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CONSTRUCTING FLAG-TRANSITIVE INCIDENCE STRUCTURES

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Abstract. The aim of this research is to develop efficient techniques to construct flag-transitive incidence structures. In this paper we describe those techniques, present the construction results and take a closer look at how some types of flag-transitive incidence structures relate to arc-transitive graphs.

1. Motivation and background

In order to better understand the motivation for exploring incidence structures, we give a short overview of different, but similar and intertwined, structures and concepts that have been studied throughout modern history. They include configurations, incidence structures, linear spaces, finite geometries, arc-transitive graphs and combinatorial designs. In the broadest sense, an *incidence structure* is any structure that consists of two types of objects, called points and blocks, and an incidence relation between them.

However, a *configuration* was the first incidence structure explicitly defined, [17,23,24]. (An interested reader may refer to [13] for even older examples of incidence structures, dating to 17th century, to Pascal and Desargues.) It is defined as a set of points, a set of lines, and an incidence relation between them such that each point is incident to the same number of lines and each line is incident to the same number of points.

A *linear space* is an incidence structure in which any two distinct points lie on exactly one common line and in which every line has at least two points. A *finite geometry* is a finite linear space that is either a finite projective plane (with no parallel lines) or a finite affine plane (with parallel lines), [2]. Probably the best known example of a finite projective plane is the Fano plane.

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Also worth mentioning is a *block design*, an incidence structure in which every two points occur together in the same number of blocks, and each block is incident to the same number of points. For instance, the Fano plane is an example of the smallest nontrivial block design.

In the prior research of these structures, there are surveys and classifications of some types and classes of structures with well-chosen properties. We decided to focus this research on *flag-transitive incidence structures.* We obtained results that relate to all of the specific incidence structures mentioned above, and believe that this methods may be applied to various further work in specific areas of incidence geometry. In this paper, flag-transitive structures are studied and constructed directly, but the aforementioned structures that are subsequently also constructed, will be mentioned wherever that will be of interest.

A *flag* in an incidence structure is an ordered pair of a point and a block which are in the incidence relation. Flag-transitivity is a property of an automorphism group of an incidence structure, which was first proposed by Tits in [26] and since then studied in geometries, linear spaces and designs [1, 6, 28].

Flag-transitivity allowed us to relate incidence structures with a specific class of graphs that will have a special place in this research. The first mathematician who explicitly related incidence structures to graph theory was Levi, [19]. He represented an incidence structure with a bipartite graph with two types of vertices, points and blocks, where edges were incidences. This graph later became known as the Levi graph (the incidence graph) [14]. Another way to represent an incidence structure is with the Menger graph (the point graph) [14], in which vertices are points, and two vertices are adjacent if the corresponding points are in the same block. The representation of one incidence structure with these two graphs is shown in Figure 1.

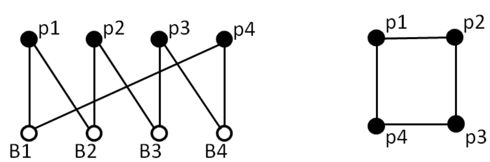

FIGURE 1. The Levi graph and the Menger graph of the same incidence structure

The connection opened up new questions and areas of research, as now various properties of graphs could be studied as properties of incidence structures those graphs represented, and vice-versa. Flag-transitivity in an incidence structure corresponds to arc-transitivity in Menger graphs [16]. In Section 5 we give more details about arc-transitive graphs and incidence structures.

The main issue of this paper is constructing flag-transitive incidence structures. After the preliminaries and foundations in Section 2, in Section 3, we proceed to define the concept of a basic flag-transitive incidence structure, and to show how the construction of all flag-transitive incidence structures comes down to the construction of the basic ones. In Section 4, the algorithms and methods of construction are presented. We use the program package Magma [3] for all constructions. Section 5 is dedicated to relating flag-transitive incidence structures to arc-transitive graphs, and the paper concludes with the results in Section 6. The full list of constructed flag-transitive incidence structures can be found on the web page [18].

2. Definitions and known results

In this section, we give an overview of definitions and notation of incidence structures and their relation to groups.

2.1. *Incidence structures.* Let P and $\mathcal{B}$ be disjoint sets and $\mathcal{I}$ a relation on $P\times\mathcal{B}$. A triple $\Gamma=(P,\mathcal{B},\mathcal{I})$ is called an *incidence structure*. Elements of sets P, $\mathcal{B}$ and $\mathcal{I}$ are called *points*, *blocks* and *flags* respectively, and if $(p,B)\in\mathcal{I}$, we say that the point p is *incident* with the block B. In this paper we only observe incidence structures with finite sets P and $\mathcal{B}$.

For the incidence structure $\Gamma=(P,\mathcal{B},\mathcal{I})$ we note the following sets. For a point $p\in P$, the set $\Gamma(p)=\{B\in\mathcal{B}:(p,B)\in\mathcal{I}\}$ of all blocks incident with the point p; and for a block $B\in\mathcal{B}$, the set $\Gamma(B)=\{p\in P:(p,B)\in\mathcal{I}\}$ of all points incident with the block B. With this, we say that $|\Gamma(p)|$ is the *degree of the point* p, and $|\Gamma(B)|$ is the *degree of the block* B. If $|\Gamma(p)|=0$ for some point $p\in P$, we say that p is an *isolated point* and analogously we define an *isolated block*.

We say that an incidence structure Γ is *point-simple* if

$$\Gamma(p_i)=\Gamma(p_j)\implies p_i=p_j$$

holds, for each $p_i,p_j\in P$. Analogously, we say that an incidence structure Γ is *block-simple* if

$$\Gamma(B_i)=\Gamma(B_j)\implies B_i=B_j$$

holds, for each $B_i,B_j\in\mathcal{B}$. If a structure is block-simple, we can identify each block B with the set of its incident points $\Gamma(B)$. As a result, we can view incidences as a relation “is an element of"” and thus denote such structure by $\Gamma=(P,\mathcal{B})$, where $\mathcal{B}\subseteq 2^P$.

