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On the generalized Fibonacci like sequences and matrices

Kalika Prasad[*], **Hrishikesh Mahato**[†]

Department of Mathematics, Central University of Jharkhand, India
klkaprsd@gmail.com
hrishikesh.mahato@cuj.ac.in

Abstract. In this paper, we study the generalized Fibonacci like sequences $\{t_{k,n}\}_{k\in\{2,3\},n\in\mathbb{N}}$ with arbitrary initial seed and give some new and well-known identities like Binet's formula, trace sequence, Catalan's identity, generating function, etc. Further, we study various properties of these generalized sequences, establish a recursive matrix and relationships with Fibonacci and Lucas numbers and sequence of Fibonacci traces. In this study, we examine the nature of identities and recursive matrices for arbitrary initial values.

Keywords: Binet's formula, Fibonacci like sequences, generating function, recursive matrix, trace sequence

AMS Subject Classification: 11B37, 11B39, 11B83, 65Q30

1. Introduction

In recent years, several papers [1, 2, 4, 18] published involving new identities and results based on Fibonacci-like sequences and their generalizations which have many interesting properties. One can refer to the book [8] of T. Koshy for more such sequences, generalizations, and rich applications.

In spite of many articles, books, and literature reviews on Fibonacci-like sequences and their generalizations [3–10, 13, 17], investigating new identities, results and their applications are interesting areas among researchers. Ongoing through the available literature review on generalizations of Fibonacci sequences, it can be noted that mainly the work may be generalized in two directions. Either the re-

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[†] Corresponding author

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cursive formula can be generalized and extended or the formula is preserved with arbitrary initial assumptions. Kalman et al. [6] discussed some well-known results of classical Fibonacci-like sequences and demonstrated that many of the properties of these sequences can be established for much more general classes.

The recursive matrices corresponding to recursive sequences always attract researchers to investigate new identities and establish some well-known results such as Binet's formula, determinants, permanents, etc. For instance, Kumari et al. [9] have proposed some new families of identities of k-Mersenne and generalized k-Gaussian Mersenne numbers and their polynomials. Tianxiao et al. [16] presented a recursive matrix for recursive sequences of order three $a_{k+3} = pa_k + qa_{k+1} + ra_{k+2}$ with arbitrary initial conditions, and discussed some special third order recurrences such as Padavon and Perrin numbers. Saba et al. [14] introduced the concept of bivariate Mersenne Lucas polynomials then established Binet's formula and obtained many well-known identities using Binet's formula. Özkan et al. [11] obtained the elements of the Lucas polynomials by using two matrices and extended the study to the n-step Lucas polynomials, whereas Testan et al. [15] given some families of generalized Fibonacci and Lucas polynomials and developed some properties of these families and established interrelationships.

1.1. Fibonacci and Lucas matrices

The well-known integer sequences, Fibonacci $\{f_{2,n}\}$ and Lucas $\{u_{2,n}\}$ sequence are defined as

$$f_{2,n+2} = f_{2,n} + f_{2,n+1} \quad \text{and} \quad u_{2,n+2} = u_{2,n} + u_{2,n+1}; \quad n \geq 0, \tag{1.1}$$

with $f_{2,0} = 0,\ f_{2,1} = 1$ for $\{f_{2,n}\}$ and $u_{2,0} = 2,\ u_{2,1} = 1$ for $\{u_{2,n}\}$. These sequences are also extendable in the negative direction which can be achieved by rearranging Eqn. (1.1). It is also noted that $f_{2,-n} = (-1)^{n+1} f_{2,n}$ and $u_{2,-n} = (-1)^n u_{2,n}$ for $n \in \mathbb{N} \cup \{0\}$.

A matrix sequence [8] corresponding to above integer sequences are given as

$$Q_2^n = \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \quad \text{and} \quad L_2^{(n)} = \begin{bmatrix} u_{2,n+1} & u_{2,n} \\ u_{2,n} & u_{2,n-1} \end{bmatrix}. \tag{1.2}$$

Further in [12], Prasad et al. have obtained some interesting properties of generalized Fibonacci matrices (Q_k^n) given in the following theorem. We use these identities to establish some new identities and results in this paper.

Theorem 1.1 ([12])**.** *Let $n, l \in \mathbb{Z},\ k(\geq 2) \in \mathbb{N}$ and Q_k^n be a generalized Fibonacci matrix of order k, then we have*

1. $(Q_k^1)^n = Q_k^n$,
2. $Q_k^0 = I_k$, *where I_k is identity matrix of order k,*
3. $Q_k^n Q_k^l = Q_k^{n+l}$,

4. $\det(Q_k^n) = (-1)^{(k-1)n}$.

Note. Throughout the paper, we adopt the notation $t_{k,n}$ to denote the nth term of the sequence $\{t_{k,n}\}$ of order k with arbitrary initial values.

2. The $\{t_{2,n}\}$ sequence and some properties

Consider the second order linear difference equation given by

$$t_{2,n+2} = t_{2,n+1} + t_{2,n}, \quad n \geq 0 \quad \text{with} \quad t_{2,0} = a \quad \text{and} \quad t_{2,1} = b. \tag{2.1}$$

Similar to the Fibonacci sequence, the sequence $\{t_{2,n}\}$ can also be extended in the negative direction by rearranging Eqn. (2.1) as $t_{2,-n} = t_{2,-n+2} - t_{2,-n+1}$; $n \in \mathbb{N}$ with the same initial values.

