb{R}$ such that $f = g_1(f_1^1, \dots, f_{n_1}^1)$ on $B(x_1, 2r_1)$ and $f = g_2(f_1^2, \dots, f_{n_2}^2)$ on $B(x_2, 2r_2)$ with g_1, g_2 continuous functions and $f_1^1, \dots, f_{n_1}^1, f_1^2, \dots, f_{n_2}^2 \in X^*$, then there exist a bounded and linear operator $T : X \to c_0(\mathbb{N})$ and a continuous function $g : c_0(\mathbb{N}) \to \mathbb{R}$ such that $f(x) = g(T(x))$ on $B(x_1, r_1) \cup B(x_2, r_2)$.

Furthermore, in Section 3 of Chapter 2, we generalize the LFC notion to functions which *locally factorize by Banach spaces* that belong to a fixed class of Banach spaces $\mathcal{F}$ (*LF-$\mathcal{F}$*, for short), i.e. for every $x \in X$ there are a neighborhood U of x, a Banach space $E \in \mathcal{F}$, a bounded and linear operator $T : U \to E$ and a continuous function $g : E \to \mathbb{R}$ such that $f(y) = g(T(y))$ on U. In this context, we obtain a similar theorem to the above one:

Theorem 1.1.2. *Let $\mathcal{F}$ be a family of separable Banach spaces and let X be a Banach space which admits a separable dual and a continuous and LF-$\mathcal{F}$ bump function. Assume that every $E \in \mathcal{F}$ admits a bump function b which is C^k-smooth (C^k-smooth and LFC). Then X admits a C^k-smooth bump function (respectively, C^k-smooth and LFC).*

In Chapter 3 we give a brief survey of recent results concerning smooth and Lipschitz approximation of Lipschitz functions. In any context, it is natural to wonder whether a function can be approximated by another one with better properties. It is well known that in finite-dimensional case, every Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ can be uniformly approximated by C^∞-smooth and Lipschitz functions considering the integral convolutions

$$f_n(x) = \int_{\mathbb{R}^n} f(y)\varphi_n(y - x)dy,$$

where φ_n are C^∞-smooth functions on $\mathbb{R}^n$ such that satisfy $\int_{\mathbb{R}^n} \varphi_n = 1$ and $\operatorname{supp} \varphi_n \subset B(0, 1/n)$. Unfortunately, the integral convolution method cannot be used in infinite-dimensional Banach spaces, due to the lack of a measure like Lebesgue's measure.

The approximation of continuous functions by smooth or analytic functions has been extensively studied in the last century (see [83], [15], [108], [48], [34], [17], [22], [40], [90], etc). In Chapter 3 we focus on some results of uniformly approximation of Lipschitz functions by smooth and Lipschitz functions, i.e. given a Lipschitz function $f : X \to \mathbb{R}$ defined on a Banach space and $\varepsilon > 0$, when there exists a C^k-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that $|f(x)-g(x)| < \varepsilon$ on X and $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$, where $C_0 \geq 1$ is a constant which only depends on the Banach space. J.M. Lasry and P.L. Lions [85] obtained the first result in this directions. They used inf-sup-convolution in order to show that every Lipschitz function defined on a Hilbert space can be uniformly approximated by C^1-smooth and Lipschitz functions with the same Lipschitz constants. For more information on inf-sup-convolution techniques, see [4], [106], [107] and [16].

In contrast to smooth approximation, we cannot use smooth partitions of unity to obtain smooth and Lipschitz approximation, since the size of the sum of their derivatives are not controlled, despite all involved functions are Lipschitz. In order to deal with this problem, R. Fry in [35] constructs a family of smooth and Lipschitz functions with uniformly bounded Lipschitz constants in all Banach space with separable dual. This family of functions satisfies the same properties as a partition of unity, except that the supremum of the functions evaluated in any point must be equal to one instead of their sum. For that reason they are called *sup-partitions of unity* some time later. Using sup-partitions of unity techniques we are able to approximate Lipschitz functions $f : X \to \mathbb{R}$ by smooth and Lipschitz functions $g : X \to \mathbb{R}$ with controlled Lipschitz constant, i.e. $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$, where $C_0 \geq 1$ is a constant that depends only on the Banach space X (see [35], [10] and [56]). Recently, these techniques have been used to uniformly approximate Lipschitz functions by Lipschitz and analytic functions in a certain class of Banach spaces (see [39], [8] and [9]).

In this chapter we also present some slightly improvements in the smooth and Lipschitz approximation results, the main one is that the constant $C_0 \geq 1$ can be obtained to be independent of the Banach space provided that its dual space is separable. In fact, we prove that the constant can be chosen less or equal to $4 + r$ for any $r > 0$. Finally, let us note that smooth and Lipschitz approximation of Lipschitz functions is strongly related to C^1-fine approximation, i.e. the problem of uniformly approximating both a C^1-smooth function and its derivative by functions with a higher order of differentiability (see [92], [11] and [56]). This chapter and its results are key to developing the rest of the thesis.

In Chapter 4 we consider a smooth extension question, which is one of the main subjects of this thesis. We can always wonder whether a function defined on a subset with a nice property can be extended to a bigger subset and the extension function keeps the same property. Two well known examples are: the Tiezte extension Theorem, which states that X is a normal space if and only if for every continuous function f defined on a closed subset of X, there exists a continuous extension of f to X; and the Hahn-Banach Theorem, which extends every bounded and linear mapping defined on a closed subspace of a Banach space to a bounded and linear mapping defined on the whole Banach space with the same norm. We can ask whether these results hold for smooth functions. In other words, if X is a Banach space, when a function defined on a closed subset of X is the restriction of a smooth function defined on X, and when a smooth function defined on a closed subspace of X can be extended to the whole Banach space by a smooth function.

Let us notice that for every C^k-smooth function defined on a complement subspace there exists a C^k-smooth extension defined on the whole space. Indeed, the C^k-smooth extension function is the composition of the original function with the bounded and linear projection of the subspace. J. Lindenstrauss and L Tzafriri showed in [89] that the only Banach space all of whose closed subspaces are complement is isomorphic to a Hilbert space. Thus, the smooth extension problem from closed subspaces is only interesting when we consider infinite-dimensional Banach spaces which are not Hilbert.

The smooth extension problem from closed subsets of finite-dimensional spaces has been exhaustively studied. H. Whitney gave a first answer in [111] and [112] when $X = \mathbb{R}$. In these papers necessary and sufficient conditions of the functions defined on closed subsets of $\mathbb{R}$ (in terms of its divided differences) are obtained for the existence of C^k-smooth extensions to $\mathbb{R}$. The $\mathbb{R}^n$ case was solved by G. Glaeser [47] when C^1-smooth functions are considered. Finally, C. Fefferman in a series of papers ([29] and [30]) establishes a characterization of the functions that are the restriction on a compact subset of a C^k-smooth function on $\mathbb{R}^n$, for all $k, n \geq 1$.

For infinite-dimensional Banach spaces, the question has been much less studied. On the one hand, C.J. Atkin in [3] extends every smooth function f defined on a finite union of open convex sets in a separable Banach space which does not admit smooth bump functions, provided that for every point in the domain of f, the restriction of f to a suitable neighborhood of the point can be extended to the whole space. On the other hand, D. Azagra, R. Fry and L. Keener in [7] show that every C^1-smooth function defined on a closed subspace of a Banach space with separable dual can be extended to a C^1-smooth function defined on the whole space. That is our starting point.

In this chapter, the problem of the smooth extension of real-valued functions and vector-valued functions from closed subsets of Banach spaces is addressed. We find a necessary condition for a mapping defined on a closed subset to admit a C^1-smooth extension to the whole space and discuss in which Banach spaces that condition is also sufficient. The condition that is considered in a pair of Banach spaces (X, Z), where X is the domain and Z is the target space, is a smooth and Lipschitz approximation property. More precisely, the pair of Banach spaces (X, Z) is said to satisfy *property (*)* if there is a constant $C_0 \geq 1$, which depends only on X and Z, such that for every subset $A \subset X$, every Lipschitz mapping $f : A \to Z$ and every $\varepsilon > 0$, there is a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that

$$||f(x) - g(x)|| < \varepsilon \qquad \text{for all } x \in A, \text{ and} \qquad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

The main result in Chapter 4 is the following:

Theorem 1.1.3. *Let (X, Z) be a pair of Banach spaces with property (*), A a closed subset of X and let $f : A \to Z$ be a mapping. There is a C^1-smooth extension of f to the whole space X if and only if there is a continuous map $D : A \to \mathcal{L}(X, Z)$ such that for every $y \in A$ and $\varepsilon > 0$, there exists $r > 0$ such that*

$$||f(z) - f(w) - D(y)(z - w)|| \leq \varepsilon ||z - w||, \qquad \text{for all } z, w \in A \cap B(y, r). \tag{1.1}$$

As we see in the above theorem, the smooth extension of mappings satisfying inequality (1.1) for a continuous map D is closely related to the smooth and Lipschitz approximation of Lipschitz mappings. In fact, we obtain the following characterization in separable Banach spaces:

Corollary 1.1.4. *Let X be a separable Banach space. The following statements are equivalent:*

(i) The pair $(X, \mathbb{R})$ satisfies property (*), i.e. there is a constant $C_0 \geq 1$ such that every real-valued Lipschitz function defined on a subset of X can be uniformly approximated by a C^1-smooth and Lipschitz function defined on X with $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$.

(ii) For every real-valued function defined in a closed subset $A \subset X$ which satisfies inequality (1.1) on A for a continuous map $D : A \to X^*$, there exists a C^1-smooth extension to the whole space X.

(iii) The dual space X^* is separable.

In Section 3 of Chapter 4, we list pairs of Banach spaces with property (*), thus pairs of Banach spaces satisfying Theorem 1.1.3. It is worth noticing that property (*) is exactly the smooth and Lipschitz approximation of Lipschitz functions studied in Chapter 3 when $Z = \mathbb{R}$.

In Section 4 of this chapter we deal with the problem of the smooth extension of mappings defined on closed subspaces. We need to add an extra-condition in the pair of Banach spaces (X, Z). The pair of Banach spaces (X, Z) is said to satisfy the *linear extension property* if there is $\lambda \geq 1$, which depends only on X and Z, such that for every closed subspace $Y \subset X$ and every bounded and linear operator $T : Y \to Z$, there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|Y} = T$ and $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda ||T||_{\mathcal{L}(Y,Z)}$. We prove that if a pair of Banach spaces satisfies the linear extension property, then every C^1-smooth mapping $f : Y \to Z$ satisfies inequality (1.1) for a continuous map D. Hence, Theorem 1.1.3 can be applied and it is obtained the following corollary:

Corollary 1.1.5. *Let (X, Z) be a pair of Banach spaces which satisfies property (*) and the linear extension property. Then for every C^1-smooth mapping $f : Y \to Z$ defined on a closed subspace of X there is a C^1-smooth extension to the whole space X.*

It is important to note that the linear extension property is the well known Hahn-Banach theorem when $Z = \mathbb{R}$. Thus, this result is an extension of the given one in [7] to both vector-valued functions and mappings defined on a certain class of non-separable Banach spaces. In this section we also list pairs of Banach spaces which satisfy both properties.

In addiction, we study both smooth extension and smooth and Lipschitz extension, throughout the chapter. In fact, we show that property (*) is equivalent to the following statement: for every Lipschitz mapping which satisfies inequality (1.1) for a continuous and bounded map D, there is a C^1-smooth and Lipschitz extension to the whole space with controlled Lipschitz constant. In this direction, the linear extension property is also necessary to obtain C^1-smooth and Lipschitz extensions (with controlled Lipschitz constant) of C^1-smooth and Lipschitz mappings defined on closed subspaces. These results have been collected in the works [67] and [68].

Apart from smooth problems on Banach spaces, the study of Banach-Finsler manifolds is one of the main subject of this thesis. Thus, in Chapter 5, 6 and 7 several properties of this class of manifolds are presented.

A Banach-Finsler manifold (from now on, Finsler manifold) is a smooth manifold modeled on a Banach space such that there exists a continuous map which assigns a norm on the tangent space (which is isomorphic to the Banach space where the manifold is modeled) to each point at the manifold. Using this structure, we can define the length of paths on the manifold in the usual way, and, therefore, the associated Finsler metric which is consistent with the topology given in the manifold. On the one hand, R.S. Palais introduced a certain class of Finsler manifolds in [100]. Precisely, he studied those Finsler manifolds whose tangent spaces locally have $(1+\varepsilon)$-equivalent norms for every $\varepsilon > 0$. Those manifolds are the so-called *Finsler manifolds in the sense of Palais* and they generalize Riemann manifolds. On the other hand, H. Upmeier in [110] and K.H. Neeb in [98] considered a weaker condition. They assume that for every point at the manifold there exist a neighborhood of the point and a constant $K \geq 1$ such that the norms of the tangent spaces are K-equivalent in that neighborhood. Those manifolds are the so-called *Finsler manifolds in the sense of Neeb-Upmeier*.

In Chapter 5 we introduce two new classes of manifolds, *Finsler manifolds in the sense of Neeb-Upmeier weak-uniform* and *in the sense of Neeb-Upmeier uniform*. They are generalizations of Finsler manifolds in the sense of Palais and Riemannian manifolds, respectively, but their properties are better than those in Finsler manifolds in the sense of Neeb-Upmeier. It is worth noting that these four concepts of manifolds are equivalent whenever they are finite-dimensional manifolds. In this chapter, several properties of these classes of manifolds and the relations between them are studied. Some results related to mean value inequalities are also provided, and a result of the existence of suitable local bi-Lipschitz charts is given as an essential tool to obtain the results in the next chapters.

In Chapter 6 we address some problems on smooth and Lipschitz approximation, smooth extension and smooth and Lipschitz extension in the context of manifolds. The study of smooth and Lipschitz approximation of Lipschitz functions is motivated by the paper [6], where D. Azagra, J. Ferrera, F. López-Mesas and Y.C. Rangel show that every Lipschitz function defined on a separable Riemannian manifold can be uniformly approximated by a C^∞-smooth and Lipschitz function with almost the same Lipschitz constant. It is not difficult to see that this fact implies that every separable Riemannian manifold is C^∞*-uniformly bumpable* (this concept is introduced in [5]), i.e. there are $r > 0$ and $R > 1$ such that for every $p \in M$ and $\delta \in (0, r)$, there exists a C^∞-smooth function $b : M \to [0, 1]$ such that $b(p) = 1$, $b(x) = 0$ whenever $d_M(x, p) \geq \delta$, and $\sup_{x\in M} ||db(x)||_x \leq R/\delta$. The uniformly bumpable character of a Riemannian manifold provides nice applications, as can be seen in [5] and [42]. In the first paper, the authors prove the Deville-Godefroy-Zizler smooth variational principle for every complete Riemannian manifold provided that the manifold is C^1-uniformly bumpable. This smooth variational principle is key in non-smooth analysis and to obtain viscosity solutions to Hamilton-Jacobi equations. In the second paper, an infinite-dimensional Myers-Nakai theorem is stated for complete, C^1-uniformly bumpable and separable Riemannian manifolds. These issues will be discussed again in the relevant sections below.

Hence, two natural questions that arise are whether the same approximation result holds in non-separable Riemannian manifolds, and more generally in Finsler manifolds. We consider the Finsler manifold setting, so we give a positive answer to both questions at once in Section 1 of Chapter 6. One of the main results that we prove is the following:

Theorem 1.1.6. *Let M be a C^k Finsler manifold in the sense of Neeb-Upmeier weak-uniform modeled on a separable Banach space which admits a C^k-smooth and Lipschitz bump function (M is modeled on a non-separable Hilbert space). Then for every Lipschitz function $f : M \to \mathbb{R}$ and every continuous function $\varepsilon : M \to (0, \infty)$, there exists a C^k-smooth (respectively, C^1-*

smooth) and Lipschitz function $g : M \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon(x) \qquad \textit{for every } x \in M, \textit{ and} \qquad \operatorname{Lip}(g) \le C \operatorname{Lip}(f),$$

where $C \ge 1$ depends only on M.

The given results provide a generalization to the non-separable setting of the approximation result given in [6] for separable Riemannian manifolds, but the function g which approximates f can only be chosen to be C^1-smooth. Recall that for separable Riemannian manifolds, the Lipschitz function g that approximates f can be obtained to be C^∞-smooth.

Corollary 1.1.7. *Let M be a Riemannian manifold. For every Lipschitz function $f : M \to \mathbb{R}$, every continuous function $\varepsilon : M \to (0, \infty)$ and $r > 0$, there exists a C^1-smooth and Lipschitz function $g : M \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \textit{for every } x \in M, \textit{ and} \qquad \operatorname{Lip}(g) \le \operatorname{Lip}(f) + r.$$

The concept of *C^k-uniformly bumpable Finsler manifold* is introduced in the same way as the Riemannian case [5] in Section 2 of Chapter 3. It can be proved that every Finsler manifold is uniformly bumpable whenever every Lipschitz function can be uniformly approximated by a smooth and Lipschitz function. Hence, the above theorem gives us many examples of Finsler manifolds which are uniformly bumpable, among them every Riemannian manifold. Furthermore, sup-partitions of unity are constructed in every separable and uniformly bumpable Finsler manifold, along the same lines as [35], [10] and [56], and we obtain the following characterization:

Theorem 1.1.8. *Let M be a separable C^k Finsler manifold in the sense of Neeb-Upmeier. Then, M is C^k-uniformly bumpable if and only if every real-valued Lipschitz function defined on M can be uniformly approximated by a C^k-smooth and Lipschitz function with controlled Lipschitz constant.*

In the last section of the chapter some results about smooth extension on Banach manifolds and smooth and Lipschitz extension on Finsler manifolds are stated. For that purpose we use the theorems obtained in Chapter 4.

In Chapter 7, several applications of these results are presented. As we already know, a Riemannian manifold is C^1-uniformly bumpable provided that every Lipschitz function can be uniformly approximated by a C^1-smooth and Lipschitz function. Thus, Corollary 1.1.7 implies that every Riemannian manifold is C^1-uniformly bumpable, which answers a question posed in [5], [6] and [42]. Hence, the Devile-Godefroy-Zizler smooth variational principle and the infinite-dimensional version of the Myers-Nakai theorem hold for every complete Riemannian manifold.

We also use the developed tools, especially the mean value inequalities and the existence of suitable local bi-Lipschitz charts, in order to solve a question of scalar derivative. Recall that if M and N are metric spaces and $f : M \to N$ is a continuous mapping, for every $x \in M$ the *upper scalar derivative* of f at x is defined as

$$D_x^+ f = \limsup_{\substack{z \to x \\ z \neq x}} \frac{d_N(f(z), f(x))}{d_M(z, x)}.$$

When M and N are either Banach spaces or connected and complete Riemannian manifolds, and $f : M \to N$ is a smooth mapping at x, then the scalar derivative of f at x is the norm of the differential of the mapping at x (see [70] and [50], respectively). O. Gutú and J.A. Jaramillo showed in [50] that if M and N are connected and complete Finsler manifolds in the sense of Neeb-Upmeier, and $f : M \to N$ is a smooth mapping in M, then $D_x^+ f \leq ||df(x)||_x$ for every x in M. We obtain that actually both values are equal, whenever M and N are connected and complete Finsler manifolds in the sense of Palais, so we answer a question set out in [50].

In Section 3 we extend the Deville-Godefroy-Zizler smooth variational principle to uniformly bumpable Finsler manifolds. In particular, we show that if M is a complete and C^1-uniformly bumpable C^1 Finsler manifold in the sense of Neeb-Upmeier weak-uniform and $f : M \to \mathbb{R} \cup \{\infty\}$ is a lower semicontinuous function which is bounded below and $f \not\equiv +\infty$, then for any $\varepsilon > 0$ there is a bounded, C^1-smooth and Lipschitz function $\varphi : X \to \mathbb{R}$ such that $f - \varphi$ attains its strong minimum in M, $||\varphi||_\infty = \sup_{x \in M} |\varphi(x)| < \varepsilon$ and $||d\varphi||_\infty = \sup_{x \in M} ||d\varphi(x)||_x < \varepsilon$. That variational principle is a key tool to develop non-smooth analysis and in order to prove theorems of existence and uniqueness of viscosity solutions to Hamilton-Jacobi equations (see [21], [22] and [5]). For this reason, we think it could be a starting point for the development of non-smooth analysis and Hamilton-Jacobi equations on Finsler manifolds, which may be one of the future applications of these results.

A different question, which we consider in Chapter 7, is a Banach-Stone type theorem. In this case we prove that the metric and smooth structure of a Finsler manifold is characterized in terms of its Banach algebra of smooth and Lipschitz functions. The Myers-Nakai theorem [94] and [96] states that a finite-dimensional Riemannian manifold M is characterized by its Banach algebra of all real-valued, bounded and C^1-smooth functions with bounded derivative defined on M endowed with the sup-norm of the function and its derivative. This result has been extended by I. Garrido, J.A. Jaramillo and Y.C. Rangel in [42] to infinite-dimensional, complete Riemannian manifolds whenever they are C^1-uniformly bumpable (as we saw already, every Riemannian manifold is C^1-uniformly bumpable, so we can remove this condition). Our aim in the last section of this chapter is to find an extension of this result to complete, C^k-uniformly bumpable Finsler manifolds in the sense of Palais. Unfortunately, the metric structure of a Finsler manifold does not determinate the smooth structure, up to our knowledge, as in the Riemannian case. That is to say, we cannot assure that every (metric) isometry between two Finsler manifolds is a diffeomorphism, that fact holds in Riemannian manifolds and it is the so-called Myers-Steenrod theorem [95]. For that reason, we obtain a slightly weaker version of the Myers-Nakai theorem. To overcome this shortcoming we study the algebras $C_b^k(M)$, with $k \in \mathbb{N}$, of all real-valued, bounded and C^k-smooth functions with bounded first derivative endowed with the sup-norm of the function and its derivative. Thus, one of the main results in this chapter is the following:

Theorem 1.1.9. *Let M and N be complete C^k Finsler manifolds in the sense of Palais modeled on separable Banach spaces that admit C^k-smooth and Lipschitz bump functions (modeled on WCG Banach spaces that admit C^1-smooth and Lipschitz bump functions). Then, the algebras $C_b^k(M)$ and $C_b^k(N)$ (respectively, $C_b^1(M)$ and $C_b^1(N)$) are equivalent as normed algebras if and only if M and N are weakly C^k equivalent as Finsler manifolds (respectively, weakly C^1 equivalent), i.e. there exists an isometry $h : M \to N$ such that h and h^{-1} are weakly C^k-smooth. In particular, h is an isometry and a C^{k-1}-diffeomorphism.*

We obtain a stronger version in complete, finite-dimensional Finsler manifolds, since the concept of weakly smoothness coincides with the concept of smoothness in this context. Thus, we recover the Myers-Nakai theorem for that class of manifolds.

Corollary 1.1.10. *Let M and N be complete, finite-dimensional C^k Finsler manifolds. Then, the algebras $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras if and only if there exists an isometry $h : M \to N$ which is a C^k-diffeomorphism.*

Finally, we obtain an interesting application of Finsler manifolds to Banach spaces.

Corollary 1.1.11. *Let X and Y be WCG Banach spaces which admit C^1-smooth bump functions. Then, X and Y are isometric if and only if $C_b^1(X)$ and $C_b^1(Y)$ are equivalent as normed alegras.*

The results presented in Chapter 5, Chapter 6 and the first part of Chapter 7 have been published in [66]. The characterization of a Finsler manifold in terms of its algebras of smooth functions has been collected in [63]. Lastly, let us indicate that some of the tools developed in these chapters have already been used in order to obtain global inversion of non-smooth mappings in finite-dimensional Finsler manifolds [64].

Each chapter ends with a brief section of Open Problems, containing some questions in the area that are - to our best knowledge - open.

1.2 Notation

Our notation is standard and we shall follow, whenever possible, the notations given in the textbooks [22], [25] and [84]. Along this work, Banach spaces are considered over the real numbers and smoothness is meant in the Fréchet sense. Let us denote by $\mathbb{N}$ the set of natural numbers.

If $(X, ||\cdot||)$ is a Banach space, then its dual is denoted by $(X^*, ||\cdot||^*)$. In some cases, when this notation can cause confusion, the norms of X and X^* will be denoted by $||\cdot||_X$ and $||\cdot||_{X^*}$, respectively. For $x \in X$ and $r > 0$, we denote the open ball of radius r centered at x by $B(x,r) = \{y \in X : ||x-y|| < r\}$, analogously, we denote $\overline{B}(x,r) = \{y \in X : ||x-y|| \leq r\}$ the closed ball of radius r centered at x, and $S(x,r) = \{y \in X : ||x-y|| = r\}$ the sphere of radius r centered at x. When there is a risk of confusion, we denote by $B_X(x,r)$ and $\overline{B}_X(x,r)$ the open ball and closed ball, respectively, of radius r centered at x of X. Moreover, we denote $B_X = \overline{B}_X(0,1)$ the closed unit ball of X.

If $M \subset X$, then $\text{span}(M)$ stands for the linear hull of M, that is, the smallest (in the sense of inclusion) linear subspace of X containing M. Similarly $\overline{\text{span}(M)}$ stands for the closed linear hull of M; the convex hull of M will be denoted by $\text{co}(M)$, and $\overline{\text{co}}(M)$ denotes the closed convex hull of M. The *annihilator* of M is defined as $M^{\perp} = \{f \in X^* : f(y) = 0 \text{ for all } y \in M\}$.

A Banach space X is called an *Asplund space* if every subspace separable Y of X has a separable dual.

A Banach space X is said to be *weakly compactly generated* (WCG) if there exists a weakly compact set $K \subset X$ with $\overline{\text{span}}(K) = X$. This class of Banach spaces includes the separable and reflexive Banach spaces.

If X and Z are Banach spaces, let us denote by $\mathcal{L}(X,Z)$ the space of all bounded and linear maps from the Banach space X to the Banach space Z; and by $\text{Isom}(X,Z)$ the space of all isomorphisms from X to Z. If $f : X \to Z$ is a differentiable mapping, we will denote the Fréchet derivative of f at $x \in X$ in the direction $h \in X$ by $f'(x)(h)$. The *support* of a mapping $f : X \to Z$ is defined by

$$\text{supp}(f) := \overline{\{x \in x : f(x) \neq 0\}}.$$

A norm $||\cdot||$ of X is said to be differentiable on X, if it is differentiable away from the origin. A *bump function* b on X is a function $b : X \to \mathbb{R}$ such that $\text{supp}(b)$ is nonempty and bounded.

Explicitly and implicitly, the following useful characterization will be used in different occasions (see [22]):

Theorem 1. *Let X be a separable Banach space. Then the following statements are equivalent:*

- *X admits a C^1-smooth norm.*
- *X admits a C^1-smooth bump function.*
- *X^* is separable.*

Let (M,d) be a metric space. As in the Banach case, we denote by $B_M(x,r)$, $\overline{B_M}(x,r)$ and $S_M(x,r)$ the open ball, closed ball and sphere in M of radius r centered at x, respectively.

The *distance* between two sets A and B, and the *distance* between a point $x \in M$ and a set A are defined by

$$\operatorname{dist}(A,B) := \inf\{d(a,b) : a \in A,\ b \in B\} \qquad \text{and} \qquad \operatorname{dist}(x,A) := \inf\{d(x,a) : a \in A\}.$$

Let (M, d_M) and (N, d_N) be metric spaces. A mapping $f : M \to N$ is *Lipschitz* if there exists a constant $L > 0$ such that

$$d_N(f(x), f(y)) \leq L d_M(x,y), \qquad \text{for all } x, y \in M.$$

We denote the Lipschitz constant of f by

$$\operatorname{Lip}(f) := \sup_{\substack{x,y \in M \\ x \neq y}} \frac{d_N(f(x), f(y))}{d_M(x,y)}.$$

A function is said to be *bi-Lipschitz* if it is Lipschitz and admits a Lipschitz inverse.

Chapter 2

LFC bumps on separable Banach spaces

The notion of an LFC function (a function that locally depends on finitely many coordinates) was introduced by J. Pechanec, J.H.M. Whitfield and V. Zizler in [101], where they showed that every Banach space which admits an LFC bump is saturated with copies of c_0. Nonetheless, the first use of LFC in the literature is the Kuiper's construction (which appeared in [15]) of a C^∞-smooth, LFC equivalent norm on c_0. The LFC notion has been exploited in many times. One of the most important applications is the use of C^∞-smooth, LFC bumps on $c_0(\Gamma)$ in the construction of C^k-smooth partitions of unity in a certain class of non-separable Banach spaces admitting a C^k-smooth bump function, due to H. Toruńczyk [108]. Since it is easier to check the differentiability properties of functions defined on finite-dimensional spaces, the notion of LFC has been successfully used (implicitly and explicitly) in a large number of papers.

The existence of an LFC bump on the space implies additional properties: It was proved in [26] (see also [69]) that it is an Asplund space, and, as we have already said at the beginning of the introduction, it was proved in [101] (see also [69]) that it is saturated with copies of c_0. However, not every Asplund, c_0-saturated space admits an LFC bump function [69].

The LFC notion is closely related to the class of polyhedral Banach spaces (introduced by V. Klee [81]; see [[71], Chapter 15] for results and references). P. Fonf [32] proved that every polyhedral Banach space is saturated with copies of c_0. P. Fonf [33] characterized separable polyhedral Banach spaces as those Banach spaces admitting an equivalent LFC norm. Later, P. Hájek [51] characterized them as those admitting an equivalent C^∞-smooth and LFC norm. More specifically, he proved that every Banach space with an LFC norm admits a C^∞-smooth and LFC norm. It remains an open problem whether every separable Banach space with a C^∞-smooth LFC bump is a polyhedral Banach space. P. Hájek and M. Johanis conjectured that the answer is negative [53]. They constructed an Orlicz space admitting a C^∞-smooth LFC bump and not satisfying Leung's sufficient condition on polyhedrality [86].

P. Hájek and M. Johanis proved in [54] that every separable Banach space with a Schauder basis and a continuous LFC bump, admits a C^∞-smooth and LFC bump function. The main result of this chapter extends the result of [54] and establishes a characterization of the class of separable Banach spaces admitting a continuous, LFC bump as those separable Banach spaces with a C^∞-smooth, LFC bump. This result answers a problem posed in [54], [57] and [26].

In Section 2.1 we introduce the concept of LFC function and some of their most important properties. Furthermore, we recall some geometric implications in the Banach spaces that admit a continuous, LFC bump function. In Section 2.2 we prove Theorem 2.2.2 which is the main result in this chapter and assures that a separable Banach space admits a C^∞-smooth, LFC bump function whenever it admits a continuous, LFC bump function. The proof of this result is supported in Lemma 2.2.1, which develops a technique of "join together" a finite family of neighborhoods, where the LFC function is locally factorized, in order to obtain a new factorization of the function on the union of these neighborhoods through the space c_0. In Section 2.3, we consider LF functions, a generalization of the concept of LFC function. An LF function is a function that is locally factorized through specific (finite or infinite-dimensional) Banach spaces. We give a class of natural examples of Banach spaces with LF norms and LF bump functions. We also see the relationship between LF functions and functions that locally depend on countably many coordinates, a concept introduced by M. Fabian and V. Zizler in [27], and prove a similar (and more general) result to Theorem 2.2.2. We finish this chapter by recalling some problems in this area which remain open.

2.1 Functions that Locally depend on Finitely many Coordinates

The notion of a function that locally depends on finitely many coordinates was first defined on Banach spaces with Schauder basis using the coordinate functionals [101]. Later, a generalization of this notion was considered by some authors using arbitrary continuous linear functionals.

Definition 2.1.1. *Let X be a Banach space, $A \subset X$ an open subset, E an arbitrary set, $M \subset X^*$ and a mapping $b : A \longrightarrow E$.*

*(a) We say that b **depends only on** M on a subset $U \subset A$ if $b(x) = b(y)$ whenever $x, y \in U$ are such that $f(x) = f(y)$ for all $f \in M$. If $M = \{f_1, ..., f_n\}$, this is equivalent to the existence of a mapping $g : \mathbb{R}^n \longrightarrow E$ such that $b(x) = g(f_1(x), ..., f_n(x))$ for all $x \in U$.*

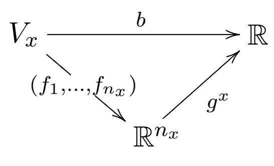
*(b) We say that b **locally depends on finitely many coordinates from** M (LFC-M, for short) if for each $x \in A$ there are a neighborhood $U_x \subset A$ of x and a finite subset $F_x \subset M$ such that b depends only on F_x on U_x. We say that b **locally depends on finitely many coordinates** (LFC, for short) if it is LFC-X^*.*

(c) A norm is said to be LFC, if it is LFC away from the origin.

A simple example is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin (where $\{e_i^*\}$ are the coordinate functionals in c_0). Indeed, for every $x \in c_0$, $x \neq 0$, there exists $n \in \mathbb{N}$ such that $|x(i)| < ||x||_\infty/2$ for every $i \geq n$. Then the norm $||\cdot||_\infty$ depends only on $\{e_1^*, \dots, e_n^*\}$ on $B(x, \|x\|_\infty/4)$.

Roughly speaking, we see a continuous LFC function as a continuous function locally factorized through finite-dimensional spaces.

Fact 2.1.2. *Let X be a Banach space, $A \subset X$ an open subset, $M \subset X^*$ and $k = 0, 1, \dots, \infty$. A C^k-smooth function $b : A \longrightarrow \mathbb{R}$ is LFC-M if and only if for every $x \in A$, there are a neighborhood $V_x \subset A$ of x, a finite subset $\{f_1, \dots, f_{n_x}\} \subset M$ and a C^k-smooth function $g^x : \mathbb{R}^{n_x} \longrightarrow \mathbb{R}$ such that $g^x(f_1(y), \dots, f_{n_x}(y)) = b(y)$ for every $y \in V_x$.*



Proof. From the definition of LFC-M, there are for every $x \in A$, a closed ball $\overline{B}(x, s_x) \subset A$ and functionals $\{f_1, ..., f_{n_x}\} \subset X^*$ such that if $f_i(y) = f_i(z)$ for $i = 1, \dots, n_x$ and $y, z \in \overline{B}(x, s_x)$, then $b(y) = b(z)$. We can assume that the functionals $\{f_1, ..., f_{n_x}\}$ are linearly independent; otherwise we can omit those that lineally depend on the others, because the condition $f(y) = f(z)$ also holds for every functional f in the linear span of $\{f_i : i = 1, \dots, n_x\}$.

Let us consider $\{e_1, \dots, e_{n_x}\}$ in X such that $f_i(e_j) = \delta_{ij}$, for $i, j = 1, \dots, n_x$. Let us define the linear mapping $L : \mathbb{R}^{n_x} \longrightarrow X$ as

$$L(t_1, \dots, t_{n_x}) = \sum_{i=1}^{n_x} t_i e_i + (x - \sum_{i=1}^{n_x} f_i(x) e_i)$$

and consider the closed and convex set $C := L^{-1}(\overline{B}(x, s_x))$ of $\mathbb{R}^{n_x}$. Now, let us define the function $G^x : C \longrightarrow \mathbb{R}$ as

$$G^x(t_1, \dots, t_{n_x}) = b(\sum_{i=1}^{n_x} t_i e_i + (x - \sum_{i=1}^{n_x} f_i(x) e_i))$$

and $s'_x = \min\{s_x, \frac{s_x}{\sum_{i=1}^{n_x} ||f_i|| \, ||e_i||}\}$.

On the one hand, the function G^x is clearly C^k-smooth on $\mathrm{int}(C) \neq \emptyset$. Now, if $y \in B(x, s'_x)$ and we take the auxiliary point $z := \sum_{i=1}^{n_x} f_i(y) e_i + (x - \sum_{i=1}^{n_x} f_i(x) e_i)$, we have that

$$||z - x|| = ||\sum_{i=1}^{n_x} f_i(y - x) e_i|| \leq \left(\sum_{i=1}^{n_x} ||f_i|| \, ||e_i|| \right) ||y - x|| < s_x.$$

Then, $(f_1(y), \dots, f_{n_x}(y)) \in C$ and

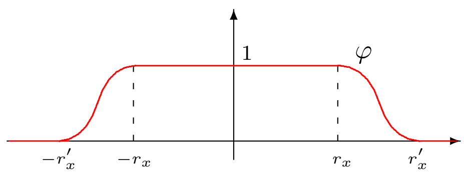
$$G^x(f_1(y), \dots, f_{n_x}(y)) = b(\sum_{i=1}^{n_x} f_i(y) e_i + (x - \sum_{i=1}^{n_x} f_i(x) e_i)).$$

On the other hand, $||y - x|| < s_x$, $\quad ||z - x|| < s_x$ and $\quad f_i(y) = f_i(z)$ for $i = 1, \dots, n_x$, which yields

$$b(y) = b(z) = b(\sum_{i=1}^{n_x} f_i(y) e_i + (x - \sum_{i=1}^{n_x} f_i(x) e_i)) = G^x(f_1(y), \dots, f_{n_x}(y)).$$

Thus, $b(y) = G^x(f_1(y), \dots, f_{n_x}(y))$ for every $y \in B(x, s'_x)$. Finally, since the point $x' = (f_1(x), \dots, f_{n_x}(x))$ belongs to $\mathrm{int}(C)$ there are $r'_x > r_x > 0$ such that $B_{\mathbb{R}^{n_x}}(x', r_x) \subset B_{\mathbb{R}^{n_x}}(x', r'_x) \subset \mathrm{int}(C)$. Let us consider a C^∞-smooth function $\varphi : \mathbb{R}^{n_x} \to [0, 1]$ such that

$$\varphi(y) = \begin{cases} 1 & \text{if } y \in B_{\mathbb{R}^{n_x}}(x', r_x), \\ 0 & \text{if } y \notin B_{\mathbb{R}^{n_x}}(x', r'_x). \end{cases}$$



Let us define $g^x : \mathbb{R}^{n_x} \to \mathbb{R}$ as $g^x(y) = \varphi(y) G^x(y)$ for all $y \in \mathbb{R}^{n_x}$. The function g^x is well defined and is C^k-smooth since $g^x(y) = 0$ for all $y \notin \text{int}(C)$, and $g^x(y) = G^x(y)$ on $B_{\mathbb{R}^{n_x}}(x', r_x)$. Thus, if let us take $V_x := (f_1, \dots, f_{n_x})^{-1}(B_{\mathbb{R}^{n_x}}(x', r_x)) \cap B(x, s'_x)$, then $x \in V_x$ and

$$b(y) = g^x(f_1(y), \dots, f_{n_x}(y)) \qquad \text{for every } y \in V_x.$$

□

We shall use the fact that for every LFC mapping $b : A \longrightarrow E$ and every mapping $h : E \longrightarrow F$ (F an arbitrary set) the composition $h \circ b$ is also LFC.

Let us recall here some useful geometrical properties (see [26], [101] and [69]), which are used throughout this chapter.

Theorem 2.1.3. *[26] Let X be a Banach space and $M \subset X^*$. Let us suppose that there exists a continuous LFC-M bump function on X. Then $\overline{\text{span}}(M) = X^*$. In particular, every Banach space which admits a continuous LFC bump function is an Asplund space.*

Theorem 2.1.4. *[101] Let X be an infinite-dimensional Banach space which admits an LFC bump function, then X is saturated by c_0.*

Finally, we also use the following lemma due to P. Hájek and M. Johanis [54, Lemma 13].

Lemma 2.1.5. *[54] Let $\varepsilon > 0$ and a sequence $\{\delta_n\}_{n=1}^{\infty}$ of strictly positive numbers. Let us consider the open subset U of $\ell_\infty(\mathbb{N})$,*

$$U = \{x \in \ell_\infty(\mathbb{N}) : |x_{j_0}| - \delta_{j_0} > \sup_{j > j_0} |x_j| + \delta_{j_0} \text{ for some } j_0 \in \mathbb{N}\}.$$

Then, there is a C^∞-smooth and LFC-$\{e_i^*\}$ function $F : U \to \mathbb{R}$ (where $\{e_i^*\}$ are the coordinate functionals on $\ell_\infty(\mathbb{N})$) such that $||x||_\infty \le F(x) \le ||x||_\infty + \varepsilon$ for every $x \in U$.

Although we shall use the above lemma to prove the results in this chapter, it is worth noting that this lemma could be replaced by the existence of an equivalent norm on $\ell_\infty(\mathbb{N})$ which is C^∞-smooth and LF-$\{e_i^*\}$ in the bigger open subset

$$U_H = \{x \in \ell_\infty(\mathbb{N}) : ||x||_\infty > \overline{\lim_{n\to\infty}} |x(n)|\}.$$

This fact can be proved using both Lemma 2.1.5 and a fundamental theorem of Haydon [58], that yields the existence of an equivalent norm $||(\cdot,\cdot)||$ on $\ell_\infty(\mathbb{N}) \times c_0(\mathbb{N})$ which is C^∞-smooth and LF-$\{e_i^*\}$ on the open subset

$$U(\mathbb{N}) = \{(x, y) \in \ell_\infty(\mathbb{N}) \times c_0(\mathbb{N}) : \max\{||x||_\infty, ||y||_\infty\} < ||\, |x| + \frac{1}{2}|y|\, ||_\infty\}.$$

Indeed, the mapping $T : \ell_\infty(\mathbb{N}) \to \ell_\infty(\mathbb{N}) \times c_0(\mathbb{N})$ defined as $T(x) = ((x(n))_n, (\frac{x(n)}{n})_n)$ is linear, bounded and injective. Furthermore, $T(U_H) \subset U(\mathbb{N})$. Thus, the function $G(x) = ||T(x)||$ is an equivalent norm on $\ell_\infty(\mathbb{N})$ and it is C^∞-smooth and LF-$\{e_i^*\}$ on U_H.

2.2 Smooth LFC bump functions

We first show that it is possible to "join together" any finite number of neighborhoods, where we have local factorizations of a given LFC function, to obtain a new factorization of the LFC function in the union of these neighborhoods by a suitable composition through the space c_0.

Lemma 2.2.1. *Let X be a Banach space such that X^* is separable and $b : X \longrightarrow \mathbb{R}$ a continuous, LFC function on X. Let us consider $p \in \mathbb{N}$, $B_j = B(x_j, r_j)$ open balls, integers $n_j \in \mathbb{N}$, continuous functions $g^j : \mathbb{R}^{n_j} \longrightarrow \mathbb{R}$ and functionals $\{f_i^j\}_{i=1}^{n_j} \subset X^*$, for $j = 1 \dots, p$. Let us assume that for every $x \in B(x_j, 2r_j)$,*

$$b(x) = g^j(f_1^j(x), \dots, f_{n_j}^j(x)).$$

Then, there exists a continuous linear map $T : X \longrightarrow c_0(\mathbb{N})$ and a continuous function $g : c_0(\mathbb{N}) \longrightarrow \mathbb{R}$ such that $b(x) = g(T(x))$ for every $x \in \bigcup_{j=1}^p B_j$.

Proof. Since X^* is a separable Banach space, there exists a one-to-one continuous linear mapping $i : X \longrightarrow c_0(\mathbb{N})$. Indeed, it is enough to take a sequence $\{g_k\}_{k=1}^\infty$ dense on S_{X^*} and define $i(x) = (g_k(x)/2^k)_{k=1}^\infty$. In addition, the linear mapping i satisfies that

$$\begin{aligned} x_n \xrightarrow{\omega} 0 \text{ (weakly) whenever} \\ \{x_n\}_{n=1}^\infty \text{ is a bounded sequence with } i(x_n) \to 0 \text{ (in norm).} \end{aligned} \tag{Δ}$$

Let us consider the continuous, LFC function $b : X \longrightarrow \mathbb{R}$. We define $n = \sum_{j=1}^p n_j$, consider $\mathbb{R}^n = \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$ and the canonical projection $p_j : \mathbb{R}^n \longrightarrow \mathbb{R}^{n_j}$ given by $p_j(v) = v_j$, for $v = (v_1, \dots, v_p) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_p}$. We can relabel the set of functionals $\{f_1^1, \dots, f_{n_1}^1, \dots, f_1^p, \dots, f_{n_p}^p\}$ as $\{f_1, \dots, f_n\}$ in such a way that $p_j(f_1(x), \dots, f_n(x)) = (f_1^j(x), \dots, f_{n_j}^j(x))$ for every $x \in X$ and $j = 1, \dots, p$. Let us define $G^j : \mathbb{R}^n \longrightarrow \mathbb{R}$ as $G^j(x) = g^j(p_j(x))$. To simplify notation, we will use g^j to denote G^j in the rest of the proof (thus, we have $g^j(f_1(x), \dots, f_n(x)) = b(x)$ for all $x \in B(x_j, 2r_j)$). We define

$$T : X \longrightarrow \mathbb{R}^n \times c_0(\mathbb{N}), \qquad T(x) = (f_1(x), \dots, f_n(x), i(x)). \tag{2.1}$$

The function T is one-to-one, linear and continuous.

Let us first show the assertion of the lemma for $p = 2$. Since T is one-to-one, $T(B_1) \cap T(B_2) = T(B_1 \cap B_2)$. If $x \in T(B_1) \cap T(B_2) = T(B_1 \cap B_2)$, there exists $y \in B_1 \cap B_2$ such that $T(y) = x$. Thus

$$\begin{aligned} b(y) &= g^1(f_1(y), \dots, f_n(y)) = g^1(\pi(x)) \\ &= g^2(f_1(y), \dots, f_n(y)) = g^2(\pi(x)), \end{aligned}$$

where π is the projection of $\mathbb{R}^n \times c_0(\mathbb{N})$ onto $\mathbb{R}^n$ given by the n first coordinates.

Let us define $g : \overline{T(B_1) \cup T(B_2)} \longrightarrow \mathbb{R}$ as

$$g(x) = \begin{cases} g^1(\pi(x)) & \text{if } x \in \overline{T(B_1)} \\ g^2(\pi(x)) & \text{if } x \in \overline{T(B_2)}. \end{cases}$$

If $x \in T(B_1) \cap T(B_2)$, we have already showed that $g^1(\pi(x)) = g^2(\pi(x))$. To show that g is well defined and continuous on $\overline{T(B_1) \cup T(B_2)}$, it suffices to prove that $g^1(\pi(x)) = g^2(\pi(x))$ whenever $x \in \overline{T(B_1)} \cap \overline{T(B_2)}$. Assume, on the contrary, that there is $z \in \overline{T(B_1)} \cap \overline{T(B_2)}$ with $g^1(\pi(z)) \neq g^2(\pi(z))$. Then, there exist two sequences $\{x_m\} \subset B_1$ and $\{y_m\} \subset B_2$ such that $T(x_m) \to z$ and $T(y_m) \to z$. Since $\lim_m ||\pi(T(x_m)) - \pi(z)||_\infty = \lim_m ||\pi(T(y_m)) - \pi(z)||_\infty = 0$ and g^1 and g^2 are continuous, we have

$$g^1(\pi(z)) = \lim_{m\to\infty} g^1(\pi(T(x_m))) = \lim_{m\to\infty} g^1(f_1(x_m), \dots, f_n(x_m)),$$
$$g^2(\pi(z)) = \lim_{m\to\infty} g^2(\pi(T(y_m))) = \lim_{m\to\infty} g^2(f_1(y_m), \dots, f_n(y_m)).$$

Let $\delta > 0$ such that $|g^1(z_1, \dots, z_n) - g^2(z_1, \dots, z_n)| \geq \delta > 0$, where z_i is the i-coordinate of z. Since the functions g^1 and g^2 are continuous on the point $(z_1, \dots, z_n)$, there exists $\eta > 0$ such that

$$|g^1(t_1, \dots, t_n) - g^1(z_1, \dots, z_n)| < \delta/4 \qquad \text{and} \qquad |g^2(t_1, \dots, t_n) - g^2(z_1, \dots, z_n)| < \delta/4$$
$$\text{whenever } t = (t_1, \dots, t_n) \in \mathbb{R}^n \text{ and } \|(t_1, \dots, t_n) - (z_1, \dots, z_n)\|_\infty < \eta.$$

Let us take $0 < \varepsilon < \min\{\eta, r_2/2\}$. There exists $n_0 \in \mathbb{N}$ such that $||\pi(T(x_m)) - \pi(z)||_\infty < \varepsilon$ and $||\pi(T(y_m)) - \pi(z)||_\infty < \varepsilon$ whenever $m \geq n_0$. To simplify, we denote $\{x_m\}$ and $\{y_m\}$ as the subsequences $\{x_m\}_{m \geq n_0}$ and $\{y_m\}_{m \geq n_0}$.

Since $T(x_m - y_m) \to 0$, by property (Δ) of the mapping i, we obtain that $x_m - y_m \xrightarrow{\omega} 0$. From the fact that $\overline{\text{co}}^\omega(\{x_m - y_m : m \in \mathbb{N}\}) = \overline{\text{co}}(\{x_m - y_m : m \in \mathbb{N}\})$, we obtain convex combinations of $\{x_m - y_m\}$ converging (in norm) to 0, i.e. there are non-negative numbers $\{\lambda_i^\varepsilon\}_{i=1}^{m_\varepsilon}$ such that $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon = 1$ and $\| \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i - \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i \| < \varepsilon$. Since $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i \in B_1$ and $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i \in B_2$, we have

$$\text{dist}(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i, B_2) \leq \| \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i - \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon y_i \| < \varepsilon.$$

Notice that $\varepsilon < r_2/2$ and then $\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i \in B(x_2, 2r_2) \cap B_1$. Therefore

$$b(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i) = g^2(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), \dots, \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) = g^2(\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i)) \tag{2.2}$$
$$= g^1(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), \dots, \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) = g^1(\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i)). \tag{2.3}$$

We know that $||\pi(T(x_m)) - \pi(z)||_\infty < \varepsilon$ for every $m \in \mathbb{N}$. Thus, by convexity, we have that $||\pi \circ T(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon x_i) - \pi(z)||_\infty < \varepsilon$. Since $\varepsilon < \eta$, we deduce

$$|g^1(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), \dots, \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) - g^1(z_1, \dots, z_n)| < \delta/4, \tag{2.4}$$
$$|g^2(\sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_1(x_i), \dots, \sum_{i=1}^{m_\varepsilon} \lambda_i^\varepsilon f_n(x_i)) - g^2(z_1, \dots, z_n)| < \delta/4. \tag{2.5}$$

From equations (2.2), (2.3), (2.4) and (2.5) we deduce that

$$|g^1(z_1,\dots,z_n) - g^2(z_1,\dots,z_n)| < \delta/2$$

which is a contradiction. This proves that the function g is well defined and continuous on the closed set $\overline{T(B_1) \cup T(B_2)}$. Now, by the Tietze theorem we can construct a continuous extension, which we shall denote also by g, on the space $\mathbb{R}^n \times c_0(\mathbb{N})$. Notice that the above arguments imply that $b|_{B_1 \cup B_2}$ is locally weakly uniformly continuous.

Finally, let us define $B(x) = g(T(x))$ for every $x \in X$. Then B is a continuous function and $B(x) = b(x)$ for every $x \in B_1 \cup B_2$.

Let us consider the general case of p balls. Since the function T defined in (2.1) is one-to-one, $\bigcap_{i\in I} T(B_i) = T(\bigcap_{i\in I} B_i)$ where $I \subset \{1,\dots,p\}$. If $x \in \bigcap_{i\in I} T(B_i) = T(\bigcap_{i\in I} B_i)$ there exists $y \in \bigcap_{i\in I} B_i$ such that $T(y) = x$. Thus

$$b(y) = g^i(f_1(y),\dots,f_n(y)) = g^i(\pi(x)) = g^j(f_1(y),\dots,f_n(y)) = g^j(\pi(x))$$

for every $i,j \in I$, where π is the projection of $\mathbb{R}^n \times c_0(\mathbb{N})$ onto $\mathbb{R}^n$ given by the n first coordinates. Let us define $g : \overline{\bigcup_{i=1}^p T(B_i)} \longrightarrow \mathbb{R}$ such that

$$g(x) = g^i(\pi(x)), \qquad \text{if } x \in \overline{T(B_i)}.$$

Let us check that g is well defined and continuous in $\overline{\bigcup_{i=1}^p T(B_i)}$. Consider $z \in \bigcap_{i\in I} \overline{T(B_i)}$ where $I \subset \{1,\dots,p\}$ and I has at least two elements. If $i, j \in I$ and $i \neq j$, it is enough to check that $g^i(\pi(x)) = g^j(\pi(x))$, whenever $x \in \overline{T(B_i) \cap T(B_j)}$. This equality is already proved for the case $p = 2$. Notice that the integer n considered in the case $p = 2$ for the two balls B_i and B_j is less or equal than the integer n considered in the general case of the p balls $B_1,\dots,B_p$, and thus the projections, both denoted as π, do not necessarily coincide. Nevertheless, this fact does not interfere in the proof, since we consider $g^k(\pi(x))$ as $g^k(p_k(x))$ in both case. Now, we can apply the Tietze theorem and find a continuous extension, which we shall denote also by g, defined on $\mathbb{R}^n \times c_0(\mathbb{N})$. Notice that the above arguments imply that $b|_{\cup_{i=1}^p B_i}$ is locally weakly uniformly continuous.

Finally, let us define $B(x) = g(T(x))$ for every $x \in X$. Then, B is a continuous function and $B(x) = b(x)$, for every $x \in \bigcup_{i=1}^p B_i$. □

The following theorem is the main result of this chapter. It establishes a characterization of the class of separable Banach spaces admitting a continuous, LFC bump as those separable Banach spaces with a C^∞-smooth LFC bump.

