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On the components of the unstable set of isolated invariant sets ☆

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Dedicated to Professor José Manuel Rodríguez Sanjurjo on the occasion of his 70th birthday

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ABSTRACT

The aim of this note is to shed some light on the topological structure of the unstable set of an isolated invariant set K. We give a bound on the number of essential quasicomponents of the unstable set of K in terms of the homological Conley index of K. The proof relies on an explicit pairing between Čech homology classes and Alexander–Spanier cohomology classes that takes the form of an integral.

1. Introduction

The Conley index is an invariant of an isolated invariant set K [3,11–13]. It captures relevant information of the dynamics and is robust under perturbations, actually it remains invariant under continuation. Roughly, the Conley index describes the unstable set $W^u(K, f)$ of K and the dynamics induced on it. Although for hyperbolic fixed points the unstable set is actually a manifold, in general the unstable set may exhibit a very complicated topological structure. Nonetheless, if we just look at dimension 0, the description

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of the unstable set and its dynamics reduces to the study of the connected components and how they are permuted by the action. The connected components of $W^u(K, f) - K$ are called *branches* of the unstable set.

The family of branches of an unstable set may be infinite and contain components that do not accumulate on K or are accumulated by other branches. We focus on the "essential" components: those which are adherent to K and extend to infinity (more precisely, their backward images are never contained in a neighborhood of K). The object of this note is to compute their number in terms of invariants related to K and the dynamics. Since connected components can be difficult to distinguish, we work with quasicomponents instead. The main theorem of the article computes the number of essential quasicomponents of the unstable set when K is an isolated invariant set:

Theorem 1. *Suppose that f is a homeomorphism of a manifold, K is an isolated invariant set for f with finitely generated Čech homology groups that is not an attractor. Then, the number of essential quasicomponents of the unstable set of K is equal to* $\dim \check{H}_1(W_\infty, K) + 1$.

In the statement, and everywhere in this note, homology and cohomology groups are assumed to have coefficients in a field. The compactified unstable set W_∞ is the Alexandroff compactification of $W^u(K, f)$ and the essential quasicomponents are those adherent to ∞ and K. An important caveat to the previous assertion is that the topology of the unstable set is not the subspace topology that it inherits from phase space but the *intrinsic* topology [13], which is finer. Very roughly, the intrinsic topology coincides with the subspace topology as long as the unstable set (excluding an initial part) is not adherent to K, so the result remains true in many instances considering the subspace topology. The theorem holds on more general phase spaces, such as locally compact metric ANR, and can be stated for non–invertible maps, but in the last case we only obtain the upper bound for the number of essential quasicomponents.

The dimension of the Čech homology (or cohomology, they are equal in our setting) group of W_∞ can be computed easily from a single index pair (N, L) for K. In fact, it turns out to be the rank of the homological Conley index of K in dimension 1. In particular, $\dim \check{H}_1(W_\infty)$ is bounded from above by $\dim \check{H}_1(N, L)$. It follows that the dimension of $\check{H}_1(W_\infty, \widetilde{K})$ does not exceed $\dim \check{H}_1(N, L) + (k - 1)$, where k is the number of connected components of K when finite.

As a quick example to illustrate the theorem, we can think of a saddle point p in the plane. The unstable manifold has two branches, so there are two essential quasicomponents (they coincide with the connected components when they come in a finite number). The Alexandroff compactification of $W^u(p)$ is a circle and the equality follows. Similarly, a n–saddle point (the coalescence of n saddle points) displays $n+1$ branches and the one–point compactification of the unstable set is the wedge sum of n circles.

Theorem 1 has already been proved in cohomological terms in [8], so this version immediately follows from a duality argument. However, the discussion we present gives a better insight on the question, as it uses a suitable description of Čech homology classes and an explicit pairing with cohomology classes that were introduced recently [6].

As the main theorem suggests, our work applies the machinery of algebraic topology to examine the topology of the unstable set. Čech theory is typically a better fit than singular theory to study bad spaces (meaning that their (local) topology may be very complicated). Instead of working with Čech cohomology, we use an alternative approach named after Alexander and Spanier that yields the same groups as Čech's under very general circumstances (for paracompact spaces [9], for example). Alexander–Spanier cohomology has the advantage of being completely intrinsic, it does not need extrinsic simplicial complexes as the nerve of a cover in Čech's original definition. For homology, we use an approach in terms of "formal" simplices, that resembles the simplices introduced in [4] and essentially recovers the Vietoris complex. In both instances, we define $\mathcal{U}$–small groups and their limit as the cover $\mathcal{U}$ gets finer yield the Čech homology and cohomology groups [6]. If we fix the cover $\mathcal{U}$ considered, that is, the scale at which we study the space, we recover partial information of the space that is enough for certain computations.

The main tool used in the proof of Theorem 1 is a bilinear pairing between Čech homology classes and Alexander cohomology classes of X which is defined at any fixed scale $\mathcal{U}$. The pairing takes the form of an integral: we are able to integrate Alexander–Spanier cohomology classes over Čech homology classes. This tool is defined from scratch in Section 5, but we refer the interested reader to the recent article in which the integral was introduced [6]. One of the properties of the integral is that it goes along well with the connecting homomorphisms of the Mayer-Vietoris sequences for homology and cohomology. This is key in the proof of the main theorem to relate zero dimensional objects, namely essential quasicomponents, to $\check{H}_1(W_\infty, \widetilde{K})$.

The paper and, especially, the proofs require a considerable technical effort. The fundamental reason is that Čech homology is defined through an inverse limit, so a Čech homology class is a sequence of homology classes. Thus, a representative of a class consists of a sequence of cycles. In our approach, the sequence is indexed by a cofinal family of open covers. We need to choose a cover fine enough so that several properties concerning the quality of the approximation from the associated Vietoris complex to the space are satisfied.

The article is organized as follows. We review Alexander–Spanier cohomology, its version at a fixed scale $\mathcal{U}$ and its relationship with the notion of quasicomponent in Section 2. A brief account on Conley theory and dynamical considerations on the unstable set is presented in Section 3. Next, the homology at scale $\mathcal{U}$ is defined and we show how it recovers the Čech homology group and a result concerning the homology of the basin of attraction of a fixed point. Section 5 introduces the integral and its main properties. We finish the paper with the proof of the main result, Theorem 23, that is a more general version of Theorem 1.

2. Alexander–Spanier cohomology

The original approach to Čech cohomology involved the construction of simplicial complexes extrinsic to the space of study. The first definition, due to Čech [18], introduced the nerve of an open cover, a simplicial complex where the vertices are the elements of the cover and a collection of vertices span a simplex if their associated open sets have a non-empty intersection. Later, Vietoris [19] gave an alternative description via a larger complex, the Vietoris complex that we shall later recall. They key feature of both complexes is that as the cover gets progressively finer, the complexes become ever more similar to their underlying space. The Čech homology and cohomology groups of a topological space X are defined as the limit of the homology and cohomology groups of the nerves (or, equivalently [1,9], the Vietoris complexes) as the cover ranges over all the covers of X, and are denoted by $\check{H}_*(X)$ and $\check{H}^*(X)$, respectively. As mentioned above, coefficients are by default taken in an abstract field $\mathbb{K}$ and they are always excluded from the notation. Čech theory coincides with singular theory in CW-complexes.

Unlike Čech cohomology, Čech homology does not satisfy Eilenberg–Steenrod axioms, as exactness does not hold in general. However, the exactness axiom is satisfied in the category of compacta when coefficients are in a field. A classical reference for the definition and properties of Čech groups is [2]. Čech homology and cohomology satisfy a continuity property: if X is obtained as a limit of compacta X_i then the direct (inverse) limit of the Čech cohomology (homology) groups of X_i is isomorphic to the Čech cohomology (homology) group of X. Furthermore, Čech homology and cohomology satisfy a strong form of excision.

There is an alternative approach to Čech cohomology due to Alexander and Spanier that is of intrinsic nature. This is the approach we use in this article. We refer the reader to [16,17] for details.

2.1. Definition

Let $\mathcal{U}$ be a fixed open cover of a topological space X. $\mathcal{U}$ shall be thought of as the "scale" below which the structure of X is disregarded. For $q \geq 0$ a q–cochain is a (not necessarily continuous) map

$\xi\colon X^{q+1} \to \mathbb{K}$. These form a vector space $C^q_{\mathcal{U}}(X)$. Say that a q–cochain ξ is $\mathcal{U}$–locally zero if $\xi(x_0, \ldots, x_q) = 0$ whenever $x_0, \ldots, x_q$ belong to some member of $\mathcal{U}$. The coboundary of a cochain is defined as follows: $\delta\xi(x_0, \ldots, x_{q+1}) := \sum_{j=0}^{q}(-1)^j \xi(x_0, \ldots, \widehat{x_j}, \ldots, x_q)$. It is straightforward to check that the coboundary of a $\mathcal{U}$–locally zero cochain is again $\mathcal{U}$–locally zero. Thus we may quotient $C^q_{\mathcal{U}}(X)$ by the subspace of $\mathcal{U}$–locally zero cochains to obtain a vector space $\overline{C}^q_{\mathcal{U}}(X)$. The coboundary operator descends to a coboundary operator on this space yielding a cochain complex $\{\overline{C}^q_{\mathcal{U}}(X), \delta\}$, which we assume is augmented by the map that sends $a \in \mathbb{K}$ to the class that contains the constant map equal to a. Cocycles in $\overline{C}^q_{\mathcal{U}}(X)$ are called $\mathcal{U}$–small. We denote the cohomology of the complex by $H^q_{\mathcal{U}}(X)$ and call $H^q_{\mathcal{U}}(X)$ the *$\mathcal{U}$–small* (reduced) cohomology group or the cohomology group *at scale* $\mathcal{U}$ of X. Notice that, by definition, the class $\overline{\xi} \in \overline{C}^q_{\mathcal{U}}(X)$ of a cochain ξ is a cocycle if and only if $\delta\xi$ vanishes over every member of $\mathcal{U}$; similarly, $\overline{\xi} = \overline{\eta}$ means that ξ and η agree when evaluated on tuples that are contained in some element of $\mathcal{U}$. Throughout this note, we drop the bar from the notation and denote equally a cochain ξ and the equivalence class in $\overline{C}^q_{\mathcal{U}}(X)$ it represents.

If $\mathcal{V}$ refines $\mathcal{U}$ then every $\mathcal{U}$–locally zero cochain is also $\mathcal{V}$–locally zero, so there is a natural homomorphism $\overline{C}^q_{\mathcal{U}}(X) \to \overline{C}^q_{\mathcal{V}}(X)$. This inclusion map commutes with the coboundary operator and hence induces an inclusion homomorphism $p_{\mathcal{V}\mathcal{U}}\colon H^q_{\mathcal{U}}(X) \to H^q_{\mathcal{V}}(X)$. There is, then, a direct system indexed by the open covers $\mathcal{U}$ of X, composed of the groups $H^q_{\mathcal{U}}(X)$ and whose bonding maps are the maps $p_{\mathcal{V}\mathcal{U}}$. As we shall see below, the limit of this system is the Alexander–Spanier cohomology group of X, which we now introduce.

The previous definitions work well without sticking to a specific open cover of X. A cochain is said to be locally zero if there exists an open cover $\mathcal{U}$ of X such that $\xi(x_0, \ldots, x_q) = 0$ for every tuple contained in the same element of $\mathcal{U}$. We denote by $\overline{C}^q(X)$ the quotient space of all cochains over the locally zero ones. Since the coboundary of a locally zero cochain is evidently locally zero, we obtain a cochain complex $\{\overline{C}^q(X), \delta\}$, that is again assumed augmented. The cohomology of the complex is, by definition, the (reduced) Alexander–Spanier cohomology of X. It is isomorphic to the Čech cohomology of X when X is paracompact ([9] and [17, Corollary 8, p. 334]) or, in general, when they are defined using the same family of coverings [1].

As the definitions suggest, the limit of $\mathcal{U}$–cohomology groups is the Alexander–Spanier cohomology group.