Thus, the first few terms of the sequence are as follows:

n	...	-3	-2	-1	**0**	**1**	2	3	4	5	6	...
$t_{2,n}$	...	-3a+2b	2a-b	-a+b	**a**	**b**	a+b	a+2b	2a+3b	3a+5b	5a+8b	...
$f_{2,n}$	...	2	-1	1	**0**	**1**	1	2	3	5	8	...
$l_{2,n}$	...	-4	3	-1	**2**	**1**	3	4	7	11	18	...

Remark 2.1. For a sequence $\{t_{2,n}\}_{n\geq 0}$ satisfying Eqn. (2.1), we have

$$t_{2,n} = af_{2,n-1} + bf_{2,n}, \quad \text{where} \quad f_{2,0} = 0 \quad \text{and} \quad f_{2,1} = 1. \tag{2.2}$$

2.1. Matrix formation

The matrix sequence $\{T_2^{(n)}\}_{n\geq 0}$ associated with the integer sequence $\{t_{2,n}\}$ is defined as

$$T_2^{(n)} = \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \quad \text{with} \quad T_2^{(0)} = \begin{bmatrix} b & a \\ a & b-a \end{bmatrix}, \tag{2.3}$$

where $\det(T_2^{(0)}) = b(-a+b) - a^2 = b^2 - ab - a^2 = K(\text{say})$.

In next theorems and results, we present some interesting recursive and explicit formulas for the matrix sequence $T_2^{(n)}$ associated with the Fibonacci matrices.

Theorem 2.2. *The determinant of matrix $T_2^{(n)}$ is given by*

$$\det(T_2^{(n)}) = (a^2 + ab - b^2)(-1)^{n-1} = K(-1)^n.$$

Proof. To prove it, we use the following result of Fibonacci numbers

$$f_{2,n+1}f_{2,n-2} - f_{2,n}f_{2,n-1} = (-1)^{n-1}. \tag{2.4}$$

Therefore,

$$\det(T_2^{(n)}) = t_{2,n+1}t_{2,n-1} - t_{2,n}^2$$

$$\begin{aligned}
&= (af_{2,n} + bf_{2,n+1})(af_{2,n-2} + bf_{2,n-1}) - (af_{2,n-1} + bf_{2,n})^2\\
&= a^2(f_{2,n}f_{2,n-2} - f_{2,n-1}^2) + b^2(f_{2,n+1}f_{2,n-1} - f_{2,n}^2)\\
&\quad + ab(f_{2,n}f_{2,n-1} + f_{2,n+1}f_{2,n-2} - 2f_{2,n}f_{2,n-1})\\
&= a^2[(-1)^{n-1}] + b^2[(-1)^n] + ab(f_{2,n+1}f_{2,n-2} - f_{2,n}f_{2,n-1})\\
&= a^2[(-1)^{n-1}] + b^2[(-1)^n] + ab[(-1)^{n-1}] \quad \text{(using Eqn. (2.4))}\\
&= (a^2 - b^2 + ab)(-1)^{n-1} = -K(-1)^{n-1} = K(-1)^n
\end{aligned}$$

as required. □

Corollary 2.3. $\det(T_2^{(n+1)}) = (-1)\det(T^{(n)})$.

Example 2.4 (Fibonacci matrix)**.** For $a = 0,\ b = 1$, we have $\det(T_2^{(n)}) = (-1)^n$.

Example 2.5 (Lucas matrix)**.** For $a = 2,\ b = 1$, we have $\det(T_2^{(n)}) = (-1)^n 5$.

Theorem 2.6. *Let $T_2^{(n)}$ be a matrix as defined in (2.3) and Q_2^n is the Fibonacci matrix, then we write*

$$T_2^{(n)} = Q_2^n T_2^{(0)} = T_2^{(0)} Q_2^n, \quad \forall n \in \mathbb{Z}.$$

Proof. We have

$$\begin{aligned}
Q_2^n T_2^{(0)} &= \begin{bmatrix} f_{2,n+1} & f_{2,n} \\ f_{2,n} & f_{2,n-1} \end{bmatrix} \begin{bmatrix} b & a \\ a & b-a \end{bmatrix} = \begin{bmatrix} bf_{2,n+1} + af_{2,n} & af_{2,n+1} + (b-a)f_{2,n} \\ bf_{2,n} + af_{2,n-1} & af_{2,n} + (b-a)f_{2,n-1} \end{bmatrix}\\
&= \begin{bmatrix} af_{2,n} + bf_{2,n+1} & bf_{2,n} + af_{2,n-1} \\ af_{2,n-1} + bf_{2,n} & bf_{2,n-1} + af_{2,n-2} \end{bmatrix} \quad \text{(using relation (1.1))}\\
&= \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \quad \text{(using relation (2.2))}\\
&= T_2^{(n)}.
\end{aligned}$$

By a similar argument, we have $T_2^{(0)} Q_2^n = T_2^{(n)}$. □

Corollary 2.7. *If $a = 0,\ b = 1$ then $T_2^{(0)} = I_2$ and $T_2^{(n)} = Q_2^n$, where I_2 is an identity matrix of order 2.*

Corollary 2.8. *For $n \in \mathbb{N}$, we have $T_2^{(n)} = Q_2 T_2^{(n-1)} = Q_2^{-1} T_2^{(n+1)}$.*

