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A NEW TRIAL EQUATION METHOD TO FIND THE EXACT TRAVELING WAVE SOLUTIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

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We propose an new trial equation method to solve nonlinear differential equations. By this method, we obtain some exact solutions to the RLW-Burgers equation and the (2+1)-dimensional KdV-Burgers equation.

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1. *Introduction*

It is important to find exact traveling wave solutions of nonlinear differential equations for many applications. Several methods have been proposed such as inverse scattering method [1], direct method [2], Backlund transformation [3], algebraic expansion method (Ref. [4] and references therein), the complete discrimination system method [5], and so on. Liu [6−9] proposed the trial equation method to find exact solutions to nonlinear differential equations. In order to describe Liu's method, we consider a differential equation of u. We always assume that its exact solution satisfies a solvable equation $u' = F(u)$ or $u'' = F(u)$. Therefore, our task is just to find the function F. Liu has obtained abundant exact solutions of many nonlinear differential equations when $F(u)$ is a polynomial or a rational function. In the present paper, we take F as a new irrational function form and propose a new trial equation. As application, we consider the RLW-Burgers equation

$$u_t + u_x + 12uu_{xx} - \alpha u_{xx} - \beta u_{xxt} = 0\,. \tag{1}$$

RLW-Burgers equation (1) is a model equation for describing the propagation of surface water in a channel and it represents a balance relation among the dispersion, dissipation and nonlinearity. Eq. (1) has been researched in some papers [10 − 14]. We also consider the (2+1)-dimensional Burgers equation

$$(u_t + uu_x + \alpha u_{xxx} - \beta u_{xx})_x + \gamma u_{yy} = 0\,, \tag{2}$$

and obtain some of its exact solutions by the same method.

The rest of the paper is organized as follows. In Section 2, the new trial equation method is described in detail. In Section 3, the application to RLW-Burgers equation and the (2+1)-dimensional KdV-Burgers equation are given. The last section is a short summary.

2. *New trial equation method*

We consider the following nonlinear partial differential equation

$$N(u, u_t, u_{tt}, \cdots, u_x, u_{xx}, \cdots, u_{tx}, \cdots) = 0\,. \tag{3}$$

Under the traveling wave transformation

$$u = u(\xi), \quad \xi = kx + \omega t\,, \tag{4}$$

Eq. (3) becomes the following ordinary differential equation,

$$P(u, u', u'', \cdots) = 0\,, \tag{5}$$

where the prime means the differentiation with respect to ξ. Sometimes, by integration, the order of Eq. (5) can be reduced. Now, our method can be described as follows.

Step 1. Take a trial equation

$$u' = \sum_{i=0}^{k_1} a_i u^i + (\sum_{i=0}^{k_2} b_i u^i)\sqrt{\sum_{i=0}^{k_3} c_i u^i}\,, \tag{6}$$

where $a_0, \cdots, a_{k_1}$,$b_0, \cdots, b_{k_2}$ and $c_0, \cdots, c_{k_3}$ are the constants to be determined. Using Eq. (6), we derive the following equation

$$u'' = (\sum_{i=1}^{k_1} ia_i u^{i-1})(\sum_{i=0}^{k_1} a_i u^i) + (\sum_{i=0}^{k_2} b_i u^i)(\sum_{i=1}^{k_2} ib_i u^{i-1})(\sum_{i=0}^{k_3} c_i u^i)$$
$$+\frac{1}{2}(\sum_{i=0}^{k_2} b_i u^i)^2(\sum_{i=1}^{k_3} ic_i u^{i-1}) + \frac{1}{2}(\sum_{i=0}^{k_1} a_i u^i)(\sum_{i=0}^{k_2} b_i u^i)(\sum_{i=1}^{k_3} ic_i u^{i-1})(\sum_{i=0}^{k_3} c_i u^i)^{-\frac{1}{2}}$$

$$+\left[(\sum_{i=1}^{k_2} ib_iu^{i-1})(\sum_{i=0}^{k_1} a_iu^i)+(\sum_{i=1}^{k_1} ia_iu^{i-1})(\sum_{i=0}^{k_2} b_iu^i)\right]\sqrt{(\sum_{i=0}^{k_3} c_iu^i)}\,, \quad (7)$$

and other derivation terms such as u''', and so on.

