UNIVERSIDAD COMPLUTENSE DE MADRID
FACULTAD DE CIENCIAS FÍSICAS

TESIS DOCTORAL

Gravitational Interactions in Dark Matter Models

Interacciones gravitacionales en modelos de materia oscura

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

José Manuel Sánchez Velázquez

Directores

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por

Jose Manuel Sánchez Velázquez

Bajo la supervisión de:

Jose Alberto Ruiz Cembranos
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Madrid, octubre de 2020

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Resumen

El modelo cosmológico estándar incluye un sector oscuro para poder explicar las observaciones que tenemos. Este sector incluye todos los constituyentes del universo no descritos por el modelo estándar de física de partículas, cuya presencia solo se infiere mediante interacciones gravitacionales. Actualmente, el modelo más aceptado incluye una constante cosmológica para explicar la expansión acelerada tardía del universo y materia oscura fría para explicar la dinámica del universo. Estas dos componentes, junto con los campos ya conocidos, constituyen el modelo ΛCDM, que ajusta con éxito un gran número de observaciones cosmológicas a diferentes escalas. Sin embargo, muy poco se conoce de las propiedades fundamentales de este sector oscuro. En particular, en esta tesis estudiaremos dos aspectos relacionados con la materia oscura:

1. *Construcción de modelos:* la falta de datos experimentales sobre las propiedades fundamentales de la materia oscura deja mucho margen para la especulación teórica sobre posibles modelos. En esta tesis exploramos cómo construir modelos para campos escalares y vectoriales que se acoplan consistentemente al tensor de energía-momento, en analogía con la teoría para el gravitón de Fierz-Pauli. La construcción de estos modelos parte de introducir un acoplo directo al tensor de energía-momento a nivel lineal en la acción del campo. Esta interacción produce correcciones en el tensor de energía-momento que deben ser tenidas en cuenta en la acción. De esta forma, cada nueva corrección de la acción introduce cambios en el tensor que deben ser incorporados. La suma de estas infinitas correcciones da lugar a una acción no lineal que obtenemos mediante la imposición de condiciones de consistencia entre el modelo lineal y la acción no lineal resultante. Este procedimiento convierte la suma de una serie infinita en la resolución de un sistema de ecuaciones diferenciales o incluso algebraicas, dependiendo del caso particular. Asimismo, hemos estudiado el impacto de posibles términos superpotenciales que provienen de las ambigüedades en la definición del tensor de energía-momento. Estos términos provocan interacciones de galileones para todos los campos estudiados. Estudiamos también la posibilidad de describir estos campos mediante funcionales generatrices en términos de una métrica efectiva,

debido a la motivación original de teorías de gravedad. Por otro lado, discutimos la fenomenología asociada a estos modelos y sus posibles señales experimentales.

2. *¿Cómo se puede haber producido?:* debido a lo débilmente que interacciona la materia oscura con los campos del modelo estándar de la física de partículas, una pregunta que surge de manera natural es cómo se puede haber producido la abundancia de materia oscura que observamos. Si la materia oscura alcanza el equilibrio térmico con los campos del modelo estándar, su abundancia se puede explicar como reliquia térmica para varios modelos. Sin embargo, cuando dicho equilibrio no se alcanza o la interacción es demasiado débil, otros mecanismos deben entrar en juego. En esta tesis investigamos si la producción gravitacional en las fases tempranas del universo es suficiente para dar lugar a la abundancia necesaria para explicar las observaciones y, en caso afirmativo, extraemos cotas sobre sus propiedades fundamentales exigiendo que no dé lugar a un exceso de producción. En particular, estudiamos la producción de partículas para un campo escalar acoplado a la curvatura de forma no mínima, teniendo en cuenta las oscilaciones de la curvatura escalar debidas a la dinámica del inflatón durante el recalentamiento. Estas oscilaciones inducen diferentes mecanismos que amplifican la producción de partículas con respecto a haber considerado el comportamiento promedio de la curvatura. Comparando la abundancia predicha por el modelo con las observaciones, vemos que la masa de la materia oscura está acotada, con un rango de masas intermedias prohibido. Por otro lado, también estudiamos el efecto de considerar acoplos derivativos del campo a la geometría de fondo en la producción de partículas. En este caso, consideramos el comportamiento promedio de la geometría. Vemos que la producción gravitacional depende apreciablemente del valor de las constantes de acoplo, incluso quedándonos en el régimen perturbativo, por lo que un estudio pormenorizado de estos acoplos tiene que llevarse a cabo para poder extraer conclusiones precisas sobre los parámetros fundamentales de la materia oscura a partir de la producción gravitacional.

Abstract

The cosmological standard model includes a dark sector in order to explain the current observations. This dark sector includes all the constituents of the universe that are not described by the standard model of particle physics, which presence is only inferred from gravitational interactions. The currently most accepted model introduces a cosmological constant to account for the late expansion of the universe and cold dark matter to explain the dynamics of the universe. These two components, along with the already known fields, constitute the ΛCDM model, which successfully fits different cosmological observations at different scales. However, there is a lot of uncertainty concerning the fundamental properties of this dark sector. In this thesis we will study two aspects related to dark matter:

1. *Model building:* there is a lot of room for theoretical speculation in model building due to the lack of experimental data on the fundamental properties of dark matter. In this thesis we explore how to build models for scalar and vector fields which couple to the energy-momentum tensor in a consistent way, in analogy to the Fierz-Pauli theory for the graviton. The starting point for building these models is to consider a direct coupling to the energy-momentum tensor at the linear level on the action for the field. This new interaction induces corrections in the energy-momentum tensor that need to be taken into account in the action. Hence, each new correction of the action introduces changes in the tensor that need to be incorporated. The sum of this infinite number of corrections gives raise to a non-linear action that we obtain by imposing consistency conditions between the linear model and non-linear resulting action. This procedure turns the problem of summing an infinite series into solving a set of differential or even algebraic equations, depending on the particular case under study. Furthermore, we have studied the importance of possible superpotential terms arising from the ambiguity in the definition of the energy-momentum tensor. These terms lead to the generation of Galileon interactions for all the considered fields. We have also studied the possibility of describing these fields by generating functionals defined in terms of an effective metric, due to the original motivation of gravity theories.

On the other hand, we discuss the phenomenology associated with these models and their possible experimental signatures.

2. *How has it been produced?:* a natural question that arises due how weakly dark matter interacts with the fields of the standard model of particle physics is how can it has been produced to yield the abundance we observe. If dark matter reaches thermal equilibrium with the standard model fields, its abundance can be explain as a thermal relic for some models. However, if this equilibrium is never attained or if the interaction is weak enough, different production mechanisms need to be taken into account. In this thesis we investigate if the gravitational production during the early phases of the universe is enough to obtain the necessary abundance to explain the observations, and, if this is the case, we extract constraints on its fundamental properties by imposing that this mechanism does not lead to overproduction. In particular, we study the particle production for a scalar field non-minimally coupled to the spacetime curvature, taking into account the oscillations of the scalar curvature due to the dynamics of the inflaton field during reheating. These oscillations induce different enhancement mechanisms for particle production as compared to the case with the averaged behavior of the curvature. Comparing the predicted abundance for the model with the observations, we find that the mass for dark matter is constrained, with an intermediate range of forbidden masses. On the other hand we also study the effect of introducing derivative couplings between the field and the background geometry. In this case, we consider the averaged behavior of the geometry. We conclude that the gravitational production depends appreciably with the value of the coupling constants, even if we work within the perturbative regime, so a detailed study of these couplings has to be carried in order to extract precise conclusions on the fundamental parameters of dark matter based on gravitational production.

Contents

1 | Introduction

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way—in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.

– Charles Dickens, *A Tale of Two Cities*

In this thesis I have studied semiclassical effects in gravity and its consequences in cosmology. During these last five years I have dealt with phenomenology associated to quantum field theory in curved spacetimes in order to explain the abundance of dark matter in the Universe and I have used the results to set constraints on its possible fundamental parameters like its mass and its couplings to gravity. Moreover, I have also been interested in model building for dark matter models. In particular we have proposed new models for both scalar and vector candidates based upon the construction of the gravitational field theory by Fierz and Pauli [FP39].

The foundations of this thesis lie on two of the main frameworks of the current physical understanding of nature: quantum field theory in curved spacetimes and cosmology. So in this introductory chapter I will briefly describe the fundamentals of quantum field theory in curved spacetimes, the basics of cosmology and the early universe stages as well as a quick review on popular dark matter models and their possible production mechanisms.

1.1 | Quantum Field Theory in Curved Spacetimes

The fundamental interactions in physics can be divided into four: electromagnetism, weak interaction, strong interaction, and gravity. Out of these four interactions, gravity is by far the weakest one, e.g., the ratio between the electric and gravitational forces of an electron is of the order of 10^{-40}.

From the physical developments of the last century it is well stablished that all the fundamental fields shaping nature must obey the laws of quantum theory. In the last few decades the construction of a unified theory of forces of nature has undergone a remarkable progress. The Weinberg-Salam theory provided a unified description of electromagnetism and weak interactions [Gla61; Wei67] which was demonstrated to be renormalizable [t H71; tV72], giving rise to a strenuous effort to find a unified theory for all the interactions. The discovery of asymptotic freedom in quantum chromodynamics [GW73], i.e., the fact that the coupling constant of the interaction decreases with increasing energy, raised hopes of including the strong interaction into a wider gauge theory along with the electromagnetic and the weak interactions, the so-called grand unified theories [GG74]. Although there is still not a single theory for grand unification, there are several proposals. Only one out of the four fundamental forces resists this unifying endeavor so far, being also elusive to be described by a complete and satisfactory quantum theory: this force is gravity.

Despite the considerable amount of effort made during the last half century in pursuing a complete quantum description of gravity, it still remains elusive. However, in the absence of a viable quantum theory for gravity, we can still wonder about the influence of gravity on quantum phenomena. In the same spirit as the early works with the classical electromagnetic field, one may hope that there exists a similar approximation for the gravitational field at least in some regimes. The framework of quantum field theory in curved spacetime deals with the study of scenarios in which both the quantum nature of the matter fields and the effect of gravity are important but in which the possible quantum nature of gravity is negligible. Hence, in these regimes, the gravity sector can be described with its classical formulation, i.e., through a pseudo-Riemannian differential manifold as described by the theory of General Relativity [HE11; Wal84; MTW73]. The quantization procedure for fields on curved-spacetimes, the striking phenomenology associated with the interplay between gravity and the quantum fields and its implications for black holes thermodynamics are described in detail in [Wal94; BD84] while its implications in the early universe are discussed in [Par68; For87; MFB92].

The early attempts of incorporating gravity as a new quantum field into quantum field theory run into conceptual problems. These problems led to radically different approaches that go beyond the established theories, the most prominent ones being string theory [Pol98] and loop quantum gravity [Rov08]. However, after some decades of effort, a satisfactory theory for quantum gravity is still missing, and there are indications that a field theoretical approach may be better suited than originally expected [Eic19]. In particular, due to the weak strength of the gravitational interaction, in first approximation, the back-reaction of the spacetime metric to the energy-momentum tensor of the quantum fields may be neglected in certain scenarios, and one is left with the problem of studying quantum field theory in Lorentzian manifolds. This approach leads to far-reaching conceptual and technical problems and to unexpected predictions.

A corner stone of this effective theory is the work of Hawking [Haw75]. His novel study of quantum effects in black holes and their thermal emission was a great breakthrough in the development of the theory, bringing hope in that a new field in which gravity, quantum theory and thermodynamics could be described together, leading to new advances in our physical knowledge. More striking results came during the mid-seventies from the effort of computing the expectation value of the energy-momentum tensor $\langle \hat{T}_{\mu\nu} \rangle$ for quantum fields in generally curved spacetimes [DeW75; DFU76]. An essential feature of the renormalized energy-momentum tensor obtained through different techniques is that it is always covariantly conserved [Wal94], being then a suitable source for the semiclassical Einstein equations. However, this result was in principle in contradiction with the particle production described earlier as demonstrated by Hawking [Haw70]. Nevertheless, this last result is only true if the energy-momentum tensor is subject to the dominant energy condition and as it was soon discovered, one of the unusual aspects of the quantum theory is that the curvature of spacetime can induce negative energy-momentum in the vacuum [Haw75]. Then, particle creation and a conserved energy-momentum tensor is compatible. Furthermore, this possibility of violating the classical energy conditions opened a wide window in looking for resolution of classical singularities.

Although for the study of back-reaction of the quantum fields their effects on the dynamics of the spacetime are relevant, for most of the applications of quantum field theory we can consider probe fields, meaning that the dynamics of spacetime will not be affected by the dynamics of the quantum field. In this thesis this will be the case. We will start discussing the quantization procedure for a scalar Klein-Gordon field in a fixed curved background [BD84; Wal94].

1.1.1 | Phase Space and Observables

The classical dynamics of the scalar field φ is encoded in the action

$$\mathcal{S} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\partial_a\varphi\partial^a\varphi + m_\varphi^2\varphi^2 + \xi R\varphi^2\right), \tag{1.1.1}$$

where a direct coupling to the scalar curvature R is allowed through a dimensionless constant ξ and g is the determinant of the metric. The Klein-Gordon equation for the field can be derived from this action by extremization and it reads

$$\nabla_a\nabla^a\varphi - m_\varphi^2\varphi - \xi R\varphi = 0, \tag{1.1.2}$$

where ∇_a is the derivative operator compatible with the metric.

We will restrict our attention to spacetimes for which the classical dynamics for the field has a well-posed initial-value formulation. Hence we should be able to construct a quantum theory for the field in an analogous way as it is done in flat spacetime. There is a set of conditions on the spacetime structure that ensures this feature of the solutions. We will require the spacetime to be time orientable and that all the manifold can be foliated in hypersurfaces intersected by inextendible causal curves. If from a given hypersurface we can reach all points of the manifold using causal curves, this hypersurface is a Cauchy one and the spacetime is said to be *globally hyperbolic*. There is a theorem [Ger70] stating that if the manifold is globally hyperbolic, it can be foliated by a one-parameter family of smooth Cauchy surfaces Σ_t defined at t constant. If the manifold is globally hyperbolic, the Klein-Gordon equation, as well as more general classes of linear wave equations, has a well posed initial value formulation. In the Klein-Gordon case, if we know the initial value of the field φ on a Cauchy hypersurface Σ_t and its normal derivative $\nabla_{\mathbf{n}}\varphi := n^a\nabla_a\varphi$, then we can obtain a unique solution defined in all the manifold.

The metric tensor on the manifold g_{ab} can now be decomposed as the metric h_{ab} on the Cauchy hypersurfaces Σ_t and their normal vector n_a as

$$g_{ab} = h_{ab} - n_a n_b. \tag{1.1.3}$$

In order to have a Hamiltonian formulation to apply the canonical quantization procedure, we need to define the phase space Γ of the system. This space encodes the information of all the possible states for the system. Phase space is a symplectic differentiable manifold, in which any point represents the value of the field and its conjugate momentum.

From the action we can obtain the canonical conjugate momentum for the field as

$$\pi_\varphi := \frac{\delta\mathcal{S}}{\delta\partial_t\varphi} = \sqrt{h}\nabla_{\mathbf{n}}\varphi, \tag{1.1.4}$$

where h is the determinant of the metric induced in the hypersurface Σ_t. Now, the phase space Γ can be understood as the vector space of pairs (φ, π_φ), meaning that a point in phase space is given by the specification of the functions $\varphi(\mathbf{x})$ and $\pi_\varphi(\mathbf{x})$ on a Cauchy surface. Furthermore, we can identify the space of solutions $\mathscr{S}$ of the Klein-Gordon equation (1.1.1) with the phase space Γ understood as the initial data space on any Cauchy surface Σ. On top of this space of solutions we introduce the symplectic structure Ω: a bilinear antisymmetric application on any two states $(\varphi_1, \pi_{\varphi_1})$, $(\varphi_2, \pi_{\varphi_2})$ on the phase space. Its application acts as

$$\Omega[(\varphi_1, \pi_{\varphi_1}), (\varphi_2, \pi_{\varphi_2})] := \int_\Sigma (\pi_{\varphi_1}\varphi_2 - \pi_{\varphi_2}\varphi_1)\,\mathrm{d}^3x. \tag{1.1.5}$$

Here the integral is performed over any Cauchy hypersurface as the symplectic structure is independent on the choice of surface if the field satisfies appropiate boundary conditions, e.g., if the value of the field vanishes in the boundary.

The Poisson brackets for the conjugate variables are defined by the symplectic structure on a (constant t) spatial hypersurface Σ as

$$\{\varphi(t,\mathbf{x}), \pi_\varphi(t,\mathbf{x}')\} = \delta^3(\mathbf{x}-\mathbf{x}');\ \{\varphi(t,\mathbf{x}), \varphi(t,\mathbf{x}')\} = \{\pi_\varphi(t,\mathbf{x}), \pi_\varphi(t,\mathbf{x}')\} = 0. \tag{1.1.6}$$

The Poisson brackets endow the set of physical observables with a Lie algebra structure.

1.1.2 | Canonical Quantization

The quantization procedure consists in representing the physical observables as operators on a Hilbert space. In this space, the state of the system is represented by a ray (an element up to a phase). Once the system is defined by a symplectic phase space (Γ, Ω), the quantization procedure is based on constructing a Hilbert space such that a closed algebra of classical observables of the theory is represented by self-adjoint linear operators on it.

We want the quantization procedure to preserve the Poisson bracket structure, hence given two observables $\mathcal{A}$ and $\mathcal{B}$ of the classical algebra, their quantized version must verify

$$[\widehat{\mathcal{A}}, \widehat{\mathcal{B}}] = i\hbar\widehat{\{\mathcal{A}, \mathcal{B}\}}. \tag{1.1.7}$$

However, not all the observables can be promoted to quantum operators satisfying this canonical commutation relations without ambiguities. We will choose only a complete subset of observables to be quantized so that the resulting representation is irreducible, i.e., any operator that commutes with all the others is a multiple of the identity.

Hence, to build a quantum field theory for the scalar field, we will look for an irreducible representation of the field and its momentum through self-adjoint operators $\widehat{\varphi}$ and $\widehat{\pi_\varphi}$ acting on a Hilbert space such as they satisfy the canonical commutation relations

$$[\varphi(t,\mathbf{x}),\pi_\varphi(t,\mathbf{x}')] = i\hbar\delta^3(\mathbf{x}-\mathbf{x}');\ [\varphi(t,\mathbf{x}),\varphi(t,\mathbf{x}')] = [\pi_\varphi(t,\mathbf{x}),\pi_\varphi(t,\mathbf{x}')] = 0. \quad (1.1.8)$$

We need to construct the Hilbert space for the field. Using the Fock quantization, the Hilbert space is built as the direct sum of the tensor product (symmetrized or anti-symmetrized depending on the spin of the field) of copies of a one-particle Hilbert space $\mathcal{H}_0$. We start by constructing the Hilbert space for one particle $\mathcal{H}_0$. Its construction is carried out using the classical solutions to the Klein-Gordon equation (1.1.1) in the case of the scalar field, or to the corresponding equation of motion for different fields.

To construct the one-particle Hilbert space $\mathcal{H}_0$, we need to define an inner product in the complexified space of solutions $\mathscr{S}^{\mathbb{C}}$. To define an inner product we will introduce a complex structure J from the space of solutions onto itself whose square is minus the identity and which preserves the symplectic structure in the sense that $\Omega(J\cdot,J\cdot) = \Omega(\cdot,\cdot)$, and which satisfies that

$$(\cdot,\cdot) = i\Omega(J\cdot,\cdot) \quad (1.1.9)$$

is a definite positive inner product in the complexified space of solutions.

The eigenvalues of the complex structure are $\pm i$, so we can use the spectral decomposition (iJ is a self-adjoint operator) of the complexified space of solutions to split it into two orthogonal subspaces, each one corresponding to each eigenvalue. The completion of these subspaces with the inner product defines the so called one-particle and one-antiparticle Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_0^*$ respectively. Very often $\mathcal{H}_0$ is identified with the positive frequency solutions subspace (eigenvalue $+i$).

Note that the only thing we need to define the one-particle Hilbert space from the space of solutions is to specify a complex structure. However, this choice is not unique and, generally in curved spacetimes, different choices of complex structure will lead to inequivalent quantizations. Actually, in general, if we do not impose further criteria to select a specific complex structure, we would deal with infinite inequivalent quantizations.

Once the one-particle Hilbert space is defined, the Fock space for the scalar field is constructed as

$$\mathcal{F} := \oplus_{n=0}^{\infty}(\otimes_S^n\mathcal{H}_0). \quad (1.1.10)$$

On this one-particle Hilbert space, we can take an orthonormal basis, maybe in a generalized sense, of complex functions ψ_i, and express any solution for the field as

$$\varphi = \sum_i (a_i \psi_i + a_i^* \psi_i^*) \tag{1.1.11}$$

where a_i^* and a_i are the creation and annihilation variables respectively and the subindex i stands for any combination of parameters needed to specify the elements of the basis. We can express the annihilation and creation variables in terms of the field and the elements of the basis as

$$a_i = (\psi_i, \varphi). \tag{1.1.12}$$

With this expression and the orthonormality of the basis, the Poisson brackets for the variables read

$$\{a_i, a_j^*\} = -i\delta_{ij}, \qquad \{a_i, a_j\} = \{a_i^*, a_j^*\} = 0. \tag{1.1.13}$$

Upon quantization, these variables become the creation and annihilation operators $\widehat{a}_i^\dagger$ and $\widehat{a}_i$ such as their only non vanishing commutator between them is

$$[\widehat{a}_i, \widehat{a}_j^\dagger] = \hbar\delta_{ij}. \tag{1.1.14}$$

The quantum field is then represented as

$$\widehat{\varphi} = \sum_i (\widehat{a}_i \psi_i + \widehat{a}_i^\dagger \psi_i^*). \tag{1.1.15}$$

We still need to identify the states on the Hilbert space. In the Fock representation all the possible states are generated through the application of the creation and annihilation operators on a vacuum state. We define the vacuum state associated to the complex structure J as the normalized state $|0\rangle$ which is annihilated by all the annihilation operators, meaning that $\widehat{a}_i |0\rangle = 0$ for all values of i. Any multi-particle state of the theory is obtained by applying the creation operators to the vacuum state.

As we have mentioned earlier, different choices of the complex structure, which determines the one-particle Hilbert space, lead to different quantizations. Let us see how this different quantizations are related to each other. Let $\mathcal{H}_0'$ be the one-particle Hilbert space defined by a different complex structure J' with a different orthonormal basis $\{\psi_i'\}$ in its corresponding inner product $(\cdot,\cdot)' = i\Omega(J'\cdot,\cdot)$.

As this new Hilbert space is contained in the complexified space of solutions, the elements of the basis can be decomposed using the projectors defined with the original complex structure and hence they can be written as a linear combination of the elements of the basis ψ_i and ψ_i^* respectively, so the elements of both basis can be written as:

$$\psi_i' = \sum_k \left(\alpha_{ik}\psi_k + \beta_{ik}\psi_k^*\right), \qquad \psi_k = \sum_i \left(\alpha_{ik}^*\psi_i' - \beta_{ik}\psi_i'^*\right). \tag{1.1.16}$$

These linear maps between both Hilbert spaces are called *Bogolyubov transformations* and the coefficients α_{ik} and β_{ik} are the *Bogolyubov coefficients.* If we express the classical field in both bases, we obtain a relation between the creation and annihilation variables for both Hilbert spaces:

$$a_k = \sum_i \left(\alpha_{ik} a'_i + \beta^*_{ik} a'^*_i\right), \qquad a'_i = \sum_k \left(\alpha^*_{ik} a_k - \beta^*_{ik} a^*_k\right). \tag{1.1.17}$$

Two different Fock quantizations $(\mathcal{F}, \hat{\varphi})$ and $(\mathcal{F}', \hat{\varphi}')$ are *unitarily equivalent* if and only if they are related by a unitary map $\widehat{U}' : \mathcal{F}' \to \mathcal{F}$ such as $\widehat{U}'\widehat{\varphi}'\widehat{U}'^{\dagger} = \hat{\varphi}$. This condition ensures that the states on one Fock space have the same physical properties, and hence, give the same physical predictions as their counterpart on the other Fock space. It can be seen that the condition for unitary equivalence can be recast in the form of a Bogolyubov transformation between the creation and annihilation operators on both Fock spaces. If the transformation $\widehat{U}'$ is unitary, it must conserve the norm of the states. This condition reduces to demand that the Bogolyubov transformation relating both quantizations is such that the β map is Hilbert-Schmidt, meaning:

$$\mathrm{tr}(\beta\beta^{\dagger}) = \sum_{ij} |\beta_{ij}|^2 < \infty. \tag{1.1.18}$$

Finally, the particle number associated with the complex structure J' is determined by the operator $\widehat{N}' = \sum_i \widehat{a}'^{\dagger}_i \widehat{a}'_i$. It is straightforward to see that the expectation value for this operator on the original vacuum $|0\rangle$ is precisely the quantity $\mathrm{tr}(\beta\beta^{\dagger})$. Hence, the condition to discern whether two different quantizations are unitarily equivalent or not can be interpreted as the condition that the vacuum state of one quantization contains a finite number of particles corresponding to the other quantization.

Until this point we have described the quantum theory of a field formulated on top of a given spacetime, hence neglecting the posible *back-reaction* of the field on the dynamical spacetime in which it is embedded. In General Relativity, the matter content affects the dynamics of spacetime through the energy-momentum tensor. Therefore, in situations in which we can still treat the spacetime classically, it seems plausible that the back-reaction effects may be described by the semiclassical Einstein equation using the expectation value of the energy-momentum tensor on physically relevant states as source.

Although its classical counterpart is a well defined and easy to compute quantity from the action of the classical field, the definition of the quantum energy-momentum tensor in curved spacetimes is plagued with subtleties and ambiguities. In particular, this tensor presents ultraviolet divergences and hence needs to be renormalized. In flat spacetime this subtraction of the infinite vacuum energy is carried imposing normal ordering. However, this prescription has no direct generalization to curved spacetimes, so a different covariant approach for defining the renormalized energy-momentum tensor is generally adopted. This covariant prescription, called *point-splitting* [DeW75; DFU76] establishes a unique form for the expected energy-momentum tensor up to the addition of conserved local curvature terms. The renormalized energy-momentum tensor obtained with this prescription satisfies all the desired properties for a quantum operator representing the energy-momentum tensor [Wal77; Wal94] as it preserves causality, is covariantly conserved, and recovers the normal ordering results in the aproppiate limits.

The fact that different complex structures determine quantum theories with different number of particles will be exploited in this thesis to explain the dark matter abundance of the universe. But first, we will give a brief review on the current status of cosmology, with special attention on the inflationary paradigm and the reheating dynamics as they will be of crucial importance for our study.

1.2 | Cosmology

Cosmology is the branch of Physics devoted to the study of the origin and evolution of the Universe as a whole. It is an observational discipline rather than an experimental one and this introduces a handicap in obtaining statistically sufficient population at very large scales. Then, improbable characteristics of the observable Universe may suggest that an underlying dynamical mechanism is causing these anomalies.

Cosmology inherited the Copernican Principle as the guiding light in almost all its models and has become a corner stone of any plausible cosmological theory. This Cosmological Principle states that there is no special point in all the Cosmos, but all of them are equally important. This principle led to the conclusion that all the mass must be distributed homogeneous and isotropically in the Universe. This assumption may not seem to be valid in the late epochs of the Universe as the distribution of stars and galaxies is clearly not homogeneous. However, this homogeneity and isotropy should be understood as a statistically emergent description at large enough distances. For the Early Universe, this assumption is a good approximation as can be inferred from the almost negligible anisotropies of the Cosmic Microwave Background. However, this extreme isotropy at the epoch of recombination constitutes also a puzzle for the hot Big Bang model as regions in the CMB separated more than 2 degrees should never have been in causal contact before. This puzzle is known as the horizon problem and is the main motivation for considering an inflationary phase prior to primordial nucleosynthesis.

Due to the large distances involved in cosmology, it must rely on a good description of the gravitational interaction. The current understanding of our Universe has its foundations in General Relativity, although there is a big effort being done by the community in looking for plausible modifications or extensions of Einstein's theory.

1.2.1 | The Standard Model of Cosmology

The standard model of Cosmology, also known as the hot Big Bang model, is based fundamentally on three observational facts: the expansion of the observable Universe, the abundance of primordial light elements and the features of the Cosmic Microwave Background. The standard model is so robust that allows to make sensible speculations about the dynamics of the Universe at time scales of the order of $10^{-43}\,s$ after the Big Bang.

The assumption of homogeneity and isotropy leads to describe the cosmological spacetime, at least on average, with a manifold equipped with the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, described by the line element:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\mathrm{d}\varphi^2\right), \tag{1.2.1}$$

where (t, r, θ, φ) are the comoving coordinates, $a(t)$ is the cosmic scale factor and k determines the curvature of the spatial hypersurfaces. It can take three different values: $k = 0$ for flat spatial sections, $k = 1$ for closed spatial sections and $k = -1$ for open spatial sections. The dynamics of the expanding Universe appears implicitly in the time-dependence of the scale factor. The explicit behavior of the scale factor is obtained by inserting the FLRW metric into the Einstein equations and solving them. However, we need to make some assumptions on the energy-momentum tensor sourcing the spacetime so we do not need to have all the detailed information on the properties of all the fundamentals fields contributing to the energy content of the Universe.

To be consistent with the symmetries of the spacetime, the total energy-momentum tensor must be diagonal and with all its spatial components equal. The simplest realization fulfilling these conditions is a perfect fluid characterized by a time-dependent energy density $\rho(t)$ and pressure $p(t)$:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{1.2.2}$$

where u^μ is the velocity field of the fluid, which in a comoving coordinate system is $u^\mu = (1, 0, 0, 0)$.

In general, we need to specify an equation of state for the perfect fluid. In Cosmology it is usually assumed that the fluids are barotropic, meaning that their pressure only depends on the energy density. The most used models for the cosmological components consider a linear relation between both quantities

$$p = w\rho, \tag{1.2.3}$$

with w being the so-called equation of state parameter. Different values of this parameter correspond to different behaviors of the constituent particles of the fluid. For non-relativistic particles $w = 0$, and hence there is no pressure. For relativistic species, $w = 1/3$, recovering the radiation pressure relation and making the energy-momentum tensor traceless. The case of a cosmological constant Λ which is assumed in the standard ΛCDM model to explain the accelerated expansion of the Universe in the late time requires a parameter $w = -1$ which violates the energy conditions of General Relativity (the weak energy condition requires $w > -1/3$).

The conservation of the energy-momentum tensor implies that the cosmological fluids must satisfy the following conservation equation:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{1.2.4}$$

where H is the Hubble parameter which is defined as $H = \dot{a}/a$ and the dot denotes differentiation with respect to the cosmological time t. From cosmological observations it is known that the early universe was dominated by a radiation fluid, the adolescent universe was dominated by a matter fluid and the current observations tend to suggest that the adult universe is dominated by a cosmological constant term.

With the FLRW metric (1.2.1) and the cosmological fluid approximation (1.2.2) we can tackle the Einstein equations in the cosmological scenario to determine the dynamics of the scale factor $a(t)$. We then get the Friedmann equations for an homogeneous and isotropic universe:

$$H^2 = \frac{\kappa^2}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{1.2.5}$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.2.6}$$

which describe the evolution of the Hubble rate and the acceleration of the universe. This set of equations are not independent of the conservation equation (1.2.4) and hence we only need to solve two out of these three equations.

It is customary to define the abundance Ω as the ratio between the energy density ρ and the critical energy density ρ_c, for which the universe has flat spatial sections:

$$\Omega = \frac{\rho}{\rho_c}, \qquad \rho_c = \frac{3H^2}{\kappa^2}. \tag{1.2.7}$$

The present day value for the critical density is $\rho_c \approx 4 \times 10^{-47}\mathrm{GeV}^4$. It is a common practice to define the abundance for each of the species permeating the universe. From observations such as the CMB, one can estimate the abundances of different constituents comparing the observed quantities with theoretical predictions. The abundances on the concordance model (ΛCDM) to fit the observations are [Agh+18]:

$$\Omega_{\mathrm{B}} \simeq 0.0486 \pm 0.0010 \tag{1.2.8}$$
$$\Omega_{\mathrm{DM}} \simeq 0.2589 \pm 0.0057 \tag{1.2.9}$$
$$\Omega_{\Lambda} \simeq 0.6911 \pm 0.0062. \tag{1.2.10}$$

where the first one is the abundance of baryons, the second one is the abundance of dark matter and the third one is the energetic abundance in the form of a cosmological constant.

The last few decades has witnessed the appearance of plenty of accurate observations with which the era of precision Cosmology begun. The principal observational sources to set the parameters of the cosmological model are: the CMB, the abundance of primordial elements, large scale structure and astrophysical observations such as gravitational lensing. These observations that once motivated the standard cosmological model have also aroused new questions on the initial conditions of the universe, the fundamental nature of the constituents of the cosmological fluid, and even on the validity of General Relativity at long scales.

The problem of initial conditions can be theoretically tackled by introducing an hypothetical epoch prior to the hot big bang Cosmology in which the universe is accelerating as we will discuss later. For this thesis, this inflationary epoch is of relevance as it constitutes the most important source of gravitational particle production along with reheating. On the other hand, the observations suggest that there is more matter in the universe than the one we can observe, matter that only seems to interact gravitationally with the rest of the constituents, hence named dark matter. We will elaborate more on this topic in the next subsection as dark matter is a corner stone of this thesis. Finally, the observations of the late universe show that it is undergoing an accelerated expansion. To explain it, some cosmologist are not satisfied with a cosmological constant Λ and suggest that a new field must be responsible for this behavior, or even, that a modified theory of gravity should be considered. In this thesis we will not worry about this last open problem.

1.2.2 | Dark Matter in Cosmology

Dark matter constitutes, along with dark energy, most of the energy content of our Universe according to the concordance model. Even if we accept that the current accelerated expansion of the Universe is due to a cosmological constant, the ΛCDM model still lacks a fundamental description of the dark matter sector which constitutes around a quarter of the energetic content of the Universe. The contribution of non-luminous matter to the dynamics of galaxies and the Universe has been postulated for almost over a century since it was first propose to explain the dynamics of the Coma galaxy cluster [Zwi33]. More recent evidences for the existence of dark matter comes from different cosmological observations such as the distribution of luminous and gravitational mass in the Bullet Cluster or the correlation in the angular power spectrum of the CMB. However, despite all the observational hints that dark matter should be out there, so far no candidate has been detected.

The standard cosmological model provides only a phenomenological description of dark matter as a perfect fluid which permeates the Universe. The only characteristic we can extract from the observations, to be more precise, from its clustering properties, is that it behaves as a pressureless fluid, meaning that it has to be made up of non-relativistic species, in other words, dark matter should be cold. If dark matter were hot during the equality epoch, i.e., when the energy density of radiation and the one of matter where equal, its clustering would be suppressed below some scale due to the free-streaming. Hence, hot dark matter posses problems in explaining the formation of structures such as galaxies. A different option is that dark matter was ultra-relativistic at the early epochs of the Universe and cooled enough during the radiation-dominated epoch, hence, there is a cut-off in structure formation at small enough scales. These kind of models receive the generic name of warm dark matter. So there is still room to have initially highly relativistic particles. In addition, there is some controversy as the cold dark matter scenario has tensions between the observations and the N-body simulations [Bag05]. Hence, lately a considerable effort has been made in running these numerical simulations for the structure formation of warm dark matter models [Lov+14; Bos+16].

Dark Matter Models

All the evidence we have so far for dark matter is based on its gravitational interactions. The fundamental identity of dark matter has great implications beyond cosmology and astrophysics: in particle physics dark matter is one of the leading empirical evidences for new particles and hence it may have implications for the puzzles that plague the

Standard Model of particles. Therefore, dark matter true nature is a topic of central importance for both Cosmology and particle physics and then it is not surprising that many models for dark matter candidates have their roots in problems of the Standard Model and theories that go beyond it. In this subsection we will go through some of the leading dark matter candidates as in this thesis we will construct theories inspired by gravity that can accommodate dark matter.

One of the most promising models because it also alleviates the strong CP problem of the Standard Model, where C and P are the discrete transformations of charge conjugation and parity, is the *axion* [PQ77; Wei78; DB09; VG09]. The axion solution introduces a new pseudoscalar field with a new mass scale: the axion decay constant. This new field reduces the electric dipole moment of the neutron so it is consistent with the current bounds. The allowed parameter space for the axions imply that they are extremely light and weakly-interacting particles yielding a well-motivated dark matter candidate. The axion mass is bounded by several independent constraints. For instance, for the axions to live longer than the age of the universe its mass must verify: $m_a \lesssim 10$ eV. Typical values for the axion mass lies in the range from μeV to meV.

The most studied dark matter candidates are the *Weakly-Interacting Massive Particles (WIMPs)* [JKG96] as they arise in many particle physics theories, they naturally lead to the correct relic density, and they can potentially be found in accelerator experiments. The mass for the WIMPs lies in the range $m_{\rm WIMP} \sim 10\,\text{GeV} - 10\text{TeV}$ and most of them present interactions at tree-level with the W and Z weak bosons. As long as the WIMP is stable, it can be produced as a thermal relic as we will discuss later. There are several candidates for WIMP dark matter that are well-motivated as they may theoretically solve some puzzles of the weak sector of particle physic:

1. *Neutralinos*
   Historically, one elegant solution to the gauge hierarchy problem in the SM, i.e., the problem of why the Higgs boson mass is so small, is to postulate supersymmetry [SS00] although the latest experimental results (mainly LHC data) seems to put into question this possibility. Supersymmetric extensions of the SM predict that every particle has a new partner particle which has the same quantum numbers and gauge interactions but differs in spin by 1/2. For dark matter it is natural to consider all the new particles that are electrically neutral. These particles are the spin 3/2 *gravitino*, the spin 1/2 *neutralinos* and the spin 0 *sneutrinos*. From all these particles the sneutrinos are not viable as dark matter in the simplest supersymmetric realizations, and the gravitino is not a WIMP.

2. *Kaluza-Klein dark matter*
   New weak-scale physics can be obtain using extra dimensions. The original idea of using extra dimensions to explain electromagnetism and general relativity in the same theory goes as far as the works by Kaluza and Klein in the 1920's [Kle26]. This idea has evolved to the scenario of universal extra dimensions (UED) [ACD01]. In the simplest UED model, there is only an extra dimension of size R compactified on a circle. Then every SM particle has infinite partner particles, each one at every Kaluza-Klein level n with mass $\sim nR^{-1}$. These new particles have the same spin as the original SM one. The simplest UED models preserve a discrete parity which implies that the lightest Kaluza-Klein particle is stable and hence may be a possible dark matter candidate [ST03; CFM02]. The typical required mass for these Kaluza-Klein dark matter candidates lies in the range $600\,\mathrm{GeV} - 1.4\,\mathrm{TeV}$.

These are just two of the most studied WIMPs candidates, but there are much more models coming from different motivations. However, all the WIMPs are produced through thermal freeze-out and are cold and collisionless, but each model implies different signatures for detection.

There are other candidates with weaker interactions with the SM that can also account for the dark matter abundance in a natural way through thermal freeze-out, these are the *superWIMPs* which, despite of their extremely weak interactions, may be observed in signals from cosmic rays, collider data and astrophysical processes. In this scenario, dark matter can be produced in late decays from previous WIMPs (unstables in this scenario) or during the reheating phase of the universe that we will discuss later. There are many particle physics models that predicts superWIMPs with the *gravitino* being the archetypical candidate [FRT03a; FRT03b]. Gravitinos are the spin $3/2$ superpartners of the gravitons and they appear in all supersymmetric theories. They would be typically produced by the decay of charged sleptons, sneutrinos or neutralinos which are hevier particles in this scenario. In addition to gravitinos another well-motivated superWIMP is the *axino*, the superpartner of the axion [RTW91; CKR99b]. One of the best motivated multi-component dark matter scenario is the one in which it is composed of both axions and axinos.

