ISSN 1330–0008
CODEN FIZAE4

EMERGENCE OF CENTRIPETAL ACCELERATION WITHIN THE FRENET–SERRET FRAME

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Received 6 October 2003; Accepted 15 December 2003
Online 19 April 2004

A clear physical description of a particle's motion in terms of the components of its acceleration is obtained if the trajectory is described in terms of the Frenet–Serret frame. Within this frame, centripetal acceleration emerges as a natural consequence of a particle's motion.

PACS numbers: 01.30.Rr, 01.40.-d, 02.40.Hw UDC 531.122
Keywords: particle's motion, Frenet–Serret frame, centripetal acceleration

1. Introduction

Classical mechanics textbooks universally address particle motion and associated quantities such as acceleration. The concept of acceleration is developed as the rate of change of velocity with special cases (e.g., centripetal acceleration) developed to describe particular types of motion (e.g., circular motion). In most textbooks, the concept of centripetal acceleration appears as a separate concept that is not tied to the general acceleration concept, and is often presented as an artifact of circular motion rather than a consequence of the generalized motion of a particle. One possible reason is that the mathematical treatment in classical mechanics texts relies on relatively simple arguments involving Cartesian or polar coordinates that are not naturally adapted to the curve describing the general motion of a particle. The author does not intend to slight these excellent pedagogical efforts, but feels an improved treatment is possible using a junior physics major's knowledge of basic calculus.

An improved physical description of the acceleration of a particle is obtained if it is viewed in terms of the Frenet–Serret frame [1–6]. This frame is defined in terms of tangent and normal vectors that are physically tied to the particle's motion and the underlying space-time geometry along the path of motion of the particle. Herein, we will demonstrate that centripetal acceleration is a natural consequence of the motion when viewed within the Frenet–Serret frame.

2. *Frenet–Serret frame*

The movement of a particle may be described in terms of its motion along some curve $\boldsymbol{\beta}(s)$. For any given point P on this curve, a set of orthonormal unit vectors may be defined:

$\boldsymbol{T}$ = unit tangent vector at each point P
$\boldsymbol{N}$ = unit normal vector at each point P
$\boldsymbol{B}$ = unit binormal vector at each point P

This set of orthogonal unit vectors, known as the Frenet–Serret frame, has the following properties

$$\boldsymbol{T}\cdot\boldsymbol{T} = \boldsymbol{N}\cdot\boldsymbol{N} = \boldsymbol{B}\cdot\boldsymbol{B} = 1, \tag{1}$$

$$\boldsymbol{T}\cdot\boldsymbol{N} = \boldsymbol{T}\cdot\boldsymbol{B} = \boldsymbol{N}\cdot\boldsymbol{B} = 0, \tag{2}$$

$$\boldsymbol{B} = \boldsymbol{T}\times\boldsymbol{N}, \tag{3}$$

$$\boldsymbol{T} = \boldsymbol{N}\times\boldsymbol{B}, \tag{4}$$

$$\boldsymbol{N} = \boldsymbol{B}\times\boldsymbol{T}. \tag{5}$$

Given this set of coordinates, let $\boldsymbol{\beta}(s)$ be a curve parameterized by the arc length (s) and let $\boldsymbol{T}(s)$ be the vector

$$\boldsymbol{T}(s) = \boldsymbol{\beta}'(s). \tag{6}$$

where the prime indicates differentiation with respect to s. While there might be other canonical parameterizations, only a parameterization by the arc length leads to a normalized vector $\boldsymbol{T}(s)$.

The vector $\boldsymbol{T}(s)$ is tangential to the curve and it has unit length. We will refer to $\boldsymbol{T}$ as the unit tangent vector. Eq. (1) $\big(\boldsymbol{T}\cdot\boldsymbol{T} = 1\big)$ may be differentiated with respect to s to yield a relationship between $\boldsymbol{T}$ and $\boldsymbol{T}'$,

$$2\,\boldsymbol{T}(s)\cdot\boldsymbol{T}'(s) = 0, \tag{7}$$

that suggests $\boldsymbol{T}$ and $\boldsymbol{T}'$ are orthogonal. If we let $\boldsymbol{N}$ be a unit vector orthogonal to $\boldsymbol{T}$, we can write

$$\boldsymbol{T}'(s) = \kappa\,\boldsymbol{N}(s), \tag{8}$$

where κ is a scalar called the curvature and $\boldsymbol{N}$ is called the unit normal to the curve. If the length of both sides of Eq. (8) is taken, then we obtain

$$\kappa \quad = \quad \big|\boldsymbol{T}'(s)\big|. \tag{9}$$

In Eq. (9), the curvature measures the rate of change of the tangent vector, which is a measure of how much the curve is curving.

