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On relevant deformations in Open String Field Theory

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I wish to thank my advisor, Martin Schnabl, for his academic guidance, countless inspiring discussions and for giving me the opportunity to work with him. This thesis is dedicated to my family for their unwavering support, especially to my kind inspiring parents and to my beloved Léňa.

Title: On relevant deformations in Open String Field Theory

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Abstract: String field theory takes string perturbation theory off-shell and since the different vacua of string theory are described by conformal field theories, it is tempting to use string field theory to describe conformal perturbation theory. We use open string field theory to reproduce the known leading order results of boundary conformal perturbation theory and we also perform the next-to-leading order calculation of the boundary degeneracy g for a generic theory. This is achieved by finding perturbative solutions to the classical equations of motion of open string field theory corresponding to weakly relevant deformations. The observables of conformal perturbation theory are then given by calculating the on-shell action or by invoking the Kudrna-Maccaferri-Schnabl correspondence. We also lay the groundwork for the investigation of the shift in the boundary spectrum. This work is largely based on the collaboration with Martin Schnabl [1].

Keywords: Conformal field theory, string field theory, relevant deformations, conformal perturbation theory

Název práce: Relevantní deformace v teorii pole otevřených strun

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Abstrakt: Strunová teorie pole umožňuje rozšířit strunovou poruchovou teorii off-shell a díky tomu, že různá vakua v teorii strun jsou popsaná konformními teoriemi pole, tak je lákavé použít strunovou teorii pole k popisu konformní poruchové teorie. Skrze teorii pole otevřených strun reprodukujeme známé výsledky z konformní poruchové teorie s hranicí v prvním netriviálním řádu a dále počítáme degeneraci hranice g do druhého netriviálního řádu pro generickou teorii. Tyto výpočty provádíme skrze hledání poruchových řešení klasických pohybových rovnic teorie pole otevřených strun odpovídajících relevantním deformacím. Pozorovatelné konformní poruchové teorie jsou poté vypočteny z hodnoty on-shell akce či skrze Kudrna-Maccaferri-Schnabl korespondenci. Také pokládáme základy pro zkoumání změn ve spektru operátorů na hranici. Tato práce je z velké části založena na kolaboraci s Martinem Schnablem [1].

Klíčová slova: Konformní teorie pole, strunová teorie pole, relevantní deformace, konformní poruchová teorie

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Introduction

Field theories have been at the heart of the development of physical theories for more than a hundred years. Their natural course of development included lifting the classical field theories to quantum field theories (QFTs). A very special class of QFTs are conformal field theories (CFTs), which possess conformal invariance, that is invariance under local rescalings. CFTs naturally arise as signposts in the landscape of QFTs, which are present at the ends of RG flows since the theories at RG flow endpoints have scaling symmetry and thanks to Polyakov's conjecture it is enhanced to the full conformal symmetry under very general conditions. In two dimensions the study of CFTs is particularly rich since the symmetry algebra becomes infinite-dimensional. A natural question is whether one can understand the off-critical theories using CFT data or whether one can use the CFT data of one CFT to characterise new CFTs. It is conformal perturbation theory that aims at answering this question in a perturbative setting.

It is natural to ask whether there are any extended objects that can be added to the bulk CFTs, such objects could embody for example certain defects and impurities. An example of such extended objects are conformal boundaries, which break the least amount of conformal symmetry, and the study of CFTs with their presence is called boundary conformal field theory (BCFT). The classification of all conformal boundary conditions that can be added to a fixed CFT is an outstanding problem. The boundary states and the spectra of boundary fields that live on the conformal boundaries have to satisfy stringent consistency conditions, which thankfully can sometimes lead to the classifications of all BCFTs with a fixed CFT in the bulk. One could again try to find new conformal boundary conditions by having a BCFT that one understands and perturb it along the boundary. Analogously to the bulk case, it is boundary conformal perturbation theory that sheds light on this problem.

Just as the classification of conformal boundary conditions is difficult, so is conformal perturbation theory both in the bulk and on the boundary. Very often only leading order results are available, which is in part due to the not so practical point-splitting regularisation that is used. This is where string theory comes in. The different CFTs act as different vacua of string theory and one might hope that conformal perturbation theory can be rephrased as a problem in string perturbation theory. We run into a problem with the conventional string perturbation theory in that it only describes the scattering of on-shell states, which would correspond only to conformal perturbation theory of marginal deformations. Clearly a framework that can take string theory off-shell is of interest.

Now the problems of classifying boundary conditions and taking string theory off-shell beautifully come together in a very particular framework called open string field theory (OSFT). It is a field theory formulated on a fixed boundary condition, but can describe all other boundary conditions as solutions to its classical equations of motion. Such solutions are string fields and the KMS correspondence enables one to find the corresponding boundary states. What is less clear is how the boundary spectrum of the BCFT described by a given string field looks like. OSFT also consistently takes the theory of open strings off-shell and so one can hope that it can take boundary conformal perturbation theory beyond the leading order. This is indeed the case since for example one does not need to use point-splitting regularisation but can interpret the divergences as arising from the propagation of tachyons and zero modes, something which one knows how to handle in OSFT.

In this thesis we construct perturbative solutions to the OSFT equations of motion corresponding to relevant deformations. These solutions are investigated in the near-marginal limit where contact with boundary conformal perturbation theory is made. We reproduce the leading order results and even extend the computation of the entropy associated with a given boundary condition to the next-to-leading order. The first step towards understanding the boundary spectrum of the BCFT described by the perturbative solution is made. In particular we find that the perturbing operator disappears from the spectrum in line with RG intuition. This work is based on the collaboration with Martin Schnabl and will be submitted to PRL [1].

The thesis is organised as follows. First we present four introductory chapters. In chapter (1), we introduce the basics of scalar QFT and investigate some consequences of formulating perturbation theory around an unstable vacuum. Then some RG ideas are sketched. After this in (2) we provide a sufficiently detailed introduction to bulk CFTs, including bulk conformal perturbation theory. At the end of this chapter, we introduce the unitary minimal models and their perturbations together with CFTs that are key for string theory, namely the free boson and the *bc*-ghost system. Then in (3), we briefly introduce the basic objects of study in BCFT, the various consistency conditions and properties and then finish with boundary conformal perturbation theory. The chapter (4) presents the general framework of Witten's OSFT and the standard analytic techniques used to solve the classical equations of motion. It ends with a discussion of two classes of classical solutions, namely the tachyon vacuum class of solutions and the exactly marginal perturbations, on which we build the intuition for more general perturbative solutions. The last chapter (5) presents the original research. We begin by sketching the general setup for investigating perturbative solutions and through the OSFT classical action elegantly compute the shift in the boundary degeneracy g to the next-to-leading order. Going beyond the leading order requires fixing a gauge and we fix Siegel gauge. The KMS correspondence also enables us to find the leading order shift in the general boundary state coefficient. Then we investigate the perturbed boundary spectrum. Lastly we explore two alternative gauge conditions, namely Schnabl gauge and pseudo-Schnabl gauge to third order in string perturbation theory. To do so, we have to learn how to manipulate oblique projectors and handle a zero-mode present in the Schnabl gauge calculation. We reproduce the leading order shift in g through the Kudrna-Maccaferri-Schnabl (KMS) correspondence, giving a nontrivial check on the Schabl gauge solution. We also reproduce this using pseudo-Schnabl gauge but we think this is a fluke since the presented pseudo-Schnabl gauge string field violates the equations of motion. We include two appendices, in (A) we present correlators useful in OSFT and in (B) we investigate the original Kaluza-Klein theory to gain a little practice with extra dimensions that show up in string theory and present an interesting instanton.

Chapter 1

An introduction to Quantum Field Theory

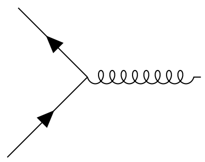
In this chapter we provide a brief introduction to quantum field theory (QFT) as necessary for providing the wider context into which later chapters belong. It goes without saying that to gain proper exposure, one should consult monographies on the subject and luckily today many are available. The monographies we find most exhaustive are Weinberg's three volumes [2, 3, 4] for the particle physics setting and Zinn-Justin [5] for applications outside of particle physics. In this chapter we draw most inspiration from the brief book of Banks [6] and the lectures of Coleman [7]. As a great first book on the subject we recommend Zee [8]. Other widely used books include [9, 10, 11, 12]. We follow the path integral approach which is nicely summarized in the appendix of [13]. As for the more specialised sections, the section on Euclidean field theory is essentially taken out of [14] and the topic is also nicely covered in [15]. The section on large order behavior of perturbation series and instantons is inspired by [16]. Renormalisation group ideas are mainly inherited from the reviews [17, 18] and the already mentioned book of Banks, see also [19] for further reading. We note that the mostly plus signature is used in contrast to most QFT literature. We set units to $\hbar = c = 1$.

1.1 Motivation

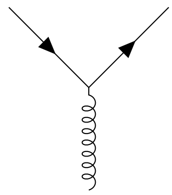
We begin by outlining the motivation for QFT. Following the historic route we wish to show that QFT naturally emerges when one considers physical situations in which both quantum mechanical (QM) and relativistic effects have to be taken into account. Naively one might expect such a theory to be rather simple since adding Lorentz invariance might simplify matters in analogy to how spherical symmetry does. To show why this is not the case, we first consider a thought experiment by Bohr. Suppose we have a box with a piston and a particle inside. We then try to locate the particle by squeezing it with the piston and eventually we locate it within its Compton wavelength λ. By the defining relation of λ

$$m = \frac{2\pi}{\lambda}, \tag{1.1}$$

where m is the particle's mass, we see that the uncertainty in momentum implied by Heisenberg's uncertainty principle is roughly of the order where pair production is energetically allowed. The uncertainty in momentum thus becomes uncertainty in particle number and so our theory must make creation and annihilation of particles possible. This requirement can also be seen in the theory of scattering. There what looks like off-shell propagation leading to scattering on an external potential (turned on only for a brief moment) in one frame can look like pair production in a boosted frame, see figure (1.1).



(a) Scattering before boost



(b) Pair production after boost

Figure 1.1 Scattering boosts into pair production

This situation can happen because in QFT propagators may be nonzero for spacelike propagation but one shouldn't confuse it with on-shell propagation.

Having an uncertainty in particle number is not the only complication that combining QM with relativity enforces. From relativity we know that information cannot travel faster than light and this is in conflict with the fact that in QM an observer can measure any observable of choice. The conflict arises because the noncommutative nature of observables implies that observation itself may have a physical effect. So having access to an observable in a distant experiment, one may affect the physics there superluminally. There is a way out, however. We can associate observables to regions of spacetime so that for any two spacelike separated regions R_1 and R_2 we have $[\mathcal{O}_1, \mathcal{O}_2] = 0$ where $\mathcal{O}_1$ can be measured in the region R_1 and $\mathcal{O}_2$ in R_2. Having encountered Maxwell's theory we know that there this is realized on a classical level by introducing local fields $A_\mu(x)$ with proper equations of motion. So we attempt to build our observables out of operator valued quantum fields $\phi(x)$, which satisfy $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 > 0$ ensuring the analogous condition on observables. We do not a priori give these fields direct physical meaning, for us they serve the role of auxiliary objects out of which physical observables are built.

1.2 Multiparticle states

In the last section we showed a need to build a state spaces which can accommodate a variable number of particles. This state space was introduced by Fock and is called the Fock space. It is defined as the direct sum $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$, where $\mathcal{H}_k$ is a k-particle Hilbert space containing three-momentum eigenstates $|\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\rangle$. In Fock space we introduce an inner product

$$\langle \boldsymbol{p}_1, \dots, \boldsymbol{p}_k | \boldsymbol{q}_1, \dots, \boldsymbol{q}_l \rangle = \delta_{kl} \sum_{\sigma} \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_{\sigma(1)}) \dots \delta^3(\boldsymbol{p}_k - \boldsymbol{q}_{\sigma(k)}), \tag{1.2}$$

where we specialised to bosonic statistics. The zero particle space $\mathcal{H}_0$ contains the unique normalised state $|0\rangle$ such that $\langle 0|0\rangle = 1$. By the definition (1.2) we realise that $|\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\rangle$ and $|\boldsymbol{p}_\sigma(1), \dots, \boldsymbol{p}_\sigma(k)\rangle$ are equivalent. Making this manifest, we simplify a lot of our computations and it is done so by introducing creation operators $a^\dagger(\boldsymbol{p})$ such that

$$|\boldsymbol{p}_1, \dots, \boldsymbol{p}_k\rangle = a^\dagger(\boldsymbol{p}_1) \dots a^\dagger(\boldsymbol{p}_k) |0\rangle, \tag{1.3}$$

where $\left[a^\dagger(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right] = 0$. One can check that (1.2) is correctly reproduced if for the hermitian conjugates $a(\boldsymbol{p})$ one has

$$\left[a(\boldsymbol{p}), a^\dagger(\boldsymbol{q})\right] = \delta^3(\boldsymbol{p} - \boldsymbol{q}), \tag{1.4}$$

with $a(\boldsymbol{p}) |0\rangle = 0$. The operators $a(\boldsymbol{p})$ are then called annihilation operators and they commute $[a(\boldsymbol{p}), a(\boldsymbol{q})] = 0$. To see that the definition (1.3) is correct, we make an explicit computation of $\langle \boldsymbol{p}_1, \boldsymbol{p}_2 | \boldsymbol{q}_1, \boldsymbol{q}_2 \rangle$

$$\begin{aligned}
\langle \boldsymbol{p}_1, \boldsymbol{p}_2 | \boldsymbol{q}_1, \boldsymbol{q}_2 \rangle &= \langle 0| a(\boldsymbol{p}_1) a(\boldsymbol{p}_2) a^\dagger(\boldsymbol{q}_1) a^\dagger(\boldsymbol{q}_2) |0\rangle \\
&= \langle 0| a(\boldsymbol{p}_1) a^\dagger(\boldsymbol{q}_1) a(\boldsymbol{p}_2) a^\dagger(\boldsymbol{q}_2) |0\rangle + \\
&\quad \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_1) \langle 0| a(\boldsymbol{p}_1) a^\dagger(\boldsymbol{q}_2) |0\rangle \\
&= \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_2) \langle 0| a(\boldsymbol{p}_1) a^\dagger(\boldsymbol{q}_1) |0\rangle + \\
&\quad \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2) \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_1) \\
&= \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_1) \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_2) + \\
&\quad \delta^3(\boldsymbol{p}_1 - \boldsymbol{q}_2) \delta^3(\boldsymbol{p}_2 - \boldsymbol{q}_1)
\end{aligned}$$

Since $a^\dagger(\boldsymbol{p})a(\boldsymbol{p})$ acts as particle number density in momentum space, we may equip $\mathcal{H}$ with generators of the Lorentz group. For example we have for the four-momentum

$$P^\mu = \int \mathrm{d}^3\boldsymbol{p}\, p^\mu a^\dagger(\boldsymbol{p})a(\boldsymbol{p}) \tag{1.5}$$

The states we have created so far do not have simple transformation properties under the action of the Lorentz group, which is represented on $\mathcal{H}$ by unitary operators $U(\Lambda)$, where Λ belongs to the Lorentz group. It is convenient to work with states $|p_1 \dots p_k\rangle$, where the momenta are now four-momenta. Up to a conventional factor of 2π, the correct normalisation is provided by taking

$$|p\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\boldsymbol{p}}}\,|\boldsymbol{p}\rangle\,, \tag{1.6}$$

where $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ and m is the mass of the particles our momentum eigenstates represent. To see this we start by building a Lorentz invariant measure on the mass-shell hyperbola $p^2 + m^2 = 0$, $p^0 > 0$. Since the sign of the time component of a timelike vector is Lorentz invariant and p^2 is also Lorentz invariant, the product $\delta(p^2+m^2)\theta(p^0)$ is Lorentz invariant and restricts p to lie on the future mass-shell hyperbola. The Lorentz invariant measure is then $\mathrm{d}^4p\,\delta(p^2+m^2)\theta(p^0)$ and with the definition (1.6) we have the completeness relation

$$1 = \int \mathrm{d}^3\boldsymbol{p}\,|\boldsymbol{p}\rangle\langle\boldsymbol{p}| = \frac{1}{(2\pi)^3}\int \mathrm{d}^4p\,\delta(p^2+m^2)\theta(p^0)\,|p\rangle\langle p| \tag{1.7}$$

This is so by

$$\int_{-\infty}^{\infty}\mathrm{d}p^0\,\left(\mathrm{d}^3\boldsymbol{p}\,\delta(p^2+m^2)\theta(p^0)\right) = \frac{\mathrm{d}^3\boldsymbol{p}}{2\omega_{\boldsymbol{p}}} \tag{1.8}$$

and writing

$$\begin{aligned}
\delta(p^2+m^2) &= \delta(-(p^0)^2 + \omega_{\boldsymbol{p}}^2)\\
&= \delta\left[(\omega_{\boldsymbol{p}} - p^0)(\omega_{\boldsymbol{p}} + p^0)\right]\\
&= \frac{\delta(\omega_{\boldsymbol{p}} - p^0)}{2\omega_{\boldsymbol{p}}} + \frac{\delta(\omega_{\boldsymbol{p}} + p^0)}{2\omega_{\boldsymbol{p}}},
\end{aligned}$$

which follows from standard delta function identities and only one sign is chosen by the step function $\theta(p^0)$. Since the measure on the mass-shell hyperbola transforms as

$$\mathrm{d}^4p\,\delta(p^2+m^2)\theta(p^0) \to \mathrm{d}^4\Lambda p\,\delta(p^2+m^2)\theta(p^0) \tag{1.9}$$

under Lorentz transformations, one has from (1.7)

$$U(\Lambda)\,|p_1,\dots,p_k\rangle = |\Lambda p_1,\dots,\Lambda p_k\rangle \tag{1.10}$$

The kets $|p_1, \ldots, p_k\rangle$ also transform very simply under translations since they are momentum eigenstates

$$U(T_x)\,|p_1, \ldots, p_k\rangle = e^{-i\sum_j p_j \cdot x}\,|p_1, \ldots, p_k\rangle\,, \tag{1.11}$$

where x is a four-vector. Taking (1.10) and (1.11) we know how $|p_1, \ldots, p_k\rangle$ transform under the action of the Poincaré group. From the definition (1.6) it is clear that the creation and annihilation operators in the relativistic normalisation should be

$$\alpha^\dagger(p) = \sqrt{2\pi^3}\sqrt{2\omega_{\boldsymbol{p}}}\,a^\dagger(\boldsymbol{p}) \tag{1.12}$$
$$\alpha(p) = \sqrt{2\pi^3}\sqrt{2\omega_{\boldsymbol{p}}}\,a(\boldsymbol{p}) \tag{1.13}$$

These inherit transformation properties from (1.10) and (1.11)

$$\alpha^\dagger(\Lambda p) = U(\Lambda)\alpha^\dagger(p)U^\dagger(\Lambda) \tag{1.14}$$
$$\alpha(\Lambda p) = U(\Lambda)\alpha(p)U^\dagger(\Lambda) \tag{1.15}$$
$$e^{-ip\cdot x}\alpha^\dagger(p) = e^{-iP\cdot x}\alpha^\dagger(p)e^{iP\cdot x} \tag{1.16}$$
$$e^{ip\cdot x}\alpha(p) = e^{-iP\cdot x}\alpha(p)e^{iP\cdot x} \tag{1.17}$$

One may derive these identities by noting that $U(\Lambda)\,|0\rangle = |0\rangle$, $P\,|0\rangle = |0\rangle$ and inserting a clever unit operator $U^\dagger(\Lambda)U(\Lambda)$ or $e^{iP\cdot x}e^{-iP\cdot x}$ before $|0\rangle$. Also taking P^μ to have only one non-zero component, namely the Hamiltonian, we see that we are working in the Heisenberg picture. Now that we have a multiparticle state space $\mathcal{H}$ and a basis in which Lorentz transformations act nicely, we may use it to build a quantum field $\phi(x)$ with the desired Lorentz transformation properties.

1.3 The scalar field

For simplicity, we are interested in the simplest possible transformation properties of $\phi(x)$ under the action of the Poincaré group. That is

$$\phi(x - y) = e^{-iP\cdot y}\phi(x)e^{iP\cdot y} \tag{1.18}$$
$$\phi(\Lambda^{-1}x) = U(\Lambda)^\dagger\phi(x)U(\Lambda) \tag{1.19}$$

as appropriate for a scalar field. These transformation properties resemble those of creation and annihilation operators and suggest that ϕ could be a linear combination of them. This linearity is also supported by the fact that eventually we would like to couple fields to sources of particles as $\int \mathrm{d}^4x\, \phi(x)J(x)$ and do perturbation theory in the small amplitudes for creation of particles J (multiple actions of J creating multiple particles). The translation property (1.18) is also guaranteed if we write ϕ as a Fourier integral. The measure in this integral should be Lorentz invariant to inherit the transformation properties of creation and annihilation operators. To build observables it is convenient to make ϕ Hermitean and putting all of these requirements together naturally leads to

$$\phi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi^3)(2\omega_{\boldsymbol{p}})}\Bigg(e^{-ip\cdot x}\alpha(p) + e^{ip\cdot x}\alpha^\dagger(p)\Bigg). \tag{1.20}$$

In the motivation section we mentioned that $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 > 0$ by causality. This now needs to be verified for (1.20) and it is convenient to bring back the original creation and annihilation operators with simple commutation relations

$$\phi(x) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{\sqrt{2\pi^3}\sqrt{2\omega_{\boldsymbol{p}}}}\Bigg(e^{-ip\cdot x}a(\boldsymbol{p}) + e^{ip\cdot x}a^\dagger(\boldsymbol{p})\Bigg) \tag{1.21}$$

so that modulo terms that are trivially zero we get using (1.4)

$$[\phi(x), \phi(y)] = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi^3)(2\omega_{\boldsymbol{p}})}\Bigg(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)}\Bigg). \tag{1.22}$$

Because we are integrating two Lorentz scalars and since spacelike vectors move outside the light cone, one can for $(x-y)^2 > 0$ turn $x-y$ into $y-x$ with a Lorentz transformation, we have

$$[\phi(x), \phi(y)] = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi^3)(2\omega_{\boldsymbol{p}})}\Bigg(e^{-ip\cdot(x-y)} - e^{-ip\cdot(x-y)}\Bigg) = 0 \tag{1.23}$$

as it should. As a consequence of the vanishing of $[\phi(x), \phi(y)]$ for $(x-y)^2 > 0$, choosing a time direction, we have the vanishing of any equal-time commutator

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = 0 \tag{1.24}$$

It is also easy to see from (1.22) and the integral representation of the delta function $\delta^3(\boldsymbol{x}) = \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\, e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$ the equal-time commutator

$$\Big[\phi(t, \boldsymbol{x}), \dot{\phi}(t, \boldsymbol{y})\Big] = i\delta^3(\boldsymbol{x} - \boldsymbol{y}) \tag{1.25}$$

It is worth a remark that (1.20) satisfies the Klein-Gordon equation

$$(-\partial^2 + m^2)\phi(x) = 0 \tag{1.26}$$

by virtue of the mass-shell condition so that one can essentially revert the entire process, that is postulate an equation of motion (EOM) and derive (1.20) as a classical solution and from the Fourier modes build $\mathcal{H}$. In the light of (1.26) the causality condition on the quantum level can be seen as a sort of preservation of the classical causality, which is almost self-evident since $-\partial^2$ is a wave operator. The Klein-Gordon equation can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{1.27}$$

by the usual formula $\partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$. In the next sections we shall work with more general, interacting Lagrangian densities

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi), \tag{1.28}$$

where V is taken to be polynomial in ϕ. The reason why we do not call the previous theory interacting is the fact that the oscillators, from which $\mathcal{H}$ was built, were uncoupled, leading to for example the additivity of energy.

1.4 Observables and perturbation theory

The quantities we would like to compute are correlation functions of the form

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = \langle 0| T\phi(x_1)\dots\phi(x_n)|0\rangle\,, \tag{1.29}$$

where T denotes time ordering for some chosen time direction, which places the operators from left to right according to an ordering by largest to smallest times, so that for example

$$T\phi(x_1)\phi(x_2) = \theta(x_1^0 - x_2^0)\phi(x_1)\phi(x_2) + \theta(x_2^0 - x_1^0)\phi(x_2)\phi(x_1) \tag{1.30}$$

These correlation functions are of interest even in ordinary QM as we now demonstrate by attempting an exercise from [6]. Suppose that we have a one-dimensional QM problem with a ground state $|0\rangle$ such that $H|0\rangle = 0$ and as a complete set of commuting observables we take x. Picking a time ordering we consider correlation functions of the form

$$\langle 0| x(t_1)\dots x(t_n)|0\rangle \tag{1.31}$$

we write these out for $n = 1, 2, 3$.

$$\langle 0| x(t)|0\rangle = \langle 0| e^{-itH} x e^{itH}|0\rangle = \langle 0| x|0\rangle \tag{1.32}$$

$$\begin{aligned}\langle 0| x(t_1)x(t_2) |0\rangle &= \langle 0| e^{-it_1 H} x e^{it_1 H} e^{-it_2 H} x e^{it_2 H} |0\rangle \\ &= \sum_n |\langle 0| x |n\rangle|^2 e^{it_{12}E_n} \end{aligned} \tag{1.33}$$

$$\begin{aligned}\langle 0| x(t_1)x(t_2)x(t_3) |0\rangle &= \langle 0| e^{-it_1 H} x e^{it_1 H} e^{-it_2 H} x e^{it_2 H} x e^{-it_3 H} x e^{it_3 H} |0\rangle \\ &= \sum_{n,m} \langle 0| x |n\rangle \langle n| x |m\rangle \langle m| x |0\rangle e^{it_{12}E_n} e^{it_{23}E_m}, \end{aligned} \tag{1.34}$$

where we inserted a complete set of energy eigenstates $|n\rangle$ and $t_{ij} \equiv t_i - t_j$. We see for example from (1.33) that the Fourier transformation of correlation functions encodes the energy levels E_n. We have shown that eigenvalues of H can be obtained from correlation functions, but what about H itself? For this we show that (1.31) contains vacuum expectation values (VEVs) of the form $\langle 0| x^m p^n |0\rangle$. To do this, we pick the usual nonrelativistic ansatz for H

$$H = \frac{p^2}{2m} + V(x), \tag{1.35}$$

and with the formula $[f(A), B] = [A, B]\frac{\partial f}{\partial A}$ valid for the commutator $[p^n, x]$ and the canonical commutation relation $[x, p] = i$ we then have

$$\begin{aligned}\langle 0| x(t_1)x(t_2) |0\rangle &= \langle 0| e^{-it_1 H} x e^{it_1 H} e^{-it_2 H} x e^{it_2 H} |0\rangle \\ &= \langle 0| x e^{it_{12}H} x |0\rangle \\ &= \langle 0| x^2 |0\rangle + \langle 0| x \left[e^{it_{12}H}, x\right] |0\rangle \\ &= \langle 0| x^2 |0\rangle + \sum_{n=0}^{\infty} \frac{(it_{12})^n}{n!} \langle 0| x[H^n, x] |0\rangle \end{aligned} \tag{1.36}$$

where we expanded the exponential. We now simply match the coefficients of t_{12}^n on the right hand side with the same coefficients in the Taylor expansion of the left hand side to obtain constraints on various VEVs. Higher point correlators give further restrictions on the VEVs and enable us to solve for them. Such VEVs with xs only give moments of the ground state wave function modulus squared $\Psi_0^2(x)$, which by a version of the moment problem determine $\Psi_0(x)$ up to a position dependent phase, which can be fixed by considering the VEVs with ps. Now that we have the ground state wave function $\Psi_0(x)$, we can calculate the potential $V(x)$ by the time-independent Schrödinger equation

$$V(x) = \frac{1}{2m\Psi_0} \frac{\mathrm{d}^2\Psi_0}{\mathrm{d}x^2} \tag{1.37}$$

determining the functional form of H. So in the end we have obtained the eigenvalues and functional form of H from the correlation functions (1.31).

In QFT the correlation functions (A) directly lead to the computation of particle masses by the Källén–Lehmann spectral representation of $\langle 0| T\phi(x_1)\phi(x_2) |0\rangle$ and their scattering amplitudes by the LSZ formula. When we compute correlation functions, it is usually very convenient to define a generating functional

$$Z[J] \equiv \langle 0| T e^{i\int d^4x J(x)\phi(x)} |0\rangle\,, \tag{1.38}$$

so that the correlation functions are easily computed once Z is known

$$\langle 0| T\phi(x_1)\ldots\phi(x_n) |0\rangle = (-i)^n \frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)} Z[J]\bigg|_{J=0}\,, \tag{1.39}$$

where for the functional derivative, we have $\frac{\delta J(x)}{\delta J(y)} = \delta^4(x-y)$. Our task thus reduces to the computation of Z and we begin by formulating the equations of motion for it. First we consider the two-point correlator, where we differentiate by the time x^0 twice

$$\begin{aligned}\partial_0^2 \langle 0| T\phi(x)\phi(y) |0\rangle \;&=\; \partial_0^2\Big(\theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \\ &\qquad \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle\Big)\end{aligned}$$

By the identity $\theta'(x) = \delta(x)$ this simplifies to

$$\partial_0\Big(\langle 0| T\partial_0\phi(x)\phi(y) + \delta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle\Big)$$

The delta function forces the commutator to be equal-time so it vanishes by (1.24). Applying the derivative a second time, we get

$$\begin{aligned}\partial_0^2 \langle 0| T\phi(x)\phi(y) |0\rangle \;&=\; \langle 0| T\partial_0^2\phi(x)\phi(y)|0\rangle - \delta(x^0-y^0)[\phi(y),\partial_0\phi(x)] \\ &=\; \langle 0| T\partial_0^2\phi(x)\phi(y)|0\rangle - i\delta^4(x-y),\end{aligned}$$

where the delta function again made the commutator equal-time and so we used (1.25). For the higher point correlators the pattern is

$$\begin{aligned}\partial_0^2 \langle 0| T\phi(x)\phi(y_1)\ldots\phi(y_n)|0\rangle = &\langle 0| T\partial_0^2\phi(x)\phi(y_1)\ldots\phi(y_n)|0\rangle \\ &-i\sum_j \delta^4(x-y_j)\langle 0| T\phi(y_1)\ldots\phi(y_{j+1})\ldots\phi(y_n)|0\rangle\end{aligned} \tag{1.40}$$

Expanding the generating function in sources J will make the following more transparent

$$Z[J] = \sum_n \frac{1}{n!}\int Z_n(x_1,\ldots,x_n)J(x_1)\ldots J(x_n) \tag{1.41}$$

First we notice that when multiplying the higher point correlators and integrating over $y_1, \dots, y_n$, one source always survives the integration. By the fact that we can bring down a $\phi(x)$ by functional differentiation of $Z[J]$ by $iJ(x)$ and that EOMs for ϕ read $\partial^2\phi = \frac{\partial V}{\partial \phi}$, we then have

$$\partial^2 \frac{\delta Z}{\delta(iJ(x))} = \left(\frac{\partial V}{\partial \phi}\left[\frac{\delta}{\delta(iJ(x))}\right] + J(x)\right) Z[J] \tag{1.42}$$

This equation is called the Schwinger-Dyson (SD) equation and we would like to find a formal solution. We see that we have two derivatives with respect to J and one explicit factor of J in (1.42). Knowing elementary Fourier analysis one gets the idea that a simplification will occur when Fourier transforming. The two functional derivatives turn into multiplication and the explicit factor of J turns into a derivative creating a first-order PDE for the Fourier transform. Writing the Fourier ansatz

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi] + iJ\phi}, \tag{1.43}$$

where $\mathcal{D}\phi$ is a formal measure on field space, the SD equation (1.42) yields

$$\int \mathcal{D}\phi \left(\partial^2\phi - \frac{\partial V}{\partial \phi} - \frac{\delta S}{\delta \phi}\right) e^{iS[\phi]+iJ\phi} = 0, \tag{1.44}$$

where integration by parts was used to obtain a functional derivative of S. This is satisfied for

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\phi\partial^2\phi - V(\phi)\right) = \int \mathrm{d}^4x \left(-\frac{1}{2}(\partial\phi)^2 - V(\phi)\right) = \int \mathrm{d}^4x \mathcal{L}, \tag{1.45}$$

where the correctness of the factor $\frac{1}{2}$ is seen a posteriori after doing a per partes to obtain $-\frac{1}{2}(\partial\phi)^2$. It can also be understood by discretizing spacetime and interpreting ∂^2 as a symmetric matrix, which replaces $\partial^2\phi$ by $K_{AB}\phi^B$. The integral S of the Lagrangian density is nothing but the classical action. Integration with the measure $\mathcal{D}\phi$ in (1.43) is then called functional integration and it has the interpretation of integrating over classical configurations of the field ϕ weighed by the exponential of the classical action. The resulting integral is called a path integral.

We will evaluate the generating functional $Z[J]$ for the case of free (also called Gaussian for reasons that soon become clear) field theory with

$$S[\phi] = \int \mathrm{d}^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right)$$

The integral (1.43) is oscillating for real m and so we make the Feynman $i\epsilon$ prescription $m^2 \to m^2 - i\epsilon$ to make it convergent. Apart from this issue it is a simple Gaussian and we evaluate it as a continuum limit of

$$I(J) = \int d^n y \, e^{-\frac{1}{2}K_{ij}y^i y^j + iy^i J^i} = e^{-\frac{1}{2}J^i (K^{-1})_{ij} J^j} I(0) \tag{1.46}$$

The proper generalisation of a matrix inverse K^{-1} to the continuum case is the Green function satisfying

$$i(\partial^2 - m^2 + i\epsilon)D(x-y) = -\delta^4(x-y), \tag{1.47}$$

where D is called the Feynman propagator and it can be expressed in momentum space

$$D(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-ip(x-y)} \frac{-i}{p^2 + m^2 - i\epsilon} \tag{1.48}$$

The Feynman propagator is evaluated by integration along the standard Jordan semicircle chosen by the imaginary part of p^0 needed to make the integral converge and the $i\epsilon$ chooses the appropriate signs of p^0 based on the sign of $x^0 - y^0$. By the virtue of choosing these appropriate signs, the equation

$$D(x-y) = \langle 0| T\phi(x)\phi(y) |0\rangle \,, \tag{1.49}$$

holds. This can be understood by the fact that the right hand side is also a Green function for the Klein-Gordon equation by (1.40) and the $i\epsilon$ prescription chooses the appropriate time ordering on the left hand side. Combining (1.46) and (1.47), we have

$$Z[J] = e^{-\frac{1}{2}\int d^4x \int d^4y \, J(x)D(x-y)J(y)}, \tag{1.50}$$

where we used the fact that $Z[0] = 1$ by (1.38). The Gaussian theory is solved.

The correlators (1.29) of the Gaussian theory can be computed using an important theorem called Wick's theorem. To formulate the theorem, we need to define normal ordering of an expression constructed from creation and annihilation operators. The definition is very simple, we simply move all annihilation operators to the right of all creation operators. Denoting normal ordering with a ::, we have for example

$$:a(\boldsymbol{p})a^\dagger(\boldsymbol{q}): = a^\dagger(\boldsymbol{q})a(\boldsymbol{p}). \tag{1.51}$$

Now we define the contraction of two fields as

$$\phi(x)\phi(y) = D(x-y) \tag{1.52}$$

Abbreviating $\phi(x_i) \equiv \phi_i$, we have Wick's theorem

$$T(\phi_1\phi_2 \ldots \phi_n) \; = :\!\exp\left\{ \frac{1}{2} \sum_{i,j=1}^{n} \wick{\c\phi_1 \c\phi_2} \frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_2} \right\} \phi_1\phi_2 \ldots \phi_n\!: \tag{1.53}$$

from which the following formula for correlation functions of the Gaussian theory follows

$$\langle 0| \, T(\phi_1 \ldots \phi_n) \, |0\rangle = \sum_{pairings} \prod_{\{i,j\}} \wick{\c\phi_i \c\phi_j} \tag{1.54}$$

This can be proven from the form of (1.50) and differentiating to get correlation functions. As an example we have for $n = 2$

$$\langle 0| \, T(\phi_1\phi_2) \, |0\rangle = \wick{\c\phi_1 \c\phi_2} = D(x_1 - x_2) \tag{1.55}$$

which is just (1.49) and for $n = 3$ we get zero, since the VEV of a normal ordered expression is zero. For $n = 4$, we find the result

$$\begin{aligned} \langle 0| \, T(\phi_1\phi_2\phi_3\phi_4) \, |0\rangle \;\; &= \;\; \wick{\c1\phi_1 \c1\phi_2 \c2\phi_3 \c2\phi_4} + \wick{\c1\phi_1 \c2\phi_2 \c1\phi_3 \c2\phi_4} + \wick{\c1\phi_1 \c2\phi_2 \c2\phi_3 \c1\phi_4} \\ &= \;\; D(x_1 - x_2)D(x_3 - x_4) + D(x_1 - x_3)D(x_2 - x_4) \\ &\quad\; + D(x_1 - x_4)D(x_2 - x_3) \end{aligned} \tag{1.56}$$

The importance of Wick's theorem is for perturbation theory. Suppose we have $S = S_0 - g \int \mathrm{d}^4x \, V_{int}(\phi)$, where S_0 is the free action (1.46) and V_{int} is an interaction potential. When we write the generating functional Z by the path integral

$$Z[J] = \int \mathcal{D}\phi \, e^{iS_0[\phi] + iJ\phi} e^{-ig \int \mathrm{d}^4x \, V_{int}(\phi)} \tag{1.57}$$

and expand the exponential with V_{int}, we get

$$Z[J] = \frac{I[J]}{I[0]}, \tag{1.58}$$

where

$$I[J] = \sum_{n=0}^{\infty} \frac{g^n}{n!} \int \mathcal{D}\phi \, e^{iS_0[\phi] + iJ\phi} \int \mathrm{d}^4x_1 \, \ldots \mathrm{d}^4x_n \, V_{int}(x_1) \ldots V_{int}(x_n) \tag{1.59}$$

If V_{int} is polynomial in ϕ, then the only expressions we need to calculate in order to find the perturbative generating functional (1.58) are the correlators

$$\int \mathcal{D}\phi \, e^{iS_0[\phi] + iJ\phi} \phi(x_1) \ldots \phi(x_m) \tag{1.60}$$

These we know how to do by Wick's theorem and so formally the problem is solved. In practice one usually develops a diagrammatic calculus of Wick diagrams, which help organise the perturbative expression (1.58) and give precise quantitative results by the virtue of Wick's theorem.

1.5 Euclidean field theory

Suppose we take the path integral

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi]+iJ\phi} \tag{1.61}$$

for general S of the form (4.41) and we analytically continue to imaginary times $t \to i\tau$. For the metric this has the effect of going from a Minkowskian to a Euclidean metric and so the resulting theory with the partition function

$$Z_E[J] = \int \mathcal{D}\phi \, e^{-S_E[\phi]+iJ\phi} \tag{1.62}$$

and with $S_E[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}(\partial\phi)^2 + V(\phi) \right)$ is called Euclidean. We again stress that $(\partial\phi)^2$ is evaluated with a Euclidean metric.

Such field theories naturally arise in the continuum limit of lattice models of statistical physics. To demonstrate this, we consider the Ising model with a partition function

$$Z = \sum_{\{\sigma_i\}} \exp\left\{ \sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i \right\} \tag{1.63}$$

with $\sigma_i \in \{-1, 1\}$ and the external field h_i providing a source. The identity

$$\int \prod_i \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4}\sum_{i,j} \varphi_i J^{-1}_{ij} \varphi_j + \sum_i \varphi_i\sigma_i \right\} \sim \exp\left\{ \sum_{i,j} J_{ij}\sigma_i\sigma_j \right\} \tag{1.64}$$

for multiple Gaussian integrals lets us rewrite (1.63) as

$$Z = \int \prod_i \mathrm{d}\varphi_i \exp\left\{ -\frac{1}{4}\sum_{i,j} (\varphi_i - h_i) J^{-1}_{ij} (\varphi_j - h_j) \right\} \sum_{\sigma_i} \exp\left\{ \sum_i \varphi_i\sigma_i \right\} \tag{1.65}$$

modulo a constant. The spins are now decoupled and we can explicitly perform the sum over spins

$$\sum_{\sigma_i} \exp\left\{ \sum_i \varphi_i\sigma_i \right\} = \prod_i (2\cosh\varphi_i) \sim \exp\left\{ \sum_i \ln\cosh\varphi_i \right\} \tag{1.66}$$

For the partition function we have after $\phi_i \to \frac{1}{2}J^{-1}_{ij}\phi_j$ the result

$$Z = \exp\left\{ -\frac{1}{4}\sum_{ij} h_i J^{-1}_{ij} h_j \right\} \int \prod_i \mathrm{d}\varphi_i \exp\left\{ -\sum_{ij} J_{ij}\varphi_i\varphi_j + \sum_i \ln\cosh 2J_{ij}\varphi_j \right\} \tag{1.67}$$

up to a constant. This means that after taking the continuum limit we will replace the spins σ_i by a scalar field ϕ. We continue by Fourier expanding

$$\varphi_i = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} \varphi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i} \tag{1.68}$$

$$J_{ij} = \frac{1}{N} \sum_{\boldsymbol{k}} J(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}, \tag{1.69}$$

where N is the number of sites of our lattice. We then have

$$\sum_{ij} J_{ij}\varphi_i\varphi_j = \sum_{\boldsymbol{k}} J(\boldsymbol{k})|\varphi(\boldsymbol{k})|^2 \tag{1.70}$$

and expanding the logarithm

$$\ln\cosh 2x \sim 2x^2 - \frac{4}{3}x^4, \tag{1.71}$$

the second quadratic term is

$$2\sum_i (J_{ij}\varphi_j)^2 = 2\sum_{\boldsymbol{k}} |J(\boldsymbol{k})|^2|\varphi(\boldsymbol{k})|^2, \tag{1.72}$$

so that the quadratic term in (1.67) becomes

$$\int \mathrm{d}^D x\, \mathcal{L}_0 = \sum_{\boldsymbol{k}} \left[J(\boldsymbol{k}) - 2|J(\boldsymbol{k})|^2 \right] |\varphi(\boldsymbol{k})|^2, \tag{1.73}$$

where $\mathcal{L}_0$ is the free Lagrangian density. Doing an expansion

$$J(\boldsymbol{k}) \sim J_0(1-\rho^2k^2), \tag{1.74}$$

and defining ρ by

$$J_0\rho^2k^2 = \frac{1}{2}\sum_{\boldsymbol{r}} J(\boldsymbol{r})(\boldsymbol{k}\cdot\boldsymbol{r})^2, \tag{1.75}$$

one then gets for the quadratic term of (1.73)

$$\int \mathrm{d}^D x\, \mathcal{L}_0 = J_0 \sum_{\boldsymbol{k}} \left[(1-2J_0) + (4J_0-1)\rho^2k^2 \right] |\varphi(\boldsymbol{k})|^2 \tag{1.76}$$

From elementary statistical mechanics we have for a nearest neighbor interaction

$$J_0 = \sum_{\boldsymbol{r}} J(\boldsymbol{r}) = \frac{1}{2}z\beta\widetilde{J}, \tag{1.77}$$

where z is the number of neighbors, $\widetilde{J}$ the coupling between neighbors and β the inverse temperature. We expect that the ks in (1.76) turn into derivatives, so the $(1-2J_0)$ term is a mass term so that critical temperature T_c can be identified by the condition that the mass term vanishes (more on this in the section (1.7) on the renormalisation group). This gives

$$T_c = z\widetilde{J} \tag{1.78}$$

Writing $1-2J_0 = \frac{T-T_c}{T_c}$, we have

$$\int \mathrm{d}^D x \mathcal{L}_0 = \frac{1}{2}\sum_{\boldsymbol{k}}\left(\frac{T-T_c}{T_c} + \rho^2 k^2\right)|\varphi(\boldsymbol{k})|^2 \tag{1.79}$$

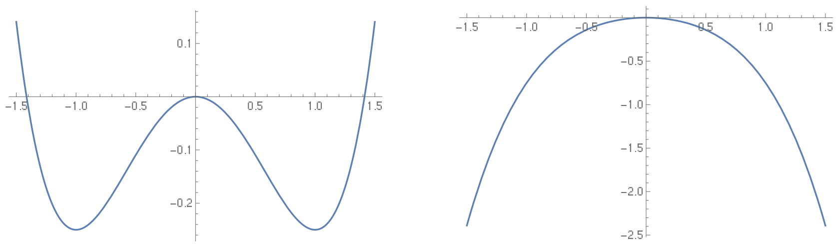
so that by defining $\phi(x) = \rho\,\varphi(x)$ and $m^2 = \frac{1}{\rho^2}\frac{T-T_c}{T_c}$ in the continuum limit $N\to\infty$, the result for the free Euclidean action is

$$\int \mathrm{d}^D x \mathcal{L}_0 = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{1.80}$$

Computing higher order terms would reveal that the Ising model is equivalent to the ϕ^4 theory with

$$S_E = \int \mathrm{d}^D x \left[\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4}\phi^4\right] \tag{1.81}$$

It is curious that $m^2 < 0$ for $T < T_c$, but this is the physics one would expect since the potential in (1.81) then has two degenerate minima reflecting the two ferromagnetic phases of the Ising, see figure (1.2). If we want to continue along this line of thought, we should interpret ϕ as an order parameter, which gets nontrivial VEVs when the Z_2 symmetry of the Ising model is broken below T_c. This indicates that we can proceed backwards by postulating an action for the order parameter with the desired symmetry properties reflecting the symmetry of the lattice model. We may then calculate correlators in the lattice theory by performing a path integral.



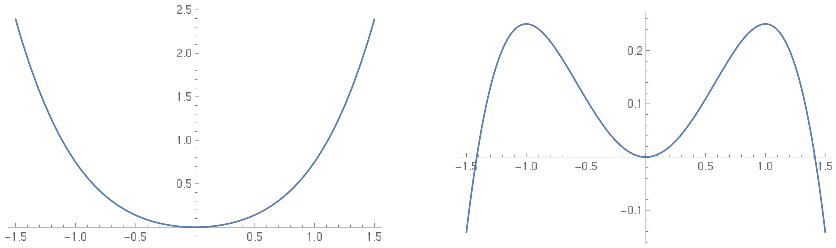
(a) Plus sign

(b) Minus sign

Figure 1.2 Plot of $V(\phi) = -\frac{1}{2}\phi^2 \pm \frac{1}{4}\phi^4$ for two signs of the ϕ^4 term

1.6 Unstable vacua and instantons

Let us now consider ϕ^4 theory in the high-temperature regime with $m^2 > 0$.



(a) Plus sign **(b)** Minus sign

Figure 1.3 Plot of $V(\phi) = \frac{1}{2}\phi^2 \pm \frac{1}{4}\phi^4$ for two signs of the ϕ^4 term

From figure (1.3), we see that the physics of (1.81) is dramatically altered when one passes from the stable regime $g > 0$ to the unstable regime $g < 0$ (assuming $m^2 > 0$) since for $g < 0$ the perturbative vacuum $\phi = 0$ is unstable in the sense that it can decay by tunneling effects. One might expect to be able to see this change in regimes in perturbation theory. Indeed this is the case and to illustrate this, we consider the zero-dimensional path integral

$$I(g) = \int_{-\infty}^{\infty} \mathrm{d}x \, e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4}, \tag{1.82}$$

which converges for $g > 0$ and diverges for $g < 0$. To see the perturbation theory manifestation of this, we expand $e^{-\frac{g}{4}x^4} = \sum_{n=0}^{\infty}(-1)^n \frac{g^n}{4^n n!} x^{4n}$ and obtain

$$I(g) = \sum_{n=0}^{\infty}(-1)^n \frac{g^n}{4^n n!} \int_{-\infty}^{\infty} \mathrm{d}x \, e^{\frac{1}{2}x^2} x^{4n} = \sum_{n=0}^{\infty}(-1)^n \frac{(4n-1)!!}{4^n n!} g^n \tag{1.83}$$

which clearly has zero radius of convergence. This is seen either by using elementary calculus or using Stirling's formula to get the asymptotics $(-1)^n 4^n n!$ of the power series coefficients. We want to emphasize that the zero radius of convergence of the zero-dimensional perturbation series is physical since if the perturbation series had nonzero radius of convergence, it would describe both the stable and unstable theories for g sufficiently small, see [20]. Although it diverges, the series (1.83) can still be very useful since it is asymptotic in the following sense.

Consider a function $f(g)$ and a formal power series $\sum_{n=0}^{\infty} a_n g^n$, we say that the series is asymptotic to $f(g)$ if

$$\lim_{g \to 0} \frac{1}{g^N} \left[f(g) - \sum_{n=0}^{N} a_n g^n \right] = 0, \tag{1.84}$$

which means that the remainder after summing $N+1$ terms is smaller than the last retained term for g very small. This usually means that the asymptotic series gives a good approximation for $f(g)$ when we truncate it for the right N since eventually its terms become large and the sum deviates from $f(g)$. A truncation scheme in which we truncate for N such that $|a_n|$ is at its smallest for $n = N$ is called the optimal truncation scheme. We estimate how close this scheme gets us to $f(g)$ for the asymptotic behavior $a_n \sim \frac{1}{A^n} n!$, which we have already seen for $A = -\frac{1}{4}$ in the zero-dimensional path integral. We thus want to minimise $\left|\frac{g}{A}\right|^n n!$ and by using Stirling's approximation, we estimate

$$\left|\frac{g}{A}\right|^n n! = \exp\left\{n\left(\ln n - 1 + \ln\left|\frac{A}{g}\right|\right)\right\} \tag{1.85}$$

and this implies that for g small the optimal n is $N = \left|\frac{A}{g}\right|$ displaying the expected behavior $N \sim \left|\frac{1}{g}\right|$. Evaluating (1.85) at the stationary point $n = N$, we get the estimate for the magnitude of near N terms of $e^{-\left|\frac{A}{g}\right|}$, meaning that the deviation of the optimally truncated small g asymptotic series is of the magnitude $e^{-\left|\frac{A}{g}\right|}$.

Since the perturbation series obtained by summing Wick diagrams usually scales factorially as hinted at in the zero-dimensional example, this means that we might encounter corrections that go like $e^{-\left|\frac{A}{g}\right|}$ in addition to the usual perturbative corrections. These corrections are the symptoms of the presence of an instability and are nonperturbative in the sense that $e^{-\left|\frac{A}{g}\right|}$ has an essential singularity at $g = 0$. It is curious that these corrections are exponentially damped for g small, but nevertheless nonperturbative. Such corrections arise when one solves the Euclidean EOMs with appropriate boundary conditions, thus finding a saddle point of the path integral since the classical EOMs are given by the stationary points of the action. When having finite action, these solutions are called instantons. We note that in string theory the analysis of large order behavior leads to nonperturbative corrections, which don't have a point particle origin. They can be explained by the inclusion of D-branes [21]. Investigating to what extend and how the usual perturbation series contains nonperturbative data is a research program onto itself [22, 23, 24]. In particular one may see that the asymptotic series around instantons resurge from the asymptotic series around the perturbative background.