There is heavy evidence that dark matter does not have appreciable strong or electromagnetic interactions. Hence it is not wild to suppose that it has no SM gauge interactions. *Hidden dark matter* has been widely explored and brings a great model building freedom [KOP66; BK82; FLV91; BDM96]. This freedom however makes these models to loose connection with the problems of the SM and due to the lack of interaction with the SM, it is difficult to obtain experimental signatures to identify the candidates.

The evidence for neutrino mass, due to the flavor oscillations already measured, requires new physics beyond the SM. This puzzle can be solve if one assumes the existence of right-handed neutrinos so the mass of the neutrinos is generated by the same mechanism as for the other fermions of the SM. However, for these mass terms to be allowed under the symmetries of the SM, the right-handed neutrinos cannot have any gauge interactions. These *sterile neutrinos* [DW94; Sel+06] may constitute the dark matter as they are not charged under the SM interactions.

In this thesis we are going to present models for dark matter inspired by gravity. We develop a family of theories for scalar and vector fields following a similar bootstrapping approach as the one leading to General Relativity from a massless spin-2 field, i.e., by prescribing a coupling to the energy-momentum tensor that remains at the full non-linear level.

Production Mechanisms

Another question of great interest due to the almost negligible cross-section between dark matter and the fields we know is how such an important abundance could have been produced:

- If the dark matter particle has interactions with the primordial plasma it can be produced through the interactions between the plasma particles. This kind of production is named *thermal production*. At some point dark matter would reach thermal equilibrium with the rest of the plasma and hence the evolution of its number density, n will be governed by the Boltzmann equation (assuming it is stable)

  $$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn - \langle v\sigma\rangle\,(n^2 - n_0^2), \tag{1.2.11}$$

  where n_0 is the number density of dark matter particles would have in thermal equilibrium at a given temperature, σ is the annihilation cross-section properly averaged over particle-antiparticle states and initial spins, v is the relative velocity of the considered particles, and $\langle\rangle$ indicates a thermal average as is explained in [SWO88], where also the full relic abundance computation is presented. The n^2 arises from destruction processes $\mathrm{DM\,DM} \to \mathrm{SM\,SM}$ while the n_0^2 comes from the reverse processes, which creates dark matter particles.

  Although the Boltzmann equation has to be solved numerically, a rough analysis can give instructive insights in the order of magnitude of the created abundance. Considering the freeze out time as the one in which $n\,\langle v\sigma\rangle = H$, the thermal relic abundance can be written as [Fen10]

$$\Omega_{\text{DM}} \sim \frac{m_{\text{DM}} T_{\text{today}}^3}{\rho_c M_{\text{P}} T_f} \langle v\sigma \rangle^{-1} , \tag{1.2.12}$$

where the subscript f denotes quantities evaluated at the freeze out time, M_{P} is the Planck mass, m_{DM} is the mass of the dark matter candidate, ρ_c is the critical density, and the annihilation cross-section $\langle v\sigma \rangle$ depends on the specific model under study.

- On the other hand, if the interactions of dark matter with the primordial plasma are not strong enough, there are still different *non-thermal production* mechanisms that can potentially explain the observed abundance of dark matter. The specific mechanism depends on the model under study. For instance, in the case of sterile right-handed neutrinos can be produced via the oscillations with the left-handed SM neutrinos, as they are mixed [DW94]. To explain the abundance of axions, the misalignment mechanism is considered, as if they were produced thermally, the required mass to match the dark matter abundance ($m_a \sim 80$eV) would make them hot dark matter instead of cold. If inflation took place after the Peccei-Quinn phase transition, which occurs when the universe cools to a temperature of the order of the axion decay constant and the axion field takes values that vary from point to point, then our observable universe lies in a patch with a single value of the axion field. When the temperature reaches the order of the GeV, the axion relaxes to its minimum. This realignment of the vacuum generates a relic abundance [BHK08]

$$\Omega_a \simeq 0.4\theta_i^2 \left(\frac{f_a}{10^{12}\text{GeV}} \right)^{1.18} , \tag{1.2.13}$$

where θ_i is the initial vacuum misalignment angle, and f_a is the decay constant of the axion.

On the other hand, if inflation took place before the Peccei-Quinn phase transition our observable universe would be filled with a mixture of many patches with different θ_i. The abundance due to the misalignment mechanism will be given by the same expression but with an effective misalignment angle of order one. Furthermore, since many different patches are observable, in the boundaries between them topological defects as axionic strings and domain walls are generated after the phase transition and they may have observable effects. The production of axions due to domain walls decays is subdominant to the one due to misalignment [CHS98], but the production from axionic string decays can be one order of magnitude larger [HCS01; YKY99].

Another production mechanism that is universal to all the dark matter candidates, but which is usually neglected in the literature, is the gravitational production due to the dynamics of the background geometry and the evolution of the fields vacuum.

The pioneering work in the arena of gravitational production of elementary particles was done by L. Parker [Par68; Par69]. L. H. Ford focused on the gravitational production at the transition between an inflationary epoch and a radiation-dominated cosmology [For87] trying to explain how the Universe could have been reheated. If the particles can be regarded as massless during the inflationary phase, i.e., if the mass of the produced particle is much smaller than the energy scale of inflation $m \ll H_{\rm inf}$, then the produced number density is proportional to the third power of the Hubble parameter $n \propto H_{\rm inf}^3$ if the phase transition is not too abrupt. However it is not clear if the production of these particles is due to the transition itself or if they are being continuously created during the quasi de Sitter expansion. In this thesis we are going to explore gravitational production of scalar dark matter candidates during the inflationary and reheating epoch taking into account the non-trivial dynamics of the geometric quantities of the spacetime and the inclusion of derivative couplings of the field.

1.2.3 | Inflation and Reheating

The hot big bang model, despite its remarkable success in describing the cosmological observations, has some drawbacks concerning the likeliness of its initial conditions. These problems arise because the configuration the universe seems to demand a very specific choice of initial values for some parameters. Let us start with the *flatness problem*: the dynamics of the cosmological model dictates that if the universe has some spatial curvature initially, its relative contribution to the Friedmann equation (1.2.5) with respect to matter and radiation will increase in time, deviating from flatness as it evolves. Nowadays, the universe seems to be flat, imposing then that at the beginning of times, the universe may have been extremely flat. This initial condition seems extremely unlikely as almost all possible initial conditions would lead either to a closed or an open universe.

On the other hand, the extreme isotropy of the CMB lead to the *horizon problem*. The CMB is the radiation coming form the last scattering surface and the comoving distance at which causal interactions at this moment could have taken place is of the order of a degree in the sky. Hence radiation coming from regions separated by more than the horizon scale at the moment of decoupling have never been in causal contact.

Then, the astonishing isotropy in the temperature measured from the CMB can not be explained within the standard cosmological model. The same situation occurs with nucleosynthesis, as it needs that the universe is homogeneous at scales far larger that the horizon scale at that time. Another related issue is that although the universe is homogeneous and isotropic in average it has some inhomogeneities responsible for structure formation. One question the standard model can not tackle is how these inhomogeneities were generated.

A mechanism capable to solve the initial condition problems of the hot Big Bang model was first suggested by A. Guth in 1981 [Gut81]: the inflationary paradigm. His scenario was based on the phase transition from a supercooled unstable phase in grand unified theories. Guth realized that in the case of extreme supercooling the energy density of an expanding Universe would be dominated by the vacuum energy of the supercooled phase and hence it would not depend on time. According to the Friedmann equations (1.2.5), the Universe expands exponentially in this situation. If there is a phase transition to the absolute minimum of the theory and all this energy is converted into heat, the Universe would be reheated to a temperature after the phase transition which only depends on the vacuum energy of the supercooled phase and not on the duration of the previous exponential growth. This first inflationary model solved the horizon, and the flatness problems as the duration of the inflationary phase could be long enough. However, this scenario also led to some consequences on the Universe after the phase transition that were in disagreement with the observations. To be precise, the problem arises because the phase transition is not instantaneous but takes a time depending on the dynamics of the specific model. The first step is the transition between the barrier separating both states, a transition that can proceed either by quantum or thermal tunneling with the nucleation of bubbles. The reheating of the Universe would result from the collision of different bubble walls and within this paradigm, the final spacetime would be highly inhomogeneous and anisotropic in contrast with the observations of the CMB [HMS82; GW83].

The old inflationary paradigm cannot get rid of its drawbacks but the idea of a primordial inflationary epoch which dynamically removes the problem of initial conditions for the Universe remains. There are currently two different scenarios for inflation: the new inflationary universe scenario based on the theory of phase-transitions in the early Universe [Lin82; AS82] and the chaotic inflationary scenario [Lin83] based on a scalar field chaotically distributed. A comprehensive introduction to the new inflationary scenario can be found in [Lin90]. Here, we are going to focus on the chaotic scenario and its consequences. In this thesis, for the sake of simplicity, we have considered the inflationary phase as a pure de Sitter geometry but as we are interested in the effects of reheating in particle production, we have implemented the dynamics of reheating coming from a chaotic inflationary model.

The definition of inflation is simply any epoch in which the scale factor is accelerating: $\ddot{a} > 0$ or equivalently an epoch in which the comoving Hubble length, which is the only important scale in an expanding universe, is decreasing in time:

$$\frac{\mathrm{d}}{\mathrm{d}t}(aH)^{-1}, \tag{1.2.14}$$

which means that the observable universe for a comoving observer becomes smaller during inflation. If inflation occurs, then all the mentioned problems of the hot big bang model are solved.

In order to drive inflation we need a source with negative pressure as the energy density is considered to be positive and from the acceleration equation (1.2.5) in order to have inflation:

$$\rho + 3p < 0. \tag{1.2.15}$$

Such a source may be a scalar field ϕ with the unusual feature that its potential energy varies slowly with the expansion of the universe. If we consider a homogeneous scalar field $\phi = \phi(t)$, its energy density and pressure are determined by:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.2.16}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \tag{1.2.17}$$

where $V(\phi)$ is the potential of the field which depends on the specific model under consideration and it is obtained from different particle physics motivations. During inflation, the energetically dominant species is the inflaton, and hence, the Hubble rate can be written as

$$H^2 = \frac{\kappa^2}{3}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \tag{1.2.18}$$

and the inflaton field fulfills the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{1.2.19}$$

From the equations above, one can see that the condition for inflation is satisfied with the scalar field as a source provided that $\dot{\phi} < V(\phi)$. This later condition can be fulfilled with either a nearly flat potential or with a field with an initial value high enough.

During inflation it is customary to use the *slow-roll* approximation, which neglects derivative terms of the inflaton field as compared with its potential. For the approximation to hold, there are two conditions that must be satisfied:

$$\epsilon(\phi) := \frac{\kappa^2}{2}\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2 \ll 1, \tag{1.2.20}$$

$$\eta(\phi) := \frac{\kappa^2}{V}\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2} \ll 1, \tag{1.2.21}$$

where the two slow-roll parameters are defined.

Note that the slow-roll condition is necessary but not sufficient to have an inflationary phase because they only restrict the form of the potential, but the value of the derivative of the field is free and can be chosen so it violates the condition for inflation. Within the slow-roll aproximation, the dynamical equations for the field and the universe greatly simplify:

$$H^2 \simeq \frac{\kappa^2}{3} V(\phi), \tag{1.2.22}$$

$$3H\dot{\phi} \simeq -\frac{\mathrm{d}V}{\mathrm{d}\phi}. \tag{1.2.23}$$

One of the successes of the inflationary model is that it also provides a natural way to generate the inhomogeneities that are imprinted in the CMB and that will eventually lead to structure formation. We will not enter into the details as they are not important for our work but we will give a brief explanation for the sake of completeness. The interested reader can consult [MFB92] for a technical description. If we consider the quantum fluctuations of the inflaton field during the inflationary phase, the evolution of these tiny stochastic deviations can explain the inhomogeneities as the power spectrum obtained is nearly scale invariant as the one observed.

The moment at which inflation ends is determined by the condition $\epsilon(\phi_{\mathrm{end}}) = 1$. Once inflation has ended we need to recover the standard hot big bang evolution, hence the universe should undergo a phase of reheating in which the inflaton decays into all the particles of the Standard Model of Particles and maybe some others if the energy at which it takes place is high enough.

Reheating after inflation

As we have discussed, inflation produces an homogeneous and flat background spacetime and set on top of that the seeds for inhomogeneities due to the quantum fluctuations of the inflaton field. How does this scenario connect with the successful hot big bang scenario? How does all the matter and radiation which drove the expansion of the universe during nucleosynthesis and the rest of epochs appear? The answer lies on a mechanism that transforms all the potential energy which drove inflation into radiation at the end of inflation in a process known as *reheating*.

We will focus on the reheating dynamics coming from the simplest inflationary model: the theory for a massive scalar field ϕ (the inflaton) which interacts with another scalar field χ and a spinor field ψ, serving as proxies for the standard model of particles. A typical Lagrangian for this reheating model can be written as

$$\begin{aligned}\mathcal{L} = \; & -\tfrac{1}{2}\partial_a\phi\partial^a\phi - V(\phi) - \tfrac{1}{2}\partial_a\chi\partial^a\chi - U(\chi) - \bar{\psi}(i\gamma^a\partial_a - m_\psi)\psi \\ & -h\phi\bar{\psi}\psi - \tfrac{1}{2}g^2\phi^2\chi^2 - g^2\sigma\phi\chi^2, \end{aligned} \tag{1.2.24}$$

where the coupling constants h, g should be small to avoid important radiative corrections, we have introduced a possible vacuum expectation value for the inflaton field σ and we have shifted $\phi - \sigma \rightarrow \phi$ so the minimum of the potential is at $\phi = 0$. The potential of the inflaton can be expanded around the minimum as

$$V(\phi) = \frac{1}{2}m_\phi^2\phi^2 + \mathcal{O}(\phi^4), \tag{1.2.25}$$

where the possible value for the mass of the inflaton at the minimum is bounded by the observations of the cosmic microwave background to be $m_\phi \sim 10^{13}$GeV. For the investigation of the perturbative decay of the inflaton we will consider that all the other masses are much smaller than the inflaton mass and that at the epoch of interest $g^2\phi^2, g^2\sigma\phi \ll m_\chi$ and $h\phi \ll m_\psi$.

The mean dynamics of the inflaton field and the background geometry can be well approximated if we neglect all the couplings or, equivalently, if we neglect the particle production effects. The field after inflation will be oscillating around the point $\phi = 0$ satisfying the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + m_\phi^2\phi = 0, \tag{1.2.26}$$

whose solution, at first consistent order in a WKB expansion, given that after inflation has ended $H < m_\phi$, can be written as

$$\phi = \frac{\Phi}{t}\sin(m_\phi t)\left[1 + \mathcal{O}\left(\frac{1}{m_\phi t}\right)\right], \tag{1.2.27}$$

where Φ is an initial amplitude which is constrained by the observations. To see the dynamics of the background for this behavior of the field, let us consider the average value of both the energy density ρ and pressure p of the inflaton field.

$$\langle\rho\rangle = \frac{1}{2}m_\phi^2\frac{\Phi^2}{t^2}\left(\langle\cos^2 m_\phi t\rangle + \langle\sin^2 m_\phi t\rangle\right), \tag{1.2.28}$$

$$\langle p\rangle = \frac{1}{2}m_\phi^2\frac{\Phi^2}{t^2}\left(\langle\cos^2 m_\phi t\rangle - \langle\sin^2 m_\phi t\rangle\right) \simeq 0. \tag{1.2.29}$$

Note that in these expressions we have neglected the change in the amplitude of the field as it will be subdominant as the condition $m \gg t^{-1}$ must hold for the approximation to be valid. The oscillating inflaton field will behave then as a pressureless fluid. Hence, the background spacetime will behave, on average, as a matter dominated cosmological epoch, i.e., $a(t) \propto t^{2/3}$, $H = 2/(3t)$. Let us stress that this is the average behavior. In this thesis, we will explore the non-average behavior of these oscillations and its impact on gravitational particle production through the scalar curvature coupling. We will explore this in detail in Chapter 3. As we have seen, a homogeneous scalar field oscillating with a frequency equal to its mass can be considered as a coherent wave of particles with zero momenta. This last statement will be important for the non-perturbative dynamics of reheating: the *preheating*.

To include the effect of the particle production due to the decay of the inflaton, we take into account the quantum corrections on the equation of motion (1.2.26). This is achieved by introducing the denominator of the propagator of the quantum theory:

$$\ddot{\phi} + 3H\dot{\phi} + \left[m_\phi^2 + \Pi(\omega)\right]\phi = 0, \tag{1.2.30}$$

where $\Pi(\omega)$ is the polarization operator for the inflaton field at a four momentum $k = (\omega, 0, 0, 0)$ evaluated at $\omega = m_\phi$.

The real part of the polarization operator will give a little contribution to the mass of the inflaton field that can be safely neglected due to the small coupling assumption. However, if the frequency of oscillation $\omega < \min(m_\chi, m_\psi)$, then the polarization operator acquires an imaginary part

$$\mathrm{Im}\Pi(m_\phi) = m_\phi \Gamma_\phi, \tag{1.2.31}$$

with Γ_ϕ the total decay rate of the inflaton. Hence, when $\Gamma_\phi \gg 3H$, the energy density of the inflaton field decays exponentially in a time less than the typical expansion time of the universe: $\rho = \rho_0 e^{-\Gamma_\phi t}$.

The total decay rate can be decomposed as a sum over partial decays on each of the available decay channels:

$$\Gamma_\phi = \Gamma(\phi \to \chi\chi) + \Gamma(\phi \to \bar{\psi}\psi) \tag{1.2.32}$$

where the partial decay ratios are known [Oku82] and are expressed as

$$\Gamma(\phi \to \chi\chi) = \frac{g^4\sigma^2}{8\pi m_\phi}; \quad \Gamma(\phi \to \bar{\psi}\psi) = \frac{h^2 m_\phi}{8\pi}. \tag{1.2.33}$$

The evolution of the inflaton energy density during the reheating phase will be due to the expansion of the background geometry until the total decay rate becomes of the order of the rate of expansion. Then the inflaton decays suddenly, releasing all its energy density into relativistic particles χ and ψ in an exponential burst. If the created particles interact strongly enough with each other or if they can decay rapidly into other species, the thermodynamic equilibrium is reached quickly, and the resulting fluid acquires a temperature $T_{\rm reh}$ which can be estimated. As we have a plasma of relativistic particles:

$$\rho(t_{\rm reh}) = \frac{\pi^2}{30} g(T_{\rm reh}) T_{\rm reh}^4 = 3\Gamma_\phi^2 \kappa^2, \tag{1.2.34}$$

where $g(T)$ is the effective numbers of relativistic degrees of freedom at a given temperature, and we recall $\kappa^2 = 8\pi G$. Hence, assuming $g(T_{\rm reh}) \sim 10^2 - 10^3$, the temperature of reheating can be estimated to be

$$T_{\rm reh} \simeq \sqrt{\Gamma_\phi \kappa}. \tag{1.2.35}$$

Note that the temperature of reheating does not depend on the initial conditions of the inflaton field but rather on the elementary particles theory governing its decay.

The perturbative reheating we have discussed is based on the assumption that the inflaton decays as if it was composed of individual quanta. However, as we mentioned earlier, the inflaton field is behaving as a coherent wave and hence collective phenomena may emerge that changes this perturbative vision. Indeed this is the case as it was first discussed by Linde, Koffman and Starobinsky [KLS97] who introduced the picture of preheating as a phase prior to reheating in which the fields coupled to inflaton are produced in bursts due to a parameric resonance. We will not discuss in detail the preheating phase in this thesis as it will not affect the background dynamics in which we are interested for our purposes.

1.3 | Scope and structure

The fundamental properties of dark matter are still a mystery as no conclusive experiment has shed light on important aspects of its nature. This uncertainty makes room for the theoretical exploration of new models for the potential candidates. Taking inspiration on the Fierz-Pauli field model for the graviton, which in the end at full non-linear level recovers General Relativity [Ort15], we have proposed different potential candidates imposing a coupling of these fields to the stress-energy tensor. We have resummed the nonlinear action using a boot-strapping method, guessing appropriate ansätze according with the analytical expressions of the first terms of the series.

On the other hand, in our interest for the interplay between gravitation and quantum fields, we have followed a semiclassical approach in which the geometric nature of classical gravity is preserved while the quantum nature of the fields is considered. These quantum fields can be considered either on top of a given geometry ignoring then their back-reaction onto the spacetime (quantum field theory in curved spacetimes) or as a source of gravity itself through their stress-energy tensor (semiclassical gravity). Regarding the first approach, we have used it to explore the possibility that the observed dark matter abundance may have been produced during the early moments of the Universe via gravitational production. We have explored this production mechanism from different approaches: via the Bogolyubov transformations induced in the field by the evolution of the classical geometric background and via the study of the properly renormalized stress-energy tensor of the field. Using both approaches we have investigated the gravitational production of a scalar dark matter candidate, imposing constraints on its parameter space in order to avoid overproduction. In the same spirit we have also considered the case of an exotic coupling between the classical curvature of spacetime and a scalar field, introducing a coupling between the field derivatives and the background geometry in addition to the non-minimal coupling to the field value.

After this introductory chapter in which we have motivated our work and revisited the two fundamental frameworks in which this thesis is based on, the corpus of the manuscript is structured in three differentiated parts:

- In chapter 2 we discuss how to formulate theories for scalar and vector fields that are consistently coupled to the energy-momentum tensor. We delve into the details of how to resum the full non-linear action starting from a linear direct coupling to the tensor in analogy to the Fierz Pauli theory for the graviton. We extend the known results for the scalar field coupled to the trace of the energy-momentum tensor to more general cases including disformal couplings and we also explore the vector field scenarios. We explore the implications of introducing superpotential terms arising from the ambiguities in the definition of the energy-momentum tensor and we also introduce the possibility of describing the same theories within a geometrical framework in terms of effective metrics. Finally we discuss the phenomenology associated with these kind of models and constraint the new parameters with collider and astrophysical data. These new gravity-inspired models can accommodate dark matter candidates.

- In chapter 3 we investigate the gravitational production of scalar dark matter during the early phases of the universe and we set constraints on its fundamental properties comparing the theoretically predicted abundance with the observed one. We investigate the effects of the nontrivial dynamics of the scalar curvature during the reheating phase on particle production for a non-minimally coupled to gravity scalar field and we discuss the enhancement mechanisms that the background dynamics induces. On the other hand, we also explore the effects of including derivative couplings between the scalar field and the background geometry. In this latter case we consider the averaged behavior of the geometrical quantities.

- Finally, in chapter 4 we present the conclusions and final remarks of this thesis.

2 | Gravity-inspired Dark Matter

“Oh, figures!" answered Ned.
"You can make figures do whatever you want.”

– Jules Verne, *Twenty Thousand Leagues Under the Sea*

As we have mentioned in the introductory chapter, General Relativity is currently the standard framework to describe the gravitational interaction. It stands out as the most suitable candidate for a classical gravitational theory among all the new modified gravity paradigm due to its astonishing agreement with observations on a wide range of scales [Wil14] after more than a century since it was first proposed. From a field theory point of view, General Relativity may be formulated as a theory describing an interaction mediated by a massless spin-2 particle as it will be discussed in the next section. The absence of mass for this particle together with the explicit Lorentz invariance, makes it to naturally couple to the energy-momentum tensor and, since it also carries energy-momentum, consistency dictates that it needs to present self-interactions. This requirement has sometimes led to regard gravity as a theory for a spin-2 particle that is consistently coupled to its own energy-momentum tensor so that the total energy-momentum tensor is the source of the gravitational field[1]. These interactions can be constructed order by order following the usual Noether procedure (see for instance [Ort15]) and one obtains an infinite series of terms. One could attempt to resum this infinite series directly or one could use Deser's procedure [Des70] of introducing auxiliary fields so that the construction of the interactions ends at the first iteration. Either way, General Relativity arises as the full non-linear theory and the

[1] Note however that the actual crucial requirement is to maintain the gauge symmetry at the non-linear level.

equivalence principle together with diffeomorphism invariance come along in a natural way.

In this chapter, we develop a family of theories for scalar and vector fields following a similar bootstrapping approach as the one leading to General Relativity, i.e., by prescribing a coupling to the energy-momentum tensor that remains at the full non-linear level. The reader interested in the details of the bootstrapping method may find interesting some recent works [Pad08; BHL09; Des10; BCG14] in which it is widely discussed. Unlike the case of gravity where the coupling to the energy-momentum tensor comes motivated from the requirement of maintaining gauge invariance, in our case there is no real necessity to incorporate a consistent coupling to the energy-momentum tensor nor self-couplings of this form. However, the construction of theories whose interactions are universally described in terms of the energy-momentum tensor (as to fulfill some form of equivalence principle) is an alluring question in relation with gravitational phenomena. Let us remind that, starting from Newton's law, the simplest (perhaps naive) relativistic completion is to promote the gravitational potential to a scalar field. However, the most leading order coupling of the scalar to the energy-momentum tensor is through its trace and, therefore, there is no bending of light. This is a major obstacle for this simple theory of gravity based on a scalar field since the bending of light is a paramount feature of the gravitational interaction. Nevertheless, the problem of finding a theory for a scalar field that couples in a self-consistent manner to the energy-momentum tensor is an interesting problem on its own that has already been considered in the literature [Kra55; FN68; DH70; SP88; SP03]. In this chapter we will extend those results for the case of more general couplings for a scalar field (adding a shift symmetry that leads to derivative couplings) and also explore the case of vector fields coupled to the energy-momentum tensor. These new fields could potentially be new mediators of the gravitational interaction at high energies. Furthermore, as we will discuss, the rich phenomenology associated to this family of scalar and vector theories can potentially make them suitable dark matter candidates. We will hence analyse the possible constraints on their energy scales set by direct and indirect dark matter searches.

2.1 | Gravity as a spin-2 massless field theory

We will see that the geometrical theory of General Relativity as it was formulated in the introductory chapter can be regarded from a special relativistic field theory point of view as the theory of a massless spin-2 field coupled to the total energy-momentum tensor. The action for a free theory of a massless spin-2 field such that it has divergenceless equations of motion is the Fierz-Pauli action:

$$\mathcal{S}_{\rm FP} = \frac{1}{2}\int \mathrm{d}^4x \left(\frac{1}{2}\partial^\mu h^{\nu\rho}\partial_\mu h_{\nu\rho} - \partial^\mu h^{\nu\rho}\partial_\nu h_{\mu\rho} + \partial^\mu h^\rho_{\ \rho}\partial^\rho h_{\rho\mu} - \frac{1}{2}\partial^\mu h^\rho_{\ \rho}\partial_\mu h^\rho_{\ \rho}\right), \quad (2.1.1)$$

where $h_{\mu\nu}$ is the graviton field, the kinetic term has been canonically normalized and we are considering a flat Minkowski spacetime with a metric $\eta_{\mu\nu}$ determined by the line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$. This action encodes the dynamics of the theory for free gravitons. It is easy to prove that the Fierz-Pauli action is invariant under gauge transformations of the field $\delta_\epsilon h_{\mu\nu} = -2\partial_{(\mu}\epsilon_{\nu)}$ with ϵ_ν a local Lorentz vector. Using this gauge symmetry, we can remove the 8 unwanted degrees of freedom of the field. There are two popular gauges to work with spin-2 fields: the *traceless transverse gauge*:

$$h^\mu_{\ \mu} = 0; \quad \partial_\mu h^{\mu\nu} = 0, \quad (2.1.2)$$

for which the equation of motion for the free theory is written as

$$\Box h_{\mu\nu} = 0. \quad (2.1.3)$$

On the other hand it is also customary to use the *De Donder* or *harmonic gauge*

$$\partial_\mu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^\rho_{\ \rho}\right) = 0, \quad (2.1.4)$$

which leads to the equation of motion

$$\Box\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^\rho_{\ \rho}\right) = 0. \quad (2.1.5)$$

The traceless transverse gauge implies the De Donder one but the reverse is not true. Note that the traceless transverse condition does not fix the gauge completely as transformations of the form $\epsilon^\mu = \partial^\mu\epsilon$ and such as $\Box\epsilon = 0$ preserve the equations of motion. The free theory is governed by the Fierz-Pauli action, however, we want gravity to be coupled to matter via its energy-momentum tensor, as it is requested by the third postulate of General Relativity. Hence, we need to include to the theory of free gravitons a Lagrangian for the matter content and a coupling term such as the total action for the interacting system reads:

$$\mathcal{S} = \mathcal{S}_{\rm FP}[h_{\mu\nu}] + \mathcal{S}_{\rm matter}[\chi] + \frac{1}{2}\int \mathrm{d}^4x\,\varphi h_{\mu\nu}T^{\mu\nu}_{\rm matter}(\chi). \quad (2.1.6)$$

This gauge identity for the graviton field implies that the matter energy-momentum tensor has to be conserved for consistency, in analogy to what happens in General Relativity due to the Bianchi identities. Also the conservation of the energy-momentum tensor is necessary for the total action to be invariant under the gauge transformations for the graviton field $\delta_\epsilon h_{\mu\nu}$. However, let us note that due to the coupling to the graviton field, the energy-momentum tensor for the matter fields will no longer be conserved in general as can be seen in detail in [Ort15]. This consistency problem is overcome by assuming that the graviton field has to couple also to its own energy-momentum tensor and hence the conserved quantity is the total energy-momentum tensor for the whole system $T^{\mu\nu}(\chi, h)$.

However a little inconvenient appears at this point as if we want to derive the equations of motion for the system with the full energy-momentum tensor as a source, we need to include new terms in the action, which will induce new terms in the energy-momentum tensor. To include these new terms in the equations of motion we will need to introduce new corrections to the action and so on and so forth. This problem is actually already present in the pure gravity scenario once we accept the self-interaction of the gravitons through their own energy-momentum tensor $t^{\mu\nu}$ with the same strength as with matter. We can say we have achieved consistency if the equations of motion

$$\mathcal{D}^{\mu\nu}(h) = \chi \left[T^{\mu\nu}_{\text{matter}}(\chi, h) + t^{\mu\nu}\right], \tag{2.1.7}$$

where $\mathcal{D}^{\mu\nu}$ is a wave operator and $T^{\mu\nu}_{\text{matter}}(\chi, h)$ reads for the energy-momentum tensor of matter and the interaction with the graviton, are consistent with the equations of motion for matter

$$\partial_\mu \left[T^{\mu\nu}_{\text{matter}}(\chi, h) + t^{\mu\nu}\right] = 0, \tag{2.1.8}$$

on-shell.

As it was demonstrated by Deser [DH70] using an elegant argument involving auxiliary fields that stop the iterative process at the first iteration, General Relativity arises from the Fierz-Pauli theory when all the corrections to the action are resumed. Furthermore, from the works of Weinberg, and Boulware and Deser, the conclusion that any interacting quantum theory of a spin-2 particle coincides with General Relativity in the infrared limit follows.

We will construct theories for scalar and vector fields taking as inspiration the consistency condition for the graviton field and imposing a coupling to the energy-momentum tensor at the linear-level of the action that we want to preserve at the full non-linear level of the resumed action. Then, we will discuss the resulting theories from this procedure and the associated phenomenology as dark matter candidates.

2.2 | Scalar self-coupled field

We will start our study with the simplest case of a scalar field theory that couples to the trace of the energy-momentum tensor to recover known results for scalar gravity. Then, we will extend these results to include derivative interactions that typically arise from disformal couplings. Such couplings will be the natural ones when imposing a shift symmetry for the scalar field, as usually happens for Goldstone bosons. We will also consider the problem from a first order point of view. Finally, couplings to matter, both derivative and non-derivative, will be constructed.

2.2.1 | Self-interactions for scalar field

Let us begin our tour on theories coupled to the energy-momentum tensor from a scalar field and focus on the self-coupling problem neglecting other fields, i.e., we will look for consistent couplings of the scalar field to its own energy-momentum tensor. Firstly, we need to properly define our procedure. Our starting point will be the action for a free massive scalar field given by

$$\mathcal{S}_{(0)} = \frac{1}{2}\int \mathrm{d}^4x \left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right). \tag{2.2.1}$$

The goal now is to add self-interactions of the scalar field through couplings to the energy-momentum tensor. This can be done in two ways, either by imposing a coupling of the scalar field to its own energy-momentum tensor at the level of the action or by imposing its energy-momentum tensor to be the source in the field equations. We will solve both cases for completeness and to show the important differences that arise in both procedures at the non-linear level. Let us start by adding an interaction of the scalar to the energy-momentum tensor of the free field in the action as follows

$$\mathcal{S}_{(1)} = -\frac{1}{M_{\rm sc}}\int \mathrm{d}^4x\, \varphi T_{(0)}, \tag{2.2.2}$$

with $M_{\rm sc}$ some mass scale determining the strength of the interaction and $T_{(0)}$ the trace of the energy-momentum tensor of the free scalar field, i.e., the one associated to $\mathcal{S}_{(0)}$. We encounter here the usual ambiguity due to the different available definitions for the energy-momentum tensor that differ either by a term of the form $\partial_\alpha\Theta^{[\alpha\mu]\nu}$ with $\Theta^{[\alpha\mu]\nu}$ some super-potential antisymmetric in the first pair of indices so that it is off-shell divergenceless, or by a term proportional to the field equations (or more generally, any rank-2 tensor whose divergence vanishes on-shell). In both cases, the form of the added piece guarantees that all of the related energy-momentum tensors give the same

Lorentz generators, i.e., they carry the same total energy and momentum. We will consider in more detail the role of such boundary terms in Sec. 2.4 and, until then, we will adopt the Hilbert prescription to compute the energy-momentum tensor in terms of a functional derivative with respect to an auxiliary metric tensor as follows[2]

$$T^{\mu\nu} \equiv \left(-\frac{2}{\sqrt{-\gamma}} \frac{\delta \mathcal{S}[\gamma_{\mu\nu}]}{\delta \gamma_{\mu\nu}} \right)_{\gamma_{\mu\nu}=\eta_{\mu\nu}}, \tag{2.2.3}$$

where in the action we need to replace $\eta_{\mu\nu} \to \gamma_{\mu\nu}$ with $\gamma_{\mu\nu}$ some background (Lorentzian) metric and γ its determinant. This definition has the advantage of directly providing a symmetric and gauge-invariant (in case of fields with spin and/or internal gauge symmetries) energy-momentum tensor. In general, this does not happen for the canonical energy-momentum tensor obtained from Noether's theorem, although the Belinfante-Rosenfeld procedure [Bel39; Bel40; Ros40] allows to *correct* it and transform it into one with the desired properties [3]. For the scalar field theory we are considering, the energy-momentum tensor is

$$T^{\mu\nu}_{(0)} = \partial^\mu \varphi \partial^\nu \varphi - \eta^{\mu\nu} \mathcal{L}_\varphi, \tag{2.2.4}$$

which is also the one obtained as Noether current so that the above discussion is not relevant here. However, in the subsequent sections dealing with vector fields this will be important since the canonical and Hilbert energy-momentum tensors differ.

After settling the ambiguity in the energy-momentum tensor, we can now write the first order corrected action for $\mathcal{S}_{(0)}$ by incorporating the coupling (2.2.2), so we obtain

$$\mathcal{S}_{(0)} + \mathcal{S}_{(1)} = \frac{1}{2} \int \mathrm{d}^4 x \left[\left(1 + 2\frac{\varphi}{M_{\rm sc}} \right) \partial_\mu \varphi \partial^\mu \varphi - \left(1 + 4\frac{\varphi}{M_{\rm sc}} \right) m^2 \varphi^2 \right]. \tag{2.2.5}$$

As usual, when we introduce the coupling of the scalar field to the energy-momentum tensor, the new energy-momentum tensor of the whole action acquires a new contribution and, therefore, the coupling $-\frac{\varphi}{M_{sc}} T$ receives additional corrections that will contribute to order $1/M_{\rm sc}^2$. The added $1/M_{\rm sc}^2$ interaction will again add a new correction that will contribute an order $1/M_{\rm sc}^3$ term and so on. This iterative process will continue indefinitely so we end up with a construction of the interactions as a perturbative expansion in powers of $\varphi/M_{\rm sc}$ and, thus, we obtain an infinite series whose

[2]This definition is not free from subtleties either since one also needs to choose a covariantisation procedure and the tensorial character of the fields. We will assume a minimal coupling prescription for the covariantisation and that all the fields keep their tensorial character as in the original Minkowski space.

[3]See also [GM92] for a method to construct an energy-momentum tensor that can be interpreted as a generalisation of the Belinfante-Rosenfeld procedure.

resummation will give the final desired action. The iterative process for the case at hand gives the following expansion for the first few terms:

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x \left[\left(1 + 2\frac{\varphi}{M_{\rm sc}} + 4\frac{\varphi^2}{M_{\rm sc}^2} + 8\frac{\varphi^3}{M_{\rm sc}^3} + \cdots\right)\partial_\mu\varphi\partial^\mu\varphi \right.$$
$$\left. - \left(1 + 4\frac{\varphi}{M_{\rm sc}} + 16\frac{\varphi^2}{M_{\rm sc}^2} + 64\frac{\varphi^3}{M_{\rm sc}^3} + \cdots\right) m^2\varphi^2\right]. \tag{2.2.6}$$

It is not difficult to identify that we obtain the first terms of a geometric progression with ratios $2\varphi/M_{\rm sc}$ and $4\varphi/M_{\rm sc}$ which can then be easily resummed. One can confirm this by realising that a term $(\varphi/M_{\rm sc})^n\partial_\mu\varphi\partial^\mu\varphi$ gives a correction $2(\varphi/M_{\rm sc})^{n+1}\partial_\mu\varphi\partial^\mu\varphi$, while a term $(\varphi/M_{\rm sc})^{m+2}$ introduces a correction $4(\varphi/M_{\rm sc})^{m+3}$. Then, the resummed series will be given by

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x \Big[\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}(\varphi)m^2\varphi^2\Big], \tag{2.2.7}$$

with

$$\mathcal{K}(\varphi) = \sum_{n=0}^{\infty}\left(2\frac{\varphi}{M}\right)^n = \frac{1}{1-2\varphi/M_{\rm sc}}, \qquad \mathcal{U}(\varphi) = \sum_{n=0}^{\infty}\left(4\frac{\varphi}{M}\right)^n = \frac{1}{1-4\varphi/M_{\rm sc}}. \tag{2.2.8}$$

This recovers the results in [SP03] in the corresponding limits. Technically, the geometric series only converges for[4] $2\varphi < M_{\rm sc}$, but the final result can be extended to values $2\varphi > M_{\rm sc}$, barring the potential poles at $2\varphi/M_{\rm sc} = 1$ and $4\varphi/M_{\rm sc} = 1$ that occur for positive values of the scalar field, assuming $M_{\rm sc} > 0$, while for $\varphi < 0$ the functions are analytic. Let us also notice that, had we started with an arbitrary potential for the scalar field $V(\varphi)$ instead of a mass term, the corresponding final action would have resulted in a re-dressed potential with the same factor, i.e., the effect of the interactions on the potential would be $V(\varphi) \to \mathcal{U}(\varphi)V(\varphi)$ and, as a particular case, if we start with a constant potential V_0 corresponding to a cosmological constant, the same re-dressing will take place so that the cosmological constant becomes a $\varphi-$dependent quantity. In any case, we find it more natural to start with a mass term in compliance with the prescribed procedure of generating the interactions through the coupling to the energy-momentum tensor, i.e., the natural starting point is the free theory.

