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$O(d,d)$ INVARIANT SOLUTIONS OF SPACE–TIME DEPENDENT STRING VACUA

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We solve the $O(d,d)$ invariant equation of motions for string vacua for $v(\varphi)$ of the form $v(\varphi) = -B_0 \mathrm{e}^{-\alpha\varphi}$.

1. *Introduction*

One of the deepest quantum symmetries of string theory is target space duality which says that a (closed) string moving on a circle of radius R is equivalent to one which moves on a circle of radius α'/R. This is known as the R-duality [1]. Recently it has been shown [2-4] that if one starts with the low energy effective action in string theory and restricts oneself to a space-translation invaiant but time-dependent background, the effective exibits an $O(d,d)$ symmetry. This is also known as scale factor duality (SFD). This result has far reaching consequences for, if given one space translation invariant solution of the equation of motion, we can generate others by applying the $O(d,d)$ transformation on the original solution.

As $O(d, d)$ is a symmetry of the effective action, not the full action, it will relate, in general, physically inequivalent cosmological solutions of the string modified Einstein-Friedmann equations which include a non-trivial dilation. Unlike the R--duality, this symmetry does not rest on compactification and conects expanding universes to contracting universes [5]. In this context one can also give a Pre-Big--Bang picture [6]. This duality aspect has been employed to relate different black holes to one another [7-9].

In this note we show that one can solve the $O(d, d)$ symmetric equations of motion obtained by variation of φ for a φ dependent $v(\varphi)$ of the form $v(\varphi) = -B_0 e^{-\alpha\varphi}$. For $\alpha \to \infty$ and $\alpha \to 0$ one recovers the cosmological solution and the 2D black hole solution respectively as obtained by Meissner and Veneziano [10].

2. *$O(d, d)$ symmetric equations of motions and their solutions*

We start with the usual genus zero low wnwrgy effective action for closed superstrings in its bosonic sector [11]

$$S = \int \mathrm{d}^D x \, \sqrt{-G} e^{-\varphi} \left[\Lambda - R - G^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{12} H_{\mu\nu\varrho} H^{\mu\nu\varrho} \right]. \tag{1}$$

Here we use the notations of Ref. 10. Note that the string tension is normalized so that $2k = 1$. Here $G^{\mu\nu}$ is the metric of the σ model [12], φ is the Fradkin-Tseytlin [13] dilation, Λ is the cosmological constant and $H_{\mu\nu\varrho}$ is the tensor defined in terms of $B_{\nu\varrho}$ as follows.

$$H_{\mu\nu\varrho} = \partial_\mu B_{\nu\varrho} + \text{cyclic terms} \tag{2}$$

where $B_{\nu\varrho}$ is the antisymmetric tensor field. If G and B are functions of time only and are invariant under general coordinate transformatons and under the transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu\Lambda_\nu]}$, one can write

$$G = \begin{pmatrix} -1 & 0 \\ 0 & G(t) \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & B(t) \end{pmatrix}. \tag{3}$$

$G(t)$ and $B(t)$ are $d \times d$ matrices where $d = D - 1$, D being the total dimensions. The curvature scalar for the σ model metric $G_{\mu\nu}$ is given by

$$R = 2\partial_0^2 \ln \sqrt{\det g} + (\partial_0 \ln \sqrt{\det G})^2 - \mathrm{Tr}(\partial_0 G)(\partial_0 G^{-1}). \tag{4}$$

Defining the field Φ by

$$\Phi = \varphi - \ln \det g \tag{5}$$

and using in (1) the value of R given in equation (3), we get

$$S = \int \mathrm{d}t\, \mathrm{e}^{-\Phi} \left\{ \Lambda + (\dot{\Phi})^2 + \frac{1}{8}\mathrm{Tr}[\dot{M}\eta\, \dot{M}\eta] \right\} \tag{6}$$

where M is a $2d \times 2d$ matrix given by

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{7}$$

and η is an $O(d,d)$ metric given by

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8}$$

1 stands for the d-dimensional unit matrix. The action is invariant under a global $O(d,d)$

$$\Phi \to \Phi, \quad M \to \Omega M \Omega^\dagger, \quad \Omega^\dagger \eta \Omega = \eta. \tag{9}$$

Reintroducing G_{00} in the action and from setting the corresponding variation to zero, the following zero energy condition is obtained.

$$\dot{\Phi}^2 + \frac{1}{8}\mathrm{Tr}(\dot{M}\eta\, \dot{M}\eta) - V(\Phi) = 0 \tag{10}$$

and V is now considered a general function of Φ.

From the variation of M we get

$$\partial_t (M\eta\dot{M}) = (M\eta\dot{M}). \tag{11}$$

Integrating we get

$$\mathrm{e}^{-\Phi} M\eta\dot{M} = \mathrm{constant} = A \;(\mathrm{say}).$$

(10) can now be written as

$$\dot{\Phi}^2 = \frac{\exp(2\Phi)}{8}\mathrm{Tr}(A\eta)^2 + V(\Phi) \tag{12}$$

where $V(\Phi)$ can be any function of Φ.