Let us now find some instantons, which nicely illustrate the process of tunneling from false to true vacua. To do this, we choose to work with the family of potentials

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{1}{2}\phi^3 + \frac{\alpha}{8}\phi^4, \qquad 0 < \alpha < 1 \tag{1.86}$$

which are plotted in figure (1.4). These potentials have a false vacuum $\phi_+ = 0$ and a true vacuum $\phi_- = \frac{3}{2\alpha} + \frac{\sqrt{9-8\alpha}}{2\alpha}$

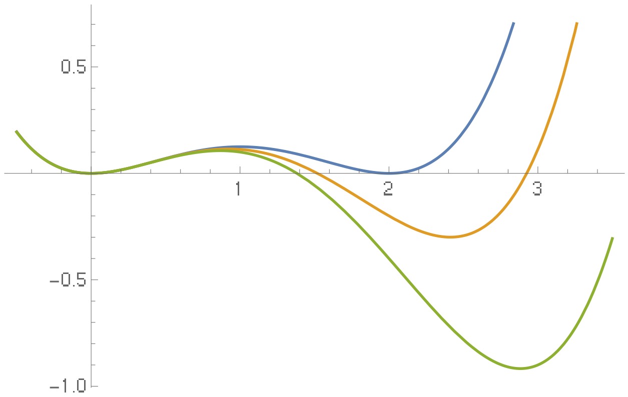


Figure 1.4 The potential (1.86) for $\alpha = 0.8, 0.9$ and 1 from bottom to top

Separating the Euclidean time τ, so that $\phi = \phi(\tau, \boldsymbol{x})$, we want to solve the Euclidean EOM

$$-\left(\partial^2 + \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\right)\phi + V'(\phi) = 0, \tag{1.87}$$

where ∂^2 is a $d-1$ dimensional Laplacian. Since we integrate over the entire Euclidean space in the action S and this action has to be finite, we need to impose the following boundary conditions at infinity

$$\begin{aligned} \phi(\pm\infty, \boldsymbol{x}) &= \phi_+ \\ \phi(\tau, \boldsymbol{x}) &= \phi_+, \qquad |\boldsymbol{x}| \to \infty \end{aligned} \tag{1.88}$$

When we write $\phi = \phi_+ + \delta$ we understand why these solutions are called instantons. Their deviations δ from ϕ_+ are localised in space explaining the particle suffix -on and localised in time explaining the instant. One can unify the boundary conditions and simplify the EOM by the virtue of V being rotationally symmetric by picking a rotationally invariant ansatz

$$\phi = \phi\left(\sqrt{\tau^2 + |\boldsymbol{x}|^2}\right) \equiv \phi(r) \tag{1.89}$$

so that $\lim_{r\to\infty}\phi(r) = \phi_+$. This leads to the EOM

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{d-1}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = V'(\phi) \tag{1.90}$$

We will be interested in studying the limit $\alpha \sim 1$ corresponding to the so-called thin-wall approximation, in which the separation between the two minima of (1.86) is small since a simple calculation gives $V(\phi_+) - V(\phi_-) = 2(1-\alpha) + O((1-\alpha)^2)$. It turns out that in this limit neglecting the first derivative term in (1.90) gives sensible results so that the thin-wall EOM is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = V_0'(\phi), \tag{1.91}$$

where V_0 is V evaluated at $\alpha = 1$. The EOM (1.91) can be solved using integral of motion techniques from elementary classical mechanics, where r plays the role of time. Recalling these techniques, one obtains

$$\frac{1}{2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2 - V_0(\phi) = 0, \tag{1.92}$$

where the integral of motion was fixed by the fact that $V_0(\phi_\pm) = 0$ and that ϕ approaches a constant at large r. The differential equation (1.92) with boundary conditions $\phi(r \to \pm\infty) = \phi_\pm$ is now easy to solve remembering that in (1.92) r can be negative since it formally plays the role of time, but for us the physical region is $r > 0$. One obtains the exact solution

$$\phi(r) = \frac{2}{1+e^{r-\bar{r}}}, \tag{1.93}$$

where $\bar{r}$ is defined so that $\phi(\bar{r}) = \frac{1}{2}(\phi_+ + \phi_-)$ and it turns out that $\bar{r} \sim \frac{1}{1-\alpha}$. In particular one easily checks that $\phi(r \to \infty) = \phi_+ = 0$ and that $\phi(r \to -\infty) = \phi_- = 2$, where we used $\phi_\pm$ with $\alpha = 1$. The solution (1.93) with $\bar{r}$ large enough to have $\alpha \sim 1$ is plotted for three values of $\bar{r}$ in figure (1.5).

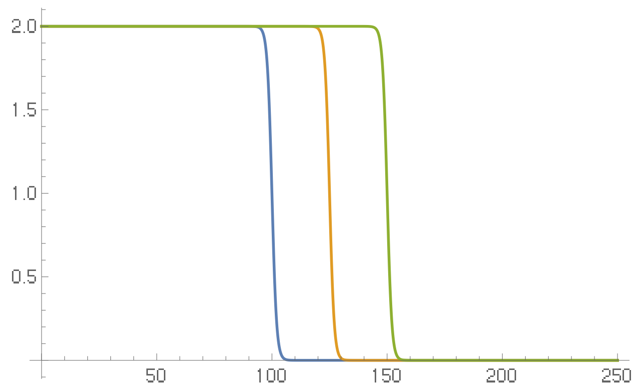


Figure 1.5 The solution (1.93) for $\bar{r} = 100, 125$ and 150 from left to right

One may interpret the solution (1.93) as a bubble of true vacuum ϕ_- centered at the origin of radius approximately $\bar{r}$ separated by a thin wall from the false vacuum ϕ_+. We would now also like to interpret the solution in spacetime. To do this, we rotate back to Minkowski spacetime and introduce a field in spacetime

$$\Phi(t, \boldsymbol{x}) = \phi\left(r = \sqrt{-t^2 + |\boldsymbol{x}|^2}\right) \tag{1.94}$$

This field satisfies the wave equation

$$\left(-\partial^2 + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\Phi + V'(\Phi) = 0, \tag{1.95}$$

by the virtue of (1.87) with the boundary conditions $\Phi(0, \boldsymbol{x}) = \phi(|\boldsymbol{x}|)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, \boldsymbol{x})\Big|_{t=0} = 0$, the latter being obtained by simple differentiation of (1.94). So from the instanton ϕ, we have a new solution (1.94) to the wave equation (1.95). Its interpretation is actually quite clear, it describes the spherically symmetric expansion of the bubble of true vacuum ϕ_- starting at time $t = 0$. At later times, when the bubble has expanded to be much larger than its initial radius R at $t = 0$ it expands almost at the speed of light as can be seen from

$$|\boldsymbol{x}|^2 = t^2 + R^2, \tag{1.96}$$

as satisfied by the boundary of the bubble from the standard wave ansatz. We conclude by mentioning that in (B.3) we analyse a famous instanton of Kaluza-Klein theory as discovered by Witten [25] admitting a similar bubble picture. See also the essay [26] on bubbles.

1.7 The renormalisation group

An essential physicist's know-how is to select a description that is precisely fit to the problem under study. One uses hydrodynamics to describe fluid dynamics and not quantum mechanics of water molecules, even though these are more fundamental. We say that one uses an effective description whose validity is usually demarcated by length scales. In order for the reductionist program in physics to be justified, we need a way of obtaining the effective macroscopic theories from the microscopic. We have no final theory of everything, so these microscopic theories are actually effective, too. So in reality we need to find the same macroscopic behavior from classes of microscopic theories that differ by one another by deformations affecting physics in the unknown and to have any hope in doing so, we need universality.

Universality is the phenomenon of different microscopic theories having the same macroscopic limit.

The key feature we want to have when passing from the microscopic to the macroscopic is the reduction in the number of degrees of freedom (DOFs) needed. One can also imagine reducing the number of DOFs in the opposite way. For example if we consider a large amount of gas, we may ask what is the smallest pocket of gas having the same properties? This question makes sense, because the gas is roughly translationally symmetric. The characteristic size of such a small pocket is the correlation length ξ, which depends on the state of the system. For ξ very small, we need to consider only a very limited number of DOFs and by symmetry extend their dynamics to the entirety of the gas. So in the end in this very specific limit, we might have simplified the system by going from the macroscopic to the microscopic.

The lesson is that with ξ very small, the physics can be described in the microscopic and possibly extrapolated, while for ξ in the other regimes, we need to integrate out the many microscopic DOFs to obtain an effective macroscopic theory. Examples with many microscopic DOFs are plentiful, for instance we can think of describing the dynamics of a field, then in any pocket of characteristic size ξ we have an infinite continuum of DOFs. We can also consider critical phenomena, in which the characteristic length ξ grows without bounds. In any case, we want to generate a sequence of theories that gives the same physics we're interested in, only with progressively more efficient DOFs.

To give an example on producing an effective description, we consider the iconic block spins. Consider the lattice partition function

$$Z = \sum_{\{c\}} \exp\Big\{-\beta H_{\{g\}}\{c\}\Big\} \tag{1.97}$$

with $\{c\}$ the different DOFs, for example the spin variables of (1.63), $\{g\}$ a collection of couplings in the Hamiltonian H. We now consider transforming the DOFs $\{c\} \to \{C\}$ so that

$$\exp\Big\{-\beta H'_{\{g\}}\{C\}\Big\} = \sum_{\{c\}} f(\{c\},\{C\}) \exp\Big\{-\beta H_{\{g\}}\{c\}\Big\} \tag{1.98}$$

with the function f satisfying

$$\sum_{\{C\}} f(\{c\},\{C\}) = 1 \tag{1.99}$$

which guarantees that the partition function doesn't change under the transformation, preserving the physics. An example of such a transformation occurs in the case where the $\{c\}$ are spins and we uniformly separate the lattice into blocks of spin. To each such block we assign a block spin C that is given by the average of spins in that block. In this way we effectively increase the magnitude of the lattice spacing

$$a' = \lambda a, \tag{1.100}$$

moving into the macroscopic. It is clear that the transformed Hamiltonian $H'_{\{g\}}$ can have additional interactions not present in $H_{\{g\}}$. When the Hamiltonian $H_{\{g\}}$ contains all the interactions allowed by the symmetry of the lattice, then no additional interactions get generated and we can describe the transformation as a change in the couplings $\{g\} \to \{g'\}$ such that $H'_{\{g\}} = H_{\{g'\}}$. We write

$$\{g'\} = \mathcal{R}\{g\} \tag{1.101}$$

where our aim is to find the transformation $\mathcal{R}$ acting in the space of couplings. The equations (1.98), (1.99), (1.100) and (1.101) specify the structure of the renormalisation (semi)group (RG). Now the RG transformations may be iterated until the lattice spacing reaches approximately the correlation length ξ and when the correlation length is infinite, as it happens to be in critical phenomena, we may do an infinity of iterations. The most common behavior is that under this infinity of iterations, we end up with a fixed point $\{g_*\}$ in the space of interactions such that

$$\{g_*\} = \mathcal{R}\{g_*\} \tag{1.102}$$

meaning that we end up with the fixed point Hamiltonian H_* which doesn't change under the RG transform $H'_{\{g_*\}} = H_{\{g_*\}} \equiv H_*$. The fixed point theory is invariant under scaling and in the next chapter we show that under reasonable conditions this scale invariance becomes conformal invariance so that the fixed point theory is described by a conformal field theory (CFT).

In the end, we expect the description to get simpler so only a few couplings should be relevant. To make this idea more precise, we consider the deviation δ from the fixed point coupling

$$\delta = g - g_*, \tag{1.103}$$

which under the RG changes to

$$\delta' = g' - g_* \tag{1.104}$$

Remembering that we have all allowed interactions turned on, it should be possible to choose a convenient basis in the space of couplings such that these basis elements are eigenvalues of the RG, meaning

$$\delta' = \lambda^y \delta \tag{1.105}$$

One then classifies all these eigenvectors by their eigenvalues. In particular those with $y > 0$ are called relevant, since from (1.105), we see that they are relevant in the infrared (IR) limit $\lambda >> 1$. Analogously we call the couplings $y < 0$ irrelevant, since they are important in the ultraviolet (UV) limit $\lambda << 1$. The couplings with $y = 0$ are called marginal, such couplings parametrise lines of fixed points. When we study deformations of fixed point theories, then turning on new relevant couplings drives us away from the fixed points, whereas turning on new irrelevant couplings simply moves us on a so-called critical surface and under the RG the theory flows into the same fixed point.

To illustrate these ideas, we demonstrate how one can estimate the relevancy of couplings based on dimensional analysis (we neglect interactions in this analysis, its just a free field dimensional analysis). The action is dimensionless and since $S = \int \mathrm{d}^D x\, \mathcal{L}$, then $\mathcal{L}$ has mass dimension D. Now we specialise to scalar field theory. Every derivative gives a mass dimension 1 and every scalar gives a $\frac{D-2}{2}$. This is so because the kinetic term has two derivatives and two fields. The coupling $\lambda_{m,n}$ of a term with m derivatives and n fields then has mass dimension $D - m - \frac{D-2}{2}n$. From (1.105) one sees that for example in $D = 4$, we have two marginal couplings to $(\partial\phi)^2, \phi^4$ and two relevant couplings to ϕ^2, ϕ^3. The reason why the mass and not length dimension of couplings is relevant for our discussion is that we understand RG scalings in a passive sense.

Chapter 2

Conformal field theory

Having gained some introductory knowledge of QFTs in chapter (1), we study a particular class of QTFs called conformal field theories (CFTs), whose study is absolutely essential for all of our later developments. There are countless sources on CFTs, the ones which inspired us most are [27, 28, 29]. Since CFTs are foundational for string theory, they are discussed in many string theory books [13, 30, 31, 15, 32, 33]. Useful review articles are among others [34, 35, 36, 37, 38]. Detailed accounts of the applications of CFTs to statistical physics include [14, 39]. Some historically key papers are Polyakov's [40], which conjectured the enhancement of scale to conformal symmetry in critical phenomena and the BPZ paper [41] realising that two-dimensional CFTs are exceptional, enabling the exact solution of the so-called minimal models. To make a connection to our own work, we introduce conformal perturbation theory (CPT) in the bulk before moving the more relevant boundary case in the next chapter. A foundational paper in this area is [42], where the c-theorem was first proved. Another discussion on bulk CPT can be found in [14, 39].

2.1 Conformal transformations

2.1.1 Arbitrary dimension

We begin by studying the mathematics of conformal transformations. Conformal transformations are coordinate transformations $x^\mu \to x'^\mu$ which preserve angles, meaning

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{2.1}$$

On an infinitesimal level this means that for a transformation generated by a vector field ϵ^μ, the conformal Killing equation

$$\mathcal{L}_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = c(x) g_{\mu\nu} \tag{2.2}$$

holds, where for simplicity the connection ∇ is Levi-Civita and the solution ϵ^μ is called a *conformal Killing vector* (CKV) field. Contracting both sides of (2.2) with $g_{\mu\nu}$, we easily obtain that $c(x) = \frac{2}{D}(\nabla \cdot \epsilon)$. Specialising to the flat case $g = \eta$ with arbitrary signature (p, q) and using the fact that the coordinate derivative provides a Levi-Civita connection for the flat metric, we have the equation

$$\epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} = \frac{2}{D}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.3}$$

Acting with a ∂^μ on both sides of (2.3),we find

$$\Box \epsilon_\nu + \left(1 - \frac{2}{D}\right)\partial_\nu(\partial \cdot \epsilon) = 0 \tag{2.4}$$

Another useful equation is obtained by acting with a $\Box$ on (2.3)

$$\partial_\nu \Box \epsilon_\mu + \partial_\mu \Box \epsilon_\nu = \frac{2}{D}\eta_{\mu\nu}\Box(\partial \cdot \epsilon) \tag{2.5}$$

Plugging (2.4) into (2.5) gives

$$\left(\eta_{\mu\nu}\Box + (D-2)\partial_\mu \partial_\nu\right)\partial \cdot \epsilon = 0 \tag{2.6}$$

which indicates that the case $D = 2$ might be special, which we will confirm shortly. In the meantime, we want to find the CKVs present in any dimension $D \geq 2$. Contracting (2.6) with $\eta^{\mu\nu}$, we get

$$(D-1)\Box(\partial \cdot \epsilon) = 0 \tag{2.7}$$

which shows that ϵ^μ is at most quadratic in the coordinates x^μ. We proceed with the ansatz

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}\, x^\nu x^\rho \tag{2.8}$$

with $c_{\mu\nu\rho} = c_{\mu(\nu\rho)}$. Since ϵ appears in the Killing equation only when differentiated, then a^μ is arbitrary and the associated transformation is a translation with the generator being the familiar momentum $P_\mu = -i\partial_\mu$. To constrain $b_{\mu\nu}$, we need to plug ϵ_μ into the linear equation (2.3) and set $c_{\mu\nu\rho} = 0$, obtaining

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{D}\left(\eta^{\lambda\rho}\, b_{\rho\lambda}\right)\eta_{\mu\nu} \tag{2.9}$$

from which we see, that $b_{\mu\nu}$ can be split into its symmetric and antisymmetric part

$$b_{\mu\nu} = \lambda\, \eta_{\mu\nu} + \omega_{\mu\nu} \tag{2.10}$$

with $\omega_{\mu\nu} = \omega_{[\mu\nu]}$. It is easy to see that the term $\lambda\, \eta_{\mu\nu}$ corresponds to dilatations $x'^{\mu} = (1+\lambda)x^{\mu}$ with the generator $D = -i(x \cdot \partial)$ (simple not to confuse with the dimension D from the context) and the term $\omega_{\mu\nu}$ to rotations $x'^{\mu} = x^{\mu} + \omega^{\mu}{}_{\nu}\, x^{\nu}$ with the rotation generators $J_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x^{\nu}\partial_{\mu})$. To fix $c_{\mu\nu\rho}$, we differentiate (2.3) a permute indices

$$\partial_{\rho}\partial_{\mu}\epsilon_{\nu} + \partial_{\rho}\partial_{\nu}\epsilon_{\mu} = \frac{2}{D}\eta_{\mu\nu}\partial_{\rho}(\partial \cdot \epsilon) \tag{2.11}$$

$$\partial_{\nu}\partial_{\rho}\epsilon_{\mu} + \partial_{\mu}\partial_{\rho}\epsilon_{\nu} = \frac{2}{D}\eta_{\rho\mu}\partial_{\nu}(\partial \cdot \epsilon) \tag{2.12}$$

$$\partial_{\mu}\partial_{\nu}\epsilon_{\rho} + \partial_{\nu}\partial_{\mu}\epsilon_{\rho} = \frac{2}{D}\eta_{\nu\rho}\partial_{\mu}(\partial \cdot \epsilon) \tag{2.13}$$

Adding (2.11) to (2.12) and subtracting (2.13), we obtain

$$2\partial_{\mu}\partial_{\nu}\epsilon_{\rho} = \frac{2}{D}(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu})(\partial \cdot \epsilon) \tag{2.14}$$

and plugging in (2.8), the equation for $c_{\mu\nu\rho}$ becomes

$$\partial \cdot \epsilon = b^{\mu}{}_{\mu} + 2c^{\mu}{}_{\mu\rho}x^{\rho} \tag{2.15}$$

and differentiating by ∂_{ν}, the resulting equation is

$$\partial_{\nu}(\partial \cdot \epsilon) = 2c^{\mu}{}_{\mu\nu} \tag{2.16}$$

and is solved by

$$c_{\mu\nu\rho} = \eta_{\mu\nu}b_{\rho} + \eta_{\mu\rho}b_{\nu} - \eta_{\nu\rho}b_{\mu} \tag{2.17}$$

with $b_{\mu} \equiv \frac{1}{D}c^{\nu}{}_{\nu\mu}$. And as in the previous cases we easily identify the infinitesimal form of the transformation $x'^{\mu} = x^{\mu} + 2(x \cdot b)x^{\mu} - x^2 b^{\mu}$ called the *special conformal transformation* (SCT) with the generator $K_{\mu} = -i(2x_{\mu}(x \cdot \partial) - x^2\partial_{\mu})$. The finite conformal transformations found for general $D \geq 2$ and their generators are found in table (2.1).

The generators in (2.1) satisfy the commutation relations

$$\left[J_{\mu\nu}, P_{\rho}\right] = -i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}) \tag{2.18}$$

$$[P_{\mu}, K_{\nu}] = 2i(J_{\mu\nu} + \eta_{\mu\nu}D) \tag{2.19}$$

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}) \tag{2.20}$$

$$[J_{\mu\nu}, K_{\rho}] = -i(\eta_{\mu\rho}K_{\nu} - \eta_{\nu\rho}K_{\mu}) \tag{2.21}$$

$$[D, K_{\mu}] = -iK_{\mu} \tag{2.22}$$

$$[D, P_{\mu}] = iP_{\mu} \tag{2.23}$$

$$[J_{\mu\nu}, D] = 0 \tag{2.24}$$

Name	Transformation	Generator
Translation	$x^\mu \to x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilatations	$x^\mu \to \lambda x^\mu$	$D = -i(x \cdot \partial)$
Rotations	$x^\mu \to R^\mu{}_\nu x^\nu, R_{\mu\nu} = R_{[\mu\nu]}$	$J_{\mu\nu} = i(x_\mu \partial_\nu - x^\nu \partial_\mu)$
SCT	$x^\mu \to \frac{x^\mu - x^2 b^\mu}{1-2(b\cdot x)+b^2x^2}$	$K_\mu = -i(2x_\mu(x\cdot\partial) - x^2\partial_\mu)$

Table 2.1 Finite conformal transformations with generators

For $D \geq 3$, these generate all conformal transformations, while for $D = 2$ we have an infinite number of generators, see (2.1.2). Let us now count the number of generators in table (2.1). We have 1 dilatation generator, $\frac{D(D-1)}{2}$ rotation generators, D SCT generators and D translation generators. Adding these up numbers up, we have a total of $\frac{D(D+1)}{2}$ generators. This is exactly the number of generators of $SO(p+1, q+1)$ with $p + q = D$ so we have a suspicion that the conformal group of $\mathbb{R}^{p,q}$ is $SO(p+1, q+1)$, for $D = 2$ this group gets further augmented. To confirm this, we construct the generators

$$M_{\mu\nu} = J_{\mu\nu} \tag{2.25}$$

$$M_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{2.26}$$

$$M_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{2.27}$$

$$M_{-1,0} = D \tag{2.28}$$

from which together with the commutation relations (2.18)-(2.24)

$$[M_{mn}, M_{rs}] = i(\eta_{ms}M_{nr} + \eta_{nr}M_{ms} - \eta_{mr}M_{ns} - \eta_{ns}M_{mr}) \tag{2.29}$$

holds (here $m, n = -1, 0, 1, \ldots, D$ and η_{mn} has signature $(p+1, q+1)$) and these are exactly the required Lie algebra commutation relations for the group $SO(p+1, q+1)$ with $p+q = D$. Observing this, we realise that embedding $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p+1,q+1}$ simplifies the action of conformal transformations. For our further purposes it is enough to specialise to $(p, q) = (D, 0)$, meaning that we embed Euclidean space $\mathbb{R}^{D,0} \equiv \mathbb{R}^D$ into Minkowski space $\mathbb{R}^{D+1,1}$. The conformal transformations in $\mathbb{R}^D$ can then be realised using Lorentz transformations in $\mathbb{R}^{D+1,1}$. To choose an embedding, we need to get rid of the two extra coordinates and denoting the embedding space coordinates as $X^{-1}, X^0, X^1, \ldots, X^D$ with $X^\mu = x^\mu$, we first restrict to the lightcone

$$X^2 = 0 \tag{2.30}$$

which is preserved by the action of the Lorentz group. Next we do a section

$$X^+ = f(X^\mu) \tag{2.31}$$

of the lightcone, where we defined the lightcone coordinate $X^+ = X^{-1} + X^0$ accompanied by $X^- = X^{-1} - X^0$. The Lorentz transformations reduce to conformal transformations as follows: a point x^μ on the section (2.31) defines a light ray that gets moved by the Lorentz transform to another ray, which we identify with x'^μ. With (2.30) and (2.31), we get the section metric

$$\eta_{sec} = \mathrm{d}x^2 - \mathrm{d}X^+ \mathrm{d}X^- |_{X^+ = f(X^\mu), X^- = \frac{x^2}{X^+}} \tag{2.32}$$

and the condition of having conformal transformations that preserve a flat metric in $\mathbb{R}^D$ gets translated into the flatness condition on the section metric. It is certainly flat for $f(X^\mu) = 1$ and we finally obtain that our Euclidean section is $(X^+, X^-, X^\mu) = (1, x^2, x^\mu)$. This *projective null cone* formalism will prove very useful in investigating conformal kinematics since then we just perform relativistic kinematics in the embedding space and then perform a lightcone section. To do so, we must convert Lorentz invariants into expressions involving only x^μ. One such example is

$$\begin{aligned} X \cdot Y &= X^\mu Y_\mu - \frac{1}{2}\left(X^+ Y^- + X^- Y^+\right) \\ &= x^\mu y_\mu - \frac{1}{2}\left(Y^- + X^-\right) \\ &= x^\mu y_\mu - \frac{1}{2}\left(y^2 + x^2\right) \\ &= -\frac{1}{2}(x-y)^2 \end{aligned} \tag{2.33}$$

2.1.2 Two dimensions

Looking back at the equation (2.3) and setting $D = 2$, we get

$$\epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} = (\partial \cdot \epsilon)\delta_{\mu\nu}, \tag{2.34}$$

where $\mu, \nu = 1, 2$ and we are starting to write $\eta \to \delta$, since as already mentioned, we specialise to Euclidean space. The equation (2.34) decomposes into two independent equations $\partial_1 \epsilon_1 = \partial_2 \epsilon_2$ and $\partial_1 \epsilon_2 = -\partial_2 \epsilon_1$ which we recognise as the Cauchy-Riemann equations. We complexify $\mathbb{R}^2 \to \mathbb{C}^2$ by introducing the coordinates $z, \bar{z}$ which become conjugate under a real section $\bar{z} = z^*$, so that after performing this section, we write $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, recovering $\mathbb{C} \simeq \mathbb{R}^2$. Often one performs calculations in the independent complex coordinates $z, \bar{z}$ and only at the end returns to $\mathbb{R}^2$. We also write $\epsilon = \epsilon^1 + i\epsilon^2$ and $\bar{\epsilon} = \epsilon^1 - i\epsilon^2$ so that in the complex coordinates, the Cauchy-Riemann equations become $\partial\bar{\epsilon} = \bar{\partial}\epsilon = 0$ where we abbreviated $\partial_z =$

$\frac{1}{2}(\partial_1 - i\partial_2) \equiv \partial$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \equiv \overline{\partial}$, meaning that ϵ is holomorphic and $\bar{\epsilon}$ is antiholomorphic. The mathematics is the same for the holomorphic and antiholomorphic sectors, so we sometimes write equations in the holomorphic sector only to avoid repetition.

In the complex coordinates the infinitesimal conformal transformation becomes $z \to z + \epsilon$ with ϵ holomorphic, since $z + \epsilon$ is then also holomorphic, we have that a transformation $z \to f(z)$ is conformal for f holomorphic. The metric in complex coordinates is

$$\delta = \mathrm{d}z\mathrm{d}\bar{z} \tag{2.35}$$

and so the associated scale factor is $\Omega = |\partial f|^2$. Laurent expanding ϵ

$$\epsilon = -\sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1}, \tag{2.36}$$

we have for infinitesimal transformations

$$z' = z + \sum_{n\in\mathbb{Z}} (-\epsilon_n z^{n+1}), \tag{2.37}$$

meaning that for a particular n we identify a generator $l_n \equiv -z^{n+1}\partial$. Together with $\bar{l}_n \equiv -\bar{z}^{n+1}\overline{\partial}$, we have the Lie algebra

$$[l_n, l_m] = (m-n)l_{m+n} \tag{2.38}$$

$$\left[\bar{l}_n, \bar{l}_m\right] = (m-n)\bar{l}_{m+n} \tag{2.39}$$

$$\left[l_n, \bar{l}_m\right] = 0 \tag{2.40}$$

as can be trivially verified. This Lie algebra is infinite-dimensional and is called the *Witt algebra*. It is easy to see that the l_n are well defined at the origin $z = 0$ only for $n \geq -1$. To study its regularity at ∞, we write $z = -\frac{1}{w}$ and then

$$l_n = -\left(-\frac{1}{w}\right)^{n-1} \partial_w \tag{2.41}$$

The point $z = \infty$ corresponds to $w = 0$ and from (2.41) we have that for l_n to be well-defined at infinity, we need $n \leq 1$. This means that the globally well-defined and invertible conformal transformations of the Riemann sphere are generated by $l_{-1} = -\partial, l_0 = -z\partial, l_1 = -z^2\partial$. It is easy to see that l_{-1} corresponds to translations and since l_0 generates $z \to az, a \in \mathbb{C}$ and multiplication by a complex number corresponds to a rotation and a dilatation, we expect to be able to construct dilatation and rotation generators from l_0 and $\bar{l}_0$. To see this, write $z = re^{i\phi}$, then $l_0 + \bar{l}_0 = -r\partial_r$ and $i(l_0 - \bar{l}_0) = -\partial_\phi$ so that $l_0 + \bar{l}_0$ is a generator of dilatations and $i(l_0 - \bar{l}_0)$ a generator of rotations. As for l_1, we simply notice that it sends $z \to -\epsilon z^2$, which is an infinitesimal form of the SCT $z \to \frac{z}{\epsilon z+1}$, meaning that l_1 generates SCTs. In summary, we have found that the group of conformal transformations of the Riemann sphere that are everywhere well-defined and invertible is the *Möbius group* $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$ of transformations

$$z \to \frac{az + b}{cz + d} \tag{2.42}$$

with $a, b, c, d \in \mathbb{C}$ satisfying $ad - bc = 1$ as per the invertibility condition (by scaling the determinant is brought to 1) and there is a $\mathbb{Z}_2$ equivalence $(a, b, c, d) \sim (-a, -b, -c, -d)$. These transformations are distinguished by the fact that they transform circles to circles and that it can map any three points to $0, 1$ and ∞. It is not a surprise that the Möbius group is isomorphic to $SO(3, 1)$, which appears in our previous analysis of arbitrary dimension.

2.2 Polyakov's conjecture

Polyakov conjectured in [40] that a local theory invariant under translations, rotations and dilatations is also invariant under the larger conformal group. We have already encountered scale invariance in our discussion of fixed point theories in (1.7). Most RG flows end in such fixed points so the enhancement of their symmetry is of great significance for the study of the space of QFTs. Suppose we have a local theory with an action S and a stress tensor $T_{\mu\nu}$ (that is, our charges come from integrals of the stress tensor), which is defined as the susceptibility of the action with respect to an infinitesimal change in the metric, meaning that using (2.2), we have

$$\delta S = \frac{1}{(2\pi)^{D-1}} \int \mathrm{d}x^D T_{\mu\nu} \nabla^\mu \epsilon^\nu \tag{2.43}$$

Choosing $\epsilon^\mu = a^\mu$ for translations, we get by integrating by parts the conservation law $\partial_\mu T^{\mu\nu} = 0$. The rotations $\epsilon^\mu = R^\mu{}_\nu x^\nu$ with R antisymmetric imply that $T_{\mu\nu} = T_{\nu\mu}$. Finally scaling $\epsilon^\mu = \lambda x^\mu$ implies that $T^\mu{}_\mu$ meaning that T is traceless. From this (classical) conformal invariance follows since we can do a sequence of steps

$$
\begin{aligned}
\delta S &= \frac{1}{(2\pi)^{D-1}}\int \mathrm{d}x^D T_{\mu\nu}\nabla^\mu\epsilon^\nu \\
&= \frac{1}{(2\pi)^{D-1}}\frac{1}{2}\int \mathrm{d}x^D (T_{\mu\nu}+T_{\nu\mu})\nabla^\mu\epsilon^\nu \\
&= \frac{1}{(2\pi)^{D-1}}\frac{1}{2}\int \mathrm{d}x^D T_{\mu\nu}(\nabla^\mu\epsilon^\nu+\nabla^\nu\epsilon^\mu) \\
&= \frac{1}{(2\pi)^{D-1}}\frac{1}{D}\int \mathrm{d}x^D T^\mu{}_\mu\,(\nabla\cdot\epsilon) \\
&= 0
\end{aligned}
$$

where (2.3) was used together with the tracelessness of T.

2.3 Operator spectrum and correlators

In the following we shall concern ourselves with scalar fields only. Such fields ϕ are characterised by their *conformal dimension* (*weight*) Δ, which encodes their response to global scaling $x \to \lambda x$

$$\phi'(\lambda x) = \lambda^{-\Delta}\phi(x) \tag{2.44}$$

Since conformal transformations are local rescalings, inspired by (2.44), we define the so-called *primary* fields V by the property that they transform under an arbitrary conformal transformations $x \to x'$ as

$$V'(x') = b(x)^{-\Delta}V(x) \tag{2.45}$$

for $b(x) = \sqrt{\Omega(x)}$ with $\Omega(x)$ the familiar scale factor of (2.2). In $D = 2$ there is a peculiarity that formally the field may be taken to depend on both z and $\bar{z}$, so that for $z \to f(z)$, (2.45) lifts to

$$V'(z,\bar{z}) = \left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^h \left(\frac{\mathrm{d}\bar{f}}{\mathrm{d}\bar{z}}\right)^{\bar{h}} V(f(z),\bar{f}(\bar{z})) \tag{2.46}$$

while we used the before derived relation between Ω and f. Such a primary is said to have a conformal dimension $(h,\bar{h})$. Another specialty of two dimensions is that there are infinitely many conformal transformations, but only finitely many that form the conformal group. If a field transforms like (2.46) for f in the conformal group, it is called *quasiprimary* (primaries are always quasiprimaries). Other fields are called *secondary.*

The correlation functions of primaries are heavily kinematically constrained. To begin, they have to be covariant with respect to conformal transformations, meaning

$$\langle V(x')\dots V(y')\rangle = b(x)^{-\Delta}\cdots b(y)^{-\Delta}\langle V(x)\dots V(y)\rangle \tag{2.47}$$

Luckily, we have developed the projective null cone, which enables us to build correlators that are automatically covariant in the sense (2.47). Consider two scalar fields $V(X)$ and $V(Y)$ of weight Δ living in the embedding space, then by $X^2 = 0$ and $Y^2 = 0$ and the right behavior under scaling, we must have

$$\langle V(X_1)V(X_2)\rangle \sim \frac{1}{(X_1\cdot X_2)^{\Delta}}, \tag{2.48}$$

where the covariance is ensured by $X\cdot Y$ being a Lorentz invariant. By (2.34), we then have

$$\langle V(x_1)V(x_2)\rangle = \frac{1}{|x_{12}|^{2\Delta}}, \tag{2.49}$$

where we normalised the fields so that 1 appears in the numerator and defined $x_{ij} \equiv x_i - x_j$. It can be shown that for different fields V_i, V_j, we can rotate the field basis so that

$$\langle V_i(x_1)V_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta}} \tag{2.50}$$

Analogously, the three-point function for three ϕ_i with weight Δ_i must be

$$\langle V_1(X_1)V_2(X_2)V_3(X_3)\rangle \sim \frac{1}{(X_1\cdot X_2)^{\alpha_{123}}(X_1\cdot X_3)^{\alpha_{132}}(X_2\cdot X_3)^{\alpha_{231}}}, \tag{2.51}$$

where from covariance under scaling, we must have

$$\begin{aligned}\alpha_{123}+\alpha_{132} &= \Delta_1\\ \alpha_{123}+\alpha_{231} &= \Delta_2\\ \alpha_{132}+\alpha_{231} &= \Delta_3\end{aligned}$$

which is solved by

$$\alpha_{ijk} = \frac{\Delta_i+\Delta_j-\Delta_k}{2} \tag{2.52}$$

so that from (2.34)

$$\langle V_1(x_1)V_2(x_2)V_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}}, \tag{2.53}$$

where the C_{123} is called a *structure constant* of the CFT. For a scalar CFT, these together with conformal weights specify the CFT data. The formula (2.53) essentially gave birth to the study of CFTs since Polyakov [40] used it to compute the $\epsilon\sigma\sigma$ correlator in the critical Ising and got a match with Onsager's solution. In this work, the only four-point function of interest will be for identical fields, and using the same techniques as before, it is easy to see that

$$\langle V(x_1)V(x_2)V(x_3)V(x_4)\rangle = G(\xi)\prod_{i<j}^{4}|x_{ij}|^{-\frac{2}{3}\Delta} \tag{2.54}$$

is indeed correct by having the proper scaling behavior and it is built from projected Lorentz invariants. We call G the invariant part of the four-point function and ξ is called the *cross-ratio* and is given by

$$\xi = \frac{x_{12}x_{34}}{x_{13}x_{24}} \tag{2.55}$$

The cross-ratio itself is also invariant in the sense that it is an $SL(2,\mathbb{C})$ invariant. The just derived forms of correlators hint at the fact that CFTs are very different from the usual QFTs. For example, their power-like behavior implies that their Fourier transforms have no isolated poles in the p^2 plane (contrast this with the mass-shell pole in (1.48) resulting from the exponential damping of the Feynman propagator) and so when one tries to compute scattering amplitudes by the LSZ formula, which gets contributions from poles of momentum space correlators, one gets zero. In $D = 2$ the exact same formulae as (2.50), (2.53) and (2.54) hold for quasiprimaries with potentially being lifted when the field has dependence on both z and $\bar{z}$, for example

$$\langle V_i(z_1,\bar{z}_1)V_j(z_2,\bar{z}_2)\rangle = \frac{\delta_{ij}}{|z_{12}|^{2h}|\bar{z}_{12}|^{2\bar{h}}}, \tag{2.56}$$

otherwise, we just replace $\Delta \to h$ and $x \to z$ for purely holomorphic (*chiral*) fields $V(z)$ and analogously for purely antiholomorphic (*antichiral*) fields. Having exploited the general D connection to the Lorentz group, we will restrict ourselves to $D = 2$ and complex coordinates only from now on. This does not mean, however, that many of our formulae do not have a general D analog. We also work with chiral fields to avoid repetition.

First we would like to know how primaries respond to infinitesimal conformal transformations. To do so, we expand

$$\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^h = 1 + h\epsilon + O(\epsilon^2), \tag{2.57}$$

so that using $V(z+\epsilon) = V(z) + \epsilon\partial V(z) + O(\epsilon^2)$, we have

$$\delta_\epsilon V(z) = (h\partial\epsilon + \epsilon\partial)V(z), \tag{2.58}$$

where $\delta_\epsilon V(z)$ is the change in V. From the energy-momentum $T_{\mu\nu}$ tensor, we can construct the conserved currents $j_\mu = T_{\mu\nu}\epsilon^\nu$ and from the proof of Polyakov's conjecture (2.2), we see that these correspond to conformal transformations. By this, we would expect $\delta_\epsilon V(z)$ to be computable from $T(z)\epsilon(z)$. We show this later after quantising, but first need to see how the $T(z)$ arises. From a simple change of coordinates, we have from the tensorial properties of T

$$\begin{aligned}
T_{zz} &= \frac{1}{4}(T_{00} - T_{11} - 2iT_{10}) \\
T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} - T_{11} + 2iT_{10}) \\
T_{\bar{z}z} &= T_{z\bar{z}} = \frac{1}{4}(T_{00} + T_{11})
\end{aligned}$$

and from the tracelessness condition $T_{00} + T_{11} = 0$, we see that in the complex coordinates the stress-energy tensor is diagonal. Rewriting the conservation law $\partial_\mu T^{\mu\nu}$ for T to complex coordinates, we get $\overline{\partial} T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$ meaning that T_{zz} is holomorphic and $T_{\bar{z}\bar{z}}$ is antiholomorphic justifying the notation $T(z) \equiv T_{zz}$, $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$.

2.4 Radial quantisation

In equation (A), we computed correlation functions as time ordered VEVs. For this we had to pick a Lorentz frame (time direction) and then we had access to a canonical structure (see (1.24), (1.25)) on the Hilbert spaces defined at constant times. Such Hilbert spaces can be connected by an evolution operator given by exponentiating the Hamiltonian. In other words, we found it advantageous to choose a foliation of spacetime by constant time surfaces that respected Poincaré invariance and define Hilbert spaces on such surfaces.

In CFT, we have scale invariance so it is tempting to foliate the Euclidean space by spheres of constant radius. Our evolution operator should then be constructed by exponentiating the generator of dilatations $D = -z\partial$. Then on each sphere a Hilbert space of states lives and an ordering by time becomes an ordering by the radius of such spheres. We will build up our Hilbert space by acting with modes of conformal fields. To do this, we consider the conformal map $z = e^w$ from the cylinder to the plane, with $w = \tau + i\sigma$, $\tau \in \mathbb{R}$

and $\sigma \sim \sigma + 2\pi$. Here τ has an interpretation of Euclidean time and a Fourier expansion of a primary V_{cyl} living on the cylinder

$$V_{cyl} = \sum_{n\in\mathbb{Z}} V_n e^{-n(\tau+i\sigma)} = \sum_{n\in\mathbb{Z}} V_n z^{-n} \tag{2.59}$$

becomes an expansion

$$V = \sum_{n\in\mathbb{Z}} V_n z^{-n-h} \tag{2.60}$$

on the plane. We see that the limit $\tau \to -\infty$ corresponds to $z \to 0$ so we define the so-called *asymptotic in-states*

$$|V\rangle = \lim_{z\to 0} V(z) |0\rangle \tag{2.61}$$

with $|0\rangle$ being the $SL(2,\mathbb{C})$ invariant vacuum. For the definition (2.61) to make sense, we need $V_n |0\rangle = 0$ for $n > -h$. This means that

$$|V\rangle = V_{-h} |0\rangle \tag{2.62}$$

The correspondence between Euclidean time τ and Minkowskian time t is $\tau = it$ so that Hermitean conjugation changes $\tau \to -\tau$, meaning that in the plane coordinates, it has the effect $z \to \frac{1}{\bar{z}}$. This means that the Hermitean conjugate of a conformal field on the plane should be defined as

$$V^\dagger(\bar{z}) = \bar{z}^{-2h} V\left(\frac{1}{\bar{z}}\right) \tag{2.63}$$

so that

$$V^\dagger(z) = \sum_{n\in\mathbb{Z}} z^{n-h} V_n \tag{2.64}$$

corresponding to $V_n^\dagger = V_{-n}$. The *asymptotic out-states* are then defined as

$$\langle V| = \lim_{z\to 0} \langle 0| V^\dagger(z) = \lim_{w\to\infty} w^{2h} \langle 0| V(w) \tag{2.65}$$

meaning that $\langle 0| V_n = 0$ for $n < h$ so we have

$$\langle V| = \langle 0| V_h \tag{2.66}$$

as expected by the Hermitean conjugation properties of modes.

We should be able to act with conformal symmetry generators on our Hilbert space so that $\delta_\epsilon V(z) = [Q, V]$ for some charge Q. As usual, such a charge is built from an integral over a constant radius (time) slice. This means that

$$Q = \frac{1}{2\pi i} \oint_C \mathrm{d}z\, T(z)\epsilon(z) \tag{2.67}$$

where C is a circle and by the holomorphicity of the integrand, we don't need to specify its radius. Since ϵ is any holomorphic function, we have an infinity of conserved charges, this is what makes two-dimensional CFT so powerful. For the change in V, we have

$$\delta_\epsilon V(w) = \frac{1}{2\pi i}\oint_C \mathrm{d}z\,[T(z)\epsilon(z), V(w)] \tag{2.68}$$

To make sense of the commutator, we need to define radial ordering

$$R(\phi(z)\chi(w)) = \begin{cases} \phi(z)\chi(w) & \text{if } |z| > |w|, \\ \chi(w)\phi(z) & \text{if } |w| > |z| \end{cases} \tag{2.69}$$

so that

$$\begin{aligned} \oint_C \mathrm{d}z[T(z)\epsilon(z), V(w)] \quad &= \quad \oint_{|z|>|w|} \mathrm{d}z\,\epsilon(z)\Big[T(z)V(w)\Big] \\ &\quad - \oint_{|z|<|w|} \mathrm{d}z\,\epsilon(z)\Big[V(w)T(z)\Big] \end{aligned} \tag{2.70}$$

Using the fact that the difference of two contour integrals as in (2.71) can be deformed into a countour integral over the contour C_w encircling the intermediate point w, see figure (2.1), we get

$$\delta_\epsilon V(w) = \frac{1}{2\pi i}\oint_{C_w} \mathrm{d}z\,\epsilon(z) R(T(z)V(w)) \tag{2.71}$$

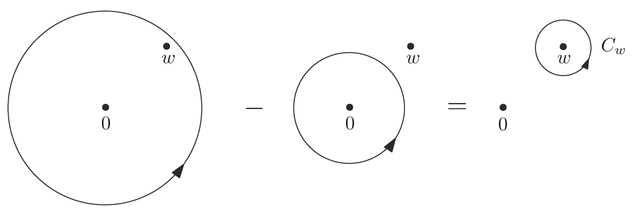


Figure 2.1 The difference of two contours turns into one contour

We observe that (2.71) can be satisfied if we make the expansion around the point w

$$R(T(z)V(w)) \sim \frac{h}{(z-w)^2}V(w) + \frac{1}{z-w}\partial V(w) \tag{2.72}$$

This is an example of an operator product expansion (OPE) of the stress-energy tensor T with a primary V. Such expansions appear in ordinary QFTs as well, encoding the UV behavior as operators approach one another. But since ordinary QFTs have a mass scale, the OPEs are merely asymptotic meaning that they cannot be easily extended beyond the UV region. Such an extension is provided by performing RG transforms and is nontrivial. In CFTs the OPEs have a finite radius of convergence enabling us to extend the short-distance behavior of the theory to larger distances. In general OPEs have the form

$$\phi_i(z)\phi_j(w) = \sum_k C_{ijk}(z-w, \partial_w)\phi_k(w) \tag{2.73}$$

where we wrote an equality emphasizing the finite radius of convergence. By dimensional analysis (∂ adds to the conformal dimension, see (2.23)), we conclude that

$$V_i(z)V_j(w) \sim \sum_k C_{ijk} \frac{V_k(w)}{|z-w|^{h_i+h_j-h_k}} \tag{2.74}$$

holds for primaries with $|z-w|$ small, with the C_{ijk} being the structure constants that appear in (2.53). The OPEs can be iteratively inserted into correlation functions enabling one to reduce any correlator to one-point functions. The order in which one performs the OPEs shouldn't matter. This is called the associativity of the OPE and it leads to stringent consistency conditions on the correlators often enabling one to bootstrap the theory.

We conclude this section by stating the state-operator correspondence. As we've already shown, one can build a Hilbert space from states of the form $|V\rangle$ created by an operator located at the origin $z=0$ acting on the vacuum $|0\rangle$. This is the operator $\to$ state mapping and it is essentially automatic. What is nontrivial is the state $\to$ operator mapping. Suppose we have some dilatation eigenstates

$$D\,|V_i\rangle = h_i\,|V_i\rangle \tag{2.75}$$

corresponding to operators of definite weight. Now we want to compute the correlators of the corresponding operators $V_i(z_i)$ using the path integral. We cut out holes B_i centered at z_i and glue the states $|V_i\rangle$ to their boundaries. This means that the fields ϕ in path integral measure now have support only outside of these holes, giving us

$$\langle V_1(z_1)\dots V_n(z_n)\rangle = \int \prod_{i=1}^{n} \mathcal{D}\phi_{S_i}\,\langle \phi_{S_i}|V_i\rangle \int_{\phi_{\partial B_i}=\phi_{S_i}} \mathcal{D}\phi(z\notin B_i)e^{-S} \tag{2.76}$$

where we still have to integrate over the boundary values at ∂B_i. The scale invariance comes into play in that the $V_i(z_i)$ have to be far enough apart for the B_i not to overlap, which would lead to overcounting. We haven't specified the radii of the holes so this is a real problem, in a sense the construction shouldn't depend on the radii. This is achieved only if we can always move the z_i away from one another and this is precisely achieved by the covariance

$$\langle V_1(z_1) \dots V_n(z_n) \rangle = \lambda^{\sum_i h_i} \langle V_1(\lambda z_1) \dots V_n(\lambda z_n) \rangle \tag{2.77}$$

Taking the radii very small, we essentially do not change the support of ϕ and thus specify an insertion of a local operator at the center of a ball. We have thus shown that states of definite conformal weight correspond to local operators.

2.5 Virasoro algebra

To further investigate the character of the stress-energy tensor T, we do a Laurent expansion

$$T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \tag{2.78}$$

where by the tensorial nature of T, we anticipate $h_T = 2$. Let us now look at the family of charges Q_n with $\epsilon = -z^{n+1}\epsilon_n$. These can be easily evaluated by the residue theorem

$$Q_n = -\epsilon_n \sum_{m \in \mathbb{Z}} \frac{1}{2\pi i} \oint_C \mathrm{d}z \, L_m z^{n-m-1} = -\epsilon_n L_n \tag{2.79}$$

Does this mean that $L_n = l_n$ for l_n in the Witt algebra (see (2.37))? In a classical theory this would be the case, but in the quantum theory the symmetry develops an anomaly. In a sense this is a good thing, since the resulting representation theory is richer. The resulting algebra is then called the *Virasoro algebra* and it can be motivated in several ways. For example, we could restrict the form of the TT OPE and from a contour argument get the commutation relations, since

$$L_n = \frac{1}{2\pi i} \oint \mathrm{d}z \, z^{n+1} T(z) \tag{2.80}$$

We motivate it by the fact that the Witt algebra doesn't possess unitary irreducible representations, so that in a quantum theory a central extension is needed, see [43]. Luckily such a central extension is unique and closely following [28] we may write the ansatz

$$[L_m, L_n] = (m-n)L_{m+n} + p(m,n)\, c \tag{2.81}$$

with c an element of the Lie algebra center called the *central charge*. An initial trivial consistency condition is $p(m,n) = -p(n,m)$. We then define the modified Virasoro generators

$$
\begin{aligned}
\hat{L}_n &= L_n + \frac{p(n,0)}{n}\,c, \;\; n \neq 0 \\
\hat{L}_0 &= L_0 + \frac{p(1,-1)}{2}\,c
\end{aligned}
\tag{2.82}
$$

It is rather simple to compute

$$
\begin{aligned}
\left[\hat{L}_n, \hat{L}_0\right] &= nL_n + p(n,0)\,c = n\hat{L}_n \\
\left[\hat{L}_1, \hat{L}_{-1}\right] &= 2L_0 + p(1,-1)\,c = 2\hat{L}_0
\end{aligned}
$$

implying that we can always redefine the generators so that $p(1,-1) = p(n,0) = 0$. We do this redefinition and drop the hats from now on. We continue by computing the Jacobi identity

$$
\begin{aligned}
0 &= [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] \\
&= (m-n)p(m+n,0)\,c + np(n,m)\,c - mp(m,n)\,c \\
&= (m+n)p(n,m)\,c
\end{aligned}
\tag{2.83}
$$

from which we see that $p(n,m) = 0$ for $n \neq m$. Another interesting Jacobi identity is

$$
\begin{aligned}
0 &= [[L_{1-n}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{1-n}] + [[L_{-1}, L_{1-n}], L_n] \\
&= 0 + (n+1)p(n-1, 1-n)\,c + (n-2)p(-n,n)\,c
\end{aligned}
$$

from which we get the recursion identity

$$
p(n,-n) = \frac{n+1}{n-2}\,p(n-1,1-n) = \frac{1}{12}(n^3 - n) \tag{2.84}
$$

which we solved in the normalisation $p(2,-2) = \frac{1}{2}$ which will give $c = 1$ for a theory of a single free boson (2.8.3). In summary, we found that the unique central extension of the Witt algebra is

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \tag{2.85}
$$

There is an antiholomorphic copy of the Virasoro algebra, where we simply perform $L_m \to \bar{L}_m$ with $\left[L_m, \bar{L}_n\right] = 0$. For this second copy there is in general a different central charge $\bar{c}$. It is interesting to see that the conformal group did not get an anomaly since the L_{-1}, L_0 and L_1 still have the same commutation relations. To prove that no nontrivial representations exist for

$c = 0$, we compute the norm of $L_{-m}|0\rangle$ for $m > -2$ (in a Hermitean product sense)

$$\|L_{-m}|0\rangle\|^2 = \langle 0| L_m L_{-m} |0\rangle = \frac{c}{12}(m^3 - m) \tag{2.86}$$

where we used the algebra (2.85) and that $L_m|0\rangle = 0$ since T has weight 2. So for $c = 0$, all states created from the vacuum by the action of the conformal generators L_m with $m \leq 1$ have zero norm leading to trivial representation theory (these Virasoro generators act as raising operators as we'll see later). It is trivial to reverse-engineer the TT OPE from the Virasoro algebra (2.85) by using

$$[L_m, L_n] = \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)] \tag{2.87}$$

and going through similar steps that led to (2.72) with an ansatz reflecting that T is bosonic, meaning $T(z)T(w) = T(w)T(z)$. This leads to

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{2.88}$$

where we dropped the radial ordering and we see consistency with the expansion (2.78) since (2.88) tells us that T has weight 2. It is interesting to investigate the transformation properties of T. From (2.67) and writing the Laurent expansion

$$\epsilon(z) = \epsilon(w) + \partial\epsilon(w)(z-w) + \frac{1}{2}\partial^2\epsilon(w)(z-w)^2 + \frac{1}{3!}\partial^3\epsilon(w)(z-w)^3 + \ldots, \tag{2.89}$$

we get for the change in T

$$\delta_\epsilon T(z) = \frac{c}{12}\partial^3\epsilon(z) + 2T(z)\partial\epsilon(z) + \epsilon(z)\partial T(z) \tag{2.90}$$

where we renamed $w \to z$. This can be exponentiated to the full conformal transformation $z \to f(z)$ under which $T \to T'$ as

$$T'(z) = \left(\frac{\partial f}{\partial z}\right)^2 T(f(z)) + \frac{c}{12} S(f)(z) \tag{2.91}$$

with $S(f)$ being the Schwarzian derivative

$$S(f) = \frac{\partial^3 f}{\partial f} - \frac{3}{2}\left(\frac{\partial^2 f}{\partial f}\right)^2 \tag{2.92}$$

The textbook procedure is now to expand the Schwarzian for an infinitesimal transformation

$$S(z+\epsilon) = \frac{\partial^3\epsilon}{1+\partial\epsilon} - \frac{3}{2}\left(\frac{\partial^2\epsilon}{1+\partial\epsilon}\right)^2 \sim \partial^3\epsilon \tag{2.93}$$

to recover (2.90) and then verify that the conformal transforms compose correctly

$$\begin{aligned}
T''(u) &= \left(\frac{\partial u}{\partial w}\right)^2 T'(w) + \frac{c}{12}S(u)(w) \\
&= \left(\frac{\partial u}{\partial w}\right)^2 \left[\left(\frac{\partial w}{\partial z}\right)^2 T(z) + \frac{c}{12}S(w)(z)\right] + \frac{c}{12}S(u)(w) \\
&= \left(\frac{\partial u}{\partial z}\right)^2 T(z) + \frac{c}{12}S(u)(z)
\end{aligned}$$

where we used a composition property of the Schwarzian

$$S(g \circ f) = (\partial f)^2 S(g) \circ f + S(f) \tag{2.94}$$

this means that $S(f)$ doesn't transform like a function, but rather like a quadratic differential. We show that the Schwarzian of a Möbius transformation is zero

$$S\left(\frac{az+b}{cz+d}\right)(z) = \frac{6c^2}{(cz+d)^2} - \frac{3}{2}\frac{4c^2}{(cz+d)^2} = 0 \tag{2.95}$$

meaning that T is a quasiprimary operator. The interpretation is that the Schwarzian $S(f)$ measures how much a map f doesn't preserve circles. We could ask why is the Schwarzian derivative called a derivative? To see this, we define a cross-ratio in a slightly different convention suited for this computation

$$[x_1, x_2; x_3, x_4] \equiv \frac{x_{13}x_{24}}{x_{12}x_{34}} \tag{2.96}$$

We then want to see how much f changes the cross-ratio meaning that we compare $[f(z), f(z+\epsilon); f(z+2\epsilon), f(z+3\epsilon)]$ with $[z, z+\epsilon; z+2\epsilon, z+3\epsilon] = 4$. Expanding the $f(z+n\epsilon)$ to third order, we get

$$[f(z), f(z+\epsilon); f(z+2\epsilon), f(z+3\epsilon)] - 4 = -2S(f)(z)\epsilon^2 + O(\epsilon^3) \tag{2.97}$$

We note that the choice of points $z, z+\epsilon, z+2\epsilon, z+3\epsilon$ was purely for simplicity, the result (2.97) is general but the prefactor of the Schwarzian may change. Another way of seeing why the Schwarzian must appear is that the CFT consistency conditions (2.94) and (2.95) determine it uniquely [44]. We are not too satisfied with simply verifying that the Schwarzian works, in particular we will explicitly make it appear in the free boson (2.8.3) following [45], where many other properties of the Schwarzian can be found.