An alternative way to resum the series that will be very useful in the less obvious cases that we will consider later is to notice that the resulting perturbative expansion (2.2.6) allows to guess the final form of the action to be of the form (2.2.7). Then,

[4] The convergence of the generated perturbative series will be a recurrent issue throughout our analysis. In fact, most of the obtained series will need to be interpreted as asymptotic series of the true underlying theory. We will discuss this issue in more detail in due time.

we can impose the desired form of our interactions to the energy-momentum tensor so that the full non-linear action must satisfy

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x \left(\frac{1}{2}\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}(\varphi)V(\varphi)\right) \\ &= \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) - \frac{\varphi}{M_{\mathrm{sc}}}T\right),\end{aligned} \tag{2.2.9}$$

with T the trace of the energy-momentum tensor of the full action, i.e.,

$$T = -\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi + 4\mathcal{U}V. \tag{2.2.10}$$

We have also included here an arbitrary potential for generality. Thus, we will need to have

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x \left(\frac{1}{2}\mathcal{K}\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}V\right) \\ &= \int \mathrm{d}^4x \left[\frac{1}{2}\Big(1 + \frac{2\varphi}{M_{\mathrm{sc}}}\mathcal{K}\Big)\partial_\mu\varphi\partial^\mu\varphi - \Big(1 + \frac{4\varphi}{M_{\mathrm{sc}}}\mathcal{U}\Big)V\right],\end{aligned} \tag{2.2.11}$$

from which we can recover the solutions for $\mathcal{K}$ and $\mathcal{U}$ given in (2.2.8). Notice that this method allows to obtain the final action without relying on the convergence of the perturbative series and, thus, the aforementioned extension of the resummed series is justified. As a final remark, it is not difficult to see that, had we started with a coupling to an arbitrary function of φ of the form $-f(\varphi/M_{\mathrm{sc}})T_{(0)}$, the final result would be the same with the replacement $\varphi/M_{\mathrm{sc}} \to f(\varphi/M_{\mathrm{sc}})$ in the final form of the function $\mathcal{K}$ and $\mathcal{U}$, recovering that way the results of [SP03].

We have then obtained the action for a scalar field coupled to its own energy-momentum tensor at the level of the action. However, as we mentioned above, we can alternatively impose the trace of the energy-momentum tensor to be the source of the scalar field equations, i.e., the full theory must lead to equations of motion satisfying

$$(\Box + m^2)\varphi = -\frac{1}{M_{\mathrm{sc}}}T, \tag{2.2.12}$$

again with T the total energy-momentum tensor of the scalar field. Before proceeding to solve this case, let us comment on some important differences with respect to the gravitational case involving a spin-2 field. The above equation is perfectly consistent at first order, i.e., we could simply add $T_{(0)}$ on the RHS, so we already have a consistent theory and there is no need to include higher order corrections. This is in high contrast with the construction in standard gravity where the Bianchi identities for the spin-2 field (consequence of the required gauge symmetry) are incompatible with the conservation of the energy-momentum tensor and one must add higher order corrections to have consistent equations of motion. For the scalar gravity case, although not imposed by the consistency of the equations, we can extend the construction in an analogous manner and impose that the source of the equation is not given in terms of the energy-momentum tensor of the free scalar field, but the total energy-momentum tensor. As before, we could proceed order by order to find the interactions, but we will directly resort to guess the final action to be of the form given in (2.2.7) and obtain the required form of the functions $\mathcal{K}$ and $\mathcal{U}$ for the field equations to be of the form given in (2.2.12). For the sake of generality, we will consider a general bare potential $V(\varphi)$ instead of a simple mass term. By varying (2.2.7) with respect to the scalar field we obtain

$$\Box\varphi = -\frac{\mathcal{K}'}{2\mathcal{K}}(\partial\varphi)^2 - \frac{(\mathcal{U}V)'}{\mathcal{K}}, \tag{2.2.13}$$

that must be compared with the prescribed form of the field equation

$$\Box\varphi + V' = -\frac{1}{M_{\rm sc}}T = \frac{1}{M_{\rm sc}}\Big[\mathcal{K}(\partial\varphi)^2 - 4\mathcal{U}V\Big]. \tag{2.2.14}$$

Thus, we see that the functions $\mathcal{K}$ and $\mathcal{U}$ must satisfy the following equations

$$\mathcal{K}' = -\frac{2\mathcal{K}^2}{M_{\rm sc}}, \qquad \mathcal{U}' + \left(\frac{V'}{V} - \frac{4\mathcal{K}}{M_{\rm sc}}\right)\mathcal{U} = \mathcal{K}\frac{V'}{V}. \tag{2.2.15}$$

The solution for $\mathcal{K}$ can be straightforwardly obtained to be

$$\mathcal{K} = \frac{1}{1 + 2\varphi/M_{\rm sc}} \tag{2.2.16}$$

where we have chosen the integration constant so that $\mathcal{K}(0) = 1$, i.e., we absorbed $\mathcal{K}(0)$ into the normalization of the free field. It might look surprising that the solution for $\mathcal{K}$ in this case is related to (2.2.8) by a change of sign of φ. This could have been anticipated by noticing that the construction of the theory so that T appears as a source of the field equations requires an extra minus sign with respect to the coupling at the level of the action to compensate for the one introduced by varying the action. Thus, the two series only differ by this extra $(-1)^n$ factor in the series that results in the overall change of sign of φ.

From the obtained equations we see that the solution for $\mathcal{U}$ depends on the form of the bare potential $V(\varphi)$. We can solve the equation for an arbitrary potential and the solution is given by

$$\mathcal{U} = \frac{(1 + 2\varphi/M_{\rm sc})^2}{V(\varphi)}\left[C_1 + \int \frac{V'(\varphi)}{(1 + 2\varphi/M_{\rm sc})^3}{\rm d}\varphi\right], \tag{2.2.17}$$

with C_1 an integration constant that must be chosen so that $\mathcal{U}(0) = 1$. If $V'(\varphi) \neq 0$, we need to set $C_1 = 0$. Remarkably, if we take a quadratic bare potential corresponding to adding a mass for the scalar field (which is the most natural choice if we start from a free theory), the above solution reduces to $\mathcal{U} = 1$. In that case, the resummed action reads

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}^4x \left[\frac{(\partial\varphi)^2}{1+2\varphi/M_{\rm sc}} - m^2\varphi^2\right], \tag{2.2.18}$$

which is the action already obtained by Freund and Nambu in [FN68], and which reduces to Nordstrøm's theory in the massless limit. We have obtained here the solution for the more general case with an arbitrary bare potential, in which case the solution for $\mathcal{U}$ depends on the form of the potential. Finally, if we have $V' = 0$, the starting action contains a cosmological constant V_0 and the obtained solution for $\mathcal{U}(\varphi)$ gives the scalar field re-dressing of the cosmological constant, which is $(1+2\varphi/M_{\rm sc})^2$ obtained after setting $C_1 = V_0$ as it corresponds to have $\mathcal{U}(0) = 1$. We will re-obtain this result in Sec. 2.2.4 when studying couplings to matter fields.

2.2.2 | Derivatively self-coupled scalar field

After warming up with the simplest coupling of the scalar field to the trace of its own energy-momentum tensor, we will now look at interactions enjoying a shift symmetry, what happens for instance in models where the scalar arises as a Goldstone boson, a paradigmatic case in gravity theories being branons, that are associated to the breaking of translations in extra dimensions [DM01]. This additional symmetry imposes that the scalar field must couple derivatively to the energy-momentum tensor and this further imposes that the leading order interaction must be quadratic in the scalar field, i.e., we will have a coupling of the form $\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}$. This is also the interaction arising in theories with disformal couplings [Bek93]. Although an exact shift symmetry is only compatible with a massless scalar field, we will leave a mass term for the sake of generality (and which could arise from a softly breaking of the shift symmetry). In fact, again and for the sake of generality, we will consider a general bare potential term. Then, the action with the first order correction arising from the derivative coupling to the energy-momentum tensor in this case is given by

$$\mathcal{S}_{(0)} + \mathcal{S}_{(1)} = \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + \frac{1}{M_{\rm sd}^4}\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}_{(0)}\right], \tag{2.2.19}$$

with $M_{\rm sd}$ some mass scale. It is interesting to notice that now the coupling is suppressed by $M_{\rm sd}^{-4}$ so that the leading order interaction corresponds to a dimension 8 operator, unlike in the previous non-derivative coupling whose leading order was a dimension 5 operator. As before, the added interaction will contribute to the energy-momentum tensor so that the interaction needs to be corrected. If we proceed with this iterative process, we find the expansion

$$\mathcal{S}_\varphi = \int \mathrm{d}^4x \left[\frac{1}{2}\Big(1 + X + 3X^2 + 15X^3 + \cdots\Big)\partial_\mu\varphi\partial^\mu\varphi - \Big(1 - X - X^2 - 3X^3 \cdots\Big)V(\varphi)\right]. \tag{2.2.20}$$

where we have defined $X \equiv (\partial\varphi)^2/M_{\rm sd}^4$. Again, we could obtain the general term of the generated series and eventually resum it. However, it is easier to use an ansatz for the resummed action by noticing that, from (2.2.20), we can guess the final form of the action to be

$$\mathcal{S}_\varphi = \int \mathrm{d}^4x \left[\frac{1}{2}\mathcal{K}(X)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}(X)V(\varphi)\right], \tag{2.2.21}$$

with $\mathcal{K}(X)$ and $\mathcal{U}(X)$ some functions to be determined from our prescribed couplings. Thus, by imposing that the final action must satisfy

$$\mathcal{S}_\varphi = \int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + \frac{1}{M_{\rm sd}^4}\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}\right], \tag{2.2.22}$$

with $T_{\mu\nu}$ the total energy-momentum tensor, we obtain the following relation:

$$\begin{aligned}\mathcal{S}_\varphi &= \int \mathrm{d}^4x\Big[\frac{1}{2}\mathcal{K}(X)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}(X)V(\varphi)\Big]\\ &= \int \mathrm{d}^4x\Big[\frac{1}{2}\Big(1 + X\mathcal{K}(X) + 2X^2\mathcal{K}'(X)\Big)\partial_\mu\varphi\partial^\mu\varphi\\ &\qquad -\Big(1 - X\mathcal{U}(X) + 2X^2\mathcal{U}'(X)\Big)V(\varphi)\Big],\end{aligned} \tag{2.2.23}$$

where the prime stands for derivative with respect to its argument. Thus, the functions $\mathcal{K}(X)$ and $\mathcal{U}(X)$ will be determined by the following first order differential equations

$$\mathcal{K}(X) = 1 + X\mathcal{K}(X) + 2X^2\mathcal{K}'(X), \tag{2.2.24}$$

$$\mathcal{U}(X) = 1 - X\mathcal{U}(X) + 2X^2\mathcal{U}'(X). \tag{2.2.25}$$

We have thus reduced the problem of resuming the series to solving the above differential equations. The existence of solutions for these differential equations will guarantee the convergence (as well as the possible analytic extensions) of the perturbative series. Although not important for us here, it is possible to obtain the explicit analytic solutions as

$$\mathcal{K} = -\frac{e^{-1/(2X)}}{2X}\mathrm{Ei}_{1/2}\big(-1/(2X)\big), \tag{2.2.26}$$

$$\mathcal{U} = -\frac{e^{-1/(2X)}}{2X}\mathrm{Ei}_{-1/2}(-1/(2X)). \tag{2.2.27}$$

where $\mathrm{Ei}_n(x)$ stands for the exponential integral function of order n and we have chosen the integration constants in order to have a well-defined solution for $X \to 0$. In principle, one might think that boundary conditions must be imposed so that $\mathcal{K}(0) = \mathcal{U}(0) = 1$. However, these boundary conditions are actually satisfied by all solutions of the above equations since they are hardwired in the own definition of the functions $\mathcal{K}$ and $\mathcal{U}$ through the perturbative series. The way to select the right solution is thus by imposing regularity at the origin $X = 0$. Even this condition is not sufficient to select one single solution and this is related to the fact that the perturbative series must be interpreted as an asymptotic expansion[5], rather than a proper series expansion. In fact, it is not difficult to check that the perturbative series is divergent, as it is expected for asymptotic expansions. Thus, the above solution is actually one of many different possible solutions. We will find these equations often and we will defer a more detailed discussion of some of their features in the discussion section.

So far we have focused on the coupling $\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}$, but, at this order, we can be more general and allow for another interaction of the same dimension so that the first correction becomes

$$\mathcal{S}_{(1)} = \frac{1}{M_{\rm sd}^4}\int \mathrm{d}^4x\Big(b_1\partial_\mu\varphi\partial_\nu\varphi + b_2\partial_\alpha\varphi\partial^\alpha\varphi\eta_{\mu\nu}\Big)T^{\mu\nu}_{(0)}, \tag{2.2.28}$$

where b_1 and b_2 are two arbitrary dimensionless parameters, one of which could actually be absorbed into $M_{\rm sd}$, but we prefer to leave it explicitly to keep track of the two different interactions. The previous case then reduces to $b_2 = 0$, which is special in that the coupling does not depend on the metric and, as we will see in Sec. 2.5, this has interesting consequences in some constructions. For this more general coupling, the perturbative series reads

$$\begin{aligned}\mathcal{S}_\varphi = \int \mathrm{d}^4x\Big[\frac{1}{2}\Big(&1 + (b_1 - 2b_2)X + 3b_1(b_1-2b_2)X^2\\ &+ 3b_1(b_1-2b_2)(5b_1+2b_2)X^3 + \cdots\Big)\partial_\mu\varphi\partial^\mu\varphi\\ -\Big(1 - (b_1+4b_2)&\left(X + (b_1-2b_2)X^2 + 3b_1(b_1-2b_2)X^3\cdots\right)\Big)V(\varphi)\Big].\end{aligned} \tag{2.2.29}$$

To resum the series we can follow the same procedure as before using the same Ansatz for the resummed action as in (2.2.21), in which case we obtain that the following relation must hold:

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x\left[\frac{1}{2}\mathcal{K}(X)\partial_\mu\varphi\partial^\mu\varphi - \mathcal{U}(X)V(\varphi)\right]\\ &= \int \mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + \frac{1}{M_{\rm sd}^4}\Big(b_1\partial_\mu\varphi\partial_\nu\varphi + b_2\partial_\alpha\varphi\partial^\alpha\varphi\eta_{\mu\nu}\Big)T^{\mu\nu}\right]\end{aligned}$$

[5] In fact, the solution resembles one of the paradigmatic examples of asymptotic expansion $e^{-1/t}\mathrm{Ei}(1/t) = \sum_{n=0}^{\infty} n! t^{n+1}$.

$$= \int d^4x \left[\frac{1}{2}\Big(1+(b_1-2b_2)X\mathcal{K}+2(b_1+b_2)X^2\mathcal{K}'\Big)\partial_\mu\varphi\partial^\mu\varphi \right.$$
$$\left. -\Big(1-(b_1+4b_2)X\mathcal{U}+2(b_1+b_2)X^2\mathcal{U}'\Big)V(\varphi)\right]. \tag{2.2.30}$$

Thus, the equations to be satisfied in this case are

$$\mathcal{K} = 1+(b_1-2b_2)X\mathcal{K}+2(b_1+b_2)X^2\mathcal{K}', \tag{2.2.31}$$
$$\mathcal{U} = 1-(b_1+4b_2)X\mathcal{U}+2(b_1+b_2)X^2\mathcal{U}'. \tag{2.2.32}$$

The additional freedom to choose the relation between the two free parameters b_1 and b_2 allows now to straightforwardly obtain some particularly interesting solutions. Firstly, for $b_1+b_2=0$, the equations become algebraic and the unique solution is given by

$$\mathcal{K}(X)=\mathcal{U}(X)=\frac{1}{1-3b_1X}. \tag{2.2.33}$$

This particular choice of parameters that make the equations algebraic is remarkable because it precisely corresponds to coupling the energy-momentum tensor to the orthogonal projector to the gradient of the scalar field $\eta_{\mu\nu}-\partial_\mu\varphi\partial_\nu\varphi/(\partial\varphi)^2$. On the other hand, we can see from the perturbative series (2.2.29) that the condition $b_1-2b_2=0$ cancels all the corrections to the kinetic term and this can also be seen from the differential equations where it is apparent that, for those parameters, $\mathcal{K}=1$ is the corresponding solution. Moreover, for that choice of parameters, we see from the perturbative expansion that $\mathcal{U}=1-6b_2X$, which can be confirmed to be the solution of the equation for $\mathcal{U}$ with $b_1=2b_2$. Likewise, for $b_1+4b_2=0$, all the corrections to the potential vanish and only the kinetic term is modified. It is worth mentioning that the iterative procedure used to construct the interactions also allows to obtain polynomial solutions of arbitrarily higher order by appropriately choosing the parameters. All these interesting possibilities are explained in more detail in section 2.7.

Finally, let us notice that a constant potential V_0 that amounts to introducing a cosmological constant in the free action leads to a re-dressing of the cosmological constant analogous to what we found above for the non-derivative coupling, but with the crucial difference that now the cosmological constant becomes kinetically re-dressed in the full theory.

In the general case we see that we obtain a particular class of K-essence theories where the φ-dependence is entirely given by the starting potential, but it receives a kinetic-dependent re-dressing. On the other hand, if we start with an exact shift symmetry, given that the interactions do not break it, the resulting theory reduces to a particular class of $P(X)$ theories.

2.2.3 | First order formalism

In the previous section we have looked at the theory for a scalar field that is derivatively coupled to its own energy-momentum tensor. The problem was reduced to solving a couple of differential equations expressed in (2.2.25). Here we will explore the same problem but from the first order formalism perspective. In the case of non-abelian gauge fields and also in the case of gravity, the first order formalism has proven to significantly simplify the problem since the iterative process ends at the first iteration [Des70]. For non-abelian theories, the first order formalism solves the self-coupling problem in one step instead of the four iterations required in the Lagrangian formalism. In the case of gravity, the simplification is even greater since it reduces the infinite iterations of the self-coupling problem to only one. This is in fact the route used by Deser to obtain the resummed action for the self-couplings of the graviton [Des70]. The significant simplifications in these cases encourages us to consider the construction of the theories with our prescription in the first order formalism in order to explore if analogous simplifications take place. As a matter of fact, the first order formalism for scalar gravitation was already explored in [DH70] for the massless theory and with a conformal coupling so that the trace of the energy-momentum tensor appears as the source of the scalar field. It was then shown the equivalence of the resulting action with Nordstrøm's theory of gravity and the massless limit of the theory obtained by Freund and Nambu [FN68] with the first order formalism (which we reproduced and extended above). We will use this formalism for the theories with derivative couplings to the energy-momentum tensor, what in the first order formalism means couplings to the canonical momentum.

The first thing we need to clarify is how we are going to define the theory in the first order formalism. The starting free theory for a massive scalar field φ can be described by the following first order action:

$$\mathcal{S}_{(0)} = \int \mathrm{d}^4x \left[\pi^\mu\partial_\mu\varphi - \frac{1}{2}\left(\pi^2 + m^2\varphi^2\right)\right], \tag{2.2.34}$$

with π^μ the corresponding momentum in phase space. Upon variations with respect to the momentum π^μ and the scalar field we obtain the usual Hamilton equations $\partial_\mu\pi^\mu + m^2\varphi = 0$ and $\pi_\mu = \partial_\mu\varphi$, which combined gives the desired equation $(\Box + m^2)\varphi = 0$. At the lowest order we then prescribe a coupling to the energy-momentum tensor as[6]

$$\mathcal{S}_{(0)} + \mathcal{S}_{(1)} = \int \mathrm{d}^4x \left[\pi^\mu\partial_\mu\varphi - \frac{1}{2}\left(\pi^2 + m^2\varphi^2\right) + \frac{1}{M_{\rm sd}^4}\pi_\mu\pi_\nu T^{\mu\nu}_{(0)}\right], \tag{2.2.35}$$

[6] Of course, we could have also added a term $\pi^2 T$, but the considered coupling will be enough to show how the use of the first order formalism leads to simpler non-differential equations.

with $T^{\mu\nu}_{(0)}$ the energy-momentum tensor corresponding to the free theory $\mathcal{S}_{(0)}$. This is the form that we will also require for the final theory replacing $T^{\mu\nu}_{(0)}$ by the total energy-momentum tensor. Following the same reasoning as in the previous section, the final theory should admit an Ansatz of the following form:

$$\mathcal{S} = \int \mathrm{d}^4x \left[\pi^\mu \partial_\mu \varphi - \mathcal{H}(\varphi, \pi^2)\right], \tag{2.2.36}$$

where the Hamiltonian[7] $\mathcal{H}$ will be some function of the phase space coordinates. Lorentz invariance imposes that the momentum can only enter through its norm. As in the Lagrangian formalism, the energy-momentum tensor admits several definitions that differ by a super-potential term or quantities vanishing on-shell. As before, we shall resort to the Hilbert energy-momentum tensor. In this approach, one needs to specify the tensorial character of the fields, which are usually assumed to be true tensors. In some cases, it is however more convenient to assume that some fields actually transform as tensorial densities. In the Deser construction, assuming that the graviton is a tensorial density simplifies the computations. At the classical level and on-shell, assuming different weights only results in terms that vanish on-shell in the energy-momentum tensor[8]. In the present case, it is convenient to assume that π^μ is a tensorial density of weight 1 such that $p^\mu = \pi^\mu/\sqrt{-\gamma}$ is a tensor of zero weight. The advantage of using this variable is twofold: on one hand, this tensorial weight for π^μ makes $\pi^\mu\partial_\mu\varphi$ already a weight-0 scalar without the need to introduce the $\sqrt{-\gamma}$ in the volume element and, consequently, this term will not contribute to the energy-momentum tensor. On the other hand, the variation of the Hamiltonian $\mathcal{H}$ with respect to the auxiliary metric $\gamma_{\mu\nu}$ gives

$$\delta\mathcal{H}(p^2) = \frac{\partial\mathcal{H}}{\partial p^2}\delta p^2 = -\frac{\partial\mathcal{H}}{\partial \pi^2}\pi^2\left(\gamma^{\mu\nu} - \frac{\pi^\mu\pi^\nu}{\pi^2}\right)\delta\gamma_{\mu\nu}, \tag{2.2.37}$$

which is proportional to the orthogonal projector to the momentum and, thus, although it does contribute to the energy-momentum tensor, it will not contribute to the interaction $T^{\mu\nu}\pi_\mu\pi_\nu$. To derive the above variation, we have taken into account that the Hamiltonian remains a scalar after the covariantisation and, because of the assumed

[7]Let us stress that this Hamiltonian function will not give, in general, the energy of the system, although that is the case for homogeneous configurations.

[8]If we re-define a given field Φ with a metric-dependent change of variables $\Phi \to \Phi' = \Phi'(\Phi, \gamma_{\mu\nu})$ (as it happens when the re-definition corresponds to a change in the tensorial weight of the field), we have the following relation for the variation of the action

$$\frac{\delta\mathcal{S}[\Phi', \gamma_{\mu\nu}]}{\delta\gamma_{\mu\nu}} = \left(\frac{\delta\mathcal{S}[\Phi', \gamma_{\mu\nu}]}{\delta\gamma_{\mu\nu}}\right)_{\Phi'} + \frac{\delta\mathcal{S}}{\delta\Phi'}\frac{\delta\Phi'}{\delta\gamma_{\mu\nu}}.$$

Since the second term on the RHS vanishes on the field equations of Φ' we obtain that both energy-momentum tensors coincide on-shell.

weight of π^μ, it will become a function of $\mathcal{H}(\varphi,\pi^2)\to\mathcal{H}\big(\varphi,p^2\big)=\mathcal{H}\big(\varphi,\pi^2/|\gamma|\big)$. The covariantised action then reads

$$\mathcal{S}=\int \mathrm{d}^4x\left[\pi^\mu\partial_\mu\varphi-\sqrt{-\gamma}\mathcal{H}\big(\varphi,\pi^2/|\gamma|\big)\right], \tag{2.2.38}$$

and the total energy-momentum tensor computed with the described prescription is given by

$$T^{\mu\nu}=-2\pi^2\frac{\partial\mathcal{H}}{\partial\pi^2}\left(\eta^{\mu\nu}-\frac{\pi^\mu\pi^\nu}{\pi^2}\right)+\mathcal{H}\eta^{\mu\nu}. \tag{2.2.39}$$

If we compare with the canonical energy-momentum tensor $\Theta^{\mu\nu}=2\frac{\partial\mathcal{H}}{\partial\pi^2}\pi^\mu\pi^\nu-\mathcal{L}\eta^{\mu\nu}$, we see that the difference is $\mathcal{H}-2\pi^2\frac{\partial\mathcal{H}}{\partial\pi^2}-\mathcal{L}$, which vanishes upon use of the Hamilton equation $\partial_\mu\varphi=\frac{\partial\mathcal{H}}{\partial\pi^\mu}$ and the relation between the Hamiltonian and the Lagrangian via a Legendre transformation $\mathcal{L}=\pi^\mu\partial_\mu\varphi-\mathcal{H}$. The final action must therefore satisfy the relation

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x\left[\pi^\mu\partial_\mu\varphi-\mathcal{H}\right]=\int \mathrm{d}^4x\left[\pi^\mu\partial_\mu\varphi-\frac{1}{2}(\pi^2+m^2\varphi^2)+\frac{1}{M_{\rm sd}^4}\pi_\mu\pi_\nu T^{\mu\nu}\right]\\ &= \int \mathrm{d}^4x\left[\pi^\mu\partial_\mu\varphi-\frac{1}{2}(\pi^2+m^2\varphi^2)+\frac{1}{M_{\rm sd}^4}\pi^2\mathcal{H}\right],\end{aligned} \tag{2.2.40}$$

where we see that the chosen weight for π^μ has greatly simplified the interaction term that simply reduces to $\pi^2\mathcal{H}$. The above relation then leads to the algebraic equation

$$\mathcal{H}=\frac{1}{2}(\pi^2+m^2\varphi^2)-\frac{1}{M_{\rm sd}^4}\pi^2\mathcal{H}, \tag{2.2.41}$$

so that the Hamiltonian of the desired action will be given by

$$\mathcal{H}=\frac{1}{2}\frac{\pi^2+m^2\varphi^2}{1+\pi^2/M_{\rm sd}^4}. \tag{2.2.42}$$

We see that the use of the first order formalism has substantially simplified the resolution of the problem since we do not encounter differential equations. Needless to say that the solutions obtained in both first and second order formalisms are different. This apparent ambiguity in the resulting theory as obtained with the first or the second order formalism actually reflects the ambiguity in the definition of the energy-momentum tensor because, as discussed above, the energy-momentum tensor Eq. (2.2.39) differs from the one used in the second order formalism by a term that vanishes on-shell (see also Footnote 8). Expressing the theory obtained here in the second order formalism is not very illuminating so we will not give it, although it would be straightforward to do it. Let us finally notice that, if the leading order term in the Hamiltonian for the limit $\pi^2/M_{\rm sd}^4\ll 1$ is assumed to be $\mathcal{H}_0$, then it is not difficult to see that the full Hamiltonian will be

$$\mathcal{H}=\frac{\mathcal{H}_0}{1+\pi^2/M_{\rm sd}^4}. \tag{2.2.43}$$

i.e., the procedure simply re-dresses the seed Hamiltonian with the factor $(1+\pi^2/M_{\rm sd}^4)^{-1}$. One interesting property of the resulting theory is that the Hamiltonian density $\mathcal{H}$ for the massless case saturates to the scale $M_{\rm sd}^4$ at large momenta.

2.2.4 | Coupling to other fields

In the previous subsections we have found the action for the self-interacting scalar field through its own energy-momentum tensor, both with ultra-local and derivative couplings. Now we will turn our analysis to the couplings of the scalar field with other matter fields following the same philosophy, i.e., the scalar will couple to the energy-momentum tensor of matter fields. For simplicity, we will only consider the case of a matter sector described by a scalar field χ. The derivative couplings will be the same as we will obtain for the vector field couplings to matter that will be treated in the next section so that, in order not to unnecessarily repeat the derivation, we will not give it here and discuss it in Sec. 2.3.4. Thus, we will only deal with the conformal couplings so that our starting action for the proxy scalar field χ including the first order coupling to φ is

$$\mathcal{S}_{\chi,(0)}+\mathcal{S}_{\chi,(1)}=\int \mathrm{d}^4x\left[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi-W(\chi)-\frac{1}{M_{\rm sc}}\varphi T_{\chi,(0)}\right], \tag{2.2.44}$$

where $T_{\chi,(0)}$ is the trace of the energy-momentum tensor of the proxy field and $W(\chi)$ the corresponding potential. Going on in the iterative process yields the following series for the total action

$$\begin{aligned}\mathcal{S}_\chi=\int \mathrm{d}^4x\Bigg[&\frac{1}{2}\left(1+\frac{2}{M_{\rm sc}}\varphi+\frac{4}{M_{\rm sc}^2}\varphi^2+\frac{8}{M_{\rm sc}^3}\varphi^3+\ldots\right)\partial_\mu\chi\partial^\mu\chi\\&-\left(1+\frac{4}{M_{\rm sc}}\varphi+\frac{16}{M_{\rm sc}^2}\varphi^2+\frac{64}{M_{\rm sc}^3}\varphi^3+\ldots\right)W(\chi)\Bigg].\end{aligned} \tag{2.2.45}$$

This expression has the form of a geometric series so it is straightforward to resum it yielding the final action for scalar gravity algebraically coupled to matter:

$$\mathcal{S}=\int \mathrm{d}^4x\left(\frac{1}{2}\alpha(\varphi)\partial_\mu\chi\partial^\mu\chi-\beta(\varphi)W(\chi)\right), \tag{2.2.46}$$

with

$$\alpha(\varphi)=\sum_{n=0}^{\infty}\left(2\frac{\varphi}{M}\right)^n=\frac{1}{1-2\varphi/M_{\rm sc}},\quad \beta(\varphi)=\sum_{n=0}^{\infty}\left(4\frac{\varphi}{M}\right)^n=\frac{1}{1-4\varphi/M_{\rm sc}}. \tag{2.2.47}$$

Not very surprisingly, we obtain the same result as for the self-couplings of the scalar field, i.e., both the kinetic and potential terms get re-dressed by the same factors as we found in Sec. 2.2.1 for φ. If the matter sector consists of a cosmological constant, which would correspond to a constant scalar field in the above solutions, the coupling procedure gives rise to an additional modification of the φ potential or, equivalently, the cosmological constant becomes a φ-dependent quantity, which is also the result that we anticipated in Sec 2.2.1 for a constant potential of φ. Some phenomenological consequences of this mechanism were explored in a cosmological context in [SP03].

As in the self-coupling case, we could have introduced the coupling so that the trace of the energy-momentum tensor appears as a source of the scalar field equations. In order to obtain the theory with the required property, we shall follow the procedure of assuming the following action for the scalar gravity field φ and the scalar proxy field χ:

$$\mathcal{S}=\int \mathrm{d}^4x\left(\frac{1}{2}\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi-\mathcal{U}(\varphi)V(\varphi)+\frac{1}{2}\tilde{\alpha}(\varphi)\partial_\mu\chi\partial^\mu\chi-\tilde{\beta}(\varphi)W(\chi)\right). \tag{2.2.48}$$

with $\mathcal{K}$, $\mathcal{U}$, $\tilde{\alpha}$ and $\tilde{\beta}$ some functions of φ that will be determined from our requirement and $W(\chi)$ is some potential for the scalar χ. In our Ansatz we have included the self-interactions of the scalar field encoded in $\mathcal{K}$ and $\mathcal{U}$. Since this sector was resolved above, we will focus here on the couplings to χ so that $\tilde{\alpha}$ and $\tilde{\beta}$ are the functions to be determined by imposing the φ field equations be of the form

$$\Box\varphi+V'(\varphi)=-\frac{1}{M_{\rm sc}}T. \tag{2.2.49}$$

The trace of the total energy-momentum tensor derived from (2.2.48) is given by

$$T=4\mathcal{U}(\varphi)V(\varphi)+4\tilde{\beta}(\varphi)W(\chi)-\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi-\tilde{\alpha}(\varphi)\partial_\mu\chi\partial^\mu\chi, \tag{2.2.50}$$

and hence we obtain that the equation of motion must be of the form

$$\Box\varphi+V'(\varphi)=-\frac{1}{M_{\rm sc}}(4\mathcal{U}(\varphi)V(\varphi)+4\tilde{\beta}(\varphi)W(\chi)-\mathcal{K}(\varphi)\partial_\mu\varphi\partial^\mu\varphi-\tilde{\alpha}(\varphi)\partial_\mu\chi\partial^\mu\chi). \tag{2.2.51}$$

On the other hand, varying (2.2.48) with respect to φ yields

$$\Box\varphi=-\frac{1}{\mathcal{K}(\varphi)}\left(\frac{1}{2}\mathcal{K}'(\varphi)\partial_\mu\varphi\partial^\mu\varphi-(\mathcal{U}(\varphi)V(\varphi))'+\frac{1}{2}\tilde{\alpha}'(\varphi)\partial_\mu\chi\partial^\mu\chi-\tilde{\beta}'(\varphi)W(\chi)\right). \tag{2.2.52}$$

Comparing (2.2.51) and (2.2.52) will give the equations that must be satisfied by the functions in our Ansatz for the action. The φ sector has already been solved in the previous subsection, so we will only pay attention to the χ sector now. Then, we see that the functions $\tilde{\alpha}$ and $\tilde{\beta}$ must satisfy the following equations

$$\frac{\tilde{\alpha}'}{\tilde{\alpha}} = \frac{2\mathcal{K}}{M_{\rm sc}}, \qquad \frac{\tilde{\beta}'}{\tilde{\beta}} = \frac{4\mathcal{K}}{M_{\rm sc}}. \tag{2.2.53}$$

The solution for these equations, taking into account the functional form of $\mathcal{K}(\varphi)$ given in 2.2.16, is then

$$\tilde{\alpha} = 1 + \frac{2\varphi}{M_{\rm sc}}, \qquad \tilde{\beta} = \left(1 + \frac{2\varphi}{M_{\rm sc}}\right)^2, \tag{2.2.54}$$

that coincides with the expression given in [FN68]. We see again that, although both procedures give the same leading order coupling to matter for the scalar field, the full theory crucially depends on whether the coupling is imposed at the level of the action or the equations. If we consider again the case of a cosmological constant as the matter sector, we see that its re-dressing with the scalar field will be different in both cases. It could be interesting to explore the differences with respect to the analysis performed in [SP03], where the coupling was assumed to occur at the level of the action.

2.3 | Vector self-coupled field

After having revisited and extended the case of a scalar field coupled to the energy-momentum tensor, we now turn to the case of a vector field. Since vectors present a richer structure than scalars due to the possibility of having a gauge invariance or not depending on whether the vector field is massless of massive, we will distinguish between gauge invariant couplings and non-gauge invariant couplings. For the latter, the existence of a decoupling limit where the dominant interactions correspond to those of the longitudinal mode will lead to a resemblance between some of the interactions obtained here and those of the derivatively coupled scalar studied above.

2.3.1 | Self-coupled Proca field

Analogously to the scalar field case, our starting point will be the action for a massive vector field given by the Proca action[9]

$$\mathcal{S}_{(0)} = \int \mathrm{d}^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A^2\right), \tag{2.3.1}$$

[9] Of course we could consider an arbitrary potential, but a mass term is the natural choice if we really assume that we start with a free theory.

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A^2 \equiv A_\mu A^\mu$ and m^2 is the mass of the vector field. The energy-momentum tensor of this field is given by

$$T^{\mu\nu}_{(0)} = -F^{\mu\alpha}F^{\nu\alpha} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{m^2}{2}\eta^{\mu\nu}A^2 + m^2 A^\mu A^\nu \,. \tag{2.3.2}$$

Unlike the case of the scalar field, this energy-momentum tensor does not coincide with the canonical one obtained from Noether's theorem and, thus, the Belinfante-Rosenfeld procedure would be needed to obtain a symmetric energy-momentum tensor, showing the importance of the choice in the definition of the energy-momentum tensor in the general case.

Along the lines of the procedure carried out in the previous sections, we will now introduce self-interactions of the vector field by coupling it to its energy-momentum tensor, so the first correction will be

$$\mathcal{S}_{(1)} = \frac{1}{M^2_{\rm vc}}\int \mathrm{d}^4x A_\mu A_\nu T^{\mu\nu}_{(0)}, \tag{2.3.3}$$

with $M^2_{\rm vc}$ the corresponding coupling scale. In this case, the leading order interaction corresponds to a dimension 6 operator. Since this interaction will also contribute to the energy-momentum tensor, we will need to add yet another correction as in the previous cases, resulting in an infinite series in $A^2/M^2_{\rm vc}$ that reads:

$$\begin{aligned}\mathcal{S} = \int \mathrm{d}^4x\Big[&- \frac{1}{4}\Big(1 - Y - Y^2 - 3Y^3 + \cdots\Big)F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2}\Big(1 + Y + 3Y^2 + 15Y^3 + \cdots\Big)m^2A^2 \\ &- \frac{1}{M^2_{\rm vc}}\Big(1 + 2Y + 9Y^2 + \cdots\Big)A_\mu A_\nu F^{\mu\alpha}F^\nu{}_\alpha\Big]\end{aligned} \tag{2.3.4}$$

where $Y \equiv A^2/M^2_{\rm vc}$. Again, to resum the iterative process we will use a guessed form for the full action. The above perturbative series makes clear that the final form of the action will take the form

$$\mathcal{S} = \int \mathrm{d}^4x\left[-\frac{1}{4}\alpha(Y)F_{\mu\nu}F^{\mu\nu} - \frac{1}{M^2_{\rm vc}}\beta(Y)A_\mu A_\nu F^{\mu\alpha}F^\nu{}_\alpha + \frac{m^2}{2}\mathcal{U}(Y)A^2\right] \tag{2.3.5}$$

where the functions α, β and $\mathcal{U}$ will be obtained by imposing the desired form of the interactions through the total energy-momentum tensor, i.e., we need to have

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x\left[-\frac{1}{4}\alpha(Y)F_{\mu\nu}F^{\mu\nu} - \frac{1}{M^2_{\rm vc}}\beta(Y)A_\mu A_\nu F^{\mu\alpha}F^\nu{}_\alpha + \frac{m^2}{2}\mathcal{U}(Y)A^2\right] \\ &= \int \mathrm{d}^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A^2 + \frac{1}{M^2_{\rm vc}}A_\mu A_\nu T^{\mu\nu}\right]\end{aligned}$$

$$= \int \mathrm{d}^4x \left[\frac{1}{4}\Big(Y\alpha(Y) + 2Y^2\alpha'(Y) - 1\Big)F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}\Big(1 + Y\mathcal{U}(Y) + 2Y^2\mathcal{U}'(Y)\Big)A^2 \right.$$
$$\left. - \frac{1}{M_{\rm vc}^2}\Big(\alpha(Y) + 3Y\beta(Y) + 2Y^2\beta'(Y)\Big)A_\mu A_\nu F^{\mu\alpha}F^\nu{}_\alpha\right]. \tag{2.3.6}$$

Thus, the coupling functions in the resummed action need to satisfy the following first order differential equations

$$\alpha = 1 - Y\alpha + 2Y^2\alpha' , \tag{2.3.7}$$
$$\beta = \alpha + 3Y\beta + 2Y^2\beta' , \tag{2.3.8}$$
$$\mathcal{U} = 1 + Y\mathcal{U} + 2Y^2\mathcal{U}' . \tag{2.3.9}$$

These equations are of the same form as the ones obtained for the derivatively coupled scalar field and the solutions will also present similar features. For instance, the perturbative series will need to be interpreted as asymptotic expansions of the solutions of the above equations. Furthermore, the integration constants that determine the desired solution are already implemented in the equations so we need to impose regularity at the origin, but this only selects a unique solution for a given semi-axis, either $Y > 0$ or $Y < 0$, that can then be matched to an infinite family of solutions in the complementary semi-axis. The particular form of the solutions is not specially relevant for us here (although it will be relevant for practical applications), but we will only remark that they correspond to the class of theories for a vector field that is quadratic in the field strength or, in other words, in which the field strength only enters linearly in the equations.