Another basic concept is the dual structure of an incidence structure. For an incidence structure $\Gamma=(P,\mathcal{B},\mathcal{I})$, the *dual structure* of Γ is the incidence structure $\Gamma^*=(\mathcal{B},P,\mathcal{I}^*)$, where $\mathcal{I}^*=\{(B,p):(p,B)\in\mathcal{I}\}\subseteq\mathcal{B}\times P$.

The incidence structure is *connected* if its Levi graph is connected.

2.2. *Automorphisms and group action.* Let $\Gamma_1 = (P_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\Gamma_2 = (P_2, \mathcal{B}_2, \mathcal{I}_2)$ be two incidence structures. A bijection $\phi : P_1 \cup \mathcal{B}_1 \to P_2 \cup \mathcal{B}_2$ such that $\phi(P_1) = P_2$, $\phi(\mathcal{B}_1) = \mathcal{B}_2$, and for each point $p \in P_1$ and each block $B \in \mathcal{B}_1$ it holds

$$(p, B) \in \mathcal{I}_1 \iff (\phi(p), \phi(B)) \in \mathcal{I}_2,$$

is called an *isomorphism* of incidence structures Γ_1 and Γ_2. If such function exists, we say that Γ_1 and Γ_2 are *isomorphic* and denote it by $\Gamma_1 \cong \Gamma_2$.

An *automorphism* of an incidence structure $\Gamma = (P, \mathcal{B}, \mathcal{I})$ is an isomorphism $\phi : \Gamma \to \Gamma$. A *full automorphism group* of Γ is a group of all automorphisms of the structure Γ and is denoted by $Aut(\Gamma)$. Each subgroup of $Aut(\Gamma)$ is called an *automorphism group*.

A bijection $\psi : P \cup \mathcal{B} \to P \cup \mathcal{B}$ is called a *duality* of the structure $\Gamma = (P, \mathcal{B}, \mathcal{I})$ if $\psi(P) = \mathcal{B}, \psi(\mathcal{B}) = P$ and for each $p \in P$ and $B \in \mathcal{B}$ the following holds:

$$(p, B) \in \mathcal{I} \Longleftrightarrow (\psi(B), \psi(p)) \in \mathcal{I}.$$

We say that *a group G acts on an incidence structure* $\Gamma = (P, \mathcal{B}, \mathcal{I})$ if G acts on the sets P, $\mathcal{B}$ and $\mathcal{I}$ in such way that for each $g \in G$ and for each $(p, B) \in \mathcal{I}$, it holds $g(p, B) = (gp, gB)$. A group G is *point-transitive* on the incidence structure $\Gamma = (P, \mathcal{B}, \mathcal{I})$ if it acts transitively on the set P; *block-transitive* if it acts transitively on the set $\mathcal{B}$, and is *flag-transitive* if it acts transitively on the set $\mathcal{I}$.

For a group G and a set X, we use the notation GX, which is defined as $GX = \{gX : g \in G\}$. A permutation representation of the group G on the set X is a homomorphism $\rho : G \to Sym(X)$. The image $\rho(G) \subseteq Sym(X)$ is a permutation group and is denoted by G^X.

2.3. *Flag-transitive incidence structures.* Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be an incidence structure without isolated points or isolated blocks. We say that Γ is a *flag-transitive* incidence structure if its automorphism group acts flag-transitively. We define incidence structures that are *point-transitive* and *block-transitive* in the same way.

We will refer to flag-transitive incidence structures as *FTIS* throughout the paper. The examples of FTIS are finite affine and finite projective planes over some finite field. For instance, the Fano plane is a projective plane over the field $\mathbb{F}_2$.

If a group G is flag-transitive on the incidence structure Γ, we say that Γ is *G-flag-transitive*.

It is worthy to note that if an incidence structure Γ is G-flag-transitive, then its dual incidence structure, Γ^*, is also G-flag-transitive.

The following characterization of FTIS will be used in the construction methods. We present it in the form of a proposition but omit the well known proof.

Proposition 2.1. *Let* $\Gamma = (P, \mathcal{B}, \mathcal{I})$ *be an incidence structure without isolated points and blocks, with a group* G *acting on it and let* $p \in P$ *and* $B \in \mathcal{B}$*. Then the following statements are equivalent:*

- Γ *is flag-transitive;*
- G *is transitive on points and* G_p *is transitive on* $\Gamma(p)$*;*
- G *is transitive on blocks and* G_B *is transitive on* $\Gamma(B)$*.*

Now let us define $\Gamma_{(m)}$ and $\Gamma^{(n)}$, two types of structures obtained from the incidence structure Γ and describe their connection to Γ.

Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be an incidence structure and $n, m \in \mathbb{N}$.

- $\Gamma_{(m)}$ is an incidence structure with the point set P and the block set $\{1, ..., m\} \times \mathcal{B}$ in which a point p is incident with a block (i, B) if and only if $(p, B) \in \mathcal{I}$.

- $\Gamma^{(n)}$ is an incidence structure with point set $\{1, ..., n\} \times P$ and block set $\mathcal{B}$ in which a point (i, p) is incident with a block B if and only if $(p, B) \in \mathcal{I}$.

Clearly, $(\Gamma^{(n)})_{(m)} = (\Gamma_{(m)})^{(n)}$, so that type of structure is denoted by $\Gamma^{(n)}_{(m)}$.

Proposition 2.2. *Let* Γ *be a FTIS. Then there exist unique* $n, m \in \mathbb{N}$ *and the unique point-simple and block-simple FTIS* Ω*, such that* Γ *is isomorphic to* $\Omega^{(n)}_{(m)}$*.*

Proof. Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be a G-FTIS. Let Ω be the incidence structure with the point set $\{\Gamma(p) : p \in P\}$, the block set $\{\Gamma(B) : B \in \mathcal{B}\}$, where a point $\Gamma(p)$ is incident with a block $\Gamma(B)$ if and only if p is incident with B. Let us notice that if p is incident with B, then all the points in $\Gamma(B)$ are incident with all of the blocks in $\Gamma(p)$, so incidence in Ω is well-defined. For any $g \in G$, it holds $g\Gamma(p) = \Gamma(gp)$ and $g\Gamma(B) = \Gamma(gB)$ for all the points $p \in P$ and all the blocks $B \in \mathcal{B}$. Thus, G preserves incidences of Ω, so Ω is a FTIS.

