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Distortion Estimates for Conformal Maps Predicated on Geometric Properties of a Domain

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ABSTRACT

Geometric properties of a domain in the complex plane reflect important information about the conformal maps to and from the domain. We examine a variety of geometric properties and use them to construct explicit global distortion bounds for both the compression and stretching of conformal map. Compressive distortion is controlled when the modulus of the derivative of a complex function is bounded from below, expansive distortion when it is bounded above.

For the initial set of results, we quantify the degree to which a convex domain is nearly round with two parameters; radii of the largest inscribed disk and smallest circumscribed disk. A third parameter captures information about curvature on the boundary. The three parameters are used to construct a global stretching bound for a conformal map of the unit disk onto the domain, or equivalently, a bound on compression in the other direction. The Möbius invariant Kulkarni-Pinkall metric is used in constructing these explicit bounds.

Next we generalize the previous results by weakening the assumption of convexity to something slightly stronger than star-shaped. The parameter giving the radius of the largest inscribed disk is replaced a more relevant radius, that of the largest disk from which every point of the domain can be seen.

Finally we turn to the bounds in the opposite direction, that is, stretching bounds on conformal maps from a convex domain onto the unit disk, and compression bounds from the disk onto a convex domain. We use the same two parameters to quantify the degree to which a domain is nearly round, but have no need of a curvature parameter in this case. The bound in this final chapter is shown to be the best possible.

Distortion Estimates for Conformal Maps Predicated on Geometric Properties of a Domain

by

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Chapter 1

Introduction

1.1 Conformal Maps and Distortion

Conformal mappings are a class of well behaved functions that locally preserve angle with only a moderate distortion of lengths away from the boundary of their domain. The compression or stretching of length or distance is referred to as *distortion* in the literature. Local distortion can be understood by thinking of the modulus of the derivative as a local scale factor. Results that estimate the modulus of the derivative are collectively known as distortion theorems.

The Riemann mapping theorem is a fundamental result in complex analysis, established first with an incomplete proof by Riemann himself in 1851. This was not realized until later, and it was not until 1900 that William Osgood published the first complete proof. A standard version of the theorem is given below. Note that throughout this dissertation $\mathbb{D}$ refers to the unit disk, and $D(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

Theorem 1.1.1 (Riemann Mapping Theorem)**.** *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain with $z_0 \in \Omega$. Then there exists a unique conformal mapping $f : \mathbb{D} \xrightarrow{\text{onto}} \Omega$ with $f(0) = z_0$, $f'(0) > 0$.*

Some intuition for conformal mapping can be found in Curtis McMullen's characterization using an *egg yolk principle* [22]. Regardless of how jagged and non-convex Ω may be, there is a conformal map from the unit disk onto Ω, and while there may be large oscillations of stretching and compression near the boundary, these are smoothed as you move away from the boundary, and tend toward a simple scale factor of $|f'(0)|$ as you move toward 0. This is analogous to cracking an egg into a pan; the footprint of the thin egg white can be wild and irregular, the thick egg white is convex, and the yolk is nearly round.

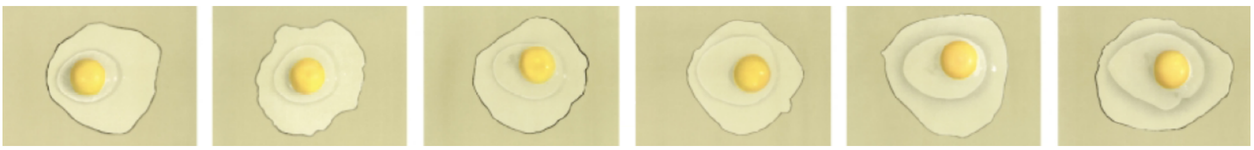

Figure 1.1: Cracked eggs as an analogy to distortion in conformal maps.

Next we move to present some well known results that demonstrate control of distortion away from the boundary. To present them in their canonical forms, we must also introduce a class of normalized conformal maps.

Definition 1.1.2 (Schlicht Functions)**.** Let $\mathscr{S}$ be the family of conformal mappings of the unit disk with $f(0) = 0$ and $f'(0) = 1$.

Theorem 1.1.3. *Define the* starlike radius *(*$\rho_S = \tanh(\pi/4) \approx .655$*) and the* radius of convexity *(*$\rho_C = 2 - \sqrt{3} \approx .267$*). For* any *$f \in \mathscr{S}$, the image of $D(0, r)$ for $0 < r \leqslant \rho_S$ is a starlike set, and the image of $D(0, r)$ for $0 < r \leqslant \rho_C$ is a convex set [5, Theorem 2.13].*

Observe that $\mathscr{S}$ contains all the conformal mappings from Theorem 1.1.1 up to a translation, and rescaling. That is, if f is as in Theorem 1.1.1, then

$$\frac{f(z) - f(0)}{f'(0)} \in \mathscr{S}.$$

One of the most famous distortion theorems is credited to Paul Koebe.

Theorem 1.1.4 (Koebe Distortion Theorem)**.** *If $f \in \mathscr{S}$, then for all $z \in \mathbb{D}$*

$$\frac{1 - |z|}{(1 + |z|)^3} \leqslant |f'(z)| \leqslant \frac{1 + |z|}{(1 - |z|)^3}. \tag{1.1.1}$$

This is one of a variety of forms the theorem can take. This form nicely expresses the control of distortion around 0. In $\mathbb{D} \setminus \{0\}$, equality is attained only by the *Koebe function* $k \in \mathscr{S}$, up to a rotation.

$$k\colon \mathbb{D} \xrightarrow{\text{onto}} \mathbb{C} \setminus (-\infty, -\tfrac{1}{4}), \qquad k(z) = \frac{z}{(1-z)^2}, \qquad k'(z) = \frac{1+z}{(1-z)^3}. \tag{1.1.2}$$

One can see that the left bound of (1.1.1) is attained by the Koebe function when $z \in [-1, 0)$ and the right when $z \in (0, 1]$. This shows the upper and lower bounds are both sharp.

1.2 Global Distortion Bounds

A global bound on distortion can be either an upper or lower bound on the modulus of the derivative, formulated as

$$\sup |1/f'| \leqslant M \quad \text{ or } \quad \sup |f'| \leqslant M,$$

or an upper or lower bound on the ratio

$$\frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|} \tag{1.2.1}$$

that holds for all z_1, z_2 in the domain of f. An upper bound on the ratio is called a Lipschitz constant. A Lipschitz function $f\colon G \to \Omega$ is one that satisfies a Lipschitz condition, defined by the inequality

$$|f(z_1) - f(z_2)| \leqslant L|z_1 - z_2| \tag{1.2.2}$$

for all z_1, z_2 in G and for some Lipschitz constant $L > 0$. For a holomorphic function on a convex domain a Lipschitz bound is equivalent to the statement $|f'(z)| \leqslant L$, but this is not true in general. Consider the conformal map $f(z) = \sqrt{z}$ sending the slit annulus $G = \{1 < |z| < 4\} \setminus (-4, -1)$ onto $\Omega = \{1 < |z| < 2\} \cap \{\text{Re } z > 0\}$. Then $f'(z) = 1/(2\sqrt{z})$

and for all $z \in G$, $\frac{1}{4} < |f'(z)| < \frac{1}{2}$. If however you consider the two points straddling the slit, like $3 + i\epsilon$ and $3 - i\epsilon$ for $\epsilon > 0$ small, the ratio

$$\frac{|f(3+i\epsilon) - f(3-i\epsilon)|}{|(3+i\epsilon) - (3-i\epsilon)|} \approx \frac{\sqrt{3}}{\epsilon}$$

which approaches ∞ as $\epsilon \to 0$. The discrepancy arises from the denominator measuring the distance between the points $3 \pm i\epsilon$ as 2ϵ, that is, along a straight line path that crosses the slit and thus is not contained in G. The requirement of convexity ensures that the straight line path between any two points is contained in the domain. Defining the domain distance in the denominator of (1.2.1) to be the infimum path length over all paths contained in G connecting z_1 and z_2, called *intrinsic distance*, is a workable way to modify the global ratio bounds to agree with global derivative bounds in the non-convex case. This approach was recently used, e.g., in [11].

There are comparatively few results related to global distortion bounds, and in fact there can be no global bounds in general. We can illustrate this point with the Koebe function (1.1.2), where one can see that $k' \to 0$ as $z \to -1$ and $k' \to \infty$ as $z \to 1$. Even among relatively simple bounded domains, it is often true that a global compression bound of $|1/f'| \leqslant \infty$ and a stretching bound of $|f'| \leqslant \infty$ are the best possible. The functions $f_1(z) = 2\sqrt{1+z} - 2$ and $f_2(z) = \frac{1}{2}(1+z)^2 - \frac{1}{2}$ are both elementary functions (normalized to be in $\mathscr{S}$), and as $z \to -1$, $|f_1'|$ and $|1/f_2'|$ both go to infinity.

One can however consider the related concept of integral bounds on distortion. This warrants some preliminary discussion of *Bergman spaces* and *Hardy spaces*.

Definition 1.2.1 (Bergman Space)**.** A *Bergman space* is function space composed of holomorphic functions defined on the unit disk. The Bergman spaces are denoted A^p where $p \in (0, \infty]$ is an index. For $0 < p < \infty$ we say $f \in A^p$ if

$$\|f\|_{A^p} \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} |f(z)|^p \, dA \right)^{1/p} < \infty. \tag{1.2.3}$$

Here $\|f\|_{A^p}$ is the Bergman p-norm, and the differential dA is normalized area. As in Lebesgue spaces, $\|f\|_{A^p}$ is not a true norm if $p < 1$, but is sometimes still useful. When $p = \infty$, define the *infinity Bergman norm* by

$$\|f\|_{A^\infty} \stackrel{\text{def}}{=} \sup_{\mathbb{D}} |f|. \tag{1.2.4}$$

Definition 1.2.2 (Integral Means)**.** Let f be analytic in the unit disk, then the integral means are defined by

$$\begin{aligned} M_p(r,f) &= \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}, \qquad 0 < p < \infty; \\ M_\infty(r,f) &= \max_{0 \leqslant \theta < 2\pi} |f(re^{i\theta})|. \end{aligned}$$

Definition 1.2.3 (Hardy Space)**.** *Hardy spaces*, denoted H^p for $0 < p \leqslant \infty$, are also function spaces composed of holomorphic functions defined on the unit disk. We say $f \in H^p$ if

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0 \leqslant r < 1} M_p(r,f) < \infty, \tag{1.2.5}$$

where the Hardy norm $\|f\|_{H^p}$ satisfies the definition of a norm when $p \geqslant 1$. When $p = \infty$, define the *infinity Hardy norm* by

$$\|f\|_{H^\infty} \stackrel{\text{def}}{=} \sup_{\mathbb{D}} |f|. \tag{1.2.6}$$

Remark 1.2.4. Observe that for a fixed $r \in (0,1)$ and $p < q$:

$$\left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} \leqslant \left(\int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} \right)^{1/q}.$$

It then follows that $\|f\|_{H^p} \leqslant \|f\|_{H^q}$, and more importantly, that

$$f \in H^q \quad \Longrightarrow \quad f \in H^p \text{ for all } p < q. \tag{1.2.7}$$

Remark 1.2.5. If $f \in H^p$ for some $0 < p \leqslant \infty$, then as $r \to 1$, $f(re^{i\theta})$ converges to a function $F \in L^p[0, 2\pi]$ a.e. [6, Section 2.3]. Moreover, the Hardy p-norm of f and the L^p norm of F agree. [12]

Brennan's conjecture [3](James Brennan, 1978) is one of the most widely recognized open question in complex analysis, and has been an area of active research since the hypothesis was made. A concise statement of the conjecture is that for a simply connected $\Omega \subsetneq \mathbb{C}$ and a conformal mapping $g\colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$, $|g'|^p$ is integrable for $4/3 < p < 4$.

The upper and lower bounds on p are of course two distinct statements, and for our purposes it will be simpler to work with equivalent statements written in terms of $f \overset{\text{def}}{=} g^{-1}$.

Conjecture 1.2.6 (Brennan's Conjecture). Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$. Then it is *known* that

$$\|f'\|_{A^p} < \infty \quad \text{for} \quad 0 < p < 2/3. \tag{1.2.8}$$

In addition it is *conjectured* that

$$\|1/f'\|_{A^p} < \infty \quad \text{for} \quad 0 < p < 2. \tag{1.2.9}$$

From the perspective of stretching and compression, we could interpret the statement in (1.2.8) as a limit on the stretching of a conformal map, and (1.2.9) as a limit on compression.

Proof of (1.2.8). It suffices to prove this for $f \in \mathscr{S}$ since translation, rotation, and rescaling have no effect on integrability. Rearranging the combined growth and distortion theorem [5, Theorem 2.7] gives

$$|f'(z)| \leqslant \frac{(1+r)}{r(1-r)}|f(z)|. \tag{1.2.10}$$

Raising each side of (1.2.10) to the power p and integrating around $|z| = r < 1$ gives

$$M_p^p(r, f') \leqslant \left(\frac{(1+r)}{r(1-r)} \right)^p M_p^p(r, f). \tag{1.2.11}$$

In [1] Baernstein showed that for all $p > 0$, $M_p(r, f) \leqslant M_p(r, k)$ where $k(z)$ is the Koebe function (1.1.2). Applying this result yields

$$M_p^p(r, f') \leqslant \left(\frac{(1+r)}{r(1-r)} \right)^p M_p^p(r, k), \tag{1.2.12}$$

and then [7, Section 3] provides a way to estimate $M_p(r, k)$. This shows $M_p^p(r, k)$ is $O((1-r)^{1-2p})$. It then follows from (1.2.12) that

$$M_p^p(r, f') \leqslant g(r) \tag{1.2.13}$$

where g is $O((1-r)^{1-3p})$ and thus $\int_0^1 M_p^p(r, f')\, dr$ converges for $1 - 3p > -1$, that is, for $0 < p < 2/3$. $\square$

Progress on the compression bound (1.2.9) has been slow, but it is still widely believed to hold. Before Brennan made the conjecture, it was known to Metzger [23] working in polynomial approximations that (1.2.9) holds for $0 < p < 1$; this can be easily shown by first rearranging and relaxing the left side of (1.1.1) in Theorem 1.1.4 to get

$$|1/f'(z)| \leqslant \frac{8}{1 - |z|}. \tag{1.2.14}$$

Raising to power p and integrating around $|z| = r < 1$ shows $M_p^p(r, 1/f')$ is $O((1-r)^{-p})$, and thus $\int_0^1 M_p^p(r, 1/f')\, dr$ converges for $0 < p < 1$. Alongside the original conjecture in [3] Brennan showed that Metzger's result could be extended to $0 < p < 1+\epsilon$. In his 1991 text [25] Pommerenke proved Theorem 8.5 which extended the confirmed range to $0 < p < 1.399$. Daniel Bertilsson extended this to $0 < p < 1.422$ in a computer assisted proof as part of his

1999 Thesis [2].

1.3 Global Distortion Bounds Predicated on Geometric Conditions

Definition 1.3.1 (Jordan Domain)**.** A *Jordan domain* is a domain bounded by a simple closed curve in $\mathbb{C}$. The boundary is called a *Jordan curve.*

Theorem 1.3.2 (Carathéodory Theorem)**.** [25, Section 2.1] *Let f map the unit disk conformally onto a domain Ω. Then f and f^{-1} extend to homeomorphisms of the closures of $\mathbb{D}$ and Ω if and only if $\partial\Omega$ is a Jordan curve.*

Corollary 1.3.3. [5, Section 1.5] *Let Ω_1 and Ω_2 be Jordan domains and let $\varphi\colon \Omega_1 \xrightarrow{\text{onto}} \Omega_2$ be a conformal map. Then φ and φ^{-1} extend to homeomorphisms between $\overline{\Omega}_1$ and $\overline{\Omega}_2$.*

Corollary 1.3.4. *Let Ω_1 and Ω_2 be two Jordan domains, and let $F, G\colon \Omega_1 \xrightarrow{\text{onto}} \Omega_2$ be conformal maps. If F and G agree on an interior point and a boundary point, then $F = G$.*

Proof. Let a be the point in Ω_1 such that $F(a) = G(a)$, let b be the point in $\partial\Omega_1$ such that $F(b) = G(b)$. Define a conformal map $h\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega_1$ such that $h(0) = a$. Consider the composition $h^{-1} \circ G^{-1} \circ F \circ h$. This is a conformal mapping of the unit disk fixing 0, therefore it is a rotation. By the Carathéodory theorem the composition extends to the boundary. The point $h^{-1}(b) \in \partial\mathbb{D}$ is also fixed, therefore the composition is the identity map on $\mathbb{D}$ and thus $F = G$. $\square$

The pool of potential geometric conditions to be imposed on Ω is limitless, but a guiding principle as always is to impose a carefully chosen minimal set of restrictions from which a comparatively strong result can be deduced.

The Riesz brothers proved an early result of this type. In 1916, Friedrich and Marcel Riesz presented a proof to the Congress of Scandinavian Mathematicians that if we restrict

Ω to Jordan domains with rectifiable boundary, then $f' \in H^1$. In fact they were able to prove that for a Jordan domain, the boundary is rectifiable if and only if $f' \in H^1$.

