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SOLUTIONS OF THE MARKOFF EQUATION IN TRIBONACCI NUMBERS

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Abstract. In this paper, we determine all of the positive integer solutions of the so-called Markoff equation $x^2 + y^2 + z^2 = 3xyz$ in the sequence of Tribonacci numbers $\{T_n\}$, i.e. $(x, y, z) = (T_i, T_j, T_k)$ such that $i, j, k \geq 2$.

1. Introduction

The Markoff equation is a well known Diophantine equation that has the form

$$x^2 + y^2 + z^2 = 3xyz, \tag{1.1}$$

where $x, y, z \in \mathbb{N}$ such that $z \geq y \geq x$, and a solution (x, y, z) of this equation is called a Markoff triple whose components are Markoff numbers. Indeed, this equation was deeply studied by Markoff in $1879 - 1880$ [12, 13] in which he proved that there are infinitely many Markoff triples that can be obtained by the base solution $(1, 1, 1)$ with the triples $(y, z, 3yz - x)$ and $(x, z, 3xz - y)$. These numbers firstly appeared to describe the minimal values of indefinite quadratic forms with which he showed that these forms are in one-to-one correspondence with the Markoff triples.

Several authors have generalized and studied the solutions of this equation in many ways over integer, natural, or complex numbers or over some finite fields, e.g. by Baer and Rosenberger [1], Baragar and Umeda [2], González-Jiménez [4], Hu and Li [7], Hurwitz [8], Jin and Schmidt [9], Rosenberger [14], and the references given there.

Recently, many studies have been devoted to investigating the solutions, that are numbers in some binary recurrence sequences, of the Markoff equation and some of its generalizations. For instance, in 2018 Luca and Srinivasan [11] determine all of the Markoff triples whose components are Fibonacci numbers, i.e. $x = F_i, y = F_j$ and $z = F_k$. In 2020, Kafle, Srinivasan and Togbé [10]

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found the Markoff triples containing components of Pell numbers. In the same year, Tengely [18] found all of the triples with Fibonacci terms of the so called Markoff-Rosenberger equation. In my recent result with Tengely [5], we studied the solutions, that are presenting Fibonacci numbers, of another generalization called the Jin-Schmidt equation. Furthermore, Szalay, Tengely and I [6] gave general results regarding the Markoff-Rosenberger triples (G_i, G_j, G_k) with generalized Lucas number components. Then, we applied these results to find the Markoff-Rosenberger triples containing only Jacobsthal numbers or balancing numbers. Such results can be found in a recent survey written by Srinivasan in [16]. Indeed, all of these results deal with the determination of the solutions of the Markoff equation and some of its generalizations in certain binary recurrence sequences. Therefore, we introduce here a technique for determining the solutions of the Markoff equation (1.1) containing terms of a certain ternary sequence presented by the Tribonacci sequence whose terms are given by

$$T_0 = 0, T_1 = T_2 = 1, \quad T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \quad n \geq 3. \tag{1.2}$$

This means we investigate all of the Markoff triples $(x, y, z) = (T_i, T_j, T_k)$ satisfying equation (1.1) with $i, j, k \geq 2$. Since $T_1 = T_2 = 1$, we assumed the indices of the components are greater than or equal to 2.

2. Auxiliary results

In order to present our main result, we present here some auxiliary results concerning the Tribonacci sequence $\{T_n\}$. We first introduce the characteristics polynomial of this sequence as follows (see e.g. [3])

$$x^3 - x^2 - x - 1 = 0,$$

with the roots

$$\alpha = \frac{1 + \delta_+ + \delta_-}{3}, \qquad \beta = \frac{2 - \delta_+ - \delta_-}{6} + i\frac{\sqrt{3}(\delta_+ - \delta_-)}{6} \qquad \text{and} \qquad \gamma = \overline{\beta},$$

where $\delta_\pm = \sqrt[3]{19 \pm 3\sqrt{33}}$. It follows that $\alpha \in (1.83, 1.84)$ and $|\beta| = |\gamma| = \alpha^{-1/2} \in (0.73, 0.74)$.

Moreover, due the result of Spickerman [15] the Binet's formula for this sequence is defined as follows

$$T_n = a\alpha^n + b\beta^n + c\gamma^n \quad \text{for} \quad n \geq 0, \tag{2.1}$$

where

$$a = ((\alpha - \beta)(\alpha - \gamma))^{-1}, b = ((\beta - \alpha)(\beta - \gamma))^{-1} \text{ and } c = \overline{b}$$

such that

$$a \in (0.18, 0.19) \text{ and } |b| = |c| \in (0.35, 0.36).$$

Furthermore, it is also know that the following inequality is satisfied for all $n \geq 1$

$$\alpha^{n-2} \leq T_n \leq \alpha^{n-1}. \tag{2.2}$$

Note that the above inequality can be proven easily using induction.