As for the derivative couplings of the scalar field, we can be more general and allow for a coupling of the form

$$\mathcal{S}_{(1)} = \frac{1}{M_{\rm vc}^2}\int \mathrm{d}^4x\Big(b_1 A_\mu A_\nu + b_2 A^2\eta_{\mu\nu}\Big)T^{\mu\nu}_{(0)}. \tag{2.3.10}$$

The iterative process in this case gives rise to

$$\mathcal{S} = \int \mathrm{d}^4x\Big[-\frac{b_1}{m^2}\Big(1 + 2(b_1+b_2)Y + (9b_1^2 + 16b_1b_2 + 8b_2^2)Y^2 + \cdots\Big)A_\mu A_\nu F^{\mu\alpha}F^\nu{}_\alpha$$
$$-\frac{1}{4}\Big(1 - b_1Y - b_1(b_1+2b_2)Y^2 - b_1(b_1+2b_2)(3b_1+4b_2)Y^3 + \cdots\Big)F_{\mu\nu}F^{\mu\nu}$$
$$+\frac{1}{2}\Big(1 + (b_1-2b_2)Y + 3b_1(b_1-2b_2)Y^2 + 3b_1(b_1-2b_2)(5b_1+2b_2)Y^3 + \cdots\Big)m^2A^2\Big]. \tag{2.3.11}$$

We can again use our Ansatz for the final action to obtain that the differential equations to be satisfied are

$$\alpha = 1 - b_1Y\alpha + 2(b_1+b_2)Y^2\alpha', \tag{2.3.12}$$

$$\beta = b_1\alpha + (3b_1 + 2b_2)Y\beta + 2(b_1 + b_2)Y^2\beta', \tag{2.3.13}$$

$$\mathcal{U} = 1 + (b_1 - 2b_2)Y\mathcal{U} + 2(b_1 + b_2)Y^2\mathcal{U}' \,. \tag{2.3.14}$$

Remarkably, we see from the perturbative expansion that the case $b_1 = 0$ exactly cancels all the corrections to the kinetic part so that $\alpha = 1$, $\beta = 0$ and only the potential sector is modified. It is not difficult to check that this is indeed a solution to the above differential equations. We can also see here again that the choice $b_1 + b_2 = 0$, which corresponds to a coupling to the orthogonal projector to the vector field given by $\eta_{\mu\nu} - A_\mu A_\nu/A^2$, reduces the equations to a set of algebraic equations whose solution is

$$\alpha = \frac{1}{1 + b_1 Y}, \tag{2.3.15}$$

$$\beta = \frac{b_1\alpha}{1 - b_1 Y} = \frac{b_1}{1 - b_1^2 Y^2}, \tag{2.3.16}$$

$$\mathcal{U} = \frac{1}{1 - 3b_1 Y}. \tag{2.3.17}$$

These solutions show that α and β present different analytic properties depending on whether the field configuration is timelike or spacelike, but, in any case, β has a pole for $|b_1 Y| = 1$ so that it seems reasonable to demand $|b_1 Y| < 1$ for this particular solution. The properties of these equations are similar to the ones we found in Sec. 2.2.2 for the derivatively coupled case and, in fact, the equation for $\mathcal{U}$ here is the same as the equation for $\mathcal{K}$ in (2.2.32), which is of course no coincidence. Thus, the more detailed analysis given in the discussion section also applies and, in particular, it will also be possible to obtain polynomial solutions by appropriately choosing the parameters, owed to the recursive procedure used to construct the interactions.

From our general solution we can also analyse what happens if our starting free theory is simply a Maxwell field, i.e., $m^2 = 0$. In that case, only the terms containing α and β will have an effect and, in fact, they will provide the vector field with a mass around non-trivial backgrounds of $F_{\mu\nu}$, signaling that the number of perturbative propagating polarisations will depend on the background configuration. However, the lack of a gauge symmetry in the full theory makes a vanishing bare mass seem like an unnatural choice.

So far we have solved the problem of a vector field coupled to its own energy-momentum tensor at the full non-linear level in the action. We could also follow the procedure of finding the theory such that the energy momentum-tensor is the source of the vector field equations of motion. This is analogous to the case of scalar gravity considered above and also the procedure that leads to GR for the spin-2 case. However, a crucial difference arises for the vector field[10] case owed to the fact that the leading order interaction is quadratic in the vector field and, thus, the energy-momentum tensor cannot act as a source of the vector field equations. We have different possibilities then on how to generalise the linear coupling to the energy-momentum tensor to the full theory at the level of the field equations. Because of the lack of a clear criterion at this point and the existence of several different inequivalent possibilities of carrying out this procedure, we will not pursue it further here and we will content ourselves with the analysis of the construction of the theories where the coupling occurs at the level of the action. We will simply mention that a possibility to get this difficulty around is by breaking Lorentz invariance. If we introduce some fixed vector u^μ (that could be identified for instance with some vev of the vector field), then we could construct our interactions as e.g. $A_\mu T^{\mu\nu}u_\nu$ so that $T^{\mu\nu}u_\nu$ would act as the source of the vector field equations and, then, we could extend this result at the non-linear level.

2.3.2 | Derivative gauge-invariant self-couplings

In the previous section we have studied the case where the vector field couples to its energy-momentum tensor without imposing gauge invariance. For a scalar field, the equations of motion do not contain any off-shell conserved current derived from some Bianchi identities, and this makes a crucial difference with respect to the vector field case where we do have the Bianchi identities derived from the $U(1)$ symmetry of the Maxwell Lagrangian. Even if we break the gauge symmetry by adding a mass term, the off-shell current leads to a constraint equation that must be satisfied. In the theories obtained in the previous section by coupling A_μ to the energy-momentum tensor, we also obtain a constraint equation by taking the divergence of the corresponding field equations. This constraint is actually the responsible for keeping three propagating degrees of freedom in the theory. In fact, starting from a massless vector field with its gauge invariance, the couplings actually generate a mass term around non-trivial backgrounds, increasing that way the number of polarisations. In this section, we will aim to re-consider our construction by maintaining the $U(1)$ gauge symmetry of Maxwell theory also in the couplings of the vector field to its own energy-momentum tensor. This also resembles somewhat the extension of the scalar field case to include derivative couplings arising from imposing a shift symmetry, although with crucial differences, for instance the scalar field interactions are based on a global symmetry while the ones considered in this section will be dictated by a gauge symmetry.

After the above clarifications on the procedure that we will follow in this section,

[10] The discussion presented here also applies to the case of derivative couplings for the scalar field.

we can write our starting action describing a massless vector field

$$\mathcal{S}_{(0)} = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu}, \tag{2.3.18}$$

and add interactions to its own energy-momentum tensor respecting the $U(1)$ symmetry. At the lowest order, these interactions can be written as

$$\mathcal{S}_{(1)} = \frac{1}{M_F^4}\int \mathrm{d}^4x\Big(b_1 F_{\mu\alpha}F_\nu{}^\alpha + b_2 F_{\alpha\beta}F^{\alpha\beta}\eta_{\mu\nu}\Big)T^{\mu\nu}_{(0)}, \tag{2.3.19}$$

with M_F the corresponding coupling scale, $b_{1,2}$ dimensionless parameters and the Maxwell energy-momentum tensor given by

$$T^{\mu\nu}_{(0)} \;=\; -F^{\mu\alpha}F^\nu{}_\alpha + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \tag{2.3.20}$$

Before proceeding further, let us digress on the form of the introduced interactions. We have followed the easiest possible path of adding couplings that trivially respect the original $U(1)$ symmetry by using the already *gauge invariant* field strength $F_{\mu\nu}$. This is along the lines of the Pauli interaction term for a charged fermion ψ given by $\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi F_{\mu\nu}$, where gauge invariance is trivially realised. Of course, this non-renormalisable term is within the effective field theory, but the usual renormalisable coupling $A_\mu J^\mu$ gives the leading order interaction. This is somehow analogous to existing constructions for the spin-2 case where one seeks for a consistent coupling of the graviton to the energy-momentum tensor respecting the original linearised diffeomorphisms invariance. Besides the usual coupling $h_{\mu\nu}T^{\mu\nu}$ that respects the symmetry upon conservation of $T^{\mu\nu}$, the realisation of the linearised diffeomorphisms can be achieved in two different ways, namely: one can either introduce a coupling of the form $h_{\mu\nu}P^{\mu\nu\alpha\beta}T_{\alpha\beta}$ with $P^{\mu\nu\alpha\beta}$ an identically divergenceless projector $\partial_\mu P^{\mu\nu\alpha\beta} = 0$ (see for instance [DL68]), or one can add a coupling of the matter fields to an exactly gauge invariant quantity. The second approach is possible by using that the linearised Riemann tensor $R^{\mathrm{L}}_{\mu\nu\alpha\beta}(h)$ is *gauge invariant*, and not only covariant (see the seminal Wald's paper [Wal86] and [Her18; HS17; BX17] for interesting discussions on these alternative couplings). This is the analogous construction scheme we are following here. One could have also tried to follow another approach and tried to construct non-derivative couplings with a field dependent realisation of the original $U(1)$ symmetry whose lowest order in fields is given by the usual transformation law. The non-linear completion of gauge symmetry can correspond either to the original $U(1)$ symmetry up to some field redefinition or to a genuine non-linear completion, what will likely require transformations involving higher derivatives of the gauge parameter (see for instance [Wal86]). This interesting path will not be pursued further here and we will focus on the simplest case.

After briefly discussing some alternatives for gauge invariant couplings, we will proceed with the construction considered here. The iterative process in this case leads to the perturbative series

$$\begin{aligned}\mathcal{S} &= -\frac{1}{4}\int \mathrm{d}^4x \Big[\Big(1+b_1Z+b_1(3b_1+4b_2)Z^2+b_1(3b_1+4b_2)^2Z^3+\cdots\Big)F_{\mu\nu}F^{\mu\nu} \\ &\quad + \frac{b_1}{M_F^4}\Big(1+(3b_1+4b_2)(Z+b_1\tilde{Z}^2)+(3b_1+4b_2)(5b_1+8b_2)Z^2+\cdots\Big)\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big)^2\Big]\end{aligned} \tag{2.3.21}$$

where $Z \equiv F_{\alpha\beta}F^{\alpha\beta}/M_F^4$, $\tilde{Z} \equiv F_{\alpha\beta}\tilde{F}^{\alpha\beta}/M_F^4$ and $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual of the field strength. Moreover, we have used the identity $4F^{\mu}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\sigma}F^{\sigma}{}_{\mu} = 2(F_{\mu\nu}F^{\mu\nu})^2 + (F_{\mu\nu}\tilde{F}^{\mu\nu})^2$. This perturbative series is not straightforwardly resummed, so we will follow the alternative procedure of making an Ansatz for the resummed action. Since we are maintaining gauge invariance the resulting action must be a function of the two independent Lorentz and gauge invariants, i.e., the action must take the form

$$\mathcal{S} = M_F^4\int \mathrm{d}^4x\,\mathcal{K}(Z,\tilde{Z}), \tag{2.3.22}$$

with $\mathcal{K}$ a function to be determined. Notice that our prescribed interactions do not break parity so that $\mathcal{K}$ will need to be an even function of $\tilde{Z}$. Given our requirement of a coupling to the energy-momentum tensor, the action also needs to take the form

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{M_F^4}\left(b_1F_{\mu\alpha}F_{\nu}{}^{\alpha}+b_2F_{\alpha\beta}F^{\alpha\beta}\eta_{\mu\nu}\right)T^{\mu\nu}\right] \\ &= M_F^4\int \mathrm{d}^4x\left[-\frac{Z}{4}-(b_1+4b_2)Z(\mathcal{K}-\tilde{Z}\mathcal{K}_{\tilde{Z}})+(2(b_1+2b_2)Z^2+b_1\tilde{Z}^2)\mathcal{K}_Z\right].\end{aligned} \tag{2.3.23}$$

From these two expressions we conclude that $\mathcal{K}(Z,\tilde{Z})$ must satisfy the partial differential equation:

$$\mathcal{K} = -\frac{Z}{4}-(b_1+4b_2)Z\Big(\mathcal{K}-\tilde{Z}\mathcal{K}_{\tilde{Z}}\Big)+\Big[2(b_1+2b_2)Z^2+b_1\tilde{Z}^2\Big]\mathcal{K}_Z. \tag{2.3.24}$$

We can easily check that for $b_1=0$ the perturbative series does not generate any interaction, which is nothing but a reflection of the conformal invariance of the Maxwell Lagrangian in 4 dimensions that leads to a traceless energy-momentum tensor. If we consider that case, the above equation reduces to $\mathcal{K} = -Z/4+4b_2(-\mathcal{K}+Z\mathcal{K}_Z+\tilde{Z}\mathcal{K}_{\tilde{Z}})$, that has the Maxwell Lagrangian as a regular solution. On the other hand, for $3b_1+4b_2=0$, we see that only the first corrections in the perturbative expansion remains so that one expects the full solution to take the simple polynomial form $\mathcal{K} = -1/4\big[(1+$

$b_1 Z)Z + b_1 \tilde{Z}^2\big]$. One can readily check that this is indeed the solution of Eq. (2.3.24) with $b_2 = -3b_1/(4b_4)$. As in the previous sections dealing with derivatively coupled scalars or non-gauge invariant couplings for vectors, the existence of this polynomial solution is a direct consequence of the recursive procedure generating the interactions and, thus, we could also choose the parameters b_1 and b_2 as to have some higher order polynomial solutions for the gauge invariant theories obtained here.

2.3.3 | First order formalism

In the previous section we have looked at the theory for a vector field that is coupled to its own energy-momentum tensor in a gauge invariant way. We have just seen that the gauge-invariant coupling leads to a partial differential equation that is, in general, not easy to solve so we will now consider the problem from the first order formalism perspective, hoping that it will simplify the resulting equations, as it happens in other contexts. Unfortunately, we will see that this does not seem to be the case here. The starting free theory for a $U(1)$ invariant vector field can be described in the first order formalism by the action

$$\mathcal{S}_{(0)} = \int \mathrm{d}^4x \left[\Pi^{\mu\nu}\partial_{[\mu}A_{\nu]} + \frac{1}{4}\Pi^{\mu\nu}\Pi_{\mu\nu}\right]. \tag{2.3.25}$$

Upon variations with respect to the momentum $\Pi^{\mu\nu}$ and the vector field, we obtain the usual De Donder-Weyl-Hamilton equations $\Pi_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu = -F_{\mu\nu}$ and $\partial_\mu \Pi^{\mu\nu} = 0$ respectively, that reproduce the usual Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. By integrating out the momentum, we reproduce the usual Maxwell theory in the second order formalism. At the lowest order we then prescribe the self-interaction be of the form

$$\mathcal{S}_{(0)} + \mathcal{S}_{(1)} = \int \mathrm{d}^4x \left[\Pi^{\mu\nu}\partial_{[\mu}A_{\nu]} + \frac{1}{4}\Pi^{\mu\nu}\Pi_{\mu\nu} + \frac{1}{M_F^4}\big(b_1\Pi_{\mu\alpha}\Pi_\nu{}^\alpha + b_2\Pi^{\alpha\beta}\Pi_{\alpha\beta}\,\eta_{\mu\nu}\big)T^{\mu\nu}_{(0)}\right], \tag{2.3.26}$$

with $T^{\mu\nu}_{(0)}$ the energy-momentum tensor corresponding to the free theory $\mathcal{S}_{(0)}$. This is the form that we will require for the final theory replacing $T^{\mu\nu}_{(0)}$ by the total energy-momentum tensor. Following the same reasoning as in the previous section, the final theory should be described by the following action

$$\mathcal{S} = \int \mathrm{d}^4x \left[\Pi^{\mu\nu}\partial_{[\mu}A_{\nu]} - \mathcal{H}(Z,\tilde{Z})\right] \tag{2.3.27}$$

where the Hamiltonian $\mathcal{H}$ must be a function of the momentum due to gauge invariance and, furthermore, Lorentz invariance imposes that the dependence must be through the only two independent Lorentz scalars, namely $Z = \Pi^{\mu\nu}\Pi_{\mu\nu}/M_F^4$ and $\tilde{Z} = \Pi^{\mu\nu}\tilde{\Pi}_{\mu\nu}/M_F^4$.

Parity will also impose it to be an even function of $\tilde{Z}$. In order to proceed to compute the energy-momentum tensor, we need to choose the tensorial character of the phase space variables and, as we discussed in Sec. 2.2.3, this may lead to substantial simplifications. As in Sec. 2.2.3, let us assume that $\Pi^{\mu\nu}$ is a density so that $P^{\mu\nu} = \Pi^{\mu\nu}/\sqrt{-\gamma}$ is a zero-weight tensor. This has the advantage that the term $\Pi^{\mu\nu}\partial_{[\mu}A_{\nu]}$ in (2.3.27) will not contribute to $T^{\mu\nu}$. The total energy-momentum tensor is thus given by

$$T^{\mu\nu} = \left[\mathcal{H} - 2\frac{\partial\mathcal{H}}{\partial Z}\Pi^2 - \frac{\partial\mathcal{H}}{\partial\tilde{Z}}\Pi^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}\right]\eta^{\mu\nu} + 4\frac{\partial\mathcal{H}}{\partial Z}\Pi^{\mu\alpha}\Pi^{\nu}{}_{\alpha}, \tag{2.3.28}$$

so that the interaction term reads

$$\frac{1}{M_F^4}\big(b_1\Pi_{\mu\alpha}\Pi_\nu{}^\alpha + b_2\Pi^{\alpha\beta}\Pi_{\alpha\beta}\,\eta_{\mu\nu}\big)T^{\mu\nu} = (b_1+4b_2)Z\left(\mathcal{H} - \tilde{Z}\frac{\partial\mathcal{H}}{\partial\tilde{Z}}\right) + \left[b_1\tilde{Z}^2 - 4b_2Z^2\right]\frac{\partial\mathcal{H}}{\partial Z}. \tag{2.3.29}$$

From this expression it is already apparent that the resummation will necessarily involve the resolution of a partial differential equation as in the first order formalism case. In fact, the resulting equation will be of the same type and, consequently, resorting to the first order formalism does not lead to any simplification. One may think that another choice of the weight for the momentum could lead to some simplifications, but that is not the case and, in fact, choosing an arbitrary weight leads to the same result.To show this more explicitly, let us assume that the momentum has an arbitrary weight w so that $P^{\mu\nu} = (\sqrt{-\gamma})^{-w}\Pi^{\mu\nu}$ is a tensor of zero weight. Then, the variation of the Hamiltonian with respect to the auxiliary metric will give

$$\delta\mathcal{H} = -\frac{\partial\mathcal{H}}{\partial Z}\Big(w\Pi^2\gamma^{\mu\nu} - 2\Pi^{\mu\alpha}\Pi^{\nu}{}_{\alpha}\Big)\delta\gamma_{\mu\nu} + \frac{1}{2}\frac{\partial\mathcal{H}}{\partial\tilde{Z}}(1-2w)\Pi^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}\gamma^{\mu\nu}\delta\gamma_{\mu\nu} \tag{2.3.30}$$

where we have taken into account that the Hamiltonian $\mathcal{H}$ becomes a function of the densities $\mathcal{H}\big(Z,\tilde{Z}\big) \to \mathcal{H}\big(|\gamma|^{-w/2}Z, |\gamma|^{-w/2}\tilde{Z}\big)$ after our covariantisation choice. The total energy-momentum tensor can be readily computed to be

$$T^{\mu\nu} = \left[(w-1)\Pi^{\alpha\beta}\partial_{[\alpha}A_{\beta]} + \mathcal{H} - 2w\frac{\partial\mathcal{H}}{\partial Z}\Pi^2 + (1-2w)\frac{\partial\mathcal{H}}{\partial\tilde{Z}}\Pi^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}\right]\eta^{\mu\nu} + 4\frac{\partial\mathcal{H}}{\partial Z}\Pi^{\mu\alpha}\Pi^{\nu}{}_{\alpha}\,. \tag{2.3.31}$$

It is easy to see that this expression directly gives the energy-momentum tensor of a Maxwell field with $\mathcal{H} = -\frac{1}{4}\Pi^2$ if we set $w = 0$ and integrate out the momentum by using $\Pi^{\mu\nu} = -F^{\mu\nu}$. For the general case, we need to express the energy-momentum tensor in phase space variables. From the equation for $\Pi^{\mu\nu}$ we have

$$\partial_{[\mu}A_{\nu]} = \frac{\partial\mathcal{H}}{\partial\Pi^{\mu\nu}} = 2\frac{\partial\mathcal{H}}{\partial Z}\Pi_{\mu\nu} + \frac{\partial\mathcal{H}}{\partial\tilde{Z}}\tilde{\Pi}_{\mu\nu}. \tag{2.3.32}$$

If we insert this expression into (2.3.31) we obtain

$$T^{\mu\nu} = \left[\mathcal{H} - 2\frac{\partial\mathcal{H}}{\partial Z}\Pi^2 - \frac{\partial\mathcal{H}}{\partial\tilde{Z}}\Pi^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}\right]\eta^{\mu\nu} + 4\frac{\partial\mathcal{H}}{\partial Z}\Pi^{\mu\alpha}\Pi^{\nu}{}_{\alpha}, \tag{2.3.33}$$

which does not depend on the weight and, therefore, it is exactly the same that we obtained in (2.3.28). Of course, this is not very surprising, since, as a consequence of the argument in footnote 8, different choices of weights only result in quantities that vanish on-shell. In this case, the use of the equation of motion of $\Pi^{\mu\nu}$ in order to express $T^{\mu\nu}$ in phase space variables precisely corresponds to the mentioned on-shell difference.

For completeness, let us give the resulting equation in this case

$$\mathcal{H} = -\frac{M_F^4}{4} Z - (b_1 + 4b_2) Z \left(\mathcal{H} - \tilde{Z} \frac{\partial \mathcal{H}}{\partial \tilde{Z}} \right) - \left[b_1 \tilde{Z}^2 - 4b_2 Z^2 \right] \frac{\partial \mathcal{H}}{\partial Z}. \tag{2.3.34}$$

We see that, as advertised, the first order formalism does not seem to give any advantage with respect to the second order formalism in this case. It may be that there is some clever choice of phase space coordinates that does reduce the difficulty of the problem.

2.3.4 | Coupling to other fields

We will end our study of the vector field case by considering couplings to matter fields through the energy-momentum tensor. As in Sec. 2.2.4 we will take a scalar field χ as a proxy for the matter. Moreover, as we mentioned in 2.2.4, the results obtained here will also give how the scalar field φ couples to matter fields when the interactions follow our prescription. Thus, our starting action will now contain the additional term

$$\mathcal{S}_{\chi,(1)} = \int \mathrm{d}^4 x \left[\frac{1}{2} \partial_\mu \chi \partial^\mu \chi - W(\chi) + \frac{1}{M_{\rm vd}^2} \Big(b_1 A_\mu A_\nu + b_2 A^2 \eta_{\mu\nu} \Big) T^{\mu\nu} \right] \tag{2.3.35}$$

where $T^{\mu\nu}$ includes the energy-momentum of the own vector field plus the contribution coming from the scalar field. The iterative process applied to this case yields the following expansion

$$\begin{aligned} \mathcal{S}_\chi = & \int \mathrm{d}^4 x \Big\{ \frac{1}{2} \mathcal{K}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \\ & - \left[1 - (b_1 + 4b_2) \left(Y + (b_1 - 2b_2)(Y^2 + 3b_1 Y^3) + \cdots \right) \right] W(\chi) \Big\}, \end{aligned} \tag{2.3.36}$$

where we have defined

$$\begin{aligned} \mathcal{K}^{\mu\nu} \equiv & \left[1 - (b_1 + 2b_2) \left(Y + b_1 Y^2 + b_1 (3b_1 + 2b_2) Y^3 \cdots \right) \right] \eta^{\mu\nu} \\ & + 2b_1 \left(1 + 2(b_1 - b_2) Y + (9b_1^2 - 8b_1 b_2 - 4b_2^2) Y^2 \cdots \right) A^\mu A^\nu. \end{aligned} \tag{2.3.37}$$

In this case, our ansatz for the resummed action is

$$\mathcal{S}_\chi = \int \mathrm{d}^4 x \left[\frac{1}{2} \Big(C(Y) \eta^{\mu\nu} + D(Y) A^\mu A^\nu \Big) \partial_\mu \chi \partial_\nu \chi - U(Y) W(\chi) \right] \tag{2.3.38}$$

so we have that the action reads

$$
\begin{aligned}
\mathcal{S}_\chi = &\int \mathrm{d}^4x \left[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\chi) + \frac{1}{M_{\rm vd}^2}(b_1 A_\mu A_\nu + b_2 A^2\eta_{\mu\nu})T^{\mu\nu}\right] \\
= &\int \mathrm{d}^4x \left[\frac{1}{2}\left(1-(b_1+2b_2)YC + 2(b_1+b_2)Y^2C'\right)\partial_\mu\chi\partial^\mu\chi \right.\\
&\quad + \frac{1}{2}\left(2b_1C + 3b_1YD + 2(b_1+b_2)Y^2D'\right)A^\mu A^\nu\partial_\mu\chi\partial_\nu\chi \\
&\quad \left. - \left(1+(b_1+4b_2)YU - 2(b_1+b_2)Y^2U'\right)W(\chi)\right], \qquad (2.3.39)
\end{aligned}
$$

and, therefore, the functions C, D and U should be the solutions of

$$C = 1-(b_1+2b_2)YC + 2(b_1+b_2)Y^2C', \qquad (2.3.40)$$

$$D = 2b_1C + 3b_1YD + 2(b_1+b_2)Y^2D', \qquad (2.3.41)$$

$$U = 1+(b_1+4b_2)YU - 2(b_1+b_2)Y^2U'. \qquad (2.3.42)$$

We see again that when coupling to the orthogonal projector, i.e., $b_2 = -b_1$, the equations become algebraic and the solution is

$$C = \frac{1}{1-b_1Y}, \qquad (2.3.43)$$

$$D = \frac{1+2b_1(1-Y)}{1-4b_1Y+3b_1^2Y^2}, \qquad (2.3.44)$$

$$U = \frac{1}{1+3b_1Y}. \qquad (2.3.45)$$

The form of the equations are similar to the ones found in the precedent sections so we will not repeat once again the same discussion, but obviously the same types of solutions will exist in this case. Let us however mention that the same results for the matter coupling can be obtained for the derivatively coupled scalar field upon the replacement $A_\mu \to \partial_\mu\varphi$.

2.4 | Superpotential terms

In the previous sections we have considered the energy-momentum tensor obtained from the usual prescription of coupling it to gravity and taking variational derivatives with respect to the metric. However, as we already explained above, the energy-momentum tensor (as any usual Noether current) admits the addition of super-potential terms with vanishing divergence either identically or on-shell. This freedom in the definition of the energy-momentum tensor can be used to *improve* the canonical energy-momentum tensor in special cases. For instance, theories involving spin 1 fields give non-symmetric canonical energy-momentum tensors that can be symmetrised by adding suitable super-potentials given in terms of the generators of the corresponding Lorentz representation (the Belinfante-Rosenfeld procedure). If the theory has a gauge symmetry, one can also add super-potential terms (on-shell divergenceless this time) to obtain a gauge invariant energy-momentum tensor[11] obtaining then the energy-momentum tensor that results from the Rosenfeld prescription. Theories featuring scale invariance admit yet another improvement to make the energy-momentum tensor traceless[12]. This traceless energy-momentum tensor is not the one obtained from the Hilbert prescription (nor with the Belinfante-Rosenfeld procedure) upon minimal coupling to gravity, but one needs to add a non-minimal coupling to the curvature, which simply tells us that the iterative coupling procedure to gravity starting from the improved energy-momentum tensor gives rise to non-minimal couplings. This example illustrates how considering different super-potential terms can result in different theories for the full action. In this section we will briefly discuss this point within our constructions for the self-interactions of scalar and vector fields to their own energy-momentum tensors.

Let us start with the scalar field and consider a particular family of terms that lead to interesting results. The lowest order object that is identically divergence-free is given by

$$\frac{1}{M_{\rm sd}}X_1^{\mu\nu}=\Box\varphi\eta^{\mu\nu}-\partial^\mu\partial^\nu\varphi, \tag{2.4.1}$$

where we have introduced the factor $M_{\rm sd}$ to match the dimension of an energy-momentum tensor for $X_1^{\mu\nu}$. This corresponds to a super-potential term of the form $\partial_\alpha(\partial^{[\alpha}\eta^{\mu]\nu})$. Now we want to study the effect on the full theory of adding this boundary term to the energy-momentum tensor. Notice that this object will give rise to an operator of lower dimensionality than the coupling to the energy-momentum tensor and, therefore, the added correction will be suppressed by one less power of the corresponding scale. In the case of the conformal coupling, we can see that the interaction $\varphi X^\mu{}_\mu$ simply amounts to a re-scaling of the kinetic term, so we will move directly to the derivative coupling. In that case, the first order correction is given by

$$\mathcal{L}_{(1)}=\frac{1}{M_{\rm sd}^4}\partial_\mu\varphi\partial_\nu\varphi X_1^{\mu\nu}=\frac{1}{M_{\rm sd}^3}\Big[(\partial\varphi)^2\Box\varphi-\partial_\mu\varphi\partial_\nu\varphi\partial^\mu\partial^\nu\varphi\Big]=\frac{3}{2M_{\rm sd}^3}(\partial\varphi)^2\Box\varphi, \tag{2.4.2}$$

where we have integrated by parts and dropped a total derivative in the last term. We can recognize here the cubic Galileon Lagrangian [NRT09] that we have obtained by

[11] Let us recall the Weinberg-Witten theorem here that prevents the construction of a Lorentz covariant and gauge invariant energy-momentum tensor for particles with spin $\geqslant 2$.

[12] With only scale invariance the trace of the energy-momentum tensor is given by the divergence of a vector and only when the theory exhibits full conformal invariance the energy-momentum tensor can be made traceless. Since theories that are scale invariant are also conformally invariant, we do not make a distinction here.

simply following our coupling prescription to the identically conserved object $X_1^{\mu\nu}$ that one can legitimately add to the energy-momentum tensor. By iterating the process we obtain the following perturbative expansion for the Lagrangian:

$$\mathcal{L} = \left(1 + 3X + 15X^2 + 105X^3 + \cdots\right)\frac{1}{M_{\rm sd}^3}\Big[(\partial\varphi)^2\Box\varphi - \partial_\mu\varphi\partial_\nu\varphi\partial^\mu\partial^\nu\varphi\Big]. \quad (2.4.3)$$

We can now use that, for an arbitrary function $\mathcal{G}(X)$ and upon integration by parts, we have

$$\mathcal{G}(X)\partial_\mu\varphi\partial_\nu\varphi\partial^\mu\partial^\nu\varphi = \frac{1}{2}M_{\rm sd}^4\partial^\mu\varphi\partial_\mu\tilde{\mathcal{G}}(X) \to -\frac{1}{2}\tilde{\mathcal{G}}(X)\Box\phi, \quad (2.4.4)$$

with $M_{\rm sd}^4\tilde{\mathcal{G}}'(X) = \mathcal{G}(X)$, to express the final Lagrangian as

$$\mathcal{L} = \frac{3}{2}\left(1 + \frac{5}{2}X + \frac{35}{3}X^2 + \frac{315}{4}X^3 + \cdots\right)\frac{(\partial\varphi)^2}{M_{\rm sd}^3}\Box\varphi. \quad (2.4.5)$$

This Lagrangian is a particular case of the so-called KGB models [Def+10] with a shift symmetry. Remarkably, we have obtained this Lagrangian from our construction, which can be understood in a similar fashion to the generation of non-minimal couplings in the case of gravity.

We can also obtain a *resummed* action by adding the superpotential term to the full theory so that its Lagrangian should read

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{M_{\rm sd}^4}\Big[b_1\partial_\mu\varphi\partial_\nu\varphi + b_2(\partial\varphi)^2\eta_{\mu\nu}\Big]\Big(T^{\mu\nu} + X_1^{\mu\nu}\Big), \quad (2.4.6)$$

being $T^{\mu\nu}$ the total energy-momentum tensor, i.e., the one computed from $\mathcal{L}$ by means of the Hilbert prescription. For the sake of generality, we have added here the more general coupling involving b_1 and b_2 as discussed in the precedent sections. In the above Lagrangian we have also included the term $\mathcal{L}_0$ that corresponds to the *free* Lagrangian, i.e., the part that survives in the decoupling limit $M_{\rm sd} \to \infty$. This term will not affect the resummation of the interactions generated from the term with $X_1^{\mu\nu}$ so we do not need to consider it here (we already solved that part in the previous sections). Now we can notice that the interactions generated from the superpotential terms are of the form $\mathcal{L} = M_{\rm sd}G(X)\Box\varphi$, up to boundary terms, so we need to have

$$\mathcal{L}_X = M_{\rm sd}G(X)\Box\varphi = \Big[b_1\partial_\mu\varphi\partial_\nu\varphi + b_2(\partial\varphi)^2\eta_{\mu\nu}\Big]\Big(X_1^{\mu\nu} + T_X^{\mu\nu}\Big), \quad (2.4.7)$$

being $T_X^{\mu\nu}$ the total energy-momentum tensor of $\mathcal{L}_X$. By computing this energy-momentum tensor we finally get

$$\mathcal{L}_X = M_{\rm sd}G(X)\Box\varphi = M_{\rm sd}\left[\frac{3}{2}X + 2(b_1 + b_2)X^2G'(X) + (b_1 - 2b_2)\tilde{G}(X)\right]\Box\varphi, \quad (2.4.8)$$

with $\tilde{G}'(X) = XG'(X)$ and we have integrated by parts. From this relation we then see that the following equation must hold

$$G(X) = \frac{3}{2}X + 2(b_1 + b_2)X^2 G'(X) + (b_1 - 2b_2)\tilde{G}(X). \tag{2.4.9}$$

Notice that this is an integro-differential equation, but we can easily transform it into the ordinary differential equation

$$2(b_1 + b_2)XF'(X) + \left(3b_1 - \frac{1}{X}\right)F(X) + \frac{3}{2} = 0, \tag{2.4.10}$$

with $F(X) \equiv XG'(X)$. As usual, the coupling to the orthogonal projector, i.e., $b_1 = -b_2$ reduces the equations to a set of algebraic equations, although in this case an integration to go from F to G is necessary. In that case we obtain $F = -\frac{3}{2}(3b_1 - 1/X)^{-1}$ so that $G = -1/(2b_1)\log(1 - 3b_1 X)$. In this case, the integration constant is irrelevant because it corresponds to a total derivative in the Lagrangian.

We have considered the simplest of the identically divergence-free superpotential terms, but we could consider the whole family given (in arbitrary dimension d) by

$$X_n^{\mu\nu} \propto M_{\rm sd}^{d-3n} \epsilon^{\mu\mu_1\cdots\mu_n\mu_{n+1}\cdots\mu_{d-1}} \epsilon^{\nu\nu_1\cdots\nu_n}{}_{\mu_{n+1}\cdots\mu_{d-1}} \partial_{\mu_1}\partial_{\nu_1}\varphi \cdots \partial_{\mu_n}\partial_{\nu_n}\varphi, \tag{2.4.11}$$

which can be seen to be trivially divergence-free by virtue of the antisymmetry of the Levi-Civita tensor. These higher order superpotentials would generate higher order versions of the Galileon Lagrangians. At the first order, each $X_n^{\mu\nu}$ will obviously generate the n-th Galileon Lagrangian (in fact, Galileon fields are precisely defined by coupling the gradients of the scalar field to identically divergen-free objects), while the higher order corrections will eventually produce sub-classes of shift-symmetric generalised Galileons fields.

Similar results to those found for the scalar field can be obtained by adding superpotential terms in the case of vector fields. We can consider the lowest order and identically conserved object

$$\frac{1}{M_{\rm vc}^2} Y_1^{\mu\nu} = \partial \cdot A \eta^{\mu\nu} - \partial^\nu A^\mu, \tag{2.4.12}$$

which corresponds to a superpotential of the form $\Theta^{\alpha\mu\nu} = 2A^{[\alpha}\eta^{\mu]\nu}$. Unlike in the scalar field case, this superpotential is not symmetric in the last two indices. While in the scalar field case, a non-derivative coupling did not produce new terms, for the vector field case already algebraic couplings generate new interesting interactions (as expected because the non-derivative coupling for the vector is related to the derivative coupling for the scalar). The leading order interaction by coupling the vector to this superpotential term gives

$$\mathcal{L}_{(0)} = A_\mu A_\nu Y_1^{\mu\nu} = \frac{3}{2} A^2 \partial \cdot A, \tag{2.4.13}$$

where we have integrated by parts and dropped a total derivative term. We see that, as expected, we recover the cubic vector Galileon Lagrangian [Tas14; Hei14; APR16; BH16]. It is remarkable that this interaction corresponds to a dimension four operator which means that it is not suppressed by the scale $M_{\rm vc}$ or, in other words, it will survive in the decoupling limit $M_{\rm vc} \to \infty$. It is not surprising the resemblance of this interaction with the case of the derivatively coupled scalar previously discussed and it is not difficult to convince oneself that the resummation will lead to the same type of equations. In addition, very much like in the scalar field case, there are higher order superpotential terms that can be constructed for the vector field case, but, given the similarities with the scalar field couplings, we will not give more details here. A more interesting class of super-potential terms would be those respecting the $U(1)$ gauge invariance. However, it is known that there are no Galileon-like interactions for abelian vector fields [Def+14; DMS16] so that we do not expect to find anything crucially new by adding gauge invariant superpotential terms, but similar interactions to the ones already worked out.

2.5 | Effective metrics and generating functionals

In the precedent sections we have studied the coupling of scalar and vector fields to the energy-momentum tensor and how this can be generalised to the full theory. The definition that we have mostly considered for the energy-momentum tensor is the Hilbert prescription, although we have briefly commented on some interesting consequences of considering superpotential terms in the previous section. Since the Hilbert prescription gives the energy-momentum tensor as a functional derivative with respect to some fiducial metric, a natural question is to what extent the full theory can be expressed in terms of an effective metric. In this section we intend to briefly discuss this aspect with special emphasis in the cases considered throughout this chapter.

For the clarity of our construction, let us go back to the beginning and consider again the case of a conformally coupled scalar field considered in 2.2.1. The starting point there was a scalar field coupled to the trace of the energy-momentum tensor as[13]

$$\mathcal{L}_{\rm int}^{(1)} = \varphi T_{(0)}. \tag{2.5.1}$$

[13]In order to alleviate the notation in this section we will drop all the scales used in the precedent sections.