In 1929, Oliver Kellogg imposed a $C^{1,\alpha}$ Hölder condition on $\partial\Omega$ for $0 < \alpha \leqslant 1$. We pause to define Hölder continuity and then a general $C^{n,\alpha}$ Hölder condition.

Definition 1.3.5 (Hölder and Lipschitz continuity)**.** Let X, Y be metric spaces and let $F \colon X \to Y$ be a continuous function between them. For $0 < \alpha < 1$ we say F is α-Hölder continuous, denoted $F \in C^\alpha$, if there exists a constant M such that for all $a, b \in X$

$$\operatorname{dist}_Y(F(a), F(b)) \leqslant M\left[\operatorname{dist}_X(a, b)\right]^\alpha. \tag{1.3.1}$$

Lipschitz continuity describes the special case of Hölder continuity when $\alpha = 1$.

Definition 1.3.6 (Hölder condition)**.** Let C be a Jordan curve and let $w \colon \mathbb{R} \to C$ be a periodic parameterization. We say that C satisfies a *$C^{n,\alpha}$-Hölder condition* (or is $C^{n,\alpha}$-Hölder continuous) for $0 < \alpha \leqslant 1$ if w is n times continuously differentiable, the first derivative is nonvanishing, and the n-th derivative is α-Hölder continuous.

With these defined, we present Kellogg's 1929 Result.

Theorem 1.3.7 (Kellogg's Theorem)**.** *If f maps $\mathbb{D}$ onto the Jordan domain Ω whose boundary is $C^{1,\alpha}$, then f' extends continuously to the boundary of $\mathbb{D}$. Furthermore, f' and $1/f'$ are in H^∞.*

This has proven to be a useful result, and was extended a few years later by Stefan Warschawski into what is now known as the Kellogg-Warschawski Theorem.

Theorem 1.3.8 (Kellogg-Warschawski Theorem)**.** *[25, Theorem 3.6] Let f map $\mathbb{D}$ conformally onto a Jordan domain with boundary satisfying a $C^{n,\alpha}$-Hölder condition for some $n \in \{1, 2, ...\}$ and $0 < \alpha < 1$. Then $f^{(n)}$ extends continuously to $\partial\mathbb{D}$ and*

$$|f^{(n)}(z_1) - f^{(n)}(z_2)| \leqslant M|z_1 - z_2|^\alpha \quad \textit{for} \quad z_1, z_2 \in \overline{\mathbb{D}}.$$

For a map $f \in \mathscr{S}$, there is no p such that $f' \in H^p$ in general, nor a q such that $1/f' \in H^q$. In the 1955 article [20] Lohwater, Piranian, and Rudin demonstrated that there exists a conformal map on the unit disk with the property that for almost all $\theta \in [0, 2\pi)$

$$\liminf_{r \to 1} |f'(re^{i\theta})| = 0, \qquad \limsup_{r \to 1} |f'(re^{i\theta})| = +\infty. \tag{1.3.2}$$

By virtue of Remark 1.2.5 such f' cannot belong to any Hardy class. With this established, it was clear any further results regarding Hardy space inclusion or Hardy norm bounds of the derivative *must* be predicated on some conditions on the domain Ω.

In 1962, Dieter Gaier published the article [10] looking at what happens if we require Ω to be bounded by a polar curve of the form $r = \rho(\theta)$ with $\log \rho$ an L-Lipschitz function, that is, $\left|\frac{\rho'(\theta)}{\rho(\theta)}\right| \leqslant L$. The condition prevents $\partial\Omega$ from having corners of opening less that $\pi - 2\arctan L$ (intruding or extruding). The condition also disallows some corners with larger opening if the angle bisector deviates too much from the radial direction. Gaier was able to show that for $0 < p < \pi/(2\arctan L)$ we get explicit bounds for $\|f'\|_{H^p}$ and $\|1/f'\|_{H^p}$. The explicit bounds can be expressed in terms of an *outer radius* and an *inner radius* (both with respect to the origin), and the Lipschitz bound L from the imposed precondition. These radii are defined below, recall the notation $D(a, r)$ indicates the set $\{z \in \mathbb{C} \colon |z - a| < r\}$.

Definition 1.3.9 (Outer and inner radii)**.** For a Jordan domain Ω containing 0, the outer radius $R_O(\Omega)$ is the smallest r such that $\Omega \subset D(0, r)$. The inner radius $R_I(\Omega)$ is the largest radius r such that $D(0, r) \subset \Omega$.

Then, for $0 < p < \pi/(2\arctan L)$ Gaier showed

$$\|f'\|_{H^p}^p \leqslant \frac{R_O^p}{\cos(p \arctan L)} \qquad \text{and} \qquad \|1/f'\|_{H^p}^p \leqslant \frac{R_I^{-p}}{\cos(p \arctan L)}.$$

There is a natural conjecture one can make based on the observation that a conformal map taking a corner of angle β to $\partial\mathbb{D}$ must behave locally like $z^{\beta/\pi}$ in order to unfold the angle and land it on the smooth boundary of the unit disk. It follows that the derivative must behave locally like $z^{(\beta/\pi)-1}$, and thus $|f'|^p \approx |z|^{p((\beta/\pi)-1))}$. We lose integrability if $p((\beta/\pi)-1))$ drops to -1. The following conjecture is then natural and intuitive, albeit only to those specialized in this area. If $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ and $\partial\Omega$ has no extruding corners of angle less than β (measured inside the domain), we should have integrability of $|f'|^p$ and thus containment in H^p when $p < \frac{1}{1-\beta/\pi}$.

Schober and Warschawski used this intuition in their 1966 article [27], capturing information about the corner angle with an analytic condition involving tangent vectors. They imposed the condition that $\partial\Omega$ be a Jordan curve with a tangent vector of bounded variation, where the argument of the tangent vector is not allowed to have a jump discontinuity larger than α^+ counterclockwise or α^- clockwise. From these assumptions they were able to show that $f' \in H^p$ for $p < \pi/\alpha^+$ and that $1/f' \in H^p$ for $p < 1/\alpha^-$. This attains the bound from the so-called natural conjecture, and indeed one can show using the Schwarz-Christoffel formula for polygons that both bounds on p are sharp.

In the 1987 paper [9] considering conformal mappings of the unit disk onto Jordan domains, Fitzgerald and Lesley tried to achieve the same conclusion using a geometric condition to regulate the corner angles. They imposed the condition that for every point ζ on the boundary of Ω, there must be a sector contained in Ω with vertex ζ and angular opening β, and with radius $r > 0$. They called this an interior wedge condition. In the paper they were able to demonstrate a partial result, proving $f' \in H^p$ for $p < \frac{2\pi-\beta}{2\pi-2\beta}$. For a concrete comparison, if $\beta = \pi/2$, then Schober and Warschawski (using the tangent vector condition) showed that $f' \in H^p$ for $p < 2$. For this value of β, Fitzgerald and Lesley were able to show that $f' \in H^p$ for $p < 3/2$. In general, they make the natural conjecture that $f' \in H^p$ for $p < \frac{1}{1-\beta/\pi}$ under the β-interior wedge condition [8, p. 153].

In his 2017 paper [18], Leonid Kovalev obtained an explicit Lipschitz bound for conformal maps onto a convex domains using R_I and R_O from Definition 1.3.9 as well as a curvature radius R_C.

Definition 1.3.10 (Curvature radius)**.** Let Ω be a Jordan domain. The curvature radius $R_C(\Omega)$ is the largest r such that Ω can be expressed as a union of open disks of radius r.

In terms of $R_O(\Omega)$, $R_I(\Omega)$, and $R_C(\Omega)$, Kovalev determined a global bound on the derivative of a conformal map from the unit disk onto a convex Jordan domain Ω as

$$\|f'\|_{H^\infty} \leqslant R_C \exp\{2(R_O - R_C)\Phi(R_I, R_C)\}$$

where

$$\Phi(a,b) \stackrel{\text{def}}{=} \begin{cases} \frac{\log a - \log b}{a-b}, & \text{if } a \neq b \\ \frac{1}{a}, & \text{if } a = b. \end{cases}$$

The explicit upper bounds on $|f'|$ obtained by Gaier and Kovalev were recently found to be useful in mathematical physics; more specifically in the spectral gap for graphene quantum dots in [21] (2019), and in the PDEs of fluid mechanics in [14] (2020). In the former, Lotoreichik and Ourmières-Bonafos found an upper bound for the first positive eigenvalue using the H^2-norm of the derivative of an underlying conformal map. They then used the Gaier and Kovalev result to express this bound in terms of geometric quantities: Gaier's bound in the case where the domain satisfies Gaier's conditions (star-shaped, et cetera) and Kovalev's bound when the domain is convex and satisfies a curvature radius.

1.4 New results

This document is focused on deriving explicit formulas for global stretching bounds for conformal maps both to and from the unit disk.

Chapter 2 reconsiders the maps in [18] from the unit disk onto a convex domain. We leave the geometric conditions on the target domain as they were in Kovalev's paper, that is, a convex bounded domain Ω obeying the inclusions $D(0, R_I) \subset \Omega \subset D(0, R_O)$ and expressible as a union of open disk with radius R_C. The derived upper bound on $|f'|$ is sharpened by improving an estimate of a hyperbolic distance on which both Kovalev's bound and this new one are based. To express the explicit formula obtained in the main result of the chapter, we first define some notation. R_O should be understood as a shorthand for $R_O(\Omega)$, the same goes for R_I, R_C, and later R_S. Let $R = \max(R_C, R_I)$, $r = \min(R_C, R_I)$, $d = R_O - R_C$, and $\theta = \arcsin \frac{R-r}{d}$. Define $F(R_O, R_I, R_C)$ as

$$\frac{d}{R} \qquad \text{if } r = R, \tag{1.4.1a}$$

when $r < R$

$$\frac{1}{2}\log\frac{R+d}{R-d} \qquad \text{if } d \leqslant R\tan\frac{\theta}{2}, \tag{1.4.1b}$$

$$\frac{1}{2}\left[\cot\frac{\theta}{2}\log\left(\frac{R}{r}\cos\theta\right) + \log\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right] \qquad \text{otherwise.} \tag{1.4.1c}$$

Then,

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_I,R_C)}.$$

The chapter closes with some examples demonstrating the improvement, in some cases there is an improvement of an order of magnitude [4].

In Chapter 3 we seek to reduce the restrictions imposed on the domain, but retain the conclusion. Instead of smooth convex domain we require only a star shaped Jordan domain. Any extruding corner in the target domain will have $|f'| = \infty$ at its preimage, so some control on the boundary is necessary, and indeed some is retained in the requirement that Ω be expressible as a union of disks of radius R_C.

We also introduce the *stellar core radius* R_S to replace the inner radius R_I. The stellar core radius $R_S(\Omega)$ is defined to be the largest r for which every point in $D(0, r)$ is a star center of Ω. The formulation of the bound in the main result of this chapter is identical to that in Chapter 2, except that R_I is replaced by R_S. In the convex case, R_S is equal to R_I and in fact the main theorem from Chapter 2 is a special case of the theorem in Chapter 3. The reformulated conditions introduce the possibility of intruding corners though, and this lack of smoothness on the boundary causes some problems that need to be dealt with.

Chapter 4 works in the opposite direction, looking at conformal maps from a bounded convex domain containing 0 onto the unit disk. This gives an explicit upper bound on the derivative in terms of only the outer radius and the inner radius. Let $f\colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ be a conformal map fixing 0. Define $\theta = \arcsin(R_I/R_O)$ and $\alpha(\theta) = 2\pi\theta/(\pi + 2\theta)$. Theorem 4.3.2 states that

$$\|f'\|_{H^\infty} \leqslant R_O^{-1} \frac{2\alpha \cot \alpha}{\theta \cos \theta}$$

and this estimate is shown to be the best possible.

Chapter 2

Lipschitz Estimates for Maps from the Unit Disk onto Convex Domains

2.1 Introduction

This chapter improves a uniform upper bound on $|f'|$ from [18]. The assumptions controlling the target domain Ω are maintained. We require that Ω be a convex Jordan domain which contains a neighborhood of 0 and is expressible as a union of disks of radius R_C. These assumptions are quantified by three radii $R_O(\Omega)$, $R_I(\Omega)$, and $R_C(\Omega)$, which were previously introduced in Definitions 1.3.9 and 1.3.10. They are bound together in the (R_O, R_I, R_C) condition which is introduced in Definition 2.3.1.

Our main result is stated below.

Theorem. *Let Ω satisfy the (R_O, R_I, R_C) condition in Definition 2.3.1. Then for any conformal map $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ fixing 0 we have*

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_I,R_C)}$$

where $F(R_O, R_I, R_C)$ is as in Theorem 2.4.8. Equality is attained whenever Ω is a disk.

For comparison, the bound in [18] is sharp only for disks centered 0, that is, only when f is a linear function. The main tool we use to estimate f' is the Kulkarni-Pinkall metric [19] which is defined in §2.2. A precise estimate for this metric is derived in Section 2.4. Section 2.5 contains the proof of the main result. The chapter concludes with examples in Section 2.6.

2.2 Hyperbolic type metrics

Throughout this section let Ω be a simply connected proper subdomain of $\mathbb{C}$. The *hyperbolic metric* [25, Section 4.6] of Ω is conformally invariant and has constant Gaussian curvature -4. The hyperbolic distance between $z, w \in \Omega$ is denoted $\rho_\Omega(z, w)$, and the density at z by $\lambda_\Omega(z)$. When Ω is a disk of radius r and z is a point at distance d from its center, we have

$$\lambda_\Omega(z) = \frac{r}{r^2 - d^2} \tag{2.2.1}$$

and therefore [25, p. 6]

$$\rho_\Omega(z, w) = \frac{1}{2} \log \frac{1 + r|z - w|/|r^2 - z\bar{w}|}{1 - r|z - w|/|r^2 - z\bar{w}|}. \tag{2.2.2}$$

In the special case $w = 0$ this formula simplifies to

$$\rho_\Omega(z, 0) = \frac{1}{2} \log \frac{r + |z|}{r - |z|}. \tag{2.2.3}$$

Note that $\rho_\Omega(z, 0) \to \infty$ as $|z| \to r$, which indicates that every radius of the disk has infinite hyperbolic length. By conformal invariance, every geodesic ray in a simply connected domain has infinite hyperbolic length.

For more general Ω however, explicit formulas for λ_Ω or ρ_Ω are tied to explicit conformal maps between Ω and the unit disk—in most cases neither exist. For this reason alternative metrics are used as approximations to the hyperbolic in the literature [13], we will look at two.

Definition 2.2.1. Distance in the *quasihyperbolic metric* between $z, w \in \Omega$ is denoted $\rho^*_\Omega(z,w)$. The density at any point $z \in \Omega$, denoted $\lambda^*_\Omega(z)$, is the hyperbolic density with respect to the largest disk centered at z that is contained in Ω. The quasihyperbolic density is comparable to the hyperbolic [16, Section 2],

$$\frac{1}{4}\lambda^*_\Omega(z) \leqslant \lambda_\Omega(z) \leqslant \lambda^*_\Omega(z).$$

Note that [16] uses a version of the hyperbolic metric with curvature -1, whereas we have opted for the -4 convention. The quasihyperbolic metric was used to attain the bound in [18]. We now introduce a more refined metric which will be used to improve this bound.

Definition 2.2.2. Distance in the *Kulkarni-Pinkall (KP) metric* between $z, w \in \Omega$ is denoted by $\mathtt{KP}_\Omega(z,w)$, density by $\mu_\Omega(z)$. If we take Δ to be the set of all disks D such that $z \in D \subset \Omega$, then

$$\mu_\Omega(z) \stackrel{\text{def}}{=} \inf_{D \in \Delta} \lambda_D(z). \tag{2.2.4}$$

The KP density is also comparable to the hyperbolic density [16, Section 3],

$$\frac{1}{2}\mu_\Omega \leqslant \lambda_\Omega \leqslant \mu_\Omega, \tag{2.2.5}$$

again shown with the -4 curvature convention. Note that the KP metric gives a better approximation to the hyperbolic metric than the quasihyperbolic does.

The KP metric was introduced by Kulkarni and Pinkall in a 1994 article [19] with an emphasis on its Möbius invariance. In different ways the KP and quasihyperbolic metrics both take advantage of the fact that the hyperbolic metric is monotone with respect to domain, that is, if $\Omega_1 \subset \Omega_2 \subset \mathbb{C}$ are simply connected domains then

$$\forall a, b \in \Omega_1, \quad \rho_{\Omega_1}(a,b) \geqslant \rho_{\Omega_2}(a,b). \tag{2.2.6}$$

This can be seen as a consequence of the Schwarz-Pick lemma.

By [15, Theorem 3.5] for each point z in a simply connected domain $\Omega \subsetneq \mathbb{C}$, there exists a unique disk that attains the infimum in (2.2.4), referred to as the *extremal disk* (this disk is understood in the sense of the Riemann sphere when Ω is unbounded). The extremal disk is determined by a subtle trade-off between the size of the disk and the proximity of its center to z.

Lemma 2.2.3. *[16, Section 2.3] For a simply connected domain $\Omega \subsetneq \mathbb{C}$ and a point $z \in \Omega$, the $\mathtt{KP}_\Omega$ extremal disk for z is the unique disk D satisfying the condition that z lies in the closure of the convex hull of $\partial D \cap \partial\Omega$ with respect to the hyperbolic metric on D.*

Remark 2.2.4. Suppose $\Omega_1 \subset \Omega_2$ are domains and D is the $\mathtt{KP}_{\Omega_2}$ extremal disk for some $z \in \Omega_2$. If $D \subset \Omega_1$, then D is also the $\mathtt{KP}_{\Omega_1}$ extremal disk for z.

