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Continued fractions for bi-periodic Fibonacci sequence*

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Abstract. In this paper, we study generalized continued fractions for the expression of bi-periodic Fibonacci ratios.

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1. Introduction

A continued fraction is an expression of the form

$$x = s_0 + \cfrac{r_1}{s_1 + \cfrac{r_2}{s_2 + \cfrac{r_3}{\ddots}}} = s_0 + \frac{r_1}{s_1} + \frac{r_2}{s_2} + \cdots,$$

where s_n and r_n are real or complex numbers with $r_n \neq 0$. The $r_1, r_2, r_3, \dots$ in this context will usually be referred to as the "partial numerators" of the continued fraction and the terms $s_0, s_1, s_2, \dots$ are the "partial denominators". The most common restriction imposed on continued fractions is to have $r_n = 1$ and then call the expression a *simple continued fraction*, denoted by $[s_0; s_1, s_2, \dots]$. A periodic continued fraction is one that repeats and has the form $[s_0; s_1, \dots, s_m, \overline{s_{m+1}, \dots, s_n}]$.

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Let F_n and L_n denote the Fibonacci and Lucas numbers, defined, respectively, by $F_n = F_{n-1} + F_{n-2}$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $F_0 = 0$, $F_1 = 1$ and $L_0 = 2$, $L_1 = 1$.

Several generalizations of the Fibonacci sequence have been presented in the literature [1–8]. One of them was given by Edson and Yayenie in [6], called the bi-periodic Fibonacci sequence defined for any nonzero real numbers a, b, and any integer $n \geq 2$, as follows

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{for } n \text{ even}, \\ bq_{n-1} + q_{n-2}, & \text{for } n \text{ odd}, \end{cases}$$

with initial values $q_0 = 0$ and $q_1 = 1$. Note that for $a = b = 1$, we get the classical Fibonacci sequence. Similarly, Bilgici [5] introduced the bi-periodic Lucas sequence, for $n \geq 2$, as follows

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{for } n \text{ even}, \\ al_{n-1} + l_{n-2}, & \text{for } n \text{ odd}, \end{cases}$$

with initial conditions $l_0 = 2$ and $l_1 = a$. It gives the classical Lucas sequence for $a = b = 1$.

Also, the bi-periodic Fibonacci and Lucas sequences satisfy, for $n \geq 4$, the same recurrence relation

$$w_n = (ab + 2)w_{n-2} - w_{n-4}.$$

The Binet's formulas of the bi-periodic Fibonacci and Lucas sequences are given by

$$q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor n/2 \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \tag{1.1}$$

$$l_n = \frac{a^{\xi(n)}}{(ab)^{\lfloor (n+1)/2 \rfloor}} (\alpha^n + \beta^n), \tag{1.2}$$

where $\lfloor x \rfloor$ is the floor function of x, $\xi(n) = n - 2\lfloor n/2 \rfloor$ is the parity function and α, β are the roots of the characteristic equation $x^2 - abx - ab = 0$ given by

$$\alpha = \frac{ab + \sqrt{ab(ab+4)}}{2} \quad \text{and} \quad \beta = \frac{ab - \sqrt{ab(ab+4)}}{2}.$$

It is well known that the limit of the ratios of consecutive Fibonacci numbers is the golden ratio ϕ. Thus

$$\phi = [\overline{1}] = [1; 1, 1, \cdots] = \frac{1 + \sqrt{5}}{2}.$$

For more details on continued fractions and their connection to the Fibonacci sequence, the reader is referred to [9–12].

