UNIVERSIDAD COMPLUTENSE DE MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS

Departamento de Matemática Aplicada

TESIS DOCTORAL

Ecuaciones adventivas-difusivas de la dinámica de poblaciones

Adventive-diffusive equations of population dynamics

MEMORIA PARA OPTAR AL GRADO DE DOCTOR

PRESENTADA POR

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Madrid, 2016

UNIVERSIDAD
COMPLUTENSE
MADRID

Doctorado en Investigación Matemática
Departamento de Matemática Aplicada
Facultad de Ciencias Matemáticas
Tesis Doctoral

ECUACIONES ADVENTIVAS-DIFUSIVAS DE LA DINÁMICA DE POBLACIONES (ADVENTIVE-DIFFUSIVE EQUATIONS OF POPULATION DYNAMICS)

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Tesis dirigida por Julián López Gómez.

MADRID, 2015

Agradecimientos

Si este trabajo ha llegado a ser posible se lo debo:

- A Julián por su dedicación y exigencia. Por saber potenciar mi formación matemática, desde la licenciatura hasta ahora, y por su manera rigurosa de hacer matemáticas.
- A Marcela por acogerme en la Carlos III y seguir abriendo las puertas de mi futuro profesional.
- A mis compañeros de la Carlos III por hacer que cada día de trabajo sea más ameno. Mencionar a Misael por su espíritu servicial.
- A mi padre y mi madre por la educación que me han dado y por su ayuda incondicional.
- A toda mi familia por todo su apoyo y confianza en mi. En especial a José Ricardo por ser mi referencia a seguir y mi primer punto motivador para empezar esta larga carrera.
- A mis amigos porque sin celebraciones el trabajo no sería el mismo.
- A mi grupo Scout por los valores que me han transmitido de autosuperación y de ser cada día mejor persona.

Gracias a todos por hacer de mi sueño una realidad, gracias de corazón.

Contents

1 Summary **1**
1.1 Introduction and goal . . . 1
1.2 Content . . . 3

2 Some paradoxical effects of the advection on a class of diffusive equations in Ecology **7**
2.1 Introduction . . . 7
2.2 The graphs of $\Sigma(\lambda, \alpha)$ as α varies from 0 to infinity . . . 10
2.3 Local bifurcations from $\lambda = 0$ and $\lambda = \lambda_\pm(\alpha)$ if $\Sigma'(0, \alpha) \neq 0$ . . . 14
2.4 Uniqueness of the positive steady state and bifurcation diagrams. Global attractivity. . . 18
2.5 Estimating λ_* and λ^* in some special cases. . . . 23
2.6 Ascertaining $\lim_{\alpha\to\infty} \Sigma_0(\lambda, \alpha, \Omega_0)$ when $\lambda > 0$ and $m(x_+) > 0$ for some $x_+ \in \Omega_0$. 28

3 Concentration through large advection **35**
3.1 Introduction . . . 35
3.2 Notations and preliminaries . . . 37
3.3 Upper estimates . . . 40
3.4 Lower estimates . . . 54

4 Dynamics of a class of advective-diffusive equations in Ecology **59**
4.1 Introduction . . . 59
4.2 Notations and preliminaries . . . 60
4.3 First variations of the principal eigenvalues $\Sigma_0(\lambda)$ . . . 64
4.4 Limiting behavior of the positive steady state solution in $\bar{\Omega}_0$ . . . 69
4.5 Existence of large solutions . . . 71
4.6 Dynamics of (1.1) when $m > 0$ . . . 77
4.7 Dynamics of (1.1) when m changes sign . . . 80

5 Conclusions **85**

6 Resumen **87**
6.1 Introducción y objetivos . . . 87
6.2 Contenido . . . 88
6.3 Conclusiones . . . 91

Chapter 1

Summary

1.1 Introduction and goal

This thesis studies the dynamics of the parabolic problem

$$\begin{cases} \partial_t u = \nabla\cdot(D\nabla u - \alpha u\nabla m) + u(\lambda m - a u^{p-1}) & \text{in } \Omega, \quad t>0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t>0, \\ u(\cdot,0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain (open and connected set) of $\mathbb{R}^N$, $N \geq 1$, $D > 0$, $\alpha > 0$, $\lambda \in \mathbb{R}$, $p \geq 2$, $a \in \mathcal{C}(\bar{\Omega})$ satisfies $a > 0$, in the sense that $a \geq 0$ and $a \neq 0$, ν stands for the outward unit normal along the boundary of Ω, $\partial\Omega$, and $m \in \mathcal{C}^2(\bar{\Omega})$ is a function such that $m(x_+) > 0$ for some $x_+ \in \Omega$. Thus, either $m \geq 0$, $m \neq 0$, in Ω, or else m changes sign in Ω. The initial data u_0 are in $L^\infty(\Omega)$.

Under these conditions, it is well known that there exists $T > 0$ such that (1.1) admits a unique classical solution, denoted by $u(x,t;u_0)$ in $[0,T]$ (see, e.g., Henry [19], Daners and Koch [14] and Lunardi [37]). Moreover, the solution is unique if it exists, and according to the parabolic strong maximum principle of Nirenberg [39], $u(\cdot,t;u_0) \gg 0$ in Ω, in the sense that

$$u(x,t;u_0) > 0 \quad \text{for all} \quad x \in \bar{\Omega} \quad \text{and} \quad t \in (0,T].$$

Thus, since $a > 0$ in Ω, we have that

$$\partial_t u = \nabla\cdot(D\nabla u - \alpha u\nabla m) + u(\lambda m - a u^{p-1}) \leq \nabla\cdot(D\nabla u - \alpha u \nabla m) + \lambda m u$$

and, hence, thanks again to the parabolic maximum principle,

$$u(\cdot,t;u_0) \ll z(\cdot,t;u_0) \quad \text{for all} \quad t \in (0,T],$$

where $z(x,t;u_0)$ stands for the unique solution of the linear parabolic problem

$$\begin{cases} \partial_t z = \nabla\cdot(D\nabla z - \alpha z\nabla m) + \lambda m z & \text{in } \Omega, \quad t>0, \\ D\partial_\nu z - \alpha z \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t>0, \\ z(\cdot,0) = u_0 > 0 & \text{in } \Omega. \end{cases}$$

As z is globally defined in time, $u(x, t; u_0)$ cannot blow up in a finite time and, therefore, it is globally defined for all $t > 0$. In applications it is imperative to characterize the asymptotic behavior of $u(x, t; u_0)$ as $t \uparrow \infty$.

A special version of this model (with $\lambda = 1$, $p = 2$ and $a(x) > 0$ for all $x \in \bar{\Omega}$) was introduced by Belgacem and Cosner [6] to "*analyze the effects of adding a term describing drift or advection along environmental gradients to reaction diffusion models for population dynamics with dispersal.* " In these models, $u(x, t; u_0)$ stands for the density of a population at the location $x \in \Omega$ after time $t > 0$, $D > 0$ is the usual diffusion rate, and "*the constant α measures the rate at which the population moves up the gradient of the growth rate $m(x)$. If $\alpha < 0$, the population would move in a direction along which m is decreasing, that is, away from the favorable habitat and toward regions of less favorable habitat.*" The parameter λ provides us with a sort of re-normalization of the drift term to add some insight and a wider perspective to the overall analysis already carried out in [6].

Although in the special case when $a(x) > 0$ for all $x \in \bar{\Omega}$ it is well known that the dynamics of (1.1) is regulated by its non-negative steady-states, when they exists, which are the non-negative solutions of the semilinear elliptic boundary value problem

$$\begin{cases} \nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) + \theta(\lambda m - a\theta^{p-1}) = 0 & \text{in } \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in the general case when the weight function $a(x)$ vanishes somewhere in Ω, the dynamics of (1.1) might be governed by the *metasolutions* of (1.2), which are large solutions supported in $\operatorname{supp} a$ and, possibly, on a finite number of components of $a^{-1}(0)$, prolonged by infinity through the edges of these components up to be defined on the whole habitat Ω. The metasolutions were introduced in the Ph. D. Thesis of Gómez-Reñasco [17] to describe the dynamics of a generalized class of semi-linear parabolic equations of logistic type. The pioneering results of [17] were published after four years in [18], and later improved in [29] and [30]. In [26, Section 8] rather complete historical bibliographic details are given.

The mathematical analysis of these *generalized diffusive logistic equations* has been tremendously facilitated by the theorem of characterization of the maximum principle of López-Gómez and Molina-Meyer [34] and the later refinements of [28] and Amann and López-Gómez [5], which have been collected in [32, Th. 7.10]. Besides it has enhanced the development of the theory of semilinear parabolic equations in the presence of spatial heterogeneities, Theorem 7.10 of [32] has shown to be a milestone for the generation of new results on wide classes of linear weighted boundary value problems. Indeed, the results of [27], Hutson et al. [21], [28] and Cano-Casanova and López-Gómez [9], substantially polished in [32, Ch. 9], provide us with extremely sharp refinements of the classical results of Manes and Micheletti [38], Hess and Kato [20], Brown and Lin [8], Senn and Hess [42] and Senn [41]. Consequently, [32, Th. 7.10] is a pivotal result which has tremendously facilitated the analysis of the effects of spatial heterogeneities in some of the most paradigmatic models of population dynamics.

The main goal of this thesis is to obtain the dynamic of (1.1). Moreover, we build up a upper estimates and lower estimates of the positive solutions of (1.2), when they exist, for α sufficiently large. As they are a global attractor of (1.1), these estimates provides us the shape of $u(\cdot, t; u_0)$ for $t \to \infty$. These results have been already published in [1], [2] and [3].

1.2 Content

Throughout this thesis, given a linear second order uniformly elliptic operators in Ω,

$$\mathfrak{L} := -\text{div}\,(A(x)\nabla\cdot) + \langle b(x), \nabla\cdot\rangle + c, \qquad A = (a_{ij})_{1\leq i,j\leq N}, \quad b = (b_j)_{1\leq j\leq N},$$

with $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$, $b_j, c \in L^\infty(\Omega)$, $1 \leq i, j \leq N$, a smooth subdomain $\mathcal{O} \subset \Omega$, two nice disjoint pieces of the boundary of $\mathcal{O}$, Γ_0 and Γ_1, such that $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$, and a boundary operator

$$\mathfrak{B} \;:\; \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\mathcal{O} \cup \Gamma_1) \to \mathcal{C}(\partial\mathcal{O})$$

of the general mixed type

$$\mathfrak{B}\psi := \begin{cases} \psi & \text{on } \Gamma_0, \\ \partial_\nu\psi + \beta\psi & \text{on } \Gamma_1, \end{cases} \qquad \psi \in \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\mathcal{O} \cup \Gamma_1),$$

where $\nu = A\mathbf{n}$ is the co-normal vector field and $\beta \in \mathcal{C}(\Gamma_1)$, we will denote by $\sigma[\mathfrak{L}, \mathfrak{B}, \mathcal{O}]$ the principal eigenvalue of $(\mathfrak{L}, \mathfrak{B}, \mathcal{O})$, i.e., the unique value of τ for which the linear eigenvalue problem

$$\begin{cases} \mathfrak{L}\varphi = \tau\varphi & \text{in } \mathcal{O}, \\ \mathfrak{B}\varphi = 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{1.3}$$

admits a positive eigenfunction $\varphi > 0$. Naturally, if $\Gamma_1 = \emptyset$, we will simply denote $\mathfrak{D} := \mathfrak{B}$ (Dirichlet), and if $\Gamma_0 = \emptyset$ and $\beta = 0$, we will write $\mathfrak{N} := \mathfrak{B}$ (Neumann).

We introduce the principal eigenvalue

$$\sigma[-\nabla \cdot (D\nabla - \alpha\nabla m) - \lambda m, D\partial_\nu - \alpha\partial_\nu m, \Omega] \tag{1.4}$$

that plays a significant role to describe the dynamic of (1.1). One of the main results of the Chapter 2 establishes that the sign of this principal eigenvalue predicts the global behavior of the species. In the special case when $a(x) > 0$ for all $x \in \bar{\Omega}$ and α is sufficiently large, we prove the following properties:

(P1) $u = 0$ is a global attractor of (1.1) if it is linearly stable as a steady state of the evolution problem (1.1).

(P2) (1.2) possesses a positive solution, necessarily unique, if $u = 0$ is linearly unstable. In such case, the unique positive steady state is a global attractor of (1.1).

(P3) (1.2) possesses a unique positive solution for all $\lambda > 0$.

On the other hand, in the case that $a > 0$, the property (P1) holds but the properties (P2) and (P3) might change. We prove the following properties:

(P4) If (1.2) does not admit a positive solution and $u = 0$ is linearly unstable, then

$$\lim_{t\to\infty} \|u(\cdot, t; u_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty. \tag{1.5}$$

(P5) There exists $\lambda^*(\alpha) \in (0, \infty]$ such that (1.2) has a unique positive solution if $\lambda \in (0, \lambda^*(\alpha))$, which is a global attractor of (1.1). Moreover, if we suppose that

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{is a nice subdomain of } \ \Omega \ \text{ with } \ \bar{\Omega}_0 \subset \Omega, \tag{1.6}$$

then (1.5) holds for all $\lambda \geq \lambda^*(\alpha)$.

(P6) Suppose (1.6) and either m does not admit any critical point in $\bar{\Omega}_0$, or it admits finitely many critical points in $\bar{\Omega}_0$, say x_j, $1 \leq j \leq q$, with $\Delta m(x_j) > 0$ for all $1 \leq j \leq q$. Then, $\lim_{\alpha \to \infty} \lambda^*(\alpha) = \infty$.

Consequently, when $a > 0$ vanishes on some open subset of Ω, a sufficiently large advection α can provoke the dynamics of (1.1) to be regulated by a positive steady state, though the solutions of (1.1) might grow up to infinity in Ω_0 as $t \uparrow \infty$ for a smaller advection. Similar results are obtained for $\lambda < 0$ in this chapter.

The main goal of Chapter 3 is to adapt the extremely elegant analysis of Lam [23] and Chen, Lam and Lou [10, Section 2] (see also Lam [24] and Lam and Ni [25]) to the more general situation when (1.6) holds, which provides us with the precise shape of the positive solution of (1.2) as $\alpha \uparrow \infty$. Consequently, as it is a global attractor for (1.1), these profiles also provide us with the shape of $u(\cdot, t; u_0)$ for sufficiently large $\alpha > 0$ and $t > 0$. Essentially, for sufficiently large α, the solutions of (1.1) are concentrated around the positive local maxima of $m(x)$ as $t \uparrow \infty$. Moreover, if z stands for any of these local maxima, the solutions are bounded around z if $a(z) > 0$, while they should be unbounded if $z \in \Omega_0$, which is a new phenomenology not previously described.

As a byproduct of these results, we will obtain a generalized version of Theorem 2.2 of Chen, Lam and Lou [10], which was stated for the special case when $\lambda = 1$, $a = 1$ and $p = 2$. Besides the results of [10, Section 2] are substantially sharpened here, some of the technical conditions imposed in [10], like $\int_\Omega m \geq 0$, will be removed here, as well as the strong barrier condition $\partial_\nu m(x) < 0$ for all $x \in \partial\Omega$, which will be relaxed to

$$\frac{\partial m}{\partial \nu}(x) \leq 0 \qquad \text{for all } \ x \in \partial\Omega.$$

In Chapter 4, under condition (1.6), we characterize the limiting behavior of $u(x, t; u_0)$ as $t \uparrow \infty$ when $u = 0$ is linearly unstable and (1.2) does not admit a positive solution. Essentially, this occurs for sufficiently large λ provided $m(x_+) > 0$ for some $x_+ \in \Omega_0$. Under these circumstances, the main result of this chapter establishes that if $a \in \mathcal{C}^2(\bar{\Omega})$, then

$$\lim_{t \uparrow \infty} u(\cdot, t; u_0) = +\infty \quad \text{uniformly in } \ \bar{\Omega}_0, \tag{1.7}$$

whereas

$$L^{\min}_{[\lambda, m, \Omega \setminus \bar{\Omega}_0]} \leq \liminf_{t \to \infty} u(\cdot, t; u_0) \leq \limsup_{t \to \infty} u(\cdot, t; u_0) \leq L^{\max}_{[\lambda, m, \Omega \setminus \bar{\Omega}_0]} \text{ in } \bar{\Omega} \setminus \bar{\Omega}_0,$$

where $L^{\min}_{[\lambda, m, \Omega \setminus \bar{\Omega}_0]}$ and $L^{\max}_{[\lambda, m, \Omega \setminus \bar{\Omega}_0]}$ stand for the minimal and maximal solutions, respectively, of the singular boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \theta(\lambda m - a\theta^{p-1}) = 0 & \text{in } \ \Omega \setminus \bar{\Omega}_0, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \ \partial\Omega, \\ \theta = +\infty & \text{on } \ \partial\Omega_0, \end{cases}$$

whose existence will be shown in Section 4.5. This is the first result of this nature available for non-self-adjoint differential operators like the ones dealt with in this chapter. One of the most novel parts of the proof consists in establishing (1.7) for the case when $m(x)$ changes of sign in Ω by perturbing the weight function m instead of the parameter λ as it is usual in the available literature. This technical device should have a huge number of applications to deal with spatially heterogeneous Reaction Diffusion equations. All the previous available results for degenerate diffusive logistic boundary value problems were established for the the Laplace operator without advection terms (see Gómez-Reñasco and López-Gómez [18], Gómez–Reñasco [17], López-Gómez [29] and Du and Huang [15]). In [26, Section 8] and [33] a rather complete account of historical bibliographic details are given.

Chapter 2

Some paradoxical effects of the advection on a class of diffusive equations in Ecology

2.1 Introduction

In this chapter we suppose that

$$m(x_+) > 0, \qquad m(x_-) < 0 \quad \text{for some } \; x_+, x_- \in \Omega. \tag{2.1}$$

The proof of the results are similar for $m > 0$.

It should be noted that the change of variable

$$u = e^{\alpha m/D} w, \qquad u_0 = e^{\alpha m/D} w_0, \tag{2.2}$$

transforms (1.1) in

$$\begin{cases} \partial_t w = D\Delta w + \alpha \nabla m \cdot \nabla w + \lambda m w - a e^{\alpha(p-1)m/D} w^p & \text{in } \; \Omega, \;\; t > 0, \\ \partial_\nu w = 0 & \text{on } \; \partial\Omega, \;\; t > 0, \\ w(\cdot, 0) = w_0 > 0 & \text{in } \; \Omega. \end{cases} \tag{2.3}$$

The steady-states of the parabolic problem (2.3) are the solutions of the semilinear elliptic boundary value problem

$$\begin{cases} -D\Delta w - \alpha \nabla m \cdot \nabla w - \lambda m w = -a e^{\alpha(p-1)m/D} w^p & \text{in } \; \Omega, \\ \partial_\nu w = 0 & \text{on } \; \partial\Omega. \end{cases} \tag{2.4}$$

As the change of variable (2.2) preserves the attractive or repulsive character of the solutions of (1.2), regarded as steady states of (1.1), the problem of analyzing the dynamics of (1.1) is equivalent to the problem of analyzing the dynamics of (2.3). Consequently, our attention in this chapter will be focused on models (2.3) and (2.4). As the variational equation of (2.3) at $w = 0$ is the linear parabolic problem

$$\begin{cases} \partial_t z = D\Delta z + \alpha \nabla m \cdot \nabla z + \lambda m z & \text{in } \Omega, \quad t > 0, \\ \partial_\nu z = 0 & \text{on } \partial\Omega, \quad t > 0, \\ z(\cdot, 0) = z_0 & \text{in } \Omega, \end{cases} \tag{2.5}$$

it becomes apparent that the principal eigenvalue of the linear operator

$$\mathfrak{L}(\lambda, \alpha) := -D\Delta - \alpha \nabla m \cdot \nabla - \lambda m \tag{2.6}$$

should play a significant role in describing the dynamics of (2.3). We introduce the eigenvalue

$$\Sigma(\lambda, \alpha) := \sigma[\mathfrak{L}(\lambda, \alpha), \mathfrak{N}, \Omega], \qquad \lambda \in \mathbb{R}, \tag{2.7}$$

which by the change of variable (2.2) is equal to (1.4). As the unique solution of (2.5) is given by the analytic semigroup generated by $-\mathfrak{L}(\lambda, \alpha)$ through the formula

$$z(\cdot, t; z_0) = e^{-t\mathfrak{L}(\lambda,\alpha)} z_0, \qquad t > 0,$$

and, according to [32, Th. 7.8], any eigenvalue of (1.3), $\tau \neq \Sigma(\lambda, \alpha)$, satisfies $\mathrm{Re}\, \tau > \Sigma(\lambda, \alpha)$, it becomes apparent that

$$\lim_{t \to \infty} z(\cdot, t; z_0) = 0$$

if $\Sigma(\lambda, \alpha) > 0$, and consequently, the trivial solution is stable in this case by the Lyapunov theorems, whereas it is unstable if $\Sigma(\lambda, \alpha) < 0$, as in this case, if $z_0 > 0$ is a principal eigenfunction associated to $\Sigma(\lambda, \alpha)$, we have that

$$z(\cdot, t; z_0) = e^{-t\mathfrak{L}(\lambda,\alpha)} z_0 = e^{-\Sigma(\lambda,\alpha)t} z_0 \to \infty \quad \text{as} \quad t \to \infty.$$

Therefore, the curve $\Sigma(\lambda, \alpha) = 0$ provides us with the set of values of the parameters λ and α where the attractive character of the zero solution changes. Actually, as we are assuming that $a > 0$, in the sense that $a \geq 0$ but $a \neq 0$, from the parabolic maximum principle it follows that the unique solution of (2.3), denoted by $w(x, t; w_0)$, satisfies

$$0 \leq w(\cdot, t; w_0) \leq e^{-t\mathfrak{L}(\lambda,\alpha)} w_0, \qquad t > 0,$$

and, therefore, $w = 0$ is a global attractor of (2.3) if $\Sigma(\lambda, \alpha) > 0$.

One of the main results of this chapter establishes that, actually, the sign of the principal eigenvalue $\Sigma(\lambda, \alpha)$ not only provides us with the local attractive, or repulsive, character of the zero solution but it also predicts the global behavior of the species. Indeed, thanks to Corollary 2.4.1, the species is permanent if $\Sigma(\lambda, \alpha) < 0$, while it becomes extinct if $\Sigma(\lambda, \alpha) > 0$. Therefore, to analyze the effects of the advection on the dynamics of (1.1), or (2.3), it is imperative to study how varies $\Sigma(\lambda, \alpha)$ as α ranges from zero to infinity. This task will be accomplished in Section 2.2, where we will use the abstract theory of Cano-Casanova and López-Gómez [9], recently refined in [32, Chapter 9], and the main theorem of Chen and Lou [11, Th. 1.1] for sharpening some of the pioneering results of Belgacem and Cosner [6] in a substantial way.

Whereas Belgacem and Cosner [6] used the method of sub and supersolutions to get most of their findings, this chapter invokes to a rather complete battery of local and global continuation techniques to perform the analysis of the existence of positive solutions of (2.4), so complementing the classical analysis of [6]. This analysis will be carried out in Sections 2.3 and 2.4, where the structure of the set of λ's for which (2.4) admits a positive solution is characterized and the uniqueness and the local attractive character of the positive steady states of (2.3) are established. Our main result shows that, for sufficiently large $\alpha > 0$, there exist $\lambda_*(\alpha) < \lambda_-(\alpha) < 0 < \lambda^*(\alpha)$ such that (2.4) has a positive solution if

$$\lambda \in \Lambda = \Lambda(\alpha) := (\lambda_*(\alpha), \lambda_-(\alpha)) \cup (0, \lambda^*(\alpha)),$$

while it cannot admit a positive solution if $\lambda_-(\alpha) \leq \lambda \leq 0$. Moreover, the positive solution is unique if it exists and all the positive steady state solutions (λ, w) conform two global real analytic arcs of curve bifurcating from zero at $\lambda = 0$ and $\lambda = \lambda_-(\alpha)$ and from infinity at $\lambda = \lambda_*(\alpha)$ and $\lambda^*(\alpha)$. It turns out that $\lambda_-(\alpha)$ and 0 are the unique zeroes of $\Sigma(\cdot, \alpha)$. Furthermore, due to Theorem 2.4.4, the unique positive steady state must be a global attractor of (2.3).

Although, in general, it remains an open problem to ascertain whether or not the model (2.4) can admit some additional positive solution for $\lambda < \lambda_*(\alpha)$ or $\lambda > \lambda^*(\alpha)$, in some special cases of interest one can determine explicitly $\lambda_*(\alpha)$ and $\lambda^*(\alpha)$ and prove that indeed (2.4) cannot admit a positive solution if $\lambda \notin \Lambda$. This analysis will be done in Section 2.5, where, in particular, we will prove that, in the classical case treated by Belgacem and Cosner [6], where $a(x) > 0$ for all $x \in \bar{\Omega}$, one has that $\lambda_*(\alpha) = -\infty$ and $\lambda^*(\alpha) = \infty$, while these extremal values must be finite if $a(x)$ vanishes on some subset of Ω with non-empty interior where m changes sign. When, in addition, the interior of $a^{-1}(0)$ is a nice open subdomain of Ω one can characterize explicitly the values of $\lambda_*(\alpha)$ and $\lambda^*(\alpha)$.

Finally, in Section 2.6 we will show how, for sufficiently large advection, the solutions of (2.3) can be regulated by its unique positive steady state, even when for sufficiently small α the corresponding solution grows up to infinity somewhere in Ω as $t \to \infty$. Consequently, though rather paradoxical, a large advection might cause a severe reduction of the productivity rates of the species in spatially heterogeneous habitats. Indeed, according to Theorems 2.5.3, when

$$\Omega_0 := \operatorname{int} a^{-1}(0)$$

is a nice subdomain of Ω with $\bar{\Omega}_0 \subset \Omega$ and α is sufficiently large, then (2.4) admits a positive solution if and only if $\lambda \in \Lambda(\alpha)$. Moreover, by Theorem 2.4.1, the positive solution is unique. On the other hand, thanks to Theorem 2.6.1, we have that

$$\lim_{\alpha \to \infty} \lambda^*(\alpha) = \infty \tag{2.8}$$

provided that $m(x_+) > 0$ for some $x_+ \in \Omega_0$ and that either m does not admit any critical point in $\bar{\Omega}_0$, or m admits finitely many critical points in $\bar{\Omega}_0$, say x_j, $1 \leq j \leq q$, with $\Delta m(x_j) > 0$ for all $1 \leq j \leq q$. Thus, under these hypotheses on $m(x)$, and for any given $\alpha > 0$, Theorem 2.4.5 shows that for every $\lambda \geq \lambda^*(\alpha)$ and $w_0 > 0$, the unique solution of (2.3), denoted by $w(\cdot, t; w_0)$, satisfies

$$\lim_{t \to \infty} \|w(\cdot, t; w_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty.$$

However, once fixed any of these λ's, according to (2.8), there exists $\alpha_1 > 0$ such that $\lambda < \lambda^*(\tilde{\alpha})$ for all $\tilde{\alpha} > \alpha_1$. Therefore, as $\lambda \in \Lambda(\tilde{\alpha})$ for $\tilde{\alpha} > \alpha_1$, Theorem 2.4.4 yields

$$\lim_{t \to \infty} \|w(\cdot, t; w_0) - w_{\lambda, \tilde{\alpha}}\|_{\mathcal{C}(\bar{\Omega})} = 0,$$

where $w_{\lambda,\tilde{\alpha}}$ stands for the unique positive solution of (2.4). The fact that the advection can stabilize towards an equilibrium the explosive solutions seems to be a new phenomenon not previously described in the literature. Another rather paradoxical effect of the advection had been observed at the end of Section 2.2, where it is shown how the species can be permanent even in the worse environmental conditions when it disperses with an extremely severe taxis down the environmental gradient. These paradoxical behaviors of the balance between diffusion and advection deserve further attention.

Throughout this chapter, we set $\mathbb{R}_+ = (0, \infty)$, and it should be noted that, thanks to the strong maximum principle, any positive solution of (2.4) must be strongly positive, i.e., $w \gg 0$, in the sense that $w(x) > 0$ for all $x \in \bar{\Omega}$.

2.2 The graphs of $\Sigma(\lambda, \alpha)$ as α varies from 0 to infinity

The next result shows the analyticity of the map $\Sigma(\lambda, \alpha)$ defined in (2.7) with respect to λ and α.

Proposition 2.2.1. *For every $\lambda \in \mathbb{R}$, the map $\alpha \mapsto \Sigma(\lambda, \alpha)$ is real analytic in $\alpha \in \mathbb{R}$; in particular, it is continuous. Similarly, $\lambda \mapsto \Sigma(\lambda, \alpha)$ is analytic for all $\alpha \in \mathbb{R}$.*

Proof. Obviously,

$$\mathfrak{L}(\lambda, \alpha) = -D\Delta - \alpha \nabla m \cdot \nabla - \lambda m = T + (\alpha_0 - \alpha) T^{(1)},$$

where we are denoting

$$T := -D\Delta - \alpha_0 \nabla m \cdot \nabla - \lambda m \quad \text{and} \quad T^{(1)} := \nabla m \cdot \nabla$$

for all $\alpha_0 \in \mathbb{R}$. Moreover, if for every $q > 1$, we regard $\mathfrak{L}(\lambda, \alpha)$ as an operator from the Sobolev space $W^{2,q}(\Omega)$ into $L^q(\Omega)$, one has that

$$\|T^{(1)} u\|_{L^q} = \|\nabla m \cdot \nabla u\|_{L^q} \leq C \|u\|_{W^{2,q}} \leq C \left(\|u\|_{W^{2,q}} + \|Tu\|_{L^q} \right)$$

for some positive constant $C > 0$. Thus, according to Kato [22, Th. 2.6 on p. 377], $\mathfrak{L}(\lambda, \alpha)$ is an holomorphic family of type (A) in α. Consequently, from [22, Rem. 2.9 on p. 379], all the algebraically simple eigenvalues of $\mathfrak{L}(\lambda, \alpha)$ vary analytically with α. In particular, owing to [32, Th. 7.8], the map $\alpha \mapsto \Sigma(\lambda, \alpha)$ is real analytic for all $\lambda \in \mathbb{R}$. This proof can be easily adapted to prove the analyticity in λ. □

According to (2.1), it is well known that

$$\lim_{\lambda \to \pm\infty} \Sigma(\lambda, \alpha) = -\infty$$

for all $\alpha \in \mathbb{R}$, and, actually, according to [32, Th. 9.1], the next result holds. Note that $\Sigma^{-1}(0)$ provides us with the principal eigenvalues of the linear weighted boundary value problem

$$\begin{cases} -D\Delta\varphi - \alpha \nabla m \cdot \nabla \varphi = \lambda m \varphi & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Subsequently, we denote

$$\Sigma' := \partial_\lambda \Sigma.$$

Proposition 2.2.2. *For every $\alpha \in \mathbb{R}$, the map $\lambda \mapsto \Sigma(\lambda, \alpha)$ is real analytic and strictly concave. Moreover, the weighted eigenvalue problem* (2.9) *admits two principal eigenvalues if, and only if, $\Sigma'(0, \alpha) \neq 0$; one among them equals zero, while the non-zero eigenvalue is negative if $\Sigma'(0, \alpha) < 0$, and positive if $\Sigma'(0, \alpha) > 0$. Furthermore, $\lambda = 0$ is the unique principal eigenvalue of* (2.9) *if $\Sigma'(0, \alpha) = 0$, and, in addition, if we denote by λ_0 the unique value of λ for which $\Sigma(\lambda_0, \alpha) = \max_{\lambda \in \mathbb{R}} \Sigma(\lambda, \alpha)$, we have that*

$$\Sigma'(\lambda, \alpha) \begin{cases} > 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ < 0 & \text{if } \lambda > \lambda_0. \end{cases}$$

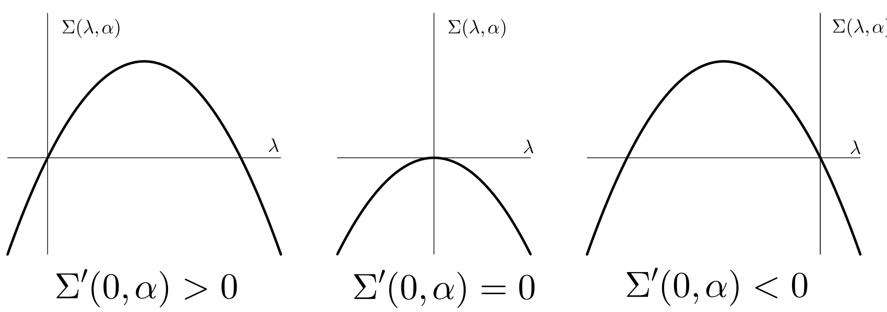

Figure 2.1: The three possible situation cases

The next result provides us with the value of $\Sigma'(0, \alpha)$. It is a substantial improvement of Belgacem and Cosner [6, Prop. 2.1].

Theorem 2.2.1. *For every $\alpha \in \mathbb{R}$,*

$$\Sigma'(0, \alpha) = -\int_\Omega m(x) e^{\alpha m(x)/D}\, dx \Big/ \int_\Omega e^{\alpha m(x)/D}\, dx. \tag{2.10}$$

Thus, the following assertions are true:

(i) $\Sigma'(0, \alpha) < 0$ *for all $\alpha > 0$ if $\int_\Omega m \geq 0$. Therefore, by Proposition 2.2.2, the problem* (2.9) *admits a unique negative eigenvalue $\lambda_-(\alpha) < 0$ for all $\alpha > 0$; necessarily, $\Sigma(\lambda_-(\alpha), \alpha) = 0$ and $\Sigma(\lambda, \alpha) > 0$ if and only if $\lambda \in (\lambda_-(\alpha), 0)$.*

(ii) Suppose $\int_\Omega m < 0$. Then, there exists a unique $\alpha_0 > 0$ such that

$$\Sigma'(0, \alpha) \begin{cases} > 0 & \text{if } 0 < \alpha < \alpha_0, \\ = 0 & \text{if } \alpha = \alpha_0, \\ < 0 & \text{if } \alpha > \alpha_0. \end{cases}$$

Consequently, (2.9) *admits a unique non-zero principal eigenvalue* $\lambda_+(\alpha) > 0$ *if* $\alpha \in (0, \alpha_0)$*, a unique non-zero principal eigenvalue* $\lambda_-(\alpha) < 0$ *if* $\alpha > \alpha_0$*, while* $\lambda = 0$ *is the unique principal eigenvalue of* (2.9) *if* $\alpha = \alpha_0$.

Proof. As already observed by Belgacem and Cosner in [6, p. 384], the adjoint operator of

$$\mathfrak{L}_0 u := \mathfrak{L}(0,\alpha)u = -D\Delta u - \alpha \nabla m \cdot \nabla u, \qquad u \in W^{2,q}(\Omega), \quad q > N,$$

subject to Neumann boundary conditions, admits the following realization

$$\mathfrak{L}_0^* v := -\nabla \cdot (D\nabla v - \alpha v \nabla m)$$

for all $v \in \mathcal{C}^2(\bar{\Omega})$ such that

$$D\partial_\nu v - \alpha v \partial_\nu m = 0 \qquad \text{on } \partial\Omega.$$

Moreover, $\varphi^* := e^{\alpha m/D}$ provides us with a positive eigenfunction associated with the zero eigenvalue of $\mathfrak{L}_0^*$.

Now, we will prove (2.10). According to Kato [22, Th. 2.6 on p. 377] and [22, Rem. 2.9 on p. 379], the perturbed eigenfunction $\varphi(\lambda)$ from $\varphi(0) = 1$ associated to the principal eigenvalue $\Sigma(\lambda,\alpha)$ as λ perturbs from 0 is real analytic as a function of λ. Thus, differentiating with respect to λ the identities

$$\begin{cases} -D\Delta\varphi(\lambda) - \alpha\nabla m \cdot \nabla\varphi(\lambda) - \lambda m(x)\varphi(\lambda) = \Sigma(\lambda,\alpha)\varphi(\lambda) & \text{in } \Omega, \\ \partial_\nu \varphi(\lambda) = 0 & \text{on } \partial\Omega, \end{cases}$$

yields

$$-D\Delta\varphi'(\lambda) - \alpha\nabla m \cdot \nabla\varphi'(\lambda) - m\varphi(\lambda) - \lambda m\varphi'(\lambda) = \Sigma'(\lambda,\alpha)\varphi(\lambda) + \Sigma(\lambda,\alpha)\varphi'(\lambda)$$

where $' := d/d\lambda$. Hence, particularizing at $\lambda = 0$, we are driven to

$$\mathfrak{L}_0\varphi'(0) = m(x) + \Sigma'(0,\alpha)$$

because $\Sigma(0,\alpha) = 0$ and $\varphi(0) = 1$. Consequently,

$$\langle m(x) + \Sigma'(0,\alpha), \varphi_0^* \rangle = \langle \mathfrak{L}_0\varphi'(0), \varphi_0^* \rangle = \langle \varphi'(0), \mathfrak{L}_0^*\varphi_0^* \rangle = 0$$

and therefore, (2.10) holds. As a byproduct, we find that

$$\operatorname{sign} \Sigma'(0,\alpha) = -\operatorname{sign} f(\alpha) \qquad \text{for all } \alpha \geq 0,$$

where

$$f(\alpha) := \int_\Omega m(x) e^{\alpha m(x)/D}\, dx.$$

As

$$f'(\alpha) = \frac{1}{D}\int_\Omega m^2(x) e^{\alpha m(x)/D}\, dx > 0$$

the function f is strictly increasing. On the other hand, setting

$$\Omega_+ := m^{-1}(0,\infty), \qquad \Omega_- := m^{-1}(-\infty,0),$$

we have that

$$f(\alpha)=\int_{\Omega_+} m(x)e^{\alpha m(x)/D}\,dx+\int_{\Omega_-} m(x)e^{\alpha m(x)/D}\,dx.$$

Consequently, since

$$\lim_{\alpha\to\infty}\int_{\Omega_+} me^{\alpha m(x)/D}\,dx=\infty,\quad \lim_{\alpha\to\infty}\int_{\Omega_-} me^{\alpha m(x)/D}\,dx=0,$$

we conclude from (2.1) that

$$\lim_{\alpha\to\infty} f(\alpha)=\infty.$$

Therefore, as $f(0)=\int_\Omega m$, we have that $f(\alpha)>0$ for all $\alpha>0$ if $\int_\Omega m\geq 0$, while in case $\int_\Omega m<0$, there exists a unique $\alpha_0>0$ such that $f(\alpha)<0$ if $\alpha<\alpha_0$ and $f(\alpha)>0$ if $\alpha>\alpha_0$. The proof is complete. □

Next, we will analyze the global behavior of $\Sigma(\lambda,\alpha)$, $\lambda\in\mathbb{R}$, as the advection α approximates $+\infty$. The next result is a direct consequence from Theorem 2.2.1.