It is not difficult to see that this interaction can be conveniently written as the following functional derivative

$$\mathcal{L}^{(1)}_{\rm int} = \frac{\delta \mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0}, \tag{2.5.2}$$

where the zeroth order action $\mathcal{S}_{(0)}$ is evaluated on the conformal effective metric $h_{\mu\nu} = \exp(-2J\varphi)\eta_{\mu\nu}$ with J an external field[14]. The equivalence of the two expressions can be easily checked by using the chain rule:

$$\frac{\delta \mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0} = \left(\frac{\delta \mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta h_{\mu\nu}}\frac{\delta h_{\mu\nu}}{\delta J}\right)_{J=0}. \tag{2.5.3}$$

It is now immediate to re-obtain the results of Sec. 2.2.1 as well as obtaining generalisations. Let us assume that the initial action is a linear combination of homogeneous functions of the metric so we have

$$\mathcal{S}_{(0)}[\lambda\eta_{\mu\nu}] = \sum_i \lambda^{w_i}\mathcal{S}_{(0),i}[\eta_{\mu\nu}], \tag{2.5.4}$$

with λ some parameter and w_i the degree of homogeneity of the corresponding term $\mathcal{S}_{(0),i}$. Then, the functional derivative can be straightforwardly computed as

$$\frac{\delta \mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0} = \sum_i \frac{\delta}{\delta J}\int {\rm d}^4x \exp(-2w_i\varphi J)\mathcal{L}_{(0),i}\Big|_{J=0} = \sum_i(-2w_i\varphi)\mathcal{L}_{(0),i}. \tag{2.5.5}$$

Since all the dependence on the metric in this first order correction is again in $\mathcal{L}_{(0),i}$ we have, for this specific case, that the n-th order interaction will be given by

$$\mathcal{L}^{(n)}_{\rm int} = \frac{\delta \mathcal{S}_{(n-1)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0} = \frac{\delta^n \mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta J^n}\Big|_{J=0} = \sum_i(-2w_i\varphi)^n\mathcal{L}_{(0),i}, \tag{2.5.6}$$

so that the resummed Lagrangian for the term of degree w_i is

$$\mathcal{L}_{(i)} = \sum_{n=0}^{\infty}\frac{\delta^n \mathcal{S}[h_{\mu\nu}(J)]}{\delta J^n}\Big|_{J=0} = \left[\sum_{n=0}^{\infty}(-2w_i\varphi)^n\right]\mathcal{L}_{(0),i} = \frac{1}{1+2w_i\varphi}\mathcal{L}_{(0),i}. \tag{2.5.7}$$

This exactly reproduces the results of Sec. 2.2.1 since the kinetic term of the scalar has degree 1 while any potential term for a scalar field has degree 2. Notice that for a term of zero degree there is no correction, in accordance with the fact that the energy-momentum tensor of a zeroth weight (i.e. conformally invariant) is traceless.

After warming up with the simplest example (which in fact allows for a full resolution of the problem) we can proceed to develop a general framework. Let us consider a leading order interaction to the energy-momentum tensor of the general form

$$\mathcal{L}^{(1)} = \Omega_{\mu\nu}T^{\mu\nu}, \tag{2.5.8}$$

[14]We have chosen here the exponential for the conformal factor for simplicity, but any conformal factor $\Omega(J)$ satisfying $\Omega(0) = 1$ and $\Omega'(0) = -2\varphi$ would do the job.

where $\Omega_{\mu\nu}$ is some rank-2 tensor built out of the field which we want to couple to $T^{\mu\nu}$, be it the scalar or the vector under consideration throughout this chapter. Then, it is easy to see that the generating effective metric will be given by

$$h_{\mu\nu} = \exp\left[-2J\omega_{\mu\nu}{}^{\alpha\beta}\right]\eta_{\alpha\beta}, \tag{2.5.9}$$

with $\omega_{\mu\nu}{}^{\alpha\beta}$ some rank-4 tensor satisfying $\omega_{\mu\nu}{}^{\alpha\beta}\eta_{\alpha\beta} = \Omega_{\mu\nu}$. This effective metric indeed generates the desired interaction from the functional derivative

$$\mathcal{L}^{(1)} = \frac{\delta\mathcal{S}_{(0)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0}, \tag{2.5.10}$$

while the higher order interactions are simply

$$\mathcal{L}^{(n)} = \frac{\delta\mathcal{S}_{(n-1)}[h_{\mu\nu}(J)]}{\delta J}\Big|_{J=0}. \tag{2.5.11}$$

This general construction can be straightforwardly applied to the cases that we have considered in the precedent sections. The coupling of the scalar field corresponds to taking $\omega_{\mu\nu}{}^{\alpha\beta}$ proportional to the identity in the space of rank-4 tensors. The algebraic vector field coupling corresponds to $\omega_{\mu\nu}{}^{\alpha\beta} = \frac{1}{4}\left(b_1 A_\mu A_\nu + b_2 A^2\eta_{\mu\nu}\right)\eta^{\alpha\beta}$, while the gauge invariant coupling is generated by $\omega_{\mu\nu}{}^{\alpha\beta} = b_1 F_\mu{}^\alpha F_\nu{}^\beta + \frac{1}{4}b_2 F^2\eta_{\mu\nu}\eta^{\alpha\beta}$.

An interesting case is when $\Omega_{\mu\nu}$ does not depend on $\eta_{\mu\nu}$, which happens when $\omega_{\mu\nu}{}^{\alpha\beta}$ is linear in the inverse metric. To see why this is particularly interesting, let us obtain the second correction to the original action, that will be given, in general, by

$$\begin{aligned}
\mathcal{L}^{(2)} &= \left(\frac{\delta\mathcal{S}_{(1)}[h_{\mu\nu}]}{\delta J}\right)_{J=0} = \left[\frac{\delta}{\delta J}\int \mathrm{d}^4x\left(\frac{\delta\mathcal{S}_{(0)}[h_{\mu\nu}]}{\delta J}\right)_{J=0}\right]_{J=0} \\
&= \left[\frac{\delta}{\delta J}\int \mathrm{d}^4x\left(-2\omega_{\mu\nu}{}^{\alpha\beta}\eta_{\alpha\beta}\frac{\delta\mathcal{S}_{(0)}[h_{\mu\nu}]}{\delta h_{\mu\nu}}\Big|_{J=0}\right)\right]_{J=0} \\
&= -2\left[\frac{\delta}{\delta J}\int \mathrm{d}^4x\,\omega_{\mu\nu}{}^{\alpha\beta}(J)h_{\alpha\beta}(J)\frac{\delta\mathcal{S}_{(0)}[h_{\mu\nu}]}{\delta h_{\mu\nu}}\right]_{J=0}.
\end{aligned} \tag{2.5.12}$$

where we have used the chain rule and the definition of the generating metric (2.5.9). Now, the last expression is substantially simplified if $\Omega_{\mu\nu} = \omega_{\mu\nu}{}^{\alpha\beta}\eta_{\alpha\beta}$ does not depend on the metric $\eta_{\mu\nu}$ because, in that case, $\omega_{\mu\nu}{}^{\alpha\beta}\eta_{\alpha\beta}$ will not acquire a dependence on J so we obtain

$$\mathcal{L}^{(2)} = (-2)^2\omega_{\mu\nu}{}^{\alpha\beta}\eta_{\alpha\beta}\omega_{\rho\sigma}{}^{\gamma\delta}\eta_{\gamma\delta}\left(\frac{\delta^2\mathcal{S}_{(0)}[h(J)]}{\delta h_{\mu\nu}\delta h_{\rho\sigma}}\right)_{J=0}. \tag{2.5.13}$$

It is then straightforward to iterate this process and arrive at the following expression for the final Lagrangian

$$\mathcal{L} = \sum_{n=0}^{\infty}(-2)^n\frac{\delta^n\mathcal{S}_{(0)}[h(J)]}{\delta h_{\mu_1\nu_1}\cdots\delta h_{\mu_n\nu_n}}\Big|_{J=0}\Omega_{\mu_1\nu_1}\cdots\Omega_{\mu_n\nu_n}. \tag{2.5.14}$$

This (asymptotic) expansion reminds of a Taylor expansion, though without the required $1/n!$. It is not difficult to motivate the appearance of the missing factorial by simply imposing that the variation should appear at the level of the field equations instead of the action, similarly to what we did for the scalar field case at the end of Sec. 2.2.1. In that case, the resulting series will be

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} \frac{\delta^n \mathcal{S}_{(0)}[h]}{\delta h_{\mu_1\nu_1} \cdots \delta h_{\mu_n\nu_n}} \Omega_{\mu_1\nu_1} \cdots \Omega_{\mu_n\nu_n}. \tag{2.5.15}$$

which admits a straightforward resummation as the Taylor expansion of the seed action $\mathcal{S}_{(0)}$ around $2\Omega_{\mu\nu}$, i.e., the total action will be $\mathcal{S} = \mathcal{S}_0[\eta_{\mu\nu} - 2\Omega_{\mu\nu}]$. Of course, this reproduces the well-known result for gravity when $\Omega_{\mu\nu}$ is identified with the metric perturbation $-2h_{\mu\nu}$. We see here that the same applies when a scalar field is derivatively coupled to matter fields as $\partial_\mu\varphi\partial_\nu\varphi T^{\mu\nu}$ or a vector field is coupled as $A_\mu A_\nu T^{\mu\nu}$. In both cases, the resulting coupling to matter fields is through a disformal metric $g_{\mu\nu} = \eta_{\mu\nu} - 2A_\mu A_\nu$ and $g_{\mu\nu} = \eta_{\mu\nu} - 2\partial_\mu\varphi\partial_\nu\varphi$ respectively. This result shows that the couplings of the precedent sections with $b_2 = 0$ are indeed special because they satisfy the above condition of having the corresponding $\Omega_{\mu\nu}$ independent of the metric and, therefore, admit a resummation in terms of an effective metric.

Finally, let us notice that the expression (2.5.15) also serves as a starting point to explore a class of "non-metric" theories, i.e., theories where the coupling is not through an effective metric, analogous to the theories explored in [Bla92] as non-metric departures from GR.

2.6 | Phenomenology

The different scalar and vector field theories discussed along this chapter give raise to a rich phenomenology depending in the specific coupling to the energy-momentum tensor prescribed. The best understood scenario corresponds to the scalar coupling to the trace of the energy momentum tensor which leads to the standard Jordan-Brans-Dicke framework and dilaton structures. In this section, we will summarize the main phenomenological aspects associated to disformal scalars and vectors fields. In particular, we would like to focus on the phenomenology shared by both fields due to the equivalence theorem[15]. For instance, in the vector case, we will focus in the regime where its mass is negligible as compared to the physical energy involved in the process. Hence, following an effective field theory approach, we can work in the perturbation

[15] The equivalence theorem states that at sufficiently high energy the longitudinal mode of a vector field behaves as a disformal scalar field except for corrections of order of the inverse of the energy.

limit. Under these assumptions, the dimension 6 interaction

$$\mathcal{L}_V = \frac{1}{M_{\rm vc}^2} A_\mu A_\nu T^{\mu\nu} \tag{2.6.1}$$

dominates the distinctive phenomenology associated to vector fields discussed along this chapter. This type of interaction appears in the vector modes associated to the metric in theories with additional spatial dimensions [DM01]. The phenomenology of these *graviphotons* have been studied for the brane world scenario in different works under the name of *brane vectors* [Cla+08b; Cla+09a; Cla+08a; Cla+09b]. In this framework, there is a Higgs-like mechanism providing mass to the *graviphotons*. At high energies with respect to this mass, the longitudinal mode of these vector fields can be identified with the branons φ, the scalar degree of freedom associated to the fluctuation of the brane along the extra dimensions [DM01]. Within this regime, the experimental signatures of these vectors can be computed by the branon disformal coupling [CM16]:

$$\mathcal{L}_D = \frac{1}{m^2\, M_{\rm vc}^2} \partial_\mu\varphi \partial_\nu\varphi T^{\mu\nu}\ , \tag{2.6.2}$$

where m is the mass of the vector field. A detailed analysis of the constraints to the above interaction has been developed in the context of branon fields [Sun99; DM01; CDM02; Alc+03]. This term dominates the distinctive observational signatures of disformal scalars. Not only potential searches in colliders have been studied in different works [CS01; Alc+03], but also astrophysical constraints have been considered in these analysis, such as the ones coming from cooling of stelar objects [KY01; CDM03a; CDM03b] or the associated to the relic abundances of this type of massive vectors [Cla+08b].

At high energies, in virtue of the equivalence theorem, the vector fields can be studied by analysing the Lagrangian (2.6.2). In this scenario, the vector mass times the vector coupling is the combination of parameters which suffers the constraints from present data. However, for the disformal scalar fields, the observational constraints apply directly to the disformal scalar coupling. They may be detected at the Large Hadron Collider (LHC) or in a future generation of accelerators [Alc+03; BBD12; BB14; Hei+05; CDM02; CRT08; CDM07; Aar+07; Cla+08b; CMP11; Ach+04; CDM04f; CDD13; CDP11; Lan15; Kha+16; CDM06a; CDM06b]. For the case of the LHC, the most sensitive production process is the gluon fusion resulting in a gluon in addition to a pair of longitudinal vectors or disformal scalars, and the quark-gluon interaction giving raise to a quark and a pair of the commented particles. These processes contribute to the mono-jet J and a transverse missing momentum and energy signal. An additional process is the quark-antiquark annihilation, resulting in a photon and the mentioned pair of new modes. In this case, the signal is a single photon in addition to the transverse missing momentum and energy. The cross-section of the subprocesses were computed in Refs. [CDM04f] and [CDP11]. The analysis of the single photon channel is simpler and cleaner but the mono-jet channel is more sensitive.

In addition to these processes, there are other complementarity constraints on the same combination of parameters corresponding to different collider data. A summary of these analyses [DM01; CDM03c; CDM04d; CDM04b; CDM04a; CDM04c; CDM04e; Mar04b; Mar04a; Cem+08; Hei+05; CDM02; CRT08; CDM07; Aar+07; Cla+08b; CMP11; CDM04f; CDP11] can be found in Table 2.1. In this Table, the limits coming from HERA, LEP-II and Tevatron are compared with the present restrictions from LHC running at a centre of mass energy (c.m.e.) of 8 TeV and the prospects for the LHC running at 14 TeV c.m.e. with full luminosity. Other missing transverse momentum and energy processes, such as those related to the mono-lepton channel [BB14], are also potential signatures of the models developed in this chapter. In the same reference, the authors discuss other different phenomenological signatures, but they are subdominant due to the important dependence of the interaction with the energy.

On the other hand, it has been shown that the new modes under study introduce radiative corrections, which generate new couplings among SM particles, which can be described by an effective Lagrangian. Although the study of such processes demands the introduction of new parameters, they can provide interesting effects in electroweak precision observables, anomalous magnetic moments, or SM four particle interactions [CDM06a; CDM06b].

On the other hand, as mentioned before, astrophysical observations can also constraint the parameters of these dark matter models. We can compute the thermal relic abundance corresponding to these new fields assuming they are stable [CDM03a; CDM03b; Cla+08b]. The larger the coupling scale M, the weaker the annihilating cross-section into SM particle-antiparticle pairs, and hence the larger the relic abundance. This is the expected conclusion since the sooner the decoupling occurs, the larger the relic abundance is. Therefore the cosmological restrictions related to the relic abundance are complementary to those coming from particle accelerators. Indeed, a constraint such as $\Omega_D < \mathcal{O}(0.1)$ means a lower limit for the value of the cross-sections in contrast with the upper limits commented above from non observation at colliders. If we assume that the DM halo of the Milky Way has an important amount of these new vectors or scalars, its flux on the Earth could be sufficiently large to be measured in direct detection experiments. These experiments measure the rate R, and energy E_R of nuclear recoils. These constraints depend on different astrophysical assumptions as it has been discussed in different analyses [CDM03a; CDM03b; Cla+08b; CM16; CDP11].

Experiment	$\sqrt{s}$(TeV)	$\mathcal{L}$(pb^{-1})	$\sqrt{mM_{\rm vc}}$(GeV)
HERA [1]	0.3	110	19
Tevatron-II [1]	2.0	10^3	304
Tevatron-II [2]	2.0	10^3	285
LEP-II [2]	0.2	600	214
LHC [2]	8	19.6×10^3	523
LHC [1]	14	10^5	1278
LHC [2]	14	10^5	948

Table 2.1: Summary table for the phenomenology of vectors and disformal scalars coupled to the energy-momentum tensor at colliders. Monojet and single photon analyses are labeled by the upper indices $1, 2$, respectively. Present bounds and prospects for the LHC [Lan15; Kha+16; CDM04f; CDP11] are compared with constraints from LEP [Alc+03; Ach+04], HERA and Tevatron [CDM04f]. $\sqrt{s}$ means the centre of mass energy associated with the total process; $\mathcal{L}$ denotes the total integrated luminosity; $\sqrt{mM_{\rm vc}}$ is the constraint at the 95 % confidence level by assuming a very light vector (in the limit $m \to 0$). The effective coupling is not valid for energy scales $\Lambda^2 \gtrsim 8\pi\sqrt{2}mM_{\rm vc}$ [CDM06a; CDM06b].

If the abundance of these new particles is significant, they cannot only be detected by direct detection experiments, but also by indirect ones. In fact, a pair of vectors or scalars can annihilate into ordinary particles such as leptons, gauge bosons, quarks or Higgs bosons. Their annihilations from different astrophysical regions produce fluxes of cosmic rays. Depending on the characteristics of these fluxes, they may be discriminated from the background. After the annihilation and propagation, the particle species that can be potentially detected by different detectors are gamma rays, neutrinos and antimatter (fundamentally, positrons and antiprotons). In particular, gamma rays and neutrinos have the advantage of maintaining their original trajectory. Indeed, this analysis are more sensitive for the detection of these signatures [Cem+05; CFS07; CS08; Cem+11a; Cem+11b; Cem+12a; CGM12; CGM14; CGM15].

Furthermore, there are astrophysical observations that are able to constraint the parameter space of the new fields studied in this chapter independently of their abundance. For example, one of the most successful predictions of the standard cosmological model is the relative abundances of primordial elements. These abundances are sensitive to several cosmological parameters and were used in Refs. [CDM03a] and [CDM03b] in order to constrain the number of light fields. These restrictions apply in this case since the new particles will behave as dark radiation for small enough masses.

For instance, the production of ^{4}He increases with an increasing rate of the expansion H and the Hubble parameter depends on the total amount of radiation. However, these restrictions are typically important for relatively strong couplings. On the contrary, if the new modes decouple above the QCD phase transition, $mM_{\rm vc} \sim 10000$ GeV^2, the limit increases so much that the restrictions become extremely weak [CDM03b]. Different astrophysical bounds can be obtained from modifications of cooling processes in stellar objects like supernovae [KY01; CDM03a; CDM03b; BB14]. These processes take place by energy loosing through light particles such as photons and neutrinos. However, if the mass of the new particles is low enough, these new particles are expected to carry a fraction of this energy, depending on their mass and the coupling to the SM fields. These constraints are restricting up to masses of order of the GeV. For heavier fields, the limits on the coupling disappear due to the short value of the mean free path of the vector particle inside the stellar object [CDM03b]. A summary of these astrophysical and cosmological constraints for a particular model of a single disformal scalar described by the branon Lagrangian can be seen in Fig. 2.1.

In the previous paragraphs, we have summarized the phenomenology associated to the abundance of these new vectors and disformal scalars by assuming it was generated by the thermal decoupling process in an expanding universe. However, if the reheating temperature $T_{\rm reh}$ after inflation was sufficiently low, then these new fields were never in thermal equilibrium with the primordial plasma [Mar04a]. However, still there is the possibility for them to be produced non-thermally, very much in the same ways as axions [PWW83; AS83; DF83; FJ92] or other bosonic degrees of freedom [Cem09; BCK10; Cem+12b; CCG13; AC20; CMN13b; CMN13a; CMN14; CMN16]. This possibility modifies some of the previous astrophysical signatures, since they have typically associated a much lower mass. In particular, the potential isotropies related to the coherent relic density of the new vectors could constitute a very distinctive signature [CMN13b; CMN13a; CMN14; CMN16].

2.7 | Discussion

In this section we will discuss the recurrent differential equations we have encountered throughout the chapter and analyse their behavior. On the other hand we will also discuss the results and main conclusions obtained from delving into gravity-inspired scalar and vector field models.

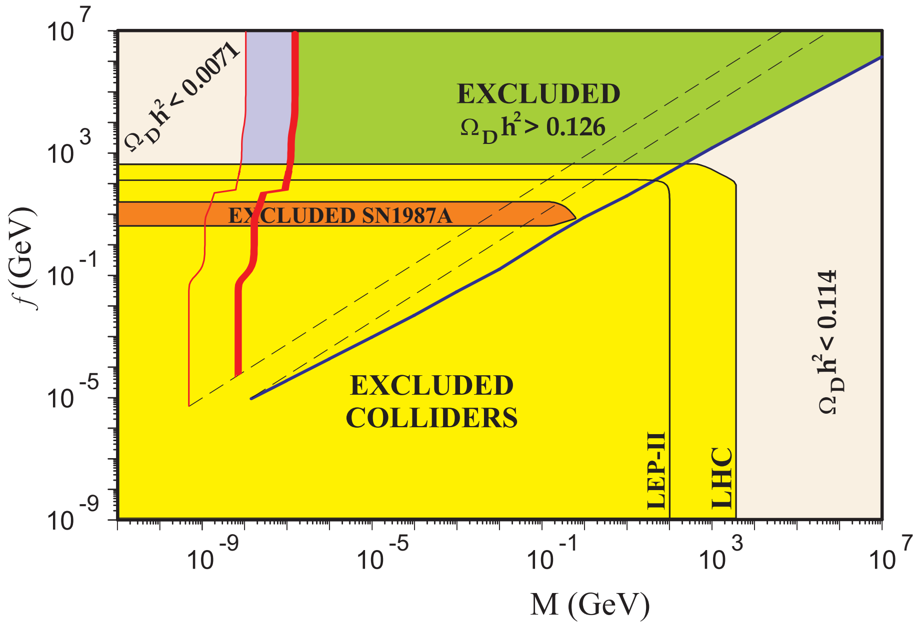

Figure 2.1: In this figure we show the combined exclusion plot for a model with a single disformal scalar described by the branon Lagrangian. M is the scalar (branon) mass, whereas f is the brane tension scale (which verifies $2f^4 = M_{\rm sd}^4$). Within the (green) upper area, the model is ruled out because these scalars are overproduced, The (purple) left region is excluded by structure formation, whereas the (yellow) lower regions are excluded by LEP-II and LHC single photon analysis. Supernovae cooling rules out the horizontal (orange) area. The (blue) solid line on the right is associated with a cold DM behavior with the proper relic density. The two (red) solid lines on the left are related to a hot DM behavior. This figure is taking from Ref. [CM16], where a more detailed explanation can be found.

2.7.1 | Recurrent differential equations

In this section we will discuss the differential equation that we have recurrently seen throughout this chapter. The equation can be expressed as

$$2x^2F'(x) + (px-1)F(x) + 1 = 0, \tag{2.7.1}$$

with p an arbitrary real constant. Taking for instance our first encounter with this type of equation in (2.2.32), that we reproduce here for convenience

$$\mathcal{K} = 1 + (b_1 - 2b_2)X\mathcal{K} + 2(b_1+b_2)X^2\mathcal{K}', \tag{2.7.2}$$

$$\mathcal{U} = 1 - (b_1 + 4b_2)X\mathcal{U} + 2(b_1+b_2)X^2\mathcal{U}', \tag{2.7.3}$$

it is easy to check that they can be brought into the form (2.7.1) by simply rescaling $X \to X/(b_1+b_2)$, which is always legitimate except for $(b_1+b_2)=0$, in which case, as we already discussed, the equations become algebraic. The values for the constant parameter p are then $p = (b_1 - 2b_2)/(b_1 + b_2)$ and $p = -(b_1 + 4b_2)/(b_1 + b_2)$ for $\mathcal{K}$ and $\mathcal{U}$ respectively. It is clear that $x = 0$ is a singular point of the equation. If we evaluate the equation at $x = 0$ we find $F(0) = 1$, as it should since the series of the functions satisfying this equation start at 1. Furthermore, by taking subsequent derivatives of the equation and evaluating at $x = 0$, we can reproduce the corresponding perturbative series, which is in turn an asymptotic expansion. Alternatively, we can seek for solutions in the form of a power series $F = \sum_{n=0} F_n x^n$. By substituting this series in the equation we find that $F_0 = 1$ and the following recurrent formula for the coefficients with $n \geqslant 1$:

$$F_n = \Big[p + 2(n-1)\Big] F_{n-1}, \quad n \geqslant 1, \tag{2.7.4}$$

which can be readily solved as

$$F_n = \prod_{j=1}^{n} \Big[p + 2(n-j)\Big], \quad n \geqslant 1. \tag{2.7.5}$$

From this general solution we see that $F_1 = p$ so that if we choose the parameters such that $p = 0$, then the solution is simply $F = 1$ because then only F_0 remains non-vanishing. This result of course corresponds to the choice of parameters that cancelled either the corrections to the kinetic terms or to the potential discussed throughout this chapter. In particular, we see that $p = 0$ corresponds to $b_1 = 2b_2$ and $b_1 = -4b_4$ in (2.7.3), as we already found in Sec. 2.2.2.

The fact that we can write the general solution in terms of a recursive relation is due to the iterative procedure prescribed to built our actions and this in turn has noteworthy consequence, namely, we can choose parameters such that the solution becomes polynomial. Since the coefficients F_n are recursively given by (2.7.4), if we have that $F_{r+1} = 0$ for some r, then $F_n = 0$ for $n > r$ and, thus, the solution is a polynomial of degree r. The result commented in the previous paragraph that $F = 1$ for $p = 0$ is a particular case of this general result with $r = 0$. In general, if we want the solution to be a polynomial of degree r we will need to impose $p + 2r = 0$. As an illustrative example, if we want $\mathcal{K}$ and $\mathcal{U}$ in (2.7.3) to be polynomials of degree r and s respectively, we need to choose the parameters b_1 and b_2 satisfying

$$\frac{b_1 - 2b_2}{b_1 + b_2} = -2r, \qquad \frac{b_1 + 4b_2}{b_1 + b_2} = 2s. \tag{2.7.6}$$

These equations do not admit a general solution for arbitrary r and s. If we eliminate for instance b_1, we end up with the equation $b_2(1 + r - s) = 0$ (for r and s integers), that imposes the relation $s = 1 + r$ and leaves b_2 as a free parameter. The solution for b_1 is then $b_1 = 2b_2(1-r)/(1+2r)$. Thus, in this particular case, if we want both functions to become polynomials, they cannot be of arbitrary degree, but $\mathcal{U}$ must be one degree higher than $\mathcal{K}$. For $r = 0$, we have $s = 1$ and the corresponding theory has $b_1 = 2b_2$, i.e., for those parameters $\mathcal{K}$ is constant and $\mathcal{U}$ is polynomial of degree 1, as we found in Sec. 2.2.2.

If we do not impose the solutions to be polynomials, we can do better than the solution expressed as the (asymptotic) series with general coefficients given in (2.7.5) and obtain the general solution of the equation in terms of known functions, which can be expressed as

$$F(x) = e^{-1/(2x)}|x|^{-p}\left[C - \frac{1}{2}\int e^{1/(2x)}|x|^{p-2}\mathrm{d}x\right], \tag{2.7.7}$$

with C a constant that must be chosen in order to have a well-defined solution at $x = 0$. The above solution can also be expressed in terms of exponential integral functions as follows

$$F(x) = C|x|^{-p/2}e^{-1/(2x)} - \frac{1}{2x}e^{-1/(2x)}\mathrm{Ei}_{p/2}(-1/(2x)). \tag{2.7.8}$$

Since the exponential integral has the asymptotic expansion $\mathrm{Ei}_n(x) \sim e^{-x}/x$ for large x, the second term in the above solution is regular at the origin and, thus, in order to have a regular solution we must impose $C = 0$ so that the desired solution is finally

$$F(x) = -\frac{e^{-1/(2x)}}{2x}\mathrm{Ei}_{p/2}(-1/(2x)). \tag{2.7.9}$$

The exponential integral presents a branch cut for the negative axis. In the final solution, it must be interpreted as the real part of the analytic continuation to the complex plane.

We can gain some more insights on the equation by considering the equivalent autonomous system

$$\begin{aligned} \dot{F}(t) &= \Big(1 - px(t)\Big)F(t) - 1, \\ \dot{x}(t) &= 2x^2(t), \end{aligned} \tag{2.7.10}$$

where the dot means derivative with respect to t. The integral curves of this autonomous system are the solutions of our original equation. The only fix point is given by $x = 0$, $F = 1$ and the associated eigenvalues are 0 and 1. Moreover, it is also easy to see that the x-axis is indeed a separatrix, ($\dot{x} = 0$ on that axis) so the only solution that can cross it must necessarily pass through $(F = 1, x = 0)$. The fact that there is a negative eigenvalue indicates the existence of a centre (slow) manifold. We will not provide a detailed analysis of the general properties of the associated autonomous

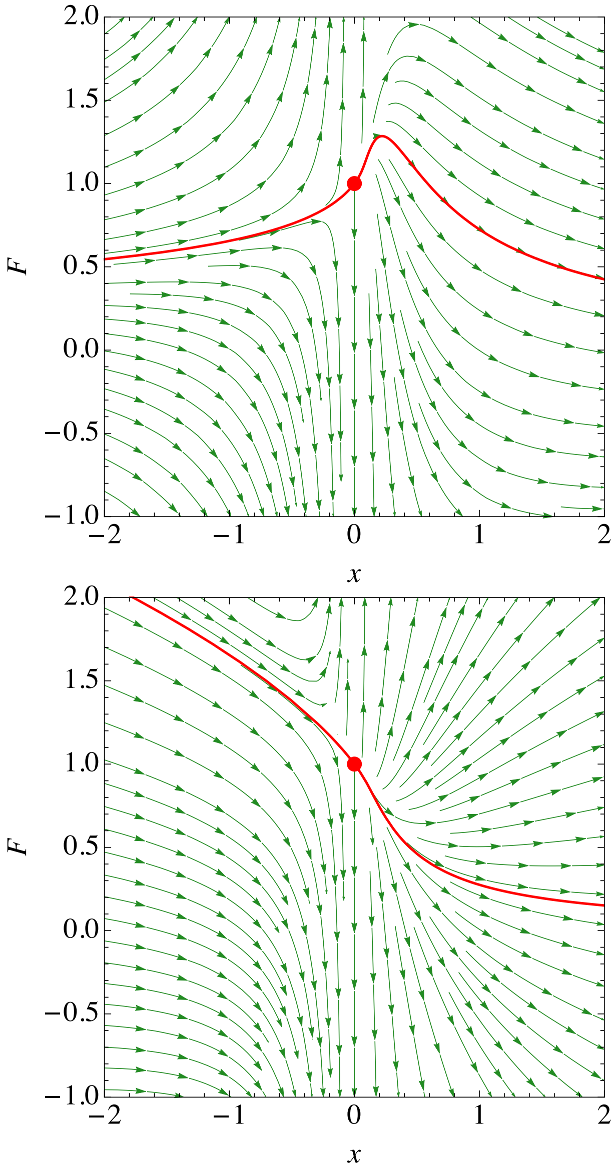

Figure 2.2: In these plots we show the phase maps corresponding to $p = 1$ (left) and $p = -1$ (right) of the associated autonomous system. We also highlight the unique solution that smoothly goes through the critical point $(x = 0, F = 1)$ (red dot) from the left. Notice however that the continuation in the right is not uniquely determined due to the existence of a centre manifold that can be clearly appreciated in the plots. We have simply singled out one of them.

system, but we will simply enumerate some important properties that are illustrated in Fig. 2.2. We can see that there is only one trajectory than can smoothly pass through $x = 0$ so the requirement to have a well-defined behavior for $x \to 0$ singles out one solution on the left half plane. However, since the trajectories approach $x = 1$ along a centre manifold from the right half-plane, one can, in principle, match the regular solution coming from the left to any of the existing solutions in the right half-plane.

2.7.2 | Results

In this chapter we have explored the construction of theories involving scalar and vector fields following a procedure inspired by gravity so that the fields are coupled to the total energy-momentum tensor. This defining property is not free from ambiguities arising from the different definitions of the energy-momentum tensor that we have fixed using the Hilbert prescription. Another ambiguity present in the construction comes from requiring a coupling at the level of the action or a coupling such that it is the source of field equations. We have mostly employed the former, although we have commented upon this subject and considered the latter in some specific cases.

We have started by reviewing some known results that exist in the literature concerning scalar fields. We have then extended these results to theories where the coupling of the scalar to the energy-momentum tensor respects a shift symmetry, i.e., the scalar couples derivatively. We have reduced the resummation of the infinite perturbative series to the resolution of some differential equations, whose properties we have described in the previous subsection. We have nevertheless highlighted some particularly remarkable parameters choices where the solutions can be readily obtained as well as the existence of polynomial solutions. The difficulty encountered in obtaining the full theory within the second order formalism encouraged us to consider the problem from a first order perspective, where it is known that the self-coupling procedure simplifies substantially. We have indeed confirmed that this is also the case for the problem at hand and we managed to reduce the resumation to the resolution of algebraic instead of differential equations. We ended our tour on scalar fields by exploring couplings to matter fields taking a scalar as a proxy.

After considering scalar fields, we delved into vector fields coupled to the energy-momentum tensor. This case offers a richer spectrum of results owed to the possibility of having a gauge symmetry. We have first constructed the theory for a vector coupled to the energy-momentum tensor without taking care of the gauge invariance. We have then considered gauge-invariant couplings where the $U(1)$ symmetry is realised in the usual way in the vector field so that the couplings are through the field strength of the vector. We have however commented on the interesting possibility of having the $U(1)$ symmetry non-linearly realised in the vector field so that the resulting action could explicitly contain the vector not necessarily through $F_{\mu\nu}$, but still keeping two propagating degrees of freedom for the vector. This path would be interesting to pursue further, specially if it can be linked to the conservation of the energy-momentum tensor (or some related conserved current) so that the theories are closer to the gravity case, i.e., the consistent couplings are imposed by symmetries rather than by a somewhat ad-hoc prescription. Similarly to the scalar field theories, the resummed actions for the vector fields reduced to solving some differential equations with similar features. However, unlike for theories with scalars, the first order formalism applied to the vector field case did not result in any simplification of the iterative problem. Finally, we also have considered couplings to a scalar as a proxy for the matter fields. Although we did not consider other types of fields throughout this chapter, it would not be difficult to include higher spin fields.

After constructing the actions for our self-interacting fields through the energy-momentum tensor, we have discussed the impact of superpotential terms arising from the ambiguity in the energy-momentum tensor. We have shown that these superpotential terms lead to the generation of Galileon interactions for both, the scalar and the vector.

Given the motivation from gravity theories that triggered our study and the prescribed couplings to the energy-momentum tensor, we have explored generating functionals defined in terms of an effective metric. This method allowed us to re-derive in a simpler way some of the results in the previous sections and give a general procedure to generate all the interactions. Moreover, we have shown that the generating functional procedure greatly simplifies if the leading order correction to the energy-momentum tensor does not depend on the metric. If that is the case, the series can be regarded as a Taylor expansion and the final resummed action is simply the original action coupled to an effective metric.

Finally, we have considered the phenomenology of these type of theories. The possible experimental signatures associated with the scalar and vector couplings discussed along this chapter is very rich and depends on the particular term under study. We have focused on the phenomenology related to disformal scalars and vectors. In fact, at high energies, the longitudinally polarized vector and the disformal scalar are related by the equivalence theorem. Following an effective theory approach, we can work in the perturbation limit. In such a case, monojet and single photon analysis at the LHC are the most sensitive signatures, constraining the dimensional couplings at the TeV scale. On the other hand, both models can support viable dark matter candidates, offering a broad range of astrophysical and cosmological observable possibilities depending of their stability. Note that given the quadratic character of the leading order interaction for the derivative couplings of the scalar or the non-gauge invariant couplings of the vector, the corresponding force will decay faster than the Newtonian behavior $1/r^2$. However, if there is some vev for the fields, be it $\langle \partial_\mu \varphi \rangle$ or $\langle A_\mu \rangle$ for the scalar and the vector respectively, then we can recover the Newtonian-like force with a coupling constant determined by the vev of the field. Indeed the non-linear nature of the interactions may give rise to screening mechanisms, with interesting phenomenology for dark energy models.

3 | Dark Matter Gravitational Production

Nature's creative power is far beyond man's instinct of destruction.

– Jules Verne, *Twenty Thousand Leagues Under the Sea*

The nature of dark matter has been an open question since F. Zwicky first proposed its existence to explain the dynamics of the Coma cluster galaxies [Zwi33]. In the last decades, dark matter has become a key ingredient to explain cosmological observations. However, we are still lacking a fundamental description of its nature. The community has followed several approaches to understand it with special effort in looking for extensions of the Standard Model (SM) of particles as many beyond-SM proposals include new different fields that could be potential candidates to explain dark matter [BHS05]. However, without any conclusive experiment so far, there is great uncertainty in the intrinsic properties of the dark matter candidates as each theoretical proposal predicts different values for its parameters. The only experimental certainty we have so far is that the cross-section of dark matter with the SM fields must be very small [Kah17; Gas16]. Hence, the question of how it was produced arises naturally. Within this thesis we study the production of particles due to gravitational effects during both the inflationary and post-inflationary epochs. We focus on discerning if this mechanism can provide enough dark matter density to explain the observed abundance. In particular, we study the case of a scalar field with a non-minimal coupling to the Ricci curvature scalar and couplings to the background curvature through its derivatives that we will describe later.

Many different authors have studied the particle production due to gravitational effects in the cosmological scenario in detail. The first works in this area focused on the production of elementary particles due to the expansion of the Universe [Par69;

BD84]. The classic article [For87] analysed the gravitational production of particles after the phase transition to a radiation dominated cosmology after inflation, obtaining a general result for the number density of created particles which is of the order of the energy scale of inflation cube ($n \sim H_0^3$) if the particles can be considered effectively massless during inflation, i.e., if the mass is much smaller than the energy scale during this phase ($m \ll H_0$). More recent papers [CKR99a; Chu+01; HY19] have considered the gravitational production as a possible mechanism to produce enough supermassive dark matter particles (WIMPZillas), obtaining then a lower bound for the possible dark matter mass in this scenario. All these works have neglected the importance of the oscillations of the background quantities due to the inflaton dynamics and their impact on the evolution of the quantum field. However, the effects of these oscillatory behaviors on the gravitational production can be important. In [Ema+16; ENT18], they analyse the impact of the scale factor oscillations on the gravitational production, and the works [BL98; MN17] study the tachyonic instability induced on the field by the oscillations of the scalar curvature. There are some recent works which deal with the gravitational production of self-interacting dark matter during the early Universe, imposing some tight constraints on the possible values of the self-coupling parameter [MRT18; Fai+19].

This chapter is devoted to the study of dark matter gravitational production. We will be considering two modifications on the usual picture of cosmological production: on the one hand we will consider the non-trivial dynamics of the background curvature during reheating and, on the other hand, we will study direct couplings of the field to the background quantities through its derivatives. For both scenarios, we will consider a massive scalar field φ with mass m and negligible interactions to the SM sector as the dark matter candidate. The dynamics of this field will be encoded in the action

$$\mathcal{S} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\partial_\mu\varphi\partial^\mu\varphi + (m^2+\xi R)\phi^2 - (\sigma R^{\mathrm{T}\mu\nu} + \gamma R g^{\mu\nu})\partial_\mu\varphi\partial_\nu\right], \qquad (3.0.1)$$

where R is the scalar curvature of the background, and $R^{\mathrm{T}\mu\nu} = R^{\mu\nu} - g^{\mu\nu}R/4$ is the traceless Ricci tensor. The coupling to these quantities is through the dimensionless constant ξ and the dimensionfull constants γ and σ with inverse squared energy units. The action has been written in this particular form in order to separate the part that describes the coupling to the curvature scalar from the part that contains the coupling to the traceless Ricci tensor. In fact, as we will see, these two contributions play an entirely different role in the particle production. During this chapter we will be neglecting the back-reaction of the scalar field on the background dynamics as we expect its energy density to be small as compared to the one associated with the relativistic degrees of freedom of the SM during the considered cosmological epochs.