If $\Omega(\Gamma(p)) = \Omega(\Gamma(p'))$ for some $p, p' \in P$, then sets $\{\Gamma(B) : (p, B) \in \mathcal{I}\}$ and $\{\Gamma(B') : (p', B') \in \mathcal{I}\}$ are equal. That gives us $\Gamma(p) = \Gamma(p')$, which means that Ω is a point-simple incidence structure. Analogously, one can easily show that Ω is also block-simple.

For any point $p \in P$, let $U(p) = \{p' \in P : \Gamma(p') = \Gamma(p)\}$, and for any block $B \in \mathcal{B}$, let $U(B) = \{B' \in B : \Gamma(B') = \Gamma(B)\}$. For any $g \in G$, it holds $gU(p) = U(gp)$ and $gU(B) = U(gB)$, for all the points $p \in P$ and all the blocks $B \in \mathcal{B}$. Because G is point-transitive, it follows that sets $U(p)$ for all $p \in P$ are of the same cardinality, so we can denote that cardinality by n. Analogously, because G is block-transitive, it follows that sets $U(B)$, for all $B \in \mathcal{B}$, are of the same cardinality, so we can denote that cardinality by m. Let us denote by P' the set of points and by $\mathcal{B}'$ the set of blocks of incidence structure $\Omega^{(n)}_{(m)}$, i.e. $P' = \{1, 2, ..., n\} \times \{\Gamma(p) : p \in P\}$ and

$\mathcal{B}' = \{1, 2, ..., m\} \times \{\Gamma(B) : B \in \mathcal{B}\}$. Now, let us define a function

$$\phi : P \cup B \to P' \cup \mathcal{B}'$$

in a way that $\phi(U(p)) = \{1, 2, ..., n\} \times \{\Gamma(p)\}$ for all $p \in P$ and $\phi(U(B)) = \{1, 2, ..., m\} \times \{\Gamma(B)\}$ for all $B \in \mathcal{B}$. Now, it is easily shown that ϕ is an isomorphism of incidence structures Γ and $\Omega^{(n)}_{(m)}$. □

If $\Gamma_1 = (P_1, \mathcal{B}_1, \mathcal{I}_1)$ and $\Gamma_2 = (P_2, \mathcal{B}_2, \mathcal{I}_2)$ are incidence structures, then by $\Gamma_1 + \Gamma_2$ we denote the incidence structure $(P_1 \sqcup P_2, \mathcal{B}_1 \sqcup \mathcal{B}_2, \mathcal{I}_1 \sqcup \mathcal{I}_2)$, where $\sqcup$ denotes a disjoint union. The same structure, $\Gamma = (P, \mathcal{B}, \mathcal{I})$, added n times this way, is denoted by $n\Gamma$.

PROPOSITION 2.3. *Let Γ be a FTIS. Then there exists the unique $n \in \mathbb{N}$ and the unique connected FTIS Ω such that Γ is isomorphic to $n\Omega$.*

PROOF. Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be a G-FTIS. The group G acts transitively on the edges of the Levi graph of Γ, so all of its connected components are isomorphic to each other. A connected component of the Levi graph naturally induces the connected incidence substructure Ω of Γ. The stabilizer of the set of all flags of Ω in G acts transitively on that set, so Ω is a FTIS. Now, it is easily shown that Γ is isomorphic to $n\Omega$. □

2.4. *Imprimitive flag-transitive incidence structures.* Let a group G act transitively on a set X. A partition $\{\Delta_i \subseteq X : i \in \{1, ..., d\}\}$ of the set X is called a *block system* if for each $i \in \{1, ..., d\}$ and for each $g \in G$ there exists $j \in \{1, ..., d\}$ such that $g\Delta_i = \Delta_j$.

For a group action of any transitive group G on any set X with at least 2 elements, there are at least two block systems, a partition to singletons and a whole set X. Those block systems are called trivial. A non-trivial block system is called a *system of imprimitivity.* If a group G has the action that allows at least one system of imprimitivity, we say that such action is *imprimitive on* X and we say that G is *imprimitive.* Elements of a system of imprimitivity are called *blocks of imprimitivity* and they are of the same cardinality. If a transitive action of a group G does not preserve any non-trivial block system, we say that G is *primitive.*

With regard to incidence structures, we will observe imprimitive groups G acting on sets of points of the structures, partitioning those sets into systems of imprimitivity. Let us note that $\Gamma(\Delta)$ naturally expands $\Gamma(p)$ in the following way:

$$\Gamma(\Delta) = \{B \in \mathcal{B} : (\exists p \in \Delta)(p, B) \in \mathcal{I}\}.$$

With this, we define the following structures.

Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be a G-FTIS, Σ a system of imprimitivity for G on the point set P and $\Delta \in \Sigma$. Then

- The incidence structure $\Gamma_\Delta = (\Delta, \Gamma(\Delta), \mathcal{I}_\Delta)$, where $\mathcal{I}_\Delta = \mathcal{I} \cap (\Delta \times \mathcal{B})$, is called a *substructure of* Γ *with regard to* Δ.

- The incidence structure $\Gamma/\Sigma = (\Sigma, \mathcal{B}, \mathcal{I}_\Sigma)$ where $\mathcal{I}_\Sigma = \{(\Delta, B) : \Delta \in \Sigma, B \in \mathcal{B}, \Delta \cap \Gamma(B) \neq \emptyset\}$ is called a *quotient structure of* Γ *by* Σ.

