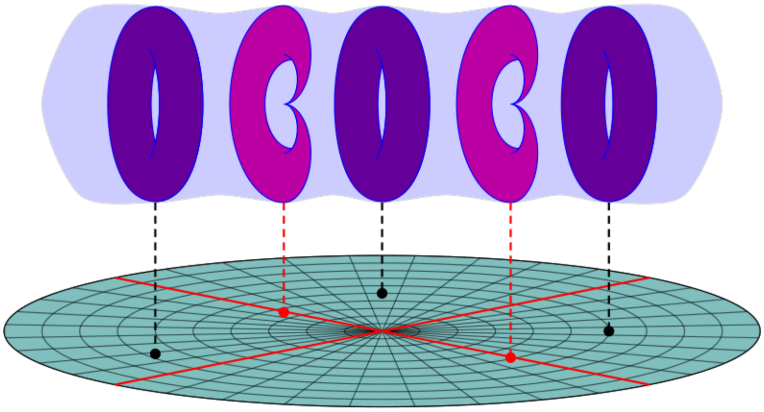

Higgs bundles and the Hitchin system

Rodrigo Pereira

Mestrado em Matemática

Departamento de Matemática

2023

Orientador

André Gama Oliveira, Doutorado de Nível Inicial,
Faculdade de Ciências da Universidade do Porto,
Professor Auxiliar, Escola de Ciências e Tecnologia
da Universidade de Trás-os-Montes e Alto Douro

Coorientador

Peter Beier Gothen, Professor Associado,
Faculdade de Ciências da Universidade do Porto

U. PORTO

FC FACULDADE DE CIÊNCIAS
UNIVERSIDADE DO PORTO

MSc

Sworn Statement

I, Rodrigo Alves dos Santos Pereira, enrolled in the Master Degree of Mathematics at the Faculty of Sciences of the University of Porto hereby declare, in accordance with the provisions of paragraph a) of Article 14 of the Code of Ethical Conduct of the University of Porto, that the content of this dissertation reflects perspectives, research work and my own interpretations at the time of its submission.

By submitting this dissertation, I also declare that it contains the results of my own research work and contributions that have not been previously submitted to this or any other institution.

I further declare that all references to other authors fully comply with the rules of attribution and are referenced in the text by citation and identified in the bibliographic references section. This dissertation does not include any content whose reproduction is protected by copyright laws.

I am aware that the practice of plagiarism and self-plagiarism constitute a form of academic offense.

Rodrigo Alves dos Santos Pereira
July 19, 2023

Acknowledgments

This work was supported by the research grant with reference EXPL/MAT-PUR/1162/2021 – Mirror symmetry on Higgs bundles moduli spaces – financed by the Portuguese Science and Technology Foundation (FCT) by national funds.

I thank my advisors, Professors André Oliveira and Peter Gothen, for giving me the opportunity to work with them in an area of Mathematics I have come to grow extremely fond of, and for all the invaluable help they have given me while writing this text.

I thank all the Professors I encountered during my academic journey at FCUP. Each and every one of them had a role to play in the mathematician I am today and aspire to be. A special thanks to Professor Maria Carvalho, for introducing me into the world of mathematical research and for believing in me from the very first Analysis exam; to Professor Jorge Rocha, for his constant availability and kindness, in and out of the classroom; and to Professor Inês Cruz, not only for the insights into integrable systems, but mostly for always going above and beyond for her students.

I thank my friend António for accompanying me in the head-scratching that came with learning advanced mathematics, especially during the rough times of the pandemic.

I thank Vítor for being there for me during all this time.

Finalmente, agradeço à minha Mãe. Sem ela nada disto seria possível.

Resumo

Os fibrados de Higgs e o seu espaço moduli têm vindo a desempenhar um papel preponderante na matemática moderna. Neste texto, depois de rever os preliminares necessários, introduzimos o espaço moduli de fibrados vetoriais sobre uma superfície de Riemann. Através do seu espaço cotangente, motivamos a definição de fibrado de Higgs e descrevemos o seu espaço moduli. De seguida, introduzimos a aplicação de Hitchin e o conceito de curva espetral. Descrevemos as fibras da aplicação de Hitchin para curvas espetrais integrais, o que inclui o caso genérico das curvas espetrais suaves. Por fim, provamos o resultado principal do texto, que é ver que a aplicação de Hitchin torna o espaço moduli de fibrados de Higgs num sistema completamente integrável.

Palavras-chave: Superfície de Riemann, fibrado vetorial, fibrado de Higgs, espaço moduli, aplicação de Hitchin, curva espetral, sistema integrável, Lagrangiano.

Abstract

Higgs bundles and their moduli space have taken a prominent role in modern mathematics. In this text, after reviewing the necessary preliminaries, we introduce the moduli space of vector bundles on a Riemann surface. Through its cotangent bundle, we motivate the definition of a Higgs bundle and describe their moduli space. Afterwards, we introduce the Hitchin map and the concept of spectral curve. We describe the fibers of the Hitchin map for integral spectral curves, which includes the generic case of smooth spectral curves. At the end, we prove the main result of the text, that the Hitchin map gives the moduli space of Higgs bundles the structure of a completely integrable system.

Keywords: Riemann surface, vector bundle, Higgs bundle, moduli space, Hitchin map, spectral curve, integrable system, Lagrangian.

x

Table of Contents

List of Figures

Introduction

Vector bundles are ubiquitous in geometry. If we consider smooth vector bundles over a compact Riemann surface X, their classification is simple – up to isomorphism, there is a single smooth vector bundle for every rank n and degree d. The holomorphic case however is quite different and leads us to consider the moduli space of holomorphic vector bundles $\mathcal{N}(n, d)$ of rank n and degree d. It is a variety of its own and so we would like to study its cotangent bundle. It turns out that cotangent vectors to a point E will be holomorphic maps $\Phi : E \to E \otimes K$, where K is the canonical bundle of X. These are instances of one of the main objects of this thesis: Higgs bundles.

A Higgs bundle on X is then a pair (E, Φ) consisting of a holomorphic vector bundle E and a holomorphic map $\Phi : E \to E \otimes K$, called the Higgs field. Such objects were introduced by Nigel Hitchin in [Hit87b], arising from certain equations in mathematical physics. As with vector bundles, we will not be concerned with single Higgs bundles, but rather with their moduli space. These spaces are known to possess a very rich topological and geometric structure, requiring tools from several branches of geometry (differential, symplectic, algebraic) to tackle its complexity, whose interplay makes its study all the more interesting.

Although their properties already make them objects worth studying in their own right, moduli spaces of Higgs bundles also show up in various, seemingly disparate, areas of mathematics, making their study quite a pertinent one. They can be found, for example, when studying the representation theory of the fundamental group of X.

More recently, excitement over Higgs bundles has built over their relevance to the Langlands program, an ambitious collection of conjectures purporting to establish vast connections between number theory and geometry. Moreover, their moduli spaces constitute a fundamental example where mirror symmetry can be tested. This is a concept arising from theoretical physics, specifically string theory, but which has proven to be a very rich theory from the mathematical point of view as well, via a conjectural equivalence of categories associated to two Calabi-Yau varieties.

A crucial fact that makes moduli spaces of Higgs bundles so relevant to mirror symmetry is that they come equipped with an integrable system structure – the so called Hitchin system. The description of said structure is the main goal of this thesis. The main tool we will use to that purpose is the Hitchin map, which, roughly speaking, takes a Higgs bundle (E, Φ) and gives us the coefficients of the characteristic polynomial of its Higgs field Φ. Stripping away the sophisticated notions of modern geometry, we are essentially following the philosophy of studying an endomorphism through its eigenvalues and eigenspaces. Through this map, the moduli space becomes an integrable system which, although we do not delve into it here, leads us into the aforementioned applications in Physics.

We will be particularly focused on the fibers of the Hitchin map. Through the construction of spectral curves (objects encoding the eigenvalues of the Higgs field), we will be able to describe the generic fibers as isomorphic to Jacobians – complex tori – which will be Lagrangian subvarieties of the moduli space. We will also venture outside the generic case and say some words on the simpler singular fibers.

Structure of the text

Section 1 is devoted to introducing the basic concepts needed to understand the rest of the text. We start with Riemann surfaces and vector bundles, the foundations of all the constructions in this text. We then review the concepts of sheaves and divisors and the several links between them and vector bundles. After we describe how holomorphic vector bundles can be seen as certain operators defined on a smooth vector bundle, a point of view that will last through the entire text, we briefly describe the cohomology of holomorphic vector bundles and the hypercohomology of complexes made up of such objects. We also review the equivalence between Riemann surfaces and smooth algebraic curves, and introduce the compactified Jacobian of a singular curve. We end this section by reviewing the basics of symplectic geometry needed to grasp the concept of a completely integrable system.

In Section 2 we first describe the construction of the moduli space of holomorphic vector bundles over a Riemann surface. The analytical point of view is preferred since it allows for a somewhat intuitive discussion of the subject, although it does have the disadvantage that we cannot be fully rigorous without bringing in tools from advanced Functional Analysis which are outside of the scope of this text. We also introduce Higgs bundles and construct their moduli space – the main object of this thesis – motivating the definition via the cotangent space of the moduli space of vector bundles. We finish the section by describing the symplectic structure of the moduli space of Higgs bundles.

Section 3 makes up the core of this text. After a brief discussion on what the characteristic polynomial of a Higgs field is, we are ready to define the Hitchin map. We then focus on spectral curves, their construction and properties. We see how to relate their Jacobians to the fibers of the Hitchin map via the BNR-correspondence theorem. Finally, we check the complete integrability of the moduli space, and along the way we also give a description of the tangent space of the smooth fibers.

To conclude, we present, in appendix A, some more technical definitions and results from algebraic geometry that are used throughout the text.

Chapter 1

Preliminaries

1.1 Riemann surfaces

The base object of all our constructions in this text will be a *compact Riemann surface* (a complex manifold of dimension 1). We follow [Mir95].

A Riemann surface is a topological space X satisfying the following properties: it is second countable and Hausdorff (these are technical conditions that we impose in order to exclude pathological objects), connected, and, most importantly, it is equipped with a *complex structure.*

A complex structure is a maximal complex atlas, i.e., a collection of charts

$$\{\Phi_\alpha : U_\alpha \to V_\alpha\}$$

where the U_α are open sets of X such that

$$\bigcup_\alpha U_\alpha = X$$

and the V_α are open sets of $\mathbb{C}$. Moreover, the charts are compatible, that is, for every α, β such that $U_\alpha \cap U_\beta \neq \varnothing$, the transition function

$$\Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \to \Phi_\beta(U_\alpha \cap U_\beta)$$

is *holomorphic.*

A map between two Riemann surfaces X and Y is holomorphic if at every point p of X we can find charts covering p and its image where the expression of the map in coordinates is holomorphic.

Using that $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, any Riemann surface can be seen as a real 2 dimensional manifold. In particular, in the case of compact Riemann surfaces, the underlying real manifold can be classified as follows.

Proposition 1.1 ([Mir95, Proposition 1.23])**.** *Every compact Riemann surface is diffeomorphic to the g-holed torus, for some unique integer $g \geq 0$.*

The number g in the proposition is known as the *topological genus* of the compact Riemann surface.

One particularly fruitful way of building Riemann surfaces is by considering zeroes of polynomials, as in the following example.

Example 1.2 (Plane algebraic curves)**.** Consider $f \in \mathbb{C}[z, w]$, a polynomial in two variables. Using the implicit function theorem, we see that, at a point p in the zero locus of f

$$X = \{(z, w) \in \mathbb{C}^2 \mid f(z, w) = 0\}$$

such that $\frac{\partial f}{\partial z}$ or $\frac{\partial f}{\partial w}$ does not vanish, X is locally a graph of a holomorphic function. If this happens for all $p \in X$, we say X is an *affine non singular plane curve.*

More precisely, it is possible, using the same theorem, to define compatible charts in X, equipping it with a complex structure. If f is irreducible, then X is connected and thus a Riemann surface. It is not however, compact. In order to compactify it, we consider the following.

Let $F \in \mathbb{C}[x, y, z]$ be an homogeneous polynomial and consider its zero locus

$$X = \{[x : y : z] \in \mathbb{CP}^2 \mid F(x, y, z) = 0\}.$$

If we define X_0 as the set of points in $\mathbb{C}^2$ where the polynomial $F(1, y, z)$ vanishes, and similarly sets X_1 and X_2 for the other coordinates, we obtain three affine plane curves, each of them sitting inside an open set of $\mathbb{CP}^2$ isomorphic to $\mathbb{C}$ (where the corresponding coordinate does not vanish).

If X does not have any point where all three partial derivatives of F vanish simultaneously, the three affine curves will be non-singular, and X, obtained by "gluing", will be a compact Riemann surface inside of $\mathbb{CP}^2$ – a *projective non-singular plane curve.* Its genus g can be determined as a function of the degree d of F via the formula

$$g = \frac{(d-1)(d-2)}{2}.$$

Remark 1.3*.* More generally, we can define complex manifolds of higher dimension (the two dimensional case will be useful to us later) as spaces locally homeomorphic to $\mathbb{C}^n$ and with holomorphic transition functions. The basic examples are the complex projective spaces $\mathbb{CP}^n$ and complex tori (quotients of $\mathbb{C}^n$ by lattices).

1.2 Vector bundles

We now consider vector bundles over Riemann surfaces. We start with the general definition of a smooth vector bundle (see [Wel08, Section I.2]).

Definition 1.4. Let X be a manifold. A *smooth complex vector bundle* of rank n over X is a manifold E equipped with a smooth surjective map $\pi : E \to X$ satisfying the following conditions.

1. For all $p \in X$, the *fiber* over p, $E_p := \pi^{-1}(p)$, has a complex vector space structure of dimension n;

2. the space E is *locally trivial*, i.e., for all $p \in X$, there is a neighborhood U of p in X and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^n$ (called a *local trivialization* of E over U), satisfying the following conditions:

- $\pi_U \circ \Phi = \pi$ (where $\pi_U : U \times \mathbb{C}^n \to U$ is the projection onto the first factor);
- for all $q \in U$, the restriction of Φ to E_q is a linear isomorphism between E_q and $\{q\} \times \mathbb{C}^n \cong \mathbb{C}^n$;

3. suppose $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^n$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{C}^n$ are two local trivializations of the bundle E such that $U \cap V \neq \varnothing$. Then the composition

$$\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{C}^n \to (U \cap V) \times \mathbb{C}^n$$

is of the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v),$$

for some smooth map $\tau : U \cap V \to \mathrm{GL}(n, \mathbb{C})$ (called a *transition function*).

E is said to be the *total space* of the vector bundle and X its *base*. If $n = 1$, E is said to be a *line bundle*.

A vector bundle on a complex manifold X is said to be *holomorphic* if the projection π and the transition functions τ are holomorphic.

Every holomorphic vector bundle over a complex manifold has a smooth counterpart (a smooth vector bundle over a smooth manifold).

Vector bundles generalize the product space $E = X \times \mathbb{C}^n$, where $\pi : E \to X$ is given by projection onto the first factor. In fact, property 2. of the definition tells us that, locally, vector bundles are indistinguishable from products.

Example 1.5. Complex projective space $\mathbb{CP}^n$ can be thought of as the space of all lines through the origin in $\mathbb{C}^{n+1}$. The *tautological line bundle* $p : E \to \mathbb{CP}^n$ has total space E defined as the subspace $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ whose elements are pairs (l, z) with $z \in l$, and $p(l, z) = l$.

Let U_i be the standard open sets of $\mathbb{CP}^n$ where the i-th coordinate is non-zero. Then $\psi_i : p^{-1}(U_i) \to U_i \times C$, defined by $\psi_i(l, z) = z_i$ gives a local trivialization of E. On $U_i \cap U_j$, we have

$$\psi_i \circ \psi_j^{-1}(l, w) = \left(l, \frac{z_i}{z_j} w\right),$$

where $l = [z_0 : \cdots : z_n]$.

The bundle E is then an example of a holomorphic vector bundle over the complex manifold $\mathbb{CP}^n$.

Definition 1.6. An *isomorphism* between smooth vector bundles $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ over the same base X is a diffeomorphism $g : E_1 \to E_2$ taking each fiber $\pi_1^{-1}(p)$ to the corresponding fiber $\pi_2^{-1}(p)$ through a linear isomorphism.

An isomorphism between holomorphic vector bundles is defined similarly, by requiring that g be biholomorphic.

A smooth/holomorphic vector bundle is said to be *trivial* if it is isomorphic (in its respective category) to the product bundle.

Definition 1.7. A smooth/holomorphic *section* of a vector bundle $\pi : E \to M$ is a smooth/holomorphic map $\sigma : M \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_M$. That is, for each $p \in M$, $\sigma(p)$ is in its respective fiber E_p.

A smooth/holomorphic *local section* is a smooth/holomorphic map $\sigma : U \to E$ defined on an open set U of M such that $\pi \circ \sigma = \mathrm{id}_U$.

Canonical constructions in linear algebra (those that not depend on a choice of basis) allow us to define new bundles starting from already existing ones, in both the smooth and holomorphic settings. Starting with vector bundles E and F, the constructions we will use in this text are

- the direct sum $E \oplus F$;
- the tensor product $E \otimes F$;
- the symmetric powers $S^i E$ and the exterior powers $\bigwedge^i E$ – the top exterior power $\bigwedge^{\mathrm{rk}\, E} E$ is called the *determinant* of E, and denoted by $\det E$ (it is a line bundle);
- the dual bundle E^*;
- the bundle of linear maps $\mathrm{Hom}(E, F)$ and of endomorphisms $\mathrm{End}(E)$;
- the bundle $\mathrm{Aut}(E)$, although not a vector bundle since its fibers are isomorphic to $\mathrm{GL}(\mathrm{rk}\, E, \mathbb{C})$, is also a natural object to consider (it is a bundle of groups).

Example 1.8. Going back to Example 1.5, the tautological bundle on $\mathbb{CP}^n$ is just one of a whole family of line bundles defined on projective space. The dual of the tautological bundle, called the *hyperplane bundle*, is denoted by $\mathcal{O}(1)$ due to the fact that its global sections are the homogeneous polynomials of degree 1 in $n+1$ variables. Tensoring $\mathcal{O}(1)$ with itself $m > 0$ times, we obtain the bundle $\mathcal{O}(m)$, whose global sections are polynomials of degree m. Dualizing we obtain the bundles $\mathcal{O}(-m)$, which have no global sections (in particular the tautological bundle would be written as $\mathcal{O}(-1)$).

1.3 The tangent and cotangent bundles of a Riemann surface

Our starting point is the idea of studying the geometry of a real manifold through its tangent and cotangent bundles – leading us to vector fields and k-forms. We follow [Huy05, Section 1.3].

Consider a Riemann surface X. Thinking of X as a real manifold, with standard coordinates x, y, at each $p \in X$ there is a tangent space T_pX, spanned by the vectors (as an $\mathbb{R}$ vector space) $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

Since X is a complex manifold, there is a natural ($\mathbb{R}$-linear) complex structure I_p on each T_pX, defined by

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x}.$$

Likewise, the cotangent space T_p^*X is spanned by the dual forms dx and dy, and the complex structure I_p acts by

$$dx \mapsto -dy, \quad dy \mapsto dx.$$

We now complexify these spaces in order to get the 2-dimensional complex vector spaces

$$T_pX \otimes \mathbb{C} \quad \text{and} \quad T_p^*X \otimes \mathbb{C}.$$

Each of these spaces has a direct sum decomposition into their i and $-i$ eigenspaces for the complex structure I_p (in fact, its $\mathbb{C}$-linear extension). We call them their $(1,0)$ and $(0,1)$ parts, respectively,

$$\begin{aligned} T_pX \otimes \mathbb{C} &= T_p^{1,0}X \oplus T_p^{0,1}X, \\ T_p^*X \otimes \mathbb{C} &= (T_p^*X)^{1,0} \oplus (T_p^*X)^{0,1}. \end{aligned} \tag{1.1}$$

By making the change of variables to z and $\bar{z}$, using that

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

and

$$dx = \frac{dz+d\bar{z}}{2}, \quad dy = \frac{dz-d\bar{z}}{2i},$$

we find that the 1-dimensional complex spaces $(T_p^*X)^{1,0}$ and $(T_p^*X)^{0,1}$ are spanned by dz and $d\bar{z}$, respectively.

Dualizing, we get the tangent vectors

$$\frac{\partial}{\partial z} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

which span the spaces $T_p^{1,0}X$ and $T_p^{0,1}X$.

With our new basis, note that a smooth function f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

since this condition is equivalent to the Cauchy-Riemann equations.

1.3.1 Types of forms and operations on them

Now letting p vary over X, we consider the (complexified) bundle T^*X. Its smooth sections are the 1-forms, and the direct sum decomposition in (1.1) allows us to define forms of type $(1,0)$ (sections of $T^*X^{1,0}$) and of type $(0,1)$ (sections of $T^*X^{0,1}$).

We now define forms of type (p,q) as smooth sections of the bundle

$$\bigwedge\nolimits^{p,q} T^*X \coloneqq \left(\bigwedge\nolimits^{p} T^*X^{1,0}\right) \otimes \left(\bigwedge\nolimits^{q} T^*X^{0,1}\right).$$

Any k-form decomposes as a sum of forms of type (p,q), with $p+q=k$, i.e., we have the following bundle decomposition

$$\bigwedge\nolimits^{k} T^*X = \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} T^*X$$

which induces a decomposition of their respective spaces of smooth sections

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

The local expressions of these forms, on a chart with coordinate z and domain U, are

$$f dz \in \Omega^{1,0}(U), \quad f d\bar{z} \in \Omega^{0,1}(U), \quad f dz \wedge d\bar{z} \in \Omega^{1,1}(U),$$

where f is a smooth function on U.

The decomposition of k-forms into sums of forms of type (p,q) also induces a decomposition of the exterior derivative operator. If $d : \Omega^k(X) \to \Omega^{k+1}(X)$ is the usual exterior derivative of forms and $\pi^{p,q} : \Omega^k(X) \to \Omega^{p,q}(X)$ is the canonical projection, we define two new operators

$$\partial := \pi^{p+1,q} \circ d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$$

and

$$\bar{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X).$$

Note that, since X has complex dimension 1, the only non-zero forms are those of type $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$, so the whole ensemble of forms and operators just described can be seen in the following diagram

$$\begin{tikzcd}
 & \Omega^{0,0}(X) \arrow[dl, "\partial"'] \arrow[d, "d"] \arrow[dr, "\bar{\partial}"] & \\
\Omega^{1,0}(X) \arrow[r, hook] \arrow[dr, "\bar{\partial}"'] & \Omega^1(X) \arrow[d, "d"] & \Omega^{0,1}(X). \arrow[l, hook'] \arrow[dl, "\partial"] \\
 & \Omega^{1,1}(X) &
\end{tikzcd}$$

Locally we have

$$\partial f = \frac{\partial f}{\partial z} dz, \quad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}, \quad df = \partial f + \bar{\partial} f;$$

and

$$\bar{\partial}(f dz) = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}, \quad \partial(f d\bar{z}) = \frac{\partial f}{\partial z} dz \wedge d\bar{z}, \quad d(f dz + g d\bar{z}) = \bar{\partial}(f dz) + \partial(g d\bar{z}),$$

where f and g are smooth functions.