Proposition 2. *The direct limit of $\{H^q_{\mathcal{U}}(X); p_{\mathcal{V}\mathcal{U}}\}$ is precisely the Alexander–Spanier cohomology of X. Therefore, under general hypothesis (described above) it is also isomorphic to the Čech cohomology of X, $\check{H}^q(X)$.*

Proof. It follows directly from its definition that $\overline{C}^q(X)$ can be identified with the direct limit of $\{\overline{C}^q_{\mathcal{U}}(X)\}$. Then the result owes to the fact that the homology functor commutes with direct limits. □

2.2. Quasicomponents

The *quasicomponent* of $x \in X$ is the intersection of all clopen (closed and open) subspaces of X that contain x. Alternatively, the quasicomponent of x can be defined as the equivalence class of x under the relation $\sim$, where $y \sim z$ iff there is no separation $\{A, B\}$ of X (A, B disjoint, open and $A \cup B = X$) such that $y \in A, z \in B$. From this interpretation, it is easy to deduce that the connected component of x is contained in the quasicomponent of x. For a brief account on quasicomponents we refer the reader to [20]. Although connected components and quasicomponents are different in general, they coincide, for instance, when one of them is open or when X is compact Hausdorff.

Lemma 3. *Suppose X has a finite number of quasicomponents. Then, connected components and quasicomponents coincide on X.*

Proof. Let us show that every quasicomponent F is connected. Suppose $\{A, B\}$ is a separation of F. Since quasicomponents are closed, the hypothesis implies that F is open as well, so A and B are clopen in X. By definition of quasicomponent, either A or B is empty, so F is connected. □

A few quick properties of quasicomponents for later use:

Lemma 4.

(i) If a quasicomponent meets a clopen set O then it is contained in O.
(ii) Denote $F(p)$ the quasicomponent of p. If $(p_n) \to p$ and $(q_n) \to q$ are two convergent sequences in X such that $F(p_n) = F(q_n)$ for every n, then $F(p) = F(q)$.
(iii) A finite family of quasicomponents can be separated by clopen sets.

Proof. (i) immediately follows from the fact that the quasicomponent of a point in O must be contained in O. For (ii), let V be a clopen neighborhood of $F(p)$. Since p_n is contained in V for n large, so is its quasicomponent $F(p_n) = F(q_n)$ by the previous remark. In particular, $q_n \in V$ and we conclude that $q \in F(p)$. For (iii), suppose F, F' are different quasicomponents. There is a clopen neighborhood V of F that does not contain F', so $F' \cap V = \emptyset$ by (i). We obtain similarly a clopen V' such that $F' \subset V' \subset X - F$. Then $\{V \cap (X - V'), V' \cap (X - V)\}$ are clopen sets that separate F and F'. The argument can be generalized easily. □

Our discussion with Čech theories requires working with homology and cohomology groups at a fixed scale, that is, with a fixed open cover $\mathcal{U}$. Thus, we typically have a fixed open cover $\mathcal{U}$ of X that bounds from below the topology that can be distinguished. Let us adapt some of the standard notions on connectedness to this framework.

We say that $A \subset X$ is $\mathcal{U}$–clopen if it is a maximal union of elements of $\mathcal{U}$, in the sense that if an element of $\mathcal{U}$ meets A then it is contained in A. A clopen set A is $\mathcal{U}$–clopen provided $\mathcal{U}$ refines the open cover $\{A, X \setminus A\}$. Recall that an open cover $\mathcal{U}$ refines an open cover $\mathcal{V}$ (denoted by $\mathcal{U} \succ \mathcal{V}$) if every element of $\mathcal{U}$ is contained in an element of $\mathcal{V}$. Evidently, if $\mathcal{U} \succ \mathcal{V}$, every $\mathcal{V}$–clopen set is automatically $\mathcal{U}$–clopen. We can define an equivalence relation on X by $y \sim z$ iff there is no separation $\{A, B\}$ of X such that $y \in A, z \in B$ and A and B are $\mathcal{U}$–clopen. The equivalence class of x under $\sim$ is called *$\mathcal{U}$–component* of x and may be thought as the (quasi)component of x at scale $\mathcal{U}$. In fact, the $\mathcal{U}$–component of x is the smallest $\mathcal{U}$–clopen set that contains x and the quasicomponent of x is equal to the intersection of all $\mathcal{U}$–components of x as $\mathcal{U}$ ranges over all open covers of X.

Define a $\mathcal{U}$–small point path as a sequence $x_0 = x, x_1, \ldots, x_n = x'$ such that for every $0 \leq i \leq n-1$, x_i and x_{i+1} are $\mathcal{U}$-close, i.e., $x_i, x_{i+1} \in U_i$ for some open set $U_i \in \mathcal{U}$. The $\mathcal{U}$–component of x can be alternatively defined as the set of endpoints of $\mathcal{U}$–small point paths that start at x.

Lemma 5. *Let $g\colon X \to Y$ be a continuous surjective map. Then, the number of quasicomponents of X is greater or equal than the number of quasicomponents of Y. The same is true for the connected components.*

Proof. The result follows from the fact that the preimage of a clopen set of Y is clopen in X. □

2.3. 0–dimensional cohomology

The relationship between the quasicomponents of a space and the elements of the 0–dimensional Čech groups goes back to the original works of Čech [18]. Here we show in an elementary fashion that the notions of quasicomponent and $\mathcal{U}$–component serve to describe the 0–dimensional Alexander–Spanier cohomology group and the $\mathcal{U}$–small cohomology group, respectively.

We start with the group $H^0_{\mathcal{U}}(X)$. Note first that classes in $\overline{C}^0_{\mathcal{U}}(X)$ can be viewed as maps $\mathcal{D} \to \mathbb{K}$ whose domain $\mathcal{D}$ consists of $(q+1)$–tuples contained in a single element of $\mathcal{U}$. Suppose ξ is a $\mathcal{U}$–small 0–cocycle, that is, a map $\xi\colon X \to \mathbb{K}$ that satisfies the cocycle condition inside each element of $\mathcal{U}$. Thus, for every

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$x, y \in U \in \mathcal{U}$, $0 = \delta\xi(x, y) = \xi(y) - \xi(x)$, i.e. ξ is constant on each element of the cover. In other words, ξ is constant on every $\mathcal{U}$–component.

From the previous characterization, it is very easy to construct $\mathcal{U}$–small 0–cocycles in X. Given a $\mathcal{U}$–clopen subset A of X, the characteristic function χ_A on A is evidently constant on each $\mathcal{U}$–component, so it is a representative of a $\mathcal{U}$–small 0-cycle and defines an element $[\chi_A] \in H^0_{\mathcal{U}}(X)$.

The discussion extends in a straightforward fashion to the Alexander–Spanier 0–cohomology group $\check{H}^0(X)$. The cocycle condition applied to a 0–cochain $\xi \colon X \to \mathbb{K}$ forces $0 = \delta\xi(x, y) = \xi(y) - \xi(x)$ for every x, y that belong to the same element of a certain open cover $\mathcal{U}$ of X. Equivalently, ξ is constant on every $\mathcal{U}$–component for some $\mathcal{U}$, that is, ξ is locally constant. Algebraically, this is a mere consequence of Proposition 2: $\check{H}^0(X)$ is the direct limit of the groups $H^0_{\mathcal{U}}(X)$, so every Alexander–Spanier 0–cocycle is a $\mathcal{U}$–small 0-cocycle for some $\mathcal{U}$.

Unlike the previous example, it is not true that the characteristic function χ_F on a quasicomponent F of X always represents a Alexander-Spanier 0-cocycle. Indeed, if F is not open, χ_F is not locally constant, so certainly not constant on $\mathcal{U}$–components for any open cover $\mathcal{U}$. This example gives a small hint on why it is preferable to work on scales fixed by an open cover than in the limit. The situation becomes evident as we move to Čech homology, which lacks a satisfying point-set topological description. Incidentally, the dynamics induced on the 0–dimensional Čech homology and cohomology groups have been studied in [7].

By Lemma 4, a finite family of quasicomponents F_i can be separated by disjoint clopen neighborhoods O_i, that may be thought of as $\mathcal{U}$–clopen neighborhoods for a sufficiently fine cover $\mathcal{U}$. The set of Alexander-Spanier 0-cocycles $\{\chi_{O_i}\}$ are clearly linearly independent in $\check{H}^0(X)$ (provided the O_i do not form a partition of X).

Corollary 6. *If $\check{H}^0(X)$ is finite dimensional then X has a finite number of quasicomponents.*

A small effort immediately yields the equality between the dimension of the (non-reduced) 0–cohomology group and the number of quasicomponents when any of them is finite.

2.4. Results on quasicomponents

In this subsection we suppose we work with a dynamical system as in the main results of the article. The dynamical motivation (and hypothesis) is postponed to (the end of) Section 3, here we give some lemmas that can be traced back to [8]. Let X be a compact metric space and $f \colon X \to X$ be a homeomorphism such that $\{a, a^*\}$ forms an attractor–repeller decomposition of X, that is, a and a^* are invariant under f and the rest of orbits of f converge uniformly to a in forward time.

For the following results the standing assumption is that $\dim \check{H}^0(X) < +\infty$, so by Corollary 6 the number of quasicomponents of X is finite and the same is true for the connected components by Lemma 3.

Lemma 7. *Every connected component of $X - \{a, a^*\}$ reaches a or a^* in the sense that it is adherent to a or a^*. The result is also true for quasicomponents because they are a union of connected components.*

Proof. Suppose C is a connected component of $X - \{a, a^*\}$ that is not adherent to a or a^*. C is also a connected component of X (cf. [20, p. 101]). The same features are shared by $f^n C$ for every $n \in \mathbb{Z}$. Since $f^n C \to a$, it follows that they are all different, which is a contradiction. □

We are interested in the topological structure of X and, in particular, in the quasicomponents of $X - \{a, a^*\}$. A quasicomponent of $X - \{a, a^*\}$ that is adherent to a and to a^* is called *essential*. Suppose in the following that O_e is a clopen (as a subset of $X - \{a, a^*\}$) neighborhood of the family of essential quasicomponents.

Lemma 8. *There exist neighborhoods V, V^* of a, a^* in X such that for every quasicomponent F of $X - \{a, a^*\}$ exactly one of the following conditions hold:*

(1) $F \subset O_e$.
(2) $F \cap V = \emptyset$.
(3) $F \cap V^* = \emptyset$.

Proof. Let $\{P_n\}_{n\geq 1}, \{P_n^*\}_{n\geq 1}$ be bases of compact neighborhoods of a and a^*, respectively. We impose that $P_i \cap P_j^* = \emptyset$ for all i, j and that the bases are nested. We proceed by contradiction. Suppose for every n there is a quasicomponent F_n disjoint from O_e that meets both P_n and P_n^*. F_n is not essential, so by Lemma 7 either all the connected components of F_n reach a or all of them reach a^*. As a consequence, there is a connected component of F_n that meets ∂P_m and ∂P_m^* for every $m \leq n$. Take points $p_n^m \in F_n \cap \partial P_m$, $q_n^m \in F_n \cap \partial P_m^*$. By passing to a subsequence inductively, we can assume $p_n^m \to p^m \in \partial P_m$ and $q_n^m \to q^m \in \partial P_m^*$ for every $m \geq 1$. By Lemma 4, p^m, q^m belong to the same quasicomponent F, independent of m, so F meets P_m and P_m^* for any m. We conclude that F is essential, so $F \subset O_e$, a contradiction. □

The lemma suggests a partition in clopen sets that is very relevant to the proofs in Section 6.

Corollary 9.

$$X - \{a, a^*\} = O_e \cup O_a \cup O_{a^*}$$

is a partition into clopen sets, where O_a, O_{a^*} are composed of the quasicomponents of $X - \{a, a^*\}$ adherent to a, a^* outside O_e, respectively, and $O_a \cap V^* = \emptyset = O_{a^*} \cap V$ from Lemma 8.