Theorem 2.2.2. *Let X be a separable Banach space. The following statements are equivalent:*

1. *X admits a continuous, LFC bump function.*
2. *X admits a C^∞-smooth, LFC bump function.*

Proof. We only need to prove (1) $\Rightarrow$ (2). Let $b : X \longrightarrow \mathbb{R}$ be a continuous, LFC bump function. We can obtain, using a composition of b with a suitable real function, a continuous, LFC bump $b : X \longrightarrow [1,2]$ such that $b(0) = 1$ and $b(x) = 2$ whenever $\|x\| \geq 1$. For every

$x \in X$, there exist $r_x > 0$, $n_x \in \mathbb{N}$, functionals $\{f_1^x, \ldots, f_{n_x}^x\} \subset X^*$ and a continuous function $g^x : \mathbb{R}^{n_x} \longrightarrow \mathbb{R}$ such that

$$b(y) = g^x(f_1^x(y), \ldots, f_{n_x}^x(y)), \qquad \text{for every } y \in B(x, 2r_x).$$

Since X is separable, there exists a sequence of points $\{x_m\}_{m=1}^{\infty} \subset X$ such that $X = \bigcup_{m \in \mathbb{N}} B_m$ (where $r_m = r_{x_m}$ and $B_m = B(x_m, r_m)$). We can assume that $0 \in B_1$ and define the increasing sequence of open sets $V_j := B_1 \cup \cdots \cup B_j$. We know by Theorem 2.1.3 that, under our assumptions, X^* is separable. From Lemma 2.2.1, we obtain for every $j \in \mathbb{N}$, a continuous linear map $T_j : X \longrightarrow c_0(\mathbb{N})$ and a continuous function $g_j : c_0(\mathbb{N}) \longrightarrow \mathbb{R}$ such that $b(x) = g_j(T_j(x))$ for every $x \in V_j$.

Following the construction given by P. Hájek and M. Johanis in [54], let us choose two sequences of real numbers ε_j and η_j decreasing to 0 and 1 respectively, $0 < \varepsilon_j < \frac{1}{4}(\eta_j - \eta_{j+1})$ with $\eta_1 < 1 + \frac{1}{4}$ and $\varepsilon_1 < \frac{1}{8}$. We can uniformly approximate the continuous function $\eta_j \, g_j$ in $c_0(\mathbb{N})$ by a C^∞-smooth and LFC-$\{e_i^*\}$ function [108], which we shall denote by h_j, satisfying

$$|h_j(x) - \eta_j \, g_j(x)| < \varepsilon_j, \qquad \text{for every } x \in c_0(\mathbb{N}).$$

Let us define $H_j : X \longrightarrow \mathbb{R}$, $H_j(x) = h_j(T_j(x))$, for every $x \in X$ and $j \in \mathbb{N}$. Since T_j is linear and continuous and h_j is C^∞-smooth and LFC-$\{e_i^*\}$, we can easily deduce that H_j is C^∞-smooth and LFC. Indeed, for every $x \in X$, let us consider $V \subset c_0(\mathbb{N})$ a neighborhood of $T_j(x)$, a natural number s, a continuous function $p : \mathbb{R}^s \longrightarrow \mathbb{R}$ and $\{e_1^*, \ldots, e_s^*\}$ (coordinate functionals on $c_0(\mathbb{N})$) such that $h_j(y) = p(e_1^*(y), \ldots, e_s^*(y))$ for every $y \in V$. Since T_j is continuous, the set $W = T_j^{-1}(V)$ is a neighborhood of x on X. Then $H_j(z) = p(e_1^* \circ T_j(z), \ldots, e_s^* \circ T_j(z))$ for all $z \in W$. Since $e_i^* \circ T_j \in X^*$, we conclude that H_j is LFC. In addition, we have

$$|H_j(x) - \eta_j \, b(x)| < \varepsilon_j, \qquad \text{for every } x \in V_j.$$

Let us define

$$\Phi : X \longrightarrow \ell_\infty(\mathbb{N}), \qquad \Phi(x) = (H_j(x))_j.$$

Φ is well defined since $\lim_j H_j(x) = b(x)$ for every $x \in X$. Let us check that Φ is continuous. Consider $x \in X$ and $\varepsilon > 0$. Since b is continuous at x, there is $\delta > 0$ such that $|b(x) - b(y)| < \frac{\varepsilon}{4}$ whenever $||x - y|| < \delta$. In addition, there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$, then $x \in V_j$ and $\varepsilon_j < \frac{\varepsilon}{4}$. Thus, for every $y \in V_{j_0}$ with $||x - y|| < \delta$, we have

$$|H_j(x) - H_j(y)| \leq |H_j(x) - \eta_j b(x)| + \eta_j |b(x) - b(y)| + |\eta_j b(y) - H_j(y)| \leq 2\varepsilon_j + \eta_j \frac{\varepsilon}{4} < \varepsilon,$$

whenever $j \geq j_0$. From the above inequality and the fact that $H_1, \ldots, H_{j_0}$ are continuous at x, we can easily deduce the continuity of Φ at x.

Let us consider the open subset U of $\ell_\infty(\mathbb{N})$ defined in Lemma 2.1.5,

$$U = \{x \in \ell_\infty(\mathbb{N}) : |x_{j_0}| - \varepsilon_{j_0} > \sup_{j > j_0} |x_j| + \varepsilon_{j_0} \text{ for some } j_0 \in \mathbb{N}\}.$$

Let us prove that $\Phi(X) \subset U$. If $x \in V_{j_0}$ for some j_0 and $j > j_0$, we have

$$H_{j_0}(x) - \varepsilon_{j_0} > \eta_{j_0} b(x) - 2\varepsilon_{j_0} > \eta_{j_0+1} b(x) + 2\varepsilon_{j_0} > (\eta_j b(x) + \varepsilon_j) + \varepsilon_{j_0} > H_j(x) + \varepsilon_{j_0}$$

and thus $\Phi(X) \subset U$. By Lemma 2.1.5, there exists a C^∞-smooth and LFC-$\{e_i^*\}$ function $F : U \to (0, \infty)$ (where $\{e_i^*\}$ are the coordinate functionals on $\ell_\infty(\mathbb{N})$) satisfying $||x||_\infty \leq F(x) \leq ||x||_\infty + \varepsilon_1$. Then the composition function defined as

$$B : X \longrightarrow \mathbb{R}, \qquad B(x) = F(\Phi(x))$$

is C^∞-smooth and LFC. In addition,

(a) Since $0 \in V_j$ for every $j \in \mathbb{N}$, $H_j(0) < \eta_j b(0) + \varepsilon_j \leq \eta_1 + \varepsilon_1$. Thus, $B(0) \leq ||\Phi(x)||_\infty + \varepsilon_1 \leq \eta_1 + 2\varepsilon_1 \leq \frac{3}{2}$.

(b) If $\|x\| \geq 1$ and $j_0 \in \mathbb{N}$ verifies $x \in V_{j_0}$, then $H_{j_0}(x) > \eta_{j_0} b(x) - \varepsilon_{j_0} \geq 2\eta_{j_0} - \varepsilon_{j_0} > 2 - \varepsilon_1$ and $B(x) \geq ||\Phi(x)||_\infty > 2 - \varepsilon_1 \geq \frac{15}{8}$.

Therefore B is a separating function on X and by composing it with a suitable C^∞-smooth, real function we obtain a C^∞-smooth, LFC bump function on X. □

2.3 Locally Factorized functions

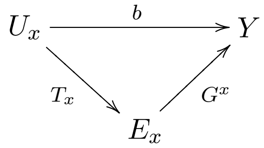
Roughly speaking, in the previous section, we have seen a LFC function as a function that is locally factorized through finite-dimensional spaces. Now, we generalize this concept to functions that are locally factorized through specific Banach spaces. Thus, Lemma 2.2.1 and Theorem 2.2.2 can be generalized using the concept of locally factorized functions.

Definition 2.3.1. *Let X, E and Y be Banach spaces, $A \subset X$ an open subset, $\mathcal{F}$ a family of Banach spaces and $b : A \to Y$ a continuous mapping.*

*(a) We say that b **is factorized by** E on a subset $U \subset A$ if there exist a continuous, linear map $T : X \longrightarrow E$ and a continuous function $G : E \longrightarrow Y$ such that $b(x) = G(T(x))$ for all $x \in U$.*

*(b) We say that b **is locally factorized by** E (b is LF-E, for short) if for each $x \in A$ there exists a neighborhood $U_x \subset A$ of x such that b is factorized by E on U_x.*

*(c) We say that b **is locally factorized by** $\mathcal{F}$ (b is LF-$\mathcal{F}$, for short) if for each $x \in A$ there are a neighborhood $U_x \subset A$ of x and a Banach space $E_x \in \mathcal{F}$ such that b is factorized by E_x on U_x.*



Now, we can see an LFC function as an LF-$\{\mathbb{R}^n : n \in \mathbb{N}\}$ function. Moreover, every continuous, LFC function is LF-c_0. However, it is clear that not every LF-$\mathcal{F}$ function is LFC.

In fact, there exist Banach spaces with LF-$\mathcal{F}$ norms but they do not admit a continuous, LFC bump function.

Example 2.3.2. *For a Banach space E with norm $||\cdot||$, let us consider*

$$c_0(E) = \sum_{c_0} E = \{(x_n)_{n=1}^{\infty} : x_n \in E \text{ and } \lim_n ||x_n|| = 0\}$$

with the norm $||x||_\infty = \sup\{||x_n|| : n \in \mathbb{N}\}$, for every $x \in c_0(E)$. It can be readily verified that the norm in $c_0(E)$ is LF-$\{E^n : n \in \mathbb{N}\}$. Moreover, if $E = \ell_p(\mathbb{N})$ with $1 \le p \le \infty$, then the norm in $c_0(\ell_p)$ is LF-ℓ_p away from the origin. Indeed, for every $x \in c_0(E)$, $x \neq 0$, there exists $n \in \mathbb{N}$ such that $||x_i|| < ||x||_\infty/2$ for every $i \ge n$. Then $||y||_\infty = \max\{||y_i|| : i = 1, \dots, n\}$ for every $y \in B(x, \|x\|_\infty/4)$. Now, if $E = \ell_p(\mathbb{N})$, let us take $\phi : (\ell_p(\mathbb{N}))^n \to \ell_p(\mathbb{N})$ and $G : \ell_p(\mathbb{N}) \to \mathbb{R}$ defined by

$$\phi(x_1, \dots, x_n) = (x_1(1), x_2(1), \dots, x_n(1), x_1(2), x_2(2), \dots, x_n(2), \dots), \text{ and}$$
$$G(z) = \max\{||\{z_{i+j\cdot n}\}_{j=0}^{\infty}||_p : i = 1, \dots, n\}.$$

Hence, $||y||_\infty = G(\phi(y_1, \dots, y_n)) = G(T(y))$ for every $y \in B(x, ||x||_\infty/4)$, where the mapping $T : \sum_{c_0} \ell_p(\mathbb{N}) \to \ell_p(\mathbb{N})$ is defined as $T(y) = \phi(y_1, \dots, y_n)$, which is continuous and linear.

However, note that in this case, $c_0(\ell_p)$ does not admit a continuous, LFC bump, because $\sum_{c_0} \ell_p$ is not c_0-saturated (recall that Theorem 2.1.4 says that every Banach space with a continuous, LFC bump function is c_0-saturated).

Example 2.3.3. *Likewise, for a family of Banach spaces $\mathcal{F} = \{(E_\gamma, ||\cdot||_\gamma) : \gamma \in \Gamma\}$, where Γ is a non-empty set, we can consider*

$$\begin{aligned} c_0(\mathcal{F}) &= \sum_{c_0(\Gamma)} E_\gamma \\ &= \{(x_\gamma)_{\gamma\in\Gamma} : x_\gamma \in E_\gamma \text{ and the set } \{\gamma \in \Gamma : ||x_\gamma||_\gamma \ge \varepsilon\} \text{ is finite for every } \varepsilon > 0\} \end{aligned}$$

with the norm $||x||_\infty = \sup\{||x_\gamma|| : \gamma \in \Gamma\}$, for every $x \in c_0(\mathcal{F})$. Then, the norm in $c_0(\mathcal{F})$ is LF-$\{\prod_{\gamma\in G} E_\gamma : G \in \mathcal{G}\}$ where $\mathcal{G}$ is the collection of all finite, non-empty subsets of Γ.

Following the proof given by M. Fabian and V. Zizler of Theorem 2.1.3 (see [26]), we can show the following proposition which gives us an expression of the dual space of a Banach space with an LF-$\mathcal{F}$ bump function in terms of its factorization.

Proposition 2.3.4. *Let X, E be Banach spaces and $\mathcal{F}$ a family of Banach spaces such that X admits a continuous, LF-E (LF-$\mathcal{F}$) bump function $b : X \to \mathbb{R}$. Let $\{U_i\}_{i\in I}$ be a family of open subsets covering X and $b = G_i \circ T_i$ on U_i for every $i \in I$, where $T_i : X \to E$ (respectively, $T_i : X \to E_i$ with $E_i \in \mathcal{F}$) is a continuous, linear map and $G_i : E \to \mathbb{R}$ (respectively, $G_i : E_i \to \mathbb{R}$) is a continuous function. Then $X^* = \overline{\bigcup_{i\in I} (\ker T_i)^\perp}$.*

Proof. Following [26], let us define the function ϕ on X by:

$$\phi(x) = \begin{cases} b^{-2}(x) & \text{if } b(x) \neq 0 \\ +\infty & \text{if } b(x) = 0. \end{cases}$$

It follows that ϕ is a bounded below, lower semicontinuous function on X and it is LF-E (respectively, LF-$\mathcal{F}$). Furthermore, the subset $\mathrm{dom}(\phi) = \{x \in X : \phi(x) < +\infty\}$ is bounded, open and non-empty.

Let $f \in X^*$ and $\varepsilon > 0$ be given. From the Ekeland variational principle (see [22, Theorem I.2.4]) it follows that there is $x_0 \in \mathrm{dom}(\phi)$ such that

$$(\phi - f)(x) \geq (\phi - f)(x_0) - \varepsilon||x - x_0||, \qquad \text{for all } x \in X.$$

Since $\{U_i\}$ covers X there is $i \in I$ such that $x_0 \in U_i$.

• If $\ker T_i = \{0\}$ then $(\ker T_i)^\perp = \{0\}^\perp = X^*$.

• If $\ker T_i \neq \{0\}$, let us take $\delta > 0$ such that $B(x_0, \delta) \subset U_i$. Then, for all $h \in \ker T_i$ with $||h|| < \delta$, we have that

$$\phi(x_0 + h) - \phi(x_0) = (G_i(T_i(x_0 + h)))^{-2} - (G_i(T_i(x_0)))^{-2} = 0.$$

Hence, for $h \in \ker T_i$ with $||h|| < \delta$

$$f(h) = f(x_0 + h) - f(x_0) \leq \phi(x_0 + h) - \phi(x_0) + \varepsilon||h|| = \varepsilon||h||.$$

Let $\tilde{f}$ be the restriction of f to $\ker T_i$. It satisfies $\tilde{f} \in (\ker T_i)^*$ and $||\tilde{f}|| < \varepsilon$. If $\tilde{f}_1$ is a norm preserving Hahn-Banach extension of $\tilde{f}$ to X, then $f - \tilde{f}_1 \in (\ker T_i)^\perp$. Hence

$$\mathrm{dist}(f, (\ker T_i)^\perp) \leq \mathrm{dist}(f, f - \tilde{f}_1) = ||\tilde{f}_1|| < \varepsilon.$$

Thus $\overline{\bigcup_{i \in I} (\ker T_i)^\perp} = X^*$. □

Corollary 2.3.5. *[26] Every Banach space which admits a continuous LFC bump function is an Asplund space.*

Proof. Assume that X is a separable Banach space. There is $\{U_i\}_{i=1}^\infty$ a countable family of open subsets which covers X and $b = G_i \circ T_i$ on U_i for every $i \in \mathbb{N}$, where $T_i : X \to \mathbb{R}^{n_i}$ are continuous, linear maps and $G_i : \mathbb{R}^{n_i} \to \mathbb{R}$ are continuous functions. Then, by Proposition 2.3.4, $X^* = \overline{\bigcup_{i \in \mathbb{N}} (\ker T_i)^\perp}$. It follows from the bipolar theorem that $(\ker T_i)^\perp = \mathrm{span}\{e_j^* \circ T_i : j = 1, \ldots, n_i\}$. Thus, X^* is separable and X is Asplund. □

Now, we shall see the relation between functions that are LF-$\mathcal{F}$ and functions that *locally depend on countably many coordinates* introduced by M. Fabian and V. Zizler in [27].

Definition 2.3.6. *Let X be a Banach space, H a subspace of X^* and $b : X \to \mathbb{R}$ a continuous function. We say that b* ***locally depends on countably many elements of*** *H if for each $x \in X$ there are a neighborhood $U_x \subset X$ of x, countably many elements $\{f_i\}_{i=1}^\infty \subset B_H$, and a continuous function $G : \ell_\infty(\mathbb{N}) \to \mathbb{R}$ such that $b(z) = G(f_1(z), f_2(z), \dots)$ for each $z \in U_x$. If $H = X^*$, we say that b* ***locally depends on countably many coordinates*** *(b is LCC, for short).*

First of all, let us note that the concept of LCC function coincides with the concept of LF-ℓ_∞ function. Indeed, on the one hand, if $b : X \to \mathbb{R}$ is a continuous, LF-ℓ_∞ function defined on a Banach space, then for every $x \in X$ there are U an open neighborhood of x on X, a continuous linear map $T : X \to \ell_\infty(\mathbb{N})$ and a continuous function $G : \ell_\infty(\mathbb{N}) \to \mathbb{R}$ such that $b = G \circ T$ on U. Let us take $f_i = \frac{1}{||T||} e_i^* \circ T : X \to \mathbb{R}$ (where $\{e_i^*\}$ are the coordinate functionals in $\ell_\infty(\mathbb{N})$) and $\tilde{G} : \ell_\infty(\mathbb{N}) \to \mathbb{R}$ given by $\tilde{G}(z) = G(||T||\, z)$. So, $b(z) = \tilde{G}(f_1(z), f_2(z), \dots)$ for every $z \in U$ and b is LCC. On the other hand, if H is a subspace of X^* and $b : X \to \mathbb{R}$ is a continuous function which locally depends on countably many elements of H, then for every $x \in X$ there are U an open neighborhood of x on X, countable many elements $\{f_i\}_{i=1}^\infty \subset B_H$ and a continuous function $G : \ell_\infty(\mathbb{N}) \to \mathbb{R}$ such that $b(z) = G(f_1(z), f_2(z), \dots)$ for every $z \in U$. We only have to define $T : X \to \ell_\infty(\mathbb{N})$ as $T(z) = (f_1(z), f_2(z), \dots)$, which is a continuous linear map, since $f_i \in B_H$. Thus, $b = G \circ T$ on U and b is LF-ℓ_∞.

Proposition 2.3.7. *Let X be a Banach space, $\mathcal{F}$ a family of separable Banach spaces and $b : X \to \mathbb{R}$ a continuous, LF-$\mathcal{F}$ function. Then b is LF-ℓ_∞ and thus LCC.*

Proof. For every $x \in X$ there are U an open neighborhood of x on X, a separable Banach space $E \in \mathcal{F}$, a continuous linear map $T : X \to E$ and a continuous function $G : E \to \mathbb{R}$ such that $b = G \circ T$ on U. It is well known that every separable Banach space is isometric to a subspace of ℓ_∞ (see, for example, [25, Proposition 5.11]). Since E is a separable Banach space, there is an isometry $S : E \to \ell_\infty(\mathbb{N})$. We can define $\tilde{G} : S(E) \to \mathbb{R}$ given by $\tilde{G}(S(z)) = G(z)$ for every $z \in E$. Then $\tilde{G}$ is a continuous function defined in a closed subspace of ℓ_∞. By the Tietze extension theorem there is a continuous extension of $\tilde{G}$ to ℓ_∞, we also denote to this continuous extension by $\tilde{G} : \ell_\infty(\mathbb{N}) \to \mathbb{R}$. So, $b = \tilde{G} \circ (S \circ T)$ on U and b is LF-ℓ_∞. □

Finally, with the same arguments employed in Lemma 2.2.1 and Theorem 2.2.2, we can show the following more general statement. For the reader's convenience, we present here the required modifications.

Theorem 2.3.8. *Let X, E be separable Banach spaces and $\mathcal{F}$ a family of separable Banach spaces such that X admits a continuous, LF-E (LF-$\mathcal{F}$) bump function. Assume that X^* is separable and E (respectively, every $E \in \mathcal{F}$) admits a bump function b satisfying one of the following properties:*

1. *b is C^k-smooth, where $k \in \mathbb{N} \cup \{\infty\}$,*
2. *b is continuous and LFC,*
3. *b is LFC and C^k-smooth, where $k \in \mathbb{N} \cup \{\infty\}$.*

Then, X admits a bump function satisfying the same property.

First of all, we must show a similar lemma to Lemma 2.2.1 for LF-E and LF-$\mathcal{F}$ functions. Here, we prove the lemma for LF-$\mathcal{F}$ functions, the result for LF-E functions follows immediately from the LF-$\mathcal{F}$ case.

Lemma 2.3.9. *Let X be a Banach space with separable dual and $\mathcal{F}$ a family of Banach spaces. Let us consider $p \in \mathbb{N}$, $B_j = B(x_j, r_j)$ open balls, $E_j \in \mathcal{F}$, continuous functions $g^j : E_j \to \mathbb{R}$, and continuous linear maps $T_j : X \to E_j$, for $j = 1 \dots, p$. Let us assume that*

$$b(x) = g^j(T_j(x)), \qquad \text{for every } x \in B(x_j, 2r_j).$$

Then, there exist a continuous linear map $T : X \to E_1 \times \cdots \times E_p \times c_0(\mathbb{N})$ and a continuous function $g : E_1 \times \cdots \times E_p \times c_0(\mathbb{N}) \to \mathbb{R}$ such that $b(x) = g(T(x))$ for every $x \in \bigcup_{j=1}^{p} B_j$.

Proof. Let us take the same function $i : X \to c_0(\mathbb{N})$ as Lemma 2.2.1. Let us define $T : X \to E_1 \times \cdots \times E_p \times c_0(\mathbb{N})$ as

$$T(x) = (T_1(x), \ldots, T_p(x), i(x)).$$

The function T is one-to-one, continuous and linear.

Let us define $g : \overline{\bigcup_{i=1}^{p} T(B_i)} \to \mathbb{R}$ such that

$$g(x) = g^i(\pi_i(x)), \qquad \text{if } x \in \overline{T(B_i)},$$

where π_i denotes the projection of $E_1 \times \cdots \times E_p \times c_0(\mathbb{N})$ onto E_i.

We can follow the steps of Lemma 2.2.1 to check that g is well defined and continuous on $\overline{\bigcup_{i=1}^{p} T(B_i)}$. Now, we apply the Tietze extension theorem and find a continuous extension of g to the whole space $E_1 \times \cdots \times E_p \times c_0(\mathbb{N})$, which we shall also denote by g, and $b(x) = g(T(x))$ for every $x \in \bigcup_{j=1}^{p} B_j$. □

Proof of Theorem 2.3.8. We only need to prove the assertion when $b : X \to \mathbb{R}$ is a continuous, LF-$\mathcal{F}$ bump function and every $E \in \mathcal{F}$ admits a bump function with one of the properties of the theorem. We shall denote that property by $\mathcal{S}$.

By composing with a suitable real function, we can obtain a continuous, LF-$\mathcal{F}$ function $b : X \longrightarrow [1, 2]$ such that $b(0) = 1$ and $b(x) = 2$ whenever $\|x\| \geq 1$. For every $x \in X$, there exist $r_x > 0$, $E_x \in \mathcal{F}$, a continuous and linear map $T_x : X \to E_x$ and a continuous function $g^x : E_x \to \mathbb{R}$ such that

$$b(y) = g^x(T_x(y)), \qquad \text{for every } y \in B(x, 2r_x).$$

Since X is separable, there exists a sequence of points $\{x_m\}_{m=1}^{\infty}$ of X such that $X = \bigcup_{m\in\mathbb{N}} B_m$ (where $r_m = r_{x_m}$ and $B_m = B(x_m, r_m)$). We can assume that $0 \in B_1$ and define the increasing sequence of open sets $V_j := B_1 \cup \cdots \cup B_j$. From Lemma 2.3.9, we obtain for every $j \in \mathbb{N}$, a continuous linear map $T_j : X \to E_1 \times \cdots \times E_j \times c_0(\mathbb{N})$ and a continuous function $g_j : E_1 \times \cdots \times E_j \times c_0(\mathbb{N}) \to \mathbb{R}$ such that $b(x) = g_j(T_j(x))$ for every $x \in V_j$.

Following the construction given by P. Hájek and M. Johanis in [54] and the proof of Theorem 2.2.2, let us choose two sequences of real numbers ε_j and η_j decreasing to 0 and 1 respectively, $0 < \varepsilon_j < \frac{1}{4}(\eta_j - \eta_{j+1})$ with $\eta_1 < 1 + \frac{1}{4}$ and $\varepsilon_1 < \frac{1}{8}$. We can uniformly approximate the continuous function $\eta_j\, g_j$ in $E_1 \times \cdots \times E_j \times c_0(\mathbb{N})$ by a function h_j with the property $\mathcal{S}$ (see [108]) satisfying

$$|h_j(x) - \eta_j\, g_j(x)| < \varepsilon_j, \qquad \text{for every } x \in E_1 \times \cdots \times E_j \times c_0(\mathbb{N}).$$

Let us define $H_j : X \longrightarrow \mathbb{R}$, $H_j(x) = h_j(T_j(x))$, for every $x \in X$ and $j \in \mathbb{N}$. Since T_j is linear and continuous and h_j satisfies the property $\mathcal{S}$, we can easily deduce that H_j satisfies the property $\mathcal{S}$. In addition, we have

$$|H_j(x) - \eta_j\, b(x)| < \varepsilon_j, \qquad \text{for every } x \in V_j.$$

Let us define

$$\Phi : X \longrightarrow \ell_\infty(\mathbb{N}), \qquad \Phi(x) = (H_j(x))_j.$$

The function Φ is well defined since $\lim_j H_j(x) = b(x)$ for every $x \in X$, and it is continuous.

Let us consider the open subset U of $\ell_\infty(\mathbb{N})$,

$$U = \{x \in \ell_\infty(\mathbb{N}) : |x_{j_0}| - \varepsilon_{j_0} > \sup_{j>j_0} |x_j| + \varepsilon_{j_0} \text{ for some } j_0 \in \mathbb{N}\}.$$

Then $\Phi(X) \subset U$. By Lemma 2.1.5, there exists a C^∞-smooth and LFC-$\{e_i^*\}$ function $F : U \to (0, \infty)$ (where $\{e_i^*\}$ are the coordinate functionals on $\ell_\infty(\mathbb{N})$) satisfying $||x||_\infty \le F(x) \le ||x||_\infty + \varepsilon_1$. Then the composition function defined as

$$B : X \longrightarrow \mathbb{R}, \qquad B(x) = F(\Phi(x))$$

satisfies the property $\mathcal{S}$. In addition, $B(0) \le \frac{3}{2}$, and $B(x) > \frac{15}{8}$ for every $\|x\| \ge 1$.

Therefore B is a separating function on X and we can obtain a bump function with the property $\mathcal{S}$ on X. $\square$

Open Problems

Let us now turn to some questions that remain open. In general, the following problems are well known in the area.

1. Let X be a separable Banach space admitting a continuous LFC bump function. Is the space X polyhedral? Or, equivalently, does there exist an LFC norm? A tentative counterexample to this question has been given by P. Hájek and M. Johanis in [53] where they have conjectured that the answer is negative. Specifically, they constructed an Orlicz space admitting a C^∞-smooth LFC bump function and not satisfying Leung's sufficient condition on polyhedrality [86].

2. Does Theorem 2.2.2 hold in the non-separable case? I.e., let X be a non-separable Banach space which admits a continuous LFC bump function. Does X admit a C^∞-smooth LFC bump function? It is worth mentioning that, in the non-separable case, the same question about LFC norms remains open.

3. Let X be a Banach space admitting a C^k-smooth and LFC bump function. Does then X admit LFC and C^k-smooth partitions of unity? This problem has a positive solution when either X is a WCG Banach space or $X = C(K)$ (see [22] and [52]).

4. Let X be a Banach space admitting a non-continuous LFC bump function. Does X admit a continuous LFC bump function?

Chapter 3

Smooth and Lipschitz approximation on Banach spaces

In this chapter we give a brief survey of some recent results concerning smooth and Lipschitz approximation of Lipschitz functions and C^1-fine approximation of smooth functions on Banach spaces. These results will be widely used in the following chapters.

We start by listing the most fundamental results in the area. First of all, let us recall what means C^1-fine approximation. Let $f : X \to Z$ be a C^1-smooth mapping between Banach spaces and $\varepsilon : X \to (0, \infty)$ a continuous function. We say that f is *C^1-fine approximated* by a C^k-smooth mapping $g : X \to Z$ (with $k \in \mathbb{N} \cup \{\infty\}$), if $||f(x) - g(x)|| < \varepsilon(x)$ and $||f'(x) - g'(x)|| < \varepsilon(x)$ for all $x \in X$. The finite-dimensional case was solved by H. Whitney in the classical paper [111].

Theorem 3.0.1. *[111] Let U be an open subset of $\mathbb{R}^n$. Then, for any C^k-smooth function $f : U \to \mathbb{R}$, with $k \in \mathbb{N}$, and any continuous function $\varepsilon : U \to (0, \infty)$, there is an analytic function $g : U \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \text{and} \qquad ||f^{(j)}(x) - g^{(j)}(x)|| < \varepsilon(x) \qquad \text{for all } x \in U \text{ and } 1 \leq j \leq k,$$

where the superscripts (j) on f and g represent the jth Fréchet derivatives.

The infinite-dimensional setting has proven to be more difficult. The most fundamental work in this direction has been made by N. Moulis in [92].

Theorem 3.0.2. *[92] Let X be $c_0(\mathbb{N})$ or $\ell_p(\mathbb{N})$ (with $1 < p < \infty$), Z a Banach space and U an open subset of X. Then, for any C^1-smooth mapping $f : U \to Z$ and any continuous function $\varepsilon : U \to (0, \infty)$, there is a C^α-smooth mapping $g : U \to Z$ (where $\alpha = \infty$ if $X = c_0$ or p is even, $\alpha = p - 1$ if p is odd, and $\alpha = [p]$ if p is not an integer number), such that f is C^1-fine approximated by g, i.e.*

$$||f(x) - g(x)|| < \varepsilon(x) \qquad \text{and} \qquad ||f'(x) - g'(x)|| < \varepsilon(x) \qquad \text{for all } x \in U.$$

D. Azagra, R. Fry, J.G. Gil, J.A. Jaramillo and M. Lovo in [11] extended this result to any Banach space that admits an unconditional Schauder basis and a smooth, Lipschitz bump function.

Theorem 3.0.3. *[11] Let X be a Banach space with an unconditional basis, Z a Banach space and U an open subset of X. Assume that X admits a C^k-smooth, Lipschitz bump function, with $k \in \mathbb{N} \cup \{\infty\}$. Then, for any C^1-smooth mapping $f : U \to Z$ and any continuous function $\varepsilon : U \to (0, \infty)$, there is a C^k-smooth mapping $g : U \to Z$ such that f is C^1-fine approximated by g, i.e.*

$$||f(x) - g(x)|| < \varepsilon(x) \qquad \textit{and} \qquad ||f'(x) - g'(x)|| < \varepsilon(x) \qquad \textit{for all } x \in U.$$

R. Fry used similar techniques in [36, 38] in order to obtain a result on smooth and Lipschitz approximation of Lipschitz functions on Banach spaces with an unconditional Schauder basis and a smooth, Lipschitz bump function. More precisely, he proved that in those spaces any Lipschitz function f can be uniformly approximated by a smooth and Lipschitz function. Moreover, the Lipschitz constant of the constructed smooth function is controlled by the Lipschitz constant of f, i.e. the ratio between their Lipschitz constants is bounded.

Theorem 3.0.4. *[36, 38] Let X be a Banach space with an unconditional basis and Z a Banach space. Assume that X admits a C^k-smooth, Lipschitz bump function, with $k \in \mathbb{N} \cup \{\infty\}$. Then, there is a constant $C_0 \geq 1$, which only depends on X, such that for every Lipschitz mapping $f : X \to Z$ and every $\varepsilon > 0$, there is a C^k-smooth and Lipschitz mapping $g : X \to Z$ such that*

$$||f(x) - g(x)|| < \varepsilon \qquad \textit{for all } x \in X, \textit{ and} \qquad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

We are mainly interested in this type of results, where any Lipschitz function can be uniformly approximated by a smooth and Lipschitz function. The finite-dimensional case is solved using the integral convolution techniques. The first theorem in the infinite-dimensional setting was given by J.M Lasry and P.L. Lions in [85] where they use the infimal convolution techniques to prove C^1-smooth and Lipschitz approximation of Lipschitz functions on Hilbert spaces (separable or non-separable).

Theorem 3.0.5. *[85] Let X be a Hilbert space. Then, for every Lipschitz function $f : X \to \mathbb{R}$ and $\varepsilon > 0$, there is a C^1-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon \qquad \textit{for all } x \in X, \textit{ and} \qquad \operatorname{Lip}(g) \leq \operatorname{Lip}(f).$$

D. Azagra, J. Ferrera, F. López-Mesas and Y.C. Rangel in [6] combined Theorem 3.0.2 and Theorem 3.0.5 to obtain C^∞-smooth and Lipschitz approximations on separable Riemannian manifolds, in particular on separable Hilbert spaces.

Theorem 3.0.6. *[6] Let X be a separable Hilbert space. Then, for any Lipschitz function $f : X \to \mathbb{R}$, any continuous function $\varepsilon : X \to (0, \infty)$ and any $r > 0$, there is a C^∞-smooth Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \textit{for all } x \in X, \textit{ and} \qquad \operatorname{Lip}(g) \leq (1 + r) \operatorname{Lip}(f).$$

On the other hand, R. Fry started studying the problem from a different point of view. In [35] he introduces new techniques (called *sup-partitions of unity*, that will be studied later) in order to obtain smooth and Lipschitz approximation of Lipschitz and bounded functions on separable Banach spaces. This method was generalized by D. Azagra, R. Fry and A. Montesinos in [10].

Theorem 3.0.7. *[35, 10] Let X be a separable Banach space which admits a C^k-smooth, Lipschitz bump function, with $k \in \mathbb{N} \cup \{\infty\}$. Then, for any $\varepsilon > 0$ there is a constant $C \geq 1$, which depends on X and ε, such that for every Lipschitz function $f : X \to [0,1]$, there is a C^k-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}(g) \leq C \operatorname{Lip}(f).$$

Recently, P. Hájek and M. Johanis have improved the above theorem in different directions in the paper [56] (see also [7]).

Theorem 3.0.8. *[56] Let X be a separable Banach space which admits a C^k-smooth, Lipschitz bump function, with $k \in \mathbb{N} \cup \{\infty\}$. Then, there is a constant $C_0 \geq 1$, which only depends on X, such that for every Lipschitz function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$, there is a C^k-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

In the next section, we show that the constant C_0 of the above theorem can be obtained to be independent of the separable Banach space.

The non-separable case has not been so well developed. In addition to Theorem 3.0.5 for Hilbert spaces, there are also two important results. P. Hájek and M. Johanis proved in [55] that every Lipschitz mapping defined on $c_0(\Gamma)$ can be uniformly approximated by C^∞-smooth and Lipschitz mappings.

Theorem 3.0.9. *[55] Let Γ be an arbitrary set and Z a Banach space. Then, for every Lipschitz mapping $f : c_0(\Gamma) \to Z$ and every $\varepsilon > 0$, there is a C^∞-smooth, Lipschitz and LFC mapping $g : c_0(\Gamma) \to Z$, such that*

$$||f(x) - g(x)|| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}(g) \leq \operatorname{Lip}(f).$$

Using this result and the techniques developed in [35], among other tools, they obtained the following characterization. Firstly, we recall the notion of *smooth and Lipschitz sup-partition of unity* on a Banach space, which was introduced by R. Fry [35] and it becomes one of the keys to obtain smooth and Lipschitz approximation. Subsequent generalizations of this result and related results were given in [10] and [56]. The symbol $c_{00}(\Gamma)$ denotes the space of sequence consisting of all $x = (x_\gamma)_{\gamma \in \Gamma}$ such that $\operatorname{supp}(x) = \{\gamma \in \Gamma : x_\gamma \neq 0\}$ is finite.

Definition 3.0.10. *Let X be a Banach space. We say that X admits **C^k-smooth and Lipschitz sup-partitions of unity** subordinated to an open cover $\mathcal{U} = \{U_r\}_{r \in \Omega}$ of X, if there is a collection of C^k-smooth and L-Lipschitz functions $\{\psi_\alpha\}_{\alpha \in \Gamma}$ (where $L > 0$ depends on X and the cover $\mathcal{U}$) such that*

(S1) $\psi_\alpha : X \to [0,1]$ *for all* $\alpha \in \Gamma$,

(S2) for each $x \in X$ *the set* $\{\alpha \in \Gamma : \psi_\alpha(x) > 0\} \in c_{00}(\Gamma)$,

(S3) $\{\psi_\alpha\}_{\alpha\in\Gamma}$ *is subordinated to* $\mathcal{U} = \{U_r\}_{r\in\Omega}$*, i.e. for each* $\alpha \in \Gamma$ *there is* $r \in \Omega$ *such that* $\operatorname{supp}(\psi_\alpha) \subset U_r$*, and*

(S4) for each $x \in X$ *there is* $\alpha \in \Gamma$ *such that* $\psi_\alpha(x) = 1$.

Theorem 3.0.11. *[56] Let* X *be a Banach space,* Γ *an infinite set and* $k \in \mathbb{N} \cup \{\infty\}$*. Then the following statements are equivalent:*

(i) There is a C^k*-smooth and Lipschitz sup-partition of unity subordinated to* $\{B(x, 1) : x \in X\}$.

(ii) X *is uniformly homeomorphic to a subset of* $c_0(\Gamma)$ *and there is* $C_0 \geq 1$ *such that for every Lipschitz function* $f : X \to \mathbb{R}$ *and* $\varepsilon > 0$*, there is a* C^k*-smooth and Lipschitz function* $g : X \to \mathbb{R}$ *such that* $|f(x) - g(x)| < \varepsilon$ *on* X *and* $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$.

(iii) There is a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ *such that the coordinate functions* $e^*_\gamma \circ \varphi$ *are* C^k*-smooth for every* $\gamma \in \Gamma$.

Finally, it is worth mentioning the close relation between smooth and Lipschitz approximation and C^1-fine approximation, as P. Hájek and M. Johanis showed in [56].

Theorem 3.0.12. *[56] Let* X *be a Banach space and* $k \in \mathbb{N} \cup \{\infty\}$*. Assume that there is* $C_0 \geq 1$ *such that for every Lipschitz function* $f : X \to \mathbb{R}$ *and* $\varepsilon > 0$ *there is a* C^k*-smooth and Lipschitz function* $g : X \to \mathbb{R}$ *such that* $|f(x) - g(x)| < \varepsilon$ *and* $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$*. Then, for every open subset* U *of* X*, every* C^1*-smooth function* $f : U \to \mathbb{R}$ *and every continuous function* $\varepsilon : U \to (0, \infty)$*, there is a* C^k*-smooth function* $g : U \to \mathbb{R}$ *such that* f *is* C^1*-fine approximated by* g*, i.e.*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \text{and} \qquad ||f'(x) - g'(x)|| < \varepsilon(x) \qquad \text{for all } x \in U.$$

Theorem 3.0.8 and Theorem 3.0.12 yield the following corollary:

Corollary 3.0.13. *[56] Let* X *be a separable Banach space which admits a* C^k*-smooth, Lipschitz bump function, with* $k \in \mathbb{N} \cup \{\infty\}$*. Then, for any* C^1*-smooth function* $f : X \to \mathbb{R}$ *and any* $\varepsilon > 0$*, there is a* C^k*-smooth and Lipschitz function* $g : X \to \mathbb{R}$ *such that*

$$|f(x) - g(x)| < \varepsilon \qquad \text{and} \qquad ||f'(x) - g'(x)|| < \varepsilon \qquad \text{for all } x \in X.$$

The purpose of Section 3.1 is to slightly improve the result on separable Banach spaces given in Theorem 3.0.8, showing that the upper bound of the ratio between the Lipschitz constant of the constructed smooth function that uniformly approximates to a function f and the Lipschitz constant of f does not depend on the Banach space. In fact, the upper bound may be taken not greater than $4 + r$, for any $r > 0$. In this section, we follow the ideas of [35] and [56].

In Section 3.2, we give an extra equivalent statement in Theorem 3.0.11 and get some sufficient conditions to have smooth and Lipschitz approximation on non-separable Banach spaces. Moreover, we construct smooth and Lipschitz partitions of unity on every (separable or non-separable) Banach space admitting smooth and Lipschitz approximation. Smooth and Lipschitz partitions of unity are fundamental tools for the following chapters.

3.1 Approximation on separable Banach spaces

In this section we are going to prove the following improvement of Theorem 3.0.8 and the result given in [35].

Theorem 3.1.1. *Let X be a separable Banach space which admits a C^k-smooth, Lipschitz bump function, with $k \in \mathbb{N} \cup \{\infty\}$. Then, for any Lipschitz function $f : X \to \mathbb{R}$, any $\varepsilon > 0$ and $r > 0$, there is a C^k-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}_{||\cdot||}(g) \leq (4 + r) \operatorname{Lip}_{||\cdot||}(f),$$

where $\operatorname{Lip}_{||\cdot||}(f) = \sup\{\frac{|f(x)-f(y)|}{||x-y||} : x, y \in X \text{ and } x \neq y\}$ *for any equivalent norm* $||\cdot||$ *on* X.

In order to prove Theorem 3.1.1, we shall follow the ideas of [35] and [56]. Let us divide the proof into several propositions.

Proposition 3.1.2. *Let X be a separable Banach space with a C^1-smooth norm $||\cdot||$. Then, for any $r > 0$ there exists a C^1-smooth and Lipschitz sup-partition of unity $\{\varphi_n\}_{n=1}^{\infty}$ subordinated to $\{B(x, 1) : x \in X\}$ such that*

$$\operatorname{Lip}(\varphi_n) \leq 2 + r \qquad \text{for all } n \in \mathbb{N},$$

where $\operatorname{Lip} = \operatorname{Lip}_{||\cdot||}$.

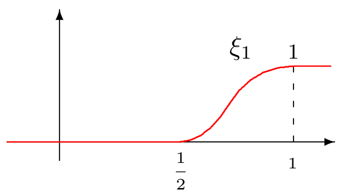
Proof. Let us take η, $\delta > 0$ such that $(1 + \eta)^5 \leq 1 + r/2$, $\delta < 1/2$ and $\frac{1}{1-2\delta} \leq 1 + \eta$, and a C^1-smooth norm $||\cdot||_1$ in $c_0(\mathbb{N})$ such that

$$||x||_\infty \leq ||x||_1 \leq (1 + \eta)||x||_\infty, \qquad \text{for every } x \in c_0(\mathbb{N})$$

(see, for example, [22]).

Let us define $\xi_1 : \mathbb{R} \to [0, 1]$ such that

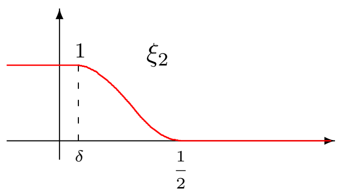
$$\xi_1(t) = \begin{cases} 0 & \text{if } t \leq 1/2, \\ 1 & \text{if } t \geq 1, \end{cases}$$



which is C^∞-smooth and $\operatorname{Lip}(\xi_1) \leq 2(1 + \eta)$.

Also, let us define $\xi_2 : \mathbb{R} \to [0, 1]$

$$\xi_2(t) = \begin{cases} 1 & \text{if } t \leq \delta, \\ 0 & \text{if } t \geq 1/2, \end{cases}$$



which is C^∞-smooth and $\operatorname{Lip}(\xi_2) \leq \frac{2(1+\eta)}{1-2\delta}$.

We take a sequence $\{x_n\}_{n=1}^\infty$ of points on X such that $\{B(x_n,\delta)\}_{n\in\mathbb{N}}$ is an open cover of X, and we define C^1-smooth and Lipschitz functions f_n, $g_n : X \to \mathbb{R}$, for every $n\in\mathbb{N}$, as

$$f_n(x) = \xi_1(||x-x_n||) \qquad \text{and} \qquad g_n(x) = \xi_2(||x-x_n||).$$

Let us take a C^∞-smooth and Lipschitz function $\theta : \mathbb{R} \to [0,+\infty)$ such that $\theta(t) = 0$ in an open neighborhood of $(-\infty, 0]$, $\theta(t) \geq t$ whenever $t \geq 1$, and $\text{Lip}(\theta) \leq 1+\eta$. Let us define the function $\psi_n : X \to \mathbb{R}$ for every $n \geq 1$ as

$$\psi_n(x) = \theta(||(g_1(x),\ldots,g_{n-1}(x), f_n(x), 0, \ldots)||_1) \qquad \text{for all } x \in X.$$

Thus, ψ_n is C^1-smooth and $\text{Lip}(\psi_n) \leq (1+\eta)^2 \max\{\text{Lip}(\xi_1), \text{Lip}(\xi_2)\} \leq 2\frac{(1+\eta)^3}{1-2\delta}$. Let us recall that we denote the Lipschitz constant $\text{Lip}_{||\cdot||}$ by Lip. Notice that the Lipschitz constant is independent of n.

Furthermore, the functions ψ_n satisfy:

- $\psi_n(x) \geq 1$ whenever $x \notin B(x_n, 1)$, since

$$\begin{aligned}\psi_n(x) \geq & ||(g_1(x),\ldots,g_{n-1}(x), f_n(x), 0, \ldots)||_1 \\ \geq & ||(g_1(x),\ldots,g_{n-1}(x), f_n(x), 0, \ldots)||_\infty \geq |f_n(x)| = 1 \qquad \text{for all } x \notin B(x_n,1).\end{aligned}$$

- For each $x \in X$, there is an n_0 such that $\psi_{n_0}(x) = 0$. Indeed, there exists $n_0 \in \mathbb{N}$ with $x \in B(x_{n_0}, \frac{1}{2})$ while $x \notin B(x_j, \frac{1}{2})$ for $j = 1, \ldots, n_0 - 1$. Thus, $f_{n_0}(x) = 0$, $g_j(x) = 0$ for $j = 1, \ldots, n_0 - 1$, and $\psi_{n_0}(x) = 0$.
- For each $x \in X$, there is an n_0 such that $\psi_n(x) \geq 1$ for all $n > n_0$. Since $\{B(x_n,\delta)\}_{n\in\mathbb{N}}$ is a cover of X, there exists $n_0 \in \mathbb{N}$ such that $x \in B(x_{n_0}, \delta)$. By the construction of ψ_n, for every $n > n_0$ we have $\psi_n(x) \geq |g_{n_0}(x)| = 1$.

Let us take another C^∞-smooth and Lipschitz function $h : \mathbb{R} \to [0,1]$ such that $h(t) = 1$ on a neighborhood of the interval $(-\infty, 0]$, $h(t) = 0$ if $t \geq 1$, and $\text{Lip}(h) \leq 1+\eta$. Finally, we define C^1-smooth and Lipschitz functions $\varphi_n : X \to [0,1]$ as

$$\varphi_n(x) = h(\psi_n(x)) \qquad \text{for every } x \in X,$$

which satisfy:

- $\text{supp}(\varphi_n) \subset B(x_n, 1)$ for all n,
- for each $x \in X$, there is an n_0 such that $\varphi_{n_0}(x) = 1$,
- for each $x \in X$, there is an n_0 such that $\varphi_n(x) = 0$ for all $n > n_0$.

Moreover,

$$\text{Lip}(\varphi_n) \leq (1+\eta)\,\text{Lip}(\psi_n) \leq 2\frac{(1+\eta)^4}{1-2\delta} \leq 2(1+\eta)^5 \leq 2 + r.$$

Hence, $\{\varphi_n\}_{n=1}^\infty$ is the C^1-smooth and Lipschitz sup-partition of unity that we were looking for. □

Proposition 3.1.3. *Let X be a separable Banach space with a C^1-smooth norm $||\cdot||$. Then, for every Lipschitz function $f : X \to \mathbb{R}$, every $\varepsilon > 0$ and $r > 0$, there is a C^1-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}(g) \leq (4 + r) \operatorname{Lip}(f),$$

where $\operatorname{Lip} = \operatorname{Lip}_{||\cdot||}$.

Proof. Let us take $\eta > 0$ such that $\frac{(2+\eta)(1+\eta)^2}{1-\eta} \leq 2 + r/2$ and $t := (1 + \eta)$.

First of all, it is not difficult to prove that there is a C^∞-smooth and Lipschitz function $M : \mathbb{R}^2 \to \mathbb{R}$ such that

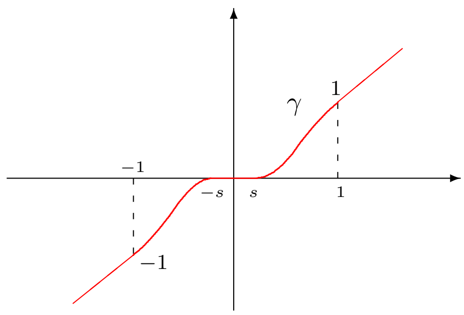
(i) $|M(x, y) - \min\{x, y\}| < \eta$ on $\mathbb{R}^2$,

(ii) $\operatorname{Lip}_{||\cdot||_\infty} M \leq 1 + \eta$, and

(iii) $M(x, y) = 0$ whenever $\min\{x, y\} = 0$.

Indeed, let us take $0 < s = \frac{\eta/2}{2+\eta/2} < \frac{\eta}{4}$. By integral convolutions, there is a C^∞-smooth, Lipschitz function $\tilde{M} : \mathbb{R}^2 \to \mathbb{R}$ such that $|\tilde{M}(x, y) - \min\{x, y\}| < s$ and $\operatorname{Lip}_{||\cdot||_\infty}(\tilde{M}) = 1$. Let us define $\gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma(u) = \begin{cases} 0 & \text{if } |u| \leq s, \\ u & \text{if } |u| \geq 1, \end{cases}$$



γ is C^∞-smooth, $\operatorname{Lip}(\gamma) \leq \frac{1+s}{1-s} \leq 1 + \eta$ and $|\gamma(u) - u| \leq 2s < \eta/2$. So, we define $M : \mathbb{R}^2 \to \mathbb{R}$ as $M(x, y) = \gamma(\tilde{M}(x, y))$ for all $(x, y) \in \mathbb{R}^2$. The function M is C^∞-smooth and $\operatorname{Lip}_{||\cdot||_\infty} M \leq (1 + \eta)$. Moreover, if $\min\{x, y\} = 0$, then $|\tilde{M}(x, y)| < s$ and thus $M(x, y) = 0$. Also, for all $(x, y) \in \mathbb{R}^2$ we have that

$$|M(x, y) - \min\{x, y\}| \leq |\gamma(\tilde{M}(x, y)) - \tilde{M}(x, y)| + |\tilde{M}(x, y) - \min\{x, y\}| \leq \frac{\eta}{2} + s < \eta.$$

Now, let us define a C^1-smooth and Lipschitz function $h : X \to \mathbb{R}$ as $h(x) = \xi(||x||)$, where $\xi : \mathbb{R} \to [0, 1]$ is a C^∞-smooth and Lipschitz function satisfying that $\xi(u) = 0$ in an open neighborhood of $(-\infty, 0]$, $\xi(u) = 1$ if $u \geq 1$, and $\operatorname{Lip}(\xi) \leq 1 + \eta$. By Proposition 3.1.2, we take a C^1-smooth and Lipschitz sup-partition of unity $\{\varphi_n\}_{n=1}^\infty$ subordinated to $\{B(x, 1) : x \in X\}$ such that $\operatorname{Lip}(\varphi_n) \leq 2 + \eta$ for all n. Following the steps of [56, Theorem 3], we construct a mapping $\Phi : X \to c_0(\mathbb{Z} \times \mathbb{N})$ such that the coordinate $(n, m) \in \mathbb{Z} \times \mathbb{N}$ of Φ is given by

$$\phi_m^n(x) = t^n M(\varphi_m\left(\frac{x}{t^n}\right), h\left(\frac{x}{t^n}\right)).$$

First of all, we shall see that $\Phi(x) \in c_0(\mathbb{Z}\times\mathbb{N})$ for all $x \in X$. For every $x \in X$ and $\varepsilon > 0$, there are n_0, $n_1 \in \mathbb{Z}$ such that $t^n < \varepsilon/(1+\eta)$ for all $n < n_0$, and $h(x/t^n) = \xi(||x||/t^n) = 0$ for all $n > n_1$. Thus, $|\phi^n_m(x)| \le t^n |M(\varphi_m\left(\frac{x}{t^n}\right), h\left(\frac{x}{t^n}\right))| \le t^n(1+\eta) < \varepsilon$ for every $m \in \mathbb{N}$ and $n < n_0$, and $|\phi^n_m(x)| = 0$ for every $m \in \mathbb{N}$ and $n > n_1$, since $\min\{\varphi_m(\frac{x}{t^n}), h(\frac{x}{t^n})\} = 0$. Since $\{\varphi_m\}_{m=1}^\infty$ is a sup-partition of unity, for each $n_0 \le n \le n_1$, $\phi^n_m(x) \neq 0$ only for finitely many $m \in \mathbb{N}$, which implies $\Phi(x) \in c_0(\mathbb{Z}\times\mathbb{N})$.

On the one hand, it is clear that the coordinate functions are C^1-smooth and $\text{Lip}(\phi^n_m) \le (1+\eta)\max\{\text{Lip}(\varphi_m), \text{Lip}(h)\} \le (1+\eta)\max\{2+\eta, 1+\eta\} = (2+\eta)(1+\eta)$ for every $(n,m) \in \mathbb{Z}\times\mathbb{N}$. Thus $\text{Lip}(\Phi) \le (2+\eta)(1+\eta)$.