One can compute how T changes by going from the plane to the cylinder, corresponding to $f(z) = \ln z$. Equation (2.91) then gives

$$T_{cylinder}(w) = z^2 T_{plane}(z) + \frac{c}{24} \tag{2.98}$$

with $w = \ln z$. This means that there is a shift in the zero mode of T so that $L_0 \to L_0 - \frac{c}{24}$ and since the plane Hamiltonian is just the dilatation operator $D = L_0$ (see the discussion under (2.41) to see the form of D), we must have for the cylinder frame Hamiltonian

$$H = L_0 - \frac{c}{24} \tag{2.99}$$

We see a Casimir energy appear and this is so because we introduced a length scale by giving a finite circumference 2π to our cylinder.

The nontrivial transformation properties of T give rise to a conformal anomaly in the trace

$$T^{\mu}{}_{\mu} = -\frac{c}{12}\mathcal{R} \tag{2.100}$$

with $\mathcal{R}$ being the Ricci scalar of the complex two-manifold on which we define our CFT. This formula can be understood by noting that for $c = 0$, we expect the anomaly to vanish and by dimensional analysis we conclude that the RHS of (2.100) has to contain a scalar with no more than two derivatives.

2.6 Representation theory

By the usual procedure, we need to identify ladder operators and a Cartan subalgebra. We take this subalgebra to consist of L_0, this is so because it essentially plays the role of energy by $D = L_0$. The ladder operators will be given by appropriate Laurent modes, which we recall to satisfy $V_n |0\rangle = 0$ for $n > -h$. Our first step is to derive a commutation relation between the Virasoro generators and V_n. To do so, we compute

$$\begin{aligned}
[L_m, V_n] &= \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n+h-1} T(z)V(w) \\
&= \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n+h-1} \left[\frac{h}{(z-w)^2} V(w) + \frac{1}{z-w} \partial V(w) \right] \\
&= \oint \frac{\mathrm{d}w}{2\pi i} \left[(m+1) w^{n+m+h-1} h V(w) + w^{n+m+h} \partial V(w) \right] \\
&= \oint \frac{\mathrm{d}w}{2\pi i} \Big[(m+1)h - (n+m+h) \Big] w^{n+m+h-1} V(w) \\
&= ((h-1)m - n) V_{m+n}
\end{aligned} \tag{2.101}$$

where we used the expansion (2.60) with (2.80) and the OPE (2.72). From (2.101), we see that

$$[L_0, V_n] = -nV_n \tag{2.102}$$

leading to

$$L_0 V_n |0\rangle = -n V_n |0\rangle \tag{2.103}$$

where we used $L_0 |0\rangle = 0$. We thus have that V_n for $n > -h$ are the annihilation operators since they annihilate $|0\rangle$ and V_n for $n \leq -h$ are the creation operators. Now we can define a normal ordering analogously to our previous definition in chapter (1), that is to move annihilation operators to the right of the creation operators. On the level of fields we write

$$\phi(z)\chi(w) - \text{sing.} = \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} :\chi \partial^n \phi: (w) \tag{2.104}$$

where sing. are the singular terms as $z \to w$. One can then show that

$$(:\chi\phi:)_n = \sum_{k>-h_\phi} \chi_{n-k}\phi_k + \sum_{k\leq -h_\phi} \phi_k \chi_{n-k} \tag{2.105}$$

for the modes of $:\chi\phi:$ and these are normal ordered in the usual sense. By differentiation

$$\partial\phi(z) = \partial \sum_n z^{-n-h}\phi_n = \sum_n (-n-h) z^{-n-(h+1)}\phi_n \tag{2.106}$$

another useful relation

$$(:\chi\partial\phi:)_n = \sum_{k>-h_\phi-1} (-h_\phi - k)\chi_{n-k}\phi_k + \sum_{k\leq -h_\phi-1} (-h_\phi - k)\phi_k\chi_{n-k} \tag{2.107}$$

easily follows.

We now build the *highest weight representations* (HWRs) of the Virasoro algebra, also called *Verma modules* corresponding to a primary V. The state $V_{-h}|0\rangle \equiv h$ satisfies

$$L_n |h\rangle = [L_n, V_{-h}]|0\rangle = (h(n+1) - n)V_{-h+n}|0\rangle = 0 \tag{2.108}$$

where $n > 0$ and we used (2.101) and the fact that V_{-h+n} acted as an annihilation operator. This means that new nontrivial states are obtained from $|h\rangle$ only when we act with the negatively moded Virasoros, obtaining the so-called *descendant* states. Minus the sum of labels of the acting Virasoros is then called the *level*, for example $L_{-1}L_{-1}|0\rangle$ and $L_{-2}|0\rangle$ both correspond to level 2. The number of states $P(N)$ at each level N is then given by the number of partitions of N for which the generating function is

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} P(N) q^N = 1 + q + 2q^2 + 3q^3 + \ldots, \tag{2.109}$$

which can be checked by expanding the individual terms $\frac{1}{1-q^n}$ into a geometric series. The Hardy-Ramanujan formula gives the large N asymptotics

$$P(N) \sim \frac{\exp\left\{\pi\sqrt{\frac{2N}{3}}\right\}}{4\sqrt{3}N} \tag{2.110}$$

which indicates a rapid growth of the number of states at each level. Because of this, we will only work out the first few levels explicitly. We start by computing the action of L_{-1} on V

$$\begin{aligned}(L_{-1}V)(z) &= \oint \frac{\mathrm{d}w}{2\pi i} T(w)V(z) \\ &= \partial V(z)\end{aligned}$$

where we used (2.80) and the OPE (2.72), meaning that L_{-1} acts as a derivative on fields. From this we find at level 1 the state $L_{-1}|h\rangle$ corresponding to the operator ∂V. At level 2, we can act with L_{-1} twice, finding a state $L_{-1}L_{-1}|h\rangle$ corresponding to the operator $\partial^2 V$. We can also act with L_{-2} and to see the corresponding operator, we consider the normal ordered product

$$:VT: = \sum_{n\in\mathbb{Z}} z^{-n-2-h}(:VT:)_n, \tag{2.111}$$

meaning that to obtain a state, we act on the vacuum only with the $n = -2-h$ mode. From equation (2.105), we have

$$(:VT:)_{-2-h} = \sum_{k>-2} V_{-2-h-k}L_k + \sum_{k\leq -2} L_k V_{-2-h-k} \tag{2.112}$$

and when acting on $|0\rangle$, only the term $L_{-2}V_{-h}|0\rangle$ survives, meaning that $L_{-2}|h\rangle$ corresponds to the operator $:VT:$. At level 3, we again have $L_{-1}L_{-1}L_{-1}|h\rangle$ corresponding to $\partial^3 V$, from the previous level, we have that $L_{-2}L_{-1}|h\rangle$ corresponds to $:\partial VT:$. There is one more state $L_{-3}|h\rangle$, which we investigate with the help of the formula (2.107). Consider

$$:V\partial T: = \sum_{n\in\mathbb{Z}} z^{-n-3-h}(:V\partial T:)_n, \tag{2.113}$$

so that only the $n = -3-h$ mode contributes when acting on $|0\rangle$. The modes are now expressed by (2.107) as

$$(:V\partial T:)_{-3-h} = \sum_{k>-3} (-2-k)V_{-3-h-k}L_k + \sum_{k\leq -3} (-2-k)L_k V_{-3-h-k} \tag{2.114}$$

meaning that only $L_{-3}\phi_{-h}|0\rangle$ survives when acting on the vacuum. We summarise our findings below in table (2.2). In general our results seem to

Level	State	Operator
0	$\vert h\rangle$	V
1	$L_{-1}\vert h\rangle$	∂V
2	$L_{-1}L_{-1}\vert h\rangle$	$\partial^2 V$
2	$L_{-2}\vert h\rangle$	$:VT:$
3	$L_{-1}L_{-1}L_{-1}\vert h\rangle$	$\partial^3 V$
3	$L_{-2}L_{-1}\vert h\rangle$	$:\partial VT:$
3	$L_{-3}\vert h\rangle$	$:V\partial T:$

Table 2.2 Verma module up to level 3

indicate that the Verma module is reflected on the operator side by using derivatives and normal ordering with T on the primary V, building an entire *conformal family* of descendants having a common *ancestor* V in the process. It would be great if the correlators of descendants were obtainable from the correlators of primaries that we worked out before. That this should be the case can be seen by thinking about the corresponding problem on the Verma module side, there one can get rid of all L_{-n} in VEVs by using the commutation relations (2.85) and (2.101), leaving only a VEV with the modes ϕ_m behind. The computation is rather simple, involving only the OPE (2.72) and some contour pulling with the result

$$\langle L_{-n}V(w)V_1(w_1)\dots V_N(w_N)\rangle = \mathcal{L}_{-n}\langle V(w)V_1(w_1)\dots V_N(w_N)\rangle, \tag{2.115}$$

where the differential operator $\mathcal{L}_{-n}$ is given by

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left[\frac{(n-1)h_i}{(w_i-w)^n} - \frac{1}{(w_i-w)^{n-1}}\partial_{w_i}\right] \tag{2.116}$$

An immediate consequence is that if two primaries are orthogonal, then the conformal families that they generate are also orthogonal. This hints at the fact that the OPE should also stay within a conformal family, indeed the equation

$$V_i(z)V_j(w) = \sum_k\sum_{\{m\}} C_{ijk}\frac{\beta^{\{m\}}{}_{ijk}V^{\{m\}}{}_k(w)}{(z-w)^{h_i+h_j-h_p-M}}, \tag{2.117}$$

where the multi-index $\{m\}$ on the field V labels all descendant fields

$$\{L_{-m_1}\dots L_{-m_n}\}V_k(w)$$

with $M=\sum_i m_i$ and the $\beta^{\{m\}}{}_{ijk}$ numbers holds. This notation means for example that $L_{-2}V = :VT:$ by table (2.2).

We may ask whether the Hermitean inner product is positive definite in the HWR. We first try to eliminate some negative norm states by extending the computation of (2.86) to obtain

$$\|L_{-m}\,|h\rangle\|^2 = 2mh + \frac{c}{12}(m^3 - m) \tag{2.118}$$

giving us the constraint $c > 0$ for nontrivial unitary theories (simply consider m large enough) and also $h \geq 0$ (simply consider m small enough). After obtaining these basic constraints, we proceed by eliminating the zero-norm states, which also indicate the reducibility of a HWR since they are orthogonal to the physical states in a HWR. The presence of a zero-norm state at level N is indicated by the root of the Kač determinant $\det M_N(h,c)$ of a level N Gram matrix $M_N(h,c)$ with entries

$$\langle h|\prod_i L_{k_i}\prod_j L_{-m_j}\,|h\rangle$$

with $\sum_i k_i = \sum_j m_j = N$. This is so, because such roots indicate zero eigenvalues of the Gram matrix. The reason why we can separate the total Gram matrix into the blocks $M_N(h,c)$ by level is that states with different levels have vanishing overlaps since only annihilation or only creation operators survive the commuting in the expectation value (2.119). We continue by computing the Gram matrices for level 1 and level 2 with the level 1 being trivial

$$M_1(h,c) = \langle h|\,L_1L_{-1}\,|h\rangle = 2h = \det M_1(h,c) \tag{2.119}$$

where we used the formula (2.118) and we see that the level 1 root $h_{1,1}(c)$ is zero. The same formula also gives for a diagonal term at level 2 $\langle h|\,L_2L_{-2}\,|h\rangle = 4h+\frac{c}{2}$. For the second diagonal term we get by the use of the Virasoro algebra (2.85)

$$\langle h|\,L_1L_1L_{-1}L_{-1}\,|h\rangle = 4h(2h+1)$$

The offdiagonal terms at level 2 are

$$\langle h|\,L_1L_1L_{-2}\,|h\rangle = \langle h|\,L_2L_{-1}L_{-1}\,|h\rangle = 3\,\langle h|\,L_1L_{-1}\,|h\rangle = 6h$$

where we commuted L_1 and L_{-2} and used $L_1\,|h\rangle = 0$. Putting this together, we have

$$M_2(h,c) = \begin{pmatrix} 4h+\frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \tag{2.120}$$

so that

$$\det M_2(h,c) = 32h\Big(h^2 + \frac{c-5}{8}h + \frac{c}{16}\Big) \tag{2.121}$$

where the two new roots defined by

$$\det M_2(h,c) = 32\Big(h - h_{1,1}(c)\Big)\Big(h - h_{1,2}(c)\Big)\Big(h - h_{2,1}(c)\Big) \tag{2.122}$$

are given by

$$\begin{aligned} h_{1,2}(c) &= \frac{5-c}{16} - \frac{1}{16}\sqrt{(1-c)(25-c)} \\ h_{2,1}(c) &= \frac{5-c}{16} + \frac{1}{16}\sqrt{(1-c)(25-c)} \end{aligned}$$

so that given c, we can expect null vectors at level 2 for Verma modules generated by $|h_{1,2}(c)\rangle$ and $|h_{2,1}(c)\rangle$ and we project these out. We see that the root $h_{1,1}$ is still present at level 2 and this turns out to be a general feature as embodied by the general formula for the Kač determinant guessed by Kač [46]

$$\det M_N(h,c) = \alpha_N \prod_{0<p,q\leq N} \Big(h - h_{r,s}(c)\Big)^{P(N-rs)}, \tag{2.123}$$

where the roots are given by

$$h_{r,s}(m) = \frac{\Big((m+1)r - ms\Big)^2 - 1}{4m(m+1)} \tag{2.124}$$

with $m(c) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$, where the sign is chosen (for $c < 1$, our case of interest) so that m is positive by convention. Apart from removing the null states, we then also have to allow only for the regions in the (h,c) plane, where the Kač determinant is nonnegative (remember that $M_N(h,c)$ is a Gram matrix). Doing this it can be shown that in theories with $c > 1$ all HWRs with $h \geq 0$ are unitary with no need to remove null states. For the degenerate case $c = 1$, we have zeros at $h_n = \frac{n^2}{4}$ with the Kač determinant remaining positive in between, meaning that we only have to remove the null states and our theory becomes unitary for $h \geq 0$. The case $0 < c < 1$ is more involved and is analysed as an example (2.8.1).

2.7 Bulk conformal perturbation theory

In general, the fixed point described by a CFT may have unstable directions, which enable us to trigger an RG flow by a deformation by a relevant operator with $y = D - h > 0$ (see the end of section (1.7)). We may then follow the RG flow perturbatively and obtain information about the off-critical theory, enabling us to see where the RG flow ends. Ideally, we would then like to be able to describe the off-critical theory using the CFT data. It may be the case that the flow ends in a new CFT or perhaps that the theory acquires mass, which should be seen by the IR limit of the off-critical correlators (they will either obey a power law or be exponentially damped). Below we analyse general RG flows in $D = 2$ and then prove the c-theorem, which elucidates the meaning of the central charge c.

Let us begin by studying the response of a correlator

$$\langle A_1(x_1)\dots A_n(x_n)\rangle = \int \mathcal{D}\phi A_1(x_1)\dots A_n(x_n)e^{-S[\phi]} \tag{2.125}$$

to an infinitesimal conformal transformation $x \to x + \epsilon$. By the exact same steps that lead to the proof of Polyakov's conjecture (2.2), we then have the Ward identity

$$\sum_{i=1}^{n}\langle A_1(x_1)\dots A_i(x_i)\dots A_n(x_n)\rangle = \frac{1}{2}\int \mathrm{d}^2x\langle\Theta(x)(\partial\cdot\epsilon)(x)A_1(x_1)\dots A_n(x_n)\rangle \tag{2.126}$$

where $\Theta(x) \equiv T^{\mu}{}_{\mu}(x)$ is the trace of T. Specialising to the case of a global dilatation and writing $\delta A(x) = \epsilon\Big(\frac{1}{2}x\cdot\partial + D\Big)A(x)$, where $\hat{D}$ is an operator encoding the possible dependence on internal indices. Substituting the change in A under the global dilatations to (2.126), we get the special case

$$\sum_{i=1}^{n}\langle A_1(x_1)\dots\Big(\frac{1}{2}x\cdot\partial+\hat{D}\Big)A_i(x_i)\dots A_n(x_n)\rangle = \int \mathrm{d}^2x\langle\Theta(x)A_1(x_1)\dots A_n(x_n)\rangle \tag{2.127}$$

Second, we analyse the dependence of correlators on various couplings g_i present in our action. Writing $S = \int \mathrm{d}^2x\mathcal{L}$ and restricting only to the interaction of scalars, we have a local scalar field belonging to the tangent space of couplings

$$\phi_a(x) = \frac{\partial\mathcal{L}}{\partial g_a} \tag{2.128}$$

If the fields $A_i(x)$ respond to the change in couplings as $\frac{\partial A_i(x)}{\partial g_a} = \hat{B}_a A_i(x)$, we then have from (2.125)

$$\begin{aligned}\frac{\partial}{\partial g_a}\langle A_1(x_1)\dots A_n(x_n)\rangle &= \sum_{i=1}^{n}\langle A_1(x_1)\dots\hat{B}_a A_i(x_i)\dots A_n(x_n)\rangle \\ &\quad -\int \mathrm{d}^2x\langle\phi_a(x)A_1(x_1)\dots A_n(x_n)\rangle\end{aligned} \tag{2.129}$$

Since the trace $\Theta(x)$ is also a scalar, it should be possible to expand it as

$$\Theta(x) = \sum_a \beta^a(\{g\})\phi_a(x), \tag{2.130}$$

where the coefficients $\beta^a(g)$ are called the *beta-functions*. This can be realised by considering the transformation $x \to (1 + \mathrm{d}t)x$, under which $\Theta(x) = \frac{\partial \mathcal{L}}{\partial t}$ and using chain rule, we have

$$\Theta(x) = \sum_k \frac{\partial \mathcal{L}}{\partial g_a}\frac{\partial g_a}{\partial t} = \sum_k \frac{\partial g_a}{\partial t}\phi_a(x), \tag{2.131}$$

meaning that the beta-functions encode the dependence of couplings on the length scale $\beta^a(\{g\}) = \frac{\partial g_a}{\partial t}$. Plugging the expansion (2.130) into the Ward identity (2.127) and using (2.130), we obtain the Callan-Symanzik equation

$$\langle \sum_{i=1}^{n} \Big(\frac{1}{2}x_i \cdot \partial_i + \hat{\gamma}^{(i)}(g)\Big) A_1(x_1)\dots A_n(x_n)\rangle = \sum_a \beta^a(g)\frac{\partial}{\partial g_a}\langle A_1(x_1)\dots A_n(x_n)\rangle, \tag{2.132}$$

where we defined the *matrix of analomalous dimensions*

$$\hat{\gamma}(g) = \hat{D} + \beta^a(g)\hat{B}_a \tag{2.133}$$

whose diagonalisation tells us about operator mixing (the eigenvalues corresponding to anomalous dimensions). What we know for certain is that the trace $\Omega(x)$ does not mix with other operators under RG since T is a conserved current, this leads to

$$\hat{\gamma}(g)\Theta = 2\Theta \tag{2.134}$$

since T has dimension 2. Using the expansion (2.130), we then have

$$\hat{\gamma}\phi_a \equiv \gamma^b{}_a\phi_b = \left(2\delta^b{}_a - \frac{\partial \beta^b}{\partial g^a}\right)\phi_b, \tag{2.135}$$

meaning that if we know the derivatives of beta-functions, we can read off $\hat{\gamma}$ acting on scalars.

We would now like to compute the beta-function of an RG flow triggered by deforming S_0 to S

$$S = S_0 + \sum_i \lambda_i \int \mathrm{d}^2x\, V_i \tag{2.136}$$

using only CFT data. Note that we use an opposite sign convention for λ to that used in CPT, such a sign convention is used in string field theory (SFT). We write $\lambda_i = g_i a^{-2(1-\Delta_i)}$, where g_i is dimensionless and a acts as a UV cutoff. We then expand the partition function

$$\begin{aligned} Z \;&=\; \int \mathcal{D}\phi \exp\left\{-S_0 - \sum_i g_i \int \frac{\mathrm{d}^2x}{a^{2(1-\Delta_i)}}V_i(x)\right\} = \qquad (2.137)\\ &Z_0\left[1 - g_i \int \frac{\mathrm{d}^2x}{a^{2(1-\Delta_i)}}\langle V_i(x)\rangle + \frac{1}{2}\sum_{i,j} g_i g_j \int_{x_{12}>a} \langle V_i(x_1)V_j(x_2)\rangle \frac{\mathrm{d}^2x_1}{a^{2(1-\Delta_i)}}\frac{\mathrm{d}^2x_2}{a^{2(1-\Delta_j)}} + \dots\right] \end{aligned}$$

We now move into the IR without changing the physics (see (1.7), this is analogous with the block spins). This is done by rescaling $a \to (1 + \mathrm{d}t)a$ and compensating by the corresponding change in couplings. The rescaling has an effect in two places since a appears both in the integration bounds and in integrands. The change in a in the integrands is trivially compensated by

$$g_i \to (1 + \mathrm{d}t)^{2(1-\Delta_i)} g_i \sim g_i + 2(1 - \Delta_i) g_i \mathrm{d}t \tag{2.138}$$

To see the effect of the rescaling on the integral bounds, we schematically have

$$\int_{x_{12} > a(1+\mathrm{d}t)} = \int_{x_{12} > a} - \int_{a < x_{12} < a(1+\mathrm{d}t)}, \tag{2.139}$$

where the first RHS contribution just gives the original integral. Since x_{12} is now roughly a in the second term on the RHS of (2.139), we perform an OPE, obtaining the contribution

$$\sum_k C_{ijk} a^{2(\Delta_k - \Delta_i - \Delta_j)} \int_{a < x_{12} < a(1+\mathrm{d}t)} \langle V_k(x_2) \rangle \frac{\mathrm{d}^2 x_1}{a^{2(1-\Delta_i)}} \frac{\mathrm{d}^2 x_2}{a^{2(1-\Delta_j)}}, \tag{2.140}$$

with the minus coming from (2.139). We see that we can explicitly perform the x_1 integration to get $2\pi a^2 \mathrm{d}t$, which is an area of an infinitesimal annulus. After performing this integration, x_2 can be any point in $\mathbb{R}^2$. We thus get the change in the partition function

$$-\pi \mathrm{d}t \sum_{ijk} C_{ijk} g_i g_j \int \langle V_k(x) \rangle \frac{\mathrm{d}^2 x}{a^{2(1-\Delta_k)}}, \tag{2.141}$$

where the various powers of a combined. Looking at the linear term in (2.137), we see that this change can be compensated by

$$g_k \to g_k - \pi \sum_{i,j} C_{ijk} g_i g_j \mathrm{d}t \tag{2.142}$$

Taking the two scale dependences (2.138) and (2.142) together, we have

$$\frac{\mathrm{d}g_k}{\mathrm{d}t} = \beta_k(g) = 2(1 - \Delta_k) g_k - \pi \sum_{ij} C_{ijk} g_i g_j + \dots \tag{2.143}$$

The first term in (2.143) corresponds to repulsion from a fixed point since we consider $\Delta_k < 1$, while the second term gives nonzero curvature to the flow meaning that we can again end up in a situation with $g_k = 0$ at the end of an RG flow. To this order, the flow in the space of couplings is a gradient flow, since

$$\frac{\mathrm{d}g_k}{\mathrm{d}t} = \frac{\partial}{\partial g_k} \tilde{C}(g) \tag{2.144}$$

with

$$\tilde{C} = \sum_i (1-\Delta_i) g_i^2 - \frac{\pi}{3} \sum_{ijk} C_{ijk} g_i g_j g_k \tag{2.145}$$

These expressions can be trusted only for deformations with weakly relevant operators with $y_i = 2(1-\Delta_i)$ much smaller than 1 since then the corresponding couplings are of the same order and it is plausible to neglect the next-to-leading corrections to (2.143). In this case, we might see a new fixed point in a very close proximity. The computation of the next-to-leading corrections is in general very involved and we haven't found a computation for a generic setup in the literature, however some next-to-leading results for particular theories are available, for example for the perturbations $\mathcal{M}_m \to \mathcal{M}_{m-1}$ considered in (2.8.2).

We now prove the Zamolodchikov's c-theorem, which states that for a unitary two-dimensional field theory with a stress tensor, there exists a function $C(\{g\})$, which decreases along the RG flows, being stationary only at fixed points $g = g_*$, where it coincides with the central charge c of the fixed point CFT. The first step it to define the functions F, G and H so that

$$\langle T(z,\bar{z})T(0,0)\rangle = \frac{F(\tau)}{z^4} \tag{2.146}$$

$$\langle T(z,\bar{z})\Theta(0,0)\rangle = \frac{G(\tau)}{z^3\bar{z}} \tag{2.147}$$

$$\langle \Theta(z,\bar{z})\Theta(0,0)\rangle = \frac{H(\tau)}{z^2\bar{z}^2}, \tag{2.148}$$

where $\tau \equiv \ln(z\bar{z})$ and T is no longer holomorphic since

$$\bar{\partial}T + \frac{1}{4}\partial\Theta = 0 \tag{2.149}$$

$$\partial\bar{T} + \frac{1}{4}\bar{\partial}\Theta = 0 \tag{2.150}$$

Overlapping these two equations with T and Θ gives two equations

$$\dot{F} + \frac{1}{4}\Big(\dot{G} - 3G\Big) = 0 \tag{2.151}$$

$$\dot{G} - G + \frac{1}{4}\Big(\dot{H} - 2H\Big) = 0 \tag{2.152}$$

where the dot means a τ derivative. If we define

$$C = 2F - G - \frac{3}{8}H, \tag{2.153}$$

then (2.151) and (2.152) give

$$\dot{C} = -\frac{3}{4}H \tag{2.154}$$

which is nonpositive in a unitary theory so that C is decreasing and at the critical point, we have $\Theta = 0$, meaning that $C = 2F$ there. Looking at the OPE (2.88) and the definition (2.146), we see that at the critical point $C = 2F = c$, which completes the proof. The interpretation is that c measures the degree of instability of a fixed point, which has to decrease along RG flows thanks to the reduction in DOFs. One can then picture the theory space as having a height function C with relevant flows going downhill. Integrating equation (2.154) and realising that $z\bar{z} = r^2$ in polar coordinates, we obtain the sum rule

$$\begin{aligned}
\Delta c &= -\frac{3}{4}\int d(r^2) r^2 \langle\Theta(r)\Theta(0)\rangle \\
&= -\frac{3}{4\pi}\int \mathrm{d}^2 r\, r^2 \langle\Theta(r)\Theta(0)\rangle \\
&= -\frac{3}{2}\int \mathrm{d}r\, r^3 \langle\Theta(r)\Theta(0)\rangle,
\end{aligned} \tag{2.155}$$

where Δc is the difference between central charges of the UV and IR fixed points (a massive theory corresponds to $c_{IR} = 0$). For perturbative purposes, it is useful to rewrite (2.155). To do this, we consider a perturbation $S_0 \to S = S_0 + \lambda \int \mathrm{d}^2 x\, V$, where V has weight Δ and we are dropping the distinction between g and λ from now on. Then we have for the expectation value of the holomorphic stress tensor

$$\langle T(z)\ldots\rangle_\lambda = \langle T(z)\ldots\rangle_0 - \lambda \int \mathrm{d}^2 z_1 \langle T(z)V(z_1)\ldots\rangle + \ldots, \tag{2.156}$$

where the subscript λ means evaluation with the perturbed action and the subscript 0 means evaluation with the unperturbed action. We expand the OPE $T(z)V(z_1, \bar{z}_1)$ around z instead of around z_1 to obtain

$$T(z)V(z_1) = \frac{\Delta}{(z - z_1)^2} V(z, \bar{z}) + \frac{1-\Delta}{z - z_1}\partial V(z, \bar{z}) + \ldots, \tag{2.157}$$

which gives that the integral (2.156) is divergent and we regularise by point-splitting, meaning that we insert the step function $\theta(|z - z_1|^2 - a^2)$ into it. Since in the critical theory, we have $\overline{\partial} T = 0$, differentiating (2.156) by $\overline{\partial}$ gives

$$\begin{aligned}
\overline{\partial} T &= -\lambda \int \mathrm{d}^2 z_1 \frac{1-\Delta}{z - z_1}(z - z_1)\delta(|z - z_1|^2 - a^2)\partial V(z, \bar{z}) \\
&= -\pi\lambda(1 - \Delta)\partial V
\end{aligned} \tag{2.158}$$

where the $\overline{\partial}$ hit the step function θ turning it into δ and the most singular term in the OPE (2.157) gave zero upon integration. Additionally a factor $\frac{1}{2}$ came from the delta function $\delta(r^2 - a^2) = \frac{\delta(r-a)}{2a}$ which effectively holds for $a, r > 0$. From the off-critical conservation law for T, we then have

$$\Theta = 4\pi\lambda(1 - \Delta)V + \ldots = 2\pi\lambda y V + \ldots \tag{2.159}$$

so that

$$\begin{aligned} \Delta c &= -6\pi^2\lambda^2 y^2 \int_0^1 \mathrm{d}r\, r^3 \langle V(r)V(0)\rangle + \ldots \\ &= -\frac{3}{2}\pi^2\lambda^2 y + \ldots, \end{aligned} \tag{2.160}$$

where we used the generic two-point function (2.50). Remember that we are still in the off-critical theory, where λ is not a fixed point coupling so that $\Delta c = C(\lambda) - c$ in (2.160). Now we observe that $C(\lambda) - c$ and $\tilde{C}$ from (2.145) should be proportional to one another to at least the first order since they have the same stationary points, leading to

$$C(\lambda) - c = \alpha\tilde{C} \tag{2.161}$$

and comparing (2.145) for one perturbing operator with (2.160), we have $\alpha = -6\pi^2$. In the case of one perturbing operator V, the vanishing of (2.143) for the coupling λ as required at a fixed point gives the fixed-point coupling

$$\lambda_* = \frac{y}{\pi C_{VVV}} + O(y^2) \tag{2.162}$$

Plugging (2.162) into (2.145), we have $\tilde{C} = \frac{y^3}{6\pi^2 C_{VVV}^2} + O(y^4)$ so that

$$\Delta c = -\frac{y^3}{C_{VVV}^2} + O(y^4) \tag{2.163}$$

is the difference between the UV central charge and the IR charge charge. In this generic setup, the result (2.163) was first written down by Cardy and Ludwig [47], but for minimal models was also derived in the independent work of Zamolodchikov [42]. The derivation here was in the spirit of [39]. From (2.163), we can see the c-theorem perturbatively since for relevant deformations with $y > 0$ gives that $\Delta c < 0$ to first order.

2.8 Examples

Below we study concrete examples of CFTs, starting with unitary minimal models and their perturbations. After this we move into CFTs that are very important for string theory, namely the free boson and the bc-ghost system.

2.8.1 Unitary minimal models $\mathcal{M}_m$

The Kač determinant (2.123) for $0 < c < 1$ was studied in [48] and the results can be summarised by saying that there is only a discrete series of unitary theories (called the *unitary minimal models* $\mathcal{M}_m$) with the central charges

$$c(m) = 1 - \frac{6}{m(m+1)} \tag{2.164}$$

with $m \geq 3$. Moreover in these theories only a discrete class of conformal families exists with ancestors of weight

$$h_{r,s}(m) = \frac{\Big((m+1)r - ms\Big)^2 - 1}{4m(m+1)}, \tag{2.165}$$

with $(1 \leq r \leq m, 1 \leq s \leq m+1)$ with the symmetry $(r,s) \sim (m-r, m+1-s)$. The simplest unitary minimal model $\mathcal{M}_2$ corresponds to the critical point of the $2D$ Ising. In this theory, there are three primaries $\mathbb{1}, \sigma$ and ϵ of weights

$$\begin{aligned} h_{\mathbb{1}} &= h_{1,1} = h_{2,3} = 0 \\ h_\sigma &= h_{1,2} = h_{2,2} = \frac{1}{16} \\ h_\epsilon &= h_{1,3} = h_{2,1} = \frac{1}{2}, \end{aligned}$$

which immediately imply the correct two-point a three-point correlators of Onsager's solution, here simply obtained from (2.50) and (2.53).

We now need to remove null states, since (2.165) coincides with (2.124), this will force correlators to obey various identities thanks to (2.115). Starting at level 2, we have the general ansatz for a null state $L_{-2}\left|h\right\rangle + aL_{-1}L_{-1}\left|h\right\rangle = 0$. By acting with L_1 on it and using the Virasoro algebra (2.85) with $L_1\left|h\right\rangle = 0$, we get

$$0 = [L_1, L_{-2}]\left|h\right\rangle + a[L_1, L_{-1}L_{-1}]\left|h\right\rangle = \Big(3 + 2a(2h+1)\Big)L_{-1}\left|h\right\rangle, \tag{2.166}$$

leading to $a = -\frac{3}{2(2h+1)}$. By applying L_2 to the null state, we obtain similarly

$$0 = [L_2, L_{-2}]\left|h\right\rangle + a[L_2, L_{-1}L_{-1}]\left|h\right\rangle = \Big(4h + \frac{c}{2} + 6ah\Big)\left|h\right\rangle, \tag{2.167}$$

leading to

$$c = \frac{2h(5-8h)}{2h+1} \tag{2.168}$$

This will help us identify to which Verma module the resulting null state

$$\Big(L_{-2} - \frac{3}{2(2h+1)} L_{-1}L_{-1}\Big) |h\rangle \tag{2.169}$$

belongs. It belongs to the Verma module generated by $V_{2,1}$ because by looking at $h_{2,1}$, we find that

$$m = \frac{3}{4h_{2,1} - 1}$$

and plugging it into (2.164) gives (2.168). The fact that (2.169) is a null state, gives the following differential equation coming from (2.115) for a generic three-point correlator with $V_{2,1}$

$$0 = \left(\sum_{i=1}^{2} \left(\frac{h_i}{(w_i - w)^2} - \frac{1}{w_i - w} \right) - \frac{3}{2(2h_{2,1}+1)} \partial_w^2 \right) \langle V_{2,1}(w) V_1(w_1) V_2(w_2) \rangle \tag{2.170}$$

Plugging in the generic form of the three-point correlator (2.53) gives the nontrivial constraint

$$2(2h_{2,1}+1)(h_{2,1}+2h_2+1) = 3(h_{2,1}+h_2-h_1)(h_{2,1}+h_2-h_1+1) \tag{2.171}$$

which has the solutions

$$h_2 = \frac{1}{6} + \frac{h_{2,1}}{3} - h_1 \pm \frac{2}{3}\sqrt{h_{2,1}^2 + 3h_{2,1}h_1 - \frac{h_{2,1}}{2} + \frac{3h_1}{2} + \frac{1}{16}} \tag{2.172}$$

It is a miracle that for $h_1 = h_{r,s}$, the plus branch gives $h_2 = h_{r+1,s}$ and the minus branch $h_2 = h_{r-1,s}$. This means that for the OPE of $V_{2,1}$ and $V_{r,s}$, we have schematically

$$\Big[V_{2,1}\Big] \times \Big[V_{r,s}\Big] = \Big[V_{r+1,s}\Big] + \Big[V_{r-1,s}\Big] \tag{2.173}$$

This is an example of a *fusion rule* and working our the consequences of having null vectors at higher levels, one has the general fusion rule

$$\Big[V_{r_1,s_1}\Big] \times \Big[V_{r_2,s_2}\Big] = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2 \text{ even}}}^{r_1+r_2-1} \; \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2 \text{ odd}}}^{s_1+s_2-1} \Big[V_{k,l}\Big] \tag{2.174}$$

As an example, we work out the fusion of two sigmas in the Ising. Since $\sigma = V_{1,2} = V_{2,2}$, we have from (2.174)

$$\Big[V_{2,2}\Big] \times \Big[V_{2,2}\Big] = \sum_{\substack{k=1 \\ k+4 \text{ even}}}^{3} \; \sum_{\substack{l=1 \\ l+4 \text{ odd}}}^{3} \Big[V_{k,l}\Big] = \Big[V_{2,1}\Big] + \Big[V_{2,3}\Big] \tag{2.175}$$

We have that $\mathbb{1} = V_{2,1}$ and $\epsilon = V_{2,3}$, from which we can then write

$$[\sigma] \times [\sigma] = [\mathbb{1}] + [\epsilon]$$

The other fusion rules of the Ising are

$$\begin{aligned} [\mathbb{1}] \times [\sigma] &= [\sigma] \\ [\mathbb{1}] \times [\epsilon] &= [\epsilon] \\ [\epsilon] \times [\epsilon] &= [\mathbb{1}] \\ [\epsilon] \times [\sigma] &= [\sigma] \end{aligned}$$

The fusion rules are often written in the form of a *fusion algebra*

$$\Big[V_i\Big] \times \Big[V_j\Big] = \sum_k N^k{}_{ij} \Big[V_k\Big] \tag{2.176}$$

with the numbers $N^k{}_{ij}$ representing how many times a given conformal family appears (if we have degeneracies in h, a given conformal family may appear multiple times). In unitary minimal models, we have $N^k{}_{ij} \in \{0, 1\}$. By the requirement of OPE associativity, one has

$$\sum_l N^l{}_{kj} N^m{}_{il} = \sum_l N^l{}_{ij} N^m{}_{lk} \tag{2.177}$$

We have seen that the unitary minimal models are highly constrained. Furthermore, the differential equations for correlation functions can often be solved (notably for the four-point case), see the work of Dotsenko and Fateev [49]. When one has the four-point correlators, one can compute the structure constants since one already knows the general OPE structure from (2.174). These structure constants will be important in the next subsection, when we study weakly relevant perturbations of $\mathcal{M}_m$.

2.8.2 Perturbations $\mathcal{M}_m \to \mathcal{M}_{m-1}$

We will consider the Zamolodchikov's perturbations (see [42]) of $\mathcal{M}_m$ in the large m limit since then there are two series of near-marginal operators. The first is given by

$$h_{n,n+2} = 1 - \frac{n+1}{m} + O\Big(\frac{1}{m^2}\Big) \tag{2.178}$$

and the second by

$$h_{n+2,n} = 1 + \frac{n+1}{m} + O\Big(\frac{1}{m^2}\Big) \tag{2.179}$$

We see that the series $h_{n,n+2}$ corresponds to weakly relevant operators and $h_{n+2,n}$ to weakly irrelevant operators. We deform by the least relevant operator

$V_{1,3}$ with weight $h_{1,3} = 1 - \frac{2}{m} + O\Big(\frac{1}{m^2}\Big)$. The operator $V_{1,3}$ has a second crucial property and that is its fusion rule

$$\Big[V_{1,3}\Big] \times \Big[V_{1,3}\Big] = \Big[V_{1,1}\Big] + \Big[V_{1,3}\Big] + \Big[V_{1,5}\Big] \tag{2.180}$$

with $V_{1,5}$ being irrelevant since $h_{1,5} = 4 - \frac{6}{m} + O\Big(\frac{1}{m^2}\Big)$. This means that the $(1,3)$ deformation doesn't turn on any other near-marginal operators and so we can reliably use the part of results of (2.7) obtained for a single operator deformation. Thanks to the weak relevance of $V_{1,3}$, for m large enough, we should be within the validity of the previously obtained results from (2.7). In particular, we would like to begin by computing the shift Δc in the central charge between $\mathcal{M}_m$ and the IR theory to which it flows. To do so, we need the structure constant $C_{(1,3)(1,3)(1,3)}$ obtainable by the methods of [49]. We simply quote the large m result from Zamolodchikov's paper

$$C_{(1,3)(1,3)(1,3)} = \frac{4}{\sqrt{3}}\Big(1 - \frac{3}{4}y\Big) + O(y^2) \tag{2.181}$$

with $y = 2(1 - h_{1,3}) = \frac{4}{m+1}$, so that from (2.163), we immediately have

$$\Delta c = -\frac{3}{16}y^3 + O(y^4) = -\frac{12}{m^3} + O\Big(\frac{1}{m^4}\Big) \tag{2.182}$$

Expanding the difference between central charges (2.164) of $\mathcal{M}_m$ and $\mathcal{M}_{m+\delta}$, we have

$$c_{m+\delta} - c_m = \delta\frac{12}{m^3} + O\Big(\frac{1}{m^4}\Big) \tag{2.183}$$

meaning that, we observe the flow $\mathcal{M}_m \to \mathcal{M}_{m-1}$ corresponding to $\delta = -1$. We may now ask questions about operator mixing leading to the analysis of the matrix of anomalous dimensions. As shown by (2.135), in order to learn about which operator $V_{1,3}$ turns into, we need to compute the derivative of the beta function (2.143) of the coupling λ to $V_{1,3}$ at the critical point. The result is easily computed to be

$$\frac{\partial\beta}{\partial\lambda}\bigg|_{\lambda_*} = -y + O(y^2) \tag{2.184}$$

where we plugged in the fixed point coupling (2.162). From (2.180), we see that $V_{1,3}$ doesn't mix with other operators so that the matrix of anomalous dimensions (2.135) is simply a number

$$\hat{\gamma}V_{1,3} = \Big(2 + y\Big)V_{1,3} + O(y^2) = 2\Big(1 + \frac{2}{m}\Big)V_{1,3} + O\Big(\frac{1}{m^2}\Big) \tag{2.185}$$

From (2.179), we have that $V_{1,3} \to V_{3,1}$, where the $V_{3,1}$ lives in $\mathcal{M}_{m-1}$. Next, we investigate the renormalisation of the diagonal fields $V_{n,n}$, which satisfy the fusion rule with $V_{1,3}$

$$\left[V_{n,k}\right] \times \left[V_{1,3}\right] = \left[V_{n,k}\right] + \left[V_{n,k+2}\right] + \left[V_{n,k-2}\right] \tag{2.186}$$

with $k = n$ and for the $k = n$ case, the relevant structure constant is given by

$$C_{(n,n)(1,3)(n,n)} = \frac{n^2-1}{32\sqrt{3}} y^2 + O(y^3) \tag{2.187}$$

Moreover, we have

$$h_{n,n} = \frac{n^2-1}{4m(m+1)} = \frac{n^2-1}{64} y^2 \Big(1 + \frac{y}{4}\Big) + O(y^4) \tag{2.188}$$

meaning that they are strongly relevant. Since only operators of comparable conformal dimensions mix at leading order, we have from (2.186) that to this order the $V_{n,n}$ do not mix. Then using the formula (2.135) and easily computing the beta-function for the coupling to $V_{n,n}$, we have

$$\begin{aligned} \hat{\gamma} V_{n,n} &= \Big(2h_{n,n} + 2\pi C_{(n,n)(1,3)(n,n)} \lambda_*\Big) V_{n,n} + \ldots \qquad (2.189) \\ &= 2\frac{n^2-1}{64} y^2 \Big(1 + \frac{3}{4} y\Big) V_{n,n} + O(y^4) = \frac{n^2-1}{4m(m-1)} V_{n,n} + O(y^4), \end{aligned}$$

where we expanded

$$\frac{C_{(1,3)(1,3)(1,3)}}{C_{(n,n)(1,3)(n,n)}} = \frac{n^2-1}{128} y^2 + O(y^3) \tag{2.190}$$

In conclusion, we see that $V_{n,n} \to V_{n,n}$ from the formula (2.188) with $m \to m-1$. As an example with a nontrivial matrix of anomalous dimensions, we consider the mixing of $V_{n,n-1}$ and $V_{n,n+1}$. The mixing is obvious from (2.186) and the fact that they have dimensions close to one another since

$$h_{n,n+1} = \frac{1}{4} - \frac{2n+1}{16} y + O(y^2) \tag{2.191}$$

and

$$h_{n,n-1} = \frac{1}{4} + \frac{2n-1}{16} y + O(y^2) \tag{2.192}$$

The relevant structure constants are

$$C_{(n,n-1)(1,3)(n,n+1)} = \frac{\sqrt{n^2-1}}{\sqrt{3}n} + O(y) \tag{2.193}$$

$$C_{(n,n-1)(1,3)(n,n-1)} = \frac{n-2}{2\sqrt{3}n} + O(y) \tag{2.194}$$

$$C_{(n,n+1)(1,3)(n,n+1)} = \frac{n+2}{2\sqrt{3}n} + O(y) \tag{2.195}$$

From which we simply obtain

$$\hat{\gamma} = \begin{pmatrix} h_{n,n+1} & 0 \\ 0 & h_{n,n-1} \end{pmatrix} + \frac{2\pi}{\sqrt{3}}\lambda_* \begin{pmatrix} \frac{n+2}{2n} & \frac{\sqrt{n^2-1}}{n} \\ \frac{\sqrt{n^2-1}}{n} & \frac{n-2}{2n} \end{pmatrix} + \dots \tag{2.196}$$

$$= \begin{pmatrix} h_{n,n+1} & 0 \\ 0 & h_{n,n-1} \end{pmatrix} + \frac{y}{2} \begin{pmatrix} \frac{n+2}{2n} & \frac{\sqrt{n^2-1}}{n} \\ \frac{\sqrt{n^2-1}}{n} & \frac{n-2}{2n} \end{pmatrix} + O(y^2) \tag{2.197}$$

where we used (2.135) and (2.143). The eigenvalues of this matrix are

$$\lambda_1 = \frac{1}{4} + \frac{2n+1}{16}y + O(y^2)$$
$$\lambda_2 = \frac{1}{4} - \frac{2n-1}{16}y + O(y^2)$$

and by comparison with

$$h_{n+1,n} = \frac{1}{4} + \frac{2n+1}{4}\frac{1}{m} + O\left(\frac{1}{m^2}\right) \tag{2.198}$$

and

$$h_{n-1,n} = \frac{1}{4} - \frac{2n-1}{4}\frac{1}{m} + O\left(\frac{1}{m^2}\right), \tag{2.199}$$

we see that schematically

$$\{V_{n,n+1}, V_{n,n-1}\} \to \{V_{n+1,n}, V_{n-1,n}\}$$

Other mixings are investigated in [42]. The next-to-leading order corrections to Δc and the mixings of [42] were obtained by R. Poghossian [50]. Poghossian found a match with the all-order mixing proposed by Gaiotto [51], who got the mixing by constructing an RG domain wall linking DOFs between the UV and IR. These mixings are dependent on the regularisation scheme, see [52], which clarifies the specifics of Zamolodchikov's scheme. See also [53] for additional leading order mixings.

2.8.3 Free boson

The free boson is a CFT with an action

$$S = \frac{1}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\partial X\bar{\partial}X, \tag{2.200}$$

where the integration is over a complex two-manifold on which the free boson lives. For now, we consider the free boson on a plane. From (2.200), it is very easy to see that the EOM for X reads

$$\partial\bar{\partial}X(z,\bar{z}) = 0 \tag{2.201}$$

The two-point function $\langle X(z,\bar{z})X(w,\bar{w})\rangle$ is obtained as a Green's function of the EOM (2.201), meaning that

$$\partial\bar{\partial}\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\pi\delta^{(2)}(z-w), \tag{2.202}$$

where

$$\int \mathrm{d}z\,\mathrm{d}\bar{z}\,\delta^{(2)}(z,\bar{z}) = 1 \tag{2.203}$$

The solution to (2.202) is known from two-dimensional electrostatics and is

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{1}{2}\ln|z-w|^2 \tag{2.204}$$

Comparing (2.204) to (2.50), it is clear that X is not a primary and to identify primaries, we look at the EOM (2.201). It tells us that there is a holomorphic current $j(z) \equiv i\sqrt{2}\partial X$ and an antiholomorphic current $\bar{j}(\bar{z}) \equiv i\sqrt{2}\,\bar{\partial}X$. These currents correspond to a global $U(1)$ translation symmetry of (2.200). From the requirement of classical scaling symmetry of (2.200), these have weights 1 since X has weight 0. By differentiation of (2.204), we easily infer that

$$\langle j(z)j(w)\rangle = \frac{1}{(z-w)^2} \tag{2.205}$$

as corresponds to a holomorphic weight 1 primary from (2.50). With j having weight 1, the corresponding Laurent expansion has the form

$$j(z) = \sum_{n\in\mathbb{Z}} \frac{j_n}{z^{n+1}}, \tag{2.206}$$

where the j_n are often written as α_n in string theory, where they play the role of unconventionally normalised oscillators, since

$$[j_n, j_m] = m\delta_{m+n,0} \tag{2.207}$$

as follows from the fact that the jj OPE contains only the identity from symmetry and a standard contour argument. We would now like to find the stress-energy tensor of the free boson, which is essentially the simplest theory of abelian currents. In theories of currents, the stress-energy tensor is given as a current bilinear via the so-called *Sugawara construction*, which enables one to intertwine the current symmetry with a conformal symmetry. The ansatz for a such a bilinear is

$$T(z) = \gamma :jj:(z), \tag{2.208}$$

where we normal ordered with respect to the z coordinate system

$$:jj:(z) = \lim_{w\to z}\left(j(z)j(w) - \frac{1}{(z-w)^2}\right) \tag{2.209}$$

to make the jj product well-defined. To find out γ, we simply compute an OPE via Wick's theorem (remember that this theory is free)

$$T(z)j(w) \sim 2\gamma j(z)\overbracket{j(z)j(w)} = 2\gamma\frac{j(z)}{(z-w)^2} \sim 2\gamma\frac{j(w)}{(z-w)^2}, \tag{2.210}$$

from which $\gamma = \frac{1}{2}$ follows from (2.72) and the fact that j has weight 1. In section (2.5), we promised that we would make the Schwarzian appear explicitly (inspired by [45]). To do so, imagine that we normal order (2.209) with respect to another coordinate system with $z \to f(z)$, then the factor $-\frac{1}{(z-w)^2} = \partial_z\partial_w \ln|z-w|^2$ turns into $\partial_z\partial_w \ln|f(z)-f(w)|^2$. We now calculate the $w \to z$ limit

$$\lim_{w\to z}\partial_z\partial_w \ln|f(z)-f(w)|^2 = \lim_{w\to z}\partial_z\partial_w \ln\Big(f(z)-f(w)\Big) \tag{2.211}$$

by explicitly writing the ∂_z and ∂_w via their definitions

$$\begin{aligned}
\lim_{w\to z}\partial_z\partial_w \ln\Big(f(z)-f(w)\Big) &= \lim_{w\to z}\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\left[\ln\left(\frac{\Big(f(z+\epsilon)-f(w+\epsilon)\Big)\Big(f(z)-f(w)\Big)}{\Big(f(z)-f(w+\epsilon)\Big)\Big(f(z+\epsilon)-f(w)\Big)}\right)\right] \\
&= \lim_{w\to z}\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\ln\Bigg[f(z), f(z+\epsilon); f(w), f(w+\epsilon)\Bigg],
\end{aligned}$$

where we used the cross-ratio definition (2.96). Since a cross-ratio appears, it doesn't surprise us that similarly to the computation of (2.97), we have after performing the limits

$$\lim_{w\to z}\partial_z\partial_w \ln\Big(f(z)-f(w)\Big) = \lim_{w\to z}\frac{1}{(z-w)^2} + \frac{1}{6}S(f)(z) \tag{2.212}$$

We see that physically the Schwarzian appears thanks to the presence of normal ordering in the sense that a coordinate transformation changes the coordinate system with respect to which we normal order. The next calculation in line is to find out the central charge of the free boson. To do so, we simply compute the TT OPE via Wick's theorem

$$\begin{aligned}
T(z)T(w) &\sim \frac{2}{4}\Big(\overbracket{j(z)j(w)}\Big)^2 + \frac{4}{4}:j(z)j(w):\overbracket{j(z)j(w)} &(2.213)\\
&\sim \frac{1}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, &(2.214)
\end{aligned}$$

where we used (2.205) and the derivative term came from Taylor expanding around w and we divided it by 2 since there are two terms in T, on which the derivative acts. By comparison with (2.88), we find that $c = 1$ (remember, this why we normalised the Virasoro algebra (2.85) the way we did) and that T constructed as a current bilinear really is a stress-tensor.