The third vector $\boldsymbol{B}$ is the binormal vector. The properties of $\boldsymbol{B}$ may be gleaned by differentiating Eq. (1) $\big(\boldsymbol{B}\cdot\boldsymbol{B}=1\big)$ with respect to s

$$2\,\boldsymbol{B}(s)\cdot\boldsymbol{B}'(s)=0. \tag{10}$$

Therefore $\boldsymbol{B}'$ is orthogonal to $\boldsymbol{B}$. We can also differentiate Eq. (2) $\big(\boldsymbol{T}\cdot\boldsymbol{B}=0\big)$

$$\boldsymbol{B}'(s)\cdot\boldsymbol{T}(s)+\boldsymbol{B}(s)\cdot\boldsymbol{T}'(s)=0. \tag{11}$$

Using Eq. (8), this equation may be rewritten as

$$\boldsymbol{B}'(s)\cdot\boldsymbol{T}(s)=-\boldsymbol{B}(s)\cdot\boldsymbol{T}'(s)=-\kappa\,\boldsymbol{N}(s)\cdot\boldsymbol{B}(s). \tag{12}$$

Since $\boldsymbol{T}$ and $\boldsymbol{T}'$ are orthogonal, and since $\boldsymbol{B}$ is perpendicular to $\boldsymbol{T}$ according to Eq. (3), Eq. (12) suggests that $\boldsymbol{B}'$ is also orthogonal to $\boldsymbol{T}$. This can only occur if $\boldsymbol{B}'$ is orthogonal to the T-B plane. Therefore, $\boldsymbol{B}'$ is proportional to $\boldsymbol{N}$

$$\boldsymbol{B}'(s)=-\tau\,\boldsymbol{N}(s). \tag{13}$$

for a quantity τ which we will call the torsion. The scalar τ is chosen such that the binormal vector is a unit vector. Torsion is similar to the curvature in that it measures the rate of change of a quantity. Torsion is an indication of the rate of change of the binormal.

The advantage of the Frenet–Serret frame is that it propagates with the particle. The tangent vector points in the direction of motion. The normal and binormal vectors point towards the directions in which the trajectory is tending to curve.

The properties of the vector $\boldsymbol{N}$ may be established by differentiating Eq. (1) $\big(\boldsymbol{N}\cdot\boldsymbol{N}=1\big)$

$$2\,\boldsymbol{N}(s)\cdot\boldsymbol{N}'(s)=0. \tag{14}$$

Therefore, $\boldsymbol{N}'$ and $\boldsymbol{N}$ are orthogonal. This suggests $\boldsymbol{N}'$ lies in the T-B plane, and therefore can be expressed as a linear combination of $\boldsymbol{T}$ and $\boldsymbol{B}$

$$\boldsymbol{N}'=a\,\boldsymbol{T}+b\,\boldsymbol{B}, \tag{15}$$

where a and b are scalars. Taking the dot product of Eq. (15) with $\boldsymbol{T}$ and $\boldsymbol{B}$, respectively yields the following results for a and b

$$a \quad = \quad \boldsymbol{N}'\cdot\boldsymbol{T}, \tag{16}$$

$$b \quad = \quad \boldsymbol{N}'\cdot\boldsymbol{B}. \tag{17}$$

When the equations $\boldsymbol{N}\cdot\boldsymbol{T}=0$ and $\boldsymbol{N}\cdot\boldsymbol{B}=0$ are differentiated and combined with Eqs. (8) and (13), we obtain

$$\boldsymbol{N}'\cdot\boldsymbol{T} \;=\; -\boldsymbol{N}\cdot\boldsymbol{T}' \;=\; -\boldsymbol{N}\cdot\big(\kappa\,\boldsymbol{N}\big) \;=\; -\kappa, \tag{18}$$

$$\boldsymbol{N}'\cdot\boldsymbol{B} \;=\; -\boldsymbol{N}\cdot\boldsymbol{B}' \;=\; -\boldsymbol{N}\cdot\big(-\tau\,\boldsymbol{N}\big) \;=\; \tau. \tag{19}$$

Eqs. (16)–(19) suggest that $a=-\kappa$ and $b=\tau$. With this result, Eq. (15) becomes

$$\boldsymbol{N}' = -\kappa\,\boldsymbol{T}+\tau\,\boldsymbol{B}, \tag{20}$$

which completes the specification of $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$.