On the other hand, fixed x, $y \in X$ with $x \neq y$, there is an $n \in \mathbb{Z}$ such that $2t^n \le ||x-y|| < 2t^{n+1}$. We can assume that $||x|| \ge t^n$ (otherwise $||y|| \ge 2t^n - ||x|| \ge t^n$). Then $h(x/t^n) = 1$ and there is $m \in \mathbb{N}$ such that $\varphi_m(x/t_n) = 1$, thus, $\phi^n_m(x) \ge t^n(\min\{\varphi_m(\frac{x}{t^n}), h(\frac{x}{t^n})\} - \eta) = t^n(1-\eta)$. Now, if $\phi^n_m(z) > 0$ for some point $z \in X$, then $\varphi_m(\frac{z}{t^n}) > 0$. Since the sup-partition of unity $\{\varphi_n\}_{n=1}^\infty$ is subordinated to $\{B(x,1) : x \in X\}$ and $\frac{x}{t^n}$, $\frac{z}{t^n} \in \text{supp}(\varphi_m)$, we have that $||\frac{x}{t^n} - \frac{z}{t^n}|| < 2$. Then $\phi^n_m(y) = 0$, since $||x-y|| \ge 2t^n$, and

$$||\Phi(x) - \Phi(y)||_\infty \ge |\phi^n_m(x) - \phi^n_m(y)| = \phi^n_m(x) \ge t^n(1-\eta) > \frac{1-\eta}{2t}||x-y||.$$

Summarizing, Φ is one-to-one and Φ^{-1} is Lipschitz. Moreover, the mapping Φ satisfies that

$$\frac{1-\eta}{2(1+\eta)}||x-y|| \le ||\Phi(x) - \Phi(y)||_\infty \le (2+\eta)(1+\eta)||x-y||, \qquad \text{for every } x, y \in X.$$

Now, if $f : X \to \mathbb{R}$ is a Lipschitz function, let us take $f \circ \Phi^{-1} : \Phi(X) \to \mathbb{R}$ which is a $2\frac{1+\eta}{1-\eta}\,\text{Lip}(f)$-Lipschitz function. We can define a Lipschitz extension of $f \circ \Phi^{-1}$ to the whole space $c_0(\mathbb{Z}\times\mathbb{N})$ with the same Lipschitz constant. Indeed, the function

$$x \in c_0(\mathbb{Z}\times\mathbb{N}) \mapsto \inf\left\{f\circ\Phi^{-1}(y) + \text{Lip}(f\circ\Phi^{-1})||x-y|| : y \in \Phi(X)\right\}$$

is a Lipschitz extension of $f \circ \Phi^{-1}$ to $c_0(\mathbb{Z}\times\mathbb{N})$ with the same Lipschitz constant. Applying Theorem 3.0.9 to the Lipschitz extension, we find a C^∞-smooth, Lipschitz and LFC function $\widetilde{g} : c_0(\mathbb{Z}\times\mathbb{N}) \to \mathbb{R}$ such that for every $x \in \Phi(X)$

$$|f\circ\Phi^{-1}(x) - \widetilde{g}(x)| < \varepsilon \qquad \text{and} \qquad \text{Lip}(\widetilde{g}) \le 2\frac{1+\eta}{1-\eta}\,\text{Lip}(f).$$

Hence, the function $g : X \to \mathbb{R}$ defined as $g(x) = \widetilde{g}\circ\Phi(x)$ is C^1-smooth, $|f(x) - g(x)| < \varepsilon$ on X and

$$\text{Lip}(g) \le 2(2+\eta)\frac{(1+\eta)^2}{1-\eta}\,\text{Lip}(f) \le (4+r)\,\text{Lip}(f).$$

□

Proof of Theorem 3.1.1. Let us take $\eta > 0$ such that $((1+\eta)^2(4+\eta)+\eta) \le 4+r$. Let us fix an equivalent norm $||\cdot||$ on X. Since X is separable and admits a C^k-smooth and Lipschitz bump function, by [22, Theorem II.4.1] there is a C^1-smooth norm $|||\cdot|||$ such that

$$\frac{1}{1+\eta}|||x||| \le ||x|| \le (1+\eta)|||x|||, \qquad \text{for every } x \in X.$$

Then $\mathrm{Lip}_{|||\cdot|||}(f) \le (1+\eta)\,\mathrm{Lip}_{||\cdot||}(f)$. Now, let us consider the C^1-smooth norm $|||\cdot|||$ on X and apply Proposition 3.1.3 to the Lipschitz function $f : X \to \mathbb{R}$. We find a C^1-smooth and Lipschitz function $h : X \to \mathbb{R}$ such that for all $x \in X$

$$|f(x) - h(x)| < \varepsilon/2 \qquad \text{and} \qquad \mathrm{Lip}_{|||\cdot|||}(h) \le (4+\eta)\,\mathrm{Lip}_{|||\cdot|||}(f).$$

Then

$$\mathrm{Lip}_{||\cdot||}(h) \le (1+\eta)^2(4+\eta)\,\mathrm{Lip}_{||\cdot||}(f).$$

Let us take $\tilde{\varepsilon} = \min\{\varepsilon/2, \eta\,\mathrm{Lip}_{||\cdot||}(f)\}$ (we may assume $\mathrm{Lip}_{||\cdot||}(f) > 0$, otherwise the assertion is trivial) and apply Corollary 3.0.13 to the C^1-smooth function $h : X \to \mathbb{R}$. Hence, we find a C^k-smooth function $g : X \to \mathbb{R}$ such that for all $x \in X$

$$|h(x) - g(x)| < \tilde{\varepsilon} \le \varepsilon/2 \qquad \text{and} \qquad ||h'(x) - g'(x)|| < \tilde{\varepsilon} \le \eta\,\mathrm{Lip}_{||\cdot||}(f).$$

Thus, $|f(x) - g(x)| < \varepsilon$ on X and

$$\begin{aligned}\mathrm{Lip}_{||\cdot||}(g) &\le \mathrm{Lip}_{||\cdot||}(h) + \eta\,\mathrm{Lip}_{||\cdot||}(f) \le ((1+\eta)^2(4+\eta)+\eta)\,\mathrm{Lip}_{||\cdot||}(f)\\ &\le (4+r)\,\mathrm{Lip}_{||\cdot||}(f).\end{aligned}$$

□

A question that arises now is whether we may construct a sup-partition of unity subordinated to $\{B(x,1) : x \in X\}$ with Lipschitz constant less that $1+r$ for any $r > 0$. We are going to show here that it is not possible in general. Let us suppose that $M \ge 1$ is the Lipschitz constant of a sup-partition of unity subordinated to the open cover $\{B(x,1) : x \in X\}$ of a separable Banach space. Then, it follows along the same lines as the proof of Proposition 3.1.3 that for every $\eta > 0$ there is a bi-Lipschitz embedding $\Phi : X \to c_0^+(\mathbb{N}\times\mathbb{Z})$ such that

$$\frac{1}{2+\eta}||x-y|| \le ||\Phi(x) - \Phi(y)||_\infty \le (M+\eta)||x-y|| \qquad \text{for every } x, y \in X,$$

where $c_0^+(\mathbb{N}\times\mathbb{Z})$ is the positive cone of $c_0(\mathbb{N}\times\mathbb{Z})$.

So, on the one hand, the separable Banach space X $(2+\eta)(M+\eta)$-embeds into c_0^+ for every $\eta > 0$.

On the other hand, N.J. Kalton and G. Lancien in [79] have obtained the best constant for the embeddings of certain Banach spaces into c_0^+. In particular, they have proved that ℓ_p $(2^p+1)^{1/p}$-embeds into c_0^+ if $1 \le p < \infty$ and the constant is the best possible.

Then, for any $\eta > 0$ and $1 < p < \infty$ we have that

$$(2^p+1)^{1/p} \le (2+\eta)(M+\eta).$$

This implies that $M \geq 3/2$. So, M cannot be taken $1+\eta$ for any separable Banach space and any $\eta > 0$. For instance, M must be greater that $\sqrt{5}/2$ for $X = \ell_2$.

3.2 Approximation and partitions of unity on Banach spaces

In this section we shall give some new statements related to Theorem 3.0.11, and obtain certain relationships between smooth and Lipschitz sup-partitions of unity, smooth and Lipschitz partitions of unity, homeomorphisms into $c_0(\Gamma)$ and smooth and Lipschitz approximation. Firstly, for $k \in \mathbb{N} \cup \{\infty\}$ we shall say that a Banach space X satisfies **property (A^k)** if there is a constant $C_0 \geq 1$, which depends only on X, such that for every Lipschitz function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a C^k-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$ and $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$.

Before carrying on with the results of this section, let us recall an useful characterization showed in [56].

Proposition 3.2.1. *[56, Proposition 1] Let X be a Banach space and $k \in \mathbb{N} \cup \{\infty\}$. The Banach space X satisfies property (A^k) if and only if there is a constant $C \geq 1$, which only depends on X, such that for each closed subset $A \subset X$ there is a C^k-smooth and C-Lipschitz function $h_A : X \to [0,1]$ such that $h_A(x) = 0$ for all $x \in A$, and $h_A(x) = 1$ for all $x \in X$ whenever* $\operatorname{dist}(x, A) \geq 1$.

3.2.1 Approximation on non-separable Banach spaces

Our purpose is to give a slightly weaker condition implying property (A^k) than the one given in Theorem 3.0.11(iii). In particular, we prove that the homeomorphism $\varphi : X \to c_0(\Gamma)$, whose coordinate functions are C^k-smooth, can be taken Lipschitz with uniformly continuous inverse.

Theorem 3.2.2. *Let X be a Banach space and $\Gamma \neq \emptyset$ a set. Assume that X is homeomorphic to a subset of $c_0(\Gamma)$ via a mapping $\varphi : X \to c_0(\Gamma)$ such that φ is Lipschitz, φ^{-1} is uniformly continuous and the coordinate functions $e_\gamma^* \circ \varphi$ are C^k-smooth for every $\gamma \in \Gamma$. Then, X satisfies property (A^k).*

Let us divide the proof into several propositions.

Proposition 3.2.3. *Under the assumptions of Theorem 3.2.2, there is a constant $d > 0$, which only depends on X, such that* $\operatorname{dist}(\varphi(A), \varphi(B)) \geq d$ *whenever A and B are closed subsets of X with* $\operatorname{dist}(A, B) \geq 1$.

Proof. Since $\varphi^{-1} : \varphi(X) \subset c_0(\Gamma) \to X$ is uniformly continuous, there is $d > 0$ such that

$$||\varphi^{-1}(x) - \varphi^{-1}(y)|| < 1 \qquad \text{for every } x, y \in \varphi(X) \text{ with } ||x - y||_\infty < d.$$

Hence, if A and B are closed subsets of X and $\operatorname{dist}(A, B) \geq 1$, then $||\varphi(x) - \varphi(y)||_\infty \geq d$ for every $x \in A$ and $y \in B$, since $||\varphi^{-1}(\varphi(x)) - \varphi^{-1}(\varphi(y))|| \geq \operatorname{dist}(A, B) \geq 1$. □

Proposition 3.2.4. *Under the assumptions of Theorem 3.2.2, there is a constant $C \geq 1$, which only depends on X, such that for each closed subset $A \subset X$, there is a C^k-smooth and C-Lipschitz function $h_A : X \to [0,1]$ such that $h_A(x) = 0$ for all $x \in A$, and $h_A(x) = 1$ for all $x \in X$ whenever* $\operatorname{dist}(x, A) \geq 1$.

Proof. Let us take a closed subset A of X and denote by B the closed subset $\{x \in X : \operatorname{dist}(x, A) \geq 1\}$. Then $\operatorname{dist}(A, B) \geq 1$ and, by Proposition 3.2.3, $\operatorname{dist}(\varphi(A), \varphi(B)) \geq d$.

Applying Theorem 3.0.9 to the function $x \in c_0(\Gamma) \mapsto \operatorname{dist}(x, \varphi(A))$, we find a C^∞-smooth, Lipschitz and LFC function $h : c_0(\Gamma) \to \mathbb{R}$ such that

$$|h(x) - \operatorname{dist}(x, \varphi(A))| \leq d/4 \qquad \text{for all } x \in c_0(\Gamma), \text{ and} \qquad \operatorname{Lip}(h) \leq 1.$$

So, $h(x) \leq d/4$ whenever $x \in \varphi(A)$, and $h(x) \geq 3d/4$ for all $x \in \varphi(B)$. Let us take a C^∞-smooth and Lipschitz function $\xi : \mathbb{R} \to [0,1]$ such that $\xi(t) = 0$ for all $t \leq d/4$, $\xi(t) = 1$ whenever $t \geq 3d/4$ and $\operatorname{Lip}(\xi) \leq 3/d$. Finally, we define $h_A : X \to [0,1]$ as

$$h_A(x) = \xi(h(\varphi(x))) \qquad \text{for all } x \in X.$$

Then h_A is C^k-smooth, $\frac{3\operatorname{Lip}(\varphi)}{d}$-Lipschitz, $h_A(x) = 0$ for $x \in A$, and $h_A(x) = 1$ for $x \in B$. □

Now, by Proposition 3.2.4 and Proposition 3.2.1, the proof of Theorem 3.2.2 is concluded, and we obtain the following slightly weaker version of Theorem 3.0.11.

Theorem 3.2.5. *Let X be a Banach space, Γ an infinite set and $k \in \mathbb{N} \cup \{\infty\}$. Then the following statements are equivalent:*

(i) There is a C^k-smooth and Lipschitz sup-partition of unity subordinated to $\{B(x,1) : x \in X\}$.

(ii) X is uniformly homeomorphic to a subset of $c_0(\Gamma)$ and satisfies property (A^k).

(iii) X is homeomorphic to a subset of $c_0(\Gamma)$ via a mapping $\varphi : X \to c_0(\Gamma)$ such that φ is Lipschitz, φ^{-1} is uniformly continuous and the coordinate functions $e_\gamma^* \circ \varphi$ are C^k-smooth for every $\gamma \in \Gamma$.

3.2.2 Smooth and Lipschitz partitions of unity

We shall show that there are C^k-smooth and Lipschitz partitions of unity whenever the Banach space satisfies property (A^k). Recall that a Banach space X admits *C^k-smooth and Lipschitz partitions of unity* if for every open cover $\mathcal{U} = \{U_r\}_{r \in \Omega}$ of X there is a collection of C^k-smooth, Lipschitz functions $\{\psi_i\}_{i \in I}$ such that

(1) $\psi_i \geq 0$ on X for every $i \in I$,

(2) the family $\{\operatorname{supp}(\psi_i)\}_{i \in I}$ is locally finite, where $\operatorname{supp}(\psi_i) = \overline{\{x \in X : \psi_i(x) \neq 0\}}$,

(3) $\{\psi_i\}_{i \in I}$ is subordinated to $\mathcal{U} = \{U_r\}_{r \in \Omega}$, i.e. for each $i \in I$ there is $r \in \Omega$ such that $\operatorname{supp}(\psi_i) \subset U_r$ and

(4) $\sum_{i \in I} \psi_i(x) = 1$ for every $x \in X$.

The following lemma gives us the tool to generalize the construction of suitable open coverings on a Banach space, which will be the key to obtain smooth and Lipschitz partitions and some of our main results in the following chapters.

Lemma 3.2.6. *(See M.E. Rudin [104]) Let E be a metric space, $\mathcal{U} = \{U_r\}_{r\in\Omega}$ be an open covering of E. Then, there are open refinements $\{V_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ and $\{W_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ of $\mathcal{U}$ satisfying the following properties:*

(i) $V_{n,r} \subset W_{n,r} \subset U_r$ *for all* $n \in \mathbb{N}$ *and* $r \in \Omega$,

(ii) $\operatorname{dist}(V_{n,r}, E \setminus W_{n,r}) \geq 1/2^{n+1}$ *for all* $n \in \mathbb{N}$ *and* $r \in \Omega$,

(iii) $\operatorname{dist}(W_{n,r}, W_{n,r'}) \geq 1/2^{n+1}$ *for any* $n \in \mathbb{N}$ *and* $r, r' \in \Omega$, $r \neq r'$,

(iv) for every $x \in E$ *there is an open ball* $B(x, s_x)$ *of* E *and a natural number* n_x *such that*

(a) if $i > n_x$, *then* $B(x, s_x) \cap W_{i,r} = \emptyset$ *for any* $r \in \Omega$,

(b) if $i \leq n_x$, *then* $B(x, s_x) \cap W_{i,r} \neq \emptyset$ *for at most one* $r \in \Omega$.

Proposition 3.2.7. *Let X be a Banach space with property (A^k). Then, for every $\{U_r\}_{r\in\Omega}$ open covering of X, there is an open refinement $\{W_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ of $\{U_r\}_{r\in\Omega}$ satisfying the properties of Lemma 3.2.6, and there is a Lipschitz and C^k-smooth partition of unity $\{\psi_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ such that*

$$\operatorname{supp}(\psi_{n,r}) \subset W_{n,r} \subset U_r \quad \textit{and} \quad \operatorname{Lip}(\psi_{n,r}) \leq C_0 2^5(2^n - 1) \quad \textit{for every } n \in \mathbb{N} \textit{ and } r \in \Omega.$$

Proof. Let us consider an open covering $\{U_r\}_{r\in\Omega}$ of X. By Lemma 3.2.6, there are open refinements $\{V_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ and $\{W_{n,r}\}_{n\in\mathbb{N},r\in\Omega}$ of $\{U_r\}_{r\in\Omega}$ satisfying the properties (i)-(iv) of Lemma 3.2.6. Consider the distance function $D_n(x) = \operatorname{dist}(x, X \setminus \bigcup_{r\in\Omega} W_{n,r})$ on X which is 1-Lipschitz. By applying property (A^k), there is a C^k-smooth, C_0-Lipschitz function $g_n : X \to \mathbb{R}$ such that $|g_n(x) - D_n(x)| < \frac{1}{2^{n+3}}$ for every $x \in X$. Thus

$$g_n(x) > \frac{1}{2^{n+2}} \quad \text{whenever } x \in \bigcup_{r\in\Omega} V_{n,r},$$
$$g_n(x) < \frac{1}{2^{n+3}} \quad \text{whenever } x \in X \setminus \bigcup_{r\in\Omega} W_{n,r}.$$

By composing g_n with a suitable C^∞-smooth function $\varphi_n : \mathbb{R} \to [0, 1]$ such that

$$\varphi_n(x) = \begin{cases} 1 & \text{if } t \leq \frac{1}{2^{n+3}} \\ 0 & \text{if } t \geq \frac{1}{2^{n+2}} \end{cases}$$

and $\operatorname{Lip}(\varphi_n) \leq 2^{n+4}$, we obtain a C^k-smooth function $h_n := \varphi_n(g_n)$ that is zero on an open set including $X \setminus \bigcup_{r\in\Omega} W_{n,r}$, $h_n|_{\bigcup_{r\in\Omega} V_{n,r}} \equiv 1$ and $\operatorname{Lip}(h_n) \leq C_0 2^{n+4}$.

Now, let us define

$$H_1 = h_1, \qquad \text{and} \qquad H_n = h_n(1 - h_1) \cdots (1 - h_{n-1}) \qquad \text{for } n \geq 2.$$

It is clear that $\sum_n H_n(x) = 1$ for all $x \in X$. Since $\operatorname{supp}(h_n) \subset \bigcup_{r\in\Omega} W_{n,r}$ and $\overline{W}_{n,r} \cap \overline{W}_{n,r'} = \emptyset$ for every $n \in \mathbb{N}$ and $r \neq r'$, we can write

$$h_n = \sum_{r\in\Omega} h_{n,r}, \qquad \text{where } h_{n,r}(x) = h_n(x) \text{ on } W_{n,r} \text{ and } \operatorname{supp}(h_{n,r}) \subset W_{n,r}.$$

Notice that $\operatorname{Lip}(h_{n,r}) \leq \operatorname{Lip}(h_n) \leq C_0 2^{n+4}$. Let us define, for every $r \in \Omega$,

$$\psi_{1,r} = h_{1,r}, \qquad \text{and} \qquad \psi_{n,r} = h_{n,r}(1-h_1)\cdots(1-h_{n-1}) \qquad \text{for each } n \geq 2.$$

The functions $\{\psi_{n,r}\}_{n\in\mathbb{N}, r\in\Omega}$ satisfy that

(i) they are C^k-smooth and Lipschitz, with $\operatorname{Lip}(\psi_{n,r}) \leq C_0 \sum_{i=5}^{n+4} 2^i = C_0 2^5(2^n - 1)$,

(ii) $\operatorname{supp}(\psi_{n,r}) \subset \operatorname{supp}(h_{n,r}) \subset W_{n,r} \subset U_r$, and

(iii) for every $x \in X$,

$$\sum_{n\in\mathbb{N}, r\in\Omega} \psi_{n,r}(x) = \sum_{r\in\Omega} \psi_{1,r}(x) + \sum_{n\geq 2} \left(\sum_{r\in\Omega} h_{n,r}(x) \right) \prod_{i=1}^{n-1} (1 - h_i(x)) = \sum_{n\in\mathbb{N}} H_n(x) = 1.$$

□

It is worth noticing that the partitions of unity yield the smooth and Lipschitz C^0-fine approximation of Lipschitz functions whenever X has property (A^k). We skip the proof, since we shall prove a stronger version in Theorems 6.1.2 and 6.1.5.

Proposition 3.2.8. *Let X be a Banach space with property (A^k). Then, for any Lipschitz function $f : X \to \mathbb{R}$, any continuous function $\varepsilon : X \to (0,\infty)$ and any $r > 0$, there is a C^k-smooth Lipschitz function $g : X \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < \varepsilon(x) \qquad \text{for all } x \in X, \text{ and} \qquad \operatorname{Lip}(g) \leq C_0(1+r)\operatorname{Lip}(f),$$

where C_0 is the constant given by (A^k).

Finally, we sum up the relationship between the properties studied in this section in Diagram 4.1.

Open Problems

Let us now turn to some questions that remain open.

1. Let X be a WCG Banach space with a C^k-smooth norm. Does X have property (A^k)? An attempt to solve this problem can be found in [37], although there is a gap in the proof and it is unknown at present if the result holds.

2. Let X and Z be Banach spaces, $f : X \to Z$ be a Lipschitz mapping and $\varepsilon > 0$. When can we assure that there is a C^k-smooth and Lipschitz mapping $g : X \to Z$ such that

$||f(x) - g(x)|| < \varepsilon$ and $\text{Lip}(g) \leq C_0 \text{Lip}(f)$, for a constant $C_0 \geq 1$ that depends only on X and Z? P. Hájek and M. Johanis [56] have given sufficient conditions so that the Banach spaces X and Z satisfy this property. In the next section, we shall see examples of pairs of Banach spaces with this property. Nevertheless, a characterization of this property in terms of the Banach spaces X and Z is unknown.

3. Let X be a Banach space and $f : X \to \mathbb{R}$ a C^k-smooth function with $k \geq 2$, and $\varepsilon : X \to (0, \infty)$ a continuous map. The function f is said to be *C^q-fine approximated* by a C^p-smooth function $g : X \to \mathbb{R}$ (with $p > k \geq q$), if $|f(x) - g(x)| \leq \varepsilon(x)$ and $||f^{(i)}(x) - g^{(i)}(x)|| \leq \varepsilon(x)$ for all $1 \leq i \leq q$ on X. In [92] N. Moulis showed that every C^{2k-1}-smooth function can be C^k-finely approximated by C^∞-smooth functions in separable Hilbert spaces. An attempt to give a more complete answer to this problem for Hilbert spaces was given by M.P. Heble in the papers [59], [60] and [61]. Unfortunately, there is a gap in [61] and it is not clear how it could be fixed. So far, the C^k-fine approximation problem with $k \geq 2$ remains open, even in the case when X is a separable Hilbert space.

4. Let X be a Banach space which satisfies property (A^k). Does X admit a C^k-smooth and Lipschitz sup-partition of unity subordinated to $\{B(x, 1) : x \in X\}$? Notice that if X is a separable Banach space and satisfies the property (A^k), a positive answer to this question is given in [35] and [10].

5. Let X be a Banach space which admits C^k-smooth and Lipschitz partitions of unity. Does X admit a C^k-smooth and Lipschitz sup-partition of unity subordinated to $\{B(x, 1) : x \in X\}$? Or, does X have property (A^k)? Again, if X is separable, a positive answer to this question is given in [35], [10] and [56].

6. Let X be a Banach space. Are the following statements equivalent?

   (i) X has property (A^1).

   (ii) For every Lipschitz function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$, there exists a C^1-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for all $x \in X$.

   If X is separable, they are equivalent to the separability of X^*.

7. The constant C_0 of the property (A^k) depends on X. In Theorem 3.1.1 we have proved that C_0 can be taken less than $4 + r$ for any Banach space with separable dual and any $r > 0$. May the constant C_0 be taken independently of the Banach space X in the non-separable case? When we consider a Hilbert space, the constant C_0 may be chosen less than $1 + r$ for any $r > 0$ (notice that C_0 is the constant of the property (A^∞) in the separable case, Theorem 3.0.6, and it is the constant of the property (A^1) in the non-separable case, Theorem 3.0.5). Could C_0 be taken less that $1 + r$ for any Banach space and any $r > 0$?

8. In Section 3.1 we have showed that for any Banach space with separable dual and any $\eta > 0$, there is a sup-partition of unity subordinated to the cover $\{B(x, 1) : x \in X\}$ with Lipschitz constant $M \leq 2 + \eta$. Nevertheless, it cannot be taken $1 + \eta$ for any separable Banach space and any $\eta > 0$. For instance, M must be greater that $\sqrt{5}/2$ for $X = \ell_2$.

So, another problem that arises is to seek the best constant of the sup-partition of unity of a particular Banach space X.

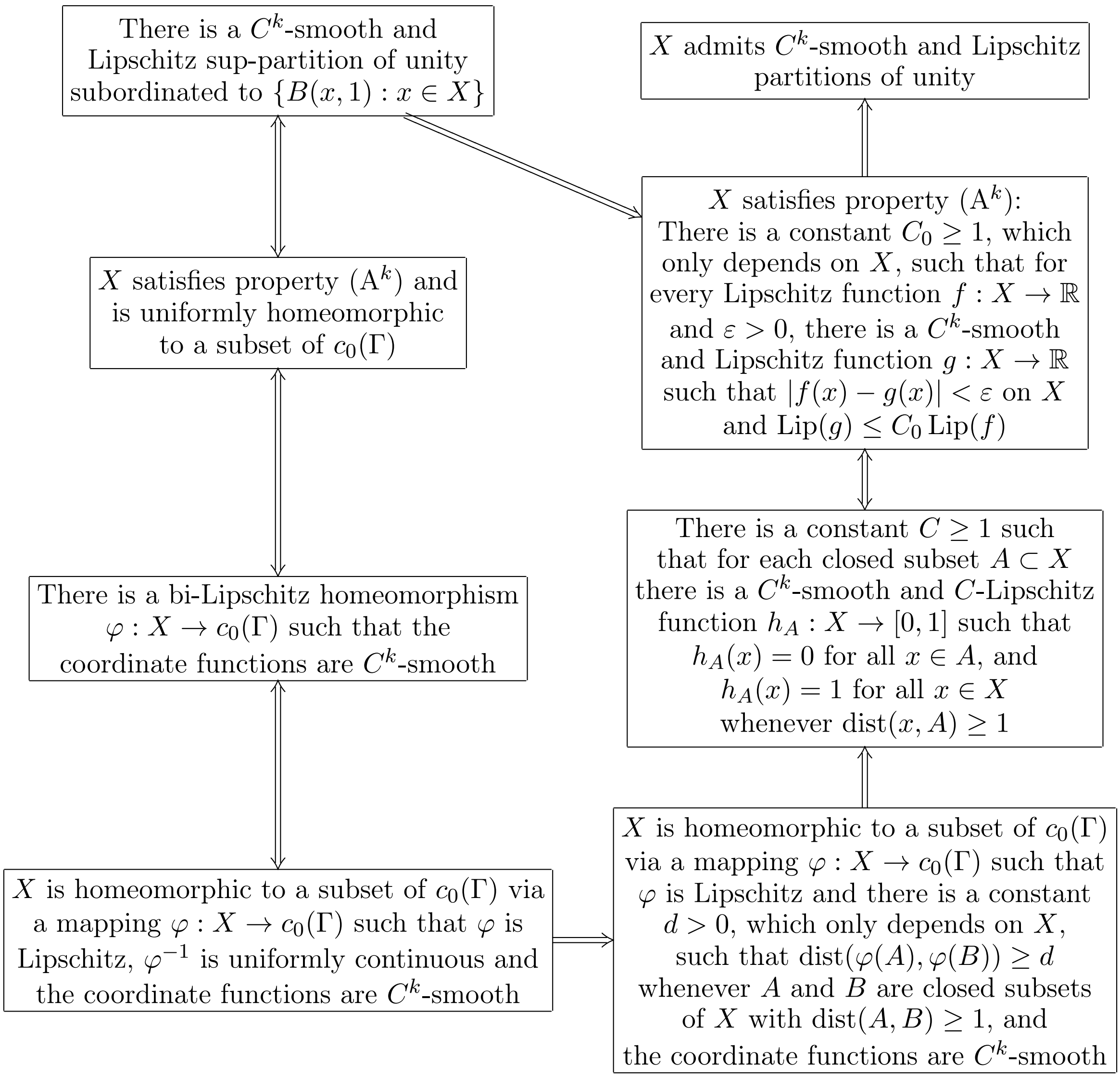


Diagram 4.1: Smooth and Lipschitz approximation

Chapter 4

Smooth extensions on Banach spaces

In this chapter, we study how the techniques given in [7] can be applied to obtain a C^1-smooth extension of a C^1-smooth and *vector-valued* function defined on a closed subset of a Banach space. More precisely, if X and Z are Banach spaces, A is a closed subset of X and $f : A \to Z$ is a mapping, under what conditions does there exist a C^1-smooth mapping $F : X \to Z$ such that the restriction of F to A is f? Under the assumption that Y is a complemented subspace of the Banach space X, an extension of a smooth mapping $f : Y \to Z$ is easily found by taking the mapping $F(x) = f(P(x))$, where $P : X \to Y$ is a continuous linear projection. Unfortunately, this extension does not solve the problem since every Banach space X has a closed subspace which is not complemented in X whenever it is not isomorphic to a Hilbert space [89].

The smooth extension problem from closed subsets of finite-dimensional spaces has been exhaustively studied. H. Whitney in [111] and [112] gave necessary and sufficient conditions for functions defined on closed subsets of $\mathbb{R}$ in order to obtain the existence of C^k-smooth extensions to $\mathbb{R}$. G. Glaeser [47] solved the problem for C^1-smooth functions defined on closed subsets of $\mathbb{R}^n$. Finally, C. Fefferman in a series of papers ([29] and [30]) establishes a characterization of the functions that are the restriction to a compact subset of a C^k-smooth function on $\mathbb{R}^n$, for all $k, n \geq 1$.

The infinite-dimensional setting has proven to be more difficult. C. J. Atkin in [3] extends every smooth function f defined on a finite union of open convex sets in a separable Banach space which does not admit smooth bump functions, provided that for every point in the domain of f, the restriction of f to a suitable neighborhood of the point can be extended to the whole space. The most fundamental result has been given by D. Azagra, R. Fry and L. Keener [7]. They have shown that if X is a Banach space with separable dual X^*, $Y \subset X$ is a closed subspace and $f : Y \to \mathbb{R}$ is a C^1-smooth function, then there exists a C^1-smooth extension $F : X \to \mathbb{R}$ of f. They proved a similar result when Y is a closed convex subset, f is defined on an open set U of X containing Y and f is C^1-smooth on Y as a function on X, i.e. $f : U \to \mathbb{R}$ is differentiable at every point $y \in Y$ and the function $Y \mapsto X^*$ defined as $y \mapsto f'(y)$ is continuous on Y.

Let us point out that the case of real analytic maps is quite different. In particular, R.M. Aron and P.D. Berner [2] have proved that for any pair of Banach spaces (X, Z) such that Z is complemented in its second dual, and any real analytic function $f : Y \to Z$ which is bounded on bounded sets and Y is a closed subspace of X, there exist an open subset U of X containing Y and an analytic extension $F : U \to Z$ of f that is bounded on bounded sets, whenever there exists a bounded and linear operator $T : Y^* \to X^*$ such that $T(g)_{|_{Y^*}} = g$ for all $g \in Y^*$.

The aim of this chapter is to extend the results in [7] to the general setting of vector-valued functions defined on (not necessarily separable) Banach spaces where every Lipschitz mapping can be approximated by a C^1-smooth, Lipschitz mapping. We shall use the techniques developed in [7], the results of Lipschitz and smooth approximation of Lipschitz mappings given by P. Hájek and M. Johanis [55] and [56] (see Chapter 3), the open coverings given by M. E. Rudin [104], and the ideas of M. Moulis [92], P. Hájek and M. Johanis [56].

4.1 Preliminaries and definitions

The existence of a C^1-smooth extension of a mapping f defined on a closed subset is characterized by the following property:

Definition 4.1.1. *Let X and Z be Banach spaces and $A \subset X$ a closed subset.*

1. *We say that the mapping $f : A \to Z$ satisfies* ***the mean value condition*** *if there exists a continuous map $D : A \to \mathcal{L}(X, Z)$ such that for every $y \in A$ and every $\varepsilon > 0$, there is an open ball $B(y, r)$ in X such that*

$$||f(z) - f(w) - D(y)(z - w)|| \leq \varepsilon ||z - w||,$$

for every $z, w \in A \cap B(y, r)$. In this case, we say that f satisfies the mean value condition on A for the map D.

2. *We say that the mapping $f : A \to Z$ satisfies* ***the mean value condition for a bounded map*** *if it satisfies the mean value condition for a bounded and continuous map $D : A \to \mathcal{L}(X, Z)$, i.e.* $\sup\{||D(y)|| : y \in A\} < \infty$.

Notice that a mapping $f : A \to Z$ satisfies the mean value condition (mean value condition for a bounded map and f is Lipschitz) whenever there is a smooth extension (respectively, smooth and Lipschitz extension) to the whole space X. Indeed, let $F : X \to Z$ be a C^1-smooth (C^1-smooth and Lipschitz) mapping with $F_{|_A} = f$. By the *Fundamental Theorem of Integral Calculus* we have that

$$F(z) - F(w) = \int_0^1 F'(sz + (1 - s)w)(z - w)ds, \qquad \text{for every } w, z \in X.$$

In addition, for every $y \in A$ and every $\varepsilon > 0$ there is an open ball $B(y, r)$ in X such that

$$||F'(w) - F'(y)|| < \varepsilon, \qquad \text{for every } w \in B(y, r).$$

Thus,

$$\begin{aligned} ||f(z) - f(w) - F'(y)(z - w)|| &= || \int_0^1 (F'(sz + (1 - s)w) - F'(y))(z - w)ds|| \\ &\leq \int_0^1 ||F'(sz + (1 - s)w) - F'(y)|| \, ||z - w||ds \leq \varepsilon ||z - w|| \end{aligned}$$

for every $z, w \in A \cap B(y,r)$. Then, f satisfies the mean value condition for $D : A \to \mathcal{L}(X,Z)$ defined by $D(y) = F'(y)$ (respectively, f satisfies the mean value condition for a bounded map and f is Lipschitz).

In this chapter we obtain, under certain conditions on the Banach spaces, C^1-smooth extensions of mappings $f : A \to Z$ satisfying the mean value condition on A. More precisely, let us consider the following properties.

Definition 4.1.2.

1. *The pair of Banach spaces (X,Z) has* **property (*)** *if there is a constant $C_0 \geq 1$, which depends only on X and Z, such that for every subset $A \subset X$, every Lipschitz mapping $f : A \to Z$ and every $\varepsilon > 0$, there is a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that $||f(x) - g(x)|| < \varepsilon$ for all $x \in A$ and $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$.*
2. *The pair of Banach spaces (X,Z) has* **property (A)** *if there is a constant $C \geq 1$, which depends only on X and Z, such that for every Lipschitz mapping $f : X \to Z$ and every $\varepsilon > 0$, there exists a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$ and $\operatorname{Lip}(g) \leq C \operatorname{Lip}(f)$.*
3. *The pair of Banach spaces (X,Z) has* **property (E)** *if there is a constant $K \geq 1$, which depends only on X and Z, such that for every subset A of X and every Lipschitz mapping $f : A \to Z$, there exists a Lipschitz extension $F : X \to Z$ such that $\operatorname{Lip}(F) \leq K \operatorname{Lip}(f)$.*
4. *A Banach space X has* **property (*)**, **property (A)** *or* **property (E)** *whenever the pair $(X,\mathbb{R})$ does.*

Remark 4.1.3.

1. *Clearly, a pair of Banach spaces (X,Z) satisfies property (A) whenever it satisfies property (*). In Section 3 we shall prove that, in general, these properties are not equivalent.*
2. *X satisfies property (*) if and only if it satisfies property (A). Indeed, this is a consequence of the fact that X always has property (E): if A is a subset of X and $f : A \to \mathbb{R}$ is a Lipschitz function, then the function F defined on X as*

$$F(x) = \inf_{a \in A} \{f(a) + \operatorname{Lip}(f)||x - a||\}$$

is a Lipschitz extension of f to X and $\operatorname{Lip}(F) = \operatorname{Lip}(f)$. Notice that we have denoted this property by property (A^1) in the previous chapter.

3. *It is easy to prove that a pair of Banach spaces (X,Z) satisfies property (*) provided that (X,Z) satisfies properties (A) and (E). Moreover, if Z is a dual Banach space, then (X,Z) satisfies property (*) if and only if (X,Z) satisfies properties (A) and (E). Indeed, let us assume that (X,Z) satisfies property (*) and consider a Lipschitz mapping $f : A \to Z$, where A is a subset of X. Then, for every $n \in \mathbb{N}$, there is a C^1-smooth, Lipschitz mapping $f_n : X \to Z$ such that $||f(x) - f_n(x)|| \leq \frac{1}{n}$ for every $x \in A$ and $\operatorname{Lip}(f_n) \leq C_0 \operatorname{Lip}(f)$. Then, for every $x \in X$, the sequence $\{f_n(x)\}_n$ is bounded. Since the closed balls in $(Z, ||\cdot||^*)$ are weak*-compact, there exists for every free ultrafilter $\mathcal{U}$ in $\mathbb{N}$, the weak*-limit*

$$\widehat{f}(x) := w^* - \lim_{\mathcal{U}} f_n(x).$$

Clearly, $\widehat{f} : X \to Z$ is an extension of $f : A \to Z$ and $\operatorname{Lip}(\widehat{f}) \leq C_0 \operatorname{Lip}(f)$.

4. It is worth mentioning that property (E) can be obtained from the following approximation property: for every subset $A \subset X$, every Lipschitz function $f : A \to Z$ and every $\varepsilon > 0$, there exists a Lipschitz mapping $g : X \to Z$ such that $||f(x) - g(x)|| < \varepsilon$ for every $x \in A$ and $\operatorname{Lip}(g) \leq (1+\varepsilon)\operatorname{Lip}(f)$*. In particular, a pair of Banach spaces (X, Z) has property (E), whenever (X, Z) satisfies property (A) with constant $C_0 = (1+\varepsilon)$ for any $\varepsilon > 0$.*

Recall that every Banach space X admits continuous partitions of unity, i.e. for every open cover $\mathcal{U} = \{U_r\}_{r\in\Omega}$ of X there is a collection of continuous functions $\{\psi_i\}_{i\in I}$ such that

(1) $\psi_i : X \to [0, 1]$ for every $i \in I$,

(2) the family $\{\operatorname{supp}(\psi_i)\}_{i\in I}$ is locally finite, where $\operatorname{supp}(\psi_i) = \overline{\{x \in X : \psi_i(x) \neq 0\}}$,

(3) $\{\psi_i\}_{i\in I}$ is subordinated to $\mathcal{U} = \{U_r\}_{r\in\Omega}$, i.e. for each $i \in I$ there is $r \in \Omega$ such that $\operatorname{supp}(\psi_i) \subset U_r$ and

(4) $\sum_{i\in I} \psi_i(x) = 1$ for every $x \in X$.

First, let us recall the vector-valued version of the Tietze theorem (see for instance [24, Theorem 6.1]) and give a proof for completeness.

Proposition 4.1.4. *[24, Theorem 6.1] Let X and Z be Banach spaces and $A \subset X$ a closed subset. Then, for every continuous mapping $f : A \to Z$, there is a continuous mapping $F : X \to Z$ such that $F_{|_A} = f$.*

Proof. First, let us give the following lemma.

Lemma 4.1.5. *Let X and Z be Banach spaces, $A \subset X$ a closed subset and $R > 0$. For every continuous mapping $f : A \to B_Z(0, R)$ and every $\varepsilon > 0$, there is a continuous mapping $g : X \to B_Z(0, R)$ such that $||f(y) - g(y)|| < \varepsilon$ for every $y \in A$.*

Let us prove this lemma. For any $a \in A$ there is a ball $B(a, r_a)$ such that $||f(z)-f(a)|| < \varepsilon$ for every $z \in A \cap B(a, r_a)$. Thus, the family $\{B(a, r_a)\}_{a\in A} \cup \{X \setminus A\}$ is an open cover of X, and there is a continuous partition of unity $\{\varphi_\gamma\}_{\gamma\in\Gamma} \cup \{\varphi\}$ such that $\operatorname{supp}(\varphi_\gamma) \subset B(a_\gamma, r_\gamma)$ and $\operatorname{supp}(\varphi) \subset X \setminus A$. The mapping $g(x) := \sum_{\gamma\in\Gamma} \varphi_\gamma(x) f(a_\gamma)$ is continuous, $||f(y) - g(y)|| \leq \sum_\gamma \varphi_\gamma(y)||f(y) - f(a_\gamma)|| < \varepsilon$ for every $y \in A$, and $||g(x)|| < R$ for all $x \in X$. This finishes the proof of the lemma.

Now, let $f : A \to Z$ be a continuous mapping. Then, there is a continuous mapping

$$g_1 : X \to Z \qquad \text{such that} \qquad ||f(y) - g_1(y)|| < 1 \qquad \text{for all } y \in A.$$

Let us apply the above lemma to $f - {g_1}_{|_A} : A \to B_Z(0, 1)$. Then, there is a continuous mapping $g_2 : X \to B_Z(0, 1)$ such that $||f(y) - g_1(y) - g_2(y)|| < 1/2$ on A. By induction, we find a sequence of continuous mappings

$$g_n : X \to B_Z(0, \frac{1}{2^{n-2}}) \qquad \text{such that} \qquad ||f(y) - \sum_{j=1}^{n} g_j(y)|| < \frac{1}{2^{n-1}} \quad \text{for all } y \in A \text{ and } n \geq 2.$$

Then, the mapping $F : X \to Z$ given by $F(x) = \sum_{j\geq 1} g_j(x)$ is continuous since the serie $\sum_{j\geq 2} g_j$ is absolutely and uniformly converge in X, and $F(y) = f(y)$ for every $y \in A$. $\square$

In Section 4.2, it is stated that, if the pair of Banach spaces (X, Z) satisfies property (*), then every mapping $f : A \to Z$, where A is a closed subset of X, is the restriction of a C^1-smooth mapping (C^1-smooth and Lipschitz mapping) $F : X \to Z$ if and only if f satisfies the mean value condition (respectively, the mean value condition for a bounded map and f is Lipschitz).

In Section 4.3 we give examples of pairs of Banach spaces (X, Z) satisfying property (*). In particular, when either (i) X and Z are Hilbert spaces with X separable, or (ii) X^* is separable and Z is a Banach space which is an absolute Lipschitz retract, or (iii) $X = L_2$ and $Z = L_p$ with $1 < p < 2$, or (iv) $X = L_p$ and $Z = L_2$ with $2 < p < \infty$. Throughout this chapter, the space L_p denotes any separable Banach space $L_p(S, \Sigma, \mu)$ with (S, Σ, μ) a σ-finite measure space. We also prove that property (*) is necessary and give an example of a pair of Banach spaces satisfying property (A) but not property (*).

In Section 4.4, it is proved that every C^1-smooth mapping $f : Y \to Z$ defined on a closed subspace of X admits a C^1-smooth extension to X, whenever the pair of Banach spaces (X, Z) satisfies property (*) and every bounded and linear operator $T : Y \to Z$ can be extended to a bounded and linear operator on X. Moreover, we obtain some results on bounded and linear extension morphisms on the Banach space $C^1_L(X, Z)$ of all C^1-smooth and Lipschitz mappings $f : X \to Z$.

4.2 On smooth extension of mappings

The main results of this chapter are the following theorems.

Theorem 4.2.1. *Let (X, Z) be a pair of Banach spaces with property (*), $A \subset X$ a closed subset of X and a mapping $f : A \to Z$. Then, f satisfies the mean value condition if and only if there is a C^1-smooth extension G of f to X.*

Theorem 4.2.2. *Let (X, Z) be a pair of Banach spaces with property (*), $A \subset X$ a closed subset of X and a mapping $f : A \to Z$. Then, f is Lipschitz and satisfies the mean value condition for a bounded map if and only if there is a C^1-smooth and Lipschitz extension G of f to X.*

Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map $D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$, then we can obtain a C^1-smooth and Lipschitz extension G with $\mathrm{Lip}(G) \leq (1 + C_0)(M + \mathrm{Lip}(f))$, where C_0 is the constant given by property (*) (which depends only on X and Z).

First of all, let us notice that if the pair (X, Z) satisfies property (*), X does too, i.e. there is a constant $C_0 \geq 1$ (which depends only on X) such that for every subset $A \subset X$, every Lipschitz function $f : A \to \mathbb{R}$ and every $\varepsilon > 0$, there is a C^1-smooth and Lipschitz function $g : X \to \mathbb{R}$ such that $|g(x) - f(x)| < \varepsilon$ for all $x \in A$ and $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f)$. Indeed, let us take $e \in Z$ with $||e|| = 1$ and $\varphi \in Z^*$ with $||\varphi|| = 1$ and $\varphi(e) = 1$. Let $f : A \to \mathbb{R}$ be an L-Lipschitz function and $\varepsilon > 0$. The mapping $h : A \to Z$ defined as $h(x) = f(x)e$ for all $x \in A$, is L-Lipschitz. Since the pair (X, Z) satisfies property (*), there exists a C^1-smooth and Lipschitz mapping $\widetilde{g} : X \to Z$ such that $||h(x) - \widetilde{g}(x)|| < \varepsilon$ for all $x \in A$ and $\mathrm{Lip}(\widetilde{g}) \leq C_0 L$. The required function $g : X \to \mathbb{R}$ can be defined as $g(x) := \varphi(\widetilde{g}(x))$. Thus, by Proposition

3.2.7, the Banach space X has C^1-smooth and Lipschitz partitions of unity (in particular, it has C^1-smooth partitions of unity) whenever the pair (X, Z) satisfies property (*).

We shall need the following lemmas.

Lemma 4.2.3. *Let (X, Z) be a pair of Banach spaces with property (*). Then, for every subset $A \subset X$, every Lipschitz mapping $f : A \to B_Z(0, R)$ (with $R \in (0, \infty)$) and every $\varepsilon > 0$, there is a C^1-smooth and Lipschitz mapping $h : X \to Z$ such that*

(i) $||f(x) - h(x)|| < \varepsilon$ *for every* $x \in A$,

(ii) $||h(x)|| < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon$ *for every* $x \in X$, *and*

(iii) $\operatorname{Lip}(h) \leq C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + \varepsilon)\operatorname{Lip}(f)^{1/2})$.

Proof. Without loss of generality we may assume that $\operatorname{Lip}(f) > 0$. By property (*) there is a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that

$$||f(x) - g(x)|| < \varepsilon \qquad \text{for all } x \in A, \text{ and} \qquad \operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f).$$

Let us define $W := \{x \in X : \operatorname{dist}(x, \overline{A}) \geq \frac{1}{\operatorname{Lip}(f)^{1/2}}\}$. Since X satisfies property (*), there is a C^1-smooth function $h_A : X \to [0, 1]$ such that $h_A(x) = 1$ whenever $x \in \overline{A}$, $h_A(x) = 0$ whenever $x \in W$ and $\operatorname{Lip}(h_A) \leq 2C_0 \operatorname{Lip}(f)^{1/2}$. Indeed, we apply property (*) to obtain a C^1-smooth and Lipschitz function $H : X \to \mathbb{R}$ such that $|H(x) - \operatorname{dist}(x, A)| < \frac{1}{5\operatorname{Lip}(f)^{1/2}}$ and $\operatorname{Lip}(\varphi) \leq C_0$. Let us take a C^∞-smooth and Lipschitz function $\varphi : \mathbb{R} \to [0, 1]$ with (i) $\varphi(t) = 1$ whenever $|t| \leq \frac{1}{5\operatorname{Lip}(f)^{1/2}}$, (ii) $\varphi(t) = 0$ whenever $|t| \geq \frac{4}{5\operatorname{Lip}(f)^{1/2}}$ and (iii) $\operatorname{Lip}(\varphi) \leq 2\operatorname{Lip}(f)^{1/2}$. Then, $h_A(x) = \varphi(H(x))$ is the required function.

Let us define $h : X \to Z$ as $h(x) := g(x)h_A(x)$, which is C^1-smooth and $||f(x) - h(x)|| < \varepsilon$ for all $x \in A$ (recall that $h_A(x) = 1$ for all $x \in A$).

Since $h_A(x) = 0$ for all $x \in W$, we have that $h(x) = 0$ for all $x \in W$. Also, $||h(x)|| \leq ||g(x)|| \leq R + \varepsilon$ for all $x \in \overline{A}$. Now, for each $x \notin W$ there is $x_0 \in \overline{A}$ such that $||x - x_0|| < \frac{1}{\operatorname{Lip}(f)^{1/2}}$ and thus,

$$||g(x)|| \leq ||g(x) - g(x_0)|| + ||g(x_0)|| \leq C_0 \operatorname{Lip}(f)||x - x_0|| + R + \varepsilon < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon.$$

Therefore, $||h(x)|| < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon$ for every $x \in X$. Now, if $x \in \operatorname{int}(W)$, then $h'(x) = 0$. Also, if $x \notin \operatorname{int}(W)$, then

$$\begin{aligned} ||h'(x)|| \leq& ||g'(x)|| \, ||h_A(x)|| + ||h'_A(x)|| \, ||g(x)|| \\ \leq& C_0 \operatorname{Lip}(f) + 2C_0 \operatorname{Lip}(f)^{1/2}(C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon) \\ \leq& C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + \varepsilon)\operatorname{Lip}(f)^{1/2}). \end{aligned}$$

Thus, $\operatorname{Lip}(h) \leq C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + \varepsilon)\operatorname{Lip}(f)^{1/2})$. □

Lemma 4.2.4. *Let (X, Z) be a pair of Banach spaces with property (*). Then, for every subset $A \subset X$, every continuous mapping $F : X \to Z$ such that $F_{|A}$ is Lipschitz, and every $\varepsilon > 0$, there exists a C^1-smooth mapping $G : X \to Z$ such that* *(i)* $||F(x) - G(x)|| < \varepsilon$ *for all* $x \in X$*,*

(ii) $\mathrm{Lip}(G_{|_A}) \leq C_0 \,\mathrm{Lip}(F_{|_A})$. *Moreover,* $||G'(y)|| \leq C_0 \,\mathrm{Lip}(F_{|_A})$ *for all* $y \in A$*, where* C_0 *is the constant given by property (*).*

(iii) In addition, if F *is Lipschitz, then there exists a constant* $C_1 \geq C_0$ *depending only on* X *and* Z*, such that the mapping* G *can be chosen to be Lipschitz on* X *and* $\mathrm{Lip}(G) \leq C_1 \,\mathrm{Lip}(F)$.

Proof. Assume that the mapping $F : X \to Z$ is continuous on X and $F_{|_A}$ is Lipschitz. Since X admits C^1-smooth partitions of unity, there is (by [22, Theorem VIII 3.2]) a C^1-smooth mapping $h : X \to Z$, such that $||F(x) - h(x)|| < \varepsilon$ for all $x \in X$. Let us apply property (*) to $F_{|_A}$ to obtain a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that

(a) $||F(x) - g(x)|| < \varepsilon/4$ for all $x \in A$, and

(b) $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(F_{|_A})$.

Consider the open sets $D = \{x \in X : ||F(x) - g(x)|| < \varepsilon/4\}$, $B = \{x \in X : ||F(x) - g(x)|| < \varepsilon/2\}$ in X and the closed set $C = \{x \in X : ||F(x) - g(x)|| \leq \varepsilon/4\}$ in X. Then $A \subset D \subset C \subset B$. By [22, Proposition VIII 3.7] there is a C^1-smooth function $u : X \to [0, 1]$ such that

$$u(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \in X \setminus B. \end{cases}$$

Let us define $G : X \to Z$ as

$$G(x) := u(x)g(x) + (1 - u(x))h(x).$$

It is clear that G is a C^1-smooth mapping. Since $u(x) = 0$ for all $x \in X \setminus B$, we deduce that

$$||F(x) - G(x)|| = ||F(x) - h(x)|| < \varepsilon \quad \text{for all } x \in X \setminus B.$$

Now, if $x \in B$, then

$$\begin{aligned} ||F(x) - G(x)|| \leq u(x)||F(x) - g(x)|| + (1 - u(x))||F(x) - h(x)|| \\ \leq u(x)\varepsilon/2 + (1 - u(x))\varepsilon \leq \varepsilon. \end{aligned}$$

Finally, since $u(x) = 1$ and $G(x) = g(x)$ for every $x \in D$, we obtain that $\mathrm{Lip}(G_{|_A}) = \mathrm{Lip}(g_{|_A}) \leq C_0 \,\mathrm{Lip}(F_{|_A})$ and $||G'(y)|| = ||g'(y)|| \leq C_0 \,\mathrm{Lip}(F_{|_A})$ for all $y \in A$.