Aside from the $U(1)$ currents, there is a very important continuum of primaries of the form $V_k \equiv :e^{ikX}:$ with $k \in \mathbb{R}$. To show that they are primaries, we need to compute their OPE with T and match it to the general form (2.72). We start by expanding

$$:e^{ikX}:(z) = 1 + ikX(z) - \frac{k^2}{2} :XX:(z) + \ldots \tag{2.215}$$

and by Wick's theorem, we have

$$\begin{aligned} T(z)V_k(w) &= -:\partial X \partial X:(z)(1 + ikX(w) - \frac{k^2}{2} :XX:(w) + \ldots) \\ &= \frac{2}{8(z-w)^2}k^2 + ik\frac{2}{2(z-w)}\partial X(z) - \frac{4}{4(z-w)}k^2 :(\partial X)X:(z) + \ldots \end{aligned}$$

from which we guess the general pattern

$$\begin{aligned} T(z)V_k(w) &\sim \frac{k^2}{4(z-w)^2}V_k(w) + \frac{ik}{z-w} :\partial X e^{ikX}:(w) \\ &= \frac{k^2}{4(z-w)^2}V_k(w) + \frac{\partial V_k(w)}{z-w} \end{aligned}$$

meaning that from (2.72) V_k has weight $\frac{k^2}{4}$. The operators V_k are called *vertex operators* and correspond to momentum eigenstates in string theory. To see this, we realise that they are charged under the $U(1)$ translation current j

$$j(z)V_k(w) = j(z)(1 + ikX(w) + \ldots) \sim \frac{k}{\sqrt{2}(z-w)} + \ldots \sim \frac{k}{\sqrt{2}(z-w)}V_k(w) \tag{2.216}$$

Thanks to this, they enable us to create physical states in string theory, that is states that satisfy appropriate mass-shell conditions (we essentially tune the conformal weight of our state to 1 by tensoring with a V_k).

Despite its simplicity, the free boson contains some very interesting physics. To illustrate, we consider the free boson compactified on a circle of radius R

$$X \sim X + 2\pi m R \tag{2.217}$$

with $m \in \mathbb{Z}$. To make most out of this symmetry, we no longer work on the complex plane, but on a torus with complex coordinates $w = \sigma_1 + \tau\sigma_2$ and

$\bar{w} = \sigma_1 + \bar{\tau}\sigma_2$, where $\sigma_1, \sigma_2 \in [0,1]$ such that $\sigma_1 \sim \sigma_1 + 1$ and $\sigma_2 \sim \sigma_2 + 1$ and τ is a leftover parameter of the complex torus metric known as the *modulus*

$$g = \frac{1}{\tau_2}|\mathrm{d}\sigma_1 + \tau \mathrm{d}\sigma_2|^2 = \frac{\mathrm{d}w\,\mathrm{d}\bar{w}}{\tau_2} \tag{2.218}$$

with $\tau = \tau_1 + i\tau_2$, $\tau_2 \geq 0$. It is trivial to see that the torus is invariant under $\tau \to \tau + 1$ and with a little bit more work, one can show that it is invariant under the *modular group* $PSl(2, \mathbb{Z})$ generated by $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$.

Our objective is to compute the torus partition function Z and read off the spectrum from it. To do this, we first realise that the torus can be thought of as a cylinder with its ends joined. From (2.4), we know that the coordinate along the cylinder can be thought of as a Euclidean time and thus to get a torus with modulus $\tau = \tau_1 + i\tau_2$, we need to twist the ends of a cylinder propagator $e^{-2\pi\tau_2 H}$, where H is the cylinder Hamiltonian (2.99). The twist is obtained by exponentiating the momentum $P = (L_0 - \frac{c}{24}) - (\bar{L}_0 - \frac{\bar{c}}{24})$. The joining of ends turns into a Hilbert space trace and thus

$$Z = \mathrm{Tr}\Big\{e^{-2\pi\tau_2 H} e^{2\pi i \tau_1 P}\Big\} = \mathrm{Tr}\Big\{q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\Big\} \tag{2.219}$$

with $q = e^{2\pi i \tau}$. Evaluating this trace (a Virasoro *character*) in a Verma module generated by an ancestor of weight h gives an enormous amount of overcounting since

$$\chi_h \equiv \mathrm{Tr}\Big\{q^{L_0 - \frac{c}{24}}\Big\} = \sum_{N=0}^{\infty} \#_{h+N} q^{N + h - \frac{c}{24}}, \tag{2.220}$$

where $\#_{h+N}$ is the number of independent states at level N and we denote the characters by χ_h. Using the generating function (2.109), we can write

$$\chi_h = \frac{q^{h + \frac{1-c}{24}}}{\eta(q)}, \tag{2.221}$$

where η defined by

$$\frac{1}{\eta(q)} = q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \tag{2.222}$$

is the *Dedekind eta function*.

The strategy is to compute the partition function using the path integral and then match it to (2.219) to obtain the spectrum. To compute the path integral, we first want to find some instantons to expand the path integral around. Recall that earlier, we had the EOM (2.201), which was the Laplace equation on the plane (a Euclidean wave equation). Now we solve the Laplace equation $\Delta X = 0$ on the torus with $\Delta = \frac{1}{\tau_2}|\tau\partial_1 - \partial_2|^2$ as derivable by per partes from the free boson action on the torus

$$S = \frac{1}{4\pi}\int \mathrm{d}^2\sigma\sqrt{g}g^{ij}\partial_i X\partial_j X = \frac{1}{4\pi}\mathrm{d}^2\sigma\frac{1}{\tau_2}|\tau\partial_1 X - \partial_2 X|^2 \tag{2.223}$$

where g^{ij} are the σ-coordinate components of the metric (2.218). The instantons we find have to be compatible with (2.217) and with the periodicity of the torus itself. Thus they are maps from $S^1 \times S^1$ to S^1 and such maps depend on two integers n, m that tell us how many times X winds around each torus cycle. From this and the quadratic nature of Δ it is not too difficult to guess the family of instantons

$$X_{m,n}(\sigma_1, \sigma_2) = 2\pi R(n\sigma_1 + m\sigma_2), \tag{2.224}$$

which are trivially compatible with the torus periodicity conditions, which read $X_{m,n}(\sigma_1 + 1, \sigma_2) \sim X_{m,n}(\sigma_1, \sigma_2)$, $X_{m,n}(\sigma_1, \sigma_2 + 1) \sim X_{m,n}(\sigma_1, \sigma_2)$ due to (2.217). The action (2.223) evaluated on the instanton (2.224) trivially gives

$$S_{m,n} = \frac{\pi R^2}{\tau_2}|m - n\tau|^2, \tag{2.225}$$

which is finite and thus we really found instantons, see (1.6). Expanding the action around instantons, we have

$$Z = \sum_{m,n\in\mathbb{Z}}\int \mathcal{D}\chi\, e^{-S_{m,n} - S(\chi)} = \sum_{m,n\in\mathbb{Z}} e^{-S_{m,n}}\int \mathcal{D}\chi\, e^{-S(\chi)}, \tag{2.226}$$

where we introduced the fluctuation field χ. Since the Laplacian is quadratic operator, we use the usual Gaussian methods to evaluate the path integral over χ. We write $\chi = \chi_0 + \delta\chi$, where $\chi_0 \in [0, 2\pi R]$ is a constant zero-mode and then as usual $\delta\chi$ is expanded into the torus Laplacian eigenfunctions ψ such that

$$\Delta\psi = -\lambda\psi \tag{2.227}$$

It is trivial to see that we have an entire family of such eigenfunctions labeled by integers m_1, m_2

$$\psi_{m_1,m_2} = e^{2\pi i(m_1\sigma_1 + m_2\sigma_2)} \tag{2.228}$$

with eigenvalues

$$\lambda_{m_1,m_2} = \frac{4\pi^2}{\tau_2}|m_1\tau - m_2|^2 \tag{2.229}$$

By writing $\delta\chi = \sum_{(m_1,m_2)\neq(0,0)} A_{m_1,m_2}\psi_{m_1,m_2}$ (we omit $m_1 = m_2 = 0$ since from (2.228) this is the constant mode) and using the fact that $S(\chi_0) = 0$, we have

$$S(\chi) = \frac{1}{4\pi}\sum_{(m_1,m_2)\neq(0,0)}\lambda_{m_1,m_2}|A_{m_1,m_2}|^2 \tag{2.230}$$

Using the fact that the ψ_{m_1,m_2} give an orthonormal basis (as one can see from elementary Fourier analysis) on field space, we have by standard Gaussian integration

$$\int \mathcal{D}\chi\, e^{-S(\chi)} = \frac{2\pi R}{\sqrt{\prod_{(m_1,m_2)\neq(0,0)} \frac{\lambda_{m_1,m_2}}{(2\pi)^2}}}, \tag{2.231}$$

where the factor $2\pi R$ comes from the integration over χ_0 and we recognise the usual square root of an operator determinant. Note that the factor of $\frac{1}{2\pi}$ in (2.230) that is different from the conventional Gaussian exponent normalisation contributes to the divison by $(2\pi)^2$. The determinant in (2.231) can be beautifully evaluated using ζ-function regularisation, where the *zeta function* is defined as $\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s}$ and we have $\zeta'(s) = -\sum_{n=1} n^{-s} \ln n$. By assigning $\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln(2\pi)$, the formulae

$$\prod_{n=-\infty}^{\infty} a = a^{2\zeta(0)+1} = 1 \tag{2.232}$$

$$\prod_{n=1}^{\infty} n^{\alpha} = e^{-\alpha\zeta'(0)} = (2\pi)^{\frac{\alpha}{2}} \tag{2.233}$$

$$\prod_{n=-\infty}^{\infty} (n+a) = a\prod_{n=1}^{\infty}(-n^2)\left(1-\frac{a^2}{n^2}\right) = 2i\sin(\pi a), \tag{2.234}$$

easily follow, where in the last equality we also used the famous Euler product formula for the sine

$$\sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right), \tag{2.235}$$

which is easily seen by recalling the roots of the sine. The following computation is then possible

$$\begin{aligned}
\prod_{(m_1,m_2)\neq(0,0)} \frac{\lambda_{m_1,m_2}}{(2\pi)^2} &= \prod_{(m_1,m_2)\neq(0,0)} \frac{4\pi^2}{\tau_2(2\pi)^2}|m_1\tau - m_2|^2 \\
&= \tau_2\left(\prod_{m_2\neq 0}^{\infty} m_2^2\right)\prod_{m_1\neq 0, m_2}(m_1\tau - m_2)(m_1\bar{\tau} - m_2) \\
&= 4\pi^2\tau_2 \prod_{m_1>0,m_2}(m_2+\tau m_1)(m_2-\tau m_1)(m_2+\bar{\tau}m_1)(m_2-\bar{\tau}m_1) \\
&= 4\pi^2\tau_2\prod_{m_1>0} 4\sin^2(\pi\tau m_1)\sin^2(\pi\bar{\tau}m_1) \\
&= 4\pi^2\tau_2\prod_{m_1>0}(q\bar{q})^{-m}(1-q^m)^2(1-\bar{q}^m)^2 \\
&= 4\pi^2\tau_2(q\bar{q})^{\frac{1}{12}}\prod_{m_1>0}(1-q^m)^2(1-\bar{q}^m)^2 \\
&= 4\pi^2\tau_2\eta^2\bar{\eta}^2
\end{aligned} \tag{2.236}$$

so that taking into account the sum over instantons

$$Z = \frac{R}{\sqrt{\tau_2}|\eta|^2} \sum_{m,n\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2}|m-n\tau|^2} \tag{2.237}$$

Finally using the special case of a *Poisson resummation formula*

$$a^{-\frac{1}{2}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi(m-b)^2}{a}} = \sum_{m\in\mathbb{Z}} e^{-\pi a m^2 + 2\pi i b m} \tag{2.238}$$

in m, we find

$$Z = \frac{1}{|\eta|^2} \sum_{m,n\in\mathbb{Z}} e^{-\pi\tau_2\left(\frac{m^2}{R^2}+n^2R^2\right)+2\pi i\tau_1 mn} = \frac{1}{|\eta|^2} \sum_{m.n\in\mathbb{Z}} q^{h_{m,n}}\bar{q}^{\bar{h}_{m,n}}, \tag{2.239}$$

where by comparison with (2.221) and the fact that $c = 1$, we identified the numbers $h_{m,n}$, $\bar{h}_{m,n}$ as conformal weights

$$h_{m,n} = \frac{1}{4}\left(\frac{m}{R} + nR\right)^2 \tag{2.240}$$

$$\bar{h}_{m,n} = \frac{1}{4}\left(\frac{m}{R} - nR\right)^2 \tag{2.241}$$

It can be checked that the partition function (2.239) is invariant under the modular group thanks to the transformation properties $\eta(\tau+1) = e^{i\frac{\pi}{12}}\eta(\tau)$ and $\eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\eta(\tau)$ of the weight $\frac{1}{2}$ modular form η. As a note, we have just also solved the non-compact free boson on a torus since in the limit $R \to \infty$, only the $m = n = 0$ contribution survives and one has $Z \sim \frac{R}{\sqrt{\tau_2}|\eta|^2} \to \frac{V}{2\pi\sqrt{\tau_2}|\eta^2|}$, where V is the non-compact volume.

The formula (2.240) can be interpreted with the help of the Kaluza-Klein point-particle spectrum (B.5). The $\frac{1}{R}$ term can be interpreted as being present due to quantisation of momentum in the compact direction. When dealing with string theory, it is present due to the fact that the center of mass (COM) of a string has point-like behavior. But the string as a whole is extended, so a term proportional to R arises. This term has the interpretation of coming from the elastic energy of a string winding around the compact direction (the larger the direction, the bigger the energy). Thus n is called the *winding number.*

From (2.240), we see that the weights are symmetric under $R \to \frac{1}{R}$ provided that $m \to n, n \to m$. This tells us that the spectra of two theories, one compactified on a circle of radius R and the second of radius $\frac{1}{R}$ are equivalent provided that winding in one theory becomes COM momentum in the other and vice versa. This extends to a full duality of theories called *T-duality.* We immediately see that there exists a self-dual theory having radius $R = 1$, this theory is very interesting, because there is an enhancement of the $U(1)$ current algebra to an $SU(2)$ current algebra. To see this, we write $h_{m,n} = \frac{1}{4}(m+n)^2$ and $\bar{h}_{m,n} = \frac{1}{4}(m-n)^2$ and thus there are two states with $h_{1,1} = h_{-1,-1} = 1$ in the holomorphic sector and two states with $\bar{h}_{1,-1} = \bar{h}_{-1,1} = 1$ in the antiholomorphic sector. In the holomorphic sector, these can be written as the vertex operators

$$j^1 = \frac{1}{\sqrt{2}} V_2 \tag{2.242}$$

$$j^2 = \frac{1}{\sqrt{2}} V_{-2}, \tag{2.243}$$

which can be understood by the fact that the weight of V_k is $\frac{k^2}{4}$. Defining $j^3 \equiv \frac{1}{\sqrt{2}} j$ with j the $U(1)$ current, we have the $SU(2)$ current algebra satisfying for example from (2.216)

$$j^3(z) j^1(w) \sim \frac{j^1(w)}{z-w} \tag{2.244}$$

$$j^3(z) j^1(w) \sim -\frac{j^2(w)}{z-w} \tag{2.245}$$

The products $j^1 j^1$ and $j^2 j^2$ are regular and

$$j^1(z) j^2(w) \sim \frac{1}{2(z-w)^2} + \frac{j^3(w)}{z-w} \tag{2.246}$$

Thus we see an enhancement of the abelian $U(1)$ symmetry to a non-abelian $SU(2)$ symmetry. Moving away from $R = 1$, the symmetry is broken to $U(1)$ and the massless currents acquire a mass. This is the Brout-Englert-Higgs-Migdal-Polyakov effect. T-duality can be seen as a discrete remnant of the $R = 1$ enhanced gauge symmetry since at $R = 1$ it's a discrete symmetry of the antiholomorphic current algebra that acts by swapping two currents.

2.8.4 *bc*-ghost system

The *bc*-ghost system is a CFT of two anticommuting fields b and c with the action

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z\, b \bar{\partial} c \tag{2.247}$$

In order to have classical scale invariance, we need b to have weight λ and c weight $1 - \lambda$. We may proceed for general λ, but for our purposes only the

$\lambda = 2$ case is of interest (such a CFT arises from gauge fixing the bosonic string). Thus we have $h_b = 2$ and $h_c = -1$, which is a little surprising since these weights look appropriate for a boson rather than a fermion. The action (2.247) has the classical $U(1)$ symmetry $b \to b - i\epsilon b, c \to c + i\epsilon c$, which says there is a conservation of *ghost number*. The b and c ghosts are assigned ghost numbers -1 and 1, respectively. The $U(1)$ symmetry develops (apart from the case $\lambda = \frac{1}{2}$) an anomaly, which turns into the fact that ghost correlation functions are nonvanishing only when [31]

$$\text{total ghost number} = \frac{2\lambda - 1}{2}\chi, \tag{2.248}$$

where χ is the Euler number of the surface on which we put the ghost CFT. For us the relevant case will be the sphere with $\chi = 2$ and also $\lambda = 2$, so that correlators will be nonzero only at ghost number 3.

The EOMs from (2.247) read

$$\overline{\partial} b = \overline{\partial} c = 0, \tag{2.249}$$

so that $b = b(z)$ and $c = c(z)$. From antisymmetry and proper scaling behavior, we then have

$$\langle b(z)c(w)\rangle = \frac{1}{z-w}, \tag{2.250}$$

so that $b(z)c(w) \sim \frac{1}{z-w}$. One also has the regularity of bb and cc OPEs $b(z)b(w) = O(z-w) = c(z)c(w)$. Even though the so-called *free fermion* case $\lambda = \frac{1}{2}$ is not directly relevant for our purposes, in this theory one trivially finds $U(1)$ currents (which develop an anomaly for $\lambda \neq \frac{1}{2}$ by creating two distinct fields b and c) $j^{mn} = i\ {:}\psi^m\psi^n{:}$. Since in this case one has regular currents, one constructs a stress-tensor proportional to ${:}\psi^m\partial\psi^m{:}$ via the Sugawara construction. This inspires us to make an ansatz

$$T = \alpha\ {:}b\partial c{:} + \beta\ {:}c\partial b{:} \tag{2.251}$$

Then we fix α and β analogously as in the free boson, that is by requiring the proper weights of b and c with the result

$$T = -2\ {:}b\partial c{:} - {:}c\partial b{:} \tag{2.252}$$

Computing the TT OPE then gives the central charge $c = -26$, which shows us that this theory is not unitary. Expanding into modes

$$b(z) = \sum_{n\in\mathbb{Z}} \frac{b_n}{z^{n+2}} \tag{2.253}$$

$$c(z) = \sum_{n\in\mathbb{Z}} \frac{c_n}{z^{n-1}}, \tag{2.254}$$

we can compute the anticommutation relation

$$\begin{aligned}
\{b_m, c_n\} &= \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n-2} b(z) c(w) \\
&= \oint \frac{\mathrm{d}z}{2\pi i} \oint \frac{\mathrm{d}w}{2\pi i} z^{m+1} w^{n-2} \frac{1}{z-w} \\
&= \oint \frac{\mathrm{d}z}{2\pi i} z^{m+n-1} \\
&= \delta_{m+n,0}
\end{aligned} \tag{2.255}$$

The b_0, c_0 algebra generates two ground states $|\downarrow\rangle$ and $|\uparrow\rangle$ such that $b_0 |\downarrow\rangle = 0$, $b_0 |\uparrow\rangle = |\downarrow\rangle$ and $c_0 |\downarrow\rangle = |\uparrow\rangle, c_0 |\uparrow\rangle = |\downarrow\rangle$ and they are annihilated by b_n and c_n for $n > 0$. But we know that in every CFT there is a unit operator, which gives us the vacuum $|0\rangle$. Is there a relation linking these two ground states to the vacuum? From the weights of b and c, we have that $b_n |0\rangle = 0$ for $n \geq -1$ and $c_n |0\rangle = 0$ for $n \geq 2$ and so the algebraically unique expression is $|0\rangle = b_{-1} |\downarrow\rangle$ since one has $b_{-1} |0\rangle = 0$ from $b_{-1}^2 = 0$ and the vanishing of $c_n |0\rangle$ for $n \geq 2$ follows from the anticommutation of such c modes with b_{-1} and $c_n |\downarrow\rangle = 0$ for $n \geq 2$. The ghost Hilbert space is thus obtained by acting with b_m and c_n on the vacuum, where $m < -1$ and $n < 2$. Finally, we can use the state-operator mapping to obtain

$$b_{-n} |0\rangle \rightarrow \frac{1}{(n-2)!} \partial^{n-2} b(0), \; n \geq 2 \tag{2.256}$$

$$c_{-n} |0\rangle \rightarrow \frac{1}{(n+1)!} \partial^{n+1} c(0), \; n \geq -1 \tag{2.257}$$

This is very simply obtained from the behavior of modes when acting on the vacuum state and realising that the state-operator mapping maps $|0\rangle$ to the unit operator, which has trivial OPEs with the ghosts. For example

$$b_{-n} = \oint \frac{\mathrm{d}z}{2\pi i} z^{-n+1} b(z) = \frac{1}{(n-2)!} \partial^{n-2} b(0), \; n \geq 2 \tag{2.258}$$

so that

$$b_{-n} |0\rangle \rightarrow \oint \frac{\mathrm{d}z}{2\pi i} z^{-n+1} b(z) 1 = \frac{1}{(n-2)!} \partial^{n-2} b(0), \; n \geq 2 \tag{2.259}$$

and

$$b_{-n} |0\rangle \rightarrow \oint \frac{\mathrm{d}z}{2\pi i} z^{-n+1} b(z) 1 = 0, \; n < 2 \tag{2.260}$$

from the holomorphicity of b.

Chapter 3

Boundary conformal field theory

In this chapter we study boundary conformal field theories (BCFTs), that is CFTs with an addition of a boundary. We mostly follow the concise reviews [54, 55] and the textbook [56]. Some attention to BCFTs is also given in the textbooks [27, 28]. In its early stages the theory was largely developed by Cardy [57, 58, 59, 60]. The thesis [61] provides an accessible introduction to boundary CPT. In boundary CPT we are often interested in the change of the g-function first defined in [62]. The g-function obeys a theorem analogous to the c-theorem, which was conjectured in [63] after a perturbative argument and proved nonperturbatively in [64]. Boundary perturbations of Cardy boundary conditions in unitary minimal models by the least relevant operator are investigated to leading order in [65]. For these a crucial input is given by the structure constants worked out by Runkel in the thesis [66]. These developments of boundary CPT are reviewed in [67].

3.1 Gluing conditions and Ishibashi states

In BCFT, we are interested in boundary conditions whose presence breaks the least amount of the bulk CFT symmetry. In particular this always means that they break only half of the conformal symmetry. It is clear that this places stringent conditions on the BCFT and the more symmetry we have in the bulk, the less conformal boundaries can exist for a given CFT. After finding such a boundary condition, the local properties embodied by the OPE are still the same away from the boundary and this gives us a chance of using our bulk CFT knowledge to understand BCFTs. However, globally the bulk theory is modified and this results in a change of bulk correlators

$$\langle \phi_1 \phi_2 \ldots \rangle_a = \langle 0 | \, \phi_1 \phi_2 \ldots \| B_a \rangle\rangle \,, \tag{3.1}$$

where $\|B_a\rangle\rangle$ is called the *boundary state* of the boundary condition a. We now derive the restrictions on the boundary state called the *gluing conditions* coming the conditions that it preserves some symmetry. For simplicity let us assume that the boundary lies along the real axis, meaning we work on the upper half-plane (UHP) and suppose we have a symmetry algebra in the bulk with quasiprimary generators W and $\bar{W}$ with $h = \bar{h}$ including T and $\bar{T}$. The condition that we break the least amount of symmetry possible translates into a condition on the real axis $z \in \mathbb{R}$

$$W(z) = \Omega_a\Big(\overline{W}(\bar{z})\Big), \tag{3.2}$$

which relates W and $\overline{W}$. The Ω_a is an ultralocal (commutes with mode expansions) symmetry algebra automorphism, which always leaves T invariant (acts trivially on T). This is motivated by the fact that the only Möbius transformations leaving the real axis (boundary) fixed are the $SL(2, \mathbb{R})$ transformation, which satisfy $f(\bar{z}) = \bar{f}(z)$, which turns into

$$T(z) = \bar{T}(\bar{z}) \tag{3.3}$$

on the boundary. Nontrivial automorphisms Ω_a are present for example in the free boson (2.8.3) that has $U(1)$ symmetry. On the $U(1)$ current, we can have the nontrivial condition

$$j(z) = -\bar{j}(\bar{z}) \tag{3.4}$$

corresponding to a Neumann boundary condition for X. The trivial automorphism in this case turns out to be the Dirichlet boundary condition for X, both being compatible with the trivial automorphism acting on T via the Sugawara construction.

From radial quantisation, we know that states naturally live on circles and so in order to restrict $\|B_a\rangle\rangle$, we make an invertible transformation from the UHP to the unit disc

$$\xi(z) = \frac{i - z}{i + z}, \tag{3.5}$$

satisfying $\xi'(z) = \frac{i}{2}(\xi - 1)^2$ and $\bar{\xi}'(\bar{z}) = -\frac{i}{2}(\bar{\xi} - 1)^2$. For our generators, the condition (3.2) then translates to

$$\left(\frac{1}{2}(\xi - 1)^2\right)^h W(\xi) = \left(-\frac{1}{2}(\bar{\xi} - 1)^2\right)^h \Omega_a\Big(\overline{W}(\bar{\xi})\Big) \tag{3.6}$$

since they are quasiprimaries of $h = \bar{h}$ and the real axis is mapped to $|\xi| = 1$. Since for $|\xi| = 1$ one has $\bar{\xi} = \xi^{-1}$, then

$$(\xi - 1)^{2h} = (\xi - \xi\bar{\xi})^{2h} = \xi^{2h}(1 - \bar{\xi})^{2h}, \tag{3.7}$$

so that (3.6) turns into

$$\xi^{2h} W(\xi) - (-1)^h \Omega_a\Big(\overline{W}(\bar{\xi})\Big) = 0 \tag{3.8}$$

After expanding in modes

$$W(z) = \sum_{n\in\mathbb{Z}} \frac{W_n}{z^{n+h}}, \tag{3.9}$$

this becomes

$$\sum_{n\in\mathbb{Z}} \xi^{h-n}\Big(W_n - (-1)^h \Omega_a\Big(\overline{W}_{-n}\Big)\Big) = 0, \tag{3.10}$$

where we again used that for $|\xi| = 1$ one has $\bar{\xi} = \xi^{-1}$. Since (3.10) holds for every ξ with $|\xi| = 1$, we must have when acting on the boundary state the gluing conditions

$$\Big(W_n - (-1)^h \Omega_a\Big(\overline{W}_{-n}\Big)\Big) \|B_a\rangle\!\rangle = 0 \tag{3.11}$$

For these gluing conditions to be nontrivial (remember, $n \in \mathbb{Z}$), clearly $\|B_a\rangle\!\rangle$ must be a *coherent state*, that is it isn't a local finite energy *Fock state*. An example of the gluing condition is $(j_n + \bar{j}_{-n}) \|B_a\rangle\!\rangle = 0$ for the Neumann condition in the $U(1)$ theory and $(j_n - \bar{j}_{-n}) \|B_a\rangle\!\rangle = 0$ for the Dirichlet condition. The resulting boundary state is then called a *D-brane*, in particular we often consider tensor copies of the free boson so that when having $p+1$ Neumann boundary conditions and the rest Dirichlet, we call it a *Dp-brane*. Another example is that for the always present trivially glued T, we get

$$\Big(L_n - \bar{L}_{-n}) \|B_a\rangle\!\rangle = 0 \tag{3.12}$$

so that only one half of the Virasoro algebra survives.

In diagonal CFTs, that is CFTs in which the Hilbert space $\mathcal{H} = \bigoplus_{i,j} N_{ij} H_i \otimes \bar{H}_j$ decomposed into irreps of the symmetry algebra H_i satisfies $N_{ij} \neq 0$ only for $i = j$ with the chiral and antichiral symmetry algebras being equal, one can find explicit solutions to the gluing conditions. These are given by the *Ishibashi states* [68], which have for the case of Ω_a trivial (we thus specialise $W \to T$) the form

$$\|i\rangle\!\rangle = \sum_{N=0}^{\infty} |i, N\rangle \otimes U\,|i, N\rangle\,, \tag{3.13}$$

where is U is an antiunitary operator on the chiral hilbert space $H_{ch} = \bigoplus_i H_i$ satisfying

$$U\overline{L}_n = (-1)^h \overline{L}_n U \tag{3.14}$$

The $\{|i,N\rangle\}$ is an orthonormal basis of the chiral representation at level N corresponding to a spinless (that is with $h=\bar{h}$) primary $|i\rangle$. The Ishibashi state can be generalised to the nontrivial Ω_a case as well, see [56], but we will present the proof that the gluing conditions hold only for the trivial case. The proof is carried out by realising that for (3.11) to be satisfied, it suffices if it is satisfied when projected onto the orthonormal basis and one also uses the antiunitarity of U and its commutation property (3.14)

$$
\begin{aligned}
&\sum_{N=0}^{\infty} \langle k,M|\, \langle k,M|\, U^{\star}\Big(L_n-(-1)^h\overline{L}_{-n}\Big)\,|i,N\rangle\otimes U\,|i,N\rangle=\\
&\sum_{N=0}^{\infty}\Bigg[\,\langle k,M|\, L_n\,|i,N\rangle\,\langle k,M|\, U^{*}U\,|i,N\rangle\\
&-(-1)^h\,\langle k,M|i,N\rangle\,\langle k,M|\,U^{*}\overline{L}_{-n}U\,|i,N\rangle\Bigg]=\\
&\sum_{N=0}^{\infty}\Bigg[\delta_{k,i}\delta_{M,N}\,\langle k,M|\,L_n\,|i,N\rangle\\
&-(-1)^h\delta_{k,i}\delta_{M,N}(-1)^{-h}\,\langle k,M|\,U^{*}U\overline{L}_{-n}\,|i,N\rangle\Bigg]=\\
&\langle i,M|\,L_n\,|i,M\rangle-\langle i,M|\,\overline{L}_n\,|i,M\rangle=0,
\end{aligned}
$$

where in the last line we used the fact that L_n acting on the chiral sector the same as $\overline{L}_n$ acting on the antichiral sector and we used the star notation for a BPZ conjugate operator to differentiate from Hermitean conjugation. Our considerations are still a little formal since we haven't specified the operator U, but it can be shown by acting with the Virasoros as in (3.12) that one can write an Ishibashi state

$$
\|i\rangle\rangle=\sum_{IJ}M^{IJ}(h)L_{-I}\bar{L}_{-J}\,|i\rangle\,, \tag{3.15}
$$

where $M^{IJ}(h)$ is an inverse of the Gram matrix $M_{IJ}(h)=\langle i|\,L_I L_{-J}\,|i\rangle$ with I,J multiindices. This allows us to write a trivial Ishibashi state level by level

$$
\begin{aligned}
\|i\rangle\rangle \;&=\; \Big(1+\frac{1}{2h}L_{-1}\bar{L}_{-1}+\frac{1}{c(2h+1)+2h(8h-5)}\Big[\qquad (3.16)\\
&2(2h+1)L_{-2}\bar{L}_{-2}-3(L_{-2}\bar{L}_{-1}^{2}+L_{-1}^{2}\bar{L}_{-2})+\frac{c+8h}{4h}L_{-1}^{2}\bar{L}_{-1}^{2}\Big]+\dots\Big)\,|i\rangle\,,
\end{aligned}
$$

by simply inverting (2.119) and (2.120).

The gluing conditions (3.11) solved by the Ishibashi states are not enough to uniquely specify the boundary state $\|B_a\rangle\rangle$. This is so because additional selections among the irreps of the symmetry algebra are made by considering various consistency conditions like the Cardy conditions [60] (analogue of modular invariance, see (2.8.3) for the discussion of modular invariance) or sewing relations [69, 70]. Nevertheless, it is true that after appropriate selections, we can write the boundary state as a superposition of Ishibashi states

$$\|B_a\rangle\!\rangle = \sum_i B_a^i \,\|i\rangle\!\rangle \tag{3.17}$$

The Ishibashi state (3.13) satisfies the important property

$$\langle j \,\|i\rangle\!\rangle = \langle j|i\rangle = \delta_{i,j} \tag{3.18}$$

since $|i\rangle$ is the unique state at level 0 in the conformal family generated by $|i\rangle$. This means that we can read off the *boundary state coefficients* B_a^i as one-point functions with spinless bulk primaries

$$B_a^i = \langle i \,\|B_a\rangle\!\rangle \tag{3.19}$$

Before we try to find some boundary states, we investigate the boundary spectrum.

3.2 Boundary fields

Looking at (3.3), we realise that an UHP correlator with $\bar{T}$ at a point specified as $(z, \bar{z})$ can be rewritten as a correlator on the entire plane using the so-called *doubling trick*, where by analytic continuation the boundary acts as a mirror that transports $\bar{T}$ at $(z, \bar{z})$ to a T at the mirror point $(\bar{z}, z)$, which is no longer on the UHP. This means in particular that as T approaches the boundary from the UHP, it finds a singularity since there is another T moving towards it, which near the boundary, where $T = \bar{T}$ this is like a $\bar{T}$ moving towards it and then we have a nontrivial OPE. From purely UHP point of view, this looks like an interaction with the boundary itself, that is with *boundary fields*. These boundary fields can be organised into irreps of the reduced symmetry algebra on the boundary, for example we have only chiral fields living at the boundary since only one half of the Virasoro algebra remains. In general because of the reduced symmetry, the boundary operator content can look quite different from the bulk. We write the state space of the boundary theory as $\mathcal{H}_a = \bigoplus_i n_a^i \mathcal{H}_i$, where the leftover symmetry algebra irreps $\mathcal{H}_i$ in general occur with nontrivial multiplicities n_a^i.

The interaction of bulk with boundary fields can be locally summarised by the *bulk-boundary OPE*

$$\phi_i(z, \bar{z}) \sim \sum_j B_{ij}^a (2y)^{h_j - \Delta_i} V_j(x) \tag{3.20}$$

where B^a_{ij} is a *bulk-boundary structure constant* and $z = x + iy$. From now on we also deploy the convention that we write bulk fields as ϕ and boundary fields as V. This is so because the machinery built in (2), where we usually write V, often purely chiral for simplicity, translates to purely chiral fields on the boundary that we'll encounter more than bulk fields from now on.

Since the boundary theory is also a CFT, a fundamental role is played by the *boundary OPE*

$$V_i(x_1)V_j(x_2) \sim \sum_k C^a_{ijk}|x_{12}|^{h_k-h_i-h_j}V_k(x_2), \tag{3.21}$$

with C^a_{ijk} being a *boundary structure constant*. On the UHP we have the correlators of a chiral field

$$\langle V_i(x_1)V_j(x_2)\rangle_{UHP} = \frac{\langle 1\rangle_a \delta_{ij}}{x_{12}^{2h}} \tag{3.22}$$

and

$$\langle V_i(x_1)V_j(x_2)V_k(x_3)\rangle_{UHP} = \frac{\langle 1\rangle_a C^a_{ijk}}{x_{12}^{h_k-h_i-h_j}x_{13}^{h_j-h_i-h_k}x_{23}^{h_i-h_j-h_k}} \tag{3.23}$$

From (3.20) and (3.22), we can very simply derive by approaching the boundary the form of the UHP bulk-boundary two-point function on the UHP

$$\langle \phi_i(z,\bar{z})V_j(r)\rangle_{UHP} = \frac{\langle 1\rangle_a B^a_{ij}}{(2y)^{\Delta_i-h_j}((x-r)^2+y^2)^{h_j}} \tag{3.24}$$

for $z = x + iy$.

Boundary fields living on a given boundary can be generalised to the so-called *boundary condition changing operators* (BCCOs). The motivation to consider this generalisation comes from studying BCFT on an infinite strip, which we position parallel to the real axis and give it height π. On this strip, nothing prevents us from having different boundary conditions on its two sides. Mapping this strip with coordinate w back to the UHP via $z = e^w$, we find that its two ends get mapped to the negative and positive real semi-lines respectively. After the mapping, we see two boundary conditions meet at the origin $z = 0$ and this produces a singularity. This singularity is a signal of the presence of a BCCO V^{ab}, which can be thought of as changing the boundary condition a to b. The generalised boundary Hilbert space is then $\mathcal{H}_{ab} = \bigoplus_i n^i_{ab}\mathcal{H}_i$ with the $a = b$ diagonal terms corresponding to the original $\mathcal{H}_a$ and the most singular $|i\rangle$ for which $n^i_{ab} \neq 0$ is the BCCO. We note that in $\mathcal{H}_{ab}$ for $a \neq b$ we don't have the vacuum $|0\rangle$ since this would correspond to a smooth transition between a and b. The generalised OPEs of BCCOs have the form

$$V_i^{ab}(x_1)V_j^{cd}(x_2) \sim \sum_k \delta_{b,c}\, C_{ijk}^{abc} x_{12}^{h_k-h_i-h_j} V_k^{ad} \tag{3.25}$$

with the $\delta_{b,c}$ being there for consistency since BCCOs have to fit together. The boundary condition index structure is similar to matrix multiplication and plays an important role in string theory, where it enables one to associate *Chan-Paton factors* to open string endpoints and in result produce gauge symmetries in the effective theory of open strings.

3.3 Cardy conditions and Cardy states

Let us begin by considering the cylinder partition function of the boundary theory

$$Z_{ab}(\tilde{q}) = \mathrm{Tr}_{\mathcal{H}_{ab}}\, e^{-\frac{2\pi T}{L} H_{ab}} = \sum_i n_{ab}^i \chi_i(\tilde{q}), \tag{3.26}$$

where we write $\tilde{q} \equiv e^{-\frac{2\pi T}{L}}$ with L being the length of the cylinder and T measuring its circumference. The χ_i is a character of $\mathcal{H}_i$, which occurs with multiplicity n_{ab}^i due to definition of $\mathcal{H}_{ab}$. The same partition function can be calculated by putting the boundary states $\|B_a\rangle\rangle$ and $\|B_b\rangle\rangle$ on the cylinder ends and evolving for a time L via a bulk hamiltonian H. This amount to swapping the space and time directions and after writing $\tau \equiv i\frac{L}{T}$ corresponds to the modular transformation $\tau \to -\frac{1}{\tau}$. One can then write $\tilde{q} = e^{-\frac{2\pi i}{\tau}}$ and $q = e^{2\pi i \tau}$, after which

$$Z_{ab}(\tilde{q}) = \langle\langle B_a \|\, e^{-\frac{2\pi L}{T} H}\, \| B_b \rangle\rangle = \sum_i B_a^{i\,*} B_b^i\, \chi_i(q), \tag{3.27}$$

where we expanded the boundary state into Ishibashi states and used a normalisation of the highly non-normalisable Ishibashi states of the form

$$\langle\langle i \|\, e^{-\frac{2\pi L}{T}}\, \| j \rangle\rangle = \delta_{ij}\chi_i(q) \tag{3.28}$$

This normalisation is possible thanks to the damping factor inserted in between. In CFTs with a finite number of conformal families (*rational* CFTs), the characters transform linearly under the modular transform $\tau \to -\frac{1}{\tau}$, that is they get multiplied by a symmetric and unitary *S-matrix*

$$\chi_i(q) = \sum_j S_{ij}\chi_j(\tilde{q}) \tag{3.29}$$

Inserting this into (3.27) and equating (3.26) with (3.27), one easily finds the Cardy conditions

$$n^i_{ab} = \sum_j B^{j\,*}_a B^j_b S_{ij}, \tag{3.30}$$

which gives the boundary field operator content after knowing the boundary state and the S-matrix. Multiplying (3.30) with an S from the right gives

$$B^{i\,*}_a B^i_b = \sum_j n^j_{ab} S_{ij} \tag{3.31}$$

thanks to the unitarity of S. Since $n^i_{ab} \in \mathbb{N}_0$, we see from both (3.30) and (3.31) that the boundary state is highly restricted by the Cardy conditions. The Cardy conditions implies that one cannot just scale the boundary state at will, one can only scale it by an integer. The boundary states solving (3.30) thus form a positive cone of a lattice, meaning that one can take positive integer combinations of the solutions of (3.30) to obtain another solution. One can ask whether there is a sort of basis in this lattice, which generates all other solutions by the positive integer combinations and indeed such a basis is believed to be characterised by

$$\sum_a B^{i\,*}_a B^j_a = \frac{\delta_{ij}}{S_{0i}} \tag{3.32}$$

where the S_{0i} is unique by the uniqueness of $|0\rangle$. Using this relation, one can find an algebra that the multiplicities n^i_{ab} obey thanks to the Verlinde formula [71]

$$N^k{}_{ij} = \sum_m \frac{S^*_{km} S_{im} S_{jm}}{S_{0m}}, \tag{3.33}$$

where the $N^k{}_{ij}$ are the fusion rules (2.176). It enables us to carry out the computation

$$\begin{aligned}
\sum_b n^i_{ab} n^j_{bc} &= \sum_b \sum_{lm} B^{l\,*}_a B^l_b S_{il}\, B^{m\,*}_b B^m_c S_{jm} \\
&= \sum_{lm} \delta_{lm} B^{l\,*}_a B^m_c \frac{S_{il} S_{jm}}{S_{0l}} \\
&= \sum_l B^{l\,*}_a B^l_c \frac{S_{il} S_{jl}}{S_{0l}} \\
&= \sum_{lkm} B^{l\,*}_a B^l_c S_{kl} \frac{S^*_{km} S_{im} S_{jm}}{S_{0m}} \\
&= \sum_{km} n^k_{ac} S^*_{km} S_{0m} S_{im} S_{jm} \\
&= \sum_k n^k_{ac} N^k{}_{ij}\,,
\end{aligned}$$

meaning that the multiplicities n_{ab}^i form a representation of a fusion algebra, called the *non-negative integer matrix representation* (NIM-rep). It is suggestive that there should be theories with the n_{ab}^i being the fusion coefficients. Using (3.33) in (3.30), one sees that such a choice of a NIM-rep indicates the form of the boundary state

$$\|B_i\rangle\rangle = \sum_j \frac{S_{ij}}{\sqrt{S_{0j}}} \|j\rangle\rangle\,, \tag{3.34}$$

meaning that there would be as many boundary conditions as there are irreps in the CFT. It indeed turns out [60] that (3.34) provides the lattice basis for diagonal rational CFTs and the states (3.34) are called *Cardy states.* A special case occurs in the minimal models $\mathcal{M}_m$, see (2.8.1), where boundary conditions are given by Kac labels (a_1, a_2) such that $1 \leq a_1 \leq m$ and $1 \leq a_2 \leq m+1$, labelling the chiral representations. The resulting states are given by the formula

$$\|a_1, a_2\rangle\rangle = \sum_{(r,s)} \frac{S_{(a_1,a_2)(r,s)}}{\sqrt{S_{(1,1)(r,s)}}} \|r, s\rangle\rangle\,, \tag{3.35}$$

where in the minimal models, the S-matrix has the explicit form

$$S_{(r,s)(r',s')} = \sqrt{\frac{8}{m(m+1)}}(-1)^{1+rs'+sr'} \sin\left(\pi\frac{m+1}{m}rr'\right) \sin\left(\pi\frac{m+1}{m}ss'\right) \tag{3.36}$$

For example, this allows us to write all boundary states of the critical two-dimensional Ising with $m = 3$

$$\left\|B_{(1,1)}\right\rangle\rangle \equiv \|+\rangle\rangle = \frac{1}{\sqrt{2}}\|\mathbb{1}\rangle\rangle + \frac{1}{\sqrt{2}}\|\epsilon\rangle\rangle + \frac{1}{\sqrt[4]{2}}\|\sigma\rangle\rangle \tag{3.37}$$

$$\left\|B_{(1,3)}\right\rangle\rangle \equiv \|-\rangle\rangle = \frac{1}{\sqrt{2}}\|\mathbb{1}\rangle\rangle + \frac{1}{\sqrt{2}}\|\epsilon\rangle\rangle - \frac{1}{\sqrt[4]{2}}\|\sigma\rangle\rangle \tag{3.38}$$

$$\left\|B_{(2,2)}\right\rangle\rangle \equiv \|f\rangle\rangle = \|\mathbb{1}\rangle\rangle - \|\epsilon\rangle\rangle \tag{3.39}$$

The boundary states (3.37) and (3.38) are identified as the boundary conditions with all spins up or down since they differ only by a sign in front of an $\|\sigma\rangle\rangle$. The third boundary state (3.39) is then interpreted as a free boundary condition since it doesn't contain a $\|\sigma\rangle\rangle$. From the primary OPEs, one can read off the fusion coefficients and thus the boundary operator content. It can be summarised by stating that on all three boundary conditions a $\mathbb{1}$ can live and in fact it is the only operator on $\|+\rangle\rangle$ and $\|-\rangle\rangle$. On $\|f\rangle\rangle$ we also have an ϵ.

3.4 Boundary entropy

The partition function (3.27) can be thought of as a partition function of a one-dimensional quantum system on an interval of length $2\pi L$ at inverse temperature $T \equiv \beta$ (remember, T here is time, not temperature). In the thermodynamic limit of $\frac{2\pi L}{\beta}$ large, the ground state $|0\rangle$ gives a dominant contribution, so that

$$Z_{ab} \sim \langle\langle B_a \| \, 0\rangle\langle 0 \, \| B_b\rangle\rangle \, e^{\frac{\pi c L}{12\beta}}, \tag{3.40}$$

where we assumed that the ground state energy is zero and used the form of the cylinder Hamiltonian (2.99). The free energy is then given by

$$F_{ab} = -\beta^{-1} \ln Z_{ab} \sim -\beta^{-1}\Big(\frac{\pi c L}{12\beta} + \ln \langle\langle B_a \| \, 0\rangle + \ln\langle 0 \, \| B_b\rangle\rangle\Big), \tag{3.41}$$

so that the entropy is

$$S_{ab} = \beta^2 \frac{\partial F_{ab}}{\partial \beta} \sim \frac{\pi c L}{6\beta} + \ln \langle\langle B_a \| \, 0\rangle + \ln\langle 0 \, \| B_b\rangle\rangle \,, \tag{3.42}$$

where apart from the extensive contribution $\frac{\pi c L}{6\beta}$, one finds a universal contribution of the zero-temperature *boundary entropies* $\ln\langle 0 \, \| B_a\rangle\rangle$ and $\ln\langle 0 \, \| B_b\rangle\rangle$ (the phase of $|0\rangle$ is chosen so that $\langle 0 \, \| B_a\rangle\rangle \in \mathbb{R}_+$). These boundary entropies can be interpreted as logarithms of the ground state degeneracy

$$g_a \equiv \langle 0 \, \| B_a\rangle\rangle \,, \tag{3.43}$$

called the g-function, so that the boundary entropy associated with a given boundary condition is $S_a = \ln g_a$. We note that these ground state degeneracies do not have to be integers a priori. The equation (3.43) means that the g-functions are disc (the cylinder has a disc on each end) one-point functions of the identity operator in the presence of a or equivalently the coefficients of the identity in B_a. In diagonal rational CFTs, we have

$$g_i = \frac{S_{i0}}{\sqrt{S_{00}}}, \tag{3.44}$$

which becomes explicit for the minimal models

$$g_{(a_1,a_2)} = \left(\frac{8}{m(m+1)}\right)^{\frac{1}{4}} \frac{\sin\left(\frac{\pi a_1}{m}\right) \sin\left(\frac{\pi a_2}{m+1}\right)}{\left(\sin\left(\frac{\pi}{m}\right) \sin\left(\frac{\pi}{m+1}\right)\right)^{\frac{1}{2}}} \tag{3.45}$$

For the Ising this gives $g_{(1,1)} = g_+ = g_{(1,3)} = g_- = \frac{1}{\sqrt{2}}$ and $g_{(2,2)} = g_f = 1$ as can be read off from (3.37)-(3.39). From (3.31) and (3.44), one reads off

$$g_i g_j = \sum_k n_{ij}^k S_{0k} = \sqrt{S_{00}} \sum_k n_{ij}^k g_k \tag{3.46}$$

This means that for example in the Ising we have

$$\begin{aligned}
g_+ g_- &= \frac{1}{\sqrt{2}} g_- \\
g_+ g_f &= \frac{1}{\sqrt{2}} g_f \\
g_- g_f &= \frac{1}{\sqrt{2}} g_f \\
g_+ g_+ &= \frac{1}{\sqrt{2}} g_+ \\
g_- g_- &= \frac{1}{\sqrt{2}} g_+ \\
g_f g_f &= \frac{1}{\sqrt{2}} (g_+ + g_-)
\end{aligned}$$

as can be verified from the values of the Ising g-functions (see the Ising fusion rules in (2.8.1) to read off the fusion coefficients and we also note that $S_{00} = \frac{1}{\sqrt{2}}$ for the Ising).