Lemma 1.9. *The operators ∂ and $\bar{\partial}$ have the following properties.*

- $d = \partial + \bar{\partial}$*;*
- $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$*;*
- *they obey the Leibniz rule*

$$\partial(\alpha \wedge \beta) = (\partial\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge (\partial\beta),$$

for $\alpha \in \Omega^{p,q}$ and $\beta \in \Omega^{r,s}$ (the same formula holds true for $\bar{\partial}$).

In particular we have that the chain

$$0 \longrightarrow \Omega^{p,0}(X) \xrightarrow{\bar{\partial}} \Omega^{p,1}(X) \xrightarrow{\bar{\partial}} 0$$

is a differential complex. Its cohomology groups

$$H^{p,q}(X) := \frac{\ker(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}$$

are called the *Dolbeault cohomology groups*. For example $H^{0,0}(X)$ is the space of holomorphic functions on X and $H^{1,0}(X)$ is the space of holomorphic differentials.

From now on, we will view $T^{1,0}X$ and $(T^*X)^{1,0}$ as holomorphic line bundles, and will denote them simply by TX and T^*X.

Remark 1.10. Let X be a compact Riemann surface with atlas $\{\varphi_i\}_i$. Since the φ_i are compatible charts, the composition $T_{ij} = \varphi_i \circ \varphi_j^{-1}$ is biholomorphic. The transition functions of the holomorphic tangent bundle TX are then $t_{ij} : U_i \cap U_j \to \mathbb{C}^*$ defined by $t_{ij} = T'_{ij} \circ \varphi_i$. Dualizing, the transition functions of the holomorphic cotangent bundle T^*X are defined by $s_{ij} = \frac{1}{T'_{ij}} \circ \varphi_i$.

Besides the regular k-forms on X, we will also need the following.

Definition 1.11. Let $\pi : E \to X$ be a smooth vector bundle over a manifold X. We denote by $\Omega^k(X, E)$ (or by $\Omega^k(E)$) the space of *smooth k-forms on X with values in E*, defined as sections of the bundle $\bigwedge^k T^*X \otimes E$.

If $\pi : E \to X$ is a complex vector bundle over a complex manifold X, then we denote by $\Omega^{p,q}(E)$ the space of smooth forms of type (p, q) with values in E, defined similarly as smooth sections of $\bigwedge^{p,q} T^*X \otimes E$.

Since $\Omega^k(X) = \Omega^k(X, X \times \mathbb{C})$, this concept generalizes k-forms on X. Note also that $\Omega^0(E)$ is the space of C^∞ sections of E and $\Omega^1(E)$ is the space of sections of $\mathrm{Hom}(TM, E)$, i.e., for every $p \in X$, s_p is a linear map between T_pM and E_p.

1.4 Sheaves, divisors and the degree of a vector bundle

In this section, we give a brief overview of sheaves, divisors, and how they relate to line bundles. We follow [Mir95], [GH94], and [Don11].

Given a topological space X, a *sheaf* $\mathcal{F}$ on X is an assignment of a set $\mathcal{F}(U)$, whose elements are called *sections*, to every open set U of X, together with a map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for every inclusion $V \subseteq U$, called a *restriction map*. This assortment of sets and maps must satisfy the following conditions.

- If $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW}\rho_{UV}$;
- whenever an open set U has an open cover U_i and we have $s_i \in \mathcal{F}(U_i)$ such that they agree on the intersections, i.e., $\rho_{U_i,U_i\cap U_j}(s_i) = \rho_{U_j,U_i\cap U_j}(s_j)$, for all i, j, then there is a unique $s \in \mathcal{F}(U)$ such that $\rho_{UU_i} = s_i$, for all i.

Examples of well known sheaves are the sheaf of continuous functions on a topological space, and the sheaves of smooth functions or of smooth k-forms on a smooth manifold. On a complex manifold X, the sheaf of holomorphic functions $\mathcal{O}_X$ is called its *structure sheaf*.

Remark 1.12. Like the examples above, most of the time the sets $\mathcal{F}(U)$ will have additional algebraic structure, as abelian groups or modules over a sheaf of rings. In that case, the restriction maps will be morphisms in the corresponding category.

An important notion associated to any sheaf $\mathcal{F}$ over a space X is that of its *stalk* over a point $p \in X$, denoted by $\mathcal{F}_p$. It is built considering the set of all sections of $\mathcal{F}$ over all open sets containing p, together with an equivalence relation that says that two sections are equivalent if they coincide on a common smaller domain. The stalk is then the set of all equivalence classes, which we call *germs* at p. For example, $\mathcal{O}_{X,p}$ is the space of germs at p of holomorphic functions of X. The *support* of a sheaf $\mathcal{F}$, denoted by $\operatorname{supp} \mathcal{F}$, is the set of points $p \in X$ such that $\mathcal{F}_p \neq 0$.

Whenever we have a holomorphic vector bundle E over a Riemann surface X, we can consider the sheaf of its holomorphic sections $\mathcal{O}(E)$. Over a trivializing open set U of E, giving a section over U is the same as giving n holomorphic functions over U (where $n = \operatorname{rk} E$), and so we have

$$\mathcal{O}(E)(U) \cong \mathcal{O}_X^{\oplus n}(U).$$

The sheaf $\mathcal{O}(E)$ is then said to be *locally free*. It is also true that any sheaf of this type corresponds to the sheaf of sections of some vector bundle (see, for example, [Wel08, Theorem 1.13, page 40]).

In most of the literature, there is in fact no distinction made between the holomorphic vector bundle E and its sheaf of holomorphic sections $\mathcal{O}(E)$, a convention which we follow in this text as well. The usefulness of this is that it allows us to use both algebraic and differential geometric methods to study these objects, depending on the situation at hand.

Restricting our attention now to line bundles, we have that, for a line bundle L and a trivializing open set U

$$\mathcal{O}(L)(U) \cong \mathcal{O}_X(U).$$

The sheaf $\mathcal{O}(L)$ is said to be *invertible*; this is because there is a natural abelian group structure on the set of (isomorphism classes of) line bundles with the tensor product operation. We call this the *Picard group* of X and denote it by $\operatorname{Pic}(X)$.

A *divisor* D on a compact Riemann surface X is a finite linear combination of points of X with integer coefficients

$$D = \sum_i n_i p_i, \quad n_i \in \mathbb{Z}, p_i \in X.$$

The set of divisors together with addition forms an abelian group denoted by $\operatorname{Div}(X)$. A divisor D is said to be *effective* if all the integer coefficients are non-negative, also denoted by $D \geq 0$.

A divisor is called *principal* if it is the divisor of a meromorphic function f on X

$$\operatorname{div}(f) = \sum_{p \in X} \operatorname{ord}_p(f) \cdot p.$$

These form a subgroup of $\operatorname{Div}(X)$ denoted by $\operatorname{PDiv}(X)$. Two divisors are said to be *linearly equivalent* if their difference is a principal divisor (i.e., they are on the same coset of $\operatorname{Div}(X)/\operatorname{PDiv}(X)$).

Divisors are related to line bundles in the following way. Let p be a point of the Riemann surface X. We associate to this point a holomorphic line bundle L_p corresponding to the invertible sheaf $\mathcal{O}(p)$ whose sections are the meromorphic functions with at worst a simple pole at p. Now, to any divisor $D = \sum_i n_i p_i$ we associate the line bundle

$$L_D = \bigotimes_i L_{p_i}^{n_i},$$

whose invertible sheaf of sections will be denoted by $\mathcal{O}(D)$. If we add a principal divisor to D, then the isomorphism class of $\mathcal{O}(D)$ does not change and so we have a well defined map $\mathrm{Div}(X)/\,\mathrm{PDiv}(X) \to \mathrm{Pic}(X)$, which turns out to be an isomorphism. To recover a divisor from a line bundle, we take any meromorphic section and record its divisor of zeros and poles, all divisors obtained in this way will be linearly equivalent, so we have a well defined inverse $\mathrm{Pic}(X) \to \mathrm{Div}(X)/\,\mathrm{PDiv}(X)$.

The *degree* of a divisor is defined to be the sum of its integer coefficients, i.e.,

$$\deg\left(\sum_i n_i p_i\right) = \sum_i n_i.$$

On a compact Riemann surface, any principal divisor has degree zero, so we have a map $\deg :\ \mathrm{Pic}(X) \to \mathbb{Z}$, which we can take as defining the degree of a line bundle. The kernel of this map, the space of isomorphism classes of degree zero line bundles, is called the *Jacobian* of X, and is denoted by $\mathrm{Jac}(X)$. An important property of the Jacobian is that it is an *abelian variety* (meaning that it is algebraic) in particular it is a complex torus. It turns out that it has dimension g (the genus of the Riemann surface X). Moreover, we denote by $\mathrm{Jac}^d(X)$ the space of isomorphism classes of degree d line bundles.

Remark 1.13. One consequence of the definition of degree is that a line bundle L can only admit global sections if $\deg L \geq 0$.

One way to extend the notion of degree to higher rank bundles E is to define $\deg E :=$ $\deg(\det E)$. The intuitive idea behind the degree is that it is a topological invariant that measures how "twisted" the vector bundle is. For instance, all trivial bundles have degree zero (there is no "twist"). We will need to use certain properties of the degree, in particular, how to calculate the degree of a bundle built from others whose degree is known.

Proposition 1.14 (Properties of the degree, [GH94, page 446], [MT97, page 185])**.**
Let E, F be two vector bundles over a compact Riemann surface X. Then

- $\deg E \in \mathbb{Z}$*;*
- $\deg(E \oplus F) = \deg E + \deg F$*;*
- $\deg(E \otimes F) = \mathrm{rk}\, F \deg E + \mathrm{rk}\, E \deg F$*.*

Remark 1.15. The notion of divisors can be generalized to higher dimensional complex manifolds. On a complex surface, a divisor is a linear combination of *integral* curves, where integral means that the curve is irreducible and reduced. The correspondence between divisors and line bundles still holds in higher dimensions (see [GH94, page 133]).

1.5 Holomorphic vector bundles and Dolbeault operators

In this section we introduce a fundamental idea, which is seeing a holomorphic vector bundle E as a pair $(\mathbb{E}, \bar{\partial}_E)$, where $\mathbb{E}$ is its underlying smooth vector bundle and $\bar{\partial}_E$ is a *Dolbeault operator*.

A Dolbeault operator on $\mathbb{E}$ is a $\mathbb{C}$-linear map $\bar{\partial}_E : \Omega^0(\mathbb{E}) \to \Omega^{0,1}(\mathbb{E})$ that satisfies a Leibniz rule, i.e., for $f \in C^\infty(X)$ and $s \in \Omega^0(\mathbb{E})$, we have

$$\bar{\partial}_E(fs) = \bar{\partial} f \otimes s + f \bar{\partial}_E s,$$

and $\bar{\partial}_E^2 = 0$ (this is always satisfied for bundles over a Riemann surface).

From a Dolbeault operator we can get holomorphic transition functions in the following manner (see [Hit89, §2]). Suppose $\operatorname{rk} \mathbb{E} = n$ and consider two non-disjoint trivializations U and V. Over U we can find n sections $(s_1, \dots, s_n)$ which are linearly independent at each point of X (called a *local frame* of E over U), and over V we find another n sections $(t_1, \dots, t_n)$ with the same property. Over $U \cap V$, these are related by

$$t_i = \sum_j a_{ij} s_j,$$

where $A = (a_{ij})_{i,j}$ is the transition function between the two trivializations. Applying the operator $\bar{\partial}_E$ to the equation, we obtain

$$\bar{\partial}_E t_i = \sum_j (\bar{\partial} a_{ij} \otimes s_j + a_{ij} \bar{\partial}_E s_j).$$

If we can guarantee that the sections s_i and t_i satisfy $\bar{\partial}_E s_i = 0 = \bar{\partial}_E t_i$ (i.e., they are holomorphic), then it follows that $\bar{\partial} a_{ij} = 0$, i.e., the transition function is holomorphic. The problem of finding such local frames s_i and t_i, guaranteed to exist by the condition $\bar{\partial}_E^2 = 0$, is one of Analysis, which can be checked, for example, in [DK90, Theorem 2.1.53 (page 45) and Section 2.2.2 (page 50)].

The point we want to emphasize here is that, once we find holomorphic local frames, then it follows that the transition functions are holomorphic.

If we already have a holomorphic vector bundle E with underlying smooth bundle $\mathbb{E}$, then we can recover the $\bar{\partial}_E$ operator as follows (see also [Huy05, Lemma 2.6.23]). Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \hookrightarrow \Omega^0(X) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X) \longrightarrow 0,$$

and tensor it by $\mathcal{O}(E)(X)$, obtaining

$$0 \longrightarrow \mathcal{O}(E)(X) \hookrightarrow \Omega^0(\mathbb{E}) \xrightarrow{\bar{\partial}_E} \Omega^{0,1}(\mathbb{E}) \longrightarrow 0.$$

This last complex also allows us to define the Dolbeault cohomology groups of the holomorphic vector bundle E as

$$H^{p,q}(X, E) \coloneqq \frac{\ker(\bar{\partial}_E : \Omega^{p,q}(\mathbb{E}) \to \Omega^{p,q+1}(\mathbb{E}))}{\operatorname{Im}(\bar{\partial}_E : \Omega^{p,q-1}(\mathbb{E}) \to \Omega^{p,q}(\mathbb{E}))}.$$

Dolbeault operators can be extended to associated bundles. For instance, starting with holomorphic vector bundles E and F, the Dolbeault operator $\bar{\partial}_{E,F}$ that gives the bundle $\mathrm{Hom}(E, F)$ its holomorphic structure is built by imposing a Leibniz rule

$$\bar{\partial}_F(f(s)) = (\bar{\partial}_{E,F} f)(s) + f(\bar{\partial}_E s),$$

for all $f \in \Omega^0(\mathrm{Hom}(E, F))$ and $s \in \Omega^0(E)$. Hence we define

$$\bar{\partial}_{E,F} f := \bar{\partial}_F \circ f - f \circ \bar{\partial}_E \in \mathrm{Hom}_{\Omega^0(X)}(\Omega^0(E), \Omega^{0,1}(F)) \cong \Omega^{0,1}(\mathrm{Hom}(E, F)).$$

It can be checked that this induces on $\mathrm{Hom}(E, F)$ the right holomorphic structure.

In this text we will be working mostly in the case where $F = E$. The associated Dolbeault operator $\bar{\partial}_{E,E}$ on $\Omega^0(\mathrm{End}\, E)$ will usually be denoted by $\bar{\partial}_E$ and it should be clear from the context which version of the operator is being used at any given moment.

Remark 1.16. We can now give a reinterpretation of an isomorphism of holomorphic vector bundles (recall Definition 1.6) as follows. Let $g : E \to F$ be a smooth bundle isomorphism. Then it is holomorphic if and only if

$$\bar{\partial}_F \circ g - g \circ \bar{\partial}_E = 0,$$

i.e., the following diagram commutes

$$\begin{tikzcd} \Omega^0(E) \arrow[r, "g"] \arrow[d, "\bar{\partial}_E"'] & \Omega^0(F) \arrow[d, "\bar{\partial}_F"] \\ \Omega^{0,1}(E) \arrow[r, "g"'] & \Omega^{0,1}(F). \end{tikzcd}$$

1.6 Cohomology and hypercohomology

Starting with a sheaf $\mathcal{F}$ on a space X, we can attach to it a series of *cohomology spaces* $H^i(X, \mathcal{F})$ (under certain conditions on $\mathcal{F}$ – more details in Appendix A.2). We start by defining $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ (the global sections of $\mathcal{F}$) and the derived functor construction gives us the rest. In this text, we will work mainly with finite rank locally free sheaves, and so these will always be finite dimensional vector spaces whose dimensions we denote by $h^i(X, E)$. We collect in this section some results about them which will be used throughout.

In the following theorems, X is a n-dimensional compact complex manifold, $\Omega^p_X := \bigwedge^p T^*X$ its sheaf of holomorphic p-forms, and E a holomorphic vector bundle over X. The line bundle $\bigwedge^n T^*X$ is called the *canonical bundle* of X and denoted by K.

Theorem 1.17 (Dolbeault)**.** *The Dolbeault cohomology of E computes the sheaf cohomology of $E \otimes \Omega^p_X$, i.e.,*

$$H^{p,q}(X, E) \cong H^q(X, E \otimes \Omega^p_X).$$

Theorem 1.18 (Serre duality)**.** *The following pairing is non-degenerate*

$$\begin{aligned} H^{p,q}(X, E) \times H^{n-p,n-q}(X, E^*) &\longrightarrow \mathbb{C} \\ (\alpha, \beta) &\longmapsto \int_X \alpha \wedge \beta. \end{aligned}$$

In particular, it follows that,

$$H^q(X,E) \cong H^{n-q}(X,E^* \otimes K)^*.$$

If $n = 1$, i.e., X is a compact Riemann surface, then $K = T^*X$ and Dolbeault's theorem gives us the following isomorphisms

$$\begin{aligned} H^{0,q}(X,E) &\cong H^q(X,E), \\ H^{1,q}(X,E) &\cong H^q(X,E \otimes K). \end{aligned}$$

Moreover, Serre duality becomes

$$\begin{aligned} H^0(X,E) &\cong H^1(X,E^* \otimes K)^*, \\ H^1(X,E) &\cong H^0(X,E^* \otimes K)^*. \end{aligned}$$

Finally, we have the following result, valid only for Riemann surfaces.

Theorem 1.19 (Riemann-Roch)**.** *Let X be a compact Riemann surface of genus g and E a holomorphic vector bundle over X. We have the following equality relating the dimensions of the sheaf cohomology of E with its rank and degree*

$$h^0(X,E) - h^1(X,E) = \deg(E) + (\operatorname{rk} E)(1-g).$$

Remark 1.20. On a compact Riemann surface, the canonical bundle has degree $2g-2$ [Mir95, Proposition V.1.14].

Consider now a complex $C^\bullet$ of sheaves on X. We can associate to it a series of *hypercohomology spaces* $\mathbb{H}^i(X,C^\bullet)$ (see [GH94, page 446]). As with cohomology (see Theorem A.6), we will be using the fact that, for any short exact sequence of complexes

$$0 \longrightarrow C^\bullet \longrightarrow D^\bullet \longrightarrow E^\bullet \longrightarrow 0$$

there is a long exact sequence of hypercohomology

$$\cdots \longrightarrow \mathbb{H}^i(C^\bullet) \longrightarrow \mathbb{H}^i(D^\bullet) \longrightarrow \mathbb{H}^i(E^\bullet) \longrightarrow \mathbb{H}^{i+1}(C^\bullet) \longrightarrow \cdots.$$

In the text, we will only work with simple complexes made up of just two sheaves, which will always be sheaves of sections of a holomorphic vector bundle, and their Dolbeault resolutions, so the hypercohomology groups of a complex $C^\bullet : E \xrightarrow{d} F$ are calculated according to the double complex

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \Omega^0(E) \arrow[r, "d"] \arrow[d, "\bar\partial_E"'] & \Omega^0(F) \arrow[r] \arrow[d, "-\bar\partial_F"] & 0 \\ 0 \arrow[r] & \Omega^{0,1}(E) \arrow[r, "d"] \arrow[d] & \Omega^{0,1}(F) \arrow[r] \arrow[d] & 0. \\ & 0 & 0 & \end{tikzcd}$$

from which we obtain the complex

$$0 \longrightarrow \Omega^0(E) \xrightarrow{f} \Omega^{0,1}(E) \oplus \Omega^0(F) \xrightarrow{g} \Omega^{0,1}(F) \longrightarrow 0.$$

The maps f and g are defined as

$$\begin{aligned} f &= \bar{\partial}_E + d, \\ g &= d - \bar{\partial}_F, \end{aligned} \tag{1.2}$$

and the three non-zero hypercohomology spaces are

$$\begin{aligned} \mathbb{H}^0(C^\bullet) &= \ker f, \\ \mathbb{H}^1(C^\bullet) &= \ker g / \operatorname{Im} f, \\ \mathbb{H}^2(C^\bullet) &= \Omega^{0,1}(F) / \operatorname{Im} g. \end{aligned} \tag{1.3}$$

1.7 Algebraic curves and the compactified Jacobian

Recall Example 1.2, where we saw that smooth plane algebraic curves are Riemann surfaces. This gives us a glimpse into a much deeper relationship between Riemann surfaces and algebraic curves. In fact, any compact Riemann surface can be embedded in a projective space $\mathbb{CP}^n$ and is thus a smooth algebraic curve (roughly speaking, the zero set of a collection of homogeneous polynomials) and vice versa (see [GH94, Chapter 2]). Via Serre's GAGA theorems, we are justified in working with Riemann surfaces in the algebraic category, and so we will follow the standard practice of blurring the distinction between the algebraic and analytic categories, as they give rise to completely parallel theories.

We will, however, want to go somewhat further in considering algebraic curves that are not necessarily smooth – they may have singular points (going back to Example 1.2, these are points where the three partial derivatives vanish simultaneously). Moreover, we will also want to consider line bundles on these types of curves and so we need to know something about their Jacobians, thought of as the set of isomorphism classes of rank 1 locally free sheaves. In order to do that, we will restrict ourselves to integral curves X and consider their normalization $\nu : \widetilde{X} \to X$, obtained by successively blowing up the singularities. Then $\widetilde{X}$ is a smooth projective curve and ν is a birational morphism (see [Liu02, Section 8.4.1] and [Har77, Exercise II.5.8, page 232]). If, for example, the singularities of X are r simple nodes, we have the following exact sequence (see a proof in [GO12, Proposition 4.1])

$$0 \longrightarrow (\mathbb{C}^*)^r \longrightarrow \operatorname{Jac}(X) \xrightarrow{\nu^*} \operatorname{Jac}(\widetilde{X}) \longrightarrow 0. \tag{1.4}$$

It tells us that $\operatorname{Jac}(X)$ is a fiber bundle over $\operatorname{Jac}(\widetilde{X})$ (the regular Jacobian of a smooth curve) with non-compact fibers. We can compactify this Jacobian by adding the *rank 1 torsion free sheaves* (of degree 0, see Definition 1.24 below) on X.

First we recall that, for a module M over an integral domain A, we define the *torsion submodule* M_{tors} to be the set of elements $m \in M$ such that there exists a non-zero $a \in A$ with $am = 0$.

Definition 1.21. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules over an integral curve X. Its torsion subsheaf $\mathcal{F}_{\text{tors}}$ is such that $\mathcal{F}_{\text{tors}}(U) = \mathcal{F}(U)_{\text{tors}}$, for every affine open set U of X. We say that $\mathcal{F}$ is *torsion free* if $\mathcal{F}_{\text{tors}} = 0$.