Theorem 2.9. *Let $T_2^{(n)}$ be a matrix as defined in (2.3), then we write*

$$T_2^{(n)} T_2^{(-n)} = (T_2^{(0)})^2.$$

Proof. By definition of $T_2^{(n)}$, we have

$$T_2^{(n)} T_2^{(-n)} = Q_2^{(n)} T_2^{(0)} Q_2^{(-n)} T_2^{(0)}$$

$$
\begin{aligned}
&= T_2^{(0)} Q_2^{(n)} Q_2^{(-n)} T_2^{(0)} \\
&= T_2^{(0)} I T_2^{(0)} = T_2^{(0)} T_2^{(0)} = (T_2^{(0)})^2
\end{aligned}
$$

as required. □

From Theorem 2.2, it is clear that the matrix $T_2^{(n)}$ is invertible if and only if $T_2^{(0)}$ is invertible i.e $\det(T_2^{(0)}) = K \neq 0$. Thus from Theorem 2.9, we have the inverse of $T_2^{(n)}$ given by

$$\mathrm{Inv}(T_2^{(n)}) = T_2^{(-n)} H^{-1}, \quad \text{where} \quad H = (T_2^{(0)})^2 \text{ and } a, b \text{ are such that } K \neq 0.$$

2.2. The trace sequence

Let us define another sequence $\{l_{2,n}\}$ of order two for the given sequence $\{t_{2,n}\}$ as follows

$$l_{2,n} = \mathrm{trace}(T_2^{(n)}) = t_{2,n+1} + t_{2,n-1}, \tag{2.5}$$

whose initial values in terms of a and b are obtained as

$$
\begin{aligned}
l_{2,0} &= t_{2,1} + t_{2,-1} = b + (b - a) = -a + 2b, \\
l_{2,1} &= t_{2,2} + t_{2,0} = (a + b) + a = 2a + b.
\end{aligned}
$$

Thus, Eqn. (2.5) can be re-stated free from $t_{2,n}$, recursively as

$$l_{2,n+2} = l_{2,n+1} + l_{2,n} \quad \text{with} \quad l_{2,0} = -a + 2b,\ l_{2,1} = 2a + b. \tag{2.6}$$

In particular, for $a = 0,\ b = 1$, $\{t_{2,n}\}$ becomes $\{f_{2,n}\}$ and its corresponding sequence of traces coincides with the standard Lucas sequence $\{u_{2,n}\}$.

Moreover, the matrix $M_2^{(n)}$ corresponding to trace sequence $\{l_{2,n}\}$ is given by

$$M_2^{(n)} = \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} \quad \text{with} \quad M_2^{(0)} = \begin{bmatrix} l_{2,1} & l_{2,0} \\ l_{2,0} & l_{2,-1} \end{bmatrix} = \begin{bmatrix} 2a+b & 2b-a \\ 2b-a & 3a-b \end{bmatrix}. \tag{2.7}$$

Theorem 2.10. *The determinant of matrix $M_2^{(n)}$ is given by*

$$\det(M_2^{(n)}) = 5K(-1)^{n+1} \quad \forall n \in \mathbb{Z}.$$

Proof. From Eqn. (2.7), we have

$$
\begin{aligned}
M_2^{(n)} &= \begin{bmatrix} l_{2,n+1} & l_{2,n} \\ l_{2,n} & l_{2,n-1} \end{bmatrix} = \begin{bmatrix} t_{2,n+2} + t_{2,n} & t_{2,n+1} + t_{2,n-1} \\ t_{2,n+1} + t_{2,n-1} & t_{2,n} + t_{2,n-2} \end{bmatrix} \\
&= \begin{bmatrix} t_{2,n+1} & t_{2,n} \\ t_{2,n} & t_{2,n-1} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = T_2^{(n)} L_2^{(0)} \quad \text{(from Eqn. (2.1) and Eqn. (1.2)).}
\end{aligned}
$$

Thus, $\det(M_2^{(n)}) = |T_2^{(n)} L_2^{(0)}| = |Q_2^n T_2^{(0)} L_2^{(0)}| = |Q_2^n||T_2^{(0)}||L_2^{(0)}| = 5K(-1)^{n+1}$. □

In particular for $n=0$, we have $\det(M_2^{(0)}) = 5a^2+5ab-5b^2 = -5K$. The first few terms of the trace sequence $\{l_{2,n}\}_{n\in\mathbb{Z}}$ are as follows:

n	...	-3	-2	-1	**0**	**1**	2	3	4	...
$l_{2,n}$	...	7a-4b	-4a+3b	3a-b	**-a+2b**	**2a+b**	a+3b	3a+4b	4a+7b	...

Remark 2.11. If $l_{2,n} = k_1a + k_2b$ for $n>0$, then we have

$$l_{2,-n+1} = (k_2a - k_1b)(-1)^n.$$

2.3. Binet's formula, identities and generating function

The characteristics equation for the second order linear difference equation (2.1) is given by

$$x^2 = x+1. \tag{2.8}$$

Equation (2.8) has two real roots, $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$, which satisfy

$$\alpha_1+\alpha_2 = 1, \quad \alpha_1-\alpha_2 = \sqrt{5}, \quad \alpha_1\alpha_2 = -1 \quad \text{and} \quad \frac{\alpha_1}{\alpha_2} = \frac{3+\sqrt{5}}{-2}. \tag{2.9}$$

And from the theory of difference equation we know that the general term of the Eqn. (2.1) can be expressed as:

$$t_{2,n} = c_1\alpha_1^n + c_2\alpha_2^n, \tag{2.10}$$

where c_1 and c_2 are arbitrary constants (to be evaluated) and α_1 and α_2 are characteristics roots.