Example 2.2.5. The $\mathtt{KP}_H$ extremal disk at $x \in \mathbb{R}$ for the infinite strip $H \stackrel{\text{def}}{=} \{|\text{Im } z| < 1\}$ is $D \stackrel{\text{def}}{=} \{z \colon |x - z| < 1\}$ with $\partial D \cap \partial H = \{x \pm i\}$. Here $(x - i, x + i)$ is a hyperbolic geodesic in D and is the hyperbolic convex hull of $\partial D \cap \partial H$ in D.

Example 2.2.6. Fix $\theta \in (0, \pi/2)$. The $\mathtt{KP}_S$ extremal disk at $x > 0$ for the sector $S \stackrel{\text{def}}{=} \{|\text{Arg } z| < \theta\}$ is the disk D with $\partial D \cap \partial S = \{xe^{\pm i\theta}\}$. The circular arc $A \stackrel{\text{def}}{=} \{xe^{it} : |t| < \theta\} \subset D$ is a hyperbolic geodesic in D and is the hyperbolic convex hull of $\partial D \cap \partial S$ in D.

Example 2.2.7. The $\mathtt{KP}$ extremal disk for every point in a domain D that is itself a disk *is* D, because the convex hull of ∂D with respect to the hyperbolic metric on D is all of D.

Example 2.2.8. Suppose a domain S contains a disk D such that $\Gamma \stackrel{\text{def}}{=} \partial D \cap \partial S$ is a circular arc. Then the convex hull of Γ in the hyperbolic metric on D is the portion of D bounded by Γ and a circle orthogonal to Γ at both of its endpoints. Here we use circle in the sense of the Riemann sphere so that if Γ is a semicircle, the orthogonal circle is a line.

The inclusion of Remark 2.2.4 and subsequent examples are to clarify the extremal disks on the segment between centers of a stadium as described in Definition 2.4.1, and in the proof of Lemma 2.4.4.

Recall from Definition 1.2.3 that the infinity Hardy norm of a holomorphic function f on $\mathbb{D}$ is given by

$$\|f\|_{H^\infty} = \sup_{\mathbb{D}} |f|.$$

As previously mentioned, we will use the KP metric to improve the derivative bound from [18], which relied on the quasihyperbolic metric and can be stated as

$$\|f'\|_{H^\infty} \leqslant R_C \exp\{2(R_O - R_C)\Phi(R_I, R_C)\} \tag{2.2.7}$$

where

$$\Phi(a,b) \stackrel{\text{def}}{=} \begin{cases} \frac{\log a - \log b}{a-b}, & \text{if } a \neq b \\ \frac{1}{a}, & \text{if } a = b. \end{cases}.$$

This restatement of the result by Lotoreichik et al can be found in [21, Proposition 19]. Our improved bound (Theorem 2.5.1) is sharp in a wider class of convex domains than (2.2.7).

2.3 Disk conditions for convex domains

Definition 2.3.1. Suppose Ω is a *convex* domain that contains 0. We say that such a domain satisfies the (R_O, R_I, R_C) condition if:

- R_O, R_I, R_C are all positive,
- R_O is the minimal r such that $\Omega \subset D(0,r)$,
- R_I is the maximal r such that $D(0,r) \subset \Omega$,

- Ω can be expressed as a union of open disks of radius R_C.

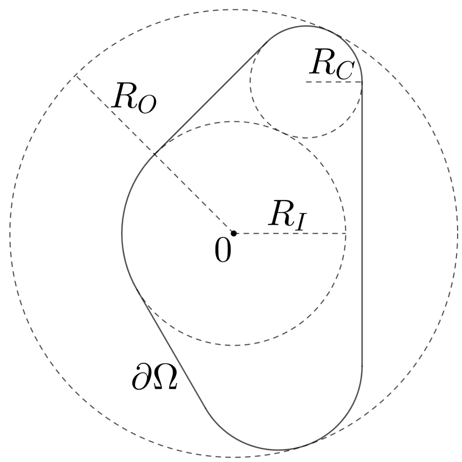

Figure 2.1: This is a sample domain Ω where the solid line denotes the boundary of Ω and the dotted lines show the disks $D(0, R_O)$, $D(0, R_I)$, and one of many disks of radius R_C. This particular disk of radius R_C is shown because where its boundary overlaps with the boundary of Ω, there is no larger disk that could touch the boundary and still be contained inside Ω.

The subscripts in Definition 2.3.1 serve to indicate that R_O is the *outer radius*, R_I the *inner radius*, and R_C a *curvature radius*. Figure 2.1 gives a visual aid to the definition. By [17, Proposition 2.4.3], the expressibility of Ω as a union of open disks of fixed radius R_C in combination with convexity implies a $C^{1,1}$ boundary. Therefore, the $C^{1,1}$ smooth boundary assumption in [18] can be removed.

Definition 2.3.2. A domain $\Omega \subset \mathbb{C}$ satisfies a *boundary* uniform interior disk condition (boundary-UIDC) with radius R_C if for all $\zeta \in \partial\Omega$ there exists a disk $D \subset \Omega$ of radius R_C such that ζ is a shared boundary point of D and Ω.

Definition 2.3.3. A domain $\Omega \subset \mathbb{C}$ satisfies a *covering* uniform interior disk condition (covering-UIDC) with radius R_C if for all $z \in \Omega$, there exists a disk D of radius R_C such that $z \in D \subset \Omega$.

Lemma 2.3.4. *Suppose a domain $\Omega \subset \mathbb{C}$ satisfies a covering-UIDC with radius R_C. Then Ω also satisfies a boundary-UIDC with radius R_C.*

Proof. We will show that for every $\zeta \in \partial\Omega$ there exists $a \in \Omega$ such that $D(a, R_C) \subset \Omega$ and $\zeta \in \partial D(a, R_C)$. Take a sequence $z_n \to \zeta$ of points $z_n \in \Omega$ and cover each z_n with a disk $D(a_n, R_C) \subset \Omega$. The sequence $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Let a be its limit. Clearly $D(a, R_C) \subset \Omega$, which implies $|a - \zeta| \geqslant R_C$. On the other hand, $|a - \zeta| = \lim_{k\to\infty} |a_{n_k} - z_{n_k}| \leqslant R_C$. $\square$

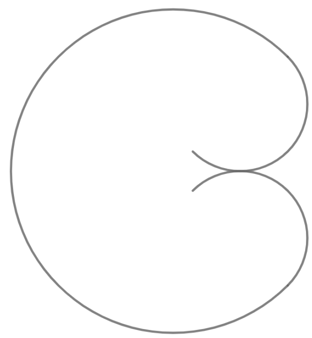

Figure 2.2: This non-Jordan domain clearly satisfies a boundary-UIDC for some radius R_C, but will not satisfy a covering-UIDC for any radius $r > 0$.

The converse does not hold, there exist non-Jordan simply connected domains like that in Figure 2.2, which satisfy a boundary-UIDC for some radius R_C, but do not satisfy a covering-UIDC for any positive radius. A non-Jordan domain satisfying a boundary-UIDC with radius R_C permits a portion of the boundary to have the domain on both sides—at each point, one side or the other must admit a boundary-UIDC disk, but it is possible that the other side is inaccessible. There is reason to believe the partial converse in Conjecture 2.3.5 holds.

Conjecture 2.3.5. If a Jordan domain satisfies a boundary-UIDC with radius R_C, then it must also satisfy a covering-UIDC with radius $R_C/\sqrt{3}$.

Example 2.3.6 (Clover Domain)**.** Define $D_0 = D(0, 1/\sqrt{3})$ (dashed in Figure 2.3). Take three equally spaced points on the circle $\{z\colon |z| = 2/\sqrt{3}\}$, construct a disk of radius 1 centered at each, call these disks D_1, D_2, and D_3. Observe that the boundaries of D_1, D_2, D_3

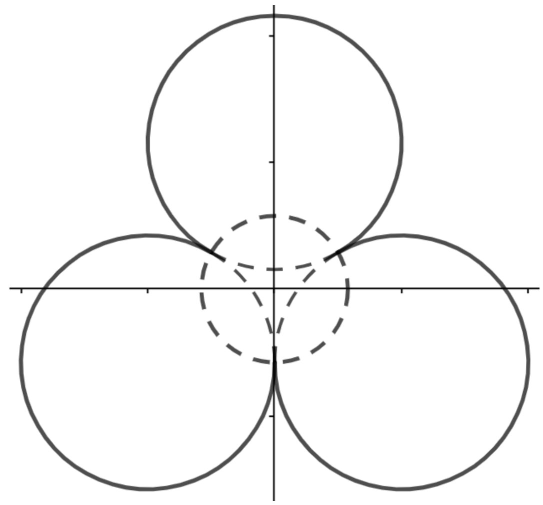

Figure 2.3: Points in the dashed hyperbolic triangle defined by $D_0 \setminus (D_1 \cup D_2 \cup D_3)$ are not contained in any sub-disk of Ω of radius 1.

have pairwise intersections of a single point, and that these three points all lie on ∂D_0. Let

$$\Omega \stackrel{\text{def}}{=} D_0 \cup D_1 \cup D_2 \cup D_3.$$

By construction, Ω satisfies the boundary-UIDC with radius 1. However, 0 cannot be covered by a disk of radius greater than $1/\sqrt{3}$ that is contained in Ω.

There is a weak converse to Lemma 2.3.4 which uses the additional assumption that Ω is convex.

Lemma 2.3.7. *Suppose a convex domain Ω satisfies a boundary-UIDC with radius R_C. Then Ω also satisfies a covering-UIDC with radius R_C.*

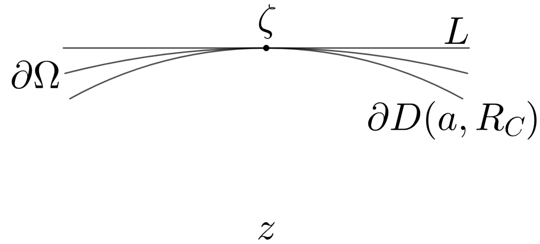

Figure 2.4: $\partial\Omega$ pinched between L and $\partial D(a, R_C)$ in a neighborhood of ζ.

Proof. Let $z \in \Omega$ be arbitrary. We will show there exists $a \in \Omega$ such that $z \in D(a, R_C) \subset \Omega$. If $\operatorname{dist}(z, \partial\Omega) \geqslant R_C$, then $a = z$ and we are done; assume $\operatorname{dist}(z, \partial\Omega) < R_C$. Since $\partial\Omega$ is a closed set, there is a point $\zeta \in \partial\Omega$ that attains the distance $\operatorname{dist}(z, \partial\Omega)$, meaning that $\operatorname{dist}(z, \partial\Omega) = |\zeta - z|$. Since Ω is convex, there exists a line L that passes through ζ and is disjoint from Ω. Let $D(a, R_C)$ be the disk satisfying the boundary-UIDC at ζ. Then in a neighborhood of ζ, $\partial\Omega$ lies between $D(a, R_C)$ inside Ω and L which is outside Ω (See Figure 2.4.) The radius of $D(a, R_C)$ terminating at ζ must then be orthogonal to L. Finally, since $|z - \zeta| = \operatorname{dist}(z, \partial\Omega)$, we must have L orthogonal to the segment (z, ζ). It follows that z, ζ, and a are collinear, and that $z \in D(a, R_C)$. □

2.4 Estimates for the hyperbolic metric in convex domains

We introduce a class of convex domains which are convenient for estimating the hyperbolic metric.

Definition 2.4.1. A *stadium* is the convex hull of the union of two open disks in the plane. It is denoted by $\mathcal{S}(r_1, r_2, d)$ where r_1 and r_2 are the radii of the two disks and d is the distance between their centers.

The notation $\mathcal{S}(r_1, r_2, d)$ in Definition 2.4.1 omits the centers of the disks that form the stadium since they are usually irrelevant to the hyperbolic geometry of the domain. The centers will be given in context when relevant.

The following lemma is a special case of [17, Proposition 2.4.3].

Lemma 2.4.2. *The boundary of a stadium is $C^{1,1}$-smooth. That is, the unit speed parameterization of its boundary has Lipschitz continuous derivative.*

Proof. The boundary of a stadium $\mathcal{S}(r_1, r_2, d)$ consists of circular arcs, possibly joined by tangent line segments. If we take w to be a unit speed parameterization of the boundary and $r = \min(r_1, r_2)$, then the inequality

$$|w'(t_1) - w'(t_2)| \leqslant \frac{1}{r}|t_1 - t_2| \tag{2.4.1}$$

holds on each of two circular arcs. It also holds on linear segments where w' is constant. It now follows that w' is Lipschitz continuous. □

Definition 2.4.3. For $0 < r < R < \infty$, let D_R and D_r denote the two open disks used to construct a stadium $\mathcal{S} \stackrel{\text{def}}{=} \mathcal{S}(R, r, d)$ as in Definition 2.4.1. If $d > R - r$, then the boundary of $\mathcal{S}$ is composed of two circular arcs and two congruent line segments. These segments can be extended to circumscribe an *infinite sector* $\widehat{\mathcal{S}}$ around $\mathcal{S}$. The opening of the sector is 2θ where $\theta \stackrel{\text{def}}{=} \arcsin \frac{R-r}{d}$. When working with a sector $\widehat{\mathcal{S}}(R, r, d)$ it is useful to assume that after a rigid motion, $\widehat{\mathcal{S}} = \{z : |\text{Arg } z| < \theta\}$.

Lemma 2.4.4. *Given a stadium $\mathcal{S}(R, r, d)$ where $r \leqslant R$, let $\theta = \arcsin \frac{R-r}{d}$ if $d > R - r$. The KP distance between the centers of D_R and D_r is given by*

$$\frac{d}{R} \qquad \textit{if } r = R, \tag{2.4.2a}$$

when $r < R$

$$\frac{1}{2}\log\frac{R+d}{R-d} \qquad \textit{if } d \leqslant R\tan(\theta/2) \tag{2.4.2b}$$

$$\frac{1}{2}\left[\cot\frac{\theta}{2}\log\left(\frac{R}{r}\cos\theta\right) + \log\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right] \quad \textit{otherwise.} \tag{2.4.2c}$$

Proof. Throughout this proof we refer to the disks D_r and D_R as well as their centers; these are the disks from Definition 2.4.1.

If $r = R$, then $\mathcal{S}$ is contained in an infinite strip of width $2r$. By Remark 2.2.4 and in light of Example 2.2.5, at every point z along the segment connecting the centers the extremal disk is $D(z, r)$. Then $\mu_{\mathcal{S}}(z) = \lambda_{D(0,r)}(0)$, from (2.2.1) the density is $1/r$, and integrating this along a segment of length d yields the result in (2.4.2a).

Next assume $d \leqslant R - r$; then $\theta = \pi/2$. In this case we clearly have $d < R\tan(\theta/2)$, and furthermore $D_r \subset D_R$ so $\mathcal{S} = D_R$. Like Example 2.2.7, for every point in $\mathcal{S}$ the $\mathtt{KP}_{\mathcal{S}}$ extremal disk will be D_R. Thus the KP distance between the centers is the KP length of a radial segment of length d, with the center of D_R as one endpoint. This distance is equivalent to $\rho_{D(0,R)}(0,d)$. The formula for hyperbolic distance in (2.2.3) gives the result in (2.4.2b).

Now assume that $R - r < d \leqslant R\tan(\theta/2)$. We will show that the segment connecting the centers is contained in the convex hull of $\partial D_R \cap \partial\mathcal{S}$ in the hyperbolic metric on D_R, and thus D_R is the extremal disk along the whole segment. Let $\widehat{\mathcal{S}} = \{|\mathrm{Arg}\, z| < \theta\}$ as in Definition 2.4.3. Then it is easily verified that $\partial D_R \cap \partial\widehat{\mathcal{S}} = \{Re^{\pm i\theta}\cot\theta\}$ and $\partial D_R \cap \partial\mathcal{S}$ has endpoints $\{Re^{\pm i\theta}\cot\theta\}$. The convex hull of $\partial D_R \cap \partial\mathcal{S}$ in the hyperbolic metric on D_R is the portion of D_R bounded by $\partial D_R \cap \partial\mathcal{S}$ and $D(0, R\cot\theta) \cap \{|\mathrm{Arg}\, z| < \theta\}$ (see Example 2.2.8). The distance from the center of D_R, located at $R\csc\theta$, to the boundary of $D(0, R\cot\theta)$ along the real axis is $R\csc\theta - R\cot\theta = R\tan(\theta/2)$. Then because $d \leqslant R\tan(\theta/2)$, the segment is contained in the convex hull, D_R is the $\mathtt{KP}_{\mathcal{S}}$ extremal disk along the segment, and the center of D_R is one endpoint of the segment. The $\mathtt{KP}_{\mathcal{S}}$ length can again be calculated as $\rho_{D(0,R)}(0,d)$. This completes the result in (2.4.2b).