PROPOSITION 2.4. *Let* Γ *be a* G*-FTIS,* Σ *a system of imprimitivity for* G *on the point set* P *and* $\Delta \in \Sigma$*. Then* Γ_Δ *is* G_Δ*-FTIS and* Γ/Σ *is* G*-FTIS.*

PROOF. For two arbitrary flags (p_1, B_1) and (p_2, B_2) in $\mathcal{I}_\Delta$, given that Γ is G-flag-transitive, there must exist $g \in G$ such that $g(p_1, B_1) = (p_2, B_2)$. Since we have $p_1, p_2 \in \Delta$ and $gp_1 = p_2$ then it must hold $g\Delta = \Delta$, which gives us $g \in G_\Delta$. Now, for two arbitrary flags (Δ_1, B_1) and (Δ_2, B_2) in $\mathcal{I}_\Sigma$ there must exist $p_1 \in \Delta_1$ and $p_2 \in \Delta_2$ such that (p_1, B_1) and (p_2, B_2) are flags in $\mathcal{I}$. Given that Γ is G-flag-transitive, there exists $g \in G$ such that $g(p_1, B_1) = (p_2, B_2)$ which means $gp_1 = p_2$. Now, we have $g\Delta_1 = \Delta_2$ which gives us $g(\Delta_1, B_1) = (\Delta_2, B_2)$. □

3. BASIC FLAG-TRANSITIVE INCIDENCE STRUCTURES AND THEIR PARAMETERS

The key aspect of the construction is a basic flag-transitive incidence structure, which we denote by BFTIS. Based on Propositions 2.2 and 2.3 and the fact that the dual structure of FTIS is also a FTIS, we are able to simplify the construction of all FTIS to their core class.

DEFINITION 3.1. *A FTIS is called basic if it is connected, point-simple, block-simple and the number of points is less or equal to the number of blocks.*

PROPOSITION 3.2. *Let* Γ *be a FTIS. Then there exist unique* $n, m, k \in \mathbb{N}$ *and the unique BFTIS* Ω *such that* Γ *is isomorphic to* $k\Omega^{(n)}_{(m)}$ *or to* $(k\Omega^{(n)}_{(m)})^*$*.*

From this, we conclude that the problem of constructing FTIS comes down to the construction of BFTIS. Let us see how we can better describe them in a numerical way, in order to develop the construction methods.

Let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be an incidence structure, transitive on points and blocks. The degrees of all points are equal, and the degrees of all blocks are equal, so we can denote by

$$\begin{aligned} v &:= |P|, \text{ the number of points;} \\ b &:= |\mathcal{B}|, \text{ the number of blocks;} \\ r &:= |\Gamma(p)|, \text{ for each } p \in P, \text{ the degree of points;} \\ k &:= |\Gamma(B)|, \text{ for each } B \in \mathcal{B}, \text{ the degree of blocks.} \end{aligned}$$

If this is the case, we will say that this incidence structure is of the *type* $r^v k^b$. It can be seen that if Γ is flag-transitive and of the type $r^v k^b$, then Γ^* is also flag-transitive and of the type $k^b r^v$.

To connect these parameters with considerations of different structures from Section 1, one can observe that this kind of incidence structures are in fact configurations, since each point is incident to the same number of blocks, and each block is incident to the same number of points. For linear spaces, $k \geq 2$ holds and the Fano plane is an incidence structure of the type $3^7 3^7$, while the Desargues configuration is an incidence structure of the type $3^{10} 3^{10}$.

In a BFTIS, $rv = kb$ and $k \leq v$. However, more on the parameters can be deduced from the definition and properties of BFTIS. From the definition directly follows

$$v \leq b \leq \binom{v}{k}.$$

Also, if $k = 1$ holds, then $v = b = r = 1$. Similarly, if $k = v$ holds, because of point and block simplicity, we have $v = b = r = 1$. So for any BFTIS, other then the one of the type $1^1 1^1$, it holds $1 < k < v$.

Since there exists a group G that acts transitively on the set of flags, which is of cardinality rv, it follows that rv divides the order of G. Because G is a subgroup of S_v, we have $|G||v!$, so it holds

$$r|(v-1)!.$$

We have shown the following:

PROPOSITION 3.3. *For the parameters of the BFTIS it holds*

- $rv = kb$;
- $v \leq b \leq \binom{v}{k}$;
- $v > 1 \implies 1 < k < v$;
- $r|(v-1)!$.

While Proposition 3.3 will be the basic tool for determining parameter sets when searching for BFTIS, we have an additional result that helped us to eliminate many sets of parameters. That result is based on Sylow theorems, from which we derived Proposition 3.8 presented in the next subsection.

3.1. *Application of Sylow theorems.*

PROPOSITION 3.4. *Let a group G act transitively on a set X, let $H \leq G$ and $x \in X$. Then H acts transitively on the set X if and only if $G = HG_x$ holds.*

PROOF. Let H act transitively on X. We have

$$|HG_x| = \frac{|H||G_x|}{|H \cap G_x|} = \frac{|H||G_x|}{|H_x|} = |X||G_x| = |G|.$$

Now it follows $G = HG_x$. For the other side, let $G = HG_x$ hold. We have

$$X = Gx = HG_x x = Hx,$$

from which the transitivity of the subgroup H follows. □

Proposition 3.5. *Let p be a prime number and H a subgroup of a finite group G. Then there exists a Sylow p-subgroup Q of G such that $Q \cap H$ is a Sylow subgroup of H.*

Proof. Every Sylow p-subgroup K of H is contained in some Sylow p-subgroup Q of G so $K = Q \cap H$ holds. ☐

Proposition 3.6. *Let X be a set for which $|X|$ is a power of some prime number p and let a group G act transitively on X. Then Sylow p-subgroups of G also act transitively on X.*

Proof. By Proposition 3.5, let Q be a Sylow p-subgroup of G such that $Q \cap G_x$ is a Sylow p-subgroup of G_x. Then there exists $m \in \mathbb{N}$ such that $p \nmid m$ and $|G_x| = |Q \cap G_x|m$. From $|G| = |X||G_x| = |X||Q \cap G_x|m$ it then follows $|Q| = |X||Q \cap G_x| = |X||Q_x|$, which means $|Qx| = \frac{|Q|}{|Q_x|} = |X|$. Now we conclude Q is transitive, and so are the rest of Sylow p-subgroups, as they are conjugated with Q. ☐