If $\mathcal{F}$ is coherent and non-zero then, by [Pot97, Lemma 2.6.1, page 28], there is a non-empty open subset U of X such that $\mathcal{F}|_U \cong \mathcal{O}_X^n|_U$, for some $n \geq 0$. The number n is called the *rank* of the torsion-free sheaf $\mathcal{F}$. This tells us that a torsion-free coherent sheaf is in fact locally free on almost the entire curve. Moreover, on a smooth curve there is in fact no distinction between being torsion-free and being locally free, a fact that we record in the following result.

Proposition 1.22 ([Pot97, Lemma 5.2.1, page 72])**.** *Torsion-free coherent sheaves on a smooth curve are locally free.*

Remark 1.23. If $\mathcal{F}_{\text{tors}} \neq 0$, then it is supported on a finite number of points of X, and the converse is true as well (by [Liu02, Exercise 1.14 (d), page 174] together with [Har77, Exercise 5.6 (c), page 124]).

To make sense of the degree of a torsion-free sheaf, we use the following definition.

Definition 1.24. The degree of a rank 1 torsion-free sheaf L on a curve X is defined such that Riemann-Roch holds, i.e.,

$$\deg(L) \coloneqq h^0(X, L) - h^1(X, L) + g - 1,$$

and it has all the usual properties of the degree of a locally free sheaf (see [Pot97, page 30] and [New78, page 130]).

The number g in the formula above is the *arithmetic genus* of the curve X, defined by

$$g = 1 - h^0(X, \mathcal{O}_X) + h^1(X, \mathcal{O}_X). \tag{1.5}$$

If X is smooth, then its arithmetic genus coincides with the topological genus defined in Proposition 1.1 (see [Mir95, page 192]).

The property of being torsion-free can also be checked at the stalks and it means that, for every $p \in X$, $\mathcal{F}_p$ is a torsion-free $\mathcal{O}_{X,p}$-module.

Example 1.25 (A rank 1 torsion free sheaf that is not locally free)**.** Let X be the singular plane algebraic curve defined by the equation $xy = 0$. Its normalization $\nu : \widetilde{X} \to X$ consists of two disjoint copies of the affine line (see Figure 1.1).

Consider then the pushforward sheaf $\nu_* \mathcal{O}_{\widetilde{X}}$. For $p \in X \setminus \{(0,0)\}$, we have the isomorphisms of stalks $\left(\nu_* \mathcal{O}_{\widetilde{X}}\right)_p \cong \left(\mathcal{O}_{\widetilde{X}}\right)_{\nu^{-1}(p)} \cong (\mathcal{O}_X)_p$, since ν is an isomorphism away from $(0,0)$. So $\nu_* \mathcal{O}_{\widetilde{X}}$ is a rank 1 locally free sheaf when restricted to $X \setminus \{(0,0)\}$.

But, if $\nu^{-1}(0,0) = \{q_1, q_2\}$, we have

$$\left(\nu_* \mathcal{O}_{\widetilde{X}}\right)_{(0,0)} \cong \mathcal{O}_{\widetilde{X},q_1} \oplus \mathcal{O}_{\widetilde{X},q_2}$$

and

$$\mathcal{O}_{X,(0,0)} = \left\{(f, g) \in \mathcal{O}_{\widetilde{X},q_1} \oplus \mathcal{O}_{\widetilde{X},q_2} \mid f(q_1) = g(q_2)\right\}.$$

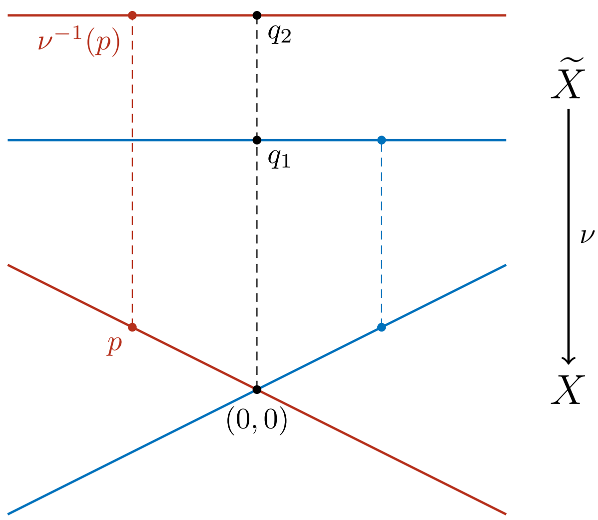

Figure 1.1: Normalization of the singular curve $xy = 0$.

Now $\left(\nu_*\mathcal{O}_{\widetilde{X}}\right)_{(0,0)}$ is torsion-free as a $\mathcal{O}_{X,(0,0)}$-module since if it were not, that would mean that $\mathcal{O}_{\widetilde{X},q_1}$ would not be torsion-free as a $\mathcal{O}_{\widetilde{X},q_1}$-module, which is absurd.

However, $\left(\nu_*\mathcal{O}_{\widetilde{X}}\right)_{(0,0)}$ is not free of rank 1 as a $\mathcal{O}_{X,(0,0)}$-module. In order to see why, we first note that

$$\left(\nu_*\mathcal{O}_{\widetilde{X}}\right)_{(0,0)} \cong \mathbb{C}[z] \oplus \mathbb{C}[z]$$

and

$$\mathcal{O}_{X,(0,0)} \cong \{(\alpha_1, \alpha_2) \in \mathbb{C}[z] \oplus \mathbb{C}[z] \mid \alpha_1(0) = \alpha_2(0)\}.$$

Suppose there exists $(f_1, f_2) \in \mathbb{C}[z] \oplus \mathbb{C}[z]$ such that

$$\mathbb{C}[z] \oplus \mathbb{C}[z] = \mathcal{O}_{X,(0,0)} \cdot (f_1, f_2),$$

then for every $g \in \mathbb{C}[z]$, $f_1|g$ which means that $f_1 \in \mathbb{C}$ and the same is true for f_2. At least one of the f_i is non-zero so suppose $f_1 \neq 0$. But then there must exist $(\alpha_1, \alpha_2) \in \mathbb{C}[z] \oplus \mathbb{C}[z]$ with $\alpha_1(0) = \alpha_2(0)$ such that

$$\begin{cases} 0 = \alpha_1 f_1 \\ 1 = \alpha_2 f_2 \end{cases}$$

which implies that $\alpha_2(0) = \alpha_1(0) = 0$, which is absurd.

In fact, it is not free of *any* rank. Take any two (non-zero) pairs (f_1, f_2) and (g_1, g_2) in $\mathbb{C}[z] \oplus \mathbb{C}[z]$. Since the polynomials in the pairs (zg_1, zg_2) and $(-zf_1, -zf_2)$ all vanish at 0, the equation

$$(zg_1, zg_2)(f_1, f_2) + (-zf_1, -zf_2)(g_1, g_2) = 0$$

tells us that there are no two linearly independent elements in $\left(\nu_*\mathcal{O}_{\widetilde{X}}\right)_{(0,0)}$ as a $\mathcal{O}_{X,(0,0)}$-module, and so it cannot be a free module of any rank ≥ 2.

We then obtain a new object, the *compactified Jacobian* of the singular curve X, denoted by $\overline{\mathrm{Jac}}(X)$ (similarly, we denote by $\overline{\mathrm{Jac}}^d(X)$ the space of rank 1 torsion-free sheaves on X of degree d). Although we will not study this object in detail in this text, we would like to mention that it is a singular variety, unlike the Jacobian of a smooth curve. However, if X is integral and embedded in a surface, then $\overline{\mathrm{Jac}}(X)$ is irreducible (see [Reg80] and [Kas13] for more on the compactified Jacobian).

1.8 Integrable systems

We shall now review the basics of symplectic geometry, adapted to the complex/holomorphic case, and having as a destination the definition of a completely integrable system. We follow [Aud96], [Sil08], and [HSW99].

Definition 1.26. A *symplectic manifold* is a pair (M, ω), where M is a complex manifold and ω is a closed non-degenerate holomorphic 2-form on M, i.e., a form of type $(2, 0)$ (called the *symplectic form*).

Remark 1.27. The non-degeneracy of ω implies that M has even dimension.

Definition 1.28. A vector field X on M is said to be *symplectic* if $i_X\omega$ is a closed 1-form, i.e., $d(i_X\omega) = 0$. X is said to be *Hamiltonian* if $i_X\omega$ is exact, i.e., there exists $f \in H^0(M, \mathcal{O}_M)$ such that $i_X\omega = df$.

The submanifolds of M can be classified according to how the symplectic form behaves on their tangent bundle. The following will be of particular importance to us.

Definition 1.29. A submanifold N of a symplectic manifold is said to be *Lagrangian* if $\dim N = \frac{\dim M}{2}$, and $\omega|_N = 0$.

The non-degeneracy of ω allows us to define an isomorphism between sections of TM (vector fields) and T^*M (1-forms) using the interior product

$$X \longmapsto i_X\omega.$$

The following is then a well-defined concept.

Definition 1.30. Let $f \in H^0(M, \mathcal{O}_M)$. The *Hamiltonian vector field* associated to f is the (unique) vector field X_f such that $i_{X_f}\omega = df$.

Hamiltonian vector fields allow us to define the following bilinear pairing on the space of holomorphic functions of M.

Definition 1.31. On any symplectic manifold (M, ω), we define the *Poisson bracket* $\{\cdot, \cdot\}$ on $H^0(M, \mathcal{O}_M)$ by

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f).$$

Two functions $f, g \in H^0(M, \mathcal{O}_M)$ are said to *Poisson-commute* if $\{f, g\} = 0$.

Definition 1.32. A symplectic manifold (M, ω) with $\dim M = 2n$ is a *completely integrable system* if there exist n functions $f_1, \dots, f_n \in H^0(M, \mathcal{O}_M)$ such that

1. the f_i are *functionally independent* – for every point p in an open dense subset U of M, the 1-forms $(df_i)_p$ are independent (equivalently, the vectors $X_{f_i}(p)$ are independent);

2. the f_i Poisson-commute with each other.

Proposition 1.33. *Let (M, ω) be a completely integrable system via the n functions $f_1, \ldots, f_n$. Letting $F = (f_1, \ldots, f_n)$, if c is a regular value of F, then $F^{-1}(c)$ is a Lagrangian submanifold of M.*

Proof. If c is a regular value of F, then $F^{-1}(c)$ is a submanifold of M with dimension equal to $\dim M - n = n$.

Moreover, at every point $p \in F^{-1}(c)$, the 1-forms $(df_i)_p$ are linearly independent and so the corresponding Hamiltonian vector fields X_i at p form a basis of $T_p F^{-1}(c)$. Since the functions Poisson-commute, we have, for all i, j,

$$\omega(X_i, X_j) = \{f_i, f_j\} = 0. \qquad \blacksquare$$

As a first example of a completely integrable system, we present the cotangent space to the Jacobian of a compact Riemann surface X of genus g. We omit most details here since it will be greatly expanded in Section 3.3.

Example 1.34. Let X be a Riemann surface and $\operatorname{Jac}(X)$ its Jacobian. It is a complex torus so it has trivial cotangent bundle, i.e., $T^* \operatorname{Jac}(X) \cong \operatorname{Jac}(X) \times H^0(X, K)$ (we shall see, in Proposition 2.7, that the tangent space at any point of $\operatorname{Jac}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X) \cong H^{0,1}(X, \mathcal{O}_X)$). The symplectic form ω coming from the Liouville 1-form will be (see Section 2.3)

$$\omega((\dot{A}, \dot{\Phi}), (\dot{B}, \dot{\Psi})) = \int_X (\dot{A} \wedge \dot{\Psi} - \dot{B} \wedge \dot{\Phi}).$$

Consider the map

$$\begin{array}{rccc} h: & T^* \operatorname{Jac}(X) & \longrightarrow & H^0(X, K) \\ & (L, \Phi) & \longmapsto & \Phi. \end{array}$$

We have $h^0(X, K) = g$, and Serre duality tells us that $H^0(X, K)^* \cong H^1(X, \mathcal{O}_X)$. Consider a basis $\alpha_1, \ldots, \alpha_g$ of $H^0(X, K)^*$, with representatives $\beta_1, \ldots, \beta_g \in \Omega^{0,1}(X, \mathcal{O}_X)$, and the functions $f_1, \ldots, f_g : T^* \operatorname{Jac}(X) \to \mathbb{C}$, defined by

$$f_i(L, \Phi) = \int_X \beta_i(\Phi).$$

It can be seen that the Hamiltonian vector fields X_i are $(\beta_i, 0)$, meaning that the f_i Poisson-commute and are functionally independent, so we have a completely integrable system. Thus, for every $\Phi \in H^0(X, K)$, $h^{-1}(\Phi)$ will be a Lagrangian subvariety.

Chapter 2

Higgs bundles and their moduli space

2.1 The moduli space of vector bundles

Let X be a compact Riemann surface of genus $g \geq 2$. Smooth complex vector bundles on X are classified by their rank n and degree d [Pot97, Corollary 3.5.4]. When turning to the holomorphic case however, the situation is not so simple, giving rise to the notion of a moduli space of vector bundles, whose construction we describe here.

Recall from Section 1.5 that a holomorphic vector bundle E can be thought of as a pair $(\mathbb{E}, \bar{\partial}_E)$, where $\mathbb{E}$ is the underlying smooth vector bundle and $\bar{\partial}_E$ a Dolbeault operator. Fixing $\mathbb{E}$, we start by considering the space of all possible Dolbeault operators on it

$$\mathcal{A}_{\bar{\partial}} = \mathcal{A}_{\bar{\partial}}(\mathbb{E}) \coloneqq \{\bar{\partial}_E \mid \bar{\partial}_E \text{ is a Dolbeault operator on } \mathbb{E}\}.$$

We call $\mathcal{A}_{\bar{\partial}}$ the *configuration space*.

For the remainder of the text, we will not make the distinction between the holomorphic vector bundle E and its underlying smooth bundle, since it will always be clear which one we are referring to (e.g., $\Omega^k(E)$ will always means $\Omega^k(\mathbb{E})$).

Proposition 2.1. *The configuration space $\mathcal{A}_{\bar{\partial}}$ is an affine space over $\Omega^{0,1}(\operatorname{End} E)$.*

Proof. Let $\bar{\partial}_1, \bar{\partial}_2 \in \mathcal{A}_{\bar{\partial}}$, $f \in C^\infty(X)$, and $s \in \Omega^0(E)$. We have

$$(\bar{\partial}_1 - \bar{\partial}_2)(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_1 s - \bar{\partial} f \otimes s - f\bar{\partial}_2 s = f(\bar{\partial}_1 - \bar{\partial}_2)(s),$$

which means that $\bar{\partial}_1 - \bar{\partial}_2 \in \Omega^{0,1}(\operatorname{End} E)$.

Conversely let $\bar{\partial}_E \in \mathcal{A}_{\bar{\partial}}$ and $A \in \Omega^{0,1}(\operatorname{End} E)$. Then

$$(\bar{\partial}_E + A)(fs) = \bar{\partial}_E(fs) + A(fs) = \bar{\partial} f \otimes s + f\bar{\partial}_E s + f(As) = \bar{\partial} f \otimes s + f(\bar{\partial}_E + A)s,$$

which means that $\bar{\partial}_E + A$ satisfies the Leibniz rule and so belongs to $\mathcal{A}_{\bar{\partial}}$. ■

Now, consider the action of the group of *gauge transformations* $\mathcal{G}^c \coloneqq \Omega^0(\operatorname{Aut} E)$ on $\mathcal{A}_{\bar{\partial}}$. It acts by conjugation on the Dolbeault operators,

$$g \cdot \bar{\partial}_E \coloneqq g^{-1} \circ \bar{\partial}_E \circ g.$$

Note that $(\mathbb{E}, \bar{\partial}_E)$ and $(\mathbb{E}, g \cdot \bar{\partial}_E)$ are isomorphic as holomorphic vector bundles, by applying Remark 1.16 to the map g^{-1}, and seeing that

$$(g \cdot \bar{\partial}_E) \circ g^{-1} - g^{-1} \circ \bar{\partial}_E = 0.$$

We could first consider the full space of orbits $\mathcal{A}_{\bar{\partial}}/\mathcal{G}^c$. However it turns out to be quite unwieldy for our purposes (it is not even Hausdorff, see [Tha96, Section 2]). What we shall do instead is “throw away the unstable bundles”, according to the following definition.

Definition 2.2. The *slope* μ of a holomorphic vector bundle E is defined as

$$\mu(E) \coloneqq \frac{\deg E}{\operatorname{rk} E}.$$

A holomorphic vector bundle E is said to be *stable* (respectively *semistable*) if for any proper non-zero holomorphic subbundle $F \subset E$, we have $\mu(F) < \mu(E)$ (respectively $\mu(F) \leq \mu(E)$).

A semistable bundle E is *polystable* if it is a direct sum of stable bundles, with all of the summands having slope equal to $\mu(E)$.

Remark 2.3. We have the following chain of inclusions

$$\{\text{stable}\} \subseteq \{\text{polystable}\} \subseteq \{\text{semistable}\}.$$

We shall denote by $\mathcal{A}^{ss}_{\bar{\partial}}$ and $\mathcal{A}^{s}_{\bar{\partial}}$ the spaces of semistable and stable holomorphic vector bundles, respectively, in $\mathcal{A}_{\bar{\partial}}$.

Remark 2.4. Stability and semistability of vector bundles are open conditions ([Pot97, Proposition 7.2.6, page 115]), i.e., $\mathcal{A}^{s}_{\bar{\partial}}$ and $\mathcal{A}^{ss}_{\bar{\partial}}$ are open sets of $\mathcal{A}_{\bar{\partial}}$.

We start by defining the smooth locus of the moduli space, considering only the orbits of the gauge group acting on the stable bundles

$$\mathcal{N}^s(n, d) \coloneqq \mathcal{A}^{s}_{\bar{\partial}}/\mathcal{G}^c.$$

This is the *moduli space of stable vector bundles* of rank n and degree d. It is smooth and projective if n and d are coprime, otherwise it is still smooth but only quasi-projective (see [Pot97, Theorem 7.2.1 and Section 8.3, Theorem 8.3.2] and [New78, Theorem 5.8 and Remark 5.9]). To compactify it, we add the (strictly) semistable bundles, but now we cannot simply identify points in the same orbit; a stronger identification must be made, which we describe as follows (see [Pot97, page 76] for details).

Given any semistable bundle E, we can associate to it a *Jordan-Hölder filtration*, i.e., a sequence of subbundles

$$0 = F_0 \subset F_2 \subset \cdots \subset F_k = E$$

such that each successive quotient F_i/F_{i-1} is stable with slope equal to $\mu(E)$. The direct sum $\bigoplus_i F_i/F_{i-1}$ is unique up to isomorphism, we denote it by $\operatorname{Gr}(E)$. Two bundles E_1 and E_2 are then said to be *S-equivalent* if $\operatorname{Gr}(E_1) \cong \operatorname{Gr}(E_2)$. Every S-equivalence class has exactly one polystable representative. If E is a stable bundle, then its filtration reduces to $0 \subset E$, which means that $\operatorname{Gr}(E) \cong E$ and so two stable bundles are S-equivalent if and only if they are isomorphic. By Remark 1.16, this means precisely that they are on the same orbit of the gauge group.

Definition 2.5. The *moduli space of vector bundles* of rank n and degree d on X is

$$\mathcal{N}(n,d) := \mathcal{A}^{ss}_{\bar{\partial}} /\!/ \mathcal{G}^c,$$

where the double slash means we are identifying the S-equivalent bundles.

Remark 2.6. Identifying S-equivalent bundles is the same as identifying orbits whose closures have non-empty intersection ([New78, Complement 5.8.1]).

One should note that our treatment of the moduli space in this work is not rigorous. Since we are dealing with the quotient of an infinite dimensional manifold by an infinite dimensional Lie group, there is some caution needed in order to make sense of such a construction, i.e., to endow the space $\mathcal{N}(n,d)$ with the structure of a manifold. In order to invoke results from the theory of Banach manifolds, we would need to complete the spaces we are considering, with respect to certain Sobolev norms (see [DK90] for details).

The space $\mathcal{N}(n,d)$ is projective and singular precisely at the points represented by strictly polystable bundles, except when $g=2$, $n=2$ and d is even ([NR69, Theorem 1, page 20]).

If n and d are coprime, then all semistable bundles are in fact stable, simply because there is no way to write the irreducible fraction n/d with a smaller denominator. So indeed $\mathcal{N}(n,d) = \mathcal{N}^s(n,d)$ if n and d are coprime.

All line bundles are stable (there are no non-zero proper subbundles) and so $\mathcal{N}(1,d)$ is the space of all line bundles of degree d, i.e., $\mathrm{Jac}^d(X)$. If $d=0$, we recover the Jacobian $\mathrm{Jac}(X)$.

2.1.1 Tangent space

We will now describe the tangent space $T_{[E]}\mathcal{N}(n,d)$, where E is a *stable* bundle, so that $[E]$ is a point in the smooth moduli space $\mathcal{N}^s(n,d)$. This amounts to describing the quotient $T_E\mathcal{A}^s_{\bar{\partial}}/T_E\mathcal{G}^c(E)$, where $T_E\mathcal{G}^c(E)$ is the tangent space at E of the orbit $\mathcal{G}^c(E)$, given by the infinitesimal action of the group of gauge transformations.

Since, by Proposition 2.1, $\mathcal{A}_{\bar{\partial}}$ is an affine space over $\Omega^{0,1}(\operatorname{End} E)$ and $\mathcal{A}^s_{\bar{\partial}}$ is an open set (Remark 2.4), we have that, for any $E \in \mathcal{A}^s_{\bar{\partial}}$,

$$T_E\mathcal{A}^s_{\bar{\partial}} = T_E\mathcal{A}_{\bar{\partial}} \cong \Omega^{0,1}(\operatorname{End} E).$$

We now determine the infinitesimal action of $\psi \in \Omega^0(\operatorname{End} E)$, the Lie algebra of $\mathcal{G}^c$, on $\bar{\partial}_E \in \mathcal{A}^s_{\bar{\partial}}$, getting

$$\left.\frac{d}{dt}\right|_{t=0} \exp(-\psi t)\bar{\partial}_E \exp(\psi t) = -\psi\bar{\partial}_E + \bar{\partial}_E\psi = \bar{\partial}_E\psi \in \bar{\partial}_E(\Omega^0(\operatorname{End} E)).$$

So we are looking at the Dolbeault cohomology group

$$H^{0,1}(\operatorname{End} E) = \Omega^{0,1}(\operatorname{End} E)/\bar{\partial}_E\Omega^0(\operatorname{End} E).$$

Together with Dolbeault's theorem, we obtain the following identifications.