Finally, assume $d > R\tan(\theta/2)$, and thus the segment connecting the centers extends beyond the convex hull of $\partial D_R \cap \partial\mathcal{S}$. We will divide the segment into a *proximal segment* $[r\csc\theta, R\cot\theta]$ and a *distal segment* $[R\cot\theta, R\csc\theta]$, where proximal and distal indicate relative position with respect to the vertex at 0. The distal segment will have as its extremal disk D_R, and as before we calculate the length as

$$\mathtt{KP}_{\mathcal{S}}(R\cot\theta, R\csc\theta) = \int_0^{R(\csc\theta - \cot\theta)} \lambda_{D(0,R)}(z)\, dz = \frac{1}{2}\log\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}. \tag{2.4.3}$$

For the proximal segment we rely on work done by Herron, Ma, and Minda [16, p. 331]. They produced a formula for the KP metric density at any point in an infinite sector. After adjusting the notation, the curvature convention, and taking advantage of the simplification that our segment is along the central axis, the formula is

$$\mu_{\widehat{\mathcal{S}}}(z) = \frac{1}{2z}\cot(\theta/2). \tag{2.4.4}$$

We need to show that the $\mathtt{KP}_{\widehat{\mathcal{S}}}$ extremal disk for every point on the proximal segment is contained in $\mathcal{S}$. Then, by Remark 2.2.4 it will also be the extremal disk in $\mathcal{S}$. This will justify using the infinite sector formula in (2.4.4) to give the KP density in our stadium $\mathcal{S}$. It will suffice to show that the extremal disk for the two endpoints of the proximal segment are in $\mathcal{S}$.

The proximal endpoint of the proximal segment is $r\csc\theta$, constructing the extremal disk in $\widehat{\mathcal{S}}$ for $r\csc\theta$ gives a disk tangent to $\partial\widehat{\mathcal{S}}$ at $\{re^{\pm i\theta}\csc\theta\}$. The disk D_r is tangent to $\partial\widehat{\mathcal{S}}$ at $\{re^{\pm i\theta}\cot\theta\}$, and because $\csc\theta < \cot\theta$ in $(0,\pi)$ the $\mathtt{KP}_{\widehat{\mathcal{S}}}$ extremal disk for the endpoint $r\csc\theta$ is far enough from the vertex to be contained in $\mathcal{S}$. The other endpoint is $R\cot\theta$, and we have already seen that $\forall z \in \mathcal{S}\colon |z| \geqslant R\cot\theta$, the $\mathtt{KP}_{\mathcal{S}}$ extremal disk is D_R.

We now calculate the KP length of the proximal segment in $\mathcal{S}$ by integrating the density given in (2.4.4):

$$\int_{r\csc\theta}^{R\cot\theta} \frac{1}{2z}\cot(\theta/2)\,dz = \frac{1}{2}\cot(\theta/2)\log\left(\frac{R}{r}\cos\theta\right). \tag{2.4.5}$$

Combining the proximal and distal lengths completes the proof,

$$\mathtt{KP}_{\mathcal{S}}(r\csc\theta, R\csc\theta) = \frac{1}{2}\left[\cot(\theta/2)\log\left(\frac{R}{r}\cos\theta\right) + \log\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right]. \tag{2.4.6}$$

□

Lemma 2.4.5. *Let $\delta(r_1, r_2, d)$ be the hyperbolic distance between the centers in a stadium $\mathcal{S}(r_1, r_2, d)$, then $\delta(r_1, r_2, d)$ is an increasing function in d.*

Proof. Fix r_1 and r_2. Let $d_1 < d_2$ and define $\lambda = d_2/d_1$. Dilating $\mathcal{S}(r_1, r_2, d_1)$ by a factor of

λ and observing the conformal invariance of the hyperbolic metric we have

$$\delta(r_1, r_2, d_1) = \delta(\lambda r_1, \lambda r_2, d_2). \tag{2.4.7}$$

Now consider $\mathcal{S}(r_1, r_2, d_2)$ and $\mathcal{S}(\lambda r_1, \lambda r_2, d_2)$. After a rigid motion, the segments connecting the centers of two stadia are coincident and $\mathcal{S}(r_1, r_2, d_2) \subset \mathcal{S}(\lambda r_1, \lambda r_2, d_2)$. Then by the monotonicity of the hyperbolic metric (2.2.6) we have

$$\delta(\lambda r_1, \lambda r_2, d_2) \leqslant \delta(r_1, r_2, d_2), \tag{2.4.8}$$

and combining (2.4.7) and (2.4.8) gives the result. □

Lemma 2.4.6. *Let* Ω *satisfy the* (R_O, R_I, R_C) *condition in Definition* 2.3.1. *Then*

$$|R_I - R_C| \leqslant R_O - R_C.$$

Proof. If $R_C \leqslant R_I$, then we are trying to show $R_I - R_C \leqslant R_O - R_C$. It is clear from the definitions that $R_I \leqslant R_O$, so the inequality is verified in this case.

Now assume $R_I < R_C$, we want to show $R_C - R_I \leqslant R_O - R_C$. There exists a point $\xi \in \partial\Omega$ such that $|\xi| = R_I$. The smoothness of $\partial\Omega$ implies that it must be share a tangent line with $\partial D(0, R_I)$ at ξ, otherwise $\partial\Omega$ has modulus less than R_I in a neighborhood of ξ contradicting that $D(0, R_I) \subset \Omega$.

By Definition 2.3.1 and Lemma 2.3.4 there exists a disk $D(a, R_C) \subset \Omega$ with ξ as a boundary point. Because $\xi \in \partial\Omega \cap \partial D(a, R_C)$ and $\partial\Omega$ is smooth, $\partial\Omega$ must share a tangent line with $\partial D(a, R_C)$ at ξ. It follows that $\partial D(0, R_I)$ and $\partial D(a, R_C)$ also share a tangent line at ξ, and this line is orthogonal to the segment $(0, \xi)$. It follows that the points ξ, 0, and a all lie on the same line L. Observe that $D(a, R_C) \subset \Omega \subset D(0, R_O)$. It follows that $2R_C \leqslant \text{diam}\ (\Omega \cap L) \leqslant R_O + R_I$, (see Figure 2.5) and thus $R_C - R_I \leqslant R_O - R_C$. □

The necessary condition in Lemma 2.4.6 turns out also to be sufficient for the existence

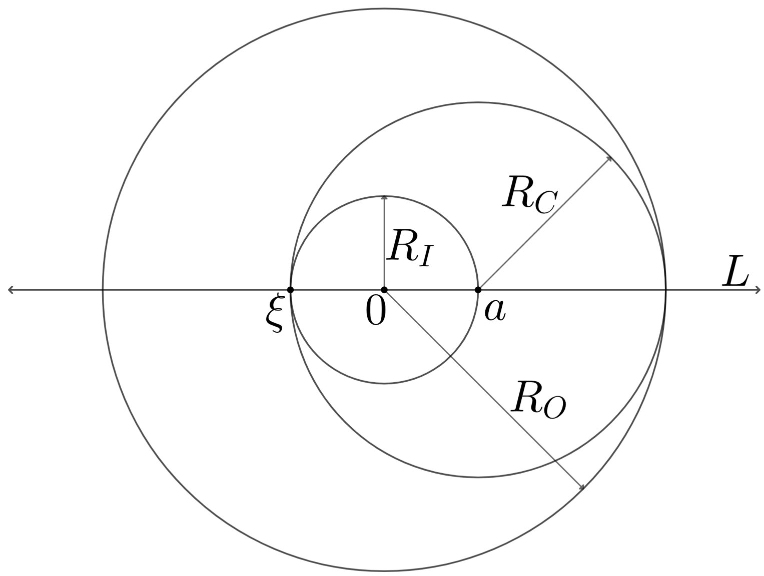

Figure 2.5: If $2R_C > R_O + R_I$, we contradict the definition of either R_O or R_I.

of such Ω.

Lemma 2.4.7. *We can construct a domain Ω satisfying the (R_O, R_I, R_C) condition for an arbitrary R_O, R_I, R_C so long as they satisfy the relationship $|R_I - R_C| \leqslant R_O - R_C$.*

Proof. If $R_I > R_C$, let K be the closed Euclidean convex hull of the set $D(0, R_I - R_C) \cup \{R_O - R_C\}$. Otherwise, let K be the line segment $[R_C - R_I, R_O - R_C]$. Define $\Omega = \bigcup_{z \in K} D(z, R_C)$.

By construction, Ω is convex and has $C^{1,1}$-smooth boundary. More specifically, Ω is a stadium in the sense of Definition 2.4.1. That Ω has the required values of R_O and R_I is a consequence of the fact that $-R_I$ and R_O are boundary points of Ω, and that $D(0, R_I) \subset \Omega \subset D(0, R_O)$. □

Our main result of this section provides an upper bound on the hyperbolic distance from the base point 0 to any point a such that $D(a, R_C) \subset \Omega$. This bound is given in terms of the outer, inner, and curvature radii of Ω.

Theorem 2.4.8. *Let Ω satisfy the (R_O, R_I, R_C) condition in Definition 2.3.1 and let a be any point such that $D(a, R_C) \subset \Omega$. Let $R = \max(R_C, R_I)$, $r = \min(R_C, R_I)$, $d = R_O - R_C$,* *and* $\theta = \arcsin \frac{R-r}{d}$. *Define* $F(R_O, R_I, R_C)$ *as*

$$\frac{d}{R} \qquad \text{if } r = R, \tag{2.4.9a}$$

when $r < R$

$$\frac{1}{2}\log\frac{R+d}{R-d} \qquad \text{if } d \leqslant R\tan\frac{\theta}{2}, \tag{2.4.9b}$$

$$\frac{1}{2}\left[\cot\frac{\theta}{2}\log\left(\frac{R}{r}\cos\theta\right) + \log\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right] \qquad \text{otherwise.} \tag{2.4.9c}$$

Then $\rho_\Omega(0, a) \leqslant F(R_O, R_I, R_C)$.

Proof. First, note that Lemma 2.4.6 allows us to define θ in this way. Observe that $D(a, R_C) \subset \Omega \subset D(0, R_O)$, thus $|a| + R_C \leqslant R_O$ and $|a| \leqslant d$. Since Ω is a convex domain containing the disks $D(0, R_I)$ and $D(a, R_C)$, it contains the corresponding stadium $\mathcal{S}(R_I, R_C, |a|)$. For containment the position of the stadium is important, so to be clear $\mathcal{S}$ is the convex hull of $D(0, R_I) \cup D(a, R_C)$. It follows from (2.2.6) regarding the domain monotonicity of the hyperbolic metric and from Lemma 2.4.5 that

$$\rho_\Omega(0, a) \leqslant \delta(R_I, R_C, |a|) = \delta(R, r, |a|) \leqslant \delta(R, r, d) \tag{2.4.10}$$

where δ is as in Lemma 2.4.5. Since the hyperbolic distance is majorized by the KP distance (2.2.5), the claim follows from the explicit formulas for KP distance from Lemma 2.4.4.

In each of the three cases we find the longest possible line segment from 0 to an allowable a, construct $\mathcal{S}(R_I, R_C, R_O - R_C) = \mathcal{S}(R, r, d) \subset \Omega$ around the segment, and find its $\mathtt{KP}_\mathcal{S}$ length. The formula in (2.4.9a) corresponds to the case where the KP metric has constant density along the segment. The formula in (2.4.9b) corresponds to the case where the $\mathtt{KP}_\mathcal{S}$ extremal disk is the same at every point of the segment. The formula in (2.4.9c) corresponds to the case where the extremal disk and density vary along the segment. $\square$

2.5 Global expansion bound for maps from the disk onto convex domains

We are now ready to prove the main result.

Theorem 2.5.1. *Let* Ω *satisfy the* (R_O, R_I, R_C) *condition in Definition* 2.3.1. *Then for any conformal map* $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ *fixing* 0 *we have*

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_I,R_C)}. \tag{2.5.1}$$

where F *is as in Theorem* 2.4.8.

Proof. By assumption, Ω has a smooth boundary and f' exist on $\partial\mathbb{D}$. Take any number $L > R_C e^{2F(R_O,R_I,R_C)}$. It suffices to show that $|f'| \leqslant L$ in $\mathbb{D}$, which we will do by proving this inequality holds on the boundary and then using the maximum principle. More specifically, it suffices to show

$$\limsup_{|z|\nearrow 1} \frac{\operatorname{dist}(f(z), \partial\Omega)}{1-|z|} \leqslant L. \tag{2.5.2}$$

Fix $z \in \mathbb{D}$. Let $d = \operatorname{dist}(f(z), \partial\Omega)$, since we are interested in the limit as $d \to 0$, we may assume $d < R_C$. We will show that

$$\frac{d}{L} \leqslant 1 - |z| \tag{2.5.3}$$

for sufficiently small d, thus establishing the inequality in (2.5.2).

We choose a point $w \in \partial\Omega$ such that $|f(z) - w| = d$. By definition of R_C there is a disk $D = D(a, R_C)$ that has w on its boundary and is contained in Ω. The smoothness of $\partial\Omega$ and ∂D at w requires that $f(z)$ lie on the radius of D that ends at w, and therefore $|f(z) - a| = R_C - d$. Keeping in mind that the hyperbolic metric is monotone with respect to domain and the formula for hyperbolic distance in a disk along a radius is well known, observe

$$\rho_\Omega(f(z), a) \leqslant \rho_{D(a,R_C)}(f(z), a) = \frac{1}{2}\log\frac{2R_C - d}{d} < \frac{1}{2}\log\frac{2R_C}{d}. \tag{2.5.4}$$

Next we estimate $\rho_\Omega(a, 0)$ using the KP estimate from Theorem 2.4.8:

$$\rho_\Omega(a, 0) \leqslant F(R_O, R_I, R_C). \tag{2.5.5}$$

Now suppose for the sake of contradiction that (2.5.3) is false, this implies

$$1 - |z| < d/L \quad \text{and} \quad 1 + |z| > 2 - d/L.$$

By conformal invariance of the hyperbolic metric we know $\rho_\Omega(f(z), 0) = \rho_{\mathbb{D}}(z, 0)$, then

$$\rho_\Omega(f(z), 0) = \frac{1}{2}\log\frac{1 + |z|}{1 - |z|} > \frac{1}{2}\log\frac{2 - d/L}{d/L}. \tag{2.5.6}$$

Using the triangle inequality to combine this with (2.5.4) and (2.5.5) we get

$$\frac{1}{2}\log\frac{2 - d/L}{d/L} < \frac{1}{2}\log\frac{2R_C}{d} + F(R_O, R_I, R_C)$$

which can be rearranged to

$$L - \frac{d}{2} < R_C e^{2F(R_O,R_I,R_C)}.$$

But $R_C e^{2F(R_O,R_I,R_C)} < L$, so we have a contradiction when d is sufficiently small. This contradiction proves (2.5.3). $\square$

There are two obvious corollaries. The first generalizes the arbitrary choice of 0 that was used as a base point throughout the chapter. The second recognizes that the stretching bound on all conformal $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ attained in Theorem 2.5.1 is equivalent to a compression bound on all $g : \Omega \xrightarrow{\text{onto}} \mathbb{D}$, that is, it formalizes the observation that if $|f'| \leqslant L$ in $\mathbb{D}$, then $|[f^{-1}]'| \geqslant L^{-1}$ in Ω.

Corollary 2.5.2. *Let Ω be a domain and z_0 a base point in the domain such that $\Omega_0 \overset{def}{=} \{z - z_0 \colon z \in \Omega\}$ satisfies the (R_O, R_I, R_C) condition in Definition* 2.3.1*. Then for any conformal map $f \colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ such that $f(0) = z_0$, we have*

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_I,R_C)}. \tag{2.5.7}$$

where F is as in Theorem 2.4.8*.*

Proof. Define a conformal map $g \colon \mathbb{D} \xrightarrow{\text{onto}} \Omega_0$ by $g(z) \overset{\text{def}}{=} f(z) - z_0$. Then for all $z \in \mathbb{D}$, we have $g'(z) = f'(z)$ and by Theorem 2.5.1 $\|g'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_I,R_C)}$. □

Corollary 2.5.3. *Let Ω satisfy the (R_O, R_I, R_C) condition in Definition* 2.3.1*. Then for any conformal map $f \colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$*

$$\sup_{z \in \Omega} |1/f'| \leqslant R_C e^{2F(R_O,R_I,R_C)}. \tag{2.5.8}$$

where F is as in Theorem 2.4.8*.*

Proof. Let $g = f^{-1}$ and let $z \in \mathbb{D}$ be arbitrary, so that $g(z) = w \in \Omega$. Then by the inverse function theorem $g'(z) = \frac{1}{f'(w)}$. It follows that

$$|1/f'(w)| = |g'(z)| \leqslant \|g'\|_{H^\infty},$$

and we know from Theorem 2.5.1 that $|g'(z)| \leqslant R_C e^{2F(R_O,R_I,R_C)}$. Since z and $f(z) = w$ were arbitrary, it follows that $\sup |1/f'| \leqslant R_C e^{2F(R_O,R_I,R_C)}$. □

2.6 Examples

Remark 2.6.1. In the absence of convexity, we would need some other condition for $\|f'\|_{H^\infty}$ to be controlled by the three radii R_O, R_I, R_C. Let $\epsilon > 0$ and define Ω as $D(0,1) \cup D(2-\epsilon, 1)$

(See Figure 2.6). Then $R_I = R_C = 1$ and $R_O = 3-\epsilon$, so all three radii stay between 1 and 3. But for a conformal map $\varphi\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ with $\varphi(0) = 0$ and $\varphi'(0) > 0$, we have $\|\varphi'\|_{H^\infty} \to \infty$ as $\epsilon \to 0$.