Step 2. Substituting u',u'' and other derivation terms into Eq. (5) yields following expression

$$G(u)+H(u)\sqrt{\sum_{i=0}^{k_3} c_iu^i}=0\,, \quad (8)$$

where $G(u)$ and $H(u)$ are two polynomials of u. According to the balance principle, we can obtain the relation of k_1, k_2 and k_3 or their values.

Step 3. Taking concrete values of k_1, k_2 and k_3, and letting all coefficients of $G(u)$ and $H(u)$ to be zero yield a system of nonlinear algebraic equations. Solving the system of nonlinear algebraic equations, we obtain the values of $a_0, \cdots, a_{k_1}$,$b_0, \cdots, b_{k_2}$ and $c_0, \cdots, c_{k_3}$.

Step 4. Integrating Eq. (6) gives the solutions of u.

3. Application

Example 1. RLW-Burgers equation (1)

Under the traveling wave transformation(4) and integration, the RLW-Burgers Eq. (1) becomes

$$u''+Au'=Bu^2-Cu+D\,, \quad (9)$$

where D is an arbitrary constant. We denote $A=\dfrac{\alpha}{\omega\beta}, B=\dfrac{6}{k\beta\omega}$ and $C=\dfrac{\omega+k}{k^2\beta\gamma}$. Substituting Eq. (6) and Eq. (7) into Eq. (9) and using the balance principle, it follows that $2k_2+k_3-1=2$ and $2k_1-1<2$. Then we obtain $k_1=k_2=k_3=1$ or $k_1=0, k_2=k_3=1$.

If $k_1=k_2=k_3=1$, Eq. (6) becomes

$$u'=a_1u+a_0+(b_1u+b_0)\sqrt{c_1u+c_0}\,, \quad (10)$$

where a_i, b_i, c_i are the parameters to be determined, for $i=0,1$. Furthermore, from Eq. (10), we have

$$u''=\left\{a_1+b_1\sqrt{c_1u+c_0}+\frac{c_1(b_1u+b_0)}{2\sqrt{c_1u+c_0}}\right\}\{a_1u+a_0+(b_1u+b_0)\sqrt{c_1u+c_0}\}\,. \quad (11)$$

Substituting u' and u'' into Eq. (9) yields

$$G(u) + H(u)\sqrt{c_1 u + c_0} = 0\,, \tag{12}$$

where

$$\begin{aligned} G(u) = {} & \Big(Ab_1c_1 + \frac{5}{2}a_1b_1c_1\Big)u^2 + \Big((A+2a_1)b_1c_0 + Ab_0c_1 \\ & + \frac{3}{2}a_1b_0c_1 + \frac{3}{2}a_0b_1c_1\Big)u + (A+a_1)b_0c_0 + a_0b_1c_0 + \frac{1}{2}a_0b_0c_1\,, \end{aligned} \tag{13}$$

$$\begin{aligned} H(u) = {} & \Big(\frac{3}{2}b_1^2c_1 - B\Big)u^2 + (2b_1c_1b_0 + b_1^2c_0 + a_1^2 + a_1A - C)u \\ & + b_1b_0c_0 + \frac{1}{2}c_1b_0^2 + a_0a_1 + a_0A - D\,. \end{aligned} \tag{14}$$

In order to find the parameters, we let $G(u) \equiv 0, H(u) \equiv 0$, and hence we get a system of algebraic equations

$$\frac{3}{2}b_1^2c_1 - B = 0\,, \tag{15}$$

$$2b_1c_1b_0 + b_1^2c_0 + a_1^2 + a_1A - C = 0\,, \tag{16}$$

$$b_1b_0c_0 + \frac{1}{2}c_1b_0^2 + a_0a_1 + a_0A - D = 0\,, \tag{17}$$

$$Ab_1c_1 + \frac{5}{2}a_1b_1c_1 = 0\,, \tag{18}$$

$$(A+2a_1)b_1c_0 + Ab_0c_1 + \frac{3}{2}a_1b_0c_1 + \frac{3}{2}a_0b_1c_1 = 0\,, \tag{19}$$

$$(A+a_1)b_0c_0 + a_0b_1c_0 + \frac{1}{2}a_0b_0c_1 = 0\,. \tag{20}$$

By solving the above algebraic equations (15) − (20), we get

$$\begin{aligned} & a_0 = -\frac{12A}{5B} - \frac{AC}{5B} - \frac{6A^3}{250B}, \quad a_1 = -\frac{2A}{5}, \quad b_1 = -2, \\ & b_0 = -\frac{C}{B} - \frac{6A^2}{25B}\,, \quad c_1 = \frac{B}{6}, \quad c_0 = 1 + \frac{C}{12} + \frac{A^2}{100}, A = \pm 10\,. \end{aligned} \tag{21}$$