Proof. It is enough to prove that O_a and O_{a^*} are closed. We show it for O_a. Suppose that (p_n) is a sequence of points in O_a that converges to $p \in X - \{a, a^*\}$. Since $F(p_n)$ is adherent to a for every n, the arguments in Lemma 8 conclude that $F(p)$ is adherent to a as well and $p \in O_a$. □

3. Dynamical setting

3.1. Isolated invariant sets

We give a quick review of Conley index theory for discrete dynamical systems. See [11] and the references therein. A similar exposition can be found in [8].

Suppose $f : U \subset M \to M$ is a map defined on an open subset of a locally compact absolute neighborhood retract for metric spaces M (such as a manifold or a simplicial complex). Note that f is not necessarily injective. Throughout this section we denote by M the ambient space. The reader shall think of X, used in the previous and upcoming sections for that purpose, as an invariant space (called compactified unstable set) constructed from an isolated invariant set in M.

A full orbit of a point $x \in M$ is a sequence $(x_i)_{i\in\mathbb{Z}}$ such that $x_0 = x$ and $f(x_i) = x_{i+1}$ for every $i \in \mathbb{Z}$. For a set $A \subset M$, define $\mathrm{Inv}(A)$ as the points for which there is a full orbit contained in A. A set K is called an *isolated invariant set* or *locally maximal* for f if it has a compact neighborhood P such that $K = \mathrm{Inv}(P)$. In this case, P is called an *isolating neighborhood* of K. Isolatedness is a condition frequently satisfied by invariant sets, for example, it is automatic under some forms of hyperbolicity.

On his work on isolated invariant sets, Conley developed invariants that describe these sets in terms of dynamically meaningful neighborhoods and the action of the dynamics on them. We introduce here the concept of *index pair*. A compact pair (N, L) is called an index pair of an isolated invariant set K if it satisfies the following conditions:

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- $\overline{N \setminus L}$ is an isolating neighborhood of K.
- $f(N \setminus L) \subset N$, that is, the points that exit N belong to L.
- $f(L) \cap N \subset L$, in words, L is positively invariant in N.

These conditions imply that if K is not an attractor then $L \neq \emptyset$.

It is always possible to find index pairs such that N/L has finitely generated Čech homology groups. Henceforth, we assume this extra property on index pairs. Actually, if the phase space is a manifold there always exist index pairs such that $N, L, \overline{N \setminus L}$ are manifolds with boundary (see the notion of filtration pair in [3]).

Let us briefly recall what is the Conley index in the discrete setting. Given an index pair (N, L) we can encapsulate the dynamics in N/L via an index map $f_\# \colon N/L \to N/L$ that sends x to $f(x)$ if $f(x) \in N \setminus L$ and to $[L]$ otherwise. The index maps that arise from different index pairs are shift equivalent in the terminology used in [3]. In homological terms this entails that the endomorphisms $(f_\#)_* \colon \check{H}_*(N, L) \to \check{H}_*(N, L)$ are all conjugate when restricted to their maximal invariant subspace (called the Leray reduction [12] of $(f_\#)_*$). The equivalence class of endomorphisms defined in that way is referred to as the homological Conley index of K. The cohomological Conley index is defined in an analogous fashion.

3.2. Compactified unstable set

Using an index pair, Robbin and Salamon [13] constructed a topological object that is intimately related to the unstable set (see also [8]). Consider the inverse sequence of pointed topological spaces:

$$\cdots \xrightarrow{f_\#} (N/L, [L]) \xrightarrow{f_\#} (N/L, [L]) \xrightarrow{f_\#} (N/L, [L])$$

The limit is denoted (W_∞, ∞) and we refer to W_∞ as the *compactified unstable set*, a terminology that will be explained later. Recall that the inverse limit is the space of sequences $(\ldots, x_{n+1}, x_n, \ldots, x_0)$ in N/L such that $f_\#(x_{i+1}) = x_i$, for every $i \geq 0$, that is, the set of backward orbits of the dynamical system $f_\# \colon N/L \to N/L$. Note that we cannot speak of *the* backward orbit of a point x because f may not be invertible, we say instead that (x_n) is *a* backward orbit of $x = x_0$. W_∞ inherits the product topology from the space of sequences. The marked point ∞ corresponds to the constant sequence of value $[L]$. Since N/L is compact, so is W_∞.

The map f induces a homeomorphism f_∞ on W_∞ by

$$f_\infty(\ldots, x_{n+1}, x_n, \ldots, x_0) = (\ldots, x_n, x_{n-1}, \ldots, x_0, f_\#(x_0)).$$

Equivalently, f_∞ applies $f_\#$ at each term of the sequence. It follows immediately that f_∞ is continuous. In fact it is a homeomorphism whose inverse is simply the shift map, $(\ldots, x_{n+1}, x_n, \ldots, x_0) \mapsto (\ldots, x_n, x_{n-1}, \ldots, x_1)$.

The description of the dynamics of $f_\infty \colon W_\infty \to W_\infty$ is fairly simple. It has an attractor-repeller decomposition given by $\{\infty, \widetilde{K}\}$, where $\widetilde{K}$ is defined as the inverse limit of

$$\cdots \xrightarrow{f} K \xrightarrow{f} K \xrightarrow{f} K$$

Let $(x_n)_{n\geq 0} \in W_\infty$ and suppose $y = x_k \neq [L]$. Clearly, y has a backward orbit contained in $N \setminus L$ given by $(x_n)_{n\geq k}$. Thus, either $y \in K$ or the forward orbit of y under f exits $N \setminus L$. In terms of the dynamics within N/L, the second alternative is equivalent to $f_\#^n(y) = [L]$ for large n. Here we use crucially that K is the maximal invariant set in $N \setminus L$. Therefore there is a dichotomy: either (x_n) belongs to $\widetilde{K}$ and its forward orbit is confined to $\widetilde{K}$ or the forward orbit of (x_n) under the action of f_∞ converges to the constant

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sequence $[L]$, that is, to the fixed point ∞. We conclude that ∞ is an attracting fixed point whose basin of attraction is $W_\infty - \widetilde{K}$ and that $\widetilde{K}$ is a repeller whose basin of repulsion equals $W_\infty - \{\infty\}$.

In the case L is empty, ∞ is an isolated point and, moreover, $W_\infty = \widetilde{K} \cup \{\infty\}$. As we assume in the main theorem that K is not an attractor this situation is not further considered.

In spite of its definition, the dynamical properties of the index pairs imply that W_∞ is a canonical object and does not depend on the choice of (N, L). Any index pair of K can be used to define W_∞.

3.3. Topology of the unstable set

The expression "compactified unstable set" used for W_∞ deserves some explanation. The next paragraphs describe the relationship between W_∞ and the standard notion of unstable set $W^u(K, f)$ of K, which is the subset of M composed of the points that have a backward orbit that converges to K. The map is hereafter omitted from the notation of the unstable set. As we made no assumption on differentiability in this note, the unstable set is nowhere close to being a manifold in general.

For technical reasons, suppose that the domain of definition of f is the whole space M (in order to ensure that the unstable set does not reach the boundary of the domain of definition of f). Let us define a map between $W_\infty - \{\infty\}$ and the unstable set $W^u(K)$. Note that $W^u(K)$ is invariant under f and inherits the subspace topology from the phase space M.

For every $(x_n) \in W_\infty - \{\infty\}$ there is $k \geq 0$ such that $x_k \neq [L]$. We define

$$h\colon W_\infty - \{\infty\} \to W^u(K), \quad h((x_n)) = f^k(x_k)$$

Evidently, the definition does not depend on the choice of k. The map h is surjective: every $x \in W^u(K)$ has a backward orbit (x_n) that converges to K. Since $N \setminus L$ is a neighborhood of K, it contains a tail of the sequence, say $(x_n)_{n \geq n_0}$. Then, $(\ldots, x_{n_0+1}, x_{n_0}, f_\#(x_{n_0}), \ldots, f_\#^{n_0}(x_{n_0})) \in W_\infty$ is a preimage of x. The map h is continuous: a neighborhood of (x_n) in W_∞ contains sequences (y_n) such that y_k is close to x_k, so, in particular $y_k \neq [L]$. The continuity of f^k implies the continuity of h in the previous neighborhood.

The subset of W_∞ defined by the condition $x_0 \neq [L]$ is sent by h onto the subset of $W^u(K)$ composed of the points of $N \setminus L$ that have a backward orbit completely contained in $N \setminus L$. Therefore, $\{x_0 \neq [L]\} \subset W_\infty$ can be regarded as an initial part of the unstable set of K.

In the case f is a homeomorphism, h is actually a bijection between $W_\infty - \{\infty\}$ and $W^u(K)$ but not necessarily a homeomorphism. The reason is that the final topology of h in $W^u(K)$ makes two points close only if their backward orbits are close at every step, so this topology (called the *intrinsic topology* by Robbin and Salamon) is finer than the subspace topology on $W^u(K)$. As shown in [13], intrinsic and subspace topologies coincide when we can find an index pair (N, L) where L is positively invariant. Let us ignore for the moment the differences between the topologies in the initial part of the unstable set in a neighborhood V of K and take (N, L) inside V. Suppose (p_n) is a sequence in $W^u(K)$ that converges to $p \in W^u(K) - V$ in the subspace topology but not in the intrinsic topology. Assume further that $p_n = f^{k_n}(x_n)$, with k_n minimal among the nonnegative integers such that $x_n \in N \setminus L$, and $k_n \to +\infty$ (otherwise, for some m, $(f^{-m}(p_n))$ is a sequence in the initial part in V which converges usually but diverges intrinsically). Then, similar conditions hold for $f^{-n}(p)$, $n \geq 0$, and also for any limit point of the backward orbit of p. Thus, we conclude that the unstable set, excluding an initial part, accumulates in K in the subspace topology.

3.4. Further considerations

The map h can be used to bound the topology on $W^u(K)$ in terms of the topology of W_∞. Suppose in what follows that K is not an attractor. In general, h restricts to a surjective map between $W_\infty - (\widetilde{K} \cup \{\infty\})$ and $W^u(K) - K$ (and these spaces are non-empty because K is not an attractor). By Lemma 5, the number of connected components of the latter, which are by definition the *branches* of the unstable set, is bounded from above by the number of connected components of the former. The same is true for quasicomponents. We are interested in the quasicomponents of $W^u(K) - K$ that start in (are adherent to) K and extend to infinity, in the sense that their backward image is never contained in a neighborhood of K. We call them *essential*. In terms of the topology of the compactified unstable set, the essential quasicomponents are those adherent to $\widetilde{K}$ and to ∞. Theorem 1 bounds their number from above by $\dim \check{H}_1(W_\infty, \widetilde{K}) + 1$. Let us recap the previous discussion:

Proposition 10. *The number of essential quasicomponents of $W^u(K) - K$ (with the subspace topology) is bounded from above by the number of essential quasicomponents of $W_\infty - (\widetilde{K} \cup \{\infty\})$. There is equality when f is a homeomorphism and the intrinsic topology coincides with the subspace topology.*

The continuity property of Čech homology gives a way to study the homology of W_∞ in terms of the homology of the index pair (N, L). Since Čech homology is continuous with respect to compact Hausdorff spaces, $\check{H}_*(W_\infty, \infty)$ is isomorphic to the (inverse) limit of

$$\cdots \xrightarrow{(f_\#)_*} \check{H}_*(N, L) \xrightarrow{(f_\#)_*} \check{H}_*(N, L) \xrightarrow{(f_\#)_*} \check{H}_*(N, L) \tag{1}$$

In particular, the Čech homology groups of W_∞ are finite dimensional and

$$\dim \check{H}_1(W_\infty) \leq \dim \check{H}_1(N, L).$$

Note that the limit of (1) is isomorphic to the maximal subspace of $\check{H}_*(N, L)$ where $(f_\#)_*$ acts as an isomorphism, so $(f_\infty)_* \colon \check{H}_*(W_\infty, \infty) \to \check{H}_*(W_\infty, \infty)$ is an isomorphism. Incidentally, the map turns out to be a representative of the homological Conley index of K. Let us also add that, similarly, if we work with cohomology we conclude that the Čech cohomology of (W_∞, ∞) is isomorphic to the direct limit of the relative cohomology group of (N, L) under the action of $(f_\#)^*$ and, in particular, it is finite dimensional.