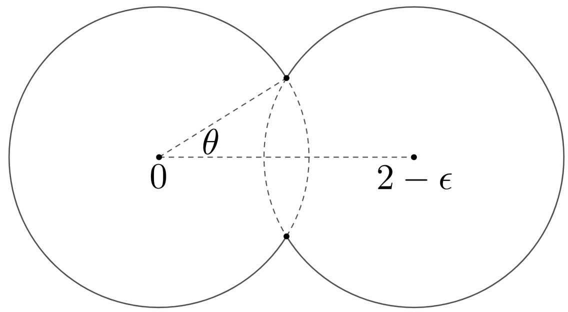

Figure 2.6: The Mastercard domain $D(0,1) \cap D(2-\epsilon, 1)$. A non-convex Ω such that for $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ with $f(0) = 0$, $\sup_{\mathbb{D}} |f'|$ is not controlled by the radii R_O, R_I, R_C.

Proof. Let $z_0 = \varphi^{-1}(2-\epsilon)$. By the Schwarz-Pick lemma [25, Corollary 1.4]

$$|\varphi'(z_0)| \geqslant \frac{\operatorname{dist}(\varphi(z_0), \partial\Omega)}{1-|z_0|^2} = \frac{1}{1-|z_0|^2}. \tag{2.6.1}$$

By conformal invariance of the hyperbolic metric

$$\rho_{\mathbb{D}}(0, z_0) = \rho_\Omega(0, 2-\epsilon). \tag{2.6.2}$$

From (2.2.3) we have

$$\rho_{\mathbb{D}}(0, z_0) = \frac{1}{2}\log\frac{1+|z_0|}{1-|z_0|}. \tag{2.6.3}$$

We will show that $\rho_\Omega(0, 2-\epsilon) \to \infty$ as $\epsilon \to 0$. This along with (2.6.2) and (2.6.3) will establish that $|z_0| \to 1$ as $\epsilon \to 0$, which along with (2.6.1) is enough to show $|\varphi'(z_0)| \to \infty$ and thus prove the claim.

The two circular arcs forming the boundary of Ω meet at $e^{\pm i\theta}$ where $\theta = \arccos(1-\frac{\epsilon}{2})$, so $\theta \to 0$ as $\epsilon \to 0$. If we define G to be the complex plane with vertical cuts going up from $e^{i\theta}$ and down from $e^{-i\theta}$, we have $\Omega \subset G$. Then by hyperbolic domain monotonicity in (2.2.6) we know $\rho_\Omega(0, 2-\epsilon) \geqslant \rho_G(0, 2-\epsilon)$.

By the linear map $f(z) = \frac{z-\cos\theta}{\sin\theta}$, G can be transformed in H which we define as the complex plane with vertical cuts up from i and down from $-i$. Then $f(0) = -\cot\theta$ and $f(2-\epsilon) = f(2\cos\theta) = \cot\theta$. The statement to be proven in Remark 2.6.1 reduces to $\rho_H(-\cot\theta, \cot\theta) \to \infty$ as $\epsilon \to 0$. Observe that as $\epsilon \to 0$, $\theta \to 0$ and thus the segments $(-\cot\theta, \cot\theta)$ cover the real line. By the symmetry of H, $\mathbb{R}$ is a hyperbolic geodesic and every hyperbolic geodesic ray in a simply connected domain has infinite hyperbolic length (see beginning of Section 2.2). $\square$

Example 2.6.2. Let $\Omega = \{|z| < r\}$, then $R_O = R_I = R_C = r$. Theorem 2.5.1 says that for all conformal $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ fixing 0, $\|f'\|_{H^\infty} \leqslant re^0 = r$. The function $f(z) = rz$ shows the bound is attained in this case.

This can be generalized to show that the bound in Theorem 2.5.1 is sharp whenever Ω is a disk containing 0.

Proposition 2.6.3. *Let $\Omega = D(a, r)$ with $0 \leqslant |a| < r$. Then the bound in Theorem 2.5.1 is sharp for a conformal map $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$.*

Proof. After a rotation about the origin, we may assume $0 \leqslant a < r$. Then Ω satisfies the condition ($R_O = r + a, R_I = r - a, R_C = r$), and one can check that Theorem 2.5.1 gives

$$\|f'\|_{H^\infty} \leqslant r\frac{r+a}{r-a}. \tag{2.6.4}$$

Take

$$f_1(z) = \frac{z - \frac{a}{r}}{1 - \frac{a}{r}z}, \quad f_2(z) = z + \frac{a}{r}, \quad \text{and } f_3(z) = rz.$$

Then the conformal mapping $f_3 \circ f_2 \circ f_1\colon \mathbb{D} \xrightarrow{\text{onto}} D(a, r)$ fixes 0, and its derivative attains the bound (2.6.4) at $z = 1$. $\square$

To illustrate the improvement over the bound given in equation (2.2.7), we close with two more examples.

Example 2.6.4. One of examples considered in [18] is a rounded triangle with $R_O = 0.6, R_I = 0.5, R_C = 0.4$. For this domain, the bound (2.2.7) is $\|f'\|_{H^\infty} \leqslant 0.977$. Theorem 2.5.1 improves this to $\|f'\|_{H^\infty} \leqslant 0.931$.

Example 2.6.5. For the domain in Proposition 2.6.3 with $0 \leqslant a < r$ the bound (2.2.7) is

$$\|f'\| \leqslant \frac{r^3}{(r-a)^2}. \tag{2.6.5}$$

The ratio of two bounds (2.6.4) and (2.6.5) tends to 0 as $a \to r$, indicating a substantial improvement. For a specific example, let $a = 1$ and $r = 2$, so $\Omega = D(1,2)$. The bound in (2.6.4) becomes $\|f'\|_{H^\infty} \leqslant 6$, which is sharp as noted above. In contrast (2.2.7) gives $\|f'\|_{H^\infty} \leqslant 8$ for this example.

Chapter 3

Lipschitz Estimates for Maps from the Unit Disk onto Stellar Core Domains

3.1 Introduction

The goal of this chapter is to move beyond convex domains. Star-shaped domains are a natural class to consider next. In Figure 3.1 we see an example of a star-shaped domain that is not convex and even has intruding corners, but is still expressible as the union of disks of some radius R_C.

Since the existence of R_C no longer guarantees smoothness (in contrast to the convex case), additional ideas are required to obtain Lipschitz estimates for such domains. Also, the inner radius of a star-shaped domain in general cannot be used in conjunction with the boundary-UIDC to construct a stadium contained in Ω.

We replace the inner radius with so-called stellar core radius R_S and show that the conclusion in Theorem 2.5.1 still holds without convexity using this new radius. Note that if Ω is convex then $R_I = R_S$, and it will be seen that Theorem 2.5.1 is a special case of the stronger Theorem 3.4.6, which is the main result in this chapter.

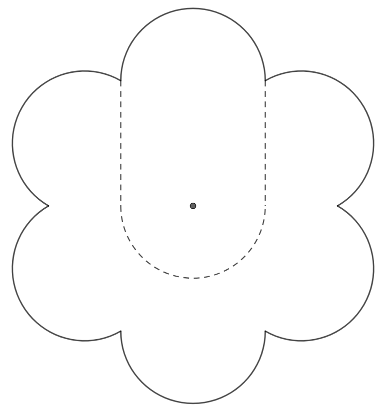

Figure 3.1: This flower domain is centered on the point 0. The dotted stadium shown is constructed using the stellar core radius (see Definition 3.2.2) and a boundary-UIDC disk. This shows that the approach used to prove Theorem 2.5.1 can be adapted for a class of non-convex domains, a motivating idea for this chapter.

3.2 Definitions and preliminary results

Definition 3.2.1. A domain Ω is starlike with respect to a point a if for all $z \in \Omega$, the line segment $[a, z]$ is contained in Ω. The point a is called a *star center* of Ω.

Definition 3.2.2 (Stellar Core Radius)**.** Let a domain Ω be starlike with respect to 0. We say Ω has a positive *stellar core radius* if there exists r such that $\forall z \in \Omega$, the convex hull of the set $\{z\} \cup D(0, r)$ is contained in Ω. If there exists such a positive number we call Ω a *stellar core domain* and define R_S to be the largest r for which it holds. If no such positive number exists, let $R_S = 0$.

Remark 3.2.3. It would be equivalent to define the stellar core radius as the largest r for which Ω is starlike with respect to every point in $D(0, r)$.

Definition 3.2.4 (Tangent Line)**.** Let Γ be a Jordan curve. By definition, there is a homeomorphic parameterization $f\colon \partial\mathbb{D} \xrightarrow{\text{onto}} \Gamma$. For a point $\zeta = f(e^{it}) \in \Gamma$ we say that there is a tangent line if the limit

$$v \stackrel{\text{def}}{=} \lim_{s \to t} \operatorname{sign}\left(\frac{f(e^{is}) - f(e^{it})}{s - t}\right) \tag{3.2.1}$$

exists, where $\operatorname{sign} z = z/|z|$ is the complex sign function. If the limit does exist, define the tangent line L to Γ at ζ as the line $\{\zeta + tv : t \in \mathbb{R}\}$.

Definition 3.2.5. Suppose Ω is a Jordan domain which is starlike with respect to 0. We say that such a domain satisfies the (R_O, R_S, R_C) condition if:

- R_O is the minimal r such that $\Omega \subset D(0,r)$.
- Ω has positive stellar core radius R_S.
- Ω satisfies the boundary uniform interior disk condition from Definition 2.3.2 with radius R_C.

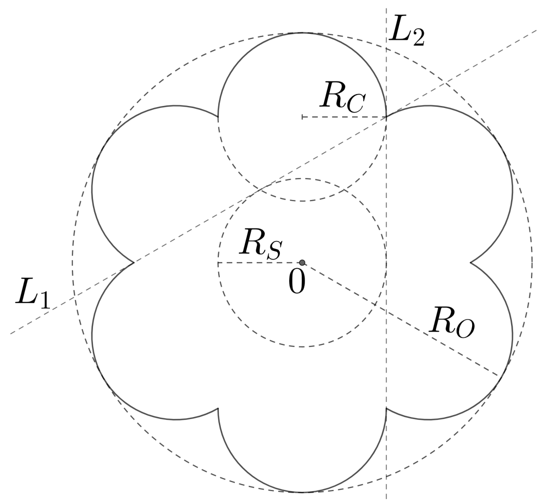

Figure 3.2: The solid lines are the boundary of a sample domain Ω. The radii R_C and R_O are shown, but are unchanged and need no further explanation. A clumsier, but visually intuitive definition of the radius R_S might be to consider the set of all lines L tangent to $\partial\Omega$ and set $R_S = \inf \operatorname{dist}(0, L)$. Two such extremal tangent lines L_1 and L_2 are shown.

The subscripts serve to indicate that R_O is the *outer radius*, R_S the *stellar core radius*, and R_C the *curvature radius*. Figure 3.2 gives a visual aid to the definition. Recall that Definition 2.3.2 says for all $\zeta \in \partial\Omega$ there exists a disk D of radius R_C such that $\zeta \in \partial D$ and $D \subset \Omega$.

Definition 3.2.6 (Corner)**.** Let Ω be a Jordan domain satisfying the boundary-UIDC (Definition 2.3.2), and let ζ be a point in $\partial\Omega$. Let Δ be the set containing the centers of all disks $D \subset \Omega$ with radius R_C such that $\zeta \in \partial D \cap \partial\Omega$. For any $a, b \in \Delta$, the ordered triple a, ζ, b defines two angles; $\beta_1(a,\zeta,b) + \beta_2(a,\zeta,b) = 2\pi$ where $\beta_1(a,\zeta,b) \leqslant \pi$ and $\beta_2(a,\zeta,b) \geqslant \pi$. Let

β denote the supremum of $\beta_1(a, \zeta, b)$ taken over all pairs $a, b \in \Delta$. Then we say Ω has a corner of exterior measure $\pi - \beta$.

Remark 3.2.7. Observe because Ω is assumed to satisfy the boundary uniform interior disk condition, all corners are necessarily intruding. If there were an extruding (jutting outward) angle at some point, Ω could not possibly satisfy the boundary-UIDC at that point.

3.3 Disk and wedge conditions without a convexity requirement

Lemma 3.3.1. *Let Ω satisfy the boundary-UIDC (Definition* 2.3.2*) and let $\zeta \in \partial\Omega$ be a point at which there is a tangent line L to $\partial\Omega$. Then there is a unique disk $D(a, R_C) \subset \Omega$ with $\zeta \in \partial D(a, R_C)$, and the segment $[a, \zeta]$ is orthogonal to L.*

Proof. Existence follows immediately from Definition 2.3.2. For uniqueness, assume not. Then there exist $a_1 \neq a_2$ such that $D(a_i, R_C) \subset \Omega$ and $\zeta \in \partial D(a_i, R_C)$. Define $0 < \alpha \leqslant \pi$ as $\alpha = \measuredangle(a_1, \zeta, a_2)$. Then $\partial\Omega$ must have a corner of exterior measure at most $\pi - \alpha$ at ζ contradicting the existence of a tangent line. This establishes that there is a unique disk $D(a, R_C)$.

Now assume that (a, ζ) is not orthogonal to L. Define $0 < \beta < \pi/2$ as the angle between the segment (a, ζ) and a line orthogonal to L. Then by Definition 3.2.6, $\partial\Omega$ must have a corner of exterior measure at most $\pi - \beta$ at ζ contradicting existence of the tangent line L. □

The converse is also true.

Lemma 3.3.2. *Let Ω satisfy the boundary-UIDC (Definition* 2.3.2*) and let $\zeta \in \partial\Omega$ be a point at which there is a unique disk $D(a, R_C) \subset \Omega$ with $\zeta \in \partial D(a, R_C)$. Then there exists a tangent line to $\partial\Omega$ at ζ.*

Proof. First, we establish continuity of the boundary-UIDC disk center at ζ by a convergence argument similar to that in the proof of Lemma 2.3.4. Let $\zeta \in \partial\Omega$ be a point at which there is a unique disk $D(a(\zeta), R_C) \subset \Omega$ such that $\zeta \in \partial D(a(\zeta), R_C)$. Since Ω is assumed to satisfy a boundary-UIDC, for every point $\xi \in \partial\Omega$ near ζ there is a disk $D(a(\xi), R_C) \subset \Omega$ such that $\xi \in \partial D(a(\xi), R_C)$. As $\xi \to \zeta$, it must be true of the disk centers that $\lim_{\xi\to\zeta} a(\xi) = a(\zeta)$. If this was not the case, then a subsequence could be chosen for which the center $a(\xi)$ converges to something other that $a(\zeta)$ which would contradict the uniqueness of $D(a(\zeta), R_C)$.

It follows that

$$\forall\epsilon > 0\, \exists\delta > 0\colon |\xi - \zeta| < \delta \implies |a(\xi) - a(\zeta)| < \epsilon. \tag{3.3.1}$$

The boundary point ξ cannot lie inside $D(a(\zeta), R_C)$, so it is constrained from the inside of the domain. Near ζ, $\partial D(a(\zeta), R_C)$ looks locally like a line L orthogonal to the segment $(\zeta, a(\zeta))$ and containing ζ. In the neighborhood $D(\zeta, \delta)$ away from $a(\zeta)$ (below ζ in Figure 3.3) there is a sector $S_\downarrow$ with vertex ζ where ξ cannot be; if it were, the contradiction $\zeta \in D(a(\xi), R_C)$ would be forced by the assumption that $a(\xi) \in D(a(\zeta), \epsilon)$. Observe that this sector is bisected by a line though ζ and $a(\zeta)$. Note also that this sector (and the next two) are created by the intersection of two circles and are only true sectors in the limit. Then ξ can only be in the two remaining sectors: $S_\leftarrow$ and $S_\rightarrow$, both with angle measure $\theta = \arctan(\epsilon/R_C)$.

In Figure 3.3, the upper arc through ζ represents a portion of $\partial D(a(\zeta), R_C)$, the two lower arcs with ζ as an endpoint represent the boundaries of two possible disks, $D(a(\xi), R_C)$ under the constraint that $|a(\xi) - a(\zeta)| < \epsilon$. The disks shown are chosen because they (diagrammatically) attain the maximal angle θ, thus setting the edge of $S_\downarrow$ as described.

As $\epsilon \to 0$, $\theta \to 0$ and the boundary point ξ squeezed onto the line L (which would be horizontal through ζ in Figure 3.3). The possibility that $\partial\Omega$ approaches ζ through $S_\leftarrow$, has a cusp at ζ, and then passes back out $S_\leftarrow$ is eliminated by the assumption of a unique boundary-UIDC disk at ζ and the continuity argument that began the proof; this is also true of $S_\rightarrow$.