Proposition 2.7. *If E is a stable vector bundle, then*

$$T_{[E]}\mathcal{N}(n,d) \cong H^{0,1}(X, \operatorname{End} E) \cong H^1(X, \operatorname{End} E).$$

We can now use this result to calculate the dimension of our moduli space.

Corollary 2.8. *The dimension of the moduli space* $\mathcal{N}(n,d)$ *is* $1+n^2(g-1)$.

Proof. By the previous proposition, what we need to know is the value of $h^1(X, \operatorname{End} E)$, where E is a stable bundle. It can be obtained via Riemann-Roch as

$$h^1(X, \operatorname{End} E) = h^0(X, \operatorname{End} E) - \deg(\operatorname{End} E) - \operatorname{rk}(\operatorname{End} E)(1-g).$$

The result follows from knowing that

$$\deg(\operatorname{End} E) = \deg(E \otimes E^*) = \operatorname{rk}(E)\deg(E^*) + \operatorname{rk}(E^*)\deg(E) = 0;$$

and that $h^0(X, \operatorname{End} E) = 1$. This is because, for a stable bundle E, we have $H^0(X, \operatorname{End} E) \cong \mathbb{C}$, i.e., every endomorphism of E is of the form $\lambda \operatorname{id}_E$, with $\lambda \in \mathbb{C}$ (a homothety), by [Pot97, Corollary 5.3.4]. ■

2.2 The moduli space of Higgs bundles

Let X be a compact Riemann surface of genus $g \geq 2$, as in the previous section. Recall from Section 1.6 that $K \coloneqq T^*X$ is called the canonical bundle of X.

Definition 2.9. A *Higgs bundle* is a pair (E, Φ), where E is a holomorphic vector bundle on X, and Φ, called the *Higgs field*, is a holomorphic section of the bundle $\operatorname{End} E \otimes K$, i.e., an element of $H^0(X, \operatorname{End} E \otimes K)$.

Remark 2.10. Recall that, by Serre duality (Theorem 1.18) the following pairing is non-degenerate (from now on, we omit the $\wedge$ symbol inside the integral)

$$\begin{aligned} H^{0,1}(X, \operatorname{End} E) \times H^{1,0}(X, \operatorname{End} E) &\longrightarrow \mathbb{C} \\ (\dot{A}, \dot{\Phi}) &\longmapsto \int_X \operatorname{tr}(\dot{A}\dot{\Phi}). \end{aligned}$$

and, together with Dolbeault's theorem, $H^1(X, \operatorname{End} E)^* \cong H^0(X, \operatorname{End} E \otimes K)$. This means that, given a stable bundle E, a Higgs field on E is an element of the cotangent space $T^*_{[E]}\mathcal{N}(n,d)$. It can then be seen in two ways: as a holomorphic map $\Phi : E \to E \otimes K$ and, since $\operatorname{End} E \otimes K \cong T^*X \otimes \operatorname{End} E$, as a holomorphic 1-form on X with values in the bundle $\operatorname{End} E$ (locally, a matrix valued form of type $(1,0)$).

We now adapt the stability conditions of Definition 2.2 to the case of Higgs bundles, the main difference being that we are only interested in Φ-*invariant* vector subbundles, i.e., subbundles F such that $\Phi(F) \subseteq F \otimes K$.

Definition 2.11. A Higgs bundle (E, Φ) is *stable* if, for every proper, non-zero, Φ-invariant subbundle $F \subset E$, $\mu(F) < \mu(E)$ (a similar adaptation is made to define semistable and polystable Higgs bundles).

Remark 2.12. Note that any stable vector bundle can be made into a stable Higgs bundle by taking $\Phi = 0$.

To describe the moduli space of Higgs bundles, we first consider the configuration space

$$\mathcal{H} = \{(\bar{\partial}_E, \Phi) \in \mathcal{A}_{\bar{\partial}} \times \Omega^{1,0}(X, \operatorname{End} E) \mid \bar{\partial}_E \Phi = 0\}$$

and its subsets

$$\mathcal{H}^{ss} = \{(\bar{\partial}_E, \Phi) \in \mathcal{H} \mid (\bar{\partial}_E, \Phi) \text{ is semistable}\},$$
$$\mathcal{H}^{s} = \{(\bar{\partial}_E, \Phi) \in \mathcal{H} \mid (\bar{\partial}_E, \Phi) \text{ is stable}\}.$$

Just like for vector bundles (recall Remark 2.4) stability of Higgs bundles is also an open condition (it follows from the much more general results in [MFK94, Proposition 1.9, Theorem 1.10]).

As before, an element $g \in \mathcal{G}^c = \Omega^0(\operatorname{Aut} E)$ acts on $\mathcal{H}^{ss}$ by conjugation

$$g \cdot (\bar{\partial}_E, \Phi) = (g^{-1} \circ \bar{\partial}_E \circ g, g^{-1} \circ \Phi \circ g).$$

As with the moduli space of vector bundles, to obtain the smooth locus of the moduli space of Higgs bundles, we take the stable bundles and identify them whenever they lie on the same orbit of the gauge group action, and get the moduli space of stable Higgs bundles

$$\mathcal{M}^s(n, d) \coloneqq \mathcal{H}^s / \mathcal{G}^c.$$

We define the Jordan-Hölder filtration for Higgs bundles in a similar manner as before, and so we also get a notion of S-equivalence, which allows us to consider the moduli space of Higgs bundles

$$\mathcal{M}(n, d) \coloneqq \mathcal{H}^{ss} /\!\!/ \mathcal{G}^c.$$

The space $\mathcal{M}^s(n, d)$ is open and dense in $\mathcal{M}(n, d)$ and they are equal if n and d are coprime ([Nit91, Theorem 5.10]). Moreover, from Remark 2.12, $\mathcal{N}(n, d) \subset \mathcal{M}(n, d)$, for any pair (n, d).

Elements of the moduli space $\mathcal{M}(n, d)$ are equivalence classes represented by some Higgs bundle (E, Φ). To alleviate notation, we will sometimes not make the distinction between the class $[(E, \Phi)]$ and the bundle (E, Φ).

Example 2.13. Choose a square root $K^{1/2}$ of the canonical bundle K (i.e., a line bundle $K^{1/2}$ such that $K^{1/2} \otimes K^{1/2} \cong K$) and a holomorphic section q of K^2. Then we get a Higgs bundle by considering $E = K^{1/2} \oplus K^{-1/2}$ and the map

$$\Phi_q = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}.$$

Note that, if $\alpha \in H^0(X, K^{1/2})$ and $\beta \in H^0(X, K^{-1/2})$, we have

$$\begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} q \otimes \beta \\ \alpha \end{pmatrix} \in H^0(X, (K^2 \otimes K^{-1/2}) \oplus K^{1/2}).$$

Since $(K^2 \otimes K^{-1/2}) \oplus K^{1/2} \cong E \otimes K$, we have that Φ_q is a Higgs field for the bundle E.

Note also that the bundle E is not stable since $\mu(K^{1/2}) = \deg(K^{1/2}) = g - 1$ and $\mu(E) = 0$, so $\mu(K^{1/2}) > \mu(E)$. But if we take a subbundle L other than $K^{1/2}$, the projection map $L \to K^{-1/2}$

is a non-zero holomorphic section of the line bundle $L^{-1} \otimes K^{-1/2}$, which means, by Remark 1.13, that $\deg(L^{-1} \otimes K^{-1/2}) \geq 0$, telling us that $\deg(L) \leq \deg(K^{-1/2}) < 0 = \deg(E)$. So $K^{1/2}$ is the only destabilizing subbundle of E, but it is not Φ_q-invariant, so the Higgs bundle (E, Φ_q) is stable.

The complement of $T^*\mathcal{N}(n,d)$ in $\mathcal{M}(n,d)$ (recall Remark 2.10) is made out of pairs (E,Φ) like the example above, where E is *not* stable by itself but becomes stable when the Higgs field is added, meaning that the destabilizing subbundles of E are not invariant by the Higgs field.

2.2.1 Tangent space

We will now describe the tangent space $T_{[(E,\Phi)]}\mathcal{M}(n,d)$, where (E,Φ) is a *stable* bundle, as the quotient $T_{(E,\Phi)}\mathcal{H}^s_{\bar\partial}/T_{(E,\Phi)}\mathcal{G}^c(E,\Phi)$, following the same method of Section 2.1.1.

In order to see what is the tangent space of $\mathcal{H}^s$ at a point $(\bar\partial_E, \Phi)$, we need to linearize the equation $\bar\partial_E\Phi = 0$, which amounts to considering the map

$$\begin{array}{rrcl} F: & \mathcal{A}_{\bar\partial} \times \Omega^{1,0}(\operatorname{End} E) & \longrightarrow & \Omega^{1,1}(\operatorname{End} E) \\ & (\bar\partial_E, \Phi) & \longmapsto & \bar\partial_E \Phi \end{array}$$

and calculating its derivative at a point $(\bar\partial_E, \Phi)$, in a direction $(\dot A, \dot\Phi) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$. Using the curve $\gamma(t) = (\bar\partial_E + t\dot A, \Phi + t\dot\Phi)$ in $\mathcal{A}_{\bar\partial} \times \Omega^{1,0}(\operatorname{End} E)$, we get

$$\begin{aligned} DF_{(\bar\partial_E,\Phi)}(\dot A, \dot\Phi) &= \frac{d}{dt}(\bar\partial_E + t\dot A)(\Phi + t\dot\Phi)\Big|_{t=0} \\ &= \frac{d}{dt}(\bar\partial_E\Phi + t\bar\partial_E\dot\Phi + t[\dot A, \Phi] + t^2[\dot A, \dot\Phi])\Big|_{t=0} \\ &= \bar\partial_E\dot\Phi - [\Phi, \dot A]. \end{aligned}$$

So we obtain a description of the tangent space of $\mathcal{H}^s$ as

$$T_{(\bar\partial_E,\Phi)}\mathcal{H}^s = \{(\dot A, \dot\Phi) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E) \mid \bar\partial_E\dot\Phi = [\Phi, \dot A]\}.$$

Now, the infinitesimal action of $\psi \in \Omega^0(\operatorname{End} E)$ on $\bar\partial_E$ has already been calculated as

$$\frac{d}{dt}\Big|_{t=0} \exp(-\psi t)\bar\partial_{E_0}\exp(\psi t) = \bar\partial_{E_0}\psi \in \Omega^{0,1}(\operatorname{End} E),$$

and the action on Φ is

$$\frac{d}{dt}\Big|_{t=0} \exp(-\psi t)\Phi\exp(\psi t) = -\psi\Phi + \Phi\psi = [\Phi, \psi] \in \Omega^{1,0}(\operatorname{End} E),$$

where the first ψ in the commutator is to be understood as a map $\Omega^{1,0}(E) \to \Omega^{1,0}(E)$. Note also that the tangent vectors coming from the infinitesimal action of $\mathcal{G}^c$ are in the tangent space, since

$$\begin{aligned} DF_{(\bar\partial_E,\Phi)}(\bar\partial_E\psi, [\Phi,\psi]) &= \bar\partial_E[\Phi,\psi] - [\Phi, \bar\partial_E\psi] \\ &= [\bar\partial_E\Phi, \psi] + [\Phi, \bar\partial_E\psi] - [\Phi, \bar\partial_E\psi] = 0. \end{aligned}$$

We are now ready to identify the tangent space of $\mathcal{M}(n,d)$ at a point (E,Φ) with a hypercohomology group of the complex

$$C^\bullet = C^\bullet_{(E,\Phi)} : \operatorname{End} E \xrightarrow{[\Phi,-]} \operatorname{End} E \otimes K. \tag{2.1}$$

We calculate the hypercohomology groups by looking at its Dolbeault resolution (recall Section 1.6)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^0(\operatorname{End} E) \arrow[r, "{[\Phi,-]}"] \arrow[d, "\bar\partial_E"'] & \Omega^{1,0}(\operatorname{End} E) \arrow[r] \arrow[d, "-\bar\partial_E"] & 0 \\
0 \arrow[r] & \Omega^{0,1}(\operatorname{End} E) \arrow[r, "{[\Phi,-]}"] \arrow[d] & \Omega^{1,1}(\operatorname{End} E) \arrow[r] \arrow[d] & 0. \\
 & 0 & 0 &
\end{tikzcd}$$

giving us the complex

$$0 \longrightarrow \Omega^0(\operatorname{End} E) \xrightarrow{f} \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E) \xrightarrow{g} \Omega^{1,1}(\operatorname{End} E) \longrightarrow 0,$$

where, by (1.2), the map f is given by the infinitesimal action

$$f(s) = (\bar\partial_E s, [\Phi, s]), \quad s \in \Omega^0(\operatorname{End} E),$$

while g is given by (minus) the derivative of F

$$g(\alpha,\beta) = -\bar\partial_E \beta + [\Phi,\alpha], \quad (\alpha,\beta) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E).$$

By (1.3) the first hypercohomology group $\mathbb{H}^1(C^\bullet)$ of the complex $C^\bullet$ in (2.1) is given by $\mathbb{H}^1(C^\bullet) = \ker g / \operatorname{Im} f$. Thus we have proved the following identification.

Proposition 2.14. *Let (E,Φ) be a stable Higgs bundle representing a point in $\mathcal{M}(n,d)$. Then*

$$T_{[(E,\Phi)]}\mathcal{M}(n,d) \cong \mathbb{H}^1(C^\bullet).$$

We can then use this to calculate the dimension of the moduli space $\mathcal{M}(n,d)$.

Corollary 2.15. *The dimension of the moduli space $\mathcal{M}(n,d)$ is $n^2(2g-2)+2$.*

Proof. We have the following short exact sequence of complexes ([BR94, Remark 2.7])

$$\begin{tikzcd}
 & D^\bullet & C^\bullet & E^\bullet & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \operatorname{End} E \arrow[r] \arrow[d, "{[\Phi,-]}"] & \operatorname{End} E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{End} E \otimes K \arrow[r] & \operatorname{End} E \otimes K \arrow[r] & 0 \arrow[r] & 0,
\end{tikzcd}$$

whose long exact sequence in hypercohomology is

$$\begin{array}{l}0 \longrightarrow \mathbb{H}^0(D^\bullet) \longrightarrow \mathbb{H}^0(C^\bullet) \longrightarrow \mathbb{H}^0(E^\bullet) \\ \quad\hookrightarrow \mathbb{H}^1(D^\bullet) \longrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow \mathbb{H}^1(E^\bullet) \\ \quad\hookrightarrow \mathbb{H}^2(D^\bullet) \longrightarrow \mathbb{H}^2(C^\bullet) \longrightarrow \mathbb{H}^2(E^\bullet) \longrightarrow 0.\end{array}$$

Calculating the hypercohomology groups of $D^\bullet$ and $E^\bullet$ and the maps $\mathbb{H}^i(D^\bullet) \to \mathbb{H}^{i+1}(E^\bullet)$ (by unwinding the definitions) the above sequence becomes

$$\begin{array}{l}0 \longrightarrow \mathbb{H}^0(C^\bullet) \longrightarrow H^0(X, \operatorname{End} E) \xrightarrow{[\Phi,-]} H^0(X, \operatorname{End}(E) \otimes K) \\ \quad\hookrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow H^1(X, \operatorname{End} E) \xrightarrow{[\Phi,-]} H^1(X, \operatorname{End}(E) \otimes K) \\ \quad\hookrightarrow \mathbb{H}^2(C^\bullet) \longrightarrow 0\end{array} \tag{2.2}$$

so we have

$$\begin{aligned}&\dim(\mathbb{H}^0(C^\bullet)) - h^0(X, \operatorname{End} E) + h^0(X, \operatorname{End}(E) \otimes K) \\ &- \dim(\mathbb{H}^1(C^\bullet)) + h^1(X, \operatorname{End} E) - h^1(X, \operatorname{End}(E) \otimes K) \\ &+ \dim(\mathbb{H}^2(C^\bullet)) = 0.\end{aligned}$$

Now we can determine the quantity $\dim(\mathbb{H}^1(C^\bullet))$ by first noting that, via Riemann-Roch, we get

$$-h^0(X, \operatorname{End} E) + h^1(X, \operatorname{End} E) = n^2(g-1),$$

and

$$h^0(X, \operatorname{End} E \otimes K) - h^1(X, \operatorname{End} E \otimes K) = n^2(g-1).$$

We then dualize sequence (2.2) to get

$$\begin{array}{l}0 \longrightarrow \mathbb{H}^2(C^\bullet)^* \longrightarrow H^0(X, \operatorname{End} E) \xrightarrow{[-,\Phi]} H^0(X, \operatorname{End}(E) \otimes K) \\ \quad\hookrightarrow \mathbb{H}^1(C^\bullet)^* \longrightarrow H^1(X, \operatorname{End} E) \xrightarrow{[-,\Phi]} H^1(X, \operatorname{End}(E) \otimes K) \\ \quad\hookrightarrow \mathbb{H}^0(C^\bullet)^* \longrightarrow 0\end{array}$$

since, using Serre duality, we have, for $i \in \{0,1\}$, $H^i(X, \operatorname{End} E \otimes K)^* \cong H^{1-i}(X, \operatorname{End} E)$, and the map $[\Phi, -]$ is anti-self-adjoint. It then follows that $\dim \mathbb{H}^0(C^\bullet)^* = \dim \mathbb{H}^2(C^\bullet)$. But $\dim \mathbb{H}^0(C^\bullet) = 1$, since, by (1.3),

$$\mathbb{H}^0(C^\bullet) = \ker f = \{s \in \Omega^0(\operatorname{End} E) \mid \bar{\partial}_E s = 0 = [\Phi, s]\}$$

i.e., it is the group of holomorphic endomorphisms of the Higgs bundle (E, Φ) and, just like the similar result for stable vector bundles used in Corollary 2.8, these are precisely the homotheties (see [Wen16, Remark 4.2.8]). ∎

Note that $\dim \mathcal{M}(n,d) = 2 \dim \mathcal{N}(n,d)$, which is consistent with the fact that the cotangent bundle of the moduli space of vector bundles is an open dense subset of the moduli space of Higgs bundles (see [Hit87b]).

2.3 Symplectic geometry of the moduli space

Let (E,Φ) represent a point in $T^*\mathcal{N}(n,d)$, i.e., (E,Φ) is a stable Higgs bundle with underlying stable bundle E. There is a canonical symplectic form on

$$\begin{aligned} T_{[(E,\Phi)]}(T^*\mathcal{N}(n,d)) &\cong T_{[E]}\mathcal{N}(n,d) \oplus T_\Phi(T^*_{[E]}\mathcal{N}(n,d)) \\ &\cong H^1(X, \operatorname{End} E) \oplus H^0(X, \operatorname{End} E \otimes K) \end{aligned}$$

which Serre duality tells us is given by

$$((\dot{A},\dot{\Phi}),(\dot{B},\dot{\Psi})) \longmapsto \int_X \operatorname{tr}(\dot{A}\dot{\Psi} - \dot{B}\dot{\Phi}).$$

Thus we obtain a holomorphic symplectic form on the cotangent bundle $T^*\mathcal{N}(n,d)$. It is actually possible to extend this to the smooth locus of the whole moduli space of Higgs bundles $\mathcal{M}(n,d)$ using the same formula (see [Hit87b] and [Bis94]). So the moduli space $\mathcal{M}(n,d)$ becomes a symplectic manifold.

Proposition 2.16. *There is a holomorphic symplectic form ω on $\mathcal{M}(n,d)$, defined by*

$$\omega((\dot{A},\dot{\Phi}),(\dot{B},\dot{\Psi})) = \int_X \operatorname{tr}(\dot{A}\dot{\Psi} - \dot{B}\dot{\Phi})$$

where $(\dot{A},\dot{\Phi}),(\dot{B},\dot{\Psi}) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$ represent tangent vectors at the point of $\mathcal{M}(n,d)$ represented by the stable Higgs bundle (E,Φ).

In the previous proposition, as in the rest of the text, anytime we mention objects that require smoothness, like the symplectic form ω, it is to be understood that we are restricting to the smooth locus of the moduli spaces.

We now explain, in an informal way, how to arrive at the symplectic form ω by working in the configuration space $\mathcal{A}_{\bar{\partial}}$. Although the calculations are easier since it is an affine space, care must be taken as it is infinite dimensional, so there are questions of Analysis that arise but will not be treated here.

Since $\mathcal{A}^s_{\bar{\partial}}$ is an open set of an affine space over $\Omega^{0,1}(\operatorname{End} E)$, its tangent bundle is trivial, i.e.,

$$T\mathcal{A}^s_{\bar{\partial}} \cong \mathcal{A}^s_{\bar{\partial}} \times \Omega^{0,1}(\operatorname{End} E).$$

The same holds true for the cotangent bundle

$$T^*\mathcal{A}^s_{\bar{\partial}} \cong \mathcal{A}^s_{\bar{\partial}} \times \Omega^{1,0}(\operatorname{End} E).$$

The Liouville 1-form λ of the cotangent bundle is given by

$$\lambda_{(E,\Phi)}(\dot{A},\dot{\Phi}) = \int_X \operatorname{tr}(\Phi\dot{A}),$$

where $(E,\Phi) \in T^*\mathcal{A}^s_{\bar\partial}$, $(\dot A,\dot\Phi) \in T_{(E,\Phi)}(T^*\mathcal{A}^s_{\bar\partial}) \cong \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$. To get the symplectic form $\omega = -d\lambda$, we take two vectors $(\dot A,\dot\Phi),(\dot B,\dot\Psi) \in T_{(E,\Phi)}(T^*\mathcal{A}^s_{\bar\partial})$ and note that they can be extended to constant vector fields Y_1 and Y_2, respectively, on $T^*\mathcal{A}^s_{\bar\partial}$ (because of its triviality). We can then calculate $d\lambda$ as

$$d\lambda_{(E,\Phi)}((\dot A,\dot\Phi),(\dot B,\dot\Psi)) = d(\lambda(Y_2))_{(E,\Phi)}(\dot A,\dot\Phi) - d(\lambda(Y_1))_{(E,\Phi)}(\dot B,\dot\Psi) + \lambda([Y_1,Y_2])(E,\Phi).$$

But since the Lie bracket of two constant vector fields is zero, the last term disappears and the above becomes

$$\int_X \operatorname{tr}(\dot\Phi\dot B) - \int_X \operatorname{tr}(\dot\Psi\dot A) = \int_X \operatorname{tr}(\dot\Phi\dot B - \dot\Psi\dot A).$$

So we get a symplectic form ω on $T^*\mathcal{A}_{\bar\partial}$ given by

$$\omega_{(E,\Phi)}((\dot A,\dot\Phi),(\dot B,\dot\Psi)) = \int_X \operatorname{tr}(\dot\Psi\dot A - \dot\Phi\dot B).$$

Now taking $g \in \mathcal{G}^c$, we see that

$$\begin{aligned} dg_{(\bar\partial_E,\Phi)}(\dot A,\dot\Phi) &= \frac{d}{dt}(g^{-1}\bar\partial_E g + tg^{-1}\dot A g, g^{-1}\Phi g + tg^{-1}\dot\Phi g)\Big|_{t=0} \\ &= (g^{-1}\dot A g, g^{-1}\dot\Phi g). \end{aligned}$$

It is clear then that $g^*\omega = \omega$, meaning we get a well defined 2-form ω on the quotient $T^*\mathcal{N}(n,d)$, which coincides with the one obtained earlier, and so it is a symplectic form.