Suppose that K has finitely many connected components, say k. Arguing as above, we deduce that the dimension of $\check{H}_0(\widetilde{K})$ is bounded from above by k (since f is surjective on K the dimension is actually equal to k.) By direct inspection of the long exact sequence in (Čech) homology relative to the pair $(W_\infty, \widetilde{K})$ we obtain that

$$\dim \check{H}_1(W_\infty, \widetilde{K}) \leq \dim \check{H}_1(N, L) + k - 1$$

We also conclude that the 0–homology group is finite dimensional. By the strong excision property of Čech homology [2, Ch. X], $\check{H}_*(W_\infty, \widetilde{K})$ is isomorphic to $\check{H}_*(W_\infty/\widetilde{K}, [\tilde{K}])$. Therefore, by the previous arguments, $W_\infty/\widetilde{K}$ has finitely generated Čech homology groups at dimensions 0 and 1.

The space $W_\infty/\widetilde{K}$ is the result of collapsing the repeller $\widetilde{K}$ to a point. Trivially, f_∞ descends to a map $\overline{f}_\infty$ on the quotient space whose dynamics is extremely simple: it has an attractor–repeller decomposition $\{\infty, [\widetilde{K}]\}$. However, the unstable set remains intact, $W_\infty - (\widetilde{K} \cup \{\infty\})$ is homeomorphic to $W_\infty/\widetilde{K} - \{[\widetilde{K}], \infty\}$ and the restriction of f_∞ and $\overline{f}_\infty$ to these subspaces are trivially conjugated. Therefore, we can study essential quasicomponents of the unstable set by examining the quotient space $W_\infty/\widetilde{K}$.

Standing hypotheses: In view of the previous discussion, let us formulate a set of hypotheses on the dynamics we study that applies to the unstable set of isolated invariant sets (that are not attractors) described above. X will be a compact metric space (that corresponds to $W_\infty/\widetilde{K}$) and $f \colon X \to X$ a homeomorphism (we emphasize that it is different to the previous f, it actually corresponds to f_∞ above) that has two fixed points a, a^* and such that the dynamics has an attractor–repeller decomposition $\{a, a^*\}$. The basin of attraction of a, $X - \{a^*\}$, will be denoted by B and the basin of repulsion of a^*, $X - \{a\}$, will be denoted

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by B^*. We will further assume that the Čech homology and (equivalently) cohomology groups of X at dimensions 0 and 1 are finite dimensional. Note that while this enumeration describes the general setting for the main theorem in Section 6, the following sections on Čech homology and the integral do not make any assumption on the topological space X unless explicitly stated.

4. Čech homology

In this section we present the approach to the Čech homology as a limit of homologies at scale $\mathcal{U}$ developed in [6]. The development of $\mathcal{U}$–small homology groups or homology groups at scale $\mathcal{U}$ was motivated by the study of [4]. In that article, the authors worked with simplices with finite image intrinsic to the space to define ϵ–homology groups that in the limit recover $\check{H}_*(X)$. It is worth to mention that discrete and intrinsic approaches to shape theory had been previously obtained in [5,15].

4.1. Homology at scale $\mathcal{U}$

Let $\mathcal{U}$ be an open cover of a topological space X. A formal q–simplex is an ordered collection $\sigma = (x_0 \dots x_q)$ of points of X. We say that σ is $\mathcal{U}$–small if its $q+1$ vertices are contained in a single element $U \in \mathcal{U}$. A $\mathcal{U}$–small formal chain in X is a linear combination of $\mathcal{U}$–small formal simplices $c = \sum a_i \sigma_i$ (with coefficients in a field as always). Denote by $C_q^{\mathcal{U}}(X)$ the vector space of $\mathcal{U}$–small formal q–chains in X. For simplicity, in what follows we understand simplices and chains as formal simplices and formal chains, respectively.

The boundary of a simplex $\sigma = (x_0 \dots x_q)$ is defined as $\partial\sigma = \sum_{i=0}^{q} (-1)^i (x_0 \dots \widehat{x_i} \dots x_q)$, where, as usual, the hat over an entry means that it is omitted from the collection. It is easy to check that $\partial\partial\sigma = 0$. The boundary operator is extended linearly to chains. Evidently, the boundary of a $\mathcal{U}$–small chain is again $\mathcal{U}$–small and $\{C_q^{\mathcal{U}}(X), \partial\}$ is a chain complex. The homology of the augmented complex is called *$\mathcal{U}$–small* (reduced) homology group or homology group *at scale* $\mathcal{U}$ and denoted by $H_*^{\mathcal{U}}(X)$.

A $\mathcal{U}$–small cycle c which defines a q–homology class $\alpha = [c] \in H_q^{\mathcal{U}}(X)$ is called a $\mathcal{U}$–small representative of α.

Example 11. Let us show that $\mathcal{U}$–components have trivial $\mathcal{U}$–small 0–homology group. Suppose that A is a $\mathcal{U}$–component of X. A $\mathcal{U}$–small 0-simplex is simply (x), where x is a point in A. Consider $x, x' \in A$. Since A is a $\mathcal{U}$–component, there is a $\mathcal{U}$–small point path $x_0 = x, x_1, \dots, x_n = x'$. Evidently, the point path can be turned into a $\mathcal{U}$–small 1–chain $c = \sum (x_i x_{i+1})$ such that $\partial c = (x') - (x)$. In particular, (x') and (x) are homologous. This argument shows that

$$C_1^{\mathcal{U}}(A) \xrightarrow{\partial} C_0^{\mathcal{U}}(A) \xrightarrow{\epsilon} \mathbb{K}$$

is exact at $C_0^{\mathcal{U}}(A)$, where ϵ is the augmentation map, and the $\mathcal{U}$–small 0–homology group of A is trivial.

The reason why $\mathcal{U}$–small homology is referred to as homology at scale $\mathcal{U}$ is that it disregards any topological structure of X within each element of $\mathcal{U}$.

Given $a \in X$ and a q–simplex $\sigma = (x_0 \cdots x_q)$, let $\mathrm{cone}_a(\sigma)$ be the $(q+1)$–simplex $(a\, x_0 \cdots x_q)$. Extend linearly this cone operator to chains. The algebraic identity $\partial\, \mathrm{cone}_a(c) = c - \mathrm{cone}_a(\partial c)$ holds for every chain c.

Lemma 12. *Every $\mathcal{U}$–small cycle c supported in $U \in \mathcal{U}$ is a boundary.*

Proof. Take $a \in U$. By the previous identity, c is the boundary of the $\mathcal{U}$–small chain $\mathrm{cone}_a(c)$. □

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Suppose $\mathcal{V}$ is a refinement of $\mathcal{U}$. Any $\mathcal{V}$–small simplex or chain is automatically $\mathcal{U}$–small and the inclusion induces a map $\pi_{\mathcal{U}\mathcal{V}}\colon H_*^{\mathcal{V}}(X) \to H_*^{\mathcal{U}}(X)$ which simply views a $\mathcal{V}$–small cycle as a $\mathcal{U}$–small cycle. Using these morphisms, we can form an inverse system of groups $\{H_*^{\mathcal{U}}(X), \pi_{\mathcal{U}\mathcal{V}}\}$ indexed by the open covers of X. The inverse limit is equal to the Čech homology group of X, as will be deduced in the next subsection. Thus, there are projection morphisms $\pi_{\mathcal{U}}\colon \check{H}_*(X) \to H_*^{\mathcal{U}}(X)$ that satisfy the relation $\pi_{\mathcal{U}\mathcal{V}}\pi_{\mathcal{V}} = \pi_{\mathcal{U}}$ for every $\mathcal{U} \prec \mathcal{V}$.

The proof of the main theorem of the article needs a considerable amount of woodworking with Čech homology classes. Any element $\gamma \in \check{H}_q(X)$ is a family of $\mathcal{U}$–small q-homology classes $(\gamma_{\mathcal{U}})$ coherent in the sense that a $\mathcal{V}$–small representative $c_{\mathcal{V}}$ of $\gamma_{\mathcal{V}}$ is also a $\mathcal{U}$–small representative of $c_{\mathcal{U}}$ for any $\mathcal{U} \prec \mathcal{V}$ (by definition $\pi_{\mathcal{V}\mathcal{U}}(\gamma_{\mathcal{V}}) = \gamma_{\mathcal{U}}$). In the arguments of Subsection 6.4, covers will only be refined outside large compact subsets and the families of covers that we will consider at once ($\mathcal{U}_{\lambda\mu}$ with $\mathcal{U}$ fixed, for later reminder) coincide in a compact set ($K_0 \cap K_0^*$ in the future notation). Therefore, for all the open covers in the family we can choose representatives of γ that essentially coincide in the compact set. Let us formalize the idea in the next lemma.

Given a $\mathcal{U}$–small chain c, if we discard all the simplices of c that are not contained in A we are left with a chain supported on A that we denote $r_A(c)$. Denote by $\mathrm{st}(A,\mathcal{U})$ the union of all elements of $\mathcal{U}$ that meet A.

Lemma 13. *Let $\mathcal{U},\mathcal{V}$ be open covers of X, $\mathcal{V}$ be a refinement of $\mathcal{U}$. Suppose that A is a subset of X such that $\mathcal{V}$ and $\mathcal{U}$ coincide on $\mathrm{st}(A,\mathcal{U})$. Let $\alpha \in H_q^{\mathcal{V}}(X)$. Given a $\mathcal{U}$–small representative c of $\pi_{\mathcal{U}\mathcal{V}}(\alpha)$, there exists a $\mathcal{V}$–small representative c' of α such that $r_A(c) = r_A(c')$.*

Proof. Let c_0 be a $\mathcal{V}$–small representative of α, $[c_0] = \alpha$. Since c_0 and c are $\mathcal{U}$–homologous cycles, $c_0 - c = \partial d$ for some $\mathcal{U}$–small chain d. Decompose d as $d_0 + d_1$, where $d_0 = r_{\mathrm{st}(A,\mathcal{U})}(d)$, and define $c' = c + \partial d_1$. We deduce from the definition of d_0 that the support of d_1 is contained in $X - A$ and $r_A(c) = r_A(c')$. Finally, note that $c_0 - c' = \partial d_0$, so c_0 and c' are homologous as $\mathcal{V}$–small cycles. □

4.2. Vietoris homology

Given an open cover $\mathcal{U}$ of X, the Vietoris complex of $\mathcal{U}$, denoted $V(\mathcal{U})$ is the simplicial complex defined as follows. The vertices are the points of X and a collection $\{x_0, \dots, x_q\}$ of vertices spans a simplex in $V(\mathcal{U})$ provided there exists $U \in \mathcal{U}$ such that $x_0, \dots, x_q \in U$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, the Vietoris complex of $\mathcal{V}$ is evidently included in the Vietoris complex of $\mathcal{U}$, $V(\mathcal{V}) \to V(\mathcal{U})$.

The Vietoris complex $V(\mathcal{U})$ is related to the nerve of $\mathcal{U}$. In fact, both simplicial complexes are homotopically equivalent [1] and, in particular, have the same homology groups. The Čech homology group of X, $\check{H}_*(X)$, is defined as the inverse limit of the homology groups of the nerves of the open covers of X or, equivalently (see the beginning of Section 2), of the homology groups of the Vietoris complexes of the open covers of X.

As announced in the previous subsection, we show that the homology group of $V(\mathcal{U})$ is isomorphic to the $\mathcal{U}$–small homology group. Therefore, the Čech homology group of X can be also defined as the limit of the $\mathcal{U}$–small homology groups as $\mathcal{U}$ ranges over all open covers of X.

Send any $\mathcal{U}$–small formal simplex $(x_0 \dots x_q)$ to the same oriented simplex $(x_0, \dots, x_q)$ of $V(\mathcal{U})$ if all the vertices are different and to zero otherwise. This assignment defines a map between $\mathcal{U}$–small chains and simplicial chains on $V(\mathcal{U})$ that basically disregards the order (up to sign) of the collection of vertices. From this remark we conclude that the map induces isomorphisms between $H_*^{\mathcal{U}}(X)$ and $H_*(V(\mathcal{U}))$ (see [6] for more details).