Corollary 3.7. *Let Γ be a FTIS for which the number of flags is a power of some prime number p. Then there exists a p-group G such that Γ is G-flag-transitive.*

Proposition 3.8. *Let p be a prime number, $r, v, k, b, \alpha \in \mathbb{N}$, $k = p^\alpha$, $k < v < p^{\alpha+1}$ and $p|b$. Then there is no simple block-transitive incidence structure of type $r^v k^b$.*

Proof. Let us assume the opposite, let Γ be a G-FTIS with given properties and let B be any block in $\mathcal{B}$. If K is a Sylow p-subgroup of the stabilizer G_B, then it acts transitively on B and there exists a Sylow p-subgroup Q of G such that $K = Q \cap G_B$. For some x incident with B it holds

$$p^\alpha = k = |B| = |K_x| \leq |Q_x| \leq v < p^{\alpha+1},$$

and then from $|Q_x|$ being a power of p it follows $|Q_x| = k$, which means $Q_x = B$. Now we have $Q \leq G_B$ and $p \nmid [G : G_B]$, which is a contradiction. ☐

For example, Proposition 3.8 is used to eliminate the possibility of existence of BFTIS of the type $16^{30}16^{30}$. Here we have $k = 2^4$, $2^4 < 30 < 2^5$ and $2|30$, which proves the elimination. Failure to solve the existence of BFTIS of exactly this type via other construction methods in a timely manner, prompted us to develop this elimination technique.

4. Methods of construction

The main goal is to construct BFTIS of type $r^v k^b$. We use several methods, depending on parameters and the computer capacity. Also, given the fact that MAGMA has a library of transitive permutation groups up to the degree

31 and a library of primitive permutation groups up to the degree 4096, we adjust the construction methods regarding the values of v and k.

For $v \leq 31$, we use the library of transitive groups to obtain the list of all possible automorphism groups of BFTIS with v points and for each of those, we run algorithms BASIC, k-REP or k-ORB (depending on the value of k), described in Subsection 4.1.

For prime values of v greater than 31, we use the library of primitive permutation groups and again for each group in the list run one of the same three algorithms. For composite values of v greater than 31, we use the same library to check for primitive groups but for the imprimitive ones we devised a technique described in Subsection 4.2.

To make things somewhat simpler, we give the diagram in Figure 2 for easier visualization of the methods of construction of BFTIS.

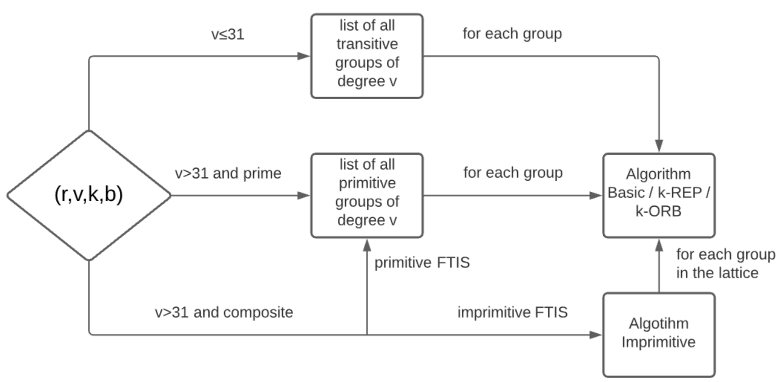

Figure 2. Methods of construction

Algorithm BASIC, with its complete theoretical background, is explained in [4], where we used it as a part of the construction method for obtaining flag-transitive designs that have $S_n \wr S_m$ as an automorphism group. Algorithms BASIC and IMPRIMITIVE were used in [5] to construct flag-transitive designs with the automorphism group $S_n \times S_m$. In that same paper, we used some ideas from algorithms k-REP and k-ORB, to solve some special cases, however, we developed them in their fullness for this research so here we present their theoretical explanation.

4.1. *Main algorithms.* We start with a set of parameters $[r, v, k, b]$. Depending on the value of v we use different methods for obtaining possible groups of automorphisms G, but once we have those, for each such group we will use algorithms BASIC, k-REP or k-ORB, whichever appropriate, for the construction of BFTIS of the type $r^v k^b$. Since not all candidates for the group G will be able to yield any FTIS, we lay this elimination proposition that we incorporate in the algorithm to eliminate such groups before we even start trying to construct BFTIS.

PROPOSITION 4.1. *Let G be a permutation group of the degree v and let Γ be a basic G -FTIS of the type $r^v k^b$. If p is a prime number for which $r < p < v$ and $p \nmid v$ holds, then $p \nmid |G|$.*

PROOF. This claim was stated in terms of bipartite graphs and proven in [15]. It can be seen that our proposition follows from applying this result to the Levi graph of the structure. □

Now we present the basic algorithm.

Algorithm BASIC

Input data is a set of parameters $[r, v, k, b]$ and a group G.
First we check if $rv \mid |G|$, and for all prime numbers p such that $r < p < v$ and $p \nmid v$ check if $p \nmid |G|$. Then we list all subgroups (up to conjugation) H of G with the index b and for each subgroup H list all its orbits O such that $|O| = k$ and $|GO| = b$. Now we have those orbits that remain for base blocks of structures, so we can construct a FTIS by acting with the group G on an orbit and thus gaining all blocks of the desired structure. Lastly, we make a list of all such FTIS, and in the end, filter that list of structures to remove all which are not connected or point-simple, and all that are isomorphic or dual isomorphic.

Even with those methods, with some parameters, the number of observed groups was very large, and given that the algorithm calculates all the subgroups with the said properties, for each of those groups, sometimes this algorithm required inefficient amount of time or too much memory. That is the reason we devised the other algorithm, called k-REP, which constructs structures in a different way. Theoretical background for this algorithm is given in the following.