Consider the bi-periodic Fibonacci sequence $\{q_n\}_{n\geq 0}$ with a and b nonnegative integers. For $a \neq b$, we have

$$\frac{q_n}{q_{n-1}} = a^{\xi(n-1)} b^{\xi(n)} + \frac{1}{\frac{q_{n-1}}{q_{n-2}}}.$$

So the ratios of the successive terms do not converge (see [6]). Therefore

$$\lim_{n\to\infty} \frac{q_{2n}}{q_{2n-1}} = \frac{\alpha}{b}, \quad \lim_{n\to\infty} \frac{q_{2n+1}}{q_{2n}} = \frac{\alpha}{a}, \quad \text{and} \quad \lim_{n\to\infty} \frac{q_n}{q_{n-2}} = \alpha + 1.$$

2. Main results

Let n, s, r, and t be positive integers with $r < s$ and $n \geq 2t$. Our aim is to derive closed form expressions for the continued fractions of $\lim\limits_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-t)+r}}$. Let's start with $t = 1$.

This theorem gives the recurrence satisfied by a subsequence of arithmetic progression.

Theorem 2.1. *For $n \geq 2$ and fixed s and r, we have the following relation*

$$q_{sn+r} = \left(\frac{b}{a}\right)^{\xi(s)\xi(n+r)} l_s q_{s(n-1)+r} + (-1)^{s+1} q_{s(n-2)+r}. \tag{2.1}$$

Proof. From Binet's formulas (1.1), (1.2), and since $\alpha\beta = -ab$, $\xi(n+m) = \xi(n) + \xi(m) - 2\xi(n)\xi(m)$, and $\lfloor n/2 \rfloor = (n - \xi(n))/2$, we can write

$$\begin{aligned}
&\left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} l_s q_{s(n-1)+r} + (-1)^{s+1} q_{s(n-2)+r} \\
&= \left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} \frac{a^{\xi(s)}}{(ab)^{\lfloor (s+1)/2 \rfloor}} (\alpha^s + \beta^s) \frac{a^{\xi(s(n-1)+r+1)}}{(ab)^{\lfloor (s(n-1)+r)/2 \rfloor}} \\
&\quad \times \left(\frac{\alpha^{s(n-1)+r} - \beta^{s(n-1)+r}}{\alpha - \beta}\right) \\
&\quad + (-1)^{s+1} \frac{a^{\xi(s(n-2)+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta}\right) \\
&= \left(\frac{b}{a}\right)^{\xi(s)\xi(sn+r)} \frac{a^{\xi(sn+r+1)+2\xi(s)\xi(sn+r)}}{(ab)^{\lfloor (sn+r)/2 \rfloor + \xi(s)\xi(sn+r)}} \\
&\quad \times \left(\frac{\alpha^{sn+r} - \beta^{sn+r} + (\alpha\beta)^s \left(\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}\right)}{\alpha - \beta}\right) \\
&\quad + (-1)^{s+1} \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta}\right)
\end{aligned}$$

$$= \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (sn+r)/2 \rfloor}} \left(\frac{\alpha^{sn+r} - \beta^{sn+r} + (-ab)^s \left(\alpha^{s(n-2)+r} - \beta^{s(n-2)+r} \right)}{\alpha - \beta} \right)$$

$$- (-1)^s \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (s(n-2)+r)/2 \rfloor}} \left(\frac{\alpha^{s(n-2)+r} - \beta^{s(n-2)+r}}{\alpha - \beta} \right)$$

$$= \frac{a^{\xi(sn+r+1)}}{(ab)^{\lfloor (sn+r)/2 \rfloor}} \left(\frac{\alpha^{sn+r} - \beta^{sn+r}}{\alpha - \beta} \right)$$

$$= q_{sn+r}. \qquad \square$$

The following theorem derives the closed form for continued fraction expressions of $\lim\limits_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-1)+r}}$, which is the ratio of two consecutive terms of the cited subsequence.