Corollary 2.2.1. *For sufficiently large* α, *one has that* $\Sigma'(0,\alpha)<0$ *and hence,* $\Sigma(\lambda,\alpha)<0$ *for all* $\lambda>0$. *Moreover, there exists a unique* $\lambda_-(\alpha)<0$ *such that* $\Sigma'(\lambda_-(\alpha),\alpha)>0$ *and* $\Sigma(\lambda,\alpha)>0$ *if, and only if,* $\lambda_-(\alpha)<\lambda<0$. *All these properties independently of the value of* $\int_\Omega m$.

In the special case when the critical points of m are non-degenerate, the behavior of $\lambda_-(\alpha)$ as $\alpha\to\infty$ can be derived from the next result of Chen and Lou [11, Th. 1.1].

Theorem 2.2.2. *Assume that all critical points of* $m(x)$ *are non-degenerate. Let* $\mathcal{M}$ *denote the set of points* $x\in\bar{\Omega}$ *where* $m(x)$ *attains a local maximum. Then,*

$$\lim_{\alpha\to\infty}\Sigma(\lambda,\alpha)=\min_{x\in\mathcal{M}}\{-\lambda m(x)\} \tag{2.11}$$

for all $\lambda\in\mathbb{R}$.

Consequently, if, in addition, we assume that

$$\mathcal{M}=\{x_1,\dots,x_h\}\quad\text{and}\quad\{m(x_j)\ :\ 1\leq j\leq h\}=\{m_1,\dots,m_k\}$$

with $m_j<m_{j+1}$, $1\leq j\leq k-1$, then, (2.11) implies that

$$\lim_{\alpha\to\infty}\Sigma(\lambda,\alpha)=\begin{cases}-\lambda m_1 & \text{if }\ \lambda<0,\\ 0 & \text{if }\ \lambda=0,\\ -\lambda m_k & \text{if }\ \lambda>0.\end{cases} \tag{2.12}$$

Owing to (2.1), we always have that $m_k>0$. Thus, the half-curve $\Sigma(\lambda,\alpha)$, $\lambda>0$, approximates the straight half-line $-\lambda m_k$, $\lambda>0$, with negative slope $-m_k<0$, as $\alpha\to\infty$. To ascertain the behavior of $\Sigma(\lambda,\alpha)$ for $\lambda<0$, we have to distinguish two different situations, according to the sign of m_1. By (2.12), in the case when $m_1>0$, the half-curve $\Sigma(\lambda,\alpha)$, $\lambda<0$, approximates a line with negative slope $-m_1<0$, much like in the previous case. In particular, this entails

$$\lim_{\alpha\to\infty}\lambda_-(\alpha)=-\infty. \tag{2.13}$$

Thus, in such case, by Corollary 2.4.1, the species becomes extinct if $\lambda < 0$ for sufficiently large α, as for these values $\Sigma(\lambda,\alpha) > 0$, as already discussed in Section 2.1. Although we did not prove it, we conjecture that (2.13) holds if $m_1 = 0$, by the strict concavity of $\Sigma(\lambda,\alpha)$. Consequently, some strong taxis down the environmental gradient is harmful for the species in these circumstances. However, the situation reverses when $m_1 < 0$, as in such case $\Sigma(\lambda,\alpha) < 0$ for all $\lambda < 0$ and sufficiently large α and therefore, according to Corollary 2.4.1, the species is permanent. Moreover, in this case,

$$\lim_{\alpha\to\infty} \lambda_-(\alpha) = 0.$$

Therefore, even with a severe taxis down the environmental gradient the model exhibits permanence, which is a rather paradoxical behavior, not well understood yet, utterly attributable to the effects of the spatial dispersion.

2.3 Local bifurcations from $\lambda = 0$ and $\lambda = \lambda_\pm(\alpha)$ if $\Sigma'(0,\alpha) \neq 0$

Throughout this section, we will asume that

$$\Sigma'(0,\alpha) \neq 0.$$

This guarantees the existence of $\lambda_-(\alpha)$, or $\lambda_+(\alpha)$, according to the sign of $\Sigma'(0,\alpha)$. The main result of this section reads as follows.

Theorem 2.3.1. *Suppose $\alpha > 0$ and (2.4) admits a positive solution. Then, $\Sigma(\lambda,\alpha) < 0$. Moreover, the following assertions are true:*

(i) If $\Sigma'(0,\alpha) > 0$, then there exists $\varepsilon > 0$ such that (2.4) possesses a positive solution $w_{\lambda,\alpha}$ for every $\lambda \in (-\varepsilon, 0) \cup (\lambda_+(\alpha), \lambda_+(\alpha)+\varepsilon)$. Actually, in a neighborhood of $(\lambda,u) = (0,0)$ and of $(\lambda,u) = (\lambda_+(\alpha),0)$ the set of positive solutions of (2.4) has the structure of a real analytic curve.

(ii) If $\Sigma'(0,\alpha) < 0$, then there exists $\varepsilon > 0$ such that (2.4) possesses a positive solution $w_{\lambda,\alpha}$ for every $\lambda \in (\lambda_-(\alpha)-\varepsilon, \lambda_-(\alpha)) \cup (0,\varepsilon)$, and in a neighborhood of $(\lambda,u) = (0,0)$ and of $(\lambda,u) = (\lambda_-(\alpha),0)$ the set of positive solutions of (2.4) has the structure of a real analytic curve.

Proof. Suppose $\alpha > 0$ and (2.4) admits a positive solution (λ, w). Then, $w \gg 0$ and, due to (2.4) and (2.6), we find that

$$\left(\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}w^{p-1}\right) w = 0.$$

Hence, by the Krein–Rutman theorem,

$$\sigma[\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}w^{p-1}, \mathfrak{N}, \Omega] = 0. \tag{2.14}$$

Consequently, by the monotonicity of the principal eigenvalue with respect to the potential, we find that

$$\Sigma(\lambda,\alpha) = \sigma[\mathfrak{L}(\lambda,\alpha), \mathfrak{N}, \Omega] < \sigma[\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}w^{p-1}, \mathfrak{N}, \Omega] = 0.$$

Therefore, $\Sigma(\lambda,\alpha) < 0$ is necessary for the existence of a positive steady-state.

Subsequently, we introduce the operator $\mathfrak{F} : \mathbb{R} \times W^{2,q}_{\mathfrak{N}}(\Omega) \to L^q(\Omega)$, $q > N$, defined by

$$\mathfrak{F}(\lambda, w) := \mathfrak{L}(\lambda,\alpha)w + ae^{\alpha(p-1)m/D}|w|^{p-1}w$$

for all $\lambda \in \mathbb{R}$ and $w \in W^{2,q}_{\mathfrak{N}}(\Omega)$, where we are denoting

$$W^{2,q}_{\mathfrak{N}}(\Omega) := \{\, w \in W^{2,q}(\Omega) \;:\; \partial_\nu w = 0 \,\}.$$

$\mathfrak{F}$ is an operator of class $\mathcal{C}^2$-regularity on $w \geq 0$, real analytic on $w \gg 0$, such that

$$\mathfrak{F}(\lambda, 0) = 0 \quad \text{and} \quad D_w\mathfrak{F}(\lambda, 0) = \mathfrak{L}(\lambda,\alpha) \quad \text{for all} \;\; \lambda \in \mathbb{R}.$$

As the positive constants are the unique positive solutions of the problem

$$\begin{cases} -D\Delta\varphi - \alpha\nabla m \cdot \nabla\varphi = 0 & \text{in } \;\Omega, \\ \partial_\nu \varphi = 0 & \text{on } \;\partial\Omega, \end{cases}$$

we have that

$$N[\mathfrak{L}(0,\alpha)] = \operatorname{span}[1].$$

We claim that

$$R[\mathfrak{L}(0,\alpha)] = \left\{ f \in L^q(\Omega) \;:\; \int_\Omega e^{\alpha m(x)/D} f(x)\,dx = 0 \right\}. \tag{2.15}$$

Let $f \in L^q(\Omega)$ be such that

$$\begin{cases} \mathfrak{L}(0,\alpha)w = f & \text{in } \;\Omega, \\ \partial_\nu w = 0 & \text{on } \;\partial\Omega, \end{cases} \tag{2.16}$$

for some $w \in W^{2,q}(\Omega)$. We already know that $\varphi^* = e^{\alpha m/D}$ is a positive eigenfunction associated to the zero eigenvalue of $\mathfrak{L}_0^*$. Thus, multiplying the differential equation of (2.16) by φ^* and integrating in Ω yields

$$\int_\Omega e^{\alpha m(x)/D} f(x)\,dx = 0.$$

Now, the identity (2.15) follows from the Fredholm alternative.

According to (2.10), since $\Sigma'(0,\alpha) \neq 0$, it becomes apparent that

$$\int_\Omega m(x) e^{\alpha m(x)/D}\,dx \neq 0$$

and, consequently, owing to (2.15), we find that

$$\partial_\lambda \mathfrak{L}(0,\alpha)1 = -m \notin R[\mathfrak{L}(0,\alpha)].$$

Therefore, the transversality condition of Crandall and Rabinowitz [12] holds (see also [31]), and, as a result, from the main theorem of [12], it is easy to infer that $(\lambda, w) = (0,0)$ is a bifurcation point to a real analytic curve of positive solutions of (2.4). Moreover, these solutions provide us with the unique positive solutions of (2.4) in a neighborhood of the bifurcation point.

As $\Sigma(\lambda_\pm(\alpha),\alpha)=0$, there exist $\varphi_\pm \gg 0$, unique up to a positive multiplicative constant, such that

$$\begin{cases} -D\Delta\varphi_\pm - \alpha\nabla m\cdot\nabla\varphi_\pm - \lambda_\pm(\alpha)m\varphi_\pm = 0 & \text{in } \Omega, \\ \partial_\nu\varphi_\pm = 0 & \text{on } \partial\Omega, \end{cases}$$

and, since the principal eigenvalue is algebraically simple, we have that

$$N[\mathfrak{L}(\lambda_\pm(\alpha),\alpha)] = \operatorname{span}[\varphi_\pm].$$

Moreover, as

$$\mathfrak{L}(\lambda_\pm(\alpha),\alpha)\varphi_\pm = 0,$$

there exist $\varphi^*_\pm \gg 0$ such that

$$\mathfrak{L}^*(\lambda_\pm(\alpha),\alpha)\varphi^*_\pm = 0,$$

where $\mathfrak{L}^*(\lambda_\pm(\alpha),\alpha)$ stand for the adjoint of $\mathfrak{L}(\lambda_\pm(\alpha),\alpha)$. By the Fredholm alternative, we have that

$$R[\mathfrak{L}(\lambda_\pm(\alpha),\alpha)] = \left\{ f\in L^q(\Omega) \;:\; \int_\Omega \varphi^*_\pm(x)f(x)\,dx = 0 \right\}. \tag{2.17}$$

According to Kato [22, Th. 2.6 on p. 377] and [22, Rem. 2.9 on p. 379], the perturbed eigenfunction $\varphi_\pm(\lambda)$ from $\varphi_\pm(\lambda_\pm(\alpha)) = \varphi_\pm$ associated to the principal eigenvalue $\Sigma(\lambda,\alpha)$ as λ perturbs from $\lambda_\pm(\alpha)$ is real analytic as a function of λ. Thus, differentiating with respect to λ yields

$$-D\Delta\varphi'_\pm(\lambda) - \alpha\nabla m\cdot\nabla\varphi'_\pm(\lambda) - m\varphi_\pm(\lambda) - \lambda m\varphi'_\pm(\lambda) = \Sigma'(\lambda,\alpha)\varphi_\pm(\lambda) + \Sigma(\lambda,\alpha)\varphi'_\pm(\lambda)$$

where we are denoting $' := d/d\lambda$. Hence, particularizing at $\lambda=\lambda_\pm(\alpha)$, shows that

$$\mathfrak{L}(\lambda_\pm(\alpha),\alpha)\varphi'_\pm = m\varphi_\pm + \Sigma'(\lambda_\pm(\alpha),\alpha)\varphi_\pm$$

since $\Sigma(\lambda_\pm,\alpha)=0$. Consequently, multiplying by $\varphi^*_\pm$ and integrating, we find that

$$\Sigma'(\lambda_\pm(\alpha),\alpha) = -\frac{\int_\Omega m\varphi_\pm\varphi^*_\pm}{\int_\Omega \varphi_\pm\varphi^*_\pm}.$$

As $\Sigma'(0,\alpha)\neq 0$, by Proposition 2.2.2, we also have that $\Sigma'(\lambda_\pm(\alpha),\alpha)\neq 0$, and therefore,

$$\int_\Omega m\varphi_\pm\varphi^*_\pm \neq 0.$$

Consequently, thanks to (2.17), we find that

$$\partial_\lambda\mathfrak{L}(\lambda_\pm(\alpha),\alpha)\varphi_\pm = -m\varphi_\pm \notin R[\mathfrak{L}(\lambda_\pm(\alpha),\alpha)].$$

In other words, the transversality condition of Crandall and Rabinowitz [12] holds at $\lambda_\pm(\alpha)$ and hence, for every $\alpha>0$, $(\lambda,w)=(\lambda_\pm(\alpha),0)$ is a bifurcation point to a real analytic curve of positive solutions of (2.4). Moreover, the solutions along these curves are the unique positive solutions of (2.4) in a neighborhood of $(\lambda_\pm(\alpha),0)$. As we already know that $\Sigma(\lambda,\alpha)<0$ is necessary for the existence of a positive solution of (2.4), the remaining assertions of the theorem follow easily from the fact that the λ–projections of the bifurcated curves are connected. □

The next result establishes that $\Sigma(\lambda_b, \alpha) = 0$ if $(\lambda, u) = (\lambda_b, 0)$ is a bifurcation point to positive solutions of (2.4). Consequently, thanks to Proposition 2.2.2, $\lambda_b \in \{\lambda_\pm(\alpha), 0\}$ if $\Sigma'(0,\alpha) \neq 0$, while $\lambda_b = 0$ if $\Sigma'(0,\alpha) = 0$. In particular, in the context of Theorem 2.3.1, $\lambda_\pm(\alpha)$ and 0 are the unique values of the parameter λ where bifurcation to positive solutions from $u = 0$ occurs.

Lemma 2.3.2. $\Sigma(\lambda_b, \alpha) = 0$ *if* $(\lambda, u) = (\lambda_b, 0)$ *is a bifurcation point to positive solutions of* (2.4).

Proof. Let (λ_n, w_n), $n \geq 1$, be a sequence of positive solutions of (2.4) such that

$$\lim_{n\to\infty} (\lambda_n, w_n) = (\lambda_b, 0).$$

Then, dividing the w_n-equation by $\|w_n\|_\infty$, $n \geq 1$, yields

$$\mathfrak{L}(\lambda_b, \alpha)\frac{w_n}{\|w_n\|_\infty} + (\lambda_b - \lambda_n) m \frac{w_n}{\|w_n\|_\infty} + a e^{\alpha(p-1)m/D} \frac{w_n^p}{\|w_n\|_\infty} = 0. \tag{2.18}$$

Note that

$$H_n := (\lambda_n - \lambda_b) m \frac{w_n}{\|w_n\|_\infty} - a e^{\alpha(p-1)m/D} \frac{w_n^p}{\|w_n\|_\infty} \to 0 \quad \text{as} \ \ n \to \infty.$$

As the operator

$$\mathfrak{L}(\lambda_b, \alpha) - \Sigma(\lambda_b, \alpha) + 1$$

is invertible, (2.18) can be equivalently written as

$$\frac{w_n}{\|w_n\|_\infty} = [\mathfrak{L}(\lambda_b, \alpha) - \Sigma(\lambda_b, \alpha) + 1]^{-1} \left\{ H_n + [1 - \Sigma(\lambda_b, \alpha)] \frac{w_n}{\|w_n\|_\infty} \right\}. \tag{2.19}$$

Since

$$H_n + [1 - \Sigma(\lambda_b, \alpha)] w_n / \|w_n\|_\infty, \qquad n \geq 1,$$

is bounded in L^∞, by compactness, there exists a subsequence of $w_n/\|w_n\|_\infty$, relabeled by n, such that

$$\lim_{n\to\infty} \frac{w_n}{\|w_n\|_\infty} = \varphi$$

in L^∞ for some $\varphi \in L^\infty$. Necessarily, $\|\varphi\|_\infty = 1$ and $\varphi > 0$. Moreover, letting $n \to \infty$ in (2.19), we find that

$$\varphi = [\mathfrak{L}(\lambda_b, \alpha) - \Sigma(\lambda_b, \alpha) + 1]^{-1} \{[1 - \Sigma(\lambda_b, \alpha)]\varphi\}.$$

By elliptic regularity, φ must be a strong solution of

$$\mathfrak{L}(\lambda_b, \alpha)\varphi = 0.$$

Therefore, since $\varphi > 0$, $\Sigma(\lambda_b, \alpha) = 0$. This ends the proof. □

2.4 Uniqueness of the positive steady state and bifurcation diagrams. Global attractivity.

The next result shows that (2.4) admits, at most, a unique positive solution.

Theorem 2.4.1. *Suppose* (2.4) *admits a positive solution for some pair* (λ, α), $\lambda \in \mathbb{R}$, $\alpha > 0$. *Then, it is unique; it will be throughout denoted by* $w_{\lambda,\alpha}$. *Moreover,* $w_{\lambda,\alpha} \gg 0$ *in the sense that* $w_{\lambda,\alpha}(x) > 0$ *for all* $x \in \bar{\Omega}$.

Proof. Let w be a positive solution of (2.4). Then, according to (2.14),

$$\sigma[\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}w^{p-1}, \mathfrak{N}, \Omega] = 0$$

with associated eigenfunction $w > 0$. Consequently, thanks to the Krein–Rutman theorem (see [32, Cor. 7.1]), $w \gg 0$ in the sense that $w(x) > 0$ for all $x \in \bar{\Omega}$.

To prove the uniqueness, let $w_1 \neq w_2$ two positive solutions and set

$$\phi(t) = [tw_1 + (1-t)w_2]^p\,, \qquad t \in [0,1].$$

Then,

$$w_1^p - w_2^p = \int_0^1 \phi'(t)dt = p\int_0^1 [tw_1 + (1-t)w_2]^{p-1}\, dt(w_1 - w_2)$$

and hence,

$$\begin{aligned}\mathfrak{L}(\lambda,\alpha)(w_1 - w_2) &= -ae^{\alpha(p-1)m/D}(w_1^p - w_2^p)\\ &= -ae^{\alpha(p-1)m/D}p\int_0^1 [tw_1 + (1-t)w_2]^{p-1}dt(w_1 - w_2).\end{aligned}$$

Thus, $w_1 - w_2 \neq 0$ is an eigenfunction of the linear differential operator

$$\mathfrak{L}(\lambda,\alpha) + a(x)e^{\alpha(p-1)m/D}pI, \qquad I := \int_0^1 [tw_1 + (1-t)w_2]^{p-1}dt,$$

in Ω under Neumann boundary conditions. Consequently, by the dominance of the principal eigenvalue (see [32, Th. 7.8]), we find that

$$\sigma[\mathfrak{L}(\lambda,\alpha) + pa(x)e^{\alpha(p-1)m/D}I, \mathfrak{N}, \Omega] \leq 0. \tag{2.20}$$

On the other hand, as $p \geq 2$ and $w_2 \gg 0$, we have that

$$[tw_1 + (1-t)w_2]^{p-1} > t^{p-1}w_1^{p-1}, \qquad t \in (0,1),$$

and hence,

$$pI > p\int_0^1 t^{p-1}w_1^{p-1}dt = w_1^{p-1}.$$

Consequently, since $a > 0$ and $w_1 \gg 0$, we find from (2.20) that

$$\begin{aligned} 0 &\geq \sigma[\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}pI, \mathfrak{N}, \Omega] \\ &> \sigma[\mathfrak{L}(\lambda,\alpha) + ae^{\alpha(p-1)m/D}w_1^{p-1}, \mathfrak{N}, \Omega] = 0, \end{aligned}$$

which is impossible. Therefore, the positive solution must be unique if it exists. □

The next result establishes that the set of positive solutions (λ, w) of (2.4) consists of analytic arcs of curve parameterized by λ and that any positive steady state of (2.3) must be a local attractor.

Theorem 2.4.2. *Let* (λ_0, w_0) *be a positive solution of* (2.4). *Then, there exists* $\varepsilon > 0$ *and a real analytic function* $w : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \to W^{2,q}_{\mathfrak{N}}(\Omega)$, $q > N$, *such that* $w(\lambda_0) = w_0$ *and* $w(\lambda) = w_{\lambda,\alpha}$ *if* $|\lambda - \lambda_0| < \varepsilon$. *Moreover,* (λ_0, w_0) *must be a local attractor as an steady-state of* (2.3).

Proof. Using the notations introduced in the proof of Theorem 2.3.1, we have that

$$D_w\mathfrak{F}(\lambda_0, w_0)w = \mathfrak{L}(\lambda_0,\alpha)w + pae^{\alpha(p-1)m/D}w_0^{p-1}w.$$

By the monotonicity of the principal eigenvalue with respect to the potential,

$$\sigma[\mathfrak{L}(\lambda_0,\alpha) + pae^{\alpha(p-1)m/D}w_0^{p-1}, \mathfrak{N}, \Omega] > \sigma[\mathfrak{L}(\lambda_0,\alpha) + ae^{\alpha(p-1)m/D}w_0^{p-1}, \mathfrak{N}, \Omega] = 0.$$

Hence, $D_w\mathfrak{F}(\lambda_0, w_0)$ is invertible. Actually, thanks to [32, Th. 7.10], the inverse must be positive. The implicit function theorem ends the proof, as, thanks to Theorem 2.4.1, we already know that the positive solution is unique. The local attractive character of w_0 is an easy consequence of the positivity of the principal eigenvalue of the linearization and its dominance, by the principle of linearized stability of Lyapunov. □

Within the setting of Theorem 2.4.2, differentiating with respect to λ, yields

$$D_\lambda\mathfrak{F}(\lambda_0, w_0) + D_w\mathfrak{F}(\lambda_0, w_0)D_\lambda w(\lambda_0) = 0$$

and hence,

$$D_\lambda w(\lambda_0) = [D_w\mathfrak{F}(\lambda_0, w_0)]^{-1}(mw_0).$$

Although $[D_w\mathfrak{F}(\lambda_0, w_0)]^{-1}$ is strongly positive, as m changes sign, it is possible that $D_\lambda w(\lambda_0)$ changes sign in Ω. Consequently, $w(\lambda)$ might increase in some subdomain Ω_+ of Ω, while simultaneously decays in $\Omega_- := \Omega \setminus \Omega_+$. Actually, the regions Ω_+ and Ω_- should vary with the parameter λ. Nevertheless, by Theorem 2.3.1, in case $\Sigma'(0,\alpha) < 0$ we already know that

$$\lim_{\lambda \uparrow \lambda_-(\alpha)} w(\lambda) = 0, \qquad \lim_{\lambda \downarrow 0} w(\lambda) = 0,$$

and, actually, $w(\lambda)$ decays for $\lambda < \lambda_-(\alpha)$, $\lambda \sim \lambda_-(\alpha)$, while it grows for $\lambda > 0$, $\lambda \sim 0$. However, far from $\lambda_-(\alpha)$ and 0, $D_\lambda w(\lambda)$ might change sign. But, even if $w(\lambda)$ decays for some range of values of λ, by Lemma 2.3.2, the curve of positive solutions cannot reach $w = 0$, unless $\lambda = \lambda_\pm(\alpha)$ or $\lambda = 0$.

Throughout the rest of this chapter we will assume that α is sufficiently large so that

$$\Sigma'(0,\alpha) < 0, \tag{2.21}$$

which is guaranteed by Corollary 2.2.1. Since the corresponding mathematical analysis is easily adaptable to cover the case $\Sigma'(0,\alpha) > 0$, the discussion of this alternative case is omitted. Under assumption (2.21), it follows from Proposition 2.2.2 and Theorem 2.3.1 that (2.4) cannot admit a positive solution if $\lambda \in (\lambda_-(\alpha), 0)$. Moreover, thanks to Theorem 2.3.1, the next extremal values of the parameter λ are well defined:

$$\lambda^* := \sup \{ \mu > 0 \ \text{ such that } \ (2.4) \ \text{ has a positive solution for each } \ \lambda \in (0,\mu) \},$$
$$\lambda_* := \inf \{ \mu < 0 \ \text{ such that } \ (2.4) \ \text{ has a positive solution for each } \lambda \in (\mu, \lambda_-(\alpha)) \},$$

and satisfy

$$\lambda_* \in [-\infty, \lambda_-(\alpha)) \quad \text{and} \quad \lambda^* \in (0, +\infty].$$

By construction and Theorem 2.4.1, (2.4) possesses a unique positive solution for every

$$\lambda \in \Lambda := (\lambda_*, \lambda_-(\alpha)) \cup (0, \lambda^*).$$

Moreover these solutions fill in two real analytic arcs of curve, which are not necessarily monotonic in λ because $m(x)$ changes sign in Ω. The next result shows that the positive solutions of (2.4) must be unbounded at λ^* (resp. λ_*) if $\lambda^* < +\infty$ (resp. $\lambda_* > -\infty$). It seems an open problem to ascertain whether or not (2.4) can admit some further positive solution for either $\lambda > \lambda^*$ or $\lambda < \lambda_*$, though we will give some positive answers to this problem in Section 2.5.

Theorem 2.4.3. *Suppose $\lambda^* < +\infty$. Then,* (2.4) *cannot admit a positive solution for $\lambda = \lambda^*$. Moreover,*

$$\lim_{\lambda \uparrow \lambda^*} \|w_{\lambda,\alpha}\|_\infty = \infty.$$

Similarly, if $\lambda_* > -\infty$, then (2.4) *cannot admit a positive solution at $\lambda = \lambda_*$ and*

$$\lim_{\lambda \downarrow \lambda_*} \|w_{\lambda,\alpha}\|_\infty = \infty.$$

Proof. Suppose $\lambda^* < +\infty$. Then, according to Theorem 2.4.2, (2.4) cannot admit a positive solution at $\lambda = \lambda^*$, as, otherwise, we would contradict the definition of λ^*. Let $\lambda_n < \lambda^*$, $n \geq 1$, be a sequence such that $\lim_{n\to\infty} \lambda_n = \lambda^*$, and let $w_n := w_{\lambda_n,\alpha}$ be the unique positive solution of (2.4) for $\lambda = \lambda_n$. Then,

$$\mathfrak{L}(\lambda^*,\alpha)w_n + (\lambda^* - \lambda_n)mw_n + ae^{\alpha(p-1)m/D}w_n^p = 0, \qquad n \geq 1. \tag{2.22}$$

As $\mathfrak{L}(\lambda^*,\alpha) - \Sigma(\lambda^*,\alpha) + 1$ is an invertible operator, with compact resolvent

$$\mathcal{K} := [\mathfrak{L}(\lambda^*,\alpha) - \Sigma(\lambda^*,\alpha) + 1]^{-1}$$

(2.22) can be equivalently expressed as

$$w_n = \mathcal{K}[-ae^{\alpha(p-1)m/D}w_n^p + (\lambda_n - \lambda^*)mw_n + (1 - \Sigma(\lambda^*,\alpha))w_n], \quad n \geq 1. \tag{2.23}$$

Consequently, if there exists a constant $C > 0$ such that $\|w_n\|_\infty \leq C$ for all $n \geq 1$, then, the sequence

$$-ae^{\alpha(p-1)m/D}w_n^p + (\lambda_n - \lambda^*)mw_n + (1 - \Sigma(\lambda^*,\alpha))w_n, \qquad n \geq 1,$$

is bounded in L^∞, and therefore, by compactness, it follows from (2.23) that one can extract a subsequence, relabeled by n, such that

$$\lim_{n\to\infty} w_n = w \in L^\infty(\Omega). \tag{2.24}$$

Letting $n \to \infty$ in (2.23) and using elliptic regularity it becomes apparent that

$$\mathfrak{L}(\lambda^*, \alpha)w + ae^{\alpha(p-1)m/D} w^p = 0.$$

Consequently, $w \geq 0$ is a solution of (2.4) for $\lambda = \lambda^*$. By Lemma 2.3.2 and (2.24), $w > 0$, because $\lambda^* > 0$. Therefore, (2.4) admits a positive solution for $\lambda = \lambda^*$, which is impossible. This proof can be easily adapted to prove the second assertion. □

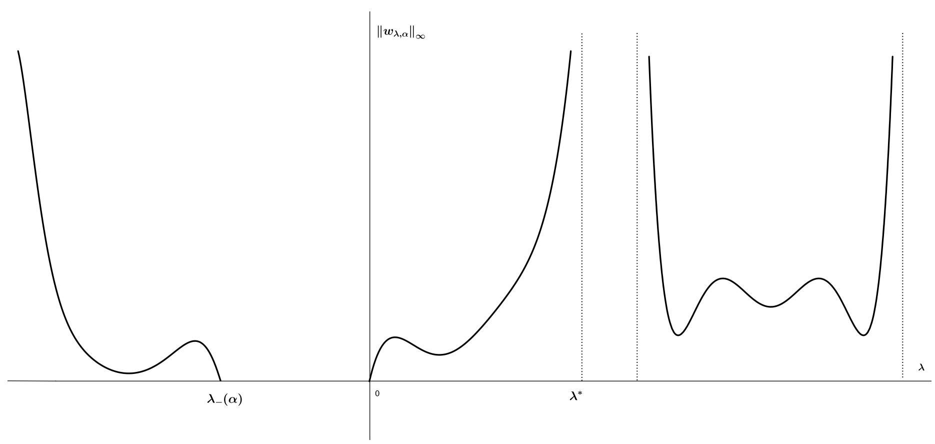

Figure 2.2: An admissible global bifurcation diagram

Figure 2.2 shows an (ideal) admissible bifurcation diagram with $\lambda_* = -\infty$ and $\lambda^* < +\infty$. We are plotting the value of the parameter λ in abscisas versus the value of the L^∞–norm of the solution in ordinates. In such an example the set of positive solutions of (2.4) consists of three (connected) components. It should be noted that the curves are far from monotonic, except in a neighborhood of the bifurcation points from zero. In the next section we shall give a number of sufficient conditions on the weight function $a(x)$ so that the solution set consist of two components.

We conclude this section by proving that the unique positive steady state of (2.3), $w_{\lambda,\alpha}$, must be a global attractor if it exists.

Theorem 2.4.4. *Suppose* (2.4) *admits a positive solution* $w_{\lambda,\alpha}$*; necessarily unique by Theorem 2.4.1. Then,*

$$\lim_{t\to\infty} \|w(\cdot, t; w_0) - w_{\lambda,\alpha}\|_{\mathcal{C}(\bar{\Omega})} = 0$$

where $w(\cdot, t; w_0)$ *stands for the unique solution of the parabolic problem* (2.3).

Proof. According to Theorem 2.3.1, we have that $\Sigma(\lambda,\alpha) < 0$. Let $\varphi \gg 0$ be the unique principal eigenfunction associated to $\Sigma(\lambda,\alpha)$ normalized so that $\|\varphi\|_\infty = 1$. Then,

$$\mathfrak{L}(\lambda,\alpha)\varphi = \Sigma(\lambda,\alpha)\varphi$$

and there exists $\beta > 0$ such that

$$\Sigma(\lambda,\alpha) + \beta < 0.$$

Thus, the function $\underline{w} = \varepsilon\varphi$ provides us with a subsolution of (2.4) provided

$$0 < \varepsilon^{p-1} < \frac{\beta}{\|a\|_\infty e^{\alpha(p-1)\|m\|_\infty/D}},$$

because

$$\varepsilon^{p-1} e^{\alpha(p-1)m(x)/D} a(x)\varphi^{p-1}(x) + \Sigma(\lambda,\alpha) < \Sigma(\lambda,\alpha) + \beta < 0,$$

and hence,

$$\mathfrak{L}(\lambda,\alpha)(\varepsilon\varphi) = \varepsilon\Sigma(\lambda,\alpha)\varphi < -e^{\alpha(p-1)m/D} a(\varepsilon\varphi)^p.$$

Also, for all $\kappa > 1$ we have that

$$\begin{aligned}\mathfrak{L}(\lambda,\alpha)(\kappa w_{\lambda,\alpha}) &= \kappa\mathfrak{L}(\lambda,\alpha) w_{\lambda,\alpha} \\ &= -\kappa a e^{\alpha(p-1)m/D} w_{\lambda,\alpha}^p > -a e^{\alpha(p-1)m/D}(\kappa w_{\lambda,\alpha})^p,\end{aligned}$$

because $\kappa < \kappa^p$ for all $p > 1$. Thus,

$$\overline{w} := \kappa w_{\lambda,\alpha}$$

provides us with a supersolution of (2.4) for all $\kappa \geq 1$.

According to the parabolic strong maximum principle, $w(x,1;w_0) > 0$ for all $x \in \bar{\Omega}$. Now, choose $\varepsilon > 0$ sufficiently small and $\kappa > 1$ sufficiently large so that

$$\underline{w} = \varepsilon\varphi \leq w_1 := w(\cdot,1;w_0) \leq \kappa w_{\lambda,\alpha} = \overline{w}.$$

Then, thanks again to the parabolic maximum principle, and using the semigroup property, we find that

$$w(\cdot,t;\underline{w}) \leq w(\cdot,t;w_1) = w(\cdot,t+1;w_0) \leq w(\cdot,t;\overline{w})$$

for all $t > 0$. According to Sattinger [40], $w(\cdot,t;\underline{w})$ must increase towards the minimal positive solution of (2.4) in $[\underline{w},\overline{w}]$, while $w(\cdot,t;\overline{w})$ must decay towards the maximal one. By the uniqueness of the positive solution, the proof is completed. □

In Section 2.1 we have already proven that 0 is a global attractor of (2.3) if $\Sigma(\lambda,\alpha) > 0$. When $\Sigma(\lambda,\alpha) < 0$ and (2.4) does not admit a positive solution, then the solutions of (2.3) must grow up to infinity as $t \to \infty$, as established by the next result.

Theorem 2.4.5. *Suppose $\Sigma(\lambda,\alpha) < 0$ and (2.4) does not admit a positive solution. Then,*

$$\lim_{t\to\infty} \|w(\cdot,t;w_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty$$

for all $w_0 \geq 0$, $w_0 \neq 0$.

Proof. Arguing as in the proof of Theorem 2.4.4, it is apparent that $\underline{w} := \varepsilon\varphi$ provides us with a subsolution of (2.4) for sufficiently small $\varepsilon > 0$. Moreover, shortening ε, if necessary, one can reach

$$\varepsilon\varphi < w_1 := w(\cdot, 1; w_0)$$

and hence,

$$w(\cdot, t; \varepsilon\varphi) \leq w(\cdot, t; w_1) = w(\cdot, t+1; w_0)$$

for all $t > 0$. As, due to Santtinger [40], the map $t \mapsto w(\cdot, t; \varepsilon\varphi)$ is increasing, $w(x, t; w_0)$ must be bounded away from zero as $t \to \infty$. If the theorem fails for some w_0, there should exist some constant $C > 0$ such that $\|w(\cdot, t; w_0)\|_{\mathcal{C}(\bar{\Omega})} \leq C$ for all $t > 0$ and, therefore, $\|w(\cdot, t; \varepsilon\varphi)\|_{\mathcal{C}(\bar{\Omega})} \leq C$. Consequently, the function

$$w(x) := \sup_{t>0} w(x, t; w_0), \qquad x \in \bar{\Omega},$$

is well defined. Moreover, it must provide us with a positive solution of (2.4), which is impossible. This ends the proof. □

Actually, under the general assumptions of Theorem 2.4.5, $w(x, t; w_0)$ must approximate a metasolution of (2.3) as $t \to \infty$, much like in [26], but this analysis will be done in Chapter 4. In any circumstances, thanks to the proofs of Theorems 2.4.4 and 2.4.5, the following result holds.

Corollary 2.4.1. *The species w is permanent if $\Sigma(\lambda, \alpha) < 0$.*

2.5 Estimating λ_* and λ^* in some special cases.

In this section, we shall ascertain the values of λ^* and λ_* for some special, but important, classes of weight functions $a(x)$. Our first result provides us with these values for the special case analyzed by Belgacem and Cosner [6].

Theorem 2.5.1. *$\lambda^* = \infty$ and $\lambda_* = -\infty$ if $a(x) > 0$ for all $x \in \bar{\Omega}$. Consequently,* (2.4) *has a (unique) positive solution if, and only if,*

$$\lambda \in \Lambda := (-\infty, \lambda_-(\alpha)) \cup (0, \infty).$$

Moreover, the set of positive solutions of (2.4) *consists of two real analytic arcs of curve bifurcating from $w = 0$ at $\lambda = \lambda_-(\alpha)$ and $\lambda = 0$.*

Proof. It suffices to show that (2.4) admits a positive solution if $\Sigma(\lambda, \alpha) < 0$, i.e., if $\lambda > 0$ and $\lambda < \lambda_-(\alpha)$. This will be accomplished by constructing appropriate sub and supersolutions. Let λ be such that $\Sigma(\lambda, \alpha) < 0$ and denote by $\varphi \gg 0$ the unique principal eigenfunction associated to $\Sigma(\lambda, \alpha)$ normalized by $\|\varphi\|_\infty = 1$. Then, arguing as in the proof of Theorem 2.4.4, it becomes apparent that $\underline{w} = \varepsilon\varphi$ provides us with a subsolution of (2.4) for sufficiently small $\varepsilon > 0$. Moreover, it is straightforward to check that the positive constants $\bar{w} := M$ provide us with supersolutions of (2.4) as soon as

$$M^{p-1} > \|\lambda m / a e^{\alpha(p-1)m/D}\|_\infty.$$

Consequently, by enlarging M so that $M > \varepsilon \geq \underline{w}$, if necessary, it follows from the main theorem of Amann [4] that (2.4) has a solution w such that $\varepsilon\varphi \leq w \leq M$. This ends the proof the existence.