Corollary 14. *$H_*^{\mathcal{U}}(X)$ is finitely generated provided $\mathcal{U}$ is finite.*

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Proof. It follows from the discussion above and the fact that the nerve of $\mathcal{U}$ has finitely many simplices. □

4.3. Dynamical coverings

Let $f\colon X \to X$ be a continuous map. We know that f induces a map in the Čech homology group of X, but we need to be more precise as we have to work with the system of $\mathcal{U}$–small homology groups of X.

Suppose that $\mathcal{U}, \mathcal{V}$ are open covers of X such that $\mathcal{U}$ refines $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$. Then f sends $\mathcal{U}$–small simplices to $\mathcal{V}$–small simplices and, consequently, it induces a map $f_*\colon H_*^{\mathcal{U}}(X) \to H_*^{\mathcal{V}}(X)$.

Clearly, for an arbitrary open cover $\mathcal{U}$ and a $\mathcal{U}$–small simplex $(x_0 \ldots x_q)$, the image of the vertex set $\{f(x_0) \ldots f(x_q)\}$ may not be, in general, contained in a single element of $\mathcal{U}$. Consequently, f does not define a map $H_*^{\mathcal{U}}(X) \to H_*^{\mathcal{U}}(X)$. However, the obstruction disappears in a familiar dynamical setting.

Definition 15. An open cover $\mathcal{U}$ of X is called *dynamical* if $f^{-1}\mathcal{U} \prec \mathcal{U}$, that is, for every $U \in \mathcal{U}$, $f(U) \subset U'$ for some $U' \in \mathcal{U}$.

In view of the previous discussion, if $\mathcal{U}$ is dynamical, $f_*\colon H_*^{\mathcal{U}}(X) \to H_*^{\mathcal{U}}(X)$ is well-defined. The downside is evident, dynamical open covers are scarce, they exist only in very simple dynamical situations. For example, dynamical open covers are a cofinal subfamily of the family of open covers of the basin of attraction of a fixed point.

Lemma 16. *Let $g\colon Y \to Y$ and $a \in Y$ a globally attracting fixed point for g. Every open cover $\mathcal{U}$ of Y can be refined to a dynamical open cover $\mathcal{V}$.*

Proof. Let $V_a \in \mathcal{U}$ be a positively invariant neighborhood of a. Since a is a global attractor, $Y = \cup_{n\geq 0} g^{-n}(V_a)$. Set $\mathcal{V}_0 = \{V_a\}$ and define inductively $\mathcal{V}_{n+1}$ by adding to $\mathcal{V}_n$ all the nonempty open sets of the form $g^{-1}(V) \cap U$, where $V \in \mathcal{V}_n$ and $U \in \mathcal{U}$. Note that $\mathcal{V}_n$ is a dynamical open cover of $\cup_{k=0}^{n} g^{-k}(V_a)$ that refines the restriction of $\mathcal{U}$. Define $\mathcal{V}$ as the union of all $\mathcal{V}_n$. □

4.4. Čech homology of the basin of attraction

Assume now that X is a compact metric space, $f\colon X \to X$ is a homeomorphism and $\{a, a^*\}$ is an attractor-repeller decomposition of X. Then, $B = X - \{a^*\}$ is the basin of attraction of a and $B^* = X - \{a\}$ is the basin of repulsion of a^*. Further, we assume that $\check{H}_1(X)$ is finite dimensional to apply the following lemma.

Lemma 17. *Let $\{E_i, \pi_{ij}\}$ be an inverse system of vector spaces indexed by a partially ordered set I that possesses a cofinal sequence. Suppose that for every $i \in I$ the image of the bonding map $\pi_{ji}\colon E_j \to E_i$ is finite dimensional if j is large enough and $E = \varprojlim E_i$ has a finite dimension as well.*

- *(i) There exists $i_0 \in I$ such that the projection $\pi_j\colon E \to E_i$ is injective for every $i \geq i_0$.*
- *(ii) For every $j \in I$, there exists $i_j \in I, i_j \geq j$ such that the image of the bonding map $\pi_{ji}\colon E_i \to E_j$ is equal to $\operatorname{im} \pi_j$ for all $i \geq i_j$ (in the language of shape theory [10], i_j is the movability index of j).*

Proof. (i) The kernels $\ker_i$ of the projections $E \to E_i$ are finite dimensional and satisfy $\ker_{i'} \subset \ker_i$ for $i' \geq i$ and $\cap_i \ker_i = \{0\}$. Therefore, $\ker_{i_0} = \{0\}$ for some $i_0 \in I$ and the conclusion follows.

(ii) Being nested and eventually finite dimensional, the subspaces $\operatorname{im} \pi_{ji}$ stabilize, $\operatorname{im} \pi_{ji} = \operatorname{im} \pi_{ji_0}$ for every $i \geq i_0$. Similarly, we can find i_1 such that $\operatorname{im} \pi_{i_0 i} = \operatorname{im} \pi_{i_0 i_1}$ for every $i \geq i_1$. Inductively, we construct a cofinal sequence of indices $(i_n)_{n\geq 0}$. Let $v \in \operatorname{im} \pi_j$ and define $w_1 \in E_{i_1}$ such that $\pi_{ji_1} w_1 = v$. Put

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$v_0 = \pi_{i_0 i_1} w_1$ and take $w_2 \in E_{i_2}$ such that $\pi_{i_0 i_2} w_2 = v_0$. Then, put $v_1 = \pi_{i_1 i_2} w_2$ and continue the same procedure. Inductively, we find an element of E (defined in a cofinal subset of indices) that projects into v. □

We can apply the previous lemma to the inverse system composed of the $\mathcal{U}$–small homology groups of X and the bonding maps $\pi_{\mathcal{U}\mathcal{V}}$. The property on the bonding maps is guaranteed by the last remark in Subsection 4.2 and the fact that every open cover has a finite refinement by the compactness of X. Alternatively, one could use the arguments in [6, p. 10]. By (i), the projections $\pi_{\mathcal{U}} \colon \check{H}_*(X) \to H_*^{\mathcal{U}}(X)$ become injective if the cover $\mathcal{U}$ is fine enough, i.e.

($\star$) $\operatorname{im} \pi_{\mathcal{U}}$ is isomorphic to $\check{H}_1(X)$ if $\mathcal{U}$ is finer than a fixed cover $\mathcal{W}$.

If $\mathcal{V} \succ \mathcal{W}$ and $\mathcal{V}' \succ f^{-1}\mathcal{V}$ then $f_* \colon H_*^{\mathcal{V}'}(X) \to H_*^{\mathcal{V}}(X)$ restricts to an isomorphism between $\operatorname{im} \pi_{\mathcal{V}'}$ and $\operatorname{im} \pi_{\mathcal{V}}$. The inverse is given by the composition of the isomorphism $\operatorname{im} \pi_{\mathcal{V}} \to \operatorname{im} \pi_{\mathcal{V}''}$ and the restriction of $(f^{-1})_* \colon H_*^{\mathcal{V}''}(X) \to H_*^{\mathcal{V}'}(X)$, for some $\mathcal{V}''$ finer than $f\mathcal{V}'$ (recall that f is a homeomorphism and, in particular, it is open). We conclude that $f_* \colon \check{H}_*(X) \to \check{H}_*(X)$ is conjugate to $f_* \colon \operatorname{im} \pi_{\mathcal{V}'} \to \operatorname{im} \pi_{\mathcal{V}}$.

The next lemma sheds some light on the relation between the topologies of $B = X - \{a^*\}$ and X. Related results in cohomological terms were obtained in [8,14]. It only deals with compact subsets of B, a more general result is beyond the scope of this note.

Lemma 18. *Suppose $\mathcal{W}$ is an open cover of X defined by ($\star$). Let $\mathcal{B}$ be a dynamical open cover of B that refines $\mathcal{W}|_B = \{U \cap B : U \in \mathcal{W}\}$ and Z be a compact neighborhood of a in B. Then, the image of the map induced by inclusion*

$$H_*^{\mathcal{B}}(Z) \to H_*^{\mathcal{W}}(X)$$

does not contain any nontrivial element of $\operatorname{im} \pi_{\mathcal{W}}$.

Proof. Since the map factors through $H_*^{\mathcal{B}}(Z')$ for any $Z \subset Z' \subset X$, we can suppose that Z is positively invariant under f. Then f induces a map f_* in $H_*^{\mathcal{B}}(Z)$ that sends $\gamma = [c]$ to $f_*\gamma = [f_\# c]$, where $f_\# c$ is also a $\mathcal{B}$–small cycle supported on Z because $\mathcal{B}$ is dynamical, $f^{-1}\mathcal{B} \prec \mathcal{B}$, and $f(Z) \subset Z$. Since a is a global attractor in B and Z is compact, for a fixed neighborhood $V_a \in \mathcal{B}$ of a, there exists m such that $f^m(Z) \subset V_a$. Therefore, $f_*^m\gamma = [f_\#^m c]$ is trivial in $H_*^{\mathcal{B}}(Z)$ by Lemma 12 because the support of $f_\#^m c$ is contained in V_a.

Since $\mathcal{B}$ is dynamical, $\mathcal{B}$ also refines the restriction to B of $\mathcal{W}_m = \mathcal{W} \vee f^{-1}\mathcal{W} \vee \cdots \vee f^{-m}\mathcal{W}$. There is a commutative diagram where the horizontal arrows are induced by the inclusion:

$$\begin{tikzcd} H_*^{\mathcal{B}}(Z) \arrow[r] \arrow[d, "f_*^m"] & H_*^{\mathcal{W}_m}(X) \arrow[d, "f_*^m"] \\ H_*^{\mathcal{B}}(Z) \arrow[r] & H_*^{\mathcal{W}}(X) \end{tikzcd}$$

Now, since the left vertical arrow is the zero morphism and the right vertical arrow restricts to an isomorphism between $\operatorname{im} \pi_{\mathcal{W}_m}$ and $\operatorname{im} \pi_{\mathcal{W}}$ by the previous remark, the conclusion follows. □

4.5. Čech homology is dual to Čech cohomology

As throughout this note we make several claims concerning the dimensions of Čech homology and cohomology groups, let us briefly address the duality in Čech theory. In contrast to singular theory, where cohomology is obtained dualizing the construction of homology, in Čech theory the relationship between

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homology and cohomology is not immediate. It turns out that Čech homology is dual to Čech cohomology, as the following algebraic argument shows.

Since $\varprojlim \mathrm{Hom}(E_i, \mathbb{K}) \cong \mathrm{Hom}(\varinjlim E_i, \mathbb{K})$ we deduce that

$$\check{H}_*(X) = \varprojlim H_*^{\mathcal{U}}(X) \cong \varprojlim \mathrm{Hom}(H^*_{\mathcal{U}}(X), \mathbb{K}) \cong \mathrm{Hom}(\varinjlim H^*_{\mathcal{U}}(X), \mathbb{K}) = \mathrm{Hom}(\check{H}^*(X), \mathbb{K})$$

5. Integral

In this section we review the integration of Alexander–Spanier cohomology classes over Čech homology classes introduced in [6]. Since the pairing happens actually at any fixed scale $\mathcal{U}$, for the moment we will assume that $\mathcal{U}$ is fixed. In the last part of the section we will define the integration in the limits.

5.1. At scale $\mathcal{U}$

Let $\xi\colon X^{q+1} \to \mathbb{K}$ be a $\mathcal{U}$–small q-cochain and let $\sigma = (x_0 \dots x_q)$ be a formal $\mathcal{U}$–small q–simplex. We can evaluate ξ over σ in the obvious way, $\xi(\sigma) := \xi(x_0, \dots, x_q)$. We can extend this definition linearly to $\mathcal{U}$–small q–chains, $\xi(\sum a_i\sigma_i) = \sum a_i\xi(\sigma_i)$, and ξ becomes a linear functional on $C_q^{\mathcal{U}}(X)$. An straightforward computation yields:

Lemma 19 *(Stokes' formula). For every $\mathcal{U}$–small q–cochain ξ and q–chain c:* $\xi(\partial c) = \delta\xi(c)$.