Theorem 2.2. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-1)+r}} = \begin{cases} \left[l_s - 1; \overline{1, l_s - 2}\right] = \dfrac{\alpha^s}{(ab)^{s/2}}, & \text{for } s \text{ even,} \\ \left[l_s; \overline{l_s}\right] = \dfrac{a\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n+r \text{ even,} \\ \left[\dfrac{b}{a} l_s; \overline{\dfrac{b}{a} l_s}\right] = \dfrac{b\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ and } n+r \text{ odd.} \end{cases}$$

Proof. From (2.1), we have

- For s even

$$\frac{q_{sn+r}}{q_{s(n-1)+r}} = l_s - \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = l_s - 1 + \frac{q_{s(n-1)+r} - q_{s(n-2)+r}}{q_{s(n-1)+r}}$$

$$= l_s - 1 + \cfrac{1}{\cfrac{q_{s(n-1)+r}}{q_{s(n-1)+r} - q_{s(n-2)+r}}}$$

$$= l_s - 1 + \cfrac{1}{1 + \cfrac{q_{s(n-2)+r}}{q_{s(n-1)+r} - q_{s(n-2)+r}}}$$

$$= l_s - 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{q_{s(n-1)+r}}{q_{s(n-2)+r}} - 1}}$$

$$= l_s - 1 + \cfrac{1}{1 + \cfrac{1}{l_s - 2 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}.$$

- For s odd and $n+r$ even

$$\frac{q_{sn+r}}{q_{s(n-1)+r}} = l_s + \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = l_s + \cfrac{1}{l_s + \cfrac{1}{l_s + \cfrac{1}{\ddots}}}.$$

- For s and $n+r$ odd

$$\frac{q_{sn+r}}{q_{s(n-1)+r}} = \frac{b}{a}l_s + \frac{q_{s(n-2)+r}}{q_{s(n-1)+r}} = \frac{b}{a}l_s + \cfrac{1}{\frac{b}{a}l_s + \cfrac{1}{\frac{b}{a}l_s + \cfrac{1}{\ddots}}}.$$

Using Binet's formula of the bi-periodic Fibonacci sequence, we obtain

$$\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-1)+r}} = \lim_{n\to\infty}\frac{a^{\xi(sn+r-1)}(ab)^{\lfloor(s(n-1)+r)/2\rfloor}}{a^{\xi(s(n-1)+r-1)}(ab)^{\lfloor(sn+r)/2\rfloor}}\alpha^s\left(\frac{1-\left(\frac{\beta}{\alpha}\right)^{sn+r}}{1-\left(\frac{\beta}{\alpha}\right)^{s(n-1)+r}}\right)$$

$$= \begin{cases} \dfrac{\alpha^s}{(ab)^{s/2}}, & \text{for } s \text{ even,} \\ \dfrac{a\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n+r \text{ even,} \\ \dfrac{b\alpha^s}{(ab)^{(s+1)/2}}, & \text{for } s \text{ odd and } n+r \text{ odd.} \end{cases} \qquad \square$$

For the classical Fibonacci sequence, when $a=b=1$, Theorem 2.2 gives the following result.

Corollary 2.3.

$$\lim_{n\to\infty}\frac{F_{sn+r}}{F_{s(n-1)+r}} = \phi^s = \begin{cases} \left[L_s-1;\overline{1, L_s-2}\right], & \textit{for } s \textit{ even,} \\ \left[L_s;\overline{L_s}\right], & \textit{for } s \textit{ odd.} \end{cases}$$

The following theorem extends Theorem 2.1 in the sense that it considers the t-periodic terms in the arithmetic progression subsequence.

Theorem 2.4. *For $n \geq 2t$, we have*

$$q_{sn+r} = \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)} l_{st}q_{s(n-t)+r} + (-1)^{st+1}q_{s(n-2t)+r}. \tag{2.2}$$