According to Theorem 2.4.1, the solution is unique if it exists. The structure of the solution set is a direct consequence from Theorem 2.4.2, as it has been already discussed in Section 2.4. □

The next result provides us with a necessary condition for the existence of a positive solution of (2.4) when

$$\Omega_0 := \text{int}\, a^{-1}(0) \neq \emptyset \quad \text{with} \quad \bar{\Omega}_0 \subset \Omega. \tag{2.25}$$

Lemma 2.5.2. *Suppose* (2.25) *holds and* (2.4) *has a positive solution. Then, for any smooth subdomain* $\mathcal{O} \subset \Omega_0$,

$$\Sigma_0(\lambda, \alpha, \mathcal{O}) := \sigma[\mathfrak{L}(\lambda, \alpha), \mathfrak{D}, \mathcal{O}] > 0.$$

Proof. Let w be a positive solution of (2.4). Then, as $w(x) > 0$ for all $x \in \bar{\Omega}$, we have that $w > 0$ on $\partial\mathcal{O}$. Moreover, under condition (2.25), condition (2.2) of [9] holds and hence, according to (3.2) of [9], we find from (2.14) that

$$\begin{aligned} 0 &= \sigma[\mathfrak{L}(\lambda, \alpha) + a e^{\alpha(p-1)m/D} w^{p-1}, \mathfrak{N}, \Omega] \\ &< \sigma[\mathfrak{L}(\lambda, \alpha) + a e^{\alpha(p-1)m/D} w^{p-1}, \mathfrak{D}, \mathcal{O}] \\ &= \sigma[\mathfrak{L}(\lambda, \alpha), \mathfrak{D}, \mathcal{O}], \end{aligned}$$

because $a = 0$ in $\mathcal{O}$. This concludes the proof. □

The next result provides us with the λ–intervals where $\Sigma_0(\lambda, \alpha, \mathcal{O}) > 0$ according to the nodal behavior of $m(x)$ in the sub-domain $\mathcal{O}$.

Proposition 2.5.1. *Suppose $\mathcal{O} \subset \Omega_0$ is a smooth domain and $\alpha > 0$. Then, the map $\lambda \mapsto \Sigma_0(\lambda, \alpha, \mathcal{O})$ is real analytic, strictly concave if $m \neq 0$ in $\mathcal{O}$, and*

$$\Sigma_0(\lambda, \alpha, \mathcal{O}) := \sigma[\mathfrak{L}(\lambda, \alpha), \mathfrak{D}, \mathcal{O}] > \sigma[\mathfrak{L}(\lambda, \alpha), \mathfrak{N}, \Omega] =: \Sigma(\lambda, \alpha) \tag{2.26}$$

for all $\lambda \in \mathbb{R}$. Moreover, $\Sigma_0(0, \alpha, \mathcal{O}) > 0$ and the following assertions are true:

(a) Suppose that m changes sign in $\mathcal{O}$. Then, there exist $\lambda_1(\mathcal{O})$ and $\lambda_2(\mathcal{O})$ such that $\lambda_1(\mathcal{O}) < \lambda_-(\alpha) < 0 < \lambda_2(\mathcal{O})$ and

$$\Sigma_0^{-1}(0) = \{\lambda_1(\mathcal{O}), \lambda_2(\mathcal{O})\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (\lambda_1(\mathcal{O}), \lambda_2(\mathcal{O})).$$

(b) Suppose that $m < 0$ in $\mathcal{O}$. Then, there exists $\lambda_1(\mathcal{O})$ such that $\lambda_1(\mathcal{O}) < \lambda_-(\alpha) < 0$ and

$$\Sigma_0^{-1}(0) = \{\lambda_1(\mathcal{O})\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (\lambda_1(\mathcal{O}), \infty).$$

(c) Suppose that $m > 0$ in $\mathcal{O}$. Then, there exists $\lambda_2(\mathcal{O})$ such that $0 < \lambda_2(\mathcal{O})$ and

$$\Sigma_0^{-1}(0) = \{\lambda_2(\mathcal{O})\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (-\infty, \lambda_2(\mathcal{O})).$$

(d) Suppose that $m = 0$ in $\mathcal{O}$. Then,

$$\Sigma_0^{-1}(0) = \emptyset, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = \mathbb{R}.$$

Proof. By definition, $\Sigma_0(\lambda, \alpha, \mathcal{O})$ is the principal eigenvalue of the linear eigenvalue problem

$$\begin{cases} -D\Delta\varphi - \alpha\nabla m \cdot \nabla\varphi - \lambda m\varphi = \sigma\varphi & \text{in } \mathcal{O}, \\ \varphi = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

Consequently, the analyticity and the concavity follow from Proposition 2.2.1 and Theorem 9.1 of [32]. The estimate (2.26) follows from (3.2) of [9], arguing as in the proof of Lemma 2.5.2. The fact that $\Sigma_0(0, \alpha, \mathcal{O}) > 0$ can be either derived from (2.26), or from Theorem 7.10 of [32], as $h := 1$ provides us with a strict positive supersolution of the tern $(-D\Delta - \alpha\nabla m \cdot \nabla, \mathfrak{D}, \mathcal{O})$. The remaining assertions of the proposition are easy consequences of the monotonicity of the principal eigenvalue with respect to the potential and of Theorem 9.1 of [32]. ☐

Figure 2.3 shows the four admissible graphs of $\Sigma_0(\lambda, \alpha, \mathcal{O})$ described by Proposition 2.5.1. We have also superimposed the graphs of $\Sigma(\lambda, \alpha)$ using a dashed curve.

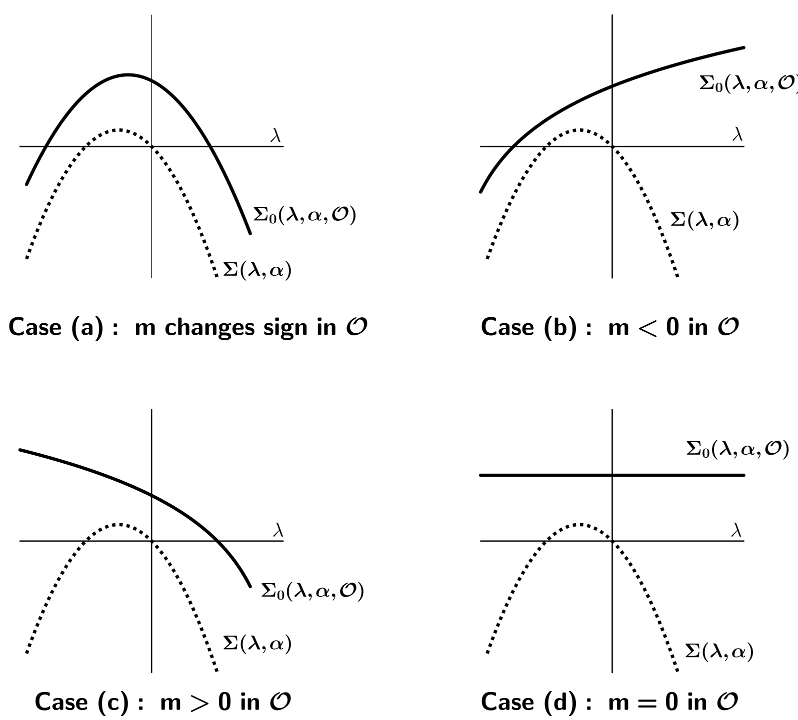

Figure 2.3: The four possible graphs of $\Sigma_0(\lambda, \alpha, \mathcal{O})$

The next theorem characterizes the existence of positive solutions of (2.4) under condition (2.25) when Ω_0 is a nice smooth domain. It should be remembered that we are taking α sufficiently large so that $\Sigma'(0,\alpha) < 0$ and hence, regarding Σ as a function of the parameter λ, we have that

$$\Sigma^{-1}(0) = \{\lambda_-(\alpha), 0\}.$$

Theorem 2.5.3. *Suppose* $\Sigma(\lambda,\alpha) < 0$ *and* (2.25) *holds for some smooth domain* Ω_0*. Then, the problem* (2.4) *admits a positive solution if, and only if,* $\Sigma_0(\lambda,\alpha,\Omega_0) > 0$*.*

Proof. Note that, due to Theorem 2.3.1, $\Sigma(\lambda,\alpha) < 0$ is necessary for the existence of a positive solution. Moreover, according to Lemma 2.5.2, we already know that $\Sigma_0(\lambda,\alpha,\Omega_0) > 0$ is necessary for the existence of a positive solution of (2.4). It remains to prove that this condition is not only necessary but also sufficient. Suppose $\Sigma_0(\lambda,\alpha,\Omega_0) > 0$. The existence of arbitrarily small positive subsolutions can be accomplished as in the proof of Theorem 2.4.4. Indeed, if φ stands for any principal eigenfunction associated to $(\mathfrak{L}(\lambda,\alpha),\mathfrak{N},\Omega)$, then $\underline{w} := \varepsilon\varphi$ is a positive subsolution of (2.4) for sufficiently small $\varepsilon > 0$.

To show the existence of arbitrarily large supersolutions we proceed as follows. Set, for sufficiently large $n \in \mathbb{N}$,

$$\Omega_n := \Omega_0 + B_{1/n}(0) = \{\, x \in \Omega \quad : \quad \mathrm{dist}(x,\Omega_0) < 1/n \,\}.$$

By (2.25), $\bar{\Omega}_n \subset \Omega$. Moreover, $\Omega_n \to \Omega_0$ from the exterior, as discussed in [9], [28] and [32, Chapter 8]. Hence,

$$\lim_{n\to\infty} \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_n] = \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_0] = \Sigma_0(\lambda,\alpha,\Omega_0) > 0.$$

Thus, since Ω_0 is a proper subdomain of Ω_n, there exists $m \in \mathbb{N}$ such that

$$0 < \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_m] < \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_0]. \tag{2.27}$$

Now, let $\varphi_m > 0$ be any a principal eigenfunction associated to $\sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_m]$ and consider any smooth function $\phi : \bar{\Omega} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} \varphi_m(x) & \text{in } \ \Omega_{2m}, \\ \psi(x) & \text{in } \ \bar{\Omega}/\Omega_{2m}, \end{cases}$$

where ψ is any smooth function satisfying

$$\inf_{\bar{\Omega}/\Omega_{2m}} \psi > 0 \quad \text{and} \quad \partial_\nu \psi = 0 \quad \text{on } \ \partial\Omega.$$

We claim that $\bar{w} := \kappa\phi$ is a supersolution of (2.4) as soon as κ is sufficiently large so that

$$\kappa^{p-1} a e^{\alpha(p-1)m/D} \psi^p > -\mathfrak{L}(\lambda,\alpha)\psi \quad \text{in } \ \bar{\Omega}\setminus\Omega_{2m}. \tag{2.28}$$

Indeed, let $x \in \Omega_{2m}$. Then, due to (2.27), we have that

$$-\sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_m] < 0 \le \kappa^{p-1} a(x) e^{\alpha(p-1)m(x)/D} \varphi_m^{p-1}(x)$$

for all $\kappa \ge 0$. Thus, multiplying by $\kappa\varphi_m$ yields

$$-\sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_m]\kappa\varphi_m(x) \le \kappa^p a(x) e^{\alpha(p-1)m(x)/D} \varphi_m^p(x),$$

or, equivalently,

$$\mathfrak{L}(\lambda,\alpha)\bar{w}(x) > -a(x)e^{\alpha(p-1)m(x)/D}\bar{w}^p(x).$$

Now, let $x \in \Omega/\Omega_{2m}$. As κ has been chosen to satisfy (2.28), we have that

$$\mathfrak{L}(\lambda,\alpha)\bar{w}(x) > -a(x)e^{\alpha(p-1)m(x)/D}\bar{w}(x)$$

since $\bar{w} = \kappa\psi$ in Ω/Ω_{2m}. Note that, by the choice of ψ, $\bar{w}$ satisfies the boundary condition. Finally, as for sufficiently small $\varepsilon > 0$, we also have that

$$\underline{w} = \varepsilon\varphi \leq \varepsilon\|\varphi\|_\infty \leq \kappa \inf_\Omega \phi \leq \kappa\phi = \bar{w}$$

the existence of a positive solution can be inferred from the main theorem of Amann [4]. This ends the proof. □

Subsequently, it should be remembered that λ_* (resp. λ^*) is the infimum (resp. supremum) of the set of $\mu < \lambda_-(\alpha)$ (resp. $\mu > 0$) such that (2.4) has a positive solution for each $\lambda \in (\mu, \lambda_-(\alpha))$ (resp. $\lambda \in (0,\mu)$). Moreover, $\lambda_1(\Omega_0) < 0$ and $\lambda_2(\Omega_0) > 0$ stand for the unique zeroes of

$$\Sigma_0(\lambda,\alpha,\Omega_0) := \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_0],$$

if some of them exists, which depends on the sign of m in Ω_0. Combining Proposition 2.5.1 with Theorems 2.3.1 and 2.5.3 yields

Theorem 2.5.4. *Suppose* (2.25) *holds for some smooth domain* Ω_0. *Then,*

(a) $\lambda_* = \lambda_1(\Omega_0) < \lambda_-(\alpha) < 0 < \lambda_2(\Omega_0) = \lambda^*$ *if* m *changes sign in* Ω_0.

(b) $\lambda_* = \lambda_1(\Omega_0) < \lambda_-(\alpha) < 0$ *and* $\lambda^* = \infty$ *if* $m < 0$ *in* Ω_0.

(c) $\lambda_* = -\infty$ *and* $\lambda^* = \lambda_2(\Omega_0)$ *if* $m > 0$ *in* Ω_0.

(d) $\lambda_* = -\infty$ *and* $\lambda^* = \infty$ *if* $m = 0$ *in* Ω_0.

Moreover, in any of these cases, (2.4) *admits a positive solution if, and only if,* $\lambda \in (\lambda_*, \lambda_-(\alpha)) \cup (0,\lambda^*)$, *and the solution is unique. Furthermore, the set of positive solutions consists of two real analytic arcs of curve bifurcating from* $w = 0$ *at* $\lambda = \lambda_-(\alpha)$ *and* $\lambda = 0$.

The previous results admit a number of extensions. Among them, the following generalization of Theorem 2.5.1, where a is allowed to vanish on a finite subset of Ω.

Theorem 2.5.5. *Suppose* $a(x_j) = 0$ *for some* $x_j \in \Omega$, $1 \leq j \leq m$, *and* $a(x) > 0$ *for all* $x \in \bar{\Omega} \setminus \{x_1,\dots,x_m\}$. *Then,* $\lambda^* = \infty$ *and* $\lambda_* = -\infty$. *Thus,* (2.4) *admits a (unique) positive solution if and only if* $\Sigma(\lambda,\alpha) < 0$; *i.e., if* $\lambda < \lambda_-(\alpha)$ *or* $\lambda > 0$. *Moreover, the set of positive solutions of* (2.4) *consists of two real analytic arcs of curve.*

Proof. Throughout this proof, for any $\delta > 0$ sufficiently small, say $\delta \leq \delta_0$, we consider a smooth subdomain of Ω, denoted by $\Omega_{0,\delta}$, satisfying $\bar{\Omega}_{0,\delta} \subset \Omega_{0,\eta} \subset \Omega$ if $0 < \delta < \eta$ and

$$\{x_1,\dots,x_m\} \subset \Omega_{0,\delta}, \qquad \lim_{\delta \downarrow 0} |\Omega_{0,\delta}| = 0. \tag{2.29}$$

These subdomains can be constructed, for example, by linking the open balls $B_\delta(x_j)$, $1 \le j \le m$, with appropriate $m-1$ thin corridors. Next, we consider the one-parameter family of weight functions

$$a_\delta := \begin{cases} a & \text{in } \bar{\Omega} \setminus \Omega_{0,2\delta}, \\ \psi & \text{in } \Omega_{0,2\delta} \setminus \Omega_{0,\delta}, \\ 0 & \text{in } \Omega_{0,\delta}, \end{cases}$$

where ψ is any continuous extension of $a|_{\bar{\Omega}\setminus\Omega_{0,2\delta}}$ to $\Omega_{0,2\delta} \setminus \Omega_{0,\delta}$ satisfying

$$0 < \psi(x) \le a(x) \quad \text{for each } \; x \in \Omega_{0,2\delta} \setminus \Omega_{0,\delta} \quad \text{and} \quad \psi = 0 \quad \text{on} \quad \partial\Omega_{0,\delta}.$$

Such a function exists because a is positive and bounded away from zero along $\partial\Omega_{0,\delta}$. By construction, we have that

$$a_\delta \le a \quad \text{in} \quad \Omega \quad \text{for all} \quad \delta \in (0, \delta_0). \tag{2.30}$$

Thus, the positive solutions of the auxiliary problem

$$\begin{cases} -D\Delta w - \alpha \nabla m \cdot \nabla w - \lambda m w = -a_\delta e^{\alpha(p-1)m/D} w^p & \text{in } \Omega, \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.31}$$

provide us with positive supersolutions of (2.4). According to Theorem 2.5.3, the problem (2.31) admits a positive solution if and only if $\Sigma_0(\lambda, \alpha, \Omega_{0,\delta}) > 0$ as soon as $\Sigma(\lambda, \alpha) < 0$. According to (2.29) and Proposition 8.6 of [32], we indeed have that $\Sigma_0(\lambda, \alpha, \Omega_{0,\delta}) > 0$ for sufficiently small $\delta > 0$. Pick one of these δ's and let $w_{\lambda,\alpha,\delta}$ denote the unique positive solution of (2.31). As $a_\delta < a$, $\bar{w} := w_{\lambda,\alpha,\delta}$ is a supersolution of (2.4). Now, let φ be a principal eigenfunction associated to $\Sigma(\lambda, \alpha)$. Then, $\underline{w} := \varepsilon\varphi$ provides us with a subsolution of (2.4) such that $\underline{w} \le \bar{w}$ for sufficiently small $\varepsilon > 0$. The existence of the positive solution follows from [4]. The uniqueness is a consequence from Theorem 2.4.1. The global structure of the set of positive steady states is a consequence from Theorem 2.4.2 and the general discussion already done in Section 2.4. □

2.6 Ascertaining $\lim_{\alpha\to\infty} \Sigma_0(\lambda, \alpha, \Omega_0)$ when $\lambda > 0$ and $m(x_+) > 0$ for some $x_+ \in \Omega_0$

Throughout this section we assume (2.21), that (2.25) holds for some smooth domain Ω_0 and that $m(x_+) > 0$ for some $x_+ \in \Omega_0$. Then, according to Theorem 2.5.3, (2.4) admits a positive solution if, and only if, $\Sigma_0(\lambda, \alpha, \Omega_0) > 0$. Moreover, by Proposition 2.5.1, the graph of the map $\lambda \mapsto \Sigma_0(\lambda, \alpha, \Omega_0)$ looks like either Case (a), or Case (c), of Figure 2.3, according to whether or not m changes sign in Ω_0. In particular, (2.4) cannot admit a positive solution for sufficiently large $\lambda > 0$ and the solutions of (2.3) blow up everywhere when $t \uparrow \infty$, as already discussed by Theorem 2.4.5. As a byproduct of the theory developed in the previous sections, it is apparent that the advection can stabilize to all these explosive solutions to an equilibrium as soon as

$$\lim_{\alpha\to\infty} \Sigma_0(\lambda, \alpha, \Omega_0) = \infty.$$

Fix λ and let $\varphi_0 > 0$ denote any principal eigenfunction associated to

$$\Sigma_0(\alpha) := \Sigma_0(\lambda, \alpha, \Omega_0).$$

Then, by definition,

$$\begin{cases} -D\Delta\varphi_0-\alpha\nabla m\cdot\nabla\varphi_0-\lambda m(x)\varphi_0=\Sigma_0(\alpha)\varphi_0 & \text{in } \Omega_0,\\ \varphi_0=0 & \text{on } \partial\Omega_0, \end{cases} \tag{2.32}$$

and, according to Berestycki et al. [7] and López-Gómez and Montenegro [35], the change of variable

$$\psi_0(x):=e^{\frac{\alpha}{2D}m(x)}\varphi_0(x), \qquad x\in\bar{\Omega}_0,$$

transforms (2.32) into the equivalent problem

$$\begin{cases} -D\Delta\psi_0+\left(\frac{\alpha}{2}\Delta m+\frac{\alpha^2}{4D}|\nabla m|^2-\lambda m\right)\psi_0=\Sigma_0(\alpha)\psi_0 & \text{in } \Omega_0,\\ \psi_0=0 & \text{on } \partial\Omega_0. \end{cases}$$

Therefore, by the uniqueness of the principal eigenvalue, we infer that

$$\Sigma_0(\alpha)=\sigma\left[-D\Delta+\frac{\alpha}{2}\Delta m+\frac{\alpha^2}{4D}|\nabla m|^2-\lambda m,\mathfrak{D},\Omega_0\right].$$

The next result provides us with two sufficient conditions so that

$$\lim_{\alpha\to\infty}\Sigma_0(\alpha)=\infty. \tag{2.33}$$

Theorem 2.6.1. *Suppose* (2.21), (2.25) *for some smooth domain* Ω_0*,* $m(x_+)>0$ *for some* $x_+\in\Omega_0$*, and one of the next two conditions is satisfied:*

(i) m *does not admit any critical point in* $\bar{\Omega}_0$*.*

(ii) m *admits finitely many critical points in* $\bar{\Omega}_0$*, say* x_j*,* $1\le j\le q$*, such that* $\Delta m(x_j)>0$ *for all* $1\le j\le q$*.*

Then, (2.33) *holds.*

Proof. As $m\in\mathcal{C}^2(\bar{\Omega}_0)$, under any of these conditions, (i) or (ii), it is apparent that

$$\frac{1}{2}\Delta m(x)+\frac{\alpha}{4D}|\nabla m(x)|^2>0 \qquad \text{for all} \quad x\in\bar{\Omega}_0,$$

for sufficiently large α, say $\alpha\ge\alpha_0$. Thus, there is a constant $C>0$ such that

$$\frac{1}{2}\Delta m(x)+\frac{\alpha}{4D}|\nabla m(x)|^2>C$$

for all $x\in\bar{\Omega}_0$ and $\alpha\ge\alpha_0$. Hence, by the monotonicity of the principal eigenvalue with respect to the potential, we find that

$$\Sigma_0(\alpha)\ge D\sigma[-\Delta,\mathfrak{D},\Omega_0]+\alpha C-\lambda m_L$$

for every $\lambda>0$ an $\alpha\ge\alpha_0$. Therefore, (2.33) holds. □

As a consequence, the next result holds. The notations introduced in the statement of Proposition 2.5.1 will be maintained.

Theorem 2.6.2. *Under the same assumptions of Theorem 2.6.1, for any given $\alpha > 0$ and every $\lambda > \lambda_2(\Omega_0) = \lambda^*(\alpha)$ and $w_0 > 0$, we have that*

$$\lim_{t\to\infty} \|w(\cdot,t;w_0)\|_{\mathcal{C}(\bar\Omega)} = \infty, \tag{2.34}$$

however, fixed any of these values of λ, there exists $\alpha_1 > \alpha$, such that $\lambda < \lambda^*(\tilde\alpha)$ for all $\tilde\alpha > \alpha_1$ and

$$\lim_{t\to\infty} \|w(\cdot,t;w_0) - w_{\lambda,\tilde\alpha}\|_{\mathcal{C}(\bar\Omega)} = 0$$

for all $\tilde\alpha > \alpha_1$. Consequently, under these circumstances, the large advection entails the existence of a unique positive steady-state of the model which, according to Theorem 2.4.4, must be a global attractor of (2.3).

Proof. By Proposition 2.5.1, $\Sigma_0(\lambda,\alpha,\Omega_0) < 0$ for all $\lambda > \lambda_2(\Omega_0)$. Thus, according to Theorem 2.5.3, (2.4) cannot admit a positive solution and, hence, (2.34) follows from Theorem 2.4.5. Moreover, owing Theorem 2.6.1, we have that $\Sigma_0(\lambda,\alpha,\Omega_0) > 0$ for sufficiently large α. Finally, Theorems 2.5.3 and 2.4.4 conclude the proof. □

Subsequently, we will analyze the sharp behavior of $\Sigma_0(\alpha)$ as $\alpha \to \infty$ in some special cases of interest. First, as in Berestycki et al. [7], we will assume that

$$\Delta m = 0 \qquad \text{in} \quad \Omega_0. \tag{2.35}$$

Then,

$$\begin{aligned}\Sigma_0(\alpha) &= \sigma\left[-D\Delta + \frac{\alpha^2}{4D}|\nabla m|^2 - \lambda m, \mathfrak{D}, \Omega_0\right] \\ &= \alpha^2\sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2 - \frac{\lambda}{\alpha^2}m, \mathfrak{D}, \Omega_0\right]\end{aligned}$$

for all $\alpha > 0$. In this case, the next result holds. Subsequently, we denote by S^{N-1} the $(N-1)$–dimensional unit sphere.

Theorem 2.6.3. *Suppose* (2.21), (2.25) *for some nice smooth domain Ω_0, and* (2.35). *Then,*

$$\lim_{\alpha\to\infty} \frac{\Sigma_0(\alpha)}{\alpha^2} = \lim_{\alpha\to\infty} \sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2, \mathfrak{D}, \Omega_0\right]. \tag{2.36}$$

Moreover,

$$\lim_{\alpha\to\infty} \sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2, \mathfrak{D}, \Omega_0\right] = \frac{1}{4D}\min_{\bar\Omega_0} |\nabla m|^2. \tag{2.37}$$

Furthermore, if m possesses a unique critical point in Ω_0, say x_0, such that

$$|\nabla m(x)|^2 = |x-x_0|^\beta g(\omega) + o(|x-x_0|^\beta) \qquad \text{as} \;\; x \to x_0, \tag{2.38}$$

for some $\beta \geq 2$, where $\omega := (x-x_0)/|x-x_0| \in S^{N-1}$ and $g : S^{N-1} \to \mathbb{R}_+$ is bounded and positive (bounded away from zero), then

$$\lim_{\alpha\to\infty} \frac{\sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2, \mathfrak{D}, \Omega_0\right]}{\left(\frac{D}{\alpha^2}\right)^{\frac{\beta}{2+\beta}}} = s_D > 0, \tag{2.39}$$

where s_D stands for the spectral bound of the Schrödinger operator

$$-\Delta + \frac{1}{4D} g(y/|y|)|y|^\beta, \qquad y \in \mathbb{R}^N.$$

Proof. By the monotonicity of the principal eigenvalue with respect to the potential, we have that

$$\sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2, \mathfrak{D}, \Omega_0\right] - \frac{\lambda}{\alpha^2} m_M \le \frac{\Sigma_0(\alpha)}{\alpha^2}$$
$$\le \sigma\left[-\frac{D}{\alpha^2}\Delta + \frac{1}{4D}|\nabla m|^2, \mathfrak{D}, \Omega_0\right] - \frac{\lambda}{\alpha^2} m_L$$

for all $\lambda > 0$, where we have denoted

$$m_L := \min_{\bar\Omega_0} m, \qquad m_M := \max_{\bar\Omega_0} m.$$

By letting $\alpha \to \infty$, (2.36) holds.

The validity of (2.37) is a direct consequence from Lemma 3.1 of Furter and López-Gómez [16] and Theorem 3.1 of Dancer and López-Gómez [13]. Identity (2.39) is a direct consequence from Theorem 4.1 of [13]. □

According to Theorem 2.6.3, we have that

$$\Sigma_0(\alpha) \sim \frac{1}{4D} \min_{\bar\Omega_0} |\nabla m|^2 \alpha^2 \qquad \text{as } \alpha \to \infty$$

if ∇m does not vanish in $\bar\Omega_0$, while if

$$(\nabla m)^{-1}(0) \cap \bar\Omega_0 = \{x_0\} \tag{2.40}$$

satisfies (2.38), then

$$\Sigma_0(\alpha) \sim s_D \alpha^2 \left(\frac{D}{\alpha^2}\right)^{\frac{\beta}{2+\beta}} = s_D D^{\frac{\beta}{2+\beta}} \alpha^{\frac{4}{2+\beta}} \qquad \text{as } \alpha \to \infty$$

and, in particular,

$$\lim_{\alpha\to\infty} \Sigma_0(\alpha) = \infty. \tag{2.41}$$

Consequently, much like Theorem 2.6.2, the next result holds.

Theorem 2.6.4. *Suppose* (2.21), (2.25) *for some smooth domain Ω_0, and m satisfies* (2.35), (2.40) *and* (2.38) *for some $\beta \ge 2$, and $m(x_+) > 0$ for some $x_+ \in \Omega_0$. Then, all the conclusions of Theorem 2.6.2 hold true.*

As discussed by Berestycki et al. [7], given a vector field v, a function $w \in H_0^1(\Omega_0)$, $w \neq 0$, is said to be a first integral of v if $\langle v, \nabla w\rangle = 0$ almost everywhere in Ω_0. Let $\mathcal{I}_0$ denote the set of first integrals of the vector field v in Ω_0. Then, according to the main theorem of [7], one has that

$$\lim_{\alpha\to\infty} \sigma[-\Delta + \alpha v\nabla, \mathfrak{D}, \Omega_0] = \infty \qquad \text{if } \mathcal{I}_0 = \emptyset,$$

whereas

$$\lim_{\alpha\to\infty}\sigma[-\Delta+\alpha v\nabla,\mathfrak{D},\Omega_0]=\inf_{w\in\mathcal{I}_0}\frac{\int_{\Omega_0}|\nabla w|^2}{\int_{\Omega_0}w^2}\qquad \text{if } \mathcal{I}_0\neq\emptyset.$$

Consequently, according to (2.41), we have actually proven that $-\nabla m$ cannot admit a first integral.

More generally, the following result holds from Corollary 4.3 of Dancer and López-Gómez [13].

Theorem 2.6.5. *Suppose* (2.21), (2.25) *for some smooth domain* Ω_0*,* $m\in\mathcal{C}^2(\bar\Omega_0)$ *satisfies* $\Delta m=0$ *in* Ω_0*,*

$$(\nabla m)^{-1}(0)\cap\bar\Omega_0=\{x_0,\dots,x_p\},\qquad p\geq 0,$$

and, for every $j\in\{0,\dots,p\}$*,*

$$|\nabla m(x)|^2=|x-x_j|^{\beta_j}g_j(\omega_j)+o(|x-x_j|^{\beta_j})\qquad \textit{as }\ x\to x_j,$$

with $\beta_j\geq 2$*,* $\omega_j=(x-x_j)/|x-x_j|$*,* $x\sim x_j$*,* $x\neq x_j$*, and* $g_j:S^{N-1}\to\mathbb{R}_+$ *is bounded and positive (bounded away from zero) for all* $j\in\{0,\dots,p\}$*. Set*

$$\beta:=\max_{0\leq j\leq p}\beta_j$$

and let $\{i_1,...,i_q\}\subset\{0,...,p\}$ *be the set of indices for which*

$$\beta_j=\beta,\qquad j\in\{i_1,...,i_q\}.$$

Then,

$$\lim_{\alpha\to\infty}\frac{\sigma\left[-\frac{D}{\alpha^2}\Delta+\frac{1}{4D}|\nabla m|^2,\mathfrak{D},\Omega_0\right]}{\left(\frac{D}{\alpha^2}\right)^{\frac{\beta}{2+\beta}}}=s_D:=\min_{1\leq j\leq q}s_{D,j}>0,$$

where $s_{D,j}$*,* $1\leq j\leq q$*, stands for the spectral bound of the Schrödinger operator*

$$-\Delta+\frac{1}{4D}g_{i_j}(y/|y|)|y|^{\beta},\qquad y\in\mathbb{R}^N.$$

Naturally, under the general assumptions of Theorem 2.6.5, it is easy to see that (2.41) holds, as well as Theorem 2.6.4, and, in particular, owing to Berestycki et al. [7], $-\nabla m$ cannot admit a first integral neither.

In the more general case when m is not harmonic in Ω_0, the following result holds

Theorem 2.6.6. *Suppose* (2.21), (2.25) *for some smooth domain* Ω_0*, and* m *satisfies* $m(x_+)>0$ *for some* $x_+\in\Omega_0$*,*

$$\Delta m\geq 0\qquad \textit{in }\ \Omega_0, \tag{2.42}$$

and it possesses finitely many critical points in $\bar\Omega_0$*, say* $\{x_0,...,x_p\}\subset\Omega_0$*. Then, for every* $\lambda>0$*, we have that*

$$\lim_{\alpha\to\infty}\Sigma_0(\alpha)=\infty.$$

Therefore, the conclusions of Theorem 2.6.2 also hold.

Proof. Let Ω_δ, $\delta>0$, be a family of smooth open connected subsets of Ω_0 such that

$$\{x_0,...,x_p\}\subset\Omega_\delta \quad \text{and} \quad \lim_{\delta\to 0}|\Omega_\delta|=0.$$

Fix $\delta>0$ and let $V_\delta>0$ a continuous function such that $V_\delta^{-1}(0)=\bar\Omega_\delta$ and $|\nabla m|^2>V_\delta$ in Ω_0. Then, due to (2.42), by the monotonicity of the principal eigenvalue with respect to the domain, we find that

$$\Sigma_0(\alpha)\geq\sigma\left[-D\Delta+\frac{\alpha^2}{4D}V_\delta,\mathfrak{D},\Omega_0\right]-\lambda m_M.$$

Therefore, according to Theorem 3.3 of [27], it follows that

$$\lim_{\alpha\to\infty}\Sigma_0(\alpha)\geq\sigma[-D\Delta,\mathfrak{D},\Omega_\delta]-\lambda m_M.$$

As this estimate holds true for all $\delta>0$ and, thanks to the Faber–Krahn inequality,

$$\lim_{\delta\to 0}\sigma[-D\Delta,\mathfrak{D},\Omega_\delta]=\infty$$

(see Theorem 9.5 of [32], if necessary). This estimate ends the proof. □

Chapter 3

Concentration through large advection

3.1 Introduction

In this chapter we suppose that

$$m(x_+) > 0, \qquad m(x_-) < 0 \qquad \text{for some} \quad x_+, x_- \in \Omega. \tag{3.1}$$

The results are analogous for $m > 0$. Moreover, we suppose that

$$\frac{\partial m}{\partial \nu}(x) \leq 0 \qquad \text{for all} \;\; x \in \partial\Omega,$$

and as in [10], in order to get the a priori bounds we need to impose that the set of critical points of m, denoted by

$$\mathcal{C}_m := \left\{ x \in \bar{\Omega} \;\; / \;\; \nabla m(x) = 0 \right\}, \tag{3.2}$$

is finite and that it consists of non-degenerate critical points. Moreover, the set of points $x \in \bar{\Omega}$ where m has some local maximum, denoted throughout this chapter by $\mathcal{M}_m$, is assumed to satisfy

$$\mathcal{M}_m = \left\{ z_i^j \;\; : \;\; 1 \leq j \leq n_i, \;\; 1 \leq i \leq q \right\} \subset \Omega \tag{3.3}$$

for some integers $q \geq 1$ and $n_i \geq 1$, $1 \leq i \leq q$, such that

$$m_1 := m(z_1^j) < m_2 := m(z_2^j) < \cdots < m_q := m(z_q^j), \qquad 1 \leq j \leq n_i, \tag{3.4}$$

$$a(z_i^j) > 0, \qquad 1 \leq i \leq q, \quad 1 \leq j \leq n_i. \tag{3.5}$$

Note that

$$m(z_i^j) = m_i, \qquad 1 \leq j \leq n_i, \quad 1 \leq i \leq q.$$

Moreover, we also need to impose

$$\Delta m(x) > 0 \qquad \text{for all} \quad x \in \mathcal{C}_m \setminus \mathcal{M}_m,$$

$$m(x) \in \mathbb{R} \setminus \{m_1, ..., m_{q-1}\} \qquad \text{for all} \quad x \in \partial\Omega \quad \text{if} \quad q \geq 2. \tag{3.6}$$

When condition (3.5) fails, the solutions are unbounded around the z_i^j's where $a(z_i^j) = 0$ and hence, the a priori bounds fail. Consequently, (3.5) is optimal to get upper estimates for the solutions of (1.2). Condition (3.6), which has an evident geometrical meaning, is imperative in order to apply the theorem of characterization of the strong maximum of Amann and López-Gómez [5], as well as the properties of the principal eigenvalues of Cano-Casanova and López-Gómez [9], which have been recently polished and collected in [32]. Very specially, is necessary for applying Proposition 3.2 of [9], which can be stated as follows.

Proposition 3.1.1. *Suppose:*

(a) $\mathfrak{L}$ *is a second order uniformly elliptic operator in a smooth bounded domain* Ω *of* $\mathbb{R}^N$, $N \geq 1$.

(b) $\partial\Omega = \Gamma_0 \cup \Gamma_1$, *where* Γ_0 *and* Γ_1 *are two disjoint open and closed subsets of* $\partial\Omega$.

(c) $\mathfrak{B}$ *is a mixed boundary operator of the form*

$$\mathfrak{B}\psi := \begin{cases} \psi & \text{on } \Gamma_0, \\ \partial_\nu \psi + \beta\psi & \text{on } \Gamma_1, \end{cases} \qquad \psi \in \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\Omega \cup \Gamma_1),$$

where $\nu = An$ *is the co-normal vector field and* $\beta \in \mathcal{C}(\Gamma_1)$.