Let us now assume that F is Lipschitz on X. Let us apply property (*) to F and $F_{|_A}$ to obtain C^1-smooth and Lipschitz mappings $g,\ h : X \to Z$ such that

(a) $||F(x) - g(x)|| < \varepsilon/4$ for all $x \in A$,

(b) $||F(x) - h(x)|| < \varepsilon$ for all $x \in X$,

(c) $\mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(F_{|_A})$ and $\mathrm{Lip}(h) \leq C_0 \,\mathrm{Lip}(F)$.

We take again the subsets D, B and C as in the previous case. Notice that $\operatorname{dist}(C, X \setminus B) \geq \frac{\varepsilon}{4(\operatorname{Lip}(F)+C_0\operatorname{Lip}(F_{|A}))} = \varepsilon'$. Let us prove that there is a C^1-smooth, Lipschitz function $u : X \to [0,1]$ such that

$$u(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \in X \setminus B \end{cases} \qquad \text{and} \qquad \operatorname{Lip}(u) \leq \frac{9C_0(\operatorname{Lip}(F) + C_0\operatorname{Lip}(F_{|A}))}{\varepsilon}.$$

Let us consider $0 < r \leq \varepsilon'/4$, and the distance function $D : X \to \mathbb{R}$, $D(x) = \operatorname{dist}(x, C)$. Since the function D is 1-Lipschitz, we apply property (*) to obtain a C^1-smooth, Lipschitz function $R : X \to \mathbb{R}$ such that $\operatorname{Lip}(R) \leq C_0$ and $|D(x) - R(x)| < r$ for all $x \in X$. Also, let us take a C^1-smooth and Lipschitz function $\varphi : \mathbb{R} \to [0,1]$ with (i) $\varphi(t) = 1$ whenever $|t| \leq r$, (ii) $\varphi(t) = 0$ whenever $|t| \geq \varepsilon' - r$ and (iii) $\operatorname{Lip}(\varphi) \leq \frac{9}{8(\varepsilon'-2r)} \leq \frac{9}{4\varepsilon'}$. Next, we define the C^1-smooth function $u : X \to [0,1]$, $u(x) = \varphi(R(x))$. Notice that $\operatorname{Lip}(u) \leq \frac{9C_0(\operatorname{Lip}(F)+C_0\operatorname{Lip}(F_{|A}))}{\varepsilon}$.

Let us now consider $G : X \to Z$ as

$$G(x) = u(x)g(x) + (1 - u(x))h(x).$$

Clearly G is C^1-smooth on X. We follow the above proof to obtain that

(i) $||F(x) - G(x)|| < \varepsilon$ on X,

(ii) $\operatorname{Lip}(G_{|A}) = \operatorname{Lip}(g_{|A}) \leq C_0\operatorname{Lip}(F_{|A})$, and $||G'(y)|| \leq C_0\operatorname{Lip}(F_{|A})$ for all $y \in A$.

Additionally, if $x \in X \setminus \overline{B}$, then $u(x) = 0$, $G(x) = h(x)$, and $||G'(x)|| = ||h'(x)|| \leq C_0\operatorname{Lip}(F)$. For $x \in \overline{B}$, we have

$$\begin{aligned} ||G'(x)|| \leq& ||g(x)u'(x) + h(x)(1-u)'(x)|| + ||u(x)g'(x) + (1-u(x))h'(x)|| \\ \leq& ||(g(x) - F(x))u'(x) + (h(x) - F(x))(1-u)'(x)|| + C_0\operatorname{Lip}(F) \\ \leq& (\varepsilon/2 + \varepsilon)||u'(x)|| + C_0\operatorname{Lip}(F) \\ \leq& \frac{3\varepsilon}{2} \cdot \frac{9C_0(\operatorname{Lip}(F) + C_0\operatorname{Lip}(F_{|A}))}{\varepsilon} + C_0\operatorname{Lip}(F) \leq \frac{C_0}{2}(29 + 27C_0)\operatorname{Lip}(F). \end{aligned}$$

We define $C_1 := \frac{C_0}{2}(29 + 27C_0)$ and obtain that $\operatorname{Lip}(G) \leq C_1\operatorname{Lip}(F)$. □

Lemma 4.2.5. *Let (X, Z) be a pair of Banach spaces with property (*), a closed subset $A \subset X$ and a mapping $f : A \to B_Z(0, R)$ (with $R \in (0, \infty]$) satisfying the mean value condition for a map $D : A \to \mathcal{L}(X, Z)$. Then, for every $\varepsilon > 0$ there exists a C^1-smooth mapping $h : X \to B_Z(0, R + \varepsilon)$ such that*

(i) $||f(y) - h(y)|| < \varepsilon$ *for all* $y \in A$,

(ii) $||D(y) - h'(y)|| < \varepsilon$ *for all* $y \in A$, *and*

(iii) $\operatorname{Lip}(f - h_{|A}) < \varepsilon$.

Proof. Since A is closed, by the vector-valued version of the Tietze theorem (Proposition 4.1.4) there is a continuous extension $F : X \to B_Z(0, R)$ of f. Since X is a Banach space, $A \subset X$

is a closed subset and f satisfies the mean value condition for $D : A \to \mathcal{L}(X, Z)$ on A, there exists $\{B(y_\gamma, r_\gamma)\}_{\gamma\in\Gamma}$ a covering of A by open balls of X, with centers $y_\gamma \in A$, such that

$$||D(y) - D(y_\gamma)|| \leq \frac{\varepsilon}{8C_0} \quad \text{and} \quad ||f(z) - f(w) - D(y_\gamma)(z - w)|| \leq \frac{\varepsilon}{8C_0}||z - w||, \tag{4.1}$$

for every $y, z, w \in B_\gamma \cap A$, where $B_\gamma := B(y_\gamma, r_\gamma)$ and C_0 is the constant given by property (*).

Let us define $T_\gamma : X \to Z$ by $T_\gamma(x) = f(y_\gamma) + D(y_\gamma)(x - y_\gamma)$, for every $x \in X$. Notice that T_γ satisfies the following properties:

(B.1) T_γ is C^∞-smooth on X,

(B.2) $T'_\gamma(x) = D(y_\gamma)$ for all $x \in X$, and

(B.3) $\text{Lip}((T_\gamma - F)|_{B_\gamma\cap A}) \leq \frac{\varepsilon}{8C_0}$, since for all $z, w \in B_\gamma \cap A$,

$$||(T_\gamma - F)(z) - (T_\gamma - F)(w)|| = ||f(w) - f(z) - D(y_\gamma)(w - z)|| \leq \frac{\varepsilon}{8C_0}||z - w||.$$

Since $F : X \to B_Z(0, R)$ is a continuous mapping and X admits C^1-smooth partitions of unity, there is a C^1-smooth mapping $F_0 : X \to Z$ such that $||F(x) - F_0(x)|| < \frac{\varepsilon}{2}$ for every $x \in X$.

Let us denote $B_0 := X \setminus A$, $\Sigma := \Gamma \cup \{0\}$ (we assume $0 \notin \Gamma$), and $\mathcal{C} := \{B_\beta : \beta \in \Sigma\}$, which is a covering of X. By Proposition 3.2.7, there are an open refinement $\{W_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ of $\mathcal{C} = \{B_\beta : \beta \in \Sigma\}$ and a C^1-smooth and Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ satisfying:

(P1) $\text{supp}(\psi_{n,\beta}) \subset W_{n,\beta} \subset B_\beta$;

(P2) $\text{Lip}(\psi_{n,\beta}) \leq C_0 2^5(2^n - 1)$ for every $(n, \beta) \in \mathbb{N} \times \Sigma$; and

(P3) for each $x \in X$ there is an open ball $B(x, s_x)$ of X with center x and radius $s_x > 0$, and a natural number n_x such that

  (1) if $i > n_x$, then $B(x, s_x) \cap W_{i,\beta} = \emptyset$ for every $\beta \in \Sigma$,

  (2) if $i \leq n_x$, then $B(x, s_x) \cap W_{i,\beta} \neq \emptyset$ for at most one $\beta \in \Sigma$.

Let us define $L_{n,\beta} := \max\{\text{Lip}(\psi_{n,\beta}), 1\}$ for every $n \in \mathbb{N}$ and $\beta \in \Sigma$. Now, for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$, we apply Lemma 4.2.4 to $T_\gamma - F$ on $B_\gamma \cap A$ to obtain a C^1-smooth mapping $\delta_{n,\gamma} : X \to Z$ so that

$$||T_\gamma(x) - F(x) - \delta_{n,\gamma}(x)|| < \frac{\varepsilon}{2^{n+2}L_{n,\gamma}} \quad \text{for every } x \in X, \tag{C.1}$$

$$\|\delta'_{n,\gamma}(y)\| \leq \frac{\varepsilon}{8} \quad \text{for every } y \in B_\gamma \cap A \tag{C.2}$$

and

$$\text{Lip}((\delta_{n,\gamma})_{|B_\gamma\cap A}) \leq \frac{\varepsilon}{8}. \tag{C.3}$$

From inequality (4.1), (B.2), (C.2) and (C.3), we have, for all $y \in B_\gamma \cap A$,

$$\|T'_\gamma(y) - D(y) - \delta'_{n,\gamma}(y)\| \leq \|T'_\gamma(y) - D(y)\| + \|\delta'_{n,\gamma}(y)\| \leq \frac{\varepsilon}{4},$$

and

$$\mathrm{Lip}((T_\gamma - F - \delta_{n,\gamma})_{|B_\gamma \cap A}) \leq \frac{\varepsilon}{4}.$$

Let us define $\Delta^n_\beta : X \to Z$,

$$\Delta^n_\beta(x) = \begin{cases} F_0(x) & \text{if } \beta = 0, \\ T_\beta(x) - \delta_{n,\beta}(x) & \text{if } \beta \in \Gamma. \end{cases} \tag{4.2}$$

Thus, $||F(x) - \Delta^n_\beta(x)|| < \frac{\varepsilon}{2}$ whenever $n \in \mathbb{N}$, $\beta \in \Sigma$ and $x \in X$. Now, we define

$$h(x) = \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)\Delta^n_\beta(x).$$

Since $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ is locally finitely nonzero, h is C^1-smooth. Now, if $x \in X$, then

$$||F(x) - h(x)|| \leq \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)||F(x) - \Delta^n_\beta(x)|| \leq \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)\frac{\varepsilon}{2} < \varepsilon.$$

Therefore, $||h(x)|| < R + \varepsilon$ for all $x \in X$ (recall that $||F(x)|| \leq R$ for all $x \in X$).

Let us now estimate the distance between the derivatives. From the definitions given above, notice that:

(D.1) Since $\sum_{\mathbb{N}\times\Sigma} \psi_{n,\beta}(x) = 1$ for all $x \in X$, we have that $\sum_{\mathbb{N}\times\Sigma} \psi'_{n,\beta}(x) = 0$ for all $x \in X$.

(D.2) Thus, we can write $D(y) = \sum_{\mathbb{N}\times\Sigma}(\psi'_{n,\beta}(y))f(y) + \sum_{\mathbb{N}\times\Sigma} \psi_{n,\beta}(y)D(y)$, for every $y \in A$.

(D.3) $\mathrm{supp}(\psi_{n,0}) \subset B_0 = X \setminus A$, for all n.

(D.4) $h'(x) = \sum_{\mathbb{N}\times\Sigma} \psi'_{n,\beta}(x)\Delta^n_\beta(x) + \sum_{\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)(\Delta^n_\beta)'(x)$, for all $x \in X$.

(D.5) Properties (1) and (2) of the open refinement $\{W_{n,\beta}\}$ imply that for every $x \in X$ and $n \in \mathbb{N}$, there is at most one $\beta \in \Sigma$, which we shall denote by $\beta_x(n)$, such that $x \in \mathrm{supp}(\psi_{n,\beta})$. In the case that $y \in A$, then $\beta_y(n) \in \Gamma$. We define $F_x := \{(n,\beta) \in \mathbb{N}\times\Sigma : x \in \mathrm{supp}(\psi_{n,\beta})\}$. In particular, $F_y \subset \mathbb{N}\times\Gamma$ whenever $y \in A$.

We obtain, for $y \in A$,

$$\begin{aligned} \|D(y) - h'(y)\| & \\ \leq \sum_{(n,\beta)\in F_y} & \|\psi'_{n,\beta}(y)\|||T_\beta(y) - f(y) - \delta_{n,\beta}(y)|| \\ & + \sum_{(n,\beta)\in F_y} \psi_{n,\beta}(y)\|T'_\beta(y) - D(y) - \delta'_{n,\beta}(y)\| \end{aligned} \tag{4.3}$$

$$\leq \sum_{\{n:(n,\beta_y(n))\in F_y\}} L_{n,\beta_y(n)}||T_{\beta_y(n)}(y) - f(y) - \delta_{n,\beta_y(n)}(y)||$$
$$+ \sum_{\{n:(n,\beta_y(n))\in F_y\}} \psi_{n,\beta_y(n)}(y)\|T'_{\beta_y(n)}(y) - D(y) - \delta'_{n,\beta_y(n)}(y)\|$$
$$\leq \sum_{\{n:(n,\beta_y(n))\in F_y\}} (L_{n,\beta_y(n)}\frac{\varepsilon}{2^{n+2}L_{n,\beta_y(n)}} + \psi_{n,\beta_y(n)}(y)\frac{\varepsilon}{4}) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.$$

Let us prove that $\text{Lip}(f - h_{|_A}) < \varepsilon$. In order to simplify the notation let us write

$$S^n_\beta(y) := \Delta^n_\beta(y) - f(y) \qquad \text{and} \qquad R(y,z) := \sum_{(n,\beta)\in F_y} \psi_{n,\beta}(z)S^n_\beta(y)$$

for $y, z \in A$. Notice that $\psi_{(n,\beta)}(z) = 0$ whenever $(n, \beta) \notin F_z$ and thus,

$$R(y,z) = \sum_{(n,\beta)\in F_y\cap F_z} \psi_{n,\beta}(z)S^n_\beta(y)$$

for $y, z \in A$. In addition, let us write

$$M(z,y) := \sum_{(n,\beta)\in F_z\setminus F_y} \psi_{n,\beta}(y)S^n_\beta(z) = 0 \qquad \text{for } y,z \in A.$$

Now, from the above and properties (D.1) to (D.5), we obtain for every $y, z \in A$

$$||(h(y){-}f(y)) - (h(z) - f(z))||$$
$$= || \sum_{(n,\beta)\in F_y} \psi_{n,\beta}(y)S^n_\beta(y) - \sum_{(n,\beta)\in F_z} \psi_{n,\beta}(z)S^n_\beta(z) - R(y,z) + R(y,z) + M(z,y)||$$
$$= ||(\sum_{(n,\beta)\in F_y} \psi_{n,\beta}(y)S^n_\beta(y) - R(y,z)) + (R(y,z) - \sum_{(n,\beta)\in F_z\cap F_y} \psi_{n,\beta}(z)S^n_\beta(z))$$
$$+ (M(z,y) - \sum_{(n,\beta)\in F_z\setminus F_y} \psi_{n,\beta}(z)S^n_\beta(z))||$$
$$\leq \sum_{(n,\beta)\in F_y} |\psi_{n,\beta}(y) - \psi_{n,\beta}(z)|\, ||S^n_\beta(y)|| + \sum_{(n,\beta)\in F_z\cap F_y} \psi_{n,\beta}(z)||S^n_\beta(y) - S^n_\beta(z)||$$
$$+ \sum_{(n,\beta)\in F_z\setminus F_y} |\psi_{n,\beta}(y) - \psi_{n,\beta}(z)|\, ||S^n_\beta(z)|| \leq \sum_{(n,\beta)\in F_y} L_{n,\beta}||y-z||\, \frac{\varepsilon}{2^{n+2}L_{n,\beta}}$$
$$+ \sum_{(n,\beta)\in F_z\cap F_y} \psi_{n,\beta}(z)\frac{\varepsilon}{4}||y-z|| + \sum_{(n,\beta)\in F_z\setminus F_y} L_{n,\beta}||y-z||\, \frac{\varepsilon}{2^{n+2}L_{n,\beta}} < \varepsilon||y-z||.$$

Thus $\text{Lip}(f - h_{|_A}) < \varepsilon$. □

Lemma 4.2.6. *Let (X, Z) be a pair of Banach spaces with property (*), a closed subset $A \subset X$ and a Lipschitz mapping $f : A \to B_Z(0,R)$ (with $R \in (0,\infty]$) satisfying the mean value condition for a bounded map $D : A \to \mathcal{L}(X,Z)$ with $M = \sup\{||D(y)|| : y \in A\} < \infty$. Then, for every $\varepsilon > 0$ there is a C^1-smooth and Lipschitz mapping $g : X \to Z$ such that* *(i)* $||f(y) - g(y)|| < \varepsilon$ *for every* $y \in A$,

(ii) $||D(y) - g'(y)|| < \varepsilon$ *for every* $y \in A$,

(iii) $\text{Lip}(f - g_{|A}) < \varepsilon$,

(iv) $||g(x)|| < C_0 \,\text{Lip}(f)^{1/2} + R + \varepsilon$ *for every* $x \in X$,

(v) $\text{Lip}(g) \leq C_0((1 + 2C_0)\,\text{Lip}(f) + 2(R + \varepsilon)\,\text{Lip}(f)^{1/2} + M) + \varepsilon$ *whenever* $R < \infty$, *and* $\text{Lip}(g) \leq C_0(M + \text{Lip}(f)) + \varepsilon$ *whenever* $R = +\infty$; *where* C_0 *is the constant given by property (*).*

Proof. Let us suppose that $R < +\infty$, and take $0 < 3\varepsilon' < \varepsilon$. By Lemma 4.2.5 there is a C^1-smooth mapping $h : X \to B_Z(0, R + \varepsilon')$ such that

(i) $||f(y) - h(y)|| < \varepsilon'$ for all $y \in A$,

(ii) $||D(y) - h'(y)|| < \varepsilon'$ for all $y \in A$, and

(iii) $\text{Lip}(f - h_{|A}) < \min\{\frac{\varepsilon'}{C_0(1+2C_0)}, (\frac{\varepsilon'}{2C_0(R+2\varepsilon')})^2\}$.

Since h is C^1-smooth on X, there exists $\{B(y_\gamma, r_\gamma)\}_{\gamma\in\Gamma}$ a covering of A by open balls of X, with centers $y_\gamma \in A$ such that

$$||h(y) - h(y_\gamma)|| \leq \frac{\varepsilon'}{8C_0} \quad \text{and} \quad ||h'(y) - h'(y_\gamma)|| \leq \frac{\varepsilon'}{8C_0}, \text{ for every } y \in B_\gamma, \tag{4.4}$$

where $B_\gamma := B(y_\gamma, r_\gamma)$ and C_0 is the constant given by property (*) (which depends only on X and Z). Let us define T_γ by $T_\gamma(x) = h(y_\gamma) + h'(y_\gamma)(x - y_\gamma)$, for $x \in X$. Notice that T_γ satisfies the following properties:

(B.1) T_γ is C^∞-smooth on X,

(B.2) $T'_\gamma(x) = h'(y_\gamma)$ for all $x \in X$,

(B.3) $\text{Lip}((T_\gamma - h)|_{B_\gamma}) \leq \frac{\varepsilon'}{8C_0}$, and

(B.4) $||T'_\gamma(x)|| = ||h'(y_\gamma)|| \leq ||D(y_\gamma)|| + \varepsilon' \leq M + \varepsilon'$ for every $x \in X$.

Let us define $B_0 := X \setminus A$, $\Sigma := \Gamma \cup \{0\}$ (we assume $0 \notin \Gamma$), and $\mathcal{C} := \{B_\beta : \beta \in \Sigma\}$, which is an open covering of X. Following the proof of Lemma 4.2.5, we obtain an open refinement $\{W_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ of $\mathcal{C} = \{B_\beta : \beta \in \Sigma\}$ and a C^1-smooth and Lipschitz partition of unity $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ satisfying conditions (P1), (P2) and (P3).

Let us define $L_{n,\beta} := \max\{\text{Lip}(\psi_{n,\beta}), 1\}$ for every $n \in \mathbb{N}$ and $\beta \in \Sigma$. Now, for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$, we apply property (*) to $T_\gamma - h$ on B_γ in order to obtain a C^1-smooth mapping $\delta_{n,\gamma} : X \to Z$ so that

$$||T_\gamma(x) - h(x) - \delta_{n,\gamma}(x)|| < \frac{\varepsilon'}{2^{n+2}L_{n,\gamma}} \qquad \text{for every } x \in B_\gamma \quad \text{and} \tag{C.1}$$

$$\text{Lip}(\delta_{n,\gamma}) \leq C_0\,\text{Lip}((T_\gamma - h)_{|B_\gamma}) \leq \frac{\varepsilon'}{8}. \tag{C.2}$$

In particular,

$$||T_\gamma(x) - \delta_{n,\gamma}(x)|| < ||h(x)|| + \frac{\varepsilon'}{2^{n+2}L_{n,\gamma}} < R + 2\varepsilon' \qquad \text{for every } x \in B_\gamma. \tag{4.5}$$

From inequality (4.4), (B.2) and (C.2) and for every $y \in B_\gamma$, we have

$$\|T'_\gamma(y) - h'(y) - \delta'_{n,\gamma}(y)\| \leq \|T'_\gamma(y) - h'(y)\| + \|\delta'_{n,\gamma}(y)\| \leq \frac{\varepsilon'}{4}.$$

Therefore,

$$\operatorname{Lip}((T_\gamma - h - \delta_{n,\gamma})_{|B_\gamma}) \leq \frac{\varepsilon'}{4}.$$

Now, since $\operatorname{Lip}(h_{|A}) \leq \operatorname{Lip}(f) + (\varepsilon'/C_0)^2$, let us apply Lemma 4.2.3 to $h_{|A} : A \to B_Z(0, R+\varepsilon')$ to obtain C^1-smooth and Lipschitz mappings $F_0^n : X \to B_Z(0, C_0 \operatorname{Lip}(f)^{1/2} + R + 3\varepsilon')$ such that $||F_0^n(z) - h(z)|| < \frac{\varepsilon'}{2^{n+2}L_{n,0}}$ for all $z \in A$ and $n \in \mathbb{N}$, and

$$\operatorname{Lip}(F_0^n) \leq C_0((1 + 2C_0)\operatorname{Lip}(h_{|A}) + 2(R + 2\varepsilon')\operatorname{Lip}(h_{|A})^{1/2}).$$

From condition (iii) above, we deduce

$$\operatorname{Lip}(F_0^n) \leq C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + 2\varepsilon')\operatorname{Lip}(f)^{1/2}) + 2\varepsilon'.$$

Let us define $\Delta_\beta^n : X \to Z$ and $g : X \to Z$ as

$$\Delta_\beta^n(x) = \begin{cases} F_0^n(x) & \text{if } \beta = 0, \\ T_\beta(x) - \delta_{n,\beta}(x) & \text{if } \beta \in \Gamma, \end{cases} \qquad \text{and} \qquad g(x) = \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)\Delta_\beta^n(x). \tag{4.6}$$

Since $\{\psi_{n,\beta}\}_{n\in\mathbb{N},\beta\in\Sigma}$ is locally finitely nonzero, the mapping g is C^1-smooth. It is clear that

$$||g(x)|| \leq \sum_{(n,\beta)\in\mathbb{N}\times\Sigma} \psi_{n,\beta}(x)||\Delta_\beta^n(x)|| < C_0 \operatorname{Lip}(f)^{1/2} + R + \varepsilon \qquad \text{for all } x \in X.$$

The proofs of $||h(y) - g(y)|| < \varepsilon'$, $||h'(y) - g'(y)|| < \varepsilon'$ for all $y \in A$ and $\operatorname{Lip}((h-g)_{|A}) < \varepsilon'$ follow along the same lines as Lemma 4.2.5. Thus, $||f(y) - g(y)|| < \varepsilon$ for all $y \in A$, $||D(y) - g'(y)|| < \varepsilon$ for $y \in A$, and $\operatorname{Lip}(f - g_{|A}) < \varepsilon$.

In addition, since $||T'_\gamma(x)|| + ||\delta'_{n,\gamma}(x)|| \leq M + 9\varepsilon'/8$,

$$\begin{aligned} ||(\Delta_\beta^n)'(x)|| &\leq \max\{C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + 2\varepsilon')\operatorname{Lip}(f)^{1/2}) + 2\varepsilon', M + 9\varepsilon'/8\} \\ &\leq C_0((1 + 2C_0)\operatorname{Lip}(f) + 2(R + 2\varepsilon')\operatorname{Lip}(f)^{1/2} + M) + 2\varepsilon'. \end{aligned}$$

Let us check that g is Lipschitz. From the fact that $\sum_{(n,\beta)\in F_x} \psi'_{n,\beta}(x) = 0$ for all $x \in X$, where $F_x := \{(n,\beta) \in \mathbb{N} \times \Sigma : x \in \operatorname{supp}(\psi_{n,\beta})\}$ and the fact (P3) we deduce that

$$\begin{aligned}
||g'(x)|| &\le \sum_{(n,\beta)\in F_x} ||\psi'_{n,\beta}(x)||\,||h(x)-\Delta^n_\beta(x)|| + \sum_{(n,\beta)\in F_x} \psi_{n,\beta}(x)||(\Delta^n_\beta)'(x)|| \\
&\le \sum_{\{n:(n,\beta_x(n))\in F_x\}} L_{n,\beta_x(n)} \frac{\varepsilon'}{2^{n+2}L_{n,\beta_x(n)}} \\
&+ \sum_{\{n:(n,\beta_x(n))\in F_x\}} \psi_{n,\beta_x(n)}(x)(C_0((1+2C_0)\operatorname{Lip}(f)+2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2}+M)+2\varepsilon') \\
&< C_0((1+2C_0)\operatorname{Lip}(f)+2(R+2\varepsilon')\operatorname{Lip}(f)^{1/2}+M)+3\varepsilon',
\end{aligned}$$

for all $x \in X$, where $\beta_x(n)$ is the only index β (if it exists) satisfying condition (P3)-(2) for x. Thus, $\operatorname{Lip}(g) \le C_0((1+2C_0)\operatorname{Lip}(f)+2(R+\varepsilon)\operatorname{Lip}(f)^{1/2}+M)+\varepsilon$. (Recall, that here we do not assume $\varepsilon < \operatorname{Lip}(f)$.)

If $R=+\infty$, we apply property (*) to $h_{|_A} : A \to Z$ in order to obtain C^1-smooth mappings $F_0^n : X \to Z$ such that $||h(x)-F_0^n(x)|| < \frac{\varepsilon'}{2^{n+2}L_{0,\beta}}$ on A, and $\operatorname{Lip}(F_0^n) \le C_0\operatorname{Lip}(h_{|_A}) \le C_0\operatorname{Lip}(f)+\varepsilon'$. Thus, $||(\Delta^n_\beta)'(x)|| \le \max\{C_0\operatorname{Lip}(f)+\varepsilon', M+9\varepsilon'/8\} \le C_0(M+\operatorname{Lip}(f))+9\varepsilon'/8$ and $||g'(x)|| \le C_0(M+\operatorname{Lip}(f))+\varepsilon$ for every $x \in X$. □

Proof of Theorems 4.2.1 and 4.2.2. Let us assume that the mapping $f : A \to Z$ satisfies the mean value condition and consider $0<\varepsilon<1$. Then, by Lemma 4.2.5 there exists a C^1-smooth mapping $G_1 : X \to Z$ such that if $g_1 := G_{1|_A}$, then

(i) $||f(y)-g_1(y)|| < \frac{\varepsilon}{2^4C_0}$ for every $y \in A$,

(ii) $||D(y)-G_1'(y)|| < \frac{\varepsilon}{2^4C_0}$ for every $y \in A$, and

(iii) $\operatorname{Lip}(f-g_1) < \min\{\frac{\varepsilon}{2^4C_0(1+2C_0)}, (\frac{\varepsilon}{2^4C_0})^2\}$.

Notice that the mapping $f-g_1$ satisfies the mean value condition for the bounded map $D-G_1' : A \to \mathcal{L}(X,Z)$ with $\sup\{||D(y)-G_1'(y)|| : y \in A\} \le \frac{\varepsilon}{2^4C_0}$. Let us apply Lemma 4.2.6 to $f-g_1$ to obtain a C^1-smooth mapping $G_2 : X \to Z$ such that if $g_2 := G_{2|_A}$, then

(i) $||(f-g_1)(y)-g_2(y)|| < \frac{\varepsilon}{2^5C_0}$ for every $y \in A$,

(ii) $||D(y)-(G_1'(y)+G_2'(y))|| < \frac{\varepsilon}{2^5C_0}$ for every $y \in A$,

(iii) $\operatorname{Lip}(f-(g_1+g_2)) < \min\{\frac{\varepsilon}{2^5C_0(1+2C_0)}, (\frac{\varepsilon}{2^5C_0})^2\}$,

(iv) $||G_2(x)|| \le C_0\frac{\varepsilon}{2^4C_0}+\frac{\varepsilon}{2^4C_0}+\frac{\varepsilon}{2^5C_0} \le \frac{\varepsilon}{2^2}$ for all $x \in X$, and

(v) $\operatorname{Lip}(G_2) \le C_0((1+2C_0)\frac{\varepsilon}{2^4C_0(1+2C_0)}+2(\frac{\varepsilon}{2^4C_0}+\frac{\varepsilon}{2^5C_0})\frac{\varepsilon}{2^4C_0}+\frac{\varepsilon}{2^4C_0})+\frac{\varepsilon}{2^5C_0} \le \frac{\varepsilon}{2^2}$.

By induction, we find a sequence $G_n : X \to Z$ of C^1-smooth mappings satisfying for $n \ge 2$, where $g_n := G_{n|_A}$,

(i) $||(f-\sum_{i=1}^{n-1} g_i)(y)-g_n(y)|| < \frac{\varepsilon}{2^{n+3}C_0}$ for every $y \in A$,

(ii) $||D(y)-\sum_{i=1}^{n} G_i'(y)|| < \frac{\varepsilon}{2^{n+3}C_0}$ for every $y \in A$,

(iii) $\text{Lip}(f - \sum_{i=1}^{n} g_i) < \min\{\frac{\varepsilon}{2^{n+3}C_0(1+2C_0)}, (\frac{\varepsilon}{2^{n+3}C_0})^2\}$,

(iv) $||G_n(x)|| \leq \varepsilon/2^n$ for all $x \in X$, and

(v) $\text{Lip}(G_n) \leq \varepsilon/2^n$.

Let us define the mapping $G : X \to Z$ as $G(x) := \sum_{n=1}^{\infty} G_n(x)$. Since $||G_n(x)|| \leq \varepsilon/2^n$ and $||G'_n(x)|| \leq \text{Lip}(G_n) \leq \frac{\varepsilon}{2^n}$ for all $x \in X$ and $n \geq 2$, the series $\sum_{n=1}^{\infty} G_n$ and $\sum_{n=1}^{\infty} G'_n$ are absolutely and uniformly convergent on X. Hence, the mapping G is C^1-smooth on X. It follows from (i) that $||f(y) - \sum_{i=1}^{n} G_i(y)|| < \varepsilon/2^{n+3}$ for every $y \in A$ and $n \geq 1$. Thus $G(y) = f(y)$ for all $y \in A$.

Let us now consider $f : A \to Z$ a Lipschitz mapping satisfying the mean value condition for a bounded map $D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$. We can assume that $\varepsilon \leq \frac{16(M+\text{Lip}(f))}{9}$ (if $\text{Lip}(f) = 0$, the extension is obvious). By Lemma 4.2.6, there exists a C^1-smooth mapping $G_1 : X \to Z$ such that if $g_1 := G_{1|_A}$

(i) $||f(y) - g_1(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$,

(ii) $||D(y) - G'_1(y)|| < \frac{\varepsilon}{2^4 C_0}$ for every $y \in A$,

(iii) $\text{Lip}(f - g_1) < \min\{\frac{\varepsilon}{2^4 C_0(1+2C_0)}, (\frac{\varepsilon}{2^4 C_0})^2\}$, and

(iv) $\text{Lip}(G_1) \leq C_0(M + \text{Lip}(f)) + \frac{\varepsilon}{2^4}$.

The mappings $G_n : X \to Z$ for $n \geq 2$ are defined as in the general case. It can be checked that the mapping $G : X \to Z$ defined as $G(x) := \sum_{n=1}^{\infty} G_n(x)$ is C^1-smooth, is an extension of f to X and

$$\text{Lip}(G) \leq C_0(M + \text{Lip}(f)) + \frac{\varepsilon}{2^4} + \sum_{n=2}^{\infty} \frac{\varepsilon}{2^n} \leq (1 + C_0)(M + \text{Lip}(f)).$$

□

4.3 On the properties (*), (A) and (E)

In this section, we present examples of pairs of Banach spaces (X, Z) satisfying property (*). The first examples are pairs of Banach spaces satisfying properties (A) and (E) and thus property (*).

Example 4.3.1. *Let X and Z be Banach spaces such that X is finite-dimensional. Then, the pair (X, Z) satisfies properties (A) and (E).* On the one hand, W.B. Johnson, J. Lindenstrauss and G. Schechtman have shown in [73] that every pair of Banach space (X, Z) with X n-dimensional satisfies property (E) with constant $K(n) \geq 1$ (which depends only on the dimension of X). On the other hand, the classical convolution techniques for smooth approximation in finite-dimensional spaces provide property (A) for (X, Z).

Example 4.3.2. *Let X and Z be Hilbert spaces with X separable. Then (X, Z) satisfies the properties (A) and (E).* M.D. Kirszbraun has shown in [80] (see [14, Theorem 1.12]) that the pair (X, Z) satisfies property (E) with $K = 1$, whenever X and Z are Hilbert spaces. Also,

R. Fry has proven in [38] (see also [56, Theorem H] and Theorem 3.0.4) that (X, Z) satisfies property (A) when X is a separable Hilbert space.

Example 4.3.3. *The pairs (L_2, L_p) for $1 < p < 2$ and (L_p, L_2) for $2 < p < \infty$ satisfy properties (A) and (E).* K. Ball showed that for every $1 < p < 2$ the pair (L_2, L_p) satisfies property (E) with constant $K(p) \geq 1$ depending only on p [13]. I. G. Tsar'kov proved that for every $2 < p < \infty$ the pair (L_p, L_2) satisfies property (E) with constant $K(p) \geq 1$ depending only on p [109]. Also, the results in [38, Theorem 1] (see also Theorem 3.0.4) yield the fact that (X, Z) satisfies property (A).

Recall that a subset A of a metric space Z is called a *Lipschitz retract* of Z if there is a *Lipschitz retraction* from Z to A, i.e. there is a Lipschitz map $r : Z \to A$ such that $r_{|A} = id_A$. A metric space Z is called an *absolute Lipschitz retract* if it is a Lipschitz retract of any metric space W containing Z. Some examples of absolute Lipschitz retracts are the following:

1. $\mathbb{R}^n$ for every $n \in \mathbb{N}$,
2. $(c_0(\mathbb{N}), ||\cdot||_\infty)$,
3. $(\ell_\infty(\mathbb{N}), ||\cdot||_\infty)$,
4. $(C_u(P), ||\cdot||_\infty)$, where P is a metric space and
$$C_u(P) = \{f : P \to \mathbb{R} : f \text{ is bounded and uniformly continuous}\},$$
5. the space $(C(K), ||\cdot||_\infty)$ for every compact Hausdorff space K, and
6. $(B_0(V), ||\cdot||_\infty)$, where V is a topological space, $v_0 \in V$ and
$$B_0(V) = \{f : V \to \mathbb{R} : f \text{ is bounded, and } f(v) \to 0 \text{ whenever } v \to v_0\}.$$

(See [14] and [87] for more information on absolute Lipschitz retracts and a proof of these examples.) An absolute Lipschitz retract space satisfies the following Lipschitz extension property.

Proposition 4.3.4. *[14, Proposition 1.2] Let Z be a metric space. The following are equivalent:*

(i) *Z is an absolute Lipschitz retract.*

(ii) *There is $K \geq 1$, which only depends on Z, such that for every metric space X, every subset $A \subset X$ and every Lipschitz mapping $f : A \to Z$, there is a Lipschitz extension $F : X \to Z$ of f such that* $\mathrm{Lip}(F) \leq K\,\mathrm{Lip}(f)$.

By combining the above characterization and the results in [56], we obtain the following proposition.

Proposition 4.3.5. *Let X be a Banach space such that there are a set $\Gamma \neq \emptyset$ and a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1-smooth coordinate functions $e_\gamma^* \circ \varphi : X \to \mathbb{R}$. Let Z be a Banach space which is an absolute Lipschitz retract. Then the pair (X, Z) satisfies properties (A) and (E).*

Proof. Property (E) trivially follows from Proposition 4.3.4. Now, let us take the mapping $f \circ \varphi^{-1} : \varphi(X) \to Z$ which is $\mathrm{Lip}(\varphi^{-1})\,\mathrm{Lip}(f)$-Lipschitz. By Proposition 4.3.4, there is a Lipschitz extension $\widetilde{f} : c_0(\Gamma) \to Z$ of $f \circ \varphi^{-1}$ with $\mathrm{Lip}(\widetilde{f}) \leq K\,\mathrm{Lip}(\varphi^{-1})\,\mathrm{Lip}(f)$ and K is the constant given in Proposition 4.3.4. Now, from Theorem 3.0.9 we can find a C^∞-smooth and Lipschitz mapping $h : c_0(\Gamma) \to Z$ which locally depends on finitely many coordinate functionals $\{e^*_\gamma\}_{\gamma\in\Gamma}$, such that $||\widetilde{f}(x) - h(x)|| < \varepsilon$ for every $x \in c_0(\Gamma)$ and $\mathrm{Lip}(h) = \mathrm{Lip}(\widetilde{f})$. Let us define $g : X \to Z$ as $g(x) := h(\varphi(x))$ for every $x \in X$. The mapping g is C^1-smooth, $||f(x) - g(x)|| < \varepsilon$ for all $x \in X$ and $\mathrm{Lip}(g) \leq C\,\mathrm{Lip}(f)$, with $C := K\,\mathrm{Lip}(\varphi)\,\mathrm{Lip}(\varphi^{-1})$. Thus (X, Z) has property (A). □

This provides the following example.

Example 4.3.6. *Let X and Z be Banach spaces such that there are a set $\Gamma \neq \emptyset$ and a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1-smooth coordinate functions, and Z is an absolute Lipschitz retract. Then, the pair (X, Z) satisfies properties (A) and (E). In particular, let X and Z be Banach spaces such that X^* is separable and Z is an absolute Lipschitz retract. Then, the pair (X, Z) satisfies properties (A) and (E).* Notice that P. Hájek and M. Johanis [56] (see the proof of Theorem 3.1.3) proved the existence of a bi-Lipschitz homeomorphism with C^k-smooth coordinate functions in every separable Banach space with a C^k-smooth and Lipschitz bump function.

Now, with these examples, Theorem 4.2.1 and Theorem 4.2.2, we obtain the following consequence.

Corollary 4.3.7. *Let X and Z be Banach spaces and assume that at least one of the following conditions holds:*

(i) *X is finite-dimensional,*

(ii) *X and Z are Hilbert spaces and X is separable,*

(iii) *$X = L_2$ and $Z = L_p$ with $1 < p < 2$,*

(iv) *$X = L_p$ and $Z = L_2$ with $2 < p < \infty$,*

(v) *there are a set $\Gamma \neq \emptyset$ and a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1-smooth coordinate functions (for example, when either X^* is separable or $X = c_0(\Gamma)$), and Z is an absolute Lipschitz retract.*

Let A be a closed subset of X and $f : A \to Z$ a mapping. Then, f satisfies the mean value condition (mean value condition for a bounded map and f is Lipschitz) on A if and only if there is a C^1-smooth (respectively, C^1-smooth and Lipschitz) extension G of f to X.

Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map $D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$, then we can obtain a C^1-smooth and Lipschitz extension G with $\mathrm{Lip}(G) \leq (1+C_0)(M+\mathrm{Lip}(f))$, where $C_0 \geq 1$ is the constant given by property (*) (which depends only on X and Z).

It is easy to see the relationship between smooth extension (smooth and Lipschitz extension) and the existence of C^1-smooth bump functions (respectively, C^1-smooth and Lipschitz bump functions). Indeed, if every real-valued function defined on a closed subset of X which satisfies the mean value condition (mean value condition for a bounded map and it is Lipschitz) admits a C^1-smooth extension to X (C^1-smooth and Lipschitz extension), then the function (see Figure 4.1)

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \in X \setminus B(0,1) \end{cases}$$

admits an C^1-smooth extension (respectively, C^1-smooth and Lipschitz extension), since f satisfies the mean value condition for the bounded map $D : \{0\} \cup (X \setminus B(0,1)) \to X^*$ defined as $D(x) = 0_{X^*}$. This extension is a C^1-smooth bump function (respectively, C^1-smooth and Lipschitz bump function).

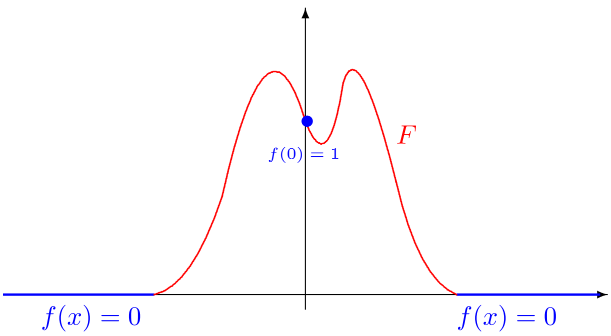


Figure 4.1: Bump function

Let us now prove that property (*), which is a closely related to admitting smooth bump functions, is necessary to obtain C^1-smooth and Lipschitz extensions. We also obtain an example of a pair of Banach spaces satisfying property (A) but not property (*). Thus, it does not admit C^1-smooth and Lipschitz extension.

Proposition 4.3.8. *Let (X, Z) be a pair of Banach spaces such that there is a constant $C \geq 1$, which only depends on X and Z, such that for every closed subset $A \subset X$ and every Lipschitz mapping $f : A \to Z$ satisfying the mean value condition for a bounded map D with $M = \sup\{||D(y)|| : y \in A\} < \infty$, there exists a C^1-smooth and Lipschitz extension G of f to X with $\text{Lip}(G) \leq C(M + \text{Lip}(f))$. Then, the pair (X, Z) satisfies property (*). Therefore, by Theorem 4.2.2, the above assumption is equivalent to property (*).*

Proof. Let A be a subset of X, $f : A \to Z$ a L-Lipschitz mapping and $\varepsilon > 0$. Let us take a $\frac{\varepsilon}{(C+1)L}$-net in A which we shall denote by N, i.e. a subset N of A such that (i) $||z-y|| \geq \frac{\varepsilon}{(C+1)L}$ for every $z, y \in N$, (ii) for every $x \in A$ there is a point $y \in N$ such that $||x - y|| \leq \frac{\varepsilon}{(C+1)L}$. Clearly, N is a closed subset of X and $f_{|_N} : N \to Z$ is a L-Lipschitz mapping on N satisfying the mean value condition for the bounded map given by $D(x) = 0 \in \mathcal{L}(X, Z)$ for every $x \in N$. Then, by assumption, there exists a C^1-smooth and CL-Lipschitz mapping $G : X \to Z$ such that $G_{|_N} = f_{|_N}$. For any $x \in A$, let us choose $y \in N$ such that $||x - y|| \leq \frac{\varepsilon}{(C+1)L}$. Then, $G(y) = f(y)$ and

$$||f(x) - G(x)|| \leq ||f(x) - f(y)|| + ||G(x) - G(y)|| \leq (L + CL)||x - y|| \leq \varepsilon.$$

□

Example 4.3.9. *Although the pairs (L_p, L_2) and (L_2, L_q) with $1 < p < 2$ and $2 < q < \infty$ satisfy property (A) (see [38]), they do not satisfy property (E) [97].* Thus, Remark 4.1.3 (3) implies that the pairs (L_p, L_2) and (L_2, L_q) do not satisfy property (*) whenever $1 < p < 2 < q < \infty$ and L_p, L_q and L_2 are infinite dimensional, and the above proposition reveals that there exist Lipschitz mappings $h : A \to L_2$ and $h' : B \to L_q$ defined on closed subsets A of L_p and B of L_2 with $1 < p < 2$ and $2 < q < \infty$ satisfying the mean value condition on A and B for a bounded map which cannot be extended to C^1 smooth and Lipschitz mappings on L_p and L_2, respectively, i.e. the conclusion of Theorem 4.2.2 does not hold for the pairs (L_p, L_2) and (L_2, L_q) with $1 < p < 2$ and $2 < q < \infty$. In particular, property (A) is a necessary condition but it is not a sufficient condition to obtain C^1-smooth and Lipschitz extensions, and properties (*) and (A) are not equivalent in general.

It is worth pointing out the following characterization that it is obtained when the problem is restricted to the separable and real-valued case.

Proposition 4.3.10. *Let X be a separable Banach space. The following statements are equivalent:*

(i) *X satisfies property (*).*

(ii) *X admits C^1-smooth and Lipschitz extension of every Lipschitz function defined on a closed subset, which satisfies the mean value condition for a bounded map, i.e. for every closed subset $A \subset X$ and every Lipschitz function $f : A \to \mathbb{R}$ satisfying the mean value condition for a bounded map $D : A \to X^*$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$, there is a C^1-smooth and Lipschitz extension of f to X, $G : X \to \mathbb{R}$, with $\text{Lip}(G) \leq C(M + \text{Lip}(f))$ (where $C \geq 1$ depends only on the space X).*

(iii) *X admits C^1-smooth extension of every function defined on a closed subset, which satisfies the mean value condition.*

(iv) *X admits a C^1-smooth bump function.*

(v) *X^* is Asplund.*

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are given by Theorem 4.2.1 and Theorem 4.2.2. We have already showed that $(ii) \Rightarrow (iv)$ and that (iii) implies that X admits a C^1-smooth bump function (see Figure 4.1), which is equivalent to (iv) (see [22]). $(iv) \Rightarrow (v)$ is well known (see again [22]) and $(v) \Rightarrow (i)$ was shown by P. Hájek and M. Johanis in [56]. □

4.4 Smooth extension from subspaces

Finally, let us make a brief comment on the extension of C^1-smooth mappings defined on a subspace. First of all, we must realize that we need to extend bounded and linear operators to extend smooth mappings from subspaces.

Proposition 4.4.1. *Let X and Z be Banach spaces and Y a closed subspace of X. If every C^1-smooth mapping $f : Y \to Z$ can be extended to a C^1-smooth mapping $F : X \to Z$, then for every bounded and linear operator $T : Y \to Z$ there is a bounded and linear operator*

$\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|Y} = T$. Moreover, assume that every C^1-smooth and Lipschitz mapping $f : Y \to Z$ can be extended to a C^1-smooth and Lipschitz mapping $F : X \to Z$ such that $\operatorname{Lip}(F) \leq C \operatorname{Lip}(f)$ with C depending only on X and Z. Then, for every bounded and linear operator $T : Y \to Z$ there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|Y} = T$ and $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq C||T||_{\mathcal{L}(Y,Z)}$.

Proof. Let $T : Y \to Z$ be a bounded and linear operator. Obviously, T is C^1-smooth on Y, $||T||_{\mathcal{L}(Y,Z)}$-Lipschitz and $T'(0) = T$. If $F : X \to Z$ is a C^1-smooth extension (and $C||T||_{\mathcal{L}(Y,Z)}$-Lipschitz), then $\widetilde{T} = F'(0)$ satisfies that $\widetilde{T}_{|Y} = F'(0)_{|Y} = T'(0) = T$ (and $||\widetilde{T}||_{\mathcal{L}(X,Z)} = ||F'(0)|| \leq \operatorname{Lip}(F) \leq C||T||_{\mathcal{L}(Y,Z)}$, respectively). □

Definition 4.4.2. *We say that the pair of Banach spaces (X, Z) satisfies the **linear extension property** if there is $\lambda \geq 1$, which depends only on X and Z, such that for every closed subspace $Y \subset X$ and every bounded and linear operator $T : Y \to Z$, there is a bounded and linear operator $\widetilde{T} : X \to Z$ such that $\widetilde{T}_{|Y} = T$ and $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda||T||_{\mathcal{L}(Y,Z)}$.*

Examples 4.4.3.

(i) Maurey's extension theorem [91] asserts that the pair of Banach spaces (X, Z) satisfies the linear extension property whenever X has type 2 and Z has cotype 2. Therefore, (L_2, L_p) for $1 < p < 2$ and (L_p, L_2) for $2 < p < \infty$ satisfy the linear extension property (recall that L_p has type 2 for $2 \leq p < \infty$ and cotype 2 for $1 < p \leq 2$, see [1]).

(ii) For every compact Hausdorff space K, every non-empty set Γ and $1 < p < \infty$, the pairs $(c_0(\Gamma), C(K))$ and $(\ell_p(\mathbb{N}), C(K))$ satisfy the linear extension property ([88, Theorem 3.1], [74] and [72, Chapter 40]).

(iii) For every compact Hausdorff space K, the pair $(X, C(K))$ satisfies the linear extension property whenever X is an Orlicz space with a separable dual [77].

(iv) The pair $(X, c_0(\mathbb{N}))$ satisfies the linear extension property whenever X is a separable Banach space [105] (see also [72, Chapter 40]).

(v) However, there are Banach spaces X so that for any K compact Hausdorff, the pair $(X, C(K))$ fails the linear extension property. For instance, W.B. Johnson and M. Zippin [75] proved that there are subspaces Y of ℓ_1 such that not every bounded and linear operator $T : Y \to C(K)$ can be extended to a bounded and linear operator $\widetilde{T} : \ell_1 \to C(K)$. Also, N.J. Kalton [78] showed that for every $1 < p < \infty$ there exist a separable and superreflexive Banach space X containing ℓ_p and a bounded and linear operator $T : \ell_p \to C(K)$ that cannot be extended to a bounded and linear operator $\widetilde{T} : X \to C(K)$. So, by Proposition 4.4.1, they are examples of pair of Banach spaces (X, Z), one of them satisfying properties (A) and (E), so that not every C^1-smooth mapping defined on a subspace of X can be extended to a C^1-smooth mapping defined on the whole space X.

We shall prove the following useful proposition.

Proposition 4.4.4. *Let (X, Z) be a pair of Banach spaces satisfying the linear extension property and Y a closed subspace of X. If $f : Y \to Z$ is a C^1-smooth mapping (C^1-smooth and Lipschitz mapping), then f satisfies the mean value condition (respectively, mean value condition for a bounded map) on Y.*

Proof. First, let us give the following lemma.

Lemma 4.4.5. *Let (X, Z) be a pair of Banach spaces satisfying the* linear extension property *and Y a closed subspace of X. Then there is a constant $\eta \geq 1$ and there is a continuous map $B : \mathcal{L}(Y, Z) \to \mathcal{L}(X, Z)$ such that $B(f)_{|_Y} = f$ and $||B(f)||_{\mathcal{L}(X,Z)} \leq \eta ||f||_{\mathcal{L}(Y,Z)}$ for every $f \in \mathcal{L}(Y, Z)$.*

The proof of this lemma follows the lines of the real-valued case [7, Lemma 2]. Indeed, let us take $W = \mathcal{L}(X, Z)$, $V = \mathcal{L}(Y, Z)$ and $T : W \to V$ the bounded and linear map given by the restriction to Y, $T(f) = f_{|_Y}$. By assumption, the map T is onto. Thus, we apply the Bartle-Graves's theorem (see [22, Lemma VII 3.2]) in order to find the map B.

Now, if $f : Y \to Z$ is a C^1-smooth mapping (C^1-smooth and Lipschitz mapping), we consider the mapping $D : Y \to \mathcal{L}(X, Z)$ defined as $D(y) := B(f'(y))$ for every $y \in Y$. Then, f satisfies the mean value condition for D (respectively, the mean value condition for the bounded map D). □

Now, we can apply Theorems 4.2.1 and 4.2.2 to obtain the following result on C^1-smooth extensions and C^1-smooth and Lipschitz extensions to X of C^1-smooth mappings defined on Y whenever (X, Z) satisfies property (*) and the linear extension property.

Corollary 4.4.6. *Let (X, Z) be a pair of Banach spaces which satisfies property (*) and the linear extension property. Let Y be a closed subspace of X. Then, every C^1-smooth mapping defined on Y has a C^1-smooth extension to the whole space X.*

Moreover, there is $C \geq 1$, which depends on X and Z, such that every Lipschitz and C^1-smooth mapping $f : Y \to Z$ has a Lipschitz and C^1-smooth extension $F : X \to Z$ to X with $\text{Lip}(F) \leq C \,\text{Lip}(f)$.

In particular:

Corollary 4.4.7. *Let (X, Z) be any of the following pairs of Banach spaces:*

(i) (L_p, L_2), $2 < p < \infty$,

(ii) $(c_0(\Gamma), C(K))$, *Γ is a non-empty set and K is a compact Hausdorff space,*

(iii) $(\ell_p(\mathbb{N}), C(K))$, $1 < p < \infty$ *and K is a compact Hausdorff space,*

(iv) $(X, C(K))$, *X is an Orlicz space with separable dual and K is a compact Hausdorff space,*

(v) $(X, c_0(\mathbb{N}))$, *X with separable dual,*

(vi) $(X, \mathbb{R})$, *such that there is a set $\Gamma \neq \emptyset$ and there is a bi-Lipschitz homeomorphism $\varphi : X \to c_0(\Gamma)$ with C^1-smooth coordinate functions (for instance, when X^* is separable).*

Let Y be a closed subspace of X. Then, every C^1-smooth mapping $f : Y \to Z$ has a C^1-smooth extension to X.

Moreover, there is $C \geq 1$, which depends only on X and Z, such that every Lipschitz and C^1-smooth mapping $f : Y \to Z$ has a Lipschitz and C^1-smooth extension $F : X \to Z$ to X with $\text{Lip}(F) \leq C \,\text{Lip}(f)$.