Stokes' formula is the key to pass from chains and cochains to homology and cohomology classes. Indeed, if ξ is a q–cocycle and c, c' homologous q–chains, $c - c'$ is a boundary and $\xi(c) - \xi(c') = \xi(\partial d) = \delta\xi(d) = 0$. In other words, the evaluation of ξ over c does not depend on c but on the ($\mathcal{U}$–small) homology class of c. Similarly, if ξ, ξ' are cohomologous and c is a q–cycle, $\xi(c) - \xi'(c) = \delta\eta(c) = \eta(\partial c) = 0$. Thus, the evaluation of ξ over c does not depend on ξ but on the ($\mathcal{U}$–small) cohomology class of ξ.

Definition 20. The integral of $z \in H^q_{\mathcal{U}}(X)$ over $\gamma \in H_q^{\mathcal{U}}(X)$ is defined as

$$\int\limits_{\gamma} z = \xi(c)$$

where ξ is a $\mathcal{U}$–small q–cocycle that represents z and c is a $\mathcal{U}$–small q–cycle representative of γ.

The above discussion proves that the integral is well–defined, it does not depend on the choices of ξ and c. The linear structure of the cohomology and homology groups and the linear extension of the evaluation implies that the integral defines a bilinear pairing $H^q_{\mathcal{U}}(X) \times H_q^{\mathcal{U}}(X) \to \mathbb{K}$.

5.2. Mayer-Vietoris

In Section 6, we shall need a version of the Mayer–Vietoris exact sequence for homology and cohomology at scale $\mathcal{U}$ to exploit a nice feature of the integral and the connecting homomorphisms. The precise definition of the connecting homomorphisms can be deduced directly from the chain complexes using the standard arguments. Here, we give a quick description.

Suppose A and B are subsets of X such that $X = \operatorname{int} A \cup \operatorname{int} B$ and $\mathcal{U}$ refines $\{\operatorname{int} A, \operatorname{int} B\}$. Then

$$\cdots \to H^{\mathcal{U}}_{q+1}(A) \oplus H^{\mathcal{U}}_{q+1}(B) \to H^{\mathcal{U}}_{q+1}(X) \xrightarrow{\Delta_*} H^{\mathcal{U}}_q(A \cap B) \to H^{\mathcal{U}}_q(A) \oplus H^{\mathcal{U}}_q(B) \to \cdots$$

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is exact. The connecting homomorphism Δ_* is defined as follows. Let c be a $\mathcal{U}$–small representative of $\alpha \in H^{\mathcal{U}}_{q+1}(X)$. It can be expressed as the sum of two $\mathcal{U}$–small chains $c_A + c_B$, where c_A is supported on A and c_B is supported on B. Then, $\Delta_*\alpha$ is the q–homology class represented by the $\mathcal{U}$–small cycle $\partial c_A = -\partial c_B$ on $A \cap B$. As usual, the choices of representative and decomposition do not affect the homology class obtained.

A similar exact sequence is obtained for cohomology at scale $\mathcal{U}$:

$$\cdots \to H^q_{\mathcal{U}}(A) \oplus H^q_{\mathcal{U}}(B) \to H^q_{\mathcal{U}}(A \cap B) \xrightarrow{\Delta^*} H^{q+1}_{\mathcal{U}}(X) \to H^{q+1}_{\mathcal{U}}(A) \oplus H^{q+1}_{\mathcal{U}}(B) \to \cdots$$

Here, the connecting homomorphism $\Delta^* \colon H^q_{\mathcal{U}}(A \cap B) \to H^{q+1}_{\mathcal{U}}(X)$ is defined as follows. Let ξ be a q–cocycle on $A \cap B$ that represents $z \in H^q_{\mathcal{U}}(A \cap B)$. Extend ξ to a q–cochain on X of the form $\xi_A - \xi_B$, where ξ_A is a cochain supported on A and ξ_B is a cochain supported on B such that $\xi = \xi_A - \xi_B$ on $A \cap B$. Then, define $\Delta^* z$ as the $(q+1)$–cohomology class on X represented by the cocycle $\eta = \delta\xi_A$ on A and $\eta = \delta\xi_B$ on B. Again, the choices in the definition do not change the outcome.

The integral behaves well with Δ_* and Δ^*. Suppose $z = [\xi] \in H^q_{\mathcal{U}}(A \cap B)$ and $\gamma = [c] \in H^{\mathcal{U}}_{q+1}(X)$. Then $\Delta^* z$ is a $\mathcal{U}$–small $(q+1)$–cohomology class in X represented by η (defined as $\delta\xi_A$ in A and as $\delta\xi_B$ in B) and $\Delta_*\gamma$ is a $\mathcal{U}$–small q–homology class in $A \cap B$ represented by $\partial c_A = -\partial c_B$.

On one hand, the integral of $\Delta^* z$ over γ is, by definition, equal to $\eta(c) = \delta\xi_A(c_A) + \delta\xi_B(c_B)$. On the other hand, the integral of z over $\Delta_*\gamma$ is equal to $\xi(\partial c_A) = \xi_A(\partial c_A) + \xi_B(\partial c_B)$. By Stokes' formula we conclude that:

Proposition 21. $\displaystyle\int_\gamma \Delta^* z = \int_{\Delta_* \gamma} z.$

5.3. Some examples

We provide an example in which the explicit computation of the integral is used to describe the cohomology class. Let $X = \mathbb{R}/\mathbb{Z}$ be the circle, $\mathcal{U}$ be the cover composed of the open intervals of diameter 2ϵ for some $0 \le \epsilon \le 1/4$ and $A = [-\overline{\epsilon}, \overline{1/2 + \epsilon}]$, $B = [\overline{1/2 - \epsilon}, \overline{\epsilon}]$ are two arcs that cover X (we use the bar to denote points on the quotient). Note that any element of $\mathcal{U}$ is contained either in A or in B.

The characteristic function χ on $[\overline{1/2 - \epsilon}, \overline{1/2 + \epsilon}]$ is $\mathcal{U}$–locally constant on $A \cap B$ and defines a $\mathcal{U}$–small 0-cohomology class $[\chi]$. Then, $\Delta^*[\chi]$ is a $\mathcal{U}$–small 1-cohomology class on X that roughly counts (with orientation) how many times a path goes around the circle. Let us check the intuition in an example.

For a given integer $k \ge 1$, let (x_i) be a strictly increasing sequence of reals such that $|x_{i+1} - x_i| < \epsilon$, $x_0 = 0$ and $x_n = k$. Then, $c_k = \sum_{i=0}^{n-1} (\overline{x}_i\, \overline{x}_{i+1})$ is a $\mathcal{U}$–small cycle in X that goes around the circle k times. By Proposition 21, $\int_{[c_k]} \Delta^*[\chi] = \int_{\Delta_*[c_k]} [\chi] = \chi(\partial c_k^A)$, where c_k^A is the sum of 1-simplices $(\overline{x}_i, \overline{x}_{i+1})$ whose vertices lie in A. Since each piece of path in c_k^A starts ϵ–close to $\overline{0}$ (in particular, inside $\chi^{-1}(\overline{0})$) and finishes ϵ–close to $\overline{1/2}$ (inside $\chi^{-1}(\overline{1})$), the term $\chi(\partial c_k^A)$ counts exactly how many times the sequence (x_i) crosses the intervals $[1/2 - \epsilon, 1/2 + \epsilon] + \mathbb{Z}$ and is equal to k, as announced.

As in the previous example, in this article we go back and forth between the zero and first (co)homology groups. Suppose now that A is a $\mathcal{U}$–component of X, as in Example 11. The augmentation map $\epsilon \colon C_0^{\mathcal{U}}(A) \to \mathbb{K}$ can be identified with the linear extension of the characteristic function on A, which we also denote χ_A. With the terminology we recently introduced, we have:

Lemma 22. *A $\mathcal{U}$–small 0–chain d in A is a cycle if and only if $\chi_A(d) = 0$. More generally, suppose Y is $\mathcal{U}$–clopen. A $\mathcal{U}$–small 0–chain d in Y is a cycle that represents the trivial class in $H_0^{\mathcal{U}}(Y)$ if and only if the evaluation of d against χ_A vanishes for every $\mathcal{U}$–component A in the support of d.*

Proof. We go directly to the proof of the second statement. Decompose $d = \sum_i d_i$, where d_i is a 0–chain in a $\mathcal{U}$–component A_i. Clearly, d is a $\mathcal{U}$–small cycle if and only if so are d_i for every i. By the first statement, the latter is equivalent to $\chi_{A_i}(d_i) = 0$ and the result follows from the fact that $\chi_{A_i}(d) = \sum_j \chi_{A_i}(d_j) = \chi_{A_i}(d_i)$. $\square$

Note that in the case d is known to be a 0–cycle in Y, $\chi_Y(d) = 0$ and it is enough to check that $\chi_A(d)$ vanishes for all $\mathcal{U}$–components except for one.

5.4. At various scales

The scale at which the elements of the integral are defined is not very relevant. Let $\mathcal{U} \prec \mathcal{V}$, $z = [\xi] \in H^q_{\mathcal{U}}(X)$ and $\gamma = [c] \in H^{\mathcal{V}}_q(X)$. We can interpret $\int_\gamma z$ in two equivalent ways. First, we can see z as a $\mathcal{V}$–small cohomology class, represented by the same q–cocycle, ξ, and integrate at scale $\mathcal{V}$. Alternatively, we can think of γ as a $\mathcal{U}$–small homology class whose representative is still c and integrate at scale $\mathcal{U}$. Evidently, both interpretations return the same result, $\xi(c)$.

The previous remark paves the way to the definition of the integral in the limits: the integral of an Alexander–Spanier cohomology class over a Čech homology class. Let $z \in \check{H}^q(X)$, ξ a q–cocycle that represents z and $\mathcal{U}$ an open cover for which $\delta\xi$ is $\mathcal{U}$–locally zero. Let $\gamma \in \check{H}_q(X)$. Since γ is an element of an inverse limit, it is a collection $\gamma = (\gamma_{\mathcal{V}})_{\mathcal{V}}$, where $\gamma_{\mathcal{V}} = \pi_{\mathcal{V}}(\gamma)$ is a $\mathcal{V}$–small homology class of X. Then, define

$$\int_\gamma z = \xi(c_{\mathcal{U}}),$$

where $c_{\mathcal{U}}$ is a $\mathcal{U}$–small representative of γ. As before, it is easy to check that the integral is well–defined. It is important to notice that, in fact, in the definition we compute $\int_{\gamma_{\mathcal{U}}} z$, an integral at scale $\mathcal{U}$. The scale is imposed by the cocycle condition of ξ and, more crucially, is the scale at which the cohomology class z first appears in the direct limit that defines $\check{H}^q(X)$.

6. Proof of the theorem

Let us recall the hypotheses formulated at the end of Section 3. Assume X is a compact metric space, $f\colon X \to X$ is a homeomorphism and X has an attractor–repeller decomposition $\{a, a^*\}$, where a is an attracting fixed point and a^* is a repelling fixed point. $B = X - \{a^*\}$ and $B^* = X - \{a\}$. Moreover, we assume that $\check{H}_0(X)$ and $\check{H}_1(X)$ are finite dimensional. Recall that the quasicomponents of $X - \{a, a^*\} = B \cap B^*$ that are adherent to a and a^* are called essential.

Theorem 23. *The number of essential quasicomponents of $B \cap B^*$ is equal to* $\dim \check{H}_1(X) + 1$.

As a corollary, when $X = W_\infty/\widetilde{K}$ for an isolated invariant set K we obtain Theorem 1. Recall (Proposition 10) that if we consider the subspace topology instead of the intrinsic topology in $W^u(K)$ or if the original dynamics was not invertible we only get the upper bound of $\dim \check{H}_1(X) + 1$ for the number of essential quasicomponents.