We take $V(\Phi)$ to be of the form

$$V(\Phi) = -B_0 \mathrm{e}^{-\alpha\Phi}. \tag{13}$$

The rational behind taking the form (13) is that in the limit $\alpha \to 0$ and $\alpha \to \infty$ the result for the cases V = constant and $V = 0$, respectively, can be recovered easily as will be shown below. For α small but not zero $V(\Phi)$ will be of the form $a + b\Phi + c\Phi^2$ which is the form of a harmonic oscillator with a damping term.

Putting (13) in (12) we get (henceforth we denote Φ by φ)

$$\dot{\varphi}^2 = \frac{1}{C^2} \mathrm{e}^{2\varphi}(1 - D\mathrm{e}^{-\beta\varphi}) \tag{14}$$

where

$$C = \frac{8}{\mathrm{Tr}(A\eta)^2}, \quad \beta = \alpha + 2 \quad \text{and} \quad D = B_0 C^2. \tag{15}$$

Solution of (14) can be found easily and turns out to be

$$\begin{aligned} E(t_0 - t) &= \int\limits_{z_0}^{z} Z^{-\left(\frac{\alpha+1}{\alpha+2}\right)} (1-z)^{-\frac{1}{2}} \mathrm{d}z \\ &= B_0^{\frac{1}{\alpha+2}} C^{\frac{2}{\alpha+2}} \mathrm{e}^{-\varphi}\, B\left(\frac{1}{\alpha+2}, 1\right)\, F\left(\frac{1}{2}, \frac{1}{\alpha+2}, \frac{\alpha+3}{\alpha+2}, Z\right) - Z \leftrightarrow Z_0 \end{aligned} \tag{16}$$

where

$$E = (\alpha+2)(B_0 \mathrm{e}^{-\alpha})^{\frac{1}{\alpha+2}}, \quad Z = D\mathrm{e}^{-\beta\varphi} \tag{17}$$

$B(\alpha, \beta)$ is the usual Beta function and $F(a, b, c, z)$ is the well known hypergeometric function.

3. Limiting cases

Case I: $\alpha \to 0$

From (16), using the result that $\lim\limits_{\alpha\to 0} Z = D\mathrm{e}^{-2\varphi} = B_0 C^2 \mathrm{e}^{-2\varphi}$ and $E \to 2B_0^{\frac{1}{2}}$, we get

$$C\mathrm{e}^{-\varphi}\, B\left(\frac{1}{2}, 1\right)\, F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, B_0 C^2 \mathrm{e}^{-2\varphi}\right) - \varphi \leftrightarrow \varphi_0 = 2(t_0 - t). \tag{18}$$

But, $B\left(\dfrac{1}{2}, 1\right) = 2$ and

$$F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, B_0 C^2 \mathrm{e}^{-2\varphi}\right) = \frac{1}{B_0 C} \mathrm{e}^{\varphi} \sin h^{-1}[B_0 C \mathrm{e}^{-\varphi}] \tag{19}$$

(see Ref. 14).

Hence from (18) we get,

$$\mathrm{e}^{\varphi} = CB_0 / \sin h(T - t) \tag{20}$$

where we have taken $t_0 = T$.

Hence, result of Meissner and Veneziano [9], $v(\varphi) = \text{constant}$, is reproduced when $\alpha \to 0$.

Case II: $\alpha \to \infty$

Using the result

$$\lim_{\alpha\to\infty} F\left(\frac{1}{2}, \frac{1}{\alpha+2}, \frac{\alpha+3}{\alpha+2}, Z\right) = 1 \tag{21}$$

$$(\text{as } Z \to 0 \quad \alpha \to \infty)$$

and

$$\lim_{\alpha\to\infty} \frac{B\left(\frac{1}{\alpha+2}, 1\right)}{\alpha+2} = 1 \tag{22}$$

we obtain from (16)

$$\mathrm{e}^{-\varphi} - \mathrm{e}^{-\varphi_0} = \frac{1}{C}(t_0 - t). \tag{23}$$