Recall that a Banach space X has the *separable complementation property* provided that every separable Banach subspace of X is contained in a separable complemented subspace of X. An important class of Banach spaces with this property is the class of WCD Banach spaces (see [22, Chapter VI]). For more examples of Banach spaces with the separable complemented property see [31]. So, using (v) and (vi) of Corollary 4.4.7, it is easy to prove the following consequence.

Corollary 4.4.8. *Let X be an Asplund WCG Banach space and let Z be either $c_0(\mathbb{N})$ or $\mathbb{R}$. Let Y be a separable and closed subspace of X. Then, every C^1-smooth mapping $f: Y \to Z$ has a C^1-smooth extension to X.*

Moreover, there is $C \geq 1$, which depends only on X and Z, such that every Lipschitz and C^1-smooth mapping $f: Y \to Z$ has a Lipschitz and C^1-smooth extension $F: X \to Z$ to X with $\mathrm{Lip}(F) \leq C\,\mathrm{Lip}(f)$.

Sometimes it is not possible to extend C^1-smooth mappings defined on any subspace of X, but there exist some subspaces from which every C^1-smooth mapping can be extended. For instance, in the example 4.4.3(v) we have seen that for every $1 < p < \infty$ there is a separable and superreflexive Banach space X containing ℓ_p such that there are C^1-smooth mappings $f: \ell_p \to C(K)$ that cannot be extended to a C^1-smooth mapping $F: X \to C(K)$. Nevertheless, it is obvious that every C^1-smooth mapping defined on a complement subspace of X can be extended to the whole space. In this way, let us consider the following definition.

Definition 4.4.9. *Let X and Z be Banach spaces and Y a closed subspace of X. We say that the pair (Y, Z) has the* ***linear X-extension property*** *if there is $\lambda \geq 1$, which depends on X, Y and Z, such that for every bounded and linear map $T: Y \to Z$ there is a bounded and linear extension $\widetilde{T}: X \to Z$ with $||\widetilde{T}||_{\mathcal{L}(X,Z)} \leq \lambda ||T||_{\mathcal{L}(Y,Z)}$.*

By Theorem 4.2.1, Theorem 4.2.2 and a slight modification of Proposition 4.4.4, we obtain the following corollary.

Corollary 4.4.10. *Let (X, Z) be a pair of Banach spaces with property (*). Let Y be a closed subspace of X such that the pair (Y, Z) has the linear X-extension property. Then, every C^1-smooth mapping $f: Y \to Z$ has a C^1-smooth extension to X.*

Moreover, there is $C \geq 1$, which depends on X, Y and Z, such that every Lipschitz and C^1-smooth mapping $f: Y \to Z$ has a Lipschitz and C^1-smooth extension $F: X \to Z$ to X with $\mathrm{Lip}(F) \leq C\,\mathrm{Lip}(f)$.

We conclude this chapter with some considerations on extension morphisms of C^1-smooth mappings. Let X and Z be Banach spaces and consider the Banach space

$$C^1_L(X, Z) := \{f : X \to Z \ : \ f \text{ is } C^1\text{-smooth and Lipschitz}\},$$

with the norm $||f||_{C^1_L} := ||f(0)|| + \mathrm{Lip}(f)$. We write $C^1_L(X) := C^1_L(X, \mathbb{R})$.

Definition 4.4.11. *Let X and Z be Banach spaces and Y a closed subspace of X. We say that a bounded and linear mapping $T: C^1_L(Y, Z) \to C^1_L(X, Z)$ ($T: Y^* \to X^*$) is an extension morphism whenever $T(f)_{|Y} = f$ for every $f \in C^1_L(Y, Z)$ (respectively, for every $f \in Y^*$).*

Lemma 4.4.12. *Let X be a Banach space and Y a closed subspace of X. If there exists an extension morphism $T : C^1_L(Y) \to C^1_L(X)$, then there exists an extension morphism $S : Y^* \to X^*$.*

Proof. Let $T : C^1_L(Y) \to C^1_L(X)$ be an extension morphism and define $D : C^1_L(X) \to X^*$ as $D(f) = f'(0)$ for every $f \in C^1_L(X)$. The mapping D is linear, bounded and $||D|| \leq 1$. Thus, $D \circ T : C^1_L(Y) \to X^*$ is linear and bounded. Also, $(D \circ T(f))_{|_Y} = (T(f)'(0))_{|_Y} = f'(0) \in \mathcal{L}(Y, Z)$. Now, let us take the restriction $S := D \circ T_{|_{Y^*}} : Y^* \to X^*$, which is linear, bounded,

$$S(\varphi)_{|_Y} = (T(\varphi))'(0)_{|_Y} = \varphi'(0) = \varphi, \text{ and}$$
$$||S(\varphi)||_{X^*} = ||D \circ T(\varphi)||_{X^*} \leq ||T(\varphi)||_{C^1_L} \leq ||T||\,||\varphi||_{C^1_L},$$

for every $\varphi \in Y^*$. □

The above lemma and the results given by H. Fakhoury in [28] provide the following characterizations.

Proposition 4.4.13. *Let X and Z be Banach spaces. The following statements are equivalent:*

(i) There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $P : C^1_L(Y, Z) \to C^1_L(X, Z)$ with $||P|| \leq M$.

(ii) There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $T : C^1_L(Y) \to C^1_L(X)$ with $||T|| \leq M$.

(iii) There is a constant $M > 0$ such that for every closed subspace $Y \subset X$, there exists an extension morphism $S : Y^* \to X^*$ with $||S|| \leq M$.

(iv) X is isomorphic to a Hilbert space.

Proof. The equivalence between *(iii)* and *(iv)* was established in [28, Théorème 3.7].

(i) ⇒ *(ii)* Let us take $z \in S_Z$ and $\varphi \in S_{Z^*}$ (where S_Z and S_{Z^*} denote the unit spheres of Z and Z^*, respectively) such that $\varphi(z) = 1$. Let Y be a closed subspace of X and $P : C^1_L(Y, Z) \to C^1_L(X, Z)$ an extension morphism with $||P|| \leq M$. For every $f \in C^1_L(Y)$, let us consider the Lipschitz and C^1-smooth mapping $f_z : Y \to Z$ defined as $f_z(y) = zf(y)$ for every $y \in Y$. Let us define $R_f := P(f_z) \in C^1_L(X, Z)$. Then, the mapping $T : C^1_L(Y) \to C^1_L(X)$ defined as $T(f) = \varphi \circ R_f$, where $f \in C^1_L(Y)$ is a linear extension mapping with $||T|| \leq M$.

(ii) ⇒ *(iii)* is given by Lemma 4.4.12 and *(iv)* ⇒ *(i)* follows from the result that every closed subspace of a Hilbert space H is complemented in H [89]. □

Recall that X is a $\mathscr{P}_\lambda$-space if X is a complemented subspace of every Banach superspace W of X (see [18, p. 95]).

Corollary 4.4.14. *Let X and Z be Banach spaces. The following statements are equivalent:*

(i) There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $P : C^1_L(X, Z) \to C^1_L(W, Z)$ with $||P|| \leq M$.

(ii) There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $T : C^1_L(X) \to C^1_L(W)$ with $||T|| \leq M$.

(iii) There is a constant $M > 0$ such that for every Banach superspace W of X, there exists an extension morphism $S : X^* \to W^*$ with $||S|| \leq M$.

(iv) X is a $\mathscr{P}_\lambda$-space.

Proof. The equivalence between *(iii)* and *(iv)* was established in [28, Corollaire 3.3]. The rest of the proof is similar to that of Proposition 4.4.13. □

It is worth noticing that the existence of an extension morphism $S : X^* \to W^*$ does not imply that X is a complement subspace of W. Indeed, there exists an extension morphism $S : X^* \to X^{***}$ for every Banach space X (see for instance [28]). Nevertheless, there is no an extension morphism $T : C^1_L(X) \to C^1_L(W)$, up to our knowledge, with X a non-complement subspace of W.

Open Problems

Let us list some questions that remain open in the area.

1. When does a Banach space admit C^2-smooth extensions?

2. Let $f : c_0(\mathbb{N}) \to \mathbb{R}$ be a C^1-smooth function. Is there a C^1-smooth function $F : \ell_\infty(\mathbb{N}) \to \mathbb{R}$ such that $F_{|c_0} = f$? Is there at least a C^1-smooth function $F : U \to \mathbb{R}$ defined on an open subset $U \subset \ell_\infty(\mathbb{N})$ such that $c_0(\mathbb{N}) \subset U$ and $F_{|c_0} = f$?

3. We have showed that there exist C^1-smooth extensions of C^1-smooth real-valued functions defined on closed subspaces whenever X satisfies property (*). Nevertheless, we do not know whether property (*) or any other condition is necessary. In other words, are there a Banach space X, a closed subspace $Y \subset X$ and a C^1-smooth function $f : Y \to \mathbb{R}$ such that there is not a C^1-smooth extension of f to the whole space X?

4. In Remark 4.1.3 we have proved that if Z is a dual Banach space, then (X, Z) satisfies property (*) if and only if (X, Z) satisfies properties (A) and (E). Is the property (*) equivalent to the properties (A) and (E) when Z is a general Banach space? I.e., is the duality condition on Z necessary?

5. Let (X, Z) be a pair of Banach spaces such that for every subset A in X and every Lipschitz mapping $f : A \to Z$, there exists a Lipschitz extension $F : X \to Z$ of f. Does the pair (X, Z) have property (E)? In other words, may the Lipschitz extension F be chosen so that $\mathrm{Lip}(F) \leq K\,\mathrm{Lip}(f)$ and K depending only on X and Z?

6. Let Y be a closed subspace of a Banach space X. H. Fakhoury [28] showed that the existence of an extension morphism $S : Y^* \to X^*$ is equivalent to the existence of an extension morphism $R : \mathrm{Lip}(Y) \to \mathrm{Lip}(X)$, where

$$\mathrm{Lip}(X) := \{f : X \to \mathbb{R} \ : \ f \text{ is Lipschitz}\},$$

with the norm $||f||_{\text{Lip}} := |f(0)| + \text{Lip}(f)$. We have proved in Lemma 4.4.12 that there exists an extension morphism $S : Y^* \to X^*$ whenever there exists an extension morphism $T : C^1_L(Y) \to C^1_L(X)$. Nevertheless, we do not know whether the reverse holds. I.e., is there an extension morphism $T : C^1_L(Y) \to C^1_L(X)$ whenever there exist both an extension morphism $S : Y^* \to X^*$ and an extension morphism $R : \text{Lip}(Y) \to \text{Lip}(X)$?

Chapter 5

On Finsler manifolds

In this Chapter we present the definitions of Banach manifolds and Finsler manifolds, as well as several well-known properties that will be used in the following chapters. We also give some new properties on Finsler manifolds. Our object is to refresh the reader with the language and the tools of manifolds, and obtain some new properties which are used in the following chapters.

In Section 5.1 we introduce the basic knowledge of Banach manifolds. We continue in Section 5.2 by recalling the concepts of C^ℓ Finsler manifold in the sense of Palais and Neeb-Upmeier introduced in [100], [98], [110] (both modeled on a Banach space X either separable or non-separable). We introduce the notions of C^ℓ Finsler manifold in the sense of Neeb-Upmeier *weak-uniform* and *uniform* as a generalization of a C^ℓ Finsler manifold in the sense of Palais and a Riemannian manifold, respectively. Some results related to the structure of C^ℓ Finsler manifolds in the sense of Palais, Neeb-Upmeier *weak-uniform* and Neeb-Upmeier *uniform* are provided. In Section 5.3 we will give some mean value inequalities for C^ℓ Finsler manifolds in the different senses, and a result on the existence of suitable local bi-Lipschitz diffeomorphisms. Both properties will be essential tools to establish the results on the following chapters.

The notation we use is standard. A more exhaustive description of these concepts can be found in classical books, for instance [23], [82], [84], [19] and [110]. We will say that the norms $||\cdot||_1$ and $||\cdot||_2$ defined on X are K-equivalent ($K \geq 1$) whether $\frac{1}{K}||v||_1 \leq ||v||_2 \leq K||v||_1$, for every $v \in X$.

5.1 Banach manifolds

Definition 5.1.1. *A C^ℓ* ***Banach manifold*** *(modeled on a Banach space $(X, ||\cdot||)$) is a set M and a family of injective mappings $\varphi_\alpha : U_\alpha \subset M \to X$ of sets U_α onto open sets $\varphi_\alpha(U_\alpha) \subset X$ such that:*

(1) $\bigcup_\alpha U_\alpha = M$.

(2) For every pair α, β with $U_\alpha \cap U_\beta = W \neq \emptyset$, the sets $\varphi_\alpha(W)$ and $\varphi_\beta(W)$ are open in X and the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^ℓ-smooth in $\varphi_\alpha(W)$.

(3) The family $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ is maximal relative to the conditions (1) and (2).

The pair $(U_\alpha, \varphi_\alpha)$ with $x \in U_\alpha$ is called a **chart** of M at x. A family $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ satisfying (*1*) and (*2*) is called an **atlas** on M. In general, the extension to the maximal structure will be done without further comment, since given an atlas on M, we can easily complete it to a maximal one, by taking the union of all the charts that, together with any of the charts of the given structure, satisfy condition (*2*). Recall that an atlas on M induces a natural topology on M: a set $A \subset M$ is open whether $\varphi_\alpha(A \cap U_\alpha)$ is an open set in X for all α.

Definition 5.1.2. *Let M and N be C^ℓ Banach manifolds modeled on X and Y, respectively. A mapping $f : M \to N$ is C^k**-smooth** at $x \in M$ ($k \leq \ell$) if there are (U, φ) a chart of M at x and (V, ψ) a chart of N at $f(x)$, such that $f(U) \subset V$ and the mapping*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U) \subset X \to Y$$

is C^k-smooth at $\varphi(x)$.

$$\begin{array}{ccc} U \subset M & \xrightarrow{\quad f \quad} & V \subset N \\ {\scriptstyle \varphi^{-1}}\uparrow & & \downarrow{\scriptstyle \psi} \\ \varphi(U) \subset X & \xrightarrow{\psi \circ f \circ \varphi^{-1}} & \psi(V) \subset Y \end{array}$$

It follows from the definition of Banach manifolds that the above definition is independent of the choice of the charts.

Definition 5.1.3. *Let M be a C^ℓ Banach manifold. A C^1-smooth function $\gamma : (-\delta, \delta) \to M$ is called a C^1 **curve** in M. Suppose that $\gamma(0) = x \in M$ and let*

$$C^1(M)_x = \{f : M \to \mathbb{R} : \ f \text{ is } C^1\text{-smooth at } x\}.$$

*The **tangent vector to the curve** γ at $t = 0$ is a function $\gamma'(0) : C^1(M)_x \to \mathbb{R}$ given by*

$$\gamma'(0)(f) = (f \circ \gamma)'(0), \qquad \text{for } f \in C^1(M)_x.$$

*A **tangent vector** at x is the tangent vector at $t = 0$ of some curve $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = x$. The set of all tangent vectors to M at x is denoted by T_xM and called the **tangent space** of M at x.*

Let $x \in X$ and $\varphi : U \subset M \to X$ be a C^ℓ-smooth chart of M at x. We easily check that the map

$$\Phi(x) : T_xM \to X \qquad \text{defined as} \qquad \Phi(x)(v) = (\varphi \circ \gamma)'(0), \tag{5.1}$$

where v is the tangent vector of a C^1 curve $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = x$, defines a bijection from T_xM onto X. By means of these bijection we can transport linear structure and topology from X to T_xM.

The set

$$TM = \bigsqcup_{p \in M} T_pM = \{(p, v) : p \in M, \ v \in T_pM\}$$

*is called the **tangent bundle** of M and it is endowed with an atlas which provides a structure of $C^{\ell-1}$ Banach manifold.*

For more information on these concepts see, for instance, [19] and [23]. Now, with the idea of tangent space we can extend to Banach manifolds the notion of the differential of a differentiable function. Let M and N be C^ℓ Banach manifolds modeled on X and Y, respectively, and let $f : M \to N$ be a C^1-smooth function. For every $x \in M$ and for every $v \in T_xM$, choose a curve $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Then $f \circ \gamma$ is a C^1 curve in N with $f \circ \gamma(0) = f(x)$. The mapping

$$df(x) : T_xM \to T_{f(x)}N \qquad \text{given by} \qquad df(x)(v) = (f \circ \gamma)'(0)$$

is a linear mapping and it does not depend on the choice of γ representing v. The linear mapping $df(x)$ is called the **differential** of f at x. Notice that the differential of a chart φ at x is the bijection $\Phi(x)$ from T_xM onto X defined in the equation (5.1).

We shall denote the differential of a C^1-smooth function at a point x between two Banach manifolds by $df(x)$, while the differential of a C^1-smooth function at a point x between two Banach spaces will be denoted by $f'(x)$.

Finally, recall that a topological space X is said to be **paracompact** if every covering $\{U_\alpha\}$ of X by open sets has a *locally finite refinement*, more precisely, there is a covering $\{V_\beta\}$ satisfying (i) $\{V_\beta\}$ refines $\{U_\alpha\}$ in the sense that each $V_\beta \subset U_\alpha$ for some α, and (ii) $\{V_\beta\}$ is locally finite, that is, each $x \in X$ has a neighborhood W which intersects only a finite number of sets V_β. From now on, we will work with paracompact Banach manifolds.

5.2 Finsler manifolds

Let us begin by introducing the class of manifolds we will consider in the following chapters.

Definition 5.2.1. *Let M be a (paracompact) C^ℓ Banach manifold modeled on a Banach space $(X, ||\cdot||)$. Let us denote by TM the tangent bundle of M and consider a continuous map $||\cdot||_M : TM \to [0, \infty)$. We say that*

(F1) $(M, ||\cdot||_M)$ *is a C^ℓ **Finsler manifold in the sense of Palais** (see [100], [19] and [102]) if $||\cdot||_M$ satisfies the following conditions:*

(P1) For every $x \in M$, the map $||\cdot||_x := ||\cdot||_{M|_{T_xM}} : T_xM \to [0, \infty)$ is a norm on the tangent space T_xM such that for every chart $\varphi : U \to X$ with $x \in U$, the norm $v \in X \mapsto ||d\varphi^{-1}(\varphi(x))(v)||_x$ is equivalent to $||\cdot||$ on X.

(P2) For every $x_0 \in M$, $\varepsilon > 0$ and every chart $\varphi : U \to X$ with $x_0 \in U$, there is an open neighborhood W of x_0 such that if $x \in W$ and $v \in X$, then

$$\frac{1}{1+\varepsilon}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \le ||d\varphi^{-1}(\varphi(x))(v)||_x \le (1+\varepsilon)||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}.$$

In terms of equivalence of norms, the above inequalities yield to the fact that the norms $||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ are $(1+\varepsilon)$-equivalent.

(F2) $(M, ||\cdot||_M)$ *is a C^ℓ **Finsler manifold in the sense of Neeb-Upmeier** ([98]; Upmeier in [110] denotes these manifolds by **normed Banach manifolds**) if $||\cdot||_M$ satisfies conditions* (P1) *and*

(NU1) For every $x_0 \in M$ there exists a chart $\varphi : U \to X$ with $x_0 \in U$ and $K_{x_0} \ge 1$ such

that for every $x \in U$ and every $v \in T_xM$,

$$\frac{1}{K_{x_0}}||v||_x \leq ||d\varphi(x)(v)|| \leq K_{x_0}||v||_x. \tag{5.2}$$

Equivalently, $(M, ||\cdot||_M)$ is a C^ℓ Finsler manifold in the sense of Neeb-Upmeier if it satisfies conditions (P1) *and*

(NU2) For every $x_0 \in M$ there exists a chart $\varphi : U \to X$ with $x_0 \in U$ and a constant $M_{x_0} \geq 1$ such that for every $x \in U$ and every $v \in X$,

$$\frac{1}{M_{x_0}}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq M_{x_0}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}. \tag{5.3}$$

The proof is explained below in Proposition 5.2.3(2).

(F3) $(M, ||\cdot||_M)$ *is a* C^ℓ ***Finsler manifold in the sense of Neeb-Upmeier weak-uniform*** *if it satisfies* (P1) *and there is $K \geq 1$ such that*

(NU3) For every $x_0 \in M$, there exists a chart $\varphi : U \to X$ with $x_0 \in U$ satisfying, for every $x \in U$ and $v \in X$,

$$\frac{1}{K}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq K||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}. \tag{5.4}$$

In this case, we will say that $(M, ||\cdot||_M)$ is ***K-weak-uniform***.

(F4) $(M, ||\cdot||_M)$ *is a* C^ℓ ***Finsler manifold in the sense of Neeb-Upmeier uniform*** *if $||\cdot||_M$ satisfies* (P1) *and*

(NU4) There is $S \geq 1$ such that for each $x_0 \in M$ there exists a chart $\varphi : U \to X$ with $x_0 \in U$ and

$$\frac{1}{S}||v||_x \leq ||d\varphi(x)(v)|| \leq S||v||_x, \qquad \text{whenever } x \in U \text{ and } v \in T_xM. \tag{5.5}$$

In this case, we will say that $(M, ||\cdot||_M)$ is ***S-uniform***.

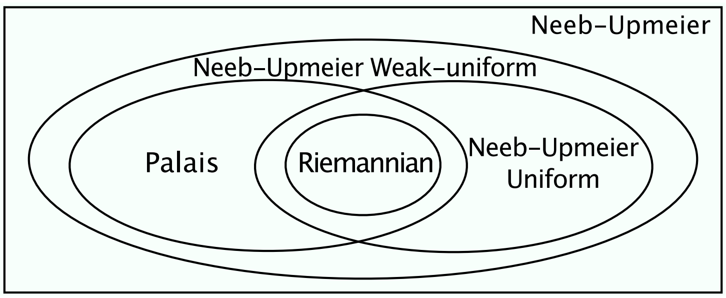


Figure 5.1: Finsler manifolds

Examples 5.2.2.

- *It is clear that Banach spaces are C^∞ Finsler manifolds in the sense of Palais and in the sense of Neeb-Upmeier uniform.*
- *Every Riemannian manifold is a C^∞ Finsler manifold in the sense of Palais (see [100]) and a C^∞ Finsler manifold in the sense of Neeb-Upmeier uniform. Recall that a Riemannian manifold (M, g) is a C^∞ Banach manifold M modeled on some Hilbert space H, such that $g(p) =< \cdot, \cdot >_p$ is a scalar product on the tangent space T_pM for every $p \in M$, so that $||x||_p := (< x, x >_p)^{1/2}$ defines an equivalent norm on T_pM for each $p \in M$, and in such away that the mapping $p \in M \mapsto g_p \in \mathcal{S}^2(M)$ is a C^∞ section of the bundle $\sigma_2 : \mathcal{S}_2 \to M$ of symmetric bilinear forms.*
- *Every paracompact C^1 Banach manifold M admits a continuous map $|| \cdot ||_M : TM \to [0, \infty)$ such that $(M, || \cdot ||_M)$ is a C^1 Finsler manifold in the sense of Palais (see [100, Theorem 2.11]).*
- *Let $(X, || \cdot ||)$ be a Banach space and $(N(X), \rho)$ the metric space of all norms on X which are equivalent to the norm $|| \cdot ||$, endowed with the metric defined for p, $q \in N(X)$ by*

$$\rho(p, q) = \sup\{|p(x) - q(x)| : ||x|| < 1\}.$$

  Let $f : X \to (N(X), \rho)$ be a continuous function. Let us take the C^∞ Banach manifold $M = X$ (then, $TM = X \times X$) and define the Finsler structure in M

$$|| \cdot ||_M : TM \to [0, \infty) \qquad \text{given by} \qquad ||(x, v)||_M = f(x)(v).$$

  Then $(M, || \cdot ||_M)$ is a C^∞ Finsler manifold in the sense of Palais.
- *Let X be a Banach space and $f : X \to \mathbb{R}$ a C^k-smooth function on X. The set $M = \{x \in X : f(x) = 0\}$ is a C^k Banach manifold whenever $f'(x) \neq 0$ for all $x \in M$. In this case the tangent space at x is $T_xM = \ker f'(x) = \{v \in X : f'(x)(v) = 0\}$ (see [19] for details) and the Finsler structure $|| \cdot ||_M \to [0, \infty)$ is given by*

$$||(x, v)||_M = ||v||.$$

  Then $(M, || \cdot ||_M)$ is a C^k Finsler manifold in the sense of Palais. In particular, the unit sphere $S(0, 1)$ of a Banach space $(X, || \cdot ||)$ is a C^k Finsler manifold in the sense of Palais whenever the norm $|| \cdot ||$ is C^k-smooth.

Properties 5.2.3.

(1) *Clearly,* (F1) $\Rightarrow$ (F3). *Also,* (F4) $\Rightarrow$ (F3) $\Rightarrow$ (F2) *(see Figure 5.1).*

(2) *If $(M, || \cdot ||_M)$ satisfies condition* (P1), *then conditions* (NU1) *and* (NU2) *are equivalent.*

Proof. Let us suppose that $(M, || \cdot ||_M)$ satisfies (NU1). For every $x_0 \in M$ there exists a chart $\varphi : U \to M$ with $x_0 \in U$ and $K_{x_0} \geq 1$ satisfying inequality (5.2). Thus, let us take $d\varphi^{-1}(\varphi(x_0))(w) \in T_{x_0}M$ and $d\varphi^{-1}(\varphi(x))(w) \in T_xM$ for every $w \in X$ and $x \in U$, and they satisfy

$$\frac{1}{K_{x_0}}||d\varphi^{-1}(\varphi(x_0))(w)||_{x_0} \leq ||w|| \leq K_{x_0}||d\varphi^{-1}(\varphi(x_0))(w)||_{x_0}, \qquad \text{and}$$
$$\frac{1}{K_{x_0}}||d\varphi^{-1}(\varphi(x))(w)||_{x} \leq ||w|| \leq K_{x_0}||d\varphi^{-1}(\varphi(x))(w)||_{x}.$$

Then, the same chart satisfies the inequality (5.3) for $M_{x_0} = K_{x_0}^2$.

Conversely, let us suppose that for $x_0 \in M$ there exists a chart $\varphi : U \to X$ with $x_0 \in U$ and $M_{x_0} \geq 1$ satisfying inequality (5.3). On the one hand, by hypothesis (P1) there is a constant $C_{x_0} \geq 1$ such that

$$\frac{1}{C_{x_0}}||d\varphi^{-1}(\varphi(x_0))(w)||_{x_0} \leq ||w|| \leq C_{x_0}||d\varphi^{-1}(\varphi(x_0))(w)||_{x_0} \qquad \text{for every } w \in X.$$

On the other hand, by (NU2), for every $x \in U$ and $v \in T_xM$, we have $d\varphi(x)(v) \in X$ and

$$\frac{1}{M_{x_0}}||v||_x \leq ||d\varphi^{-1}(\varphi(x_0))(d\varphi(x)(v))||_{x_0} \leq M_{x_0}||v||_x.$$

Hence, the chart $\varphi : U \to X$ satisfies the inequality (5.2) for $K_{x_0} = M_{x_0}C_{x_0}$. □

(3) *The concepts of C^ℓ Finsler manifold in the sense of Palais and C^ℓ Finsler manifold in the sense of Neeb-Upmeier are equivalent for finite-dimensional manifolds. In fact, for a finite-dimensional C^ℓ Banach manifold M and a map $||\cdot||_M : TM \to [0,\infty)$, the following statements are equivalent:*

- *(i) $(M, ||\cdot||_M)$ is a C^ℓ Finsler manifold in the sense of Palais.*
- *(ii) The map $||\cdot||_M : TM \to [0,\infty)$ is continuous and $||\cdot||_x := ||\cdot||_{M|_{T_xM}} : T_xM \to [0,\infty)$ is a norm on the tangent space T_xM for every $x \in M$.*

Proof. Let M be a C^ℓ finite-dimensional Banach manifold modeled on X satisfying the condition (ii), $x_0 \in M$ and (U,φ) a chart of M at x_0. We can define the following continuous structure

$$|||\cdot||| : U \times X \to [0,\infty) \qquad \text{as} \qquad |||(x,v)||| = |||v|||_x := ||d\varphi^{-1}(\varphi(x))(v)||_x,$$

which gives an equivalent norm on X for any $x \in U$.

Let us take $\varepsilon > 0$, $A = |||\cdot|||^{-1}(\frac{1}{1+\varepsilon}, 1+\varepsilon) \subset U \times X$ and $S_{x_0} = \{v \in X : |||v|||_{x_0} = 1\}$. Since $\{x_0\} \times S_{x_0} \subset A$ and S_{x_0} is compact, using the Tube Lemma (see for instance [93, Lemma 26.8]) we obtain an open neighborhood of x_0, $U_{x_0}^\varepsilon \subset U$ such that $U_{x_0}^\varepsilon \times S_{x_0} \subset A$. Thus,

$$\frac{1}{1+\varepsilon} < |||v|||_x < 1+\varepsilon, \qquad \text{for every } x \in U_{x_0}^\varepsilon \text{ and } v \in S_{x_0}.$$

Since $|||v|||_x = ||d\varphi^{-1}(\varphi(x))(v)||_x$ and $|||v|||_{x_0} = 1$, we obtain that

$$\frac{1}{1+\varepsilon}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} < ||d\varphi^{-1}(\varphi(x))(v)||_x < (1+\varepsilon)||d\varphi^{-1}(\varphi(x))(v)||_{x_0},$$

for every $x \in U_{x_0}^{\varepsilon}$ and $|||v|||_{x_0} = 1$. Hence,

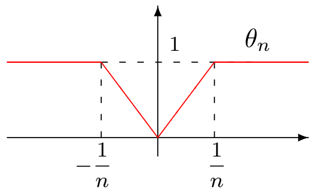
$$\frac{1}{1+\varepsilon}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq (1+\varepsilon)||d\varphi^{-1}(\varphi(x))(v)||_{x_0},$$

for every $x \in U_{x_0}^{\varepsilon}$ and $v \in X$. □

(4) Nevertheless, in the infinite-dimensional setting, there are examples of C^ℓ Finsler manifolds in the sense of Neeb-Upmeier uniform (so, in the sense of Neeb-Upmeier weak-uniform) that do not satisfy the Palais condition (P2) *(see [41, Example 10]). Also, there are examples of C^ℓ Finsler manifolds in the sense of Neeb-Upmeier that are not weak-uniform.*

Proof. The ideas of this example follow those of [41, Example 10]. Let $(X, ||\cdot||)$ be an infinite-dimensional Banach space. For each n we consider the map $\theta_n : X \to [0,1]$ defined by

$$\theta_n(x) = \inf\{1, n||x||\} = \begin{cases} 1 & \text{if } ||x|| \geq \frac{1}{n}, \\ n||x|| & \text{if } ||x|| \leq \frac{1}{n}. \end{cases}$$



Let us take $X' = \mathbb{R} \times X$ with the norm $||(t,x)||_{X'} = |t| + ||x||$ and consider the Banach manifold $M = X'$. Thus, $TM = X' \times X'$. We can choose a normalized sequence $\{v_n^*\}_{n=1}^{\infty}$ in the dual space $(X')^*$ such that the sequence $\{v_n^*\}_{n=1}^{\infty}$ converges to 0 in the ω^*-topology (see [76] and [99]). For each n we choose $v_n \in X'$ such that

$$1 \leq ||v_n||_{X'} \leq 2 \qquad \text{and} \qquad v_n^*(v_n) = 1.$$

We shall denote $\mathbf{x} = (t,x) \in X'$ whenever $t \in \mathbb{R}$ and $x \in X$. Now, let us define a structure $||\cdot||_M : TM \to [0,\infty)$ by

$$||((t,x),v)||_M := ||v||_{X'} + |t| \sup_n\{\theta_n(x)|v_n^*(v)|\},$$

and denote it by $||v||_{\mathbf{x}}$.

- The map $||\cdot||_M$ is continuous. Indeed, let us fix $((t_0,x_0),v_0) \in TM$ and $\varepsilon > 0$,

$$\begin{aligned}\big|||((t,x),v)||_M - ||((t_0,x_0),v_0)||_M\big| &\leq \\ &\leq ||v-v_0||_{X'} + \big||t|\sup\{\theta_n(x)|v_n^*(v)|\} - |t_0|\sup\{\theta_n(x_0)|v_n^*(v_0)|\}\big| \\ \leq ||v-v_0||_{X'} + |t-t_0|\sup\{\theta_n(x)|v_n^*(v)|\} &+ |t_0|\,\big|\sup\{\theta_n(x)|v_n^*(v)|\} - \sup\{\theta_n(x_0)|v_n^*(v_0)|\}\big|.\end{aligned}$$

Let us take $v \in X'$ and $t \in \mathbb{R}$ such that $||v-v_0||_{X'} < \varepsilon/(1+|t_0|)$ and $|t-t_0| < \varepsilon$, so it is enough to prove that

$$|t_0|\,\big|\sup\{\theta_n(x)|v_n^*(v)|\} - \sup\{\theta_n(x_0)|v_n^*(v_0)|\}\big| < \varepsilon$$

for x and v close enough to x_0 and v_0, respectively.

Since the sequence $\{v_n^*(v_0)\}_{n=1}^{\infty}$ tends to 0, there exists n_0 such that $|v_n^*(v_0)| < \varepsilon/(1+|t_0|)$ for every $n \geq n_0$. Thus,

$$|v_n^*(v)| \leq |v_n^*(v_0)| + |v_n^*(v-v_0)| < \varepsilon/(1+|t_0|) + ||v_n^*|| ||v-v_0||_{X'} < 2\varepsilon/(1+|t_0|),$$

for every $n \geq n_0$ and $||v-v_0||_{X'} < \varepsilon/(1+|t_0|)$, and therefore

$$0 \leq |t_0| \left(\sup_n \{\theta_n(x)|v_n^*(v)|\} - \sup_{1\leq n\leq n_0} \{\theta_n(x)|v_n^*(v)|\} \right) \leq 2\varepsilon.$$

Using the continuity of the map

$$(x,v) \in X \times X' \mapsto \sup_{1\leq n\leq n_0} \{\theta_n(x)|v_n^*(v)|\},$$

we deduce the continuity of $||\cdot||_M$ at $((t_0,x_0),v_0)$ and, thus, on M.

• $(M, ||\cdot||_M)$ is a Finsler manifold in the sense of Neeb-Upmeier. Indeed, for every $\mathbf{x}_0 = (t_0,x_0) \in M$ we take $U = B_M(\mathbf{x}_0,1)$, $K_{\mathbf{x}_0} = 2+|t_0|$ and the smooth chart $\mathrm{id}: U \to X'$ defined as $\mathrm{id}(\mathbf{x}) = \mathbf{x}$. Then its differential is the "identity" mapping, i.e. $d\,\mathrm{id}(\mathbf{x})(v) = v$ for every $\mathbf{x} \in U$ and $v \in T_{\mathbf{x}}M$. Thus, for $\mathbf{x} = (t,x) \in U$ and $v \in T_{\mathbf{x}}M$, we have that

$$||v||_{X'} \leq ||v||_{\mathbf{x}} \leq (1+|t|)||v||_{X'} \leq (2+|t_0|)||v||_{X'} = K_{\mathbf{x}_0}||v||_{X'},$$

since $|t| \leq 1+|t_0|$ whenever $(t,x) \in U$.

• However, the manifold $(M, ||\cdot||_M)$ is not a Finsler manifold in the sense of Neeb-Upmeier uniform. Let us suppose that there is a constant $K \geq 1$ such that $(M, ||\cdot||_M)$ is K-uniform in the sense of Neeb-Upmeier and take $0 \in X$ and $t_0 \in \mathbb{R}$ such that $t_0/2 > K(K+1)$. By assumption, there is a chart $\varphi: U \to X'$ with $\mathbf{x}_0 = (t_0,0) \in U$ such that

$$\frac{1}{K}||v||_{\mathbf{x}} \leq ||d\varphi(\mathbf{x})(v)||_{X'} \leq K||v||_{\mathbf{x}}$$

for every $\mathbf{x} \in U$ and $v \in T_{\mathbf{x}}M = X'$. Since $\varphi \circ \mathrm{id}^{-1} : \mathrm{id}(U) \subset X' \to X'$ is C^1-smooth, we can assume that $||d(\varphi\circ\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x})) - d(\varphi\circ\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}_0))||_{X'} < 1$ for every $\mathbf{x} \in U$. Thus, for every $\mathbf{x} \in U$ and $v \in X'$ we obtain that

$$\begin{aligned} ||d\varphi(\mathbf{x})(v) - d\varphi(\mathbf{x}_0)(v)||_{X'} &= ||d\varphi(\mathbf{x})\circ d(\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}))(v) - d\varphi(\mathbf{x}_0)\circ d(\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}_0))(v)||_{X'} \\ &= ||d(\varphi\circ\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}))(v) - d(\varphi\circ\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}_0))(v)||_{X'} \leq ||v||_{X'}, \end{aligned}$$

since $d(\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}))(v) = v \in T_{\mathbf{x}}M$ and $d(\mathrm{id}^{-1})(\mathrm{id}(\mathbf{x}_0))(v) = v \in T_{\mathbf{x}_0}M$.

Then, for every $\mathbf{x} \in U$ and $v \in T_{\mathbf{x}}M = X'$

$$\begin{aligned} ||v||_{\mathbf{x}} \leq K||d\varphi(\mathbf{x})(v)||_{X'} &\leq K(||d\varphi(\mathbf{x}_0)(v)||_{X'} + ||v||_{X'}) \\ &\leq K(K||v||_{\mathbf{x}_0} + ||v||_{X'}) = K(K+1)||v||_{X'}, \end{aligned} \tag{5.6}$$

since $||v||_{\mathbf{x}_0} = ||v||_{(t_0,0)} = ||v||_{X'}$. Now, we can choose n large enough, $x_n \in X$ with $||x_n|| = \frac{1}{n}$ and $t_n \in \mathbb{R}$ such that $|t_n - t_0| < 1$ and $\mathbf{x}_n$ denotes the point $(t_n,x_n) \in U$. Then $\theta_n(x_n) = 1$

and

$$||v_n||_{\mathbf{x}_n} \geq ||v_n||_{X'} + |t_n|\theta_n(x_n)|v_n^*(v_n)| \geq \left(1 + \frac{|t_n|}{2}\right)||v_n||_{X'} \geq \left(\frac{1}{2} + \frac{|t_0|}{2}\right)||v_n||_{X'},$$

which contradicts the inequality (5.6), because $t_0/2 > K(K+1)$. □

(5) *Note that if M is a C^ℓ Finsler manifold in the sense of Palais, then it is K-weak-uniform for every $K > 1$. Also, if M is S-uniform, then M is S^2-weak-uniform.*

Proof. The first part is obvious. For the second one, notice that the inequality (5.5) is equivalent to the fact that $||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ is S-equivalent to $||\cdot||$ on X, for every $x \in U$. Now, if $x \in U$ and $v \in X$, then $d\varphi^{-1}(\varphi(x))(v) \in T_xM$ and $d\varphi^{-1}(\varphi(x_0))(v) \in T_{x_0}M$, and we apply the inequality (5.5) to obtain

$$\frac{1}{S^2}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq \frac{1}{S}||v|| \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq S||v|| \leq S^2||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}.$$

□

(6) *The condition* (NU2) *in the definition of C^ℓ Finsler manifold in the sense of Neeb-Upmeier yields the following condition:*

(NU2') For every $x_0 \in M$, there exists a constant $M_{x_0} \geq 1$ such that for every $R_{x_0} > M_{x_0}$ and every chart $\psi : V \to X$ with $x_0 \in V$, there is an open subset W with $x_0 \in W \subset V$ such that for every $x \in W$ and $v \in X$

$$\frac{1}{R_{x_0}}||d\psi^{-1}(\psi(x_0))(v)||_{x_0} \leq ||d\psi^{-1}(\psi(x))(v)||_x \leq R_{x_0}||d\psi^{-1}(\psi(x_0))(v)||_{x_0}. \quad (5.7)$$

Thus, the condition (NU3) *in the definition of C^ℓ Finsler manifold in the sense of Neeb-Upmeier weak-uniform yields the following condition:*

(NU3') For every $R > K$, $x_0 \in M$ and every chart $\psi : V \to X$ with $x_0 \in V$, there is an open subset W with $x_0 \in W \subset V$ such that

$$\frac{1}{R}||d\psi^{-1}(\psi(x_0))(v)||_{x_0} \leq ||d\psi^{-1}(\psi(x))(v)||_x \leq R||d\psi^{-1}(\psi(x_0))(v)||_{x_0}, \quad (5.8)$$

whenever $x \in W$ and $v \in X$.

Proof. Let us fix $x_0 \in M$. By the condition (NU2) there exist a constant $M_{x_0} \geq 1$ and a chart $\varphi : U \to X$ of M at x_0 such that

$$\frac{1}{M_{x_0}}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq M_{x_0}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}$$

for every $x \in U$ and $v \in X$. Let us take $R_{x_0} > M_{x_0}$ and a chart $\psi : V \to X$ with $x_0 \in V$.

Let us apply conditions (P1) and (NU3) to φ and (P1) to ψ to obtain $L_1, L_2 \geq 1$ such that

$$||d\varphi^{-1}(\varphi(x))(v)||_x \leq L_1||v|| \text{ and } ||v|| \leq L_2||d\psi^{-1}(\psi(x_0))(v)||_{x_0} \quad (5.9)$$

for every $x \in U \cap V$ and $v \in X$. Notice that L_1 and L_2 depend on x_0.

Let us take $\varepsilon > 0$ such that $M_{x_0}/(1-\varepsilon) \leq R_{x_0}$. Since $\varphi \circ \psi^{-1}$ is C^ℓ-smooth and ψ is continuous, there is $W \subset U \cap V$ a neighborhood of x_0 such that

$$||(\varphi \circ \psi^{-1})'(\psi(x_0)) - (\varphi \circ \psi^{-1})'(\psi(x))|| < \frac{\varepsilon}{M_{x_0} L_1 L_2} \tag{5.10}$$

for every $x \in W$. Thus, for $x \in W$ and $v \in X$ we denote $w = d\varphi(x_0)(d\psi^{-1}(\psi(x_0))(v)) \in X$, and using the equations (NU2), (5.9) and (5.10), we obtain

$$\begin{aligned}
\frac{1}{M_{x_0}}||d\psi^{-1}(\psi(x_0))(v)||_{x_0} &= \frac{1}{M_{x_0}}||d\varphi^{-1}(\varphi(x_0))(w)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(w)||_x \\
&\leq ||d\varphi^{-1}(\varphi(x))(d\varphi(x)(d\psi^{-1}(\psi(x))(v)))||_x \\
&\qquad + ||d\varphi^{-1}(\varphi(x))(d\varphi(x_0)(d\psi^{-1}(\psi(x_0))(v)) - d\varphi(x)(d\psi^{-1}(\psi(x))(v)))||_x \\
&\leq ||d\psi^{-1}(\psi(x))(v)||_x + L_1||((\varphi \circ \psi^{-1})'(\psi(x_0)) - (\varphi \circ \psi^{-1})'(\psi(x)))(v)|| \\
&\leq ||d\psi^{-1}(\psi(x))(v)||_x + \frac{\varepsilon}{M_{x_0} L_2}||v|| \leq ||d\psi^{-1}(\psi(x))(v)||_x + \frac{\varepsilon}{M_{x_0}}||d\psi^{-1}(\psi(x_0))(v)||_{x_0}.
\end{aligned}$$

Thus

$$\frac{1}{R_{x_0}}||d\psi^{-1}(\psi(x_0))(v)||_{x_0} \leq \frac{1-\varepsilon}{M_{x_0}}||d\psi^{-1}(\psi(x_0))(v)||_{x_0} \leq ||d\psi^{-1}(\psi(x))(v)||_x$$

for every $x \in W$ and $v \in X$. The other inequality can be proved in the same way.

(NU3') follows from the (NU2') and the fact that if M is Neeb-Upmeier K-weak-uniform, then $M_{x_0} = K$ for every $x_0 \in M$. □

It is worth mentioning that we do not know whether there exist C^ℓ Finsler manifolds in the sense of Palais which are not C^ℓ Finsler manifolds in the sense of Neeb-Upmeier uniform. In fact, we do not know whether the class of C^ℓ Finsler manifolds in the sense of Neeb-Upmeier uniform coincides with the class of C^ℓ Finsler manifolds in the sense of Neeb-Upmeier weak-uniform.

Let M be a C^ℓ Finsler manifold and $f : M \to \mathbb{R}$ a differentiable function at $p \in M$. The norm of $df(p) \in T_pM^*$ is given by

$$||df(p)||_p = \sup\{|df(p)(v)| : v \in T_pM, ||v||_p \leq 1\},$$

where T_pM^* is the dual space of T_pM. Let us consider a differentiable function $f : M \to N$ between Finsler manifolds M and N. The norm of the derivative at the point $p \in M$ is defined as

$$\begin{aligned}
||df(p)||_p &= \sup\{||df(p)(v)||_{f(p)} : v \in T_pM, ||v||_p \leq 1\} \\
&= \sup\{\xi(df(p)(v)) : \xi \in T_{f(p)}N^*,\ v \in T_pM \text{ and } ||v||_p = 1 = ||\xi||^*_{f(p)}\},
\end{aligned}$$

where $||\cdot||^*_{f(p)}$ is the dual norm of $||\cdot||_{f(p)}$.

Finally, in any Finsler manifold there is a metric consistent with its topology, which is defined as follows: If $(M, ||\cdot||_M)$ is a Finsler manifold in the sense of Neeb-Upmeier, the

length of a piecewise C^1-smooth path $c : [a, b] \to M$ is defined as

$$\ell(c) := \int_a^b ||c'(t)||_{c(t)}\, dt.$$

Besides, if M is connected, then it is connected by piecewise C^1-smooth paths, and the associated **Finsler metric** d_M on M is defined as

$$d_M(p, q) = \inf\{\ell(c) : c \text{ is a piecewise } C^1\text{-smooth path connecting } p \text{ to } q\}.$$

The Finsler metric is consistent with the topology given in M (see [100] or [110, Proposition 12.22]). The open ball of center $p \in M$ and radius $r > 0$ is denoted by $B_M(p, r) := \{q \in M : d_M(p, q) < r\}$. The Lipschitz constant $\mathrm{Lip}(f)$ of a Lipschitz function $f : M \to N$, where M and N are Finsler manifolds, is defined as

$$\mathrm{Lip}(f) = \sup\{\frac{d_N(f(x), f(y))}{d_M(x, y)} : x, y \in M,\ x \neq y\}.$$

Throughout we shall always assume that all Finsler manifolds are connected, thus they are metric spaces. Otherwise we may consider its connected components.

5.3 Mean value inequalities

In the following proposition we obtain some “mean value" inequalities. The ideas of the proof follow those of the Riemannian case (see [5]).

Proposition 5.3.1. *(Mean value inequalities). Let M and N be C^1 Finsler manifolds in the sense of Neeb-Upmeier, and let $f : M \to N$ be a C^1-smooth mapping.*

- *(i) If $\sup\{||df(x)||_x : x \in M\} < \infty$, then f is Lipschitz and $\mathrm{Lip}(f) \leq \sup\{||df(x)||_x : x \in M\}$.*
- *(ii) If f is Lipschitz and the manifold N is P-weak-uniform, then $\sup\{||df(x)||_x : x \in M\} \leq P\,\mathrm{Lip}(f)$.*
- *(ii') If f is Lipschitz and the manifold N is P-uniform, then $\sup\{||df(x)||_x : x \in M\} \leq P^2\,\mathrm{Lip}(f)$.*
- *(iii) Thus, if f is Lipschitz and the manifold N is Palais, then $\sup\{||df(x)||_x : x \in M\} = \mathrm{Lip}(f)$.*

Proof. (i) Let us consider $p, q \in M$ with $d_M(p, q) < \infty$, and $\varepsilon > 0$. Then there is a piecewise C^1-smooth path $\gamma : [0, T] \to M$ joining p and q with $\ell(\gamma) \leq d_M(p, q) + \varepsilon/C$, where $C = \sup\{||df(x)||_x : x \in M\}$ (notice that it can be assumed that $C > 0$, otherwise the assertion is trivial). In order to simplify the proof, let us assume that γ is C^1-smooth (the general case follows straightforward). Now, we define $\beta : [0, T] \to N$ as $\beta(t) = f(\gamma(t))$. Then, β joins the points $f(p)$ and $f(q)$, and

$$\begin{aligned} d_N(f(p),f(q)) &\leq \ell(\beta) \leq \int_0^T ||df(\gamma(t))(\gamma'(t))||_{\beta(t)}dt \leq \int_0^T ||df(\gamma(t))||_{\gamma(t)}||\gamma'(t)||_{\gamma(t)}\,dt \\ &\leq C\int_0^T ||\gamma'(t)||_{\gamma(t)}\,dt \leq Cd_M(p,q)+\varepsilon. \end{aligned}$$

Thus, $d_N(f(p),f(q)) \leq Cd_M(p,q)+\varepsilon$ for every $\varepsilon>0$. Then, $d_N(f(p),f(q)) \leq Cd_M(p,q)$ and $\mathrm{Lip}(f)\leq C$.

(*ii*) First, let us consider the case $N=\mathbb{R}$. Let us denote by $L:=\mathrm{Lip}(f)$. Let us take $x_0\in M$ and $v\in T_{x_0}M$ with $||v||_{x_0}=1$ and choose a C^1-smooth curve $\gamma:(-\delta,\delta)\to M$ with $\gamma(0)=x_0$ and $\gamma'(0)=v$. Then the function $f\circ\gamma:(-\delta,\delta)\to\mathbb{R}$ is C^1-smooth with $f\circ\gamma(0)=f(x_0)$ and $df(x_0)(v)=(f\circ\gamma)'(0)$. Since the map $t\mapsto||\gamma'(t)||_{\gamma(t)}$ is continuous on $(-\delta,\delta)$, for every $\varepsilon>0$ there exists $r>0$ with $r<\delta$ such that

$$\big|||\gamma'(t)||_{\gamma(t)}-||\gamma'(0)||_{\gamma(0)}\big|<\varepsilon \qquad \text{for every } t\in[-r,r].$$

Hence,

$$\begin{aligned} |df(x_0)(v)| = |(f\circ\gamma)'(0)| &= \left|\lim_{t\to0}\frac{(f\circ\gamma)(t)-(f\circ\gamma)(0)}{t}\right| \leq \mathrm{Lip}(f)\limsup_{t\to0}\frac{d_M(\gamma(t),\gamma(0))}{|t|} \\ &\leq \mathrm{Lip}(f)\,\mathrm{Lip}(\gamma_{|_{[-r,r]}}) \leq \mathrm{Lip}(f)\sup\left\{||\gamma'(s)||_{\gamma(s)}:s\in[-r,r]\right\} \\ &\leq \mathrm{Lip}(f)(||\gamma'(0)||_{\gamma(0)}+\varepsilon) = \mathrm{Lip}(f)(||v||_{x_0}+\varepsilon) \leq (1+\varepsilon)\,\mathrm{Lip}(f). \end{aligned}$$

Since this conclusion holds for every $x_0\in M$, $v\in T_{x_0}M$ with $||v||_{x_0}=1$ and $\varepsilon>0$, we deduce that

$$\sup_{x\in M}||df(x)||_x \leq L.$$

Now, let us consider the general case, i.e. $f:M\to N$ where N is a C^1 Finsler manifold in the sense of Neeb-Upmeier P-weak-uniform. If there is a point $x_0\in M$ such that $||df(x_0)||_{x_0}>PL$, then there are $\xi\in T_{f(x_0)}N^*$ and $v\in T_{x_0}M$ such that

$$||v||_{x_0}=||\xi||^*_{f(x_0)}=1 \qquad \text{and} \qquad |\xi(df(x_0)(v))|>PL.$$

Let us take a chart of N at $f(x_0)$ satisfying inequality (5.4) with constant P, which we shall denote by $\psi:V\to Y$, where N is modeled on the Banach space Y. Also, let us take $r>0$ such that $f(x_0)\in B_N(f(x_0),r/4)\subset B_N(f(x_0),r)\subset V$ and define the function

$$g:f^{-1}(B_N(f(x_0),r/4))\to\mathbb{R}, \qquad \text{as} \qquad g(x)=\xi\circ d\psi^{-1}(\psi(f(x_0)))(\psi(f(x))).$$

Then, on the one hand,

$$|dg(x_0)(v)| = |\xi\circ d\psi^{-1}(\psi(f(x_0)))(d\psi(f(x_0))df(x_0)(v))| = |\xi(df(x_0)(v))| > PL. \tag{5.11}$$

On the other hand, let us check that the function g is PL-Lipschitz with the distance d_M.