3. Main result

THEOREM 3.1. *If $x = T_i, y = T_j$ and $z = T_k$ with $i, j, k \geq 2$, then the set of solutions of equation* (1.1) *is completely given by*

$$(x, y, z) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1)\}.$$

REMARK 3.2. In order to prove Theorem 3.1 by detemining the complete set of the solutions $(x, y, z) = (T_i, T_j, T_k)$ of equation (1.1), we start the procedure by studying the set of solutions of this equation under the assumption $k \geq j \geq i \geq 2$. Then, we permute the components of each triple of solution to obtain the complete set of solutions of the Markoff equation.

PROOF OF THEOREM 3.1. The main technique for studying the solutions (T_i, T_j, T_k) of equation (1.1) under the assumption $k \geq j \geq i \geq 2$ is summarized in the following steps:

- *Determining an upper bound for i.* Substituting $(x, y, z) = (T_i, T_j, T_k)$ in equation (1.1) gives that
$$T_i^2 + T_j^2 + T_k^2 = 3T_iT_jT_k,$$
which can be further written as
$$T_k - 3T_iT_j = -\frac{T_i^2 + T_j^2}{T_k}.$$
Inserting the corresponding Binets's formulas defined in (2.1) in the terms of the left hand side of the above equation gives that
$$\begin{aligned} a\alpha^k - 3a^2\alpha^{i+j} = &-\frac{T_i^2 + T_j^2}{T_k} - b\beta^k - c\gamma^k + 3(ab\alpha^i\beta^j \\ &+ ac\alpha^i\gamma^j + ab\alpha^j\beta^i + b^2\beta^{i+j} + bc\beta^i\gamma^j \\ &+ ac\alpha^j\gamma^i + bc\beta^j\gamma^i + c^2\gamma^{i+j}).\end{aligned}$$
The absolute values of both sides of the above equation leads to the inequality
$$\begin{aligned} \left|a\alpha^k - 3a^2\alpha^{i+j}\right| \leq &\left|\frac{T_i^2 + T_j^2}{T_k}\right| + \left|b\beta^k + c\gamma^k\right| + |3(ab\alpha^i\beta^j \\ &+ ac\alpha^i\gamma^j + ab\alpha^j\beta^i + b^2\beta^{i+j} + bc\beta^i\gamma^j \\ &+ ac\alpha^j\gamma^i + bc\beta^j\gamma^i + c^2\gamma^{i+j})|.\end{aligned}$$

Based on the assumption that $2 \leq i \leq j \leq k$ (or $-k \leq -j$) and inequality (2.2), we get that

$$\left|\frac{T_i^2+T_j^2}{T_k}\right| \leq \left|\frac{2T_j^2}{T_k}\right| \leq \left|2\alpha^{2j-k}\right| \leq 2\alpha^j. \tag{3.1}$$

By using the facts that $|\beta| = |\gamma| = \alpha^{-1/2}$ and $|b| = |c|$ with the assumption of $-k \leq -j < j$, we obtain that

$$\left|b\beta^k + c\gamma^k\right| \leq (|b|+|c|)\alpha^{-k/2} \leq (|b|+|c|)\alpha^{-j/2} < 2\,|c|\,\alpha^j, \tag{3.2}$$

and similarly we get that

$$\begin{aligned}&|3(ab\alpha^i\beta^j + ac\alpha^i\gamma^j + ab\alpha^j\beta^i + b^2\beta^{i+j} + bc\beta^i\gamma^j + ac\alpha^j\gamma^i \\ &\qquad + bc\beta^j\gamma^i + c^2\gamma^{i+j})| < 12(|ac| + \left|b^2\right|)\alpha^j.\end{aligned} \tag{3.3}$$

Based on the inequities (3.1)-(3.3) and the facts that $a \in (0.18, 0.19)$ and $|b| = |c| \in (0.35, 0.36)$, we have that

$$\begin{aligned}\left|a\alpha^k - 3a^2\alpha^{i+j}\right| <&(2 + 2(0.36) + 12[(0.19)(0.36) \\ &+ (0.36)^2])\alpha^j < 5.096\alpha^j.\end{aligned}$$

Multiplying the above inequality by $\frac{1}{a\alpha^{i+j}}$, we get that

$$\left|\alpha^{k-i-j} - 3a\right| < \frac{5.096}{a\alpha^i} < \frac{5.096}{0.18\alpha^i} < 28.4\alpha^{-i} \qquad \text{as} \qquad a > 0.18. \tag{3.4}$$

Suppose that

$$B = \min_{I\in\mathbb{Z}} \left|\alpha^I - 3a\right|.$$

Lemma 3.3. *For all $\alpha \in (1.83, 1.84)$ and $a \in (0.18, 0.19)$, we have $B > 0.007$.*