The Frenet–Serret frame equations may be simplified by writing them in matrix form

$$\begin{bmatrix}\boldsymbol{T}'\\ \boldsymbol{N}'\\ \boldsymbol{B}'\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{bmatrix}. \tag{21}$$

3. *Motion within the Frenet–Serret frame*

The motion associated with a curve in the Frenet–Serret frame may be used to compute the velocity and acceleration along that curve. We assume the arc length s is a function of time (t) and consider the motion along the curve $\boldsymbol{\beta}(s)$.

Using Eq. (6), the velocity $\boldsymbol{v}$ may be written as

$$\boldsymbol{v} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\beta}\big(s(t)\big) \;=\; \frac{\mathrm{d}\boldsymbol{\beta}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}s}{\mathrm{d}t} \;=\; \boldsymbol{\beta}'(s)v \;=\; v\,\boldsymbol{T}, \tag{22}$$

where

$$v \;=\; \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{23}$$

The acceleration $\boldsymbol{a}$ is obtained in a similar manner

$$\boldsymbol{a} \;=\; \frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\beta}\big(s(t)\big) \;=\; \frac{\mathrm{d}v}{\mathrm{d}t}\,\boldsymbol{T}+v\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\,\frac{\mathrm{d}s}{\mathrm{d}t}, \tag{24}$$

$$\boldsymbol{a} \;=\; a\,\boldsymbol{T}+\kappa v^2\,\boldsymbol{N}, \tag{25}$$

where Eqs. (8) and (24) have been used to simplify Eq. (25) and

$$a \;=\; \frac{\mathrm{d}v}{\mathrm{d}t}. \tag{26}$$

Eq. (25) demonstrates the logical emergence of the centripetal acceleration. The equation states that a particle moving along a curve in space–time feels a component of acceleration along the direction of motion whenever there is a change in speed. The second component in Eq. (25) is the centripetal acceleration in the direction of the normal.

In order to more fully demonstrate the relationship between the second term in Eq. (25) and the centripetal acceleration, we will evaluate the relationship between κ and τ and the curve $\boldsymbol{\beta}(s)$. We let $\boldsymbol{\beta}(s)$ be a curve with curvature κ and torsion τ. Using Eqs. (6) and (8), we can write

$$\boldsymbol{T}'(s) = \boldsymbol{\beta}''(s) = \kappa\,\boldsymbol{N}(s). \tag{27}$$

The relationship between κ and $\boldsymbol{\beta}''(s)$ may be obtained by taking the dot product of $\boldsymbol{\beta}''(s)$ with itself

$$\boldsymbol{\beta}''(s)\cdot\boldsymbol{\beta}''(s) = \kappa\,\boldsymbol{N}(s)\cdot\kappa\,\boldsymbol{N}(s) = \kappa^2, \tag{28}$$

or

$$\kappa = \big|\boldsymbol{\beta}''(s)\big|. \tag{29}$$

In a similar manner, τ may be obtained from $\boldsymbol{\beta}'''(s)$ and Eq. (27)

$$\boldsymbol{\beta}'''(s) = \kappa'\,\boldsymbol{N}(s) + \kappa\,\boldsymbol{N}'(s). \tag{30}$$

Using Eq. (20), we obtain

$$\boldsymbol{\beta}'''(s) = \kappa'\,\boldsymbol{N}(s) + \kappa\,\big(-\kappa\boldsymbol{T}(s) + \tau\,\boldsymbol{B}(s)\big), \tag{31}$$

$$\boldsymbol{\beta}'''(s) = \kappa'\,\boldsymbol{N}(s) - \kappa^2\,\boldsymbol{T}(s) + \kappa\tau\,\boldsymbol{B}(s). \tag{32}$$