3.5 Boundary conformal perturbation theory

Let us consider a setup, where we deform the boundary theory by boundary operators inserted along the unit disc boundary, that is $S_{boundary} \to S_{boundary} + \sum_i \lambda_i \int \mathrm{d}\theta\, V_i$ and observe an RG flow of the boundary condition. Immediately we see that the beta-function can be obtained for free from the bulk result by replacing (for dimensional reasons) $a^2 \to a$ and the annulus area $2\pi a^2 \mathrm{d}t \to 2a\mathrm{d}t$ in the procedure that lead to (2.143). After adding an appropriate boundary index a, it is easy to see that this results in the leading order beta-function

$$\beta_k^a(\lambda) = (1 - h_k)\lambda_k - \sum_{ij} C_{ijk}^a \lambda_i \lambda_j + \ldots = y_k \lambda_k - \sum_{ij} C_{ijk}^a \lambda_i \lambda_j, \tag{3.47}$$

so that for a deformation by a single operator V, we have the fixed point coupling

$$\lambda_* = \frac{y}{C_{VVV}^a} + O(y^2) \tag{3.48}$$

Now we try to reproduce the leading order shift in the g-function after perturbing with a single V, first calculated by Affleck and Ludwig [63]. To do so, we compute the leading order shift in the one-point function of the identity on a disc and for this purpose, we reintroduce the cutoff a into the coupling, so that we write the deformation as $a^{h-1}\lambda \int \mathrm{d}\theta\, V$. Since $g_{a'} = \langle \mathbb{1} \rangle_{a'} = \langle \mathbb{1} e^{-S_{boundary} - a^{h-1}\lambda \int \mathrm{d}\theta\, V} \rangle_a$, we have for the shift Δg_a in the g-function when flowing from the boundary condition a to a' the equation

$$\begin{aligned} \Delta g_a &= -a^{h-1}\lambda \int \mathrm{d}\theta\, \langle V(\theta) \rangle_a && (3.49) \\ &\quad + \frac{1}{2!} a^{2(h-1)} \lambda^2 \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \langle V(\theta_1) V(\theta_2) \rangle_a && (3.50) \\ &\quad - \frac{1}{3!} a^{3(h-1)} \lambda^3 \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 \langle V(\theta_1) V(\theta_2) V(\theta_3) \rangle_a + \ldots && (3.51) \end{aligned}$$

The contribution (3.49) vanishes since in the boundary theory, one has $\langle V \rangle_a = 0$ for all $V \neq 1$. The other correlators on the disc are given by transforming (3.22) and (3.23) to the disc by $f(w) = i\frac{1-w}{1+w}$

$$\langle V(\theta_1) V(\theta_2) \rangle_a = g_a \left| 2 \sin \frac{\theta_{12}}{2} \right|^{-2h} \tag{3.52}$$

$$\langle V(\theta_1) V(\theta_2) V(\theta_3) \rangle_a = g_a C^a_{VVV} \left| 8 \sin \frac{\theta_{12}}{2} \sin \frac{\theta_{13}}{2} \sin \frac{\theta_{23}}{2} \right|^{-h} \tag{3.53}$$

Expanding the correlator (3.52) in y, we have

$$\langle V(\theta_1) V(\theta_2) \rangle_a \sim g_a \left| 2 \sin \frac{\theta_{12}}{2} \right|^{-2} + g_a \frac{\ln \left| 2 \sin \frac{\theta_{12}}{2} \right|}{2 \left| \sin \frac{\theta_{12}}{2} \right|} y + O(y^2) \tag{3.54}$$

and similarly (3.53) gives

$$\langle V(\theta_1) V(\theta_2) V(\theta_3) \rangle_a = g_a C^a_{VVV} \left| 8 \sin \frac{\theta_{12}}{2} \sin \frac{\theta_{13}}{2} \sin \frac{\theta_{23}}{2} \right|^{-1} + O(y), \tag{3.55}$$

where we only needed to expand to zeroth order. To evaluate (3.50), we need to calculate an integral by substituting $\theta_1 \to \theta_1, \theta_1 - \theta_2 \to \theta$ and using the fact that θ_1 and θ_2 are separated by at least the cutoff a, we have

$$\begin{aligned} \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \left| 2 \sin \frac{\theta_{12}}{2} \right|^{-2h} &= \frac{\pi}{2} \int_a^{2\pi - a} \mathrm{d}\theta \left(\sin^{-2}\left(\frac{\theta}{2}\right) + 2 \frac{\ln\left(2 \sin \frac{\theta}{2}\right)}{\sin \frac{\theta}{2}} y \right) + O(y^2) \\ &= \frac{4\pi}{a} + \left(\frac{32\pi(1 + \ln a)}{a} - 2\pi^2 \right) y + O(a, y^2) && (3.56) \end{aligned}$$

For the contribution (3.51), we need to evaluate the integral

$$\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 \left| 8 \sin\frac{\theta_{12}}{2} \sin\frac{\theta_{13}}{2} \sin\frac{\theta_{23}}{2} \right|^{-h} = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 \left| 8 \sin\frac{\theta_{12}}{2} \sin\frac{\theta_{13}}{2} \sin\frac{\theta_{23}}{2} \right|^{-1} + O(y) \tag{3.57}$$

by substitution $\theta_{13} \to \theta_1, \theta_{23} \to \theta_2, \theta_3 \to \theta_3$ and trivially integrating over θ_3 to get 2π. After doing so, we observe a residual permutation symmetry on θ_1 and θ_2 giving an extra factor of 2, after which

$$\begin{aligned} \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \mathrm{d}\theta_3 \left| 8 \sin\frac{\theta_{12}}{2} \sin\frac{\theta_{13}}{2} \sin\frac{\theta_{23}}{2} \right|^{-1} &= \\ 4\pi \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{\theta_1} \mathrm{d}\theta_2 \left| 8 \sin\frac{\theta_{12}}{2} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \right|^{-1} &= \\ \pi \int_0^{2\pi} \mathrm{d}\theta_1 \left[\frac{\ln \sin\frac{\theta_2}{2} - \ln \sin\frac{\theta_{12}}{2}}{\sin^2 \frac{\theta_1}{2}} \right]_{\theta_2 \to 0}^{\theta_2 \to \theta_1} & \end{aligned} \tag{3.58}$$

We choose to apply the cut-off on both bounds, this somewhat arbitrary choice (we could impose a cut-off only on the divergent terms) changes only cut-off dependent terms

$$\begin{aligned} &\pi \int_0^{2\pi} \mathrm{d}\theta_1 \left[\frac{\ln \sin\frac{\theta_2}{2} - \ln \sin\frac{\theta_{12}}{2}}{\sin^2 \frac{\theta_1}{2}} \right]_{\theta_2 \to a}^{\theta_2 \to \theta_1 - a} = \\ &\pi \int_a^{2\pi - a} \mathrm{d}\theta_1 \csc^2\left(\frac{\theta_1}{2}\right) \Bigg(\ln\left(-\sin\left(\frac{a-\theta_1}{2}\right)\right) + \\ &\ln\left(\sin\left(\frac{a-\theta_1}{2}\right)\right) - \ln\left(-\sin\frac{a}{2}\right) - \ln\left(\sin\frac{a}{2}\right)\Bigg) = \\ &\frac{4\pi \left(\ln(-a) - \ln\left(-\frac{a}{2}\right) + 3\ln(2)\right)}{a} - 4\pi^2 + O(a) \end{aligned} \tag{3.59}$$

Adding (3.58) with (3.59) together with appropriate prefactors as in (3.52) and (3.53) gives the universal cut-off independent contribution to the shift Δg_a at the leading order

$$\Delta g_a = g_a \left(1 - \pi^2 \lambda^2 y - \frac{2\pi^2}{3} \lambda^3 C^a_{VVV} \right) + O(y^4), \tag{3.60}$$

since no poles in y in structure constants occur, there are not further contributions at the leading order and thus (3.60) is exact. After plugging in the fixed point coupling (3.48), we have the wanted result of Affleck and Ludwig

$$\frac{\Delta g_a}{g_a} = -\frac{\pi^2}{3}\frac{y^3}{\left(C^a_{VVV}\right)^2} + O(y^4) \tag{3.61}$$

An alternative form can be obtained from $e^x = 1 + x + \dots$

$$\Delta \ln g_a = -\frac{\pi^2}{3}\frac{y^3}{\left(C^a_{VVV}\right)^2} + O(y^4) \tag{3.62}$$

From the form of (3.61), we see that an analogue of the c-theorem holds perturbatively for g (compare with (2.163)) as well. This is known as the perturbative g-theorem. Affleck and Ludwig conjectured that it also holds nonperturbatively [63], which was proved by Friedan and Konechny [64], see also [72]. Physically this can be interpreted as saying that when going to IR, we see more and more gaps in the spectrum since they appear larger and larger compared to our scale and these gaps remove degeneracies.

One can also obtain the leading order correction ΔB^ϕ_a to an arbitrary boundary state coefficient of $\|\phi\rangle\rangle \neq \|\mathbb{1}\rangle\rangle$. To do so, we compute the perturbed one-point function of a bulk insertion at the center of a disc $r = 0$. The generic form of the one-point function of a bulk insertion on a unit disc is easily read off from the bulk-boundary OPE (3.20) and going to disc coordinates via the previously used f

$$\langle \phi(r,\theta)\rangle_a = \frac{g_a B^a_{\phi\mathbb{1}}}{(1-r^2)^\Delta}, \tag{3.63}$$

so that

$$\langle \phi(0)\rangle_a = g_a B^a_{\phi\mathbb{1}} = B^\phi_a, \tag{3.64}$$

where we used [56]

$$B^a_{\phi\mathbb{1}} = \frac{B^\phi_a}{g_a} \tag{3.65}$$

Now we just need to compute

$$B^\phi_{a'} = B^\phi_a - a^{h-1}\lambda \int \mathrm{d}\theta\, \langle \phi(0)V(\theta)\rangle_a + \dots \tag{3.66}$$

Transforming (3.24) to the unit disc, we have [61]

$$\langle \phi(r,\theta_1)V(\theta_2)\rangle_a = B^a_{\phi V} g_a (1-r^2)^{-\Delta}\left(\frac{1-r^2}{1-2r\cos\theta_{12}+r^2}\right)^h \tag{3.67}$$

so that

$$\langle \phi(0)V(\theta)\rangle_a = B^a_{\phi V} g_a \tag{3.68}$$

This makes the angular integration trivial so it simply gives 2π, leading to (the term depending on the cut-off gets absorbed into the higher order contributions)

$$B^{\phi}_{a'} = B^{\phi}_{a} - 2\pi\lambda B^{a}_{\phi V} g_a + \dots \tag{3.69}$$

so that we have

$$\frac{\Delta B^{\phi}_{a}}{g_a} = -2\pi\lambda B^{a}_{\phi V} + \dots \tag{3.70}$$

After plugging in the fixed point coupling (3.48), we obtain the final result

$$\frac{\Delta B^{\phi}_{a}}{g_a} = -2\pi \frac{B^{a}_{\phi V}}{C^{a}_{VVV}} y + O(y^2) \tag{3.71}$$

consistent with the absence of a $O(y)$ correction to g_a since $B^{a}_{\mathbb{1}V} = 0$. We haven't found the result (3.71) in the CFT literature, but we shall confirm it using open string field theory (5.2.3).

To give an example of boundary RG flows, we investigate perturbations of pure Cardy boundary conditions $a = (a_1, a_2), a_2 > 1$ in the unitary minimal models $\mathcal{M}_m$ for m large by the least relevant operator $V_{(1,3)} \equiv V$, see [65] and also [73] for a more general setup, where superpositions of Cardy boundary conditions are deformed. Following [65], we work with non-normalised two-point functions in which there is an extra $C^{a}_{VV\mathbb{1}}$ in (3.62). Recall from (2.8.2) that for this deformation, we have $y = \frac{2}{m}$, so that from (3.62)

$$\Delta \ln g_a = -\frac{\pi^2}{3} \frac{C^{a}_{VV\mathbb{1}}}{\left(C^{a}_{VVV}\right)^2} \frac{8}{m^3} + O\Big(\frac{1}{m^4}\Big) \tag{3.72}$$

The needed Runkel's structure constants [66] are given in [65] so that the authors obtain after expanding in $\frac{1}{m}$

$$\frac{C^{a}_{VV\mathbb{1}}}{\left(C^{a}_{VVV}\right)^2} = \frac{1}{8}(a_2^2 - 1) + O\Big(\frac{1}{m}\Big) \tag{3.73}$$

for $a = (a_1, a_2)$. This gives

$$\Delta \ln g_a = -\frac{\pi^2}{3}(a_2^2 - 1)\frac{1}{m^3} + O\Big(\frac{1}{m^4}\Big) \tag{3.74}$$

Expanding the ratio of two g-functions g_a and g_b with $a = (a_1, a_2)$ and $b = (b_1, b_2)$ given by (3.45), one has

$$\ln \frac{g_b}{g_a} = \ln \frac{b_1 b_2}{a_1 a_2} + \frac{\pi^2}{6}(a_1^2 + a_2^2 - b_1^2 - b_2^2)\frac{1}{m^2} + \frac{\pi^2}{3}(a_1^2 - b_1^2)\frac{1}{m^3} + O\Big(\frac{1}{m^4}\Big), \tag{3.75}$$

so that comparing with (3.74), we must have

$$
\begin{aligned}
a_1 a_2 &= b_1 b_2 && (3.76)\\
a_1^2 + a_2^2 &= b_1^2 + b_2^2 && (3.77)\\
b_1^2 - a_1^2 &= a_2^2 - 1 && (3.78)
\end{aligned}
$$

This has a pure Cardy state solution for $a_1 = 1$ given by a flow $a = (1, r) \to b = (r, 1)$. To obtain flows from more general boundary conditions, one needs to consider superposition $\sum_k b^k = \sum_k (b_1^k, b_2^k)$ of Cardy states in the IR. Then one has for the superposition g-function g_{sup}

$$
\begin{aligned}
\ln \frac{g_{sup}}{g_a} &= \ln \sigma + \frac{\pi^2}{6}\Big(a_1^2 + a_2^2 - \sum_k \frac{s_k}{\sigma}((b_1^k)^2 + (b_2^k)^2)\Big)\frac{1}{m^2} \\
&\quad + \frac{\pi^2}{3}\Big(a_1^2 - \sum_k \frac{s_k}{\sigma}(b_1^k)^2\Big)\frac{1}{m^3} + O\Big(\frac{1}{m^4}\Big), \qquad (3.79)
\end{aligned}
$$

where $s_k \equiv \frac{b_1^l b_2^l}{a_1 a_2}$ and $\sigma \equiv \sum_k s_k$. Again comparing with (3.74), we have

$$
\begin{aligned}
\sigma &= 1 && (3.80)\\
\sum_k s_k\Big((b_1^k)^2 + (b_2^k)^2\Big) &= a_1^2 + a_2^2 && (3.81)\\
\sum_k s_k (b_2^k)^2 &= 1 && (3.82)
\end{aligned}
$$

From these one reads off that necessarily $b_2^k = 1$ so the final boundary state is a superposition of Cardy states $b^k = (b_1^k, 1)$. One can confirm that the superposition of the form $b^k = (b_1^k, 1)$ with $b_1^k = a_1 + a_2 + 1 - 2k$ for $k = 1, \ldots, \min(a_1, a_2)$ is a generic solution. There exist other non-generic solutions, but their number gets reduced by studying shifts of other boundary state coefficients or possibly going to higher order. In summary, we have identified the perturbative flow

$$
a = (a_1, a_2) \to b = \bigoplus_{k=1}^{\min(a_1, a_2)} (a_1 + a_2 + 1 - 2k, 1) \qquad (3.83)
$$

from a pure Cardy state in the UV to a superposition of Cardy states in the IR.

Chapter 4

Open string field theory

Open string field theory (OSFT) [74] is a theory of conformal boundary conditions. A major impetus for the development of OSFT was to obtain an understanding of nonperturbative phenomena in string theory, such as tachyon condensation [75, 76]. There has been a lot of numerical progress in this direction [77, 78, 79, 80], see also the thesis [81] for the state of the art. An analytic solution to OSFT EOMs describing tachyon condensation was found by Schnabl [82] by observing that the star product behaves supraadditively in a basis of wedge states [78, 83] with local operator insertions. This solution initiated a flurry of analytic work [84, 85, 86, 87, 88, 89, 90], see the reviews [91, 92, 93] (we draw mostly from [91]) and the textbook [33] for a broader exposition. In this thesis we are mostly interested in applications of perturbative analytic methods [85, 86, 94, 95] to BCFT. For applications of OSFT to BCFT a major milestone is the KMS correspondence [96] building on the work of [97, 98, 99]. It enables one to search for new boundary states using OSFT [100, 101, 81, 95, 102], sometimes this may even inspire one to find the exact boundary state [103, 104].

4.1 Hilbert space

OSFT is formulated in the BCFT Hilbert space $\mathcal{H} = \mathcal{H}_m \otimes \mathcal{H}_{gh}$, where $\mathcal{H}_m$ is a *matter Hilbert space* of a $c_m = 26$ BCFT and $\mathcal{H}_{gh}$ is a *ghost Hilbert space* of a BCFT corresponding to the bc-ghost system (2.8.4) of central charge $c_{gh} = -26$ in the bulk. Since one is not always interested in the $c_m = 26$ BCFT itself a further decomposition of $\mathcal{H}_m = \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$ so that $c_{m_1} + c_{m_2} = 26$ is often done. The boundary condition on ghosts is the trivial one

$$\begin{aligned} b(x) &= \bar{b}(x) && (4.1) \\ c(x) &= \bar{c}(x), && (4.2) \end{aligned}$$

for $x \in \mathbb{R}$, so that the doubling trick can be used in exactly the same way as for T, see (3.2). The ghost Hilbert space naturally provides a gradation to $\mathcal{H}$ because b and c are anticommuting. In bosonic theories, we then have for elements of $\mathcal{H}$

$$\text{Grassmann parity} = \text{ghost number mod } 2,$$

with Grassmann parity 1 and 0 meaning anticommuting and commuting respectively. The Grassmann parity of A being denoted as $|A|$. The total product BCFT has central charge 0 as required for the vanishing of the conformal anomaly (2.100). This means that the total stress tensor $T = T^m + T^{gh}$ is a primary of weight 2. Such a product structure arises in bosonic string theory after fixing reparametrisation invariance playing the role of a gauge symmetry. After this gauge fixing, a residual symmetry remains and that is *BRST symmetry*, which is generated by a ghost number 1 holomorphic current

$$j_B(z) = cT^m(z) + :bc\partial c:(z) + \frac{3}{2}\partial^2 c(z) \tag{4.3}$$

with its antiholomorphic counterpart and on the real axis we again have trivial gluing so that using the doubling trick, we may write a *BRST charge*

$$Q = \oint \frac{\mathrm{d}z}{2\pi i} j_B(z) \tag{4.4}$$

It is a very rewarding exercise to show that Q is nilpotent, meaning $Q^2 = 0$, if and only if $c_m = 26$. One may compute BRST variations of local operators by surrounding them with the contour from (4.4)

$$QV(x) = \oint_x \frac{\mathrm{d}z}{2\pi i} j_B(z)V(x) \tag{4.5}$$

By employing the appropriate OPEs, one can show that

$$Qb(z) = T(z) \tag{4.6}$$
$$QT(z) = 0 \tag{4.7}$$
$$Qc(z) = c\partial c(z) \tag{4.8}$$
$$QV(z) = c\partial V(z) + h\partial c V(z), \tag{4.9}$$

where V is a weight h matter primary. We note that (4.6) readily implies the very important anticommutator

$$\{Q, b_n\} = L_n \tag{4.10}$$

We see that the action of Q doesn't change the conformal dimension, but shifts the ghost number by a $+1$. The BRST charge obeys the graded Leibniz rule $Q(AB) = (QA)B + (-1)^{|A|}A(QB)$, so that

$$Q(cV)(z) = c\partial cV(z) - c^2\partial V(z) - hc\partial cV(z) = (1-h)c\partial cV(z), \tag{4.11}$$

where we used $c^2 = 0$.

The total Hilbert space $\mathcal{H}$ naturally comes endowed with a physical structure so that we call a ghost number 1 state $|\Psi\rangle$ such that $Q|\Psi\rangle = 0$ a physical state. We say that this state is *BRST closed*. We observe that since $Q^2 = 0$, BRST closed states are defined only up to a *BRST exact* state $Q|\Lambda\rangle$, such that we have the physical equivalence

$$|\Psi\rangle \sim |\Psi\rangle + Q|\Lambda\rangle \tag{4.12}$$

This BRST exact state is trivially closed and so the space of physically distinct nontrivially closed physical states is the *cohomology* of Q at ghost number 1, that is the space

$$H^1(Q) = \{|\Psi\rangle : Q|\Psi\rangle = 0\}/\sim \tag{4.13}$$

where we mod by the physical equivalence (4.12). After using the state-operator correspondence, an example of a physical state would be $|\Psi\rangle = cV|0\rangle$ for V marginal since (4.11) holds. The state-operator correspondence also justifies calling the $|\Psi\rangle \in \mathcal{H}$ as *string fields* since they correspond to fluctuation fields of the conformal boundary condition (for example fluctuations fields of a D-brane when $\mathcal{H}_m$ is the free boson, see (3.1) for the definition of D-branes). To emphasize the fluctuation nature of $|\Psi\rangle$, we often drop the ket.

To justify the nomenclature for physical fields, we Fourier expand the string field on a Dp-brane up to level k^2 (see [91, 92, 33] for details on the following computations)

$$\Psi = \int \frac{\mathrm{d}^{p+1}k}{(2\pi)^{p+1}} \left[T(k)c_1 + A_\mu(k)\alpha^\mu_{-1}c_1 + \phi_a(k)\alpha^a_{-1}c_1 + \frac{i}{\sqrt{2}}\beta(k)c_0 + \dots \right] |k_\mu\rangle\,, \tag{4.14}$$

where the $|k_\mu\rangle$ is a tensor product of the vertex operators V_k acting on $b_{-1}|0,\downarrow\rangle$ with $\mu = 1,\dots,p$, $a = 1,\dots,25-p$. Then we want to extract EOMs for the fluctuation fields in momentum space by imposing $Q\Psi = 0$. To do so, one expands Q in modes

$$Q = c_0 L_0 - b_0 M + \hat{Q} \tag{4.15}$$

with

$$\hat{Q} = \sum_{m\neq 0} c_{-m}L^m_m - \frac{1}{2}\sum_{\substack{m,n\neq 0\\ m+n\neq 0}} (m-n) :c_{-m}c_{-n}b_{m+n}: \tag{4.16}$$

$$M = \sum_{m\neq 0} m c_{-m}c_m \tag{4.17}$$

Up to higher level contributions, one can truncate $M \to 2c_{-1}c_1$, $\hat{Q} \to c_1 L^m_{-1} + c_{-1}L^m_1$ and $L^m_1 \to \alpha_0\alpha_1$, $L^m_{-1} \to \alpha_0\alpha_{-1}$. The result are the position space EOMs

$$\begin{aligned}(\partial^2 + 1)T &= 0 \qquad (4.18)\\ \partial^2 A_\mu - \partial_\mu \beta &= 0 \qquad (4.19)\\ \partial^2 \phi_a &= 0 \qquad (4.20)\\ \beta - \partial \cdot A &= 0, \qquad (4.21)\end{aligned}$$

from which we see for example Maxwell's equations

$$\partial^2 A_\mu - \partial_\mu \Big(\partial \cdot A\Big) = 0 \qquad (4.22)$$

for the gauge potential A or the tachyonic massive Klein-Gordon equation with $m^2 = -1$ for the tachyon T and the massless Klein-Gordon equation for the fields ϕ_a, which get interpreted as giving the position of the D-brane. The tachyon T indicates an unstable direction of the *perturbative D-brane* $\Psi = 0$ with all fluctuation fields turned off (see the analogy with figure (1.2)). To see where the theory ends up after turning on such a tachyon is the problem of tachyon condensation in OSFT. The equivalence $\Psi \sim \Psi + Q\Lambda$ gets interpreted as gauge invariance since after expanding

$$\Lambda = \int \frac{\mathrm{d}^{p+1}k}{(2\pi)^{p+1}} \Big[i\lambda(k) + \dots \Big] |k_\mu\rangle \qquad (4.23)$$

we can read off from the change of Ψ the equivalence relation of the momentum space fluctuation fields $T \sim T, A_\mu \sim A_\mu + \partial_\mu \lambda, \phi_a \sim \phi_a, \beta \sim \beta + \partial^2\lambda$, which gives the expected gauge transformation properties.

By having chosen the BCFT on which we formulate OSFT, we have lost manifest background independence. However, we can see that Ψ contains an infinite tower of fields, which often have nontrivial gauge transformation properties despite being massive (for example β). This means that we expect a massive redundance in our description, which gives us hope that with so many DOFs, we can make them rearrange themselves to DOFs of any other BCFT consistent with a given CFT by an appropriate dynamical principle. It is akin to being given Einstein's equations evaluated on $g \equiv \eta + h$, being told that $h = 0$ is a solution and wanting to prove that the resulting theory actually describes much more than linearised fluctuations around a Minkowski background. Miraculously, such a proof for OSFT is given by the Erler-Maccaferri solution [89, 90], which given two BCFTs with the same bulk CFT, establishes a connection within OSFT formulated on one of them. In particular it implies that OSFT can go against the RG flow and find boundary conditions of higher g-functions.

Apart from $\mathbb{Z}_2$ gradatation and the physical structure, the OSFT Hilbert space carries a real structure. To show this, we first define the *BPZ product*

$$\langle V_1, V_2 \rangle \equiv \langle I(V_1(0))V_2(0)\rangle_{UHP}, \tag{4.24}$$

where $I(z) = -\frac{1}{z}$ is an inversion, which moves a unit half-disc into its UHP complement. In this sense the BPZ product corresponds to the gluing of a unit half-disc to its complement, which gives a number. Since I^2 is the identity and I leaves the correlator invariant (it is a real Möbius transformation), one can show that

$$\begin{aligned} \langle V_1, V_2 \rangle &= \langle I\Big[I(V_1(0))V_2(0)\Big]\rangle_{UHP} \\ &= \langle V_1(0)I(V_2(0))\rangle_{UHP} \\ &= (-1)^{|V_1||V_2|}\langle V_2, V_1\rangle \end{aligned} \tag{4.25}$$

so that the BPZ product is graded symmetric. One can then define the BPZ conjugate $V^\star$ of an operator V by

$$\langle V_1, VV_2\rangle = (-1)^{|V||V_1|}\langle V^\star V_1, V_2\rangle \tag{4.26}$$

Extending BPZ conjugation from operators to states, we have $|V\rangle^\star \equiv \langle V^\star|$, so that

$$\langle V_1, V_2\rangle = \langle V_1^\star|V_2\rangle \tag{4.27}$$

We will often suppress the star and simply write $\langle V_1|V_2\rangle$ by which we from now on mean the BPZ product and not the Hermitean product (unless stated otherwise). One should be careful about thinking of the BPZ product or of the Hermitean product as if they were inner products since in $\mathcal{H}$ negative norm states exist (for $\mathcal{H}_{gh}$ this is obvious since the theory is not unitary). Despite not being an inner product, the BPZ product is nondegenerate in the sense that if $\langle V_1, V_2\rangle = 0$ for all V_2, then $V_1 = 0$. BPZ conjugation can be shown to have the following properties

$$V^{\star\star} = V \tag{4.28}$$

$$(a_1V_1 + a_2V_2)^\star = a_1V_1^\star + a_2V_2^\star \tag{4.29}$$

$$(V_1V_2)^\star = (-1)^{|V_1||V_2|}V_2^\star V_1^\star, \tag{4.30}$$

with $a_1, a_2 \in \mathbb{C}$ so that $a^\star = a, a \in \mathbb{C}$. Expanding V in modes, one can easily verify by performing an inversion that

$$V_n^\star = (-1)^{n+h}V_{-n}, \tag{4.31}$$

which one can compare to $V_n^\dagger = V_{-n}$, see (2.4). After having defined BPZ conjugation, we can define the *reality conjugation* as

$$V^\ddagger = V^{\dagger\star}, \tag{4.32}$$

which has the properties

$$\begin{aligned} V^{\ddagger\ddagger} &= V & (4.33)\\ (a_1 V_1 + a_2 V_2)^\ddagger &= a_1^* V_1^\ddagger + a_2^* V_2^\ddagger & (4.34)\\ (V_1 V_2)^\ddagger &= (-1)^{|V_1||V_2|} V_2^\ddagger V_1^\ddagger, & (4.35) \end{aligned}$$

as can be verified from the standard Hermitean conjugation properties

$$\begin{aligned} V^{\dagger\dagger} &= V & (4.36)\\ (a_1 V_1 + a_2 V_2)^\dagger &= a_1^* V_1^\dagger + a_2^* V_2^\dagger & (4.37)\\ (V_1 V_2)^\dagger &= V_2^\dagger V_1^\dagger, & (4.38) \end{aligned}$$

together with the ones for BPZ conjugates. BPZ conjugation and Hermitean conjugation map $\mathcal{H}$ into their respective dual spaces, but the reality conjugation maps $\mathcal{H} \to \mathcal{H}$ so that it enables us to say which elements of $\mathcal{H}$ we call *real*.

4.2 Interactions

One can see that the physical state condition $Q\Psi = 0$ arises from an action

$$S_0[\Psi] = -\frac{1}{2}\langle \Psi, Q\Psi \rangle, \tag{4.39}$$

where Ψ is at ghost number 1 (as required for the nonvanishing BPZ product), by employing the usual variational techniques

$$\begin{aligned} \delta S_0 &= -\frac{1}{2}\Big[\langle \delta\Psi, Q\Psi\rangle + \langle \Psi, Q\delta\Psi\rangle\Big] \\ &= -\frac{1}{2}\Big[\langle \delta\Psi, Q\Psi\rangle + \langle Q\Psi, \delta\Psi\rangle\Big] \\ &= -\langle \delta\Psi, Q\Psi\rangle, & (4.40) \end{aligned}$$

where by the nondegeneracy of the BPZ product, we conclude that $Q\Psi = 0$ and we also used $Q^\star = -Q$ since its given by a zero mode of a $h = 1$ field, see (4.31) together with the gradation of the BPZ product. One also easily sees that the action (4.39) has a gauge invariance $\Psi \to \Psi + Q\Lambda$.

Now we would like to extend the action (4.39) off-shell in the sense that we also want to describe for example relevant deformations of boundary conditions, for which $h \neq 1$. This extension also has to follow the stringent condition of being able to consistently take string theory off-shell, see [105, 106]. Such an extension was found by Witten [74]

$$S[\Psi] = -\frac{1}{2}\,\mathrm{Tr}\{\Psi Q\Psi\} - \frac{1}{3}\,\mathrm{Tr}\Big\{\Psi^3\Big\}, \tag{4.41}$$

where we have the following consistency conditions on the trace, $\Psi \in \mathcal{H}$ and Q, see [91],

1. Grading:
   (a) $\mathrm{gh}(QV) = \mathrm{gh}(V) + 1$
   (b) $\mathrm{gh}(V_1V_2) = \mathrm{gh}(V_1) + \mathrm{gh}(V_2)$
   (c) $\mathrm{Tr}\{V\} = 0$ if $\mathrm{gh}(V) \neq 3$
2. Ghost number: $\mathrm{gh}\Psi = 1$
3. Reality: $\Psi = \Psi^{\ddagger}$ with $\mathrm{Tr}\Big\{\Psi^{\ddagger}\Big\} = \mathrm{Tr}\{\Psi\}^*$
4. Nilpotency: $Q^2 = 0$
5. Derivation property: $Q(V_1V_2) = (QV_1)V_2 + (-1)^{|V_1|}V_1(QV_2)$
6. Associativity: $V_1(V_2V_3) = (V_1V_2)V_3$
7. Integration by parts: $\mathrm{Tr}\{QV\} = 0$
8. Cyclicity: $\mathrm{Tr}\{V_1V_2\} = (-1)^{|V_1||V_2|}\,\mathrm{Tr}\{V_2V_1\}$
9. Nondegeneracy: If $\mathrm{Tr}\{V_1V_2\} = 0$ for all V_2, then $V_1 = 0$

With these conditions, the EOMs

$$Q\Psi + \Psi^2 = 0 \tag{4.42}$$

follow by doing the same procedure as for (4.39)

$$\begin{aligned} \delta S &= -\langle\delta\Psi, Q\Psi\rangle - \frac{1}{3}\Big(\langle\delta\Psi, \Psi^2\rangle + \langle\Psi, \delta\Psi\Psi\rangle + \langle\Psi^2, \delta\Psi\rangle\Big) \\ &= -\langle\delta\Psi, Q\Psi\rangle - \langle\delta\Psi, \Psi^2\rangle \\ &= -\langle\delta\Psi, Q\Psi + \Psi^2\rangle \end{aligned} \tag{4.43}$$

The solutions to (4.42) represent other BCFTs with $\Psi = 0$ representing the perturbative vacuum, that is the BCFT on which our OSFT is formulated. One can show that

$$\delta\Psi = Q\Lambda + [\Psi, \Lambda] \tag{4.44}$$

is a gauge invariance for Λ at ghost number 0. We do so at the level of EOMs

$$\begin{aligned}
\delta(Q\Psi + \Psi^2) &= Q(Q\Lambda + [\Psi,\Lambda]) + \Psi(Q\Lambda + [\Psi,\Lambda]) + (Q\Lambda + [\Psi,\Lambda])\Psi \\
&= Q(\Psi\Lambda - \Lambda\Psi) + \Psi(Q\Lambda + \Psi\Lambda - \Lambda\Psi) + (Q\Lambda + \Psi\Lambda - \Lambda\Psi)\Psi \\
&= Q(\Psi\Lambda - \Lambda\Psi) + \Psi(Q\Lambda + \Psi\Lambda - \Lambda\Psi) + (Q\Lambda + \Psi\Lambda - \Lambda\Psi)\Psi \\
&= Q(\Psi)\Lambda - \Psi Q(\Lambda) - Q(\Lambda)\Psi - \Lambda Q(\Psi) \\
&\quad + \Psi(Q\Lambda + \Psi\Lambda - \Lambda\Psi) + (Q\Lambda + \Psi\Lambda - \Lambda\Psi)\Psi \\
&= Q(\Psi)\Lambda - \Lambda Q(\Psi) + \Psi(\Psi\Lambda - \Lambda\Psi) + (\Psi\Lambda - \Lambda\Psi)\Psi \\
&= -\Psi^2\Lambda + \Lambda\Psi^2 + \Psi^2\Lambda - \Psi\Lambda\Psi + \Psi\Lambda\Psi - \Lambda\Psi^2 \\
&= 0
\end{aligned} \tag{4.45}$$

The reality condition on Ψ is imposed so that the action is real

$$\begin{aligned}
S^* &= -\frac{1}{2}\operatorname{Tr}\Big\{(\Psi Q\Psi)^\ddagger\Big\} - \frac{1}{3}\operatorname{Tr}\Big\{(\Psi^3)^\ddagger\Big\} \\
&= -\frac{1}{2}\operatorname{Tr}\Big\{(Q\Psi)^\ddagger\Psi^\ddagger\Big\} - \frac{1}{3}\operatorname{Tr}\Big\{(\Psi^\ddagger)^3\Big\} \\
&= -\frac{1}{2}\operatorname{Tr}\Big\{Q(\Psi^\ddagger)\Psi\Big\} - \frac{1}{3}\operatorname{Tr}\Big\{\Psi^3\Big\} \\
&= S
\end{aligned} \tag{4.46}$$

One immediately sees that the two-vertex should be given by the BPZ product $\operatorname{Tr}\{\Psi Q\Psi\} = \langle\Psi, Q\Psi\rangle$ as in (4.39), but what about the three-vertex ? In order to define the three-vertex, we need to understand what state Ψ^2 is, since $\operatorname{Tr}\{\Psi^3\} = \langle\Psi^2, \Psi\rangle$. That is, we need to understand the *star product* of string fields (one sometimes writes $\Psi^2 = \Psi * \Psi$). These products can be given a not so practical functional integral definition, for which we represent the string field as a wave functional in the *Schrödinger representation* (we assume the matter sector to be a free boson) $\Psi[x(\sigma)]$ in which the string field is a function of the curve representing a string. Splitting the string into two halves $l(\sigma), r(\sigma)$ at the midpoint, we have $\Psi[x(\sigma)] = \Psi[l(\sigma), r(\sigma)]$. It is then easy to define the star product

$$V_1 V_2[l(\sigma), r(\sigma)] = \int \mathcal{D}w(\sigma) V_1[l(\sigma), w(\sigma)] V_2[w(\sigma), r(\sigma)], \tag{4.47}$$

which represents the gluing of two strings along their halves. The trace is then

$$\operatorname{Tr}\{V\} = \int \mathcal{D}w(\sigma) V[w(\sigma), w(\sigma)], \tag{4.48}$$

so that we glue the string onto itself by folding it at the midpoint.

This definition provides the right intuition but is too formal, we need a definition that uses the CFT data of $\mathcal{H}$. To do so, recall that when we represent a string field as an UHP BPZ in-state, we think of it as a half-disc with an insertion of an operator at the origin. This half-disc is easily split into left and right halves so that the midpoint is at i. Now we make a coordinate transformation from the half-disc with the coordinate ξ to the *sliver coordinate frame* with the coordinate

$$z = f_S(\xi) = \frac{2}{\pi} \arctan \xi \tag{4.49}$$

The image is then a semi-infinite vertical strip $\operatorname{Im} z \geq 0$ and $\operatorname{Re} z \in [-\frac{1}{2}, \frac{1}{2}]$ with the origin getting mapped to $z = 0$ and the midpoint $\xi = i$ getting mapped to $i\infty$. Under this transformation we have the origin insertion mapped as $V(0) \to f_S \circ V(0)$. Since the left half gets mapped to $\operatorname{Re} z = \frac{1}{2}$ and the right half to $\operatorname{Re} z = -\frac{1}{2}$, we have that the star product $V_1 V_2$ creates a strip of width 2 with the operator insertions $T_1 \circ f_S \circ V_1(0)$ and $f_S \circ V_2(0)$, where $T_a(z) = z + a$ is just a translation by a. This means that we have put the operator V_2 in the origin so that the total strip spans $\operatorname{Re} z \in [-\frac{1}{2}, \frac{3}{2}]$. If we want to map this star product back to the half-disc, it would no longer be described by a half-disc with an insertion of a local operator at the origin, in fact the star product produces a nonlocal state (we have a strip of length 1 inserted between the two insertions). To compute $\operatorname{Tr}\{V_1 V_2\}$, we simply realise that the self-gluing operation that a trace is, simply calculates a cylinder correlator (glue the ends of a vertical strip together to produce a cylinder)

$$\begin{aligned} \operatorname{Tr}\{V_1 V_2\} &= \langle \big(T_1 \circ f_S \circ V_1(0)\big)\big(f_S \circ V_2(0)\big)\rangle_{C_2} \\ &= \langle \big(C_2^{-1} \circ T_1 \circ f_S \circ V_1(0)\big)\big(C_2^{-1} \circ f_S \circ V_2(0)\big)\rangle_{UHP}, \end{aligned} \tag{4.50}$$

where C_L^{-1} maps a cylinder of length L to the UHP

$$C_L^{-1} = \frac{L}{\pi} \tan \frac{\pi z}{L} \tag{4.51}$$

We now evaluate the composition

$$\begin{aligned} g_{L,a}(\xi) &= C_L^{-1} \circ T_a \circ f_S(\xi) = C_L^{-1}\Big(\frac{2}{\pi} \arctan \xi + a\Big) \\ &= \frac{L}{\pi} \tan\Big(\frac{2}{L} \arctan \xi + \frac{\pi a}{L}\Big) \end{aligned} \tag{4.52}$$

It is very simple to compute

$$
\begin{aligned}
g_{L,a}(0) &= \frac{L}{\pi}\tan\frac{a\pi}{L} && (4.53)\\
\frac{\mathrm{d}g_{L,a}}{\mathrm{d}\xi}(0) &= \frac{2}{\pi}\frac{1}{\cos^2\frac{a\pi}{L}} && (4.54)
\end{aligned}
$$

so that

$$
\begin{aligned}
\mathrm{Tr}\{V_1V_2\} &= \left(\frac{2}{\pi}\frac{1}{\cos^2\frac{\pi}{2}}\right)^{h_1}\Big(\frac{2}{\pi}\Big)^{h_2}\langle V_1\Big(\frac{2}{\pi}\tan\frac{\pi}{2}\Big)V_2(0)\rangle_{UHP}\\
&= \left(\frac{2}{\pi}\frac{1}{\cos^2\frac{\pi}{2}}\right)^{h_1}\Big(\frac{2}{\pi}\Big)^{h_2} g_0\delta_{1,2}\Big(\frac{\pi}{2}\Big)^{2h_1}\frac{1}{\tan^{2h_1}\frac{\pi}{2}}\\
&= g_0\delta_{1,2}\\
&= \langle V_1, V_2\rangle && (4.55)
\end{aligned}
$$

meaning that we indeed reproduce the expected BPZ product of two primaries (note the cavalier cancellation of infinities). Our next next task is to compute the three-vertex

$$
\begin{aligned}
\mathrm{Tr}\{V_1V_2V_3\} &= \langle\Big(g_{3,2}\circ V_1(0)\Big)\Big(g_{3,1}\circ V_2(0)\Big)\Big(g_{3,0}\circ V_3(0)\Big)\rangle_{UHP}\\
&= \left(\frac{2}{\pi}\frac{1}{\cos^2\frac{2\pi}{3}}\right)^{h_1}\left(\frac{2}{\pi}\frac{1}{\cos^2\frac{\pi}{3}}\right)^{h_2}\Big(\frac{2}{\pi}\Big)^{h_3}\langle V_1\Big(-\frac{3\sqrt{3}}{\pi}\Big)V_2\Big(\frac{3\sqrt{3}}{\pi}\Big)V_3(0)\rangle_{UHP}\\
&= \left(\frac{8}{\pi}\right)^{h_1}\left(\frac{8}{\pi}\right)^{h_2}\Big(\frac{2}{\pi}\Big)^{h_3} g_0C_{123}\Big(2\frac{3\sqrt{3}}{\pi}\Big)^{h_3-h_1-h_2}\Big(\frac{3\sqrt{3}}{\pi}\Big)^{h_2-h_1-h_3}\Big(\frac{3\sqrt{3}}{\pi}\Big)^{h_1-h_2-h_3}\\
&= g_0C_{123}K^{-h_1-h_2-h_3}, && (4.56)
\end{aligned}
$$

where $K \equiv \frac{3\sqrt{3}}{4}$ and we have suppressed boundary condition indices and g_0 is the g-function of the perturbative vacuum. Higher vertices are constructed analogously. The two vertices (4.55) and (4.56) lie at the heart of the cubic OSFT with the action (4.41), since one can reduce even vertices involving descendants to them via the so-called conservation laws, see [78, 81]. Thus when one expands Ψ into some basis given by the Verma modules of $\mathcal{H}$, one can at least in principle evaluate for example the action $S[\Psi]$. We consider it a bit of a miracle that OSFT, whose vertices use only CFT data, correctly encodes the dynamics of conformal boundary conditions, which satisfy nontrivial global conditions such as the Cardy conditions (3.30). On the other hand, one would expect this to be the case by the uniqueness of consistent string interactions.

We have defined the star product geometrically as gluing slivers together, but what about its non-geometrical definition? To define it non-geometrically, we consider a BPZ-orthonormal Fock space basis $\langle V_i, V_j\rangle = \delta_{i,j}$, so that we resolve the identity $1 = \sum_i |V_i\rangle\langle V_i^\star|$. This simply gives us

$$\begin{aligned} V_1 V_2 &= \sum_i |V_i\rangle \langle V_i, V_1 V_2\rangle \\ &= \sum_i |V_i\rangle \operatorname{Tr}\{V_i V_1 V_2\}, \end{aligned} \tag{4.57}$$

which uses only BCFT correlators. The equation (4.57) defines the star product via an expansion into Fock states. It is nontrivial that this is equivalent to the manifestly associative gluing of surfaces [107, 108] for the free boson or a general BCFT for that matter [109].

4.3 Observables

In this section we list some of the most notable observables of OSFT.

4.3.1 The action

The OSFT action (4.41) evaluated on a classical solution Ψ_* to the EOMs (4.42) was conjectured [76] and later proved [82, 89] by others to encode the informations about a shift in the g-function between the perturbative vacuum and the BCFT described by Ψ_*. The correspondence is as follows, consider the on-shell value of the action

$$S[\Psi_*] = -\frac{1}{2}\operatorname{Tr}\{\Psi_* Q \Psi_*\} - \frac{1}{3}\operatorname{Tr}\Big\{\Psi_*^3\Big\} = -\frac{1}{6}\operatorname{Tr}\{\Psi_* Q\Psi_*\}, \tag{4.58}$$

then we have

$$\Delta g = -2\pi^2 S[\Psi_*] = \frac{\pi^3}{3}\langle \Psi_*, Q\Psi_*\rangle \tag{4.59}$$

The value of the OSFT action appeared famously in Sen's first conjecture [76], which says that when one constructs an effective potential $V(T)$ for the tachyonic mode introduced in (4.1) from the action, there exists a local minimum $T = T_*$ such that $V(0) - V(T_*) = \frac{1}{2\pi^2}$. This means that at the endpoint of tachyon condensation, we end up in the *tachyon vacuum* of zero energy, which represents the nothing an unstable D-brane decays into. Note that this statement is independent of the underlying matter BCFT since T lives in the Verma modules of the identity. This can be rephrased at the level of the tachyon vacuum solution Ψ_{tv} as requiring that $S[\Psi_{\text{tv}}] = \frac{g_0}{2\pi^2}$, where g_0 is the perturbative vacuum's g-function, giving $\Delta g = -g_0$ as required for a cancellation. A representant of such a class of solutions was found by Schnabl [82] thus proving Sen's first conjecture.

4.3.2 Linearised fluctuations around a solution

Given a solution Ψ_*, one can take it to be the new perturbative vacuum and formulate an OSFT around it. The action of this OSFT (expressed in the old Hilbert space) is constructed by plugging $\Psi = \Psi_* + \psi$ into the action (4.41)

$$\begin{aligned}
S[\Psi_* + \psi] &= -\frac{1}{2}\operatorname{Tr}\{(\Psi_* + \psi)Q(\Psi_* + \psi)\} - \frac{1}{3}\operatorname{Tr}\Big\{(\Psi_* + \psi)^3\Big\} \\
&= -\frac{1}{2}\operatorname{Tr}\{\Psi_* Q\Psi_* + \psi Q\Psi_* + \Psi_* Q\psi + \psi Q\psi\} \\
&\quad -\frac{1}{3}\operatorname{Tr}\Big\{\Psi_*^3 + \Psi_*\psi\Psi_* + \psi\Psi_*^2 + \psi^2\Psi_* + \Psi_*^2\psi + \Psi_*\psi^2 + \psi\Psi_*\psi + \psi^3\Big\} \\
&= -\frac{1}{2}\operatorname{Tr}\{\Psi_* Q\Psi_* + 2\psi Q\Psi_* + \psi Q\psi\} - \frac{1}{3}\operatorname{Tr}\Big\{\Psi_*^3 + 3\psi\Psi_*^2 + 3\psi^2\Psi_* + \psi^3\Big\} \\
&= S[\Psi_*] - \frac{1}{2}\operatorname{Tr}\Big\{\psi Q\psi + 2\psi^3\Psi_*\Big\} - \frac{1}{3}\operatorname{Tr}\Big\{\psi^3\Big\} \\
&= S[\Psi_*] - \frac{1}{2}\operatorname{Tr}\{\psi Q_{\Psi_*}\psi\} - \frac{1}{3}\operatorname{Tr}\Big\{\psi^3\Big\}
\end{aligned} \tag{4.60}$$

where we defined the shifted kinematic operator $Q_{\Psi_*} \equiv Q + [\Psi_*, \cdot]$, where the commutator is graded (an anticommutator at ghost number 1). The Q_{Ψ_*} can be understood as the BRST charge in the new background, we show the crucial part of nilpotency

$$\begin{aligned}
Q_{\Psi_*}Q_{\Psi_*}\psi &= Q_{\Psi_*}\Big(Q\psi + \Psi_*\psi + \psi\Psi_*\Big) \\
&= Q\Big(Q\psi + \Psi_*\psi + \psi\Psi_*\Big) + \Big(\Psi_* Q\psi + \Psi_*^2\psi + \Psi_*\psi\Psi_*\Big) \\
&\quad -\Big(Q(\psi)\Psi_* + \Psi_*\psi\Psi_* + \psi\Psi_*^2\Big) \\
&= \Big(Q(\Psi_*)\psi - \Psi_* Q\psi + Q(\psi)\Psi_* - \psi Q\Psi_*\Big) + \Big(\Psi_* Q\psi + \Psi_*^2\psi + \Psi_*\psi\Psi_*\Big) \\
&\quad -\Big(Q(\psi)\Psi_* + \Psi_*\psi\Psi_* + \psi\Psi_*^2\Big) \\
&= Q(\Psi_*)\psi - \psi Q\Psi_* + \Psi_*^2\psi - \psi\Psi_*^2 \\
&= 0,
\end{aligned} \tag{4.61}$$

where in the last step we used the EOMs for Ψ_*. This means that the spectrum of linearised fluctuations around Ψ_* is described as the ghost number 1 cohomology of Q_{Ψ_*}. The spectrum of such fluctuations played a key role in Sen's third conjecture, which says that the cohomology of $Q_{\Psi_{\text{tv}}}$ is empty, as required for Ψ_{tv} describing the leftover nothing after D-brane decay. Schnabl and Ellwood [84] proved this by constructing a so-called *homotopy operator* A such that $Q_{\Psi_{\text{tv}}}A = 1$, where 1 is the *identity string field* such that $\Psi 1 = 1\Psi = \Psi$. The idea is that then

$$\psi = 1\psi = (Q_{\Psi_{\text{tv}}}A)\psi = Q_{\Psi_{\text{tv}}}(A\psi) \tag{4.62}$$

for ψ Q_Ψ-closed, from which one deduces that ψ closed implies ψ exact and thus the cohomology is empty.

4.3.3 Scattering amplitudes

In order to obtain the propagator, we have to fix the gauge. The standard choice is *Siegel gauge* $b_0\Psi = 0$, which projects out states containing c_0. The propagator then takes the form

$$\frac{b_0}{L_0} = b_0 \int_0^1 \frac{\mathrm{d}s}{s} s^{L_0}, \tag{4.63}$$

as can be deduced from $[Q, b_0] = L_0$, which follows from (4.6) and Schwinger parametrising

$$\frac{1}{L_0} = \int_0^1 \frac{\mathrm{d}s}{s} s^{L_0} \tag{4.64}$$

The scattering amplitudes are then given by gluing propagators at the three-vertex (4.56) as for example in $\mathrm{Tr}\Big\{\Psi\frac{b_0}{L_0}\Psi^2\Big\}$. An alternative gauge choice called $\mathcal{B}_0$-gauge or *Schnabl gauge* for the definition of scattering amplitudes is given by $\mathcal{B}_0\Psi = 0$, where $\mathcal{B}_0 \equiv f_S \circ b_0$ is a sliver frame analogue of Siegel gauge. An advantage of the Schnabl gauge propagator $\frac{\mathcal{B}_0}{\mathcal{L}_0}$ with $\mathcal{L}_0 \equiv f_S \circ L_0$ is that its action on star products is simpler. One may also study scattering around a solution Ψ_* by defining the shifted Hamiltonian $H_{\Psi_*} \equiv \{Q_{\Psi_*}, b_0\}$, so that the new shifted Siegel gauge propagator is $\frac{b_0}{H_{\Psi_*}}$.

Thus far, we have only studied the boundary degrees of freedom but we know that in BCFT, bulk fields can interact with the boundary. Thus we also consider open-closed amplitudes (in string theory, open strings end on boundaries and closed strings make up the bulk). One such amplitude is the closed string tadpole, which is given by *Ellwood invariant* [97]

$$\mathrm{Tr}_\phi\,\Psi \equiv \langle I|\,\tilde{\phi}(i)\,|\Psi\rangle = \langle\tilde{\phi}(i) f\circ\Psi(0)\rangle_{UHP} = \langle\tilde{\phi}(i\infty) f_S\circ\Psi(0)\rangle_{C_1} \tag{4.65}$$

where $\Psi(0)$ is an insertion at the half-disc origin and $\tilde{\phi}$ is a weight $(1,1)$ spinless matter primary ϕ dressed with ghosts $\tilde{\phi} = c\bar{c}\phi$ such that it is Q-invariant. We have also made use of the identity string field $|I\rangle = U_f^\dagger\,|0\rangle$, where U_f generates the finite transformation f via

$$f\circ\Psi = U_f \Psi U_f^\dagger \tag{4.66}$$

and the scaled (a matter of convention, the correlator is scale-invariant) one-vertex f is given as (4.52)

$$f(\xi) \equiv \pi g_{1,0}(\xi) = \frac{2\xi}{1-\xi^2} \tag{4.67}$$

The Elwood invariant is gauge invariant under (4.44) and to show this, we need to show that $\operatorname{Tr}_\phi Q\Lambda = \operatorname{Tr}_\phi [\Psi, \Lambda] = 0$, which is done a follows

$$\operatorname{Tr}_\phi Q\Lambda = \langle Q[\tilde{\phi}(i) f \circ \Psi(0)]\rangle_{UHP} = 0, \tag{4.68}$$

where we used the fact that $\tilde{\phi}$ is Q-invariant and the residual BRST-invariance of amplitudes. The second term vanishes from cyclicity of the Ellwood invariant

$$\operatorname{Tr}_\phi \Psi\Lambda = \operatorname{Tr}_\phi \Lambda\Psi, \tag{4.69}$$

since one can rotate the boundary fields around the bulk insertion at cylinder infinity. Given a solution Ψ_* to the EOMs (4.42), the Ellwood invariant is conjectured to compute the shift of the on-shell closed string tadpole between the BCFT described by Ψ_* and the perturbative vacuum

$$\langle I|\, \tilde{\phi}(i)\, |\Psi_*\rangle - \langle I|\, \tilde{\phi}(i)\, |\Psi_{\rm tv}\rangle = \frac{1}{2\pi i}\langle\phi\rangle_* = -\frac{1}{4\pi i}\, \langle I|\, \tilde{\phi}(i) c_0^- \, \|B_{\Psi_*}\rangle\!\rangle\,, \tag{4.70}$$

where we used that

$$\langle 0|\, c_{-1}\bar{c}_{-1}c_0^- \,\|B_{gh}\rangle\!\rangle = -2 \tag{4.71}$$

for $c_0^- \equiv c_0 - \bar{c}_0$, see appendix C of [96], and $\|B_{\Psi_*}\rangle\!\rangle$ is the boundary state described by Ψ_* and $\|B_{gh}\rangle\!\rangle$ its trivial universal ghost part. We also used that the disc one-point function vanishes at the tachyon vacuum (no boundary to interact with) $\frac{1}{2\pi i}\langle\phi\rangle_{\rm tv} = 0$, hence the term with $\Psi_{\rm tv}$ cancelling the perturbative vacuum contribution.