Proof. We first consider the case where $I = 0$. Here, we have that

$$|1 - 3a| > 0.43.$$

On the other hand, if $I < 0$ then $\alpha^I \leq \alpha^{-1} < 0.547$. This implies that $-\alpha^I > -0.547$. Therefore,

$$\min_{I<0} \left|\alpha^I - 3a\right| > \min |3a - 0.547| > 0.007.$$

Similarly, if $I > 0$ then

$$\min_{I>0} \left|\alpha^I - 3a\right| \geq \min |\alpha - 3a| > 1.26.$$

Thus, we conclude that $B \geq 0.007$. ∎

Therefore, inequality (3.4) gives that

$$\alpha^i < \frac{28.4}{B} < \frac{28.4}{0.007} < 4057.2,$$

which implies that

$$i < \frac{\ln(4057.2)}{\ln(\alpha)} < \frac{\ln(4057.2)}{\ln(1.83)} < 13.75.$$

Hence, $i \leq 13$.

- *Determining an upper bound for $k-j$.* Starting with inequality (3.4), we get that

$$\begin{aligned} \left| \left|\alpha^{k-i-j}\right| - |3a| \right| \leq \left|\alpha^{k-i-j} - 3a\right| < 28.4\alpha^{-i} < \frac{28.4}{\alpha^2} \\ < \frac{28.4}{1.83^2} < 8.5 \quad \text{with} \quad i \geq 2. \end{aligned}$$

This implies that

$$\left|\alpha^{k-i-j}\right| < 8.5 + 3(0.19) = 9.07.$$

Hence,

$$k - j < \frac{\ln(9.07)}{\ln(\alpha)} + i < \frac{\ln(9.07)}{\ln(1.83)} + 13 < 16.7.$$

Thus, $k \leq j + 16$.

- *Eliminating the values of i.* Here, we eliminate every value of i with $2 \leq i \leq 13$ with which the Markoff equation (1.1) is not satisfied in case of $(x, y, z) = (T_i, T_j, T_k)$ where $k \geq j \geq i \geq 2$. In other words, we solve the following Diophantine equation in y and z

$$T_i^2 + y^2 + z^2 - 3T_i yz = 0, \tag{3.5}$$

with $2 \leq i \leq 13$. Indeed, this can be done using the SageMath software [17] with an algorithm implemented as `solve_diophantine()`. Note that equation (1.1) has no solution of the form (T_i, T_j, T_k) if there is no i with which equation (3.5) is solvable. It follows that equation (3.5) is solvable for y and z only in case of $i \in \{2, 3, 6\}$.

- *Determining the corresponding values of j and k.* For each $i \in \{2, 3, 6\}$, we present here a technique for determining the corresponding values of j and k for which equation (1.1) is solvable in $(x, y, z) = (T_i, T_j, T_k)$ with $2 \leq i \leq j \leq k$. Indeed, we present the procedure of this technique in details in case of $i = 2$, and the other cases will be handled similarly. So, let us start with $i = 2$. Thus, equation (1.1) becomes as

$$T_j^2 + T_k^2 - 3T_j T_k + 1 = 0, \tag{3.6}$$

with $k \in \{j, j+1, \ldots, j+16\}$ and $j \geq 2$. Therefore,

– If $k = j$, then from equation (3.6) leads to $T_j = 1$ where $j \geq 2$. Hence, $j = 2 = k$. Therefore, $(x, y, z) = (T_i, T_j, T_k) = (T_2, T_2, T_2) = (1, 1, 1)$ is a solution for equation (1.1).
– If $k = j + 1$, then from equation (3.6) we obtain that

$$T_j^2 + T_{j+1}^2 - 3T_jT_{j+1} + 1 = 0, \tag{3.7}$$

where $j \geq 2$. Here, we claim that the above equation holds only with $j = 2$. In fact, it is clear that equation (3.7) holds with $j = 2$. So, $(x, y, z) = (T_2, T_2, T_3) = (1, 1, 2)$ is a solution of equation (1.1). It remains to prove that equation (3.7) is not satisfied with $j \geq 3$. Here, our technique mainly depends on the substitution of the Tribonacci sequence formula defined in (1.2) in the left of equation (3.7) (this could be done several times) in order to show that

$$T_j^2 + T_{j+1}^2 - 3T_jT_{j+1} + 1 < 0,$$

where $j \geq 3$. Starting with the left hand side of the above inequality, we have that

$$\begin{aligned}
&T_j^2 + T_{j+1}^2 - 3T_jT_{j+1} + 1 \\
&\quad = T_j^2 + (T_j + T_{j-1} + T_{j-2})^2 - 3T_j(T_j + T_{j-1} + T_{j-2}) + 1 \\
&\quad = -T_j^2 + T_{j-1}^2 + T_{j-2}^2 - T_{j-1}T_j - T_{j-2}T_j + 2T_{j-2}T_{j-1} + 1 \\
&\quad = -T_j^2 + T_{j-1}^2 - T_{j-1}T_j + T_{j-2}T_{j-1} - T_{j-2}T_{j-3} + 1 \\
&\quad = -T_j^2 - T_{j-1}T_{j-3} - T_{j-2}T_{j-3} + 1 \\
&\quad < 0 \quad \text{for all} \quad j \geq 3.
\end{aligned} \tag{3.8}$$