Theorem 2.12 (Binet's formula)**.** *For $n \geq 0$, we have*

$$t_{2,n} = \frac{-A\alpha_1^n + B\alpha_2^n}{\sqrt{5}}, \tag{2.11}$$

where $A = a\alpha_2 - b$ and $B = a\alpha_1 - b$.

Proof. To establish the result, we eliminate arbitrary constants c_1 and c_2 from Eqn. (2.10). Now, putting the values of α_1 and α_2 in Eqn. (2.10), we get

$$t_{2,n} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2.12}$$

To determine the values of c_1 and c_2, we set $t_{2,0} = a$ and $t_{2,1} = b$ in Eqn. (2.12). Therefore,

$$t_{2,0} = a = c_1 + c_2 \quad \text{and} \quad t_{2,1} = b = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\implies b = \frac{1}{2}[a + \sqrt{5}(c_1 - c_2)],$$

which gives $c_1 + c_2 = a$ and $c_1 - c_2 = (2b - a)/\sqrt{5}$ and on solving we get

$$c_1 = \frac{a\sqrt{5} - (a-2b)}{2\sqrt{5}} \quad \text{and} \quad c_2 = \frac{a\sqrt{5} + (a-2b)}{2\sqrt{5}}.$$

Thus, from Eqn. (2.12), we have

$$\begin{aligned} t_{2,n} &= \frac{1}{2\sqrt{5}}\left[(a\sqrt{5} - (a-2b))\left(\frac{1+\sqrt{5}}{2}\right)^n + (a\sqrt{5} + (a-2b))\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \\ &= \frac{1}{\sqrt{5}}[-A\alpha_1^n + B\alpha_2^n] \end{aligned}$$

as required. □

Theorem 2.13. *For $n \in \mathbb{N}$, we have*

$$t_{2,-n} = (-1)^n \frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}}.$$

Proof. Replacing n by $-n$ in the Binet's formula (2.11), we get

$$\begin{aligned} t_{2,-n} &= \frac{-A\alpha_1^{-n} + B\alpha_2^{-n}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{-A}{\alpha_1^n} + \frac{B}{\alpha_2^n}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{-A\alpha_2^n + B\alpha_1^n}{\alpha_1^n\alpha_2^n}\right) \\ &= \frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}(-1)^n} = (-1)^n\frac{-A\alpha_2^n + B\alpha_1^n}{\sqrt{5}} \quad (\text{using } \alpha_1\alpha_2 = -1) \end{aligned}$$

as required. □

Theorem 2.14 (Catalan's identity)**.** *For the sequence $\{t_{2,n}\}$, we have*

$$t_{2,n-r}t_{2,n+r} - t_{2,n}^2 = \frac{(-1)^n(b^2 - a^2 - ab)}{2^r.5}[2^{r+1} - (\sqrt{5}-3)^r - (-\sqrt{5}-3)^r].$$

Proof. Using the Binet's formula (2.11), we write

$$\begin{aligned} & t_{2,n-r}t_{2,n+r} - t_{2,n}^2 \\ &= \left(\frac{-A\alpha_1^{n-r} + B\alpha_2^{n-r}}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^{n+r} + B\alpha_2^{n+r}}{\sqrt{5}}\right) - \left(\frac{-A\alpha_1^n + B\alpha_2^n}{\sqrt{5}}\right)^2 \\ &= \frac{1}{5}\left[AB(2\alpha_1^n\alpha_2^n - \alpha_1^{n-r}\alpha_2^{n+r} - \alpha_1^{n+r}\alpha_2^{n-r})\right] \\ &= \frac{1}{5}AB\alpha_1^n\alpha_2^n\left[(2 - \alpha_1^{-r}\alpha_2^r - \alpha_1^r\alpha_2^{-r})\right] \\ &= \frac{AB\alpha_1^n\alpha_2^n}{5}\left[2 - \left(\frac{3-\sqrt{5}}{-2}\right)^r - \left(\frac{3+\sqrt{5}}{-2}\right)^r\right] \quad (\text{using } (2.9)) \end{aligned}$$

$$= \frac{(-1)^n(b^2-a^2-ab)}{2^r.5}[2^{r+1}-(\sqrt{5}-3)^r-(-\sqrt{5}-3)^r]$$

as required. □

Corollary 2.15 (Cassini's identity)**.** *For the sequence* $\{t_{2,n}\}_{n\in\mathbb{N}}$*, we have*

$$t_{2,n-1}t_{2,n+1}-t_{2,n}^2=(-1)^n(b^2-a^2-ab).$$

Theorem 2.16 (d'Ocagne's identity)**.** *For positive integers* r *and* n*, we have*

$$t_{2,n}t_{2,r+1}-t_{2,n+1}t_{2,r}=\frac{(b^2-a^2-ab)}{\sqrt{5}}[\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n].$$