DEFINITION 4.2. *Let $G \leq S_v$ be a transitive group. We say that the sets $A, B \subseteq \{1, ..., v\}$ are G-conjugated if there exists $g \in G$ such that $gA = B$.*

G-conjugation is an equivalence relation and its equivalence classes are orbits of the G-action on the partitive set of $\{1, ..., v\}$. It is easy to see that for each $A \subseteq \{1, ..., v\}$, the simple incidence structure $\Gamma(\{1, ..., v\}, GA)$ is point and block transitive. Clearly, if sets A and B are G-conjugated, incidence structures $\Gamma(\{1, ..., v\}, GA)$ and $\Gamma(\{1, ..., v\}, GB)$ are the same. The algorithm is based on constructing representatives of classes of G-conjugation.

DEFINITION 4.3. *Let $\alpha \in \mathbb{N}$, $\alpha \leq v$. We say that a set R, of subsets of $\{1, ..., v\}$ is a set of α-representatives if each two different elements of R are* *not G-conjugated and each subset of $\{1,...,v\}$ of the cardinality α is conjugated with some element of R.*

PROPOSITION 4.4. *Let R be a set of α-subsets of $\{1,...,v\}$, such that no two elements in R are G-conjugated. Then, R is a set of α -representatives if and only if*

$$\sum_{A\in R}[G:G_A]=\binom{v}{\alpha}.$$

PROOF. If R is a set of α-representatives, then every α-subset is contained in the exactly one orbit ω_A, for some $A\in R$. Since $[G:G_A]=|\omega_A|$, then $\sum_{A\in R}[G:G_A]=\binom{v}{\alpha}$. □

PROPOSITION 4.5. *Let R be a set of α-representatives, for some $\alpha<v$, and let $U=\{A\cup\{i\}:A\in R, i\in\{1,...,v\}, |A\cup\{i\}|=\alpha+1\}$. Then there exists some subset of $(\alpha+1)$-representatives in U.*

PROOF. Let X be any $(\alpha+1)$-subset of $\{1,...,v\}$ and let $i\in X$ be any element of X. Now, the set $X\setminus\{i\}$ is an α-subset of $\{1,...,v\}$ and therefore has a representative in R, i.e. there exists $A\in R$ such that for some $g\in G$ it holds $g(X\setminus\{i\})=A$. Now, $gX=A\cup\{gi\}$ which is in U. □

So the main idea is to construct a set of k-representatives for some group G of the degree v. For small values of k this is easily achieved by induction, described in the following algorithm. After that, we take elements of the set of k-representatives and use them as base blocks in the construction.

Algorithm k-REP

For this algorithm input data is a transitive group $G\le S_v$ and a set of parameters $[r,v,k,b]$.
The first step is to construct the set R of k-representatives. We do this by induction, based on Proposition 4.5. We start with $R=\emptyset$ and in each step construct a new set R in the following way:
For some element in U, we check if it is conjugated with some element from R, if not, then we add it to R and calculate $\sigma(R)=\sum_{A\in R}[G:G_A]$. If $\sigma(R)=\binom{v}{\alpha+1}$, the process is finished, and if not, we continue with a new element from U.
The second step is to choose those $\Gamma(\{1,...,v\},GA)$, $A\in R$ which are BFTIS with given parameters. Finally, those structures are analyzed for isomorphisms or dual isomorphisms.

This algorithm is used for small values of k, or for groups which have a small number of α-representatives.

REMARK 4.6. It can be seen that if R is a set of α-representatives, then $\{A^C:A\in R\}$ is a set of $(v-\alpha)$-representatives. It follows that for the sets of parameters for which $k > \frac{v}{2}$ holds, it is better to calculate $(v - \alpha)$-representatives and then further proceed with their complements.

EXAMPLE 4.7. Let us observe described methods in search for BFTIS of the type $6^{24}6^{24}$. Here we have $v = 24$ and there are 25000 transitive groups of the degree 24. Group by group, Algorithm BASIC runs, and saves constructed BFTIS. When it gets to the group which MAGMA identifies with the number 10272, the memory runs out, and for that group we then use Algorithm k-REP. We proceed by observing group by group with Algorithm BASIC, and if needed use k-REP again. This method yields 8 BFTIS of the type $6^{24}6^{24}$.

One might wonder why not use k-REP for all the groups. The reason is that it is generally slower, as some groups are faster discarded by Algorithm BASIC.

We developed Algorithm k-REP to help us out with the construction of those FTIS with a small k. Equally important, we had to find a way to deal with those FTIS with a large k. So we devised Algorithm k-ORB.

The idea behind Algorithm k-ORB is to filter down the set $\{1, ..., v\}$ to orbits of the size k in a somewhat different way than checking for the action of all the subgroups of a group G of the index b on that set.

Algorithm k-ORB

This algorithm has the transitive group $G \leq S_v$ and a set of parameters $[r, v, k, b]$ as its input data. We start with two lists, A and B, where A starts as an empty list and B containing only the set $\{1, ..., v\}$. As long as there are any elements still left in the list B, for each element $X \in B$, we do the following:

- form a set of orbits, up to a G-conjugation, of all groups that we get by going down the lattice of subgroups of $(G_X)^X$ and choosing those that are non-transitive and maximal in some transitive subgroup

- for each orbit in that set we do the following:

- if $|O| = k$ and $|GO| = b$, then we store the orbit O in the list A

- if $|O| > k$, then we store O in the list B

- remove X from the list B.

Finally, when the list B empties out, we end up with the list A which is a list of base blocks of the desired BFTIS. So we construct a list of those BFTIS and filter out those that are isomorphic to some of the other ones in the list or their duals.

It can be seen that the problem of a large number of subgroups of index b of a group G can be bypassed by switching the focus on orbits rather than groups. Because we check for the orbit size, and discard those of the size less than k, a lot of subgroups of the index b will not even be taken into consideration. In cases with a large k, this algorithm proved to be faster than algorithms BASIC and k-REP.

4.2. *Construction of imprimitive flag-transitive structures.* Firstly, we lay the theoretical background for the construction of imprimitive structures.

4.2.1. *Parameters of substructures and quotient structures.* For this subsection, let $\Gamma = (P, \mathcal{B}, \mathcal{I})$ be a G-FTIS of the type $r^v k^b$. Let Σ be a system of imprimitivity for G on the point set P with v_1 blocks of size v_2.