Proof. Using Binet's formula, we get

$$q_{sn+r} = C_1\chi_{sn+r}\alpha^{sn+r} + C_2\chi_{sn+r}\beta^{sn+r},$$

with $\chi_{sn+r}=\frac{a^{\xi(sn+r-1)}}{(ab)^{\lfloor (sn+r)/2\rfloor}}$ and $C_1=-C_2=1/(\alpha-\beta)$. The matrix form gives

$$\begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} = \begin{pmatrix} \chi_{s(n-t)+r}\alpha^{s(n-t)+r} & \chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ \chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Thus

$$\begin{aligned} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} &= \begin{pmatrix} \chi_{s(n-t)+r}\alpha^{s(n-t)+r} & \chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ \chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} \end{pmatrix}^{-1} \begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} \\ &= \frac{1}{D}\begin{pmatrix} \chi_{s(n-2t)+r}\beta^{s(n-2t)+r} & -\chi_{s(n-t)+r}\beta^{s(n-t)+r} \\ -\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r} & \chi_{s(n-t)+r}\alpha^{s(n-t)+r} \end{pmatrix} \begin{pmatrix} q_{s(n-t)+r} \\ q_{s(n-2t)+r} \end{pmatrix} \\ &= \frac{1}{D}\begin{pmatrix} \chi_{s(n-2t)+r}\beta^{s(n-2t)+r}q_{s(n-t)+r} & -\chi_{s(n-t)+r}\beta^{s(n-t)+r}q_{s(n-2t)+r} \\ -\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r}q_{s(n-t)+r} & \chi_{s(n-t)+r}\alpha^{s(n-t)+r}q_{s(n-2t)+r} \end{pmatrix} \end{aligned}$$

where $D=\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\left(\alpha^{s(n-t)+r}\beta^{s(n-2t)+r}-\alpha^{s(n-2t)+r}\beta^{s(n-t)+r}\right)$.

Therefore

$$C_1=\frac{\chi_{s(n-2t)+r}q_{s(n-t)+r}-\chi_{s(n-t)+r}\beta^{st}q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r}(\alpha^{st}-\beta^{st})}$$

and

$$C_2=-\frac{\chi_{s(n-2t)+r}q_{s(n-t)+r}-\chi_{s(n-t)+r}\alpha^{st}q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\beta^{s(n-2t)+r}(\alpha^{st}-\beta^{st})}.$$

Plugging into $q_{sn+r}=C_1\chi_{sn+r}\alpha^{sn+r}+C_2\chi_{sn+r}\beta^{sn+r}$, we get

$$\begin{aligned} q_{sn+r} &= \chi_{sn+r}\frac{\chi_{s(n-2t)+r}q_{s(n-t)+r}-\chi_{s(n-t)+r}\beta^{st}q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\alpha^{s(n-2t)+r}(\alpha^{st}-\beta^{st})}\alpha^{sn+r} \\ &\quad -\chi_{sn+r}\frac{\chi_{s(n-2t)+r}q_{s(n-t)+r}-\chi_{s(n-t)+r}\alpha^{st}q_{s(n-2t)+r}}{\chi_{s(n-t)+r}\chi_{s(n-2t)+r}\beta^{s(n-2t)+r}(\alpha^{st}-\beta^{st})}\beta^{sn+r} \\ &= \frac{\chi_{sn+r}q_{s(n-t)+r}\left(\alpha^{2st}-\beta^{2st}\right)}{\chi_{s(n-t)+r}(\alpha^{st}-\beta^{st})}-\frac{\chi_{sn+r}q_{s(n-2t)+r}(\alpha\beta)^{st}(\alpha^{st}-\beta^{st})}{\chi_{s(n-2t)+r}(\alpha^{st}-\beta^{st})} \\ &= \frac{a^{\xi(sn+r-1)}(ab)^{\lfloor (s(n-t)+r)/2\rfloor}}{a^{\xi(s(n-t)+r-1)}(ab)^{\lfloor (sn+r)/2\rfloor}}q_{s(n-t)+r}\left(\alpha^{st}+\beta^{st}\right) \\ &\quad -\frac{a^{\xi(sn+r-1)}(ab)^{\lfloor (s(n-2t)+r)/2\rfloor}}{a^{\xi(s(n-2t)+r-1)}(ab)^{\lfloor (sn+r)/2\rfloor}}(-ab)^{st}q_{s(n-2t)+r} \\ &= \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)}\frac{a^{\xi(st)}}{(ab)^{\lfloor (st+1)/2\rfloor}}\left(\alpha^{st}+\beta^{st}\right)q_{s(n-t)+r}+(-1)^{st+1}q_{s(n-2t)+r} \\ &= \left(\frac{b}{a}\right)^{\xi(st)\xi(sn+r)}l_{st}q_{s(n-t)+r}+(-1)^{st+1}q_{s(n-2t)+r}. \end{aligned}$$

□

We will now calculate $\lim\limits_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-t)+r}}$ explicitly, the ratio of two consecutive terms in the t-periodic subsequence.