This is a contradiction for equation (3.7) and hence proves our claim.
– If $k = j + 2$, then from equation (3.6) we get that

$$T_j^2 + T_{j+2}^2 - 3T_jT_{j+2} + 1 = 0. \tag{3.9}$$

First, if $j = 2$, then the above equation leads to $T_2^2 + T_4^2 - 3T_2T_4 + 1 = 6$, which is clearly a contradiction. Following the same approach described in (3.8), one can easily prove that $T_j^2 + T_{j+2}^2 - 3T_jT_{j+2} + 1 > 0$ for all $j \geq 3$. Indeed, this can be done as follows:

$$
\begin{aligned}
&T_j^2 + T_{j+2}^2 - 3T_jT_{j+2} + 1 \\
&\quad = T_j^2 + (T_{j+1} + T_j + T_{j-1})^2 - 3T_j(T_j + T_{j-1} + T_{j-2}) + 1 \\
&\quad = -T_j^2 + T_{j-1}^2 + T_{j+1}^2 - T_{j-1}T_j - T_jT_{j+1} + 2T_{j-1}T_{j+1} + 1 \\
&\quad = -T_j^2 + 3T_{j-1}^2 + T_{j+1}^2 + T_{j-1}T_j - T_jT_{j+1} + 2T_{j-1}T_{j-2} + 1 \\
&\quad = T_{j-2}^2 + 4T_{j-1}^2 + 3T_{j-1}T_j - T_jT_{j+1} + 2T_{j-2}(2T_{j-1} + T_j) + 1 \\
&\quad = 3T_{j-1}^2 + 2T_{j-1}T_j + T_jT_{j-2} + 2T_{j-1}T_{j-2} \\
&\qquad - T_jT_{j-3} - T_{j-1}T_{j-3} - T_{j-2}T_{j-3} + 1 \\
&\quad > 0 \quad \text{for all} \quad j \geq 3.
\end{aligned} \tag{3.10}
$$

Therefore, equation (3.9) does not hold for all of the values of $j \geq 2$.

– If $k \in \{j+3, j+4, \ldots, j+16\}$ with $j \geq 2$. Here, we show that equation (3.6) does not hold by a contradiction. From equation (3.6), we have that

$$T_j^2 + 1 = 3T_jT_k - T_k^2 = T_k(3T_j - T_k),$$

which mean that $T_k < 3T_j$ and hence $k \leq j+2$. This is a contradiction since $k \geq j+3$.

Now, it remains to deal with the other cases where $i = 3$ or 6 in which the equation (1.1) becomes

$$T_j^2 + T_k^2 - 6T_jT_k + 4 = 0 \tag{3.11}$$

or

$$T_j^2 + T_k^2 - 39T_jT_k + 169 = 0, \tag{3.12}$$

with $j \geq 3$ or $j \geq 6$ (and $k \in \{j, j+1, \ldots, j+16\}$), respectively. In fact, one can follow exactly the same approach described above in case of $i = 2$ to show that there is no solution $(3, j, k)$ with $3 \leq j \leq k$ or $(6, j, k)$ with $6 \leq j \leq k$ for which equation (3.11) or (3.12) is satisfied. Therefore, we omit the details of computations.

- *Permuting the components of the solutions.* The obtained solutions (i.e. $(x, y, z) = (T_i, T_j, T_k) = (1, 1, 1)$ and $(1, 1, 2)$) represent solutions of the Markoff equation (1.1) under the assumption that $2 = i \leq j \leq k$. As noted in Remark 3.2, we have to permute the components of these solutions to obtain the other distinct solutions of the Markoff equation without this assumption. It follows that the complete set of solutions of the Markoff equation is presented by

$$(x, y, z) \in \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1)\}.$$

Hence, Theorem 3.1 is completely proved. ∎

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Rješenja Markovljeve jednadžbe u Tribonaccijevim brojevima

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SAŽETAK. U ovom članku određujemo sva rješenja u prirodnim brojevima Markovljeve jednadžbe $x^2+y^2+z^2=3xyz$ u nizu Tribonaccijevih brojeva $\{T_n\}$, tj. $(x,y,z)=(T_i,T_j,T_k)$ tako da je $i,j,k \geq 2$.

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