4.3.4 The boundary state

Remembering that the disc one-point functions calculate boundary state coefficients, one can see from (4.70) that the Ellwood invariant probes the on-shell part of the Boundary state $\|B_{\Psi_*}\rangle\!\rangle$ described by the solution Ψ_*. Can one generalise it to the off-shell part as well ? Such a generalisation was provided by KMS [96], who generically found an auxiliary BCFT with central charge 0, which supports fields w with $\langle w\rangle_* = 1$ that can make an insertion $c\bar{c}\phi$ with ϕ of weight (h, h) physical. That is, the tensor product $c\bar{c}\phi w$ is physical. If one has the splitting $\mathcal{H}_m = \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$ and studies the dynamics of $\mathcal{H}_{m_1}$, one simply tensors $\mathcal{H}_{m_2}$ with the auxiliary CFT and doesn't see a difference in the theory restricted to $\mathcal{H}_{m_1}$. Using the fact that w has a unit one-point function, we can write

$$\begin{aligned} B^\phi_{\Psi_*} &= \langle I|\, \phi(i)\, \|B_{\Psi_*}\rangle\!\rangle = \langle I|\, \phi w(i)\, \|B_{\Psi_*}\rangle\!\rangle \\ &= -\frac{1}{2}\, \langle 0|\, c_{-1}\bar{c}_{-1}c_0^- \,\|B_{gh}\rangle\!\rangle\, \langle I|\, \phi w(i)\, \|B_{\Psi_*}\rangle\!\rangle = -\frac{1}{2}\, \langle I|\, \tilde{\phi} w(i) c_0^- \,\|B_{\Psi_*}\rangle\!\rangle \end{aligned} \tag{4.72}$$

and using (4.70), which is valid since we have a physical insertion, we find

$$B^{\phi}_{\Psi_*} = 2\pi i \langle I| \tilde{\phi} w(i) |\Psi_* - \Psi_{\text{tv}}\rangle = 2\pi i \langle I| \tilde{\phi}(i) |\Psi_* - \Psi_{\text{tv}}\rangle \tag{4.73}$$

Thus given a solution Ψ_*, we can probe it with bulk operators to get the full off-shell boundary state (of the BCFT one is restricted to). In (4.73) Ψ_{tv} can be any representant of the tachyon vacuum class of solutions and we note that when we don't write it, the perturbative vacuum factor doesn't get cancelled and we are computing the shift in the boundary state

$$\Delta B^{\phi}_{\Psi_*} = 2\pi i \langle I| \tilde{\phi}(i) |\Psi_*\rangle \tag{4.74}$$

4.4 Wedge states with insertions

A very useful family of states in the universal sector are the *wedge states* [83]. These wedge states are naturally represented in the sliver coordinate frame as strips with no operator insertions. The simplest wedge state Ω is just the scaled vacuum

$$\Omega = |0\rangle\,, \tag{4.75}$$

whose strip representation has width 1. We can also conceive of strips with different lengths and thus we form an entire family of wedge states Ω^α with $\alpha \geq 0$, whose the star product is abelian

$$\Omega^\alpha \Omega^\beta = \Omega^{\alpha+\beta} = \Omega^\beta \Omega^\alpha \tag{4.76}$$

with the zero-width wedge state being that star algebra identity $\Omega^0 = 1$. Since the trace can be interpreted as computing cylinder correlators, we naturally have

$$\operatorname{Tr}\{V\Omega^\alpha\} = \langle f_S \circ V(0)\rangle_{C_{\alpha+1}} = \langle C^{-1}_{\alpha+1} \circ f_S \circ V(0)\rangle_{UHP} = \langle g_{\alpha+1,0} \circ V(0)\rangle_{UHP} \tag{4.77}$$

from which we can read off $\Omega^\alpha = U^\dagger_{g_{\alpha+1,0}} |0\rangle$, where

$$g_{\alpha+1,0}(\xi) = \frac{\alpha+1}{\pi} \tan\Big(\frac{2}{\alpha+1} \arctan \xi\Big) \tag{4.78}$$

so that indeed $\Omega^1 = \Omega$ since $g_{2,0}(\xi) = \frac{2}{\pi}\xi$ as is consistent with (4.75) since the vacuum is $Sl(2,\mathbb{R})$ invariant. One can also represent wedge states on the unit disc and finds that they correspond to wedges of angle $\frac{2\pi}{\alpha+1}$, hence justifying their name. One may anticipate the existence of a wedge state generator since they are exponential in nature and such a generator should be seen by taking derivatives of wedge states so that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Omega^\alpha \equiv -K\Omega^\alpha \tag{4.79}$$

By studying the α dependence of (4.77), one can show [110, 91] that

$$K = \int_{-i\infty}^{i\infty} \frac{\mathrm{d}x}{2\pi i} T(x), \tag{4.80}$$

where the total stress-energy tensor T is vertically integrated over the imaginary direction. We need not specify the real part of the contour since K is topological because the integrand vanishes at strip infinity. Since $\Omega^0 = 1$, we can integrate (4.79) to obtain

$$\Omega^\alpha = e^{-\alpha K} \tag{4.81}$$

Since the string fields we are most interested in are at ghost number 1, we need to generalise the ghost number 0 wedge states to the so-called *wedge states with insertions*. In their most basic form they are defined by inserting an operator transformed to sliver frame to the middle of the real boundary of a unit strip as in $\sqrt{\Omega}V\sqrt{\Omega}$. One may of course generalise this to shift the operator insertion away from the middle $\Omega^{\alpha_1}V\Omega^{\alpha_2}$ with $\alpha_1+\alpha_2 = 1$. Relaxing the unit strip constraint, one may visualise the operator insertion itself as a wedge state with insertion $V = \Omega^0 V\Omega^0$. Taking star products of such general states, we have

$$\Omega^{\alpha_1}V_1 \dots \Omega^{\alpha_i}V_i \dots \Omega^{\alpha_n}V_n\Omega^{\alpha_{n+1}} \tag{4.82}$$

From the point of view of the ansatz $\sqrt{\Omega}V\sqrt{\Omega}$, the product (4.82) may contain a nonlocal state. We may ask how such a nonlocal state looks like on the UHP and as a simplest example we take, see [91]

$$\Omega^\alpha = \sqrt{\Omega}\Omega^{\alpha-1}\sqrt{\Omega} = \sqrt{\Omega}e^{-(\alpha-1)K}\sqrt{\Omega} \tag{4.83}$$

so that by transforming K we can transform the entire nonlocal state $\Omega^{\alpha-1}$ with the result

$$\Omega^\alpha(0) = \sum_{n=0}^{\infty} \frac{1}{n!}\bigg(-\frac{\pi}{2}\int_{-i}^{i} \frac{\mathrm{d}\xi}{2\pi i}(1+\xi^2)T(\xi)\bigg)^n, \tag{4.84}$$

where we expressed K on the UHP (see next section detailing the transformation of operators)

$$K = \frac{\pi}{2}\int_{-i}^{i} \frac{\mathrm{d}\xi}{2\pi i}(1+\xi^2)T(\xi) \tag{4.85}$$

As expected, (4.84) contains an infinite number of Virasoro insertions that distort the geometry of the unit disc.

4.5 Operators in sliver frame

Operators which look simple when inserted in one frame are often complicated in another. To show this, we consider the sliver frame primary $\tilde{V}$ of weight h, such that

$$\tilde{V}(z) = \sum_{n=-\infty}^{\infty} \frac{\tilde{V}_n}{z^{n+h}}, \tag{4.86}$$

with z the sliver frame coordinate. From this, we have

$$\begin{aligned}
\tilde{V}_n &= f_S^{-1} \circ \oint \frac{\mathrm{d}z}{2\pi i} \tilde{V}(z) z^{n+h-1} \\
&= \oint \frac{\mathrm{d}\xi}{2\pi i} \frac{2}{\pi} \frac{1}{1+\xi^2} \Big(\frac{\pi}{2}(1+\xi^2)\Big)^h V(\xi) \Big(\frac{2}{\pi} \arctan \xi\Big)^{n+h-1} \\
&= \oint \frac{\mathrm{d}\xi}{2\pi i} \Big(1+\xi^2\Big)^{h-1} V(\xi) \Big(\frac{2}{\pi}\Big)^n \Big(\arctan \xi\Big)^{n+h-1}
\end{aligned} \tag{4.87}$$

where we used $\mathrm{d}z = \frac{2}{\pi}\frac{\mathrm{d}\xi}{1+\xi^2}$ and $f_S^{-1}(z) = \tan\left(\frac{\pi}{2}z\right)$ with the transformation properties of a primary. A concrete example would be

$$\begin{aligned}
\tilde{c}_{-1} &= \oint \frac{\mathrm{d}\xi}{2\pi i} \Big(1+\xi^2\Big)^{-2} c(\xi) \Big(\frac{2}{\pi}\Big)^{-1} \Big(\arctan \xi\Big)^{-3} \\
&= \oint \frac{\mathrm{d}\xi}{2\pi i} c(\xi) \Big(\frac{2}{\pi}\Big)^{-1} (\xi^{-3} - \xi^{-1} + \ldots) = \frac{\pi}{2}(c_{-1} - c_1) + \ldots
\end{aligned} \tag{4.88}$$

so that $\tilde{c}_{-1}\,|0\rangle = \frac{\pi}{2}(c_{-1}+c_1)\,|0\rangle$. To give an example where the correspondence between different coordinates is simple, consider a near marginal operator with $y = 1-h$ small so we can write

$$\tilde{V}_0 = \oint \frac{\mathrm{d}\xi}{2\pi i} V(\xi) + O(y) = V_0 + O(y) \tag{4.89}$$

meaning that to leading order we see no transformation of the zero-mode. From the nontrivial transformation (4.87), we deduce that if a Hilbert space basis is BPZ-orthonormal on the UHP, it ceases to be so after its transport (put a tilde on everything) to the sliver. For us this will have the important consequence that we'll have to deal with oblique projectors when working in sliver frame instead of the usual orthogonal projectors familiar from QM.

We have seen that sliver frame operators act non-transparently on the UHP, but we would expect them to act simply in sliver frame. To investigate this further, consider (see [91] for details on derivations)

$$\mathcal{L}_0 = \int \frac{\mathrm{d}z}{2\pi i} z T(z) = L_0 + \frac{2}{3}L_2 - \frac{2}{15}L_4 + \ldots, \tag{4.90}$$

which has the BPZ conjugate

$$\mathcal{L}_0^\star = L_0 + \frac{2}{3}L_2 - \frac{2}{15}L_4 + \ldots, \tag{4.91}$$

with which we form the BPZ even and odd combinations

$$\mathcal{L}^+ = \mathcal{L}_0 + \mathcal{L}_0^\star \tag{4.92}$$
$$\mathcal{L}^- = \mathcal{L}_0 - \mathcal{L}_0^\star \tag{4.93}$$

The BPZ even part is expressible by K

$$\mathcal{L}^+V = KV + VK \tag{4.94}$$

What is more interesting is the scaled BPZ odd part

$$\frac{1}{2}\mathcal{L}^- = \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}zT(z+l) - \int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}zT(z+r), \tag{4.95}$$

which expresses $\frac{1}{2}\mathcal{L}^-$ acting on a strip, whose left edge intersects the real axis at l and right edge at r. From this geometrical intuition, one has by inserting an intermediate pair of contours in between V_1, V_2 that $\mathcal{L}^-$ is a derivation of the star algebra

$$\mathcal{L}^-(V_1V_2) = (\mathcal{L}^-V_1)V_2 + V_1(\mathcal{L}^-V_2) \tag{4.96}$$

The $\frac{1}{2}\mathcal{L}^-$ acts as a generator of scale transformations on the cylinder. By deforming the two opposite vertical contours from (4.95) to a circle, we easily see that

$$\frac{1}{2}\mathcal{L}^-V = hV \tag{4.97}$$

on a weight h primary V by the OPE (2.72). One can further show that $\frac{1}{2}\mathcal{L}^-K = K$ by the TT OPE (2.88), implying that K has scaling dimension 1 as one would expect since its an integral of a dimension 2 field. Since $\frac{1}{2}\mathcal{L}^-$ is a derivation of the star algebra, we know how it acts on wedge states

$$\frac{1}{2}\mathcal{L}^-\Omega^\alpha = \frac{1}{2}\mathcal{L}^-e^{-\alpha K} = -\alpha K\Omega^\alpha, \tag{4.98}$$

which is integrated to

$$\lambda^{\frac{1}{2}\mathcal{L}^-}\Omega^\alpha = \Omega^{\lambda\alpha} \tag{4.99}$$

Since we also know how it acts on local operator insertions, we know how its action on wedge states with insertions

$$\lambda^{\frac{1}{2}\mathcal{L}^-}\Omega^{\alpha_1}V_1\ldots\Omega^{\alpha_i}V_i\ldots\Omega^nV_n\Omega^{n+1} \quad = \tag{4.100}$$
$$\lambda^{\sum_{k=1}^n h_k}\Omega^{\lambda\alpha_1}V_1\ldots\Omega^{\lambda\alpha_i}V_i\ldots\Omega^{\lambda\alpha_n}V_n\Omega^{\lambda\alpha_{n+1}} \tag{4.101}$$

From (4.92), (4.93) and (4.94), we have

$$\mathcal{L}_0 V = \frac{1}{2}\mathcal{L}^- V + \frac{1}{2}(KV + VK), \tag{4.102}$$

which lets us express $\mathcal{L}_0$ on a wedge with insertions

$$\begin{aligned}\mathcal{L}_0(\sqrt{\Omega}V\sqrt{\Omega}) &= \Big(\frac{1}{2}\mathcal{L}^-\sqrt{\Omega})V\sqrt{\Omega} + \sqrt{\Omega}\Big(\frac{1}{2}\mathcal{L}^-\Big)V\sqrt{\Omega} + \sqrt{\Omega}V\Big(\frac{1}{2}\mathcal{L}^-\sqrt{\Omega}\Big)\\ &\quad + \frac{1}{2}KV\sqrt{\Omega}V\sqrt{\Omega} + \frac{1}{2}VK\sqrt{\Omega}V\sqrt{\Omega}\\ &= \sqrt{\Omega}\Big(\frac{1}{2}\mathcal{L}^- V\Big)\sqrt{\Omega},\end{aligned} \tag{4.103}$$

where in the last line we used (4.98). The relation (4.103) is very useful since it reduces determining the action of $\mathcal{L}_0$ to pure scaling analysis

$$\mathcal{L}_0\sqrt{\Omega}cV\sqrt{\Omega} = (h-1)\sqrt{\Omega}cV\sqrt{\Omega}, \tag{4.104}$$

with V a primary of weight h.

Analogously to $\mathcal{L}_0$, one can study the sliver frame analogue of the b-ghost zero mode b_0

$$\mathcal{B}_0 = \oint \frac{\mathrm{d}z}{2\pi i} z b(z) = b_0 + \frac{2}{3}b_2 - \frac{2}{15}b_4 + \dots \tag{4.105}$$

since both b and T are of weight 2. The only difference is in the occasional fermionic sign and the different OPEs. To summarise, one has

$$\mathcal{B}_0 V = \frac{1}{2}\mathcal{B}^- V + \frac{1}{2}(BV + (-1)^{|V|}VB) \tag{4.106}$$

$$\mathcal{B}_0\sqrt{\Omega}V\sqrt{\Omega} = \sqrt{\Omega}\frac{1}{2}\mathcal{B}^- V\sqrt{\Omega} \tag{4.107}$$

$$\frac{1}{2}\mathcal{B}^- K = B \tag{4.108}$$

$$\frac{1}{2}\mathcal{B}^- B = 0 \tag{4.109}$$

$$\frac{1}{2}\mathcal{B}^- c = 0 \tag{4.110}$$

with B being an analogue to K.

4.6 KBc subalgebra

From the real string fields K,B and c introduced earlier one can form a universal closed subalgebra of the star product algebra. Why one should have a K of ghost number 0 is obvious since it generates wedge states, which are needed to give the solution a nonzero width as required for cylinder correlators to make sense. The c is introduced since it is the universal tachyon, see (4.14), and makes creation of ghost number 1 string fields possible. In the EOMs (4.42), we have an action of Q, which would be problematic after imposing Siegel gauge $b_0\Psi = 0$ because $\{Q, b_0\} = L_0$ and the star products of L_0 are not transparent. One is thus lead to consider Schnabl gauge $\mathcal{B}_0\Psi = 0$, from which we have the nicely controlled $\{Q, \mathcal{B}_0\} = \mathcal{L}_0$. Because of this we add the field B, which has ghost number -1 meaning that we can now have multiple insertions of c. We note that sometimes one adds other operators to the KBc algebra since by itself it only describes universal physics.

We have the commutation relations

$$\begin{aligned} [K, B] &= 0 && (4.111) \\ \{B, c\} &= 1 && (4.112) \end{aligned}$$

with the trivial $B^2 = c^2 = 0$ from anticommutation. From (4.6)-(4.8), we have

$$\begin{aligned} QK &= 0 && (4.113) \\ QB &= K && (4.114) \\ Qc &= cKc = c\partial c, && (4.115) \end{aligned}$$

where the last line follows from $\partial c = [K, c]$ as one trivially sees from the OPE (2.72). One can now play around with finding solutions to (4.42), for example $\Psi = -cK$ is a solution

$$Q(-cK) + (-cK)^2 = -cKcK + cKcK = 0$$

or $\Psi = c(1-K)$ is

$$\begin{aligned} Q\Big(c(1-K)\Big) + \Big(c(1-K)\Big)^2 &= cKc(1-K) + c(1-K)c(1-K) \\ &= cKc - cKcK - cKc + cKcK = 0 \end{aligned}$$

with the second solution actually being a tachyon vacuum (it is too singular to verify Sen's first conjecture, however since it doesn't contain a surface). This is so because a homotopy operator $A = B$ exists

$$\begin{aligned} Q_\Psi B &= QB + \{c(1-K), B\} = K + c(1-K)B + Bc(1-K) \\ &= K + cB + Bc - cKB - BcK = 1 + K - cKB - BcK \\ &= 1 + K - cBK - BcK = 1 + K - K = 1 \end{aligned}$$

In order to get the much needed practice, we will consider more solutions in (4.7). For those, various correlators, often involving the elements of the KBc algebra, are needed, see the appendix (A).

4.7 Examples of analytic solutions

In this section we study two analytic solutions, first we analyse the tachyon vacuum for its general importance (notice that for example (4.3.4) uses its universal properties) and the fact that it corresponds to a (nonperturbative) relevant deformation. We also study exactly marginal deformations, which serve as the simplest example of perturbative solutions, which we'll generalise in the next chapter.

4.7.1 Tachyon vacuum

We shall study the simple tachyon vacuum [87] obtained by performing a (singular) finite gauge transformation (see (4.44)) of the perturbative vacuum [110]

$$\Psi = UQU^{-1} \tag{4.116}$$

with $U = 1 - FBcF$ with $F = F(K)$ an appropriate function [93] of K. One can invert U with a geometric series

$$\begin{aligned} U^{-1} &= 1 + \sum_{n=1}^{\infty} (FBcF)^{n-1} FBcF \\ &= 1 + \sum_{n=1}^{\infty} (F^2)^{n-1} FBcF \\ &= 1 + \frac{1}{1-F^2} FBcF \end{aligned} \tag{4.117}$$

so that

$$QU^{-1} = \frac{1}{1-F^2}(FKcF - FBcKcF) = \frac{1}{1-F^2} FcBKcF, \tag{4.118}$$

which gives the solution

$$\begin{aligned} \Psi &= (1 - F^2 + FcBF)\frac{1}{1-F^2} FcBKcF = FcBKcF + FcB\frac{F^2}{1-F^2}KcF \\ &= FcB\Big(1 + \frac{F^2}{1-F^2}\Big)KcF = FcB\frac{1}{1-F^2}KcF, \end{aligned} \tag{4.119}$$

which is the Okawa ansatz [110]. One may rewrite this by $F \to \sqrt{F}$ to have a form which looks more like a wedge with an insertion

$$\Psi = \sqrt{F}cB\frac{1}{1-F}Kc\sqrt{F} \tag{4.120}$$

This solution is in *dressed Schnabl gauge*

$$\mathcal{B}_{\sqrt{F},\sqrt{F}}\Psi = \sqrt{F}\frac{1}{2}\mathcal{B}^{-}\Big(\frac{1}{\sqrt{F}}\Psi\frac{1}{\sqrt{F}}\Big)\sqrt{F} = 0 \tag{4.121}$$

as can be trivially verified since when $\frac{1}{2}\mathcal{B}^{-}$ hits a K it annihilates with a B, see (4.108)-(4.110). For $F = \sqrt{\Omega}$ one has the usual Schnabl gauge and the solution reduces to Schnabl's solution [82]. In order for Ψ not to be in the gauge orbit of the perturbative vacuum, the gauge transformation must be singular. Looking at $\frac{K}{1-F}$ one realises that this is the case if and only if $F(0) = 1$ and we want $F'(0) \neq 0$ to avoid a pole [111] (observe that this is the case for Schnabl's solution).

We will now show that under these conditions Ψ has empty cohomology. We do so by proving that $A = B\frac{1-F}{K}$ is a homotopy operator

$$\begin{aligned} Q_\Psi A &= QA + \{\Psi, A\} \\ &= K\frac{1-F}{K} + \sqrt{F}cB\frac{1}{1-F}Kc\sqrt{F}B\frac{1-F}{K} + B\frac{1-F}{K}\sqrt{F}cB\frac{1}{1-F}Kc\sqrt{F} \\ &= 1 - F + \sqrt{F}cB\sqrt{F} + \sqrt{F}Bc\sqrt{F} = 1 - F + F = 1 \end{aligned} \tag{4.122}$$

When $\frac{1-F}{K} = F$, one obtains the simple tachyon vacuum with $F = \frac{1}{1+K}$

$$\Psi_s = \frac{1}{\sqrt{1+K}}c(1+K)Bc\frac{1}{\sqrt{1+K}} \tag{4.123}$$

For the next calculations a different form of the solution is useful

$$\Psi_s = \frac{1}{\sqrt{1+K}}c\frac{1}{\sqrt{1+K}} + Q\bigg(\frac{1}{\sqrt{1+K}}Bc\frac{1}{\sqrt{1+K}}\bigg), \tag{4.124}$$

since Q-exact terms don't contribute to neither the Ellwood invariant or the action which we're about to calculate. We start with the Ellwood invariant, see (4.3.3), and by cyclicity and Schwinger parametrisation we have

$$\begin{aligned} \mathrm{Tr}_\phi\, c\frac{1}{1+K} &= \int_0^\infty \mathrm{d}\alpha e^{-\alpha}\,\mathrm{Tr}_\phi\, c\Omega^\alpha = \int_0^\infty \mathrm{d}\alpha e^{-\alpha}\langle\tilde{\phi}(i\infty)c(0)\rangle_{C_\alpha} \\ &= \int_0^\infty \mathrm{d}\alpha e^{-\alpha}\alpha\langle\tilde{\phi}(i\infty)c(0)\rangle_{C_1} = \langle\tilde{\phi}(i\infty)c(0)\rangle_{C_1} \\ &= \frac{1}{2\pi i}\langle c\bar{c}\phi(0)c(1)\rangle_{disc} = -\frac{1}{2\pi i}\langle\phi\rangle_0, \end{aligned} \tag{4.125}$$

where we have also mapped the cylinder to the unit disc via $f(z) = e^{2\pi i z}$. By the Ellwood conjecture, this computes the shift in the closed string tadpole but since there is no nonzero term besides the perturbative vacuum contribution in (4.125), we conclude that Ψ describes a background where the boundary has disappeared.

Now we prove Sen's first conjecture, see (4.3.1), by inserting (4.124) to the on-shell value of the action (4.58), which gives

$$\begin{aligned} S[\Psi_s] &= -\frac{1}{6}\operatorname{Tr}\Big\{\frac{1}{\sqrt{1+K}}c\frac{1}{\sqrt{1+K}}\frac{1}{\sqrt{1+K}}cKc\frac{1}{\sqrt{1+K}}\Big\} \\ &= -\frac{1}{6}\operatorname{Tr}\Big\{\frac{1}{1+K}c\frac{1}{1+K}cKc\Big\} \\ &= -\frac{1}{6}\int_0^\infty \mathrm{d}\alpha\int_0^\infty \mathrm{d}\beta e^{-\alpha-\beta}\operatorname{Tr}\Big\{\Omega^\alpha c\Omega^\beta cKc\Big\}, \end{aligned} \tag{4.126}$$

we continue by calculating the trace

$$\begin{aligned} \operatorname{Tr}\Big\{\Omega^\alpha c\Omega^\beta cKc\Big\} &= -\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\gamma}\operatorname{Tr}\Big\{\Omega^\alpha c\Omega^\beta c\Omega^\gamma c\Big\} \\ &= -\lim_{\gamma\to 0}\frac{\mathrm{d}}{\mathrm{d}\gamma}\langle c(\beta+\gamma)c(\gamma)c(0)\rangle_{C_{\alpha+\beta+\gamma}} \\ &= -g_0\frac{(\alpha+\beta)^2\sin^2\left(\frac{\pi\beta}{\alpha+\beta}\right)}{\pi^2} \end{aligned} \tag{4.127}$$

where we used the correlator (A.16) and didn't forget that we also have a matter part to our CFT, which gives a g_0. Thus the action is reduced to the integral

$$S[\Psi_s] = \frac{g_0}{6}\int_0^\infty \mathrm{d}\alpha\int_0^\infty \mathrm{d}\beta\, e^{-\alpha-\beta}\frac{(\alpha+\beta)^2\sin^2\left(\frac{\pi\beta}{\alpha+\beta}\right)}{\pi^2}, \tag{4.128}$$

which gives by elementary analysis

$$S[\Psi_s] = \frac{g_0}{2\pi^2} \tag{4.129}$$

as required by Sen's first conjecture.

4.7.2 Exactly marginal deformations

Perturbative methods of solving (4.42) originate from the observation that we are dealing with a quadratic equation in infinitely many variables and we know how to solve quadratic equations perturbatively. Suppose we are given the quadratic equation

$$qx + x^2 = qx_0 \tag{4.130}$$

This is formally solved by $x = -q^{-1}x^2$ and the ansatz $x = x_0 + \ldots$ with the source term on the RHS of (4.130) needed for the equation to be satisfied to $O(x_0)$. To second order one has

$$x = x_0 - q^{-1}x_0^2 + \ldots, \tag{4.131}$$

where the correction is present to account for the nonlinearity of (4.130). We recursively go to higher orders

$$\begin{aligned} x &= x_0 - q^{-1}x_0^2 - q^{-1}\Big(x_0(-q^{-1}x_0^2) + (-q^{-1}x_0^2)x_0\Big) - q^{-1}(-q^{-1}x_0^2)^2 \\ &\quad -q^{-1}\Bigg(x_0(-q^{-1})\Big(x_0(-q^{-1}x_0^2) + (-q^{-1}x_0^2)x_0\Big) \\ &\quad +(-q^{-1})\Big(x_0(-q^{-1}x_0^2) + (-q^{-1}x_0^2)x_0\Big)x_0\Bigg) + \ldots \end{aligned} \tag{4.132}$$

One notices that the number of terms at each order is given by the Catalan numbers C_n: $C_1 = 1, C_2 = 1, C_3 = 2, C_4 = 1 + 4 = 5$. These have the asymptotics

$$C_n \sim \frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}}, \tag{4.133}$$

so that the number of contributions grows exponentially. Since numbers commute, (4.132) simplifies to

$$x = x_0 - q^{-1}x_0^2 + 2q^{-2}x_0^3 - 5q^{-3}x_0^4 + \ldots = x_0(1 - q^{-1}x_0 + 2q^{-2}x_0^2 - 5q^{-3}x_0^3 + \ldots) \tag{4.134}$$

From this we see that this series converges for $\frac{4x_0}{q} \leq 1$, meaning that we either have to have an almost linear equation with q large or we have to guess the solution close to 0, which is a solution even if the quadratic part in (4.130) is relevant. This result makes sense when one considers the exact solutions

$$x_\pm = -\frac{q}{2}\Big(1 \pm \sqrt{1 + \frac{4x_0}{q}}\Big) \tag{4.135}$$

and a Taylor expansion of x_- gives (4.134).

We have a similar situation for exactly marginal deformations [85, 86], that is we make the sliver frame ansatz $\Psi = \lambda\sqrt{\Omega}cV\sqrt{\Omega} + \ldots$ with $Q(\sqrt{\Omega}cV\sqrt{\Omega}) = 0$ and the VV OPE regular. Since $Q(\sqrt{\Omega}cV\sqrt{\Omega}) = 0$, we don't need to add a source term as in (4.130), which is exactly what we need since (4.42) has no source term. For relevant deformations we'll get rid of the source term by properly tuning the SFT coupling λ. To third order, we have

$$\begin{aligned}\Psi &= \lambda\sqrt{\Omega}cV\sqrt{\Omega} - \lambda^2 Q^{-1}(\sqrt{\Omega}cV\sqrt{\Omega})^2 - \\ &\quad \lambda^3 Q^{-1}\Big\{-Q^{-1}(\sqrt{\Omega}cV\sqrt{\Omega})^2, \sqrt{\Omega}cV\sqrt{\Omega}\Big\} + \dots \\ &\equiv \Psi_1 + \Psi_2 + \Psi_3 + \dots \\ &= \Psi_1 - Q^{-1}\Psi_1^2 - Q^{-1}\{\Psi_2, \Psi_1\} + \dots \end{aligned} \tag{4.136}$$

We'll write $Q^{-1} = \frac{\mathcal{B}_0}{\mathcal{L}_0}$ since for a Ψ in Schnabl gauge, we then have

$$Q^{-1}Q\Psi = \frac{\mathcal{B}_0}{\mathcal{L}_0}Q\Psi = \frac{1}{\mathcal{L}_0}\mathcal{B}_0 Q\Psi = \frac{1}{\mathcal{L}_0}\mathcal{L}_0\Psi = \Psi, \tag{4.137}$$

where we used $\{Q, \mathcal{B}_0\} = \mathcal{L}_0$. We were a little bit formal with the division by $\mathcal{L}_0$ but in our case Q^{-1} acts only on states which are included in the VV OPE and since it's regular, there are no states of levels less than or equal to 0 so the division is well defined (the less than zero levels are problematic for Schwinger parametrisation). Since Q^{-1} encounters no subtleties, the full EOMs (4.42) are satisfied and not just their projection to a gauge subspace, as will be clear in the next chapter.

Now we evaluate the second order contribution Ψ_2 to (4.136)

$$\begin{aligned}\Psi_2 &= -\lambda^2 Q^{-1}(\sqrt{\Omega}cV\sqrt{\Omega})^2 = -\lambda^2\frac{\mathcal{B}_0}{\mathcal{L}_0}\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} \\ &= -\lambda^2\int_0^1 \frac{\mathrm{d}s}{s}s^{\mathcal{L}_0}\Big(-\sqrt{\Omega}cBV\Omega cV\sqrt{\Omega}\Big) \\ &= \lambda^2\int_0^1\frac{\mathrm{d}s}{s}s\sqrt{\Omega}cBV\Omega^s cV\sqrt{\Omega} \\ &= \lambda^2\sqrt{\Omega}cBV\frac{1-\Omega}{K}cV\sqrt{\Omega}, \end{aligned} \tag{4.138}$$

where we Schwinger parametrised the inverse of $\mathcal{L}_0$

$$\frac{1}{\mathcal{L}_0} = \int_0^1\frac{\mathrm{d}s}{s}s^{\mathcal{L}_0}, \tag{4.139}$$

and used the integral

$$\int_0^1 \mathrm{d}s\, e^{-sK} = \frac{1-e^{-sK}}{K} \tag{4.140}$$

In the computation of Ψ_2 we've seen that the width of an intermediate wedge state Ω^s ranged from 0 to 1. If we have operator collisions between the two Vs, this causes problems and one needs to regularise. When we have only two operator collisions, this is rather simple because we can introduce an analogue of normal ordering. At higher orders, where collisions of more than two Vs occur, one needs a generalisation of normal ordering to a situation where multiple operators collide at the same time. We shall develop this generalisation in the next chapter. The regularisation method will be based on expanding operator products in the collision limit, from which we identify states which contribute to singularities and subtract them.

At third order, we have (notice the anticommutation pattern in the expansion (4.136))

$$\begin{aligned}
\Psi_3 &= -Q^{-1}\{\Psi_2,\Psi_1\} = -\lambda^3\frac{\mathcal{B}_0}{\mathcal{L}_0}\Big\{\int_0^1 \mathrm{d}s\sqrt{\Omega}cBV\Omega^s cV\sqrt{\Omega},\sqrt{\Omega}cV\sqrt{\Omega}\Big\} \\
&= -\lambda^3\int_0^1 \mathrm{d}s\frac{1}{\mathcal{L}_0}\sqrt{\Omega}cBV\Big(\Omega^s V\Omega+\Omega V\Omega^s\Big)cV\sqrt{\Omega} \\
&= -\lambda^3\int_0^1 \mathrm{d}s\int_0^1\frac{\mathrm{d}t}{t}t^2\sqrt{\Omega}cBV\Big(\Omega^{st}V\Omega^t+\Omega^t V\Omega^{st}\Big)cV\sqrt{\Omega} \\
&= -\lambda^3\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\, t\sqrt{\Omega}cBV\Big(\Omega^{st}V\Omega^t+\Omega^t V\Omega^{st}\Big)cV\sqrt{\Omega} \\
&= -\frac{\lambda^3}{2!}\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v\sqrt{\Omega}cBV\Big(\Omega^u V\Omega^v+\Omega^v V\Omega^u\Big)cV\sqrt{\Omega} \\
&= -\lambda^3\sqrt{\Omega}cBV\Big(\frac{1-\Omega}{K}V\Big)^2 c\sqrt{\Omega},
\end{aligned} \tag{4.141}$$

where we substituted $u = st, v = t$ and used the $u \to v, v \to u$ symmetry of the integrand to produce a $\frac{1}{2!}$ instead of an integration over a triangle as would be usual in path ordering. This symmetry is present only for marginal deformations, the integrals can't be done explicitly for relevant deformations as we'll see in the next chapter. At higher orders one finds a pattern (we checked this to fifth order)

$$\Psi_{n+1} = (-\lambda)^{n+1}\sqrt{\Omega}cVB\Big(\frac{1-\Omega}{K}V\Big)^n c\sqrt{\Omega}, \tag{4.142}$$

for $n \geq 1$, which sums up to

$$\Psi = \lambda\sqrt{\Omega}cV\frac{B}{1+\frac{1-\Omega}{K}\lambda V}c\sqrt{\Omega} \tag{4.143}$$

via a geometric series. We find it very interesting that all C_n terms combine to a single term at each order, something we haven't been able to do for relevant deformations (postponing regularisation to the very end) by the lack of symmetry of the integrals.

We can now try to evaluate the shift in the boundary state, which is to first order, see (4.74)

$$\begin{aligned}
\Delta B^{\phi}_{\Psi_1} &= 2\pi i\langle\tilde{\phi}(i\infty)\Psi_1\rangle_{C_1} = 2\pi i\lambda\langle\tilde{\phi}(i\infty)\Omega cV\rangle_{C_1} \\
&= 2\pi i\lambda\langle\tilde{\phi}(i\infty)\Big(\int_0^1 \mathrm{d}t\,\Omega^{1-t}cV\Omega^t\Big)\rangle_{C_1} \\
&= 2\pi i\lambda\langle\tilde{\phi}(i\infty)\Big(\int_0^1 \mathrm{d}t\,\Omega^{1-t}cV\Omega^t\Big)\{B,c\}\rangle_{C_1} \\
&= 2\pi i\lambda\langle\tilde{\phi}(i\infty)\Big(\int_0^1 \mathrm{d}t\,\Omega^{1-t}V\Omega^t c\Big)\rangle_{C_1} \\
&= 2\pi i\lambda\langle\tilde{\phi}(i\infty)\Big(\int_0^1 \mathrm{d}t\,V(t)\Big)c(0)\rangle_{C_1} \\
&= 2\pi i\lambda\frac{1}{2\pi i}\langle c\bar{c}\phi(0)\Big(\int_0^{2\pi} \mathrm{d}\theta\,V(\theta)\Big)c(1)\rangle_{disc} \\
&= -\lambda\langle\phi(0)\Big(\int_0^{2\pi} \mathrm{d}\theta\,V(\theta)\Big)\rangle_{disc}
\end{aligned} \tag{4.144}$$

where we mapped the cylinder to the unit disc via $f(z) = e^{2\pi iz}$ and used cyclicity of the Ellwood invariant. This is exactly the leading order result one would expect from conformal perturbation theory (see (3.5)), where

$$\Delta B^{\phi}_{\Psi} = \langle\phi(0)\Big[e^{-\lambda\int_0^{2\pi}\mathrm{d}\theta\,V(\theta)} - 1\Big]\rangle_{disc} \tag{4.145}$$

and indeed this is the result one obtains to all orders [112]. We see that the SFT coupling λ which is a sort of normalisation of the string field is here equal to all orders in perturbation theory to the CFT coupling (unambiguously defined since for exactly marginal deformations one doesn't need to regularise). Notice also the naturally emerging convention in SFT (which we adopted in CPT) that one deforms the action by $\lambda\int_0^{2\pi}\mathrm{d}\theta\,V$ and not $-\lambda\int_0^{2\pi}\mathrm{d}\theta\,V$ as is more usual in CFT.

An interesting example of exactly marginal deformations is the lightlike deformation of Schnabl and Hellerman [113] with $V(x) = e^{X_+(x)}$ and the OPE $e^{X_+(x)}e^{X_+(0)} = e^{2X_+(0)} + \ldots$. The field is made marginal by turning on a nontrivial linear dilaton background. This solution describes lightlike propagation of an instability, which decays the unstable D-brane and leaves behind the tachyon vacuum, which is similar in spirit to the computations of (1.6) and (B.3).

Chapter 5

Relevant deformations in open string field theory

In this chapter we explore perturbative solutions of OSFT corresponding to relevant deformations. The material is divided into five sections

1. The general framework for perturbative solutions is summarised. We work in Siegel gauge for simplicity but the procedure can be generalised to other gauges. That the out-of-gauge equations of motion are satisfied is proved using a fixed-point argument due to Schnabl [1]. This section simply provides a slightly different perspective on [94].

2. We explore relevant deformations to leading order where no gauge fixing is required. In the near-marginal limit, we reproduce the result of Affleck and Ludwig (3.61) by using the correspondence between the OSFT action and Δg. We also validate our result (3.71) for ΔB_a^ϕ by invoking the KMS correspondence. These results, which hold for a perturbation by a single operator are then easily generalised to the case of multiple perturbing operators. After this, we investigate the spectrum of linearised fluctuations around the perturbative solution. We show that the perturbing operator is removed from the cohomology after flowing to the IR.

3. After fixing Siegel gauge, we use the Berkovits-Schnabl trick [114] to derive a completely new result in CPT, which is the next-to-leading correction to Δg for a generic weakly relevant perturbation by a single operator. The result is expressed by a regularised off-shell zero-momentum four-point amplitude of open string tachyons in the on-shell limit and we discuss how the entire moduli space is covered. We note that a drastic cancellation between the auxiliary OSFT factors such as $K = \frac{3\sqrt{3}}{4}$

occurs, hinting at possible simple all-order prescriptions. The result was tested on a theory of a nontrivial $c = 2$ free boson and we got a high precision match with [104].

4. In Schnabl gauge, we derive a closed form for the up to third order perturbative solution corresponding to relevant deformations. We encounter a technical obstacle in that projectors in sliver frame are not orthogonal but oblique, which leads us to developing a method of level expanding star products of an arbitrary number of singular operator insertions. By using the KMS correspondence, we check the result by reproducing (3.61). Together with the result of section (5.3), it appears that OSFT provides a consistent framework for doing CPT calculations to all orders.

5. This is the most experimental of the five sections. We discuss what a solution analogous to (5.4) in pseudo-Schnabl gauge [86, 115] might look like. Our guess gives the correct leading order result (3.61), but violates the equations of motion in the sense that we don't know how the coupling is to be tuned for source terms to disappear. We think this match is because of gauge invariance and cyclicity of the Ellwood invariant through which we compute Δg.

The work is based on collaboration with Martin Schnabl and the first three sections are available in preprint [1] (with the exception of (5.2.4)).

5.1 The general framework

We have seen in the solution (4.134) of the quadratic equation (4.130) that the perturbative solution is dominated by a single contribution x_0 receiving higher order corrections. To do something similar in solving (4.42), we split the string field

$$\Psi = R + X, \tag{5.1}$$

where $R = \sum_i \lambda_i c V_i |0\rangle$ is the leading order contribution corresponding to a boundary deformation by $\sum_i \lambda_i \oint V_i$ with the X being higher order corrections. To solve (4.130), we fix Siegel gauge $b_0 \Psi = 0$ and by acting with b_0 on (4.130) and using $\{Q, b_0\} = L_0$, one has

$$L_0 \Psi + b_0 \Psi^2 = 0 \tag{5.2}$$

If we introduce a projector P such that $P\Psi = R$ and a projector $\bar{P} = 1 - P$ such that $\bar{P}\Psi = X$, we can split the EOMs into two sets

$$R + \frac{b_0}{L_0} P(R+X)^2 = 0 \tag{5.3}$$

$$X + \frac{b_0}{L_0} \bar{P}(R+X)^2 = 0, \tag{5.4}$$

where we also required that the projectors commute with Q. The equations (5.4) can be solved perturbatively

$$X = \Delta R^2 + \Delta\Big\{\Delta R^2, R\Big\} + \dots, \tag{5.5}$$

where the propagator is

$$\Delta = -\frac{b_0}{L_0}\bar{P} \tag{5.6}$$

In Feynman-diagrammatic language (5.5) is given by a sum over binary diagrams with vertices meaning star multiplication, internal lines an action of Δ and external lines an R. At a given order n, the number of such diagrams is C_n so their number grows exponentially as 4^n. This implies that R must be small enough in some sense with an action of Δ not producing divergences. Since we Schwinger parametrise the action of $\frac{1}{L_0}$, such divergences come from states of $h \leq 1$ and these are projected out by $\bar{P}$ (if there are operators of $h \leq 1$ in the $V_i V_j$ OPEs not contained in R, they need to be included in there since they are turned on by the other perturbing operators). Actually weakly irrelevant operators have to projected out as well despite strictly speaking not producing divergences in $\frac{1}{L_0}$ since they have $h > 1$. This is so because in the near-marginal limit their contributions appear as poles, same as with weakly relevant operators.

In the following, we consider relevant deformations, which don't turn on marginal or weakly irrelevant terms so we don't need to worry about the case $h \geq 1$. In this case, for the validity of the perturbative approach, we must have for all SFT couplings $\lambda_i = O(y)$ with y being the RG eigenvalue of the least relevant field since schematically $\frac{1}{L_0} \to \frac{1}{h_i - 1} \sim \frac{1}{y_i}$. Once we have the solution X to (5.4), we can plug it into (5.3) to obtain a set of polynomial equations for the couplings λ_i. To leading order these polynomial equations are actually independent of X and so independent of the gauge since X is second order. This can be seen by realising that (5.3) is equivalent to

$$QR + P(R+X)^2 = QR + PR^2 + O(R^3) = 0 \tag{5.7}$$

This equivalence follows from acting with Q on (5.3)

$$\begin{aligned} QR + Q\Big(\frac{b_0}{L_0} P(R+X)^2\Big) &= 0 = QR + P(R+X)^2 + \frac{b_0}{L_0} Q\Big(P(R+X)^2\Big) \\ &= QR + P(R+X)^2 \end{aligned} \tag{5.8}$$

and seeing that P when acting at ghost number 2 projects to Q-exact states $c\partial c V_i |0\rangle$. One may ask whether the full EOMs (4.42) are satisfied and not only their projection to Siegel gauge. The equivalence of (5.3) and (5.7) means that this statement needs to be verified for the $\bar{P}$ projected equations (5.4). One then finds after using the splitting (5.1)

$$\begin{aligned} Q\Psi + \Psi^2 &= \frac{b_0}{L_0} \bar{P} Q(\Psi^2) = \frac{b_0}{L_0} \bar{P} [Q\Psi, \Psi] \\ &= \frac{b_0}{L_0} \bar{P} \left[\frac{b_0}{L_0} \bar{P} Q(\Psi^2), \Psi \right] = \dots, \end{aligned} \tag{5.9}$$

where we used that $Q\Psi + \Psi^2 = \frac{b_0}{L_0} \bar{P} Q(\Psi^2)$ and $[\Psi^2, \Psi] = 0$. Assuming that Ψ is parametrically small, this turns into a fixed-point argument showing that the full EOMs are satisfied. The fixed-point argument works only if the repeated action of Δ on exact states at ghost number 3 is not divergent. This is indeed the case since the problematic weight 0 ghost number 3 state $c\partial c \partial^2 c$ is not exact. The states $c\partial c\partial^2 cV$ are irrelevant so they pose no problem if they are not continuously connected to the identity (we should have a gap between the $h = 0$ identity and the next most relevant operator).

In the following, we write $\Psi = \Psi_1 + \Psi_2 + \dots$, where $\Psi_n = O(y^n)$ with y being the RG eigenvalue of the least relevant operator (this will become clear after we prove $\lambda_i = O(y)$ in (5.2.1)). In this notation one has $R = \Psi_1$ and $X = \Psi_2 + \dots$.

5.2 No gauge fixing

In this section we first analyse the perturbation with $R = \Psi_1 = \lambda cV |0\rangle$ inserted to the unit half-disc of weight h with RG eigenvalue $y = 1 - h$ to leading order. The operator V is assumed to have a two-point function normalised to unity. First the EOMs are solved, giving the SFT couplings and then Δg and $\Delta B_{\phi V}$ are computed in the small y limit, obtaining a correspondence with CPT (3.61), (3.71). We also provide a generalisation to multiple perturbing operators.

5.2.1 The equations of motion

The leading order equations of motion for a perturbation by $R = \lambda cV |0\rangle$ are simply given by plugging the perturbation into (5.7)

$$\lambda Q(cV) + \lambda^2 P(cV)^2 + \ldots = 0 \tag{5.10}$$

using

$$\begin{aligned} Q(cV) &= -yc\partial cV && (5.11) \\ P(cV)^2 &= \frac{1}{g_0}\langle cV, cV^2\rangle c\partial cV = \frac{1}{g_0}\langle cV, cV, cV\rangle c\partial cV \\ &= -C_{VVV}K^{-3y}c\partial cV, && (5.12) \end{aligned}$$

where we used (4.11), the three-vertex (4.56) and the ghost vertex $\langle c, c, c\rangle = -1$. The equation (5.10) then becomes

$$\lambda(y - \lambda C_{VVV}K^{-3y})c\partial cV + \ldots = 0 = \lambda(y - \lambda C_{VVV}) + O(y^2), \tag{5.13}$$

with the nontrivial $\lambda \neq 0$ solution

$$\lambda = \frac{y}{C_{VVV}} + O(y^2) \tag{5.14}$$

which to leading order matches the BCFT result (3.48).

The generalisation to multiple perturbations is simple, since one simply has as many equations as projectors and sums over all fields that give a nonzero OPE

$$\begin{aligned} \lambda_i y_i - \sum_{jk}\lambda_j\lambda_k C_{ijk}K^{-y_i-y_j-y_k} &= \lambda_i\tilde{y}_i y - \sum_{jk}\lambda_j\lambda_k C_{ijk}K^{(-\tilde{y}_i-\tilde{y}_k-\tilde{y}_l)y} \\ &= \lambda_i y_i - \sum_{jk}\lambda_j\lambda_k C_{ijk} + O(y^2) = 0 \quad (5.15) \end{aligned}$$

where we denoted $y_i = \tilde{y}_i y$ with y being the smallest RG eigenvalue serving as a common expansion parameter. This gives a coupled set of quadratic equations.

5.2.2 The g-function

We now plug $\Psi = \Psi_1 + \ldots$ into the g-function formula (4.59) to obtain

$$\Delta g = \frac{\pi^2}{3}\langle\Psi_1, Q\Psi_1\rangle + \ldots \tag{5.16}$$

and using $\langle V, V\rangle = g_0$ and $\langle c, c\partial c\rangle = -1$, we have

$$\Delta g = -\frac{\pi^2}{3} g_0 y \lambda^2 + \dots \tag{5.17}$$

Using the fixed-point SFT coupling (5.14), we simply have

$$\frac{\Delta g}{g_0} = -\frac{\pi^2}{3} \frac{y^3}{C_{VVV}^2} + O(y^4), \tag{5.18}$$

which is the result (3.61) of Affleck and Ludwig. It is astonishing how much simpler the OSFT calculation is compared to the boundary CPT one. This inspired us to extend this calculation to the next-to-leading order, see (5.3), which would be very difficult in CPT and so far nobody else calculated Δg to $O(y^4)$ as far as we know (although see [116] for some next-to-leading results). The leading order result can also be reproduced by level truncating the string field to level less than zero state Ψ_1 and evaluating the off-shell action of Ψ_1, which gives it as a function of λ

$$S[\Psi_1](\lambda) = -\frac{1}{2} y \lambda^2 \langle cV, c\partial cV\rangle - \frac{1}{3}\lambda^3 \langle cV, cV, cV\rangle = \frac{1}{2} y\lambda^2 - \frac{1}{3}\lambda^3 C_{VVV} K^{3y} \tag{5.19}$$

Finding an extremum with respect to λ gives $\lambda_* = \frac{K^{-3y} y}{C_{VVV}} = \frac{y}{C_{VVV}} + O(y^2)$, which gives the extremal value

$$S[\Psi_1](\lambda_*) = \frac{2^{-1+12y} 3^{-1-9y} y^3}{C_{VVV}} = \frac{y^3}{6C_{VVV}} + O(y^4) \tag{5.20}$$

One can also check this result by setting $y = 1$ as appropriate for the universal tachyon, which then gives $\frac{2^{11}}{3^{10}}$. This is the well-known first approximation to the value of the tachyon vacuum action.

The generalisation to multiple perturbing operators is very simple by the BPZ-orthogonality $\langle V_i, V_j\rangle = g_0 \delta_{i,j}$

$$\frac{\Delta g}{g_0} = -\frac{\pi^2}{3} \sum_i y_i \lambda_i^2 + \dots, \tag{5.21}$$

where we just plug in the couplings obtained by solving the quadratic equations (5.15).

5.2.3 The general boundary state coefficient

For the shift in the general boundary state coefficient $\Delta B_{\phi V}$, we invoke the KMS correspondence (4.74)

$$
\begin{aligned}
\Delta B_{\phi V} &= 2\pi i \langle I| \tilde{\phi}(i) |\Psi_1\rangle + \dots \\
&= 2\pi i \lambda 2 i 2^{\Delta-2} 2^{h-1} \langle \phi(i) V(0)\rangle + \dots \\
&= 2\pi i \lambda 2 i 2^{\Delta-2} 2^{h-1} \frac{g_0 B_{\phi V}}{2^{\Delta-h}} + \dots \\
&= -\pi \lambda 2^{2h-1} g_0 B_{\phi V} + \dots ,
\end{aligned}
\tag{5.22}
$$

where we used that $|I\rangle = U_f^\dagger |0\rangle$ with $f = \frac{2}{1-z^2}$ and the correlators (A.3),(A.16). Plugging in the fixed point coupling (5.14) gives

$$
\frac{\Delta B_{\phi V}}{g_0} = -2\pi \frac{B_{\phi V}}{C_{VVV}} + O(y^2) \tag{5.23}
$$

as derived by CPT (3.71). The generalisation to multiple perturbing operators is again simple

$$
\frac{\Delta B_{\phi V_i}}{g_0} = -\pi \sum_j \lambda_j 2^{2h_j-1} B_{\phi V_j} + \dots , \tag{5.24}
$$

where we plug in the solution of (5.15).

