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ON QUASI-EXACTLY SOLVABLE POTENTIALS

CAO XUAN CHUAN

01 Parvis du Breuil, 92160 Antony, France

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It is shown that for any exactly solvable potential considered here, belonging to the class of the first generation, there are always posibilities to construct the class of second generation potentials, which are either exactly solvable or quasi-exactly solvable. The alternatives exactly solvable or quasi-exactly solvable depend on the choice of the "base" related to the set of excited eigenstates corresponding to the first generation potentials.

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1. Introduction

Exactly solvable potentials are referred to those for which the Schrödinger equation can be solved analytically, generating the sets of eigenvalues, E_n, and eigenfunctions, ϕ_n.

For quasi-exactly solvable potentials, only part of these sets can be reached analytically, the remaining must be dealt by other means, mostly numerical.

The interest in investigating the second-generation potentials is to wonder what would be the possible links between these two classes of potentials, or, in other words, would it be possible to construct the second generation potentials which are exactly or quasi-exactly solvable from the first ones?

During the past decade, one may observe the first attempts in that direction, undertaken from different angles of attack, by taking advantage of the recent advances in supersymmetry ($SU(2)$) [1–6].

Assumig that the couple of partner potentials are exactly solvable and defined by

$$V^{(1)\mp} = u^2 \mp u' ,$$

where $u(x)$ is the usual superpotential, the second generation potentials are defined by

$$V^{(2)\mp} = v^{(2)^2} \mp v^{(2)'} , \tag{1}$$

where $v^{(2)}$ is the second generation superpotential,

$$v^{(2)} = u - \frac{\phi_n'}{\phi_n} , \tag{2}$$

while ϕ_n represents the n^{th} excited eigenstate related to $V^{(1)-}$.

When $n = 0$ (the ground state), this definition does not bring anything new, but when $n > 0$, the second-generation potential $v_n^{(2)}$ must contain singularities which reflect the presence of the "nodes" of the eigenstates (theorem of Sturm-Liouville). As a consequence, the second-generation potentials $V^{(2)\mp}$ will also involve singularities. Thus, the next question which arises naturally is what can one expect from the construction in Eq. (1)?

The present work will focus on two aspects which may serve for future reseach work:

(1) How and to which extent the second-generation potential $V^{(2)\mp}$ is solvable?

(2) In the context of $SU(2)$, how the concept of shape invariance and double degeneracy, which is associated with the first generation, can be conserved or broken with the second generation?

Since it will be convenient to approach this problem within the frame of the theory of mixing function, it is useful to devote the first part of this work to the methodological aspect with a brief reminder of the guidelines. The details can be found in earlier references [7–10].

In the second part, these results will serve to analyse the situation by showing that one must discern two categories of the "bases", for which the second-generation potentials may either be exactly or non-exactly solvable, with partial or total breaking of the symmetry, and conservation or non-conservation of the shape invariance.

In order to show how the theory can be implemented, the third part describes the details of an example. The case of the harmonic oscillator is chosen because of its potential role in various models of many fields of physics.

Next, returning to more general considerations, in the last part, the proof of a theorem will be given according to which, under certain conditions, it is always possible to construct a new type of quasi-exactly solvable potentails, regardless of the choice of the "base".

2. *Theory*

Let the first-order differential matrix equation

$$\phi' + F\phi = 0, \quad \text{with} \quad \phi = (\phi_1, \phi_2)^+ \quad \text{and} \quad F = \begin{pmatrix} u & d \\ 0 & u \end{pmatrix}, \tag{3}$$

$u(x)$ and $d(x)$ being arbitrary analytic functions. By construction,

$$\phi_2 \simeq \exp\left(\int -u\,\mathrm{d}x\right).$$

Let the mixing function be

$$X(x), \qquad \text{with} \qquad \phi_1 = X\phi_2 \,. \tag{4}$$

It has already been shown that if X is a solution of the second-order differential equation

$$X'' - 2uX' - E_n X_n = 0\,, \tag{5}$$

with E_n constant, the Schrödinger equation corresponding to the first component ϕ_1

$$\phi_1'' - V^{(1)-}\phi_1 = E_n\phi_1\,, \qquad \text{with} \qquad \phi_1 = X_n \exp\left(-\int u\mathrm{d}x\right) \tag{6}$$

must be exactly solvable with eigenspectrum $\{E_n\}$ and eigenfunctions $\{\phi_{1,n}\}$.