Indeed, first let us show that ψ is P-Lipschitz with the norm

$$|||\cdot||| := ||d\psi^{-1}(\psi(f(x_0)))(\cdot)||_{f(x_0)} \qquad \text{on } Y.$$

Since $\psi : V \to Y$ satisfies inequality (5.4) with constant P, we have for every $z \in V$ and $v \in T_zN$,

$$|||d\psi(z)(v)||| = ||d\psi^{-1}(\psi(f(x_0)))(d\psi(z)(v))||_{f(x_0)} \leq P||d\psi^{-1}(\psi(z))(d\psi(z)(v))||_z = P||v||_z.$$

Hence, $|||d\psi(z)||| := \sup\{|||d\psi(z)(v)||| : v \in T_zN \text{ and } ||v||_z \leq 1\} \leq P$ for every $z \in V$, and thus $\sup\{|||d\psi(z)||| : z \in V\} \leq P$. Also, let us check that, for every $z, z' \in B_N(f(x_0), r/4)$,

$$d_N(z, z') = \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } C^1\text{-smooth path connecting } z \text{ and } z' \text{ and } \gamma \subset V\}.$$

Indeed, if there are $z, z' \in B_N(f(x_0), r/4)$ with

$$\begin{aligned} d_N(z, z') &= \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } C^1\text{-smooth path connecting } z \text{ and } z'\} \\ &< \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } C^1\text{-smooth path connecting } z \text{ and } z' \text{ with } \gamma \subset V\}, \end{aligned}$$

then there is a path $\gamma : [0,1] \to N$ such that $\gamma(0) = z$, $\gamma(1) = z'$, $\ell(\gamma) < d_N(z, z') + r/4$ and $\gamma(t) \notin V$ for some $t \in [0,1]$. Then $d_N(f(x_0), \gamma(t)) \geq r$ and $d_N(z, \gamma(t)) \leq \ell(\gamma) \leq d_N(z, z') + r/4$. This yields,

$$r \leq d_N(f(x_0), \gamma(t)) \leq d_N(f(x_0), z) + d_N(z, \gamma(t)) \leq d_N(f(x_0), z) + d_N(z, z') + r/4 < r,$$

which is a contradiction. Now, we can follow the proof of part (i) to deduce that ψ is P-Lipschitz on $B_N(f(x_0), r/4)$ with the norm $|||\cdot||| = ||d\psi^{-1}(\psi(f(x_0)))(\cdot)||_{f(x_0)}$ on Y. Finally, for every $x, y \in f^{-1}(B(f(x_0), r/4))$, we have

$$\begin{aligned} |g(x) - g(y)| &= |\xi \circ d\psi^{-1}(\psi(f(x_0)))(\psi \circ f(x) - \psi \circ f(y))| \\ &\leq ||d\psi^{-1}(\psi(f(x_0)))(\psi \circ f(x) - \psi \circ f(y))||_{f(x_0)} \\ &= |||\psi \circ f(x) - \psi \circ f(y)||| \leq P d_N(f(x), f(y)) \leq PL d_M(x, y). \end{aligned}$$

Then $g : f^{-1}(B(f(x_0), r/4)) \to \mathbb{R}$ is PL-Lipschitz, and by the real case we have that $\sup\{||dg(x)||_x : x \in f^{-1}(B(f(x_0), r/4))\} \leq PL$, which contradicts (5.11).

(ii') and (iii) follow from (ii) and Properties 5.2.3(5). □

The following lemma provides a local bi-Lipschitz behavior of the charts of a C^1 Finsler manifold.

Lemma 5.3.2. *Let us consider a C^1 Finsler manifold M in the sense of Neeb-Upmeier.*

(1) For every $x_0 \in M$ there exists a constant $K_{x_0} \geq 1$ such that for every chart (U, φ) with $x_0 \in U$ satisfying inequality (5.3)*, there exists an open neighborhood $V \subset U$ of x_0 satisfying*

$$\frac{1}{K_{x_0}} d_M(p, q) \leq |||\varphi(p) - \varphi(q)||| \leq K_{x_0} d_M(p, q), \qquad \textit{for every } p, q \in V, \tag{5.12}$$

where $|||\cdot|||$ *is the (equivalent) norm* $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ *defined on* X.

(2) If M *is* K*-weak-uniform, then for every* $x_0 \in M$ *and every chart* (U, φ) *with* $x_0 \in U$ *satisfying inequality* (5.4)*, there exists an open neighborhood* $V \subset U$ *of* x_0 *satisfying*

$$\frac{1}{K} d_M(p,q) \leq |||\varphi(p) - \varphi(q)||| \leq K d_M(p,q), \qquad \text{for every } p, q \in V, \tag{5.13}$$

where $|||\cdot|||$ *is the (equivalent) norm* $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ *defined on* X.

(3) If the manifold M *is* K*-uniform, then for every* $x_0 \in M$ *and every chart* (U, φ) *with* $x_0 \in U$ *satisfying condition* (5.5)*, there exists an open neighborhood* $V \subset U$ *of* x_0 *satisfying*

$$\frac{1}{K} d_M(p,q) \leq ||\varphi(p) - \varphi(q)|| \leq K d_M(p,q), \qquad \text{for every } p, q \in V. \tag{5.14}$$

Proof. Let us assume the hypothesis in (1) holds. By Property 5.2.3(6) there exists a constant $K_{x_0} \geq 1$ such that for every chart (W, φ) with $x_0 \in W$, there is an open subset U with $x_0 \in U \subset W$ such that φ satisfies inequality (5.3) in U.

The arguments given in the proof of Proposition 5.3.1 yield the existence of $r > 0$ with $B_M(x_0, r/4) \subset B_M(x_0, r) \subset U \subset M$ such that if $p, q \in B_M(x_0, r/4)$, then

$$d_M(p,q) = \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } C^1\text{-smooth path connecting } p \text{ and } q \text{ with } \gamma \subset U\}.$$

Let us consider $p, q \in B_M(x_0, r/4)$, $\varepsilon > 0$ and a piecewise C^1-smooth path $\gamma : [0,T] \to U$ joining p and q with $\ell(\gamma) \leq d_M(p,q) + \varepsilon/K_{x_0}$. Let us define $\beta : [0,T] \to X$ as $\beta(t) = \varphi(\gamma(t))$. Then, β joins the points $\varphi(p)$ and $\varphi(q)$, and from inequality (5.3) we obtain

$$\begin{aligned} |||\varphi(p) - \varphi(q)||| \leq \ell(\beta) \leq \int_0^T |||d\varphi(\gamma(t))(\gamma'(t))|||\, dt \leq & K_{x_0} \int_0^T ||\gamma'(t)||_{\gamma(t)} dt \\ = & K_{x_0} \ell(\gamma) \leq K_{x_0} d_M(p,q) + \varepsilon. \end{aligned}$$

Now, let us consider φ^{-1} and $s > 0$ such that $B(\varphi(x_0), s) \subset \varphi(B_M(x_0, r/4))$. For $x, y \in B(\varphi(x_0), s)$, let us define the path $\gamma : [0,1] \to M$ as $\gamma(t) := \varphi^{-1}(ty + (1-t)x) \in B_M(x_0, r/4)$. Then,

$$\begin{aligned} d_M(\varphi^{-1}(x), \varphi^{-1}(y)) \leq \ell(\gamma) = \int_0^1 ||\gamma'(t)||_{\gamma(t)} dt = \int_0^1 ||d\varphi^{-1}(\varphi(\gamma(t)))(y-x)||_{\gamma(t)} dt \\ \leq \int_0^1 K_{x_0} ||d\varphi^{-1}(\varphi(x_0))(y-x)||_{x_0} = K_{x_0} |||x - y|||. \end{aligned}$$

Finally, let us define the open set $V := \varphi^{-1}(B(\varphi(x_0), s))$.

(2) follows from (1) and the fact that if M is Neeb-Upmeier K-weak-uniform, then $K_{x_0} = K$ for every $x_0 \in M$. The proof under the hypothesis given in (3) follows along the same lines. □

Finally, let us see that every C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform admits an *equivalent* structure such that with this structure M is a C^ℓ Finsler manifold in the sense of Palais and the associated metric is K-equivalent to the original one.

Proposition 5.3.3. *If $(M, ||\cdot||_M)$ is a C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform. Then there exists a continuous map $|||\cdot|||_M : TM \to [0,\infty)$ satisfying (P1) and (P2), i.e. $(M, |||\cdot|||_M)$ is a C^ℓ Finsler manifold in the sense of Palais. Moreover, for every $x \in M$ and $v \in T_xM$*

$$\frac{1}{K}|||(x,v)|||_M \le ||(x,v)||_M \le K|||(x,v)|||_M, \qquad \textit{and thus} \tag{5.15}$$

$$\frac{1}{K} d_{|||\cdot|||_M}(p,q) \le d_{||\cdot||_M}(p,q) \le K d_{|||\cdot|||_M}(p,q), \qquad \textit{for every } p,\, q \in M,$$

where $d_{|||\cdot|||_M}$ and $d_{||\cdot||_M}$ are the associated distances to $(M, |||\cdot|||_M)$ and $(M, ||\cdot||_M)$, respectively.

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Delta}$ be a family of charts of M satisfying inequality (5.4) and $\{U_\alpha\}_{\alpha\in\Delta}$ a covering of M. Let us take $x_\alpha \in U_\alpha$ for any $\alpha \in \Delta$, and let $\{\theta_\alpha\}_{\alpha\in\Delta}$ be a partition of unity of M subordinated to $\{U_\alpha\}_{\alpha\in\Delta}$. The map $|||\cdot|||_M : TM \to [0,\infty)$ given by

$$|||(x,v)|||_M = |||v|||_x = \sum_{\alpha\in\Delta} \theta_\alpha(x)||d\varphi_\alpha^{-1}(\varphi_\alpha(x_\alpha)) \circ d\varphi_\alpha(x)(v)||_{x_\alpha}$$

for every $x \in M$ and $v \in T_xM$, satisfies (P1) and (P2). Indeed, for every $x \in M$ there is an open neighborhood U of x and a finite number of indexes $\alpha_1, \dots, \alpha_n \in \Delta$ such that θ_α vanishes at U for all $\alpha \notin \{\alpha_1, \dots, \alpha_n\}$. So, if $\varphi : V \to X$ is a chart of M with $x \in V \subset U$, the map

$$\begin{aligned} v \in X \mapsto |||d\varphi^{-1}(\varphi(x))(v)|||_x &= \sum_{j=1}^n \theta_{\alpha_j}(x)||d\varphi_{\alpha_j}^{-1}(\varphi_{\alpha_j}(x_{\alpha_j})) \circ d\varphi_{\alpha_j}(x) \circ d\varphi^{-1}(\varphi(x))(v)||_{x_{\alpha_j}} \\ &= \sum_{j=1}^n \theta_{\alpha_j}(x)||d\varphi_{\alpha_j}^{-1}(\varphi_{\alpha_j}(x_{\alpha_j}))((\varphi_{\alpha_j} \circ \varphi^{-1})'(\varphi(x))(v))||_{x_{\alpha_j}} \end{aligned}$$

is an equivalent norm to $||\cdot||$ on X. Property (P2) follows from the continuity of $(\varphi_{\alpha_j} \circ \varphi^{-1})'$ and θ_{α_j}. Condition (5.15) follow from inequality (5.4). □

Chapter 6

Smooth approximation and smooth extension on Finsler manifolds

In this chapter we address some problems discussed in Chapter 3 and Chapter 4 in the context of manifolds, particularly, whether every Lipschitz function $f : M \to \mathbb{R}$ defined on a *non-separable* Riemannian manifold can be uniformly approximated by a Lipschitz, C^∞-smooth function $g : M \to \mathbb{R}$. The study of this problem was motivated by the work in [6], where this result of approximation is stated for *separable* Riemannian manifolds. The question whether this result holds for every Riemannian manifold is posed in [5], [6] and [42]. A positive answer to this question provides nice applications such as: (i) the uniformly bumpable character of every Riemannian manifold, (ii) Deville-Godefroy-Zizler smooth variational principle holds for every complete Riemannian manifold [5] and (iii) the infinite-dimensional version of the Myers-Nakai theorem given in [42] holds for every complete Riemannian manifold (either separable or non-separable) as well, which will be seen in the next chapter.

We study the smooth and Lipschitz approximation problem in the context of C^ℓ Finsler manifolds. Our aim is to study sufficient conditions on a C^ℓ Finsler manifold M so that the above result on uniform approximation of Lipschitz functions defined on M holds. We consider the setting of C^ℓ Finsler manifolds so that we can obtain a unified approach to this problem for both Riemannian and non-Riemannian manifolds, such as smooth submanifolds of a Banach space X, that is a natural continuation of the study done in infinite-dimensional Banach spaces.

In Section 6.1, we prove that every real-valued and Lipschitz function defined on a C^ℓ Finsler manifold M (in the sense of Neeb-Upmeier) *weak-uniform* modeled on a Banach space X can be uniformly approximated by a C^k-smooth and Lipschitz function provided that the Banach space X satisfies a similar approximation property, which we have denoted by (A^k) in Chapter 3, in a *uniform way*: for every Lipschitz function $f : X \to \mathbb{R}$ and every $\varepsilon > 0$, there is a C^k-smooth and Lipschitz function g such that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$ and $\operatorname{Lip}(g) \leq C_0 \operatorname{Lip}(f)$, where the constant C_0 only depends on the Banach space X and C_0 *does not depend on a certain class of equivalent norms* considered in X. This class of norms is closely related to the set of norms defined in the tangent spaces to M. We prove that, for $\ell = 1$, the above assertion holds whenever the manifold M is modeled on a Banach space X with separable dual or M is a (separable or non-separable) Riemannian manifold. In the proof of this assertion, we use the results given in the previous chapter as well as the existence of smooth and Lipschitz partitions of unity subordinated to suitable open covers of the manifold M and the ideas of the separable Riemannian case [6]. A similar result is provided in the case that M is a C^ℓ Finsler manifold (in the sense of Neeb-Upmeier) *uniform*. It is worth mentioning that, in this case, the constant C_0 does not required to be independent of a certain class of norms considered in X. We also obtain some results of uniformly approximation of Lipschitz mappings by Lipschitz and smooth mappings, when the target space is a Banach space.

In Section 6.2, under the above assumptions on the manifold M (in particular, if M is a Riemannian manifold), it is deduced that M is *uniformly bumpable* (this concept was defined in [5]), which will be used to obtain some corollaries in Chapter 7. Furthermore, we follow the ideas of [35], [10] and [56] to establish a characterization of the class of separable C^ℓ Finsler manifolds M in the sense of Neeb-Upmeier which are *C^k-smooth uniformly bumpable* (see Definition 6.2.1) as those having the property that every real-valued, Lipschitz function f defined on M can be uniformly approximated by a C^k-smooth and Lipschitz function g such that $\mathrm{Lip}(g) \leq C\,\mathrm{Lip}(f)$ and C only depends on M.

In Section 6.3, we use Theorem 4.2.1 to get smooth extensions of mappings defined on closed subsets of Banach manifolds. In particular, we show that if M is a paracompact C^1 Banach manifold modeled on a Banach space X and Z is a Banach space such that the pair of Banach spaces (X,Z) satisfies property (*) and the linear extension property (see Chapter 4) then for every closed submanifold P of M and every C^1-smooth mapping $f : P \to Z$, there is a C^1-smooth extension of f to the whole manifold M.

In Section 6.4 the following extension result is established on a C^1 Finsler manifold M in the sense of Neeb-Upmeier *weak-uniform* modeled on a Banach space X: for every C^1-smooth and real-valued function f defined on a closed submanifold P of M, such that f is Lipschitz with respect to the metric of the manifold M, there is a C^1-smooth and Lipschitz extension of f defined on M, provided the Banach space X satisfies the approximation property (*) and C_0 *does not depend on a certain class of equivalent norms* considered in X. A similar result is provided in the case that M is a C^ℓ Finsler manifold (in the sense of Neeb-Upmeier) *uniform*. It is worth mentioning that, in this case, the constant C_0 does not required to be independent of a certain class of norms considered in X. The proof relies on a related result established in Chapter 4. We also obtain some results of smooth and Lipschitz extension of mappings when the target space is a Banach space.

6.1 Smooth approximation of mappings on manifolds

Before starting the main result of this section, let us recall that a Banach space $(X, ||\cdot||)$ satisfies property (A^k) (see Chapter 3) if there is a constant $C_0 \geq 1$, which only depends on the space $(X, ||\cdot||)$, such that for any Lipschitz function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$ there is a Lipschitz, C^k-smooth function $g : X \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in X, \text{ and} \qquad \mathrm{Lip}(g) \leq C_0\,\mathrm{Lip}(f).$$

Recall that if a Banach space X satisfies property (A^k), then for every Lipschitz function $f : A \to \mathbb{R}$ (where A is a subset of X) and every $\varepsilon > 0$ there is a Lipschitz, C^k-smooth function $g : X \to \mathbb{R}$ such that

$$|f(x) - g(x)| < \varepsilon \qquad \text{for all } x \in A, \text{ and} \qquad \mathrm{Lip}(g) \leq C_0\,\mathrm{Lip}(f).$$

Indeed, there exists a Lipschitz extension $F : X \to \mathbb{R}$ of f such that $\text{Lip}(F) = \text{Lip}(f)$ (for instance $x \mapsto \inf_{y\in A}\{f(y) + \text{Lip}(f)||x - y||\}$), and applying property (A^k) to F the assertion is obtained.

In this chapter, the Banach spaces satisfying property (A^k) with constant C_0 independent of the equivalent norm considered on X play an important role. For this reason, let us give some examples of Banach spaces satisfying this property.

Remark 6.1.1.

1. *Every finite-dimensional Banach space X admits property (A^∞). Since the functions $g(\cdot)$ are constructed by means of integral convolutions, it can be easily checked that the constant C_0 can be taken as 1 for every equivalent norm $||\cdot||$ considered in X.*
2. *Every Hilbert space H admits property (A^1) (see [85]). Also, from the construction of the functions $g(\cdot)$ with inf-sup-convolution formulas, it can be easily checked that the constant C_0 can be taken as 1 for every Hilbertian norm $||\cdot||$ considered in H.*
3. *Every separable Banach space with a C^k-smooth and Lipschitz bump function satisfies property (A^k) (see [10], [35], [56] and [7]). Moreover, the constant C_0 can be obtained to be independent of the equivalent norm considered in X, in fact, C_0 can be chosen less than $4 + r$ for any $r > 0$ (see Theorem 3.1.1).*

In the following, we will extend to a certain class of C^ℓ Finsler manifolds, the result on approximation of Lipschitz functions by smooth and Lipschitz functions defined on separable Riemannian manifolds given in [6]. This result is new, even in the case when M is a non-separable Riemannian manifold.

Theorem 6.1.2. *Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform modeled on a Banach space X which admits property (A^k) and the constant C_0 does not depend on the (equivalent) norm. Then, for every Lipschitz function $f : M \to \mathbb{R}$, any $r > 0$ and any continuous function $\varepsilon : M \to (0, \infty)$ there is a Lipschitz, C^m-smooth function $g : M \to \mathbb{R}$ ($m := \min\{\ell, k\}$) such that*

$$|g(p) - f(p)| < \varepsilon(p) \qquad \textit{and} \qquad ||dg(p)||_p \le C_1 \,\text{Lip}(f) \qquad \textit{for every } p \in M,$$

and therefore, $\text{Lip}(g) \le C_1 \,\text{Lip}(f)$, where $C_1 := (1 + r)C_0K^2$.

Proof. We can assume that $L := \text{Lip}(f) > 0$ (otherwise the assertion is trivial) and $0 < \varepsilon(p) < rC_0K^2L$, for all $p \in M$. For every $p \in M$, there is $\delta_p > 0$ such that $\varepsilon(p)/3 < \varepsilon(q)$ for every $q \in B_M(p, 3\delta_p)$ and a C^ℓ-smooth chart $\varphi_p : B_M(p, 3\delta_p) \to X$ with $\varphi_p(p) = 0$, satisfying (P1) of Definition 5.2.1, inequality (5.4) and inequality (5.13) for all points of the ball $B_M(p, 3\delta_p)$. In particular, φ_p and φ_p^{-1} are Lipschitz with the (equivalent) norm $||d\varphi_p^{-1}(0)(\cdot)||_p$ considered on X, $\text{Lip}(\varphi_p) \le K$ and $\text{Lip}(\varphi_p^{-1}) \le K$.

Let us consider an open cover $\{B_M(p_\gamma, \delta_\gamma)\}_{\gamma\in\Gamma}$ of M, where $\delta_\gamma := \delta_{p_\gamma}$ for some set of indexes Γ. Also, let us write $\varphi_\gamma := \varphi_{p_\gamma}$, $\varepsilon_\gamma := \varepsilon(p_\gamma)$ and $|||\cdot|||_\gamma := ||d\varphi_\gamma^{-1}(0)(\cdot)||_{p_\gamma}$. Let us define, for every $\gamma \in \Gamma$,

$$f_\gamma : \varphi_\gamma(B_M(p_\gamma, 3\delta_\gamma)) \subset X \to \mathbb{R} \qquad \textit{as} \qquad f_\gamma(x) := f(\varphi_\gamma^{-1}(x)),$$

which is KL-Lipschitz with the norm $|||\cdot|||_\gamma$ on X. By Lemma 3.2.6, there are open refinements $\{V_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ and $\{W_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ of $\{B_M(p_\gamma, 2\delta_\gamma)\}_{\gamma\in\Gamma}$ satisfying properties $(i)-(iv)$ of Lemma 3.2.6.

Now, we need the following lemma related to the existence of smooth and Lipschitz partitions of unity on a manifold M. Firstly, let us recall the definition of a smooth and Lipschitz partitions of unity on a C^ℓ Finsler manifold.

Definition 6.1.3. *A collection of real-valued, C^k-smooth and Lipschitz functions $\{\psi_i\}_{i\in I}$ defined on a C^ℓ Finsler manifold M is a C^k-smooth and Lipschitz partition of unity subordinated to the open cover $\mathcal{U} = \{U_r\}_{r\in\Omega}$ of M whether (1) $\psi_i \geq 0$ on M for every $i \in I$, (2) the family $\{\operatorname{supp}(\psi_i)\}_{i\in I}$ is locally finite, where $\operatorname{supp}(\psi_i) = \overline{\{x \in M : \psi_i(x) \neq 0\}}$, i.e. for every $x \in M$ there is an open neighborhood U of x and a finite subset $J \subset I$ such that $\operatorname{supp}(\psi_i) \cap U = \emptyset$ for every $i \in I \setminus J$, (3) for every $i \in I$ there is $r \in \Omega$ such that $\operatorname{supp}(\psi_i) \subset U_r$, and (4) $\sum_{i\in I} \psi_i(x) = 1$ for every $x \in M$.*

Lemma 6.1.4. *Under the assumptions of Theorem 6.1.2, there is a C^m-smooth partition of unity $\{\psi_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ of M such that $\operatorname{supp}(\psi_{n,\gamma}) \subset W_{n,\gamma}$ and $\psi_{n,\gamma}$ is Lipschitz for every $n \in \mathbb{N}$, $\gamma \in \Gamma$. In fact, $||d\psi_{n,\gamma}(p)||_p \leq n15C_0K^2 2^{n+1}$ for all $p \in M$, and thus $\operatorname{Lip}(\psi_{n,\gamma}) \leq n15C_0K^2 2^{n+1}$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$.*

Let us assume that Lemma 6.1.4 has been proved. Let us denote by

$$L_{n,\gamma} := \max\{1, \sup\{||d\psi_{n,\gamma}(p)||_p : p \in M\}\}.$$

Since X admits property (A^k) and the constant C_0 does not depend on the equivalent norm considered on X, there is a C^k-smooth, Lipschitz function $g_{n,\gamma} : X \to \mathbb{R}$ such that

$$|g_{n,\gamma}(x) - f_\gamma(x)| \leq \frac{\varepsilon_\gamma/3}{2^{n+2}L_{n,\gamma}} \qquad \text{for all } \ x \in \varphi_\gamma(B_M(p_\gamma, 3\delta_\gamma))$$

and $\operatorname{Lip}(g_{n,\gamma}) \leq C_0 \operatorname{Lip}(f_\gamma) \leq C_0KL$ with the norm $|||\cdot|||_\gamma$ on X. Let us define the function $g : M \to \mathbb{R}$ as

$$g(p) := \sum_{n\in\mathbb{N},\gamma\in\Gamma} \psi_{n,\gamma}(p) g_{n,\gamma}(\varphi_\gamma(p)), \qquad p \in M.$$

Now, if $p \notin B_M(p_\gamma, 2\delta_\gamma)$, then $\psi_{n,\gamma}(p) = 0$ and $\psi_{n,\gamma}(p)g_{n,\gamma}(\varphi_\gamma(p)) = 0$. Since $\operatorname{supp}(\psi_{n,\gamma}) \subset W_{n,\gamma} \subset B_M(p_\gamma, 2\delta_\gamma)$, it is clear that $p \mapsto \psi_{n,\gamma}(p)g_{n,\gamma}(\varphi_\gamma(p))$ is C^m-smooth on M, for each $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Moreover, $\{\operatorname{supp}(\psi_{n,\gamma})\}_{n\in\mathbb{N},\gamma\in\Gamma}$ is locally finite, and thus g is well defined and C^m-smooth on M.

Note that, if $\psi_{n,\gamma}(p) \neq 0$, then $p \in \operatorname{supp}(\psi_{n,\gamma}) \subset B_M(p_\gamma, 2\delta_\gamma)$ and thus $f(p) = f_\gamma(\varphi_\gamma(p))$. Hence,

$$\begin{aligned}|g(p) - f(p)|& \\ = |\sum_{n\in\mathbb{N},\gamma\in\Gamma} \psi_{n,\gamma}(p) g_{n,\gamma}(\varphi_\gamma(p)) - f(p)| &= |\sum_{n\in\mathbb{N},\gamma\in\Gamma} \psi_{n,\gamma}(p)(g_{n,\gamma}(\varphi_\gamma(p)) - f(p))|\end{aligned}$$

$$= |\sum_{\{(n,\gamma):\psi_{n,\gamma}(p)\neq 0\}} \psi_{n,\gamma}(p)(g_{n,\gamma}(\varphi_\gamma(p)) - f_\gamma(\varphi_\gamma(p)))|$$
$$\leq \sum_{\{(n,\gamma):\psi_{n,\gamma}(p)\neq 0\}} \psi_{n,\gamma}(p)\frac{\varepsilon_\gamma/3}{2^{n+2}L_{n,\gamma}} < \sum_{\{(n,\gamma):\psi_{n,\gamma}(p)\neq 0\}} \psi_{n,\gamma}(p)\,\varepsilon(p) = \varepsilon(p).$$

Let us check that g is $2C_0K^2L$-Lipschitz on M. Recall that

- Since $\sum_{\mathbb{N}\times\Gamma}\psi_{n,\gamma}(p) = 1$ for all $p \in M$, we have that $\sum_{\mathbb{N}\times\Gamma} d\psi_{n,\gamma}(p) = 0$ for all $p \in M$.
- Property (iv) of the open refinement $\{W_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ implies that for every $p \in M$ and $n \in \mathbb{N}$, there is at most one $\gamma \in \Gamma$, which we shall denote by $\gamma_p(n)$, such that $p \in \operatorname{supp}(\psi_{n,\gamma})$. Let us define the finite set $F_p := \{(n,\gamma) \in \mathbb{N}\times\Gamma : p \in \operatorname{supp}(\psi_{n,\gamma})\} = \{(n,\gamma_p(n)) \in \mathbb{N}\times\Gamma : p \in \operatorname{supp}(\psi_{n,\gamma_p(n)})\}$.
- If we consider the norm $|||\cdot|||_\gamma$ on X, then $\operatorname{Lip}(g_{n,\gamma}) \leq C_0KL$ and thus $|||g'_{n,\gamma}(x)||| := \sup\{|g'_{n,\gamma}(x)(v)| : |||v|||_\gamma \leq 1\} \leq C_0KL$ for all $x \in X$.
- Also, $|||d\varphi_\gamma(p)||| := \sup\{|||d\varphi_\gamma(p)(v)|||_\gamma : ||v||_p \leq 1\} \leq K$ whenever $p \in B(p_\gamma, 3\delta_\gamma)$.

Therefore, we obtain that $||d(g_{n,\gamma}\circ\varphi_\gamma)(p)||_p \leq C_0K^2L$ whenever $p \in B(p_\gamma, 3\delta_\gamma)$ and

$$||dg(p)||_p = ||\sum_{(n,\gamma)\in F_p} g_{n,\gamma}(\varphi_\gamma(p))d\psi_{n,\gamma}(p) + \sum_{(n,\gamma)\in F_p}\psi_{n,\gamma}(p)d(g_{n,\gamma}\circ\varphi_\gamma)(p)||_p$$
$$= ||\sum_{(n,\gamma)\in F_p}(g_{n,\gamma}(\varphi_\gamma(p)) - f(p))d\psi_{n,\gamma}(p) + \sum_{(n,\gamma)\in F_p}\psi_{n,\gamma}(p)d(g_{n,\gamma}\circ\varphi_\gamma)(p)||_p$$
$$\leq \sum_{(n,\gamma)\in F_p}|g_{n,\gamma}(\varphi_\gamma(p)) - f_\gamma(\varphi_\gamma(p))|\,||d\psi_{n,\gamma}(p)||_p + \sum_{(n,\gamma)\in F_p}\psi_{n,\gamma}(p)C_0K^2L$$
$$\leq \sum_{\{n:\,(n,\gamma_p(n))\in F_p\}}\frac{\varepsilon(p)}{2^{n+2}L_{n,\gamma_p(n)}}L_{n,\gamma_p(n)} + C_0K^2L \leq \varepsilon(p)/4 + C_0K^2L < (1+r)C_0K^2L.$$

Finally, by Proposition 5.3.1(i), $\operatorname{Lip}(g) \leq \sup\{||dg(p)||_p : p \in M\} \leq (1+r)C_0K^2L$ which finishes the proof of Theorem 6.1.2.

Now, let us prove Lemma 6.1.4. Let us consider the two refinements $\{V_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ and $\{W_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ of $\{B_M(p_\gamma, 2\delta_\gamma)\}_{\gamma\in\Gamma}$ satisfying the properties $(i)-(iv)$ of Lemma 3.2.6. Recall that $\operatorname{dist}_M(V_{n,\gamma}, M\setminus W_{n,\gamma}) \geq 1/2^{n+1}$ and $\operatorname{dist}_M(W_{n,\gamma}, W_{n,\gamma'}) \geq 1/2^{n+1}$ for every γ, $\gamma' \in \Gamma$, $\gamma \neq \gamma'$, and every $n \in \mathbb{N}$. Also, recall that $\varphi_\gamma : B_M(p_\gamma, 3\delta_\gamma) \to \varphi_\gamma(B_M(p_\gamma, 3\delta_\gamma)) := \widetilde{B}_\gamma \subset X$ satisfies

$$\frac{1}{K}d_M(p,q) \leq |||\varphi_\gamma(p) - \varphi_\gamma(q)|||_\gamma \leq K d_M(p,q), \qquad \text{for } p,q \in B_M(p_\gamma, 3\delta_\gamma).$$

Let us denote $\widetilde{V}_{n,\gamma} := \varphi_\gamma(V_{n,\gamma})$ and $\widetilde{W}_{n,\gamma} := \varphi_\gamma(W_{n,\gamma})$. Clearly,

$$\widetilde{V}_{n,\gamma} \subset \widetilde{W}_{n,\gamma} \subset \varphi_\gamma(B_M(p_\gamma, 3\delta_\gamma)) = \widetilde{B}_\gamma \subset X.$$

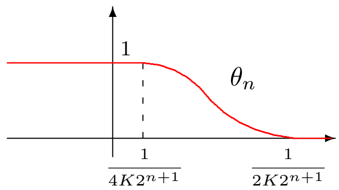
Also, for every $x \in \widetilde{V}_{n,\gamma} \subset X$ and $y \in \widetilde{B}_\gamma \setminus \widetilde{W}_{n,\gamma} \subset X$, there are $p \in V_{n,\gamma}$ and $q \in B_M(p_\gamma, 3\delta_\gamma) \setminus W_{n,\gamma}$ such that $\varphi_\gamma(p) = x$ and $\varphi_\gamma(q) = y$. Thus,

$$|||x-y|||_\gamma = |||\varphi_\gamma(p) - \varphi_\gamma(q)|||_\gamma \geq \frac{1}{K} d_M(p,q) \geq \frac{1}{K2^{n+1}}.$$

Let us define $\operatorname{dist}_\gamma(A,B) := \inf\{|||x-y|||_\gamma : x \in A \text{ and } y \in B\}$ for any pair of subsets $A, B \subset X$. Then, $\operatorname{dist}_\gamma(\widetilde{V}_{n,\gamma}, \widetilde{B}_\gamma \setminus \widetilde{W}_{n,\gamma}) \geq \frac{1}{K2^{n+1}}$.

Let us define $\phi_{n,\gamma} : X \to \mathbb{R}$ as $\phi_{n,\gamma}(x) := \operatorname{dist}_\gamma(x, \widetilde{V}_{n,\gamma})$. Then, $\phi_{n,\gamma}(x) = 0$ for every $x \in \widetilde{V}_{n,\gamma}$, and $\phi_{n,\gamma}(x) \geq \frac{1}{K2^{n+1}}$ whenever $x \in \widetilde{B}_\gamma \setminus \widetilde{W}_{n,\gamma}$. Let us take a Lipschitz function $\theta_n : \mathbb{R} \to [0,1]$ such that

$$\theta_n(t) = \begin{cases} 1 & \text{if } t < \frac{1}{4K2^{n+1}}, \\ 0 & \text{if } t > \frac{1}{2K2^{n+1}}, \end{cases}$$



with $\operatorname{Lip}(\theta_n) \leq 5K2^{n+1}$. Then, $(\theta_n \circ \phi_{n,\gamma})(\widetilde{V}_{n,\gamma}) = 1$, $(\theta_n \circ \phi_{n,\gamma})(\widetilde{B}_\gamma \setminus \widetilde{W}_{n,\gamma}) = 0$ and $\operatorname{Lip}(\theta_n \circ \phi_{n,\gamma}) \leq 5K2^{n+1}$ (with the norm $|||\cdot|||_\gamma$).

Now, by property (A^k), we can find C^k-smooth and Lipschitz functions $\xi_{n,\gamma} : X \to \mathbb{R}$ such that

$$\sup_{y \in X}\{|\xi_{n,\gamma}(y) - (\theta_n \circ \phi_{n,\gamma})(y)|\} < 1/4 \qquad \text{and} \qquad \operatorname{Lip}(\xi_{n,\gamma}) \leq C_0 \operatorname{Lip}(\theta_n \circ \phi_{n,\gamma}),$$

with the norm $|||\cdot|||_\gamma$, for every $\gamma \in \Gamma$ and $n \in \mathbb{N}$.

Let us take a C^∞-smooth Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that $\theta(t) = 0$ whenever $t < \frac{1}{4}$, $\theta(t) = 1$ whenever $t > \frac{3}{4}$ and $\operatorname{Lip}(\theta) \leq 3$. Let us define

$$\widetilde{h}_{n,\gamma} : X \to [0,1] \qquad \text{as} \qquad \widetilde{h}_{n,\gamma}(x) = \theta(\xi_{n,\gamma}(x)),$$

for every $n \in \mathbb{N}$ and $\gamma \in \Gamma$. Then, $\widetilde{h}_{n,\gamma}(x)$ is C^k-smooth, $\operatorname{Lip}(\widetilde{h}_{n,\gamma}) \leq 15C_0K2^{n+1}$ (with the norm $|||\cdot|||_\gamma$), $\widetilde{h}_{n,\gamma}(\widetilde{V}_{n,\gamma}) = 1$ and $\widetilde{h}_{n,\gamma}(\widetilde{B}_\gamma \setminus \widetilde{W}_{n,\gamma}) = 0$.

Now, let us define $h_{n,\gamma} : M \to [0,1]$ as

$$h_{n,\gamma}(p) = \begin{cases} \widetilde{h}_{n,\gamma}(\varphi_\gamma(p)) & \text{if } p \in B_M(p_\gamma, 3\delta_\gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Then, the function $h_{n,\gamma}$ is C^m-smooth, $\operatorname{supp}(h_{n,\gamma}) \subset W_{n,\gamma} \subset B_M(p_\gamma, 2\delta_\gamma)$, $||dh_{n,\gamma}(p)||_p \leq 15C_0K^2 2^{n+1}$ for every $p \in M$ and thus $\operatorname{Lip}(h_{n,\gamma}) \leq 15C_0K^2 2^{n+1}$.

Let us define $h_n : M \to \mathbb{R}$ as $h_n(p) = \sum_{\gamma \in \Gamma} h_{n,\gamma}(p)$, for every $n \in \mathbb{N}$. Since $\operatorname{dist}(W_{n,\gamma}, W_{n,\gamma'}) > 0$ whenever $\gamma \neq \gamma'$, we deduce that h_n is C^m-smooth. Also,

$$h_n(p) = \begin{cases} 1 & \text{if } p \in \bigcup_{\gamma \in \Gamma} V_{n,\gamma}, \\ 0 & \text{if } p \in M \setminus \bigcup_{\gamma \in \Gamma} W_{n,\gamma}. \end{cases}$$

In addition, $||dh_n(p)||_p \leq 15C_0K^2 2^{n+1}$ for every $p \in M$ and thus $\mathrm{Lip}(h_n) \leq 15C_0K^2 2^{n+1}$. Finally, let us define

$$\psi_{1,\gamma} = h_{1,\gamma} \quad \text{and} \quad \psi_{n,\gamma} = h_{n,\gamma}(1-h_1)\cdots(1-h_{n-1}), \quad \text{for } n \geq 2.$$

Clearly the functions $\{\psi_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ are C^m-smooth functions, $\mathrm{supp}(\psi_{n,\gamma}) \subset \mathrm{supp}(h_{n,\gamma}) \subset W_{n,\gamma} \subset B_M(p_\gamma, 2\delta_\gamma)$,

$$||d\psi_{n,\gamma}(p)||_p \leq n15C_0K^2 2^{n+1} \qquad \text{for all } p \in M$$

and thus $\mathrm{Lip}(\psi_{n,\gamma}) \leq n15C_0K^2 2^{n+1}$. In addition, for every $p \in M$,

$$\begin{aligned}\sum_{n\in\mathbb{N},\gamma\in\Gamma} \psi_{n,\gamma}(p) &= \sum_{\gamma\in\Gamma} \psi_{1,\gamma}(p) + \sum_{n\geq 2}\left(\sum_{\gamma\in\Gamma} h_{n,\gamma}(p)\right)\prod_{i=1}^{n-1}(1-h_i(p)) \\ &= h_1(p) + \sum_{n\geq 2} h_n(p)\prod_{i=1}^{n-1}(1-h_i(p)) = 1.\end{aligned}$$

Hence, $\{\psi_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ is a C^m-smooth partition of unity subordinated to the open cover $\{W_{n,\gamma}\}_{n\in\mathbb{N},\gamma\in\Gamma}$ of M with $||d\psi_{n,\gamma}(p)||_p \leq n15C_0K^2 2^{n+1}$ and $\mathrm{Lip}(\psi_{n,\gamma}) \leq n15C_0K^2 2^{n+1}$ for all $p \in M$, $n \in \mathbb{N}$ and $\gamma \in \Gamma$. This finishes the proof of Lemma 6.1.4. □

If we do not assume that the constant C_0 is independent of the (equivalent) norm considered in the Banach space X, a similar result to Theorem 6.1.2 can be obtained for C^ℓ Finsler manifolds in the sense of Neeb-Upmeier K-uniform modeled on a Banach space X.

Theorem 6.1.5. *Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-uniform, modeled on a Banach space $(X, ||\cdot||)$ which admits property (A^k). Then, for every Lipschitz function $f: M \to \mathbb{R}$, any $r > 0$ and any continuous function $\varepsilon: M \to (0,\infty)$ there is a Lipschitz, C^m-smooth function $g: M \to \mathbb{R}$ ($m := \min\{\ell, k\}$) such that*

$$|g(p) - f(p)| < \varepsilon(p) \qquad \text{and} \qquad ||dg(p)||_p \leq C_1 \,\mathrm{Lip}(f) \qquad \text{for every } p \in M,$$

and thus $\mathrm{Lip}(g) \leq C_1 \,\mathrm{Lip}(f)$, where $C_1 := (1+r)C_0K^2$ and C_0 is the constant given by property (A^k).

The proof of Theorem 6.1.5 follows along the same lines as that for Theorem 6.1.2. Let us indicate that, in this case, throughout the proof the norm considered in X is $||\cdot||$ (instead of $|||\cdot|||_\gamma$).

In the same way as the proof of Theorem 6.1.2 and Theorem 6.1.5, we can prove the following vector-valued results.

Proposition 6.1.6. *Let Z be a Banach space. Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform modeled on a Banach space X such that the pair of Banach spaces (X, Z) satisfies property ($*^k$), i.e. for every subset A of X, every Lipschitz mapping $f: A \to Z$ and any $\varepsilon > 0$, there is a Lipschitz, C^k-smooth mapping $g: X \to Z$ such that*

$$||f(x) - g(x)|| < \varepsilon \qquad \text{for every } x \in A, \text{ and} \qquad \mathrm{Lip}(g) \leq C_0 \,\mathrm{Lip}(f),$$

where C_0 depends only on X and Z, and moreover C_0 is independent of the (equivalent) norm considered in X. Then, for every Lipschitz mapping $f : M \to Z$, any $r > 0$ and any continuous function $\varepsilon : M \to (0, \infty)$, there is a Lipschitz, C^m-smooth mapping $g : M \to Z$ such that

$$||g(p) - f(p)|| < \varepsilon(p) \qquad \text{and} \qquad ||dg(p)||_p \leq (1 + r)C_0K^2 \operatorname{Lip}(f) \qquad \text{for every } p \in M.$$

Proposition 6.1.7. *Let Z be a Banach space. Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier K-uniform modeled on a Banach space X such that the pair of Banach spaces (X, Z) satisfies property $(*^k)$ with C_0 depending only on X and Z. Then, for every Lipschitz mapping $f : M \to Z$, any $r > 0$ and any continuous function $\varepsilon : M \to (0, \infty)$, there is a Lipschitz, C^m-smooth mapping $g : M \to Z$ such that*

$$||g(p) - f(p)|| < \varepsilon(p) \qquad \text{and} \qquad ||dg(p)||_p \leq (1 + r)C_0K^2 \operatorname{Lip}(f) \qquad \text{for every } p \in M.$$

6.2 Uniformly bumpable and smooth approximation

Let us recall that the existence of smooth and Lipschitz bump functions on a Banach space is an essential tool to obtain approximation of Lipschitz functions by Lipschitz and smooth functions defined on Banach spaces (see Chapter 3). A generalization of this concept to manifolds is the notion of *uniformly bumpable manifold*, which was introduced by D. Azagra, J. Ferrera and F. López-Mesas [5] for Riemannian manifolds. A natural extension to every C^ℓ Finsler manifold can be defined in the same way, as follows.

Definition 6.2.1. *A C^ℓ Finsler manifold M in the sense of Neeb-Upmeier is C^k-**uniformly bumpable** (with $k \leq \ell$) whenever there are $R > 1$ and $r > 0$ such that for every $p \in M$ and $\delta \in (0, r)$ there exists a C^k-smooth function $b : M \to [0, 1]$ such that:*

1. $b(p) = 1$,
2. $b(q) = 0$ *whenever* $d_M(p, q) \geq \delta$,
3. $\sup_{q \in M} ||db(q)||_q \leq R/\delta$.

Note that this is not a restrictive definition. In fact, D. Azagra, J. Ferrera, F. López-Mesas and Y.C. Rangel [6] proved that every separable Riemannian manifold is C^∞-uniformly bumpable. Now, we can show that a rich class of Finsler manifolds, which includes every Riemannian manifold (separable or non-separable), is uniformly bumpable. This result answers a problem posed in [5], [6] and [42].

Proposition 6.2.2. *Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier satisfying one of the following conditions:*

1. *M is K-weak-uniform and it is modeled on a Banach space X which admits property (A^k) and the constant C_0 does not depend on the (equivalent) norm.*
2. *M is K-uniform and it is modeled on a Banach space X which admits property (A^k).*

Then, M is C^m-uniformly bumpable with $m := \min\{\ell, k\}$.

Proof. The assertion follows from Theorem 6.1.2 and Theorem 6.1.5 in a similar way to the Riemannian case. Let us give the proof under the assumption given in (1) for completeness. For every $r > 0$, $0 < \delta < r$ and $p \in M$, let us define the function $f : M \to [0,1]$ such that

$$f(q) = \begin{cases} 1 - \frac{d_M(q,p)}{\delta} & \text{if } d_M(q,p) \leq \delta, \\ 0 & \text{if } d_M(q,p) \geq \delta. \end{cases}$$

The function f is $\frac{1}{\delta}$-Lipschitz, $f(p) = 1$ and $f(q) = 0$ whenever $q \notin B_M(p,\delta)$. Let us take $C_1 = 2C_0K^2$ the constant given in Theorem 6.1.2 and $\varepsilon := \frac{1}{4}$. By Theorem 6.1.2, there is a C^m-smooth function $g : M \to \mathbb{R}$ such that

$$|g(q) - f(q)| < \frac{1}{4} \qquad \text{and} \qquad ||dg(q)||_q \leq C_1 \operatorname{Lip}(f)$$

for every $q \in M$. Thus $\operatorname{Lip}(g) \leq C_1 \operatorname{Lip}(f) = \frac{C_1}{\delta}$.

Let us take a suitable C^∞-smooth and Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that $\theta(t) = 0$ whenever $t \leq \frac{1}{4}$ and $\theta(t) = 1$ for $t \geq \frac{3}{4}$ (with $\operatorname{Lip}(\theta) \leq 3$). Let us define $b(q) = \theta(g(q))$ for $q \in M$. It is clear that

- b is C^m-smooth,
- $\sup_{q\in M} ||db(q)||_q \leq \frac{3C_1}{\delta}$, and thus $\operatorname{Lip}(b) \leq \frac{3C_1}{\delta}$,
- $b(p) = 1$ and $b(q) = 0$ for $q \notin B_M(p,\delta)$.

Finally, we define $R := 3C_1 = 6C_0K^2$ and this finishes the proof. □

Now, we establish a characterization of the class of separable C^ℓ Finsler manifolds in the sense of Neeb-Upmeier which are uniformly bumpable as those separable C^ℓ Finsler manifolds in the sense of Neeb-Upmeier admitting approximation of Lipschitz functions by Lipschitz and smooth functions.

First of all, we need to define the main tool to obtain smooth and Lipschitz approximation. The notion of *smooth sup-partitions of unity* on Banach spaces was introduced by R. Fry [35] to solve the problem of smooth and Lipschitz approximation on Banach spaces with separable dual, and it has been studied in Chapter 3. This concept can be considered in the context of C^ℓ Finsler manifolds as well.

Definition 6.2.3. *Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier. M admits **C^k-smooth and Lipschitz sup-partitions of unity** subordinated to an open cover $\mathcal{U} = \{U_r\}_{r\in\Omega}$ of M, if there is a collection of C^k-smooth and L-Lipschitz functions $\{\psi_\alpha\}_{\alpha\in\Gamma}$ (where $L > 0$ depends on M and the cover $\mathcal{U}$) such that*

(S1) $\psi_\alpha : M \to [0,1]$ *for all* $\alpha \in \Gamma$,

(S2) *for each* $x \in M$ *the set* $\{\alpha \in \Gamma : \psi_\alpha(x) > 0\} \in c_{00}(\Gamma)$,

(S3) $\{\psi_\alpha\}_{\alpha\in\Gamma}$ *is subordinated to* $\mathcal{U} = \{U_r\}_{r\in\Omega}$, *i.e. for each* $\alpha \in \Gamma$ *there is* $r \in \Omega$ *such that* $\operatorname{supp}(\psi_\alpha) \subset U_r$, *and*

(S4) *for each* $x \in M$ *there is* $\alpha \in \Gamma$ *such that* $\psi_\alpha(x) = 1$.

Following the proofs given in [35] and [10], it can be stated the existence of C^k-smooth sup-partitions of unity on separable, C^ℓ Finsler manifolds that are C^k-uniformly bumpable. As in the case of Banach spaces, the existence of C^k-smooth sup-partitions of unity in a C^ℓ Finsler manifold M provides the tool to prove the uniform approximation of (real-valued) Lipschitz functions by smooth Lipschitz functions on M. Thus, it can be stated the following characterization.

Theorem 6.2.4. *Let M be a separable C^ℓ Finsler manifold in the sense of Neeb-Upmeier. The following conditions are equivalent:*

(1) M is C^k-uniformly bumpable ($k \le \ell$).

(2) There exists $r' > 0$ such that for every $\delta < r'$ the manifold M admits a C^k-smooth and Lipschitz sup-partition of unity $\{\varphi_n\}_{n\in\mathbb{N}}$ subordinated to the open cover $\{B_M(p,\delta)\}_{p\in M}$ of M such that $\operatorname{Lip}(\varphi_n) \le \sup\{||d\varphi_n(x)||_x : x \in M\} \le \frac{C_1}{\delta}$ for all $n \in \mathbb{N}$, with $C_1 \ge 1$ a constant depending only on M.

(3) There is $C_2 \ge 1$ (which only depends on M) such that for any Lipschitz function $f : G \to \mathbb{R}$ defined on an open subset of M and any $\varepsilon > 0$, there exists a Lipschitz and C^k-smooth function $g : M \to \mathbb{R}$ such that $|f(x) - g(x)| < \varepsilon$ for every $x \in G$, $||dg(x)||_x \le C_2 \operatorname{Lip}(f)$ for all $x \in M$, and thus $\operatorname{Lip}(g) \le C_2 \operatorname{Lip}(f)$.

Sketch of the Proof. The proof of (3) ⇒ (1) follows along the same lines as the proof of Proposition 6.2.2. The proof of (1) ⇒ (2) ⇒ (3) is analogous to the Banach space case [35], [10] and [56] (see also the proof of Theorem 3.1.1) with some modifications. Let us sketch the steps of the proofs for the readers convenience.

(1) ⇒ (2) Let us mention the necessary modifications to be made in [35] and [10] to prove this assertion. Let us take $\eta > 0$ such that $(1+\eta)^2 \le 2$, and a C^∞-smooth norm $||\cdot||$ on c_0, such that $||\cdot||_\infty \le ||\cdot|| \le (1+\eta)||\cdot||_\infty$. Since M is C^k-uniformly bumpable, there are constants $r > 0$ and $R > 1$ such that for every point $p \in M$ and $\delta \in (0, r')$ (where $r' := \min\{r, \frac{1}{1+\eta}\}$) we can obtain two families of C^k-smooth functions, $\{b_p\}_{p\in M}$ and $\{\widetilde{b}_p\}_{p\in M}$, where $b_p, \widetilde{b}_p : M \to [0,1]$ are such that

(1) $b_p(p) = 1$, $\widetilde{b}_p(p) = 1$,

(2) $b_p(x) = 0$ whenever $d_M(x,p) \ge \delta$, $\widetilde{b}_p(x) = 0$ whenever $d_M(x,p) \ge \delta/2R$, and

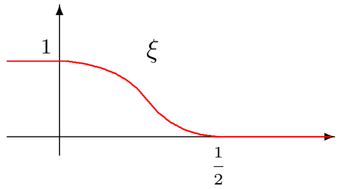
(3) $\operatorname{Lip}(b_p) \le \sup_{x\in M} ||db_p(x)||_x \le R/\delta$, and $\operatorname{Lip}(\widetilde{b}_p) \le \sup_{x\in M} ||d\widetilde{b}_p(x)||_x \le 2R^2/\delta$.

Notice that

$$\begin{cases} b_p(x) \ge 1/2 & \text{if } d_M(x,p) \le \delta/2R \\ b_p(x) = 0 & \text{if } d_M(x,p) \ge \delta \end{cases}, \quad \text{and} \quad \begin{cases} \widetilde{b}_p(x) \ge 1/2 & \text{if } d_M(x,p) \le \delta/4R^2 \\ \widetilde{b}_p(x) = 0 & \text{if } d_M(x,p) \ge \delta/2R \end{cases}.$$

Let us define a C^∞-smooth and 3-Lipschitz function $\xi : \mathbb{R} \to [0,1]$ such that

$$\xi(t) = \begin{cases} 0 & \text{if } t \geq 1/2, \\ 1 & \text{if } t \leq 0, \end{cases}$$



Now, by composing $\{b_p\}_{p\in M}$ and $\{\widetilde{b}_p\}_{p\in M}$ with ξ and $1-\xi$, we obtain C^k-smooth and Lipschitz functions $f_p = \xi(b_p)$, $g_p = 1 - \xi(\widetilde{b}_p) : M \to [0,1]$ such that

$$f_p(x) = \begin{cases} 0 & \text{if } d_M(x,p) \leq \delta/2R \\ 1 & \text{if } d_M(x,p) \geq \delta \end{cases}, \quad \text{and} \quad g_p(x) = \begin{cases} 1 & \text{if } d_M(x,p) \leq \delta/4R^2 \\ 0 & \text{if } d_M(x,p) \geq \delta/2R \end{cases}.$$

$$\mathrm{Lip}(f_p) \leq \sup\{||df_p(x)||_x : x \in M\} \leq 3R/\delta, \text{ and}$$
$$\mathrm{Lip}(g_p) \leq \sup\{||dg_p(x)||_x : x \in M\} \leq 6R^2/\delta.$$

Since M is separable, let us fix a dense countable sequence $\{p_n\}_{n\in\mathbb{N}}$ of M and let us denote $f_n := f_{p_n}$ and $g_n := g_{p_n}$. Let us define a function $h : \mathbb{R} \to [0,1]$ as a C^∞-smooth and Lipschitz function with $h(t) = 1$ on a neighborhood of the interval $(-\infty, 0]$, $h(t) = 0$ for $t \geq 1$, and $\mathrm{Lip}(h) \leq (1+\eta)$.