Proof. Using the Binet's formula (2.11), we write

$$\begin{aligned}
&t_{2,n}t_{2,r+1}-t_{2,n+1}t_{2,r}\\
&=\left(\frac{-A\alpha_1^n+B\alpha_2^n}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^{r+1}+B\alpha_2^{r+1}}{\sqrt{5}}\right)\\
&\quad-\left(\frac{-A\alpha_1^{n+1}+B\alpha_2^{n+1}}{\sqrt{5}}\right)\left(\frac{-A\alpha_1^r+B\alpha_2^r}{\sqrt{5}}\right)\\
&=\frac{AB}{5}(\alpha_1^{n+1}\alpha_2^r+\alpha_2^{n+1}\alpha_1^r-\alpha_1^n\alpha_2^{r+1}-\alpha_1^{r+1}\alpha_2^n)\\
&=\frac{AB}{5}[\alpha_1^n\alpha_2^r(\alpha_1-\alpha_2)-\alpha_1^r\alpha_2^n(\alpha_1-\alpha_2)]\\
&=\frac{AB}{5}[(\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n)(\alpha_1-\alpha_2)] \quad \text{(substituting the value of A and B)}\\
&=\frac{(b^2-a^2-ab)}{\sqrt{5}}[\alpha_1^n\alpha_2^r-\alpha_1^r\alpha_2^n] \quad (\text{using } \alpha_1-\alpha_2=\sqrt{5})
\end{aligned}$$

as required. □

Now, we aim to give the generating function for $\{t_{2,n}\}$ and $\{l_{2,n}\}$ sequences in terms of a and b.

Generating function

Let $g(x)=\sum_{n=0}^{\infty}t_{2,n}x^n$ be a generating function for the sequence $\{t_{2,n}\}$. Now, multiplying Eqn. (2.1) by x^{n+2} and then taking summation over 0 to ∞, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}x^{n+2}t_{n+2}-\sum_{n=0}^{\infty}x^{n+2}t_{n+1}-\sum_{n=0}^{\infty}x^{n+2}t_n=0\\
&\Longrightarrow (g(x)-t_0-t_1x)-(g(x)-t_0)x-g(x)x^2=0\\
&\Longrightarrow g(x)(1-x-x^2)-(t_0+t_1x-t_0x)=0\\
&\Longrightarrow g(x)=\frac{a+(b-a)x}{(1-x-x^2)}. \qquad (2.13)
\end{aligned}$$

Theorem 2.17. *Let $q(x)$ be the generating function for trace sequence $\{l_{2,n}\}$ (2.6), then we have*

$$q(x) = -g(x) + 2\left(\frac{g(x) - a}{x}\right).$$

Proof. Lat $A = -a + 2b$ and $B = 2a + b$ (initial value of trace sequence), then in Eqn. (2.13) replace a by A and b by B, we get

$$\begin{aligned} q(x) &= \frac{A + (B - A)x}{(1 - x - x^2)} = \frac{(-a + 2b) + (2a + b - (-a + 2b))x}{(1 - x - x^2)} \\ &= \frac{(-a + 2b) + (3a - b)x}{(1 - x - x^2)} \\ &= \frac{-a - (b - a)x}{(1 - x - x^2)} + 2\frac{[b + (a + b - b)x]}{(1 - x - x^2)} \\ &= -g(x) + 2\left(\frac{g(x) - a}{x}\right) \end{aligned}$$

as required. □

For $a = 0,\ b = 1$ and $a = 2,\ b = 1$, Eqn. (2.13) gives the generating function for Fibonacci and Lucas sequence, respectively.

3. The $\{t_{3,n}\}$ sequence and some properties

Let us consider the sequence $\{t_{3,n}\}_{n\geq 0}$ given by a third order linear difference equation as follows

$$t_{3,n+3} = t_{3,n+2} + t_{3,n+1} + t_{3,n} \quad \text{with} \quad t_{3,0} = a,\ t_{3,1} = b,\ t_{3,2} = c. \tag{3.1}$$

The recurrence relation (3.1) can also be extended in negative direction and it can be achieved by rearranging the relation as $t_{3,n} = t_{3,n+3} - t_{3,n+2} - t_{3,n+1},\quad n \leq 0$.

In particular for $a = b = 0,\ c = 1$, Eqn. (3.1) gives tribonacci sequence while for $a = 3,\ b = 1,\ \ c = 3$, same is known as trucas (Tribonacci-Lucas) sequence [8].

The first few terms of sequence $\{t_{3,n}\}$ are given in the following table:

Index (n)	$t_{3,n}$	Value	Index ($-n$)	$t_{3,-n}$	Value
0	$t_{3,0}$	a	0	$t_{3,0}$	a
1	$t_{3,1}$	b	-1	$t_{3,-1}$	$c - a - b$
2	$t_{3,2}$	c	-2	$t_{3,-2}$	$2b - c$
3	$t_{3,3}$	$a + b + c$	-3	$t_{3,-3}$	$2a - b$
4	$t_{3,4}$	$a + 2b + 2c$	-4	$t_{3,-4}$	$2c - 3a - 2b$
5	$t_{3,5}$	$2a + 3b + 4c$	-5	$t_{3,-5}$	$5b - 3c + a$
6	$t_{3,6}$	$4a + 6b + 7c$	-6	$t_{3,-6}$	$4a - 4b + c$