Keeping in mind that both the substructure and the quotient structure are flag-transitive, the number of blocks of the substructure is denoted by s, i.e. $|\Gamma(\Delta)| = s$, and notice that s is also the degree of a point Δ in the quotient structure and does not depend on the choice of Δ.

We also denote the number of blocks of imprimitivity that have non-empty intersections by $\Gamma(B)$, for any given $B \in \mathcal{B}$, by k_1 , i.e.

$$k_1 = |\{\Delta \in \Sigma : \Delta \cap \Gamma(B) \neq \emptyset\}|,$$

and the size of those intersections with k_2,

$$k_2 = |\Delta \cap \Gamma(B)|, \text{ for } \Delta \cap \Gamma(B) \neq \emptyset.$$

With this denotations, it holds $v = v_1 v_2$, $k = k_1 k_2$, Γ/Σ is of type $s^{v_1} k_1^b$ and Γ_Δ is of the type $r^{v_2} k_2^s$.

4.2.2. *Basic theoretical background for construction.* Two following propositions, though trivial, allow us to significantly reduce the running time and the memory usage of the Algorithm IMPRIMITIVE.

Proposition 4.8. *If Γ is point-simple, then Γ_Δ is point-simple and $k_2 < v_2$.*

Proof. If $v_2 = k_2$, then Γ is isomorphic to $(\Gamma/\Sigma)^{(v_2)}$. ∎

Proposition 4.9. *If Γ is connected, then Γ/Σ is connected and $k_1 \neq 1$.*

Proof. If $k_1 = 1$, then every block in Γ/Σ is incident with only one point, so Γ/Σ is not connected and thus Γ is not connected. ∎

Algorithm IMPRIMITIVE

We start with a quadruplet $[r, v, k, b]$ and we need to obtain two new quadruplets, which are the sets of parameters of the substructure and the quotient structure. We do this by following the steps in the next paragraph.
The first step is to find all factorizations $v = v_1 v_2$ such that $v_i \neq 1$. Next, we find factorizations $k = k_1 k_2$, such that $k_1 \neq 1$, (Proposition 4.9). Now we check if $\frac{k_1 b}{v_1}$ is an integer and if $k_2 < v_2$ holds (Proposition 4.8). If both are true, we denote $s = \frac{k_1 b}{v_1}$, and thus we obtain quadruplets $[r, v_2, k_2, s]$ and $[s, v_1, k_1, b]$. Now we construct all point-simple FTIS (Proposition 4.8) of the type $r^{v_2} k_2^s$ which are candidates for substructures and all connected FTIS (Proposition 4.9) of type $s^{v_1} k_1^b$ which are candidates for quotient structures. We do this using one of three main algorithms. After that, we search for all automorphism groups that act flag-transitively on those structures and store

their group action on points into two lists. Then we search for subgroups $G \leq H \wr K$, for each H from the list of automorphism groups of the substructure, and for each K from the list of automorphism groups of the quotient structure. We start with the group $H \wr K$ and go down the lattice of maximal subgroups while filtering and keeping only those groups G that are transitive, act on blocks of imprimitivity in the same way as H(i.e. $(G_\Delta)^\Delta = H$), and act on the system of imprimitivity in the same way as K(i.e. $G^\Sigma = K$).

5. Arc-transitive graphs as incidence structures

At this point we observe arc-transitive graphs more closely, as well as their relation to FTIS and BFTIS we are analyzing.

Commonly, a *simple graph* Λ is defined as a pair (V, E), where V is a non-empty set and $E \subseteq \binom{V}{2}$. Elements of V are called vertices and elements of E are called edges. For a simple graph Λ we define the set $A = \{(x, y) : \{x, y\} \in E\}$ which we call a *set of arcs* of the graph Λ. We can imagine this as replacing edges of Λ, each with two arcs of the opposite orientation.

Let Λ be a simple graph without isolated vertices, and G a group. We say that Λ is *G-arc-transitive* or *G-symmetric*, if G acts transitively on the set A. Analogously, we define a *G-vertex-transitive graph* and a *G-edge-transitive graph*. We say that a graph Λ is *arc-transitive* if there exists a group G such that Λ is G-arc-transitive. In that case it follows that Λ is both vertex-transitive and edge-transitive.

Let $\Lambda = (V, E)$ be a simple graph. A simple incidence structure with point set V, and block set E is denoted by $is(\Lambda)$. We can immediately tell that in $is(\Lambda)$, the degree (cardinality) of each block is 2.

It can be shown that Λ is G-arc-transitive, for some group G, if and only if $is(\Lambda)$ is a G-FTIS. If Λ is G-arc-transitive r-regular graph with v vertices then the incidence structure $is(\Lambda)$ is of the type $r^v 2^{\frac{rv}{2}}$.

Above shown is one way how graphs lead to incidence structures, but what about the opposite? We have already seen that an incidence structure can be represented as a graph in two ways, and here we observe its representation as the Menger graph. The Menger graph of the incidence structure $is(\Lambda)$ is Λ. For example, the Menger graph of the Fano plane is the complete graph K_7, as in the Fano plane for each two points there is a block to which they are both incident.

If Γ is a block-simple FTIS with blocks of degree 2, then its Menger graph is arc-transitive. Furthermore, a graph Λ is connected if and only if $is(\Lambda)$ is connected. An incidence structure $is(\Lambda)$ is point-simple and block-simple if the graph K_2 is not a component of Λ.

To summarize, we give the following proposition.

Proposition 5.1. *If Γ is a BFTIS of the type $r^v 2^{\frac{rv}{2}}$, then its Menger graph is a connected r-regular arc-transitive graph with v vertices.*

This is why the results in the construction of BFTIS are a contribution to construction and classification of arc-transitive graphs. We will dedicate a part of Section 6, the Results, to list all, until now known, classifications of arc-transitive graphs and show how we expanded the known census.

Because we are dealing with a small k, in the construction of these FTIS, Algorithm k-REP was mainly used.