Define s and t such that $\xi = f(e^{is})$ and $\zeta = f(e^{it})$. Then $s \to t$ as $\xi \to \zeta$, and the limit in Definition 3.2.4 of a tangent line exists. This tangent line is the line L alluded to just below (3.3.1). Observe that L is orthogonal to the segment $(\zeta, a(\zeta))$. $\square$

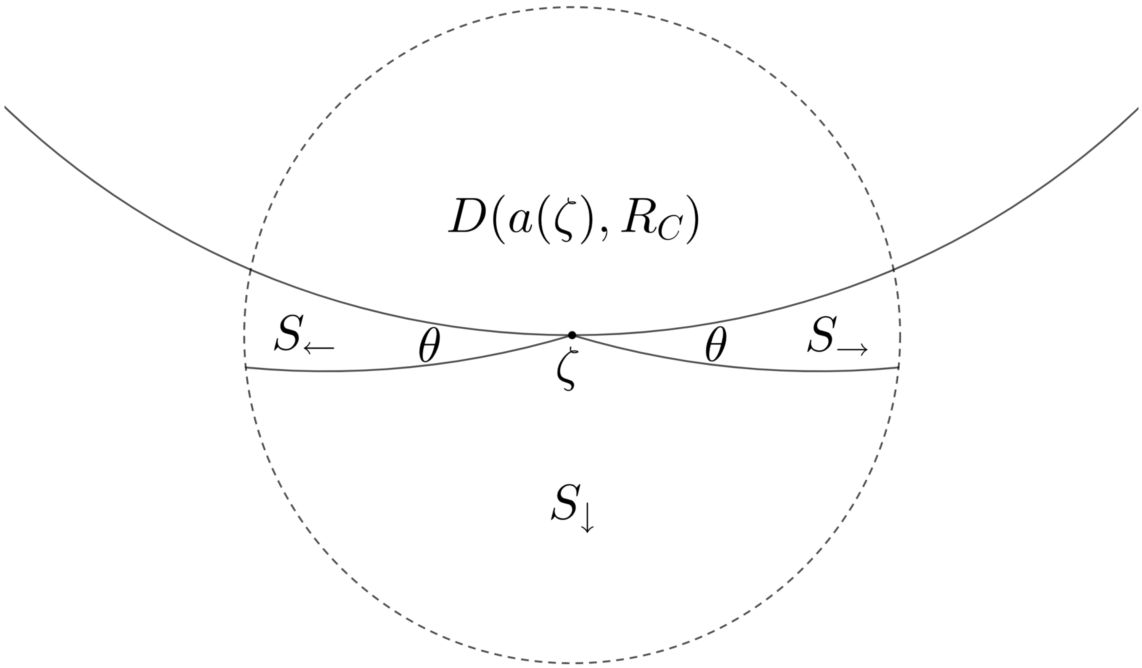

Figure 3.3: The dotted circle is the neighborhood $D(\zeta, \delta)$.

As a corollary, we obtain a partial result related to Conjecture 2.3.5.

Corollary 3.3.3. *If a Jordan domain satisfies a boundary-UIDC (Definition* 2.3.2*) with a unique disk at every boundary point, then it satisfies a covering-UIDC (Definition* 2.3.3*) for the same radius* R_C*.*

Proof. Let z be an arbitrary point in Ω. If $\operatorname{dist}(z, \partial\Omega) \geqslant R_C$, then $D(z, R_C) \subset \Omega$ covers z. Assume $d = \operatorname{dist}(z, \partial\Omega) < R_C$, because $\partial\Omega$ is closed there is a point $\zeta \in \partial\Omega$ such that $|z-\zeta| = d$. By assumption there is a unique disk $D(a, R_C) \subset \Omega$ with $\zeta \in \partial D(a, R_C) \cap \partial\Omega$, and by Lemma 3.3.2 there is a line L tangent to both $\partial\Omega$ and $\partial D(a, R_C)$ at ζ. Since $D(z, d) \subset \Omega$ and $\zeta \in \partial D(z, d)$, the line L is also tangent to $\partial D(z, d)$ at ζ. Then $\partial D(a, R_C)$ and $\partial D(z, d)$ are mutually tangent at ζ, and $z \in D(z, d) \subset D(a, R_C)$. Because z was arbitrary we have shown that for all $z \in \Omega$, there exists $D(a(z), R_C) \subset \Omega$ containing z and the statement is proven. $\square$

Lemma 3.3.4 (Wedge Condition)**.** *Let Ω be a domain satisfying the boundary-UIDC (Definition 2.3.2) and with a line L tangent to $\partial\Omega$ at ζ. Then for all $\theta < \pi/2$ there exists a sector $G \subset \Omega$ with radius $r > 0$, vertex ζ, angular opening 2θ, and angle bisector inward normal to L at ζ. The radius r depends on R_C and θ only.*

Proof. By Definition 3.2.5 there exists $D(a, R_C) \subset \Omega$ with $\zeta \in \partial D(a, R_C) \cap \partial\Omega$. By Lemma 3.3.1 the segment $[a, \zeta] \perp L$. For $r < 2R_C \cos\theta$, define a sector G with vertex ζ, radius r, angular opening 2θ, and angle bisector collinear with $[a, \zeta]$. Then by construction, $G \subset D(a, R_C) \subset \Omega$. □

3.4 Global expansion bound for maps from the disk onto stellar core domains

Lemma 3.4.1. *Let Ω be a domain satisfying the (R_O, R_S, R_C) condition. Then $|R_C - R_S| \leqslant R_O - R_C$.*

Proof. The inequality to be proven reduces to the statements $R_S \leqslant R_O$ or $R_C \leqslant R_O$ if $R_S > R_C$ or $R_S = R_C$ respectively. It is inherent to Definition 3.2.5 that R_O is the largest of the three radii, so we may concern ourselves only with the case where $R_S < R_C$. In this case, the inequality to be verified is equivalent to $R_S \geqslant 2R_C - R_O$.

Observe that if $2R_C \leqslant R_O$, then $2R_C \leqslant R_O + R_S$ and the statement is proven. Now assume $2R_C > R_O$ and take an arbitrary point $z \in \Omega$. To complete the proof we need to show that the convex hull of $\{z\} \cup D(0, 2R_C - R_O)$ is contained in Ω. Let ζ be the projection of z from 0 onto the boundary. If $z = 0$, an arbitrary choice of ζ will suffice. Definition 3.2.5 furnishes a disk $D(a, R_C) \subset \Omega$ with ζ contained in $\partial D(a, R_C) \cap \partial\Omega$. We know $|\zeta| \leqslant R_O$ and by assumption $2R_C > R_O$, so it follows that the disk $D(0, 2R_C - R_O) \subset D(a, R_C)$ (See Figure 3.4). Because $0 \in D(a, R_C)$, $\zeta \in \partial D(a, R_C)$, and by construction z in on the line segment $[0, \zeta]$, we may conclude $z \in D(a, R_C)$. Finally since $\{z\} \cup D(0, 2R_C - R_O)$ is a subset of $D(a, R_C)$ and $D(a, R_C)$ is convex, the convex hull of $\{z\} \cup D(0, 2R_C - R_O)$ is contained in $D(a, R_C)$ and is therefore contained in Ω. Since z was arbitrary, we showed that the convex hull of $\{z\} \cup D(0, 2R_C - R_O)$ is contained in Ω for all $z \in \Omega$. It follows from Definition 3.2.2 that $R_S \geqslant 2R_C - R_O$. □

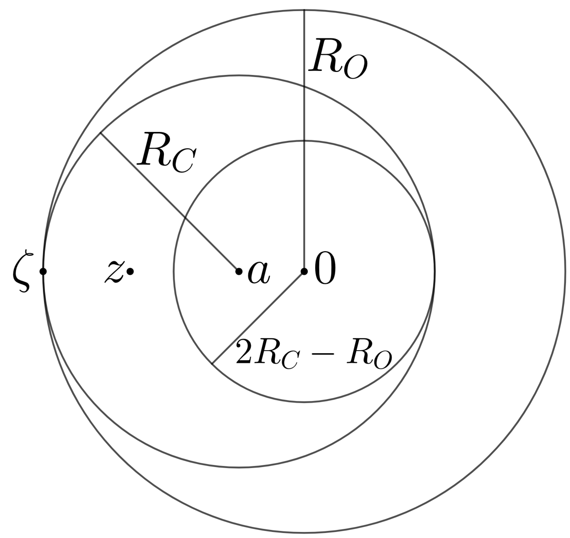

Figure 3.4: If $2R_C > R_O$, then for any disk $D(a, R_C) \subset \Omega$ it must be the case that $D(0, 2R_C - R_O) \subset D(a, R_C)$.

Theorem 3.4.2. *Let Ω satisfy the (R_O, R_S, R_C) condition in Definition 3.2.5 and let a be any point such that $D(a, R_C) \subset \Omega$. Let $R = \max(R_C, R_S)$, $r = \min(R_C, R_S)$, $d = R_O - R_C$, and $\theta = \arcsin \frac{R-r}{d}$. Define $F(R_O, R_S, R_C)$ as*

$$\frac{d}{R} \qquad \textit{if } r = R, \tag{3.4.1a}$$

when $r < R$

$$\frac{1}{2} \log \frac{R+d}{R-d)} \qquad \textit{if } d \leqslant R \tan \frac{\theta}{2}, \tag{3.4.1b}$$

$$\frac{1}{2}\left[\cot \frac{\theta}{2} \log \left(\frac{R}{r} \cos \theta\right) + \log \frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right] \quad \textit{otherwise.} \tag{3.4.1c}$$

Then $\rho_\Omega(0, a) \leqslant F(R_O, R_S, R_C)$.

Proof. We know that Ω contains $D(a, R_C)$ and has stellar core $D(0, R_S)$. Then for all

$z \in D(a, R_C)$, the convex hull of $\{z\} \cup D(0, R_S)$ is contained in Ω. It follows that the convex hull of $D(a, R_C) \cup D(0, R_S)$ is contained in Ω. This convex hull is the stadium $\mathcal{S}(R_S, R_C, |a|)$ from Definition 2.4.1.

It follows from the domain monotonicity of the hyperbolic metric that

$$\rho_\Omega(0, a) \leqslant \delta(R_S, R_C, |a|) \tag{3.4.2}$$

where δ is the hyperbolic distance between centers, as in Lemma 2.4.5. Observe that $|a| \leqslant R_O - R_C$, otherwise we contradict $D(a, R_C) \subset D(0, R_O)$. Applying Lemma 2.4.5 shows

$$\delta(R_S, R_C, |a|) \leqslant \delta(R_S, R_C, R_O - R_C). \tag{3.4.3}$$

Since the hyperbolic distance is majorized by KP distance (see (2.2.5)), the claim follows by combining inequalities (3.4.2) and (3.4.3) with the explicit formula for KP distance from Lemma 2.4.4. $\square$

Lemma 3.4.3. *Let Ω be a Jordan domain and suppose there exist a tangent line to $\partial\Omega$ at a point ζ. If $w \to \zeta$ along the interior normal half line, then*

$$\frac{|w - \zeta|}{\operatorname{dist}(w, \partial\Omega)} \to 1.$$

Proof. By Lemma 3.3.4, the existence of a tangent line at ζ implies that for all $\theta < \pi/2$ there exists an $r > 0$ such that the sector with vertex ζ, angular opening 2θ, radius r, and bisector inward normal to the tangent line is contained in Ω. Call this sector G.

When w is on the bisector of G is sufficiently close to ζ ($|\zeta - w| < r/2$), then

$$\operatorname{dist}(w, \partial\Omega) \geqslant \operatorname{dist}(w, \partial G) = |\zeta - w| \sin\theta.$$

As $w \to \zeta$ along the bisector,

$$\limsup \frac{|w-\zeta|}{\operatorname{dist}(w, \partial\Omega)} \leqslant 1/\sin\theta,$$

and this for all $\theta < \pi/2$ yields

$$\limsup \frac{|w-\zeta|}{\operatorname{dist}(w, \partial\Omega)} \leqslant 1.$$

Since $|w-\zeta| \geqslant \operatorname{dist}(w, \partial\Omega)$, we have the desired result. □

Lemma 3.4.4. *Let Ω be a Jordan Domain, and let $\zeta \in \partial\Omega$, and let $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ be a conformal map. Then f extends continuously to the boundary and if $f'(\zeta)$ exists and is nonzero, then $\partial\Omega$ admits a tangent line at ζ.*

Proof. Continuous extension to the boundary follows from Theorem 1.3.2 (Carathéodory's). Define t such that $\zeta = f(e^{it})$ and consider the limit

$$\lim_{s\to t} \frac{f(e^{is}) - f(e^{it})}{s-t} = \lim_{s\to t} \frac{f(e^{is}) - f(e^{it})}{e^{is} - e^{it}} \cdot \frac{e^{is} - e^{it}}{s-t} = f'(e^{it})\, ie^{it}. \tag{3.4.4}$$

Observe that on the right we have a product of a nonzero quantity $f'(\zeta)$ with a unimodular constant. Since the limit exists and is nonzero we are justified in defining a vector v as

$$v = \lim_{s\to t} \operatorname{sign} \left(\frac{f(e^{is}) - f(e^{it})}{s-t} \right),$$

Then the line $L \stackrel{\text{def}}{=} \{\zeta + tv : t \in \mathbb{R}\}$ satisfies Definition 3.2.4 of a tangent line. □

Lemma 3.4.5. *Let $\Omega \subset \mathbb{C}$ be a Jordan domain. Suppose that for a conformal map $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ there exists a constant $L > 0$ such that*

$$\forall \epsilon > 0\, \exists \delta > 0 : \forall z \in \mathbb{D}, \quad 1 - |z| < \delta \implies \frac{\operatorname{dist}(f(z), \partial\Omega)}{1-|z|} < L + \epsilon. \tag{3.4.5}$$

Then $|f'| \leqslant L$ *in* $\mathbb{D}$.

Proof. Fix $\epsilon > 0$. Observe that from the Koebe covering theorem [25, Corollary 1.4] we get

$$|f'(z)| \leqslant \frac{4}{1+|z|}\frac{\operatorname{dist}(f(z),\partial\Omega)}{1-|z|} < \frac{4}{2-\delta}(L+\epsilon)$$

whenever $|z| > 1-\delta$. By the maximum principle, $|f'(z)| \leqslant \frac{4}{2-\delta}(L+\epsilon)$ for all $z \in \mathbb{D}$. Since δ can be made smaller without violating (3.4.5), we have $|f'(z)| \leqslant 2(L+\epsilon)$ for all $z \in \mathbb{D}$. Letting $\epsilon \to 0$ gives $|f'(z)| \leqslant 2L$ in $\mathbb{D}$. Then the derivative of f is bounded in $\mathbb{D}$, and this is the definition of the Hardy class H^∞. Hardy spaces are nested, see Remark 1.2.7, and in particular $f' \in H^\infty$ implies $f' \in H^1$. We may then apply Theorem 6.8 from [25], yielding that

$$f'(\zeta) \stackrel{\text{def}}{=} \lim_{z\to\zeta, z\in\overline{\mathbb{D}}} \frac{f(z)-f(\zeta)}{z-\zeta}$$

exists and is nonzero a.e. on $\partial\mathbb{D}$. If we prove that $|f'| \leqslant L$ a.e. on $\partial\mathbb{D}$, it will follow that $\|f'\|_{H^\infty} \leqslant L$, because the Hardy norm is equal to the Lebesgue norm when defined on the boundary by Remark 1.2.5.

Now choose a point $\zeta \in \partial\mathbb{D}$ where f' exists. Assume for the sake of contradiction that $|f'(\zeta)| > L$, then

$$\liminf_{z\to\zeta,\, z\in\mathbb{D}} \frac{|f(z)-f(\zeta)|}{1-|z|} \geqslant \lim_{z\to\zeta,\, z\in\mathbb{D}} \frac{|f(z)-f(\zeta)|}{|z-\zeta|} > L. \tag{3.4.6}$$

We know $\partial\Omega$ is smooth at $f(\zeta)$ because $0 < |f'(\zeta)| < \infty$, and therefore must admit a tangent line at ζ by Lemma 3.4.4.

Let $z \to \zeta$ so that $f(z) \to f(\zeta)$ along the interior normal line to the boundary, then

$$\frac{\operatorname{dist}(f(z),\partial\Omega)}{1-|z|} = \frac{\operatorname{dist}(f(z),\partial\Omega)}{|f(z)-f(\zeta)|}\frac{|f(z)-f(\zeta)|}{1-|z|}$$

where the first factor tends to 1 by Lemma 3.4.3. It follows from (3.4.6) that

$$\limsup_{z\to\zeta} \frac{\operatorname{dist}(f(z),\partial\Omega)}{1-|z|} > L.$$

This is in contradiction with the initial assumption of the lemma. $\square$

Theorem 3.4.6. *Let Ω satisfy the (R_O, R_S, R_C) condition in Definition* 3.2.5. *Then for any conformal map $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ fixing* 0 *we have*

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_S,R_C)}. \tag{3.4.7}$$

where F is as in Theorem 3.4.2.

Proof. First, observe that the boundary uniform interior disk condition with radius R_C implies that for any $\zeta \in \partial\mathbb{D}$, there exists a sector with vertex $f(\zeta)$, angular opening $\pi/2$, radius $R_C/2$, and containment in $\overline{\Omega}$. Then Ω satisfies what is referred to as an $(\alpha = \frac{1}{2}, r = R_C/2)$ *wedge condition* in [9], and the authors show this implies $f' \in H^p$ for all $p < 3/2$ in [9, Theorem 2]. They proceed to show that $\partial\Omega$ is rectifiable [9, Proposition on p.279].

Theorem 6.8 in [25] gives that sets of Lebesgue measure zero in $\partial\mathbb{D}$ correspond to sets of Lebesgue measure zero in $\partial\Omega$ under f. Additionally, the theorem states that almost everywhere in $\partial\mathbb{D}$, f' exists and is nonzero. By Lemma 3.4.4 this implies that for almost all $\zeta \in \partial\mathbb{D}$ a tangent line exists at $f(\zeta) \in \partial\Omega$.