For our purposes, we can describe the inflationary phase prior to reheating using a de Sitter solution in which the initial vacuum solution for the field is well defined in the situations we are going to study and that are discussed below. Due to the maximal symmetry of the de Sitter solution, all the coupling terms in the action only give a contribution to the effective mass so we can analyse the inflationary phase for both scenarios at the same time. Then, we will dedicate a section to discuss the effects on the production induced by the oscillatory behavior of the curvature scalar during reheating without any derivative couplings, i.e., $\sigma = \gamma = 0$ [CGS20], and another section devoted to the effects of considering the derivative couplings but using just the averaged behavior of the background quantitites [Gut+20]. In summary, we will start studying the production in de Sitter and then we will study separately the effects of the oscillations of the curvature and the inclusion of derivative couplings to the background.

3.1 | Quantization and gravitational production

Let us start discussing the dynamics and quantization procedure for the field appearing in the action (3.0.1) considering all the couplings to the background we want to discuss throughout the chapter. For the discussion of the reheating non-trivial dynamics we will just need to set the derivative couplings to zero. In cosmological scenarios the background geometry is well described, in average, by the FLRW metric as we have already discussed in the introductory section of cosmology. For simplicity and in agreement with the current cosmological observations [Ade+16], we will be considering flat spatial sections on the geometry. Moreover, for later convenience in the quantization scheme, we will write the FLRW metric using conformal time:

$$ds^2 = a(\eta)^2(-d\eta^2 + \delta_{ij}x^i x^j). \tag{3.1.1}$$

For this specific geometry, the Ricci tensor is diagonal and its spatial components, due to homogeneity and isotropy, are all equal. It can be expressed in terms of the scale factor $a(\eta)$, the Hubble parameter $H = a'/a^2$, and the curvature scalar $R = 6a''/a^3$ as

$$R_{00} = (3H^2 - R/2)a^2, \qquad R_{ii} = (H^2 + R/6)a^2, \tag{3.1.2}$$

where the prime denotes derivative with respect to the conformal time η.

From the action (3.0.1) we obtain the equation of motion for the field φ:

$$(1 + A)\varphi'' - (1 + B)\nabla^2\varphi + C\varphi' + D\varphi = 0, \tag{3.1.3}$$

where, for convenience, we have defined the functions

$$\begin{aligned} &A = 3\sigma H^2 - (\gamma + \sigma/4)R, \qquad &&C = 2aH(1+E), \qquad &&E = A + A'/(2aH), \\ &B = -\sigma H^2 + (\sigma/12 - \gamma)R, \qquad &&D = (m^2 + \xi R)a^2. && \end{aligned} \tag{3.1.4}$$

An appropriate rescaling of the field variable allows us to eliminate the friction term. Indeed, in terms of the rescaled field $\chi(\mathbf{x}, \eta) = f(\eta)\varphi(\mathbf{x}, \eta)$ with

$$f(\eta) = a(\eta)\sqrt{1 + A(\eta)}, \tag{3.1.5}$$

the equation of motion for χ reads

$$\chi'' - \frac{1+B}{1+A}\nabla^2\chi + F\chi = 0, \tag{3.1.6}$$

where

$$F = \frac{(1+A)(4D - 2C') + 2CA' - C^2}{4(1+A)^2}. \tag{3.1.7}$$

In the case with no derivative couplings, the above rescaling is well-motivated from the quantization procedure point of view as in cosmological scenarios, the quantization procedure suffers from an ambiguity which can be traced back to the choice of the canonical pair of variables that we are going to quantize [CMV15]. In the case with no derivative couplings, there is a preferred choice for the canonical pair of variables if one demand that the quantum theory is invariant under the spatial symmetries of the background and that its quantum dynamics admits an unitary implementation [Cas+12]. This criterion entails that the field to be quantized must be

$$\chi = a(\eta)\varphi. \tag{3.1.8}$$

Exploiting the fact that the spatial sections are flat, we can expand the field χ in Fourier amplitudes

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\mathbf{x}\, \chi(\mathbf{x}, \eta)e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{3.1.9}$$

which satisfy harmonic oscillator-like equations of motion

$$\chi''_{\mathbf{k}} + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0, \tag{3.1.10}$$

where $k = |\mathbf{k}|$, with time-dependent frequency

$$\omega_k(\eta) = \sqrt{\frac{1+B(\eta)}{1+A(\eta)}k^2 + F(\eta)}. \tag{3.1.11}$$

We can write any solution of the equation of motion as a linear combination of two complex conjugate solutions $\{v_k(\eta), v_k^*(\eta)\}$. Due to the isotropy of the background geometry, we can choose bases of solutions which only depend on k. Then the Fourier amplitude $\chi_{\mathbf{k}}$ can be expressed as a linear combination of the mode function v_k and its complex conjugate, the coefficients being the creation and annihilation variables $a_{\mathbf{k}}$ and $a^*_{-\mathbf{k}}$:

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}\left[a_{\mathbf{k}} v_k(\eta) + a^*_{-\mathbf{k}} v_k(\eta)^*\right]. \tag{3.1.12}$$

In order to preserve the standard Poisson bracket structure for the annihilation and creation variables, the mode functions must be normalised. We will follow the convention of Ref. [MW07]:

$$W[v_k, v_k^*] = v_k' v_k^* - v_k (v_k^*)' = 2i. \tag{3.1.13}$$

The Fock quantization procedure can be carried out as it was presented in the introduction by promoting the annihilation and creation variables $a_{\mathbf{k}}$ and $a^*_{\mathbf{k}}$ to creation and annihilation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ satisfying the canonical commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} - \mathbf{k}'), \qquad [\hat{a}_{\mathbf{k}}, a_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0. \tag{3.1.14}$$

The Hilbert space for the states of the field is spanned in the usual way by the action of finite linear combinations of products of these operators on the vacuum state, defined as the one which is annihilated by all the annihilation operators, i.e.,

$$\hat{a}_{\mathbf{k}} \left|0\right\rangle = 0, \qquad \forall\, \mathbf{k}. \tag{3.1.15}$$

In curved spacetimes the selection of the vacuum state is not unique but depends on the choice of the set of mode functions. The definition of the creation an annihilation operators depends on the choice of basis and hence it introduces an ambiguity in the quantization procedure. In general, different choices will define inequivalent quantizations for the same field. In cosmology, gravitational production occurs because of the time evolution of the background geometry. In these scenarios, the vacuum state for a quantum field is not stationary. Particles are produced because the evolved vacuum state at a given instant of time $\eta_\star$ is in general an excited state with respect to the instantaneous vacuum defined at that same instant of time.

A comoving observer will have an instantaneous notion of vacuum. We will use the so-called adiabatic prescription to define this instantaneous vacuum, which is commonly used in cosmology. This prescription is appropriate and well defined whenever the geometry evolves slowly enough as compared with the characteristic time scales of the field, i.e., if the so-called adiabatic condition is satisfied:

$$\left|\frac{\omega_k'}{\omega_k^2}\right| \ll 1. \tag{3.1.16}$$

This condition will not be fulfilled in general during all the phases of the cosmological evolution for all the modes of the field. However, as we will be interested in the production during the whole reheating phase, we only need to have a good prescription for the adiabatic vacuum after reheating has ended. The modes $u_k(\eta)$ defining the adiabatic vacuum at a time $\eta_\star$ instantaneously behave as the positive-frequency plane wave modes for the vacuum in Minkowski spacetime at $\eta_\star$:

$$u_k(\eta_\star) := \frac{1}{\sqrt{\omega_k(\eta_\star)}}, \tag{3.1.17}$$

$$u'_k(\eta_\star) := -\frac{1}{\sqrt{\omega_k(\eta_\star)}}\left(i\omega_k(\eta_\star) + \frac{1}{2}\frac{\omega'_k(\eta_\star)}{\omega_k(\eta_\star)}\right), \tag{3.1.18}$$

where $\omega_k(\eta_\star)$ is given in (3.1.11).

Once the adiabatic vacuum has been defined for each instant of time, we can compare the mode expansion associated with the Fock space spanned by an initial vacuum state defined at a time η_0 and the one associated with the Fock space spanned by a vacuum state defined at $\eta_\star$ by computing the Bogolyubov coefficients as explained in the introductory chapter. These coefficients define the linear transformation between two different mode expansions evaluated at the same time. Comparing the evolved modes v_k (defined by the vacuum at η_0) until the time $\eta_\star$, with the set of modes u_k associated with the vacuum at each time $\eta_\star$, we obtain a time-dependent Bogolyubov transformation. To compute the particle production, we are only interested in the coefficient β_k which relates the negative-frequency modes of one expansion with the positive-frequency modes of the other one. This coefficient can be easily calculated and turns out to be [BD84]:

$$\beta_k = \frac{1}{2i}(u_k v'_k - v_k u'_k) = \frac{1}{2i\sqrt{\omega_k}}\left[v'_k + \left(i\omega_k + \frac{1}{2}\frac{\omega'_k}{\omega_k}\right)v_k\right]. \tag{3.1.19}$$

Through the computation of the expectation value for the number operator defined with the instantaneous vacuum modes in the original vacuum, the gravitationally produced density of particles can be obtained directly from the β_k Bogolyubov coefficient. Indeed the total number of particles with momentum $\mathbf{k}$ created at a time η is given by $|\beta_k|^2/[2(2\pi)^3]$, which integrated over all possible momenta and divided by the spatial volume a^3, gives

$$n = \frac{1}{2\pi^2 a^3}\int_0^\infty k^2\,|\beta_k(\eta)|^2\,\mathrm{d}. \tag{3.1.20}$$

3.2 | Inflation as a de Sitter solution

As we have discussed in the introduction, to explain the current cosmological observations with the standard cosmological model, people usually introduce an inflationary epoch prior to the usual big bang model. For our purposes, i.e., computing the gravitational particle production, we can consider this inflationary epoch as a pure de Sitter geometry as a first approximation. This solution can be described with a scale factor $a(\eta) = -1/(H_0\eta)$. Here the Hubble parameter H_0 is constant and the conformal time η lies on the interval $(-\infty, 0)$. In more realistic scenarios, the background geometry is governed by a slowly varying Hubble parameter in a quasi-de Sitter expansion described by the slow-roll parameters. It is also possible to compute analytically the vacuum state for a scalar field in this slow-roll approximation [Agu+10]. However, for simplicity, we will consider inflation as a pure de Sitter geometry from now on.

As the de Sitter geometry is a maximally symmetric solution, the Ricci scalar is proportional to the metric, $R_{\mu\nu} \propto R g_{\mu\nu}$. Therefore, in this background, the coupling to the derivatives of the field depends only on the parameter γ. This is not surprising because for the de Sitter geometry the traceless Ricci tensor is identically zero. Computing the curvature related quantities for the considered scale factor and substituting them in (3.1.2) and (3.1.11), we obtain the time-dependent frequency for the field during the inflationary phase:

$$\omega_k^{\text{dS}}(\eta) = \sqrt{k^2 + \mu^2/\eta^2}, \tag{3.2.1}$$

where we have defined

$$\mu = \frac{1}{H_0}\sqrt{\frac{m^2 + 12(\xi - 1/6)H_0^2 + 24\gamma H_0^2}{1 - 12\gamma H_0^2}}. \tag{3.2.2}$$

Note that μ behaves as a (constant) effective mass. Thus we see that the effect of all the different considered couplings to the curvature is to modify the effective mass of the field during the inflationary phase. Inspecting the possible values of the coupling constants, we note that:

1. The effective mass μ becomes imaginary for certain combinations of ξ and γ. In fact, the potential in these cases is not even bounded from below. These scenarios correspond to tachyonic states for the field and correspond to instabilities of the quantum vacuum.

2. For $m = 0,\ \xi = 1/6,\ \gamma = 0$ the effective mass vanishes and we recover the well-known result that there is no production of purely conformal particles.

Effectively, we have only a single parameter to characterise the scalar field, meaning that we cannot discriminate between different sets of the parameters m, ξ and γ leading to the same μ in pure de Sitter spacetime.

The adiabatic condition (3.1.16) in this scenario is fulfilled either for large effective masses ($\mu \gg 1$) or whenever the combination $k|\eta| \gg 1$. From (3.2.2) we see that a positive value of the ξ parameter tends to move the system towards the adiabatic regime, whereas nonvanishing values of the derivative coupling parameter γ will move it away from this regime.

The solution to the mode equation (3.1.10) with time-dependent frequency (3.2.1), can be written in terms of Hankel functions [AS64]:

$$\chi_{\mathbf{k}}(\eta) = \sqrt{k|\eta|}\left[A_{\mathbf{k}} H_\nu^{(1)}(k|\eta|) + B_{\mathbf{k}} H_\nu^{(2)}(k|\eta|)\right], \tag{3.2.3}$$

where $\nu = \sqrt{1/4 - \mu^2}$.

The de Sitter background admits a well defined initial vacuum state for the field: the so-called Bunch-Davies vacuum. It is defined by mode functions that in the asymptotic past $\eta \to -\infty$ behave as plain waves [PT09] in analogy with the definition of vacuum in Minkowski spacetime. Demanding that in the asymptotic past the vacuum state only has positive-frequency modes, and taking into account the asymptotic behavior of the Hankel functions for $|\eta| \to \infty$ [AS64], the mode function v_k must have the form (3.2.3) with

$$A_{\mathbf{k}} = \sqrt{\frac{\pi}{2k}e^{i\pi\nu}}, \qquad B_{\mathbf{k}} = 0. \tag{3.2.4}$$

In order to simplify the computations and for later convenience, we will perform a change of variables from the set (k, η) to the new set (y, η), where the new variable is defined as $y = k|\eta|/|\nu|$. In this new set of variables, the modes defining the Bunch-Davies vacuum read:

$$\chi_y(\eta) = \sqrt{\frac{\pi|\eta|}{2}e^{i\pi\nu}} H_\nu^{(1)}(y|\nu|). \tag{3.2.5}$$

It is now easy to see, after some algebra, that in terms of the independent variables (y, η) the Bogolyubov coefficients $\beta_k(\eta) := \beta(y)$ depend only on the variable y and not on y and η independently.

Once the $\beta(y)$ Bogolyubov coefficient is computed, the created number density of particles during the de Sitter phase is obtained using the definition (3.1.20):

$$n(\eta) = \frac{1}{2\pi^2 a(\eta)^3}\int_0^\infty |\beta_k(\eta)|^2 k^2 \mathrm{d}k = \frac{H_0^3|\nu|^3}{2\pi^2}\int_0^\infty |\beta(y)|^2 y^2 \mathrm{d}y. \tag{3.2.6}$$

Note that this expression is independent of the conformal time η. This is a remarkable result: the density of produced particles in cosmological de Sitter spacetime is constant over time for a comoving observer in the adiabatic regime.

In figure 3.1, we show the spectral particle production for small and large masses as compared with H_0 at the time $\eta = -1/H_0$. Note that we are again using the set of variables (k, η) for the figures because we want to compare them with the results at the end of reheating. In the left panel we see that as the mass increases within the sub-Hubble regime, the amplitude of the production peak decreases while the resulting spectrum broadens, in such a way that the total production grows as the mass increases. This result can be easily understood because the particle creation is a consequence of the conformal symmetry breaking, and the parameter controlling the degree of symmetry breaking is precisely the effective mass μ. On the other hand, for masses beyond the Hubble parameter H_0 (right panel), the peak amplitude also decreases as the mass increases but the spectrum broadens much less as compared with the left panel (not enough to overcome the damping of the amplitude). The global result is quite different: the net production decreases with increasing masses. We can understand this effect if we think of the Hubble parameter as an indicator of the energy stored in the gravitational field. Then, creating particles more massive than the energy of the background is more difficult.

We can analyse the asymptotic behavior of the number density (3.2.6) for the regime of large effective masses $\mu \gg 1$. In this regime, ν is pure imaginary, i.e. $\nu = i|\nu|$. Hence, we will perform a series expansion of the mode solutions in $1/|\nu|$. One could be tempted to ignore the contribution $1/4$ to ν but it turns out to be crucial. Then, the asymptotic series for the Hankel function reads [AS64]

$$H^{(1)}_{i|\nu|}(y|\nu|) = \sqrt{\frac{2}{i\pi|\nu|}\frac{e^{\pi|\nu|/2}e^{i|\nu|\zeta}}{(1+y^2)^{1/4}}}\left[1 + i\frac{3p-5p^3}{24|\nu|} - \frac{81p^2-462p^4+385p^6}{1152|\nu|^2} + \mathcal{O}\left(|\nu|^{-3}\right)\right], \tag{3.2.7}$$

where

$$\zeta(y) = \sqrt{1+y^2} + \log\left(\frac{y}{1+\sqrt{1+y^2}}\right), \qquad [p(y) = -(1+y^2)^{-1/2}]. \tag{3.2.8}$$

When we compute the coefficient $\beta(y)$, the first and second terms of the series expansion cancel out, so the leading contribution to the particle density comes from the third order terms in (3.2.7). Then, the leading order for $|\beta(y)|^2$ is

$$|\beta(y)|^2 = \frac{(1+6y^2)^2}{2^8(1+y^2)^6|\nu|^4} + \mathcal{O}(|\nu|^{-6}). \tag{3.2.9}$$

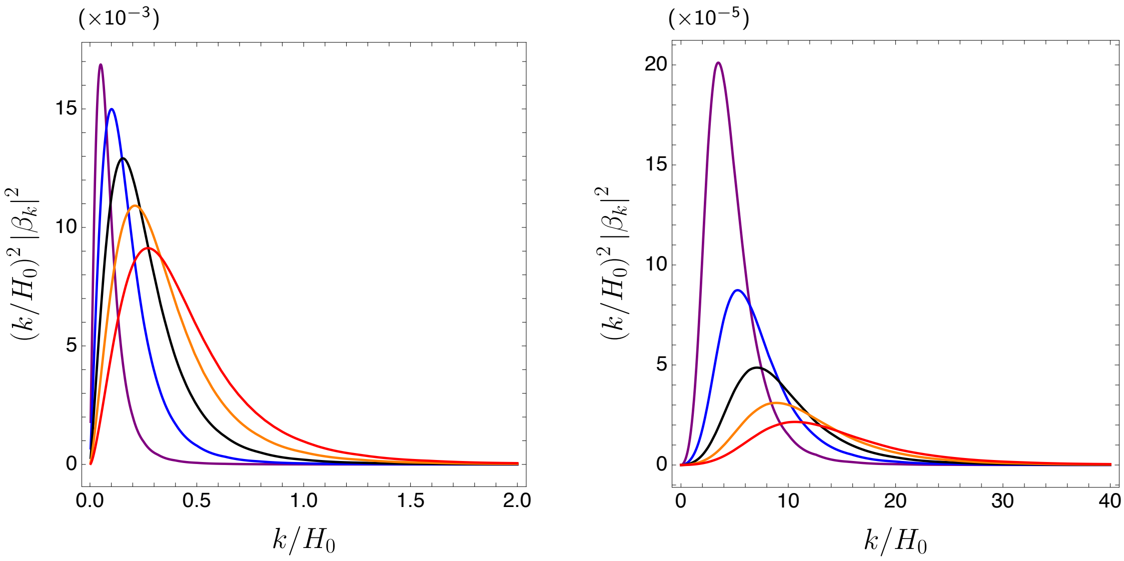

Figure 3.1: Spectral particle distribution created during the de Sitter epoch for different effective masses computed at $\eta = -1/H_0$. The left panel shows the regime of sub-Hubble masses with the following colour code: Purple, $\mu = 0.1H_0$; Blue, $\mu = 0.2H_0$; Black, $\mu = 0.3H_0$; Orange, $\mu = 0.4H_0$; and Red, $\mu = 0.5H_0$. The right panel shows the regime for super-Hubble masses with the following colour code: Purple, $\mu = 4H_0$; Blue, $\mu = 6H_0$; Black, $\mu = 8H_0$; Orange, $\mu = 10H_0$; and Red, $\mu = 12H_0$.

It should be noted that for masses larger than a few times H_0 the agreement with the exact spectrum is almost perfect.

Finally, integrating (3.2.6) using the residue theorem, we find that the behavior of the particle density in the large effective mass regime is

$$n(\eta) = \frac{151}{2^{18}\pi}\frac{H_0^3}{|\nu|} + \mathcal{O}(|\nu|^{-3}). \tag{3.2.10}$$

3.3 | Effects of reheating background dynamics

In this section we analyse the effects on gravitational production of the non-trivial dynamics of the curvature scalar R sourced by an inflaton field during reheating. To isolate the effects of the scalar of curvature oscillations, during this section we are going to ignore the effects of the disformal couplings introduced in the action (3.0.1), hence we set $\gamma = \sigma = 0$ in what follows. Therefore, we will focus on a scalar field with negligible interactions with the SM but with a direct coupling to the scalar curvature through a term in the action $\xi R\varphi^2$. Analyses based on quantum cosmology considerations suggest that the coupling constant ξ should be of order 1 (between $1/6$ and $1/3$, to be more precise) [WYV19]. For this reason together with the numerical difficulties encountered when computing the production for higher values, we will restrict the range of the parameter ξ on our discussion to lie between the values $1/6$ and 1.

3.3.1 | Background evolution

As we have discussed in the previous section, we have modeled the inflationary epoch using a de Sitter solution which is a good first approximation to single-field inflation paradigm except for the transition to reheating. Once inflation has ended the behavior of the scale factor is determined by the potential of the inflaton field ϕ near the minimum value of its potential. We will consider the reheating coming from a massive chaotic inflationary model with potential

$$V = \frac{1}{2}m_\phi^2\phi^2, \tag{3.3.1}$$

where m_ϕ is the mass of the inflaton field. With this potential, it is well known that the background geometry will behave in average as a matter dominated Universe [KLS97]. For our study, it suffices to deal only with the average behavior of both the scale factor and the Hubble parameter, for which we keep the symbols $a(\eta)$ and $H(\eta)$. On the other hand, we will take into account the oscillating terms in the curvature scalar sourced by the dynamics of the inflaton. As we will discuss, the gravitational particle production is enhanced by the instabilities provoked by these oscillations in the effective time-dependent mass term of the field.

If we set the end of inflation at $\eta = 0$, the averaged scale factor for these two epochs can be modeled as [Gar93]

$$a(\eta) = \begin{cases} \dfrac{1}{1 - H_0\eta}, & \eta \leqslant 0, \\ \left(1 + \dfrac{H_0\eta}{2}\right)^2, & \eta > 0, \end{cases} \tag{3.3.2}$$

where $H_0 = \sqrt{12\pi}m_\phi$ is the value of the Hubble parameter at the onset of the single-field inflationary epoch for the model we are considering (3.3.1).

Although this parameterization provides both a continuous scale factor and a continuous Hubble parameter $H(\eta)$ for all values of η, it fails to provide a continuous Ricci scalar at $\eta = 0$. For the approximations we are going to use to solve the mode equations for the field we will need R to be at least a $\mathcal{C}^4$ function. To construct such a function we used an eight order polynomial to interpolate between the value of R at the end of the de Sitter phase and the one coming from the dynamics of the inflaton field during reheating. We start the interpolation then at $\eta = 0$ and finish it at $\eta_{\rm osc}$, which is defined as the time in which the Hubble parameter from the background parameterization (3.4.1) coincides with the averaged one from the considered dynamics of the inflaton.

The dynamics of the inflaton is governed, in cosmological time (related to conformal time via $\mathrm{d}t = a(\eta)\mathrm{d}\eta$), by the equation of motion

$$\ddot{\phi} + 3H(t)\dot{\phi} + m_\phi^2 \phi = 0. \tag{3.3.3}$$

Inflation ends when $H(t) \sim m_\phi$ and hence during reheating the friction term is subdominant with respect to the mass term. Henceforth, the solution for the inflationary field can be approximated at first consistent order in a Wentzel-Kramers-Brillouin (WKB) expansion as

$$\phi(t) = \frac{\Phi_0}{t} \sin(m_\phi t) \left[1 + \mathcal{O}\left(\frac{1}{m_\phi t}\right)\right], \tag{3.3.4}$$

where Φ_0 is the amplitude of the field at the end of inflation and depends on the specific inflationary model. In the case of massive chaotic inflation that we are analysing, the observations of the CMB fluctuations [Ade+16] set the following values for both the mass and the amplitude of the inflaton field at the end of inflation:

$$m_\phi \simeq 1.2 \times 10^{13}\,\mathrm{GeV}, \qquad \Phi_0 \simeq \frac{4\sqrt{\pi}}{\kappa^2}, \tag{3.3.5}$$

where $\kappa^2 = 8\pi M_P^{-2}$, and M_P is the Planck mass $M_P \simeq 1.22 \times 10^{19}\,\mathrm{GeV}$.

The dynamics of spacetime during the considered epochs is dominated by the energy density of the inflaton field. Hence, the behavior of the curvature scalar during reheating is given by the trace of the Einstein equations sourced by the energy-momentum tensor of the inflaton:

$$R_{\rm osc} = \kappa^2 (2m_\phi^2 \phi^2 - \dot{\phi}^2). \tag{3.3.6}$$

Using the solution for the inflaton field (3.3.4), and keeping all the terms up to the first order WKB approximation, the curvature scalar can be written as

$$R_{\rm osc} \simeq \frac{\kappa^2 m_\phi^2 \Phi_0^2}{t^2} \left[2\sin^2(m_\phi t) - \left(\cos(m_\phi t) - \frac{\sin(m_\phi t)}{m_\phi t}\right)^2\right]. \tag{3.3.7}$$

The resulting curvature scalar from the above approximations is shown in figure 3.2.

With this background dynamics, the equation of motion of the modes of the scalar field reads

$$\chi_k'' + \omega_k^2 \chi_k = 0, \tag{3.3.8}$$

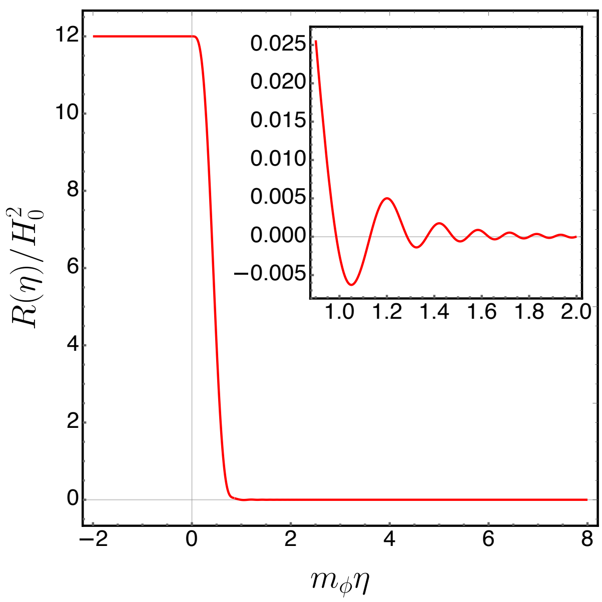

Figure 3.2: Curvature scalar $R(\eta)$ as a function of conformal time. During de Sitter inflation ($\eta < 0$) the Ricci scalar remains constant, which is in good agreement with all the inflationary models. The most appreciable difference between our approach and a realistic inflationary model should occur near $\eta = 0$. We have modeled the transition between the end of inflation and the onset of oscillations using a polynomial function so the scalar curvature is a $\mathcal{C}^4$ function. For $\eta \geqslant \eta_{\rm osc}$ we have considered the curvature sourced by the oscillations of the inflaton field (3.3.7).

with the time-dependent frequency being

$$\omega_k^2 = k^2 + a^2[m^2 + (\xi - 1/6)R] \tag{3.3.9}$$

and R is the junction of the interpolation polynomial and the oscillatory scalar of curvature $R_{\rm osc}$.

We are interested in computing the particle density produced at the end of reheating. Hence, we will be able to define the adiabatic vacuum (3.1.17) as the evolution of the background is smooth enough once reheating has ended. We will obtain the Bogolyubov transformation between the evolved initial vacuum and the adiabatic one, computing then the produced density of particles. Afterwards, we will evolve this number density taking into account that the evolution of the Universe is isoentropic after this moment. Therefore, the gravitational production of particles is negligible after reheating has ended. With the evolved number density we can compute the predicted abundance for dark matter from our model as it is explained in section 3.3.4.

3.3.2 | Analytical approximations

The differential equation for the modes (3.3.8) can be approximated in two different scenarios: either for masses of the field well above the energy scale of inflation $m \gg H_0$ or for the ultraviolet regime of the spectrum for all the parameter space. In this section we will analyse both regimes using a WKB approximation.

Large mass regime

In the scenario in which the mass of the field is well above the energy scale of inflation ($m \gg H_0$), we can approximate the solution to the mode equation (3.3.8) using a second order WKB expansion. Within this approximation, we assume an ansatz for the mode equation of the form

$$v_k = \frac{1}{\sqrt{W_k}} \left(A_k e^{-i \int_0^\eta W_k \mathrm{d}s} + B_k e^{i \int_0^\eta W_k \mathrm{d}s} \right). \tag{3.3.10}$$

Here, W_k is determined by introducing this ansatz in the equation of motion for the modes. If we can neglect the fourth and higher order derivatives of the field frequency, i.e., we can keep the approximation up to second order, this function turns out to be

$$W_k^2 = \omega_k^2 - \frac{1}{2} \left[\frac{\omega_k''}{\omega_k} - \frac{3}{2} \left(\frac{\omega_k'}{\omega_k} \right)^2 \right]. \tag{3.3.11}$$

On the other hand, A_k and B_k are coefficients obtained demanding that the modes are $\mathcal{C}^1$ functions in the transition from the de Sitter phase to the reheating one:

$$A_k = \frac{i}{2\sqrt{W_k}} \left[v_k'^{\mathrm{dS}} - \left(iW_k - \frac{W_k'}{2W_k} \right) v_k^{\mathrm{dS}} \right] \Bigg|_{\eta=0}, \tag{3.3.12}$$

$$B_k = -\frac{i}{2\sqrt{W_k}} \left[v_k'^{\mathrm{dS}} + \left(iW_k + \frac{W_k'}{2W_k} \right) v_k^{\mathrm{dS}} \right] \Bigg|_{\eta=0}, \tag{3.3.13}$$

with v_k^{dS} being the mode solution during the de Sitter era, given by

$$v_k^{\mathrm{dS}} = \sqrt{\frac{\pi|\eta - 1/H_0|}{2}} e^{i\pi\nu} H_\nu^{(1)}(k|\eta - 1/H_0|), \tag{3.3.14}$$

with $\nu = \sqrt{1/4 - m^2/H_0^2 - 12(\xi - 1/6)}$ as we have already discussed in the previous section.

The Bogolyubov coefficient encoding the spectral production can be straightforwardly computed by introducing this WKB approximation for the modes in (3.1.19), and it turns to be

$$\beta_k^{\rm WKB} = -\frac{e^{-i\int_0^\eta W_k {\rm d}s}}{4W_k^{3/2}\omega_k^{3/2}}\Big\{2W_k^2\omega_k\left(A_k - B_k e^{2i\int_0^\eta W_k {\rm d}s}\right)$$
$$- \left[2W_k\omega_k^2 + i(\omega_k W_k' - \omega_k' W_k)\right]\left(A_k + B_k e^{2i\int_0^\eta W_k {\rm d}s}\right)\Big\}. \quad (3.3.15)$$

Note that had we considered the lowest order of the WKB expansion, i.e., $W_k = \omega_k$, the expression (3.3.15) would has been trivial. Indeed, the mode expansion describing the field in this case would be the same one which defines the adiabatic vacuum at each instant of time and hence the only spectral production would be the one due to the change of vacuum at the end of inflation, encoded in the B_k coefficients of the mode expansion (3.3.10). Hence, the effects of the evolution of the background during reheating would be neglected.

We can obtain then an analytical approximation to the number density of particles in this large mass regime by integrating the spectral particle production for all the modes, and dividing by the spatial volume (see (3.1.20)):

$$n^{\rm WKB} = \frac{1}{2\pi^2 a^3(t)}\int_0^\infty {\rm d}k\, k^2 \Bigg|\frac{1}{4W_k^{3/2}\omega_k^{3/2}}\Bigg\{2W_k^2\omega_k\left(A_k - B_k e^{2i\int_0^\eta W_k {\rm d}s}\right)$$
$$- \left[2W_k\omega_k^2 + i(\omega_k W_k' - \omega_k' W_k)\right]\left(A_k + B_k e^{2i\int_0^\eta W_k {\rm d}s}\right)\Bigg\}\Bigg|^2. \quad (3.3.16)$$

In this expression it is simple to see that in order to include the non-trivial effects of the background evolution in the number density, we need to approximate the solution to (3.3.10) with a higher order WKB approximation that the one used to define the adiabatic vacuum (otherwise $W_k = \omega_k$). The integration in modes can be performed with any numerical method and the agreement with the number density obtained from numerical solutions to (3.3.10) is exceptionally good.

Ultraviolet regime

Let us now turn to the study of the ultraviolet regime ($k \gg m_\phi$). In this scenario numerical computations show a significant increase in the particle production at the time when the time dependent frequency ω_k becomes comparable with the characteristic frequency at which the frequency itself oscillates. The bare WKB approximation does not capture this resonance as shown in figure 3.3.

In order to analytically deal with this resonance, we have followed a different approximation scheme for this case. Let us start by defining an auxiliary set of mode

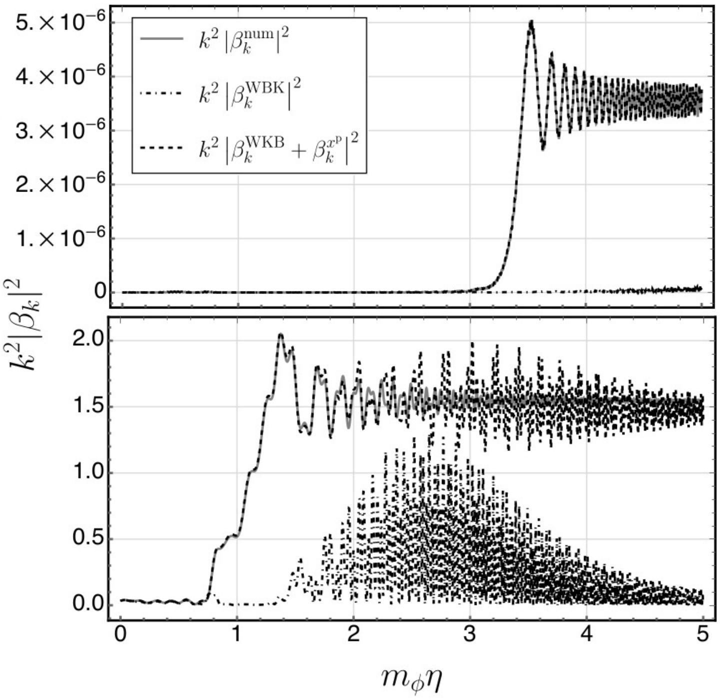

Figure 3.3: Particle production as a function of time for different kinds of computation: numerical, WKB, and refined WKB. In this figure, $\xi = 1/4$ and $m = 0.1m_\phi$. The upper panel is calculated for $k = 100m_\phi$ and the lower panel for $k = 10m_\phi$. The WKB approximation (black dot dashed line at the bottom) does not capture the resonance present in the numerical result (gray continuous line) while the refined WKB approximation (black dashed line) agrees with the numerical results.

functions x_k defined as the difference between the exact solution and the WKB approximation:

$$x_k = v_k^{\mathrm{exact}} - v_k^{\mathrm{WKB}}. \tag{3.3.17}$$

These new modes are also normalized according to (3.1.13) and satisfy a forced time-dependent harmonic oscillator equation

$$x_k'' + \omega_k^2 x_k = F_k, \tag{3.3.18}$$

where the source term turns out to be:

$$F_k = \frac{v_k^{\mathrm{WKB}}}{4W_k^{5/2}} \left(-3W_k'^2 + 4W_k^4 - 4W_k^2\omega_k^2 + 2W_kW_k''\right). \tag{3.3.19}$$

The reason to define this new set of modes is to extract the resonant behavior explicitly in the equations of motion for the modes and convert it into a classical parametric resonance induced by a source term. This source term involves two different frequencies: the one coming from the mode function oscillatory behavior and another one coming from the time-dependent oscillation frequency of the scalar curvature. The general solution to the equation (3.3.18) can be written as

$$x_k = b_1 x_k^{\mathrm{h}} + b_2 x_k^{\mathrm{h}*} + x_k^{\mathrm{p}}, \tag{3.3.20}$$

where x_k^{h} is a solution to the homogeneous equation, x_k^{p} is a particular solution of the full equation, and b_1, b_2 are arbitrary constants. This particular solution for the differential equation can be written as

$$x_k^{\mathrm{p}} = \frac{1}{2i}\int_0^{\eta}\left[x_k^{\mathrm{h}}(\tau)x_k^{\mathrm{h}*}(\eta) - x_k^{\mathrm{h}*}(\tau)x_k^{\mathrm{h}}(\eta)\right]F_k(\tau)\mathrm{d}\tau. \tag{3.3.21}$$

Before the resonance is triggered, the WKB solution v_k^{WKB} approximates very well the exact solution v_k^{exact}, i.e., initially and before the resonance is reached, $x_k = 0$. Therefore, we set b_1 and b_2 to zero. For the solution x_k^{h} to the homogeneous equation, necessary to calculate the particular solution (3.3.21), we can simply use the WKB approximation we were using before because we have now isolated the parametric resonant frequency of interest.

To summarize, the approximate mode functions can be written, in this refined WKB approximation scheme, in the form

$$v_k = v_k^{\mathrm{WKB}} + x_k^{\mathrm{p}}, \tag{3.3.22}$$

where x_k^{p} is given by (3.3.21) and, in this expression,

$$x_k^{\mathrm{h}} = \frac{1}{\sqrt{W_k}}\exp\left(-i\int_0^{\eta} W_k \mathrm{d}s\right). \tag{3.3.23}$$

The Bogolyubov coefficients are linear in the mode functions, so the corrections to the β_k coefficients obtained through this refined WKB approximations can be added straightforwardly:

$$\beta_k = \beta_k^{\mathrm{WKB}} + \frac{1}{2i\sqrt{\omega_k}}\left[x_k^{\mathrm{p}\prime} - \left(i\omega_k - \frac{1}{2}\frac{\omega_k'}{\omega_k}\right)x_k^{\mathrm{p}}\right]. \tag{3.3.24}$$

As it is shown in figure 3.3 this approximation method gives analytical estimations that agree extremely well with the numerical results and capture the resonant enhancement perfectly. Using this analytical approximation we can also obtain the number density of particles by replacing (3.3.24) into (3.1.20) in analogy with the latter scenario.

3.3.3 | Numerical analysis

In general, the computation of the mode functions during the reheating epoch cannot be performed analytically, although as we have discussed in the previous section 3.3.2, there are certain regions of the parameter space in which we can carry out an analytical approximations that are in good agreement with the numerical results.

In this section we will solve (3.3.8) numerically for a wide range of the scalar field parameters. We will then introduce the numerical solutions for the mode functions v_k in (3.1.19), and via (3.1.20) we will obtain the total particle production. We present and discuss the numerical results for the particle production spectra obtained in this way for different parameters of the field, namely for different values of the mass m and the coupling to curvature ξ.