Likewise, let

$$\bar{\phi}' + \bar{F}\bar{\phi} = 0, \qquad \text{with} \qquad \bar{\phi} = (\bar{\phi}_1, \phi_2) \qquad \text{and} \qquad \bar{F} = \begin{pmatrix} -u & \bar{d} \\ 0 & u \end{pmatrix}, \tag{7}$$

and introduce a second mixing function Y, with $\bar{\phi}_1 = Y\phi_2$, then the Schrödinger equation

$$\bar{\phi}_1'' - V^{(1)+}\bar{\phi}_1 = \bar{E}\bar{\phi}_1 \tag{8}$$

is also exactly solvable if Y satisfies the second differential equation

$$Y'' - 2uY' - (2u' + \bar{E})Y = 0\,. \tag{9}$$

Noting that Eq. (9) is simply the result of differentiation of Eq. (5), one can infer

$$Y = X_n' \qquad \text{and} \qquad \bar{E} = E_n\,. \tag{10}$$

In the context of $SU(2)$, $u(x)$ is playing the role of the superpotential of the first generation. The second condition in Eq. (10) merely expresses the double degeneracy of $V^{(1)-}$ and $V^{(1)+}$.

3. *The second generation*

Let $|n\rangle$ be any excited state of the first component $V^{(1)-}$. The second generation couple can be constructed according to Eqs. (1) and (2). The following results are obtained and already described in earlier references:

(a) The first component of the second generation couple $V^{(2)-}$, up to a constant, is similar to $V^{(1)-}$, i.e.,

$$V^{(2)-} = V^{(1)-} + const\,,$$

which means that regardless of the choice of the base, if $V^{(1)-}$ is exactly solvable, so will be $V^{(2)-}$, too.

(b) The eigenspectrum of the second generation couple is

$$E_m^{(2)} = E_m^{(1)} - E_n^{(1)}\,, \tag{11}$$

where $E_m^{(1)}$ and $E_n^{(1)}$ are eigenvalues corresponding to the first component.

(c) The second component $V^{(2)+}$ of this second generation couple is of the form

$$V^{(2)+} = V^{(1)+} + 2\frac{X_n'}{X_n}\left(\frac{X_n'}{X_n} - 2u\right) - E_n\,. \tag{12}$$

Its eigenfunction can be written as

$$\bar{\phi}_{1,m}^{(2)} = Y_m \,\exp\left(-\int u\mathrm{d}x\right)\,, \tag{13}$$

where

$$Y_m = X_m' - \frac{X_n'}{X_n}X_m\,. \tag{14}$$

The solution is not defined for $m = n$, and normalisation requirements will put further constraints on this solution.

For instance, the presence of the singularity contained in the analytic expressing of Eq. (14) can be removed only if some kind of "factorisation" is possible in the sense

$$X_m = X_n F_n(x)\,.$$

$F_n(x)$ is an unspecified function.

When this is not the case, while the first component $V^{(2)-}$ is always exactly solvable for the reasons invoked in (a), the second component $V^{(2)+}$ is not because relation (14) becomes invalid. The singularities appearing in $V^{(2)+}$ are generally "strong" in the sense of Eq. (12), which, on the other hand, require that at the locations of these singularities ($x = x_i$), $\bar{\phi}^{(2)}(x_i) = 0$.

The whole situation radically changes furthermore when one introduces another parameter t and constructs the second-generation superpotential $v_t^{(2)}$ as

$$v_t^{(2)} = u - t\frac{X_n'}{X_n}\,.$$

The couple becomes

$$\begin{aligned} V^{(2)-} &= V^{(1)-} + t(t-1)\left(\frac{X_n'}{X_n}\right)^2 + tE_n\,, \\ V^{(2)+} &= V^{(1)+} + t(t+1)\left(\frac{X_n'}{X_n}\right)^2 - 4tu\frac{X_n'}{X_n} - tE_n\,. \end{aligned} \tag{15}$$

Note that when $t = 1$, the relation (12) is recovered. The problem becomes even more complicated because neither potential is exactly solvable, except perhaps for some very special choices of the base.