The proof of Theorem 23 is organized as follows. First, we introduce an adapted family of covers of $B \cap B^*$ and X and describe the connecting homomorphisms from the Mayer–Vietoris exact sequences associated to these covers from dimensions 0 to 1. Then, we prove the upper bound for the number of essential quasicomponents in Proposition 24 and we finish the proof with Proposition 25.

A small notational remark useful in the future: henceforth the letters $\mathcal{U}, \mathcal{V}$ will be used exclusively to denote covers of $B \cap B^*$ and covers of X will carry two subindices as explained below.

6.1. Choice of a suitable family of open covers

We now define a family of open covers $\mathcal{U}$ of $B\cap B^*$. Let $\mathcal{V}$ be an open cover of $B\cap B^*$ and O_e a $\mathcal{V}$–clopen neighborhood of the essential quasicomponents. We can apply Lemma 8 and Corollary 9 to obtain open neighborhoods V_0, V_0^* of a, a^*, respectively, such that $\overline{V}_0\cap\overline{V}_0^* = \emptyset$, $V_0\backslash O_e$ does not meet any quasicomponent adherent to a^* and $V_0^*\setminus O_e$ does not meet any quasicomponent adherent to a. Define an open cover $\mathcal{U}$ of $B\cap B^*$ that refines $\mathcal{V}$ in the following fashion. The restriction of $\mathcal{U}$ to O_e is the family of sets of the form $V\cap(X-\overline{V}_0)$ or $V\cap(X-\overline{V}_0^*)$, where $V\in\mathcal{V}$ and $V\subset O_e$, while in $O_a\cup O_{a^*}$ the only additional condition for $\mathcal{U}$ is that O_a and O_{a^*} become $\mathcal{U}$–clopen. Note that no element of $\mathcal{U}$ meets both $\overline{V}_0$ and $\overline{V}_0^*$, a requirement for the Mayer–Vietoris exact sequences. The family of open covers $\mathcal{U}$ of $B\cap B^*$ obtained in this fashion is denoted $Cov(B\cap B^*)$.

Clearly, $Cov(B\cap B^*)$ is cofinal among the family of open covers of $B\cap B^*$, in the sense that any open cover of $B\cap B^*$ has a refinement in $Cov(B\cap B^*)$. Let us state as a remark that if $\mathcal{U}$ is additionally required to be finer than some given open cover of $B\cap B^*$ we obtain a subfamily of $Cov(B\cap B^*)$ that is still cofinal.

Given any $\mathcal{U}\in Cov(B\cap B^*)$, let $\{V_\lambda\}$ and $\{V_\mu^*\}$ be bases of open neighborhoods of a and a^* that are contained in V_0 and V_0^*, respectively. The indices are (partially) ordered as the element of the basis: $\lambda'\geq\lambda$ iff $V_{\lambda'}\subset V_\lambda$, with 0 being the minimal element, and similarly for μ. Define

$$\mathcal{U}_{\lambda\mu} = \mathcal{U}\cup\{V_\lambda, V_\mu^*\}$$

and denote $Cov(X)$ the set of covers of X obtained in this way. Again, $Cov(X)$ is cofinal among the family of open covers of X and if we further require $\mathcal{U}$ to be finer than some given open cover it will still be cofinal.

Let us emphasize that the form of $\mathcal{U}$, and that of $\mathcal{U}_{\lambda\mu}$, depends on the choice of the $\mathcal{V}$–clopen set O_e. This is not reflected in the notation directly, but note that a different O_e leads to different neighborhoods V_0 and V_0^*. There is a somehow canonical choice of O_e, the union of $\mathcal{V}$–components that contain essential quasicomponents. However, in principle, until we prove Proposition 24, there could be infinitely many of them and that set may not be clopen.

From Lemma 8 it follows that for every $\mathcal{U}_{\lambda\mu}\in Cov(X)$

$$B = (O_e\cup O_a\cup\{a\})\cup O_{a^*}, \qquad B^* = (O_e\cup O_{a^*}\cup\{a^*\})\cup O_a$$

are separations of B and B^* in two $\mathcal{U}_{\lambda\mu}$–clopen sets (of B and B^*, respectively). In order to be more precise, let us restrict the first partition to $K_0 = X - V_0^*\subset B$. Since O_{a^*} is $\mathcal{U}$–clopen and $O_{a^*}\cap V_\lambda = \emptyset$ for every λ, we deduce that $O_{a^*}\cap K_0$ is $\mathcal{U}_{\lambda\mu}|_{K_0}$–clopen for every λ,μ. So, we have the following separation of K_0 in $\mathcal{U}_{\lambda\mu}|_{K_0}$–clopen sets:

$$K_0 = ((O_e\cup O_a\cup V_\lambda)\cap K_0)\cup(O_{a^*}\cap K_0) \tag{2}$$

Similarly, we obtain a separation of $K_0^* = X - V_0$ in $\mathcal{U}_{\lambda\mu}|_{K_0^*}$–clopen sets:

$$K_0^* = \big((O_e\cup O_{a^*}\cup V_\mu^*)\cap K_0^*\big)\cup(O_a\cap K_0^*) \tag{3}$$

6.2. An interpretation of the Mayer–Vietoris sequence

For any $\mathcal{U}_{\lambda\mu}\in Cov(X)$, denote $K_\lambda^* = X - V_\lambda$ and $K_\mu = X - V_\mu^*$. Consider the following fragments of the Mayer–Vietoris sequence for homology

$$\cdots\to H_1^{\mathcal{U}_{\lambda\mu}}(X)\xrightarrow{\Delta_*} H_0^{\mathcal{U}_{\lambda\mu}}(K_\lambda^*\cap K_\mu)\to H_0^{\mathcal{U}_{\lambda\mu}}(K_\lambda^*)\oplus H_0^{\mathcal{U}_{\lambda\mu}}(K_\mu)\to\cdots$$

and the Mayer–Vietoris sequence in cohomology

$$\cdots \leftarrow H^1_{\mathcal{U}_{\lambda\mu}}(X) \xleftarrow{\Delta^*} H^0_{\mathcal{U}_{\lambda\mu}}(K^*_\lambda \cap K_\mu) \leftarrow H^0_{\mathcal{U}_{\lambda\mu}}(K^*_\lambda) \oplus H^0_{\mathcal{U}_{\lambda\mu}}(K_\mu) \leftarrow \cdots$$

associated to $\{K^*_\lambda, K_\mu\}$.

Let us see in a very particular case how Δ^* works. Let $\mathcal{V} \in Cov(B \cap B^*)$ and A be a $\mathcal{V}$–clopen subset of $B \cap B^*$. The characteristic function on A, χ_A, is $\mathcal{V}$–locally constant on $B \cap B^*$ so defines 0–cohomology classes $[\chi_A]_{\mathcal{U}_{\lambda\mu}}$ in $K^*_\lambda \cap K_\mu$ for every $\mathcal{U}$ finer than $\mathcal{V}$. In view of the Mayer–Vietoris sequence above we obtain 1–cohomology classes $\Delta^*[\chi_A]_{\mathcal{U}_{\lambda\mu}}$ in $H^1_{\mathcal{U}_{\lambda\mu}}(X)$. In the following paragraph we show that all these classes are essentially the same, in the sense that they are represented by the same 1–cocycle and, consequently, define the same class in $\check{H}^1(X)$.

A $\mathcal{U}_{\lambda\mu}$–small 1–cocycle that represents $\Delta^*[\chi_A]_{\mathcal{U}_{\lambda\mu}}$ can be obtained in the following way. Take a 0–cocycle in K^*_λ, say the restriction of χ_A to K^*_λ, and a 0–cocycle in K_μ, say the zero function, whose difference is equal to the restriction of χ_A to $K^*_\lambda \cap K_\mu$. Define $\xi^{\mathcal{U}_{\lambda\mu}}$ as $\delta\chi_A$ in $K^*_\lambda \times K^*_\lambda$ and set $\xi^{\mathcal{U}_{\lambda\mu}} = 0$ elsewhere. Then, $\xi^{\mathcal{U}_{\lambda\mu}}$ is a $\mathcal{U}_{\lambda\mu}$–small cocycle that represents $\Delta^*[\chi_A]_{\mathcal{U}_{\lambda\mu}}$.

Note that if x, x' are $\mathcal{U}_{\lambda\mu}$–close, $\xi^{\mathcal{U}_{\lambda\mu}}$ vanishes on (x, x') unless $x \in A$ and $x' \notin A$ or viceversa. Since A is $\mathcal{V}$–clopen and $\mathcal{U}$ refines $\mathcal{V}$, $\mathcal{U}_{\lambda\mu}$–close points that are separated by A must belong to V_λ or V^*_μ. Thus, $\xi^{\mathcal{U}_{\lambda\mu}}$ is nonzero only if $x, x' \in V^*_\mu$ and exactly one of x and x' belongs to A. Note that this description remains valid if we start with $\mathcal{U}' \succ \mathcal{U}$ as long as V^*_μ does not change. Therefore, $\xi^{\mathcal{U}_{\lambda\mu}}$ is the restriction to $\mathcal{U}_{\lambda\mu}$–close pairs of the 1–cocycle ξ defined as follows: $\xi(x, x') = 1$ if $x \in V_0 - A, x' \in V_0 \cap A$, $\xi(x, x') = -1$ if $x \in V_0 \cap A, x' \in V_0 - A$ and $\xi(x, x') = 0$ otherwise. The cocycle ξ is $\mathcal{V}_{00}$–small and may be used to compute the integral of $\Delta^*[\chi_A]$ at any finer scale $\mathcal{U}_{\lambda\mu}$.

6.3. Upper bound on the number of essential quasicomponents

Proposition 24. *The number of essential quasicomponents of $B \cap B^*$ does not exceed* $\dim \check{H}_1(X) + 1$.

Proof. From n essential quasicomponents $F_1, \ldots, F_n$ of $B \cap B^*$ we will construct $n - 1$ linearly independent homology classes in $\check{H}_1(X)$.

Consider O_j pairwise disjoint clopen neighborhoods of F_j (recall Lemma 4) and let $\mathcal{F}$ be the open cover of $B \cap B^*$ composed of $O_1, O_2, \ldots, O_n$ and $B \cap B^* - \cup O_j$. Take a refinement $\mathcal{V}$ of $\mathcal{F}$ in $Cov(B \cap B^*)$.

Denote by χ_j the characteristic function on O_j. By the arguments of the preceding Subsection 6.2, $\Delta^*[\chi_j]$ is a well–defined element of $H^1_{\mathcal{U}_{\lambda\mu}}(X)$ for every $\mathcal{U} \succ \mathcal{V}$ and any λ, μ, in particular for $\mathcal{V}_{00} = \mathcal{V} \cup \{V_0, V^*_0\}$.

By Lemma 17, the image of $\pi_{\mathcal{V}_{00}} \colon \check{H}_1(X) \to H^{\mathcal{V}_{00}}_1(X)$ is equal to the image of $H^{\mathcal{U}_{\lambda\mu}}_1(X) \to H^{\mathcal{V}_{00}}_1(X)$ for some $\mathcal{U}_{\lambda\mu} \in Cov(X)$ finer than $\mathcal{V}_{00}$. Suppose without loss of generality that $\mathcal{U}$ refines $\mathcal{V}$. Since O_i is adherent to a and a^* and $O_i \cap K^*_\lambda \cap K_\mu$ is $\mathcal{U}$–clopen ($\mathcal{U}|_{K^*_\lambda \cap K_\mu}$–clopen, to be precise) we can find points $x_i \in O_i$ such that there is a $\mathcal{U}_{\lambda\mu}$–small point path inside K^*_λ from x_i to a^* and a $\mathcal{U}_{\lambda\mu}$–small point path inside K_μ from x_i to a.