Throughout the numerical exploration of the space of parameters for the field, we have found that there are four different spectral behaviors as the dynamics of the scalar curvature induces three different effects on the particle production as we will discuss.

On the one hand, its constant value during the de Sitter phase contributes to the effective mass of the field $m^2 + 12H_0^2(\xi - 1/6)$. If this effective mass is below the energy scale of inflation H_0^2, the production grows but if it is above it, then the production decreases as we have seen in detail in the section devoted to the de Sitter solution.

On the other hand, the oscillations of the scalar curvature during the reheating phase induces two new phenomena in the gravitational production. Since the frequency of the field is oscillating, there will be resonant effects whenever the two frequencies (the frequency ω_k itself and the frequency at which it oscillates) are equal. These resonances will be important mostly in the ultraviolet regions of the spectra. Furthermore, these oscillations can make the field suffer from tachyonic instabilities, because the effective mass for the field becomes tachyonic for certain times if the bare mass of the field is small enough. This latter instability affects different bands of momenta as the instability amplitude decreases in for successive oscillations. Let us now discuss how these three effects affect the different regions of the parameter space.

For the region of masses much smaller than the interaction term in the precise sense that $m^2 \ll (\xi - 1/6)|\min(R_{\text{osc}})|$, the most important production enhancement mechanism is the tachyonic instability already studied by T. Markkanen and S. Nurmi in [MN17]. This instability affects mostly the infrared region of the spectrum as it is shown in figure 3.4. As the coupling constant ξ grows, the first infrared peak is damped because the interaction term during the de Sitter epoch makes the effective mass larger and the enhancement occurs on top of the Planckian spectral production at the end

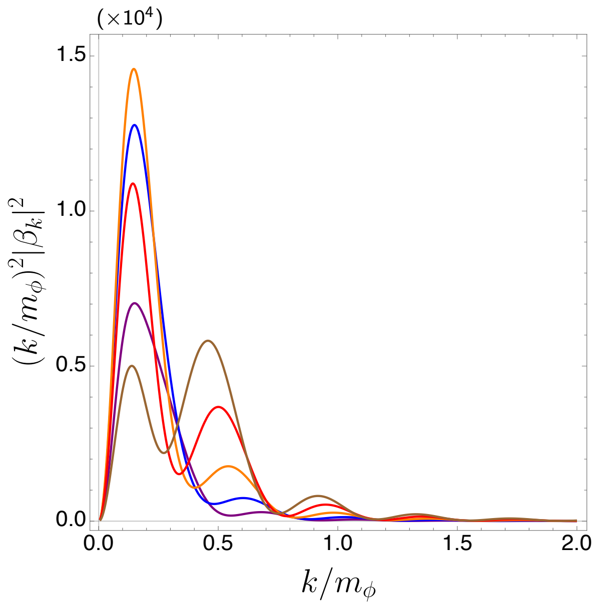

Figure 3.4: Spectral particle density for $m = 1.2 \times 10^6$ GeV as a representative case of the regime $m^2 \ll (\xi - 1/6)|\min(R_{\rm osc})|$. The different amplified bands appear because each oscillation of the Ricci scalar will excite different modes of the field due to the time dependence of the equations of motion. In this figure the colour code is: Purple, $\xi = 17/60$; Blue, $\xi = 1/3$; Orange, $\xi = 23/60$; Red, $\xi = 13/30$; and Brown, $\xi = 29/60$.

of inflation. Furthermore, the larger the interaction term, the larger the ultraviolet peaks, as the instability regions during the oscillatory regime contributes with a larger imaginary effective mass.

If the mass term is larger than the energy scale of inflation, i.e, $m^2 > H_0^2$ and hence larger than the interaction term during the oscillatory phase, then, the resulting spectrum does not differ much from the one obtained in the de Sitter phase. In this scenario, the coupling term increases the effective mass during the inflationary epoch, giving rise to a different Planckian spectrum with no instability-enhancements as it is shown in figure 3.5. Therefore, in this regime, as the coupling increases, the particle production decreases.

There is a third regime, shown in figure 3.6, in which the mass is below the energy scale of inflation and it is of the order of the coupling term. Within this regime, increasing the coupling constant increases the spectral production as long as the effective mass during inflation is still below the energy scale of inflation. Furthermore, there are resonances due to the fact that the frequency at which the field oscillates coincides at some times with the frequency at which the frequency itself oscillates. This occurs in the ultraviolet region of the spectrum.

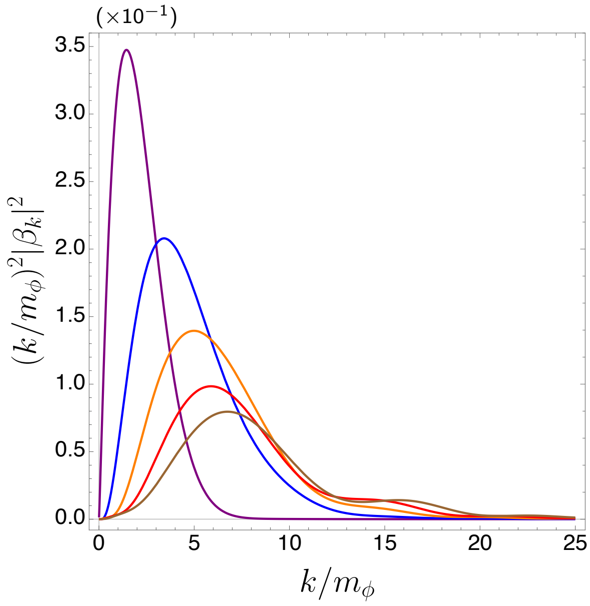

Figure 3.5: Spectral particle density for $m = 1.8 \times 10^{13}$ GeV as a representative case of the regime $m > H_0$. In these cases the production diminishes as the coupling term grows due to its contribution to the effective mass during inflation. In this figure the colour code is: Purple, $\xi = 1/6$; Blue, $\xi = 7/30$; Orange, $\xi = 3/10$; Red, $\xi = 11/30$; and Brown, $\xi = 13/30$.

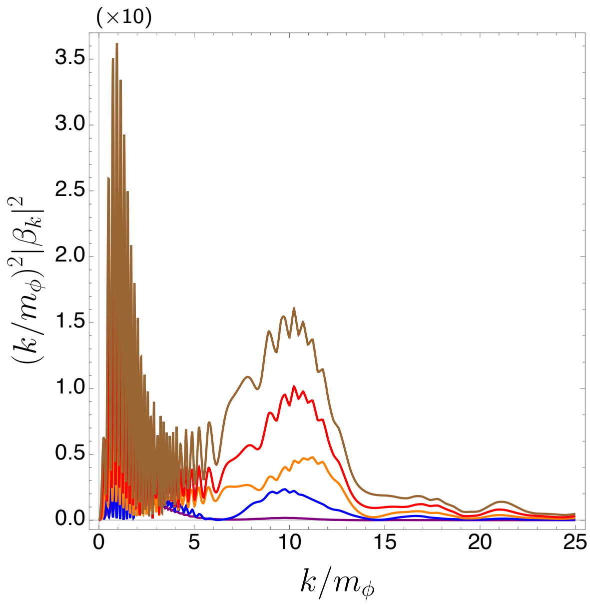

Figure 3.6: Spectral particle density for $m = 7 \times 10^{11}$ GeV as a representative case for the regime 10^9 GeV $< m < H_0$. In this scenario the spectral production increases with the coupling term as the resonance effect becomes more important. In this figure the colour code is: Purple, $\xi = 11/60$; Blue, $\xi = 4/15$; Orange, $\xi = 7/20$; Red, $\xi = 13/30$; and Brown, $\xi = 31/60$.

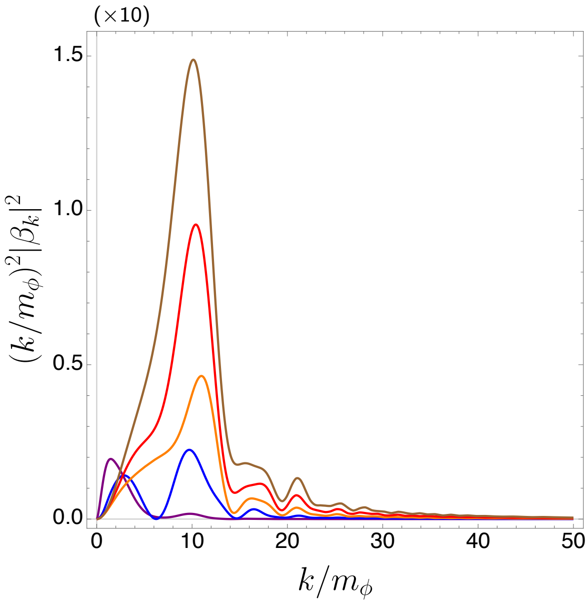

Figure 3.7: Spectral particle density for $m = 3.8 \times 10^9$ GeV as a representative case of the transition regime between the tachyonic instability dominated region and the resonance dominated one. In this regime there are contributions of both the tachyonic instability and the resonant effect with the frequency of the Ricci scalar. In this figure the colour code is: Purple, $\xi = 11/60$; Blue, $\xi = 4/15$; Orange, $\xi = 7/20$; Red, $\xi = 13/30$; and Brown, $\xi = 31/60$.

Finally, there is a fourth regime for masses $m \sim 10^9$ GeV in which there is a transition between the region dominated by the tachyonic instability and the resonance dominated one. The typical spectra for this transition regime are shown in figure 3.7.

Once we have obtained the spectral production density, we can integrate it to obtain the produced particle density at the end of reheating following Equation (3.1.20). In figure 3.8 this production is shown as a function of the field parameters m and ξ. In this plot, we can distinguish the four regions discussed above. The first one is situated in the lightest masses, where the particle production depends only on the value of the coupling to the scalar curvature except for coupling constants in the neighbourhood or the conformal one. The dominant production mechanism in this region is the tachyonic instability induced by the oscillations of the scalar curvature. Around $m \sim 10^9$ GeV there is a transition to the resonant enhancement dominated region. In the range from $m \sim 10^{10}$ GeV to 10^{13} GeV there is a dependence of the production on both field parameters, with a monotonic growth in ξ and a monotonic decrease with m. Finally, once the mass is above the energy scale of inflation and the coupling to gravity is not too large, we recover the usual behavior for the production, as it decays rapidly with both the mass and the coupling term for the considered range. It is worth noting that for

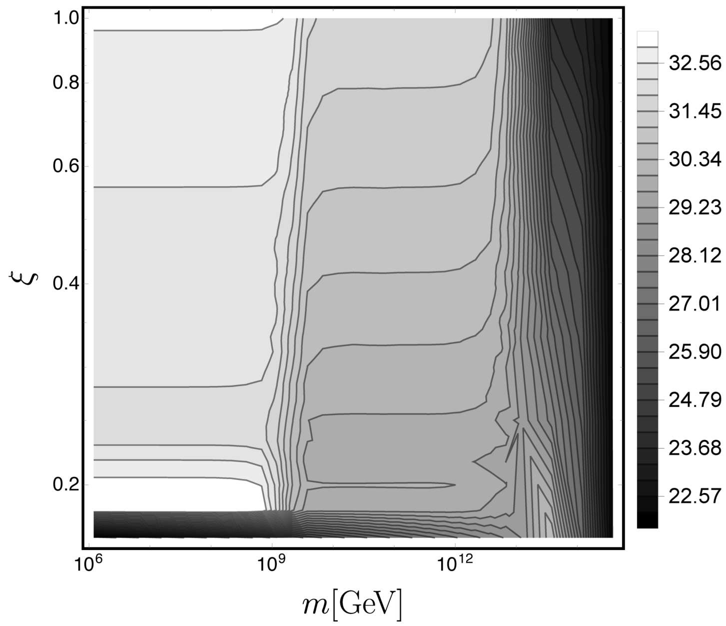

Figure 3.8: In this figure we are showing the particle density in logarithmic scale at the end of reheating, i.e., the contours of $\log_{10} n(t_{\rm rh})$ as a function of the mass m and the coupling to curvature ξ. We can see clearly the different behaviors of the production. For the lightest masses, the enhancement mechanism is the tachyonic instability. There is a transition regime for masses of about 10^9 GeV in which there are contributions from the tachyonic instability and the resonant effect. For masses lower than the energy scale of inflation the only enhancement mechanism is the resonance between the frequency of the field and the one of the Ricci scalar. For masses above the energy scale of inflation, there are no enhancement mechanisms and the typical production from a de Sitter phase is recovered.

larger couplings in this regime a resonance enhanced region would be reached eventually and the production would raise again monotonically with the coupling constant value.

3.3.4 | Constraints for Dark Matter

Once we have computed the gravitational particle production at the end of the reheating phase, we can use this result to set constraints on the field parameters if it was to be a dark matter candidate. The constraints are obtained imposing that the predicted abundance equals the one obtained from the observations. In order to make this comparison, we need to evolve the computed production from the end of reheating to the present day. This evolution is computed taking into account that the scalar field is decoupled from the rest of the constituents of the Universe as we have neglected its interactions. Therefore, the evolution of its particle density is solely due to the isoentropic expansion of the Universe. We can rewrite the time dependence of the number density as a dependence on the temperature of the radiation that fills the Universe. This relation arises from the statistical interpretation of the temperature and the evolution of the particle geodesics in a cosmological background [Pad93]. Hence, the evolved number density can be rewritten in terms of the temperature of the background radiation as

$$n(\eta_{\text{today}}) = \frac{a^3(\eta_{\text{rh}})}{a^3(\eta_{\text{today}})} n(\eta_{\text{rh}}) = \frac{g^0_{*S}}{g^{\text{rh}}_{*S}} \left(\frac{T_{\text{today}}}{T_{\text{rh}}}\right)^3 n(T_{\text{rh}}), \tag{3.3.25}$$

where g^{rh}_{*S} denotes the effective entropic relativistic degrees of freedom of the Standard Model of particles at the end of reheating, g^{today}_{*S} denotes the same quantity today, T_{today} is the temperature today and T_{rh} is the temperature at the end of reheating. The actual temperature at which reheating took place is still unknown and depends heavily on the particular model considered. Hence, our constraints will have an uncertainty coming from the indetermination on the specific reheating temperature.

Note that we have evolved the number density instead of the energy density of the field. There are two advantages in doing so. On the one hand, we do not need to take into account the effective mass which depends on the curvature term and, on the other hand, we do not need to take into account when each region of the parameter space becomes non-relativistic. Hence, it is simpler to compute its abundance as the energy density today is obtained using just the mass of the particles as they are non-relativistic. The predicted abundance is then obtained through the following expression:

$$\Omega(m,\xi) = \frac{\kappa^2}{3H^2_{\text{today}}} \frac{g^{\text{today}}_{*S}}{g^{\text{rh}}_{*S}} \left(\frac{T_{\text{today}}}{T_{\text{rh}}}\right)^3 m\, n(m,\xi), \tag{3.3.26}$$

where the nought quantities are evaluated at present time and we have emphasized that the density number of particles at the end of reheating depends on the model parameters. The predicted abundance from gravitational production during the early stages of the Universe is shown in figure 3.9 for the maximum allowed value of the reheating temperature: $T_{\text{rh}} = 10^{15}\,\text{GeV}$. This upper bound for the reheating temperature comes from the lack of observation of primordial gravitational waves [MR10]. Although our constraints depend on the reheating temperature, we can set solid constraints if we consider this maximum possible value. Also in figure 3.9, we are showing the bounds on the parameter space for the maximum allowed reheating temperature. Note that in this figure we are not showing the region near the conformal coupling to gravity, as the computed abundance depends highly on the coupling value within this region and we have not explored it numerically in sufficient detail. In figure 3.10 we show how the constraints on the parameter space depend heavily on the specific reheating temperature considered. We see that there are two allowed regions to describe dark matter: one for *light* candidates and another one for *heavy* candidates. It is a general result that there is a forbidden mass gap between these two regions for physically meaningful reheating temperatures. This forbidden region is larger as the considered temperature of reheating is lower.

We have treated the conformal case in its own and the bounds for the mass in this case are shown in figure 3.11 for different reheating temperatures. It is important to note that, as the production is minimal for the conformal case, the less stringent bounds will come from it. For the maximum allowed reheating temperature, we can see that for the conformal coupling case, the maximum allowed mass for the light sector would be $m \sim 10^{11}\,\mathrm{GeV}$ while for couplings far from the conformal the typical maximum allowed mass would be $m \sim 10^{5}\,\mathrm{GeV}$. So in conclusion, the maximum allowed mass for the light candidates sector depends heavily on the coupling constant to curvature ξ due to all the enhancement mechanisms we have discussed throughout this work. Note that the maximum mass coming from the conformal case in this scenario is the upper bound on all the possible masses for the light sector while the minimum mass for this same case is the minimum mass for a WIMPzilla like particle. Furthermore, we can give an empirical law to determine which mass will be the upper bound for the created dark matter particle as a function of the reheating temperature. Fitting the particle production n to an empirical law, we found that it is proportional to the mass in the conformal case. Hence, the upper bound on the mass as a function of the reheating temperature can be written as:

$$m_{\max}^{\text{light}} \simeq 9.4 \times 10^{10} \left(\frac{T_{\text{rh}}}{10^{15}\,\mathrm{GeV}} \right)^{3/2} \mathrm{GeV}. \tag{3.3.27}$$

For large enough masses we see that the predicted abundance becomes independent of the coupling and has a maximum value for a mass around the energy scale of inflation. Depending on the temperature at which reheating ended, for high enough masses the abundance can again match the observed one in accordance with [CKR99a], giving a possible allowed superheavy region. From our computations, this minimum mass would be $m \simeq 5 \times 10^{13}\,\mathrm{GeV}$ for the maximum allowed reheating temperature.

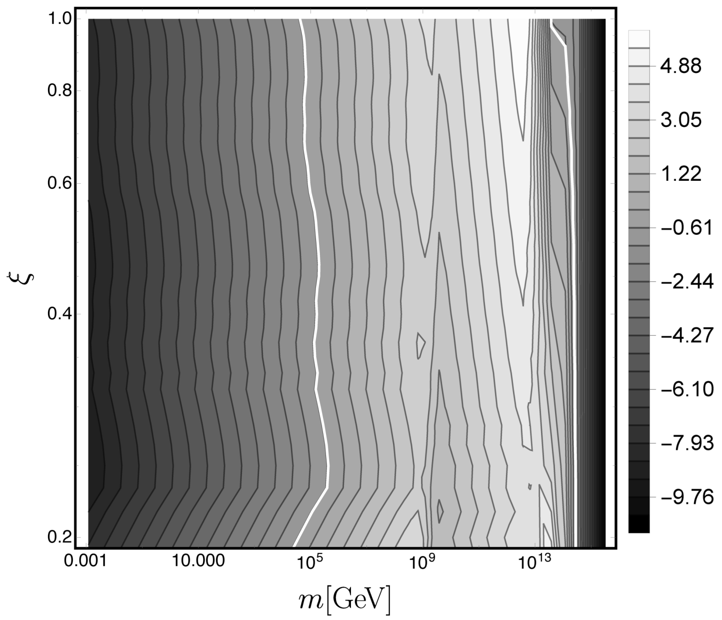

Figure 3.9: Predicted abundance for the scalar field in logarithmic scale, i.e., the contours of $\log_{10}\Omega$ considering the maximum allowed reheating temperature, $T_{\rm rh} = 10^{15}\,\text{GeV}$. The white contours represents the observed abundance of dark matter $\Omega_{\rm DM} = 0.268$. The allowed parameter space of the scalar field is to the left of the first white contour and to right of the second one.

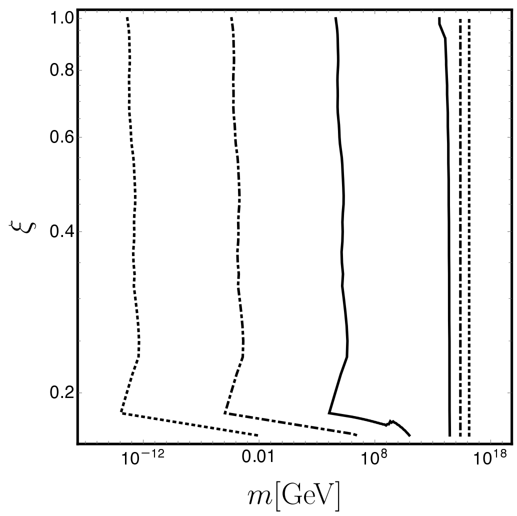

Figure 3.10: The constraints on the field parameter space are represented through the contours corresponding to the observed abundance: $\Omega = 0.268$ for different reheating temperatures. The ones obtained for a specific reheating temperature $T_{\rm rh} = 10^{15}\,\text{GeV}$ are displayed in solid line. In dot-dashed line we plot the same constraints but for $T_{\rm rh} = 10^{12}\,\text{GeV}$ and finally they are displayed in dotted line for $T_{\rm rh} = 10^{9}\,\text{GeV}$.

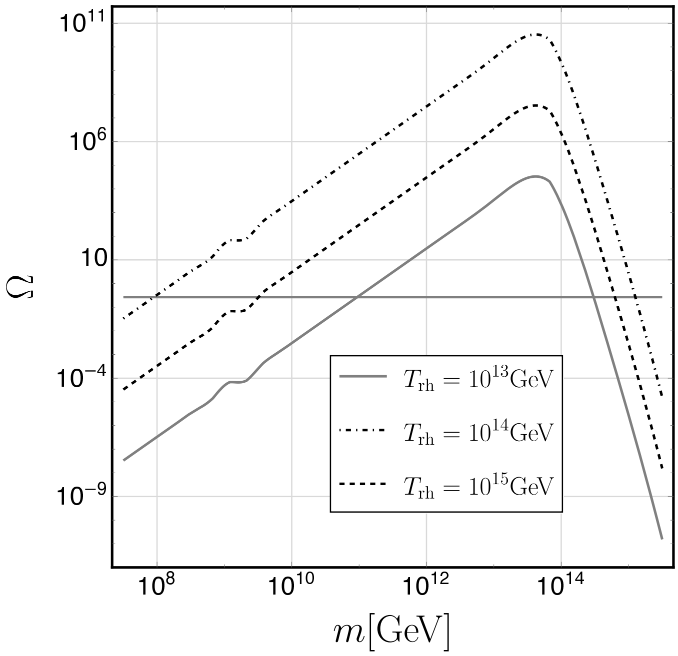

Figure 3.11: Abundance of dark matter for the conformal coupling case ($\xi = 1/6$) for different reheating temperatures. The solid horizontal line represents the observed abundance $\Omega = 0.268$. The gap between the allowed masses grows larger as the reheating temperature is lower.

3.4 | Effects of derivative couplings

In this section we are going to study the effects of the derivative couplings on gravitational production of the field described by the action (3.0.1). In order to isolate the effects of the derivative terms in this section we will consider the average behavior of the background quantities, i.e., we neglect the oscillations of the geometrical quantities that we have just studied in the previous section.

3.4.1 | Background evolution

We are mimicking the background spacetime dynamics using a de Sitter phase for inflation and a matter-dominated universe as the reheating phase as we have previously discussed. This choice of a matter-dominated solution comes from the mean behavior of the background during the oscillations of the inflaton field ϕ in the massive chaotic inflationary scenario with a potential $\propto m_\phi^2\phi^2$ [KLS97].

We can use again the parameterization for the scale factor introduced used in the previous section which provides a continuous and differentiable function:

$$a(\eta)=\begin{cases}\dfrac{1}{1-H_0\eta}, & \eta\leqslant 0,\\ \left(1+\dfrac{H_0\eta}{2}\right)^2, & \eta>0,\end{cases} \tag{3.4.1}$$

where $H_0=\sqrt{12\pi}m_\phi$ is the value of the Hubble parameter at the onset of the single-field inflationary epoch for the model we are considering and we have set the end of inflation at $\eta=0$ (with m_ϕ being the inflaton mass). During this section we will use H_0 as the basic quantity for units.

This parameterization is good enough to get a continuous Hubble parameter. However, it fails to provide continuous and differentiable curvature scalar and tensor. In order to have a satisfactory model, we need the frequency of our field to be sufficiently smooth. Hence, we need to introduce an interpolation regime for the curvature quantities between the end of inflation (i.e., the end of the de Sitter phase) and a moment η_* in the reheating epoch. This time η_* is chosen by equating the Hubble parameter from the above parameterization and the one obtained through the average dynamics of the inflaton field, which for any single-inflation model can be written during the oscillatory phase of the field around the minimum of its potential, as we already discussed in the previous section, as

$$\phi = \frac{\Phi_0}{\eta+\frac{H_0\eta^2}{2}+\frac{H_0^2\eta^3}{12}}\sin\left[m_\phi\left(\eta+\frac{H_0\eta^2}{2}+\frac{H_0^2\eta^3}{12}\right)\right], \tag{3.4.2}$$

$$H^2 = \frac{8\pi}{6M_{\mathrm{P}}^2}\left(\frac{1}{a^2}\left\langle\phi'^2\right\rangle_\eta+m_\phi^2\left\langle\phi^2\right\rangle_\eta\right), \tag{3.4.3}$$

where the inflaton field has been expanded at first consistent WKB order, and $\langle\rangle_\eta$ denotes time averaging. In these expressions, $M_{\mathrm{P}}=1.22\cdot10^{19}$GeV is the Planck mass, $m_\phi=10^{13}$GeV is the inflaton mass, and $\Phi_0=M_{\mathrm{P}}/(2\sqrt{\pi})$ is the amplitude of the field at the onset of the oscillations. These values for the inflaton mass and amplitude are constrained by the CMB observations [Akr+18]. Equating (3.4.2) and the Hubble parameter from the parameterization of the scalar factor, we determine $\eta_*=2.4/H_0$.

After η_* the evolution of the curvature scalar R is determined by the average dynamics of the inflaton field given by eq. (3.4.2) and can be computed through the Hubble parameter and its time derivative as:

$$R=12H^2+6H'/a. \tag{3.4.4}$$

In a model driven by an inflaton field, all the background quantities would be sufficiently smooth functions. Hence, we want to interpolate R and $R_{\mu\nu}$ so they reproduce this differentiability to sufficiently high order. In order to do so, we introduce polynomials of degree 24 for the value of the functions as well as for each of their derivatives independently, so each of them is a $\mathcal{C}^{12}$ function. The degree of the polynomials has been chosen so that it is the minimum possible one so the results are independent of the interpolation order. Note that without the interpolation, the frequency would exhibit a discontinuity when the de Sitter epoch finishes. This discontinuity would abruptly excite modes of arbitrarily high momenta on the field and hence the expected production of particles would be greater than the one obtained considering the background evolution due to an inflaton field.

3.4.2 | Analytical considerations

The equation of motion for the modes during the post inflationary epoch, after the junction time η_*, acquires the form

$$\chi_k'' + \omega_k^{\mathrm{pi}}(\eta)^2 \chi_k = 0, \tag{3.4.5}$$

where the frequency of the modes has a rather complicated functional form in terms of the background quantities:

$$\omega_k^2 = \frac{1+B}{1+A} k^2 + F, \tag{3.4.6}$$

where A, B, and F are given in (3.1.4) particularised for the background described above.

The expression of the frequency as a function of the conformal time η is obtained using the Hubble parameter, the Ricci tensor, and the curvature scalar coming from the averaged behavior of the inflaton field during the reheating epoch. We will solve the equation for the modes of the field numerically, using as initial conditions the value of the field and its first derivative at the end of the de Sitter phase, given by (3.2.3) with the constants given by (3.2.4). Despite the cumbersome expression for the frequency, a direct evaluation of the adiabatic parameter $|\omega_k'/\omega_k^2|$ yields that we safely lie in an adiabatic regime on the reheating phase for all the region of the parameter space we will consider. Therefore the definition of the adiabatic vacuum is well suited to define a proper instantaneous vacuum to compare with the evolved initial state.

Now we are ready to compute the gravitational particle production during the early universe using the model we have discussed, delving into the influence of the derivative couplings in this process. Once we have computed the numerical mode functions which define the initial vacuum state, we can compute the Bogolyubov coefficients relating this initial vacuum state and the adiabatic ones during the post inflationary epoch via equation (3.1.19).

We will consider the derivative couplings as perturbations, meaning that their contribution is subdominant with respect to the one coming from the mass term and the coupling to the Ricci scalar. Furthermore, even for perturbative values of these derivative couplings there is a strong limitation coming from the de Sitter phase, as each case in which their contributions lead to imaginary effective masses, i.e., $\mu^2 < 0$, the vacuum state of the field would become unstable [BD84]. Hence, we have an additional constraint on the derivative couplings given by the inequality:

$$-2[m^2/H_0^2 + 12(\xi - 1/6)] \leqslant 12\gamma H_0^2 < 1. \tag{3.4.7}$$

Perturbative analysis

For sufficiently small values of the derivative couplings γ and σ, we can perform an expansion of the particle production. As we argue in the following this requires two different regimes of approximation: a perturbative expansion of the action that can still induce nonlinear effects due to the nontrivial kinetic term, and an additional expansion for small derivative couplings as compared with the mass (in appropriate units). In this subsection, we will highlight the main aspects of these perturbative and linear expansions for the particle production.

There are reasons that suggest that we should restrict the values of the derivative couplings to the perturbative regime. For instance, these derivative couplings affect the kinetic term of the field, modifying the propagator in a highly nontrivial way. In particular, the derivative couplings of the scalar field to the Riemann tensor at first order are not renormalisable. If such terms are present, higher order terms are expected. This means that if the first order contributions are not small, neglecting higher terms may not be well-motivated.

In view of the action (3.0.1), we can qualitatively advance that, in the perturbative regime, the coupling σ multiplied by the largest value of the traceless Ricci tensor along the whole period relevant to particle production must be much smaller than unity. Similarly, γ multiplied by the largest value of the curvature scalar must be much smaller than 1. More explicitly, it is easy to see that for metrics of the form (3.1.1) the factor multiplying to φ'^2 is proportional to $1 + A$ and the factor multiplying to $\partial_i\varphi\partial^i\varphi$ is proportional to $1 + B$, where A and B are given by in (3.1.4). The perturbative regime is then achieved when $|A|$ and $|B|$ are much smaller than one during the whole evolution. Considering the maximum values of the background quantities multiplying γ and σ, we can write the perturbative conditions in terms of conditions on the maximum absolute value of the derivative couplings:

$$|\bar{\gamma}| \ll 1, \qquad |\bar{\sigma}| \ll 15. \tag{3.4.8}$$

Here and from now on we will use the rescaled dimensionless derivative couplings $\bar{\gamma} = 12H_0^2\gamma$ and $\bar{\sigma} = H_0^2\sigma$, and the dimensionless mass parameter $\bar{m} = m/H_0$.

Even in this perturbative regime, the way in which the quantity A enters the definition of the quantization field χ makes the final result nonlinear in the parameters $\bar{\gamma}$ and $\bar{\sigma}$. In particular, the derivative coupling $\bar{\gamma}$ contributes to the effective mass of the field, even during the de Sitter phase, while $\bar{\sigma}$ does not (see e.g. Eq. (3.2.2)). In the following we focus on the case $\xi = 1/6$ both for simplicity and to isolate the effects of the derivative couplings.

Let us check the conditions on the derivative couplings that are necessary for the changes on the particle production to be small. To begin with, we expand the frequency to first order in the derivative couplings,

$$\omega_k^2 = \omega_{0k}^2 + \Omega_k, \tag{3.4.9}$$

$$\omega_{0k}^2 = k^2 + a^2 m^2, \tag{3.4.10}$$

$$\Omega_k = (B - A)k^2 + a^2(\bar{m}^2 H_0^2 + R/6)A - aHA' - A''/2. \tag{3.4.11}$$

If we impose the condition that the linear term Ω_k is small compared with the unperturbed frequency squared ω_{0k}^2, once we evaluate the background geometrical quantities accompanying the derivative coupling at their maximum value during the whole evolution, we can extract a necessary condition on the derivative parameters:

$$|\bar{\gamma}| \ll \bar{m}^2, \qquad |\bar{\sigma}| \ll \frac{15\bar{m}^2}{2 + \bar{m}^2}, \tag{3.4.12}$$

These conditions are necessary but not sufficient to have small changes in the particle production $|\beta_k|^2$ as we discuss in the following.

Let us start with the zeroth order $|\beta_{0k}|^2$. This is calculated using the zeroth order term u_{0k} of the adiabatic modes u_k (i.e., (3.1.17) with ω_{0k} instead of ω_k) and the zeroth order v_{0k} of the vacuum modes v_k. These zeroth order modes are solutions to the equation $\ddot{v}_{0k} + \omega_{0k}^2 v_{0k} = 0$.

The contribution to $|\beta_k|^2$ linear in the derivative couplings can also be easily found, although the calculations are quite tedious and not very illuminating. Let us describe the ingredients involved in the final expression other than the zeroth order quantities already mentioned, namely, ω_{0k}, u_{0k}, v_{0k}, and their derivatives. These extra ingredients are the already calculated linear order contribution to the frequency Ω_k and its derivatives, the corresponding perturbations to the adiabatic modes, and the perturbation ζ_k to the modes v_k, defined as $v_k = v_{0k}(1+\zeta_k)$. This is the only nontrivial quantity that we have to calculate. Taking into account the equation that it must satisfy (namely, (3.1.10) with ω_k expanded to first order), it is easy to see that ζ_k is given by the quadrature

$$\zeta_k(\eta) = -\int_{-\infty}^{\eta} d\eta' \frac{1}{v_{0k}(\eta')^2} \int_{-\infty}^{\eta'} d\eta'' v_{0k}(\eta'')^2 \Omega_k(\eta''). \tag{3.4.13}$$

In view of this expression, we can assume that $\zeta_k(\eta)$ is small throughout the evolution provided that $\bar{\gamma}$ and $\bar{\sigma}$ satisfy some extra conditions.

For illustrative purposes let us study the large mass regime. Taking into account the form of Ω_k and a quick numerical estimation of the values of the time integrals involved, we obtain the following rough estimation of the maximum value throughout the evolution for $\zeta_k(\eta)$:

$$|\zeta_k(\eta)| \lesssim \frac{\bar{m}}{12}\big(|\bar{\gamma}| + \bar{m}^{-7/2}|\bar{\sigma}|\big), \tag{3.4.14}$$

which apply for large masses. The mass-dependent factors come from the double time integral. The huge difference in their values can be traced back to the fact that the dominant evolution quantity accompanying $\bar{\gamma}$ is proportional to $H(\eta)^2\bar{m}^2$ and the one accompanying $\bar{\sigma}$ is proportional to $H'(\eta)\bar{m}^2$: the former has a nonvanishing value during the de Sitter phase and decays afterwards while the latter vanishes in the de Sitter phase and grows in absolute value from there to vanish again. The double integration with the highly oscillatory function $v_{0k}(\eta)^2$ (with a mass dependent frequency) then reduces the amplitude of the $\bar{\gamma}$ term by a factor $1/\bar{m}$, while that of $\bar{\sigma}$ has as an additional factor $1/\bar{m}^{7/2}$. The linear modification of $|\beta_k|^2$ also depends on the derivative of the mode but nothing new comes from there. We learn two things from this upper bound to ζ_k. First, the contribution of the term linear in $\bar{\gamma}$ is small in the perturbative regime. Second, in the same regime, the contribution of $\bar{\sigma}$ is negligible. It turns out that, for large masses, although the linear contributions are small in the perturbative regime as we have argued, this does not mean that the quadratic terms in the derivative couplings are negligible because the mass-dependent factors accompanying them can be very large and in fact dominate. As we will see in the next section, this is indeed the case for the $\bar{\gamma}$ contribution. On the other hand, we will also see that the quadratic terms in $\bar{\sigma}$ are small in the large mass regime, provided that (3.4.12) are satisfied, becoming absolutely negligible.

For small masses, the linear analysis is less conclusive. However it seems to indicate that small values of $\bar{\sigma}$ can lead to significant changes in the production.

Performing the second order expansion in order to check the conditions for the linear regime to be valid is calculationally very expensive, more so taking into account that we can provide numerical results to all orders and directly see the role of each of the couplings to the particle production. In fact, as already suggested above, the most interesting results are obtained within the perturbative regime but for nonlinear contributions to the production. This is done in what follows.

3.4.3 | Numerical analysis

We will start our analysis discussing the effects of the different couplings on the spectral particle distribution for the field. We will cover the perturbative range for the derivative couplings limited by the restriction (3.4.7) that ensures that there are no tachyons in the theory but allowing for nontrivial behaviors of the particle production.

Spectral particle production

Figure 3.12 shows the spectral particle production for small masses (we use $\bar{m} = 0.1$ as the working value). The left panel shows the production for various values of the derivative coupling $\bar{\gamma}$ keeping $\bar{\sigma} = 0$, while the right panel shows the production for various values of $\bar{\sigma}$ keeping $\bar{\gamma} = 0$, so we can study the effect of each coupling separately. Both panels are very different as far as the considered range of the derivative couplings is concerned. In the left panel, the main graphics correspond to values of the $\bar{\gamma}$ coupling (not much) smaller than the boundaries (3.4.12) of the regime in which we expect small variations in the production. The inset displays the production for couplings beyond this regime but still well inside the perturbative limit (3.4.8). In the right panel, however, the main graphics correspond to values of the $\bar{\sigma}$ coupling close to 1. These values are inside both the perturbative regime and the range of (3.4.12). The inset displays the production for couplings also inside this regime but with significant nonlinear changes. This choice of values for the derivative coupling $\bar{\sigma}$ (associated with the traceless Ricci tensor) is motivated by the fact that $\bar{\sigma}$ has a much smaller effect on the particle production (as compared with $\bar{\gamma}$). This is suggested by the arguments and bounds discussed in the previous subsection and confirmed by the numerical results.