Thus, before engaging oneself in this direction, it will be useful to return to the simplest case when $t = 1$, in order to see how the theory can be implimented in practice.

4. *Example*

Consider the case of the harmonic oscillator potential for which the superpotential of the first generation can be defined as

$$u(x) = \frac{1}{2}x\,.$$

Obviously, the solutions of Eq. (5) can be represented by the usual Hermite polynomials $X_n = H_n(z)$. Since the study of the first component $V^{(2)-}$ does not bring anything new, the second component, $V^{(2)+}$, is more interesting. Its analytic expression for the first values of n of the bases are

$$\begin{aligned} n = 1 \quad V_1^{(2)+} &= \frac{1}{4}x^2 + \frac{2}{x^2} - \frac{1}{2} \\ n = 2 \quad V_2^{(2)+} &= \frac{1}{4}x^2 + \frac{4x^2}{x^2-1}\left(\frac{2}{x^2-1} - 1\right) - \frac{3}{2} \\ n = 3 \quad V_3^{(2)+} &= \frac{1}{4}x^2 + 6\frac{x^2-1}{x^2-3}\left(\frac{3}{x^2}\,\frac{x^2-1}{x^2-3} - 1\right) - \frac{5}{2}\ldots \end{aligned} \tag{16}$$

Below are some interesting remarks which may be useful in practice. Some of them have already been mentioned in Ref. [6].

(a) $V_n^{(2)+}(-x) = V_n^{(2)+}(x)$ (even parity).

(b) At infinity, they all tend to the classical harmonic-oscillator potential.

(c) If one chooses the base $n = 1$, then it can be verified that $V_1^{(2)-}$ and $V_1^{(1)-}$ differ only by a constant.

(d) The couple $V_1^{(2)-}$ and $V_1^{(2)+}$ is shape-invariant when $t \neq 1$, while it is not when $t = 1$.

The discussion will now be focused on the special case $t = 1$, with $n = 1$ (i.e., $X_n = H_n$). The eigenfunctions of $V_1^{(2)-}$ can be written as

$$\phi_m = N_m X_m \exp\left(-\int u \, \mathrm{d}x\right) ,$$

where N_m is the normalization constant, with the eigenspectrum $E_m^{(2)} = E_m^{(1)} - E_n^{(1)}$ in a general case. Depending on the choice of the base $|n\rangle$, one thus may have negative eigenvalues. For instance, if $n = 1$, one has only one such state. The existence of such states was pointed out earlier [1], and it is beleived to have an impact on the construction of certain field theories.

For $V_1^{(2)-}$, the running index m in Eq. (16) may be any integer ($m = 0, 1, 2, \ldots$), but for $V_1^{(2)+}$, only the solutions with m odd ($m = 1, 3, 5, \ldots$) can be accepted, because the validity of Eq. (14) can be assured only in this case.

Therefore, the pairings of the eigenstates of $V_1^{(2)-}$ and $V_1^{(2)+}$ are reduced to one half.

This constitutes an example of partial pairing of the states, for which the usual definition of the Witten index, specifying a clear-cut distinction between breaking and non-breaking of symmetry, would become inadequate. On the other hand, for this case, the principle of "nodal structure" [12,13] will be more appropriate.

Keeping the same base ($n = 1$), but with $t \neq 1$, the principle of shape invariance mentioned above in (d) can be explicitly written as

$$V_1^{(2)+}(t-1, x) = V_1^{(2)-}(t, x) + t - 1 \, . \tag{17}$$

Both $V_1^{(2)-}$ and $V_1^{(2)+}$ are exactly solvable with the solutions expressed in terms of the generalized Laguerre polynomials L_m^n, providing the parameter t be an integer [14,15].

When both t and n are different from 1, the presence of singularities of type "strong" mentioned above, will split the domain of the definition of them into separated and unrelated regions, implying, therefore, different Hilbert spaces. One can, therefore, expect total breaking of symmetry.

However, the situation is not so dim since some information can still be extracted with the proof of the following theorem concerning the quasi-exact solvability of the first component $V^{(2)-}$ under certain conditions, while the second component must be approached in another way.