Theorem 2.5. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-t)+r}} = \begin{cases} [l_{st}-1; \overline{1, l_{st}-2}] = \dfrac{\alpha^{st}}{(ab)^{st/2}}, & \text{for } st \text{ even,} \\ [l_{st}; \overline{l_{st}}] = \dfrac{a\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } st \text{ odd and } n+r \text{ even,} \\ \left[\dfrac{b}{a} l_{st}; \overline{\dfrac{b}{a} l_{st}}\right] = \dfrac{b\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } s,\ t,\ \text{and } n+r \text{ odd.} \end{cases}$$

Proof. Using (2.2), we have

- For st even

$$\frac{q_{sn+r}}{q_{s(n-t)+r}} = l_{st} - \frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}} = l_{st} - 1 + \frac{q_{s(n-t)+r} - q_{s(n-2t)+r}}{q_{s(n-t)+r}}$$

$$= l_{st} - 1 + \cfrac{1}{1 + \cfrac{q_{s(n-2t)+r}}{q_{s(n-t)+r} - q_{s(n-2t)+r}}} = l_{st} - 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}} - 1}}.$$

- For st odd and $n+r$ even

$$\frac{q_{sn+r}}{q_{s(n-t)+r}} = l_{st} + \frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}} = l_{st} + \cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}}}.$$

- For s, t, and $n+r$ odd

$$\frac{q_{sn+r}}{q_{s(n-t)+r}} = \frac{b}{a} l_{st} + \frac{q_{s(n-2t)+r}}{q_{s(n-t)+r}} = \frac{b}{a} l_{st} + \cfrac{1}{\cfrac{q_{s(n-t)+r}}{q_{s(n-2t)+r}}}.$$

Using Binet's formula of the bi-periodic Fibonacci sequence, we obtain

$$\lim_{n\to\infty} \frac{q_{sn+r}}{q_{s(n-t)+r}} = \lim_{n\to\infty} \frac{a^{\xi(sn+r-1)} (ab)^{\lfloor (s(n-t)+r)/2 \rfloor}}{a^{\xi(s(n-t)+r-1)} (ab)^{\lfloor (sn+r)/2 \rfloor}} \alpha^{st} \left(\frac{1 - \left(\frac{\beta}{\alpha}\right)^{sn+r}}{1 - \left(\frac{\beta}{\alpha}\right)^{s(n-t)+r}} \right)$$

$$= \begin{cases} \dfrac{\alpha^{st}}{(ab)^{st/2}}, & \text{for } st \text{ even,} \\ \dfrac{a\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } st \text{ odd and } n+r \text{ even,} \\ \dfrac{b\alpha^{st}}{(ab)^{(st+1)/2}}, & \text{for } s,\ t,\ \text{and } n+r \text{ odd.} \end{cases}$$

□

Note that if we take $t = 2$ in Theorem 2.5, we get the following result.

Corollary 2.6.

$$\lim_{n\to\infty}\frac{q_{sn+r}}{q_{s(n-2)+r}}=\left(\frac{\alpha^2}{ab}\right)^s=(\alpha+1)^s=\left[l_{2s}-1;\overline{1,l_{2s}-2}\right].$$

As a consequence of Theorem 2.5, for $a=b=1$, we have the following result.