The existence of the previous point paths guarantee that the 0–cycles $d_i = (x_i) - (x_n)$ in $K^*_\lambda \cap K_\mu$ represent the trivial homology class at scale $\mathcal{U}_{\lambda\mu}$ as 0–cycles in K^*_λ and as 0–cycles in K_μ. In view of the Mayer–Vietoris exact sequence:

$$\ldots \to H^{\mathcal{U}_{\lambda\mu}}_1(X) \xrightarrow{\Delta_*} H^{\mathcal{U}_{\lambda\mu}}_0(K^*_\lambda \cap K_\mu) \to H^{\mathcal{U}_{\lambda\mu}}_0(K^*_\lambda) \oplus H^{\mathcal{U}_{\lambda\mu}}_0(K_\mu) \to \ldots$$

we deduce that $[d_i] \in H^{\mathcal{U}_{\lambda\mu}}_0(K^*_\lambda \cap K_\mu)$ belongs to $\mathrm{im}(\Delta_*)$, $[d_i] = \Delta_*[c_i]$ for some 1-cycle $c_i \in C^{\mathcal{U}_{\lambda\mu}}_1(X)$.

From the property that defines $\mathcal{U}_{\lambda\mu}$, we obtain classes $\alpha_i \in \check{H}_i(X)$ that at scale $\mathcal{V}_{00}$ are equal to $[c_i]$ (viewed as a $\mathcal{V}_{00}$–small homology class). By Proposition 21, we have that

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$$\int_{\alpha_i} \Delta^*[\chi_j] = \int_{[c_i]} \Delta^*[\chi_j] = \int_{[d_i]} [\chi_j] = \chi_j(x_i) - \chi_j(x_n) = \delta_{ij}$$

where the integrals have been computed at scale $\mathcal{V}_{00}$. Since the integral defines a bilinear pairing, we conclude that $\{\alpha_i : 1 \leq i \leq n-1\}$ are linearly independent Čech homology classes of X and the result follows. □

6.4. Computation of the number of essential quasicomponents

An immediate consequence of the previous subsection is that the number of essential quasicomponents is finite provided the first Čech homology group of X is finite dimensional. We take it as hypothesis and denote the essential quasicomponents by $F_1, \ldots, F_n$. Let $O_1, \ldots, O_n$ be pairwise disjoint clopen neighborhoods of $F_1, \ldots, F_n$ in $B \cap B^*$, $\mathcal{F} \in Cov(B \cap B^*)$ and the functions χ_j as in the proof of Proposition 24. We show that elements of $\check{H}_1(X)$ are characterized by the values of the integrals of the cohomology classes $\Delta^*[\chi_j]$ over them.

Proposition 25. *For any $\alpha \in \check{H}_1(X)$,*

$$\int_{\alpha} \Delta^*[\chi_j] = 0 \text{ for every } 1 \leq j \leq n-1 \text{ if and only if } \alpha = 0.$$

As a consequence, there cannot be more than $n-1$ linearly independent Čech 1-homology classes of X and $\dim \check{H}_1(X) \leq n-1$.

The rest of the subsection is devoted to the proof Proposition 25, which in turn ends the proof of Theorem 23. The ensuing arguments require further hypothesis on the covers considered. Recall the open covers constructed in Subsection 4.4:

- $\mathcal{W}$ is an open cover of X such that $\operatorname{im} \pi_{\mathcal{V}}$ is isomorphic to $\check{H}_1(X)$ for every $\mathcal{V} \succ \mathcal{W}$.
- $\mathcal{B}$ is a dynamical open cover of B that refines $\mathcal{W}|_B$.
- Further, let $\mathcal{B}^*$ be an open cover of B^* that refines $\mathcal{W}|_{B^*}$ and is dynamical for f^{-1}.

Denote by $Cov^*(B \cap B^*)$ the subfamily of open covers of $B \cap B^*$ that refine the restriction of $\mathcal{B}, \mathcal{B}^*$, and automatically also of $\mathcal{W}$, to $B \cap B^*$ and denote $Cov^*(X)$ the associated subfamily of open covers of X. Assume further that the marked neighborhoods V_0 and V_0^* are small so that every element of $Cov^*(X)$ refines $\mathcal{W}$. Clearly, $Cov^*(B \cap B^*)$ is cofinal among the family of all open covers of $B \cap B^*$.

Henceforth, fix an open cover $\mathcal{U} \in Cov^*(B \cap B^*)$ finer than $\mathcal{F}$. Suppose without loss of generality (in view of the finiteness of the number of essential quasicomponents and the comments in Subsection 6.1) that in the partition of $B \cap B^*$ in $\mathcal{U}$–clopen sets from Corollary 9

$$B \cap B^* = O_e \cup O_a \cup O_{a^*}, \tag{4}$$

associated to the definition of $\mathcal{U}$, the set O_e is equal to the union of the $\mathcal{U}$–components O'_j that contain the F_j.

Let $\alpha \in \check{H}_1(X)$. Let V_0, V_0^* be the neighborhoods of a, a^* defined from $\mathcal{U}$ and O_e and put $K_0 = X - V_0^*$ and $K_0^* = X - V_0$ as usual. By Lemma 13, α has $\mathcal{U}_{\lambda\mu}$–small representatives $c_{\lambda\mu}$ for all λ, μ that coincide when restricted to $K_0^* \cap K_0$. More formally, there exist $\mathcal{U}_{\lambda\mu}$–small 1–cycles $c_{\lambda\mu}$ such that $[c_{\lambda\mu}] = \pi_{\mathcal{U}_{\lambda\mu}} \alpha$ and $r_{K_0^* \cap K_0}(c_{\lambda\mu})$ is independent of λ, μ (recall that $r_{K_0 \cap K_0^*}$ discards the simplices that are not completely contained in $K_0 \cap K_0^*$). The cycles $c_{\lambda\mu}$ can be decomposed as $c_{\lambda\mu} = c_{\lambda\mu}^{K_0} + c_{\lambda\mu}^{K_0^*}$, where the cycles are supported in K_0 and K_0^*, respectively, in a way that $\partial c_{\lambda\mu}^{K_0}$ is independent of λ, μ. Denote $d = \partial c_{\lambda\mu}^{K_0} = -\partial c_{\lambda\mu}^{K_0^*}$. It follows that d is a $\mathcal{U}_{\lambda\mu}$–small representative of $\Delta_*[c_{\lambda\mu}] \in H_0^{\mathcal{U}_{\lambda\mu}}(K_0 \cap K_0^*)$ for all λ, μ.

Use (4) to decompose d as $d_e + d_a + d_{a^*}$. The next step is to show that d_a, d_{a^*} are boundaries of $\mathcal{U}_{\lambda\mu}$–small 1–chains for any λ, μ.

Denote $c_{00}^{K_0}$ simply as c^K. Following the separation (2) of K_0, the chain c^K splits as $c_a^K + c_{a^*}^K$. Taking boundaries preserves the decomposition, so $d_e + d_a = \partial c_a^K, d_{a^*} = \partial c_{a^*}^K$. Since $c_{a^*}^K$ is a chain supported on O_{a^*} and $O_{a^*} \cap V_0 = \emptyset$, we conclude that d_{a^*} is the boundary of a $\mathcal{U}_{\lambda\mu}$–small chain in $K_0 \cap K_0^*$. A similar argument involving c^{K^*} and (3) reaches the same conclusion for d_a.

Let us proceed to study d_e. It can be decomposed as $\sum d_i$, where each d_i is supported in O_i'. Recall from Subsection 6.2 that $\Delta^*[\chi_j]$ can be interpreted as a class in $H^1_{\mathcal{U}_{\lambda\mu}}(X)$ for every λ, μ. We can compute the integral of $\Delta^*[\chi_j]$ over α at scale $\mathcal{U}_{00}$:

$$\int_\alpha \Delta^*[\chi_j] = \int_{[c_{00}]} \Delta^*[\chi_j] = \int_{\Delta_*[c_{00}]} [\chi_j] = \int_{[d]} [\chi_j] = \int_{[d_e]} [\chi_j] = \chi_j(d_e) = \chi_{O_j'}(d_j) \tag{5}$$

By Lemma 22, $\chi_{O_j'}(d_j) = 0$ if and only if d_j is the boundary of a $\mathcal{U}$–small 1–chain in $O_j' \subset B \cap B'$ or, equivalently, d_j is the boundary of a $\mathcal{U}$–small 1–chain in $K_{\lambda_j}^* \cap K_{\mu_j}$ for some large λ_j, μ_j. Thus, the integrals in (5) vanish for all $j = 1, \dots, n$ if and only if there exist λ, μ such that d_e is the boundary of a $\mathcal{U}_{\lambda\mu}$–small 1–chain in $K_\lambda^* \cap K_\mu$. The latter condition can be rephrased as $[d_e]$ is the trivial element in $H_0^{\mathcal{U}_{\lambda\mu}}(K_\lambda^* \cap K_\mu)$.

Note that

$$\sum_j \chi_{O_j'}(d_j) = (\sum_j \chi_j)(d_e) = \int_{[d_e]} \sum_j [\chi_j] = \int_{[d_e]} [\chi_X] = 0$$

because χ_X, the characteristic function on X, coincides with $\sum \chi_j$ on O_e and is the constant map of value 1 on X, so defines the trivial class in $\check{H}^0(X)$. We conclude that if the integrals in (5) vanish for $n-1$ values of j then it vanishes also for the missing j (see the remark after Lemma 22).

Since d_a, d_{a^*} are boundaries, d_e is a $\mathcal{U}_{\lambda\mu}$–small representative of $\Delta_*[c_{\lambda\mu}]$ and we conclude that the following statements are equivalent:

(1) The integral of $\Delta^*[\chi_j]$ over α vanishes for all $j = 1, \dots, n-1$.
(2) $[c_{\lambda\mu}]$ belongs to the kernel of the connecting homomorphism $\Delta_*\colon H_1^{\mathcal{U}_{\lambda\mu}}(X) \to H_0^{\mathcal{U}_{\lambda\mu}}(K_\lambda^* \cap K_\mu^*)$ for some λ, μ.

Let us elaborate a bit more on the second equivalent condition, which we take as hypothesis. Suppose λ, μ fixed as in the statement of (2). By the construction of $Cov^*(X)$, $\mathcal{U}_{\lambda\mu}$ refines $\mathcal{W}$ from Subsection 4.4 and by Lemma 18, $H_1^{\mathcal{U}_{\lambda\mu}}(K_0) \oplus H_1^{\mathcal{U}_{\lambda\mu}}(K_0^*) \to H_1^{\mathcal{W}}(X)$ is the trivial morphism. However, this is equal to the composition of $i_*\colon H_1^{\mathcal{U}_{\lambda\mu}}(K_0) \oplus H_1^{\mathcal{U}_{\lambda\mu}}(K_0^*) \to H_1^{\mathcal{U}_{\lambda\mu}}(X)$ and $\pi_{\mathcal{W}\mathcal{U}_{\lambda\mu}}\colon H_1^{\mathcal{U}_{\lambda\mu}}(X) \to H_1^{\mathcal{W}}(X)$. Since $\pi_{\mathcal{W}\mathcal{U}_{\lambda\mu}}$ restricts to an isomorphism from $\operatorname{im} \pi_{\mathcal{U}_{\lambda\mu}}$ to $\operatorname{im} \pi_{\mathcal{W}}$, we conclude that $\operatorname{im} i_* = \ker\Delta_*$ has trivial intersection with $\operatorname{im} \pi_{\mathcal{U}_{\lambda\mu}}$. But $[c_{\lambda\mu}] \in \ker\Delta_* = \operatorname{im} i_*$ and $[c_{\lambda\mu}] = \pi_{\mathcal{U}_{\lambda\mu}}\alpha$ so we deduce that $[c_{\lambda\mu}]$ is the trivial $\mathcal{U}_{\lambda\mu}$–small homology class.

In sum, condition (1) is equivalent to $\pi_{\mathcal{U}_{\lambda\mu}}\alpha = 0$. This holds for any $\mathcal{U} \in Cov^*(B \cap B^*)$ finer than $\mathcal{F}$ and all λ, μ. Therefore, we conclude that $\alpha = 0 \in \check{H}^1(X)$ and the proof of Proposition 25 is finished.

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