5. *Quasi-exact solvability*

Theorem: If equation $X'' - 2uX' - E_n X_n = 0$ (Eq. (5)) is exactly solvable with the set of solutions $\{X_n\}$, then, regardless of the choice of the base, the same equation in which u is substituted by $v^{(2)}(t, x)$, t being a parameter, will have at least two exact solutions.

Proof: The equation can be written in the form

$$X'' - 2\left(u - t\,\frac{X_n'}{X_n}\right) X' - \bar{E}\,X = 0\,, \tag{18}$$

$\bar{E}$ being a constant.

(a) Obviously, the first solution is $X = const$ which leads to $\bar{E} = 0$.

(b) The second solution has the form $X = X_n^{1-2t}$. By substitution into Eq. (18) and after simplification, one obtains

$$(1-2t)X_n^{1-2t}\left[E_n - \frac{\bar{E}}{1-2t}\right] = 0\,,$$

which implies that $\bar{E} = (1-2t)E_n$.

Note that this derivation is independent of the choice of $|n\rangle$.

Alternatively, one may also use the method described earlier [8] by letting $X = F(x)/X_n^s$, $F(x)$ being an unspecified function and s a parameter. Then, Eq. (5) can be cast into the form

$$\frac{X'' - 2uX'}{X} = A(x) + B(x)\,, \tag{19}$$

where

$$A(x) = \frac{1}{X_n^s}\left[F'' - 2uF'\right] - \frac{sF}{X_n^{s+1}}\left[X_n'' - 2uX_n'\right],$$

and

$$B(x) = 2(t-s)\frac{F'}{F}\frac{X_n'}{X_n} - s(s+1-2t)\left(\frac{X_n'}{X_n}\right)^2.$$

Requiring $B(x) = 0$, one obtains a relationship between s and t, $F(x) = X_n^q$, where q is arbitrary. Therefore, the simplest case will be $q = 1$, or $F(x) = X_n$. The two parameters, s and t, must satisfy the constraint

$$s(s+1-2t) = 2(t-s)\,, \tag{20}$$

which can be solved with the two following real solutions

$$s_+ = 2t, \qquad \text{and} \qquad s_- = 1\,.$$

In this case, the eigenvalues are

$$E_+ = (1-2t)E_n \qquad \text{and} \qquad E_- = 0\,,$$

with

$$A_+ = 2t\,,$$

corresponding to the two solutions mentioned above.

6. *Application*

In the context of the present discussion, the above theorem can be used by returning to the system (1) in which the quantity $u(x)$ is replaced by $v_n^{(2)}(x)$.

With the above results, one can see that the Schrödinger equation is solvable if the function X satisfies Eq. (19)

$$\phi''_{1,\pm} - V^{(2)-}\,\phi_{1,\pm} = E\phi_{1,\pm}\,.$$

The two eigenfunctions can be written as

$$\phi_{1,+} = X_n^{1-t}\,\exp\left(\int -u\,\mathrm{d}x\right) \quad \text{and} \quad \phi_{1,-} = X_n^{t}\,\exp\left(\int -u\,\mathrm{d}x\right)\,.$$

Normalisation requires that $0 < t < 1$.

Extension. The condition of exact solvability of Eq. (5) is sufficient but not necessary, which means that the function X must not be limited to the set of usual orthogonal functions of mathematical physics. The details of this aspect will be presented later in another report.

7. *Conclusion*

Several facets of the use of the method of mixing functions are discussed. Particularly, it is quite adapted for analysing the problem of singular potentials of the second generation, which are constructed from the first ones that are assumed supersymmetric and exactly solvable.

For the special choice of the base, both components of the second generation, $V^{(2)-}$ and $V^{(2)+}$, are exactly solvable, but with the loss of shape and a partial pairing of the states.

For that example, but with the parameter $t \neq 1$, the concept of the shape invariance remains valid, and for the same choice, both components are also exactly solvable under specific conditions of this parameter.

But for other choices of the base, both components are not solvable exactly with a complete breaking of the symmetry.

However, under certain different specific constraints on this parameter, the first component $V^{(2)-}$ can be made quasi-exactly solvable, with an exact "doublet", regardless of the base.

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