The matrix representation corresponding to Eqn. (3.1) is given by a square matrix $T_3^{(n)}$ of order 3 defined as

$$T_3^{(n)} = \begin{bmatrix} t_{3,n+2} & t_{3,n+1}+t_{3,n} & t_{3,n+1} \\ t_{3,n+1} & t_{3,n}+t_{3,n-1} & t_{3,n} \\ t_{3,n} & t_{3,n-1}+t_{3,n-2} & t_{3,n-1} \end{bmatrix} \text{with } T_3^{(0)} = \begin{bmatrix} c & a+b & b \\ b & c-b & a \\ a & b-a & c-a-b \end{bmatrix} \quad (3.2)$$

and the determinant of $T_3^{(0)}$ is given as

$$\det(T_3^{(0)}) = a^3 + 2a^2b + a^2c + 2ab^2 - 2\,a\,b\,c - a\,c^2 + 2\,b^3 - 2\,b\,c^2 + c^3 \ (= K, \text{say}).$$

Theorem 3.1. *Let $\{f_{3,k}\}_{n\geq 0}$ be tribonacci sequence [A000073] with initial values $0, 0, 1,$ then*

$$t_{3,n} = b(f_{3,n+1} - f_{3,n}) + af_{3,n-1} + cf_{3,n}, \quad \forall n \in \mathbb{Z}.$$

Proof. We prove it using mathematical induction on n. For $n = 0$, the result obviously holds. For $n = 1$, we have

$$t_{3,1} = b(f_{3,2} - f_{3,1}) + af_{3,0} + cf_{3,1} = b + a0 + c0 = b.$$

Now assuming the result is true for $n = k$. For $n = k + 1$, we write

$$\begin{aligned} t_{k+1} &= t_k + t_{k-1} + t_{k-2} \\ &= [b(f_{k+1} - f_k) + af_{k-1} + cf_k] + [b(f_k - f_{k-1}) + af_{k-2} + cf_{k-1}] \\ &\quad + [b(f_{k-1} - f_{k-2}) + af_{k-3} + cf_{k-2}] \\ &= b(f_{k+1} - f_{k-2}) + a(f_{k-1} + f_{k-2} + f_{k-3}) + c(f_k + f_{k-1} + f_{k-2}) \\ &= b(f_{k+2} - f_{k+1}) + af_k + cf_{k+1} \qquad \text{(using tribonacci sequence)} \end{aligned}$$

as required. □

Theorem 3.2. *Let $T_3^{(0)}$ be the initial matrix defined in Eqn. (3.2) and Q_3^n be tribonacci matrix, then we have $T_3^{(n)} = Q_3^n T_3^{(0)}$, $\forall n \in \mathbb{Z}$.*

Proof. It can be easily proved using mathematical induction on n and Theorem 3.1. □

Corollary 3.3. *For $n \in \mathbb{N}$, we have, $T_3^{(n)} = Q_3 T_3^{(n-1)} = Q_3^{-1} T_3^{(n+1)}$.*

Remark 3.4. Matrices Q_3^n and $T_3^{(0)}$ commutes i.e. $Q_3^n T_3^{(0)} = T_3^{(0)} Q_3^n, \quad \forall n \in \mathbb{Z}$.

Theorem 3.5. *For recursive matrix $T_3^{(n)}$, we write*

$$T_3^{(n)} T_3^{(-n)} = (T_3^{(0)})^2, \quad \forall n \in \mathbb{Z}.$$

Proof. Using definition of $T_3^{(n)}$, we have

$$\begin{aligned} T_3^{(n)}T_3^{(-n)} &= Q_3^nT_3^{(0)}Q_3^{-n}T_3^{(0)} \\ &= Q_3^nQ_3^{-n}T_3^{(0)}T_3^{(0)} = IT_3^{(0)}T_3^{(0)} = (T_3^{(0)})^2 \end{aligned}$$

as required. $\square$

Remark 3.6. Determinant of $T_3^{(n)}$ is invariant of n, i.e. $\det(T_3^{(n)}) = \det(T_3^{(0)}) = K$. Since by the properties of determinant, we write

$$\begin{aligned} \det(T_3^{(n)}) &= \det(Q_3^nT_3^{(0)}) = \det(Q_3^n)\det(T_3^{(0)}) \\ &= (-1)^{2n}\det(T_3^{(0)}) = \det(T_3^{(0)}) = K. \end{aligned}$$

Thus, $T_3^{(n)}$ is invertible if and only if $T_3^{(0)}$ is invertible, so for the existence of inverse of $T_3^{(n)}$, we consider only those values of a, b, c such that $\det(T_3^{(0)}) \neq 0$.

Example 3.7 (Tribonacci)**.** Let $a = b = 0$ and c =1 then $\det(T_3^{(n)}) = 1$.

Example 3.8 (Trucas)**.** Let $a = 3, b = 1$ and $c = 3$ then $\det(T_3^{(n)}) = 44$.

Remark 3.9. $\mathrm{Inv}(T_3^{(n)}) = T_3^{(-n)}H^{-1}$ provided $\det(T_3^{(0)}) \neq 0$, where $H = (T_3^{(0)})^2$.