Now, take any number $L > R_C e^{2F(R_O,R_S,R_C)}$ and any point $\zeta \in \partial\mathbb{D}$ where $f'(\zeta)$ exists. If we prove $|f'(\zeta)| \leqslant L$ it will show $|f'| \leqslant L$ a.e. on $\partial\mathbb{D}$ and then it will follow from Remark 1.2.5 regarding Hardy-Lebesgue equivalence on the boundary that $\|f'\|_{H^\infty} \leqslant L$.

It will suffice to show that:

$$\lim_{z\to\zeta} \frac{\operatorname{dist}(f(z),\partial\Omega)}{1-|z|} \leqslant L. \tag{3.4.8}$$

Choose $z \in \mathbb{D}$ such that such that $f(z)$ is along the normal vector to $\partial\Omega$ at $f(\zeta)$. Let

$d = \mathrm{dist}(f(z), \partial\Omega)$. Our goal is to estimate $\rho_\Omega(0, f(z))$, this will yield

$$\frac{d}{L} \leqslant 1 - |z| \tag{3.4.9}$$

for sufficiently small d proving (3.4.8).

Because there is a tangent line at $f(\zeta)$, by Lemma 3.3.1 there is a unique disk $D(a, R_C)$ such that $f(\zeta) \in \partial\Omega \cap \partial D(a, R_C)$ and $D(a, R_C) \subset \Omega$. By Lemma 3.4.3, if $f(z) \to f(\zeta)$ along the normal vector to $\partial\Omega$, then $|f(z) - f(\zeta)|/d \to 1$. Moreover, when $d < R_C$, the point $f(z)$ lies on the radius $[a, f(\zeta))$ of $D(a, R_C)$, and therefore $|f(z) - f(\zeta)| = d$ and $|f(z) - a| = R_C - d$. Keeping in mind that the hyperbolic metric is monotone with respect to domain and the formula for hyperbolic distance in a disk along a radius is (2.2.3), we have

$$\begin{aligned} \rho_\Omega(f(z), a) &\leqslant \rho_{D(a,R_C)}(f(z), a) \\ &= \frac{1}{2} \log \frac{R_C + |f(z) - a|}{R_C - |f(z) - a|} \\ &= \frac{1}{2} \log \frac{2R_C - d}{d} < \frac{1}{2} \log \frac{2R_C}{d}. \end{aligned} \tag{3.4.10}$$

Next we estimate $\rho_\Omega(a, 0)$ using the KP derived bound from Theorem 3.4.2:

$$\rho_\Omega(a, 0) \leqslant F(R_O, R_S, R_C). \tag{3.4.11}$$

Now suppose for the sake of contradiction that equation (3.4.9) is false, this implies

$$1 - |z| < d/L \quad \text{and} \quad 1 + |z| > 2 - d/L.$$

By the conformal invariance of the hyperbolic metric we know $\rho_\Omega(f(z), 0) = \rho_{\mathbb{D}}(z, 0)$, then

$$\rho_\Omega(f(z), 0) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|} > \frac{1}{2} \log \frac{2 - d/L}{d/L}. \tag{3.4.12}$$

Using the triangle inequality to combine this with equations (3.4.10) and (3.4.11) we get

$$\frac{1}{2}\log\frac{2-d/L}{d/L} < \frac{1}{2}\log\frac{2R_C}{d} + F(R_O, R_S, R_C)$$

which can be be rearranged to

$$L - \frac{d}{2} < R_C e^{2F(R_O,R_S,R_C)}.$$

But $R_C e^{2F(R_O,R_S,R_C)} < L$, so we have a contradiction when d is sufficiently small. This contradiction establishes that $|f'| \leqslant L$ almost everywhere on $\partial\mathbb{D}$ and completes the proof. $\square$

As with Theorem 2.5.1 in the previous chapter, there are some immediate consequences of Theorem 3.4.6 included as corollaries. The proofs are similar to those included for Corollaries 2.5.2 and 2.5.3 in Chapter 2.

Corollary 3.4.7. *Let Ω be a domain and z_0 a base point in Ω such that $\Omega_0 \overset{def}{=} \{z - z_0 \colon z \in \Omega\}$ satisfies the (R_O, R_S, R_C) condition in Definition* 3.2.5. *Then for any conformal map $f \colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ such that $f(0) = z_0$, we have*

$$\|f'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_S,R_C)}. \tag{3.4.13}$$

where F is as in Theorem 3.4.2.

Corollary 3.4.8. *Let Ω satisfy the (R_O, R_S, R_C) condition in Definition* 3.2.5. *Then for any conformal map $\varphi \colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$*

$$\|1/\varphi'\|_{H^\infty} \leqslant R_C e^{2F(R_O,R_S,R_C)}. \tag{3.4.14}$$

where F is as in Theorem 3.4.2.

Chapter 4

Lipschitz Estimates for Maps from Convex Domains onto the Unit Disk

4.1 Introduction

Thus far we have concerned ourselves with explicit expansion bounds for conformal maps $f\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ of the form $\|f'\|_{H^\infty} \leqslant M$, and for each expansion bound we get a compression bound of the form $\|1/f'\|_{H^\infty} \leqslant M$ for maps satisfying the same conditions, but going in the $\Omega \to \mathbb{D}$ direction.

In this chapter we turn to consider the opposing bounds. We will construct explicit expansion bounds for conformal maps $\varphi\colon \Omega \to \mathbb{D}$. We are not concerned with extruding corners of Ω because in this direction they give $\varphi'(z) \to 0$ as z approaches the vertex. Restricting to domains with no extruding corners was the major reason a curvature radius R_C was needed in the previous chapters, and here it can be dispensed with. We do however need to be concerned with intruding corners, where $\varphi'(z) \to \infty$, but this will be controlled by a requirement that Ω be convex.

4.2 Definitions and preliminary results

The following result can be found in a short paper by Robert Osserman [24, Corollary 1] where it is described as a well known, elementary consequence of the Schwarz Lemma.

Lemma 4.2.1 (Boundary Schwarz Lemma)**.** *Let $F\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function fixing* 0*. If F extends continuously to some boundary point b with $|b| = 1$ and $|F(b)| = 1$, and if $F'(b)$ exists, then $|F'(b)| \geqslant 1$ with equality only if F is a rotation.*

We next extend this result from self-mappings of the unit disk to more general self-mappings of Jordan domains.

Lemma 4.2.2 (Generalized Boundary Schwarz Lemma)**.** *Let Ω be a Jordan domain with boundary satisfying a $C^{1,\alpha}$-Hölder condition (as in Definition* 1.3.6*) in a neighborhood of $b \in \partial\Omega$, and let $F\colon \Omega \to \Omega$ be a holomorphic function fixing $a \in \Omega$ and b. Then either $|F'(b)| \geqslant 1$, or $F'(b)$ does not exist.*

Proof. Define a conformal map $G\colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ such that $G(a) = 0$ and $G(b) = 1$. Then the composition $G \circ F \circ G^{-1}$ maps the unit disk into itself fixing the points 0 and 1. The local form of Kellogg's theorem [26, Theorem 1] tells us that the derivative of G extends continuously to the boundary at b, and the derivative of G^{-1} extends continuously at 1. Assuming that $F'(b)$ exists, Lemma 4.2.1 gives

$$\left|(G \circ F \circ G^{-1})'(1)\right| \geqslant 1.$$

Applying the chain rule chain rule and observing that $|G'(b)|^{-1} = |[G^{-1}]'(1)|$ gives the desired result: $|F'(b)| \geqslant 1$. □

Definition 4.2.3 (Truncated Disk)**.** For positive numbers $r \leqslant R$, define the truncated disk

G and the chord Γ as follows:

$$G(R,r) = \{z \in D(0,R) : \text{Re } z < r\}$$
$$\Gamma(R,r) = \{z \in D(0,R) : \text{Re } z = r\}.$$

Observe that if $r = R$, then $\Gamma = \emptyset$ and $G = D(0,R)$.

In the following lemma we show that for a conformal map from $G(R,r)$ onto $\mathbb{D}$, the maximum modulus of the derivative restricted to the chord Γ is attained at the chords midpoint r. The inverse map seems to offer the most concise proof.

Lemma 4.2.4 (Maximum Modulus of Derivative on Γ attained at Midpoint)**.** *For* $0 < r < R$, *define* G *and* Γ *as in Definition* 4.2.3*. Let* $\varphi\colon G \xrightarrow{\text{onto}} \mathbb{D}$ *be a conformal map fixing* 0*. Then the derivative* φ' *extends continuously to* Γ*, and for all* $\zeta \in \Gamma : |\varphi'(\zeta)| \leqslant |\varphi'(r)|$.

Proof. Because $G(R,r)$ is a Jordan domain (Definition 1.3.1), Carathéodory's Theorem 1.3.2 tells us that φ extends continuously to the boundary. Since multiplication of φ by a unimodular constant would have no impact on the modulus of the derivative, we assume that $\varphi(r) = 1$. It is a consequence of the Schwarz reflection principle that the derivative extends continuously to the boundary segment Γ. Fix $\zeta \in \Gamma$, define $\psi\colon \mathbb{D} \xrightarrow{\text{onto}} G$ as φ^{-1}, and define $w \in \partial\mathbb{D}$ by $\varphi(\zeta) = w$ (see Figure 4.1). The statement we set out to prove, $|\varphi'(\zeta)| \leqslant |\varphi'(r)|$, can then be written in terms of the inverse function ψ as

$$|\psi'(w)| \geqslant |\psi'(1)|. \tag{4.2.1}$$

First, we claim that the function φ is symmetric with respect to the real axis. To prove this claim it is sufficient to show $\overline{\varphi(\bar{z})} = \varphi(z)$. Observe that the two conformal maps agree at one interior point and one boundary point: $\varphi(0) = \overline{\varphi(\overline{0})} = 0$ and $\varphi(r) = \overline{\varphi(\overline{r})} = 1$. Then by Corollary 1.3.4 $\overline{\varphi(\bar{z})} = \varphi(z)$. We conclude that φ and its inverse ψ are symmetric functions with respect to the real axis.

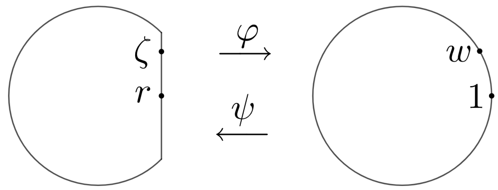

Figure 4.1: $G(R, r)$ is on the left with $\mathbb{D}$ on the right. The functions φ and ψ are conformal inverses each fixing 0, $\varphi(\zeta) = w$, and $\varphi(r) = 1$.

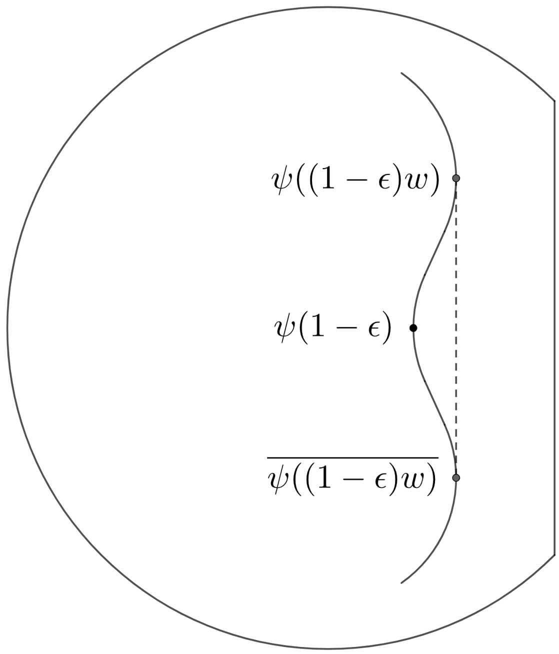

Figure 4.2: The solid curved line inside $G(R, r)$ is $\psi(\partial\mathbb{D}_\epsilon)$ which contains three points. The dotted line segment connecting $\psi((1-\epsilon)w)$ and $\overline{\psi((1-\epsilon)w)}$ illustrates the contradiction of convexity.

Now fix $\varepsilon > 0$ small and let $\mathbb{D}_\varepsilon \overset{\text{def}}{=} D(0, 1-\varepsilon)$. By the hereditary property of convexity for conformal maps [5, Ch. 2], $\psi(\mathbb{D})$ convex implies $\psi(\mathbb{D}_\varepsilon)$ is also convex. Because ψ is convex and symmetric with respect to the real axis, we may conclude that $\operatorname{Re}\{\psi((1-\varepsilon)w)\} \leqslant \operatorname{Re}\{\psi(1-\varepsilon)\}$. Observe that if $\operatorname{Re}\{\psi((1-\varepsilon)w)\} > \operatorname{Re}\{\psi(1-\varepsilon)\}$, then the segment connecting $\psi((1-\varepsilon)w)$ and $\overline{\psi((1-\varepsilon)w)}$ would not be contained in $\overline{\psi(\mathbb{D}_\varepsilon)}$ contradicting convexity (see Figure 4.2). Since both $\psi(w) = \zeta$ and $\psi(1) = r$ are points in Γ, $\operatorname{Re}\psi(w) =$

Re $\psi(1) = r$. It then follows that

$$\begin{aligned} \big|\psi(w) - \psi\big((1-\varepsilon)w\big)\big| &\geqslant \mathrm{Re}\,\big\{\psi(w) - \psi\big((1-\varepsilon)w\big)\big\} \\ &\geqslant \psi(1) - \psi(1-\varepsilon) \\ &= |\psi(1) - \psi(1-\varepsilon)|. \end{aligned}$$

Dividing both sides by ε and letting $\varepsilon \to 0$ gives the desired result. □

Recall from Definition 4.2.3 that for positive real numbers $r \leqslant R$, we defined the truncated disk by $G(R,r) = \{z \in D(0,R) \colon \mathrm{Re}\ z < r\}$ and the chord by $\Gamma(R,r) = \{z \in D(0,R) : \mathrm{Re}\ z = r\}$.

Lemma 4.2.5 (Explicit Formula for Derivative Modulus at Midpoint of Γ)**.** *Let* $\varphi\colon G(R,r) \xrightarrow{\text{onto}} \mathbb{D}$ *be a conformal map fixing* 0. *Define* $\theta = \arcsin(r/R)$ *and* $\alpha = 2\pi\theta/(\pi+2\theta)$. *Then* $|\varphi'(r)| = M(\theta)/R$ *where*

$$M(\theta) = \frac{2\alpha\cot\alpha}{\theta\cos\theta}. \tag{4.2.2}$$

Proof. Scaling $G(R,r)$ by a factor of $1/R$ makes $G \stackrel{\text{def}}{=} G(1, r/R)$, a truncated *unit* disk, which simplifies the calculations ahead. The function $M(\theta)$ will be shown to be the modulus of the derivative of a conformal map $G(1,r/R) \xrightarrow{\text{onto}} \mathbb{D}$ fixing 0, at the midpoint of $\Gamma(1, r/R)$. This proof consists of constructing the map explicitly as a composition of three maps, collecting derivative modulus factors from the chain rule at each step, and then dividing the result by R to account for the scaling. Because the composition is conformal, it will be the unique conformal map from G onto $\mathbb{D}$ fixing 0, up to multiplication by a unimodular constant.

The sequences a_k and b_k for $k = 0, 1, 2$ will be used to trace the image of 0 and the midpoint of the chord respectively. For each of the composed functions, we use the modulus of the derivative at the image of the midpoint, along with the chain rule to achieve the conclusion as follows

$$R \cdot |\varphi'(r)| = \left|[f_3 \circ f_2 \circ f_1]'(b_0)\right|$$
$$= |f_1'(b_0)| \cdot |f_2'(b_1)| \cdot |f_3'(b_2)| = M(\theta).$$

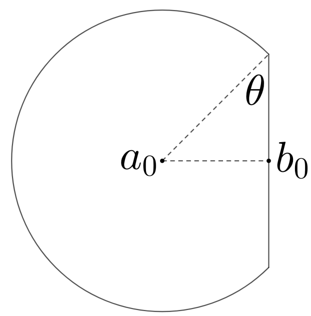

Figure 4.3: This is the scaled truncated disk G defined by $G(1, r/R)$. The larger arc is a subset of the unit circle, the vertical chord on right is the set $\{\mathrm{Re}\ z = r/R : z \in D(0,1)\}$. Observe that $a_0 = 0$, that $b_0 = r/R$. The dotted right triangle illustrates θ.

We start with $a_0 = 0$ and $b_0 = \sin\theta$ in the scaled domain (Figure 4.3), the first function

$$f_1(z) = \frac{\sin\theta + i\cos\theta - z}{z - \sin\theta + i\cos\theta}$$

maps G onto the set $\{0 < \arg z < \frac{\pi}{2} + \theta\}$. The modulus of $f_1'(b_0)$ is $2/\cos\theta$, $a_0 \mapsto a_1 = e^{2i\theta}$, and $b_0 \mapsto b_1 = 1$. Recalling that $\alpha = 2\pi\theta/(\pi + 2\theta)$, the second function

$$f_2(z) = z^{\alpha/\theta}$$

maps $\{0 < \arg z < \frac{\pi}{2} + \theta\}$ onto the upper half plane. The modulus $|f_2'(b_1)| = \alpha/\theta$, $a_1 \mapsto a_2 = e^{2i\alpha}$, and $b_1 \mapsto b_2 = 1$. The last function

$$f_3(z) = \frac{z - a_2}{z - \overline{a_2}}$$

takes the half plane to the unit disk, $f_3(a_2) = 0$ and $|f_3'(b_2)| = \cot\alpha$.