(d) $D \subset \Omega$ *is a proper smooth subdomain of* Ω *such that*

$$\text{dist}\,(\Gamma_1, \partial D \cap \Omega) > 0 \tag{3.7}$$

and $\mathfrak{B}_0$ *stands for the boundary operator*

$$\mathfrak{B}_0\psi := \begin{cases} \psi & \text{on } \partial D \cap \Omega, \\ \partial_\nu \psi + \beta\psi & \text{on } \partial D \cap \partial\Omega. \end{cases}$$

Then, the associated principal eigenvalues satisfy

$$\sigma[\mathfrak{L}, \mathfrak{B}, \Omega] < \sigma[\mathfrak{L}, \mathfrak{B}_0, D].$$

This result is pivotal in building up the upper estimates for the positive solutions of (1.2) as $\alpha \to \infty$, because it should be used in a number of occasions along the proof of the main theorem to get the positivity of $\sigma[\mathfrak{L}, \mathfrak{B}_0, D]$ from the identity $\sigma[\mathfrak{L}, \mathfrak{B}, \Omega] = 0$. The positivity of $\sigma[\mathfrak{L}, \mathfrak{B}_0, D]$ ensures that $(\mathfrak{L}, \mathfrak{B}_0, D)$ satisfies the strong maximum principle, by Theorem 2.4 of Amann and López-Gómez [5].

In the proof of the main results of this chapter, for some specific domains D, condition (3.6) entails (3.7), as it guarantees that any component Γ of ∂D either it is a component of $\partial\Omega$, or $\Gamma \subset \Omega$. In particular, (3.7) holds. Incidentally, as (3.6) was never imposed by Chen, Lam and Lou in [10] whenever they had to invoke to Proposition 3.1.1, the proofs of [10] contain some gaps. Indeed, for applying Proposition 3.1.1 one needs condition (3.7) to avoid singularities on the boundary, or components where the boundary conditions are mixed in a non-regular way, where Theorem 2.4 of Amann and López-Gómez [5] cannot be applied straightaway. Consequently, giving complete technical details in all the proofs of this chapter seems a categorical imperative.

The distribution of this chapter is the following. Section 3.2 collects some important results from Chapter 2, where the dynamics of (1.1) was analyzed. Section 3.3 adapts the upper estimates of [10] to our general setting. Finally, Section 3.4 derives some extremely useful lower estimates. These estimates show that condition (3.5) is necessary to get the upper estimates of Section 3.2 and that, actually, the problem (1.2) cannot admit a positive solution for sufficiently large α if (3.5) fails, which, in particular, sharpens the existence results of Chapter 2 (see (P6) above).

3.2 Notations and preliminaries

The next theorems collect the main existence results available for (1.2).

Theorem 3.2.1. *Suppose $a(x) > 0$ for all $x \in \bar{\Omega}$. Then, the following assertions are equivalent:*

- *(a)* (1.2) *admits a positive solution.*
- *(b)* $\Sigma(\lambda,\alpha) := \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{N},\Omega] < 0$*, where* $\mathfrak{L}(\lambda,\alpha) := -D\Delta - \alpha\nabla m\cdot\nabla - \lambda m$.
- *(c)* $u = 0$ *is linearly unstable.*

Moreover, the positive solution, denoted by $u_{\lambda,\alpha}$, is unique if it exists and it is a global attractor for (1.1)*. Furthermore,* $\lim_{t\to\infty}\|u(\cdot,t;u_0)\|_{\mathcal{C}(\bar{\Omega})} = 0$ *for all $u_0 > 0$ if $\Sigma(\lambda,\alpha) > 0$.*

Theorem 3.2.2. *Suppose $\Omega_0 := \operatorname{int} a^{-1}(0)$ is a smooth subdomain of Ω with $\bar{\Omega}_0 \subset \Omega$. Then,* (1.2) *admits a positive solution if, and only if,*

$$\Sigma(\lambda,\alpha) := \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{N},\Omega] < 0 \quad \textit{and} \quad \Sigma_0(\lambda,\alpha) := \sigma[\mathfrak{L}(\lambda,\alpha),\mathfrak{D},\Omega_0] > 0.$$

Moreover, the positive solution, denoted by $u_{\lambda,\alpha}$, is unique if it exists and it is a global attractor for (1.1)*. Furthermore, for every $u_0 > 0$, we have that*

$$\lim_{t\to\infty}\|u(\cdot,t;u_0)\|_{\mathcal{C}(\bar{\Omega})} = 0 \qquad \textit{if} \quad \Sigma(\lambda,\alpha) > 0,$$

whereas

$$\lim_{t\to\infty}\|u(\cdot,t;u_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty \qquad \textit{if} \quad \Sigma_0(\lambda,\alpha) < 0.$$

Theorem 3.2.1 goes back to Belgacem and Cosner [6], while Theorem 3.2.2 was found by the authors in Chapter 2, where it was also established that, for sufficiently large α, say $\alpha \geq \alpha_0$, there exists $\lambda_-(\alpha) < 0$ such that $\Sigma(\lambda,\alpha) < 0$ if and only if $\lambda > 0$ or $\lambda < \lambda_-(\alpha)$. Throughout the rest of this chapter we will assume that $\alpha \geq \alpha_0$.

The next result, which is Proposition 3 of Chapter 2, provides us with the λ–intervals where $\Sigma_0(\lambda,\alpha) > 0$ according to the nodal behavior of $m(x)$ in Ω_0.

Proposition 3.2.1. *The map $\lambda \mapsto \Sigma_0(\lambda,\alpha)$ is real analytic, strictly concave if, in addition, $m \not\equiv 0$ in Ω_0, and it satisfies $\Sigma_0(\lambda,\alpha) > \Sigma(\lambda,\alpha)$ for all $\lambda \in \mathbb{R}$ and $\alpha > 0$. Thus, $\Sigma_0(0,\alpha) > \Sigma(0,\alpha) = 0$ and, setting*

$$\Sigma_0^{-1} := \Sigma_0(\cdot,\alpha)^{-1},$$

the following assertions are true:

(a) *Suppose* m *changes sign in* Ω_0*. Then, there exist* $\lambda_*(\alpha)$ *and* $\lambda^*(\alpha)$ *such that* $\lambda_*(\alpha) < \lambda_-(\alpha) < 0 < \lambda^*(\alpha)$ *and*

$$\Sigma_0^{-1}(0) = \{\lambda_*(\alpha), \lambda^*(\alpha)\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (\lambda_*(\alpha), \lambda^*(\alpha)).$$

(b) *Suppose* $m < 0$ *in* Ω_0*. Then, there is* $\lambda_*(\alpha) < \lambda_-(\alpha) < 0$ *such that*

$$\Sigma_0^{-1}(0) = \{\lambda_*(\alpha)\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (\lambda_*(\alpha), \infty).$$

(c) *Suppose* $m > 0$ *in* Ω_0*. Then, there is* $\lambda^*(\alpha) > 0$ *such that*

$$\Sigma_0^{-1}(0) = \{\lambda^*(\alpha)\}, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = (-\infty, \lambda^*(\alpha)).$$

(d) *Suppose* $m = 0$ *in* Ω_0*. Then,*

$$\Sigma_0^{-1}(0) = \emptyset, \qquad \Sigma_0^{-1}(\mathbb{R}_+) = \mathbb{R}.$$

Figure 3.1 shows each of the possible graphs of $\Sigma_0(\lambda, \alpha)$ according to the sign of $m(x)$ in Ω_0. In all cases, we have superimposed the graph of $\Sigma(\lambda, \alpha)$ by using a dashed line, while the graph of $\Sigma_0(\lambda, \alpha)$ has been plotted with a continuous line.

The next result provides us with $\lim_{\alpha\uparrow\infty} \Sigma(\lambda, \alpha)$ in the special case when all the critical points of $m(x)$ are non-degenerate. It is a very sharp result going back to Chen and Lou [11].

Theorem 3.2.3. *Assume that all critical points of* $m(x)$ *are non-degenerate. Then,*

$$\lim_{\alpha\to\infty} \Sigma(\lambda, \alpha) = \min_{x\in\mathcal{M}_m} \{-\lambda m(x)\} \qquad \text{for all } \lambda \in \mathbb{R}. \tag{3.8}$$

As all the critical points of $m(x)$ are non-degenerate, $\mathcal{C}_m$ must be finite. Suppose

$$\mathcal{M}_m = \{x_1, \ldots, x_h\} \quad \text{and} \quad \{m(x_j) \ : \ 1 \le j \le h\} = \{m_1, \ldots, m_k\}$$

with $m_j < m_{j+1}$, $1 \le j \le k-1$. Then, according to (3.8), we have that

$$\lim_{\alpha\to\infty} \Sigma(\lambda, \alpha) = \begin{cases} -\lambda m_1 & \text{if } \lambda < 0, \\ 0 & \text{if } \lambda = 0, \\ -\lambda m_k & \text{if } \lambda > 0. \end{cases} \tag{3.9}$$

By (3.1), we always have that $m_k > 0$. Thus, the curve $\Sigma(\lambda, \alpha)$, $\lambda > 0$, approximates the line $-\lambda m_k$, $\lambda > 0$, with negative slope $-m_k < 0$, as $\alpha \to \infty$. To ascertain the behavior of $\Sigma(\lambda, \alpha)$ for $\lambda < 0$, we have to distinguish two different cases, according to the sign of m_1. By (3.9), when $m_1 > 0$, the half-curve $\Sigma(\lambda, \alpha)$, $\lambda < 0$, approximates a line with negative slope $-m_1 < 0$, much like in the previous case, which entails

$$\lim_{\alpha\to\infty} \lambda_-(\alpha) = -\infty. \tag{3.10}$$

Thus, according to Theorems 3.2.1 and 3.2.2, u becomes extinct for sufficiently large α if $\lambda < 0$, as for such values $\Sigma(\lambda, \alpha) > 0$.

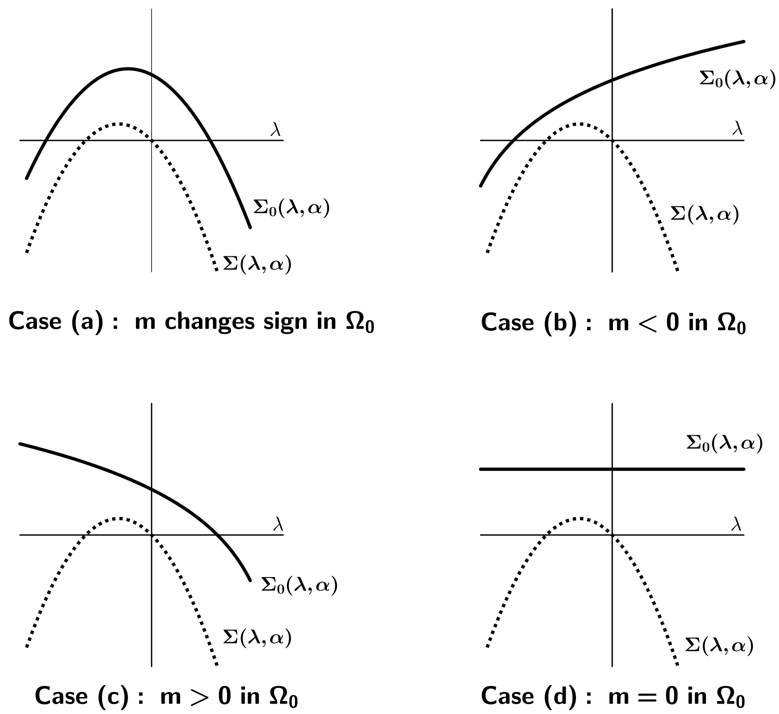

Figure 3.1: The four possible graphs of $\Sigma_0(\lambda, \alpha)$

However, the situation reverses when $m_1 < 0$, as in such case

$$\lim_{\alpha\to\infty} \lambda_-(\alpha) = 0 \tag{3.11}$$

and, hence, for any $\lambda < 0$, we have that $\Sigma(\lambda, \alpha) < 0$ for sufficiently large α. Consequently, at least in the special case when $a(x) > 0$ for all $x \in \bar{\Omega}$, due to Theorem 3.2.1, we have that

$$\lim_{t\to\infty} \|u(\cdot, t; u_0) - u_{\lambda,\alpha}\|_\infty = 0$$

for $\lambda < 0$ and sufficiently large α.

To ascertain what's going on under the general assumptions of Theorem 3.2.2, one must invoke to Theorem 2.6.1, which can be stated as follows.

Theorem 3.2.4. *Suppose* $\Omega_0 := \operatorname{int} a^{-1}(0)$ *is a smooth subdomain of* Ω *with* $\bar{\Omega}_0 \subset \Omega$ *and that either*

$\mathcal{C}_m \cap \bar{\Omega}_0 = \emptyset$, or $\mathcal{C}_m \cap \bar{\Omega}_0 = \{y_1, \dots, y_q\}$ with $\Delta m(y_j) > 0$ for all $1 \le j \le q$. Then,

$$\lim_{\alpha \to \infty} \Sigma_0(\lambda, \alpha) = \infty \qquad \text{for all} \;\; \lambda \in \mathbb{R}.$$

In particular, for any $\lambda \in \mathbb{R}$, $\Sigma_0(\lambda, \alpha) > 0$ for sufficiently large $\alpha > 0$.

According to (3.11) and Theorem 3.2.4, for every $\lambda \in \mathbb{R} \setminus \{0\}$, we have that $\Sigma(\lambda, \alpha) < 0$ and $\Sigma_0(\lambda, \alpha) > 0$ for sufficiently large α, provided $m_1 < 0$. Therefore, in such case, the dynamics of (1.1) is regulated by $u_{\lambda,\alpha}$. The next result collects all the possible results in this direction.

Theorem 3.2.5. *Suppose $\alpha \ge \alpha_0$ and that either $a(x) > 0$ for all $x \in \bar{\Omega}$, or $\Omega_0 := \operatorname{int} a^{-1}(0)$ is a smooth subdomain of Ω with $\bar{\Omega}_0 \subset \Omega$ and either $\mathcal{C}_m \cap \bar{\Omega}_0 = \emptyset$, or $\mathcal{C}_m \cap \bar{\Omega}_0 = \{y_1, \dots, y_q\}$ with $\Delta m(y_j) > 0$ for all $1 \le j \le q$. Then:*

- *(a) For any given $\lambda \neq 0$, the dynamics of (1.1) is regulated by $u_{\lambda,\alpha}$ for sufficiently large α if $m_1 < 0$.*
- *(b) For any given $\lambda > 0$, the dynamics of (1.1) is regulated by $u_{\lambda,\alpha}$ for sufficiently large α if $m_1 > 0$.*
- *(c) For any given $\lambda < 0$, the dynamics of (1.1) is regulated by 0 for sufficiently large α if $m_1 > 0$.*

3.3 Upper estimates

The main result of this section can be stated as follows.

Theorem 3.3.1. *Suppose:*

- *(a) $\Omega_0 := \operatorname{int} a^{-1}(0)$ is a smooth subdomain of Ω with $\bar{\Omega}_0 \subset \Omega$.*
- *(b) Any critical point of m is non-degenerate.*
- *(c) $\frac{\partial m(x)}{\partial \nu} \le 0$ for all $x \in \partial\Omega$.*
- *(d) $\mathcal{M}_m$ satisfies (3.3) and (3.4) with $a(z_i^j) > 0$ for all $1 \le i \le q$ and $1 \le j \le n_i$.*
- *(e) $\Delta m(x) > 0$ for all $x \in \mathcal{C}_m \setminus \mathcal{M}_m$.*
- *(f) $m(x) \in \mathbb{R} \setminus \{m_1, \dots, m_{q-1}\}$ for all $x \in \partial\Omega$ if $q \ge 2$.*

Then, for any given $\lambda \neq 0$ and $D > 0$ where (1.2) admits a positive solution, there exist $\delta \in (0,1)$, $\tilde{\alpha} \ge \alpha_0$, $C > 0$, $r_1 > 0$ and $\beta > 0$ such that

$$u_{\lambda,\alpha}(x) \le \begin{cases} Ce^{\alpha\delta[m(x)-m_i)]/D} & \text{in } \bigcup_{1 \le j \le n_i} B_{r_1}(z_i^j), \qquad 1 \le i \le q, \\ Ce^{-\frac{\beta}{D}\alpha} & \text{in } \bar{\Omega} \setminus \bigcup_{\substack{1 \le i \le q \\ 1 \le j \le n_i}} B_{r_1}(z_i^j), \end{cases} \tag{3.12}$$

for all $\alpha \ge \tilde{\alpha}$. In particular,

$$\lim_{\alpha \to \infty} \|u_{\lambda,\alpha}\|_{\mathcal{C}(\bar{\Omega} \setminus \bigcup_{\substack{1 \le i \le q \\ 1 \le j \le n_i}} B_{r_1}(z_i^j))} = 0. \tag{3.13}$$

Suppose in addition that $m_\kappa \leq 0$, $m_{\kappa+1} > 0$, *for some* $1 \leq \kappa \leq q-1$. *Then, for every* $\lambda > 0$, (3.12) *can be sharpened up to get*

$$u_{\lambda,\alpha}(x) \leq Ce^{-\frac{\beta}{D}\alpha} \quad \textit{in} \quad \cup_{\substack{1\leq j\leq n_j \\ 1\leq i\leq \kappa}} B_{r_1}(z_i^j). \tag{3.14}$$

Proof. The proof will be divided into four steps. In Step 1 we will obtain a useful estimate involving m. In Step 2 we will use the strong maximum principle, in a rather sophisticated way, to compare $u_{\lambda,\alpha}$ with certain exponentials involving $\|u_{\lambda,\alpha}\|_\infty$. In Step 3 we will get uniform a priori bounds for $\|u_{\lambda,\alpha}\|_\infty$ by using the Harnack inequality. Finally, in step 4 we will show (3.14).

Step 1: This step shows that there exist $L > 0$, $\delta \in (0,1)$, $h > 0$ and $\alpha_1 \geq \alpha_0$ such that, for every $\alpha \geq \alpha_1$,

$$\delta\frac{\alpha}{D}|\nabla m(x)|^2 + \Delta m(x) \geq L \quad \text{for all} \;\; x \in \bar{\Omega} \setminus \cup_{z\in\mathcal{M}_m} \bar{B}_{h\sqrt{\frac{D}{\alpha}}}(z). \tag{3.15}$$

As z_i^j is non-degenerate, there exist two positive constants $K > 0$ and $r > 0$ such that

$$\bigcup_{\substack{1\leq i\leq q \\ 1\leq j\leq n_i}} \bar{B}_r(z_i^j) \subset \Omega, \qquad \bar{B}_r(z_i^j) \cap \bar{B}_r(z_{\hat{i}}^{\hat{j}}) = \emptyset \quad \text{if} \;\; (i,j) \neq (\hat{i},\hat{j}),$$

and, for all $x \in \bar{B}_r(z_i^j) \subset \Omega$, $1 \leq i \leq q$, $1 \leq j \leq n_i$,

$$K^{-1}|x - z_i^j|^2 \leq m_i - m(x) \leq K|x - z_i^j|^2, \tag{3.16}$$

$$K^{-1}|x - z_i^j| \leq |\nabla m(x)| \leq K|x - z_i^j|. \tag{3.17}$$

Note that K can be taken arbitrarily large.

By the hypothesis (f), $m(x) \neq m_i$ for all $1 \leq i \leq q-1$ and $x \in \partial\Omega$. Hence, there is $\tau > 0$ such that

$$\min_{1\leq i\leq q-1} |m(x) - m_i| \geq \tau \qquad \text{for all} \quad x \in \partial\Omega. \tag{3.18}$$

Set $m_0 := \min_{\bar{\Omega}} m$ and pick $\eta > 0$ satisfying

$$\eta < \min_{1\leq i\leq q} \left\{m_i - m_{i-1}, r^2/K, \tau\right\} \tag{3.19}$$

and such that $m_i - \eta$, $1 \leq i \leq q$, are regular values of m; such value exists because $\mathcal{C}_m$ is finite. Pick an arbitrary $0 < \delta_1 < 1$ and consider, recursively,

$$\delta_{i+1} := \frac{\eta}{m_{i+1} - m_i + \eta}\delta_i \qquad i = 1, 2, .., q-1. \tag{3.20}$$

As $m_{i+1} > m_i$, we have that $\delta_{i+1} < \delta_i$. Moreover, the δ of the theorem is given by

$$\delta := \delta_q = \delta_1 \prod_{i=1}^{q-1} \frac{\eta}{m_{i+1} - m_i + \eta}.$$

Set

$$L := \min_{y \in \mathcal{C}_m \setminus \mathcal{M}_m} \Delta m(y)/2.$$

According to the hypothesis (e), $L > 0$. Moreover, as $\mathcal{C}_m$ is finite, for every $\varepsilon \in (0, r)$, there exists $\alpha_1 = \alpha_1(\varepsilon) \geq \alpha_0$ such that, for every $\alpha \geq \alpha_1$,

$$\delta \frac{\alpha}{D} |\nabla m(x)|^2 + \Delta m(x) \geq L \qquad \forall\, x \in \bar{\Omega} \setminus \bigcup_{z \in \mathcal{M}_m} B_\varepsilon(z). \tag{3.21}$$

Indeed, as $m \in \mathcal{C}^2(\bar{\Omega})$, m admits a $\mathcal{C}^2(\mathbb{R}^N)$–extension, also denoted by m. Then, by continuity, there is $R > 0$ such that

$$\bar{B}_R(y) \cap \bar{B}_r(z) = \emptyset \qquad \text{and} \qquad \bar{B}_R(y) \cap \bar{B}_R(\tilde{y}) = \emptyset$$

for every $z \in \mathcal{M}_m$, $y, \tilde{y} \in \mathcal{C}_m \setminus \mathcal{M}_m$, $y \neq \tilde{y}$, and

$$\Delta m \geq L \qquad \text{in} \quad \bigcup_{y \in \mathcal{C}_m \setminus \mathcal{M}_m} \bar{B}_R(y).$$

Thus, for every $\alpha > 0$, we have that

$$\delta \frac{\alpha}{D} |\nabla m(x)|^2 + \Delta m(x) \geq L \qquad \forall\, x \in \bigcup_{y \in \mathcal{C}_m \setminus \mathcal{M}_m} \bar{B}_R(y).$$

As $\inf_{\tilde{\Omega}_\varepsilon} |\nabla m| > 0$ in the region

$$\tilde{\Omega}_\varepsilon := \bar{\Omega} \setminus \left[\bigcup_{z \in \mathcal{M}_m} B_\varepsilon(z) \bigcup_{y \in \mathcal{C}_m \setminus \mathcal{M}_m} B_R(y) \right],$$

it is apparent that there exists $\alpha_1 \geq \alpha_0$ such that (3.21) holds for all $\alpha \geq \alpha_1$. Actually, there exists $h > 0$ such that we can take

$$\alpha_1(\varepsilon) = D(h/\varepsilon)^2 \qquad \text{for all} \quad \varepsilon > 0. \tag{3.22}$$

Indeed, according to (3.17), for every $x \in \bar{B}_r(z) \setminus B_\varepsilon(z)$ with $z \in \mathcal{M}_m$, we have that

$$\delta \frac{\alpha}{D} |\nabla m(x)|^2 + \Delta m(x) \geq \delta \frac{\alpha}{D} K^{-2} |x - z|^2 + \min_{\bar{\Omega}} \Delta m \geq \delta \frac{\alpha}{D} K^{-2} \varepsilon^2 + \min_{\bar{\Omega}} \Delta m.$$

Thus,

$$\delta \frac{\alpha}{D} |\nabla m(x)|^2 + \Delta m(x) \geq L \qquad \forall\, x \in \bar{B}_r(z) \setminus B_\varepsilon(z) \tag{3.23}$$

provided

$$\delta \frac{\alpha}{D} K^{-2} \varepsilon^2 + \min_{\bar{\Omega}} \Delta m \geq L \quad \iff \quad \alpha \geq (L - \min_{\bar{\Omega}} \Delta m) K^2 \varepsilon^{-2} D \delta^{-1}.$$

Hence, choosing (3.22) with

$$h := K \sqrt{(L - \min_{\bar{\Omega}} \Delta m)/\delta}, \tag{3.24}$$

it is apparent that (3.23) holds for all $\alpha \geq \alpha_1$. Note that h is independent of α and ε. Similarly, for each $x \in \bar{\Omega}_r$, we have that

$$\delta \frac{\alpha}{D}|\nabla m(x)|^2 + \Delta m(x) \geq \delta \frac{\alpha}{D} \min_{\bar{\Omega}_r} |\nabla m|^2 + \min_{\bar{\Omega}} \Delta m$$

and hence,

$$\delta \frac{\alpha}{D}|\nabla m(x)|^2 + \Delta m(x) \geq L \qquad \forall\, x \in \bar{\Omega}_r$$

provided

$$\delta \frac{\alpha}{D} \min_{\bar{\Omega}_r} |\nabla m|^2 + \min_{\bar{\Omega}} \Delta m \geq L \quad \Longleftrightarrow \quad \alpha \geq \frac{L - \min_{\bar{\Omega}} \Delta m}{\delta \min_{\bar{\Omega}_r} |\nabla m|^2} D.$$

Choosing K sufficiently large so that

$$\alpha_1 = (L - \min_{\bar{\Omega}} \Delta m) K^2 \varepsilon^{-2} D \delta^{-1} \geq \frac{L - \min_{\bar{\Omega}} \Delta m}{\delta \min_{\bar{\Omega}_r} |\nabla m|^2} D,$$

which can be accomplished by taking

$$K \geq \frac{r}{\min_{\bar{\Omega}_r} |\nabla m|},$$

because $\varepsilon \leq r$, it becomes apparent that (3.21) holds for all $\alpha \geq \alpha_1$, where α_1 is defined by (3.22). Therefore, (3.15) holds true for all $\alpha \geq \alpha_1$ where h is given by (3.24). Indeed, setting $\varepsilon := h\sqrt{D/\alpha}$, we have that $\alpha_1(\varepsilon) = \alpha$ and, consequently, the result follows from our previous analysis.

Step 2: This step shows that there exists $\alpha_2 \geq \alpha_1$ such that

$$u_{\lambda,\alpha}(x) \leq \phi_i(x) := \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\alpha\delta_i(m(x)-m_i)/D}, \quad x \in \Omega_i, \ \ 1 \leq i \leq q, \tag{3.25}$$

where

$$\Omega_1 := \Omega, \quad \Omega_{i+1} := \{x \in \Omega \ : \ \ m(x) > m_i - \eta\} \setminus \bigcup_{j=1}^{n_i} \bar{B}_r(z_i^j), \quad 1 \leq i \leq q-1.$$

By the continuity of m, Ω_i is an open set for each $1 \leq i \leq q$. As $m(z_i^j) = m_i > m_i - \eta$, there is a neighborhood of z_i^j contained in the open set $\{x \in \Omega : m(x) > m_i - \eta\}$, $1 \leq j \leq n_i$. By definition, these points have been excluded from Ω_{i+1}. Moreover, according to (3.16) and (3.19), for each $x \in \partial B_r(z_i^j)$ we have that $r^2/K \leq m_i - m(x)$ and hence, $m(x) - m_i \leq -r^2/K < -\eta$. As a byproduct, $\{x \in \Omega : m(x) > m_i - \eta\}$ possesses, at least, $n_i + 1$ components. Actually, for sufficiently small $\xi > 0$, one has that

$$[\partial B_r(z_i^j) + \bar{B}_\xi(0)] \cap \{x \in \Omega \ : \ m(x) \geq m_i - \eta\} = \emptyset, \qquad 1 \leq j \leq n_i.$$

In the definition of Ω_{i+1}, we have removed, at least, n_i of those components by subtracting $\bigcup_{j=1}^{n_i} \bar{B}_r(z_i^j)$. Now, we will show that

$$\Omega_{i+1} \subset \Omega_i, \qquad 1 \leq i \leq q-1. \tag{3.26}$$

Indeed, pick $x \in \Omega_{i+1}$. Then, $m(x) > m_i - \eta > m_{i-1} - \eta$. Consequently, to show that $x \in \Omega_i$ it suffices to check that $x \notin \bigcup_{j=1}^{n_{i-1}} \bar{B}_r(z_{i-1}^j)$. On the contrary, suppose that $x \in \bar{B}_r(z_{i-1}^j)$ for some $1 \leq j \leq n_{i-1}$. Then, according to (3.16), $m_{i-1} - m(x) \geq 0$, which implies $m(x) \leq m_{i-1}$. On the other hand, by (3.19),

$$m(x) > m_i - \eta > m_i - m_i + m_{i-1} = m_{i-1},$$

which contradicts $m(x) \leq m_{i-1}$ and ends the proof. Note that

$$z_\ell^j \in \Omega_{i+1}, \qquad i+1 \leq \ell \leq q, \qquad 1 \leq j \leq n_\ell,$$

because $m(z_\ell^j) = m_\ell \geq m_{i+1} > m_i > m_i - \eta$.

Let Γ be a component of $\partial\Omega$. According to (3.18) and (3.19), for each $x \in \Gamma$ and $1 \leq i \leq q-1$, either $m(x) - m_i \geq \tau > \eta$, or $m(x) - m_i \leq -\tau < -\eta$. Thus, by the hypothesis (f), for every $1 \leq i \leq q-1$, either $m(x) - m_i > \eta$ for all $x \in \Gamma$, or $m(x) - m_i < -\eta$ for all $x \in \Gamma$.

For a given $1 \leq i \leq q-1$, let Γ_{i+1} be a component of $\partial\Omega_{i+1}$. Suppose $\Gamma_{i+1} \subset \Omega$. Then, necessarily $m(x) = m_i - \eta$ for all $x \in \Gamma_{i+1}$. Actually, if $m(x) > m_i - \eta$, then x must be in the interior of Ω_{i+1}. Suppose $\Gamma_{i+1} \cap \partial\Omega \neq \emptyset$, pick a point $x_{i+1} \in \Gamma_{i+1} \cap \partial\Omega$ and let Γ be any component of $\partial\Omega$ such that $x_{i+1} \in \Gamma$. As $x_{i+1} \in \Gamma_{i+1} \subset \bar{\Omega}_{i+1}$, we have that $m(x_{i+1}) \geq m_i - \eta$. Thus, necessarily, $m(x) > m_i - \eta$ for all $x \in \Gamma$. Consequently, there exists $\xi > 0$ such that $[\Gamma + \bar{B}_\xi(0)] \cap \Omega \subset \Omega_{i+1}$ and therefore, Γ must be a component of $\partial\Omega_{i+1}$; necessarily $\Gamma = \Gamma_{i+1}$. Any component of $\partial\Omega_{i+1}$ either it is a component of $\partial\Omega$, or it is entirely contained in Ω. Thanks to this property and since to the fact that $m_i - \eta$, $1 \leq i \leq q$, are regular values of m, we also infer that Ω_{i+1} is an open set of class $\mathcal{C}^2$ of Ω; not necessarily connected.

Consider the linear operators

$$\mathfrak{M}(\psi) := -\nabla \cdot (D\nabla\psi - \alpha\psi\nabla m) - (\lambda m - a u_{\lambda,\alpha}^{p-1})\psi \quad \text{in } \Omega,$$
$$\mathfrak{B}(\psi) := D\frac{\partial\psi}{\partial\nu} - \alpha\psi\frac{\partial m}{\partial\nu} \quad \text{on } \partial\Omega,$$

for any $\psi \in X := \bigcap_{q>1} W^{2,q}(\Omega)$. As $u_{\lambda,\alpha}$ solves (1.2), we have that

$$\mathfrak{M}(u_{\lambda,\alpha}) = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathfrak{B}(u_{\lambda,\alpha}) = 0 \quad \text{on } \partial\Omega. \tag{3.27}$$

By differentiating, yields

$$\nabla\phi_i = \frac{\alpha\delta_i}{D}\phi_i\nabla m, \quad \Delta\phi_i = \frac{\alpha\delta_i}{D}\phi_i\left(\frac{\alpha\delta_i}{D}|\nabla m|^2 + \Delta m\right), \quad 1 \leq i \leq q. \tag{3.28}$$

Thus, by (3.27) and (3.28) and rearranging terms, we find that

$$\mathfrak{M}(\phi_i - u_{\lambda,\alpha}) = \mathfrak{M}(\phi_i) = \phi_i\left[\alpha(1-\delta_i)\left(\frac{\delta_i\alpha}{D}|\nabla m|^2 + \Delta m\right) - \lambda m + a u_{\lambda,\alpha}^{p-1}\right]$$

for all $1 \leq i \leq q$. Hence, thanks to (3.15), we obtain that

$$\mathfrak{M}(\phi_i - u_{\lambda,\alpha}) > \phi_i\left[\alpha(1-\delta_1)L - \lambda m\right] \quad \text{in } \bar{\Omega} \setminus \bigcup_{z \in \mathcal{M}_m} B_{h\sqrt{\frac{D}{\alpha}}}(z)$$

for all $\alpha \geq \alpha_1$ and $1 \leq i \leq q$. As $\phi_i > 0$ is bounded away from zero, $\delta_1 < 1$ and $L > 0$, there exists $\alpha_2 \geq \alpha_1$ such that

$$\mathfrak{M}(\phi_i - u_{\lambda,\alpha}) > 0 \;\text{ in }\; \bar{\Omega} \setminus \bigcup_{z \in \mathcal{M}_m} B_{h\sqrt{\frac{D}{\alpha}}}(z) \;\text{ for all }\; \alpha \geq \alpha_2, \;\; 1 \leq i \leq q. \tag{3.29}$$

On the other hand, thanks to the hypothesis (c), the following estimate holds

$$\mathfrak{B}(\phi_i - u_{\lambda,\alpha}) = \mathfrak{B}(\phi_i) = \alpha(\delta_i - 1)\phi_i \frac{\partial m}{\partial \nu} \geq 0 \quad \text{on} \quad \partial\Omega, \quad 1 \leq i \leq q. \tag{3.30}$$

To prove (3.25) we will first prove that in the neighborhood of the points of $\mathcal{M}_m$

$$u_{\lambda,\alpha}(x) \leq \phi_i(x), \qquad x \in \bigcup_{\substack{i \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} \bar{B}_{h\sqrt{D/\alpha}}(z_\ell^j) \subsetneq \Omega_i. \tag{3.31}$$

Note that α_2 can be enlarged, if necessary, so that

$$\bigcup_{\substack{i \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} \bar{B}_{h\sqrt{D/\alpha}}(z_\ell^j) = \bigcup_{\substack{1 \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} \bar{B}_{h\sqrt{D/\alpha}}(z_\ell^j) \cap \Omega_i, \qquad 1 \leq i \leq q,$$

for all $\alpha \geq \alpha_2$. To show (3.31), we argue as follows. Pick

$$1 \leq i \leq q, \quad i \leq \ell \leq q, \qquad x \in \bigcup_{1 \leq j \leq n_\ell} \bar{B}_{h\sqrt{D/\alpha}}(z_\ell^j) \subset \bigcup_{1 \leq j \leq n_\ell} \bar{B}_r(z_\ell^j).$$

According to (3.16),

$$m(x) - m_i \geq m(x) - m_\ell \geq -Kh^2 D/\alpha.$$

Thus,

$$\phi_i(x) \geq \|u_{\lambda,\alpha}\|_\infty e^{(1-\delta_i)Kh^2} > \|u_{\lambda,\alpha}\|_\infty \geq u_{\lambda,\alpha}(x),$$

which ends the proof of (3.31).

Subsequently, we introduce the open subdomain of Ω of class $\mathcal{C}^2$

$$\hat{\Omega}_i := \Omega_i \setminus \bigcup_{\substack{i \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} \bar{B}_{h\sqrt{D/\alpha}}(z_\ell^j), \qquad 1 \leq i \leq q.$$

The boundary $\partial\hat{\Omega}_i$ consists of $\partial\Omega_i$, which is $\mathcal{C}^2$, plus the boundaries of all the balls $B_{h\sqrt{D/\alpha}}(z_\ell^j)$. Consider the boundary operator

$$\hat{\mathfrak{B}}_i u := \begin{cases} \mathfrak{B}u & \text{on } \partial\hat{\Omega}_i \cap \partial\Omega, \\ u & \text{on } \partial\hat{\Omega}_i \cap \Omega, \end{cases} \qquad 1 \leq i \leq q.$$

By Proposition 3.1.1, it follows from (3.27) that

$$\sigma[\mathfrak{M}, \hat{\mathfrak{B}}_i, \hat{\Omega}_i] > \sigma[\mathfrak{M}, \mathfrak{B}, \Omega] = 0, \qquad 1 \leq i \leq q.$$

On the other hand, according to (3.29), (3.30) and (3.31), we have that

$$\begin{cases} \mathfrak{M}(\phi_1 - u_{\lambda,\alpha}) > 0 & \text{in } \hat{\Omega}_1, \\ \hat{\mathfrak{B}}_1(\phi_1 - u_{\lambda,\alpha}) \geq 0 & \text{on } \partial\hat{\Omega}_1, \end{cases}$$

for all $\alpha \geq \alpha_2$. Therefore, thanks to Amann and López-Gómez [5, Th. 2.4], or [32, Th. 7.5.2], $(\mathfrak{M}, \hat{\mathfrak{B}}_1, \hat{\Omega}_1)$ satisfies the strong maximum principle and $\phi_1 - u_{\lambda,\alpha} \gg 0$ in $\hat{\Omega}_1$. Consequently, by (3.31), $u_{\lambda,\alpha} \leq \phi_1$ in $\Omega_1 = \Omega$, as claimed in (3.25). Suppose we have proven (3.25) for $1 \leq i \leq \kappa \leq q-1$, i.e.,

$$u_{\lambda,\alpha}(x) \leq \phi_i(x), \qquad x \in \Omega_i, \;\; 1 \leq i \leq \kappa. \tag{3.32}$$

To complete the proof of (3.25), it suffices to show that $u_{\lambda,\alpha} \leq \phi_{\kappa+1}$ in $\Omega_{\kappa+1}$. It should be noted that

$$\partial\hat{\Omega}_{\kappa+1} = \partial\Omega_{\kappa+1} \cup \bigcup_{\substack{\kappa+1 \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} \partial B_{h\sqrt{D/\alpha}}(z_\ell^j). \tag{3.33}$$

This can be accomplished by enlarging α, if necessary. It should be remembered that $\partial\Omega_{\kappa+1}$ consists of a finite number of entire components of $\partial\Omega$, where (3.18) holds, plus some additional components in Ω, where necessarily $m(x) = m_\kappa - \eta$.