3.1. Matrix representation for sequence of traces

The Lucas sequence of order 3 (also known as trucas, ref. A001644, A007486) is given by following recurrence relation

$$l_{3,n+3} = l_{3,n+2} + l_{3,n+1} + l_{3,n}, \quad \text{with} \quad l_{3,0} = 3,\ l_{3,1} = 1,\ l_{3,2} = 3. \tag{3.3}$$

In terms of tribonacci sequence, trucas is given by $l_{3,n} = trace(Q_3^n) = f_{3,n+2} + f_{3,n} + 2f_{3,n-1}$. Now, redefining the trucas (3.3) for $\{t_{3,n}\}$ sequence with the relation

$$l_{3,n} = \mathrm{trace}(T_3^{(n)}).$$

Since $\mathrm{trace}(T_3^{(n)}) = t_{3,n+2} + t_{3,n} + 2t_{3,n-1}$, so from Theorem 3.1, we have

$$\begin{aligned} \mathrm{trace}(T_3^{(n)}) &= [b(f_{n+3} - f_{n+2}) + af_{n+1} + cf_{n+2}] + [b(f_{n+1} - f_n) + af_{n-1} + cf_n] \\ &\quad + 2[b(f_n - f_{n-1}) + af_{n-2} + cf_{n-1}] \\ &= b(f_{3,n+3} + f_{3,n+1} + f_{3,n} - f_{3,n+2} - 2f_{3,n-1}) \\ &\quad + a(f_{3,n+1} + f_{3,n-1} + 2f_{3,n-2}) + c(f_{3,n+2} + f_{3,n} + 2f_{3,n-1}) \\ &= 2b(f_{3,n+3} - f_{3,n+2} - f_{3,n-1}) + al_{3,n-1} + cl_{3,n}. \end{aligned} \tag{3.4}$$

Remark 3.10. For $a = b = 0,\ c = 1$, Eqn. (3.4) gives the standard trucas sequence.

The corresponding matrix sequence $\{M_3^{(n)}\}$ for the sequence $\{l_{3,n}\}$ is given by

$$M_3^{(n)} = \begin{bmatrix} l_{3,n+2} & l_{3,n+1} + l_{3,n} & l_{3,n+1} \\ l_{3,n+1} & l_{3,n} + l_{3,n-1} & l_{3,n} \\ l_{3,n} & l_{3,n-1} + l_{3,n-2} & l_{3,n-1} \end{bmatrix}.$$

Theorem 3.11. *Let $L_3^{(0)}$ be the initial trucas matrix (it can be obtained by putting $a = 3,\ b = 1,\ c = 3$ in $T_3^{(0)}$ in Eqn. (3.2)), then we have*

$$M_3^{(n)} = T_3^{(n)} L_3^{(0)}. \tag{3.5}$$

Proof. It can be easily proved with mathematical induction on n. ☐

Theorem 3.12. *If K is determinant of $T_3^{(0)}$, then $\det(M_3^{(n)}) = 44K$.*

Proof. Using properties of the determinant and Eqn. (3.5), we have

$$\begin{aligned} \det(M_3^{(n)}) &= |T_3^{(n)} L_3^{(0)}| = |T_3^{(n)}||L_3^{(0)}| = |Q_3^n||T_3^{(0)}||L_3^{(0)}| \\ &= (-1)^{2n} K 44 = 44K \end{aligned}$$

as required. ☐

Thus, it is concluded that if $T_3^{(n)}$ is invertible implies inverse for $M_3^{(n)}$ exists for all $n \in \mathbb{Z}$, i.e. $M_3^{(n)}$ is invertible if and only if $T_3^{(0)}$ is invertible.

Generating function

Let $g(x) = \sum_{n=0}^{\infty} t_{3,n} x^n$ be a generating function for $\{t_{3,n}\}$ sequence. On multiplying each term of Eqn. (3.1) with x^{n+3} and then taking summation over $n = 0$ to ∞, we get

$$\sum_{n=0}^{\infty} x^{n+3} t_{n+3} - \sum_{n=0}^{\infty} x^{n+3} t_{n+2} - \sum_{n=0}^{\infty} x^{n+3} t_{n+1} - \sum_{n=0}^{\infty} x^{n+3} t_n = 0.$$

Thus, we have

$$\begin{aligned} &(g(x) - t_0 - t_1 x - t_2 x^2) - (g(x) - t_0 - t_1 x)x - (g(x) - t_0)x^2 - g(x)x^3 = 0 \\ &\implies g(x)(1 - x - x^2 - x^3) - t_0(1 - x - x^2) - t_1(x - x^2) - t_2 x^2 = 0 \\ &\implies g(x) = \frac{a(1 - x - x^2) + b(x - x^2) + cx^2}{(1 - x - x^2 - x^3)} \\ &\implies g(x) = \frac{a + (b - a)x + (c - b - a)x^2}{(1 - x - x^2 - x^3)}. \end{aligned} \tag{3.6}$$

In particular, setting $a = b = 0,\ c = 1$ and $a = 3,\ b = 1,\ c = 3$ in Eqn. (3.6) give the generating functions for tribonacci and trucas sequence, respectively.

3.2. Binet's formula

To establish any identity involving nth term of the sequence, the Binet's formula plays an important role. Here, we derive an explicit formula for generalized third order sequences $\{t_{3,n}\}$.