Corollary 2.7.

$$\lim_{n\to\infty}\frac{F_{sn+r}}{F_{s(n-t)+r}}=\phi^{st}=\begin{cases}\left[L_{st}-1;\overline{1,L_{st}-2}\right], & \textit{for } st \textit{ even},\\ \left[L_{st};\overline{L_{st}}\right], & \textit{for } st \textit{ odds}.\end{cases}$$

In the following theorem, we give a relation between some bi-periodic Fibonacci and Lucas sequences.

Theorem 2.8. *For $n\geq 1$, we obtain*

$$q_s l_{sn+r}=\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}q_{s(n+1)+r}+(-1)^{s+1}\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}q_{s(n-1)+r}.$$

Proof. From Binet's formulas (1.1), (1.2), and since $\alpha\beta=-ab$, we write

$$\begin{aligned}
&\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}\left(q_{s(n+1)+r}+(-1)^{s+1}q_{s(n-1)+r}\right)\\
&=\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}\frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor(s(n+1)+r)/2\rfloor}}\left(\frac{\alpha^{s(n+1)+r}-\beta^{s(n+1)+r}}{\alpha-\beta}\right)\\
&\quad+(-1)^{s+1}\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}\frac{a^{\xi(s(n-1)+r+1)}}{(ab)^{\lfloor(s(n-1)+r)/2\rfloor}}\left(\frac{\alpha^{s(n-1)+r}-\beta^{s(n-1)+r}}{\alpha-\beta}\right)\\
&=\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}\frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor(s(n+1)+r)/2\rfloor}(\alpha-\beta)}\\
&\quad\times\left(\alpha^{sn+r}\left(\alpha^s-\left(\frac{-ab}{\alpha}\right)^s\right)-\beta^{sn+r}\left(\beta^s-\left(\frac{-ab}{\beta}\right)^s\right)\right)\\
&=\left(\frac{a}{b}\right)^{\xi(r)\xi(s+1)}\frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor(s(n+1)+r)/2\rfloor}}\left(\frac{\alpha^s-\beta^s}{\alpha-\beta}\right)\left(\alpha^{sn+r}+\beta^{sn+r}\right).
\end{aligned}$$

Since

$$\begin{aligned}
\frac{a^{\xi(s(n+1)+r+1)}}{(ab)^{\lfloor(s(n+1)+r)/2\rfloor}}&=\frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(sn+r)}}{(ab)^{\lfloor(sn+r+1)/2\rfloor+\lfloor s/2\rfloor+\xi(s)\xi(sn+r)-\xi(sn+r)}}\\
&=\frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(r)}}{(ab)^{\lfloor(sn+r+1)/2\rfloor+\lfloor s/2\rfloor-\xi(s+1)\xi(sn+r)}}\\
&=\frac{a^{\xi(sn+r)+\xi(s+1)-2\xi(s+1)\xi(r)}}{(ab)^{\lfloor(sn+r+1)/2\rfloor+\lfloor s/2\rfloor-\xi(s+1)\xi(r)}}\\
&=\left(\frac{b}{a}\right)^{\xi(s+1)\xi(r)}\frac{a^{\xi(s+1)}}{(ab)^{\lfloor s/2\rfloor}}\frac{a^{\xi(sn+r)}}{(ab)^{\lfloor(sn+r+1)/2\rfloor}},
\end{aligned}$$

we obtain the result. □

Note that for $s = 1$ and $r = 0$ in Theorem 2.8, we get the following result.

Corollary 2.9. *For $n \geq 1$, we obtain*

$$l_n = q_{n+1} + q_{n-1}.$$

In the following theorems, we give two continued fractions involving the bi-periodic Fibonacci and Lucas sequences.