Chapter 3

The Hitchin map and its fibers

3.1 The Hitchin map

Our starting point is a basic idea in linear algebra, the study of an endomorphism through its eigenvalues and eigenspaces.

Let (E, Φ) be a Higgs bundle of rank n over X, and consider a point $x \in X$. The characteristic polynomial $P(\lambda)_x$ of Φ_x is $\det(\Phi_x - \lambda \operatorname{id}_{E_x})$, a map $\wedge^n E_x \to \wedge^n E_x \otimes K_x^n$. This can be written as

$$P(\lambda)_x = \sum_{i=0}^{n} s_i(x) \lambda^{n-i}$$

where $s_i(x) = (-1)^i \operatorname{tr}(\wedge^i \Phi_x) \in K_x^i$.

The key point here is that the coefficients s_i are canonically defined and so, letting x vary, these become well defined sections of the bundles K^i. It then makes sense to define the following map on the moduli space $\mathcal{M}(n,d)$, collecting the characteristic coefficients of the Higgs field Φ.

Definition 3.1. The *Hitchin map* h is defined as

$$\begin{aligned} h: \quad \mathcal{M}(n,d) &\longrightarrow \bigoplus_{i=1}^{n} H^0(X, K^i) \\ [(E, \Phi)] &\longmapsto (s_1, \ldots, s_n), \end{aligned}$$

with $s_i = (-1)^i \operatorname{tr}(\wedge^i \Phi)$.

Remark 3.2. Any basis of the algebra of invariant polynomials for the Lie algebra of $\mathrm{GL}(n, \mathbb{C})$ could be used to define a Hitchin map (see [Hit87a, §4]). For example we could take the basis $\operatorname{tr} \Phi^i$ which, by Newton's relations, are related to the s_i via

$$i \operatorname{tr}(\wedge^i \Phi) = \sum_{j=1}^{i} (-1)^{j+1} \operatorname{tr}(\Phi^j) \operatorname{tr}(\wedge^{i-j} \Phi).$$

Proposition 3.3. *The space*

$$\mathcal{B} \coloneqq \bigoplus_{i=1}^{n} H^0(X, K^i)$$

(called the Hitchin base*) is a complex vector space with dimension equal to $n^2(g-1)+1$.*

Proof. Using Riemann-Roch we get that

$$h^0(X, K^i) - h^0(X, K^{1-i}) = \deg(K^i) + 1 - g.$$

Now, if $i = 1$, we get $h^0(X, K) = g$, and if $i > 1$, $h^0(X, K^i) = (2i-1)(g-1)$. So

$$\dim \mathcal{B} = \sum_{i=1}^{n} h^0(X, K^i) = n^2(g-1) + 1. \qquad \blacksquare$$

Our goal in what follows will be to understand the fibers $h^{-1}(s)$ for generic $s \in \mathcal{B}$, as well as some of the non-generic s.

Remark 3.4. The Hitchin map h is proper (see [Hit87b, Theorem 8.1] for the case of rank 2 and [Sim94, Theorem 6.11] for the general case), so we can be sure that the fibers are compact.

3.2 Spectral curves

For any $s \in \mathcal{B}$, we can build a *spectral cover* of X, a curve X_s together with a finite morphism $\pi : X_s \to X$, which, roughly speaking, records the eigenvalues of Φ.

We denote by $|K|$ the total space of the canonical bundle $K \to X$, which is a non-compact complex surface. Let $p : |K| \to X$ be its canonical projection. We then consider the pullback of K onto its total space, together with its tautological section $\lambda \in H^0(|K|, p^*K)$, $\lambda(v) = v \in (p^*K)_x$, as the diagram shows

$$\begin{tikzcd}
p^*K \arrow[r] \arrow[d] & K \arrow[d, "p"] \\
{|K|} \arrow[r, "p"] \arrow[u, bend left, "\lambda"] & X.
\end{tikzcd}$$

Note that the pullback bundle p^*K is such that its fiber over $v \in |K|$ is the fiber of K over $p(v)$, i.e., $(p^*K)_v = K_{p(v)}$ (see Figure 3.1).

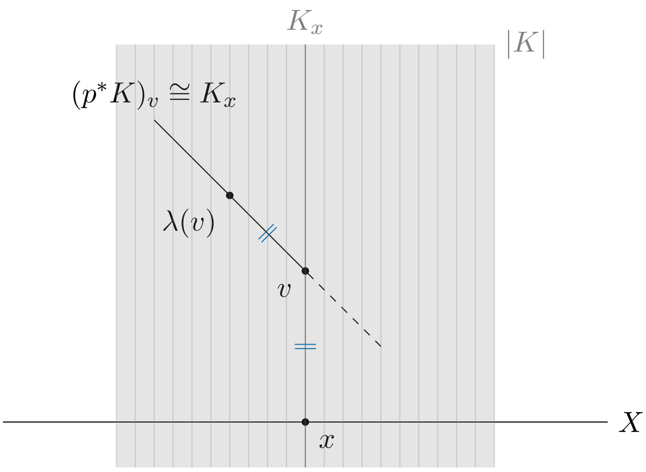

Figure 3.1: A fiber of the pullback bundle p^*K on the total space $|K|$.

Let $s = (s_1, \ldots, s_n) \in \mathcal{B}$. Then we define the following section of p^*K^n

$$P_s(\lambda) := \lambda^n + (p^*s_1)\lambda^{n-1} + (p^*s_2)\lambda^{n-2} + \cdots + p^*s_n \in H^0(|K|, p^*K^n).$$

Definition 3.5. The *spectral curve* X_s associated to $s \in \mathcal{B}$ is the zero locus in $|K|$ (or zero divisor) of the section $P_s(\lambda)$.

We now let $\pi : X_s \to X$ be the restriction of the projection p to the spectral curve X_s. It is an n-cover. In fact, consider a point $x \in X$ and trivialize K around it. Then $s_i(x) = a_i \in K_x^i \cong \mathbb{C}$, and $\pi^{-1}(x)$ is the zero locus of $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ in the fiber K_x or, equivalently, the zero locus of $z^n + a_1 z^{n-1} + \cdots + a_n$ in $\mathbb{C}$. Since a generic polynomial has n distinct complex roots, we have an n-cover, ramified whenever $P_s(\lambda)$ has multiple roots.

Remark 3.6. Since π is a n-cover, it is a proper map and so the spectral curves are all compact.

In general the curve X_s can be complicated, depending on the section P_s. It might be *non-reduced* (if $P_s = Q_s^m$, for some polynomial Q_s and $m > 1$, for example $P_0 = \lambda^n$) or *reducible* (if $P_s = Q_s R_s$, for some polynomials Q_s and R_s of degree ≥ 1, for example, if $n = 2$, $s = (0, \omega^2)$, for $\omega \in H^0(X, K)$). However, for generic $s \in \mathcal{B}$, we shall see that X_s is smooth, so generically we can avoid all of these problems. Moreover, the generic singular spectral curves are *integral*.

Remark 3.7. Note that each X_s is a divisor on the surface $|K|$. Thus they are all linearly equivalent, since they are all zeroes of sections of the same bundle, namely p^*K^n. In particular, they all belong to the linear system of $X_0 = nX$, where nX is the zero locus of $P_0(\lambda) = \lambda^n \in H^0(X, K^n)$ (i.e., nX is the non-reduced curve with multiplicity n, with X as the underlying reduced curve).

We will now see the total space $|K|$ as a *scheme* by making use of the relative spectrum construction (details in Appendix A.1). Indeed,

$$|K| = \underline{\mathrm{Spec}}(\mathrm{Sym}(K^{-1})),$$

where we are seeing K as an $\mathcal{O}_X$-module. $\mathrm{Sym}(K^{-1})$ is a sheaf of algebras on X defined by $\mathrm{Sym}(K^{-1})(U) = \mathrm{Sym}(K^{-1}(U))$, for any open set U of X. So if $p : |K| \to X$ is the canonical projection, then $p^{-1}(U) = \mathrm{Spec}(\mathrm{Sym}(K^{-1})(U))$, for every open set U of X.

Indeed, we can look at this construction as a relative (fiberwise) version of the fact that, for a vector space $E \cong \mathbb{C}^n$, we have

$$E = \mathrm{Spec}(\mathrm{Sym}(E^*))$$

and

$$\mathrm{Sym}(E^*) = \bigoplus_{i \geq 0} \mathrm{Sym}^i(E^*),$$

where

$$\begin{aligned} &\mathrm{Sym}^0(E^*) = \mathbb{C}, \\ &\mathrm{Sym}^1(E^*) = E^* \cong \mathrm{Span}_{\mathbb{C}}\{z_1, \ldots, z_n\} && \text{(homogeneous polynomials of degree 1)}, \\ &\mathrm{Sym}^2(E^*) \cong \mathrm{Span}_{\mathbb{C}}\{z_1^2, z_1 z_2 \ldots, z_n^2\} && \text{(homogeneous polynomials of degree 2)}, \end{aligned}$$

and so on.

It follows from this that we have the following direct sum decomposition

$$\mathrm{Sym}(K^{-1}) = \mathcal{O}_X \oplus K^{-1} \oplus K^{-2} + \cdots = \bigoplus_{i=0}^{\infty} K^{-i}. \tag{3.1}$$

Hence the structure sheaf of $|K|$ at an open set of the form $p^{-1}(U) \subset |K|$, consisting of the regular functions on $p^{-1}(U)$, is

$$\mathcal{O}_{|K|}(p^{-1}(U)) \cong \mathrm{Sym}(K^{-1})(U) = \bigoplus_{i=0}^{\infty} K(U)^{-i}.$$

This tells us that

$$p_* \left(\mathcal{O}_{|K|}\right) = \mathrm{Sym}(K^{-1}).$$

Now, given $s = (s_1, \dots, s_n) \in \mathcal{B}$, the spectral curve is defined as

$$X_s := \underline{\mathrm{Spec}}(\mathrm{Sym}(K^{-1})/\mathcal{J}_s), \tag{3.2}$$

where $\mathcal{J}_s$ is the ideal subsheaf of $\mathrm{Sym}(K^{-1})$ generated by the image of the morphism of sheaves

$$\begin{array}{rccl} u: & K^{-n} & \longrightarrow & \mathcal{O}_X \oplus K^{-1} \oplus \cdots \oplus K^{-n} \subset \mathrm{Sym}\, K^{-1} \\ & \alpha & \longmapsto & \alpha(s_n + s_{n-1} + \cdots + 1). \end{array}$$

Let us check that this is indeed the spectral curve X_s as we first defined it.

Proposition 3.8. *The spectral curve X_s as defined in* (3.2) *agrees with Definition 3.5.*

Proof. It is enough to show that both descriptions agree over an open cover of X. In a trivializing open set U of K^{-1}, take a non-vanishing section $\eta \in H^0(U, K^{-1})$. At every $x \in U$, $\eta(x) : K_x \to \mathbb{C}$ is a coordinate on the fiber K_x.

The map u above becomes

$$u(\eta^n) = \eta^n s_n + \eta^n s_{n-1} + \cdots + \eta^n.$$

Now, for every i, we have

$$\eta^n s_i = \eta^{n-i+i} s_i = \eta^{n-i} a_i,$$

where $a_i = \eta^i s_i \in K^{-i} \otimes K^i \cong \mathcal{O}_X(U)$ is the expression of the s_i in the induced local trivialization of K^i over U. So

$$u(\eta^n) = a_n + \eta a_{n-1} + \cdots + \eta^n.$$

Since $K|_U = \mathrm{Spec}(\mathcal{O}_X(U)[\eta])$, we have

$$X_s|_U = \mathrm{Spec}(\mathcal{O}_X(U)[\eta]/\langle a_n + a_{n-1}\eta + \cdots + \eta^n \rangle).$$

But $\mathcal{O}_X(U)[\eta]/\langle a_n + a_{n-1}\eta + \cdots + \eta^n \rangle$ is precisely the coordinate ring of the spectral curve over U, as we first defined it. Hence both definitions coincide. ■

The canonical projection $\pi : X_s \to X$ is an open map and we have, using the decomposition of K^{-1} in (3.1)

$$\begin{aligned} \mathcal{O}_{X_s}(\pi^{-1}(U)) &\cong (\mathrm{Sym}(K^{-1})/\mathcal{J}_s)(U) \\ &= \mathcal{O}_X(U) \oplus K^{-1}(U) \oplus K^{-2}(U) \oplus \cdots \oplus K^{-(n-1)}(U). \end{aligned}$$

As a consequence of this discussion we have the following.

Corollary 3.9. *The pushforward of the structure sheaf of the spectral curve X_s satisfies*

$$\pi_*\mathcal{O}_{X_s} \cong \mathcal{O}_X \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)} \cong \operatorname{Sym}(K^{-1})/\mathcal{J}_s.$$

Proof. Let $U \subseteq X$ be an open set. Then

$$\pi_*(\mathcal{O}_{X_s})(U) = \mathcal{O}_{X_s}(\pi^{-1}(U)) = \mathcal{O}_X(U) \oplus K^{-1}(U) \oplus \cdots \oplus K^{-(n-1)}(U). \qquad \blacksquare$$

The isomorphism

$$\operatorname{Sym}(K^{-1})/\mathcal{J}_s \longrightarrow \mathcal{O}_X \oplus K^{-1} \oplus \cdots \oplus K^{-(n-1)}$$

can be described as follows. Take a trivializing open set U. Then the map becomes

$$\mathcal{O}_X(U)[\lambda]/\langle a_n + a_{n-1}\lambda + \cdots + \lambda^n \rangle \longrightarrow \mathcal{O}_X(U) \oplus \mathcal{O}_X(U)\lambda \oplus \cdots \oplus \mathcal{O}_X(U)\lambda^{n-1}. \tag{3.3}$$

It sends a class $[P(x,\lambda)]$ to the remainder obtained when dividing by $a_n + a_{n-1}\lambda + \cdots + \lambda^n$ (considering both as polynomials in the variable λ).

The algebraic description we gave of the spectral curve allows us to prove the following simple but powerful result, which will be crucial for the calculations that follow (see definitions in Appendix A.2 and A.3).

Corollary 3.10. *The map $\pi : X_s \to X$ is finite, hence affine.*

Proof. Let U be an affine open set of X. Then $U = \operatorname{Spec}(B)$, where $B = \mathcal{O}_X(U)$. Choosing U to be a trivializing open set for K, we get that $\pi^{-1}(U) = \operatorname{Spec}(A)$, with

$$A \cong \mathcal{O}_X(U) \oplus \mathcal{O}_X(U)z \oplus \cdots \oplus \mathcal{O}_X(U)z^{n-1},$$

where z is the image of λ under the map (3.3). It is then clear that A is a rank n module over B, and so the map π is finite. $\blacksquare$

For some of the results that follow, we will need to work over a compact surface, so we will now describe how to projectivize the total space $|K|$. Intuitively, we add a point at infinity to each fiber of K, turning it into a copy of $\mathbb{P}^1$.

We then consider the space

$$\mathbb{K} := \mathbb{P}(K \oplus \mathcal{O}_X) \tag{3.4}$$

equipped with its canonical projection $p : \mathbb{K} \to X$ (for ease of notation, we call it the same as the projection $p : |K| \to X$).

The projective surface $\mathbb{K}$ comes equipped with the invertible sheaf $\mathcal{O}_{\mathbb{K}}(1)$, constructed by associating to each fiber $p^{-1}(x) \cong \mathbb{P}^1$ its hyperplane bundle $\mathcal{O}(1)$ (recall Example 1.8).

The usefulness of the above construction comes from the fact that it is generally preferable to have our objects sitting inside some projective variety. For example, knowing that the spectral curve X_s sits inside $\mathbb{K}$ (without intersecting the points at infinity), tells us again that it is compact (which we already checked in Remark 3.6).

3.2.1 Properties

In this section, we prove that, as we mentioned earlier, the spectral curve is generically "nice", i.e., reduced, irreducible and smooth. We also calculate, in more than one way, the genus of the curve and how it relates to both the moduli space of Higgs bundles $\mathcal{M}(n,d)$ and the genus of the base curve X. This will be a useful fact to know later. We finish by giving a description of the ramification divisor of the smooth spectral curves.

Proposition 3.11. *The set of sections $s \in \mathcal{B}$ such that X_s is reduced and irreducible is Zariski open and non-empty.*

Proof. Suppose that X_s is reducible. Then $X_s = Y \cup Z$, where Y and Z are two curves on $|K|$. Now by the arguments of Section 4 of [Fra22], any projective curve inside the total space $|K| \subset \mathbb{K}$ is a spectral curve. This allows us to write both Y and Z as zero loci of sections $P_{s'}$ and $P_{s''}$. And so we can write $P_s = P_{s'}P_{s''}$, with $\deg P_s = n$, $\deg P_{s'} = k$, $\deg P_{s''} = l$, with $k + l = n$ and $0 < k, l < n$. The dimension of the space of such s equals the sum of the dimensions of the Hitchin bases for ranks k and l, i.e.,

$$\Big(l^2(g-1)+1\Big) + \Big(k^2(g-1)+1\Big) = (k^2+l^2)(g-1)+2.$$

This is less than the dimension of the full Hitchin base for rank n, i.e., less than $n^2(g-1)+1$, if and only if $2kl(g-1) > 1$. Since $g \geq 2$, the inequality is true for any k, l in the conditions mentioned above, and so it is always possible to find an s such that P_s can not be factorized.

Openness of the set is clear since being reducible is a closed condition. ■

Proposition 3.12 ([Hit87a, Section 5.1])**.** *For generic elements $s \in \mathcal{B}$, the spectral curve X_s is smooth.*

Proof. Consider the linear system

$$\mathfrak{d} = \{X_s \mid s \in \mathcal{B}\}$$

of all spectral curves, in particular, of the curve $X_0 = nX$ (recall Remark 3.7), on $\mathbb{K}$. If we can then prove that $\mathfrak{d}$ has no *base-points* (points that belong to all elements of $\mathfrak{d}$), Bertini's theorem as presented in [Har77, III.10.9, page 274] tells us that its generic element is smooth.

Assume that $y \in \mathbb{K}$ is a base-point for $\mathfrak{d}$. Since every spectral curve X_s stays away from the points at infinity, we have in fact $y \in |K|$. Being a base-point means, in particular, that $y \in X_0$, i.e., $\lambda^n(y) = 0$, so $\lambda(y) = 0$ and $y \in X$. But then, for any $s \in \mathcal{B}$,

$$0 = P_s(\lambda)(y) = (p^* s_n)(y) = s_n(y).$$

We conclude that y is a base-point for the line bundle K^n on X. But K is base-point free ([Mir95, Lemma 1.14, page 200]), so K^n is base-point free as well, and such y cannot exist. ■

In the remainder of the text, all of our spectral curves will be integral, but we will allow them to be singular at times, in order to obtain some more general results.

Proposition 3.13. *Let $s \in \mathcal{B}$ be such that X_s is integral. The arithmetic genus of X_s is $g_s = n^2(g-1)+1$.*

Proof. By [Har77, Exercise III.5.3], if X_s is integral then $h^0(X_s, \mathcal{O}_{X_s}) = 1$ and so its arithmetic genus g_s is equal to $h^1(X_s, \mathcal{O}_{X_s})$ (recall (1.5)).

We use the fact that π_* preserves the cohomology dimensions (see Proposition A.5) to write

$$1 - g_s = h^0(X_s, \mathcal{O}_{X_s}) - h^1(X_s, \mathcal{O}_{X_s}) = h^0(X, \pi_*\mathcal{O}_{X_s}) - h^1(X, \pi_*\mathcal{O}_{X_s}).$$

By Corollary 3.9, this is equal to

$$\sum_{i=0}^{n-1}(h^0(X, K^{-i}) - h^1(X, K^{-i})),$$

which, using Riemann-Roch, becomes

$$\sum_{i=0}^{n-1}(\deg K^{-i} + 1 - g) = n^2(1-g)$$

and the result follows. ■

Suppose X_s is a *smooth* and irreducible curve on $|K|$, the *adjunction formula* ([GH94, page 471]) then tells us that

$$K_{X_s} = (K_{|K|} + X_s)|_{X_s},$$

where K_{X_s} and $K_{|K|}$ denote, respectively, the canonical bundles of the curve X_s and of the surface $|K|$.

Now $K_{|K|}$ is trivial (the canonical symplectic form on $T^*|K|$ is a nowhere vanishing section of $K_{|K|} := \wedge^2 T^*|K|$), and $X_s \sim nX$ (as divisors), so the formula becomes

$$K_{X_s} = (p^*K)^n|_{X_s} = \pi^* K^n. \tag{3.5}$$

Using that π is a degree n morphism and so $\deg(\pi^* L) = n\deg(L)$ (see [Har77, Propostion II.6.9, page 138]), we can take the degree of both sides of the equation to obtain

$$2g_s - 2 = n^2(2g-2),$$

and so we have another way to obtain the genus when the spectral curve is smooth.

Given that the spectral cover we have defined will be ramified, it will be useful to us to know something about its ramification divisor.

Definition 3.14. If X_s is smooth, the *ramification divisor* R of X_s is defined to be the zero locus of the derivative of the map π, $d\pi\colon TX_s \to \pi^*TX$.

Since $d\pi$ is a section of the line bundle $T^*X_s \otimes \pi^*TX$, we have, using (3.5),

$$\mathcal{O}(R) = K_{X_s} \otimes \pi^* K^{-1} = \pi^* K^{n-1}$$

and so the degree of the ramification divisor is $n(n-1)(2g-2)$.

If X_s is singular, we can use its normalization to bound the number of singularities. We shall consider here the simplest singular case, when X_s has only simple nodes as singularities. We then consider the normalization $\nu : \widetilde{X_s} \to X_s$, and the map $\widetilde{\pi}$ making the following diagram commute

$$\begin{tikzcd} \widetilde{X_s} \arrow[dd, "\widetilde{\pi}"'] \arrow[dr, "\nu"] & \\ & X_s. \arrow[dl, "\pi"] \\ X & \end{tikzcd}$$

Proposition 3.15. *If X_s is nodal with r simple nodes, then $r \leq n(n-1)(g-1)$.*

Proof. We can consider the ramification divisor $\widetilde{R}$ of the map $\widetilde{\pi}$ (even though it is *not* a spectral cover) using the same definition, i.e., as the zeroes of the derivative $d\widetilde{\pi}$, which is a section of the bundle $K_{\widetilde{X_s}} \otimes \widetilde{\pi}^* K^{-1}$. Looking at sequence (1.4), we see that the genus $\widetilde{g_s}$ of the normalization is equal to $g_s - r$. Using Proposition 3.13, we then have

$$\deg\left(K_{\widetilde{X_s}}\right) = 2\widetilde{g_s} - 2 = 2n^2(g-1) - 2r.$$

Since $\widetilde{R}$ is an effective divisor, we have

$$0 \leq \deg(\widetilde{R}) = 2n(n-1)(g-1) - 2r,$$

from where the result follows. ■

3.2.2 The BNR-correspondence

In this section we will prove the correspondence theorem of BNR, named after the mathematicians Beauville, Narasimhan and Ramanan, who gave a proof of it in [BNR89], following earlier work by Hitchin in [Hit87a] and [Hit87b]. The result gives us a close link between Higgs bundles on the base curve X and line bundles on spectral curves X_s. It will allow us to see some fibers of the Hitchin map (most of them in fact) as isomorphic to Jacobians of the spectral curve.