The modulus of the derivative of the composition can be expressed as a function of θ in the form

$$M(\theta) = \frac{2}{\cos\theta} \cdot \frac{\alpha}{\theta} \cdot \cot\alpha.$$

It follows that for any conformal map taking the (unscaled) domain $G(R, r)$ to the unit disk while fixing 0, the modulus of the derivative at the midpoint of the chord Γ is $M(\theta)/R$. $\square$

Lemma 4.2.6. *If* $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$, *then* $M(\theta_1) > M(\theta_2)$.

Proof. For $i = 1, 2$ define $R_i = \csc\theta_i$ so that $R_1 > R_2$ and $M(\theta_i) = M(\arcsin(1/R_i))$. We will prove $M(\theta_1) > M(\theta_2)$ using properties of the nested domains $G(R_2, 1) \subset G(R_1, 1)$ (notation introduced in Definition 4.2.3). We define three conformal mappings each fixing the points 0 and 1: $\varphi_i \colon G(R_i, 1) \to \mathbb{D}$ and $f \colon G(R_1, 1) \to G(R_2, 1)$. Then $\varphi_1 = \varphi_2 \circ f$ by Corollary 1.3.4 because they agree on an interior point and a boundary point, and in particular $|\varphi_1'(1)| = |\varphi_2'(1)| \cdot |f'(1)|$.

Since the point 1 is the midpoint of the boundary chord in both domains $G(R_i, 1)$, we can use the explicit formula from Lemma 4.2.5 to see that $|\varphi_i'(1)| = M(\theta_i)/R_i$. Applying the boundary Schwarz lemma (Lemma 4.2.2) to the mapping f with $a = 0, b = 1$ assures that $|f'(1)| \geqslant 1$ and we can write

$$M(\theta_1)/R_1 \geqslant M(\theta_2)/R_2.$$

Observing that $R_1 > R_2$, we can conclude $M(\theta_1) > M(\theta_2)$. $\square$

4.3 Global expansion bound for maps from a convex domain onto the disk

In this section we prove the main result of the chapter (Theorem 4.3.2). The proof requires a lemma about continuity of radii.

Lemma 4.3.1. *Let $(\Omega_n)_{n=1}^{\infty}$ be a sequence of domains containing* 0 *such that $\Omega_n \subset \Omega_{n+1}$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Then $R_I(\Omega_n) \to R_I(\Omega)$ and $R_O(\Omega_n) \to R_O(\Omega)$.*

Proof. Observe that $R_I(\Omega_n)$ is increasing in n and bounded by $R_I(\Omega)$, thus $R_I(\Omega_n)$ converges to some number less than or equal to $R_I(\Omega)$. Choose $r < R_I(\Omega)$, then $(\Omega_n)_{n=1}^{\infty}$ is an open cover of the compact set $\overline{D(0,r)}$. There exists a finite subcover, which means there exists N such that $\bigcup_{n=1}^{N} \Omega_n$ is a finite subcover, and it follows that $R_I(\Omega_N) \geqslant r$. For any $r < R_I(\Omega)$ then, the limit of $R_I(\Omega_n)$ as $N \to \infty$ is at least r. It follows that $R_I(\Omega_n) \to R_I(\Omega)$ as $n \to \infty$.

Observe that $R_O(\Omega_n)$ is increasing in n and bounded by $R_O(\Omega)$, thus $R_O(\Omega_n)$ converges to some number less than or equal to $R_O(\Omega)$. Assume for the sake of contradiction that $R_O(\Omega_n)$ converges to some number $R < R_O(\Omega)$. Then there exists $z \in \Omega$ with $|z| > R$, and since the Ω_n exhaust Ω, there must exist N such that $z \in \Omega_N$. This would imply that $R_O(\Omega_N) > R$. This is a contradiction, proving that $R_O(\Omega_n) \to R_O(\Omega)$ as $n \to \infty$. $\square$

Theorem 4.3.2. *Suppose Ω is a bounded convex domain in $\mathbb{C}$ containing* 0 *with $R_I \overset{def}{=} \sup\{r > 0 : D(0,r) \subset \Omega\}$ and $R_O \overset{def}{=} \inf\{r > 0 : \Omega \subset D(0,r)\}$. Let $f \colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ be a conformal map fixing* 0*. Then*

$$\sup_{z\in\Omega} |f'(z)| \leqslant \frac{1}{R_O} M\big(\arcsin(R_I/R_O)\big)$$

with equality attained if $\Omega = G(R_0, R_I)$ up to multiplication by a unimodular constant.

Proof. First, suppose $\partial\Omega$ is C^∞ smooth. By Kellogg's Theorem (Theorem 1.3.7) f' extends continuously to the boundary. Fix $\zeta \in \partial\Omega$. Since Ω is convex there exists a line l through ζ such that $l \cap \Omega = \emptyset$. Define G to be the part of the disk $D(0, R_O)$ that contains 0 and lies on one side of the line l; define $r = \operatorname{dist}(0, l)$. Now $G = G(R_O, r)$ up to multiplication by a unimodular constant, and we should observe that $\Omega \subset G$, $\zeta \in \partial\Omega \cap \partial G$, and $R_I \leqslant r \leqslant R_O$.

If $r = R_O$, define a conformal map $h \colon D(0, R_O) \xrightarrow{\text{onto}} \Omega$ fixing 0 and ζ. Then h is a conformal self mapping of $D(0, R_O)$ into itself, and by the general boundary Schwarz lemma (Lemma 4.2.2) $|h'(\zeta)| \geqslant 1$. Define $g \colon D(0, R_O) \xrightarrow{\text{onto}} \mathbb{D}$ by $g(z) = z/R_O$. Observe that by Corollary 1.3.4 which says two conformal maps agreeing on an interior point (0) and a boundary point (ζ) are equal, there exists $t \in [0, 2\pi)$ such that $f = e^{it}[g \circ h^{-1}]$. It follows that $|f'(\zeta)| = 1/(R_O|h'(\zeta)|)$, and that $|f'(\zeta)| \leqslant 1/R_O$. Finally observe that in $0 < \theta < \pi/2$, $M(\theta)$ is strictly decreasing by Lemma 4.2.6, and using L'Hôpital's rule one can show that

$$\lim_{\theta \to \frac{\pi}{2}} M(\theta) = \lim_{\theta \to \frac{\pi}{2}} \frac{4\pi}{(\pi + 2\theta)\sin(\frac{2\pi\theta}{\pi+2\theta})} \cdot \frac{\cos(\frac{2\pi\theta}{\pi+2\theta})}{\cos\theta} = 1.$$

We conclude that $M(\theta) > 1$ on $(0, \pi/2)$, and thus $|f'(\zeta)| \leqslant (1/R_O)M(\arcsin(R_I/R_O))$ when $r = R_O$.

Now assume $r < R_O$. Let $g\colon G \to \Omega$ be a conformal mapping fixing 0 and ζ, and let $\varphi\colon G \to \mathbb{D}$ be a conformal mapping fixing 0. Then $\varphi = f \circ g$ up to multiplication by a unimodular constant and by the chain rule $|\varphi'(\zeta)| = |f'(\zeta)| \cdot |g'(\zeta)|$. Lemma 4.2.5 shows that $|\varphi'(r)| = \frac{1}{R_O}M(\arcsin(r/R_O))$ and Lemma 4.2.4 shows $|\varphi'(\zeta)| \leqslant |\varphi'(r)|$. Combining these, we conclude

$$|\varphi'(\zeta)| \leqslant \frac{1}{R_O}M(\arcsin(r/R_O)).$$

Using the fact that $r/R_O \geqslant R_I/R_O$ and applying the monotonicity result in Lemma 4.2.6, we see that $M(\arcsin(r/R_O)) \leqslant M(\arcsin(R_I/R_O))$. So far we have

$$\frac{1}{R_O}M(\arcsin(R_I/R_O)) \geqslant |\varphi'(\zeta)| = |g'(\zeta)| \cdot |f'(\zeta)|.$$

Applying the boundary Schwarz lemma (Lemma 4.2.2) to g gives $|g'(\zeta)| \geqslant 1$ which yields $|f'(\zeta)| \leqslant \frac{1}{R_O}M(\arcsin(R_I/R_O))$ for $r < R_O$. We have now shown that for an arbitrary choice of $\zeta \in \partial\Omega$, $|f'(\zeta)| \leqslant \frac{1}{R_O}M(\arcsin(R_I/R_O))$, and the maximum principle extends this to all $z \in \Omega$.

Next, we drop the assumption that $\partial\Omega$ is smooth. Fix $\varepsilon \in (0, 1)$. Make the definitions $\Omega_\varepsilon = f^{-1}(D(0, 1-\varepsilon))$ and $F_\varepsilon\colon \Omega_\varepsilon \to \mathbb{D}$, $F_\varepsilon = (1-\varepsilon)^{-1}f\big|_{\Omega_\varepsilon}$. Choose an arbitrary $z \in \partial\Omega_\varepsilon$.

Since $\partial\Omega_\varepsilon$ is smooth we know

$$|F_\varepsilon'(z)| \leqslant \frac{1}{R_O(\Omega_\varepsilon)} M\big(\arcsin(R_I(\Omega_\varepsilon)/R_O(\Omega_\varepsilon))\big)$$

and it follows that

$$|f'(z)| \leqslant (1-\varepsilon)^{-1}|f'(z)| = |F_\varepsilon'(z)| \leqslant \frac{1}{R_O(\Omega_\varepsilon)} M\big(\arcsin(R_I(\Omega_\varepsilon)/R_O(\Omega_\varepsilon))\big).$$

Here the right hand side has no dependence on z, so we can take the supremum over all $z \in \partial\Omega_\varepsilon$ on the left. Then by the maximum principle we can extend this to

$$\sup_{z\in\Omega_\varepsilon} |f'(z)| \leqslant \frac{1}{R_O(\Omega_\varepsilon)} M\big(\arcsin(R_I(\Omega_\varepsilon)/R_O(\Omega_\varepsilon))\big).$$

Observe that by Lemma 4.3.1, $R_I(\Omega_\varepsilon) \to R_I(\Omega)$ and $R_O(\Omega_\varepsilon) \to R_O(\Omega)$ as $\varepsilon \to 0$. Looking at (4.2.2), the continuity $M(\theta)$ in $(0, \pi/2)$ is transparent from the formula. Then as $\varepsilon \to 0$ we get

$$\sup_{z\in\Omega} |f'(z)| \leqslant \frac{1}{R_O} M\big(\arcsin(R_I/R_O)\big). \qquad \square$$

Again there are two immediate corollaries. The first formalizes the translation invariance of the derivative, and hence the bound in Theorem 4.3.2. The second interprets the theorem in the reverse direction, as a compression bound for maps from the disk onto convex domains.

Corollary 4.3.3. *Suppose Ω is a bounded convex domain in $\mathbb{C}$ containing z_0 with $R_I \overset{def}{=} \sup\{r > 0 : D(z_0, r) \subset \Omega\}$ and $R_O \overset{def}{=} \inf\{r > 0 : \Omega \subset D(z_0, r)\}$. Let $f\colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ be a conformal map such that $f(z_0) = 0$. Then*

$$\sup_{z\in\Omega} |f'(z)| \leqslant \frac{1}{R_O} M\big(\arcsin(R_I/R_O)\big).$$

Corollary 4.3.4. *Suppose Ω is a bounded convex domain in $\mathbb{C}$ containing 0 with $R_I \overset{def}{=}$*

$\sup\{r > 0 : D(0,r) \subset \Omega\}$ *and* $R_O \overset{def}{=} \inf\{r > 0 : \Omega \subset D(0,r)\}$. *Let* $g\colon \mathbb{D} \xrightarrow{\text{onto}} \Omega$ *be a conformal map fixing* 0. *Then*

$$\|1/g'(z)\|_{H^\infty} \leqslant \frac{1}{R_O} M\big(\arcsin(R_I/R_O)\big).$$

4.4 Counterexample to continuity of the derivative

The possibility that conclusion in Theorem 4.3.2 could be strengthened to include a continuous extension of f' to the boundary was considered. The following counterexample shows this cannot be done without strengthening the assumptions as well.

Example 4.4.1. Let $D = D(\frac{1}{3}, \frac{2}{3})$. We construct Ω such that $D \subset \Omega \subset \mathbb{D}$ by choosing a countable, strictly clockwise sequence $\{\zeta_n\}_{n=1}^{\infty}$ on the unit circle accumulating at 1, and each connected to the next by a segment. The idea is to construct a convex domain with a countably infinite number of extruding corners accumulating at 1. Let $h\colon D \xrightarrow{\text{onto}} \mathbb{D}$ and $g\colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ be conformal maps, each fixing 0 and 1.

Suppose for the sake of contradiction that g' is continuous on $\partial\mathbb{D}$. It is a consequence of Theorem 3.9 in Pommerenke's book [25] that in each corner we have $g'(\zeta_n) = 0$. Then by our assumption we have $g'(1) = 0$.

The composition $g \circ h^{-1}$ is a conformal mapping of $\mathbb{D}$ into itself fixing 0 and 1. We apply Lemma 4.2.1 to conclude that $\big|[g \circ h^{-1}]'(1)\big| \geqslant 1$. Using the chain rule, this implies $|g'(1)| \geqslant |h'(1)|$. To achieve a contradiction it is enough to show $|h'(1)| > 0$ since this will imply $|g'(1)| > 0$. Observe that the conformal map $h\colon D \to \mathbb{D}$ fixing 0 and 1 is unique and easily constructed. If we let $f_1 = \frac{3}{2}[z - \frac{1}{3}]$ and $f_2 = \frac{z+.5}{1+.5z}$ then $h = f_2 \circ f_1$ where f_1 translates and rescales the disk and f_2 is a Möbius transformation assuring that $0 \mapsto 0$. The derivative of the composition at 1 is $\frac{1}{2}$.

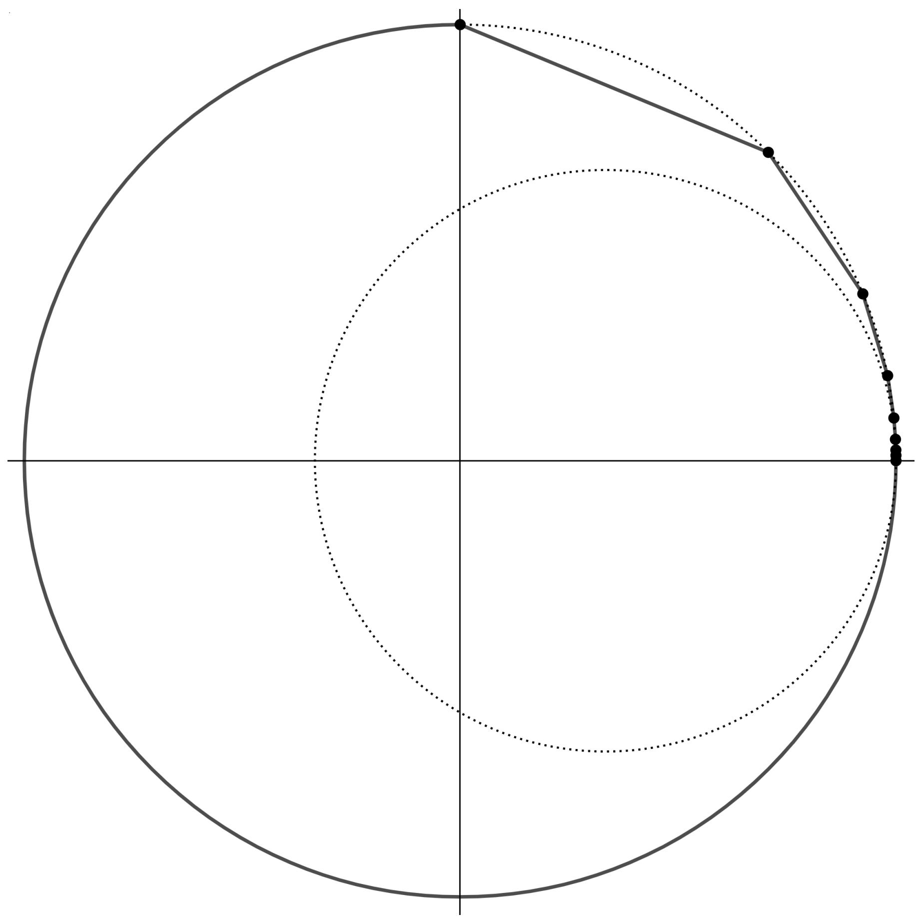

Figure 4.4: Counterexample to continuity of the derivative of g. The smaller dotted disk is $D \stackrel{\text{def}}{=} D(\frac{1}{3}, \frac{2}{3})$, the larger is $\mathbb{D}$, and $\partial\Omega$ has a solid line boundary with $D \subset \Omega \subset \mathbb{D}$. The corners of Ω accumulate at 1, the derivative of $g \colon \Omega \xrightarrow{\text{onto}} \mathbb{D}$ is 0 in each corner, and it is shown in Example 4.4.1 that $|g'(1)| \geqslant \frac{1}{2}$.

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