Theorem 2.10.

$$\lim_{n\to\infty} \frac{l_n}{q_{n+1}} = [1; ab+1, \overline{1, ab}] = \beta + 2$$

and

$$\lim_{n\to\infty} \frac{q_{n+1}}{l_n} = [0; 1, ab+1, \overline{1, ab}] = \frac{\alpha+2}{ab+4}.$$

Proof. The bi-periodic Fibonacci and Lucas sequences satisfy the equation

$$l_n = q_{n+1} + q_{n-1}.$$

Thus, we obtain

$$\frac{l_n}{q_{n+1}} = \frac{q_{n+1} + q_{n-1}}{q_{n+1}} = 1 + \frac{1}{\frac{q_{n+1}}{q_{n-1}}}.$$

Using Corollary 2.6 by taking $s = 1$ and $r = 1$, we get the result.

Furthermore,

$$\lim_{n\to\infty} \frac{l_n}{q_{n+1}} = \lim_{n\to\infty} (\alpha-\beta)\frac{\alpha^n - \beta^n}{\alpha^{n+1} - \beta^{n+1}} = \lim_{n\to\infty} \frac{\alpha-\beta}{\alpha} \frac{1 - \left(\frac{\beta}{\alpha}\right)^n}{1 - \left(\frac{\beta}{\alpha}\right)^{n+1}} = \beta + 2.$$

Taking the reciprocal of this value, we get

$$\frac{q_{n+1}}{l_n} = \frac{1}{\frac{l_n}{q_{n+1}}} \quad \text{and} \quad \lim_{n\to\infty} \frac{q_{n+1}}{l_n} = \frac{1}{\beta+2} = \frac{\alpha+2}{ab+4}. \qquad \square$$

For the arithmetic progression situation, we get

Theorem 2.11. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{l_{sn+r}}{q_{s(n+1)+r}} =$$

$$\begin{cases} \frac{1}{q_s} + \frac{1/q_s}{l_{2s}-1+} \frac{1}{1+} \frac{1}{l_{2s}-2+} \frac{1}{1+} \frac{1}{l_{2s}-2} \cdots = \frac{(ab)^{(s-1)/2}(\alpha-\beta)}{\alpha^s}, & \text{for } s \text{ odd,} \\ 0 + \frac{1/q_s}{1} + \frac{1}{l_{2s}-2+} \frac{1}{1+} \frac{1}{l_{2s}-2+} \frac{1}{1} \cdots = \frac{b(ab)^{(s-2)/2}(\alpha-\beta)}{\alpha^s}, & \text{for } s \text{ and } r \text{ even,} \\ 0 + \frac{a/(bq_s)}{1} + \frac{1}{l_{2s}-2+} \frac{1}{1+} \frac{1}{l_{2s}-2+} \frac{1}{1} \cdots = \frac{a(ab)^{(s-2)/2}(\alpha-\beta)}{\alpha^s}, & \text{for } s \text{ even and } r \text{ odd.} \end{cases}$$

Proof. By taking $r \to r+s$ in Corollary 2.6 and using Theorem 2.8, we have

- For s odd

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{1}{q_s} + \frac{1}{q_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = 1/q_s + \cfrac{1/q_s}{\cfrac{q_{s(n+1)+r}}{q_{s(n-1)+r}}}.$$

- For s and r even

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{1}{q_s} - \frac{1}{q_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = \frac{1}{q_s}\frac{q_{s(n+1)+r} - q_{s(n-1)+r}}{q_{s(n+1)+r}} = \cfrac{1/q_s}{1 + \cfrac{1}{\cfrac{q_{s(n+1)+r}}{q_{s(n-1)+r}} - 1}}.$$

- For s even and r odd

$$\frac{l_{sn+r}}{q_{s(n+1)+r}} = \frac{a}{bq_s} - \frac{a}{bq_s}\frac{q_{s(n-1)+r}}{q_{s(n+1)+r}} = \frac{a}{bq_s}\frac{\left(q_{s(n+1)+r} - q_{s(n-1)+r}\right)}{q_{s(n+1)+r}} = \cfrac{a/(bq_s)}{1 + \cfrac{1}{\cfrac{q_{s(n+1)+r}}{q_{s(n-1)+r}} - 1}}.$$