With these parameters, the solutions of Eq. (10) give the solution to the RLW-Burgers equation (1),

$$u_1 = \frac{k\alpha}{10}\left\{\frac{4\exp(4(kx + \frac{10\beta}{\alpha}t - \xi_0))}{(1 - \exp(\pm 2(kx + \frac{10\beta}{\alpha}t - \xi_0)))^2} - 2 - \frac{1}{12k^2\beta} - \frac{5}{6k\alpha}\right\}, \tag{22}$$

and

$$u_2 = -\frac{k\alpha}{10}\left\{\frac{4\exp(4(kx - \frac{10\beta}{\alpha}t - \xi_0))}{(1+\exp(\pm 2(kx - \frac{10\beta}{\alpha}t - \xi_0)))^2} - 2 - \frac{1}{12k^2\beta} + \frac{5}{6k\alpha}\right\}, \tag{23}$$

where k and ξ_0 are two arbitrary constants.

When $k_1 = 0$ and $k_2 = k_3 = 1$, the corresponding results of Eq. (1) are included as special cases in the solutions (22) and (23).

Example 2. (2+1)-dimensional KdV-Burgers equation (2)

With the traveling wave transformation

$$u = u(\xi), \quad \xi = kx + ly + \omega t\,, \tag{24}$$

and integrating two times, Eq. (2) becomes

$$u'' - \frac{\beta}{k\alpha}u' = -\frac{1}{2k^3\alpha}u^2 - \left(\frac{\omega}{2k^3\alpha} + \frac{l^2\gamma}{k^4\alpha}\right)u + D_1\xi + D\,, \tag{25}$$

where D_1 and D are two arbitrary constants. We let $D_1 = 0$, and denote $A = -\frac{\beta}{k\alpha}, B = -\frac{1}{2k^3\alpha}$ and $C = -\frac{\omega}{2k^3\alpha} - \frac{l^2\gamma}{k^4\alpha}$. Then Eq. (25) beomes Eq. (9). Using the same procedure as in the case of example 1, we obtain the exact solutions of the (2+1)-dimensional KdV-Burgers equation (2) as follows

$$u_1 = \frac{3\beta^3}{250\alpha^2}\left\{\frac{4\exp(4(-\frac{\beta}{10\alpha}x + ly + \omega t - \xi_0))}{(1-\exp(\pm 2(-\frac{\beta}{10\alpha}x + ly + \omega t - \xi_0)))^2} - 2 + \frac{125\alpha^2\omega}{3\beta^3} - \frac{l^2\gamma\alpha^3}{12\beta^4}\right\}, \tag{26}$$

and

$$u_2 = -\frac{3\beta^3}{250\alpha^2}\left\{\frac{4\exp(4(\frac{\beta}{10\alpha}x + ly + \omega t - \xi_0))}{(1+\exp(\pm 2(\frac{\beta}{10\alpha}x + ly + \omega t - \xi_0)))^2} - 2 - \frac{125\alpha^2\omega}{3\beta^3} - \frac{l^2\gamma\alpha^3}{12\beta^4}\right\}, \tag{27}$$

where l, ω and ξ_0 are three arbitrary constants.

4. *Conclusion*

We propose a new trial-equation method. As application of the method, we give some exact traveling wave solutions of RLW-Burgers equation and (2+1)-dimensional KdV-Burgers equation. The solutions of RLW-Burgers equation in references [13 − 14] are included in ours. Furthermore, our method is simpler than theirs. The method can also be applied to other diffusion equations, such as BBM-Burgers equation, Fisher equation, and so on.

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NOVA METODA S PROBNOM JEDNADŽBOM ZA TOČNA RJEŠENJA NELINERNIH DIFERENCIJALNIH JEDNADŽBI ZA PUTUJUĆE VALOVE

Predlažemo novu metodu s probnom jednadžbom za rješavanje diferencijalnih jednadžbi. Tom metodom nalazimo neka točna rješenja RLW-Burgersove i (2+1)-dimenzijske KdV-Burgersove jednadžbe.