Indeed, for values of $\bar{\sigma}$ comparable to those of $\bar{\gamma}$ in the left panel, the result would be just that all curves are superimposed. Even more, as we can see in figure 3.12 the change in the spectral production for values of $\bar{\sigma}$ close to 1 are of the same order as for values of $\bar{\gamma}$ inside the regime (3.4.12). The specific ranges that we have used for these couplings lie in the intervals $\bar{\sigma} \in [-1/2, 1/2]$ in the main graphics. For the inset we have considered the values $\bar{\sigma} = \pm 2$, still inside both the perturbative limit and the regime (3.4.12), but for which we obtain nontrivial contributions. We have

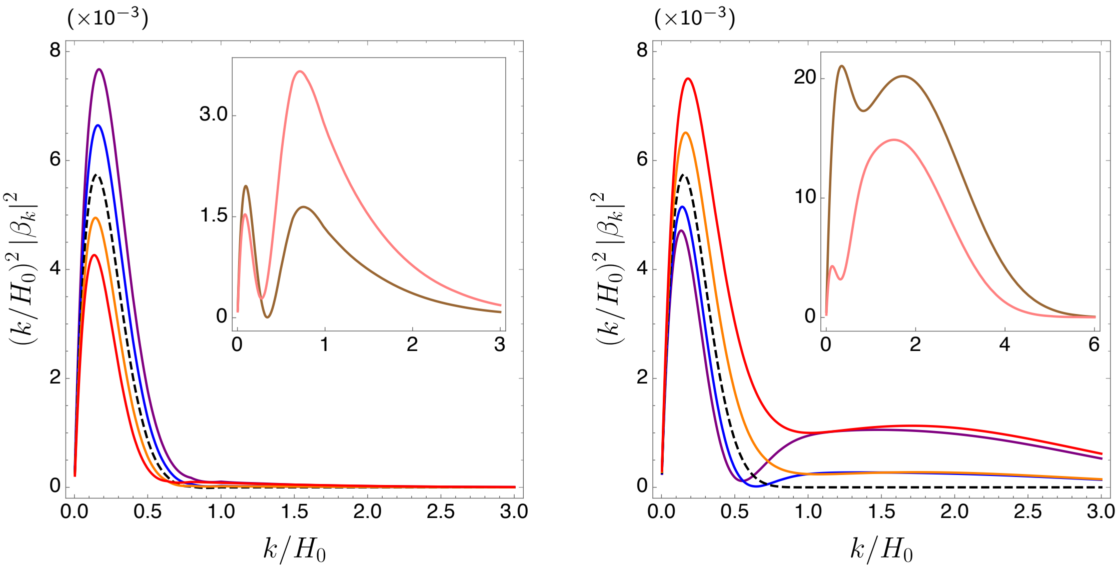

Figure 3.12: Spectral particle density for the case $\xi = 1/6$ and $\bar{m} = 0.1$ as a representative case for small masses at the end of reheating. On the left panel, different curves correspond to different values of $\bar{\gamma}$ keeping $\bar{\sigma} = 0$ with the following colour code: Purple, $\bar{\gamma} = -\bar{m}^2/2$; Blue, $\bar{\gamma} = -\bar{m}^2/4$; Black (dashed), $\bar{\gamma} = 0$; Orange, $\bar{\gamma} = \bar{m}^2/4$; Red, $\bar{\gamma} = \bar{m}^2/2$; Brown, $\bar{\gamma} = 2\bar{m}^2$; and Pink, $\bar{\gamma} = 3\bar{m}^2$. The companion curves corresponding to $\bar{\gamma} = -2\bar{m}^2, -3\bar{m}^2$ are not shown because these values of $\bar{\gamma}$ are beyond the tachyonic limit (3.4.7). On the right panel different curves correspond to different values of $\bar{\sigma}$ keeping $\bar{\gamma} = 0$ with the following colour code: Purple, $\bar{\sigma} = -1/2$; Blue, $\bar{\sigma} = -1/4$; Black (dashed), $\bar{\sigma} = 0$; Orange, $\bar{\sigma} = 1/4$; Red, $\bar{\sigma} = 1/2$; Brown, $\bar{\sigma} = 2$; and Pink, $\bar{\sigma} = -2$.

considered $\bar{\gamma} \in [-\bar{m}^2/2, \bar{m}^2/2] \subset (-1, 1)$ (with the regime in which we expect no significant changes being $|\bar{\gamma}| \ll \bar{m}^2$ as deduced from (3.4.12)). Note that we cannot go beyond $\bar{\gamma} = -\bar{m}^2$ due to the restriction (3.4.7) even being within the perturbative regime. For the curves on the inset we have considered the values $\bar{\gamma} = (2\bar{m}^2, 3\bar{m}^2)$ for which we expect nontrivial contributions to particle production, though still well inside the perturbative limits.

Let us start with the main graphics on the left panel (varying $\bar{\gamma}$). It is not surprising that within the considered range for $\bar{\gamma}$ the changes in the amplitude of the spectra are linear in the value of the coupling. We see that for any value of $\bar{\gamma}$ the spectral production has the same dominant peak as in the conformal case and that its amplitude grows for negative values of $\bar{\gamma}$ and diminishes for positive values. There are two a priori factors behind these changes in the peak amplitude. First, the curvature scalar (coupled through $\bar{\gamma}$) contributes to the value of the effective mass of the field during the de Sitter phase diminishing and broadening the spectral production for larger effective masses (see left panel in figure 3.1). Second, $\bar{\gamma}$ also contributes through an additional pure curvature effect, present only after the de Sitter phase, when the curvature scalar evolves with time. Let us briefly discuss this mechanism in more detail. If we consider the evolution after the de Sitter phase with only a mass term equivalent to the effective mass (including the effect of $\bar{\gamma}$ on the mass, i.e., $\mu(\bar{m}, \bar{\gamma})$, but neglecting any other effect during the evolution) we see that the behavior of the amplitude of the spectra with $\bar{\gamma}$ is inverted with respect to that in figure 3.1, with all the peaks centred around the same momentum and with similar width. This is precisely the opposite behavior to that in the left panel of figure 3.12. For instance, note that the purple curve (corresponding to $\bar{\gamma} = -\bar{m}^2/2$) gives an effective conformal behavior ($\mu = 0$) in the de Sitter epoch, hence no particle production occurs during that epoch, and still the spectral production is significantly augmented during the transition phase.

The inset in the left panel shows the spectra for situations beyond the regime (3.4.12) but still for perturbative values of the coupling $\bar{\gamma}$. We can see that the effect of the coupling to the curvature scalar induces significant changes to the spectral production as a secondary peak emerges at higher momenta. This secondary peak dominates the production for values of the coupling beyond the linear regime.

The main graphics on the right panel (varying $\bar{\sigma}$ close to 1 and with $\bar{\gamma} = 0$) also show a dominant production peak for momenta that were sub-Hubble during inflation. In this case, there is no contribution of the derivative couplings to the effective mass μ and the whole effect is due to pure tensor (i.e. traceless) Ricci curvature enhancements. We also see that the production for large momenta is significantly enhanced by the very same mechanism over a broad region of the spectra, augmenting the production even further. This effect is specially important on the red and purple curves which correspond to the larger values of $\bar{\sigma}$ within the considered range. In the inset we can see that the spectral production grows wildly, despite $\bar{\sigma}$ being inside the regime (3.4.12), as discussed in the previous subsection. For positive values of $\bar{\sigma}$ there are two contributions: the primary peak corresponding to the de Sitter production enhanced by the subsequent evolution and a broad band on larger momenta, while for the negative values all the production is concentrated on the latter broad band.

As a final remark, we can conclude that the coupling to the traceless Ricci tensor contributes significantly less than the coupling to the curvature scalar as we have already inferred from the perturbative analysis of the contributions to the modes.

In figure 3.13 we show the effects of the derivative couplings on the spectral particle production for large masses of the field (we have considered $\bar{m} = 10$ as a representative value). The left panel shows the production for various values of the derivative coupling $\bar{\gamma}$ for $\bar{\sigma} = 0$, while the right panel shows the production for different values of $\bar{\sigma}$

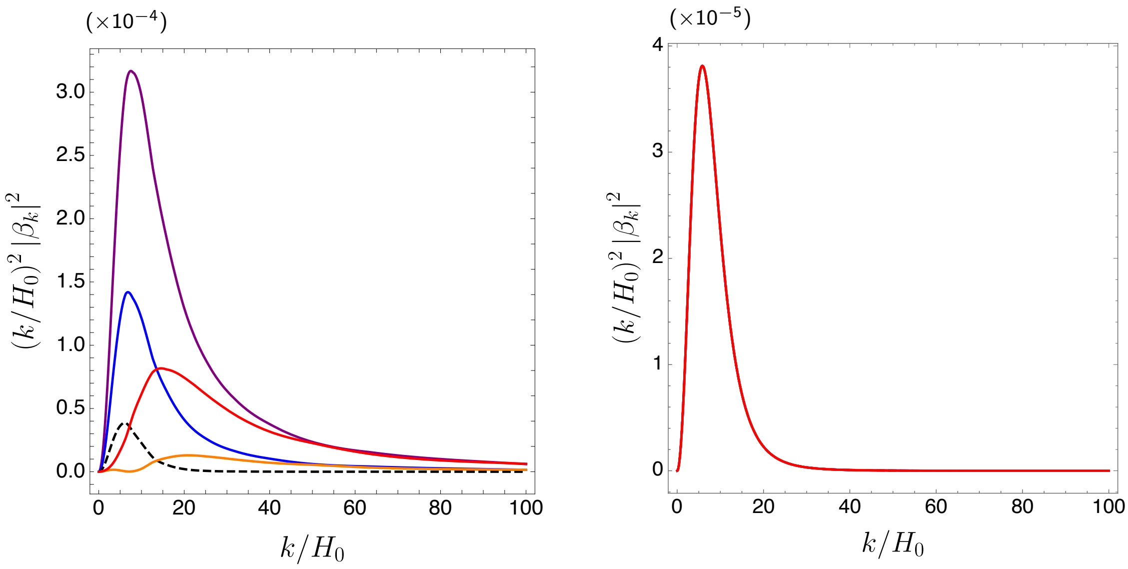

Figure 3.13: Spectral particle density for the case $\xi = 1/6$ and $\bar{m} = 10$ as a representative case for large masses at the end of reheating. On the left panel, different curves correspond to different values of $\bar{\gamma}$ keeping $\bar{\sigma} = 0$ with the following colour code: Purple, $\bar{\gamma} = -0.01/2$; Blue, $\bar{\gamma} = -0.01/4$; Black (dashed), $\bar{\gamma} = 0$; Orange, $\bar{\gamma} = 0.01/4$; Red, $\bar{\gamma} = 0.01/2$. On the right panel, different curves correspond to different values of $\bar{\sigma}$ keeping $\bar{\gamma} = 0$ with the following colour code: Purple, $\bar{\sigma} = -1/2$; Blue, $\bar{\sigma} = -1/4$; Black (dashed), $\bar{\sigma} = 0$; Orange, $\bar{\sigma} = 1/4$; Red, $\bar{\sigma} = 1/2$.

keeping $\bar{\gamma} = 0$. In both cases, the main graphics correspond to values of the considered derivative coupling smaller than the perturbative boundaries (3.4.8). In particular, the parameters lie in the interval $\bar{\gamma} \in [-0.01/2, 0.01/2]$, and $\bar{\sigma} \in [-1/2, 1/2]$, respectively (the same as in figure 3.12). Note that these values are inside the regime (3.4.12), where we expect small linear effects as discussed in subsection 3.4.2.

The figure on the left panel depicts the dependence of the spectral particle distribution on different values of the derivative coupling $\bar{\gamma}$, within the perturbative regime, for $\bar{\sigma} = 0$. We see that for negative values of the coupling constant $\bar{\gamma}$, the contribution of the curvature scalar is to enhance the particle distribution obtained at the end of the de Sitter phase, as the spectral production is dominated by the same single peak. On the other hand, for positive values of the coupling $\bar{\gamma}$ to the curvature scalar we see the same effect as for couplings beyond the linear regime for small masses (inset of figure 3.12-left), meaning that the de Sitter peak is damped and that there appears a broad band of production on larger momenta, becoming the dominant contribution to the spectral production. As happened for small masses, there are two different factors to explain these contributions: the changes in the de Sitter effective mass and the pure curvature effects during the transition phase. It is not difficult to see, by direct comparison with the right panel of figure 3.1, that the wild variation on the amplitude of the peaks has its major contribution from the effect of the curvature variation during the transition between inflation and reheating. It is important to note that even though we are considering values of the coupling constant well inside the perturbative regime and much smaller than the mass of the field, the resulting contributions to the particle distribution are nonlinear in the strength of $\bar{\gamma}$, as advanced in subsection 3.4.2.

On the right panel, the main figure shows that the effect of the traceless Ricci tensor mediated by the $\bar{\sigma}$ coupling on the spectral production is negligible for values of the coupling within the perturbative limit inasmuch as all the curves lie one on top of the other. This is precisely the behavior expected from (3.4.14), where the $\bar{\sigma}$ linear contribution to the mode perturbation is very much suppressed by high powers of the mass.

Comparing the results for large masses with our prior analysis of the small mass regime we can see two important differences: on the one hand, the importance of the derivative coupling $\bar{\sigma}$ diminishes, going from giving important contributions to the spectral density (figure 3.12-right) to becoming negligible in the large mass scenario (figure 3.13-right). On the other hand, we see that for the same values of the coupling $\bar{\gamma}$ to the curvature scalar, the contributions of this term go from being linear for small masses to become nonlinear in the large mass regime. Hence, we can conclude from the analysis of the spectral distribution that, for large masses, the produced density becomes very sensitive to the value of $\bar{\gamma}$, while it is not affected by changes in $\bar{\sigma}$.

Total density of produced particles

Once we have discussed the effect of the derivative couplings on the spectral distribution of produced particles, we are ready to focus on the total produced density at a time $\eta > \tilde{\eta}$ after reheating

$$n(\eta) = \frac{H_0^3}{2\pi^2 a(\eta)^3}\bar{N}(\tilde{\eta}), \qquad \bar{N}(\eta) = \frac{1}{H_0^3}\int_0^\infty |\beta_k(\eta)|^2\, k^2 \mathrm{d}k. \tag{3.4.15}$$

We take $\tilde{\eta} = 50/H_0$ as a working value. The subsequent evolution of $\bar{N}(\eta)$ is constant for $\eta > \tilde{\eta}$. Therefore, the evolution of the density $n(\eta)$ will be solely due to the expansion of the background as the gravitational production is already stabilised.

Let us start discussing the effects of the derivative couplings in the small mass regime for the field. In figure 3.14 we show the number density of particles computed numerically at $\tilde{\eta}$ for the small mass case ($\bar{m} = 0.1$, $\xi = 1/6$) as a function of $\bar{\sigma}$ and $\bar{\gamma}$.

The graphics in the left panel depict the density of produced particles as a function

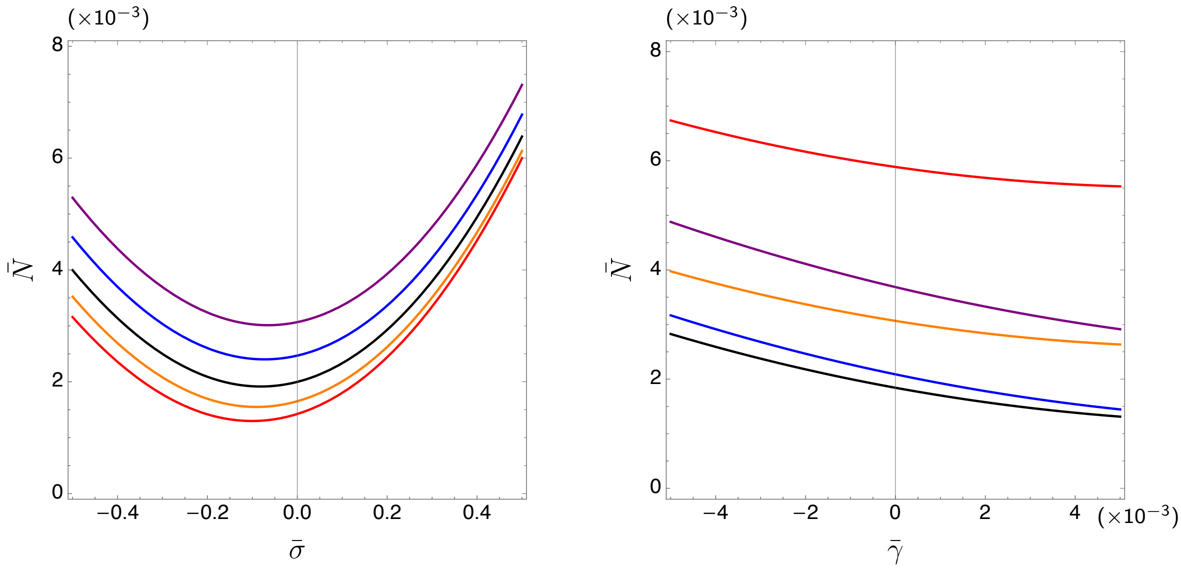

Figure 3.14: Density of produced particles for the small mass ($\bar{m} = 0.1$, $\xi = 1/6$) scenario computed at $\tilde{\eta} = 50/H_0$. On the left panel, we show the density as a function of $\bar{\sigma}$ for different values of $\bar{\gamma}$ in the perturbative regime. The colour code is the following: Purple, $\bar{\gamma} = -\bar{m}^2/2$; Blue, $\bar{\gamma} = -\bar{m}^2/4$; Black, $\bar{\gamma} = 0$; Orange, $\bar{\gamma} = \bar{m}^2/4$; Red, $\bar{\gamma} = \bar{m}^2/2$. The right panel shows the density as a function of $\bar{\gamma}$ for different values of $\bar{\sigma}$ (in the perturbative regime) with the following colour code: Purple, $\bar{\sigma} = -1/2$; Blue, $\bar{\sigma} = -1/4$; Black, $\bar{\sigma} = 0$; Orange, $\bar{\sigma} = 1/4$; Red, $\bar{\sigma} = 1/2$.

of the derivative coupling $\bar{\sigma}$ for several different constant values of $\bar{\gamma}$. As we can see and have already mentioned in the previous analysis, we are in the nonlinear regime of this parameter except for values around $|\bar{\sigma}| \simeq 0.1$. One of the main features of the dependence with $\bar{\sigma}$ is that the density attains a minimum value for some negative value of the coupling (actually the specific value varies with $\bar{\gamma}$) and grows significantly for positive and larger negative values, which is not surprising as we are already near the limit (3.4.12). Although these curves may seem parabolic, they actually have higher order contributions. Another important observation at the light of this figure is that the same density is produced for different pairs $(\bar{\sigma}, \bar{\gamma})$. In particular, if we take the production in the case with no derivative couplings, we can see that the same total production can be obtained for different sets of nonvanishing values of the derivative couplings, meaning that their respective contributions compensate each other. Although the production may be same, the spectral distribution is certainly different and this provides a mechanism for distinguishing between the case with vanishing derivative couplings and any other with the same total production.

The right panel of figure 3.14 shows the density for small masses of the field as a function of $\bar{\gamma}$ for several different constant values of $\bar{\sigma}$. As the values of the coupling

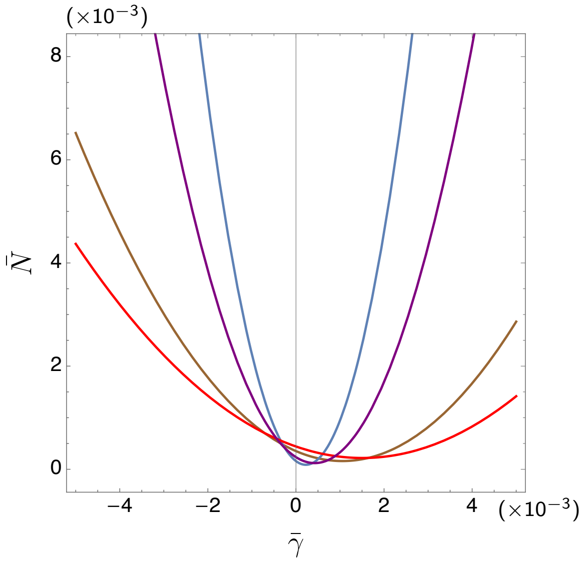

Figure 3.15: Density of produced particles for the large mass scenario computed at $\tilde{\eta} = 50/H_0$. We show the density as a function of $\bar{\gamma}$ for the single value $\bar{\sigma} = 0$ due to its negligible effect on the production (see figure 3.13-right). The colour code is the following: Red, $\bar{m} = 8$; Brown, $\bar{m} = 10$; Purple, $\bar{m} = 15$; Blue, $\bar{m} = 20$.

we are considering lie well inside the regime (3.4.12), it is not surprising that we obtain a linear behavior of the density with the strength of the coupling $\bar{\gamma}$. The intensity of the linear effect of the $\bar{\gamma}$ term increases with increasing absolute value of the coupling constant $\bar{\sigma}$. It is worth noting that for positive values of $\bar{\gamma}$ the production diminishes, which is the opposite effect of just changing the effective mass in the de Sitter phase, giving a strong indication that the curvature effects during the transition are of vital importance for understanding the derivative couplings.

For the large mass regime, the contribution of the $\bar{\sigma}$ coupling is negligible, as can be deduced from the spectral distribution (figure 3.13). Hence in figure 3.15, we have focused only on the nonlinear dependence of the total production (for $\bar{m} = 10$) on the $\bar{\gamma}$ coupling. The production is very sensitive to the value of $\bar{\gamma}$. It attains a minimum value for a certain strength of the coupling. Again we see that the total production may have the same value for different $\bar{\gamma}$, although the spectral production is very different. The total particle production is a quadratic function of the derivative coupling $\bar{\gamma}$ to very good approximation with mass dependent coefficients.

Finally, let us see how the particle production depends on the mass of the field $\bar{m}$ for given values of the derivative couplings, as shown in figure 3.16. We have chosen these values so they are representative of the regimes we have studied above. More explicitly, we consider the limiting cases used in the discussions and figures of the

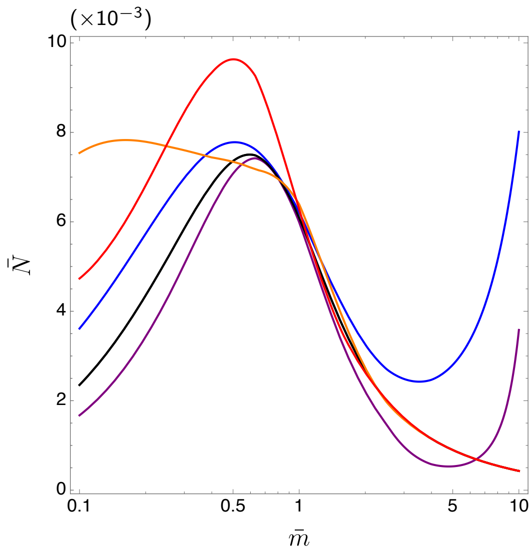

Figure 3.16: Density of produced particles computed at $\tilde{\eta} = 50/H_0$ as a function of the mass of the field for different combinations $(\bar{\gamma}, \bar{\sigma})$ with $\xi = 1/6$ with the following colour code: Purple, $(0.01/2, 0)$; Blue, $(-0.01/2, 0)$; Black, $(0, 0)$; Orange, $(0, 1/2)$; Red, $(0, -1/2)$.

spectral production both in $\bar{\gamma}$ and in $\bar{\sigma}$. Smaller values of $\bar{\sigma}$ do not provide additional information other than inducing small perturbative changes in the total production. We can see from figure 3.16 that the two regimes we have discussed in the previous subsections are indeed representative of the behavior for large and small masses, so we will analyse the figure in these terms.

Let us start with the small mass regime. The effect of negative $\bar{\sigma}$ couplings close to 1 in absolute value on the total production is less prominent than the effect of positive values. Indeed the contribution for positive values of $\bar{\gamma}$ grows rapidly as the mass decreases, giving larger total production for small masses. In the rest of cases the maximum production in the small mass regime is attained for masses of order $\bar{m} = 1$. In this region of small mass, perturbative values of $\bar{\gamma}$ induce linear changes in the production (as also would small values of $\bar{\sigma}$, although not shown in the figure) that grows in importance as the mass decreases.

The large mass regime presents some nontrivial characteristic as we have already mentioned. To begin with, the purple and blue curves describing the production for perturbative values of $\bar{\gamma}$ quickly grow for large masses. The contribution of positive values of $\bar{\gamma}$ changes from diminishing the total production for intermediate masses to wildly enhancing it after a threshold mass that depend on the specific value of the coupling. We can understand this effect using the previous analysis of the spectral production: as the mass becomes larger, nonlinear effects enter the game. The linear effect for positive values of this coupling is to damp the amplitude of the de Sitter peak. The nonlinearity introduces a broad band of excited modes that grows with the strength of the coupling and, as we have already mentioned, becomes the principal contribution to the production. These effects are generic for any value of $\bar{\gamma}$. Note that the growth in importance of the γ coupling with larger mass is already suggested by the linear analysis of the mode as the linear contribution grows with the mass (3.4.14). A similar behavior could be expected for the $\bar{\sigma}$ coupling for the same reason. However, we see a total production that is entirely unaffected by this coupling to the traceless Ricci curvature tensor. This negligible behavior continues for masses greater than the ones represented. This is in agreement with the study performed in subsection 3.4.2, where we concluded that the $\bar{\sigma}$ contribution to the perturbed mode is very much suppressed (see (3.4.14)). This is also in consonance with the spectral analysis above. The production is dominated by the de Sitter phase and it is not surprising that the production decays as $\bar{N} \propto m^{-1}$, given the adiabatic evolution of the background geometry.

The behavior of the total production in the intermediate mass region is simply an interpolation of the two regimes (small and large mass) already described.

In conclusion, we have seen that including derivative couplings to the background curvature changes in a highly nontrivial way the gravitational production in the early universe for the scalar field even in the perturbative regime.

3.5 | Conclusions

In this chapter we have studied the effects on the gravitational particle production of a scalar field in the early Universe of both considering the non-average behavior of the curvature and of including derivative couplings to the background. We have considered a de Sitter phase prior to the reheating scenario, mimicking the behavior of a geometry sourced by a massive inflationary field. Using this approach we can mimic any inflationary model by adjusting the de Sitter phase parameters and inflaton mass accordingly. We have focused in mimicking a chaotic massive inflation with potential (3.3.1).

Through section 3.3, we have realized that the oscillations of the curvature scalar R affect the gravitational particle production of a scalar field in different ways. In addition to the already known tachyonic instability [BL98; MN17] produced by making the effective mass term imaginary, there is a resonance mechanism that has not been studied before. This resonance affects the ultraviolet region of the spectra and dominates the particle production enhancement for $m > 10^9$GeV. This resonant behavior is related to the micro-oscillations in the frequency of the field due to the Ricci oscillations, in analogy to what happens in the Mathieu equation [AS64]. This result is in disagreement with the previous literature where the effect of having many oscillations of the scalar curvature has been ignored and hence the resonant effect induced by these oscillations was not included.

One of the main conclusions of this section is that gravitational production can account for the observed dark matter abundance. Furthermore, this gravitational production has allowed us to set constraints on the possible values of the mass of the dark matter particle if any possible direct coupling with the SM is negligible. In particular we have found that there is a forbidden band of masses which split the dark matter into either light or supermassive candidates. The width of this band is very sensitive to the reheating temperature. We have seen that the largest possible mass for the light candidate is obtained in the conformal coupling case and that for masses above the inflationary energy scale, the production become independent of the coupling constant ξ. Hence, analysing the conformal case is sufficient to set the constraints on the possible masses for the scalar dark matter candidate. However, the most stringent constrains come from the cases in which $\xi > 1/6$ as the possible maximum mass is much smaller than in the conformal case. For values of the coupling constant ξ not covered in this work, i.e., $\xi > 1$, the gravitational production would be further enhanced, widening the range of forbidden masses for a dark matter candidate. The most affected region would be the large mass one because the field would suffer from tachyonic instabilities for large enough values of ξ, enhancing heavily the production and dragging the allowed lightest heavy mass to larger values. For the small and intermediate mass regions, considering larger values of ξ would increase monotonically the production, dragging the allowed values of the mass for the dark matter candidate to even lighter ones.

In section 3.4 we have studied the role of derivative couplings to the background curvature in the gravitational production for a quantum scalar field. We have considered a coupling to the curvature scalar mediated through a coupling constant γ and a term proportional to the traceless Ricci tensor through a constant σ. During the inflationary epoch of the Universe, mimicked by the de Sitter solution, only the term proportional to γ is nonvanishing, so there is no contribution to the production due to different values of σ. We have analytically calculated both the spectral and the total production for the large mass regime, obtaining that the total production decreases with the inverse of the effective mass, which depends on the mass itself and the couplings to the curvature scalar ξ and γ.

After inflation we have considered a reheating phase dominated by the oscillations of the inflaton field around the minimum of its potential. We have interpolated the behavior of all the background quantities from their values at the end of the de Sitter phase to their values at the onset of the inflaton oscillations using sufficiently smooth functions so the results are independent of the considered order of the interpolation functions. The production in the transition phase is strongly dependent on the value of the couplings γ and σ. To focus only on the relevance of the derivative couplings we have set $\xi = 1/6$. The gravitational production quickly stabilizes during the reheating phase (we are neglecting the oscillations of the background quantities during reheating).

The perturbative regime for the derivative couplings comes from requiring that the derivative terms in the action are perturbative in the sense that they are much smaller than one. However, small values for the coupling constants can induce nonlinear effects because of nontrivial contributions as compared with the field mass. In addition, the validity of the linear approximation for the spectral production requires additional conditions that depend on the mass. The summary is that the perturbative regime requires that the (dimensionless) couplings be much smaller than one, while the linear regime for the production requires conditions that depend on the mass. In this sense, it is not surprising to have nonlinear effects in the production even though the couplings are perturbative, as happens for sufficiently large masses.

The effects of both derivative terms are of different relative importance for a given mass of the field. Moreover, their contributions to the gravitational production depend appreciably on the mass of the field. We have seen that for small masses, values of γ lying within the perturbative regime give contributions to the particle production that are comparable to the ones due to significantly higher values of σ. In this regime, the gravitational production can change notably for perturbative values of the coupling constants so their effects cannot be neglected.

The behavior of the production drastically changes for large masses. For values of γ still well within the perturbative regime, the effects on the production become nonlinear while the effects of the σ term are negligible. Hence, for large masses the production becomes very sensitive to the value of the derivative coupling γ. This can be understood in terms of the different conditions that γ must satisfy to remain in the linear regime and which are violated for sufficiently large masses. On the other hand note that the σ coupling becomes irrelevant for large masses.

To conclude, we have seen that the presence of derivative couplings to the Riemann tensor can significantly affect the gravitational production. Therefore, they can change the current picture of gravitational production of dark matter [MN17; Chu+01; ENT18; CGS20] as the production changes in a highly nontrivial way even when the coupling constants are in the perturbation regime.

Derivative couplings are often neglected in all the discussions on gravitational production but, for certain theoretical field models, they can be important, and hence their gravitational production will be dominated by the effects we have discussed. This situation will be general for (pseudo-)Nambu-Goldstone bosons, whose disformal couplings dominate their phenomenology. In this context, the presence of disformal scalars, such as branons, are well-motivated from a quantum approach, and its gravitational production will differ from a traditional scalar field with nonderivative couplings.

4 | Summary and conclusions

But do you remember Gandalf's words: Even Gollum may have something yet to do? But for him, Sam, I could not have destroyed the Ring. The Quest would have been in vain, even at the bitter end. So let us forgive him! For the Quest is achieved, and now all is over. I am glad you are here with me. Here at the end of all things, Sam.

– J.R.R. Tolkien, *The Return of the King*

In this thesis we have explored different aspects of the interplay between gravitation and quantum field theory. We have built new models that can potentially accommodate dark matter candidates inspired by the features of the Fierz-Pauli field theory for describing gravitation. We have also delved into how the dark matter component of the universe could have been created during its early epochs via gravitational production due to the expansion of the background geometry and possible direct couplings between the dark matter field and the curvature terms. We have explored the implications of considering the non-averaged behavior of the curvature in this production mechanism and we have obtained novel results that differ with previous results in the literature [BL98; CKR99a; Chu+01; MN17; ENT18]. We have also explored the consequences of including the most general first order coupling of the curvature quantities to the derivatives of the field, obtaining a striking result: no matter how small these terms are, they may have a great impact on gravitational particle production.

On the first part of this thesis we have developed new models for scalar and vector dark matter inspired by the Fierz-Pauli field approach to gravitation. We have introduced different couplings to the energy-momentum tensor at the linear level and then obtained the full non-linear actions by imposing self-consistency conditions on these new couplings. Instead of resumming the entire series we have followed a different route: we have seen the form of the first terms of the series and we have guessed ansätze based on the general functional form of the full action. We have then reduced the problem of resuming infinite terms to solve a set of differential equations. To obtain such system, we have imposed the condition that the resummed action can be written in the same form as the linear order one. We started recovering the known results from M. Sami and T. Padmanabhan for the scalar field coupled linearly to the trace of the energy-momentum tensor [SP03]. Then, we have extended our analysis to a scalar field coupled to the energy-momentum tensor through its derivatives, respecting a shift symmetry, and to vector fields with both gauge-invariant self-couplings and Proca-like terms. We have explored these new theories both in the Lagrangian formalism and in the first order formalism which in some cases made the differential equations coming from the consistency conditions to obtain the resummed action to be algebraic equations, simplifying the process.

In addition to the self-coupling problem, we have also considered the coupling to other fields as the coupling to the energy-momentum tensor is universal following the Fierz-Pauli model. We have tackled this problem considering a proxy scalar field that mimics the SM degrees of freedom to investigate how the couplings to the new dark matter field will affect the total action. Our study is not free from ambiguities although we have fixed some of them using the Hilbert prescription for the energy-momentum tensor. However, the requirement of imposing the coupling of the field at the level of the action or as a source term in the equations of motion still remains and we have explored both options although we have focused more on the former.

After constructing the actions for the self-interacting fields we have discussed the impact of possible superpotential terms arising from the ambiguity in the prescription of the energy-momentum tensor. We have shown that these terms lead to the generation of Galileon interactions for all the fields we have studied.

We have also studied, due to the motivation coming from gravitation theories, generating functionals defined in terms of an effective metric. Using this method we have re-derived some of the previous results we had obtained in a simpler way and have given a general procedure to generate all the interactions.

To complete our study, we have explored the phenomenology associated with these new models as well as their possible experimental signatures. This phenomenology is very rich and depends on the particular term under study. We have focused on the study of the phenomenology for disformal scalar and vector fields. At energies high enough, the longitudinal polarized vector and the disformal scalar are related by the equivalence theorem and following an effective theory approach we have worked in the perturbation limit. In this scenario, the monojet and single photon analysis at the LHC are the most sensitive signatures, constraining the energy scale of the dimensional couplings at the TeV scale. As we are interested in dark matter models, there is a broad range of astrophysical and cosmological observable possibilities depending of their stability. Note that given the quadratic character of the leading order interaction for the derivative couplings of the scalar or the non-gauge invariant couplings of the vector, the corresponding force will decay faster than the Newtonian behaviour $1/r^2$. However, if there is some vacuum expectation value for the fields, be it $\langle \partial_\mu \varphi \rangle$ or $\langle A_\mu \rangle$ for the scalar and the vector respectively, then we can recover the Newtonian-like force with a coupling constant determined by the vacuum expectation value of the field.

Another interesting question we have started to explore in this thesis is the relation of our resulting actions with geometrical frameworks. We have already given some arguments in Sec. 2.5 as to what extent our constructed theories could be regarded as couplings to an effective metric. For the massless scalar field case, it is known that the resulting action can be interpreted in terms of the Ricci scalar of a metric conformally related to the Minkowski metric (which is nothing but Nordstrøm's theory). It would be interesting to check if analogous results exist for the derivatively coupled case and/or the vector field case, for instance in terms of curvature objects of disformal metrics. Perhaps a suited framework for these theories would be the arena provided by Weyl geometries or generalised Weyl geometries which naturally contain an additional vector field, which could also be reduced to the gradient of a scalar, in which case it is known as Weyl integrable spacetimes or WIST [BK14; BK16; BHK16]. Another extension of our study can be associated with the analysis of multi-field theories and the study of the role played by internal symmetries within these constructions.

The second part of this thesis has been devoted to the study of gravitational particle production during the early stages of the universe, namely, inflation and reheating and the effects of different phenomena on this production. We have investigated the gravitational production of a scalar field with several non standard couplings to the background curvature as a viable dark matter candidate in two different scenarios. On the one hand we have studied the case of a scalar field with a non-minimal coupling $\xi\phi^2 R$ while considering the non-averaged behavior of the scalar of curvature during the reheating phase and on the other hand, the impact of introducing couplings between the derivatives of the field and the averaged background curvature invariants.

First, we have explored how the oscillations of the scalar of curvature induce different phenomena on the gravitational production of the field, giving rise to important differences with respect to the case in which only the averaged behavior of the curvature is considered. We have developed a new analytical scheme to study the resonant enhancement on the field large momenta region. We performed a redefinition of the field so the resulting equation of motion isolated the resonance as a source term. Then, we employed a WKB ansatz for the solution of the homogeneous differential equation and we obtained a particular solution using a general method. With this scheme we have obtained an approximate solution for the modes of the field which yields an analytical approximation to the spectral particle production that coincides perfectly with the numerical results. The other asymptotical region we have been able to study analytically is the regime of large masses. In this regime we have obtained the usual behavior of gravitational production with a damping on the production as the mass of the field grows larger, there is no unusual enhancement due to the oscillations of the scalar curvature.

We have explored numerically the gravitational production at the end of reheating for a large space of parameters for the field. In this region we have identified four different behaviors of the spectral distribution depending on the mass and coupling strength of the field: For masses such that $m^2 \ll (\xi - 1/6)|\min(R_{\rm osc})|$, the most important enhancement mechanism is the tachyonic instability which affects mostly the long distance part of the spectrum of the corresponding field, namely $k \ll H_0$. For masses well beyond the energy scale of inflation ($m \gg H_0$), the resulting spectral production does not differ appreciably from the one obtained in the de Sitter phase and no enhancements due to the oscillations of the curvature are present. Finally there is a regime of parameters in which the principal enhancement mechanism is the resonance of the field frequency and the curvature frequency. The transition from the tachyonic to the resonant dominated regions is sit around $m = 10^9$GeV and present very rich and complex production spectra.

In order to set constraints on the field fundamental properties for it to be viable as a dark matter candidate, we have evolved the produced density at the end of the reheating phase to the present day, so we can compare the predicted abundance with the observed one. We have been able to discard a wide region of the parameter space as viable dark matter as it would lead to overproduction. In particular, there will be two allowed regions: one for light candidates and other one for superheavy ones; the particular excluded masses depend on the value of the coupling constant ξ. The less restrictive case is the one with $\xi = 1/6$, allowing for heavier light dark matter candidates. We have found that except for couplings nearly close to the conformal one, the scalar field in order to be considered as a dark matter candidate has to be either lighter than 10MeV or heavier than 10^{15}GeV for a reheating temperature of $T_{\rm reh} = 10^{15}$GeV. These bounds become more restrictive as lower reheating temperatures are considered.

On the other hand we have also explored the implications of including direct couplings between the curvature terms and the derivatives of a massive scalar field. These couplings are often neglected but they can be really important, changing completely the predictions on fundamental properties of dark matter inferred through the comparison of the predicted abundance from gravitational production with the observed one. We have considered the most general first order coupling term between the derivatives of a scalar field and the background curvature. This term is described by a coupling to the scalar curvature mediated by a coupling constant γ and a coupling to the traceless Ricci tensor through a coupling constant σ. In this scenario we have considered the averaged behavior for the background quantities. We have analyzed the perturbative regime for these couplings as well as the linear one, as perturbative values of the coupling terms can lead to non-linear effects in the production.

The effect of both derivative terms are of different relative importance for each mass of the field. We have seen that the contribution of each term depends heavily on the considered mass for the field, going from enhancing the production to diminishing it with respect to the case with no derivative couplings. For small masses, i.e. $m < H_0$, values of γ well within the perturbative regime give contributions to particle production that are comparable to the ones coming form significative larger values of σ. In this regime, gravitational production is heavily affected by small changes in the value of the coupling constants. For large masses, i.e. $m > H_0$, the behavior of the production changes drastically, the effects for the γ term become nonlinear, while the σ term contribution becomes negligible. The production becomes extremely sensitive to the value of γ for large masses of the field.

In conclusion for the second part, the present picture of gravitational production in the early universe changes drastically either if the non-averaged behavior of the background curvature is considered or if derivative couplings to the background are introduced. In order to use gravitational production to obtain bounds on the fundamental properties of dark matter, an exhaustive study of the possible derivative couplings of dark matter and the realistic behavior of the geometry is needed.

Publications

The research work done during this thesis has given raise to the following publications in peer-review journals:

- J. Beltrán Jiménez, J. A. R. Cembranos and J. M. Sánchez Velázquez, "On scalar and vector fields coupled to the energy-momentum tensor", *JHEP* **05** (2018) 100.
- J. A. R. Cembranos, L. J. Garay and J. M. Sánchez Velázquez, "Gravitational production of scalar dark matter", *JHEP*, **06** (2020) 084.
- D. E. Borrajo Gutiérrez, J. A. R. Cembranos, L. J. Garay and J. M. Sánchez Velázquez, "Derivative couplings in gravitational production in the early universe", *JHEP*, **09** (2020) 069.

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