Let us now give an example to better illustrate this connection. As the results will show, for $k = 2$ and for $v \in \{3, 4, 5\}$, the only BFTIS, and therefore the only arc-transitive graphs, are cycles and complete graphs, the structures of the type $2^3 2^3$; $2^4 2^4, 3^4 2^6$; $2^5 2^5, 4^5 2^{10}$. Whereas for $v = 6$, we have four BFTIS, and Menger graphs of two structures that are not a cycle or a complete graph are shown in Figure 3. Their types are $3^6 2^9$ and $4^6 2^{12}$ and their graphs are $K_{3,3}$ and $K_{2,2,2}$.

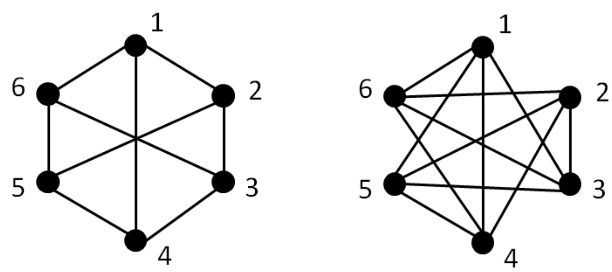

FIGURE 3. $K_{3,3}$ and $K_{2,2,2}$

6. RESULTS

In this section, we present tables with numbers of existing BFTIS for different parameter conditions. All constructed BFTIS referenced in our tables can be found in our database [18].

It is worthy to note that incidence structures with small values of parameter r are interesting on its own and are researched in the below mentioned papers we compared our own results to. For incidence structures with $r = 2$ it holds $k = 2$, so it follows $v = b$. The observed BFTIS are of type $2^n 2^n$ and their Levi graph is a cycle. Incidence structures with $r = 3$ and $k \in \{2, 3\}$ are studied as cubic semi-symmetric and cubic symmetric graphs in [9, 11]. For $r = 4$ and $k \in \{2, 3, 4\}$, incidence structures are studied as edge-transitive tetravalent graphs in [20].

In Table 1 we give the number of existing BFTIS on v points for each $v \leq 23$.

TABLE 1. The number of BFTIS for each v

v	#FTIS	v	#FTIS	v	#FTIS	v	#FTIS
1	1	7	10	13	29	19	27
2	0	8	22	14	71	20	412
3	1	9	23	15	105	21	201
4	3	10	38	16	314	22	140
5	4	11	21	17	33	23	44
6	10	12	111	18	302		

Those BFTIS for which the number of points and blocks combined is less than or equal to 79 are listed in Table 2. We omit those values of $v + b$ for which there are no BFTIS.

TABLE 2. The number of BFTIS for values of $v + b$

$v+b$	#FTIS	$v+b$	#FTIS	$v+b$	#FTIS	$v+b$	#FTIS
2	1	26	9	45	30	63	53
6	1	27	4	46	6	64*	$\geq$86
8	2	28	12	48	73	65	11
10	3	30	17	49	5	66	37
12	2	32	22	50	32	68	17
14	5	33	3	51	6	69	1
15	4	34	6	52	23	70	59
16	5	35	8	54	58	72*	$\geq$157
18	6	36	31	55	5	74	9
20	7	38	5	56	87	75	46
21	5	39	5	57	6	76	20
22	6	40	40	58	6	77	22
24	13	42	37	60	108	78	51
25	3	44	13	62	14		

* As can be seen in Table 2, we did not manage to construct all BFTIS for $v + b < 80$. At least, we are not sure if we did. For $v + b \in \{64, 72\}$ we are yet to find out if there are any BFTIS of the type $k^{32}k^{32}$ where $k \in \{8, 12, 16\}$ or any basic FTIS of the type $k^{36}k^{36}$ where $k \in \{6, 8, 10, 12, 16, 20\}$.

Finally, in Table 3, we give numbers of BFTIS for each $v \leq 63$, where each block consists of exactly two points, i.e. $k = 2$. Those BFTIS correspond to arc-transitive graphs which were classified in [12] for all $v \leq 47$. Also, all such graphs, with a prime number of points, were found in [7]. Cases where the number of points were $2p$ and $3p$ for some prime number p, were solved in [8, 27]. Lastly, in [21, 22], all cases where v equals the product of two prime numbers were described. We used the construction methods to construct all those previously described cases and to try to find all the missing ones for $48 \leq v \leq 63$.

TABLE 3. The number of BFTIS for each v, where $k = 2$

v	#FTIS	v	#FTIS	v	#FTIS	v	#FTIS
3	1	19	3	35	15	51	13
4	2	20	22	36	67	52	32
5	2	21	13	37	6	53	4
6	4	22	8	38	10	54*	≥23
7	2	23	2	39	12	55	24
8	5	24	34	40	68	56*	≥40
9	4	25	11	41	6	57	14
10	8	26	13	42	56	58	11
11	2	27	20	43	4	59	2
12	11	28	26	44	16	60*	≥50
13	4	29	4	45	42	61	8
14	8	30	41	46	7	62	17
15	10	31	4	47	2	63*	≥36
16	15	32	42	48*	≥39		
17	4	33	8	49	18		
18	14	34	10	50	40		

For values of v marked with * we managed to construct BFTIS, but only for some parameters r. The number listed in Table 3 is the number of BFTIS constructed so far. We give those values of r for which the cases remain unsolved in Table 4.

TABLE 4. Unsolved cases from Table 3

v	values of r for which BFTIS of type $r^v 2^{\frac{rv}{2}}$ were not constructed
48	6, 8, 9, 10, 12, 16, 18, 20, 24, 32
54	6, 8, 9, 12, 16, 18, 24, 32
56	6, 7, 8, 12, 14, 16, 24
60	6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 32
63	6, 12, 18

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Konstrukcija flag-tranzitivnih incidencijskih struktura

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Sažetak. Cilj ovog istraživanja je razvoj učinkovitih tehnika konstrukcije flag-tranzitivnih incidencijskih struktura. U članku su opisane tehnike i metode konstrukcije te izneseni rezultati. Istaknuta je veza nekih tipova flag-tranzitivnih incidencijskih struktura i lučno-tranzitivnih grafova.

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