We first start with a small result which will be useful later.

Lemma 3.16. *Let (E, Φ) be a Higgs bundle and F a Φ-invariant subbundle of E, i.e., $\Phi(F) \subseteq F \otimes K$. Then, the characteristic polynomial of $\Phi|_F$ divides the characteristic polynomial of Φ.*

Proof. Let $n = \mathrm{rk}(E)$ and $m = \mathrm{rk}(F)$. Since F is Φ-invariant, we can write

$$P_s = P_{s'} \cdot P_{s''},$$

where $s = h(E, \Phi)$ (so P_s is the characteristic polynomial of Φ), $s' = h(F, \Phi|_F)$ (so $P_{s'}$ is the characteristic polynomial of $\Phi|_F$) and $s'' = h(E/F, \overline{\Phi})$, with $\overline{\Phi} : E/F \to E/F \otimes K$ being the map induced by Φ on the quotient E/F. ■

Theorem 3.17 ([BNR89, Proposition 3.6], [Hit87a, Section 5.1])**.** *Let $s \in \mathcal{B}$ be such that X_s is integral. Then there is a 1-1 correspondence, up to their respective notions of equivalence, between (see Figure 3.2)*

- *Higgs bundles (E, Φ) on X of rank n and degree d such that the characteristic polynomial of Φ equals P_s,* *and*

- *rank 1 torsion-free sheaves L on X_s of degree $d + n(n-1)(g-1)$.*

Proof. Let L be a rank 1 torsion free sheaf on X_s (as an $\mathcal{O}_{X_s}$-module). Then $\pi_* L$ is a torsion-free sheaf on X as a $\pi_*\mathcal{O}_{X_s}$-module (hence torsion free as a $\mathcal{O}_X$-module because $\mathcal{O}_X \hookrightarrow \pi_*\mathcal{O}_{X_s}$, by Corollary 3.9) and since X is smooth, it is a locally free sheaf (a vector bundle) of rank n (recall Proposition 1.22).

As a sheaf, $\pi_* L$ is a $\pi_*\mathcal{O}_{X_s}$-module, which means we have a map

$$\pi_*\mathcal{O}_{X_s} \longrightarrow \operatorname{End}(\pi_* L),$$

and since, again by Corollary 3.9, $K^{-1} \subset \pi_*\mathcal{O}_{X_s}$, we get a map $\Phi : K^{-1} \longrightarrow \operatorname{End}(\pi_* L)$ defined by restriction.

Since $\pi_*\mathcal{O}_{X_s} \cong \operatorname{Sym}(K^{-1})/\mathcal{J}_s$, the map Φ vanishes at $\mathcal{J}_s$. Putting $E := \pi_* L$, we get a map $\Phi : E \to E \otimes K$ such that $P_s(\Phi) = 0$. Since X_s is irreducible, P_s is irreducible and so, by Cayley-Hamilton, it must be the characteristic polynomial of Φ, since $P_s(\Phi) = 0$.

Conversely, let (E, Φ) be a Higgs bundle such that the characteristic polynomial of Φ is P_s. Seeing Φ as a map $K^{-1} \to \operatorname{End}(E)$, it determines an algebra morphism

$$\operatorname{Sym}(K^{-1}) \longrightarrow \operatorname{End}(E)$$

that factors through $\mathcal{J}_s$ since the Cayley-Hamilton theorem tells us that $P_s(\Phi) = 0$. So E is a $\pi_*\mathcal{O}_{X_s}$-module of rank 1 since both E and $\pi_*\mathcal{O}_{X_s}$ have rank n as $\mathcal{O}_X$-modules.

This means that $E \cong \pi_* L$ for some rank 1 $\mathcal{O}_{X_s}$-module L on X_s since π_* is an equivalence of categories between $\mathcal{O}_{X_s}$-modules and $\pi_*\mathcal{O}_{X_s}$-modules, by Corollary 3.10 and Proposition A.4. Suppose that $L_{\text{tors}} \neq 0$, then, by Remark 1.23, it is a sheaf whose support is a finite number of points of X_s. But $\pi_*(L_{\text{tors}}) \neq 0$, and so we have

$$\varnothing \neq \operatorname{supp}(\pi_*(L_{\text{tors}})) \subseteq \pi(\operatorname{supp} L_{\text{tors}}).$$

This means that $\pi_*(L_{\text{tors}})$ is a $\pi_*\mathcal{O}_{X_s}$-module with finite support, hence a $\mathcal{O}_X$-module of finite support, and so $\pi_*(L_{\text{tors}}) \subset E_{\text{tors}}$. But $E_{\text{tors}} = 0$, since E is locally free (in particular it is torsion free). We then conclude that $L_{\text{tors}} = 0$ and so L is torsion-free.

The degree of L (as a function of the degree d and rank n of $E = \pi_* L$) comes from the equality of the cohomology dimensions

$$h^0(X_s, L) - h^1(X_s, L) = h^0(X, \pi_* L) - h^1(X, \pi_* L),$$

that is,

$$\deg(L) + 1 - g_s = d + n(1-g).$$

From Proposition 3.13, we conclude that

$$\deg(L) = d + n(n-1)(g-1).$$

The uniqueness (modulo equivalence) is also a consequence of the equivalence of categories induced by π_*. ∎

Remark 3.18. We can also understand how the map Φ arises in the proof of Theorem 3.17 in the following manner. Take $p \in X$ such that X_s is unramified over it. Thus $\pi^{-1}(p) = \{\lambda_1, \dots, \lambda_n\}$ and $L_i := L_{\lambda_i} = \mathbb{C}e_i$. Let $\lambda : K_p \to \mathbb{C}$ be the tautological section at p. The module structure of $(\pi_* \mathcal{O}_{X_s})_p$ on $(\pi_* L)_p$ can then be written as

$$\left(\sum_{j=0}^{n} a_j \lambda^j\right) \cdot \left(\sum_{i=0}^{n} v_i e_i\right) = \sum_{j=0}^{n} \left(\sum_{i=0}^{n} a_j \lambda_i^j v_i e_i\right).$$

Restricting it to K_p^{-1} we get

$$(a_1 \lambda) \cdot \left(\sum_{i=0}^{n} v_i e_i\right) = \sum_{i=0}^{n} a_1 \lambda_i v_i e_i,$$

so the map Φ_p becomes

$$\begin{aligned} \Phi_p : \quad K_p^{-1} &\longrightarrow \operatorname{End}(\pi_* L)_p \\ \lambda &\longmapsto (e_i \mapsto \lambda_i e_i), \end{aligned}$$

or, equivalently

$$\begin{aligned} \Phi_p : \quad (\pi_* L)_p &\longrightarrow (\pi_* L)_p \otimes K_p \\ e_i &\longmapsto \lambda_i e_i. \end{aligned}$$

In fact, we are building Φ_p by imposing its action on the L_i to be simply scaling by λ_i.

Since the condition we imposed on p is generic, the map Φ is then the pushforward of multiplication by the tautological section λ, as in the following diagram

$$\begin{tikzcd} L \arrow[r, "\otimes\lambda"] \arrow[d, "\pi_*"'] & L \otimes \pi^* K \arrow[d, "\pi_*"] \\ E \arrow[r, "\Phi"'] & E \otimes K. \end{tikzcd}$$

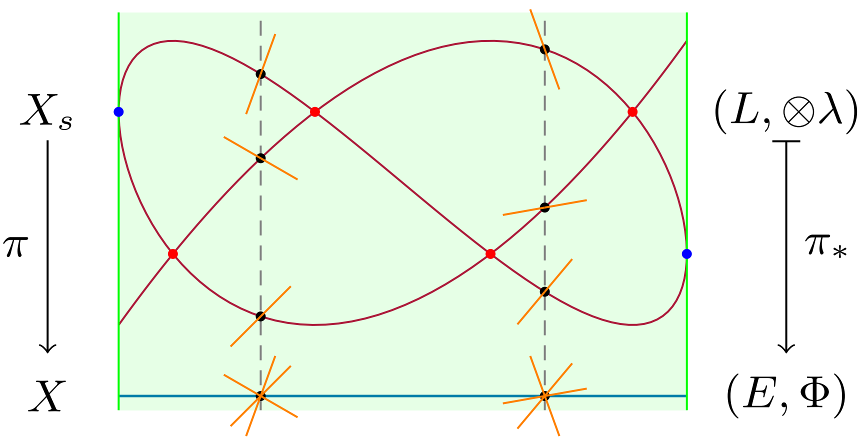

Figure 3.2: A spectral curve in rank 3. We have highlighted its singular (red) and smooth (blue) ramification points. Away from the ramification locus we have the eigenspaces forming the rank 1 torsion-free sheaf L on X_s that gets pushed forward to give the vector bundle E on X. The Higgs field comes from the multiplication by λ.

When X_s is smooth, then, by Proposition 1.22, "rank 1 torsion-free sheaves" may be replaced by "line bundles", so we have the following description of the fibers of the Hitchin map.

Corollary 3.19. *For s as above such that X_s is integral, we have (Figure 3.3)*

$$\begin{aligned} &h^{-1}(s) \cong \mathrm{Jac}(X_s) && \textit{if } X_s \textit{ is smooth,} \\ &h^{-1}(s) \cong \overline{\mathrm{Jac}}(X_s) \ \textit{(the compactified Jacobian)} && \textit{if } X_s \textit{ is singular.} \end{aligned}$$

Proof. We first check that (E, Φ) as constructed in the proof of the theorem is semistable. This follows from the fact that E does not have any Φ-invariant subbundles (and so it is, in fact, stable), since the existence of such a subbundle would tell us, by Lemma 3.16, that P_s is reducible, contradicting the irreducibility of X_s.

Note also that from the theorem we get $h^{-1}(s) \cong \mathrm{Jac}^{d+n(n-1)(g-1)}(X_s)$, if X_s is smooth. We can, however, establish a (non-canonical) isomorphism with $\mathrm{Jac}(X_s)$ by tensoring with a fixed line bundle F of degree $-d - n(n-1)(g-1)$. For X_s singular, the same argument works to establish that $\overline{\mathrm{Jac}}^{d+n(n-1)(g-1)}(X_s) \cong \overline{\mathrm{Jac}}(X_s)$. ∎

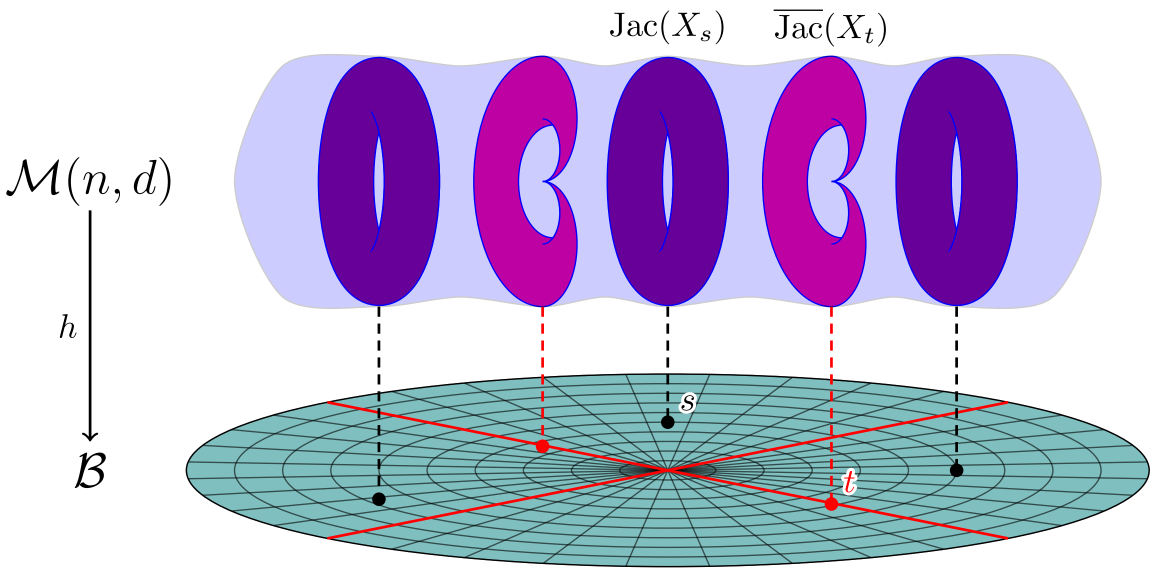

Figure 3.3: Fibers of the Hitchin map when X_s is integral. (Picture inspired by [Sch20, Figure 1] and [KR22, Figure 8])

Our goal now will be to prove the existence of an exact sequence which will also enable us to see, in case X_s is smooth, the line bundle L in Theorem 3.17 as giving us the eigenspaces of the map Φ (as in Remark 3.18). The result we shall prove requires that we once again consider the projectified total space $\mathbb{K} = \mathbb{P}(K \oplus \mathcal{O}_X)$, introduced in (3.4), along with two special sections on it.

The first is a section μ of $\mathcal{O}_{\mathbb{K}}(1)$. At a point $(x, [v]) \in \mathbb{K}$, $\mu(x, [v])$ will be the map

$$L_{x,[v]} \to \mathbb{C}, \quad (w, z) \mapsto z,$$

where $L_{x,[v]} \subset K_x \oplus \mathbb{C}$ is the line spanned by v. Its zero locus, at each fiber, will be the point $[1:0]$, i.e., the point at infinity of the fiber $\mathbb{K}_x$. In particular, μ is a non vanishing section on $|K|$, which means the bundle $\mathcal{O}(1)$ is trivial when restricted to $|K|$ or any of its subsets (thus also over the spectral curve X_s).

The other is a section λ of $p^*K \otimes \mathcal{O}(1)$ which, as the name implies, will be an extension of the tautological section λ on $|K|$. So $\lambda_{x,[v]}$ is the map

$$L_{x,[v]} \to \mathbb{C}, \quad (w,z) \mapsto w.$$

It is then clear that, when restricting λ to lines through $(w,1)$, and identifying them with the point $w \in K_x$, we get the tautological section λ of p^*K on $|K|$, as expected. Notice that the zero locus at each fiber will now be the point $[0:1]$, the point on the base curve X. So while μ vanishes on the section at infinity, λ vanishes on the zero section of the bundle $\mathbb{K}$.

Another intuitive way to understand the sections μ and λ is by noticing that they induce, fibrewise, rational maps $\widetilde{\mu}, \widetilde{\lambda} : \mathbb{K}_x \cong \mathbb{P}^1 - - \to \mathbb{C}$, defined by (see Figure 3.4)

$$\widetilde{\mu}([w:z]) = z/w; \quad \widetilde{\lambda}([w:z]) = w/z.$$

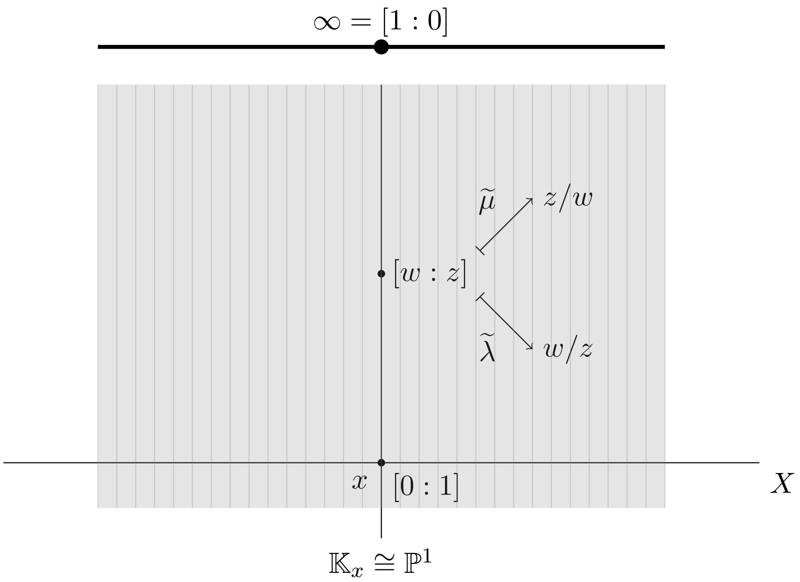

Figure 3.4: A fiber of the projectified total space.

Proposition 3.20 ([Hit87a][BNR89, Remark 3.7])**.** *Let $s \in \mathcal{B}$ be such that $\pi : X_s \to X$, its associated spectral cover, is smooth. Let $(E,\Phi) \in h^{-1}(s)$ be a Higgs pair on X such that $E = \pi_* L$ (under Theorem 3.17). Then the following sequence*

$$0 \longrightarrow L(-R) \longrightarrow \pi^*E \xrightarrow{\pi^*\Phi - \lambda} \pi^*E \otimes \pi^*K \longrightarrow L \otimes \pi^*K \longrightarrow 0,$$

is exact, where R is the ramification divisor of X_s, i.e., $\mathcal{O}(R) = \pi^*(K^{1-n})$ (recall Definition 3.14).

Proof. We follow the proof given in [Dal17, Proposition 4.1]. There is a short exact sequence on $|K|$,

$$0 \longrightarrow p^*(E \otimes K^{-1}) \xrightarrow{p^*\Phi - \lambda \otimes \mathrm{id}} p^*E \longrightarrow \mathcal{Q} \longrightarrow 0, \tag{3.6}$$

where $\mathcal{Q}$ is a sheaf supported on $X_s \subset |K|$. We then consider the compactification $p : \mathbb{K} \to X$ and the sections λ and μ described previously. Then $\mu \otimes p^*\Phi - \lambda \otimes \mathrm{id}$ is a global section of $p^*(\mathrm{End}\, E \otimes K) \otimes \mathcal{O}_{\mathbb{K}}(1)$. Since $\mathcal{O}_{\mathbb{K}}(1)$ is trivial on $|K| \subset \mathbb{K}$, the sequence above is the restriction to $|K|$ of the exact sequence

$$0 \longrightarrow p^*(E \otimes K^{-1}) \otimes \mathcal{O}_{\mathbb{K}}(-1) \xrightarrow{\mu \otimes p^*\Phi - \lambda \otimes \mathrm{id}} p^*E \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{3.7}$$

on $\mathbb{K}$.

We now determine $p_*\mathcal{Q}$ by pushing this last sequence down to X. Using the long exact sequence of the higher direct images $R^\bullet \pi_*$ (see Definition A.8) and the projection formula (in Proposition A.10), we get the long exact sequence (see Theorem A.6)

$$\begin{tikzcd} 0 \arrow[r] & (E \otimes K^{-1}) \otimes p_*\mathcal{O}_{\mathbb{K}}(-1) \arrow[r] & E \otimes p_*\mathcal{O}_{\mathbb{K}} \arrow[r] & p_*\mathcal{Q} \arrow[dll, rounded corners] \\ & (E \otimes K^{-1}) \otimes R^1 p_*\mathcal{O}_{\mathbb{K}}(-1) \arrow[r] & \cdots . & \end{tikzcd} \tag{3.8}$$

We know, from [Har77, Propostion II.7.11, page 162], that $p_*\mathcal{O}_{\mathbb{K}}(-1) = 0$ and $p_*\mathcal{O}_{\mathbb{K}} \cong \mathcal{O}_X$. Moreover, Proposition A.9 tells us that $R^1 p_*\mathcal{O}_{\mathbb{K}}(-1) = 0$, since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. So we get the exact sequence

$$0 \longrightarrow E \longrightarrow p_*\mathcal{Q} \longrightarrow 0,$$

which means that $E \cong p_*\mathcal{Q}$.

Now, restricting sequence (3.7) to $X_s \subset |K| \subset \mathbb{K}$ yields

$$0 \longrightarrow \mathfrak{K} \longrightarrow \pi^*(E \otimes K^{-1}) \longrightarrow \pi^*E \longrightarrow L \longrightarrow 0, \tag{3.9}$$

with $L \coloneqq \mathcal{Q}|_{X_s}$ satisfying $\pi_*L = p_*\mathcal{Q}$ and $\mathfrak{K}$ being the kernel gained by restricting the map.

It is easy to see that, given an exact sequence of bundles

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

after splitting and taking determinants, we get

$$\det A \otimes \det C = \det B \otimes \det D.$$

So, knowing that $\mathfrak{K}$ and L are line bundles ($\mathrm{rk}\, L = 1$ since it gives us the generic 1 dimensional eigenspaces of Φ and $\mathrm{rk}\, \mathfrak{K} = \mathrm{rk}\, L$ since $\mathrm{rk}\, \pi^*(E \otimes K^{-1}) = \mathrm{rk}\, \pi^*E$), we get

$$\mathfrak{K} = \det(\pi^*E \otimes \pi^*K^{-1}) \otimes L \otimes (\det \pi^*E)^{-1} = L \otimes \pi^*K^{-n}.$$

Tensoring sequence (3.9) by π^*K (tensoring by a vector bundle is an exact operation), we obtain the desired sequence. ∎

To end this subsection, we record a result on the relation between the pushforward of L and of its dual L^{-1}, which will be useful later. It is nothing more than an instance of relative duality (see [Kle80]) but it can also be derived using more elementary means, as in [Hit16, Section 4], whose proof we follow here.

Lemma 3.21. *Let $s \in \mathcal{B}$ be such that its associated spectral cover X_s is smooth, and let L be a line bundle on X_s. Then we have the following isomorphism of bundles*

$$(\pi_* L)^* \cong \pi_*(L^{-1}(R)),$$

where R is the ramification divisor on X_s.