Pick $x \in \partial\Omega_{\kappa+1} \cap \Omega$. Then, $m(x) = m_\kappa - \eta$ and hence, owing to (3.20),

$$\delta_{\kappa+1}\left(m(x) - m_{\kappa+1}\right) - \delta_\kappa\left(m(x) - m_\kappa\right) = \frac{\delta_\kappa \eta\left(m(x) - m_{\kappa+1}\right)}{m_{\kappa+1} - m_\kappa + \eta} + \delta_\kappa \eta = 0.$$

Thus,

$$\frac{\phi_{\kappa+1}(x)}{\phi_\kappa(x)} = e^{\frac{\alpha}{D}\left[\delta_{\kappa+1}(m(x)-m_{\kappa+1}) - \delta_\kappa(m(x)-m_\kappa)\right]} = 1.$$

Hence, due to the induction hypothesis (3.32), and using (3.26), since $\partial\Omega_{\kappa+1} \cap \Omega \subset \bar{\Omega}_{\kappa+1} \subset \bar{\Omega}_\kappa$, we find that

$$u_{\lambda,\alpha}(x) \leq \phi_\kappa(x) = \phi_{\kappa+1}(x), \qquad x \in \partial\Omega_{\kappa+1} \cap \Omega. \tag{3.34}$$

Thus, according to (3.31), (3.33) and (3.34), we find that

$$u_{\lambda,\alpha}(x) \leq \phi_{\kappa+1}(x), \qquad x \in \partial\hat{\Omega}_{\kappa+1} \cap \Omega \tag{3.35}$$

for all $\alpha \geq \alpha_2$. Consequently, thanks to (3.29), (3.30) and (3.35), we obtain that

$$\begin{cases} \mathfrak{M}(\phi_{\kappa+1} - u_{\lambda,\alpha}) > 0 & \text{in } \hat{\Omega}_{\kappa+1}, \\ \hat{\mathfrak{B}}_{\kappa+1}(\phi_{\kappa+1} - u_{\lambda,\alpha}) \geq 0 & \text{on } \partial\hat{\Omega}_{\kappa+1}, \end{cases}$$

Therefore, thanks to Amann and López-Gómez [5, Th. 2.4], or [32, Th. 7.5.2], $(\mathfrak{M}, \hat{\mathfrak{B}}_{\kappa+1}, \hat{\Omega}_{\kappa+1})$ satisfies the strong maximum principle and $\phi_{\kappa+1} - u_{\lambda,\alpha} \gg 0$ in $\hat{\Omega}_{\kappa+1}$. Consequently, by (3.31), $u_{\lambda,\alpha} \leq \phi_{\kappa+1}$ in $\Omega_{\kappa+1}$, which ends the proof of (3.25).

As for any $1 \leq i \leq q$, Ω_i is open and $z_\ell^j \in \Omega_i$ for all $i \leq \ell \leq q$, $1 \leq j \leq n_\ell$, there is $r_1 \in (0, r)$ such that

$$\bigcup_{\substack{i \leq \ell \leq q \\ 1 \leq j \leq n_\ell}} B_{r_1}(z_\ell^j) \subset \Omega_i, \qquad 1 \leq i \leq q.$$

Thus, owing to (3.25), we have that, for every $1 \le i \le q$,

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\alpha\delta_i (m(x)-m_i)/D}, \qquad x \in \bigcup_{1\le j\le n_i} B_{r_1}(z_i^j) \subset \Omega_i, \tag{3.36}$$

which provides us with the first q estimates of (3.12). Moreover, by hypothesis (b), there is $\gamma_1 > 0$ such that

$$m_q - m(x) = m(z_q^j) - m(x) > \gamma_1, \qquad x \in \bar\Omega \setminus \bigcup_{1\le j\le n_q} B_{r_1}(z_q^j),$$

and, hence, we find from (3.25) that

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\delta_q \gamma_1 \frac{\alpha}{D}}, \qquad x \in \bar\Omega_q \setminus \bigcup_{1\le j\le n_q} B_{r_1}(z_q^j). \tag{3.37}$$

Now, pick $2 \le i \le q$ and

$$x \in \Omega_{i-1} \setminus \left(\bar\Omega_i \bigcup_{1\le j\le n_{i-1}} B_{r_1}(z_{i-1}^j) \right). \tag{3.38}$$

Then, by the definition of Ω_{i-1}, we have that

$$m(x) > m_{i-2} - \eta, \qquad x \notin \bigcup_{1\le j\le n_{i-2}} \bar B_r(z_{i-2}^j).$$

Either $m(x) > m_{i-1} - \eta$, or $m(x) \le m_{i-1} - \eta$. Suppose $m(x) > m_{i-1} - \eta$. Then, x is candidate to be in Ω_i, but it does not belong to Ω_i, because of (3.38). Thus, we find that

$$x \in \bigcup_{j=1}^{n_{i-1}} \bar B_r(z_{i-1}^j) \setminus \bigcup_{j=1}^{n_{i-1}} B_{r_1}(z_{i-1}^j).$$

In such case, $r_1 \le |x - z_{i-1}^j| \le r$ for some $1 \le j \le n_{i-1}$. Then, due to (3.16),

$$m(x) - m_{i-1} \le -r_1^2/K.$$

Consequently, in any circumstances,

$$m(x) - m_{i-1} \le \max\{-\eta, -r_1^2/K\} = -\min\{\eta, r_1^2/K\}.$$

Thus, setting $\gamma_2 := \min\{\eta, r_1^2/K\}$ and taking into account that $\delta_q < \delta_{i-1}$, it becomes apparent from $(i-1)$-th estimate of (3.25) that

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\delta_q \gamma_2 \frac{\alpha}{D}}, \qquad x \in \Omega_{i-1} \setminus \left(\bar\Omega_i \bigcup_{1\le j\le n_{i-1}} B_{r_1}(z_{i-1}^j) \right). \tag{3.39}$$

Therefore, setting $\gamma = \min\{\gamma_1, \gamma_2\}$, we conclude from (3.37) and (3.39), that

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\delta\gamma\frac{\alpha}{D}}, \qquad x \in \bar{\Omega} \setminus \bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{r_1}(z_i^j). \tag{3.40}$$

It should be noted that $\delta := \delta_q$, $\Omega_1 := \Omega$ and

$$\Omega_{i-1} \setminus \left(\bar{\Omega}_i \bigcup_{1\le j\le n_{i-1}} B_{r_1}(z_{i-1}^j) \right) = [\Omega_{i-1} \setminus \bar{\Omega}_i] \setminus \bigcup_{1\le j\le n_{i-1}} B_{r_1}(z_{i-1}^j), \quad 2 \le i \le q.$$

Step 3: This step shows that there exist $\alpha_3 \ge \alpha_2$ and a constant $C > 0$, independent of α, such that

$$\|u_{\lambda,\alpha}\|_\infty \le C \qquad \text{for all } \alpha \ge \alpha_3. \tag{3.41}$$

The proof combines a blowing up argument with the Harnack inequality. Set

$$R_0 := \sqrt{DK(Kh^2 + \log 2)/\delta},$$

and enlarge α_2, if necessary, so that $R_0/\sqrt{\alpha} < r_1$ if $\alpha \ge \alpha_2$. Then, by (3.16) and (3.25), and since $\delta \le \delta_i$, $1 \le i \le q$, we find that, for every

$$x \in B_{r_1}(z_i^j) \setminus B_{R_0/\sqrt{\alpha}}(z_i^j), \qquad 1 \le j \le n_i, \quad 1 \le i \le q.$$

we have that

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\frac{\delta_i\alpha}{DK}|x-z_i^j|^2} \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\frac{-\delta}{DK}R_0^2} = \|u_{\lambda,\alpha}\|_\infty/2.$$

In addition, by (3.40), we can enlarge α_2, if necessary, so that, for every $\alpha \ge \alpha_2$,

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\frac{\delta\gamma}{D}\alpha} \le \|u_{\lambda,\alpha}\|_\infty/2, \qquad x \in \bar{\Omega} \setminus \bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{r_1}(z_i^j).$$

Therefore, as soon as $\alpha \ge \alpha_2$, we have that

$$u_{\lambda,\alpha}(x) \le \|u_{\lambda,\alpha}\|_\infty/2 \qquad \text{for all } x \in \bar{\Omega} \setminus \bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{R_0/\sqrt{\alpha}}(z_i^j).$$

As a byproduct, $\|u_{\lambda,\alpha}\|_\infty$ is taken in $\bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{R_0/\sqrt{\alpha}}(z_i^j)$. Moreover, if we pick any

$$x_\alpha \in \bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{R_0/\sqrt{\alpha}}(z_i^j)$$

such that $u(x_\alpha)=\|u_{\lambda,\alpha}\|_\infty$, there is a unique z_i^j such that $x_\alpha\in B_{R_0/\sqrt{\alpha}}(z_i^j)$. Subsequently, we denote $z_\alpha=z_i^j$. Next, we will perform the change of variable

$$y=\sqrt{\alpha}(x-z_\alpha),\ v_{\lambda,\alpha}(y)=u_{\lambda,\alpha}(x),\ n(y)=m(x),\ b(y)=a(x),\ x\in B_{\frac{4R_0}{\sqrt{\alpha}}}(z_\alpha). \tag{3.42}$$

Let $R_1>0$ be such that $\bar{B}_{R_1}(z_\alpha)\subset\Omega$ and $a(x)>0$ for all $x\in\bar{B}_{R_1}(z_\alpha)$. Note that α_2 can be enlarged, if necessary, so that $\bar{B}_{4R_0/\sqrt{\alpha}}(z_\alpha)\subset\Omega\cap B_{R_1}(z_\alpha)$ for all $\alpha\geq\alpha_2$. Moreover,

$$y\in B_{4R_0}(0),\qquad x=z_\alpha+y/\sqrt{\alpha}.$$

Differentiating yields

$$\nabla_x u_{\lambda,\alpha}=\sqrt{\alpha}\nabla_y v_{\lambda,\alpha},\quad \Delta_x u_{\lambda,\alpha}=\alpha\Delta_y v_{\lambda,\alpha},\quad \nabla_x m=\sqrt{\alpha}\nabla_y n,\quad \Delta_x m=\alpha\Delta_y n.$$

Thus, by substituting in (1.2), we find that

$$-D\Delta v_{\lambda,\alpha}+\alpha\langle\nabla n,\nabla v_{\lambda,\alpha}\rangle+\alpha^{-1}\left(\alpha^2\Delta n-\lambda n+bv_{\lambda,\alpha}^{p-1}\right)v_{\lambda,\alpha}=0 \text{ in } B_{4R_0}(0). \tag{3.43}$$

By (3.42), $y_\alpha:=\sqrt{\alpha}(x_\alpha-z_\alpha)\in B_{R_0}:=B_{R_0}(0)$ satisfies

$$v_{\lambda,\alpha}(y_\alpha)=\|v_{\lambda,\alpha}\|_{L^\infty(B_{4R_0}(0))},\quad \nabla v_{\lambda,\alpha}(y_\alpha)=0,\quad \Delta v_{\lambda,\alpha}(y_\alpha)\leq 0.$$

Thus, particularizing (3.43) at y_α shows that

$$0\leq -D\Delta v_{\lambda,\alpha}(y_\alpha)=-\alpha^{-1}\left[\alpha^2\Delta n(y_\alpha)-\lambda n(y_\alpha)+b(y_\alpha)v_{\lambda,\alpha}^{p-1}(y_\alpha)\right]v_{\lambda,\alpha}(y_\alpha),$$

and hence, since $v_{\lambda,\alpha}(y_\alpha)>0$ and $b(y_\alpha)>0$, we find that

$$v_{\lambda,\alpha}^{p-1}(y_\alpha)\leq b^{-1}(y_\alpha)\left(\lambda n(y_\alpha)-\alpha^2\Delta n(y_\alpha)\right)$$

Consequently,

$$\|v_{\lambda,\alpha}\|_{L^\infty(B_{4R_0})}^{p-1}\leq b^{-1}(y_\alpha)\left(|\lambda|\,\|n\|_{L^\infty(B_{4R_0})}+\alpha^2\,\|\Delta n\|_{L^\infty(B_{4R_0})}\right)$$

and, therefore,

$$\begin{aligned}\left\|\alpha^{-1}\left(\alpha^2\Delta n-\lambda n+bv_{\lambda,\alpha}^{p-1}\right)\right\|_{L^\infty(B_{4R_0})}&\leq C\left(\alpha\|\Delta n\|_{L^\infty(B_{4R_0})}+\frac{|\lambda|}{\alpha}\|n\|_{L^\infty(B_{4R_0})}\right)\\&\leq C\left(\|\Delta m\|_{L^\infty(\Omega)}+\frac{|\lambda|}{\alpha}\|m\|_{L^\infty(\Omega)}\right)\leq\hat{C},\end{aligned}$$

where $C>0$ is a constant depending on $a|_{B_{R_1}(z_\alpha)}$ and independent of $\alpha\geq\hat{\alpha}_2$, and $\hat{C}>0$ is a constant, depending on m and a but independent of $\alpha\geq\hat{\alpha}_2$. In order to estimate the transport term $\alpha\nabla n=\sqrt{\alpha}\nabla m$, it should be noted that there exists $\tau\in(0,4R_0)$ such that

$$\left|\nabla m(x)-D^2m(z_\alpha)(x-z_\alpha)\right|<|x-z_\alpha|\quad\text{for all }\ x\in B_\tau(z_\alpha).$$

Let $\alpha_3 \geq \hat{\alpha}_2$ such that $4R_0/\sqrt{\alpha} < \tau$ for all $\alpha \geq \alpha_3$. Then, there exists a constant $C = C(\tau, m) > 0$, independent of $\alpha \geq \alpha_3$, such that,

$$|\nabla m(x)| \leq C|x - z_\alpha| \leq C4R_0/\sqrt{\alpha}$$

for all $x \in B_{4R_0/\sqrt{\alpha}}(z_\alpha)$. Therefore,

$$\alpha \|\nabla n\|_{L^\infty(B_{4R_0})} = \sqrt{\alpha}\|\nabla m\|_{L^\infty(B_{4R_0/\sqrt{\alpha}}(z_\alpha))} \leq 4R_0 C.$$

Consequently, the coefficients of the differential operator

$$-D\Delta + \alpha\langle \nabla n, \nabla\rangle + \alpha^{-1}\left(\alpha^2 \Delta n - \lambda n + b v_{\lambda,\alpha}^{p-1}\right) \tag{3.44}$$

are uniformly bounded in B_{4R_0} for $\alpha \geq \alpha_3$. Thus, thanks to the Harnack inequality, we find from (3.44) that there exist a constant $C := C(N, R_0) > 0$ such that

$$C \max_{\bar{B}_{R_0}} v_{\lambda,\alpha} \leq \min_{\bar{B}_{R_0}} v_{\lambda,\alpha}.$$

Consequently,

$$C\|u_{\lambda,\alpha}\|_\infty = C u_{\lambda,\alpha}(x_\alpha) \leq \min_{\bar{B}_{\frac{R_0}{\sqrt{\alpha}}}(z_\alpha)} u_{\lambda,\alpha} \leq u_{\lambda,\alpha}(x) \qquad \forall\, x \in B_{\frac{R_0}{\sqrt{\alpha}}}(z_\alpha)$$

and hence,

$$\left(\frac{R_0}{\sqrt{\alpha}}\right)^N \omega_N C^p \|u_{\lambda,\alpha}\|_\infty^p = \int_{B_{\frac{R_0}{\sqrt{\alpha}}}(z_\alpha)} C^p \|u_{\lambda,\alpha}\|_\infty^p \leq \int_{B_{\frac{R_0}{\sqrt{\alpha}}}(z_\alpha)} u_{\lambda,\alpha}^p, \tag{3.45}$$

where ω_N stands for volume of $B_1(0) \subset \mathbb{R}^N$. Next, we will use (1.2) to estimate the right hand side of (3.45). Integrating by parts in Ω yields

$$\int_\Omega \left(a u_{\lambda,\alpha}^p - \lambda m u_{\lambda,\alpha}\right) = \int_{\partial\Omega} \left(D\frac{\partial u_{\lambda,\alpha}}{\partial \nu} - \alpha u_{\lambda,\alpha}\frac{\partial m}{\partial \nu}\right) = 0.$$

Thus,

$$\int_{B_{R_0/\sqrt{\alpha}}(z_\alpha)} a u_{\lambda,\alpha}^p \leq \int_\Omega a u_{\lambda,\alpha}^p = \int_\Omega \lambda m u_{\lambda,\alpha}. \tag{3.46}$$

As $a(x) > 0$ for all $x \in \bar{B}_{R_1}(z_\alpha)$ and $\bar{B}_{R_0/\sqrt{\alpha}}(z_\alpha) \subset B_{R_1}(z_\alpha)$ for all $\alpha \geq \alpha_3$, (3.46) implies that

$$\int_{B_{R_0/\sqrt{\alpha}}(z_\alpha)} u_{\lambda,\alpha}^p \leq \frac{|\lambda|\|m\|_\infty}{\min_{\bar{B}_{R_1}(z_\alpha)} a} \int_\Omega u_{\lambda,\alpha}. \tag{3.47}$$

On the other hand, we have that

$$\int_\Omega u_{\lambda,\alpha} = \int_{\Omega \setminus \bigcup_{\substack{1\leq i\leq q\\ 1\leq j\leq n_i}} B_{r_1}(z_i^j)} u_{\lambda,\alpha} + \int_{\bigcup_{\substack{1\leq i\leq q\\ 1\leq j\leq n_i}} B_{r_1}(z_i^j)} u_{\lambda,\alpha}$$

and, due to (3.40),

$$\int_{\Omega\setminus\bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{r_1}(z_i^j)} u_{\lambda,\alpha} \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\frac{\delta\gamma}{D}\alpha}|\Omega\setminus B_r| < \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\frac{\delta\gamma}{D}\alpha}|\Omega|,$$

where we have denoted

$$B_r := \bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{r_1}(z_i^j).$$

By enlarging α_3, if necessary, we can also assume that

$$e^{-\frac{\delta\gamma}{D}\alpha} \le \alpha^{-\frac{N}{2}} \qquad \forall\, \alpha \ge \alpha_3,$$

and hence,

$$\int_{\Omega\setminus\bigcup_{\substack{1\le i\le q\\ 1\le j\le n_i}} B_{r_1}(z_i^j)} u_{\lambda,\alpha} \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} \alpha^{-\frac{N}{2}}|\Omega|. \tag{3.48}$$

Moreover, by (3.36) and (3.16), and using the change of variable for $1 \le i \le q$ and $1 \le j \le n_i$,

$$y = \sqrt{\delta\alpha/(KD)}(x - z_i^j),$$

we find that

$$\begin{aligned}\int_{B_{r_1}(z_i^j)} u_{\lambda,\alpha} &\le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} \int_{B_{r_1}(z_i^j)} e^{-\delta\frac{\alpha}{D}\frac{1}{K}|x-z_i^j|^2} \\ &< \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} \left(\sqrt{\frac{KD}{\delta\alpha}}\right)^N \int_{\mathbb{R}^N} e^{-|y|^2}\end{aligned}$$

and therefore,

$$\int_{B_{r_1}(z_i^j)} u_{\lambda,\alpha} \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} \left(\sqrt{\frac{KD}{\delta\alpha}\pi}\right)^N. \tag{3.49}$$

Thus, according to (3.48) and (3.49), we have that

$$\int_\Omega u_{\lambda,\alpha} \le \|u_{\lambda,\alpha}\|_\infty P\alpha^{-N/2}, \quad P := e^{Kh^2}\left(|\Omega| + \operatorname{Card}\mathcal{M}_m\left[KD\pi\delta^{-1}\right]^{N/2}\right), \tag{3.50}$$

where

$$\operatorname{Card}\mathcal{M}_m = n_1 + \cdots + n_q.$$

Consequently, substituting (3.50) in (3.47) and using (3.45), we find that

$$\left(\frac{R_0}{\sqrt{\alpha}}\right)^N \omega_N C^p \|u_{\lambda,\alpha}\|_\infty^p \le \int_{B_{\frac{R_0}{\sqrt{\alpha}}}(z_\alpha)} u_{\lambda,\alpha}^p \le \frac{|\lambda|\|m\|_\infty}{\min_{\bar B_{R_1}(z_\alpha)} a}\|u_{\lambda,\alpha}\|_\infty P\alpha^{-N/2}$$

and therefore,

$$\|u_{\lambda,\alpha}\|_\infty \leq \max_{z\in\mathcal{M}_m}\left(\frac{|\lambda|\|m\|_\infty P}{R_0^N C^p \omega_N \min_{\bar{B}_{R_1}(z)} a}\right)^{\frac{1}{p-1}}$$

which is a bound independent of $\alpha \geq \alpha_3$. This concludes (3.41).

According to (3.40) and Step 3, the proof of (3.12) and (3.13) is completed. Finally, in Step 4, we will prove (3.14), which is a substantial refinement of the previous estimates.

Step 4: Proof of (3.14). Note that $1 \leq \kappa \leq q-1$ is the unique κ such that $m_\kappa \leq 0$ and $m_{\kappa+1} > 0$. Subsequently, we consider the function

$$\phi_0(x) = \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\alpha(m(x)-m_\kappa-\hat{\eta})/D} \quad \text{for all} \quad x \in \bar{\Omega}$$

for some $\hat{\eta}$ satisfying

$$0 < \hat{\eta} < \min\left\{\eta, \frac{\delta_{\kappa+1}(m_{\kappa+1}-m_\kappa)}{2-\delta_{\kappa+1}}\right\} \tag{3.51}$$

and such that $m_\kappa - \hat{\eta}$ is a regular value of $m(x)$. We also consider the open set

$$\Omega_0 := \{\, x \in \Omega \;:\; m(x) < m_\kappa - \hat{\eta}\,\} \bigcup_{1\leq j\leq n_k} B_r(z_\kappa^j)$$

and observe that

$$\bigcup_{\substack{1\leq j\leq n_i\\ 1\leq i\leq \kappa}} B_{r_1}(z_i^j) \subset \Omega_0. \tag{3.52}$$

Indeed, as $r_1 < r$,

$$\bigcup_{1\leq j\leq n_\kappa} B_{r_1}(z_\kappa^j) \subset \bigcup_{1\leq j\leq n_\kappa} B_r(z_\kappa^j).$$

Also, by (3.19), (3.51) and (3.16), we have that, for every $x \in \bigcup_{\substack{1\leq j\leq n_i\\ 1\leq i\leq \kappa-1}} B_{r_1}(z_i^j)$,

$$m_\kappa - m_{\kappa-1} > \eta > \hat{\eta}, \qquad m(x) \leq m_i \leq m_{\kappa-1}.$$

Thus,

$$m(x) < m_\kappa - \hat{\eta}, \qquad x \in \bigcup_{\substack{1\leq j\leq n_i\\ 1\leq i\leq \kappa-1}} B_{r_1}(z_i^j),$$

which ends the proof of (3.52). Moreover,

$$\bigcup_{1\leq j\leq n_\kappa} \partial B_r(z_\kappa^j) \subset \{x \in \Omega \;:\; m(x) < m_\kappa - \hat{\eta}\}\,.$$

Indeed, for any $x \in \bigcup_{1\leq j\leq n_\kappa} \partial B_r(z_\kappa^j)$, it follows from (3.16) and the definitions of η and $\hat{\eta}$ that

$$m(x) - m_k \leq -\frac{r^2}{K} < -\eta < -\hat{\eta}.$$

Thus, there exists $\zeta > 0$ such that

$$\bigcup_{1\leq j\leq n_\kappa} \partial B_r(z_\kappa^j) + B_\zeta(0) \subset \{x \in \Omega \;:\; m(x) < m_\kappa - \hat{\eta}\}$$

and hence,

$$\partial\Omega_0 \cap \Omega = \left\{ x \in \Omega \setminus \bigcup_{1\leq j\leq n_\kappa} B_r(z_\kappa^j) \;:\; m(x) = m_\kappa - \hat{\eta} \right\}.$$

By the hypothesis (f), we already know that, for any connected component Γ of $\partial\Omega$, either $m(x) < m_\kappa - \tau$ for all $x \in \Gamma$, or $m(x) > m_\kappa + \tau$ for all $x \in \Gamma$. Thus, if $m(y) > m_\kappa + \tau$ for some $y \in \Gamma \subset \partial\Omega$, then

$$m(x) > m_\kappa + \tau$$

for all $x \in \Gamma$ and, therefore, $x \notin \partial\Omega_0$. On the contrary, if $m(y) < m_\kappa - \tau$ for some $y \in \Gamma$, then $m(x) < m_\kappa - \tau$ for all $x \in \Gamma$ and hence,

$$m(x) < m_\kappa - \hat{\eta} \quad \text{for all} \;\; x \in \Gamma,$$

because $\hat{\eta} < \eta < \tau$. Consequently, by the continuity of m, we can infer that Γ must be a component of $\partial\Omega_0$. Therefore,

$$\partial\Omega_0 \cap \partial\Omega = \{x \in \partial\Omega \;:\; m(x) < m_\kappa - \hat{\eta}\}.$$

According to all these features, Ω_0 is an open set of class $\mathcal{C}^2$ of Ω such that, for any $x \in \partial\Omega_0 \cap \partial\Omega$, the component Γ_x of $\partial\Omega$ such that $x \in \Gamma_x$ is a component of $\partial\Omega_0$.

As in Step 2, we have that

$$\mathfrak{M}\left[\phi_0 - u_{\lambda,\alpha}\right] = \mathfrak{M}\left[\phi_0\right] = -\lambda m\phi_0 + a u_{\lambda,\alpha}^{p-1}\phi_0.$$

As $\lambda > 0$ and $m(x) < 0$ for any $x \in \Omega_0 \setminus \left\{z_\kappa^j\right\}_{j=1}^{n_\kappa}$, because any critical point of m is non-degenerate, we conclude that

$$\mathfrak{M}\left[\phi_0 - u_{\lambda,\alpha}\right] > 0 \quad \text{in} \quad \Omega_0 \tag{3.53}$$

Subsequently, we consider the boundary operator

$$\mathfrak{B}_0 u := \begin{cases} \mathfrak{B}u & \text{on} \;\; \partial\Omega \cap \partial\Omega_0, \\ u & \text{on} \;\; \Omega \cap \partial\Omega_0. \end{cases}$$

As in Step 2, on $\partial\Omega \cap \partial\Omega_0$, we have that

$$\mathfrak{B}_0[\phi_0 - u_{\lambda,\alpha}] = \mathfrak{B}[\phi_0 - u_{\lambda,\alpha}] = \mathfrak{B}[\phi_0] = D\frac{\partial\phi_0}{\partial\nu} - \alpha\phi_0\frac{\partial m}{\partial\nu} = 0. \tag{3.54}$$

On the other hand, we have that

$$\partial\Omega_0 \cap \Omega = \left\{ x \in \Omega \setminus \bigcup_{1\leq j\leq n_\kappa} B_r(z_\kappa^j) \;:\; m(x) = m_\kappa - \hat{\eta} \right\} \subset \Omega_{\kappa+1},$$

because $\hat{\eta} < \eta$. Thus, for every $x \in \partial\Omega_0 \cap \Omega$, we find that

$$\phi_0(x) = \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\alpha(m(x)-m_\kappa-\hat{\eta})/D} = \|u_{\lambda,\alpha}\| e^{Kh^2} e^{-2\hat{\eta}\alpha/D}.$$

Moreover, due to (3.51), we have that

$$\hat{\eta} < \frac{\delta_{\kappa+1}(m_{\kappa+1} - m_\kappa)}{2 - \delta_{\kappa+1}} \quad \iff \quad -2\hat{\eta} > \delta_{k+1}(m_k - \hat{\eta} - m_{k+1}),$$

and therefore,

$$\begin{aligned}\phi_0(x) &> \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\delta_{\kappa+1}\alpha(m_\kappa-\hat{\eta}-m_{\kappa+1})/D} \\ &= \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\delta_{\kappa+1}\alpha(m(x)-m_{\kappa+1})/D} = \phi_{\kappa+1}(x).\end{aligned}$$

Consequently, as $\partial\Omega_0 \cap \Omega \subset \Omega_{\kappa+1}$, it is apparent that, for any $x \in \partial\Omega_0 \cap \Omega$,

$$u_{\lambda,\alpha}(x) \le \phi_{\kappa+1}(x) < \phi_0(x). \tag{3.55}$$

Thanks to Proposition 3.1.1, we have that

$$\sigma[\mathfrak{M}, \mathfrak{B}_0, \Omega_0] > \sigma[\mathfrak{M}, \mathfrak{B}, \Omega] = 0.$$

Moreover, according to (3.53), (3.54) and (3.55), we already know that

$$\begin{cases} \mathfrak{M}(\phi_0 - u_{\lambda,\alpha}) > 0 & \text{in} \quad \Omega_0, \\ \mathfrak{B}(\phi_0 - u_{\lambda,\alpha}) \ge 0 & \text{on} \quad \partial\Omega_0, \end{cases}$$

for sufficiently large. Therefore, thanks to Amann and López-Gómez [5, Th. 2.4], or [32, Th. 7.5.2], $(\mathfrak{M}, \mathfrak{B}_0, \Omega_0)$ satisfies the strong maximum principle and $\phi_0 - u_{\lambda,\alpha} \gg 0$ in Ω_0. By (3.52), we also have that, for any $x \in \bigcup_{\substack{1\le j\le n_i \\ 1\le i\le k}} B_{r_1}(z_i^j)$, the next estimate holds

$$u_{\lambda,\alpha}(x) \le \phi_0(x) = \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{\alpha(m(x)-m_k-\hat{\eta})/D} \le \|u_{\lambda,\alpha}\|_\infty e^{Kh^2} e^{-\hat{\eta}\alpha/D}$$

This ends the proof, as $\|u_{\lambda,\alpha}\|_\infty$ is uniformly bounded for all $\alpha \ge \alpha_3$ by Step 3. □

3.4 Lower estimates

The main result of this section can be stated as follows.

Theorem 3.4.1. *Suppose:*

(a) $\Omega_0 := \operatorname{int} a^{-1}(0)$ *is a smooth subdomain of* Ω *with* $\bar{\Omega}_0 \subset \Omega$.

(b) Any critical point of m *is non-degenerate.*

(c) $\Delta m(x) > 0$ *for all* $x \in \mathcal{C}_m \setminus \mathcal{M}_m$.

(d) $\mathcal{M}_m$ *satisfies* (3.3) *and* (3.4) *with* $m_{\kappa-1} \leq 0$ *and* $m_\kappa > 0$ *for some* $2 \leq \kappa \leq q$, *or* $m_\kappa > 0$ *for all* $1 \leq \kappa \leq q$.

(e) $a(z_i^j) > 0$ *for all* $\kappa \leq i \leq q$ *and* $1 \leq j \leq n_i$.

Then, for every $\lambda > 0$, *there exist* $\tilde{\alpha} > 0$ *and* $\varepsilon > 0$ *such that*

$$u_{\lambda,\alpha} \geq C_i^j e^{\alpha(m-m_i)/D} \qquad \text{in } \; B_\varepsilon(z_i^j), \qquad \kappa \leq i \leq q, \quad 1 \leq j \leq n_i.$$

for all $\alpha \geq \tilde{\alpha}$, *where we have denoted*

$$C_i^j := \left(\frac{\lambda \min_{\bar{B}_\varepsilon(z_i^j)} m}{\max_{\bar{B}_\varepsilon(z_i^j)} a} \right)^{\frac{1}{p-1}}, \qquad \kappa \leq i \leq q, \quad 1 \leq j \leq n_i. \tag{3.56}$$

Proof. By Hypotheses (d) and (e), there exists $\varepsilon > 0$ such that

$$m(x) > 0 \quad \text{and} \quad a(x) > 0 \quad \text{for all} \quad x \in \bigcup_{\substack{k \leq i \leq q \\ 1 \leq j \leq n_i}} \bar{B}_\varepsilon(z_i^j).$$

Now, short ε, if necessary, so that $2\varepsilon < r$, where $r > 0$ is the one already constructed in the beginning of the proof of Theorem 3.3.1, and consider any radially symmetric function $\xi : \mathbb{R}^N \to [0, \infty)$ of class $\mathcal{C}^\infty$ such that

$$0 \leq \xi \leq 1, \qquad \xi(x) = \begin{cases} 1 & \text{if } \; |x| \leq \varepsilon, \\ 0 & \text{if } \; |x| \geq 2\varepsilon, \end{cases} \tag{3.57}$$

and

$$\langle \nabla \xi(\cdot - z_i^j), \nabla m \rangle > 0 \qquad \text{in} \qquad \bigcup_{\substack{\kappa \leq i \leq q \\ 1 \leq j \leq n_i}} B_{2\varepsilon}(z_i^j) \setminus B_\varepsilon(z_i^j), \tag{3.58}$$

which can be accomplished because m has a quadratic maximum at each of the z_i^j's, by shortening $\varepsilon > 0$, if necessary. Subsequently, we fix $\kappa \leq i \leq q$ and $1 \leq j \leq n_i$ to look for a constant $C > 0$ such that

$$\underline{u}_{\lambda,\alpha} := C\xi(\cdot - z_i^j)E, \qquad E := e^{\alpha(m-m_i)/D},$$

is a subsolution of (1.2). As $\underline{u}_{\lambda,\alpha} = 0$ in $\Omega \setminus B_{2\varepsilon}(z_i^j)$, it is apparent that

$$D\partial_\nu \underline{u}_{\lambda,\alpha} - \alpha \underline{u}_{\lambda,\alpha} \partial_\nu m = 0 \qquad \text{on} \quad \partial\Omega.$$

Thus, $\underline{u}_{\lambda,\alpha}$ provides us with a subsolution of (1.2) if, and only if,

$$\begin{aligned} \nabla \cdot \Big(DCE\nabla\xi(\cdot - z_i^j) + DC\xi(\cdot - z_i^j)\frac{\alpha}{D}E\nabla m - \alpha C\xi(\cdot - z_i^j)E\nabla m \Big) \\ + \lambda m C\xi(\cdot - z_i^j)E - a[C\xi(\cdot - z_i^j)E]^p \geq 0, \end{aligned}$$

or, equivalently, simplifying, differentiating and dividing by CE,

$$\alpha\langle \nabla m, \nabla\xi(\cdot - z_i^j)\rangle + D\Delta\xi(\cdot - z_i^j) + \lambda m\xi - aC^{p-1}\xi^p(\cdot - z_i^j)E^{p-1} \geq 0. \tag{3.59}$$

By (3.57), (3.59) holds in $\Omega \setminus \bar{B}_{2\varepsilon}(z_i^j)$, and it is satisfied in $B_\varepsilon(z_i^j)$ if, and only if,

$$\lambda m - aC^{p-1}E^{p-1} \geq 0 \qquad \text{in} \quad B_\varepsilon(z_i^j).$$

As $E \leq 1$ in $B_\varepsilon(z_i^j)$, it is apparent that $C = C_i^j$ satisfies this property. Finally, by (3.58), we have that

$$\langle \nabla m, \nabla \xi(\cdot - z_i^j) \rangle + \alpha^{-1} \left(D\Delta\xi(\cdot - z_i^j) + \lambda m \xi - aC^{p-1}\xi^p(\cdot - z_i^j)E^{p-1} \right) > 0$$

in $B_{2\varepsilon}(z_i^j) \setminus B_\varepsilon(z_i^j)$ for sufficiently large α. Consequently, for such α's, $\underline{u}_{\lambda,\alpha}$ is a strict positive subsolution of (1.2).

According to the hypotheses (a), (b), (c) and (d), it follows from Theorem 3.2.5 that there exists $\alpha_0 > 0$ such that (1.2) admits a (unique) positive solution, denoted by $u_{\lambda,\alpha}$, for each $\alpha > \alpha_0$. Enlarge $\tilde{\alpha}$, if necessary, so that $\tilde{\alpha} > \alpha_0$. Then, it is easy to see that $\bar{u}_{\lambda,\alpha} := Mu_{\lambda,\alpha}$ provides us with a strict positive supersolution of (1.2) for all $M > 1$. As $u_{\lambda,\alpha} \gg 0$ in Ω, it follows from Amann [4] that

$$\underline{u}_{\lambda,\alpha} \leq u_{\lambda,\alpha} \leq \bar{u}_{\lambda,\alpha}$$

for sufficiently large M provided $\alpha > \tilde{\alpha}$, which ends the proof of the theorem. □

It should be noted that the closer z_i^j is to $\Omega_0 := \operatorname{Int} a^{-1}(0)$, the smaller $\max_{\bar{B}_\varepsilon(z_i^j)} a$ and, hence, the larger C_i^j (see (3.56)), and, so, $u_{\lambda,\alpha}$. This strongly suggests that, under conditions (3.3) and (3.4), the hypothesis $a(z_i^j) > 0$ for all $1 \leq i \leq q$ and $1 \leq j \leq n_i$ is necessary for the validity of Theorems 3.2.5 and 3.3.1. This will follow from the next result of technical nature.

Proposition 3.4.1. *Suppose* (a)-(d) *of Theorem 3.4.1 and there exist $i \in \{\kappa, ..., q\}$ and $j \in \{1, ..., n_i\}$ such that $z_i^j \in \Omega_0$. Then, the following assertions are true:*

(i) There exists $\alpha_0 > 0$ such that, for every $\lambda > 0$, $\eta > 0$ and $\alpha > \alpha_0$, the problem

$$\begin{cases} \nabla \cdot [D\nabla u - \alpha u \nabla m] + \lambda m u - (a + \eta)u^p = 0 & \text{in } \Omega, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.60}$$

admits a unique positive solution. It will be denoted by $u_{\lambda,\alpha,\eta}$.