5.2.4 The spectrum of linearised fluctuations

We would now like to investigate the cohomology of the shifted kinetic operator Q_Ψ, see (4.3.2), for Ψ the perturbative solution described in (5.1). Such a computation should give one access to the operator mixings and perhaps may open a way to study shifts in operator dimensions via diagonalising the shifted Hamiltonian H_Ψ. We first want to see which states A are closed in the new background and to do this, we have to solve

$$
Q_\Psi A = QA + \{\Psi, A\} = 0 \tag{5.25}
$$

We make the ansatz $A = A_0 + \delta A$, where A_0 is a member of the ghost number 1 cohomology of the undeformed background, meaning $QA_0 = 0, A_0 \neq Q\Lambda$. We split δA by a projector P such that $P\delta A = \delta A_R$ and $\bar{P}\delta A = \delta A_X$. This projector projects on the near-marginal states, which are contained in the OPE of A_0 and Ψ_1 analogously as before when we projected on the states contained in the OPE of Ψ_1 and Ψ_1. Thus the equation (5.25) can be split into two

$$
\begin{aligned}
Q\delta A_R + P\{\Psi, A\} &= 0 &\quad (5.26)\\
Q\delta A_X + \bar{P}\{\Psi, A\} &= 0, &\quad (5.27)
\end{aligned}
$$

where we used $QA_0 = 0$. We solve (5.27) by fixing Siegel gauge (can be extended to other gauges just as in (5.1)) $b_0 \delta A_X = 0$. We can fix a gauge because we have the linearised gauge invariance $A \to A + Q_\Psi \Lambda$, which is completely fixed by choosing Siegel gauge. The solution to (5.27) then becomes

$$\delta A_X = -\frac{b_0}{L_0} \bar{P} \{\Psi, A\} \tag{5.28}$$

We now have to check that the full equations of motion (5.25) are satisfied. To do so, we again use a fixed point argument since

$$QA + \{\Psi, A\} = -\frac{b_0}{L_0} \bar{P} Q \{\Psi, A\} \tag{5.29}$$

and one has

$$\begin{aligned} Q\{\Psi, A\} &= (Q\Psi)A - \Psi QA + (QA)\Psi - AQ\Psi \\ &= -\Psi^2 A + [QA, \Psi] + A\Psi^2 \\ &= -\Psi^2 A - [\{\Psi, A\}, \Psi] + A\Psi^2 - \left[\frac{b_0}{L_0} \bar{P} Q \{\Psi, A\}, \Psi\right] \\ &= -\left[\frac{b_0}{L_0} \bar{P} Q \{\Psi, A\}, \Psi\right] \end{aligned} \tag{5.30}$$

Since the propagator acts on exact states at ghost number 3 as in (5.1), there are no divergences from $\frac{1}{L_0}$ and since Ψ is small, the fixed point argument is complete. Note that there was a cancellation needed for the fixed point argument to go through in contrast with (5.1). In fact we'll get an even more surprising cancellation trying to solve the equation

$$A = Q_\Psi B, \tag{5.31}$$

where B is at ghost number 0. If the equation (5.31) has a solution, the string field A_0 flows to a trivial member of the cohomology of Q_Ψ. We could again try to solve (5.31) perturbatively by writing $B = B_0 + \delta B_R + \delta B_X$, where $QB_0 = A|_{y=0}$ and $\delta B_R = PB, \delta B_X = \bar{P}B$, where P projects onto the near-marginal fields in the OPE of B_0 and Ψ_1. We note that the equation for B_0 need not be the contradicting $QB_0 = A_0$ (remember, A_0 is a nontrivial member of the cohomology of Q) since δA_R may be $O(1)$. On the other hand if one has $\delta A_R = O(y)$, we immediately know that A_0 stays in the cohomology. We again split (5.31) into two

$$PA = QPB + P[\Psi, B] \tag{5.32}$$

$$\bar{P}A = Q\bar{P}B + \bar{P}[\Psi, B] \tag{5.33}$$

Now we solve (5.33) by fixing the gauge

$$\bar{P}B = \frac{b_0}{L_0}\bar{P}\Big(A - [\Psi, B]\Big) \tag{5.34}$$

and we again employ a fixed point argument to show that (5.31) is solved

$$A - (QB + [\Psi, B]) = -\frac{b_0}{L_0}\bar{P}Q\Big(A - [\Psi, B]\Big) \tag{5.35}$$

where we use

$$\begin{aligned} Q[\Psi, B] &= (Q\Psi)B - \Psi QB - (QB)\Psi - BQ\Psi \\ &= -\Psi^2 B - \{QB, \Psi\} + B\Psi^2 \\ &= -\Psi^2 B + \{[\Psi, B], \Psi\} - \{\Psi, A\} + B\Psi^2 + \left\{\frac{b_0}{L_0}\bar{P}Q\Big(A - [\Psi, B]\Big), \Psi\right\} \\ &= -\{\Psi, A\} + \left\{\frac{b_0}{L_0}\bar{P}Q\Big(A - [\Psi, B]\Big), \Psi\right\}, \end{aligned} \tag{5.36}$$

so that

$$-\frac{b_0}{L_0}\bar{P}Q\Big(A - [\Psi, B]\Big) = -\frac{b_0}{L_0}\bar{P}\left\{-\frac{b_0}{L_0}\bar{P}Q\Big(A - [\Psi, B]\Big), \Psi\right\}, \tag{5.37}$$

where we used the equation of motion (5.25) for A. The propagator acts at ghost number 2 and we may encounter contributions from the problematic Q-exact states $c\partial cV\,|0\rangle$ for V near-marginal but not marginal. We are saved by the fact that $QcV\,|0\rangle = yc\partial cV\,|0\rangle$ with $y = 1 - h$ small for such states so when the problematic states arise by the action of Q they are accompanied by y. Thus these potentially problematic contributions are actually $O(1)$ and don't spoil the fixed point argument.

We now use the formalism built above to show that the perturbing operator vanishes from the cohomology when flowing to the IR, a result that can be obtained without gauge fixing. To do this, we consider the simplest case $\Psi_1 = \lambda cV\,|0\rangle$, $\lambda = \frac{y}{C_{VVV}} + O(y^2)$ with $[V] \times [V] = [\mathbb{1}] + [V]$. The operator in the cohomology A_0 will be the $cV\,|0\rangle$ but tensored with an appropriate exponential to make it weight 1 so that its physical

$$A_0 \sim cVe^{ikX}\,|0\rangle \tag{5.38}$$

This means that $[A_0] \times [\Psi_1] = [c\partial ce^{ikX}] + [c\partial cVe^{ikX}]$, where $c\partial ce^{ikX}$ is not nearly marginal so that P at ghost number 1 projects onto cVe^{ikX}, that is onto A_0. This means that

$$\delta A_R \sim A_0 \sim cVe^{ikX}\,|0\rangle \tag{5.39}$$

and the P projected equation (5.26) is

$$Q\delta A_R + P\{\Psi, A\} = 0, \tag{5.40}$$

which becomes

$$P\{\Psi_1, A_0 + \delta A_R\} + O(y^2) = 0, \tag{5.41}$$

where we used $Q\delta A_R = 0$ as is specific to this setup. Since the projector is nontrivial because of the OPE structure of A_0 and Ψ_1, we must have

$$A_0 + \delta A_R = O(y), \tag{5.42}$$

which gives $\delta A_X = O(y^2)$ because of (5.28). But then one has $A_0 + \delta A_R = O(y^2)$ since δA_X contributes as $O(y^3)$ in (5.41). Repeating this argument we have $A_0 + \delta A_R = O(y^\infty) = 0$ so that also $\delta A_X = 0$. Taken together, we have $A = 0$ so that the perturbing tachyon is not present in the Q_Ψ cohomology. A similar phenomenon was observed in the unitary minimal models [65], where the boundary flow triggered by the least relevant operator leads to a stable boundary condition with no nontrivial relevant operators present. This should not be confused with Zamolodchikov's bulk flows [42], which present a chain of least relevant bulk perturbations.

5.3 Siegel gauge

We now fix Siegel gauge $b_0\Psi = 0$ meaning that the propagator is $\Delta = -\frac{b_0}{L_0}\bar{P}$ just as in (5.1). The equation (5.7) for the coupling then becomes

$$Q\Psi_1 + P\Psi_1^2 + P\{\Psi_2, \Psi_1\} + \ldots = 0, \tag{5.43}$$

with $\Psi_2 = -\frac{b_0}{L_0}\bar{P}$. After defining the four-point amplitude

$$\begin{aligned}\mathcal{A} &\equiv \langle cV * cV, \frac{b_0}{L_0}\bar{P}cV * cV\rangle \\ &= \langle cV(-\sqrt{3})cV(\sqrt{3})U_3\big(\frac{b_0}{L_0}\bar{P}\big)U_3^\dagger cV\left(\frac{1}{\sqrt{3}}\right)cV\left(-\frac{1}{\sqrt{3}}\right), \end{aligned}\tag{5.44}$$

where U_n implements the transformation $z \to \tan\left(\frac{2}{n}\arctan z\right)$, the equation (5.46) becomes

$$\lambda(y - \lambda C_{VVV}K^{-3y} - 2\lambda^2\mathcal{A})c\partial cV + \ldots = 0, \tag{5.45}$$

which is just (5.13) with $\mathcal{A}$ added. This is solved by

$$\lambda = \frac{y}{C_{VVV}} + \frac{1}{C_{VVV}}\left(\frac{2\mathcal{A}}{C_{VVV}^2} - \ln K^3\right)y^3 + O(y^4) \tag{5.46}$$

The g-function to the next-to-leading order has the form

$$\Delta g = \frac{\pi^2}{3}\Big(\langle\Psi_1, Q\Psi_1\rangle + \langle\Psi_2, Q\Psi_2\rangle\Big) + \dots, \tag{5.47}$$

and one can write

$$\langle\Psi_2, Q\Psi_2\rangle = -\langle\Psi_2, \bar{P}\Psi_1 * \Psi_1\rangle = -\langle\Psi_2, \Psi_1 * \Psi_1\rangle = -\lambda^4\mathcal{A}, \tag{5.48}$$

so that after plugging in the coupling (5.46), we get

$$\frac{\Delta g}{g_0} = -\frac{\pi^2}{3}\left(\frac{y^3}{(C_{VVV})^2} + \left(3\frac{\mathcal{A}}{g_0 C_{VVV}^4} - \frac{6}{C_{VVV}^2}\ln K\right)y^4\right) + O(y^5), \tag{5.49}$$

all that is left is to evaluate the four-point amplitude $\mathcal{A}$. We evaluate it using the Berkovits-Schnabl trick [114], that is we use the Hodge-Kodaira decomposition

$$1 = \left\{Q, \frac{b_0}{L_0}\bar{P}\right\} + P \tag{5.50}$$

behind the factor $U_3^\dagger$ in (5.44). Taking note of the fact that $Q(cV) = O(y)$, we then have

$$\begin{aligned}\mathcal{A} &= \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) U_3\bar{P}U_3^\dagger\left(\frac{b_0}{L_0}\bar{P}\right) cV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle \\ &+ \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) U_3\left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle + O(y)\end{aligned} \tag{5.51}$$

In the second term we use the Berkovits-Schnabl trick again, but this time behind the U_3

$$\begin{aligned}&\langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) U_3\left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle = \\ &\langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) PU_3\left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle + \\ &\langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right)\left(\frac{b_0}{L_0}\bar{P}\right) U_3\bar{P}U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle,\end{aligned} \tag{5.52}$$

where we used that $QP = 0$ with P acting at ghost number 2 together with $\left\{Q, \frac{b_0}{L_0}\right\} = 1$. Using BPZ conjugation, we bring together the first term of

(5.51) and the second term of (5.52)

$$\mathcal{A} = \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right)(1+P)U_3\bar{P}U_3^\dagger\left(\frac{b_0}{L_0}\bar{P}\right) cV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle$$
$$+ \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) PU_3\left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle + O(y) \tag{5.53}$$

Now we use $\bar{P}U_3^\dagger\bar{P} = U_3^\dagger\bar{P}$, which follows from $PU_3^\dagger\bar{P} = 0$ as can be seen by BPZ conjugating $\bar{P}U_3P = 0$, which holds because primaries transform to primaries under conformal transformations. This gives

$$\mathcal{A} = \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right)(1+P)U_3U_3^\dagger\left(\frac{b_0}{L_0}\bar{P}\right) cV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle$$
$$+ \langle cV\left(-\sqrt{3}\right) cV\left(\sqrt{3}\right) PU_3\left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right)\rangle + O(y) \tag{5.54}$$

Focusing on the second term of (5.54), we first compute

$$PcV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right) = -C_{VVV}\left(\frac{2}{\sqrt{3}}\right)^{1-h} c\partial cV\left|0\right\rangle, \tag{5.55}$$

as follows from the OPE

$$cV(x)cV(-x) \sim -\sum_{V'} C_{VVV'}\frac{c\partial cV'(0)}{(2x)^{2h-h'-1}}, \tag{5.56}$$

where V' are relevant operators (not weakly relevant since we are considering perturbation by a single operator V). The second term then becomes

$$C_{VVV}^2\left(\frac{4}{3}\right)^y \left\langle c\partial cV\right| U_3\frac{b_0}{L_0}\bar{P}U_3^\dagger\left|c\partial cV\right\rangle \tag{5.57}$$

since we use the projection (5.55) twice. Now we split the projector in (5.57) as $\bar{P} = 1 - P$, after which we use

$$PU_3^\dagger\left|c\partial cV\right\rangle = \left(\frac{2}{3}\right)^{h-1}\left|c\partial cV\right\rangle, \tag{5.58}$$

in which we made $U_3^\dagger$ act on the left, where it acts on a primary giving the scale factor $\left(\frac{2}{3}\right)^{h-1}$. Then we use the Berkovits-Schnabl trick again

$$C_{VVV}^2\left(\frac{4}{3}\right)^y \left\langle c\partial cV\right| U_3\frac{b_0}{L_0}U_3^\dagger\left|c\partial cV\right\rangle = C_{VVV}^2\left(\frac{4}{3}\right)^y \left\langle c\partial cV\right| U_3U_3^\dagger\frac{b_0}{L_0}\left|c\partial cV\right\rangle, \tag{5.59}$$

and commuting the U factors, see [83]

$$U_3 U_3^\dagger = U_{\frac{8}{3}}^\dagger U_{\frac{8}{3}}, \tag{5.60}$$

we have by performing the conformal transformation implemented by the Us

$$\begin{aligned} &- g_0 C_{VVV}^2 \left(\frac{4}{3}\right)^y \frac{1}{y} \left(\left(\frac{3}{4}\right)^{-2y} - \left(\frac{2}{3}\right)^{-2y}\right) \\ &= g_0 C_{VVV}^2 \ln \frac{81}{64} + O(y) \end{aligned} \tag{5.61}$$

Maccaferri conjectured [117] that there exists a geometric constant γ independent of the underlying matter CFT such that

$$\langle c\partial cV |\, U_3 \left(\frac{b_0}{L_0}\bar{P}\right) U_3^\dagger \,| c\partial cV\rangle = \gamma \,\langle c\partial cV |\, b_0 \,| c\partial cV\rangle \tag{5.62}$$

and we have proved by (5.61) that $\gamma = \ln \frac{81}{64}$ which agrees with level truncation [117] to seven decimal places. This computation was made possible by the fact that our field was not marginal making intermediate expressions such as $\frac{1}{h-1}$ well-defined. That is, we do not rely on $\bar{P}$ to make $\frac{1}{L_0}$ well-defined, it is present only for power counting purposes so that no poles in y occur.

We continue with the first term in (5.54) and to do this we first calculate

$$\begin{aligned} &U_{\frac{8}{3}} \frac{b_0}{L_0} \bar{P} cV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right) |0\rangle = \\ &U_{\frac{8}{3}} \frac{b_0}{L_0} \bar{P}\left(cV\left(\frac{1}{\sqrt{3}}\right) cV\left(-\frac{1}{\sqrt{3}}\right) \pm \sum_{V'} C_{VVV'} \left(\frac{2}{\sqrt{3}}\right)^{1+h'-2h} c\partial cV'(0)\right) |0\rangle\,, \end{aligned} \tag{5.63}$$

where the plus-minus notation means that we add and subtract the same term. For this the OPE (5.56) was used so that we can Schwinger parametrise the action of L_0 as

$$\frac{1}{L_0} = \int_0^1 \frac{ds}{s} s^{L_0} \tag{5.64}$$

meaning that we use the Schwinger parametrisation on the plus branch and we explicitly divide by weight on the minus branch. Doing so we obtain

$$\begin{aligned} &\int_0^1 ds \Bigg[\left(\frac{\mathrm{d}\mu}{\mathrm{d}s}\right)^{2h-1} \frac{s^{2h-2}}{\sqrt{3}} \left(c(\mu) + c(-\mu)\right) V(\mu) V(-\mu) \\ &- \sum_{V' \neq V} C_{VVV'} \left(\frac{2}{\sqrt{3}}\right)^{1+h'-2h} \left(\frac{4}{3}\right)^{1-h'} \left(\frac{1}{s^{2-h'}} + \frac{1}{1-h'}\right) \\ &cV'(0) - C_{VVV} \left(\frac{2}{\sqrt{3}}\right)^{1-h} \left(\frac{4}{3}\right)^{1-h} \frac{1}{s^{2-h}} cV(0) \Bigg] |0\rangle \end{aligned} \tag{5.65}$$

where $\mu(s) = \tan\frac{3}{4}\arctan\frac{s}{\sqrt{3}}$ is present since $U_{\frac{8}{3}}$ implements the conformal transformation $z \to \tan\frac{3}{4}\arctan z$. The subterm in the first term of (5.54) containing P can then be calculated as

$$\begin{aligned}
&- C_{VVV}\left(\frac{8}{3\sqrt{3}}\right)^{1-h} \\
&\langle c\partial cV| \int_0^1 \mathrm{d}s\Bigg[\left(\frac{\mathrm{d}\mu}{\mathrm{d}s}\right)^{2h-1}\frac{s^{2h-2}}{\sqrt{3}}\left(c\left(\mu\right)+c\left(-\mu\right)\right)V\left(\mu\right)V\left(-\mu\right) \\
&- \sum_{V'\neq V} C_{VVV'}\left(\frac{2}{\sqrt{3}}\right)^{1+h'-2h}\left(\frac{4}{3}\right)^{1-h'}\left(\frac{1}{s^{2-h'}}+\frac{1}{1-h'}\right) \\
&cV'(0) - C_{VVV}\left(\frac{2}{\sqrt{3}}\right)^{1-h}\left(\frac{4}{3}\right)^{1-h}\frac{1}{s^{2-h}}cV(0)\Bigg]|0\rangle \\
&= g_0 C_{VVV}^2 \ln\frac{4}{\sqrt{3}}a + O(y)
\end{aligned} \tag{5.66}$$

where $a \equiv \tan\frac{\pi}{8} = \sqrt{2}-1$ and we used the orthogonality $\langle c\partial cV|cV'\rangle = 0$.

The final term to calculate is then

$$\begin{aligned}
&\langle cV\left(\sqrt{3}\right)cV\left(-\sqrt{3}\right)U^*_{\frac{8}{3}} \\
&\int_0^1 \mathrm{d}s\Bigg[\left(\frac{\mathrm{d}\mu}{\mathrm{d}s}\right)^{2h-1}\frac{s^{2h-2}}{\sqrt{3}}\left(c\left(\mu\right)+c\left(-\mu\right)\right)V\left(\mu\right)V\left(-\mu\right) \\
&- \sum_{V'\neq V} C_{VVV'}\left(\frac{2}{\sqrt{3}}\right)^{1+h'-2h}\left(\frac{4}{3}\right)^{1-h'}\left(\frac{1}{s^{2-h'}}+\frac{1}{1-h'}\right)cV'(0) \\
&- C_{VVV}\left(\frac{2}{\sqrt{3}}\right)^{1-h}\left(\frac{4}{3}\right)^{1-h}\frac{1}{s^{2-h}}cV(0)\Bigg]\rangle \\
&= \int_0^1 ds\Bigg[\frac{4}{\sqrt{3}a}\left(\frac{1}{a^2}-\mu^2\right)\frac{\mathrm{d}\mu}{\mathrm{d}s}\langle V\left(-\frac{1}{a}\right)V\left(\frac{1}{a}\right)V\left(\mu\right)V\left(-\mu\right)\rangle \\
&- g_0\sum_{V'\neq V} C_{VVV'}^2\left(\sqrt{3}a\right)^{h'-1}\left(\frac{1}{s^{2-h'}}+\frac{1}{1-h'}\right) - C_{VVV}^2\frac{g_0}{s}\Bigg] \\
&+ O(y)
\end{aligned} \tag{5.67}$$

It turns out that a tremendous simplification occurs when we use the generic form of the four-point function (2.54)

$$\begin{aligned}
&\int_0^1 ds\Bigg[\frac{4}{\sqrt{3}a}\left(\frac{1}{a^2}-\mu^2\right)\frac{\mathrm{d}\mu}{\mathrm{d}s}\langle V\left(-\frac{1}{a}\right)V\left(\frac{1}{a}\right)V\left(\mu\right)V\left(-\mu\right)\rangle\Bigg], \\
&= \int_0^{\frac{1}{2}} d\xi\,\langle V|\,V(1)V(\xi)\,|V\rangle
\end{aligned} \tag{5.68}$$

with the cross-ratio being explicitly given by

$$\xi(s) = 4a\frac{\mu(s)}{\Big(1 + a\mu(s)\Big)^2} \tag{5.69}$$

What is left to do is to perform the integral over the subtractions in (5.67). Doing the integral in the s variable is complicated but one can use a trick. We imagine employing an auxiliary cutoff $\tilde{\epsilon}$ on the manifestly finite contribution (5.67) to separate the integral into two, one containing the four-point function and the other the subtractions. After we calculate the integral over the subtractions, we get a function of $\tilde{\epsilon}$ and then we take $\epsilon = \xi(\tilde{\epsilon})$ to zero. We have

$$\tilde{\epsilon}(\epsilon) = \sqrt{1 + \frac{2\sqrt{2}}{3}}\epsilon + \sqrt{\frac{1}{4} + \frac{1}{3\sqrt{2}}}\epsilon^2 + O(\epsilon^3) \tag{5.70}$$

Integration over the subtractions in (5.67) is then straightforward and yields

$$\begin{aligned}
&\sum_{V' \neq V} C^2_{VVV'} \left(\sqrt{3}a\right)^{h'-1} \frac{\tilde{\epsilon}^{h'-1}}{h'-1} + C^2_{VVV} \ln\tilde{\epsilon} \\
&= \sum_{V' \neq V} C^2_{VVV'} \left(\frac{\epsilon^{h'-1}}{h'-1} + \frac{1}{2}\delta_{h'=0}\right) \\
&+ C^2_{VVV} (\ln\epsilon + \ln\frac{\sqrt{2}+1}{\sqrt{3}}) + O(\epsilon)
\end{aligned} \tag{5.71}$$

The integral over the cross-ratio ξ has the cutoff ϵ (without the tilde) and we can regulate it by the OPE structure of (5.56)

$$\begin{aligned}
&\int_\epsilon^{\frac{1}{2}} d\xi \Bigg(\langle V|\, V(1)V(\xi)\, |V\rangle - g_0 \sum_{V'} C^2_{VVV'} \frac{1}{\xi^{2-h'}} \Bigg) \\
&+ g_0 \sum_{V' \neq V} C^2_{VVV'} \left(\frac{2^{1-h'}}{h'-1} - \frac{\epsilon^{h'-1}}{h'-1} \right) \\
&+ g_0 C^2_{VVV} \left(\ln 2 - \ln\epsilon\right)
\end{aligned} \tag{5.72}$$

We see that the divergent terms of (5.71) and (5.72) cancel each other out and we can take $\epsilon \to 0$.

Putting all the contributions together we obtain

$$\begin{aligned}\mathcal{A} = \int_0^{\frac{1}{2}} d\xi \Bigg(& \langle V|\, V(1)V(\xi)\, |V\rangle - g_0 \sum_{V'} C^2_{VVV'} \frac{1}{\xi^{2-h'}} \Bigg) \\ & + g_0 \sum_{V' \neq V} C^2_{VVV'} \left(\frac{2^{1-h'}}{h'-1} + \frac{1}{2}\delta_{h'=0} \right) \\ & + g_0 C^2_{VVV} \left(\ln \frac{\sqrt{2}+1}{\sqrt{3}} + \ln \frac{81}{64} + \ln \frac{4a}{\sqrt{3}} \right) + O(y)\end{aligned} \tag{5.73}$$

For convenience we add subtractions in $1-\xi$ (they are not needed when $\xi \in [0, \frac{1}{2}]$)

$$\begin{aligned}\mathcal{A} = \int_0^{\frac{1}{2}} d\xi \Bigg(& \langle V|\, V(1)V(\xi)\, |V\rangle - g_0 \sum_{V'} C^2_{VVV'} \Big(\frac{1}{\xi^{2-h'}} + \frac{1}{(1-\xi)^{2-h'}} \Big) \Bigg) \\ & + g_0 \sum_{V' \neq V} C^2_{VVV'} \left(\frac{1}{h'-1} + \frac{1}{2}\delta_{h'=0} \right) \\ & + g_0 C^2_{VVV} \left(\ln \frac{\sqrt{2}+1}{\sqrt{3}} + \ln \frac{81}{64} + \ln \frac{4a}{\sqrt{3}} \right) + O(y)\end{aligned} \tag{5.74}$$

As a trivial illustration of this formula we can take the exactly marginal free boson, see (2.8.3), $V = \partial X$ for which

$$\langle \partial X|\, \partial X\,(1)\, \partial X\,(\xi)\, |\partial X\rangle = 1 + \frac{1}{\xi^2} + \frac{1}{(1-\xi)^2} \tag{5.75}$$

so that the four-point amplitude vanishes

$$\frac{\mathcal{A}}{g_0} = \left(\int_0^{\frac{1}{2}} \mathrm{d}\xi \right) - 1 + \frac{1}{2} = 0, \tag{5.76}$$

where we used $C_{\partial X \partial X \partial X} = 0$ and that the only V' over which we sum is the identity with $h' = 0$. This vanishing is expected since $\mathcal{A}$ gives a nonzero term to the equation for λ, but for exactly marginal deformation we expect λ to be unrestricted and thus the equation for λ schematically has the form $\lambda 0 = 0$. See also [118] for Sen's evaluation of this amplitude with the same result, where in that context ∂X is associated with the collective degrees of freedom of a D-instanton. Sen passes from the worldsheet to SFT by transforming from the ξ coordinate to the s coordinate, where we know how to handle divergences. We go the other way around since we want to make contact with CPT which is naturally formulated on the worldsheet.

We see a lot of numerical factors in the result (5.78), but what is astonishing is that they all cancel in Δg, see (5.49)

$$\frac{\Delta g}{g_0} = -\frac{\pi^2}{3}\left(\frac{y^3}{C_{VVV}^2} + 3\frac{\tilde{\mathcal{A}}}{C_{VVV}^4}y^4\right) + O(y^5), \tag{5.77}$$

where

$$\begin{aligned}\tilde{\mathcal{A}} = \int_0^{\frac{1}{2}} d\xi\Bigg(\frac{1}{g_0}\left\langle V\right| V(1)V(\xi)\left|V\right\rangle - \sum_{V'} C_{VVV'}^2\Big(\frac{1}{\xi^{2-h'}} + \frac{1}{(1-\xi)^{2-h'}}\Big)\Bigg)\\ + \sum_{V'\neq V} C_{VVV'}^2\left(\frac{1}{h'-1} + \frac{1}{2}\delta_{h'=0}\right)\end{aligned} \tag{5.78}$$

We note that the four-point correlator in (5.78) can be calculated in the near-marginal limit $y \to 0$, that is to this order we can take the zeroth term in its Taylor expansion in y. At first sight it is not clear that the entire moduli space of the four-point amplitude is covered since we have three main regions of ξ: $\xi \in [0,1]$, $\xi \in [1,\infty]$ and $\xi \in [-\infty, 0]$ and the $\xi \in [0, \frac{1}{2}]$ makes a sixth of this moduli space. However, we have a factor of three in (5.49) and we are working to second order and this gives a $\frac{1}{2!}$ from CPT. Thus together, we actually have a $\frac{1}{2!}6\tilde{\mathcal{A}}$ in the result (5.77), thus covering the entire moduli space. It is possible to extend the result to manifestly cover the entire moduli space, that is to have an integral over $\xi \in [-\infty, \infty]$. This is where additional subtractions are needed, for example we need the $1-\xi$ subtractions to extend the integral to $\xi \in [0, 1]$. We won't present the result here since it's not particularly illuminating.

We note that K does not decouple from the expression (5.46) for the SFT coupling

$$\lambda = \frac{y}{C_{VVV}} + \frac{1}{C_{VVV}}\left(\frac{2\tilde{\mathcal{A}}}{C_{VVV}^2} + \ln K\right) y^3 + O(y^4) \tag{5.79}$$

This is expected since λ is just an auxilliary normalisation factor, which is heavily scheme dependent.

5.4 Schnabl gauge

We work in Schnabl gauge $\mathcal{B}_0\Psi = 0$ and consider the deformation by a single weakly relevant operator with $\Psi_1 = \lambda\sqrt{\Omega}cV\sqrt{\Omega}$, which is expressed in sliver frame. For simplicity we assume that the only relevant operators contained in the VV OPE are $\mathbb{1}$ and V.

5.4.1 Oblique projectors

Since we work in sliver frame, we can no longer compute action of the projector P by BPZ projecting as we've done so far, see for example the nontrivial operator transformation properties (4.88), which break BPZ orthogonality. This means that P is an oblique and not orthogonal projector. To deal with this, we invent a method of identifying the level expansion (from which we read off the projection) of an arbitrary number of singular operators from correlators. To do this, we consider the elementary identity

$$\mathcal{L}_0 = \frac{\mathrm{d}}{\mathrm{d}z} z^{\mathcal{L}_0}\Big|_{z=1}, \tag{5.80}$$

which means that level expansion can be read off by reading powers of z after scaling with $z^{\mathcal{L}_0}$. This becomes clear after considering an example of projecting onto the ghost number 2 state $\sqrt{\Omega}c\partial cV\sqrt{\Omega}$, where the projection of a ghost number 2 state Φ is

$$P(\Phi) = -\frac{1}{g_0}\left(\frac{\pi}{2}\right)^{2h-2}\left(\frac{1}{z^{h-1}}\operatorname{Tr}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}(z^{\mathcal{L}_0}\phi)\Big\}\right)\Big|_{z=0}\sqrt{\Omega}c\partial cV\sqrt{\Omega} \tag{5.81}$$

To see why this is the correct definition, we put $\Phi = \sqrt{\Omega}cV\Omega cV\sqrt{\Omega}$, which we know should contain a $-C_{VVV}c\partial cV$ by the VV OPE (this will be shown in more detail in (5.4.2)) and now we use scaling

$$z^{\mathcal{L}_0}\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} = z^{2h-2}\sqrt{\Omega}cV\Omega^z cV\sqrt{\Omega} \tag{5.82}$$

Then the overlap of (5.82) with $\sqrt{\Omega}cV\sqrt{\Omega}$ is

$$-z^{2h-2}\operatorname{Tr}\Big\{\sqrt{\Omega}cV\Omega cV\Omega^z cV\sqrt{\Omega}\Big\} = -C_{VVV}g_0\left(\frac{\pi}{2}\right)^{2-2h} z^{h-1} + O(z^h), \tag{5.83}$$

which which one sees that after properly normalising, we have

$$P(\sqrt{\Omega}cV\Omega cV\sqrt{\Omega}) = -C_{VVV}\sqrt{\Omega}c\partial cV\sqrt{\Omega} \tag{5.84}$$

as it should. From this calculation it is clear that the projector formula works only if there is only one state of weight h that has nonzero overlap with cV. If this is violated, one would try to remove this degeneracy by considering products with operators that differentiate between the degenerate states, but this won't be necessary for our purposes. After using (5.84), the equation (5.7) then becomes

$$\lambda(y - \lambda C_{VVV})\sqrt{\Omega}c\partial cV\sqrt{\Omega} + \ldots = 0, \tag{5.85}$$

so that $\lambda = \frac{y}{C_{VVV}} + O(y^2)$. It is interesting to compare (5.13) to (5.85), which are different but at the leading order give the same λ as they should since we haven't fixed a gauge yet and (4.89) establishes leading order frame independence. It is interesting that the equation (5.85) is the same as the vanishing of the beta function (3.47). This correspondence shouldn't be taken too literally since the renormalisation procedures are different in OSFT and CFT so only the leading order matches meaning that we are not computing the usual CPT beta functions in some conventional scheme.

5.4.2 Second order string field

Since we fixed Schnabl gauge, we have

$$\Psi_2 = -\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\Psi_1^2, \tag{5.86}$$

where the star product is

$$\Psi_1^2 = \lambda^2\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} \tag{5.87}$$

We now need to level expand (5.87) in order to be able to work with the oblique projector $\bar{P}$. We do so in two ways: first we use the VV OPE, which is viable only to second order, where we have only two V insertions and thus further motivate the projectors of (5.4.1), which work to all orders and then as a second way we use the projectors analogous to the one in (5.4.1) as a shortcut.

As for the first way, inspired by the VV OPE containing only $\mathbb{1}$ and V, we write

$$\Psi_1^2 = \lambda^2\left(\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} \mp \sqrt{\Omega}c\Omega c\sqrt{\Omega} \mp C_{VVV}\sqrt{\Omega}c\Omega cV\sqrt{\Omega}\right) \tag{5.88}$$

We then Taylor expand the wedge state

$$\Omega = e^{-K} = 1 - K + \frac{1}{2!}K^2 - \dots \tag{5.89}$$

and collect the terms of weight less than or equal to zero with the rest reabsorbed into the previous form (5.88) obtaining

$$\Psi_1^2 = \lambda^2\left(\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} \mp \sqrt{\Omega}(-cKc + \frac{1}{2!}cK^2c)\sqrt{\Omega} \pm C_{VVV}\sqrt{\Omega}cKcV\sqrt{\Omega}\right) \tag{5.90}$$

We observe that a weight zero state $\sqrt{\Omega}cK^2c\sqrt{\Omega}$ emerges. This zero weight state is in Schnabl gauge so formally one encounters an expression $\frac{0}{0}$. This means that the solution cannot be found in Schnabl gauge [91]. A slight modification of the gauge to account for the zero mode solves this issue. To see this, we rewrite the weight zero state by a chain of identities

$$\begin{aligned}
\sqrt{\Omega}cK^2c\sqrt{\Omega} &= Q\sqrt{\Omega}\Big(BcKc + cKcB + Q(\ldots)\Big)\sqrt{\Omega} && (5.91)\\
\{B, c\} &= 1 && (5.92)\\
\sqrt{\Omega}cK^2c\sqrt{\Omega} &= Q\sqrt{\Omega}\Big(\{K, c\} - 2cKBc + Q(\ldots)\Big)\sqrt{\Omega} && (5.93)\\
\sqrt{\Omega}cKBc\sqrt{\Omega} &= Q\sqrt{\Omega}Bc\sqrt{\Omega} && (5.94)\\
\sqrt{\Omega}cK^2c\sqrt{\Omega} &= Q\sqrt{\Omega}\Big(\{K, c\} + Q(\ldots)\Big)\sqrt{\Omega} && (5.95)
\end{aligned}$$

Fixing our gauge so that the Q-exact term $Q(\ldots)$ is absent one obtains for Ψ_2 after explicitly diving by Q on the weight zero state the expression

$$\begin{aligned}
\Psi_2 = -\lambda^2\bar{P}\bigg(\frac{\mathcal{B}_0}{\mathcal{L}_0}\Big(&\sqrt{\Omega}cV\Omega cV\sqrt{\Omega} + \sqrt{\Omega}\Big(\mp(-cKc) - \frac{1}{2!}cK^2c\Big)\sqrt{\Omega}\\
&\pm C_{VVV}\sqrt{\Omega}cKcV\sqrt{\Omega}\Big) + \frac{1}{2!}\sqrt{\Omega}\{K, c\}\sqrt{\Omega}\bigg)
\end{aligned} \tag{5.96}$$

Which after the action of $\mathcal{B}_0$ and use of the anticommutator (5.92) takes the form

$$\begin{aligned}
\Psi_2 = -\lambda^2\bar{P}\bigg(\frac{1}{\mathcal{L}_0}\Big(&\sqrt{\Omega}cBV\Omega cV\sqrt{\Omega} + \sqrt{\Omega}(\mp c + cKBc)\sqrt{\Omega}\\
&\mp C_{VVV}\sqrt{\Omega}cV\sqrt{\Omega}\Big) + \frac{1}{2!}\sqrt{\Omega}\{K, c\}\sqrt{\Omega}\bigg)
\end{aligned} \tag{5.97}$$

We now use explicit division by $\mathcal{L}_0$ on the plus branch of the states with $\mp$ in front and use (4.139) on the rest (remembering that $\mathcal{L}_0$ is a dilatation generator) and get

$$\begin{aligned}
\Psi_2 &= -\lambda^2\bar{P}\bigg(\int_0^1 \frac{ds}{s}\,\Big(s^{2h-1}\sqrt{\Omega}cBV\Omega^s cV\sqrt{\Omega} - \sqrt{\Omega}(s^{-1}c - cKBc)\sqrt{\Omega} \qquad (5.98)\\
&\quad -C_{VVV}s^{h-1}\sqrt{\Omega}cV\sqrt{\Omega}\Big) - \sqrt{\Omega}c\sqrt{\Omega} + \frac{1}{h-1}C_{VVV}\sqrt{\Omega}cV\sqrt{\Omega} + \frac{1}{2!}\sqrt{\Omega}\{K, c\}\sqrt{\Omega}\bigg)
\end{aligned}$$

Applying the projector and using the identity (5.94) we get the final expression for the second order string field

$$\Psi_2 = -\lambda^2\Bigg(\int_0^1 \frac{ds}{s}\,\Big(s^{2h-1}\sqrt{\Omega}cBV\Omega^s cV\sqrt{\Omega} - \sqrt{\Omega}\Big(s^{-1}c - Q(Bc)\Big)\sqrt{\Omega}$$
$$- C_{VVV}s^{h-1}\sqrt{\Omega}cV\sqrt{\Omega}\Big) - \sqrt{\Omega}c\sqrt{\Omega} + \frac{1}{2!}\sqrt{\Omega}\{K,c\}\sqrt{\Omega}\Bigg) \tag{5.99}$$

and this is essentially a generalisation of the KORZ result [85] for the marginal case.

The level zero state is a peculiarity of Schnabl gauge. To show how this state emerges let us consider a family of gauges containing Schnabl gauge as a limit with propagators of the form

$$h = -\bar{P}\frac{\mathcal{B}_0 - \delta\mathcal{B}_0^\star}{\mathcal{L}_0 + \epsilon} = -\bar{P}\frac{(1-\delta)\mathcal{B}_0 + \delta\mathcal{B}^-}{\mathcal{L}_0 + \epsilon} \tag{5.100}$$

where δ and ϵ are parameters. We now act with this propagator on the term $-\sqrt{\Omega}cKc\sqrt{\Omega}$ which in the previous analysis turned into a tachyon subtraction. The interest is in seeing whether the states at level 0 emerge from this term by action of the propagator in Schnabl gauge limit since this is the only subtraction relevant for other gauges (one does not subtract level zero states). We start by applying the numerator of (5.100)

$$-\Big((1-\delta)\mathcal{B}_0 + \delta\mathcal{B}^-\Big)\sqrt{\Omega}cKc\sqrt{\Omega} = \sqrt{\Omega}c\sqrt{\Omega} + \frac{\delta}{2}\Big\{B, \sqrt{\Omega}cKc\sqrt{\Omega}\Big\} \tag{5.101}$$

so that dividing by $\mathcal{L}_0 + \epsilon$ (remember, B commutes with K) and taking the limit $\delta \to 0_+$ with δ and ϵ equal (this is an ambiguity, the level zero state emerges only for $\delta = O(\epsilon)$) gives

$$\frac{1}{\epsilon - 1}\sqrt{\Omega}c\sqrt{\Omega} + \frac{\delta}{2\epsilon}\Big\{B, \sqrt{\Omega}cKc\sqrt{\Omega}\Big\} \to$$
$$-\sqrt{\Omega}c\sqrt{\Omega} + \frac{1}{2}\sqrt{\Omega}\Big(\{K,c\} - 2Q(Bc)\Big)\sqrt{\Omega} \tag{5.102}$$

where we have used (5.92) and (5.94). Which is precisely the subtraction outside the integral in (5.99) modulo a Q-exact term which we can again gauge fix.

Now we show the alternative way of finding the regular string field Ψ_2 based on (5.80). We can simply use (5.81) which means repeating (5.82) and (5.83) to see that one should subtract $-C_{VVV}\sqrt{\Omega}cV\sqrt{\Omega}$ from Ψ_1^2. For the tachyon subtraction we analogously compute

$$-z^{2h-2}\frac{1}{g_0}\left(\frac{\pi}{2}\right)^{-2}\mathrm{Tr}\Big\{\sqrt{\Omega}c\Omega cV\Omega^z cV\sqrt{\Omega}\Big\} = -z^{-1} - z^0 + O(z^1) \tag{5.103}$$

where the normalisation was chosen such that the projection of $\sqrt{\Omega}c\sqrt{\Omega}$ onto $c\partial c$ has coefficient one. The previous analysis immediately lets us interpret (5.103) as the apperance of states $\sqrt{\Omega}cKc\sqrt{\Omega}$ and $\sqrt{\Omega}cK^2c\sqrt{\Omega}$ that were present in Ψ_1^2 with the correct coefficients. This is so because

$$\mathrm{Tr}\Big\{\sqrt{\Omega}c\Omega cK^2c\sqrt{\Omega}\Big\} = \lim_{\alpha\to 0}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,\mathrm{Tr}\Big\{\sqrt{\Omega}c\Omega c\Omega^\alpha c\sqrt{\Omega}\Big\} = 2\Big(\frac{2}{\pi}\Big)^2 \tag{5.104}$$

shows that $\sqrt{\Omega}cK^2c\sqrt{\Omega}$ is present with a $\frac{1}{2}$ in front. This motivates the definition of ghost number 2 projectors T onto the tachyon and $\hat{T}$ onto its level zero descendant.

$$T(\Phi) = -\frac{1}{g_0}\left(\frac{\pi}{2}\right)^{-2}\Big(z\,\mathrm{Tr}\Big\{\sqrt{\Omega}c\sqrt{\Omega}(z^{\mathcal{L}_0}\Phi)\Big\}\Big)\Big|_{z=0}\sqrt{\Omega}c\partial c\sqrt{\Omega} \tag{5.105}$$

$$\hat{T}(\Phi) = \frac{1}{2g_0}\left(\frac{\pi}{2}\right)^{-2}\Big(\mathrm{Tr}\Big\{\sqrt{\Omega}c\sqrt{\Omega}(z^{\mathcal{L}_0}\bar{T}(\Phi))\Big\}\Big)\Big|_{z=0}\sqrt{\Omega}cK^2c\sqrt{\Omega} \tag{5.106}$$

One may check that with the explicit expressions for projectors $P(\Psi_2)$ holds as required for the consistency of our perturbative procedure (Q has to commute with P).

5.4.3 Third order string field

For the higher order calculation we need to regularize a product of at least three Vs. Since our theory is not free, one would use something like the generalised Wick theorem [27] in order to guess how to subtract singular states. We will not follow this route in the present paper since we do not yet have a satisfactory string field normal ordering and besides we have $\frac{1}{\mathcal{L}_0}$ acting on products like $\sqrt{\Omega}cV\Omega cV\Omega cV\sqrt{\Omega}$ which means that we have three insertions going together simultaneously and we are not sure whether the generalised Wick theorem and a satisfactory string field normal ordering would be enough to regularise such a product. Instead we will continue using the trick based on (5.80).

Continuing with our perturbative procedure we have for the third order string field the expression

$$\Psi_3 = -\bar{P}\frac{\mathcal{B}_0}{\mathcal{L}_0}\{\Psi_1,\Psi_2\} \tag{5.107}$$

We start evaluating it by writing down the anticommutator

$$
\begin{aligned}
\{\Psi_1, \Psi_2\} = -\lambda^3 \Bigg(\int_0^1 \frac{ds}{s} \, \Bigg(& s^{2h-1} \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c B V \Omega^s c V \sqrt{\Omega} \Big\} \\
& - s^{-1} \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c \sqrt{\Omega} \Big\} - \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} Q(Bc) \sqrt{\Omega} \Big\} \\
& - C_{VVV} s^{h-1} \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c V \sqrt{\Omega} \Big\} \Bigg) - \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c \sqrt{\Omega} \Big\} \\
& + \frac{1}{2!} \Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} \{K, c\} \sqrt{\Omega} \Big\} \Bigg)
\end{aligned}
\tag{5.108}
$$

and we continue by writing down the projections needed to regularise

$$
\begin{aligned}
& P\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c B V \Omega^s c V \sqrt{\Omega} \Big\} \Bigg) \equiv -\Upsilon(s) \sqrt{\Omega} c \partial c V \sqrt{\Omega} = \\
& -\frac{2}{g_0} \left(\frac{\pi}{2} \right)^{2h-2} \left(\frac{1}{z^{h-1}} z^{3h-2} \, \mathrm{Tr} \Big\{ \sqrt{\Omega} c V \Omega c V \Omega^z c B V \Omega^{sz} c V \sqrt{\Omega} \Big\} \right) \Bigg|_{z=0} \sqrt{\Omega} c \partial c V \sqrt{\Omega} \\
& T\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c B V \Omega^s c V \sqrt{\Omega} \Big\} \Bigg) = -2 C_{VVV} \Bigg(s(s+1) \Bigg)^{-h} \sqrt{\Omega} c \partial c \sqrt{\Omega} \\
& \hat{T}\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c B V \Omega^s c V \sqrt{\Omega} \Big\} \Bigg) = -C_{VVV} \frac{1}{s} \Bigg(s(s+1) \Bigg)^{1-h} \sqrt{\Omega} c K^2 c \sqrt{\Omega} \\
& P\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c \sqrt{\Omega} \Big\} \Bigg) = -2 \sqrt{\Omega} c \partial c V \sqrt{\Omega} \\
& T\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c \sqrt{\Omega} \Big\} \Bigg) = \hat{T}\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c \sqrt{\Omega} \Big\} \Bigg) = 0 \\
& P\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} Q(Bc) \sqrt{\Omega} \Big\} \Bigg) = T\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} Q(Bc) \sqrt{\Omega} \Big\} \Bigg) \\
& = \hat{T}\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} Q(Bc) \sqrt{\Omega} \Big\} \Bigg) = 0 \\
& P\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c V \sqrt{\Omega} \Big\} \Bigg) = -2 C_{VVV} \sqrt{\Omega} c \partial c V \sqrt{\Omega} \\
& T\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c V \sqrt{\Omega} \Big\} \Bigg) = -2 \sqrt{\Omega} c \partial c \sqrt{\Omega} \\
& \hat{T}\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} c V \sqrt{\Omega} \Big\} \Bigg) = -\sqrt{\Omega} c K^2 c \sqrt{\Omega} \\
& P\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} \{K, c\} \sqrt{\Omega} \Big\} \Bigg) = T\Bigg(\Big\{ \sqrt{\Omega} c V \sqrt{\Omega}, \sqrt{\Omega} \{K, c\} \sqrt{\Omega} \Big\} \Bigg)
\end{aligned}
$$

$$= \hat{T}\Big(\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}\{K,c\}\sqrt{\Omega}\Big\}\Big) = 0 \tag{5.109}$$

Now (5.108) can be rewritten as

$$\begin{aligned}
&\{\Psi_1, \Psi_2\} = -\lambda^3\Bigg(\int_0^1 ds\ \Bigg(s^{2h-2}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}cBV\Omega^s cV\sqrt{\Omega}\Big\}\\
&-s^{-2}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}c\sqrt{\Omega}\Big\} - s^{-1}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}Q(Bc)\sqrt{\Omega}\Big\}\\
&-C_{VVV}s^{h-2}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}cV\sqrt{\Omega}\Big\} \pm s^{h-2}\Big(C_{VVV}(s+1)^{1-h} - 1\Big)\sqrt{\Omega}cK^2c\sqrt{\Omega}\\
&\pm\Big(s^{2h-2}\Upsilon(s) - 2s^{-2} - 2\left(C_{VVV}\right)^2 s^{h-2}\Big)\sqrt{\Omega}c\partial cV\sqrt{\Omega}\Big)\\
&\pm 2C_{VVV}s^{h-2}\Big((s+1)^{-h} - 1\Big)\sqrt{\Omega}c\partial c\sqrt{\Omega}\Big)\\
&-\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}c\sqrt{\Omega}\Big\} \pm 2\sqrt{\Omega}c\partial cV\sqrt{\Omega} + \frac{1}{2!}\Big\{\sqrt{\Omega}cV\sqrt{\Omega}, \sqrt{\Omega}\{K,c\}\sqrt{\Omega}\Big\}\Bigg)
\end{aligned}$$

so that after properly applying the propagator as in the second order case one obtains

$$\begin{aligned}
&\Psi_3 = \lambda^3\int_0^1 dt\int_0^1 ds\ \Big(t^{3h-2}s^{2h-2}\sqrt{\Omega}cB\Big(V\Omega^tV\Omega^{st}V + V\Omega^{st}V\Omega^tV\Big)c\sqrt{\Omega}\\
&-t^{h-2}(s^{-2}+1)\sqrt{\Omega}cB\Big\{V,\Omega^t\Big\}c\sqrt{\Omega} - t^{h-1}s^{-1}\sqrt{\Omega}cB\Big\{V,K\Omega^t\Big\}c\sqrt{\Omega}\\
&-2C_{VVV}t^{2h-2}s^{h-2}\sqrt{\Omega}cBV\Omega^tVc\sqrt{\Omega} - 2C_{VVV}(t^{-2}+1)s^{h-2}\Big((s+1)^{-h} - 1\Big)\sqrt{\Omega}c\sqrt{\Omega}\\
&-t^{h-1}\Big(s^{2h-2}\Upsilon(s) - 2(s^{-2}+1) - 2\left(C_{VVV}\right)^2 s^{h-2}\Big)\sqrt{\Omega}cV\sqrt{\Omega}\Big)\\
&-2C_{VVV}s^{h-2}\Big((s+1)^{1-h} - 1\Big)\sqrt{\Omega}\Big(t^{-1}Q(Bc) + \{K,c\}\Big)\sqrt{\Omega}\\
&+\frac{1}{2!}t^{h-1}\sqrt{\Omega}\Big(cV\Omega^t[c,B] - [c,B]\Omega^tcV + cBV\Omega^t\{K,c\} + \{K,c\}\Omega^tBcV\Big)\sqrt{\Omega}\Big)
\end{aligned} \tag{5.110}$$

where we brought everything under the t-integral since $\int_0^1 dt = 1$.

5.4.4 Ellwood invariant

We calculate the shift in the g-function using the KMS correspondence (4.74), where we use the fact that the g-function is the coefficient of $\mathbb{1}$, giving

$\Delta g = \mathrm{Tr}_{\mathbb{1}}\, \Psi$. In the calculation we use the following identities

$$\begin{aligned} \mathrm{Tr}_\phi \sqrt{\Omega} c B \Omega^n c \sqrt{\Omega} &= \mathrm{Tr}_\phi \sqrt{\Omega} c \sqrt{\Omega} &(5.111)\\ \mathrm{Tr}_\phi \sqrt{\Omega} \Omega^n K c \sqrt{\Omega} &= -\frac{1}{n+1} \mathrm{Tr}_\phi \sqrt{\Omega} \Omega^n c \sqrt{\Omega}, &(5.112)\end{aligned}$$

which follow from the fact that the Ellwood invariant is invariant under the action of $\mathcal{B}^-$ and $\mathcal{L}^-$. Taking the solution (5.99) and using Q-invariance (the terms with $Q(Bc)$ give a zero contribution) and cyclicity (the term with $\frac{1}{2!}\{K, c\}$ gives the same contribution as with Kc) of the Ellwood invariant we get with the help of (5.111),(5.112) the expression

$$\begin{aligned} g\left(\Psi_2\right) = -2\pi i \lambda^2 \frac{i}{2\pi} \Bigg(\int_0^1 \frac{ds}{s} \; \Big(s^{2h-1} \langle V\left(-\frac{s}{2}\right) V\left(\frac{s}{2}\right) \rangle_{C_{s+1}} - g_0 s^{-1} - \\ - C_{VVV} s^{h-1} \langle V \rangle_{C_1} \Big) - g_0 - g_0 \Bigg), \end{aligned} \quad (5.113)$$

where $g\left(\Psi_2\right)$ means the contribution of Ψ_2 to the shift of the g-function. Using the correlator (A.11) and the vanishing of the 1-pt function of V we get

$$g\left(\Psi_2\right) = g_0 \lambda^2 \left(\int_0^1 \frac{ds}{s} \; \left(s^{2h-1} \left(\csc\left(\frac{\pi s}{s+1}\right) \frac{\pi}{s+1} \right)^{2h} - s^{-1} \right) - 2 \right) \quad (5.114)$$

expanding in y one finds

$$\begin{aligned} g\left(\Psi_2\right) &= g_0 \frac{1}{\left(C_{VVV}\right)^2} \left(\int_0^1 ds \; \left(\left(\csc\left(\frac{\pi s}{s+1}\right) \frac{\pi}{s+1} \right)^2 - s^{-2} \right) - 2 \right) y^2 + O(y^3) \\ &= O(y^3) &(5.115)\end{aligned}$$

where a cancellation of divergences occured in the coefficient of y^2 as the integral gave zero as is needed for consistency with (3.61). Notice that the zero-mode of $\mathcal{L}_0$ gave a nontrivial contribution.