Proof. We will show that there is a non-degenerate pairing between $\pi_* L$ and $\pi_*(L^{-1}(R))$. We start by defining the pairing away from the ramified points so let $p \in X$ be a regular value of π. Since X_s is smooth, $R = \operatorname{div}(d\pi)$, and $\mathcal{O}_{X_s}(R) \cong \pi^* K^{n-1}$. So the following is a well defined pairing

$$\begin{aligned} (\pi_* L)_p \times (\pi_*(L^{-1}(R)))_p &\longrightarrow \mathbb{C} \\ (v, \xi) &\longmapsto \langle v, \xi \rangle := \sum_{\pi(u)=p} \frac{\xi(v)_u}{d\pi_u}. \end{aligned}$$

More explicitly, let $\pi^{-1}(p) = \{u_1, \ldots, u_n\}$ and choose a basis $(e_i)_i$ of $(\pi_* L)_p$ such that $L_{u_i} = \mathbb{C} e_i$, with dual basis $(e_i^*)_i$, then $v = \sum_i v_i e_i$, $\xi = \sum_i \xi_i e_i^* \otimes \eta_i \zeta$ and $d\pi_{u_i} = \pi_i \zeta$, where $\mathcal{O}(R) = \mathbb{C}\zeta$. The pairing then becomes

$$\langle v, \xi \rangle = \sum_i \frac{v_i \xi_i \eta_i}{\pi_i}$$

and its non-degeneracy becomes clear.

Now for the ramification points, we will take the limit of the pairing above and check that it remains non-degenerate. Locally we can write $\pi(z) = z^k$, with zero being our ramification point where k branches of the spectral curve meet (see Figure 3.5). On a small enough open set U around 0 such that both L and $L^{-1}(R)$ become trivial, i.e., isomorphic to $\mathcal{O}_{X_s}$, we have

$$\pi_* \mathcal{O}_{X_s}(U) = \mathcal{O}_{X_s}(\pi^{-1}(U)) \cong \mathbb{C}[z, w]/(w - z^k).$$

We can then write local sections of both bundles as

$$F(w) = \sum_{i=0}^{k-1} \underbrace{\left(\sum_{j=0}^{k-1} (\omega^i z)^j b_j(w) \right)}_{f(\omega^i z)} \in H^0(U, \pi_* L),$$

and

$$G(w) = \sum_{i=0}^{k-1} \underbrace{\left(\sum_{l=0}^{k-1} (\omega^i z)^l c_l(w) \right)}_{g(\omega^i z)} \in H^0(U, \pi_*(L^{-1}(R))),$$

where z is a branch of the k-th root of w and ω is a primitive k-th root of unity.

For $w \neq 0$, we are at a regular value, and so the pairing we defined becomes

$$\begin{aligned} \langle F(w), G(w) \rangle &= \sum_i \frac{f(\omega^i z) g(\omega^i z)}{k \omega^i z^{k-i}} \\ &= \sum_{i,j,l=0}^{k-1} \frac{1}{k} \omega^{i(j+l+1)} z^{j+l-(k-1)} b_j c_l \\ &= \sum_{j,l} \frac{1}{k} z^{j+l-(k-1)} \left(\sum_{i=0}^{k-1} (\omega^{j+l+1})^i \right) b_j c_l. \end{aligned}$$

Now note that, if $j + l < k - 1$, ω^{j+l+1} is a k-th root of unity different than one, which means that the sum $\sum_{i=0}^{k-1}(\omega^{j+l+1})^i$ is zero. Moreover, if $j + l = k - 1$, $\omega^{j+l+1} = \omega^k = 1$, which means the same sum is equal to k. Finally, if $j + l > k - 1$, $z^{j+l-(k-1)}$ goes to zero as w approaches zero. In conclusion, taking the limit we get

$$\lim_{w \to 0} \langle F(w), G(w) \rangle = \sum_{j+l=k-1} b_j c_l,$$

and so the pairing remains non-degenerate. ■

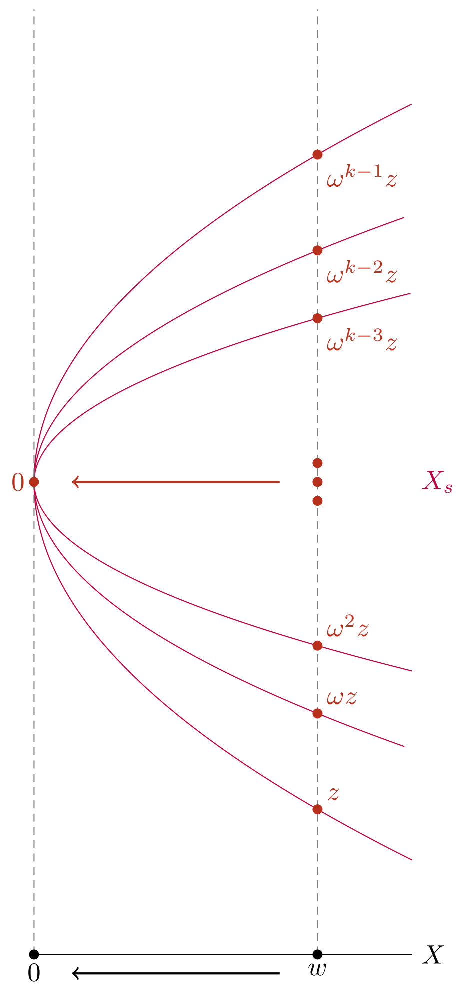

Figure 3.5: Taking the limit to the branch point via regular values.

3.3 Complete integrability of the moduli space

Using the Hitchin map, we shall see how to define $\frac{\dim \mathcal{M}(n,d)}{2}$ Poisson-commuting functions on $\mathcal{M}(n, d)$, giving the moduli space of Higgs bundles a structure of a completely integrable system.

In [Hit87b], the case of rank 2 and traceless Higgs bundles is treated, where the Hitchin map is simply the determinant $\det : \mathcal{M}(2, d) \to H^0(X, K^2)$.

Attempting a direct generalization of the methods used there to general Higgs bundles will not work so easily. What we shall do instead is use the fact that any basis for the invariant polynomials can be used to define a Hitchin map (recall Remark 3.2). As such, the one most suitable for generalization will be the basis composed of $\operatorname{tr}\Phi^i$. So we consider the following version of the Hitchin map

$$\begin{aligned} \tilde{h}:\ \mathcal{M}(n,d) &\longrightarrow \bigoplus_{i=1}^{n} H^0(X,K^i) \\ (E,\Phi) &\longmapsto (\operatorname{tr}\Phi, \operatorname{tr}\Phi^2,\dots,\operatorname{tr}\Phi^n). \end{aligned}$$

Proposition 3.22. *There are $n^2(g-1)+1$ Poisson-commuting functions $f_i^l:\mathcal{M}(n,d)\to\mathbb{C}$.*

Proof. Let l be any of the numbers $1,\dots,n$. Consider a basis $(\alpha_i^l)_i$ of $H^1(X,K^{1-l})$. Since $H^1(X,K^{1-l})\cong H^{0,1}(X,K^{1-l})$, each α_i^l is represented by an element $\beta_i^l\in\Omega^{0,1}(X,K^{1-l})$. Via Serre duality, since $H^0(X,K^l)^*\cong H^1(X,K^{1-l})$, we consider the functions

$$f_i^l(E,\Phi)=\int_X \beta_i^l\operatorname{tr}(\Phi^l).$$

Now to calculate the corresponding Hamiltonian vector fields X_i^l, we first calculate the derivative of f_i^l at a direction represented by $(\dot{B},\dot{\Psi})\in\Omega^{0,1}(\operatorname{End}E)\oplus\Omega^{1,0}(\operatorname{End}E)$. This is given by

$$df_i^l(\dot{B},\dot{\Psi})=\frac{d}{dt}\int_X\beta_i^l\operatorname{tr}(\Phi+t\dot{\Psi})^l\bigg|_{t=0}=\int_X\beta_i^l\operatorname{tr}\left(\sum_{m=0}^{l-1}\Phi^m\dot{\Psi}\Phi^{l-1-m}\right)=\int_X\beta_i^l l\operatorname{tr}(\Phi^{l-1}\dot{\Psi}).$$

By comparing with the symplectic form in Proposition 2.16, we find that a possible choice for the Hamiltonian vector field is $X_i^l=[(l\beta_i^l\Phi^{l-1},0)]$. So we can now check that

$$df_i^l(X_j^k)=0,\quad \forall i,j,k,l.$$

There is, however, one more thing to check before we can safely say the functions Poisson-commute. We got our Hamiltonian vector fields by getting representatives in $\Omega^{0,1}(\operatorname{End}E)\oplus\Omega^{1,0}(\operatorname{End}E)$ (note that, since $\beta_i^l\in\Omega^{0,1}(X,K^{1-l})$ and $\Phi^{l-1}\in\Omega^0(X,\operatorname{End}E\otimes K^{l-1})$, $l\beta_i^l\Phi^{l-1}\in\Omega^{0,1}(\operatorname{End}E)$). However, in order for a pair $(\dot{A},\dot{\Phi})$ to represent an element in the tangent space of $\mathcal{M}(n,d)$, from Proposition 2.14 it must satisfy $[\Phi,\dot{A}]=\bar{\partial}_E\dot{\Phi}$. But since

$$[\Phi,l\beta_i^l\Phi^{l-1}]=l\Phi\beta_i^l\Phi^{l-1}-l\beta_i^l\Phi^l=0=\bar{\partial}_E 0,$$

our pair satisfies the equation, as needed.

The number of functions we obtained is equal to the dimension of the Hitchin base, which, by Proposition 3.3, is $n^2(g-1)+1$. ■

Note that the number of functions is precisely half of the dimension of $\mathcal{M}(n,d)$, by Corollary 2.15.

Remark 3.23. Although we have used the Hitchin map defined by the coefficients of the characteristic polynomial to define the spectral curve, it is easy to see that there is no issue in choosing any other basis of the invariant polynomials in order to study the fibers of the map.

In fact, let T be a change of coordinates of $\mathcal{B}$ (such as the one in Remark 3.2) such that, if $b = (b_i)_i \in \mathcal{B}$ is expressed using any basis for the invariant polynomials (with Hitchin map $\tilde{h}$), then $T(b)$ gives us its coordinates in the basis of characteristic coefficients (with Hitchin map h), according to the diagram

$$\begin{tikzcd} & \mathcal{M}(n,d) \arrow[dl, "h"'] \arrow[dr, "\tilde{h}"] & \\ \mathcal{B} \arrow[rr, "T^{-1}"] & & \mathcal{B}. \arrow[ll, bend left, "T"] \end{tikzcd}$$

Then

$$h^{-1}(T(b)) = \{(E,\Phi) \mid \mathrm{char}(\Phi) = T(b)\} = \tilde{h}^{-1}(b).$$

Essentially, switching the basis of the invariant polynomials just swaps the fibers of the corresponding Hitchin maps, and so we can choose the one that works best for our purposes.

One should note, however, that we needed to use the Hitchin map h when building the spectral curve and proving the BNR correspondence, since the Cayley-Hamilton theorem played a crucial role there.

In the remainder of this text, we will be focusing on the map $\tilde{h}$ since this is one we used to find the Poisson-commuting functions on the moduli space. To ease notation, we will refer to this map as simply h.

3.3.1 The tangent space to a smooth fiber

Our goal will now be to prove that the Hitchin map h is a submersion at the points of its smooth fibers, in order to see that the functions f_i^l are independent. Along the way we will get a description of the tangent space at a point of a generic fiber as a subspace of $\mathbb{H}^1(C^\bullet)$, for the corresponding complex $C^\bullet$.

Let $s \in \mathcal{B}$ be such that the spectral curve X_s is smooth, i.e., the fiber $h^{-1}(s) \cong \mathrm{Jac}(X_s)$ is smooth. If $p = (E,\Phi) \in h^{-1}(s)$ we want to see that dh_p is a surjective linear map, or, equivalently, that $\ker dh_p = T_p h^{-1}(s)$. Recall that, by Corollary 3.19, (E,Φ) is stable since X_s is smooth.

Given that the definition of the Hitchin map does not explicitly depend on the vector bundle E, it would seem natural to assume that the tangent vectors at a point of the fiber would be the ones that can be represented by elements $(\dot{A}, 0) \in \Omega^{0,1}(\mathrm{End}\, E) \oplus \Omega^{1,0}(\mathrm{End}\, E)$, since they represent a zero infinitesimal deformation on the side of the Higgs field. In order to work further on this idea, we define the following subspace of $T_p\mathcal{M}(n,d)$,

$$\mathcal{T}_p := \{[(\dot{A}, 0)] \mid \dot{A} \in \Omega^{0,1}(\mathrm{End}\, E), [\Phi, \dot{A}] = 0\}.$$

We will then establish the equality $\ker dh_p = T_p h^{-1}(s)$ by proving that each of the spaces is isomorphic to $\mathcal{T}_p$, as in the following diagram

$$\begin{tikzcd} \ker dh_p \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow, "(1)"'] & & T_p h^{-1}(s). \arrow[dl, leftrightarrow, "(2)"] \\ & \mathcal{T}_p & \end{tikzcd} \tag{3.10}$$

To establish (1), we first record the derivative of h at p, calculated along a tangent vector represented by $(\dot{A}, \dot{\Phi})$

$$\begin{array}{rrcl} dh_p: & T_p\mathcal{M}(n,d) & \longrightarrow & \mathcal{B} \\ & [(\dot{A}, \dot{\Phi})] & \longmapsto & (\operatorname{tr}\dot{\Phi}, 2\operatorname{tr}(\Phi\dot{\Phi}), \ldots, n\operatorname{tr}(\Phi^{n-1}\dot{\Phi})). \end{array} \tag{3.11}$$

Now it is clear that $\mathcal{T}_p \subseteq \ker dh_p$, since, if we take $[(\dot{A}, 0)] \in \mathcal{T}_p$, the above formula gives us $dh_p(\dot{A}, 0) = (0, \ldots, 0)$. To see the reciprocal inclusion, we first prove the following characterization of $\mathcal{T}_p$.

Lemma 3.24. *We have the following equality*

$$\mathcal{T}_p = \{[(\dot{A}, \dot{\Phi})] \in \mathbb{H}^1(C^\bullet) \mid \dot{\Phi} \in \operatorname{Im}[\Phi, -]\}.$$

Proof. We need to check that a tangent vector represented by $(\dot{A}, \dot{\Phi})$, with $\dot{\Phi} \in \operatorname{Im}[\Phi, -]$, can be represented by a pair of the form $(\dot{B}, 0)$.

If $\dot{\Phi} = [\Phi, s]$, for $s \in \Omega^0(\operatorname{End} E)$ then, by the definition of $\mathbb{H}^1(C^\bullet)$,

$$[(\dot{A}, \dot{\Phi})] = [(\dot{A} - \bar{\partial}_E s, \dot{\Phi} - [\Phi, s])] = [(\dot{A} - \bar{\partial}_E s, 0)]. \qquad \blacksquare$$

The remaining inclusion then translates into the following result.

Proposition 3.25. *Let $(\dot{A}, \dot{\Phi})$ represent a tangent vector in $T_p\mathcal{M}(n,d)$. Then, if $\dot{\Phi} \notin \operatorname{Im}[\Phi, -]$, there is a number $k \in \{1, \ldots, n\}$ such that $\operatorname{tr}(\Phi^{k-1}\dot{\Phi}) \neq 0$. In other words, if $[(\dot{A}, \dot{\Phi})] \notin \mathcal{T}_p$, then $[(\dot{A}, \dot{\Phi})] \notin \ker dh_p$.*

Proof. Since X_s is smooth, there is a point $x \in X$ such that $\Phi_x = A\, dz$, where $A \in \mathfrak{gl}(n, \mathbb{C})$ is such that its eigenvalues $\lambda_1, \ldots, \lambda_n$ are all distinct (in fact, the generic points in X have this property). Now let $\dot{\Phi}_x$ be represented by an element $B \in \mathfrak{gl}(n, \mathbb{C})$ as well. The hypothesis $\dot{\Phi} \notin \operatorname{Im}[\Phi, -]$ means that x can also be chosen such that $B \notin \operatorname{Im}[A, -]$.

The endomorphism $[A, -] : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ induces the following direct sum decomposition

$$\mathfrak{gl}(n, \mathbb{C}) = \ker[A, -] \oplus \operatorname{Im}[A, -]$$

(to see that $\ker[A, -] \cap \operatorname{Im}[A, -] = 0$, take Y such that $[A, Y] = 0$ and $Y = [A, X]$. Since the eigenvalues of A are all distinct, and Y commutes with A, it is also diagonalizable with the same eigenvectors as A and so, working with a basis of eigenvectors for both endomorphisms, we get

$$Y = [A, X] \iff \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} = \left[\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, X \right] = \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix}$$

$$\implies \alpha_1 = \cdots = \alpha_n = 0 \implies Y = 0).$$

It follows that we can write $B = B' + B''$, with $B' \in \ker[A, -]$, $B'' \in \mathrm{Im}[A, -]$, and $B' \neq 0$. We have that A and B' commute and therefore, since the eigenvalues of A are all distinct, B' is also diagonalizable, with the same eigenvectors as A.

Assume, by contradiction, that

$$\operatorname{tr}(A^{k-1}B) = 0, \quad \forall k = 1, \ldots, n.$$

Noting that, since $B'' = [A, C]$, for some $C \in \mathfrak{gl}(n, \mathbb{C})$,

$$\operatorname{tr}(A^{k-1}B) = \operatorname{tr}(A^{k-1}B') + \operatorname{tr}(A^{k-1}[A, C]) = \operatorname{tr}(A^{k-1}B'),$$

we then have

$$\operatorname{tr}(A^{k-1}B') = 0, \quad k = 1, \ldots, n.$$

This translates into

$$\sum_{i=1}^{n} \lambda_i^{k-1} \beta_i = 0, \quad k = 1, \ldots, n,$$

where the $\beta_1, \ldots, \beta_n$ are the eigenvalues of B'. So the β_i are solutions to a system of linear equations whose matrix is

$$\begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \lambda_1 & \cdots & \cdots & \lambda_n \\ \vdots & \cdots & \cdots & \vdots \\ \lambda_1^{n-1} & \cdots & \cdots & \lambda_n^{n-1} \end{pmatrix}.$$

This is a Vandermonde matrix, with determinant equal to $\prod_{i<j}(\lambda_i - \lambda_j)$, which is non-zero, since all the λ_i are distinct. But this is impossible, since then the β_i would all be zero and $B' = 0$, which is absurd. ■

We now establish (2) in diagram 3.10 via the following result, which is derived from the exact sequence in Proposition 3.20.

Proposition 3.26 ([Mar94, Lemma 8.1]). *Consider $s \in \mathcal{B}$ such that X_s is smooth and let $(E, \Phi) \in h^{-1}(s)$. We have the following exact sequence on X*

$$0 \longrightarrow \pi_*\mathcal{O}_{X_s} \longrightarrow \operatorname{End}(E) \xrightarrow{[\Phi,-]} \operatorname{End}(E) \otimes K \longrightarrow \pi_*\mathcal{O}_{X_s} \otimes K^n \longrightarrow 0.$$

Proof. Let $E = \pi_* L$. From Proposition 3.20 we have

$$0 \longrightarrow \underbrace{L \otimes \pi^*(K^{1-n})}_{L(-R)} \longrightarrow \pi^*E \xrightarrow{\pi^*\Phi - \lambda} \pi^*E \otimes \pi^*K \longrightarrow L \otimes \pi^*K \longrightarrow 0.$$

Twisting it by $L^{-1}(R) = L^{-1} \otimes \pi^*K^{n-1}$ yields

$$0 \longrightarrow \mathcal{O}_{X_s} \longrightarrow \pi^*E \otimes L^{-1}(R) \xrightarrow{\pi^*\Phi - \lambda} \pi^*E \otimes L^{-1}(R) \otimes \pi^*K \longrightarrow \pi^*K^n \longrightarrow 0.$$

Next, we apply the pushforward functor π_* to obtain the exact sequence

$$0 \longrightarrow \pi_*\mathcal{O}_{X_s} \longrightarrow E \otimes \pi_*L^{-1}(R) \xrightarrow{f} E \otimes \pi_*L^{-1}(R) \otimes K \longrightarrow \pi_*\mathcal{O}_{X_s} \otimes K^n \longrightarrow 0$$

on X (since π is a finite map, the pushforward is exact, by [Har77, Exercise III.8.2, page 252]).

From Lemma 3.21, we know that $\pi_*(L^{-1}(R)) \cong E^*$ and so it remains to see that the map f is equal to $[\Phi, -]$. To that purpose, consider the following diagram

$$\begin{tikzcd}
\pi^*E \otimes L^{-1}(R) \arrow[r, "\pi^*\Phi\otimes\mathrm{id} - \mathrm{id}\otimes\lambda"] \arrow[d, "\pi_*"'] & \pi^*E \otimes \pi^*K \otimes L^{-1}(R) \\
E \otimes E^* \arrow[r, "f"] \arrow[d, leftrightarrow, "\cong"'] & E \otimes K \otimes E^* \\
\mathrm{End}(E) \arrow[r, "f"] & \mathrm{End}(E) \otimes K.
\end{tikzcd}$$

Now fix $p \in X$ such that Φ_p has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ (such p is generic) and consider $A \in \mathrm{End}(E)_p \cong \mathfrak{gl}(n, \mathbb{C})$. If $(e_i)_i$ is a basis of E_p of eigenvectors for Φ_p with dual basis $(e_i^*)_i$, then $A = \sum_{i,j} a_{ij} e_i \otimes e_j^*$ (see Figure 3.6).

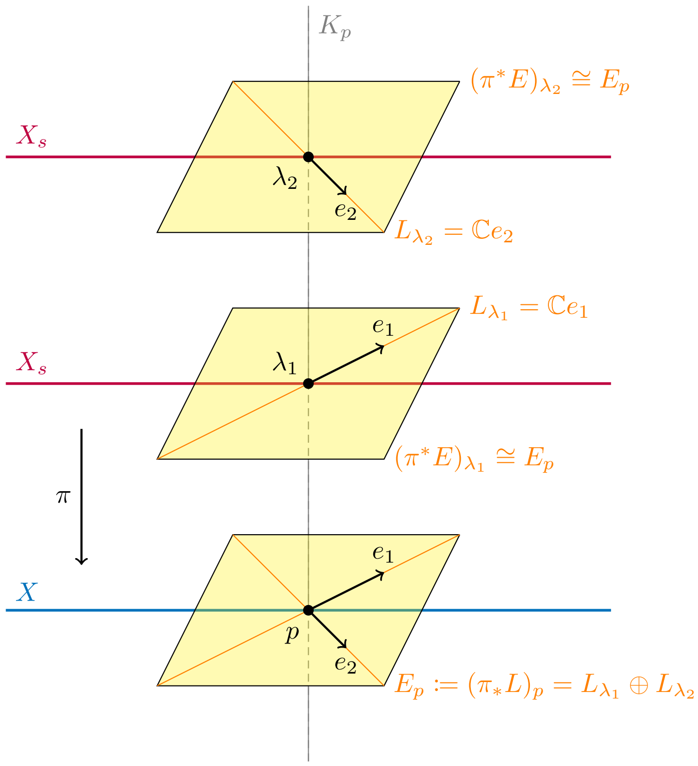

Figure 3.6: The pushforward of the line bundle L under the map $\pi : X_s \to X$ at an unramified point $p \in X$ in rank 2.