(ii) For every $\lambda > 0$, there exist $\tilde{\alpha} > \alpha_0$ and $\varepsilon > 0$ such that

$$u_{\lambda,\alpha,\eta} \geq \left(\frac{\lambda \min_{\bar{B}_\varepsilon(z_i^j)} m}{\eta} \right)^{\frac{1}{p-1}} e^{\alpha(m - m_i)/D} \qquad \text{in } B_\varepsilon(z_i^j), \tag{3.61}$$

for all $\eta > 0$ and $\alpha > \tilde{\alpha}$. In particular,

$$\lim_{\eta \downarrow 0} u_{\lambda,\alpha,\eta} = \infty \qquad \text{uniformly in } \bar{B}_\varepsilon(z_i^j). \tag{3.62}$$

Proof. Subsequenty, we consider the differential operator $\mathfrak{L}(\lambda, \alpha)$ and the principal eigenvalue $\Sigma(\lambda, \alpha)$ introduced in the statement of Theorem 3.2.1. According to Propositions 1 and 2 of Chapter 2, the map $\lambda \mapsto \Sigma(\lambda, \alpha)$ is real analytic and strictly concave. Moreover,

$$\Sigma'(0, \alpha) = -\int_\Omega m(x)e^{\alpha m(x)/D}\, dx \Big/ \int_\Omega e^{\alpha m(x)/D}\, dx.$$

Consequently, thanks to Theorem 2.2.1 of Chapter 2, there exists $\alpha_0 > 0$ such that $\Sigma'(0,\alpha) < 0$. Thus, $\Sigma(\lambda,\alpha) < 0$ for all $\alpha > \alpha_0$ and $\lambda > 0$ and, therefore, thanks to Theorem 3.2.1, the problem (3.60) admits a unique positive solution for each $\lambda > 0$ and $\eta > 0$. Subsequently, we will denote it by $u_{\lambda,\alpha,\eta}$. This concludes the proof of Part (i).

To prove Part (ii), we fix a $\lambda > 0$ and choose $\varepsilon > 0$ such that

$$m(x) > 0 \quad \text{and} \quad a(x) = 0 \quad \text{for all } \; x \in B_{2\varepsilon}(z_i^j).$$

It exists, because $z_i^j \in \Omega_0$. Then, short ε, if necessary, so that $2\varepsilon < r$, where $r > 0$ is the one already constructed in the beginning of the proof of Theorem 3.3.1, we consider a test function $\xi \in \mathcal{C}^\infty(\mathbb{R}^N)$ satisfying (3.57) and such that

$$\langle \nabla \xi(\cdot - z_i^j), \nabla m\rangle > 0 \qquad \text{in} \quad B_{2\varepsilon}(z_i^j) \setminus B_\varepsilon(z_i^j), \tag{3.63}$$

and look for a constant $C > 0$ such that

$$\underline{u}_{\lambda,\alpha,\eta} := C\xi(\cdot - z_i^j)E, \qquad E := e^{\alpha(m-m_i)/D},$$

is a subsolution of (3.60). Reasoning as in the proof of Theorem 3.4.1, it is apparent that $\underline{u}_{\lambda,\alpha,\eta}$ is a subsolution of (3.60) if, and only if,

$$\alpha\langle \nabla m, \nabla \xi(\cdot - z_i^j)\rangle + D\Delta\xi(\cdot - z_i^j) + \lambda m\xi - (a+\eta)C^{p-1}\xi^p(\cdot - z_i^j)E^{p-1} \geq 0. \tag{3.64}$$

As in the proof of Theorem 3.4.1, (3.64) holds in $\Omega \setminus \bar{B}_{2\varepsilon}(z_i^j)$, and it is satisfied in $B_\varepsilon(z_i^j)$ if, and only if,

$$\lambda m - \eta C^{p-1}E^{p-1} \geq 0 \qquad \text{in} \quad B_\varepsilon(z_i^j),$$

which is accomplished by choosing

$$C := \left(\frac{\lambda \min_{\bar{B}_\varepsilon(z_i^j)} m}{\eta} \right)^{\frac{1}{p-1}}.$$

Finally, for this choice, in $B_{2\varepsilon}(z_i^j) \setminus B_\varepsilon(z_i^j)$ we have that

$$(a+\eta)C^{p-1} = \eta C^{p-1} = \lambda \min_{\bar{B}_\varepsilon(z_i^j)} m$$

and therefore, due to (3.63), there exists $\tilde{\alpha} > \alpha_0 > 0$ such that

$$\langle \nabla m, \nabla \xi(\cdot - z_i^j)\rangle + \alpha^{-1}\left(D\Delta\xi(\cdot - z_i^j) + \lambda m\xi - (a+\eta)C^{p-1}\xi^p(\cdot - z_i^j)E^{p-1}\right) > 0$$

for all $\alpha > \tilde{\alpha}$ and $\eta > 0$. According to Part (i), (3.60) possesses a unique positive solution $u_{\lambda,\alpha,\eta}$ for all $\eta > 0$ and it is easy to see that $Mu_{\lambda,\alpha,\eta}$ is a supersolution of (3.60) for sufficiently large $M > 0$. Consequently, (3.61) holds. As a by-product, (3.62) is also satisfied. This ends the proof. □

As an immediate consequence from Proposition 3.4.1, the next results hold.

Theorem 3.4.2. *Suppose* (a)-(d) *of Theorem 3.4.1 and there exist* $i \in \{\kappa, ..., q\}$ *and* $j \in \{1, ..., n_i\}$ *such that* $z_i^j \in \Omega_0$. *Fix* $\lambda > 0$. *Then,* (1.2) *cannot admit a positive solution for sufficiently large* α.

Proof. Suppose there exist $\lambda > 0$ and a sequence α_n, $n \geq 1$, such that $\lim_{n\to\infty} \alpha_n = \infty$ and

$$\begin{cases} \nabla \cdot [D\nabla u - \alpha_n u \nabla m] + \lambda m u - a u^p = 0 & \text{in } \Omega, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.65}$$

admits a positive solution u_n for each $n \geq 1$. Necessarily, by Theorem 2.4.1 of Chapter 2, u_n is the unique solution of (3.65). For sufficiently large $n \geq 1$, we have that $\alpha_n \geq \tilde{\alpha}$ (the one constructed in the proof of Proposition 3.4.1) and hence, u_n provides us with a supersolution of

$$\begin{cases} \nabla \cdot [D\nabla u - \alpha_n u \nabla m] + \lambda m u - (a+\eta) u^p = 0 & \text{in } \Omega, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases}$$

for all $\eta > 0$. Therefore, according to Proposition 3.4.1, we find that

$$u_n \geq u_{\lambda,\alpha_n,\eta} \geq \left(\frac{\lambda \min_{\bar{B}_\varepsilon(z_i^j)} m}{\eta} \right)^{\frac{1}{p-1}} e^{\alpha_n (m - m_i)/D} \quad \text{in } B_\varepsilon(z_i^j)$$

for sufficiently large $n \geq 1$ and all $\eta > 0$, which is impossible.

□

Theorem 3.4.3. *Suppose* (a)-(d) *of Theorem 3.4.1 and there exist* $i \in \{\kappa, ..., q\}$ *and* $j \in \{1, ..., n_i\}$ *such that* $z_i^j \in \Omega_0$. *Fix* $\lambda > 0$ *and let* $\tilde{\alpha} > \alpha_0$ *and* $\varepsilon > 0$ *the constants whose existence is guaranteed by Proposition 3.4.1(ii). Then,*

$$\liminf_{t\to\infty} u(\cdot, t; u_0) = \infty \quad \textit{uniformly in } B_\varepsilon(z_i^j) \tag{3.66}$$

for all $\alpha > \tilde{\alpha}$.

Proof. Subsequently, for every $\eta > 0$, we consider the evolution problem

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda m - (a+\eta) u^{p-1}) & \text{in } \Omega, & t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, & t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \Omega. & \end{cases} \tag{3.67}$$

Let $u_\eta(\cdot, t; u_0)$ denote the unique solution of (3.67). As $u(\cdot, t; u_0)$ is a supersolution of (3.67), it follows from the parabolic maximum principle that

$$u_\eta(\cdot, t; u_0) \leq u(\cdot, t; u_0)$$

for all $\eta > 0$ and $t > 0$. Thus, by Theorem 2.4.4 of Chapter 2 it is apparent that

$$\liminf_{t\to\infty} u(\cdot, t; u_0) \geq \lim_{t\to\infty} u_\eta(\cdot, t; u_0) = u_{\lambda,\alpha,\eta}.$$

Therefore, by letting $\eta \to 0$ in this estimate, (3.66) holds from Proposition 3.4.1(ii). This ends the proof. □

These results prove that Theorem 3.2.1 is optimal, in the sense that the a priori bounds are lost if $a(z) = 0$ for some $z \in \mathcal{M}_m$.

Chapter 4

Dynamics of a class of advective-diffusive equations in Ecology

4.1 Introduction

In this chapter we will focus our attention into the degenerate case where $a^{-1}(0)$ is the closure of some smooth nonempty subdomain of Ω,

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{with} \quad \bar{\Omega}_0 \subset \Omega.$$

The distribution of this chapter is as follows. In section 4.2 we give some extensions of the previous findings of Chapter 2 which are going to be used in this chapter. In Section 4.3 we generalize the Hadamard formula of López-Gómez and Sabina de Lis [36] to the non-self-adjoint context of this chapter. In Section 4.4 we will use the Hadamard formula to establish that

$$\lim_{\lambda \uparrow \lambda^*} \theta_\lambda = +\infty \quad \text{uniformly in } \bar{\Omega}_0,$$

where $\lambda^* > 0$ is the finite limiting value of λ, if it exists, for which (1.2) admits a positive solution, θ_λ. These are the main ingredients to get (1.7) in Section 4.6 when $m > 0$ in Ω and $\lambda \geq \lambda^*$. In Section 4.5 we establish the existence of the minimal and the maximal large solutions $L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ and $L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ of (1.2). Finally, in Section 4.7 we get (1.7) for $\lambda \geq \lambda^*$ in the general case when m changes of sign in Ω by perturbing the weight function m, instead of the parameter λ, as it is usual in the available literature. This technical device should have a huge number of applications to deal with spatially heterogeneous Reaction Diffusion equations.

4.2 Notations and preliminaries

In this section, instead of (1.1), we consider the following generalized parabolic problem

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda n - a u^{p-1}) & \text{in } \Omega, \quad t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{4.1}$$

where $n \in \mathcal{C}(\bar{\Omega})$ is arbitrary. Naturally, in the special case $n = m$, (4.1) provides us with (1.1). Throughout this chapter we denote by $u_{[\lambda,n]}(x, t; u_0)$ the unique (positive) solution of (4.1). Also, we set

$$u_\lambda(x, t; u_0) := u_{[\lambda,m]}(x, t; u_0).$$

So, $u_\lambda(x, t; u_0)$ stands for the unique (positive) solution of (1.1). The main goal of this section is to adapt the abstract theory of Chapter 2 to the problem (4.1). As the proofs of these results are straightforward modifications of those of Chapter 2, they will be omitted here.

The dynamic of (4.1) is regulated by its non-negative steady-states, if they exist, which are the non-negative solutions, θ, of the semi linear elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla \theta - \alpha \theta \nabla m) - \theta(\lambda n - a\theta^{p-1}) = 0 & \text{in } \Omega, \\ D\partial_\nu \theta - \alpha \theta \partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The principal eigenvalue

$$\sigma[-\nabla \cdot (D\nabla - \alpha \nabla m) - \lambda n, D\partial_\nu - \alpha \partial_\nu m, \Omega],$$

plays a significant role to describe the dynamic of (4.1). Let denote by $\psi_0 > 0$ its associated principal eigenfunction normalized so that $\|\psi_0\|_\infty = 1$. By performing the change of variable

$$\phi_0 := e^{-\alpha m/D}\psi_0,$$

differentiating and rearranging terms in the ψ_0-equation, it is easily seen that $\psi_0 > 0$ provides us with a principal eigenfunction of

$$\sigma[-D\Delta - \alpha \nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega],$$

and that, actually,

$$\sigma[-\nabla \cdot (D\nabla - \alpha \nabla m) - \lambda n, D\partial_\nu - \alpha \partial_\nu m, \Omega] = \sigma[-D\Delta - \alpha \nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega] \tag{4.3}$$

for all $\lambda \in \mathbb{R}$ and $\alpha > 0$. The next result extends Theorem 2.2.1 to cover the general case when $n \in \mathcal{C}(\bar{\Omega})$ is arbitrary. By the sake of completeness we are including a short self-contained proof of it.

Theorem 4.2.1. *For every $\alpha > 0$ the map*

$$\lambda \mapsto \Sigma(\lambda) := \sigma[-D\Delta - \alpha \nabla m \cdot \nabla - \lambda n, \mathfrak{N}, \Omega]$$

is real analytic and strictly concave if $n \neq 0$. Moreover:

(a) *If* $n \equiv 0$, *then, for any* $\lambda \in \mathbb{R}$,

$$\Sigma(\lambda) = \sigma[-D\Delta - \alpha\nabla m \cdot \nabla, \mathfrak{N}, \Omega] = 0.$$

(b) *If* $n > 0$ *(*$n \geq 0$ *but* $n \neq 0$*), then* $\Sigma(\lambda)\lambda < 0$ *for all* $\lambda \in \mathbb{R} \setminus \{0\}$. *Moreover, by Part (a),* $\Sigma(0) = 0$.

(c) *If* $\int_\Omega n \geq 0$, *there exists* $x_- \in \Omega$ *such that* $n(x_-) < 0$, $n \leq m$ *and* $n = m$ *if* $n \leq 0$, *then there exists* $\lambda_- := \lambda_-(\alpha, n) < 0$ *such that*

$$\Sigma(\lambda) \begin{cases} < 0 & \textit{if } \lambda \in (-\infty, \lambda_-) \cup (0, \infty), \\ = 0 & \textit{if } \lambda \in \{\lambda_-, 0\}, \\ > 0 & \textit{if } \lambda \in (\lambda_-, 0). \end{cases} \tag{4.4}$$

(d) *If* $\int_\Omega n < 0$ *and* $n(x_+) > 0$ *for some* $x_+ \in \Omega$, $n \leq m$ *and* $n = m$ *if* $n \leq 0$, *then there exists* $\alpha_0 := \alpha_0(n) > 0$ *such that if* $0 < \alpha < \alpha_0$ *there exists* $\lambda_+ := \lambda_+(\alpha, n) > 0$ *such that*

$$\Sigma(\lambda) \begin{cases} < 0 & \textit{if } \lambda \in (-\infty, 0) \cup (\lambda_+, \infty), \\ = 0 & \textit{if } \lambda \in \{0, \lambda_+\}, \\ > 0 & \textit{if } \lambda \in (0, \lambda_+), \end{cases}$$

if $\alpha = \alpha_0$ *then* $\Sigma(\lambda) < 0$ *for each* $\lambda \in \mathbb{R} \setminus \{0\}$, *while* $\Sigma(0) = 0$, *and if* $\alpha > \alpha_0$ *then there exists* $\lambda_- := \lambda_-(\alpha, n) < 0$ *such that* (4.4) *holds.*

Figure 4.1 shows $\Sigma(\lambda)$ in each of the cases (b) and (d) of Theorem 4.2.1. In case (c), $\Sigma(\lambda)$ has the same graph as in case (d) for $\alpha > \alpha_0$.

Proof. According to [32, Th. 9.1], for every $\alpha \in \mathbb{R}$ the map $\lambda \mapsto \Sigma(\lambda)$ is real analytic and strictly concave if $n \neq 0$. In case $n \equiv 0$, it is obvious that the constant 1 is an eigenfunction associated to the zero eigenvalue of $-D\Delta - \alpha\nabla m \cdot \nabla$ in Ω under Neumann boundary conditions. Hence, $\Sigma(\lambda) = 0$ and Part (a) holds.

Part (b) is a direct consequence from $\Sigma(0) = 0$ taking into account that $\lambda \mapsto \Sigma(\lambda)$ is decreasing in λ, because $n > 0$.

In the remaining cases, (c) and (d), the function n changes sign in Ω. Thus, by [32, Th. 9.1],

$$\lim_{\lambda \to \pm\infty} \Sigma(\lambda) = -\infty.$$

Next we will show that

$$\Sigma'(0) = -\int_\Omega n(x) e^{\alpha m(x)/D}\, dx \Big/ \int_\Omega e^{\alpha m(x)/D}\, dx, \tag{4.5}$$

which is reminiscent of (2.10). According to Kato [22, Th. 2.6 on p. 377] and [22, Rem. 2.9 on p. 379], the perturbed eigenfunction $\varphi(\lambda)$ from the constant $\varphi(0) = 1$ associated to the principal eigenvalue $\Sigma(\lambda)$ as λ perturbs from 0 is real analytic as a function of λ. Thus, differentiating with respect to λ the identities

$$\begin{cases} -D\Delta\varphi(\lambda) - \alpha\nabla m \cdot \nabla\varphi(\lambda) - \lambda n \varphi(\lambda) = \Sigma(\lambda)\varphi(\lambda) & \text{in } \Omega, \\ \partial_\nu \varphi(\lambda) = 0 & \text{on } \partial\Omega, \end{cases}$$

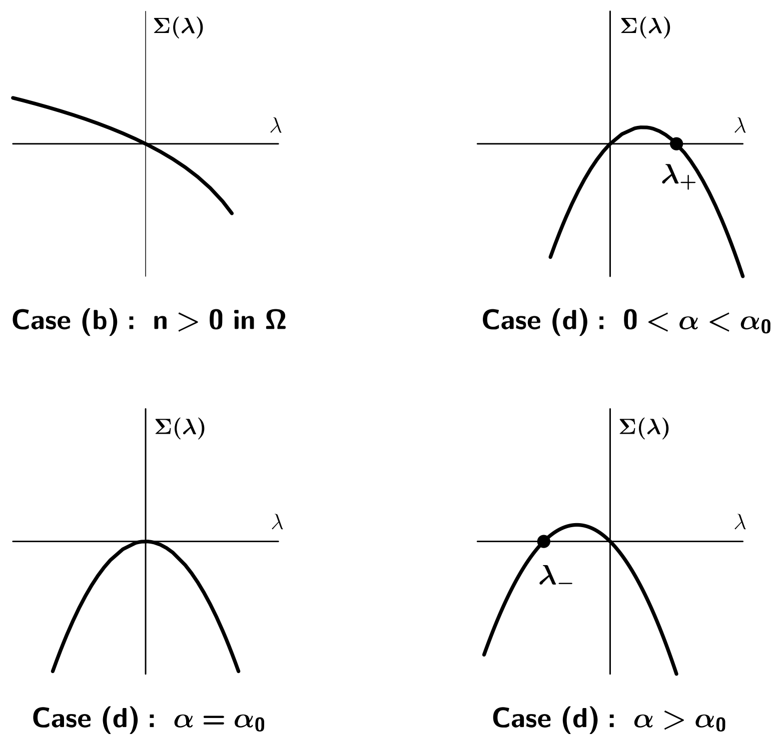

Figure 4.1: Some possible graphs of $\Sigma(\lambda)$

yields

$$-D\Delta\varphi'(\lambda) - \alpha\nabla m \cdot \nabla\varphi'(\lambda) - n\varphi(\lambda) - \lambda n\varphi'(\lambda) = \Sigma'(\lambda)\varphi(\lambda) + \Sigma(\lambda)\varphi'(\lambda).$$

and particularizing at $\lambda = 0$, we are driven to

$$\mathfrak{L}_0\varphi'(0) := -D\Delta\varphi'(0) - \alpha\nabla m \cdot \nabla\varphi'(0) = n + \Sigma'(0).$$

Consequently, as the adjoint operator of $\mathfrak{L}_0$ subject to Neumann boundary conditions admits the following realization

$$\mathfrak{L}_0^* v := -\nabla \cdot (D\nabla v - \alpha v\nabla m)$$

for all $v \in \mathcal{C}^2(\bar{\Omega})$ such that

$$D\partial_\nu v - \alpha v\partial_\nu m = 0 \qquad \text{on } \partial\Omega,$$

and $\varphi_0^* := e^{\alpha m/D}$ satisfies

$$\mathfrak{L}_0^* \varphi_0^* = 0 \text{ in } \Omega \quad \text{and} \quad D\partial_\nu \varphi_0^* - \alpha \varphi_0^* \partial_\nu m = 0 \text{ on } \partial\Omega,$$

we conclude that

$$\langle n + \Sigma'(0), \varphi_0^* \rangle = \langle \mathfrak{L}_0 \varphi'(0), \varphi_0^* \rangle = \langle \varphi'(0), \mathfrak{L}_0^* \varphi_0^* \rangle = 0$$

and therefore, (4.5) holds. As a byproduct, we find that

$$\operatorname{sign} \Sigma'(0) = -\operatorname{sign} f(\alpha) \qquad \text{for all } \alpha \geq 0,$$

where

$$f(\alpha) := \int_\Omega n(x) e^{\alpha m(x)/D}\, dx.$$

As

$$\begin{aligned} f'(\alpha) &= \frac{1}{D} \int_\Omega n(x) m(x) e^{\alpha m(x)/D}\, dx \\ &= \frac{1}{D} \int_{n \leq 0} n^2(x) e^{\alpha m(x)/D}\, dx + \frac{1}{D} \int_{n \geq 0} n(x) m(x) e^{\alpha m(x)/D}\, dx \\ &\geq \frac{1}{D} \int_{n \leq 0} n^2(x) e^{\alpha m(x)/D}\, dx + \frac{1}{D} \int_{n \geq 0} n^2(x) e^{\alpha m(x)/D}\, dx \\ &= \frac{1}{D} \int_\Omega n^2(x) e^{\alpha m(x)/D}\, dx > 0, \end{aligned}$$

because $n \neq 0$, the function f is strictly increasing. Moreover, in the region where $n > 0$ we have that $m \geq n > 0$, whereas $m = n$ if $n \leq 0$. Therefore,

$$\lim_{\alpha \to \infty} f(\alpha) = \infty.$$

Lastly, note that $f(0) = \int_\Omega n$. Therefore, we have that $f(\alpha) > 0$ for all $\alpha > 0$ if $\int_\Omega n \geq 0$, while in case $\int_\Omega n < 0$, there exists a unique $\alpha_0 > 0$ such that $f(\alpha) < 0$ if $\alpha < \alpha_0$ and $f(\alpha) > 0$ if $\alpha > \alpha_0$. The proof is complete. ☐ ☐

According to Belgacem and Cosner [6], it is well known that in the special case when $a(x) > 0$ for all $x \in \bar{\Omega}$, the dynamics of (4.1) is determined by the following theorem:

Theorem 4.2.2. *Suppose* $\alpha > 0$ *and* $a(x) > 0$ *for all* $x \in \bar{\Omega}$. *Then,* (4.2) *admits a positive solution,* $\theta_{[\lambda,n]}$, *if, and only if,* $\Sigma(\lambda) < 0$. *Moreover, it is unique if it exists and it is a global attractor for* (4.1). *Furthermore,*

$$\lim_{t \to \infty} \| u_{[\lambda,n]}(\cdot, t; u_0) \|_\infty = 0$$

if $\Sigma(\lambda) \geq 0$.

As this chapter focuses attention in the more general degenerate case when

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{is a smooth subdomain with} \quad \bar{\Omega}_0 \subset \Omega, \tag{4.6}$$

the principal eigenvalue,

$$\Sigma_0(\lambda) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, \Omega_0], \qquad \alpha > 0, \quad \lambda \in \mathbb{R},$$

will also play a significant role to characterize the dynamic of (4.1). The next theorem collects some of the main properties of $\Sigma_0(\lambda)$.

Theorem 4.2.3. *Suppose* (4.6) *and* $\alpha > 0$. *Then,* $\Sigma_0(\lambda)$ *is real analytic, strictly concave if* $n \not\equiv 0$ *in* Ω_0, *and*

$$\Sigma(\lambda) < \Sigma_0(\lambda) \quad \textit{for all} \;\; \lambda \in \mathbb{R}.$$

Moreover, the following assertions are true:

(a) *There exists* $\lambda_1 := \lambda_1(\alpha, n) < 0$ *such that* $\Sigma_0(\lambda_1) = 0$ *if, and only if,* $n(x_-) < 0$ *for some* $x_- \in \Omega_0$.

(b) *There exists* $\lambda_2 := \lambda_2(\alpha, n) > 0$ *such that* $\Sigma_0(\lambda_2) = 0$ *if, and only if,* $n(x_+) > 0$ *for some* $x_+ \in \Omega_0$.

Figure 4.2 shows each of the possible graphs of $\Sigma_0(\lambda)$ according to the sign of $n(x)$ in Ω_0. In all cases, we have superimposed the graphs of $\Sigma(\lambda)$, with a dashed line, and the graph of $\Sigma_0(\lambda)$, with a continuous line. It should be noted that we have chosen a particular type of $\Sigma(\lambda)$ for all cases.

The next theorem provide us with the dynamic of (4.1). It generalizes the theory of Chapter 2 to cover the general case when $n \neq m$.

Theorem 4.2.4. *Suppose* (4.6) *and* $\alpha > 0$. *Then,* (4.2) *admits a positive solution,* $\theta_{[\lambda,n]}$, *if and only if* $\Sigma_0(\lambda) > 0$ *and* $\Sigma(\lambda) < 0$. *Moreover, is unique if it exists and in such case* $\theta_{[\lambda,n]}$ *is a global attractor for* (4.1). *Furthermore,* u *is driven to extinction in* $L^\infty(\Omega)$ *if* $\Sigma(\lambda) \geq 0$, *whereas*

$$\lim_{t\to\infty} \|u_{[\lambda,n]}(\cdot, t; u_0)\|_\infty = \infty \qquad \textit{if} \quad \Sigma_0(\lambda) \leq 0.$$

One of the main goals of this chapter is to show that, actually, in case $\Sigma_0(\lambda) \leq 0$ the solution $u_{[\lambda,n]}(\cdot, t; u_0)$ can approximate the minimal metasolution supported in $\bar{\Omega} \setminus \bar{\Omega}_0$ of (4.2) as $t \uparrow \infty$.

4.3 First variations of the principal eigenvalues $\Sigma_0(\lambda)$

Throughout this section the domain Ω_0 is assumed to be of class $\mathcal{C}^1$ and consider $T[\delta] : \bar{\Omega}_0 \to \bar{\Omega}_\delta$, with $\delta \simeq 0$ and $\Omega_\delta := T[\delta](\Omega_0)$, an holomorphic family of $\mathcal{C}^2$-diffeormorphisms that can be expressed in the form

$$T[\delta](x) = x + \sum_{k=1}^{\infty} \delta^k T_k(x) \quad \text{for all} \;\; x \in \bar{\Omega}_0, \tag{4.7}$$

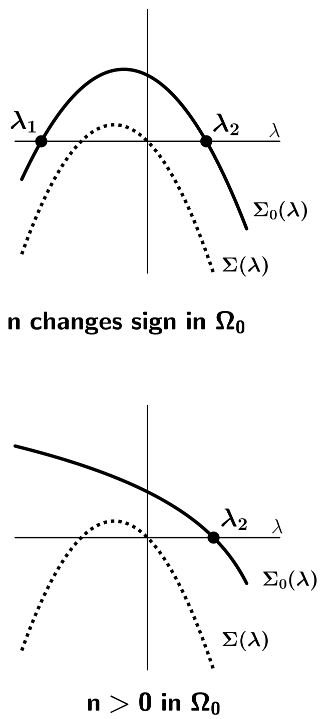

Figure 4.2: The four possible graphs of $\Sigma_0(\lambda)$

where $T_k \in \mathcal{C}^2(\bar{\Omega}_0; \mathbb{R}^N)$ for each $k \geq 1$ and

$$\limsup_{k\to\infty} \left\{ \|T_k\|_{\infty,\Omega_0} + \|D_x T_k\|_{\infty,\Omega_0} + \|D_x^2 T_k\|_{\infty,\Omega_0} \right\}^{1/k} < +\infty. \tag{4.8}$$

According to T. Kato [22, Ch. VII], it is easily seen that the family of eigenvalue problems

$$\begin{cases} -D\Delta\Phi - \alpha\nabla m \cdot \nabla\Phi - \lambda n\Phi = \tau\Phi & \text{in } \Omega_\delta, \\ \Phi = 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{4.9}$$

is real holomorphic in δ. Thus, setting

$$S(\delta) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, \Omega_\delta]$$

for $\lambda \in \mathbb{R}$ and $\alpha > 0$ fixed, $S(\delta)$ admits the next expansion as $\delta \downarrow 0$

$$S(\delta) = \sum_{k=0}^{\infty} \delta^k S_k, \qquad S_k = S^{k)}(0), \tag{4.10}$$

where $S^{k)}(0)$ stands for the k–th derivative of $S(\delta)$ with respect to δ at $\delta = 0$. Similarly, the associated perturbed eigenfunction, $\phi[\delta]$, admits the expansion

$$\phi[\delta] = \sum_{k=0}^{\infty} \delta^k \phi_k, \qquad \phi_k := \phi^{k)}[0],$$

where ϕ_0 is the principal eigenfunction associated to $S(0) = \Sigma_0(\lambda)$ normalized so that

$$\int_{\Omega_0} e^{\frac{\alpha}{D} m} \phi_0^2 = 1. \tag{4.11}$$

The next result provides us with an extension of Theorem 2.1 of [36] which provides us with the value of S_1 in (4.10).

Theorem 4.3.1. *Let Ω_0 be a bounded domain of class $\mathcal{C}^1$ and Ω_δ, $\delta \simeq 0$, a perturbed family of domains from Ω_0 given by a family of $\mathcal{C}^2$-diffeomorphisms $T[\delta]$ satisfying* (4.7) *and* (4.8). *Then, for every $\lambda \in \mathbb{R}$ and $\alpha > 0$, the eigenvalue problem* (4.9) *is real holomorphic in δ and the first variation of the principal eigenvalue, $S(\delta)$, is given by*

$$S_1 := S'(0) = -D \int_{\partial\Omega_0} e^{\frac{\alpha}{D} m} \left(\frac{\partial \phi_0}{\partial \nu} \right)^2 \langle T_1, \nu \rangle, \tag{4.12}$$

where ϕ_0 is the principal eigenfunction associated with $S(0) = \Sigma_0(\lambda)$, normalized so that (4.11) *holds.*

Proof. Subsequently, for each $\delta \simeq 0$ we set

$$\varphi[\delta] := e^{\frac{\alpha}{2D} m} \phi[\delta] \quad \text{in } \Omega_\delta.$$

This function is positive and satisfies

$$\begin{cases} -D\Delta\varphi[\delta] - W\varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{4.13}$$

with

$$W := \lambda n - \frac{\alpha}{2}(\Delta m + \frac{\alpha}{2D} |\nabla m|^2).$$

By (4.11),

$$\int_{\Omega_0} \varphi^2[0] = 1.$$

Next, we consider

$$\psi[\delta](x) := \varphi[\delta](y), \qquad Q[\delta](x) := W(y) \qquad \text{with } T[\delta](x) = y.$$

Then, setting

$$(T[\delta])^{-1}(y) = (x_1(y), x_2(y), ..., x_N(y)).$$

and substituting in (4.13) yields

$$\begin{cases} -D\sum_{k,l=1}^{N}\langle \nabla_y x_k, \nabla_y x_l\rangle \dfrac{\partial^2 \psi[\delta]}{\partial x_k \partial x_l} - D\sum_{l=1}^{N}\Delta_y x_l \dfrac{\partial \psi[\delta]}{\partial x_l} - Q[\delta]\psi[\delta] = S(\delta)\psi[\delta] & \text{in } \Omega_0, \\ \psi[\delta] = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{4.14}$$

Now, arguing as in Section 2 of [36] and setting $T_1 = (T_{1,1}, T_{1,2}, ..., T_{1,N})$ we can conclude from (4.14) that

$$-D\Delta_x \psi[\delta] + 2D\delta \sum_{k,l=1}^{N} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial^2 \psi[\delta]}{\partial x_k \partial x_l} + D\delta \sum_{l=1}^{N} \Delta_x T_{1,l}\frac{\partial \psi[\delta]}{\partial x_l} - Q[\delta]\psi[\delta] = S(\delta)\psi[\delta] + O(\delta^2). \tag{4.15}$$

Setting

$$\psi[\delta] = \sum_{k=0}^{\infty} \delta^k \psi_k, \quad \psi_k = \frac{d^k \psi}{d\delta^k}[0], \tag{4.16}$$

and

$$Q[\delta] = \sum_{k=0}^{\infty} \delta^k Q_k, \quad Q_k = \frac{d^k Q}{d\delta^k}[0], \tag{4.17}$$

with

$$\psi_0 = \psi[0] = \varphi[0], \quad Q_0 = W \quad \text{and} \quad Q_1 = \langle \nabla W, T_1\rangle = \langle \nabla Q_0, T_1\rangle, \tag{4.18}$$

substituting (4.10), (4.16) and (4.17) in (4.15), dividing by δ and letting $\delta \to 0$ gives

$$-D\Delta_x \psi_1 + 2D \sum_{k,l=1}^{N} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial^2 \psi_0}{\partial x_k \partial x_l} + D \sum_{l=1}^{N} \Delta_x T_{1,l}\frac{\partial \psi_0}{\partial x_l} - Q_0\psi_1 - Q_1\psi_0 = S_0\psi_1 + S_1\psi_0, \tag{4.19}$$

where we have used that

$$-D\Delta_x \psi_0 - Q_0\psi_0 = S_0\psi_0. \tag{4.20}$$

Hence, multiplying (4.19) by ψ_0, integrating by parts in the first factor and using (4.20), we find that

$$S_1 = 2D \sum_{k,l=1}^{N} \int_{\Omega_0} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial^2 \psi_0}{\partial x_k \partial x_l}\psi_0 + D \sum_{l=1}^{N} \int_{\Omega_0} \Delta_x T_{1,l}\frac{\partial \psi_0}{\partial x_l}\psi_0 - \int_{\Omega_0} Q_1 \psi_0^2. \tag{4.21}$$

A further integration by parts in the first two terms of (4.21), complete in the second term and partial in the first one after a cancelation of its half, gives

$$S_1 = -2D \sum_{k,l=1}^{N} \int_{\Omega_0} \frac{\partial T_{1,k}}{\partial x_l}\frac{\partial \psi_0}{\partial x_k}\frac{\partial \psi_0}{\partial x_l} - D \sum_{k,l=1}^{N} \int_{\Omega_0} \frac{\partial^2 T_{1,k}}{\partial x_k \partial x_l}\frac{\partial \psi_0}{\partial x_l}\psi_0 - \int_{\Omega_0} Q_1 \psi_0^2. \tag{4.22}$$

A further integration by parts in the first term of (4.22) yields

$$-2D\sum_{k,l=1}^{N}\int_{\Omega_0}\frac{\partial T_{1,k}}{\partial x_l}\frac{\partial \psi_0}{\partial x_k}\frac{\partial \psi_0}{\partial x_l} = -2D\int_{\partial\Omega_0}\langle\nabla_x\psi_0, T_1\rangle\langle\nabla_x\psi_0,\nu\rangle + 2D\sum_{k,l=1}^{N}\int_{\Omega_0}T_{1,k}\frac{\partial\psi_0}{\partial x_l}\frac{\partial^2\psi_0}{\partial x_l\partial x_k} + 2D\int_{\Omega_0}\Delta_x\psi_0\langle\nabla_x\psi_0, T_1\rangle.$$

Similarly, integrating by parts with respect to x_l and then with respect to x_k, the second term of (4.22) can be expressed as

$$-D\sum_{k,l=1}^{N}\int_{\Omega_0}\frac{\partial^2 T_{1,k}}{\partial x_k\partial x_l}\frac{\partial\psi_0}{\partial x_l}\psi_0 = D\int_{\partial\Omega_0}|\nabla_x\psi_0|^2\langle T_1,\nu\rangle - 2D\sum_{k,l=1}^{N}\int_{\Omega_0}T_{1,k}\frac{\partial\psi_0}{\partial x_l}\frac{\partial^2\psi_0}{\partial x_k\partial x_l} + D\sum_{k=1}^{N}\int_{\Omega_0}\Delta_x\psi_0\frac{\partial T_{1,k}}{\partial x_k}\psi_0.$$

Consequently, substituting these identities into (4.22) we obtain that

$$S_1 = -2D\int_{\partial\Omega_0}\langle\nabla_x\psi_0,T_1\rangle\langle\nabla_x\psi_0,\nu\rangle + D\int_{\partial\Omega_0}|\nabla_x\psi_0|^2\langle T_1,\nu\rangle - 2\int_{\Omega_0}(Q_0\psi_0+S_0\psi_0)\langle\nabla_x\psi_0,T_1\rangle - \sum_{k=1}^{N}\int_{\Omega_0}(Q_0\psi_0+S_0\psi_0)\frac{\partial T_{1,k}}{\partial x_k}\psi_0 - \int_{\Omega_0}Q_1\psi_0^2$$

where we have used (4.20). Finally, thanks to (4.18) and

$$-S_0\sum_{k=1}^{N}\int_{\Omega_0}\frac{\partial T_{1,k}}{\partial x_k}\psi_0^2 = 2S_0\int_{\Omega_0}\langle\nabla_x\psi_0,T_1\rangle\psi_0,$$
$$-\sum_{k=1}^{N}\int_{\Omega_0}Q_0\frac{\partial T_{1,k}}{\partial x_k}\psi_0^2 = 2\int_{\Omega_0}Q_0\langle\nabla_x\psi_0,T_1\rangle\psi_0 + \int_{\Omega_0}\langle\nabla_x Q_0,T_1\rangle\psi_0^2,$$

and

$$\nabla_x\varphi[0] = \frac{\partial\varphi[0]}{\partial\nu}\nu \quad \text{on } \partial\Omega_0,$$

we may infer that

$$S_1 = -D\int_{\partial\Omega_0}\left(\frac{\partial\varphi[0]}{\partial\nu}\right)^2\langle T_1,\nu\rangle.$$

Finally, taking into account that

$$\varphi[0] = e^{\frac{\alpha}{2D}m}\phi[0] = e^{\frac{\alpha}{2D}m}\phi_0,$$

(4.12) holds. □ □

A particular class of perturbations Ω_δ of Ω_0 is given by

$$\Omega_\delta := \Omega_0 \cup \left\{ x \in \mathbb{R}^N \setminus \Omega_0 \ : \ \text{dist}\,(x, \partial\Omega_0) < \delta \right\}. \tag{4.23}$$

The associated holomorphic family $T[\delta]$ with $\delta \simeq 0$ can be defined through the next theorem going back to [36, Th. 3.1],

Theorem 4.3.2. *Assume that Ω_0 is a bounded domain of $\mathbb{R}^N$ of class $\mathcal{C}^3$. If Ω_δ is given by* (4.23)*, then for each $\delta > 0$ sufficiently small, there exists a mapping $T[\delta] : \bar{\Omega}_0 \to \mathbb{R}^N$, such that*

(i) *$T[\delta] \in \mathcal{C}^2(\bar{\Omega}_0; \mathbb{R}^N)$ and $T[\delta] : \bar{\Omega}_0 \to \bar{\Omega}_\delta$ is a bijection.*

(ii) *$T[\delta]$ is real holomorphic in δ for $\delta \simeq 0$, in the sense that* (4.7) *and* (4.8) *hold.*

(iii) *$T_1|_{\partial\Omega_0} = \nu$ where ν is the outward unit normal along $\partial\Omega_0$.*

For (4.23), we obtain from (4.12) that

$$S_1 := S'(0) = -D \int_{\partial\Omega_0} e^{\frac{\alpha}{D} m} \left(\frac{\partial \phi_0}{\partial \nu} \right)^2 < 0. \tag{4.24}$$

In particular, the decay of the principal eigenvalue with respect to the domain is always linear independently of the size of the advection, $\alpha > 0$, though the linear decay rate is affected by the size of α and the nodal behavior of $m(x)$, of course.