Finally, we define for $n \geq 1$ the function $\varphi_n : X \to [0,1]$ as

$$\varphi_n(x) := h(||(g_1(x), \dots, g_{n-1}(x), f_n(x), 0, 0, \dots)||),$$

and it is easy to check that it is a countable C^k-smooth and Lipschitz sup-partition of unity subordinated to $\{B_M(p,\delta)\}_{p\in M}$ with

$$\mathrm{Lip}(\varphi_n) \leq \sup\{||d\varphi_n(x)||_x : x \in M\} \leq (1+\eta)^2 \max\{3R/\delta,\, 6R^2/\delta\} \leq 12R^2/\delta.$$

(2) $\Rightarrow$ (3) *Step 1. There is a constant $\widetilde{C} \geq 1$, that only depends on M, such that for every Lipschitz function $f : M \to [0,1]$ with $\mathrm{Lip}(f) \geq 1$, there exists a Lipschitz and C^k-smooth function $g : M \to \mathbb{R}$ such that*

$$|f(x) - g(x)| < 1/4 \;\; \textit{for every } x \in M, \;\; \textit{and } \mathrm{Lip}(g) \leq \sup\{||dg(x)||_x : x \in M\} \leq \widetilde{C}\,\mathrm{Lip}(f).$$

Notice that the constants $r' > 0$ and $C_1 \geq 1$ given by condition (2) depend only on M. Let us take $\eta > 0$ such that $(1+\eta)^3 \leq 2$, a C^∞-smooth norm $||\cdot||$ on c_0 that depends on finitely many coordinates on $c_0(\mathbb{N}) \setminus \{0\}$, such that $||\cdot||_\infty \leq ||\cdot|| \leq (1+\eta)||\cdot||_\infty$, and a constant $B > \max\{4, \frac{1}{r'}\}$, which only depends on M. Now, if $\mathrm{Lip}(f) := L \geq 1$, let us define $\delta := \frac{1}{(1+\eta)BL} < r'$. By (2), there exists a C^k-smooth and Lipschitz sup-partition of unity $\{\varphi_n\}_{n=1}^\infty$ subordinated to $\{B_M(p,\delta)\}_{p\in M}$ such that

$$\mathrm{Lip}(\varphi_n) \leq \sup\{||d\varphi_n(x)||_x : x \in M\} \leq C_1/\delta = (1+\eta)C_1BL.$$

Therefore, we can define the function $g : M \to \mathbb{R}$ as

$$g(x) := \frac{||\{f(p_n)\varphi_n(x)\}||}{||\{\varphi_n(x)\}||} \qquad x \in M.$$

As $||\cdot||$ is C^∞-smooth and depends locally on finitely many coordinates on $c_0(\mathbb{N}) \setminus \{0\}$, the family of functions $\{\varphi_n\}_{n=1}^\infty$ is C^k-smooth and locally finite, and $||\{\varphi_n(x)\}|| \geq 1$ on M. Then g is well defined and it is C^k-smooth on M.

Let us see that $||dg(x)||_x \leq \widetilde{C}L$ for every $x \in M$, and thus it is $\widetilde{C}L$-Lipschitz, where $\widetilde{C} := 4C_1B$ ($\widetilde{C}$ only depends on M). For any $x \in M$

$$\begin{aligned}||dg(x)||_x \leq & ||\{\varphi_n(x)\}|| \frac{|| \, ||\cdot||'(\{f(p_n)\varphi_n(x)\})(\{f(p_n)d\varphi_n(x)\})||_x}{||\{\varphi_n(x)\}||^2} \\ & + ||\{f(p_n)\varphi_n(x)\}|| \frac{|| \, ||\cdot||'(\{\varphi_n(x)\})(\{d\varphi_n(x)\})||_x}{||\{\varphi_n(x)\}||^2} \leq 2(1+\eta)^3 C_1BL \leq 4C_1BL.\end{aligned}$$

Finally $|g(x) - f(x)| < 1/4$ on M. Indeed, for $x \in M$

$$|g(x)-f(x)| \leq \frac{||\{(f(p_n) - f(x))\varphi_n(x)\}||}{||\{\varphi_n(x)\}||} \leq (1+\eta)L \cdot \left(\sup_{\substack{n\in\mathbb{N} \\ x\in B_M(p_n,\delta)}} \{d_M(x,p_n)\} \right) \leq 1/B < 1/4.$$

Step 2. Either Lemma [7, Lemma 1] or [56, Proposition 1 and Theorem 3] provides the final step to ensure the existence of a constant $C_2 \leq 3\widetilde{C}$ (C_2 only depends on M) such that every real-valued and Lipschitz function $f : M \to \mathbb{R}$ can be uniformly approximated by a C^k-smooth function g such that $\mathrm{Lip}(g) \leq \sup\{||dg(x)||_x : x \in M\} \leq C_2 \,\mathrm{Lip}(f)$. The proofs in [7] and [56] are given for Banach spaces, but they also work for a C^ℓ Finsler manifold provided Step 1 holds. □

Remark 6.2.5. *Recall that if M is a C^ℓ Finsler manifold (separable or non-separable) in the sense of Neeb-Upmeier modeled on a Banach space X, then condition (3) in Theorem 6.2.4 yields the fact that X has property (A^k). A proof of this fact can be obtained by applying the techniques of N. Moulis [92], P. Hájek and M. Johanis [56]. Unfortunately, the results given in the above section require additional assumptions on the manifold M to prove the converse, i.e. to prove that if X has property (A^k), then M satisfies condition (3) in Theorem 6.2.4.*

6.3 Smooth extensions of mappings on manifolds

Let us apply Theorem 4.2.1 in order to obtain a smooth extension theorem for mappings between a Banach manifold M and a Banach space Z. First of all, as in the Banach space case, we define the following *mean value condition.*

Definition 6.3.1. *Let us consider a C^1 Banach manifold M modeled on the Banach space X, a Banach space Z, a subset A of M and a continuous map $f : A \to Z$. We say that the mapping $f : A \to Z$ satisfies* **the mean value condition** *on A if for every $x \in A$ there is (equivalent, for all) a chart $\varphi : U \to X$ with U an open subset of M and $x \in U$, such that the* *mapping* $f \circ \varphi^{-1}$ *satisfies the mean value condition on* $\varphi(A \cap U)$*. In other words, there exists a continuous map* $D : \varphi(A \cap U) \to \mathcal{L}(X, Z)$ *such that for every* $y \in \varphi(A \cap U)$ *and every* $\varepsilon > 0$*, there is an open ball* $B(y, r)$ *in* X *such that*

$$||f \circ \varphi^{-1}(z) - f \circ \varphi^{-1}(w) - D(y)(z - w)|| \leq \varepsilon ||z - w||,$$

for every $z, w \in \varphi(A \cap U) \cap B(y, r)$*.*

The proof of the following proposition is similar to the real-valued case [7]. Recall that a paracompact C^1 manifold M modeled on a Banach space X admits C^1-smooth partitions of unity whenever the Banach space X does.

Proposition 6.3.2. *Let* M *be a paracompact* C^1 *Banach manifold modeled on the Banach space* X *and let* Z *be a Banach space. Assume that the pair* (X, Z) *satisfies property (*). Let* A *be a closed subset of* M *and* $f : A \to Z$ *a mapping. Then,* f *satisfies the mean value condition on* A *if and only if there is a* C^1*-smooth extension* $G : M \to Z$ *of* f*.*

Proof. Let $\{U_a\}_{a \in A}$ be a covering of A by open sets in M, so that there are C^1-smooth charts $\varphi_a : U_a \to X$ in M with $f \circ \varphi_a^{-1}$ satisfying the mean value condition on $\varphi_a(A \cap U_a)$, for every $a \in A$. Let us take O_a an open neighborhood of a in M with $\overline{O_a} \subset U_a$ for any $a \in A$. Thus, $\varphi_a(A \cap \overline{O_a})$ is a closed subset of X and $f \circ \varphi_a^{-1}$ satisfies the mean value condition on $\varphi_a(A \cap \overline{O_a})$. By Theorem 4.2.1 there is a C^1-smooth extension $\widetilde{F}_a : X \to Z$ of $f \circ \varphi_a^{-1}|_{\varphi_a(A \cap \overline{O_a})}$ to X for any $a \in A$. Since M is a paracompact C^1 manifold modeled on a Banach space which admits C^1-smooth partitions of unity, we can take $\{\theta_a\}_{a \in A} \cup \{\theta\}$ a C^1-smooth partition of unity subordinated to the open cover $\{O_a\}_{a \in A} \cup \{M \setminus A\}$ of M. Thus, the mapping $G : M \to Z$ defined as

$$G(x) = \sum_{a \in A} \theta_a(x)(\widetilde{F}_a \circ \varphi_a)(x)$$

is a C^1-smooth extension of the mapping f to the whole manifold M. □

Let us recall that $P \subset M$ is a C^1 **submanifold** of M modeled on a closed subspace Y of X if for every $p \in P$ there is a chart (V_p, φ_p) of M at p, such that $p \in V_p$ and $\varphi_p(V_p \cap P) = A \cap Y$, where A is an open subset of X with $\varphi_p(p) \in A \cap Y$. Notice that we do not require in this definition that Y is complemented in X. Thus, this definition of submanifold is more general than the one considered in some texts for Banach manifolds modeled on infinite-dimensional Banach spaces.

Corollary 6.3.3. *Let* M *be a paracompact* C^1 *Banach manifold modeled on a Banach space* X *and let* Z *be a Banach space. Assume that the pair* (X, Z) *satisfies property (*) and the linear extension property. Let* P *be a* C^1 *submanifold of* M*. Then, every* C^1*-smooth mapping* $f : P \to Z$ *has a* C^1*-smooth extension to* M*.*

The proof of this corollary follows along the same lines as the proof of Proposition 6.3.2. Here, we apply Corollary 4.4.6 to yield the conclusion.

Using the examples given in Section 3 and Section 4 of Chapter 4, we obtain a variety of examples of pairs consisting of a Banach manifold and a Banach space, which satisfy the smooth extension results stated in Proposition 6.3.2 and Corollary 6.3.3. Among these examples we can include any pair consisting of a Riemannian manifold M and a Hilbert space Z with either M separable or $Z = \mathbb{R}$.

6.4 Smooth and Lipschitz extensions of mappings on manifolds

In this section we give a smooth extension result on a certain class of C^1 Finsler manifolds M for functions $f : A \to \mathbb{R}$ defined on a closed subset $A \subset M$, whenever f is Lipschitz with respect to the Finsler metric of the manifold M and f satisfies the following property.

Definition 6.4.1. *Let M be a C^1 Finsler manifold in the sense of Neeb-Upmeier and let Z be a Banach space. Let A be a subset of M and a continuous map $f : A \to Z$.*

(1) If M is a C^1 Finsler manifold in the sense of Neeb-Upmeier weak-uniform, we say that a mapping $f : A \to Z$ satisfies ***the mean value condition for a bounded map*** *on A if there exists a constant $C > 0$ such that for every $x \in A$ there is a chart $\varphi : U \to X$ with U an open subset of M and $x \in U$, such that the mapping $f \circ \varphi^{-1}$ satisfies the mean value condition on $\varphi(A \cap U)$ for a bounded map $D : \varphi(A \cap U) \to \mathcal{L}(X, Z)$ such that*

$$\sup\{|D(y)|_x : y \in \varphi(A \cap U)\} \leq C, \tag{6.1}$$

where $|D(y)|_x = \sup\{||D(y)(v)|| : v \in X \text{ with } ||d\varphi^{-1}(\varphi(x))(v)||_x \leq 1\}$.

(2) If M is a C^1 Finsler manifold in the sense of Neeb-Upmeier uniform, we say that a mapping $f : A \to Z$ satisfies ***the uniform mean value condition for a bounded map*** *on A if there exists a constant $C > 0$ such that for every $x \in A$ there is a chart $\varphi : U \to X$ with U an open subset of M and $x \in U$ satisfying inequality* (5.5)*, such that the mapping $f \circ \varphi^{-1}$ satisfies the mean value condition on $\varphi(A \cap U)$ for a bounded map $D : \varphi(A \cap U) \to \mathcal{L}(X, Z)$ such that*

$$\sup\{||D(y)|| : y \in \varphi(A \cap U)\} \leq C, \tag{6.2}$$

where $||D(y)|| = \sup\{||D(y)(v)|| : v \in X \text{ with } ||v|| \leq 1\}$.

Lemma 6.4.2. *Let (X, Z) be a pair of Banach spaces with the property (*) and let $A \subset X$ be a closed subset of X. Let $f : A \to Z$ be a Lipschitz mapping which satisfies the* mean value condition for a bounded map *$D : A \to \mathcal{L}(X, Z)$ with $M := \sup\{||D(y)|| : y \in A\} < \infty$. Let us consider $\varepsilon > 0$ and a Lipschitz extension of f to X, which we shall denote by $F : X \to Z$ (i.e. $F : X \to Z$ is Lipschitz and $F(y) = f(y)$, for all $y \in A$). Then, there exists a C^1-smooth and Lipschitz mapping $G : X \to Z$ such that*

(i) $G_{|A} = f$,

(ii) $||F(x) - G(x)|| < \varepsilon$ *for all* $x \in X$, *and*

(iii) $\mathrm{Lip}(G) \leq R(\mathrm{Lip}(F) + M + \mathrm{Lip}(f))$,

where $R := (1 + \frac{27}{2}C_0)(1 + C_0)$, is a constant that depends only on X and Z.

Proof. Since the pair (X, Z) admits the property (*), from Theorem 4.2.2, we know that there exists a Lipschitz and C^1-smooth extension $g : X \to Z$ of f to X such that

$$g_{|_A} = f \qquad \text{and} \qquad \text{Lip}(g) \leq (1 + C_0)(M + \text{Lip}(f)),$$

where C_0 depends on X and Z (C_0 is the constant given by property (*)). Also, from property (*), there is a C^1-smooth and Lipschitz mapping $h : X \to Z$ such that

$$||F(x) - h(x)|| < \varepsilon \qquad \text{for } x \in X, \text{ and} \qquad \text{Lip}(h) \leq C_0 \,\text{Lip}(F).$$

Consider the sets $D = \{x \in X : ||F(x) - g(x)|| < \varepsilon/4\}$, $C = \{x \in X : ||F(x) - g(x)|| \leq \varepsilon/4\}$ and $B = \{x \in X : ||F(x) - g(x)|| < \varepsilon/2\}$ in X. Then $A \subset D \subset C \subset B$.

As in the proof of Lemma 4.2.4, let us consider a C^1-smooth and Lipschitz function

$$u(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \in X \setminus B \end{cases} \qquad \text{and} \qquad \text{Lip}(u) \leq \frac{9C_0(\text{Lip}(F) + \text{Lip}(g))}{\varepsilon}.$$

Now, let us define $G : X \to Z$ as

$$G(x) = u(x)g(x) + (1 - u(x))h(x), \qquad x \in X.$$

Clearly, the mapping G is C^1-smooth and $G(y) = f(y)$ for $y \in A$.

- If $x \in X \setminus B$, then $||F(x) - G(x)|| = ||F(x) - h(x)|| < \varepsilon$.
- If $x \in B$, then $||F(x) - G(x)|| \leq u(x)||F(x) - g(x)|| + (1 - u(x))||F(x) - h(x)|| < \varepsilon$.

Let us prove that G is Lipschitz on X.

(i) If $x \in X \setminus \overline{B}$, then $||G'(x)|| = ||h'(x)|| \leq C_0 \,\text{Lip}(F)$;

(ii) if $x \in \overline{B}$, then

$$\begin{aligned} ||G'(x)|| &\leq ||g(x)u'(x) + h(x)(1 - u)'(x)|| + ||g'(x)u(x) + h'(x)(1 - u(x))|| \\ &\leq ||(g(x) - F(x))u'(x) + (h(x) - F(x))(1 - u)'(x)|| \\ &\qquad\qquad + u(x)\,\text{Lip}(g) + (1 - u(x))C_0 \,\text{Lip}(F) \\ &\leq \frac{3\varepsilon}{2} \text{Lip}(u) + (1 + C_0)(M + \text{Lip}(f)) + C_0 \,\text{Lip}(F) \\ &\leq \frac{29}{2} C_0 \,\text{Lip}(F) + (1 + \frac{27}{2} C_0)(1 + C_0)(M + \text{Lip}(f)). \end{aligned}$$

Now, let us define

$$R := \max\{\frac{29}{2}C_0, (1 + \frac{27}{2}C_0)(1 + C_0)\} = (1 + \frac{27}{2}C_0)(1 + C_0),$$

which yields to $\text{Lip}(G) \leq R(\text{Lip}(F) + M + \text{Lip}(f))$. □

Proposition 6.4.3. *Let M be a C^1 Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform modeled on a Banach space X with property (*) such that the constant C_0 does not depend on the (equivalent) norm considered on X. Let $A \subset M$ be a closed subset of M and $f : A \to \mathbb{R}$ a function. Then, f is Lipschitz and satisfies the mean value condition for a bounded map on A if and only if there is a C^1-smooth and Lipschitz extension $G : M \to \mathbb{R}$ of f to the whole manifold M.*

Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map with constant $C > 0$, then we can obtain a C^1-smooth and Lipschitz extension G of f with $\operatorname{Lip}(G) \leq S(C + \operatorname{Lip}(f))$ where $S := \frac{1}{2} + 2RK^2$ and R is the constant given in Lemma 6.4.2.

Proof. First, let us extend f to M as $F(x) := \inf_{y \in A}\{f(y) + Ld_M(x, y)\}$, for every $x \in M$, where $L = \operatorname{Lip}(f) = \sup\{\frac{|f(p)-f(q)|}{d_M(p,q)} : p, q \in A,\, p \neq q\}$. The function F is a Lipschitz extension of f to M, with the same Lipschitz constant $\operatorname{Lip}(F) = \operatorname{Lip}(f) = L$.

By applying Properties 5.2.3(6), let us take $\{(U_a, \varphi_a)\}_{a \in A}$ a family of charts of M at the points $a \in A$ such that $\{U_a\}_{a \in A}$ is a covering of A by open sets in M, the mapping $f \circ \varphi_a^{-1}$ satisfies the mean value condition on $\varphi_a(A \cap U_a)$ for a bounded map $D_a : \varphi_a(A \cap U_a) \to X^*$ with $\sup\{|D_a(y)|_a : y \in \varphi_a(A \cap U_a)\} \leq C$, and $\varphi_a : U_a \to X$ satisfies the inequality (5.4). Let us denote $|||w|||_a := ||d\varphi_a^{-1}(\varphi_a(a))(w)||_a$ for every $a \in A$ and $w \in X$. Thus, we have

$$|||d\varphi_a(y)|||_a := \sup\{|||d\varphi_a(y)(v)|||_a : ||v||_y = 1\} \leq K \text{ for all } y \in U_a, \text{ and}$$
$$||d\varphi_a^{-1}(\varphi_a(y))||_a := \sup\{||d\varphi_a^{-1}(\varphi_a(y))(v)||_y : |||v|||_a = 1\} \leq K \text{ for all } y \in U_a.$$

Moreover, due to Lemma 5.3.2 we may assume the charts $\varphi_a : U_a \to X$ are K-bi-Lipschitz in U_a with the norms $||| \cdot |||_a$.

Let us take O_a an open neighborhood of a in M with $\overline{O_a} \subset U_a$ for every $a \in A$. Thus, $\varphi_a(A \cap \overline{O_a})$ is a closed subset of X and $f \circ \varphi_a^{-1}$ is a Lipschitz function that satisfies the mean value condition on the closed subset $\varphi_a(A \cap \overline{O_a})$ for the bounded map $D_a : \varphi_a(A \cap \overline{O_a}) \to X^*$ with constant $C > 0$.

Since $\{O_a\}_{a \in A} \cup \{M \setminus A\}$ is an open covering of M, by Lemma 6.1.4, there is an open refinement $\{W_{n,a}\}_{n \in \mathbb{N}, a \in A} \cup \{W_{n,0}\}_{n \in \mathbb{N}}$ of $\{O_a\}_{a \in A} \cup \{M \setminus A\}$ satisfying properties $(i) - (iv)$ of Lemma 3.2.6, and there is $\{\psi_{n,a}\}_{n \in \mathbb{N}, a \in A} \cup \{\psi_{n,0}\}_{n \in \mathbb{N}}$ a C^1-smooth and Lipschitz partition of unity of M such that $\operatorname{supp}(\psi_{n,a}) \subset W_{n,a} \subset O_a$ for every $n \in \mathbb{N}$ and $a \in A$, and $\operatorname{supp}(\psi_{n,0}) \subset W_{n,0} \subset M \setminus A$ for every $n \in \mathbb{N}$. Let us write $L_{n,a} := \max\{1, \sup\{||d\psi_{n,a}(x)||_x : x \in M\}\}$ for every $n \in \mathbb{N}$ and $a \in \widetilde{A} := A \cup \{0\}$.

Let us define $f_a : \varphi_a(A \cap \overline{O_a}) \to \mathbb{R}$ (for all $a \in A$) and $F_a : \varphi_a(U_a) \to \mathbb{R}$ (for all $a \in A$) as

$$f_a(y) := f \circ \varphi_a^{-1}(y) \qquad \text{and} \qquad F_a(x) := F \circ \varphi_a^{-1}(x),$$

for every $y \in \varphi_a(A \cap \overline{O_a})$ and $x \in \varphi_a(U_a)$. The functions f_a and F_a are KL-Lipschitz with the norm $||| \cdot |||_a$ in X, $f_a : \varphi_a(A \cap \overline{O_a}) \to \mathbb{R}$ is a Lipschitz function satisfying the mean value condition on the closed subset $\varphi_a(A \cap \overline{O_a})$ for a bounded map, with constant C, and F_a is a Lipschitz extension of f_a to $\varphi_a(U_a)$ for every $a \in A$. Since X admits property (*) (with the same constant C_0, for every equivalent norm), we can apply Lemma 6.4.2 to obtain C^1-smooth and Lipschitz functions $G_{n,a} : X \to \mathbb{R}$, for all $n \in \mathbb{N}$ and $a \in A$, such that

(a) $G_{n,a}|_{\varphi_a(A\cap\overline{O_a})} = f_a$,

(b) $|G_{n,a}(x) - F_a(x)| < \frac{L}{2^{n+1}L_{n,a}}$ for every $x \in \varphi_a(U_a)$, and

(c) $\text{Lip}(G_{n,a}) \leq R(\text{Lip}(F_a) + C + \text{Lip}(f_a)) \leq KR(2L + C)$, for the norm $|||\cdot|||_a$ on X, where R is the constant given in Lemma 6.4.2.

Let us apply Theorem 6.1.2 to $F : M \to \mathbb{R}$ to obtain a family $\{G_{n,0} : M \to \mathbb{R}\}_{n\in\mathbb{N}}$ of C^1-smooth and Lipschitz functions such that

(a) $|G_{n,0}(x) - F(x)| < \frac{L}{2^{n+1}L_{n,0}}$ for every $x \in M$, and

(b) $\text{Lip}(G_{n,0}) \leq 2C_0K^2\,\text{Lip}(F) = 2C_0K^2L$.

Now, let us define $G : M \to \mathbb{R}$ as

$$G(x) := \sum_{n\in\mathbb{N}, a\in A} \psi_{n,a}(x)G_{n,a}(\varphi_a(x)) + \sum_{n\in\mathbb{N}} \psi_{n,0}(x)G_{n,0}(x).$$

Since $\text{supp}(\psi_{n,a}) \subset W_{n,a} \subset O_a$ for every $(n, a) \in \mathbb{N} \times A$, $\text{supp}(\psi_{n,0}) \subset W_{n,0} \subset M \setminus A$ for every $n \in \mathbb{N}$, and $\{\psi_{n,a}\}_{n\in\mathbb{N}, a\in\widetilde{A}}$ is a C^1-smooth partition of unity of M, the function G is well defined and C^1-smooth on M. Now, if $y \in A$ and $\psi_{n,a}(y) \neq 0$, then $y \in A \cap O_a$ and $\varphi_a(y) \in \varphi_a(A\cap\overline{O_a})$. Therefore, $G_{n,a}(\varphi_a(y)) = f(y)$ and $G(y) = \sum_{n\in\mathbb{N},a\in A} \psi_{n,a}(y)f(y) = f(y)$.

Let us prove that G is Lipschitz on M. Recall that

- $\sum_{n\in\mathbb{N},a\in\widetilde{A}} d\psi_{n,a}(x) = 0$ for all $x \in M$, and thus $\sum_{n\in\mathbb{N},a\in\widetilde{A}} d\psi_{n,a}(x)F(x) = 0$.
- Property (iv) of the open refinement $\{W_{n,a}\}_{n\in\mathbb{N},a\in\widetilde{A}}$ implies that for every $p \in M$ and $n \in \mathbb{N}$, there is at most one $a \in \widetilde{A}$, which we shall denote by $a_p(n)$, such that $p \in \text{supp}(\psi_{n,a})$. Let us define the finite set $F_p := \{(n, a) \in \mathbb{N} \times \widetilde{A} : p \in \text{supp}(\psi_{n,a})\} = \{(n, a_p(n)) \in \mathbb{N} \times \widetilde{A} : p \in \text{supp}(\psi_{n,a_p(n)})\}$.

In addition, if $x \in M$, $a \in A$ and $d\psi_{n,a}(x) \neq 0$, then $F(x) = F(\varphi_a^{-1}(\varphi_a(x)))$. Therefore,

$$\begin{aligned}
||dG(x)||_x &\leq || \sum_{n\in\mathbb{N},a\in A} G_{n,a}(\varphi_a(x))\, d\psi_{n,a}(x) + \sum_{n\in\mathbb{N}} G_{n,0}(x)d\psi_{n,0}(x)||_x \\
&\quad +|| \sum_{n\in\mathbb{N},a\in A} \psi_{n,a}(x)\, dG_{n,a}(\varphi_a(x))\, d\varphi_a(x) + \sum_{n\in\mathbb{N}} \psi_{n,0}(x)dG_{n,0}(x)||_x \\
&\leq \sum_{n\in\mathbb{N},a\in A} ||d\psi_{n,a}(x)||_x|G_{n,a}(\varphi_a(x)) - F(x)| + \sum_{n\in\mathbb{N}} ||d\psi_{n,0}(x)||_x|G_{n,0}(x) - F(x)| \\
&\quad + \sum_{n\in\mathbb{N},a\in\widetilde{A}} \psi_{n,a}(x) \max\{K^2R(2L + C), 2C_0K^2L\} \\
&\leq \sum_{n\in\mathbb{N}} L_{n,a_p(n)} \frac{L}{2^{n+1}L_{n,a_p(n)}} + K^2R(2L + C) \leq \frac{L}{2} + K^2R(2L + C).
\end{aligned}$$

Thus, if we define $S := \frac{1}{2}+2RK^2$, it follows from Proposition 5.3.1 that $\text{Lip}(g) \leq \sup\{||dg(x)||_x : x \in M\} \leq S(C + \text{Lip}(f))$ and the constant S only depends on M. □

Let us recall that Theorem 3.1.1 implies that every Banach space with separable dual admits property (*) and the constant C_0 does not depend on the norm (notice that a Banach space X satisfies property (*) if and only if satisfies property (A^1)). Unfortunately, we do not know if the conclusion of the Proposition 6.4.3 holds if we drop the assumption that the Banach space X admits the property (*) with the same constant C_0 for every (equivalent) norm. If we do not assume that hypothesis, a similar result to the above proposition can be obtained for C^ℓ Finsler manifolds in the sense of Neeb-Upmeier K-uniform modeled on a Banach space X. The proof follows along the same lines as that for Proposition 6.4.3.

Proposition 6.4.4. *Let M be a C^1 Finsler manifold in the sense of Neeb-Upmeier K-uniform modeled on a Banach space X with property (*). Let $A \subset M$ be a closed subset of M and $f : A \to \mathbb{R}$ a function. Then, f is Lipschitz and satisfies the uniform mean value condition for a bounded map on A if and only if there is a C^1-smooth and Lipschitz extension $G : M \to \mathbb{R}$ of f to the whole manifold M.*

Moreover, if f is Lipschitz and satisfies the uniform mean value condition for a bounded map with constant $C > 0$, then we can obtain a C^1-smooth and Lipschitz extension G of f with $\operatorname{Lip}(G) \leq S(C + \operatorname{Lip}(f))$ where $S := \frac{1}{2} + 2RK^2$ and R is the constant given in Lemma 6.4.2.

As in Section 6.3 and by means of Corollary 4.4.6, we obtain the following results for closed submanifolds. Recall that if M is a C^1 Finsler manifold and P is a C^1 submanifold of M, then $||\cdot||_{|TP}$ is a Finsler structure for P [100, Theorem 3.6].

Corollary 6.4.5. *Let M be a C^1 Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform such that it is modeled on a Banach space X that admits property (*) and the constant C_0 does not depend on the (equivalent) norm considered in X. Let $P \subset M$ be a closed C^1 submanifold and let $f : P \to \mathbb{R}$ be a C^1-smooth function. If f is Lipschitz as a function on M (i.e., there is $L \geq 0$ such that $|f(p) - f(q)| \leq L d_M(p, q)$, for every $p, q \in P$), then there is a C^1-smooth and Lipschitz extension $g : M \to \mathbb{R}$ such that $\operatorname{Lip}(g) \leq S \operatorname{Lip}(f)$, where $S \geq 1$ depends only on M.*

The proof relies on the fact that the function f satisfies the mean value condition for a bounded map whenever f is C^1-smooth and Lipchitz as a function on M defined on a closed C^1 submanifold.

Corollary 6.4.6. *Let M be a C^1 Finsler manifold in the sense of Neeb-Upmeier K-uniform such that it is modeled on a Banach space X with property (*). Let $P \subset M$ be a closed C^1 submanifold and let $f : P \to \mathbb{R}$ be a C^1-smooth function. If f is Lipschitz as a function on M, then there is a C^1-smooth and Lipschitz extension $g : M \to \mathbb{R}$ such that $\operatorname{Lip}(g) \leq S \operatorname{Lip}(f)$, where $S \geq 1$ depends only on M.*

The proof relies on the fact that the function f satisfies the uniform mean value condition for a bounded map whenever f is C^1-smooth and Lipchitz as a function on M defined on a closed C^1 submanifold.

Similarly to Proposition 6.4.3, we can show the following vector-valued results:

Proposition 6.4.7. *Let Z be a Banach space. Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier weak-uniform modeled on a Banach space X such that the pair of Banach* *spaces (X, Z) admits property (*) and the constant C_0 does not depend on the (equivalent) norm on X. Furthermore, let us suppose that there is a constant $K \geq 1$ depending only on M and Z, such that for every subset A of M and every Lipschitz mapping $f : A \to Z$, there is a Lipschitz extension $F : M \to Z$ of f to M with* $\mathrm{Lip}(F) \leq K\,\mathrm{Lip}(f)$ *(for instance, if Z is an absolute Lipschitz retract).*

Let $A \subset M$ be a closed subset of M and $f : A \to Z$ a mapping. Then, f is Lipschitz and satisfies the mean value condition for a bounded map on A if and only if there is a C^1-smooth and Lipschitz extension $G : M \to Z$ of f to the whole manifold M. Moreover, if f is Lipschitz and satisfies the mean value condition for a bounded map with constant $C > 0$, then we can obtain a C^1-smooth and Lipschitz extension G of f with $\mathrm{Lip}(G) \leq S(C + \mathrm{Lip}(f))$ *where $S \geq 1$ depends only on M and Z.*

Proposition 6.4.8. *Let Z be a Banach space. Let M be a C^ℓ Finsler manifold in the sense of Neeb-Upmeier uniform modeled on a Banach space X such that the pair of Banach spaces (X, Z) admits property (*). Furthermore, let us suppose that there is a constant $K \geq 1$ depending only on M and Z, such that for every subset A of M and every Lipschitz mapping $f : A \to Z$, there is a Lipschitz extension $F : M \to Z$ of f to M such that* $\mathrm{Lip}(F) \leq K\,\mathrm{Lip}(f)$.

Let $A \subset M$ be a closed subset of M and $f : A \to Z$ a mapping. Then, f is Lipschitz and it satisfies the uniform mean value condition for a bounded map on A if and only if there is a C^1-smooth and Lipschitz extension $G : M \to Z$ of f to the whole manifold M. Moreover, if f is Lipschitz and it satisfies the uniform mean value condition for a bounded map with constant $C > 0$, then we can obtain the C^1-smooth and Lipschitz extension G with $\mathrm{Lip}(G) \leq S(C + \mathrm{Lip}(f))$ *where $S \geq 1$ depends only on M and Z.*

Open Problems

Let us list some unsolved problems related to this chapter.

1. Does Theorem 6.1.2 hold for every C^ℓ Finsler manifold in the sense of Neeb-Upmeier provided the same hypothesis on X?

2. It is an open problem whether the constant C_0 of the property (A^k) can be taken to be independent of the equivalent norm considered in X, when X is either non-separable or non-Hilbert. Thus, we cannot assure that Theorem 6.1.2 holds for all C^ℓ Finsler manifolds in the sense of Neeb-Upmeier weak-uniform modeled on either non-separable or non-Hilbert Banach spaces.

3. Let M be a non-separable C^ℓ Finsler manifold in the sense of Neeb-Upmeier. Can every function defined on M be uniformly approximated by a C^k-smooth and Lipschitz function whenever M is C^k-uniformly bumpable?

4. Is there any C^ℓ Finsler manifold in the sense of Neeb-Upmeier M modeled on a Banach space X which admits C^k-smooth bump functions, such that M is non-C^k-uniformly bumpable?

5. We do not know whether Propositions 6.1.6, 6.1.7, 6.3.2, 6.4.7 and 6.4.8 hold when the target space is a C^ℓ Finsler manifold N.

6. When does a Banach manifold M admit C^2-smooth extensions of C^2-smooth functions $g : P \to \mathbb{R}$ defined on a C^1 submanifold P of M?

Chapter 7

Applications on Finsler manifolds

In this chapter we obtain nice applications of the previous results on Riemannian manifolds and Finsler manifolds such as: a Deville-Godefroy-Zizler smooth variational principle and a Myers-Nakai theorem.

The Deville-Godefroy-Zizler smooth variational principle (see [22]) asserts, roughly speaking, that a lower bounded, lower semicontinuous function defined on a Banach space can be perturbed by a C^k-smooth and Lipschitz function in order to attain a perturbed minimum, provided the Banach space admits a C^k-smooth bump function with bounded derivatives up to order k. This principle has proven to be very useful in various areas of nonlinear analysis, in particular, in optimization and applications to viscosity solutions of Hamilton-Jacobi equations. D. Azagra, J. Ferrera and F. López-Mesas proved a version of this variational principle for uniformly bumpable complete Riemannian manifolds [5].

In this chapter, we are also interested in characterizing the Finsler structure of a Finsler manifold M in terms of the algebra of real-valued, bounded and C^k-smooth functions defined on M with bounded first derivative. The problem of the interrelation of the topological, metric and smooth structure of a space X and the algebraic and topological structure of the space $C(X)$ (the set of real-valued continuous functions defined on X) has been largely studied. These results are usually referred to as *Banach-Stone type theorems*. Recall the celebrated Banach-Stone theorem, asserting that the compact spaces K and L are homeomorphic if and only if the Banach spaces $C(K)$ and $C(L)$ endowed with the sup-norm are isometric. For more information about Banach-Stone type theorems see the survey [44] and references therein.

The Myers-Nakai theorem states that the structure of a complete Riemannian manifold M is characterized in terms of the Banach algebra $C_b^1(M)$ of all real-valued, bounded and C^1-smooth functions defined on M with bounded derivative endowed with the sup-norm of the function and its derivative. More specifically, two complete Riemannian manifolds M and N are equivalent as Riemannian manifolds, i.e. there is a C^1 diffeomorphism $h : M \to N$ such that

$$\langle dh(x)(v), dh(x)(w)\rangle_{h(x)} = \langle v, w\rangle_x$$

for every $x \in M$ and $v, w \in T_xM$ if and only if the Banach algebras $C_b^1(M)$ and $C_b^1(N)$ are isometric. This result was first proved by S. B. Myers [94] for a compact and Riemannian manifold and by M. Nakai [96] for a finite-dimensional Riemannian manifold. Very recently, I. Garrido, J.A. Jaramillo and Y.C. Rangel [42] have given an extension of the Myers-Nakai theorem for every infinite-dimensional, complete Riemannian manifold, provided that they are uniformly bumpable. A similar result for the so-called finite-dimensional Riemannian-

Finsler manifolds is given in [43] (see also [103]). An interesting result for Banach manifolds was obtained in [46], where it was proved that the smooth structure of a C^∞-smooth Banach manifold M is determined by the algebra $C^\infty(M)$ (the space of all real-valued and C^∞-smooth functions defined on M) whenever M is modeled on a Banach space with a C^∞-smooth bump function.

In Section 7.1 several applications on Riemannian manifolds are given. It is deduced that a Riemannian manifold M is *uniformly bumpable* and thus the Deville-Godefroy-Zizler smooth variational principle holds whenever M is complete. This generalizes the result given in [5] for separable and complete Riemannian manifolds. Moreover, it is deduced that the infinite-dimensional version of the Myers-Nakai theorem for separable Riemannian manifolds given in [42] holds for every infinite-dimensional complete Riemannian manifold.

In Section 7.2, we consider the upper and lower scalar Dini derivatives of a mapping f between Finsler manifolds at x, denoted respectively by $D_x^+ f$ and $D_x^- f$, and we prove that when M and N are Finsler manifolds in the sense of Palais and $f : M \to N$ is a C^1-smooth mapping on M then $D_x^+ f = ||df(x)||_x$ and, if in addiction $df(x) \in \text{Isom}(T_x M, T_{f(x)} N)$, then $D_x^- f = ||[df(x)]^{-1}||_{f(x)}^{-1}$.

In Section 7.3 the algebra $C_b^1(M)$ of all real-valued, bounded and C^1-smooth functions defined on M with bounded derivative endowed with the sup-norm of the function and its derivative is shown to be a Banach algebra, provided M is a Finsler manifold in the sense of Neeb-Upmeier weak-uniform. Thus, following the steps given in [5], the Deville-Godefroy-Zizler variational principle is extended to the class of complete and uniformly bumpable Finsler manifolds in the sense of Neeb-Upmeier weak-uniform.

Our aim in Section 7.4 is to extend the Myers-Nakai theorem to the context of Finsler manifolds. On the one hand, we obtain the Myers-Nakai theorem for (i) finite-dimensional and complete Finsler manifolds, and (ii) WCG Banach spaces with a C^1-smooth bump function. On the other hand, we study for $k \geq 1$ the algebra $C_b^k(M)$ of all real-valued, bounded and C^k-smooth functions with bounded first derivative defined on a complete Finsler manifold M. We prove that these algebras determine the weak Finsler structure of a complete Finsler manifold when $k = 1$ and the Finsler structure when $k \geq 2$. In particular, we obtain a weaker version of the Myers-Nakai theorem for (i) separable and complete Finsler manifolds in the sense of Palais modeled on a Banach space with a Lipschitz and C^k-smooth bump function, and (ii) C^1-uniformly bumpable and complete Finsler manifolds in the sense of Palais modeled on WCG Banach spaces. In the proof of these results we will use the ideas of the Riemannian case [42].

7.1 Consequences on Riemannian manifolds

The results given in the previous chapter provide some interesting consequences on Riemannian manifolds. The following corollary provides a generalization (in the C^1-smoothness case) to the non-separable setting of the result given in [6] for separable Riemannian manifolds (see Theorem 3.0.6).

Corollary 7.1.1. *Let M be a Riemannian manifold. Then, for every Lipschitz function $f : M \to \mathbb{R}$, every continuous function $\varepsilon : M \to (0, \infty)$ and $r > 0$ there is a C^1-smooth and Lipschitz function $g : M \to \mathbb{R}$ such that*

$$|g(p) - f(p)| < \varepsilon(p) \qquad \textit{for every } p \in M, \textit{ and} \qquad \mathrm{Lip}(g) \le \mathrm{Lip}(f) + r.$$

Let us notice that for separable Riemannian manifolds, the Lipschitz function g that approximates f can be obtained to be C^∞-smooth (see [6]). One of the tools of their proof is the result of N. Moulis [92], asserting that every separable Hilbert space has property (A^∞). It is an open problem whether the non-separable Hilbert space has property (A^∞). We do not know whether the proof of N. Moulis can be adapted to the non-separable case. Thus, in the non-separable case, we can only ensure that the Hilbert space has property (A^1) (see Theorem 3.0.5). Consequently, we only obtain that the approximating function g is C^1-smooth. Moreover, from Proposition 6.2.2 we obtain the following corollary:

Corollary 7.1.2. *Every Riemannian manifold is C^1-uniformly bumpable.*

Let M be a Riemannian manifold. Let us denote by $C_b^1(M)$ the algebra of all bounded, Lipschitz and C^1-smooth functions $f : M \to \mathbb{R}$, i.e.

$$C_b^1(M) := \{f : M \to \mathbb{R} : f \text{ is } C^1\text{-smooth, } ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty\}.$$

It is easy to check that $C_b^1(M)$ is a Banach space endowed with the norm $||f||_{C_b^1(M)} := \max\{||f||_\infty, ||df||_\infty\}$ (where $||f||_\infty = \sup_{p\in M} |f(p)|$ and $||df||_\infty = \sup_{p\in M} ||df(p)||_p$). Moreover, it is a *Banach algebra* with the norm $2||\cdot||_{C_b^1(M)}$ (see [5] and [42]). Recall that two *normed algebras* $(A, ||\cdot||_A)$ and $(B, ||\cdot||_B)$ are said to be *equivalent as normed algebras* whenever there exists an *algebra isomorphism* $T : A \to B$ such that $||T(a)||_B = ||a||_A$ for every $a \in A$. Also, the Riemannian manifolds M and N are said to be *equivalent* whenever there is a *Riemannian isometry* $h : M \to N$, i.e. h is a C^1-diffeomorphism from M onto N satisfying

$$\langle dh(x)(v), dh(x)(w)\rangle_{h(x)} = \langle v, w\rangle_x$$

for every $x \in M$ and every $v, w \in T_xM$ (where $\langle \cdot, \cdot\rangle_p$ is the scalar product defined in T_pM). I. Garrido, J.A. Jaramillo and Y.C. Rangel proved in [42] a version of the Myers-Nakai theorem (see [94], [96]) for infinite-dimensional Riemannian manifolds under the assumption that the Riemannian manifold is C^1-uniformly bumpable. Therefore, from [42] and Corollary 7.1.2, we can deduce the following assertion.

Corollary 7.1.3. *Let M and N be complete Riemannian manifolds. Then M and N are equivalent Riemannian manifolds if, and only if, $C_b^1(M)$ and $C_b^1(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^1(N) \to C_b^1(M)$ is of the form $T(f) = f \circ h$, where $h : M \to N$ is a Riemannian isometry.*

A version for uniformly bumpable complete Riemannian manifolds of the *Deville-Godefroy-Zizler smooth variational principle* [22] (DGZ smooth variational principle, for short) was proved in [5]. Thus, from [5] and Corollary 7.1.2 we deduce the following corollary. Recall that a function $f : M \to \mathbb{R} \cup \{\infty\}$ *attains its strong minimum* on M at $x \in M$ if $f(x) = \inf\{f(z) : z \in M\}$ and $d_M(x_n, x) \to 0$ whenever $\{x_n\}_{n=1}^\infty$ is a sequence of points of M such that $f(x_n) \to f(x)$. A function $f : M \to \mathbb{R} \cup \{\infty\}$ is said to be *proper* whether $f \not\equiv \infty$.

Corollary 7.1.4. *(DGZ smooth variational principle for Riemannian manifolds). Let M be a complete Riemannian manifold and let $f : M \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous (lsc) function which is bounded below and proper. Then, for each $\varepsilon > 0$ there is a bounded, Lipschitz and C^1-smooth function $\varphi : M \to \mathbb{R}$ such that*

1. *$f - \varphi$ attains its strong minimum on M,*
2. *$||\varphi||_\infty < \varepsilon$ and $||d\varphi||_\infty < \varepsilon$.*

7.2 Scalar derivatives on Finsler manifolds

Let M and N be metric spaces, $f : M \to N$ a continuous mapping and let $x \in M'$, where M' is the set of accumulation points of M. The **lower and upper scalar derivatives** of f at x are defined as

$$D_x^- f = \liminf_{z \to x} \frac{d_N(f(z), f(x))}{d_M(z, x)}, \qquad D_x^+ f = \limsup_{z \to x} \frac{d_N(f(z), f(x))}{d_M(z, x)}.$$

When M and N are Banach spaces and $f : M \to N$ is differentiable at x, then $D_x^+ f = ||f'(x)||$ and, if in addition $f'(x)$ is invertible, then $D_x^- f = ||f'(x)^{-1}||^{-1}$ (see [70]). An analogous statement holds for smooth mappings between complete Riemannian manifolds: if $f : M \to N$ is a C^1-smooth mapping between complete Riemannian manifolds, then $D_x^+ f = ||df(x)||_x$. In addition, if $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, then $D_x^- f = ||[df(x)]^{-1}||_{f(x)}^{-1}$ (see [50]). In the same paper, the authors prove that if $f : M \to N$ is a C^1-smooth mapping between C^1 Finsler manifolds in the sense of Palais, then $D_x^+ f \leq ||df(x)||_x$ for every $x \in M$. In addition, if $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, then $D_x^- f \geq ||[df(x)]^{-1}||_{f(x)}^{-1}$.

We are going to prove that, actually, the equality holds for C^1 Finsler manifolds in the sense of Palais.

Proposition 7.2.1. *Let M and N be C^1 Finsler manifolds in the sense of Neeb-Upmeier, and let $f : M \to N$ be a C^1-smooth mapping.*

(i) For every $x \in M$ we have that

$$D_x^+ f \leq K_x P_{f(x)} ||df(x)||_x \qquad \textit{and} \qquad ||df(x)||_x \leq P_{f(x)} D_x^+ f,$$

where K_x, $P_{f(x)} \geq 1$ are the constants given in Lemma 5.3.2(1). In addition, if $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, then

$$D_x^- f \leq K_x ||[df(x)]^{-1}||_{f(x)}^{-1} \qquad \textit{and} \qquad ||[df(x)]^{-1}||_{f(x)}^{-1} \leq K_x P_{f(x)} D_x^- f.$$

(ii) If the manifold M is K-weak-uniform and the manifold N is P-weak-uniform, then for every $x \in M$ we have that $D_x^+ f \leq KP||df(x)||_x$ and $||df(x)||_x \leq PD_x^+ f$. In addition, if $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, then $D_x^- f \leq K||[df(x)]^{-1}||_{f(x)}^{-1}$ and $||[df(x)]^{-1}||_{f(x)}^{-1} \leq KPD_x^- f$.

(iii) If the manifold M is K-uniform and the manifold N is P-uniform, then for every $x \in M$

we have that $D_x^+ f \leq K^2P^2||df(x)||_x$ *and* $||df(x)||_x \leq P^2 D_x^+ f$. *In addition, if* $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, *then* $D_x^- f \leq K^2||[df(x)]^{-1}||_{f(x)}^{-1}$ *and* $||[df(x)]^{-1}||_{f(x)}^{-1} \leq K^2P^2D_x^- f$.

(iv) Thus, if the manifolds M *and* N *are Finsler manifolds in the sense of Palais, then for every* $x \in M$ *we have that* $D_x^+ f = ||df(x)||_x$. *In addition, if* $df(x) \in \mathrm{Isom}(T_xM, T_{f(x)}N)$, *then* $D_x^- f = ||[df(x)]^{-1}||_{f(x)}^{-1}$.

Proof. (i) Let us denote by X and Y the Banach spaces where M and N are modeled, respectively. Let us fix $x \in M$ and $y = f(x) \in N$. By Lemma 5.3.2, there are constants K_x, $P_y \geq 1$ and charts (U, φ) of M at x and (V, ψ) of N at y, such that $\varphi(x) = 0$, $\psi(y) = 0$, $f(U) \subset V$, φ is K_x-bi-Lipschitz with the equivalent norm $|||\cdot|||_x := ||d\varphi^{-1}(0)(\cdot)||_x$ and ψ is P_y-bi-Lipschitz with the equivalent norm $|||\cdot|||_y = ||d\psi^{-1}(0)(\cdot)||_y$.

Let us prove the first inequality. Using the inequality (5.7), it is easy to prove that there exists $r > 0$ such that for every $z \in B_M(x, 2r)$

$$||df(z)||_z \leq K_xP_y||df(x)||_x,$$

where $||df(z)||_z = \sup\{||df(z)(v)||_{f(z)} : ||v||_z \leq 1\}$. Now, for each $z \in B_M(x, r)$ with $z \neq x$ there exists a C^1-smooth path $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = z$ and $\ell(\gamma) \leq d_M(x, z) + \min\{r, \varepsilon\, d_M(x, z)\}$. So, $\gamma(t) \in B_M(x, 2r)$ for every $t \in [0, 1]$ and

$$\begin{aligned} d_N(f(x), f(z)) \leq \ell(f \circ \gamma) = \int_0^1 ||df(\gamma(t))(\gamma'(t))||_{f(\gamma(t))} dt &\leq \int_0^1 ||df(\gamma(t))||_{\gamma(t)} ||\gamma'(t)||_{\gamma(t)} dt \\ &\leq K_xP_y||df(x)||_x\, \ell(\gamma) \leq (1 + \varepsilon)K_xP_y||df(x)||_x\, d_M(x, z). \end{aligned}$$

Thus, $D_x^+ f \leq K_xP_y||df(x)||_x$.

Let us prove the reverse inequality. First, let us consider the case $N = \mathbb{R}$. Let us fix $v \in T_xM$ with $||v||_x = 1$, choose a C^1-smooth curve $\gamma : (-\delta, \delta) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Then the function $f \circ \gamma : (-\delta, \delta) \to \mathbb{R}$ is C^1-smooth with $f \circ \gamma(0) = f(x)$ and $df(x)(v) = (f \circ \gamma)'(0)$. Since the map $t \mapsto ||\gamma'(t)||_{\gamma(t)}$ is continuous on $(-\delta, \delta)$, for every $\varepsilon > 0$ there exists $r > 0$ with $r < \delta$ such that

$$\big| ||\gamma'(t)||_{\gamma(t)} - ||\gamma'(0)||_{\gamma(0)} \big| < \varepsilon \qquad \text{for every } t \in [-r, r].$$

Hence,

$$\begin{aligned} |df(x)(v)| = |(f \circ \gamma)'(0)| &= \left| \lim_{t \to 0} \frac{(f \circ \gamma)'(t) - (f \circ \gamma)'(0)}{t} \right| \\ &= \left| \lim_{t \to 0} \frac{(f \circ \gamma)'(t) - (f \circ \gamma)'(0)}{d_M(\gamma(t), \gamma(0))} \cdot \frac{d_M(\gamma(t), \gamma(0))}{t} \right| \\ &\leq \left| \limsup_{z \to x} \frac{d_N(f(z), f(x))}{d_M(z, x)} \right| \cdot \left| \limsup_{t \to 0} \frac{d_M(\gamma(t), \gamma(0))}{t} \right| \\ &\leq D_x^+ f \cdot \mathrm{Lip}(\gamma_{|_{[-r,r]}}) \leq D_x^+ f \cdot \sup\left\{ ||\gamma'(s)||_{\gamma(s)} : s \in [-r, r] \right\} \\ &\leq D_x^+ f \cdot (||\gamma'(0)||_{\gamma(0)} + \varepsilon) = D_x^+ f \cdot (||v||_x + \varepsilon) \leq (1 + \varepsilon) D_x^+ f. \end{aligned}$$

This holds for every $v \in T_xM$ with $||v||_x = 1$ and $\varepsilon > 0$, thus $||df(x)||_x \leq D_x^+ f$.

Now, let us consider the general case, i.e. $f : M \to N$ where N is a C^1 Finsler manifold in the sense of Neeb-Upmeier. Let us denote $y = f(x) \in N$. By Lemma 5.3.2, there is a constant $P_y \geq 1$ and a chart (V, ψ) of N at y, such that $\psi(y) = 0$ and ψ is P_y-bi-Lipschitz with the equivalent norm $|||\cdot|||_y = ||d\psi^{-1}(0)(\cdot)||_y$. If $||df(x)||_x > P_y L$, then there are $\xi \in T_{f(x)}N^*$ and $v \in T_xM$ such that

$$||v||_x = ||\xi||^*_{f(x)} = 1 \qquad \text{and} \qquad |\xi(df(x)(v))| > P_y D^+_x f.$$

Let us define the function

$$g : f^{-1}(V) \to \mathbb{R}, \qquad \text{as} \qquad g(z) = \xi \circ d\psi^{-1}(0)(\psi(f(z))).$$

Then, on the one hand,

$$|dg(x)(v)| = |\xi \circ d\psi^{-1}(0)(d\psi(f(x))df(x)(v))| = |\xi(df(x)(v))| > P_y D^+_x f. \tag{7.1}$$

On the other hand, by the real case

$$\begin{aligned} ||dg(x)(v)|| \leq D^+_x g &= \limsup_{z\to x} \frac{\left|\xi \circ d\psi^{-1}(0)(\psi(f(z))) - \xi \circ d\psi^{-1}(0)(\psi(f(x)))\right|}{d_M(z,x)} \\ &\leq ||\xi||^*_{f(x)} \limsup_{z\to x} \frac{||d\psi^{-1}(0)(\psi(f(z))) - d\psi^{-1}(0)(\psi(f(x)))||_y}{d_M(z,x)} \\ &= \limsup_{z\to x} \frac{|||\psi \circ f(z) - \psi \circ f(x)|||_y}{d_M(z,x)} \leq P_y \limsup_{z\to x} \frac{d_N(f(z), f(x))}{d_M(z,x)} = P_y\, D^+_x f, \end{aligned}$$

which contradicts (7.1).

The proof of the second part follows along the same lines as that for the Riemannian case (see [50]).

(ii), (iii) and (iv) follow from (i), Properties 5.2.3(5) and the fact that if M and N are Neeb-Upmeier K-weak-uniform and P-weak-uniform, respectively, then $K_x = K$ and $P_{f(x)} = P$ for every $x \in M$. □

7.3 Deville-Godefroy-Zizler variational principle on Finsler manifolds

In this section, the Deville-Godefroy-Zizler smooth variational principle will be extended to the class of C^1 Finsler manifolds in the sense of Neeb-Upmeier weak-uniform, provided that they are C^1-uniformly bumpable. This proof follows the ideas of the Riemannian case [5].

Proposition 7.3.1. *(DGZ smooth variational principle for Finsler manifolds). Let M be a complete and C^1-uniformly bumpable C^1 Finsler manifold in the sense of Neeb-Upmeier K-weak-uniform, and let $f : M \to \mathbb{R} \cup \{\infty\}$ be a lsc function which is bounded below and proper. Then, for each $\varepsilon > 0$ there is a bounded, Lipschitz and C^1-smooth function $\varphi : M \to \mathbb{R}$ such that:*

1. *$f - \varphi$ attains its strong minimum on M,*
2. *$||\varphi||_\infty < \varepsilon$ and $||d\varphi||_\infty < \varepsilon$.*

Proof. D. Azagra, J. Ferrera and F. López-Mesas showed the above result for Riemannian manifolds in [5]. They split the proof into three lemmas. Following their steps, we must only prove the following lemma:

Lemma 7.3.2. *Let M be a complete C^1 Finsler manifold in the sense of Neeb-Upmeier K-weak uniform. Then, the vector space*

$$C_b^1(M) = \{f : M \to \mathbb{R} : f \text{ is } C^1\text{-smooth}, \ ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty\},$$

endowed with the norm $||f||_{C_b^1} = \max\{||f||_\infty, ||df||_\infty\}$ is a Banach space.