Rewriting A as

$$A = \sum_j \left(\sum_i a_{ij} e_i \right) \otimes e_j^*,$$

we see it comes from pushing forward the elements

$$\left(\sum_i a_{ij} e_i \right) \otimes e_j^* \in (\pi^*E \otimes L^{-1}(R))_{\lambda_j}.$$

Applying the map $\pi^*\Phi \otimes \mathrm{id} - \mathrm{id} \otimes \lambda$ we get

$$(\pi^*\Phi \otimes \mathrm{id} - \mathrm{id} \otimes\lambda)_{\lambda_j}\left(\left(\sum_i a_{ij}e_i\right) \otimes e_j^*\right) = \left(\sum_i a_{ij}\Phi_p(e_i)\right) \otimes e_j^* - \left(\sum_i a_{ij}e_i\right) \otimes e_j^* \otimes \lambda_j$$
$$= \left(\sum_i (\lambda_i - \lambda_j) a_{ij} e_i\right) \otimes e_j^*.$$

Pushing forward again, we get

$$f_p(A) = \sum_j \left(\sum_i (\lambda_i - \lambda_j) a_{ij} e_i\right) \otimes e_j^*.$$

But since

$$[\Phi_p, A](e_j) = \Phi_p(A(e_j)) - A(\Phi_p(e_j)) = \sum_i a_{ij}\Phi_p(e_i) - \lambda_j A(e_j) = \sum_i (\lambda_i - \lambda_j) a_{ij} e_i,$$

we conclude that $f_p = [\Phi_p, -]$. Since the condition imposed on p is generic, the equality $f = [\Phi, -]$ holds. ■

Corollary 3.27. *The space $\mathcal{T}_p$ is isomorphic to $T_p h^{-1}(s)$.*

Proof. The map $[(\dot{A}, 0)] \mapsto \dot{A}$ establishes an isomorphism between $\mathcal{T}_p$ and $H^1(X, \ker[\Phi, -])$. By Lemma 3.26, $\ker[\Phi, -] = \pi_*\mathcal{O}_{X_s}$, so $\mathcal{T}_p \cong H^1(X, \pi_*\mathcal{O}_{X_s}) \cong H^1(X_s, \mathcal{O}_{X_s})$. By the BNR correspondence (Corollary 3.19), $h^{-1}(s) \cong \mathrm{Jac}(X_s)$, and Proposition 2.7 tells us that any of its tangent spaces is isomorphic to $H^1(X_s, \mathcal{O}_{X_s})$. ■

We have finally shown the result we were looking for in this section, since the arrows in diagram (3.10) are all isomorphisms.

Theorem 3.28. *Let $s \in \mathcal{B}$ be such that $h^{-1}(s)$ is smooth. Then h is a submersion at every point $p \in h^{-1}(s)$ (i.e., dh_p is a surjective linear map).*

As we mentioned, in the course of proving the surjectivity of the derivative of the Hitchin map, we found the following characterization of the tangent space at the smooth fibers.

Theorem 3.29. *Let $s \in \mathcal{B}$ be as above. Then, for $p = [(E, \Phi)] \in h^{-1}(s)$, we have*

$$T_p h^{-1}(s) \cong \{[(\dot{A}, 0)] \mid \dot{A} \in \Omega^{0,1}(\operatorname{End} E), [\Phi, \dot{A}] = 0\} \subseteq \mathbb{H}^1(C^\bullet).$$

From Lemma 3.26 we can also obtain the following exact sequence, which can be shown to include the derivative of the Hitchin map (see [Hit19, Proposition 1] for the rank 2 case), and so it could be another way to show its surjectivity.

Corollary 3.30 ([Mar94, Proposition 8.2],[LM10, Lemma 3.13])**.** *Consider $s \in \mathcal{B}$ such that X_s is smooth and $(E, \Phi) \in h^{-1}(s)$. Then the short exact sequence*

$$0 \longrightarrow H^1(X, \pi_*\mathcal{O}_{X_s}) \longrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow H^0(X, \pi_*\mathcal{O}_{X_s} \otimes K^n) \longrightarrow 0$$

holds.

Proof. We use the sequence in Lemma 3.26 to obtain the following short exact sequence of complexes

$$\begin{array}{ccccccccc} & & C_1^\bullet & & C^\bullet & & C_2^\bullet & & \\ 0 & \longrightarrow & \pi_*\mathcal{O}_{X_s} & \longrightarrow & \operatorname{End} E & \xrightarrow{[\Phi,-]} & \operatorname{Im}[\Phi,-] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle [\Phi,-]} & & \downarrow & & \\ 0 & \longrightarrow & 0 & \hookrightarrow & \operatorname{End} E\otimes K & \xrightarrow{=} & \operatorname{End} E\otimes K & \longrightarrow & 0. \end{array}$$

From its long exact sequence in hypercohomology we get

$$\mathbb{H}^0(C_2^\bullet) \longrightarrow \mathbb{H}^1(C_1^\bullet) \longrightarrow \mathbb{H}^1(C^\bullet) \longrightarrow \mathbb{H}^1(C_2^\bullet) \longrightarrow \mathbb{H}^2(C_1^\bullet).$$

It is straightforward to see that $\mathbb{H}^2(C_1^\bullet) = 0$ and $\mathbb{H}^1(C_1^\bullet) = H^1(X, \pi_*\mathcal{O}_{X_s})$. For the remaining spaces, consider this other short exact sequence of complexes

$$\begin{array}{ccccccccc} & & C_3^\bullet & & C_2^\bullet & & C_4^\bullet & & \\ 0 & \longrightarrow & \operatorname{Im}[\Phi,-] & \xrightarrow{=} & \operatorname{Im}[\Phi,-] & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Im}[\Phi,-] & \hookrightarrow & \operatorname{End} E\otimes K & \longrightarrow & \pi_*\mathcal{O}_{X_s}\otimes K^n & \longrightarrow & 0. \end{array}$$

Since $\mathbb{H}^*(C_3^\bullet) = 0$, we have $\mathbb{H}^*(C_2^\bullet) = \mathbb{H}^*(C_4^\bullet)$. In particular, $\mathbb{H}^0(C_2^\bullet) = \mathbb{H}^0(C_4^\bullet) = 0$ and $\mathbb{H}^1(C_2^\bullet) = \mathbb{H}^1(C_4^\bullet) = H^0(X, \pi_*\mathcal{O}_{X_s} \otimes K^n)$. ■

3.3.2 Final conclusions

The previous subsection allows us to quickly prove the remaining condition required to obtain a completely integrable system.

Corollary 3.31. *The functions f_i^l defined in Proposition 3.22 are functionally independent.*

Proof. For simplicity of notation we order the functions α_i^l and f_i^l lexicographically on the pairs (l, i) and shall refer to them as simply α_i and f_i, $1 \le i \le N = \dim \mathcal{B}$.

Let $p \in \mathcal{M}(n, d)$ such that $X_{h(p)}$ is smooth. Then, by virtue of the α_i forming a basis of $\mathcal{B}^*$, we have

$$(d\alpha_1 \wedge \ldots \wedge d\alpha_N)_{h(p)} \neq 0.$$

This means that there exist $v_1, \ldots, v_N \in T_{h(p)}\mathcal{B}$ such that

$$(d\alpha_1 \wedge \ldots \wedge d\alpha_N)_{h(p)}(v_1, \ldots, v_N) \neq 0.$$

By Theorem 3.28, dh_p is surjective, so there exist $u_1, \ldots, u_N \in T_p\mathcal{M}(n, d)$ such that

$$h^*(d\alpha_1 \wedge \ldots \wedge d\alpha_N)_p(u_1, \ldots, u_N) \neq 0.$$

Finally, note that $h^*(d\alpha_i) = df_i$, and so

$$(df_1 \wedge \ldots \wedge df_N)_p \neq 0.$$

Since the set U of $s \in \mathcal{B}$ such that X_s is smooth is open and the moduli space $\mathcal{M}(n,d)$ is irreducible (by [Sim94, Theorem 11.1]), the set $h^{-1}(U)$ of p satisfying the above condition is open and dense in $\mathcal{M}(n,d)$ and so the f_i are functionally independent (recall part 1 of Definition 1.32). ■

We can finally write the main result of this text.

Theorem 3.32. *The moduli space $\mathcal{M}(n,d)$, together with the functions f_i^l of Proposition 3.22, is a completely integrable system, called the* Hitchin system*.*

So now that we have a completely integrable system, the following is immediate from Proposition 1.33.

Corollary 3.33. *The smooth fibers of the Hitchin map h are Lagrangian subvarieties of the moduli space $\mathcal{M}(n,d)$.*

It should be noted, however, that this corollary could also be deduced directly from the BNR correspondence, since it tells us that $h^{-1}(s) \cong \mathrm{Jac}(X_s)$, an abelian variety of dimension equal to the genus of X_s, already calculated to be

$$1 + n^2(g-1) = \frac{1}{2}\dim \mathcal{M}(n,d),$$

together with Theorem 3.29, since it is obvious, from the description of the tangent space of a smooth fiber as the space $\mathcal{T}_p$, that the symplectic form ω of Proposition 2.16 vanishes on any two tangent vectors. We also checked directly one of the consequences of the Arnold-Liouville theorem ([Sil08, Theorem 18.12]), which tells us that, since h is a proper map (recall Remark 3.4), the generic fiber will be a torus, and we have seen that they are precisely Jacobians of spectral curves.

Appendix A

Definitions and results from Algebraic Geometry

A.1 Schemes and morphisms

In Section 3.2, we built the spectral curve using the notion of a *relative spectrum*. Essentially, we described the curve by saying what its structure sheaf should look like. The precise definition is as follows.

Definition A.1 ([Har77, page 128, Exercise 5.17])**.** Let Y be a scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. There is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \hookrightarrow V$ of open affine sets of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$. This scheme X is called $\underline{\operatorname{Spec}}\,\mathcal{A}$.

The algebraic point of view tells us that the map $\pi : X_s \to X$, behaves quite well. In fact, in Proposition 3.10, we show it is finite (hence affine), in the sense of the following definitions.

Definition A.2. A morphism $f : X \to Y$ of schemes is said to be *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

Definition A.3. A morphism $f : X \to Y$ of schemes is said to be *finite* if there is an open affine cover $\{V_i\}$ of Y such that $V_i = \operatorname{Spec} B_i$ and each $f^{-1}(V_i)$ is affine, equal to $\operatorname{Spec} A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.

This property of π allows us to use the following two results, which are essential for the calculations we perform in the text.

Proposition A.4 ([Har77, page 128, Exercise 5.17])**.** *Let $f : X \to Y$ be an affine morphism of schemes. Then f_* induces an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{quasi-coherent} \\ \mathcal{O}_X\textit{-modules} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{quasi-coherent} \\ f_*\mathcal{O}_Y\textit{-modules} \end{array} \right\}.$$

Proposition A.5 ([Har77, page 222, Exercise 4.1]). *Let $f : X \to Y$ be an affine morphism of noetherian separated schemes. For any quasi-coherent sheaf $\mathcal{F}$ on X, there are natural isomorphisms for all $i \geq 0$*

$$H^i(X, \mathcal{F}) \cong H^i(Y, f_*\mathcal{F}).$$

In particular, this means that $h^i(X, \mathcal{F}) = h^i(Y, f_*\mathcal{F})$.

A.2 Derived functors

Throughout the text we are constantly referencing cohomology spaces associated to sheaves. There are many ways of defining cohomology of sheaves, we give here a brief description using the concept of a derived functor, following [Har77, Section III.1], which requires some acquaintance with the language of category theory. However it should be noted that it is not the best way to work with cohomology if we actually want to calculate what the spaces are, Dolbeault or Čech cohomology would be more appropriate in that case. For example, from Theorem 1.17, we know that $H^{p,q}(X, E) \cong H^q(X, E \otimes K^p)$, so we can determine the cohomology of $E \otimes K^p$ via a Dolbeault resolution.

Let $\mathfrak{A}$ be an abelian category. An object I of $\mathfrak{A}$ is *injective* if the functor $\mathrm{Hom}(\cdot, I)$ is exact. An *injective resolution* of an object A of $\mathfrak{A}$ is a complex $I^\bullet$, defined in degrees $i \geq 0$, together with a morphism $\epsilon : A \to I^0$, such that I^i is an injective object of $\mathfrak{A}$ for each $i \geq 0$, and such that the sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

is exact.

If every object of $\mathfrak{A}$ is isomorphic to a subobject of an injective object of $\mathfrak{A}$, we say that $\mathfrak{A}$ *has enough injectives.* If $\mathfrak{U}$ is an abelian category with enough injectives and $F : \mathfrak{A} \to \mathfrak{B}$ a covariant left exact functor, we construct the *right derived functors* of F, denoted by R^iF, $i \geq 0$ as follows. For each object A of $\mathfrak{A}$, choose once and for all an injective resolution $I^\bullet$ of A, then define $R^iF(A) = h^i(F(I^\bullet))$ (the cohomology objects of the complex $F(I^\bullet)$).

The following property is key to many calculations performed with derived functors.

Theorem A.6. *Let $\mathfrak{A}$ be an abelian category with enough injectives, and let $F : \mathfrak{A} \to \mathfrak{B}$ be a covariant left exact functor to another abelian category $\mathfrak{B}$. Then, for each short exact sequence*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

and for each $i \geq 0$, there is a natural morphism $\delta^i : RF^i(A'') \to RF^{i+1}(A')$, such that we obtain a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & R^iF(A') \arrow[r] & R^iF(A) \arrow[r] & R^iF(A'') \arrow[dll, rounded corners] \\ & R^{i+1}F(A') \arrow[r] & R^{i+1}F(A) \arrow[r] & \cdots . \end{tikzcd}$$

A.2.1 The cohomology and the direct image (pushforward) functors

$\mathfrak{Ab}(X)$ will denote the (abelian) category of sheaves of abelian groups on a topological space X. Following [Har77, Section III.2], we finally arrive at sheaf cohomology.

Definition A.7. Let $\Gamma(X, \cdot)$ be the global sections functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$. We define the cohomology functors $H^i(X, \cdot)$ to be the right derived functors of $\Gamma(X, \cdot)$.

There is also another series of derived functors, appearing in the proof of Proposition 3.20, the *higher direct image* functors.

Definition A.8 ([Har77, Section III.8])**.** Let $f : X \to Y$ be a continuous map of topological spaces. Then we define the *higher direct image* functors $R^i f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_* (this makes sense since f_* is left exact and $\mathfrak{Ab}(X)$ has enough injectives).

The higher direct images are closely related to sheaf cohomology via the following result.

Proposition A.9. *For each $i \geq 0$ and each $\mathcal{F} \in \mathfrak{Ab}(X)$, $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf*

$$V \mapsto H^i\left(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}\right).$$

Finally, we mention a result that is also used throughout the text (mostly the $i = 0$ case) for calculations.

Proposition A.10 ([Har77, Exercise III.8.3, page 253])**.** *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. The following* projection formula *is valid*

$$R^i f_*(\mathcal{F} \otimes f^*\mathcal{E}) \cong R^i f_*\mathcal{F} \otimes \mathcal{E}.$$

Bibliography

[Aud96] M. Audin. *Spinning Tops A Course on Integrable Systems*. Cambridge studies in advanced mathematics 51. Cambridge University Press, 1996.

[Bis94] I. Biswas. "A remark on a deformation theory of Green and Lazarsfeld." In: *Journal für die reine und angewandte Mathematik* 449 (1994), pp. 103–124. URL: http://eudml.org/doc/153607.

[BNR89] A. Beauville, M. S. Narasimhan, and S. Ramanan. "Spectral curves and the generalised theta divisor." In: *Journal für die reine und angewandte Mathematik* 1989.398 (1989), pp. 169–179. DOI: 10.1515/crll.1989.398.169.

[BR94] I. Biswas and S. Ramanan. "An Infinitesimal Study of the Moduli of Hitchin Pairs". In: *Journal of the London Mathematical Society* 49.2 (1994), pp. 219–231. DOI: 10.1112/jlms/49.2.219.

[Dal17] P. Dalakov. "Lectures on Higgs moduli and abelianisation". In: *Journal of Geometry and Physics* 118 (2017), pp. 94–125. DOI: 10.1016/j.geomphys.2017.01.001.

[DK90] S. K. Donaldson and P. B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Oxford: Clarendon Press, 1990.

[Don11] S. K. Donaldson. *Riemann Surfaces*. Oxford Graduate Texts in Mathematics 22. Oxford University Press, 2011.

[Fra22] E. Franco. *O'Grady spaces and symplectic resolution of moduli spaces of Higgs bundles*. 2022. DOI: 10.48550/ARXIV.2210.04823.

[GH94] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, Ltd, 1994.

[GO12] P. B. Gothen and A. G. Oliveira. "The Singular Fiber of the Hitchin Map". In: *International Mathematics Research Notices* 2013.5 (Feb. 2012), pp. 1079–1121. DOI: 10.1093/imrn/rns020.

[Har77] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics 52. Springer New York, NY, 1977.

[Hit16] N. J. Hitchin. "Higgs Bundles and Characteristic Classes". In: *Arbeitstagung Bonn 2013: In Memory of Friedrich Hirzebruch*. Ed. by W. Ballmann, C. Blohmann, G. Faltings, P. Teichner, and D. Zagier. Cham: Springer International Publishing, 2016, pp. 247–264. DOI: 10.1007/978-3-319-43648-7_8.

[Hit19] N. J. Hitchin. “Critical Loci for Higgs Bundles”. In: *Communications in Mathematical Physics* 366.2 (2019), pp. 841–864. DOI: 10.1007/s00220-019-03336-4.

[Hit87a] N. J. Hitchin. “Stable bundles and integrable systems”. In: *Duke Mathematical Journal* 54.1 (1987), pp. 91–114. DOI: 10.1215/S0012-7094-87-05408-1.

[Hit87b] N. J. Hitchin. “The self-duality equations on a Riemann surface”. In: *Proceedings of the London Mathematical Society* S3-55.1 (July 1987), pp. 59–126. DOI: 10.1112/plms/s3-55.1.59.

[Hit89] N. J. Hitchin. “Gauge theory on Riemann surfaces”. In: *Lectures on Riemann Surfaces*. Ed. by M. Cornalba, X. Gómez-Mont, and A. Verjovsky. World Scientific, 1989, pp. 99–118. DOI: 10.1142/9789814503365_0003.

[HSW99] N. J. Hitchin, G. B. Segal, and R. S. Ward. *Integrable Systems Twistors, Loop Groups, and Riemann Surfaces*. Oxford University Press, 1999.

[Huy05] D. Huybrechts. *Complex Geometry An Introduction*. Universitext. Springer Berlin, 2005.

[Kas13] J. L. Kass. “Singular curves and their compactified Jacobians”. In: *A Celebration of Algebraic Geometry*. Ed. by B. Hassett, J. McKernan, J. Starr, and R. Vakil. Clay Mathematics Proceedings 18. Providence, RI: American Mathematical Society, 2013, pp. 391–427.

[Kle80] S. L. Kleiman. “Relative duality for quasi-coherent sheaves”. In: *Compositio Mathematica* 41.1 (1980), pp. 39–60. URL: http://www.numdam.org/item/CM_1980__41_1_39_0/.

[KR22] E. Kienzle and S. Rayan. “Hyperbolic band theory through Higgs bundles”. In: *Advances in Mathematics* 409 (Nov. 2022), p. 108664. DOI: 10.1016/j.aim.2022.108664.

[Liu02] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. Trans. by R. Erné. Oxford Graduate Texts in Mathematics 6. Oxford University Press, 2002.

[LM10] M. Logares and J. Martens. “Moduli of parabolic Higgs bundles and Atiyah algebroids”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2010.649 (Jan. 2010). DOI: 10.1515/crelle.2010.090.

[Mar94] E. Markman. “Spectral curves and integrable systems”. In: *Compositio Mathematica* 93.3 (1994), pp. 255–290. URL: http://www.numdam.org/item/CM_1994__93_3_255_0/.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Third Enlarged Edition. Ergebnisse der Mathematik und ihrer Grenzgebiete 34. Springer-Verlag, 1994.

[Mir95] R. Miranda. *Algebraic Curves and Riemann Surfaces*. Graduate Studies in Mathematics 5. American Mathematical Society, 1995.

[MT97] I. H. Madsen and J. Tornehave. *From Calculus to Cohomology De Rham Cohomology and Characteristic Classes*. Cambridge University Press, Mar. 1997.

[New78] P. E. Newstead. *Introduction to Moduli Problems and Orbit Spaces*. Tata Institute of Fundamental Research Lectures on Mathematics and Physics 51. Springer-Verlag, 1978.

[Nit91] N. Nitsure. "Moduli Space of Semistable Pairs on a Curve". In: *Proceedings of the London Mathematical Society* s3-62.2 (1991), pp. 275–300. DOI: `10.1112/plms/s3-62.2.275`.

[NR69] M. S. Narasimhan and S. Ramanan. "Moduli of Vector Bundles on a Compact Riemann Surface". In: *Annals of Mathematics* 89.1 (1969), pp. 14–51. URL: `http://www.jstor.org/stable/1970807`.

[Pot97] J. L. Potier. *Lectures on Vector Bundles*. Trans. by A. Maciocia. Cambridge studies in advanced mathematics 54. Cambridge University Press, 1997.

[Reg80] C. J. Rego. "The compactified Jacobian". In: *Annales scientifiques de l'École Normale Supérieure*. 4th ser. 13.2 (1980), pp. 211–223. DOI: `10.24033/asens.1380`.

[Sch20] L. P. Schaposnik. "Higgs Bundles — Recent Applications". In: *Notices of the American Mathematical Society* 67.05 (May 2020), pp. 625–634. DOI: `10.1090/noti2074`.

[Sil08] A. C. da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics 1764. Berlin Heidelberg: Springer-Verlag, 2008.

[Sim94] C. T. Simpson. "Moduli of representations of the fundamental group of a smooth projective variety II". In: *Publications Mathématiques de l'IHÉS* 80 (1994), pp. 5–79. URL: `http://www.numdam.org/item/PMIHES_1994__80__5_0/`.

[Tha96] M. Thaddeus. "An Introduction to the Topology of the Moduli Space of Stable Bundles on a Riemann Surface". In: *Geometry and Physics*. Ed. by J. E. Andersen, J. Dupont, H. Pedersen, and A. Swann. Lecture notes in pure and applied mathematics 184. Boca Raton: CRC Press, 1996, pp. 71–99.

[Wel08] R. O. Wells Jr. *Differential Analysis on Complex Manifolds*. 3rd ed. Graduate Texts in Mathematics 65. New appendix by Oscar García-Prada. Springer New York, 2008.

[Wen16] R. A. Wentworth. "Higgs Bundles and Local Systems on Riemann Surfaces". In: *Geometry and Quantization of Moduli Spaces*. Ed. by L. A. Consul, J. E. Andersen, and I. M. i Riera. Advanced Courses in Mathematics - CRM Barcelona. Switzerland: Birkhäuser Cham, 2016. Chap. 4, pp. 165–219.