4.4 Limiting behavior of the positive steady state solution in $\bar{\Omega}_0$

Throughout this section we will assume that $\lambda > 0$ and that

$$\text{there exists} \quad x_+ \in \Omega_0 \quad \text{such that} \quad m(x_+) > 0. \tag{4.25}$$

Should it not be the case, i.e. $m \leq 0$ in Ω_0, then $\Sigma_0(\lambda) > 0$ for all $\lambda > 0$. Hence, by Theorems 4.2.1 and 4.2.4, there exists $\lambda_+(\alpha) \geq 0$ such that (1.2) admits a (unique) positive solution, θ_λ, if and only if $\lambda > \lambda_+(\alpha)$. Moreover, thanks to Theorem 4.2.4, θ_λ is a global attractor for the positive solutions of (1.1). Contrarily, under condition (4.25), owing to Theorem 4.2.3, there exists a λ^*,

$$0 \leq \lambda_+(\alpha) < \lambda^* := \lambda_2(\alpha, m),$$

such that (1.2) has a positive solution if and only if $\lambda \in (\lambda_+, \lambda^*)$. Moreover, by Theorem 4.2.1, we already know that $\lambda_+ > 0$ if and only if $\int_\Omega m < 0$ and $\alpha \in (0, \alpha_0)$. The main goal of this section is to establish the next result.

Theorem 4.4.1. *Suppose* (4.25) *and $a(x)$ is of class $\mathcal{C}^1$ in a neighborhood of $\partial\Omega_0$. Then,*

$$\lim_{\lambda \uparrow \lambda_2} \theta_\lambda(x) = \infty \quad \textit{uniformly in} \ \bar{\Omega}_0. \tag{4.26}$$

Proof. For sufficiently small $\delta > 0$, we consider Ω_δ as in (4.23). By Theorem 4.3.2, Ω_δ is an holomorphic perturbation from Ω_0. Now, consider the principal eigenvalues

$$S(\delta) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda^* m, \mathfrak{D}, \Omega_\delta]$$

with associated principal eigenfunctions $\phi[\delta] > 0$. By (4.10) and (4.24),

$$S(\delta) = S_1\delta + O(\delta^2) \quad \text{as} \quad \delta \downarrow 0 \quad \text{with} \ \ S_1 < 0. \tag{4.27}$$

The function $\varphi[\delta]$ defined by

$$\varphi[\delta] := e^{\alpha m/D}\phi[\delta] > 0,$$

where $\phi[\delta]$ stands for the (unique) principal eigenfunction of $S(\delta)$ normalized so that

$$\|\varphi[\delta]\|_{\infty,\Omega_\delta} = 1,$$

satisfies

$$\begin{cases} -\nabla \cdot (D\nabla\varphi[\delta] - \alpha\varphi[\delta]\nabla m) - \lambda^* m\varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \ \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \ \partial\Omega_\delta. \end{cases} \tag{4.28}$$

Moreover, there exists $\lambda \in (\lambda_+, \lambda^*)$ sufficiently close λ^* such that

$$S(\delta) < S(\delta/2) < (\lambda - \lambda^*) \max_{\bar{\Omega}_\delta} m < 0. \tag{4.29}$$

Let $\vartheta_\delta \in \mathcal{C}(\bar{\Omega})$ be the function defined by

$$\vartheta_\delta(y) = \begin{cases} C\varphi[\delta](y) & \text{for } \ y \in \bar{\Omega}_\delta, \\ 0 & \text{for } \ y \notin \Omega_\delta, \end{cases}$$

where $C > 0$ is a constant to be chosen later. If we suppose that

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) \leq S(\delta/2) - S(\delta) \quad \text{for all } \ y \in \Omega_\delta \tag{4.30}$$

then ϑ_δ is a subsolution of problem

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \lambda m\theta = -a\theta^p & \text{in } \ \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{4.31}$$

Indeed, by (4.30) and (4.29), we have that

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y) < (\lambda - \lambda^*)m(y) - S(\delta) \quad \text{for each } \ y \in \Omega_\delta.$$

So, multiplying by $C\varphi[\delta]$, we find that

$$S(\delta)\vartheta_\delta + (\lambda^* - \lambda)m\vartheta_\delta < -a\vartheta_\delta^p \qquad \text{in} \qquad \Omega_\delta.$$

Hence, owing to (4.28), we find that

$$-\nabla \cdot (D\nabla\vartheta_\delta - \alpha\vartheta_\delta\nabla m) - \lambda^* m\vartheta_\delta + (\lambda^* - \lambda)m\vartheta_\delta < -a\vartheta_\delta^p \qquad \text{in} \qquad \Omega_\delta$$

which proves the previous assertion. Now, thanks to (4.27), it is apparent that (4.30) holds provided

$$C = C(\delta) := \frac{1}{\sup\limits_{\Omega_\delta\setminus\Omega_0} \varphi[\delta]} \left(\frac{-S_1/2 + O(\delta)}{\sup\limits_{\Omega_\delta\setminus\Omega_0} a/\delta} \right)^{1/(p-1)}. \tag{4.32}$$

According to (4.12) and (4.13) of [36], one can easily infer that

$$\sup_{\Omega_\delta\setminus\Omega_0} \varphi[\delta] = C_0\delta + o(\delta) \text{ with } C_0 > 0 \quad \text{and} \quad \lim_{\delta\downarrow 0} \sup_{\Omega_\delta\setminus\Omega_0} a/\delta = 0,$$

by the Hopf boundary lemma. Since $S_1 < 0$, we find that

$$\lim_{\delta\downarrow 0}(\delta C(\delta)) = +\infty. \tag{4.33}$$

By (4.33),

$$\lim_{\delta\downarrow 0} \vartheta_\delta(y) = +\infty \quad \text{for all } y \in \Omega_0, \tag{4.34}$$

uniformly on compact subsets of Ω_0. To complete the proof of the theorem it suffices to show that

$$\lim_{\delta\downarrow 0} \vartheta_\delta(y) = +\infty \quad \text{for all } y \in \partial\Omega_0. \tag{4.35}$$

We use the proof of [36, Th. 4.3] and we obtain that

$$\inf_{\partial\Omega_0} \varphi[\delta] = C_1\delta + o(\delta) \quad \text{with } C_1 > 0$$

as $\delta \downarrow 0$. Therefore, for every $y \in \partial\Omega_0$,

$$\vartheta_\delta(y) = C(\delta)\varphi[\delta](y) \geq C(\delta) \inf_{\partial\Omega_0} \varphi[\delta] = \delta C(\delta)(C_1 + o(1)).$$

Lastly, by (4.33), (4.35) holds.

Finally, since (1.2) admits a unique positive solution θ_λ for each $\lambda \in (\lambda_+, \lambda^*)$, we find that

$$\vartheta_\delta(y) \leq \theta_\lambda(y) \quad \text{for all } y \in \bar{\Omega}$$

for λ sufficiently close to λ^* and for sufficiently small $\delta > 0$ satisfying (4.30). Therefore, the growth to infinity of ϑ_δ leads to the corresponding behavior for θ_λ and (4.26) holds. □ □

4.5 Existence of large solutions

Throughout this section we consider a smooth subdomain $\mathcal{O} \subset \Omega \setminus \bar{\Omega}_0$ such that, for some $\eta > 0$,

$$\{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) < \eta\} \subset \mathcal{O}. \tag{4.36}$$

Note that this entails

$$\operatorname{dist}(\partial\Omega, \Omega \cap \partial\mathcal{O}) > 0. \tag{4.37}$$

Also, for every $M \in (0, \infty)$, we consider the next family of parabolic problems

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda n - a u^{p-1}) & \text{in } \mathcal{O},\ t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{on } \partial\Omega,\ t > 0, \\ u = M & \text{on } \partial\mathcal{O} \setminus \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{in } \mathcal{O}, \end{cases} \tag{4.38}$$

whose unique (positive) solution is denoted by $u := u_{[\lambda,n,M,\mathcal{O}]}(x, t; u_0)$. The dynamics of (4.38) is regulated by the non-negative solutions, $\theta_{[\lambda,n,M,\mathcal{O}]}$, of the semilinear elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (D\nabla \theta - \alpha \theta \nabla m) - \theta(\lambda n - a\theta^{p-1}) = 0 & \text{in } \mathcal{O}, \\ D\partial_\nu \theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega, \\ \theta = M & \text{on } \partial\mathcal{O} \setminus \partial\Omega. \end{cases} \tag{4.39}$$

The following result characterizes the existence of positive solutions of (4.39).

Theorem 4.5.1. *Suppose $\mathcal{O} \subset \Omega \setminus \bar{\Omega}_0$ is sufficiently smooth, $n \in \mathcal{C}(\bar{\Omega})$, $\lambda \in \mathbb{R}$ and $M > 0$. Then,* (4.39) *has a unique positive solution, $\theta_{[\lambda,n,M,\mathcal{O}]}$. Moreover, for every $x \in \bar{\mathcal{O}}$, $\theta_{[\lambda,n,M,\mathcal{O}]}(x) > 0$ and*

$$\theta_{[\lambda,n,M_1,\mathcal{O}]}(x) < \theta_{[\lambda,n,M_2,\mathcal{O}]}(x) \quad \textit{if } M_1 < M_2. \tag{4.40}$$

Furthermore, $\theta_{[\lambda,n,M,\mathcal{O}]}$ is a global attractor for (4.38)*.*

Proof. First, we introduce the change of variable

$$\theta = e^{\alpha m/D} w$$

in order to transform the problem (4.39) in

$$\begin{cases} -D\Delta w - \alpha \nabla m \cdot \nabla w - \lambda n w = -a e^{\alpha(p-1)m/D} w^p & \text{in } \mathcal{O}, \\ \partial_\nu w = 0 & \text{on } \partial\Omega, \\ w = M e^{-\alpha m/D} & \text{on } \partial\mathcal{O} \setminus \partial\Omega. \end{cases} \tag{4.41}$$

To establish the existence of arbitrarily large supersolutions of (4.41) we proceed as follows. For sufficiently large $k \in \mathbb{N}$, we will set

$$A_k := (\partial\mathcal{O} \setminus \partial\Omega) + B_{\frac{1}{k}}(0) = \left\{ x \in \Omega : \text{dist}(x, \partial\mathcal{O} \setminus \partial\Omega) < \frac{1}{k} \right\}.$$

Figure 4.3 shows an scheme of their construction in the special case when $\Omega \setminus \bar{\Omega}_0$ is connected. In such case also A_k is connected.

As the Lebesgue measure of A_k goes to zero as $k \uparrow \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_k\right] > 0 \quad \text{for all } k \geq k_0. \tag{4.42}$$

Let φ_{k_0} be any a principal eigenfunction associated to

$$\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right]$$

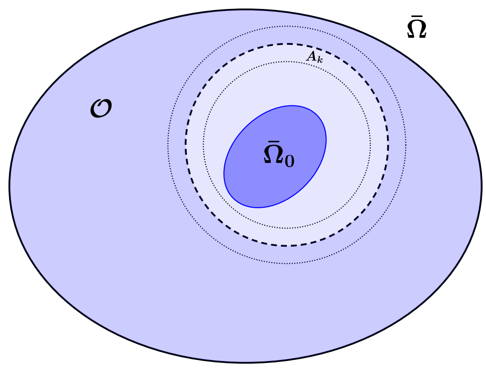

Figure 4.3: The neighborhoods A_k around $\partial\mathcal{O} \setminus \partial\Omega$

and consider any smooth function $\phi : \bar{\mathcal{O}} \to \mathbb{R}$ defined by

$$\phi(x) = \begin{cases} \varphi_{k_0}(x) & \text{in } A_{2k_0} \cap \bar{\mathcal{O}}, \\ g(x) & \text{in } \bar{\mathcal{O}} \setminus A_{2k_0}, \end{cases}$$

where g is any smooth function such that

$$\inf_{\bar{\mathcal{O}} \setminus A_{2k_0}} g > 0 \quad \text{and} \quad \partial_\nu g = 0 \quad \text{on } \partial\Omega.$$

In the general case when A_k is not connected, it must possess finitely many components, because Ω_0 is smooth. In such case, one should take the corresponding principal eigenfunction on each of these components.

Subsequently, we consider $\bar{w} := C\phi$, where $C > 0$ is sufficiently large so that

$$C^{p-1} a e^{\alpha(p-1)m/D} g^p > D\Delta g + \alpha \nabla m \cdot \nabla g + \lambda n g \quad \text{in } \mathcal{O} \setminus A_{2k_0} \tag{4.43}$$

and

$$C \geq \frac{M e^{-\alpha m/D}}{\min_{\partial\mathcal{O} \setminus \partial\Omega} \varphi_{k_0}} \qquad \text{for all } x \in \partial\mathcal{O} \setminus \partial\Omega. \tag{4.44}$$

We claim that $\bar{w} = C\phi$ is a supersolution of (4.41). Indeed, let $x \in A_{2k_0} \cap \mathcal{O}$. Then, by (4.42), we have that

$$-\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right] < 0 \leq C^{p-1} a(x) e^{\alpha(p-1)m(x)/D} \varphi_{k_0}^{p-1}(x).$$

Hence, multiplying by $C\varphi_{k_0}$ yields

$$-\sigma\left[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda n, \mathfrak{D}, A_{k_0}\right] C\varphi_{k_0}(x) \leq C^p a(x) e^{\alpha(p-1)m(x)/D} \varphi_{k_0}^p(x)$$

or, equivalently,

$$-D\Delta\bar{w} - \alpha\nabla m\cdot\nabla\bar{w} - \lambda n\bar{w} \geq -ae^{\alpha(p-1)m/D}\bar{w}^p \quad \text{in } A_{2k_0}\cap\mathcal{O}. \tag{4.45}$$

Moreover, thanks to (4.43), (4.45) also holds in $\mathcal{O}\setminus A_{2k_0}$, and, due to (4.44),

$$\bar{w}(x) \geq Me^{-\alpha m/D} \qquad \text{for all } x\in\partial\mathcal{O}\setminus\partial\Omega.$$

Note that, by the choice of g, $\bar{w}$ satisfies the boundary condition on $\partial\Omega$. Finally, we choose $\underline{w}=0$ as subsolution of (4.41). As $\underline{w}\leq\bar{w}$, the existence of a positive solution, $w_{[\lambda,n,M,\mathcal{O}]}$ can be inferred from the main theorem of Amann [4]. Necessarily,

$$0\leq w_{[\lambda,n,M,\mathcal{O}]}\leq\bar{w}.$$

Moreover, $w_{[\lambda,n,M,\mathcal{O}]}\neq 0$ because $w_{[\lambda,n,M,\mathcal{O}]}=Me^{-\alpha m/D}>0$ in $\partial\mathcal{O}\setminus\partial\Omega$. Therefore, the function

$$\theta_{[\lambda,n,M,\mathcal{O}]} := e^{\alpha m/D}w_{[\lambda,n,M,\mathcal{O}]}$$

provides us with a positive solution of (4.39). The remaining assertions of the theorem are a direct consequence from the maximum principle, arguing as in the proof of Theorem 2.4.1. The global attractiveness is a consequence from the uniqueness of the positive solution, by the abstract theory of D. Sattinger [40]. So, we will omit the technical details here. □ □

As a consequence from Theorem 4.5.1, the following limit is well defined

$$\theta_{[\lambda,n,\infty,\mathcal{O}]}(x) := \lim_{M\uparrow\infty}\theta_{[\lambda,n,M,\mathcal{O}]}(x) \quad \text{for each } x\in\bar{\mathcal{O}} \tag{4.46}$$

though it might be everywhere infinity. If we can show that $\theta_{[\lambda,n,M,\mathcal{O}]}(x)$ is bounded above by some function $U_{[\lambda,n,\mathcal{O}]}(x)$, uniformly bounded on compact subsets of $\mathcal{O}\cup\partial\Omega$ and independent of M, then, by a simple regularity and compactness argument, $\theta_{[\lambda,n,\infty,\mathcal{O}]}$ should be a solution of

$$\begin{cases} -\nabla\cdot(D\nabla\theta-\alpha\theta\nabla m)-\theta(\lambda n - a\theta^{p-1})=0 & \text{in } \mathcal{O},\\ D\partial_\nu\theta-\alpha\theta\partial_\nu m=0 & \text{on } \partial\Omega,\\ \theta=+\infty & \text{on } \partial\mathcal{O}\setminus\partial\Omega. \end{cases} \tag{4.47}$$

The existence of $U_{[\lambda,n,\mathcal{O}]}(x)$ is guaranteed by the next theorem.

Theorem 4.5.2. *Suppose $\mathcal{O}\subset\Omega\setminus\bar{\Omega}_0$ is a smooth subdomain satisfying* (4.36)*, $n\in\mathcal{C}(\bar{\Omega})$, $\lambda\in\mathbb{R}$ and $a\in\mathcal{C}^2(\bar{\Omega})$. Then, there exists a function $U_{[\lambda,n,\mathcal{O}]}\in\mathcal{C}^2(\mathcal{O}\cup\partial\Omega)$ such that*

$$\theta_{[\lambda,n,M,\mathcal{O}]}(x)\leq U_{[\lambda,n,\mathcal{O}]}(x) \quad \textit{for each } x\in\mathcal{O}\cup\partial\Omega \quad \textit{and} \quad \textit{for all } M>0.$$

Proof. The change of variable

$$\theta = e^{\frac{\alpha}{2D}m}v$$

transforms the problem (4.39) in

$$\begin{cases} -D\Delta v = W(\lambda,n)v - ae^{\frac{\alpha(p-1)}{2D}m}v^p & \text{in } \mathcal{O},\\ 2D\partial_\nu v-\alpha v\partial_\nu m=0 & \text{on } \partial\Omega,\\ v=e^{-\frac{\alpha}{2D}m}M & \text{on } \partial\mathcal{O}\setminus\partial\Omega, \end{cases} \tag{4.48}$$

where
$$W(\lambda,n) := \lambda n - \frac{\alpha}{2}\left(\Delta m + \frac{\alpha}{2D}|\nabla m|^2\right).$$
By Theorem 4.5.1, for each $M>0$
$$v_{[\lambda,n,M,\mathcal{O}]}(x) := e^{-\frac{\alpha}{2D}m}\theta_{[\lambda,n,M,\mathcal{O}]}(x).$$
is the unique positive solution of (4.48).

As $a\in\mathcal{C}^2(\bar\Omega)$, there exists $b\in\mathcal{C}^2(\bar\Omega)$ such that
$$b\equiv 0 \quad \text{in} \quad \Omega\setminus\mathcal{O}\supseteq\bar\Omega_0 \quad \text{and} \quad 0<b\leq a \quad \text{in} \quad \mathcal{O}\cup\partial\Omega. \tag{4.49}$$
In case $\mathcal{O}=\Omega\setminus\bar\Omega_0$, we can choose $b(x)=a(x)$, of course.

Now, we fix a $M_0>0$ and consider the function $\tilde V_{[\lambda,n,\mathcal{O}]}\in\mathcal{C}^2(\mathcal{O}\cup\partial\Omega)$ defined by

(i) $\tilde V_{[\lambda,n,\mathcal{O}]}(x)=v_{[\lambda,n,M_0,\mathcal{O}]}(x)$ for $x\in\bar\Omega$ and near $\partial\Omega$.

(ii) $\tilde V_{[\lambda,n,\mathcal{O}]}(x)=b^\beta(x)$ with $\beta=3/(1-p)<0$ for $x\in\mathcal{O}$ near $\partial\mathcal{O}\setminus\partial\Omega$.

(iii) $\tilde V_{[\lambda,n,\mathcal{O}]}(x)>0$ for $x\in\mathcal{O}\cup\partial\Omega$.

We claim that, for any sufficiently large positive constant $C>0$,
$$V_{[\lambda,n,\mathcal{O}]} := C\tilde V_{[\lambda,n,\mathcal{O}]}$$
is a supersolution of (4.48) for all $M>M_0$. Therefore,
$$\theta_{[\lambda,n,M,\mathcal{O}]}(x) = e^{\frac{\alpha}{2D}m(x)}v_{[\lambda,n,M,\mathcal{O}]}(x) \leq e^{\frac{\alpha}{2D}m(x)}V_{[\lambda,n,\mathcal{O}]}(x) =: U_{[\lambda,n,\mathcal{O}]}(x)$$
for each $x\in\mathcal{O}\cup\partial\Omega$, which ends the proof of the theorem. Indeed, since $\tilde V_{[\lambda,n,\mathcal{O}]}=v_{[\lambda,n,M_0,\mathcal{O}]}$ near $\partial\Omega$, we have that
$$2D\partial_\nu V_{[\lambda,n,\mathcal{O}]} - \alpha V_{[\lambda,n,\mathcal{O}]}\partial_\nu m = 0 \quad \text{on} \quad \partial\Omega,$$
and
$$\begin{aligned}-D\Delta V_{[\lambda,n,\mathcal{O}]} &= -CD\Delta\tilde V_{[\lambda,n,\mathcal{O}]} = CW(\lambda,n)\tilde V_{[\lambda,n,\mathcal{O}]} - aCe^{\frac{\alpha(p-1)}{2D}m}\tilde V^p_{[\lambda,n,\mathcal{O}]}\\ &= W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - aC^{-(p-1)}e^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}\\ &\geq W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}\end{aligned}$$
in a neighborhood of $\partial\Omega$, provided $C>1$.

On the other hand, for each $M>M_0$, if $x\in\mathcal{O}$ is sufficiently close to $\partial\mathcal{O}\setminus\partial\Omega$, we have that
$$V_{[\lambda,n,\mathcal{O}]}(x) = Cb^\beta(x) > e^{-\frac{\alpha}{2D}m(x)}M,$$
because $b=0$ on $\partial\mathcal{O}\setminus\partial\Omega$ and $\beta<0$. Using (4.49), a direct calculation shows that in a neighborhood of $\partial\mathcal{O}\setminus\partial\Omega$,
$$\begin{aligned}&-D\Delta V_{[\lambda,n,\mathcal{O}]} - W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} + ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}\\ &\quad = Cb^{p\beta+1}\left[-D\beta b\Delta b - D\beta(\beta-1)|\nabla b|^2 - W(\lambda,n)b^2 + \frac{a}{b}e^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\right]\\ &\quad \geq Cb^{p\beta+1}\left[-D\beta b\Delta b - D\beta(\beta-1)|\nabla b|^2 - W(\lambda,n)b^2 + e^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\right],\end{aligned}$$

since $a/b \geq 1$ in $\mathcal{O}$. Thus, there exists $C_0 > 1$ such that

$$-D\Delta V_{[\lambda,n,\mathcal{O}]} \geq W(\lambda,n)V_{[\lambda,n,\mathcal{O}]} - ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}$$

in a neighborhood of $\partial\mathcal{O} \setminus \partial\Omega$, for all $C > C_0$.

Finally, for any $x \in \mathcal{O}$ separated away from $\partial\mathcal{O}$, we have that

$$\begin{aligned}&-D\Delta V_{[\lambda,n,\mathcal{O}]}(x) - W(\lambda,n)(x)V_{[\lambda,n,\mathcal{O}]}(x) + ae^{\frac{\alpha(p-1)}{2D}m}V^p_{[\lambda,n,\mathcal{O}]}(x)\\ &= C\left[-D\Delta\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) - W(\lambda,n)(x)\tilde{V}_{[\lambda,n,\mathcal{O}]}(x) + ae^{\frac{\alpha(p-1)}{2D}m}C^{p-1}\tilde{V}^p_{[\lambda,n,\mathcal{O}]}(x)\right] \geq 0\end{aligned}$$

for sufficiently large $C > C_0$. Therefore, $V_{[\lambda,n,\mathcal{O}]}$ is a supersolution of (4.48) and the proof is complete.
□ □

In the previous proof we have used a technical device of Y. Du and Q. Huang [15]. Owing to Theorem 4.5.2, the limit (4.46) is finite in $\mathcal{O} \cup \partial\Omega$ and $\theta_{[\lambda,n,\infty,\mathcal{O}]}$ is a solution of (4.47). The following result characterizes the existence of solutions of the singular problem (4.47).

Theorem 4.5.3. *Under the same conditions of Theorem 4.5.2,* (4.47) *possesses a minimal and a maximal positive solution, denoted by* $L^{\min}_{[\lambda,n,\mathcal{O}]}$ *and* $L^{\max}_{[\lambda,n,\mathcal{O}]}$*, respectively, in the sense that any other positive solution* L *of* (4.47) *must satisfy*

$$L^{\min}_{[\lambda,n,\mathcal{O}]} \leq L \leq L^{\max}_{[\lambda,n,\mathcal{O}]} \qquad \text{in } \mathcal{O} \cup \partial\Omega.$$

Proof. Subsequently, for each $k \in \mathbb{N}$, we consider

$$\mathcal{O}_k := \{x \in \mathcal{O} \ : \ \operatorname{dist}(x, \partial\mathcal{O} \setminus \partial\Omega) > 1/k\}.$$

For k large, $\mathcal{O}_k$ satisfies (4.36) and we can consider (4.47) in each of these $\mathcal{O}_k$'s, instead of $\mathcal{O}$. By our previous analysis, (4.47) in $\mathcal{O}_k$ admits a positive solution, $\theta_{[\lambda,n,\infty,\mathcal{O}_k]}$, for sufficiently large k. Then, the minimal and maximal solutions of (4.47) in $\mathcal{O}$ are given by

$$L^{\min}_{[\lambda,n,\mathcal{O}]} := \theta_{[\lambda,n,\infty,\mathcal{O}]} \quad \text{and} \quad L^{\max}_{[\lambda,n,\mathcal{O}]} := \lim_{k\to\infty} \theta_{[\lambda,n,\infty,\mathcal{O}_k]}.$$

Indeed, let L be a positive solution of (4.47). By the maximum principle

$$\theta_{[\lambda,n,M,\mathcal{O}]} \leq L \quad \text{for all } \ M > 0$$

and letting $M \to \infty$ yields

$$L^{\min}_{[\lambda,n,\mathcal{O}]} := \theta_{[\lambda,n,\infty,\mathcal{O}]} := \lim_{M\to\infty} \theta_{[\lambda,n,M,\mathcal{O}]} \leq L.$$

Similarly, for sufficiently large $k > 0$,

$$L \leq \theta_{[\lambda,n,\infty,\mathcal{O}_k]} \quad \text{in } \ \mathcal{O}_k \cup \partial\Omega$$

and letting $k \to \infty$ shows that

$$L \leq \lim_{k\to\infty} \theta_{[\lambda,n,\infty,\mathcal{O}_k]} =: L^{\max}_{[\lambda,n,\mathcal{O}]} \quad \text{in } \ \mathcal{O} \cup \partial\Omega.$$

This ends the proof. □ □

4.6 Dynamics of (1.1) when $m > 0$

In this section, we suppose (4.6) and $m > 0$, in the sense that $m \geq 0$ but $m \neq 0$. Our main goal is to ascertain the dynamics of (1.1) in this particular case.

As in the special case when $m = 0$ in Ω_0, the dynamics of (1.1) it has been already described by Theorem 4.2.4, throughout this section we will assume that $m(x_+) > 0$ for some $x_+ \in \Omega_0$. Then, according to Theorems 4.2.1 and 4.2.3, there exists $\lambda^* := \lambda_2(\alpha, m) > 0$ such that

$$\Sigma(\lambda) = \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m, \mathfrak{N}, \Omega] \begin{cases} > 0 & \text{if } \lambda < 0, \\ = 0 & \text{if } \lambda = 0, \\ < 0 & \text{if } \lambda > 0. \end{cases}$$

and

$$\Sigma_0(\lambda) = \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m, \mathfrak{D}, \Omega_0] \begin{cases} > 0 & \text{if } \lambda < \lambda^*, \\ = 0 & \text{if } \lambda = \lambda^*, \\ < 0 & \text{if } \lambda > \lambda^*. \end{cases}$$

Then, thanks to Theorem 4.2.4, for every $\lambda \in (0, \lambda^*)$ the problem (1.2) possesses a unique positive solution, $\theta_\lambda := \theta_{\lambda,m}$. Moreover, owing to Theorem 4.4.1, if, in addition, a is of class $\mathcal{C}^1$ in a neighborhood of $\partial\Omega_0$, then

$$\lim_{\lambda \uparrow \lambda^*} \theta_\lambda = \infty \quad \text{uniformly in } \bar{\Omega}_0.$$

The next result provides us the convergence to ∞ of the solution of (1.1), $u_\lambda(x, t; u_0)$, as $t \uparrow \infty$, for all $x \in \bar{\Omega}_0$ and $\lambda \geq \lambda^*$.

Theorem 4.6.1. *Suppose $m \geq 0$, $m(x_+) > 0$ for some $x_+ \in \Omega_0$, and $a(x)$ is of class $\mathcal{C}^1$ in a neighborhood of $\partial\Omega_0$. Then, for every $\lambda \geq \lambda^*$,*

$$\lim_{t \to \infty} u_\lambda(\cdot, t; u_0) = \infty \quad \textit{uniformly in } \bar{\Omega}_0.$$

Proof. As $m > 0$, by the parabolic maximum principle, for each $\varepsilon > 0$ and $t \geq 0$,

$$u_\lambda(\cdot, t; u_0) \geq u_{\lambda^* - \varepsilon}(\cdot, t; u_0) \quad \text{in } \Omega$$

since $\lambda > \lambda^* - \varepsilon$. Hence, thanks to Theorem 4.2.4,

$$\liminf_{t \to \infty} u_\lambda(\cdot, t; u_0) \geq \lim_{t \to \infty} u_{\lambda^* - \varepsilon}(\cdot, t; u_0) = \theta_{\lambda^* - \varepsilon} \quad \text{in } \Omega.$$

Consequently, by Theorem 4.4.1,

$$\liminf_{t \to \infty} u_\lambda(\cdot, t; u_0) \geq \lim_{\varepsilon \downarrow 0} \theta_{\lambda^* - \varepsilon} = \infty \quad \text{uniformly in } \bar{\Omega}_0.$$

The proof is complete. □ □

The next result provides us with the dynamics of (1.1).

Theorem 4.6.2. *Suppose $m \geq 0$, $m(x_+) > 0$ for some $x_+ \in \Omega_0$, and $a \in \mathcal{C}^2(\bar{\Omega})$. Then, the following assertions are true:*

(a) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = 0$ *in* $\mathcal{C}(\bar\Omega)$ *if* $\lambda \leq 0$.

(b) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = \theta_\lambda$ *in* $\mathcal{C}(\bar\Omega)$ *if* $0 < \lambda < \lambda^*$.

(c) *In case* $\lambda \geq \lambda^*$, *we have that:*

  (i) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = \infty$ *uniformly in* $\bar\Omega_0$.

  (ii) *In* $\bar\Omega \setminus \bar\Omega_0$ *the next estimates hold*

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \leq \liminf_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq \limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq L^{\max}_{[\lambda,m,\Omega\setminus\bar\Omega_0]}.$$

  (iii) *If, in addition,* u_0 *is a subsolution of* (1.2), *then*

$$\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \textit{in } \bar\Omega\setminus\bar\Omega_0.$$

Proof. Parts (a) and (b) follow as a direct consequence from Theorem 4.2.4 in case $n = m > 0$. Part (c)(i) is given by Theorem 4.6.1. So, it remains to prove Parts (c)(ii) and (c)(iii). Suppose $\lambda \geq \lambda^*$. By Theorem 4.6.1, for each $M > 0$ there exists a constant $T_M > 0$ such that

$$u_\lambda(x,t;u_0) \geq M \quad \text{for each } (x,t) \in \partial\Omega_0 \times [T_M,\infty).$$

By the parabolic maximum principle, for each $(x,t) \in \bar\Omega\setminus\bar\Omega_0 \times (0,\infty)$, we find that

$$u_\lambda(x,t+T_M;u_0) \geq u_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x,t;u_\lambda(\cdot,T_M;u_0)).$$

Therefore, for each $x \in \bar\Omega\setminus\bar\Omega_0$,

$$\liminf_{t\to\infty} u_\lambda(x,t;u_0) \geq \lim_{t\to\infty} u_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x,t;u_\lambda(\cdot,T_M;u_0)) = \theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]}(x). \tag{4.50}$$

Hence, owing to Theorem 4.5.3 and letting $M \to \infty$ in (4.50) yields

$$\liminf_{t\to\infty} u_\lambda(\cdot,t;u_0) \geq L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega\setminus\bar\Omega_0,$$

which ends the proof of one of the inequalities.

Next, we will asume that u_0 is a subsolution of (1.2). Then, for each $t > 0$ the function $u_\lambda(\cdot,t;u_0)$ is a subsolution of (1.2) in Ω, since $t \to u_\lambda(\cdot,t;u_0)$ is increasing. Fix $t > 0$ and set

$$M_t := \max_{\partial\Omega_0} u_\lambda(\cdot,t;u_0).$$

Then, for every $M \geq M_t$,

$$u_\lambda(\cdot,t;u_0) \leq \theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega\setminus\bar\Omega_0,$$

because $u_\lambda(\cdot,t;u_0)$ is a subsolution of the problem (4.39). Therefore,

$$u_\lambda(\cdot,t;u_0) \leq \lim_{M\to\infty} \theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_0]} = L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega\setminus\bar\Omega_0,$$

and letting $t \to \infty$,

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \le L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega\setminus\bar\Omega_0.$$

Therefore, Part (c)(iii) get proven.

To conclude the proof, it remains to obtain the upper estimates for an arbitrary $u_0 > 0$. The strategy adopted here to get these estimates consists in obtaining bounds in $\bar\Omega\setminus\bar\Omega_0$ for $u_\lambda(\cdot,t;u_0)$. Those bounds can be derived as follows. Fix $\lambda \ge \lambda^*$, set $m_M := \max_{\bar\Omega} m > 0$, and let φ_λ denote the principal eigenfunction associated with the eigenvalue problem

$$\begin{cases} -\nabla\cdot(D\nabla\psi - \alpha\psi\nabla m) - \lambda m_M\psi = \sigma\psi & \text{in } \Omega, \\ D\partial_\nu\psi - \alpha\psi\partial_\nu m = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized so that with $\|\varphi_\lambda\|_\infty = 1$. Since $\varphi_\lambda(x) > 0$ for all $x \in \bar\Omega$, there exists $\kappa > 1$ such that

$$u_0 < \kappa\varphi_\lambda. \tag{4.51}$$

Subsequently, we denote

$$\Sigma_{m_M}(\lambda) := \sigma[-\nabla\cdot(D\nabla - \alpha\nabla m) - \lambda m_M, D\partial_\nu - \alpha\partial_\nu m, \Omega].$$

Thanks to (4.3), $\Sigma_{m_M}(\lambda) < 0$ because $m_M > 0$ and $\lambda \ge \lambda^* > 0$. Let $\Lambda > \lambda$ be such that

$$\|a\|_\infty \kappa^{p-1} + \Sigma_{m_M}(\lambda) \le (\Lambda - \lambda)m_M.$$

For this choice, we find that for each $x \in \Omega$

$$\Sigma_{m_M}(\lambda)\kappa\varphi_\lambda + (\lambda - \Lambda)m_M\kappa\varphi_\lambda \le -a(x)\kappa^p\varphi_\lambda \le -a(x)\kappa^p\varphi_\lambda^p$$

because $\varphi_\lambda \le 1$. Equivalently,

$$-\nabla\cdot[D\nabla(\kappa\varphi_\lambda) - \alpha\kappa\varphi_\lambda\nabla m] - \Lambda m_M\kappa\varphi_\lambda \le -a(x)\kappa^p\varphi_\lambda^p.$$

Moreover, $\kappa\varphi_\lambda$ satisfies

$$D\partial_\nu(\kappa\varphi_\lambda) - \alpha\kappa\varphi_\lambda\partial_\nu m = 0 \quad \text{on } \partial\Omega.$$

Therefore, $\kappa\varphi_\lambda$ provides us with a subsolution of the problem

$$\begin{cases} -\nabla\cdot(D\nabla\theta - \alpha\theta\nabla m) - \theta(\Lambda m_M - a\theta^{p-1}) = 0 & \text{in } \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, by the parabolic maximum principle, it follows from (4.51) that

$$u_\lambda(x,t;u_0) \le u_\lambda(x,t;\kappa\varphi_\lambda) \tag{4.52}$$

for all $(x,t) \in \Omega\times(0,\infty)$. Similarly, since $\lambda m \le \Lambda m_M$,

$$u_\lambda(x,t;\kappa\varphi_\lambda) \le u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda) \quad \text{for each } (x,t) \in \Omega\times(0,\infty). \tag{4.53}$$

Therefore, by (4.52) and (4.53), for each $(x,t) \in \Omega \times (0,\infty)$, we have that

$$\lim_{t\to\infty} u_\lambda(x,t;u_0) \leq \lim_{t\to\infty} u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda).$$

Lastly, since $m \leq m_M$,

$$\lambda_2(\alpha, m_M) \leq \lambda_2(\alpha, m) = \lambda^* \leq \lambda < \Lambda$$

and, according to Part (c)(iii), we find that

$$\lim_{t\to\infty} u_\lambda(x,t;u_0) \leq \lim_{t\to\infty} u_{[\Lambda,m_M]}(x,t;\kappa\varphi_\lambda) = L^{\min}_{[\Lambda,m_M,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega \setminus \bar\Omega_0. \tag{4.54}$$

Consequently, $u_\lambda(x,t;u_0)$ is uniformly bounded above in any compact subset of $\bar\Omega \setminus \bar\Omega_0$ for each $t > 0$, which provides us with the necessary a priori bounds to complete the proof of the theorem.