Let us assume that the three characteristic roots of difference Eqn. (3.1) are r_1, r_2 and r_3. Clearly, r_1, r_2 and r_3 satisfy the relations

$$r_1 + r_2 + r_3 = 1, \quad r_1r_2 + r_2r_3 + r_3r_1 = -1 \quad \text{and} \quad r_1r_2r_3 = 1. \tag{3.7}$$

Theorem 3.13 (Binet's formula)**.** *For $n \geq 0$, we have*

$$t_{3,n} = \frac{P{r_1}^n + Q{r_2}^n}{r_1 - r_2} + R{r_3}^n, \tag{3.8}$$

where $P = (r_2 - r_3)R - ar_2 + b$, $Q = (r_3 - r_1)R + ar_1 - b$, $R = \frac{c-(r_1+r_2)b+r_1r_2a}{r_3^2-(r_1+r_2)r_3+r_1r_2}$.

Proof. Using the relation between roots and the coefficients of a polynomial, rewriting Eqn. (3.1) as

$$t_{k,n+3} = (r_1 + r_2 + r_3)t_{k,n+2} - (r_1r_2 + r_2r_3 + r_3r_1)t_{k,n+1} + r_1r_2r_3t_{k,n}.$$

It can also be written as,

$$\begin{aligned} &t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} \\ &= r_3t_{k,n+2} - r_3(r_1 + r_2)t_{k,n+1} + r_1r_2r_3t_{k,n} \\ &= r_3[t_{k,n+2} - (r_1 + r_2)t_{k,n+1} + r_1r_2t_{k,n}]. \end{aligned} \tag{3.9}$$

Similarly, we have

$$t_{k,n+2} - (r_1 + r_2)t_{k,n+1} + r_1r_2t_{k,n} = r_3[t_{k,n+1} - (r_1 + r_2)t_{k,n} + r_1r_2t_{k,n-1}]. \tag{3.10}$$

Substitute Eqn. (3.10) in Eqn. (3.9), we get

$$t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} = r_3^2[t_{k,n+1} - (r_1 + r_2)t_{k,n} + r_1r_2t_{k,n-1}].$$

Continuing this substitution process, we obtain a recursive relation

$$t_{k,n+3} - (r_1 + r_2)t_{k,n+2} + (r_1r_2)t_{k,n+1} = r_3^{n+1}[t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0}].$$

Now, divide both side of the above equation by r_3^{n+3}, we get

$$\frac{t_{k,n+3}}{r_3^{n+3}} - \frac{(r_1 + r_2)}{r_3^{n+3}}t_{k,n+2} + \frac{(r_1r_2)}{r_3^{n+3}}t_{k,n+1} = \frac{1}{r_3^2}[t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0}]. \tag{3.11}$$

For simplicity, consider $t_{k,2} - (r_1 + r_2)t_{k,1} + r_1r_2t_{k,0} = K$ and $\frac{t_{k,n+3}}{r_3^{n+3}} = H_{k,n+3}$ in Eqn. (3.11), we write

$$H_{k,n+3} - \frac{(r_1 + r_2)}{r_3}H_{k,n+2} + \frac{(r_1r_2)}{r_3^2}H_{k,n+1} = \frac{1}{r_3^2}K, \tag{3.12}$$

which is a second order non-homogeneous linear difference equation and its solution is given by $H_{k,n} = H(C)+H(P)$, where $H(C)$ represents the solution corresponding homogeneous part and $H(P)$ is particular solution.

Since, roots of the characteristic equation for homogeneous part of Eqn. (3.12) are $\alpha_1 = \frac{r_1}{r_3}$ and $\alpha_2 = \frac{r_2}{r_3}$. So, the solution for homogeneous part is given by

$$H(C) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n, \text{ where A and B are arbitrary constants.}$$

Furthermore, the non-homogeneous part of Eqn. (3.12) is a constant, so particular solution is also a constant and it is given by $H(P) = \frac{K}{r_3^2-(r_1+r_2)r_3+r_1r_2}$. Thus, general solution of Eqn. (3.12) is

$$H_{k,n} = H(C) + H(P) = A\left(\frac{r_1}{r_3}\right)^n + B\left(\frac{r_2}{r_3}\right)^n + \frac{K}{r_3^2 - (r_1+r_2)r_3 + r_1r_2}.$$

Replacing $H_{k,n}$ by $\dfrac{t_{k,n}}{r_3^n}$ and K by $t_{k,2}-(r_1+r_2)t_{k,1}+r_1r_2t_{k,0}$ in the above equation, we get

$$t_{k,n} = A{r_1}^n + B{r_2}^n + {r_3}^n R, \quad \text{where } R = \left[\frac{t_{k,2} - (r_1+r_2)t_{k,1} + r_1r_2t_{k,0}}{r_3^2 - (r_1+r_2)r_3 + r_1r_2}\right]. \quad (3.13)$$

Hence, using initial values from Eqn. (3.1) in Eqn. (3.13), we have

$$A = \frac{(r_2 - r_3)R - ar_2 + b}{r_1 - r_2} \quad \text{and} \quad B = \frac{(r_3 - r_1)R + ar_1 - b}{r_1 - r_2},$$

where $R = \frac{c-(r_1+r_2)b+r_1r_2a}{r_3^2-(r_1+r_2)r_3+r_1r_2}$, as required. ☐

Remark 3.14. Setting $a = b = 0$ and $c = 1$ in Eqn. (3.8) gives the Binet's formula for the standard tribonacci sequence (the Fibonacci sequence of order three).

Remark 3.15. Setting $a = 3$, $b = 1$ and $c = 3$ in Eqn. (3.8) gives the Binet's formula for the Tribonacci-Lucas sequence.

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