We begin by extracting the term in (5.115) proportional to y^3 by writing down the next term in the expansion. The result is

$$\begin{aligned} g\left(\Psi_2\right) &= -2 g_0 \frac{1}{\left(C_{VVV}\right)^2} \left(\int_0^1 ds \; \left(\csc\left(\frac{\pi s}{s+1}\right) \frac{\pi}{s+1} \right)^2 \ln \left(\frac{\pi s}{s+1} \csc\left(\frac{\pi s}{s+1}\right) \right) \right) y^3 + \\ &\quad + O(y^4) &(5.116)\end{aligned}$$

and it is finite giving further credit to the regularity of Ψ_2.

Now we need to consider the contribution from Ψ_3, concretely we plug in (5.110) into the g-function obtaining with the use of equations (A.11), (A.12), (5.111), (5.112)

$$
\begin{aligned}
&g\left(\Psi_3\right) = -2g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt\ \int_0^1 ds\ t^{3h-2}s^{2h-2}\pi^{3h}\\
&\left(\left(\frac{1}{1+t+st}\right)^3 \csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h\\
&-\pi^{2h}t^{2h-2}s^{h-2}\left(\frac{1}{t+1}\csc\left(\frac{\pi t}{t+1}\right)\right)^{2h} - (t^{-2}+1)\Big(s^{2h-2}(s(s+1))^{-h} - s^{h-2}\Big)\\
&+2s^{h-2}\Big((s+1)^{1-h}-1\Big)\Bigg]
\end{aligned}
\tag{5.117}
$$

Combining (5.116) with (5.117) and expanding in y while plugging in the fixed point SFT coupling (5.13) we find for the $O(y^3)$ contribution to the g-function the result

$$
\begin{aligned}
&-2g_0\frac{1}{\left(C_{VVV}\right)^2}\Bigg[\int_0^1 dt\ \int_0^1 ds\\
&\left(\frac{\pi^3 t\csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(s+1)t}{1+t+st}\right)}{(1+t+st)^3} - \frac{\pi^2\csc^2\left(\frac{\pi t}{t+1}\right)}{s(t+1)^2} + \frac{1}{s+1}(t^{-2}+1)\right)\\
&+\int_0^1 ds\ \left(\csc\left(\frac{\pi s}{s+1}\right)\frac{\pi}{s+1}\right)^2 \ln\left(\frac{\pi s}{s+1}\csc\left(\frac{\pi s}{s+1}\right)\right)\Bigg]y^3 = -g_0\frac{\pi^2}{3}\frac{y^3}{\left(C_{VVV}\right)^2}
\end{aligned}
\tag{5.118}
$$

reproducing (3.61).

We could have proceeded differently and used the fact that the divergence structure is very indicative of the structure of subtractions. Such an approach would effectively lead to regularisation by analytic continuation as done for example in [115]. We start by writing down the contribution without subtractions

$$
\begin{aligned}
&g\left(\Psi_3^{singular}\right) = 2\pi i\lambda^3\frac{i}{2\pi}\Bigg[\int_0^1 dt\ t^{3h-2}\int_0^1 ds\ s^{2h-2}\Big(\langle V(0)\,V(st)\,V(t)\rangle_{C_{1+t+st}}\\
&+\langle V(0)\,V(t)\,V(st)\rangle_{C_{1+t+st}}\Big)\Bigg] = -2g_0\lambda^3 C_{VVV}\left(\int_0^1 dt\ t^{3h-2}\int_0^1 ds\ s^{2h-2}\right.\\
&\left.\pi^{3h}\left[\left(\frac{1}{1+t+st}\right)^3\csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h\right]
\end{aligned}
$$

Asymptotically expanding around $s \sim 0$ we see that the following subtraction in the s-channel should be made

$$g\left(\Psi_3^{singular}\right) = -2g_0\lambda^3 C_{VVV}\Bigg[\Bigg(\int_0^1 dt \int_0^1 ds\; t^{3h-2}s^{2h-2}\pi^{3h}$$
$$\left(\left(\frac{1}{1+t+st}\right)^3 \csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h$$
$$\mp \pi^{2h}s^{h-2}t^{2h-2}\left(\frac{1}{t+1}\csc\frac{\pi t}{t+1}\right)^{2h}\Bigg]$$
$$= -2g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt \int_0^1 ds\; t^{3h-2}s^{2h-2}\pi^{3h}$$
$$\left(\left(\frac{1}{1+t+st}\right)^3 \csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h$$
$$- \pi^{2h}t^{2h-2}s^{h-2}\left(\frac{1}{t+1}\csc\left(\frac{\pi t}{t+1}\right)\right)^{2h}\Bigg] \tag{5.119}$$

where we used the fact that the plus branch of the subtraction should be set to zero. This is easily seen by realising that this subtraction comes from the $\sqrt{\Omega}cV\sqrt{\Omega}$ in (5.99) which has the corresponding $\sqrt{\Omega}cV\sqrt{\Omega}$ outside the integral projected out. This result is effectively given by the following analytic continuation

$$\int_0^1 ds\; \frac{1}{s} = 0 \tag{5.120}$$

We note that in the context of [118, 119, 120], this continuation corresponds to keeping path integration over D-instanton collective coordinates last. We continue by regularising in the t-channel by making an asymptotic expansion around $t \sim 0$ of the $s-t$ integrand to obtain

$$g\left(\Psi_3^{singular}\right) = -2g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt \int_0^1 ds\; t^{3h-2}s^{2h-2}\pi^{3h}$$
$$\left(\left(\frac{1}{1+t+st}\right)^3 \csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h$$
$$- \pi^{2h}t^{2h-2}s^{h-2}\left(\frac{1}{t+1}\csc\left(\frac{\pi t}{t+1}\right)\right)^{2h} \mp \frac{1}{t^2}\left(s^{2h-2}(s(s+1))^{-h} - s^{h-2}\right)\Bigg] \tag{5.121}$$

Using the in SFT well-known analytic continuation

$$\int_0^1 dt\; \frac{1}{t^2} = -1 \tag{5.122}$$

which can be motivated by realising that it is the Schwinger parametrisation of $\frac{1}{\mathcal{L}_0}$ on level -1 states, we obtain the regularised result

$$g\left(\Psi_3^{singular}\right) = -2g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt\ \int_0^1 ds\ t^{3h-2}s^{2h-2}\pi^{3h}$$
$$\left(\left(\frac{1}{1+t+st}\right)^3 \csc\left(\frac{\pi t}{1+t+st}\right)\csc\left(\frac{\pi st}{1+t+st}\right)\csc\left(\frac{\pi(1+s)t}{1+s+st}\right)\right)^h$$
$$-\pi^{2h}t^{2h-2}s^{h-2}\left(\frac{1}{t+1}\csc\left(\frac{\pi t}{t+1}\right)\right)^{2h} - (t^{-2}+1)\left(s^{2h-2}(s(s+1))^{-h} - s^{h-2}\right)\Bigg] \tag{5.123}$$

Which is (5.117) up to an $O(y^4)$ contribution from zero modes. It is not clear to us how to obtain the zero mode contribution by asymptotically expanding since there is a mixing between the contributions of descendants which makes the procedure less transparent for non-leading singularities. We continue with pseudo-Schnabl gauge where the zero modes are absent so the regularisation by analytic continuation does not encounter such subtleties.

5.5 Pseudo-Schnabl gauge

In (5.3), we have seen the usefulness of the Berkovits-Schnabl trick to evaluate the amplitude $\mathcal{A}$. The idea behind the Berkovits-Schnabl trick is to have a propagator acting on a string of local operator insertions. Writing the general ansatz (see [115]) $\Psi = \sum_{n=1}\hat{U}_{n+1}\hat{\Psi}_n\left|0\right\rangle$, the equations of motion (4.42) become

$$\left(Q\hat{U}_2\hat{\Psi}_1 + Q\hat{U}_3\hat{\Psi}_2 + \hat{U}_3\hat{\Psi}_1^2 + \dots\right)\left|0\right\rangle = 0, \tag{5.124}$$

where the $\hat{U}_3\hat{\Psi}_1^2\left|0\right\rangle$ is a shorthand for $\hat{U}_2\hat{\Psi}_1\left|0\right\rangle * \hat{U}_2\hat{\Psi}_1\left|0\right\rangle$. Having the propagator act behind the U factors amounts to the gauge fixing $\mathcal{B}_0\hat{\Psi}_n = 0$, which is *pseudo-Schnabl gauge*. For the propagator acting in a well-defined way, we should split the equation (5.124) using projectors

$$\left(Q\hat{U}_2P\hat{\Psi}_1 + Q\hat{U}_3P\hat{\Psi}_2 + \hat{U}_3P\hat{\Psi}_1^2 + \dots\right)\left|0\right\rangle = 0 \tag{5.125}$$
$$\left(Q\hat{U}_2\bar{P}\hat{\Psi}_1 + Q\hat{U}_3\bar{P}\hat{\Psi}_2 + \hat{U}_3\bar{P}\hat{\Psi}_1^2 + \dots\right)\left|0\right\rangle = 0 \tag{5.126}$$

Using $\bar{P}\hat{\Psi}_1 = 0$ and $P\hat{\Psi}_2 = 0$, we have

$$\left(Q\hat{U}_2\hat{\Psi}_1 + \hat{U}_3P\hat{\Psi}_1^2 + \dots\right)\left|0\right\rangle = 0 \tag{5.127}$$
$$\left(Q\hat{U}_3\hat{\Psi}_2 + \hat{U}_3\bar{P}\hat{\Psi}_1^2 + \dots\right)\left|0\right\rangle = 0 \tag{5.128}$$

We can solve (5.128) thanks to the pseudo-Schnabl gauge condition and the commuting of Q and wedges

$$\hat{\Psi}_2 = -\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2, \tag{5.129}$$

but this leaves (5.127) unsolved, which is now not a single equations but an infinity of equations thanks to the different wedge lengths present. This means that we can't simply determine the coupling λ by solving a single equations. We thus see that simply moving propagators behind wedge states doesn't give a solution to (5.124). We note that for exactly marginal deformations, where we don't need projectors and $Q\hat{\Psi}_1 = 0$, pseudo-Schnabl gauge gives a perfectly fine solution [86, 115]. We are not sure to what extend the pseudo-Schnabl gauge string field can be modified to give a proper solution, but we now demonstrate that despite being anomalous, it gave us the correct g-function (3.61). To show why this might happen, we consider the shift in the g function $\mathrm{Tr}_{\mathbb{1}}\,\Psi$, where Ψ is the Schnabl gauge solution and we start applying the Berkovits-Schnabl trick to it order by order. We start with the contribution $g(\Psi_2)$

$$\begin{aligned} g(\Psi_2) &= \mathrm{Tr}_{\mathbb{1}}\,\Psi_2 = -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\hat{\Psi}_1^2 = -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3 P\hat{\Psi}_1^2 \\ &\quad -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\Big\{\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P},Q\Big\}\hat{\Psi}_1^2 = -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\Big\{\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P},Q\Big\}\hat{\Psi}_1^2 \\ &= -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}Q\hat{\Psi}_1^2 - \mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3 Q\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2 \\ &= -\,\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3 Q\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2 = -\,\mathrm{Tr}_{\mathbb{1}}\,\hat{U}_3\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2, \end{aligned}$$

where we used

$$\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3 P\hat{\Psi}_1^2 = 0, \tag{5.130}$$

since $\langle V\rangle = 0$ and

$$\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}Q\hat{\Psi}_1^2 = \mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\Big[Q\hat{\Psi}_1,\hat{\Psi}_1\Big] = 0, \tag{5.131}$$

as follows from the cyclicity of the Ellwood invariant. Also in

$$\mathrm{Tr}_{\mathbb{1}}\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{U}_3 Q\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2 = \mathrm{Tr}_{\mathbb{1}}\,\hat{U}_3\frac{\mathcal{B}_0}{\mathcal{L}_0}\bar{P}\hat{\Psi}_1^2,$$

one acts with Q to the left and uses the fact that the bulk insertion is physical due to dressing. Thus we have proved that given the same coupling λ, the Schnabl gauge and the pseudo-Schnabl gauge string field give the same g-function at second order. We continue by constructing the pseudo gauge string fields and then compute the shift in the g-function.

5.5.1 Second order string field

By the general formula for star products of wedges with insertions (can be seen geometrically)

$$\hat{U}_r\Psi(x)\,|0\rangle * \hat{U}_s\phi(y)\,|0\rangle = \hat{U}_{r+s-1}\Psi\left(x+\frac{s-1}{2}\right)\phi\left(y-\frac{r-1}{2}\right)|0\rangle \quad (5.132)$$

and by our perturbative construction (5.5) we have

$$\Psi_2 = -\lambda^2\hat{U}_3\left(\bar{P}\frac{\mathcal{B}_0}{\mathcal{L}_0}cV\left(\frac{1}{2}\right)cV\left(-\frac{1}{2}\right)\right)|0\rangle \equiv \hat{U}_3\hat{\Psi}_2\,|0\rangle \quad (5.133)$$

We use the OPE (5.56), twist symmetry and taylor expansion around zero to regularise as

$$cV(x)cV(-x) = cV(x)cV(-x) \pm \frac{c\partial c(0)}{(2x)^{2h-1}} \pm C_{VVV}\frac{c\partial cV(0)}{(2x)^{h-1}} \quad (5.134)$$

Using the identities derived by contour methods in sliver frame ($b(z)c(w) \sim \frac{1}{z-w}$)

$$[\mathcal{B}_0, c(w)] = \oint_w \frac{\mathrm{d}z}{2\pi i}\, zb(z)c(w) = w \quad (5.135)$$

$$[\mathcal{B}_0, \partial c(w)] = \oint_w \frac{\mathrm{d}z}{2\pi i}\, zb(z)\partial c(w) = 1 \quad (5.136)$$

one has

$$\Psi_2 = -\lambda^2\hat{U}_3\Bigg(\bar{P}\frac{1}{\mathcal{L}_0}\Big(\frac{1}{2}\left(c\left(\frac{1}{2}\right)+c\left(-\frac{1}{2}\right)\right)V\left(\frac{1}{2}\right)V\left(-\frac{1}{2}\right)$$
$$\mp c(0) \mp C_{VVV}cV(0)\Big)\Bigg)|0\rangle$$

and using explicit division by $\mathcal{L}_0$ on the plus branch and (4.139) on the rest while projecting out the contribution to a y-pole we get the final expression for the second order string field in pseudo-Schnabl gauge

$$\Psi_2 = -\lambda^2\hat{U}_3\Bigg(\int_0^1\frac{ds}{s}\left(s^{2h-1}\frac{1}{2}\left(c\left(\frac{s}{2}\right)+c\left(-\frac{s}{2}\right)\right)V\left(\frac{s}{2}\right)V\left(-\frac{s}{2}\right)\right.$$
$$\left. - s^{-1}c(0) - C_{VVV}s^{h-1}cV(0) - c(0)\right)\Bigg)|0\rangle \quad (5.137)$$

The same result can be obtained by using previously defined projectors P and T. In contrast to Schnabl gauge, these projectors now act on unit wedges with insertions so that on these states the general expressions (5.81) and (5.105) simplify to

$$\begin{aligned}
P(\Phi\,|0\rangle) &= -\frac{1}{g_0}\left(\frac{\pi}{2}\right)^{2h-2}\left(\frac{1}{z^{h-1}}\langle cV(1)z^{\mathcal{L}^-}\Phi\rangle_{C_2}\right)\Big|_{z=0} c\partial cV\,|0\rangle \quad (5.138)\\
T(\Phi\,|0\rangle) &= -\frac{1}{g_0}\left(\frac{\pi}{2}\right)^{-2}\left(z\langle c(1)z^{\mathcal{L}^-}\Phi\rangle_{C_2}\right)\Big|_{z=0} c\partial c\,|0\rangle \quad (5.139)
\end{aligned}$$

where Φ is a ghost number two string of insertions of local operators. For this order the relevant subtractions are given in (5.134) which is consistent with the expressions (5.138) and (5.139) giving

$$\begin{aligned}
P\left(cV\left(\frac{1}{2}\right)cV\left(-\frac{1}{2}\right)|0\rangle\right) &= -C_{VVV}c\partial cV\,|0\rangle \quad (5.140)\\
T\left(cV\left(\frac{1}{2}\right)cV\left(-\frac{1}{2}\right)|0\rangle\right) &= -c\partial c\,|0\rangle \quad (5.141)
\end{aligned}$$

5.5.2 Third order string field

We now continue with the third order calculation beginning with the expression generated by our perturbative procedure

$$\Psi_3 = \hat{U}_4\hat{\Psi}_3\,|0\rangle = -\hat{U}_4\bar{P}\frac{\mathcal{B}_0}{\mathcal{L}_0}\left[\left[\lambda cV(0),\hat{\Psi}_2\right]\right]_{(2,3)}|0\rangle \quad (5.142)$$

where we have defined a graded commutator-like symbol following [115]

$$\begin{aligned}
\left[\left[\Psi(x),\phi(y)\right]\right]_{(r,s)} \equiv\ & \Psi\left(x+\frac{s-1}{2}\right)\phi\left(y-\frac{r-1}{2}\right)\\
& -(-1)^{|\Psi||\phi|}\phi\left(y+\frac{r-1}{2}\right)\Psi\left(x-\frac{s-1}{2}\right) \quad (5.143)
\end{aligned}$$

which is motivated by (5.132). We begin by calculating the graded commutator of $\lambda cV|0\rangle$ and $\hat{\Psi}_2$

$$\begin{aligned}
\left[\left[\lambda cV(0),\hat{\Psi}_2\right]\right]_{(2,3)} &= -\lambda^3\int_0^1 ds\,\Bigg[\frac{s^{2h-2}}{2} \\
&\Bigg(cV(1)\left(c\left(\frac{s-1}{2}\right)+c\left(-\frac{s+1}{2}\right)\right)V\left(\frac{s-1}{2}\right)V\left(-\frac{s+1}{2}\right) \\
&+\left(c\left(\frac{s+1}{2}\right)+c\left(\frac{1-s}{2}\right)\right)V\left(\frac{s+1}{2}\right)V\left(\frac{1-s}{2}\right)cV(-1)\Bigg) \\
&-\left(s^{-2}+1\right)\left(c(1)cV\left(-\frac{1}{2}\right)+cV\left(\frac{1}{2}\right)c(-1)\right)\Bigg) \\
&-C_{VVV}s^{h-2}\left(cV(1)\,cV\left(-\frac{1}{2}\right)+cV\left(\frac{1}{2}\right)cV(-1)\right)\Bigg]
\end{aligned} \tag{5.144}$$

so that after applying the projectors (5.138) and (5.139) we get

$$\begin{aligned}
\Psi_3 &= \lambda^3\hat{U}_4\frac{\mathcal{B}_0}{\mathcal{L}_0}\int_0^1 ds\,\Bigg[\frac{s^{2h-2}}{2} \\
&\Bigg(cV(1)\left(c\left(\frac{s-1}{2}\right)+c\left(-\frac{s+1}{2}\right)\right)V\left(\frac{s-1}{2}\right)V\left(-\frac{s+1}{2}\right) \\
&+\left(c\left(\frac{s+1}{2}\right)+c\left(\frac{1-s}{2}\right)\right)V\left(\frac{s+1}{2}\right)V\left(\frac{1-s}{2}\right)cV(-1)\Bigg) \\
&-\left(s^{-2}+1\right)\left(c(1)cV\left(-\frac{1}{2}\right)+cV\left(\frac{1}{2}\right)c(-1)\right)\Bigg) \\
&-C_{VVV}s^{h-2}\left(cV(1)\,cV\left(-\frac{1}{2}\right)+cV\left(\frac{1}{2}\right)cV(-1)\right) \\
&\pm\left(s^{2h-2}\upsilon(s)-3\left(s^{-2}+1\right)-\left(C_{VVV}\right)^2 2^h 3^{1-h}s^{h-2}\right)c\partial cV(0) \\
&\pm 3C_{VVV}4^h s^{h-2}\left(\left(9-s^2\right)^{-h}-9^{-h}\right)c\partial c(0)\Bigg]|0\rangle
\end{aligned} \tag{5.145}$$

5.5.3 Ellwood invariant

Plugging (5.137) into the Ellwood invariant, we immediately find

$$g\left(\Psi_2\right) = -2\pi i\lambda^2\frac{i}{2\pi}2\left(\int_0^1\frac{ds}{s}\left(s^{2h-1}\langle V\left(-\frac{s}{2}\right)V\left(\frac{s}{2}\right)\rangle_{C_2}-g_0 s^{-1}\right)-g_0\right) \tag{5.146}$$

With the use of (A.11) this becomes

$$g\left(\Psi_2\right)=2g_0\lambda^2\left(\int_0^1\frac{ds}{s}\left(s^{2h-1}\left(\csc\left(\frac{\pi s}{2}\right)\frac{\pi}{2}\right)^{2h}-s^{-1}\right)-1\right) \tag{5.147}$$

and expanding in powers of y we get

$$\begin{aligned} g\left(\Psi_2\right) &= 2g_0\frac{1}{\left(C_{VVV}\right)^2}\left(\int_0^1 ds\left(\left(\csc\left(\frac{\pi s}{2}\right)\frac{\pi}{2}\right)^2-s^{-2}\right)-1\right)y^2+O(y^3) \\ &= O(y^3) \end{aligned} \tag{5.148}$$

in accordance with (3.61).

Expanding (5.147) to third order one gets the finite contribution

$$g\left(\Psi_2\right)=-g_0\frac{1}{\left(C_{VVV}\right)^2}\pi^2\left(\int_0^1 ds\csc^2\left(\frac{\pi s}{2}\right)\ln\left(\frac{\pi s}{2}\csc\left(\frac{\pi s}{2}\right)\right)\right)y^3+O(y^4) \tag{5.149}$$

The finiteness gives credit to the regularity of (5.137) and one may observe that numerically this is the same as the analogous Schnabl gauge contribution (5.116) which agrees with our discussion at the beginning of the section.

We continue by calculating the contribution from the third order string field (5.145)

$$\begin{aligned} &g\left(\Psi_3\right)=-9g_0\lambda^3C_{VVV}\int_0^1 dt\int_0^1 ds \\ &\left(t^{3h-2}s^{2h-2}\pi^{3h}3^{-3h}\csc^h\left(\frac{1}{6}\pi(3-s)t\right)\csc^h\left(\frac{\pi st}{3}\right)\csc^h\left(\frac{1}{6}\pi(s+3)t\right)-\right. \\ &\left.t^{2h-2}s^{h-2}\pi^{2h}3^{-2h}\csc^{2h}\left(\frac{\pi t}{2}\right)-4^hs^{h-2}\left(s^h\left(-s\left(s^2-9\right)\right)^{-h}-9^{-h}\right)\left(t^{-2}+1\right)\right) \end{aligned} \tag{5.150}$$

where we used the cyclicity of the Ellwood invariant and correlators (A.11), (A.12) on a cylinder of circumference 3. Combining (5.149) with (5.150) while expanding in y gives the result for the third-order contribution to the g-function

$$\begin{aligned} &-\frac{g_0}{\left(C_{VVV}\right)^2}\left[\int_0^1\!dt\int_0^1\!ds\left(\frac{1}{3}\pi^3t\csc\left(\frac{1}{6}\pi(3-s)t\right)\csc\left(\frac{\pi st}{3}\right)\csc\left(\frac{1}{6}\pi(s+3)t\right)-\right.\right. \\ &\left.\left.\frac{\pi^2\csc^2\left(\frac{\pi t}{2}\right)}{s}+\frac{4s}{(s^2-9)}(t^{-2}+1)\right)+\int_0^1\!ds\,\pi^2\csc^2\left(\frac{\pi s}{2}\right)\ln\left(\frac{\pi s}{2}\csc\left(\frac{\pi s}{2}\right)\right)\right]y^3 \\ &=-g_0\frac{\pi^2}{3}\frac{y^3}{\left(C_{VVV}\right)^2} \end{aligned} \tag{5.151}$$

as was to be reproduced by (3.61). The integrals are somewhat simpler than in Schnabl gauge since we work with cylinders of fixed length.

We can also proceed by an analytic continuation regularisation similarly to the Schnabl gauge case and we obtain for the contribution without subtractions

$$
\begin{aligned}
g\left(\Psi_3^{singular}\right) &= -g_0\lambda^3 C_{VVV}\Bigg(\int_0^1 dt\ t^{3h-2}\int_0^1 ds\ s^{2h-2} \\
&\pi^{3h}3^{2-3h}\csc^h\left(\frac{1}{6}\pi(3-s)t\right)\csc^h\left(\frac{\pi st}{3}\right)\csc^h\left(\frac{1}{6}\pi(s+3)t\right)\Bigg)
\end{aligned}
\tag{5.152}
$$

Asymptotically expanding around $s\sim 0$ we must make the following subtraction

$$
\begin{aligned}
g\left(\Psi_3^{singular}\right) &= -9g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt\ \int_0^1 ds \\
&\Bigg(t^{3h-2}s^{2h-2}\pi^{3h}3^{-3h}\csc^h\left(\frac{1}{6}\pi(3-s)t\right)\csc^h\left(\frac{\pi st}{3}\right)\csc^h\left(\frac{1}{6}\pi(s+3)t\right) \\
&- t^{2h-2}s^{h-2}\pi^{2h}3^{-2h}\csc^{2h}\left(\frac{\pi t}{2}\right)\Bigg)\Bigg]
\end{aligned}
\tag{5.153}
$$

Where as for Schnabl gauge there is no plus branch of the subtraction since the subtraction comes from $cV\,|0\rangle$ in (5.137) and thus it is projected out by $\bar{P}$. Again this argument corresponds to the analytic continuation (5.120).

Asymptotically expanding around $t\sim 0$ we subtract and add a tachyonic singularity while using analytic continuation (5.122) in the same way as for Schnabl gauge with the result

$$
\begin{aligned}
g\left(\Psi_3^{singular}\right) &= -9g_0\lambda^3 C_{VVV}\Bigg[\int_0^1 dt\ \int_0^1 ds \\
&\Bigg(t^{3h-2}s^{2h-2}\pi^{3h}3^{-3h}\csc^h\left(\frac{1}{6}\pi(3-s)t\right)\csc^h\left(\frac{\pi st}{3}\right)\csc^h\left(\frac{1}{6}\pi(s+3)t\right) \\
&- t^{2h-2}s^{h-2}\pi^{2h}3^{-2h}\csc^{2h}\left(\frac{\pi t}{2}\right) - \frac{4^h s^{h-2}\left(s^h\left(-s\left(s^2-9\right)\right)^{-h}-9^{-h}\right)}{t^2}\Bigg) \\
&- \int_0^1 ds\ 4^h s^{h-2}\left(s^h\left(-s\left(s^2-9\right)\right)^{-h}-9^{-h}\right)\Bigg]
\end{aligned}
\tag{5.154}
$$

and since the zero modes are not present, this is precisely the contribution (5.151) which was obtained from a regular string field.

Conclusion

After providing exposure to field theoretic ideas in (1), we reviewed aspects of (mostly two-dimensional) conformal field theory in (2). In this chapter we also introduced conformal perturbation theory in the bulk, which serves as a basis for the study of boundary conformal perturbation theory in (3). In our context we used boundary conformal perturbation theory to compute shifts in boundary state coefficients. Boundary states characterise a given conformal boundary condition, which can be seen as having dynamics of its own thanks to the presence of boundary fields.

We introduced Witten's open string field theory as a very particular framework for the study of conformal boundary conditions in (4). The solutions of its classical equations of motion correspond to such boundary conditions, which can be seen from the KMS correspondence [96] giving a map between solutions to the classical equations of motion and boundary states. We reviewed the basic analytic tools used to find solutions to the classical equations of motion.

In the last chapter (5) we presented the research based on the collaboration with Martin Schnabl of the perturbative classical solutions corresponding to relevant deformations. In particular, we were interested in the limit where the operators triggering such deformations are weakly relevant. In this limit, we reproduced the leading order results of boundary conformal perturbation theory. In particular, we reproduced the boundary degeneracy g result of Affleck and Ludwig through a very simple computation of the action and the shift of the general boundary state coefficient through the Ellwood invariant. These results together with a remarkably simple formula for the next-to-leading correction to the shift of g obtained in Siegel gauge are to be submitted to PRL [1]. We also laid the groundwork for the study of the spectrum of linearised fluctuations around these perturbative solutions. The presented framework is illustrated on an example where we observe the perturbing operator disappearing from the cohomology. Then we constructed the perturbative solution in Schnabl gauge to third order. In doing so, we had to tackle the very important issue that in the perturbative solutions we often used projectors were oblique and we also had to modify Schnabl gauge to account for a zero mode. The correct definition of the oblique projectors and the appropriate handling of the zero mode was verified by reproducing the leading order correction to g through the Ellwood invariant. We concluded by presenting an argument why a naive attempt at a pseudo-Schnabl gauge solution violates the equations of motion, but thanks to the properties of observables may accidentaly give the correct result. This was explicitly shown by constructing the third order pseudo-Schnabl gauge string field and again reproducing the leading order correction to g with the use of the Ellwood invariant.

A natural future direction would be to extend the computation of observables to higher orders. In particular, we might compute the next-to-leading order shift to the general boundary state coefficient or even extend the computation of g. One would hope to see a pattern in such high order results and perhaps even make contact with on-shell string theory since we work in the near-marginal limit. It seems difficult to carry out the calculation beyond the next-to-leading order in Siegel gauge and that's where the solution in Schnabl gauge may come useful. Another direction would be to compute other observables relevant for conformal perturbation theory such as shifts in structure constants, conformal weights and the operator mixings. We think that diagonalising the shifted Hamiltonian and computing the spectrum of linearised fluctuations will be useful for this purpose. One might also try to investigate solutions corresponding to weakly irrelevant deformations, which go against the RG flow, something that was already observed in level truncation [100]. Perhaps the perturbative framework can prove itself useful in the study of the tachyon vacuum of the noncubic closed string field theory, a first step might be to rephrase the result of [121].

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Appendix A

A list of useful correlators in OSFT

Below we list some correlators that one finds useful in the study of OSFT. See [61, 91] and (3) for more details.

A.1 Upper half-plane

On the UHP we have the following correlators of primary boundary fields as fixed by conformal invariance

$$\begin{aligned} \langle V_i(x_1)V_j(x_2)\rangle_{UHP} &= \frac{g_0\delta_{ij}}{x_{12}^{2h}} && \text{(A.1)} \\ \langle V_i(x_1)V_j(x_2)V_k(x_3)\rangle_{UHP} &= \frac{g_0 C_{ijk}}{x_{12}^{h_k-h_i-h_j}x_{13}^{h_j-h_i-h_k}x_{23}^{h_i-h_j-h_k}} && \text{(A.2)} \end{aligned}$$

with g_0 being the g-function of the perturbative vacuum, h_i being conformal weights and we suppress boundary condition indices on the boundary structure constant C_{ijk}. For an insertion of a single bulk field ϕ, we have

$$\langle \phi(x+iy, x-iy)\rangle_{UHP} = g_0 B_{\phi\mathbb{1}}(2y)^{-\Delta}, \tag{A.3}$$

with Δ being the weight of ϕ, and the bulk-boundary correlator

$$\langle \phi_i(x+iy, x-iy)V_j(r)\rangle_{UHP} = \frac{g_0 B_{ij}}{(2y)^{\Delta_i-h_j}((x-r)^2+y^2)^{h_j}}, \tag{A.4}$$

where B_{ij} is a bulk-boundary structure constant. A special case is the three-ghost correlator

$$\langle c(x_1)c(x_2)c(x_3)\rangle_{UHP} = x_{12}x_{13}x_{23} \tag{A.5}$$

A.2 Disc frame

To get correlators in disc frame, we simply transform those on the UHP by the map $f(z) = i\frac{1-z}{1+z}$, to obtain the boundary correlators

$$\langle V_i(\theta_1)V_j(\theta_2)\rangle_{disc} = \frac{g_0\delta_{ij}}{\left|2\sin\left(\frac{\theta_{12}}{2}\right)\right|^{2h}} \tag{A.6}$$

$$\langle V_i(\theta_1)V_j(\theta_2)V_k(\theta_3)\rangle_{disc} = \frac{g_0 C_{ijk}}{\left|2\sin\left(\frac{\theta_{12}}{2}\right)\right|^{h_k-h_i-h_j}\left|2\sin\left(\frac{\theta_{13}}{2}\right)\right|^{h_j-h_i-h_k}\left|2\sin\left(\frac{\theta_{23}}{2}\right)\right|^{h_i-h_j-h_k}} \tag{A.7}$$

and the bulk one-point function

$$\langle\phi(r,\theta)\rangle_{disc} = \frac{g_0 B_{\phi\mathbb{1}}}{(1-r^2)^{\Delta}}, \tag{A.8}$$

with the bulk-boundary correlator

$$\langle\phi(r,\theta_1)V(\theta_2)\rangle_{disc} = g_0 B_{\phi V}(1-r^2)^{-\Delta}\left(\frac{1-r^2}{1-2r\cos\theta_{12}+r^2}\right)^h \tag{A.9}$$

The three-ghost correlator becomes

$$\langle c(\theta_1)c(\theta_2)c(\theta_3)\rangle_{disc} = 8\sin\frac{\theta_{12}}{2}\sin\frac{\theta_{13}}{2}\sin\frac{\theta_{23}}{2} \tag{A.10}$$

A.3 Cylinder frame

To get correlators on the cylinder of length L, we simply transform those on the UHP by the map $C_L^{-1}(z) = \frac{L}{\pi}\tan\frac{\pi z}{L}$, to obtain the boundary correlators

$$\langle V_i(z_1)V_j(z_2)\rangle_{C_L} = \left(\frac{\pi}{L}\right)^{2h}\frac{g_0\delta_{ij}}{\sin\left(\frac{\pi z_{12}}{L}\right)^{2h}} \tag{A.11}$$

$$\langle V_i(z_1)V_j(z_2)V_k(z_3)\rangle_{C_L} = \left(\frac{\pi}{L}\right)^{h_i+h_j+h_k}\frac{g_0 C_{ijk}}{\sin\left(\frac{\pi z_{12}}{L}\right)^{h_k-h_i-h_j}\sin\left(\frac{\pi z_{13}}{L}\right)^{h_j-h_i-h_k}\sin\left(\frac{\pi z_{23}}{L}\right)^{h_i-h_j-h_k}} \tag{A.12}$$

The bulk one-point function becomes

$$\langle\phi(z,\bar{z})\rangle_{C_L} = \left(\frac{\pi}{L}\right)^{\Delta} g_0 B_{\phi\mathbb{1}}\sin^{-\Delta}\left(\frac{\pi}{L}|z-\bar{z}|\right) \tag{A.13}$$

and the bulk-boundary correlator is

$$\langle\phi_i(z_1,\bar{z_1})V_j(z_2)\rangle_{C_L} = \left(\frac{\pi}{L}\right)^{\Delta} g_0 B_{ij}\sin^{-h_j}\left(\frac{\pi}{L}z_{12}\right)\sin^{-\Delta_i+h_j}\left(\frac{\pi}{L}(z_1-\bar{z}_1)\right) \tag{A.14}$$

The three-ghost correlator can be written in two ways

$$\begin{aligned}\langle c(z_1)c(z_2)c(z_3)\rangle_{C_L} &= \left(\frac{L}{\pi}\right)^3 \sin\left(\frac{\pi}{L}z_{12}\right)\sin\left(\frac{\pi}{L}z_{13}\right)\sin\left(\frac{\pi}{L}z_{23}\right) \qquad \text{(A.15)}\\ &= \frac{1}{4}\left(\frac{L}{\pi}\right)^3\left(\sin\left(\frac{2\pi z_{12}}{L}\right)-\sin\left(\frac{2\pi z_{13}}{L}\right)+\sin\left(\frac{2\pi z_{23}}{L}\right)\right)\end{aligned}$$

We will also need correlators with the B ghost

$$\begin{aligned}\langle c(z_1)c(z_2)c(z_3)B(z_4)\rangle_{C_L} = \frac{L^2}{4\pi^2}\Bigg(& z_{14}\sin\left(\frac{2\pi z_{23}}{L}\right)+z_{23}\sin\left(\frac{2\pi z_{14}}{L}\right)-z_{13}\sin\left(\frac{2\pi z_{24}}{L}\right)\\ & -z_{24}\sin\left(\frac{2\pi z_{13}}{L}\right)+z_{12}\sin\left(\frac{2\pi z_{34}}{L}\right)+z_{34}\sin\left(\frac{2\pi z_{12}}{L}\right)\Bigg)\end{aligned} \tag{A.16}$$

Sometimes one encounters correlators involving K, for those it is useful to imagine inserting a wedge state $\Omega^\alpha = e^{-\alpha K}$, differentiating the correlator without a K but with the wedge inserted with respect to α and setting $\alpha \to 0$. An example would be

$$\begin{aligned}\mathrm{Tr}\Big\{\sqrt{\Omega}c\Omega cK^2c\sqrt{\Omega}\Big\} &= \lim_{\alpha\to 0}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\,\mathrm{Tr}\Big\{\sqrt{\Omega}c\Omega c\Omega^\alpha c\sqrt{\Omega}\Big\}\\ &= \lim_{\alpha\to 0}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\langle c(1+\alpha)c(\alpha)c(0)\rangle_{C_{2+\alpha}} = \frac{8}{\pi^2}\end{aligned} \tag{A.17}$$

Appendix B

Kaluza-Klein reduction on S^1

In this appendix we study the Kaluza-Klein theory with compactification on a circle S^1. This means that we consider a product spacetime $\mathcal{M} \times S^1$ and observe the $U(1)$ symmetry of S^1 turning into a $U(1)$ gauge symmetry of an effective theory. The metric of this spacetime is generally parametrisable as

$$G = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \chi^2(\mathrm{d}x^d + A_\mu \mathrm{d}x^\mu)^2, \tag{B.1}$$

with $g_{\mu\nu} = g_{\mu\nu}(x^\mu, x^d), \chi = \chi(x^\mu, x^d)$ and $A_\mu = A_\mu(x^\mu, x^d)$. In addition one has $x^d \sim x^d + 2\pi R$, R being the radius of S^1.

B.1 The spectrum

We consider a massless scalar field Φ and for simplicity we have $\chi = 1$. By the compactness of the d-direction we have the usual quantisation of the momentum in this direction

$$p_d = \frac{n}{R}, \tag{B.2}$$

where n is an integer. From the periodicity we can expand

$$\Phi(x^\mu, x^d) = \sum_{n=-\infty}^{\infty} \Phi_n(x^\mu) \exp\left\{-ix^d \frac{n}{R}\right\} \tag{B.3}$$

and the Klein-Gordon equation (1.26) gives

$$\partial^2 \Phi_n = \frac{n^2}{R^2}\Phi_n, \tag{B.4}$$

where ∂ is in the x^μ coordinates. This means that the mass of the scalar in the d dimensions becomes parametrised by n

$$m_d^2 = \frac{n^2}{R^2} \tag{B.5}$$

giving rise to an infinite tower of massive states. For small enough R, these are so heavy that we observe only the massless $n = 0$ mode so that $\Phi(x^\mu, x^d) = \Phi_0(x^\mu)$.

In general, the massless modes of Kaluza-Klein theory are characterised by the effective $D = d + 1$ dimensional metric

$$G_{eff} = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \chi^2(\mathrm{d}x^d + A_\mu \mathrm{d}x^\mu)^2, \tag{B.6}$$

with $g_{\mu\nu} = g_{\mu\nu}(x^\mu), \chi = \chi(x^\mu)$ and $A_\mu = A_\mu(x^\mu)$. This ansatz is the most general one with translation invariance in the d-direction, reflecting that $n = 0$. Reparametrisation invariance enforces that A_μ is a gauge potential since $x^d \sim x^d + \Lambda(x^\mu)$ enforces $A_\mu \sim A_\mu - \partial_\mu \Lambda(x^\mu)$. We remark that A_μ has nontrivial dynamics even in the case when it is constant since a compact direction is present. Seeing this gauge invariance, we think that the Maxwell theory should appear in the action describing the spacetime dynamics of G_{eff}. This is indeed the case as we show in the next section.

B.2 The Ricci scalar

We begin by calculating the Ricci scalar $\mathcal{R}$ of the Levi-Civita curvature associated to the metric (B.6). The computation serves as a nice exercise in Cartan's formalism. We rewrite the metric (B.6) in an orthonormal frame

$$G_{eff} = \eta_{mn} e^m e^n + e^d e^d, \tag{B.7}$$

where η_{mn} is the Minkowski metric whose indices go from 0 to $d - 1$ and

$$e^d = \chi(\mathrm{d}x^d + A) \tag{B.8}$$

Using the definition of the curvature form $F = dA$ we have

$$\mathrm{d}e^d = \partial_k \ln \chi \, e^k \wedge e^d + \frac{\chi}{2} F_{kl} \, e^k \wedge e^l \tag{B.9}$$

By the first Cartan's equation of structure $\mathrm{d}e^K + \omega^K_L \wedge e^L = 0$, where we use the convention that the uppercase letter goes over the lowercase ones and d, we find

$$\omega^d_k = \partial_k \ln \chi \, e^d + \frac{\chi}{2} F_{kl} \, e^l \tag{B.10}$$

We continue by writing out the structure equation for $\mathrm{d}e^k$

$$\mathrm{d}e^k = -\omega^k_L \wedge e^L = -\omega^k_l \wedge e^l - \omega^k_d \wedge e^d \tag{B.11}$$

By $\omega_{KL} = \omega_{[KL]}$ and (B.10) we have $\omega^k_d \wedge e^d = \frac{\chi}{2} F^k_l e^l \wedge e^d$. Using this together with (B.11) and the fact that we could have computed $\mathrm{d}e^k$ for the metric without the d-direction (quantities refering to this metric we denote with a tilde) we get

$$\omega^k_l = \tilde{\omega}^k_l - \frac{\chi}{2} F^k_l e^d \tag{B.12}$$

Now we write out the second Cartan's equation of structure $\Omega^K_L = \mathrm{d}\omega^K_L + \omega^K_M \wedge \omega^M_L$ for the lowercase directions

$$\Omega^k_l = \mathrm{d}\omega^k_l + \omega^k_m \wedge \omega^m_l + \omega^k_d \wedge \omega^d_l \tag{B.13}$$

With some foresight we now neglect terms containing $e^k \wedge e^d$ since these do not contribute to $\mathcal{R}$ (this is denoted by $\simeq$) and we then find

$$\Omega^k_l \simeq \tilde{\Omega}^k_l - \frac{\chi^2}{4} \left(F^k_l F_{mn} + \frac{1}{2}(F^k_m F_{ln} - F^k_n F_{lm}) \right) e^m \wedge e^n \tag{B.14}$$

Another curvature two-form that we need to compute is Ω^d_m

$$\begin{aligned} \Omega^d_m &= \mathrm{d}\omega^d_m + \omega^d_n \wedge \omega^n_m && \text{(B.15)} \\ &\simeq \left(\partial_{mo} \ln \chi + (\partial_m \ln \chi)(\partial_o \ln \chi) - \frac{\chi^2}{4} F^n_m F_{no} \right) e^o \wedge e^d && \text{(B.16)} \end{aligned}$$

where we neglected a term $\partial_n \ln \chi \, e^d \wedge \tilde{\omega}^n_a$ since it gets traced out in $\mathcal{R}$. This is forced by the symmetry properties of ω.

From the curvature two-forms we find the necessary components of the Riemann tensor R by comparison with $\Omega^M_N = \frac{1}{2} R_{KL}{}^M{}_N e^K \wedge e^L$ so that

$$\begin{aligned} R_{kl}{}^m{}_n &\simeq \widetilde{R}_{kl}{}^m{}_n - \frac{\chi^2}{2} \left(F^m_n F_{kl} + \frac{1}{2}(F^m_k F_{nl} - F^m_l F_{nk}) \right) && \text{(B.17)} \\ R_{dk}{}^d{}_l &\simeq -\left(\partial_{lk} \ln \chi + (\partial_k \ln \chi)(\partial_l \ln \chi) - \frac{\chi^2}{4} F^n_l F_{nk} \right) && \text{(B.18)} \end{aligned}$$

Now using the definition of the Ricci tensor $\mathrm{Ric}_{KL} = R_{MK}{}^M{}_L$ we have

$$\mathrm{Ric}_{kl} = \widetilde{\mathrm{Ric}}_{kl} - \frac{\chi^2}{4} \left(F^n_l F_{nk} + F^n_k F_{nl} \right) - \partial_{lk} \ln \chi - (\partial_k \ln \chi)(\partial_l \ln \chi) \tag{B.19}$$

What is left is the computation of Ric_{dd} but this is straightforward with the use of cyclicity of R

$$\begin{aligned}
\mathrm{Ric}_{dd} &= R_{nd\ d}^{\ \ n} &\text{(B.20)}\\
&= \eta^{mn} R_{ndmd} &\text{(B.21)}\\
&= \eta^{mn} R_{dndm} &\text{(B.22)}\\
&= \eta^{mn} R_{dn\ m}^{\ \ d} &\text{(B.23)}\\
&\simeq -\eta^{mn}\left(\partial_{mn}\ln\chi + (\partial_m \ln\chi)(\partial_n \ln\chi) - \frac{\chi^2}{4}F_m^o F_{on}\right) &\text{(B.24)}\\
&\simeq -\left(\partial_n\partial^n \ln\chi + (\partial_n \ln\chi)^2 - \frac{\chi^2}{4}F^{mn}F_{mn}\right) &\text{(B.25)}
\end{aligned}$$

Using the definition of the Ricci scalar $\mathcal{R} = \eta^{AB}\mathrm{Ric}_{AB}$ and writing invariants in the original coordinates x^μ we have the result

$$\mathcal{R} = \tilde{\mathcal{R}} - 2\frac{\nabla^2\chi}{\chi} - \frac{\chi^2}{4}F^{\mu\nu}F_{\mu\nu} \tag{B.26}$$

where ∇^2 is the Beltrami Laplacian. We confirm our suspicion, the Maxwell Lagrangian indeed appears.

B.3 The instanton

We specialise to the Kaluza-Klein theory with $d = 4$ and the spacetime manifold $\mathcal{M}_4 \times S^1$, $\mathcal{M}_4$ being the four-dimensional Minkowski space, with the metric

$$G = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}\phi^2 \tag{B.27}$$

where we renamed $x^5 \equiv \phi$. It is trivial to see that G is a solution to Einstein's equations just as $\mathcal{M}_5$ is. We show that the spacetime corresponding to G is unstable due to the presence of an instanton, see the section (1.6) for the necessary background and references. Analogously to the procedure outlined in the section on instantons, we begin with the Euclidean theory with $t = i\tau$ and the metric

$$\mathcal{G} = \mathrm{d}\tau^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}\phi^2 = \mathrm{d}r^2 + r^2\mathrm{d}\Theta_3^2 + \mathrm{d}\phi^2 \tag{B.28}$$

where the latter form is obtained by switching to spherical coordinates and there $\mathrm{d}\Theta_3^2$ is a line element of S^3. We continue by considering a solution to the vacuum Euclidean Einstein's equations (a rotated five-dimensional Schwarzschild solution)

$$g = \frac{\mathrm{d}r^2}{1-\frac{\alpha}{r^2}} + r^2\mathrm{d}\Theta^2 + \left(1 - \frac{\alpha}{r^2}\right)\mathrm{d}\phi^2 \tag{B.29}$$

which asymptotically approaches (B.28) so we see a potential for interpretation as an instantonic solution connecting the original $\mathcal{M}_4 \times S^1$ to some other vacuum. The solution (B.29) seems to be singular at the point $r = \sqrt{\alpha}$, but this is not the case because it is just a coordinate singularity analogous to $\rho = 0$ in $\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\phi^2$. We would like to rewrite (B.29) so that near $r = \sqrt{\alpha}$ it looks similar to the metric in polar coordinates near $\rho = 0$. To do this, we consider an ansatz

$$\rho = c\left(1 - \frac{\alpha}{r^2}\right)^{\beta} \tag{B.30}$$

so that with some algebra

$$\frac{\mathrm{d}r^2}{1 - \frac{\alpha}{r^2}} = \frac{\alpha}{4c^2\beta^2}\left(\frac{\rho}{c}\right)^{\frac{1-2\beta}{\beta}} \frac{\mathrm{d}\rho^2}{\left(1 - (\frac{\rho}{c})^{\frac{1}{\beta}}\right)^3} \tag{B.31}$$

To avoid the singularity we need this to look like $\mathrm{d}\rho^2$ near $\rho = 0$, which forces us to take $\beta = \frac{1}{2}$ and $c = \sqrt{\alpha}$. The periodic part of (B.29) can then be rewritten

$$\left(1 - \frac{\alpha}{r^2}\right)\mathrm{d}\phi^2 = \frac{\rho^2}{\alpha}\mathrm{d}\phi^2 \tag{B.32}$$

showing that ϕ should be periodic with the period of $2\pi\sqrt{\alpha}$ so that $\alpha = R^2$. With this the instanton (B.29) becomes

$$g = \frac{\mathrm{d}r^2}{1 - (\frac{R}{r})^2} + r^2\mathrm{d}\Theta_3^2 + \left[1 - \left(\frac{R}{r}\right)^2\right]\mathrm{d}\phi^2 \tag{B.33}$$

Since ρ starts at 0, then by (B.30) we have $r \geq R$. We wish to rotate the instanton to Minkowski space to find its spacetime interpretation, in particular we want to rotate $\mathrm{d}r^2 + r^2\mathrm{d}\Theta_3^2 \to \mathrm{d}x^2 + x^2\mathrm{d}\Theta_2^2 - \mathrm{d}t^2$. To do so, we write

$$\mathrm{d}\Theta_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\Theta_2^2 \tag{B.34}$$

Introducing the coordinates

$$x = r\cosh\psi \tag{B.35}$$

$$t = r\sinh\psi \tag{B.36}$$

with $\theta = \frac{\pi}{2} + i\psi$, we indeed have $\mathrm{d}r^2 + r^2\mathrm{d}\Theta_3^2 = \mathrm{d}x^2 + x^2\mathrm{d}\Theta_2^2 - \mathrm{d}t^2$ by virtue of

$$\mathrm{d}\Theta_3^2 = -\mathrm{d}\psi^2 + \cosh^2\psi\, \mathrm{d}\Theta_2^2 \tag{B.37}$$

Finally using (B.37), the rotated instanton becomes

$$g_{rot} = \frac{\mathrm{d}r^2}{1-(\frac{R}{r})^2} + r^2(-\mathrm{d}\psi^2 + \cosh^2\psi\,\mathrm{d}\Theta_2^2) + \left[1 - \left(\frac{R}{r}\right)^2\right]\mathrm{d}\phi^2 \tag{B.38}$$

We may finally interpret the result and to do so, we observe that by the definitions (B.35) and (B.36) we have $r^2 = x^2 - t^2$ and since $r \geq R$ the metric g_{rot} is defined only outside the hyperboloid

$$x^2 - t^2 \leq R^2 \tag{B.39}$$

From the perspective of an observer in $\mathcal{M}_4$ a hole of radius R forms at $t = 0$ and then its boundary expands according to

$$x^2 = t^2 + R^2 \tag{B.40}$$

The boundary is accelerated uniformly and soon it expands at the speed of light. So the region of space in which g_{rot} is defined shrinks and the Kaluza-Klein vacuum $\mathcal{M}_4 \times S^1$ decays into nothing. In the light of section (1.6), this is quite shocking since we would expect a false vacuum to decay into some other true ground state.