Let us recall that $||df||_\infty = \sup_{x\in M} ||df(x)||_x$ and $||df(x)||_x = \sup\{|df(x)(v)| : v \in T_xM, \ ||v||_x \leq 1\}$. We only have to show that $(C_b^1(M), ||\cdot||_{C_b^1})$ is complete. Let $(\phi_n)_{n=1}^\infty$ be a Cauchy sequence of $C_b^1(M)$, then the sequences $(\phi_n)_{n=1}^\infty$ and $(d\phi_n)_{n=1}^\infty$ uniformly converge to continuous functions $\phi : M \to \mathbb{R}$ and $\psi : M \to TM^*$, respectively, where ψ is defined as

$$\psi(x) = \lim_{n\to\infty} d\phi_n(x) \in T_xM^*, \qquad \text{for every } x \in M.$$

First of all, we shall prove that $d\phi = \psi$. For a fixed $p \in M$, there is a C^1-smooth chart (U, φ) of M at p satisfying conditions (P1) and (NU3) of Definition 5.2.1, $\varphi(p) = 0$ and $\varphi : U \to V := \varphi(U) \subset X$. By Lemma 5.3.2 the chart φ is K-bi-Lipschitz, i.e. φ and φ^{-1} are K-Lipschitz with the (equivalent) norm $|||\cdot||| := ||d\varphi^{-1}(0)(\cdot)||_p$ considered on X. Let us define

$$\widetilde{\phi}_n = \phi_n \circ \varphi^{-1} : V \subset X \to \mathbb{R} \qquad \text{and} \qquad \widetilde{\phi} = \phi \circ \varphi^{-1} : V \subset X \to \mathbb{R}.$$

Fix $r > 0$ such that $B_r := \{x \in X : |||x||| < r\} \subset V \subset X$. For every $x \in B_r$ we have

$$\begin{aligned}
&|||d\widetilde{\phi}_m(x) - d\widetilde{\phi}_n(x)|||\\
&\quad = \sup\left\{|d\widetilde{\phi}_m(x)(v) - d\widetilde{\phi}_n(x)(v)| : v \in X, \ |||v||| = 1\right\}\\
&\quad = \sup\left\{|(d\phi_m(\varphi^{-1}(x)) - d\phi_n(\varphi^{-1}(x)))(d\varphi^{-1}(x)(v))| : v \in X, \ |||v||| = 1\right\} \qquad (7.2)\\
&\quad \leq \sup\left\{||d\phi_m(\varphi^{-1}(x)) - d\phi_n(\varphi^{-1}(x))||_{\varphi^{-1}(x)} ||d\varphi^{-1}(x)(v)||_{\varphi^{-1}(x)} : v \in X, \ |||v||| = 1\right\}\\
&\quad \leq K||d\phi_m - d\phi_n||_\infty \sup\{||d\varphi^{-1}(0)(v)||_p : v \in X, \ |||v||| = 1\} = K||d\phi_m - d\phi_n||_\infty.
\end{aligned}$$

Thus, for every $h \in B_r$

$$\begin{aligned}
|(\widetilde{\phi}_m(h) - \widetilde{\phi}_m(0)) - (\widetilde{\phi}_n(h) - \widetilde{\phi}_n(0))| &\leq \mathrm{Lip}_{|||\cdot|||}\,(\widetilde{\phi}_m - \widetilde{\phi}_n)_{|B_r}\, |||h||| \qquad (7.3)\\
&\leq \left(\sup_{x\in B_r} |||d\widetilde{\phi}_m(x) - d\widetilde{\phi}_n(x)|||\right) |||h||| \leq K||d\phi_m - d\phi_n||_\infty |||h|||.
\end{aligned}$$

On the one hand, $(d\phi_m)_{m=1}^\infty$ is a Cauchy sequence with the norm $||\cdot||_\infty$, and, by inequality (7.3), for every $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$|(\widetilde{\phi}_m(h) - \widetilde{\phi}_m(0)) - (\widetilde{\phi}_n(h) - \widetilde{\phi}_n(0))| < \frac{\varepsilon}{3}|||h|||,$$

for every $h \in B_r$ and every $m, n \in \mathbb{N}$ with $m, n \geq n_0$. Since $\{\widetilde{\phi}_m\}_{m=1}^\infty$ uniformly converges to

$\widetilde{\phi}$, we obtain that

$$|(\widetilde{\phi}(h)-\widetilde{\phi}(0))-(\widetilde{\phi}_n(h)-\widetilde{\phi}_n(0))| \leq \frac{\varepsilon}{3}|||h|||, \qquad \text{for } n \geq n_0 \text{ and } h \in B_r. \tag{7.4}$$

On the other hand, it is easy to prove that

$$d\widetilde{\phi}_n(0) \xrightarrow{n\to\infty} \psi(p)\circ d\varphi^{-1}(0) \qquad \text{since} \qquad d\phi_n(p) \xrightarrow{n\to\infty} \psi(p). \tag{7.5}$$

In addition,

$$\left|\frac{\widetilde{\phi}_{n_0}(h)-\widetilde{\phi}_{n_0}(0)}{|||h|||} - d\widetilde{\phi}_{n_0}(0)\left(\frac{h}{|||h|||}\right)\right| \xrightarrow{|||h|||\to 0} 0. \tag{7.6}$$

Hence, by the inequalities (7.4), (7.5) and (7.6), we can take h small enough such that

$$\begin{aligned}
&\left|\frac{\widetilde{\phi}(h)-\widetilde{\phi}(0)-\psi(p)\circ d\varphi^{-1}(0)(h)}{|||h|||}\right| \\
&\quad \leq \left|\frac{(\widetilde{\phi}(h)-\widetilde{\phi}(0))-(\widetilde{\phi}_{n_0}(h)-\widetilde{\phi}_{n_0}(0))}{|||h|||}\right| + \left|\frac{\widetilde{\phi}_{n_0}(h)-\widetilde{\phi}_{n_0}(0)}{|||h|||} - d\widetilde{\phi}_{n_0}(0)\left(\frac{h}{|||h|||}\right)\right| \\
&\qquad + \left|(d\widetilde{\phi}_{n_0}(0)-\psi(p)\circ d\varphi^{-1}(0))\left(\frac{h}{|||h|||}\right)\right| < \varepsilon.
\end{aligned}$$

So, ϕ is differentiable at p and $d\phi(p)=\psi(p)$. Indeed,

$$\begin{aligned}
d\phi(p) &= d(\widetilde{\phi}\circ\varphi)(p) = d\widetilde{\phi}(0)\circ d\varphi(p) \\
&= \psi(p)\circ d\varphi^{-1}(0)\circ d\varphi(p) = \psi(p)\circ d(\varphi^{-1}\circ\varphi)(p) = \psi(p).
\end{aligned}$$

Now, we will show that $d\phi=\psi$ is continuous. For every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that $||d\phi_n(x)-d\phi_m(x)||_x \leq \varepsilon/3K$ for every $m,n\geq n_0$ and $x\in M$. Then, $||d\phi_n(x)-d\phi(x)||_x \leq \varepsilon/3K$ for every $x\in M$ and $n\geq n_0$. Let us define

$$\widetilde{\phi}_n=\phi_n\circ\varphi^{-1}: V\subset X\to\mathbb{R} \qquad \text{and} \qquad \widetilde{\phi}=\phi\circ\varphi^{-1}: V\subset X\to\mathbb{R},$$

which satisfy

$$\begin{aligned}
|||d\widetilde{\phi}(x)-d\widetilde{\phi}(0)||| &\leq |||d\widetilde{\phi}(x)-d\widetilde{\phi}_{n_0}(x)||| + |||d\widetilde{\phi}_{n_0}(x)-d\widetilde{\phi}_{n_0}(0)||| + |||d\widetilde{\phi}_{n_0}(0)-d\widetilde{\phi}(0)||| \\
&\leq K||d\phi-d\phi_{n_0}||_\infty + |||d\widetilde{\phi}_{n_0}(x)-d\widetilde{\phi}_{n_0}(0)||| + K||d\phi_{n_0}-d\phi||_\infty,
\end{aligned}$$

for every $x\in B_r\subset V\subset X$ (the last inequality is consequence of (7.2)). Since $||d\phi-d\phi_{n_0}||_\infty \leq \varepsilon/3K$ and $d\widetilde{\phi}_{n_0}$ is continuous at 0, there exists $\delta>0$ such that $|||d\widetilde{\phi}(x)-d\widetilde{\phi}(0)|||<\varepsilon$ whenever $x\in X$ with $|||x|||<\delta$. Thus, $d\widetilde{\phi}$ is continuous at 0 and $d\phi$ is continuous at p. □

Remark 7.3.3. *It is worth noting that if a C^1 Finsler manifold M in the sense of Neeb-Upmeier satisfies the* DGZ smooth variational principle, *then it is necessarily modeled on a*

Banach space X with a C^1-smooth and Lipschitz bump function. Indeed, by the DGZ smooth variational principle, there exists a C^1-smooth and Lipschitz function $\phi : M \to \mathbb{R}$ such that

- *$g = 1 - \phi$ attains its strong minimum at $x_0 \in M$,*
- *$||\phi||_\infty < \frac{1}{4}$ and $||d\phi||_\infty < \frac{1}{4}$.*

Since x_0 is the strong minimum of g on M, for every $\delta > 0$ there exists $a > 0$ such that

$$g(y) \geq a + g(x_0) \qquad \text{for every } y \in M \text{ with } d_M(y, x_0) > \delta. \tag{7.7}$$

Let us take $\varphi : U \to X$ a chart with $x_0 \in U$ satisfying inequality (5.2) with the constant $K_{x_0} \geq 1$. Let us choose $\delta > 0$ such that $B_M(x_0, 2\delta) \subset U$, $\varphi(B_M(x_0, 2\delta))$ is bounded in X, and the corresponding constant $a > 0$ satisfies the above inequality (7.7) for δ. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth and Lipschitz function with

$$\theta(t) = \begin{cases} 1 & \text{if } t \leq g(x_0) \\ 0 & \text{if } t \geq g(x_0) + a \end{cases} \qquad \text{and} \qquad \mathrm{Lip}(\theta) \leq 2/a.$$

Let us define $b : X \to \mathbb{R}$ by $b(x) = \theta(g(\varphi^{-1}(x)))$ for every $x \in \varphi(B_M(x_0, 2\delta))$, and $b(x) = 0$ whenever $x \notin \varphi(B_M(x_0, 2\delta))$. Then, it is clear that b is a C^1-smooth bump function on X and $\mathrm{Lip}(b) \leq \frac{K_{x_0}}{2a}$.

7.4 A Myers-Nakai theorem on Finsler manifolds

In this section, we are interested in characterizing the Finsler structure of a Finsler manifold M in terms of the space of real-valued, bounded and C^k-smooth functions defined on M with bounded derivative. Hence, we will extend the Myers-Nakai theorem to the class of C^k Finsler manifolds in the sense of Palais, provided that they are C^k-uniformly bumpable.

From now on, we shall refer to C^k Finsler manifolds in the sense of Palais as C^k Finsler manifolds, and $k \in \mathbb{N} \cup \{\infty\}$. We shall use the standard notation of $C^k(U, Y)$ for the set of all k-times continuously differentiable mappings defined on an open subset U of a Banach space (Finsler manifold) taking values into a Banach space (Finsler manifold) Y. We shall write $C^k(U)$ whenever $Y = \mathbb{R}$.

7.4.1 On weakly smooth mappings and the algebra $C_b^k(M)$

Now, let us recall the concept of weakly C^k-smooth mapping.

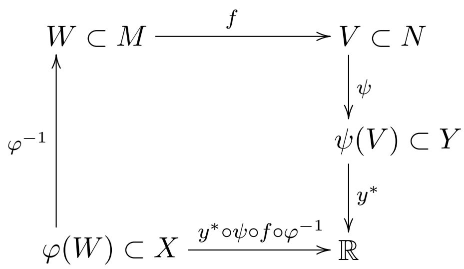
Definition 7.4.1. *Let X and Y be Banach spaces and consider a mapping $f : U \to Y$, where U is an open subset of X. The mapping f is said to be **weakly C^k-smooth** at the point x_0 whenever there is an open neighborhood U_{x_0} of x_0 such that $y^* \circ f$ is C^k-smooth at U_{x_0}, for every $y^* \in Y^*$. The mapping f is said to be **weakly C^k-smooth** on U whenever f is weakly C^k-smooth at every point $x \in U$.*

On the one hand, J. M. Gutiérrez and J.L. G. Llavona [49] proved that if $f : U \to Y$ is weakly C^k-smooth on U, then $g \circ f \in C^k(U)$ for all $g \in C^k(Y)$. They also proved that if $f : U \to Y$ is weakly C^k-smooth on U, then $f \in C^{k-1}(U)$. For $k = 1$, the above yields that every weakly C^1-smooth mapping on U is continuous on U. Also, for $k = \infty$, every weakly C^∞-smooth mapping on U is C^∞-smooth on U. M. Bachir and G. Lancien [12] proved that, if the Banach space Y has the Schur property, then the concept of weakly C^k-smoothness coincides with the concept of C^k-smoothness. On the other hand, there are examples of weakly C^1-smooth mappings that are not C^1-smooth (see [49] and [12]).

Definition 7.4.2. *Let M and N be C^k Finsler manifolds modeled on X and Y, respectively, and let $U \subset M$, $O \subset N$ be open subsets of M and N. A mapping $f : U \to N$ is said to be* ***weakly C^k-smooth*** *at the point x_0 of U if there exist charts (W, φ) of M at x_0 and (V, ψ) of N at $f(x_0)$ such that $W \subset U$, $f(W) \subset V$ and*

$$\psi \circ f \circ \varphi^{-1} : \varphi(W) \subset X \to Y$$

is weakly C^k-smooth at $\varphi(W)$. We say that $f : U \to N$ is ***weakly C^k-smooth*** *in U if f is weakly C^k-smooth at every point $x \in U$. We say that a bijection $f : U \to O$ is a weakly C^k-diffeomorphism if f and f^{-1} are weakly C^k-smooth on U and O, respectively. Notice that these definitions do not depend on the chosen charts.*



Let us note that there are homeomorphisms which are weakly C^1-smooth but not differentiable. Indeed, we follow [49, Example 3.9] and define $g : \mathbb{R} \to c_0(\mathbb{N})$ and $h : c_0(\mathbb{N}) \to c_0(\mathbb{N})$ as

$$g(t) = (0, \frac{1}{2}\sin(2t), \dots, \frac{1}{n}\sin(nt), \dots) \qquad \text{and} \qquad h(x) = x + g(x_1)$$

for every $t \in \mathbb{R}$ and $x = (x_1, \dots, x_n, \dots) \in c_0$. The mapping h is an homeomorphism, $h^{-1}(y) = y - g(y_1)$ for every $y = (y_1, \dots, y_n, \dots) \in c_0$, and h is weakly C^1-smooth on $c_0(\mathbb{N})$. Notice that if h were differentiable at a point $x \in c_0$ with $x_1 = 0$, then

$$h'(x)(1, 0, 0, \dots) = (1, 1, 1, \dots) \in \ell_\infty \setminus c_0,$$

which is a contradiction.

Now, let us consider different definitions of isometries between C^k Finsler manifolds.

Definition 7.4.3. *Let $(M, ||\cdot||_M)$ and $(N, ||\cdot||_N)$ be C^k Finsler manifolds and a bijection $h : M \to N$.*

(**MI**) *We say that h is a* ***metric isometry*** *for the Finsler metrics, if*

$$d_N(h(x), h(y)) = d_M(x, y), \qquad \text{for every } x, y \in M.$$

(FI) *We say that h is a C^k* ***Finsler isometry*** *if it is a C^k-diffeomorphism satisfying*

$$||dh(x)(v)||_{h(x)} = ||(h(x), dh(x)(v))||_N = ||(x,v)||_M = ||v||_x,$$

for every $x \in M$ and $v \in T_xM$. We say that the Finsler manifolds M and N are C^k ***equivalent as Finsler manifolds*** *if there is a C^k Finsler isometry between M and N.*

(ω-**FI**) *We say that h is a* ***weak*** *C^k* ***Finsler isometry*** *if it is a weakly C^k-diffeomorphism and a metric isometry for the Finsler metrics. We say that the Finsler manifolds M and N are* ***weakly*** *C^k* ***equivalent as Finsler manifolds*** *if there is a weak C^k Finsler isometry between M and N.*

Proposition 7.4.4. *Let M and N be C^k Finsler manifolds. Let us assume that there is a C^k-diffeomorphism and metric isometry (for the Finsler metrics) $h : M \to N$. Then h is a C^k Finsler isometry.*

Proof. Let us fix $x \in M$ and $y = h(x) \in N$. For every $\varepsilon > 0$, there are $r > 0$ and charts $\varphi : B_M(x,r) \subset M \to X$ and $\psi : B_N(y,r) \subset N \to Y$ satisfying property (P2) of Definition 5.2.1 and inequality (5.13). Since $h : M \to N$ is a metric isometry, h is a bijection from $B_M(x,r)$ onto $B_N(y,r)$.

Let us consider the equivalent norms on X and Y defined as $|||\cdot|||_x := ||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $|||\cdot|||_y = ||d\psi^{-1}(\psi(y))(\cdot)||_y$, respectively.

Since h is a metric isometry, we obtain from Lemma 5.3.2, for p, q in an open neighborhood of $\varphi(x)$,

$$\begin{aligned}|||\psi \circ h \circ \varphi^{-1}(p) - \psi \circ h \circ \varphi^{-1}(q)|||_y &\le (1+\varepsilon) d_N(h \circ \varphi^{-1}(p), h \circ \varphi^{-1}(q)) \\ &= (1+\varepsilon) d_M(\varphi^{-1}(p), \varphi^{-1}(q)) \le (1+\varepsilon)^2 |||p-q|||_x.\end{aligned}$$

Thus, $\sup\{|||d(\psi \circ h \circ \varphi^{-1})(\varphi(x))(w)|||_y : |||w|||_x \le 1\} \le (1+\varepsilon)^2$. Now, for every $v \in T_xM$ with $v \neq 0$, let us write $w = d\varphi(x)(v) \in X$. We have

$$\begin{aligned}||dh(x)(v)||_y &= ||d\psi^{-1}(\psi(y)) d\psi(y) dh(x)(v)||_y = |||d(\psi \circ h)(x)(v)|||_y \\ &= |||d(\psi \circ h)(x) d\varphi^{-1}(\varphi(x))(w)|||_y = |||d(\psi \circ h \circ \varphi^{-1})(\varphi(x))(w)|||_y \\ &\le (1+\varepsilon)^2 |||w|||_x = (1+\varepsilon)^2 ||v||_x.\end{aligned}$$

Since this inequality holds for every $\varepsilon > 0$ and the same argument works for h^{-1}, we conclude that $||dh(x)(v)||_y = ||v||_x$ for all $v \in T_xM$. Thus, h is a C^k Finsler isometry. □

Let us now turn our attention to the *Banach algebra* $C_b^1(M)$, the algebra of all real-valued, C^1-smooth and bounded functions with bounded derivative defined on a C^1 Finsler manifold M, i.e.

$$C_b^1(M) = \{f : M \to \mathbb{R} : f \in C^1(M),\ ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty\},$$

where $||f||_\infty := \sup\{|f(x)| : x \in M\}$ and $||df||_\infty := \sup\{||df(x)||_x : x \in M\}$, introduced in the previous section. Recall that the usual norm considered on $C_b^1(M)$ is $||f||_{C_b^1} = \max\{||f||_\infty, ||df||_\infty\}$ for every $f \in C_b^1(M)$ and $(C_b^1(M), ||\cdot||_{C_b^1(M)})$ is a Banach space (see

Lemma 7.3.2). In fact $(C_b^1(M), 2||\cdot||_{C_b^1(M)})$ is a Banach algebra. Let us notice that, by Proposition 5.3.1, we have $||df||_\infty = \mathrm{Lip}(f)$.

For $2 \le k \le \infty$ and a C^k Finsler manifold M, let us consider the algebra $C_b^k(M)$ of all real-valued, C^k-smooth and bounded functions that have bounded first derivative, i.e.

$$C_b^k(M) = \{f : M \to \mathbb{R} : f \in C^k(M),\ ||f||_\infty < \infty \text{ and } ||df||_\infty < \infty\} = C^k(M) \cap C_b^1(M).$$

with the norm $||\cdot||_{C_b^1}$. Thus, $C_b^k(M)$ is a subalgebra of $C_b^1(M)$. Nevertheless, it is not a Banach algebra.

A function $\varphi : C_b^k(M) \to \mathbb{R}$ $(1 \le k \le \infty)$ is said to be an *algebra homomorphism* whether for all $f, g \in C_b^k(M)$ and $\lambda, \eta \in \mathbb{R}$,

(i) $\varphi(\lambda f + \eta g) = \lambda\varphi(f) + \eta\varphi(g)$, and

(ii) $\varphi(f \cdot g) = \varphi(f)\varphi(g)$.

Let us denote by $H(C_b^k(M))$ the set of all nonzero algebra homomorphisms, i.e.

$$H(C_b^k(M)) = \{\varphi : C_b^k(M) \to \mathbb{R} : \varphi \text{ is an algebra homomorphism and } \varphi(1) = 1\}.$$

Let us list some of the basic properties of the algebra $C_b^k(M)$ and the algebra homomorphisms $H(C_b^k(M))$. They can be checked as in the Riemannian case (see [42], [45] and [62]).

(a) If $\varphi \in H(C_b^k(M))$, then $\varphi \ne 0$ if and only if $\varphi(1) = 1$.

(b) If $\varphi \in H(C_b^k(M))$, then φ is positive, i.e. $\varphi(f) \ge 0$ for every $f \ge 0$.

(c) If the C^k Finsler manifold M is modeled on a Banach space that admits a Lipschitz and C^k-smooth bump function, then $C_b^k(M)$ is a *unital algebra that separates points and closed sets* of M. Indeed, let us take $x \in M$, and $C \subset M$ a closed subset of M with $x \notin C$. Let us take $r > 0$ small enough so that $C \cap B_M(x,r) = \emptyset$ and a chart $\varphi : B_M(x,r) \to X$ satisfying property (P2) of Definition 5.2.1. Let us take $s > 0$ small enough so that $\varphi(x) \in B(\varphi(x), s) \subset \varphi(B(x, r/2)) \subset X$ and a Lipschitz and C^k-smooth bump function $b : X \to \mathbb{R}$ with $b(\varphi(x)) = 1$ and $b(z) = 0$ for every $z \notin B(\varphi(x), s)$. Let us define $h : M \to \mathbb{R}$ as $h(p) = b(\varphi(p))$ for every $p \in B_M(x,r)$ and $h(p) = 0$ otherwise. Then $h \in C_b^k(M)$, $h(x) = 1$ and $h(c) = 0$ for every $c \in C$.

(d) The space $H(C_b^k(M))$ is closed as a topological subspace of $\mathbb{R}^{C_b^k(M)}$ with the product topology. Moreover, since every function in $C_b^k(M)$ is bounded, it can be checked that $H(C_b^k(M))$ is compact in $\mathbb{R}^{C_b^k(M)}$.

(e) If $C_b^k(M)$ separates points and closed subsets, then M can be embedded as a topological subspace of $H(C_b^k(M))$ by identifying every $x \in M$ with the *point evaluation homomorphism* δ_x given by $\delta_x(f) = f(x)$ for every $f \in C_b^k(M)$. Also, it can be checked that the subset $\delta(M) = \{\delta_x : x \in M\}$ is dense in $H(C_b^k(M))$. Indeed, let us take $\varphi \in H(C_b^k(M))$, $f_1, \ldots, f_n \in C_b^k(M)$ and $\varepsilon > 0$. Then there exists $p \in M$ such that $|\delta_p(f_i) - \varphi(f_i)| < \varepsilon$ for all $i = 1, \ldots, n$. Otherwise, the function $g = \sum_{i=1}^n (f_i - \varphi(f_i))^2 \in C_b^k(M)$ would satisfy $g \ge \varepsilon$ and $\varphi(g) = 0$, which is impossible since φ is positive. Therefore, it follows that $H(C_b^k(M))$ is a compactification of M.

(f) Every $f \in C_b^k(M)$ admits a continuous extension $\widehat{f}$ to $H(C_b^k(M))$, where $\widehat{f}(\varphi) = \varphi(f)$ for every $\varphi \in H(C_b^k(M))$. Notice that this extension $\widehat{f}$ coincides in $H(C_b^k(M))$ with the projection $\pi_f : \mathbb{R}^{C_b^k(M)} \to \mathbb{R}$, given by $\pi_f(\varphi) = \varphi(f)$, i.e. $\pi_f|_{H(C_b^k(M))} = \widehat{f}$. In the following, we shall identify M with $\delta(M)$ in $H(C_b^k(M))$.

Proposition 7.4.5. *Let M be a complete C^k Finsler manifold that is C^k-uniformly bumpable. Then, $\varphi \in H(C_b^k(M))$ has a countable neighborhood basis in $H(C_b^k(M))$ if and only if $\varphi \in M$.*

Proof. The assertion can be proved in a similar way to the Riemannian case [42]. Let us give the proof for completeness. Let us take $\varphi \in H(C_b^k(M)) \setminus M$ with a countable neighborhood basis. Since M is dense in $H(C_b^k(M))$ (recall that M is identified with $\delta(M)$) there is a sequence of points $(p_n)_{n=1}^\infty$ in M converging at φ, i.e. $\delta_{p_n} \to \varphi$. Since $\varphi \notin M$ and M is complete, the sequence $(p_n)_{n=1}^\infty$ has none Cauchy subsequence. So there exist $\varepsilon > 0$ and $(p_{n_j})_{j=1}^\infty$ a subsequence of $(p_n)_{n=1}^\infty$ such that $d_M(p_{n_k}, p_{n_j}) \geq \varepsilon$ for every $k \neq j$.

Since M is C^k-uniformly bumpable, there exist $R > 1$ and $0 < \delta < \varepsilon/2$ such that we can construct a sequence $\{b_j\}_{j=1}^\infty$ of C^k-smooth bump functions satisfying, for any $j \geq 1$, the following conditions:

1. $b_j(p_{n_{2j}}) = 1$,
2. $b_j(q) = 0$ whenever $d_M(q, p_{n_{2j}}) \geq \delta$, and
3. $||db_j||_\infty \leq \frac{R}{\delta}$.

Let us define $f : M \to \mathbb{R}$ by $f(x) = \sum_j b_j$. It is clear that $f \in C_b^k(M)$, $f(p_{n_{2j}}) = 1$ and $f(p_{n_{2j+1}}) = 0$ for every $j \geq 1$. Therefore, the continuous extended function $\hat{f} : H(C_b^k(M)) \to \mathbb{R}$ is such that

$$\hat{f}(x) = \begin{cases} 1 & \text{if } x \in \overline{A}^{H(C_b^k(M))}, \\ 0 & \text{if } x \in \overline{B}^{H(C_b^k(M))}, \end{cases}$$

where $A = \{p_{n_{2j}} : j \geq 1\}$ and $B = \{p_{2j+1} : j \geq 1\}$. However, $\varphi \in \overline{A}^{H(C_b^k(M))} \cap \overline{B}^{H(C_b^k(M))}$ and thus $\hat{f}(\varphi) = 0$ and $\hat{f}(\varphi) = 1$, which is a contradiction.

Conversely, if $\varphi \in M$ we can consider the open ball in M, $B_n = B_M(\varphi, \frac{1}{n})$. Then, $\{\overline{B_n}^{H(C_b^k(M))} : n \in \mathbb{N}\}$ is a countable neighborhood basis in $H(C_b^k(M))$. □

7.4.2 A Myers-Nakai Theorem

Our main result is the following Banach-Stone type theorem for a certain class of Finsler manifolds. It states that the algebra structure of $C_b^k(M)$ determines the C^k Finsler manifold. Let us begin by defining the class of Banach spaces where the Finsler manifolds shall be modeled.

Definition 7.4.6. *A Banach space $(X, ||\cdot||)$ is said to be **k-admissible** if for every equivalent norm $|\cdot|$ and $\varepsilon > 0$, there are an open subset $B \supset \{x \in X : |x| \leq 1\}$ of X and a C^k-smooth function $g : B \to \mathbb{R}$ such that* *(i)* $|g(x) - |x|| < \varepsilon$ *for* $x \in B$, *and*

(ii) $\mathrm{Lip}(g) \le (1 + \varepsilon)$ *for the norm* $|\cdot|$.

It is easy to prove the following lemma.

Lemma 7.4.7. *Let* X *be a Banach space with one of the following properties:*

(A.1) Density of the set of equivalent C^k*-smooth norms: every equivalent norm on* X *can be approximated in the Hausdorff metric by equivalent* C^k*-smooth norms [22], i.e. for every equivalent norm* $|\cdot|$ *on* X *and every* $\varepsilon > 0$*, there exists an equivalent* C^k*-smooth norm* $|||\cdot|||$ *on* X *such that*

$$(1-\varepsilon)|x| \le |||x||| \le (1+\varepsilon)|x| \qquad \text{on } X.$$

(A.2) C^1*-fine approximation by* C^k*-smooth functions* ($k \ge 2$) *and density of the set of equivalent* C^1*-smooth norms: For every* C^1*-smooth function* $f : X \to \mathbb{R}$ *and every* $\varepsilon > 0$*, there is a* C^k*-smooth function* $g : X \to \mathbb{R}$ *satisfying* $|f(x) - g(x)| < \varepsilon$ *and* $||f'(x) - g'(x)|| < \varepsilon$ *for all* $x \in X$ *(see [56], [11], [92] and Chapter 3); also, every equivalent norm defined on* X *can be approximated in the Hausdorff metric by equivalent* C^1*-smooth norms (see [22, Theorem II 4.1]).*

Then X *is* k*-admissible.*

Proof. For the reader's convenience, we present the proof of (A.2). Fix $|\cdot|$ an equivalent norm on X and $\varepsilon > 0$, let us take $\alpha > 0$ such that $2\alpha + \alpha^2 < \varepsilon$ and the open subset $B := \{x \in X : |x| < 1 + \alpha\}$. It is clear that $\{x \in X : |x| \le 1\} \subset B$. Since the set of equivalent C^1-smooth norms in X is dense in the set of equivalent norms in X, there exists an equivalent C^1-smooth norm $|||\cdot|||$ on X such that

$$(1-\alpha)|x| \le |||x||| \le (1+\alpha)|x| \qquad \text{on } X.$$

In addition, by the C^1-fine approximation property, there exists a C^k-smooth function $g : X \to \mathbb{R}$ such that

$$\big||||x||| - g(x)\big| < \alpha \qquad \text{and} \qquad \big|||||\cdot|||'(x) - g'(x)\big|| < \alpha.$$

Hence,

$$\big||x| - g(x)\big| \le \big||||x||| - g(x)\big| + \big||x| - |||x|||\big| < \alpha + \alpha|x| < 2\alpha + \alpha^2 < \varepsilon \qquad \text{on } B,$$

and $\mathrm{Lip}(g) \le \alpha + (1+\alpha) \le (1+\varepsilon)$ with the norm $|\cdot|$. □

Banach spaces satisfying condition (A.2) are, for instance, separable Banach spaces with a Lipschitz C^k-smooth bump function. Banach spaces satisfying condition (A.1) for $k = 1$ are, for instance, Weakly Compactly Generated (WCG) Banach spaces with a C^1-smooth bump function.

Theorem 7.4.8. *Let M and N be complete C^k Finsler manifolds that are C^k-uniformly bumpable and are modeled on k-admissible Banach spaces. Then M and N are weakly C^k equivalent as Finsler manifolds if and only if $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^k Finsler isometry. In particular, h is a C^{k-1} Finsler isometry whenever $k \geq 2$.*

In order to prove Theorem 7.4.8, we shall follow the ideas of the Riemmanian case [42]. Let us divide the proof into several propositions.

Proposition 7.4.9. *Let M and N be C^k Finsler manifolds such that N is modeled on a k-admissible Banach space Y. Let $h : M \to N$ be a map such that $T : C_b^k(N) \to C_b^k(M)$ given by $T(f) = f \circ h$ is continuous. Then h is $||T||$-Lipschitz for the Finsler metrics.*

Proof. For every $y \in N$, let us take a chart $\psi_y : V_y \to Y$ with $\psi_y(y) = 0$. Let us consider the equivalent norm on Y, $|||\cdot|||_y := ||d\psi_y^{-1}(0)(\cdot)||_y$ and fix $\varepsilon > 0$. Let us define the ball $B_{|||\cdot|||_y}(z,t) := \{w \in Y : |||w - z|||_y < t\}$.

Fact. For every $r > 0$ such that $B_{|||\cdot|||_y}(0,r) \subset \psi_y(V_y)$ and every $\widetilde{\varepsilon} > 0$, there exists a C^k-smooth and Lipschitz function $f_y : Y \to \mathbb{R}$ such that

1. $f_y(0) = r$,
2. $||f_y||_\infty := \sup\{|f_y(z)| : z \in Y\} = r$,
3. $\mathrm{Lip}(f_y) \leq (1+\varepsilon)^2$ for the norm $|||\cdot|||_y$,
4. $f_y(z) = 0$ for every $z \in Y$ with $|||z|||_y \geq r$, and
5. $|||z|||_y \leq r - f_y(z) + \widetilde{\varepsilon}$ for every $|||z|||_y \leq r$.

Let us prove the Fact. First of all, let us take $r > 0$, $\widetilde{\varepsilon} > 0$ and $0 < \alpha < \min\{1, \frac{\varepsilon}{4}, \frac{2\widetilde{\varepsilon}}{5r}\}$. Since N is a C^k Finsler manifold modeled on a k-admissible Banach space Y, there are an open subset $B \supset \{x \in Y : |||x|||_y \leq 1\}$ of Y and a C^k-smooth function $g : B \to \mathbb{R}$ such that

(i) $|g(x) - |||x|||_y| < \alpha/2$ on B, and

(ii) $\mathrm{Lip}(g) \leq (1 + \alpha/2)$ for the norm $|||\cdot|||_y$.

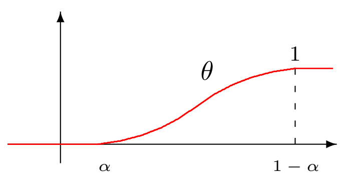
Now, let us take a C^∞-smooth and Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that

(i) $\theta(t) = 0$ whenever $t \leq \alpha$,

(ii) $\theta(t) = 1$ whenever $t \geq 1 - \alpha$,

(iii) $\mathrm{Lip}(\theta) \leq (1+\varepsilon)$, and

(iv) $|\theta(t) - t| \leq 2\alpha$ for every $t \in [0, 1+\alpha]$.



Let us define

$$f(x) = \begin{cases} \theta(g(x)) & \text{if } x \in B, \\ 1 & \text{if } x \in Y \setminus B. \end{cases}$$

It is straightforward to verify that f is well defined, C^k-smooth, $f(x) = 1$ whenever $|||x|||_y \geq 1$, $f(x) = 0$ whenever $|||x|||_y \leq \alpha/2$, and $\mathrm{Lip}(f) \leq (1+\varepsilon)(1+\alpha/2)$ for the norm $|||\cdot|||_y$. Let us now consider $f_y : Y \to [0, r]$ as $f_y(z) = r(1 - f(\frac{z}{r}))$, which is C^k-smooth, Lipschitz and satisfies:

(i) $f_y(0) = r$,

(ii) $||f_y||_\infty = r$,

(iii) $|f_y(z) - f_y(x)| \leq (1+\varepsilon)(1+\alpha/2)|||z - x|||_y \leq (1+\varepsilon)^2|||z - x|||_y$,

(iv) $f_y(z) = 0$ for every $z \in Y$ with $|||z|||_y \geq r$,

(v) $|||\frac{z}{r}|||_y \leq \frac{\alpha}{2} + g(\frac{z}{r}) \leq \frac{\alpha}{2} + 2\alpha + f(\frac{z}{r})$ for every $|||z|||_y \leq r$. Thus, $|||z|||_y \leq r(\frac{\alpha}{2} + 2\alpha) + r - f_y(z) \leq \widetilde{\varepsilon} + r - f_y(z)$ for every $|||z|||_y \leq r$.

Let us now prove Proposition 7.4.9. Let us fix $p_1, p_2 \in M$ and $\varepsilon > 0$. Let us consider $\sigma : [0,1] \to M$ a piecewise C^1-smooth path in M joining p_1 and p_2, with $\ell(\sigma) \leq d_M(p_1, p_2) + \varepsilon$. Since $h : M \to N$ is continuous, the path $\widehat{\sigma} := h \circ \sigma : [0,1] \to N$, joining $h(p_1)$ and $h(p_2)$, is continuous as well. For every $q \in \widehat{\sigma}([0,1])$, there is $0 < r_q < 1$ and a chart $\psi_q : V_q \to Y$ such that $\psi_q(q) = 0$, $B_N(q, r_q) \subset V_q$ and the bijection $\psi_q : V_q \to \psi_q(V_q)$ is $(1+\varepsilon)$-bi-Lipschitz for the norm $||d\psi_q^{-1}(0)(\cdot)||_q$ in Y (see Lemma 5.3.2). Since $\widehat{\sigma}([0,1])$ is a compact set of N, there is a finite family of points $0 = t_1 < t_2 < \cdots < t_m = 1$ and a family of open intervals $\{I_k\}_{k=1}^m$ covering the interval $[0,1]$ so that, if we define $q_k := \widehat{\sigma}(t_k)$ and $r_k := r_{q_k}$, for every $k = 1, \ldots, m$, we have

(a) $\widehat{\sigma}(I_k) \subset B_N(q_k, r_k/(1+\varepsilon))$,

(b) $I_j \cap I_k \neq \emptyset$ if, and only if, $|j - k| \leq 1$.

It is clear that $\widehat{\sigma}([0,1]) \subset \bigcup_{k=1}^m B_N(q_k, \frac{r_k}{1+\varepsilon})$. Now, let us select a point $s_k \in I_k \cap I_{k+1}$ such that $t_k < s_k < t_{k+1}$, for every $k = 1, \ldots, m-1$. Let us write $a_k := \widehat{\sigma}(s_k)$, for every $k = 1, \ldots, m-1$, $\psi_k := \psi_{q_k}$, $V_k := V_{q_k}$ and $|||\cdot|||_k := ||d\psi_k^{-1}(0)(\cdot)||_{q_k}$, for every $k = 1, \ldots, m$. Notice that $a_k \in B_N(q_k, \frac{r_k}{1+\varepsilon}) \cap B_N(q_{k+1}, \frac{r_{k+1}}{1+\varepsilon})$, for every $k = 1, \ldots, m-1$. Since $\psi_k : V_k \to \psi_k(V_k)$ is $(1+\varepsilon)$-bi-Lipschitz for the norm $|||\cdot|||_k$ in Y, we deduce that $\psi_k(a_k) \in B_{|||\cdot|||_k}(0, r_k)$, for every $k = 1, \ldots, m-1$.

Now, let us we apply the above Fact to r_k, ε and $\widetilde{\varepsilon} = \varepsilon/2m$ to obtain functions $f_k : Y \to [0, r_k]$ satisfying properties (1)–(5), $k = 1, \ldots, m$. Let us define the C^k-smooth and Lipschitz functions $g_k : N \to [0, r_k]$ as

$$g_k(z) = \begin{cases} f_k(\psi_k(z)) & \text{if } z \in V_k, \\ 0 & \text{if } z \notin V_k, \end{cases}$$

$k = 1, \ldots, m$. Then,

(i) $g_k \in C_b^k(N)$,

(ii) $g_k(q_k) = r_k$,

(iii) $|g_k(z) - g_k(x)| \leq (1+\varepsilon)^3 d_N(z, x)$ for all $z, x \in N$,

(iv) If $z \in \psi_k^{-1}(B_{|||\cdot|||_k}(0,r_k))$, then $|||\psi_k(z)|||_k \leq r_k$ and from condition (5) on the Fact, we obtain

$$d_N(z,q_k) \leq (1+\varepsilon)|||\psi_k(z)-\psi_k(q_k)|||_k = (1+\varepsilon)|||\psi_k(z)|||_k \leq (1+\varepsilon)(r_k - g_k(z)+\varepsilon/2m).$$

The Lipschitz constant of $g_k \circ h$, for $k=1,\dots,m$, is the following

$$\begin{aligned}\mathrm{Lip}(g_k\circ h) &\leq ||g_k\circ h||_{C_b^1(M)} = ||T(g_k)||_{C_b^1(M)} \leq ||T||||g_k||_{C_b^1(N)} \\ &= ||T||\max\{||g_k||_\infty, ||dg_k||_\infty\} \leq ||T||(1+\varepsilon)^3.\end{aligned}$$

Now, since $r_k = g_k(q_k) = g_k(h(\sigma(t_k)))$ and $\psi_k(h(\sigma(s_k))) \in B_{|||\cdot|||_k}(0,r_k)$, we have

$$\begin{aligned}d_N(h(p_1),h(p_2)) &\leq \sum_{k=1}^{m-1}[d_N(h(\sigma(t_k)),h(\sigma(s_k))) + d_N(h(\sigma(s_k)),h(\sigma(t_{k+1})))] \\ &\leq \sum_{k=1}^{m-1}(1+\varepsilon)[g_k(q_k)-g_k(h(\sigma(s_k))) \\ &\qquad + g_{k+1}(q_{k+1}) - g_{k+1}(h(\sigma(s_k))) + \varepsilon/m] \\ &\leq \sum_{k=1}^{m-1}(1+\varepsilon)[\mathrm{Lip}(g_k\circ h)d_M(\sigma(t_k),\sigma(s_k)) \\ &\qquad + \mathrm{Lip}(g_{k+1}\circ h)d_M(\sigma(t_{k+1}),\sigma(s_k)) + \varepsilon/m] \\ &\leq \sum_{k=1}^{m-1}||T||(1+\varepsilon)^4[d_M(\sigma(t_k),\sigma(s_k)) + d_M(\sigma(t_{k+1}),\sigma(s_k))] + \varepsilon(1+\varepsilon) \\ &\leq \sum_{k=1}^{m-1}||T||(1+\varepsilon)^4\ell(\sigma_{|_{[t_k,t_{k+1}]}}) + \varepsilon(1+\varepsilon) = ||T||(1+\varepsilon)^4\ell(\sigma)+\varepsilon(1+\varepsilon) \\ &\qquad \leq ||T||(1+\varepsilon)^4(d_M(p_1,p_2)+\varepsilon)+\varepsilon(1+\varepsilon)\end{aligned}$$

for every $\varepsilon > 0$. Thus, h is $||T||$-Lipschitz. □

Lemma 7.4.10. *Let M and N be C^k Finsler manifolds such that N is modeled on a Banach space with a Lipschitz C^k-smooth bump function. Let $h: M \to N$ be a homeomorphism such that $f \circ h \in C_b^k(M)$ for every $f \in C_b^k(N)$. Then, h is a weakly C^k-smooth mapping on M.*

Proof. Let us fix $x \in M$ and $\varepsilon = 1$. There are charts $\varphi : U \to X$ of M at x and $\psi : V \to Y$ of N at $h(x)$ satisfying property (P2) of Definition 5.2.1 and inequality (5.13) on U and V, respectively. We can assume that $h(U) \subset V$. Since Y admits a Lipschitz and C^k-smooth bump function and $\psi(h(U))$ is an open neighborhood of $\psi(h(x))$ in Y, there are real numbers $0 < s < r$ such that $B(\psi(h(x)),s) \subset B(\psi(h(x)),r) \subset \psi(h(U))$ and a Lipschitz and C^k-smooth function $\alpha : Y \to \mathbb{R}$ such that $\alpha(y) = 1$ for $y \in B(\psi(h(x)),s)$ and $\alpha(y) = 0$ for $y \notin B(\psi(h(x)),r)$. Let us define $U_0 := h^{-1}(\psi^{-1}(B(\psi(h(x)),s))) \subset U$, which is an open neighborhood of x in M.

Let us check that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is C^k-smooth on $\varphi(U_0) \subset X$ for all $y^* \in Y^*$. Following the proof of [46, Theorem 4], we define $g : N \to \mathbb{R}$ as

$$g(y) = \begin{cases} 0 & \text{whenever } y \notin V \\ \alpha(\psi(y)) \cdot y^*(\psi(y)) & \text{whenever } y \in V. \end{cases}$$

It is clear that $g \in C_b^k(N)$ and, by assumption, $g \circ h \in C_b^k(M)$. Now, it follows that $\psi(h(\varphi^{-1}(z))) \in B(\psi(h(x)), s)$ for every $z \in \varphi(U_0)$. Thus

$$\begin{aligned} y^* \circ (\psi \circ h \circ \varphi^{-1})(z) &= y^*(\psi(h(\varphi^{-1}(z)))) = \alpha(\psi(h(\varphi^{-1}(z))))y^*(\psi(h(\varphi^{-1}(z)))) \\ &= g(h(\varphi^{-1}(z))) = g \circ h \circ \varphi^{-1}(z), \end{aligned}$$

for every $z \in \varphi(U_0)$. Since $(g \circ h) \circ \varphi^{-1}$ is C^k-smooth on $\varphi(U_0)$, we have that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is C^k-smooth on $\varphi(U_0)$. Thus $\psi \circ h \circ \varphi^{-1}$ is weakly C^k-smooth on $\varphi(U_0)$ and h is weakly C^k-smooth on M. □

Proof of Theorem 7.4.8. If $h : M \to N$ is a weak C^k Finsler isometry, we can define the operator $T : C_b^k(N) \to C_b^k(M)$ by $T(f) = f \circ h$. Let us check that T is well defined. For every $x \in M$, there are charts $\varphi : U \to X$ of M at x and $\psi : V \to Y$ of N at $h(x)$, such that $h(U) \subset V$ and $\psi \circ h \circ \varphi^{-1}$ is weakly C^k-smooth on $\varphi(U) \subset X$. Also, $f \circ \psi^{-1}$ is C^k-smooth on $\psi(V) \subset Y$. Thus, by [49, Proposition 4.2], $(f \circ \psi^{-1}) \circ (\psi \circ h \circ \varphi^{-1}) = f \circ h \circ \varphi^{-1}$ is C^k-smooth on $\varphi(U)$. Therefore, $f \circ h$ is C^k-smooth on U. Since this holds for every $x \in M$, we deduce that $f \circ h$ is C^k-smooth on M. Moreover, T is an algebra isomorphism with $||T(f)||_{C_b^1(M)} = ||f \circ h||_{C_b^1(M)} = ||f||_{C_b^1(N)}$ for every $f \in C_b^k(N)$.

Conversely, let $T : C_b^k(N) \to C_b^k(M)$ be a normed algebra isometry. Then, we can define the mapping $h : H(C_b^k(M)) \to H(C_b^k(N))$ by $h(\varphi) = \varphi \circ T$ for every $\varphi \in H(C_b^k(M))$. The mapping h is a bijection. Moreover, h is an homeomorphism. Recall that we identify $x \in M$ with $\delta_x \in H(C_b^k(M))$. Thus, $h(x) = h(\delta_x) = \delta_x \circ T$. Since h is an homeomorphism, by Proposition 7.4.5, we obtain that a point $\varphi \in H(C_b^k(M))$ has a countable neighborhood basis in $H(C_b^k(M))$ if, and only if, $\varphi \in M$ and the same holds in N. Therefore for every $p \in N$, there is a unique point $x \in M$ such that $h(\delta_x) = \delta_p$. Let us check that $T(f) = f \circ h$ for all $f \in C_b^k(N)$. Indeed, for every $x \in M$ and every $f \in C_b^k(N)$,

$$T(f)(x) = \delta_x(T(f)) = (\delta_x \circ T)(f) = h(\delta_x)(f) = \delta_{h(x)}(f) = f(h(x)) = f \circ h(x).$$

Now, from Proposition 7.4.9 and Lemma 7.4.10 we deduce that h is a weak C^k Finsler isometry. □

Remark 7.4.11. *It is worth mentioning that, for Riemannian manifolds, every metric isometry is a C^∞ Finsler isometry. This result was proved by S. Myers and N. Steenrod [95] in the finite-dimensional case and by I. Garrido, J.A. Jaramillo and Y.C. Rangel [42] in the general case. Also, S. Deng and Z. Hou [20] obtained a version for finite-dimensional Riemannian-Finsler manifolds. Nevertheless, there is no a generalization, up to our knowledge, of the Myers-Steenrod theorem for all Finsler manifolds. Thus, for $k = 1$ we can only assure that the metric isometry obtained in Theorem 7.4.8 is weakly C^1-smooth.*

Let us finish this note with some interesting corollaries of Theorem 7.4.8. First, recall that every separable Banach space with a Lipschitz C^k-smooth bump function satisfies condition (A.2) and every WCG Banach space with a C^1-smooth bump function satisfies condition (A.1) for $k = 1$.

Corollary 7.4.12. *Let M and N be complete, C^1 Finsler manifolds that are C^1-uniformly bumpable and are modeled on WCG Banach spaces. Then M and N are weakly C^1 equivalent as Finsler manifolds if, and only if, $C_b^1(M)$ and $C_b^1(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^1(N) \to C_b^1(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^1 Finsler isometry.*

Corollary 7.4.13. *Let M and N be complete, separable C^k Finsler manifolds that are modeled on Banach spaces with a Lipschitz and C^k-smooth bump function. Then M and N are weakly C^k equivalent as Finsler manifolds if and only if $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^k Finsler isometry. In particular, h is a C^{k-1} Finsler isometry whenever $k \geq 2$.*

Since every weakly C^k-smooth function with values in a finite-dimensional normed space is C^k-smooth and, by Remark 6.1.1 and Proposition 6.2.2, every finite-dimensional C^k Finsler manifold is C^k-uniformly bumpable, we obtain the following Myers-Nakai result for finite-dimensional C^k Finsler manifolds.

Corollary 7.4.14. *Let M and N be complete and finite-dimensional C^k Finsler manifolds. Then M and N are C^k equivalent as Finsler manifolds if, and only if, $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a C^k Finsler isometry.*

We obtain an interesting application of Finsler manifolds to Banach spaces. Recall the well known Mazur-Ulam Theorem establishing that every surjective isometry between two Banach spaces is affine.

Corollary 7.4.15. *Let X and Y be WCG Banach spaces with C^1-smooth bump functions. Then X and Y are isometric if, and only if, $C_b^1(X)$ and $C_b^1(Y)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^1(Y) \to C_b^1(X)$ is of the form $T(f) = f \circ h$ where $h : X \to Y$ is a surjective isometry. In particular, h and h^{-1} are affine isometries.*

Open Problems

1. Corollary 7.1.1 states that every Lipschitz function defined in a non-separable Riemannian manifold can be uniformly approximated by a C^1-smooth and Lipschitz function with almost the same Lipschitz constant. Can this function be constructed to be C^∞-smooth? Equivalently, do Hilbert spaces satisfy property (A^∞)? N. Moulis [92] showed that every separable Hilbert space has property (A^∞). It is an open problem whether the non-separable Hilbert space has property (A^∞).

2. The Deville-Godefroy-Zizler smooth variational principle has proven to be very useful in various areas of nonlinear analysis, in particular, to obtain viscosity solutions of

Hamilton-Jacobi equations (see [21] and [22]). D. Azagra, J. Ferrera and F. López-Mesas in [5] developed a subdifferential theory in Riemannian manifolds and used this variational principle in order to obtain viscosity solutions of Hamilton-Jacobi equations defined in Riemannian manifolds. An interesting application of these chapters could be the development of the subdifferential theory in Finsler manifolds and the study of viscosity solutions of Hamilton-Jacobi equations defined on Finsler manifolds.

3. Is the isometry $h : M \to N$ obtained in Theorem 7.4.8 not only a weak C^k Finsler isometry but also a C^k Finsler isometry? In other words, let M and N be complete C^k Finsler manifolds that are C^k-uniformly bumpable and are modeled on k-admissible Banach spaces. Are $C_b^k(M)$ and $C_b^k(N)$ equivalent as normed algebras if and only if M and N are C^k equivalent as Finsler manifolds?

4. For Riemannian manifolds, every metric isometry is a C^∞ Finsler isometry [95] and [42]. Nevertheless, there is no a generalization, up to our knowledge, of this result for all Finsler manifolds. In fact, we do not know whether every weak C^1 metric isometry between C^k Finsler manifolds in the sense of Palais is a C^1 Finsler isometry. Let us note that this assertion does not hold in the Neeb-Upmeier uniform case. If we take M the Banach space $c_0(\mathbb{N})$ with the Finsler structure $||(x,v)||_M := ||v||_\infty$, and $(N, ||\cdot||_N)$ the C^∞ Finsler manifold in the sense of Neeb-Upmeier uniform with $N = c_0(\mathbb{N})$ and $||(x,v)||_N = ||(v_1, v_2 + v_1\cos(2x_1), \dots, v_n + v_1\cos(nx_1), \dots)||_\infty$. Then, the mapping $h : M \to N$ defined as

$$h(x) = (x_1, x_2 + \frac{1}{2}\sin(2x_1), \dots, x_n + \frac{1}{n}\sin(nx_1), \dots)$$

is weakly C^1-smooth on M but it is not differentiable.

Also, it is worth noticing that the map $h : \mathbb{R} \to (\mathbb{R}^2, ||\cdot||_1)$ defined as $h(t) = (t, 0)$ whenever $t \leq 0$ and $h(t) = (0, t)$ whenever $t \geq 0$, is an non-surjective isometry between C^∞ Finsler manifolds in the sense of Palais which is not C^1-smooth.

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