With $\varphi_0 \to \infty$ we get back the result fo Meissner and Veneziano

$$\mathrm{e}^{\varphi} = \frac{C}{T-t}. \tag{24}$$

III. General case:

The solution for finite α is given by

$$E(t_0 - t) = B_0^{\overline{\beta}}\, \mathrm{e}^{2\overline{\beta}}\, B(\overline{\beta}, 1)\, F\left(\frac{1}{2}, \overline{\beta}, \overline{\beta}+1, B_0 C^2 \mathrm{e}^{-\varphi/\overline{\beta}}\right) \tag{25}$$

where

$$\overline{\beta} = \frac{1}{\alpha+2}$$

The dilatation time is obtained from the relation

$$\tau = \int\limits_{t_0}^{t} \mathrm{e}^{\varphi}\, \mathrm{d}t = \int\limits_{\varphi_0}^{\varphi} f(\varphi)\, \mathrm{d}\varphi \tag{26}$$

where

$$f(\varphi) = \mathrm{e}^{\varphi} \frac{\mathrm{d}t}{\mathrm{d}\varphi}.$$

Using (25) and (26), we obtain after some algebric manipulations

$$\tau = \frac{C}{\alpha+2}\, B\left(\frac{1}{\alpha+2}, 1\right) \left[\int\limits_{\varphi_0}^{\varphi} F \mathrm{d}\varphi - F(\varphi) + F(\varphi_0)\right] \tag{27}$$

where

$$F(\varphi) = F\left(\frac{1}{2}, \frac{1}{\alpha+2}, \frac{\alpha+3}{\alpha+2}, B_0 C^2 \mathrm{e}^{-(\alpha+2)\varphi}\right) \tag{28}$$

For large α, Z is small. By expanding $F(a,b,c,z)$ and keeping terms upto $O(z^2)$ we have

$$F(a,b,c,z) \simeq 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \ldots \tag{29}$$

From (27), (28) and (29), after some straightforward but tedious calculations it can be shown that

$$\begin{aligned}\tau =& \frac{CB\left(\frac{1}{\alpha+2}, 1\right)}{\alpha+2}\left\{(\varphi-\varphi_0) - \frac{B_0 C^2}{2(\alpha+2)}\right\}\left[\mathrm{e}^{-(\alpha+2)\varphi} - \mathrm{e}^{-(\alpha+2)\varphi_0}\right] - \\ & - B^2 C^4 \left[\mathrm{e}^{-2(\alpha+2)\varphi} - \mathrm{e}^{-2(\alpha+2)\varphi_0}\right] \frac{1}{8(\alpha+2)}\,. \end{aligned} \tag{30}$$

For $\alpha \to \infty$ only the first term contributes and we get

$$\tau = C(\varphi - \varphi_0) = C \ln \frac{T - t_0}{T - t} \tag{31}$$

which was obtained by Meissner and Veneziano (1991a).

4. Conclusion

In this paper we have found the general solution of $O(d,d)$ invariant equations of motion for low energy effective action of string gravity coupled to dilation and antisymmetric tensor. For the particular cases our solutions exactly reproduce the earlier results.

For the general case $V(\varphi) \simeq \mathrm{e}^{-\alpha\varphi}$ the analytical expression for dilation time has been given explicitly.

References

1) K. Kikkawa and M. Yamasaki, Phys. Lett. B **149** (1984);
K. S. Narain, Phys. Lett. B **169** (1986) 41;

2) K. A. Meissner and G. Veneziano, Symmetries of Cosmological Superstring Vacua, Mod. Phys. Lett. A **6** (1991) 3397;

3) A. Sen, Preprint TIFR/TH/91-35 (July);

4) G. Veneziano, *Scale Factor Density for Classical and Quantum Strings*, CERN-TH--6077/91;

5) K. A. Meissner and G. Veneziano, Phys. Lett. B **267** (1991) 33;
M. Gasperini and G. Veneziano, *Pre-Big Bang in String Cosmology* – CERN-TH-6572/92;

6) A. A. Tseytlin, Mod. Phys. Lett. A **6** (1991) 1721;

7) E. Witten, *String Theory and Black Holes*, IASSNS-HEP-91/12;

8) G. Mandal, A. M. Sengputa and S. R. Wadia, Preprint IASSNS-HEP-91/10;

9) A. Sen, Preprint TIFR/TH/91/37 (Aug. 1991);

10) M. Gasperini and A. G. Veneziano, *Manifesty $O(d,d)$ Invariant Approach to Space-time Dependent String Vacua* – CERN-TH-6235/91 (Sept.);

11) M. Gasperini and G. Veneziano, *$O(d,d)$ Covariant String Cosmology* – CERN-TH 6321/91;

12) E. Abdalla, M. C. B. Abdalla, K. D. Rothe, *Non-perturbative Methods in 2-Dimensional Quantum Field Theory* (World Scientific) 1991;

13) E. S. Fradlkin and A. A. Tseytlin, Phys. Lett. **158B** (1985) 316;

14) A. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover N. Y. 1970);

$O(d,d)$ INVARIJANTNO RJEŠENJE PROSTORNO-VREMENSKI OVISNIH VAKUUMA STRUNA

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Originalni znanstveni rad

Riješena je $O(d,d)$ invarijantna jednadžba gibanja za vakuume struna za $v(\varphi)$ oblika $v(\varphi) = -B\mathrm{e}^{-\alpha\varphi}$.