The desired expression for τ is obtained by computing the quantity $\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big]$. Using Eqs. (3), (4), (6), (27) and (32), we can write

$$\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big] = \boldsymbol{T}\cdot\big[\kappa\,\boldsymbol{N}\times\big(\kappa'\,\boldsymbol{N} - \kappa^2\,\boldsymbol{T} + \kappa\tau\,\boldsymbol{B}\big)\big], \tag{33}$$

$$\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big] = \boldsymbol{T}\cdot\big[\kappa^3\,\boldsymbol{B} + \kappa^2\tau\,\boldsymbol{T}\big], \tag{34}$$

$$\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big] = \kappa^2\tau. \tag{35}$$

Solving for τ we obtain

$$\tau = \frac{\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big]}{\kappa^2}. \tag{36}$$

Using Eq. (28), Eq. (36) becomes

$$\tau = \frac{\boldsymbol{\beta}'(s)\cdot\big[\boldsymbol{\beta}''(s)\times\boldsymbol{\beta}'''(s)\big]}{\boldsymbol{\beta}''(s)\cdot\boldsymbol{\beta}''(s)}. \tag{37}$$

The assertions regarding the centripetal acceleration may be verified by considering the specific case of circular motion. For circular motion

$$\boldsymbol{\beta}(s) \quad = \quad \big(r\cos\theta, r\sin\theta, 0\big)\,, \tag{38}$$

where r is the radius of the circle and θ is the polar angle, which is a function of both s and r

$$s \quad = \quad r\,\theta. \tag{39}$$

Using Eq. (39), Eq. (38) may be written in terms of s

$$\boldsymbol{\beta}(s) \quad = \quad \big(r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0\big). \tag{40}$$

The derivatives $\boldsymbol{\beta}'$ and $\boldsymbol{\beta}''$ follow from Eq. (40)

$$\boldsymbol{\beta}'(s) \quad = \quad \big(-\sin\frac{s}{r}, \cos\frac{s}{r}, 0\big), \tag{41}$$

$$\boldsymbol{\beta}''(s) \quad = \quad \big(-\frac{1}{r}\cos\frac{s}{r}, -\frac{1}{r}\sin\frac{s}{r}, 0\big). \tag{42}$$

Determining a value for κ will complete the specification of the acceleration defined by Eq. (25). Using Eq. (29) κ is determined

$$\kappa \quad = \quad \big|\boldsymbol{\beta}''(s)\big| = \sqrt{\big(-\frac{1}{r}\cos\frac{s}{r}\big)^2 + \big(-\frac{1}{r}\sin\frac{s}{r}\big)^2 + 0}, \tag{43}$$

$$\kappa \quad = \quad \sqrt{\big(\frac{1}{r}\big)^2\big(\sin^2\frac{s}{r} + \cos^2\frac{s}{r}\big)} \quad = \quad \frac{1}{r}. \tag{44}$$

Using Eq. (44), Eq. (25) defined the total acceleration $\boldsymbol{a}$,

$$\boldsymbol{a} \quad = \quad a\,\boldsymbol{T} + \frac{v^2}{r}\,\boldsymbol{N}. \tag{45}$$

The first term is the acceleration associated with the change in velocity in the direction of the tangent to the curve describing the particle's motion, and the second term is the acceleration normal to the curve or the centripetal acceleration having the expected v^2/r form for circular motion. Eq. (45) can be derived in a more compact and elegant manner using differential geometry through the use of Christoffel symbols [6,7]. However, that approach is beyond the knowledge level of most students in a junior level physics course. Such an approach would be a welcome addition to advanced undergraduate or graduate students as an initial application illustrating the usefulness of affine connection coefficients.

4. Conclusions

The motion of a particle within the Frenet–Serret frame leads to the development of the components of acceleration that logically lead to the introduction of centripetal acceleration. This approach presents the concept of acceleration in a logical and consistent manner, and provides a junior physics major with a unified description of acceleration and also provides the more advanced student with a natural tie to differential geometry.

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NASTANAK CENTRIPETALNOG UBRZANJA U FRENET–SERRETOVOM SUSTAVU

Ako se putanja čestice opisuje u Frenet–Serretovom sustavu, postiže se jasan fizički opis njenog gibanja preko komponenata njenog ubrzanja. U tom se sustavu centripetalno ubrzanje javlja kao prirodna posljedica gibanja čestice.