Using Binet's formulas for the bi-periodic Fibonacci and Lucas sequences, we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{l_{sn+r}}{q_{s(n+1)+r}} \\
&= \lim_{n\to\infty} \frac{a^{\xi(sn+r)}(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}}{a^{\xi(s(n+1)+r-1)}(ab)^{\lfloor (sn+r+1)/2 \rfloor}}(\alpha-\beta)\frac{\alpha^{sn+r} - \beta^{sn+r}}{\alpha^{s(n+1)+r} - \beta^{s(n+1)+r}} \\
&= \lim_{n\to\infty} \frac{a^{\xi(sn+r)}(ab)^{\lfloor (s(n+1)+r)/2 \rfloor}}{a^{\xi(s(n+1)+r-1)}(ab)^{\lfloor (sn+r+1)/2 \rfloor}} \frac{\alpha-\beta}{\alpha^s} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{sn+r}}{1 - \left(\frac{\beta}{\alpha}\right)^{s(n+1)+r}} \\
&= \begin{cases} (ab)^{(s-1)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ odd}, \\ b(ab)^{(s-2)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ and } r \text{ even}, \\ a(ab)^{(s-2)/2}(\alpha-\beta)/\alpha^s, & \text{for } s \text{ even and } r \text{ odd}. \end{cases}
\end{aligned}$$

□

Taking the reciprocal of Theorem 2.11, the next result follows.

Theorem 2.12. *For $r < s$, we have*

$$\lim_{n\to\infty} \frac{q_{s(n+1)+r}}{l_{sn+r}} =$$

$$\begin{cases}
0 + \frac{1}{1/q_s} + \frac{1/q_s}{l_{2s}-1} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{\alpha^s}{(ab)^{(s-1)/2}(\alpha-\beta)}, & \text{for } s \text{ odd}, \\
0 + \frac{1}{0} + \frac{1/q_s}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{a\alpha^s}{(ab)^{s/2}(\alpha-\beta)}, & \text{for } s \text{ and } r \text{ even}, \\
0 + \frac{1}{0} + \frac{a/(bq_s)}{1} + \frac{1}{l_{2s}-2} + \frac{1}{1} + \frac{1}{l_{2s}-2} + \cdots = \frac{b\alpha^s}{(ab)^{s/2}(\alpha-\beta)}, & \text{for } s \text{ even and } r \text{ odd}.
\end{cases}$$

Proof. Knowing that

$$\frac{q_{s(n+1)+r}}{l_{sn+r}} = \frac{1}{\dfrac{l_{sn+r}}{q_{s(n+1)+r}}} \quad \text{and} \quad \lim_{n\to\infty} \frac{q_{s(n+1)+r}}{l_{sn+r}} = \lim_{n\to\infty} \frac{1}{\dfrac{l_{sn+r}}{q_{s(n+1)+r}}},$$

using Theorem 2.11, we get the result. □

Note that, for $a = b = 1$, we get the following result.

Corollary 2.13.

$$\lim_{n\to\infty} \frac{L_{sn+r}}{F_{s(n+1)+r}} = \frac{\alpha-\beta}{\alpha^s} = \begin{cases} \dfrac{1}{F_s} + \dfrac{1/F_s}{L_{2s}-1} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \text{for } s \text{ odd},\\ 0 + \dfrac{1/F_s}{1} + \dfrac{1}{L_{2s}-2} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \text{for } s \text{ even}, \end{cases}$$

and

$$\lim_{n\to\infty} \frac{F_{s(n+1)+r}}{L_{sn+r}} = \frac{\alpha^s}{\alpha-\beta} = \begin{cases} 0 + \dfrac{1}{1/F_s} + \dfrac{1/F_s}{l_{2s}-1} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \text{for } s \text{ odd},\\ 0 + \dfrac{1}{0} + \dfrac{1/F_s}{1} + \dfrac{1}{L_{2s}-2} + \dfrac{1}{1} + \dfrac{1}{L_{2s}-2} + \cdots, & \text{for } s \text{ even}. \end{cases}$$

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