Subsequently, for sufficiently large $k \in \mathbb{N}$ we consider

$$\Omega_k := \left\{ x \in \Omega \setminus \bar\Omega_0 \ : \ d(x,\Omega_0) < 1/k \right\}.$$

Fix one of these values of k. Since $\partial\Omega_k \subset \Omega \setminus \bar\Omega_0$, it follows from (4.54) that there exists a constant $M_0 > 0$ such that, for each $M \geq M_0$ and $t > 0$,

$$u_\lambda(\cdot,t;u_0) \leq M \quad \text{on } \partial\Omega_k,$$

and hence, the parabolic maximum principle shows that

$$u_\lambda(\cdot,t;u_0) \leq u_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]}(\cdot,t;u_0) \quad \text{in } \bar\Omega \setminus \Omega_k \tag{4.55}$$

for all $t > 0$. By Theorem 4.5.1, (4.39) has a unique positive solution, $\theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]}$, which is a global attractor of (4.38). Letting $t \to \infty$ in (4.55) yields

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq \theta_{[\lambda,m,M,\Omega\setminus\bar\Omega_k]} \quad \text{in } \bar\Omega \setminus \Omega_k.$$

Consequently, taking limits as $M \to \infty$ gives

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq L^{\min}_{[\lambda,m,\Omega\setminus\bar\Omega_k]} \quad \text{in } \bar\Omega \setminus \Omega_k. \tag{4.56}$$

Finally, thanks to the proof of Theorem 4.5.3, letting $k \to \infty$ in (4.56) yields

$$\limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq L^{\max}_{[\lambda,m,\Omega\setminus\bar\Omega_0]} \quad \text{in } \bar\Omega \setminus \bar\Omega_0$$

and the upper estimate is proven. □ □

4.7 Dynamics of (1.1) when m changes sign

Throughout this section, we suppose that m changes sign in Ω, i.e., there are $x_- \in \Omega$ and $x_+ \in \Omega$ such that

$$m(x_-) < 0 \quad \text{and} \quad m(x_+) > 0.$$

The main goal of this section is to obtain the dynamics of (1.1) for all $\lambda > 0$. In the special case when that $m \leq 0$ in Ω_0, according to Theorems 4.2.1 and 4.2.4, the existence of θ_λ is guaranteed for $\lambda > \lambda_+ \geq 0$ and it is a global attractor for (1.1). So, we suppose that

$$x_+ \in \Omega_0.$$

By Theorem 4.2.3, there exists $\lambda^* := \lambda_2(\alpha, m) > 0$ such that

$$\Sigma_0(\lambda) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m, \mathfrak{D}, \Omega_0] \begin{cases} > 0 & \text{if } \lambda \in [0, \lambda^*), \\ = 0 & \text{if } \lambda = \lambda^*, \\ < 0 & \text{if } \lambda > \lambda^*. \end{cases}$$

Subsequently, for sufficiently small $\varepsilon > 0$, we introduce the truncated functions

$$m_\varepsilon(x) := \begin{cases} \varepsilon & \text{if } m(x) \geq \varepsilon, \\ m(x) & \text{if } m(x) < \varepsilon. \end{cases}$$

By Theorem 4.2.4, the limiting behavior of $u_{[\lambda, m_\varepsilon]}(x, t; u_0)$ as $t \to \infty$ is regulated by the positive solution $\theta_{[\lambda, m_\varepsilon]}$ whenever

$$0 \leq \tilde{\lambda}_+(\alpha, m_\varepsilon) < \lambda < \lambda_2(\alpha, m_\varepsilon) \equiv \lambda^*_\varepsilon \tag{4.57}$$

where $\tilde{\lambda}_+(\alpha, m_\varepsilon) := \lambda_+(\alpha, m_\varepsilon)$ if $\lambda_+(\alpha, m_\varepsilon)$ exists, while it equals zero if not. It should be remembered that $\lambda_+(\alpha, m_\varepsilon)$ is the unique positive zero of the principal eigenvalue

$$\Sigma(\lambda, m_\varepsilon) := \sigma[-D\Delta - \alpha\nabla m \cdot \nabla - \lambda m_\varepsilon, \mathfrak{N}, \Omega]$$

if it exists. Note that

$$\lim_{\varepsilon \to 0} \tilde{\lambda}_+(\alpha, m_\varepsilon) = \lim_{\varepsilon \to 0} \lambda_+(\alpha, m_\varepsilon) = \infty.$$

Fix $\hat{\lambda} \geq \lambda^*$. By the continuity of the principal eigenvalue with respect to ε, there exists $\varepsilon_0 := \varepsilon_0(\hat{\lambda}) > 0$ such that

$$\lambda_+(\alpha, m_{\varepsilon_0}) = \hat{\lambda}.$$

Similarly, since

$$\lim_{\varepsilon \to 0} \lambda_2(\alpha, m_\varepsilon) = \infty \quad \text{and} \quad \hat{\lambda} \geq \lambda^*,$$

there exists $\varepsilon_1 := \varepsilon_1(\hat{\lambda}) > \varepsilon_0$ such that $\lambda^*_{\varepsilon_1} = \hat{\lambda}$. Note that $\varepsilon_1 = \max m$, or, equivalently, $m_{\varepsilon_1} = m$, if $\hat{\lambda} = \lambda^*$. According to (4.57), for each $\varepsilon \in (\varepsilon_0, \varepsilon_1)$, the problem (4.2) possesses a unique positive solution, $\theta_{[\hat{\lambda}, m_\varepsilon]}$. Figure 4.4 shows the graphs of $\Sigma(\lambda, m_\varepsilon)$ and $\Sigma_0(\lambda, m_\varepsilon)$ for $\varepsilon = 0$ and some $\varepsilon \in (\varepsilon_0, \varepsilon_1)$.

The next result provides us with the limiting behavior of $\theta_{[\hat{\lambda}, m_\varepsilon]}$ in $\bar{\Omega}_0$ as $\varepsilon \uparrow \varepsilon_1$.

Theorem 4.7.1. *Suppose $a(x)$ is of class $\mathcal{C}^1$ in a neighborhood of $\partial\Omega_0$. Then*

$$\lim_{\varepsilon \uparrow \varepsilon_1} \theta_{[\hat{\lambda}, m_\varepsilon]}(x) = +\infty \quad \textit{uniformly in } x \in \bar{\Omega}_0.$$

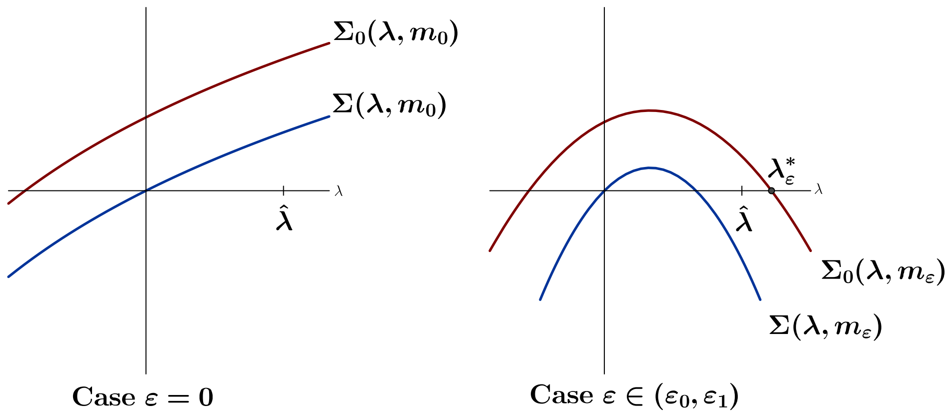

Figure 4.4: The graphs of $\Sigma(\lambda, m_\varepsilon)$ and $\Sigma_0(\lambda, m_\varepsilon)$

Proof. We will argue as in the proof of the Theorem 4.4.1. For $\delta > 0$ with $\delta \simeq 0$, consider the holomorphic perturbation from Ω_0, Ω_δ, defined in (4.23), as well as the principal eigenvalue

$$\begin{aligned} S(\delta) &:= \sigma[-D\Delta - \alpha \nabla m \cdot \nabla - \hat{\lambda} m_{\varepsilon_1}, \mathfrak{D}, \Omega_\delta] \\ &= \sigma[-\nabla \cdot (D\nabla - \alpha \nabla m) - \hat{\lambda} m_{\varepsilon_1}, \mathfrak{D}, \Omega_\delta]. \end{aligned}$$

Let $\varphi[\delta] > 0$ be the unique positive solution of

$$\begin{cases} -\nabla \cdot (D\nabla \varphi[\delta] - \alpha \varphi[\delta] \nabla m) - \hat{\lambda} m_{\varepsilon_1} \varphi[\delta] = S(\delta)\varphi[\delta] & \text{in } \Omega_\delta, \\ \varphi[\delta] = 0 & \text{on } \partial\Omega_\delta, \end{cases} \tag{4.58}$$

satisfying

$$\|\varphi[\delta]\|_{\infty, \Omega_\delta} = 1.$$

Applying (4.10) and (4.24) to the weight function $n = m_{\varepsilon_1}$ yields

$$S(\delta) = S_1 \delta + O(\delta^2) \quad \text{with} \;\; S_1 < 0.$$

Let $\varepsilon \in (\varepsilon_0, \varepsilon_1)$ such that

$$S(\delta) < S(\delta/2) < \hat{\lambda}(\varepsilon - \varepsilon_1) < 0 \tag{4.59}$$

and consider the function $\vartheta_\delta \in \mathcal{C}(\bar{\Omega})$ defined by

$$\vartheta_\delta(y) = \begin{cases} C\varphi[\delta](y) & \text{for } \; y \in \bar{\Omega}_\delta, \\ 0 & \text{for } \; y \notin \Omega_\delta, \end{cases}$$

where $C > 0$ is a constant to be chosen later. If we suppose that

$$a(y) C^{p-1} \varphi^{p-1}[\delta](y) \leq S(\delta/2) - S(\delta) \quad \text{for all } \; y \in \Omega_\delta \tag{4.60}$$

then ϑ_δ is a subsolution of problem

$$\begin{cases} -\nabla\cdot(D\nabla\theta-\alpha\theta\nabla m)-\hat{\lambda}m_\varepsilon\theta=-a\theta^p & \text{in } \Omega, \\ D\partial_\nu\theta-\alpha\theta\partial_\nu m=0 & \text{on } \partial\Omega. \end{cases} \tag{4.61}$$

Indeed, by (4.59), for every $y\in\Omega_\delta$

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y)<\hat{\lambda}(\varepsilon-\varepsilon_1)-S(\delta).$$

Hence, since $\varepsilon-\varepsilon_1\le m_\varepsilon-m_{\varepsilon_1}$,

$$a(y)C^{p-1}\varphi^{p-1}[\delta](y)<\hat{\lambda}(m_\varepsilon-m_{\varepsilon_1})-S(\delta).$$

Multiplying by $C\varphi[\delta]$, it follows from (4.58) that ϑ_δ is a subsolution of problem (4.61).

As in the proof of Theorem 4.4.1, (4.60) holds if $C=C(\delta)$ is given by (4.32). Therefore,

$$\lim_{\delta\downarrow 0}\vartheta_\delta(y)=+\infty \quad \text{for all } \ y\in\bar{\Omega}_0. \tag{4.62}$$

Finally, by the maximum principle, for each $\delta>0$, $\delta\simeq 0$, satisfying (4.59), we have that

$$\vartheta_\delta(y)\le u_{[\hat{\lambda},m_\varepsilon]}(y) \quad \text{for all } \ y\in\bar{\Omega}.$$

Consequently, (4.62) ends the proof. □ □

The next result, provides us the behavior of $u_{\hat{\lambda}}(x,t;u_0)$ in $\bar{\Omega}_0$ as $t\uparrow\infty$. It is a counterpart of Theorem 4.6.1 for the case dealt with in this section.

Theorem 4.7.2. *If $a(x)$ is of class $\mathcal{C}^1$ in a neighborhood of $\partial\Omega_0$, then*

$$\lim_{t\to\infty}u_{\hat{\lambda}}(x,t;u_0)=+\infty \quad \textit{uniformly in } \ \bar{\Omega}_0.$$

Proof. Let $\varepsilon\in(\varepsilon_0,\varepsilon_1)$. Since $\hat{\lambda}>0$ and $m\ge m_\varepsilon$ in Ω, by the parabolic maximum principle, we find that

$$u_{\hat{\lambda}}(\cdot,t;u_0)\ge u_{[\hat{\lambda},m_\varepsilon]}(\cdot,t;u_0) \quad \text{in } \ \Omega$$

for all $t>0$. Thus, letting $t\to\infty$ yields

$$\liminf_{t\to\infty}u_{\hat{\lambda}}(\cdot,t;u_0)\ge\lim_{t\to\infty}u_{[\hat{\lambda},m_\varepsilon]}(\cdot,t;u_0)=\theta_{[\hat{\lambda},m_\varepsilon]} \quad \text{in } \ \Omega.$$

As this holds for all $\varepsilon\in(\varepsilon_0,\varepsilon_1)$, from Theorem 4.7.1 it becomes apparent that

$$\liminf_{t\to\infty}u_{\hat{\lambda}}(\cdot,t;u_0)\ge\lim_{\varepsilon\uparrow\varepsilon_1}\theta_{[\hat{\lambda},m_\varepsilon]}=\infty \quad \text{uniformly in } \ \bar{\Omega}_0$$

and the proof of the theorem is complete. □ □

Finally, the next theorem provides us with the dynamics of (1.1) for the class of m's dealt with in this section. As the proof follows the general patterns of the proof of Theorem 4.6.2, we will omit the technical details here. It should be emphasized that Theorem 4.7.2 holds true for all $\hat{\lambda}\ge\lambda^*$.

Theorem 4.7.3. *If* $a \in \mathcal{C}^2(\bar{\Omega})$*, then:*

(a) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = 0$ *in* $\mathcal{C}(\bar{\Omega})$ *if* $0 \leq \lambda \leq \tilde{\lambda}_+(\alpha,m)$.

(b) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = \theta_\lambda$ *in* $\mathcal{C}(\bar{\Omega})$ *if* $\tilde{\lambda}_+(\alpha,m) < \lambda < \lambda^*$.

(c) *In case* $\lambda \geq \lambda^*$*, the following assertions are true:*

  (i) $\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = +\infty$ *uniformly in* $\bar{\Omega}_0$.

  (ii) *In* $\bar{\Omega} \setminus \bar{\Omega}_0$ *the following estimate holds*

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \leq \liminf_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq \limsup_{t\to\infty} u_\lambda(\cdot,t;u_0) \leq L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}.$$

  (iii) *If, in addition,* u_0 *is a subsolution of* (1.2)*, then*

$$\lim_{t\to\infty} u_\lambda(\cdot,t;u_0) = L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \quad \text{in } \bar{\Omega}\setminus\bar{\Omega}_0.$$

Chapter 5

Conclusions

Thanks to the theorem of characterization of the maximum principle of López-Gómez and Molina-Meyer [34] and the later refinements of [28] and Amann and López-Gómez [5], which have been collected in [32, Th. 7.10], and the main Theorem of Crandall and Rabinowitz [12], we have characterized the existence and uniqueness of the positive solutions of (1.2), and, in addition, have obtained that the set of positive steady states (λ, u) consists of two global real analytic arcs of curve bifurcating from zero at $\lambda = 0$ and $\lambda = \lambda_-(\alpha) < 0$ and from infinity at $\lambda = \lambda_*(\alpha) < 0$ and $\lambda = \lambda^*(\alpha) > 0$ for α sufficiently large. Although, in general, it remains an open problem to ascertain whether or not the model (1.2) can admit some additional positive solution for $\lambda < \lambda_*(\alpha)$ or $\lambda > \lambda^*(\alpha)$, we have been able to prove that (1.2) cannot admit a positive solution for this range of values of λ when $a(x) > 0$ for all $x \in \bar{\Omega}$ or when (1.6) holds, as well as determine explicitly the values of $\lambda_*(\alpha)$ and $\lambda^*(\alpha)$.

The analysis carried out in Chapter 2 is imperative to determine the upper and lower estimates of the positive steady states in Chapter 3. Under the hypothesis of Theorem 2.6.1, which ensures the existence of positive solutions of (1.2) for α sufficiently large, and some other more technical hypotheses, we have proved that these positive solutions are bounded when $\alpha \to \infty$ and tend to zero far from the maximums of m, which describes how the individuals of the species tend to accumulate in the areas with the greatest local population growth.

On the other hand, under the hypotheses of Theorem 2.6.1, it should be noted that the results of this Thesis are rather paradoxical and unexpected, because, under the appropriate conditions on $a(x)$, the larger the advection, measured by α, the smaller the population as times grows. Actually, when $a(x)$ does not vanish nearby the maxima of m, an increase of the advection can lead to the emergence of a positive steady state that did not exist for smaller advection values. As a by-product, explosive solutions can stabilize towards an equilibrium by increasing the advection. Consequently, increasing the advection might provoke a decay in the density of the population as time passes by, which might be considered as the most innovative result of this Thesis. If, instead, we suppose that there is exponential growth in a neighborhood of a positive maximum of m, Theorem 3.4.3 dictates that the species grows up in the areas with the greatest local population growth as α increases.

In Chapter 4 the dynamics of (1.1) are fully described when $a(x) > 0$ for all $x \in \bar{\Omega}$ or (1.6) holds. As predicted in Chapter 1, in the absence of a positive steady state, when $u = 0$ is linearly unstable as a solution of (1.2), the dynamics of (1.1) is governed by the metasolutions of (1.2) which are infinite in

$\bar{\Omega}_0$. These are the first available results for non-self-adjoint differential operators.

Finally, we should emphasize the difficulty of dealing with the case when m changes sign in Ω. One of the hardest results in this sense has been (1.7). This proof consists in perturbing the function m in order to be able to use comparison techniques since we cannot perturb the parameter λ, as it is usually done in the literature, because m changes sign. Undoubtedly, the latest result has been posible thanks to the information about the model previously found in this thesis.

Chapter 6

Resumen

6.1 Introducción y objetivos

Esta tesis estudia el problema parabólico

$$\begin{cases} \partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda m - a u^{p-1}) & \text{en } \Omega, \quad t > 0, \\ D\partial_\nu u - \alpha u \partial_\nu m = 0 & \text{sobre } \partial\Omega, \quad t > 0, \\ u(\cdot, 0) = u_0 > 0 & \text{en } \Omega, \end{cases} \tag{6.1}$$

donde Ω es un dominio (abierto y conexo) acotado regular de $\mathbb{R}^N$, $N \geq 1$, $D > 0$, $\alpha > 0$, $\lambda \in \mathbb{R}$, $p \geq 2$, $a \in \mathcal{C}(\bar{\Omega})$ cumple $a > 0$, en el sentido de que $a \geq 0$ y $a \neq 0$, ν es el vector exterior unitario a lo largo de la frontera de Ω, $\partial\Omega$, y $m \in \mathcal{C}^2(\bar{\Omega})$ es una función tal que $m(x_+) > 0$ para algún $x_+ \in \Omega$. Así, o $m \geq 0$, $m \neq 0$, en Ω o m cambia de signo en Ω. El dato inicial u_0 está en $L^\infty(\Omega)$.

Bajo estas condiciones, es bien conocido que existe $T > 0$ tal que (6.1) admite una única solución clásica, denotada por $u(x, t; u_0)$, en $[0, T]$ (ver, e.g., Henry [19], Daners y Koch [14], y Lunardi [37]). Además, la solución es única si existe, y por el principio del máximo parabólico fuerte de Nirenberg [39], $u(\cdot, t; u_0) \gg 0$ en Ω, en el sentido de que

$$u(x, t; u_0) > 0 \quad \text{para todo} \quad x \in \bar{\Omega} \quad \text{y} \quad t \in (0, T].$$

Así, ya que $a > 0$ en Ω, tenemos que

$$\partial_t u = \nabla \cdot (D\nabla u - \alpha u \nabla m) + u(\lambda m - a u^{p-1}) \leq \nabla \cdot (D\nabla u - \alpha u \nabla m) + \lambda m u$$

y, por tanto, gracias otra vez al principio del máximo parabólico

$$u(\cdot, t; u_0) \ll z(\cdot, t; u_0) \quad \text{para todo} \quad t \in (0, T],$$

donde $z(x, t; u_0)$ representa la única solución del problema lineal parabólico

$$\begin{cases} \partial_t z = \nabla \cdot (D\nabla z - \alpha z \nabla m) + \lambda m z & \text{en } \Omega, \quad t > 0, \\ D\partial_\nu z - \alpha z \partial_\nu m = 0 & \text{sobre } \partial\Omega, \quad t > 0, \\ z(\cdot, 0) = u_0 > 0 & \text{en } \Omega. \end{cases}$$

Como z está globalmente definida en tiempo, $u(x,t;u_0)$ no puede explotar en tiempo finito y, por tanto, está globalmente definida para todo $t > 0$. En las aplicaciones es imprescindible caracterizar el comportamiento asintótico de $u(x,t;u_0)$ cuando $t \uparrow \infty$.

Una versión especial de este modelo (con $\lambda = 1$, $p = 2$ y $a(x) > 0$ para todo $x \in \bar{\Omega}$) fue introducida por Belgacem y Cosner [6] para "*analizar el efecto de incorporar un término que describe la dirección o advección a lo largo de un gradiente medioambiental para modelos de reacción difusión en dinámica de poblaciones.* " En estos modelos, $u(x,t;u_0)$ representa la densidad de población en $x \in \Omega$ después de un tiempo $t > 0$, $D > 0$ es la constante de difusión, y "*la constante α mide la tasa con la que se desplaza la población en dirección al gradiente de la tasa de crecimiento $m(x)$. Si $\alpha < 0$, la población se movería en la dirección en la que m decrece, o sea, lejos de las regiones más favorables del habitat, hacia las regiones más desfavorables.*" El parámetro λ es una especie de re-normalización del término de transporte con objeto de adquirir una perspectiva lo más amplia posible sobre el análisis llevado a cabo en [6].

Aunque en el caso especial en que $a(x) > 0$ para todo $x \in \bar{\Omega}$ es bien conocido que la dinámica de (6.1) esta regulada por sus estados estacionarios no negativos, cuando existen, que son las soluciones no negativas del problema de contorno elíptico semilineal

$$\begin{cases} \nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) + \theta(\lambda m - a\theta^{p-1}) = 0 & \text{en } \Omega, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{sobre } \partial\Omega, \end{cases} \tag{6.2}$$

en el caso general en que la función peso $a(x)$ se anula en algún lugar de Ω, la dinámica de (6.1) puede estar gobernada por las *metasoluciones* de (6.2), que son prolongaciones por infinito de determinadas soluciones largas con soporte en supp a y, posiblemente, en un número finito de componentes de $a^{-1}(0)$. Las metasoluciones fueron introducidas en la Ph. D. Thesis de Gómez-Reñasco [17] para describir la dinámica de una clase generalizada de ecuaciones parabólicas semilineales de tipo logístico. Los pioneros resultados de [17] fueron publicados al cabo de cuatro años en [18], y más tarde mejorados en [29] y [30]. En [26, Section 8] se dieron detalles más completos.

El principal objetivo de esta tesis es obtener la dinámica de (6.1). Además, construimos unas estimaciones superiores y inferiores de las soluciones positivas de (6.2), cuando existen, para α suficientemente grande. Como son atractores globales de (6.1), estas estimaciones nos proporcionan el comportamiento de $u(\cdot,t;u_0)$ para $t \to \infty$. Estos resultados ya han sido publicados en [1], [2] y [3].

6.2 Contenido

En esta tesis, dado un operador lineal de segundo orden uniformemente elíptico en Ω,

$$\mathfrak{L} := -\text{div}\,(A(x)\nabla\cdot) + \langle b(x), \nabla\cdot\rangle + c, \qquad A = (a_{ij})_{1\le i,j\le N}, \quad b = (b_j)_{1\le j\le N},$$

con $a_{ij} = a_{ji} \in W^{1,\infty}(\Omega)$, $b_j, c \in L^\infty(\Omega)$, $1 \le i,j \le N$, un subdominio regular $\mathcal{O} \subset \Omega$, dos subconjuntos regulares abiertos y cerrados de la frontera de $\mathcal{O}$, Γ_0 y Γ_1, tales que $\partial\mathcal{O} = \Gamma_0 \cup \Gamma_1$, y un operador de frontera

$$\mathfrak{B} \;:\; \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\mathcal{O} \cup \Gamma_1) \to \mathcal{C}(\partial\mathcal{O})$$

de tipo mixto general

$$\mathfrak{B}\psi := \begin{cases} \psi & \text{sobre } \Gamma_0, \\ \partial_\nu\psi + \beta\psi & \text{sobre } \Gamma_1, \end{cases} \qquad \psi \in \mathcal{C}(\Gamma_0) \otimes \mathcal{C}^1(\mathcal{O} \cup \Gamma_1),$$

donde $\nu = A\mathbf{n}$ es el vector co-normal y $\beta \in \mathcal{C}(\Gamma_1)$, denotaremos por $\sigma[\mathfrak{L}, \mathfrak{B}, \mathcal{O}]$ al autovalor principal de $(\mathfrak{L}, \mathfrak{B}, \mathcal{O})$; es decir, al único valor de τ para el cual el problema lineal de autovalores

$$\begin{cases} \mathfrak{L}\varphi = \tau\varphi & \text{en } \mathcal{O}, \\ \mathfrak{B}\varphi = 0 & \text{sobre } \partial\mathcal{O}, \end{cases} \tag{6.3}$$

admite una autofunción positiva $\varphi > 0$. Naturalmente, si $\Gamma_1 = \emptyset$, denotaremos $\mathfrak{D} := \mathfrak{B}$ (Dirichlet), y si $\Gamma_0 = \emptyset$ y $\beta = 0$, escribiremos $\mathfrak{N} := \mathfrak{B}$ (Neumann).

El autovalor principal

$$\sigma[-\nabla \cdot (D\nabla - \alpha\nabla m) - \lambda m, D\partial_\nu - \alpha\partial_\nu m, \Omega] \tag{6.4}$$

juega un papel muy relevante para describir la dinámica de (6.1). Uno de los principales resultados del Capítulo 2 establece que el signo de este autovalor principal predice el comportamiento global de la especie. En efecto, en el caso especial en que $a(x) > 0$ para todo $x \in \bar{\Omega}$ y α suficientemente grande, se obtienen las siguientes propiedades:

(P1) $u = 0$ es atractor global de (6.1) si es linealmente estable como estado estacionario del problema de evolución (6.1).

(P2) (6.2) posee una solución positiva, necesariamente única, si $u = 0$ es linealmente inestable. En tal caso, el único estado estacionario positivo es un atractor global de (6.1).

(P3) (6.2) posee una única solución positiva para todo $\lambda > 0$.

Por otro lado, en el caso general en que $a > 0$, se cumple la propiedad (P1) pero las propiedades (P2) y (P3) pueden cambiar. Probamos las siguiente propiedades:

(P4) Si (6.2) no admite solución positiva y $u = 0$ es linealmente inestable entonces,

$$\lim_{t\to\infty} \|u(\cdot, t; u_0)\|_{\mathcal{C}(\bar{\Omega})} = \infty. \tag{6.5}$$

(P5) Existen $\lambda^*(\alpha) \in (0, \infty]$ tal que (6.2) tiene una única solución positiva si $\lambda \in (0, \lambda^*(\alpha))$ y la solución positiva es un atractor global de (6.1). Además, si suponemos que

$$\Omega_0 := \operatorname{int} a^{-1}(0) \neq \emptyset \quad \text{es un subdominio suficientemente regular de } \Omega \text{ con } \bar{\Omega}_0 \subset \Omega, \tag{6.6}$$

entonces se cumple (6.5) para todo $\lambda \geq \lambda^*(\alpha)$.

(P6) Supongamos (6.6) y que m carece de puntos críticos en $\bar{\Omega}_0$ o bien admite un número finito de puntos críticos en $\bar{\Omega}_0$, x_j, $1 \leq j \leq q$, con $\Delta m(x_j) > 0$ para todo $1 \leq j \leq q$. Entonces,

$$\lim_{\alpha\to\infty} \lambda^*(\alpha) = \infty.$$

Consecuentemente, cuando $a > 0$ se anula en algún subconjunto abierto de Ω, una advención α suficientemente grande puede provocar que la dinámica de (6.1) esté regulada por un estado estacionario positivo, aunque las soluciones de (6.1) pueden crecer a infinito en Ω_0 cuando $t \uparrow \infty$ para una advección más pequeña. En este capítulo también se obtienen resultados similares para el caso $\lambda < 0$.

El principal objetivo del Capítulo 3 es adaptar el análisis de Lam [23] y Chen, Lam y Lou [10, Section 2] (ver también Lam [24] y Lam y Ni [25]) a la situación más general en que se cumple (6.6). Este análisis nos proporciona el comportamiento fino de la solución positiva de (6.2) cuando $\alpha \uparrow \infty$. Consecuentemente, al ser un atractor global para (6.1), tales perfiles también nos proporcionan el comportamiento de $u(\cdot, t; u_0)$ para $\alpha > 0$ y $t > 0$ suficientemente grandes. Esencialmente, para α suficientemente grande, las soluciones de (6.1) se concentran alrededor de los máximos locales positivos de $m(x)$ cuando $t \uparrow \infty$. Además, si z representa algúnos de estos máximos locales, las soluciones están acotadas alrededor de z si $a(z) > 0$, mientras que deben estar no acotadas si $z \in \Omega_0$, que constituye un nuevo fenómeno no descrito previamente.

Como consecuencia de tales resultados, obtenemos una versión generalizada del Theorem 2.2 de Chen, Lam y Lou [10], derivado originalmente en el caso especial en que $\lambda = 1$, $a = 1$ y $p = 2$. Además de los resultados de [10, Section 2] que mejoramos substancialmente en este capítulo, nosotros prescindiremos de algunas de las condiciones técnicas impuestas en [10], como $\int_\Omega m \geq 0$, que erradicaremos de nuestro análisis, así como de la condición de barrera fuerte $\partial_\nu m(x) < 0$ para todo $x \in \partial\Omega$, que relajaremos a

$$\frac{\partial m}{\partial \nu}(x) \leq 0 \qquad \text{para todo } \; x \in \partial\Omega.$$

En el Capítulo 4, caracterizamos el comportamiento límite de $u(x, t; u_0)$ cuando $t \uparrow \infty$ bajo la condición (6.6), cuando $u = 0$ es linealmente inestable y (6.2) carece de soluciones positivas. Esencialmente, esto ocurre para λ suficientemente grande y $m(x_+) > 0$ para algún $x_+ \in \Omega_0$. Bajo tales circunstancias, el principal resultado de este capítulo establece que si $a \in \mathcal{C}^2(\bar{\Omega})$, entonces

$$\lim_{t\uparrow\infty} u(\cdot, t; u_0) = +\infty \quad \text{uniformemente en } \; \bar{\Omega}_0, \tag{6.7}$$

mientras que

$$L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \leq \liminf_{t\to\infty} u(\cdot, t; u_0) \leq \limsup_{t\to\infty} u(\cdot, t; u_0) \leq L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]} \text{ en } \bar{\Omega} \setminus \bar{\Omega}_0,$$

donde $L^{\min}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ y $L^{\max}_{[\lambda,m,\Omega\setminus\bar{\Omega}_0]}$ representan la solución minimal y maximal, respectivamente, del problema de contorno singular

$$\begin{cases} -\nabla \cdot (D\nabla\theta - \alpha\theta\nabla m) - \theta(\lambda m - a\theta^{p-1}) = 0 & \text{en } \; \Omega \setminus \bar{\Omega}_0, \\ D\partial_\nu\theta - \alpha\theta\partial_\nu m = 0 & \text{sobre } \; \partial\Omega, \\ \theta = +\infty & \text{sobre } \; \partial\Omega_0, \end{cases}$$

cuya existencia será probada en la Sección 4.5. Este es el primer resultado de esta naturaleza para operadores diferenciales no autoadjuntos. Una de las partes más novedosas de la prueba consiste en establecer (6.7) para el caso en que $m(x)$ cambie de signo en Ω perturbando la función peso m, en lugar del parámetro λ, como es habitual en la literatura existente sobre el tema. Esta técnica debería tener un alto número de aplicaciones. Todos los resultados previos para problemas de contorno logísticos difusivos degenerados fueron obtenidos para el operador de Laplace sin término adventivo (ver Gómez-Reñasco y López-Gómez [18], Gómez–Reñasco [17], López-Gómez [29] y Du y Huang [15]). En [26, Section 8] y [33] se puede consultar una bibliografía más exhaustiva.

6.3 Conclusiones

El análisis matemático de las *ecuaciones logísticas generalizadas* ha sido tremendamente facilitado por el teorema de caracterización del principio del máximo de López-Gómez y Molina-Meyer [34], y los posteriores refinamientos de [28] y Amann y López-Gómez [5], que han sido recogidos en [32, Th. 7.10]. Además de incentivar el desarrollo de la teoría de ecuaciones parabólicas semilineales en presencia de hetereogenidades espaciales, el Teorema 7.10 de [32] ha demostrado ser muy importante para la generación de nuevos resultados en amplias clases de problemas lineales de contorno con peso. En efecto, los resultados de [27], Hutson et al. [21], [28] y Cano-Casanova y López-Gómez [9], sustancialmente pulido en [32, Ch. 9], proporcionan refinamientos de los resultados clásicos de Manes y Micheletti [38], Hess y Kato [20], Brown y Lin [8], Senn y Hess [42] y Senn [41]. Consecuentemente, [32, Th. 7.10] es un resultado central que ha facilitado enormemente el análisis de los efectos de las heterogeneidades espaciales en algunos de los modelos más paradigmáticos de la dinámica de poblaciones.

Gracias a estos resultados y al teorema principal de Crandall y Rabinowitz [12], hemos podido determinar la existencia y unicidad de las soluciones positivas de (6.2) y, además, hemos demostrado que, para α suficientemente grande, el conjunto de soluciones positivas de (6.2) consta de dos curvas analíticas bifurcadas desde cero en $\lambda = 0$ y $\lambda = \lambda_-(\alpha) < 0$ y desde infinito en $\lambda = \lambda_*(\alpha) < 0$ y $\lambda = \lambda^*(\alpha) > 0$. Aunque en el caso general queda abierto el problema de determinar si el modelo (6.2) posee solución positiva para $\lambda < \lambda_*(\alpha)$ o $\lambda > \lambda^*(\alpha)$, en esta memoria hemos podido demostrar que cuando $a(x) > 0$ para todo $x \in \bar{\Omega}$ o cuando se cumple (6.6), no existen soluciones positivas en este rango de valores de λ, además de determinar explícitamente los valores de $\lambda_*(\alpha)$ y $\lambda^*(\alpha)$.

El análisis efectuado en el Capítulo 2 es de vital importancia para obtener las estimaciones superiores e inferiores de los estados estacionarios en el Capítulo 3. Bajo las hipótesis del Teorema 2.6.1, que garantiza la existencia de soluciones positivas de (6.2) para α suficientemente grande, y algunas hipótesis adicionales de carácter más técnico, hemos probado que estas soluciones positivas están acotadas cuando $\alpha \to \infty$ y que se van a cero lejos de los máximos de m, lo que nos permite describir el comportamiento de la especie y cómo los individuos tienden a acumularse en las zonas de mayor crecimiento teórico local de la población.

Por otro lado, tanto si se cumplen como si no las hipótesis del Teorema 2.6.1, obtenemos resultados muy paradójicos para $\lambda > 0$. En el caso de que impongamos un crecimiento exponencial lejos de los máximos de m, es decir, que $a(x)$ no se anule cerca de los máximos de m, un aumento de la advección puede dar lugar a la aparición de un estado estacionario que para advenciones pequeñas no existía, y por lo tanto, a que soluciones que eran explosivas se equilibren a ese estado estacionario. En cambio, si suponemos que hay crecimiento exponencial en un entorno de un máximo positivo de m, el Teorema 3.4.3 dicta que la especie acaba explotando en las zonas de mayor crecimiento local de la población cuando aumentamos α.

En el Capítulo 4 queda totalmente descrita la dinámica de (6.1) en el caso en que se cumpla $a(x) > 0$ para todo $x \in \bar{\Omega}$, o bien se cumpla (6.6). Como predijimos en la introdución, caso de no existir estado estacionario positivo y que $u = 0$ sea linealmente inestable, la dinámica está gobernada por las metasoluciones de (6.2) que son infinito en $\bar{\Omega}_0$. Estos son los primeros resultados de esta naturaleza en un contexto no autoadjunto.

Finalmente destacar la dificultad que entraña que m cambie de signo en la presente tesis. Uno de los resultados más complicados de obtener ha sido (6.7) para el caso que m cambie de signo. Esta demostración consiste en perturbar la función m para poder utilizar técnicas de comparación ya que perturbando el parámetro λ, como es habitual en la literatura, no era posible, justamente por el hecho de cambiar de signo m.

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