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ORIGINAL ARTICLE

An extension of the complex–real (C–R) calculus to the bicomplex setting, with applications

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Abstract

In this paper, we extend notions of complex $\mathbb{C}-\mathbb{R}$-calculus to the bicomplex setting and compare the bicomplex polyanalytic function theory to the classical complex case. Applications of this theory include two bicomplex least mean square algorithms, which extend classical real and complex least mean square algorithms.

KEYWORDS

BLMS algorithm, bicomplex gradient operator, bicomplex numbers, C–R calculus, $\mathbb{BC}-\mathbb{R}$ calculus

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1 | INTRODUCTION

The last few years saw a resurgence of applications of hypercomplex analysis and algebra in neural networks. These new approaches have been especially successful in using particular examples of hypercomplex numbers such as bicomplex numbers, quaternions, and Clifford ones to develop different machine learning techniques, see, for example, [18–20, 23, 25], and references therein. The authors have also extended the complex perceptron algorithm to the bicomplex case in [5, 6]. As we have already discussed, the LMS algorithm discovered by Widrow and Hoff was extended to the complex domain in [37] for the first time and the gradient descent technique was derived with respect to the real and imaginary part. The theory of gradient descent was further generalized in [11] using Wirtinger (or $\mathbb{C}-\mathbb{R}$) calculus such that the gradient is considered with respect to complex variables instead. We note that Wirtinger calculus provides a framework for obtaining the gradient with respect to complex-valued functions [35].

In his work, Rosenblatt [28] introduced the perceptron, the first trainable linear classifier. Inspired by his work, a first implementation of a trainable neural network was the ADALINE machine introduced by Widrow and Hoff in[36]. This application uses the techniques of the least mean square (LMS) and of the stochastic gradient descent for deriving optimal weights. In [11], Brandwood studied properties of the complex gradient operator using a technique called *"Complex-Real-Calculus"* (i.e., $\mathbb{C}-\mathbb{R}$-calculus or Wirtinger calculus, see also [21]). In his paper, Brandwood considered different applications of these mathematical concepts to adaptive array theory including the well-known complex least mean square (LMS) algorithm. This algorithm was first discovered by Widrow and his student Hoff in the real-valued case in 1960 (see [36]).

In Section 2.1, we give an overview of the complex–real ($\mathbb{C}-\mathbb{R}$) calculus, starting with Definition 2.1, followed by examples and relationships with the theory of polyanalytic functions.

As stated in the work of Brandwood [11], mentioned above, if the desired and predicted output of a complex neural network are given by $d \in \mathbb{C}^N$ and $o \in \mathbb{C}^N$, respectively, then the error can be defined by $e := d - o$ and the complex mean square loss is a non-negative scalar which has the following expression:

$$\mathcal{L}(e) = \sum_{k=0}^{N-1} |e_k|^2.$$

Although in many types of analysis involving complex numbers the individual components of the complex number have been treated independently as real numbers, it would not make sense to apply the same concept to complex-valued neural networks (CVNNs) by assuming that a CVNN is equivalent to a two-dimensional real-valued neural network. In fact, it has been shown that this is not the case [16], because the operation of complex multiplication limits the degrees of freedom of the CVNNs at the synaptic weighting. This suggests that the phase-rotational dynamics strongly underpins the process of learning.

On the other hand, a fundamental concern is whether the activation function is differentiable everywhere, differentiable around certain points, or not differentiable at all. Complex functions that are holomorphic at every point are known as "entire functions." However, in the complex domain, one cannot have a bounded complex activation function that is complex-differentiable at the same time. This is a direct consequence of the well-known Liouville's theorem in complex analysis stating that every entire functions which is bounded is constant. Hence, it is not possible to have a CVNN that uses squashing activation functions that are entire.

In this research paper, inspired from the work of Brandwood [11, 37], we propose to develop the bicomplex counterpart of the $\mathbb{C}-\mathbb{R}$-calculus and its application to bicomplex gradient operators. We also apply these new techniques to derive two bicomplex LMS algorithms extending both the classical real and complex LMS algorithms of Widrow. For a relevant reference on applications of the $\mathbb{C}-\mathbb{R}$-calculus and adaptive array theory to random vibrations control tests, see [24].

The structure of our paper is as follows. In Section 2, we review different notions of the classical complex case including polyanalytic functions, $\mathbb{C}-\mathbb{R}$-calculus and briefly discuss the complex gradient operators. We also explain several connections between polyanalytic functions and $\mathbb{C}-\mathbb{R}$-regular functions. Section 3 collects different definitions and notations which will be useful in the rest of the paper. Here, we review different notions related the bicomplex algebra, the hyperbolic-valued modulus and bicomplex differential operators. In Section 4, we introduce the bicomplex counterpart of the complex $\mathbb{C}-\mathbb{R}$ calculus, namely the newly defined "$\mathbb{BC}-\mathbb{R}$ calculus" in one bicomplex variable. Here, we also study the basic properties of classes of functions associated with the $\mathbb{BC}-\mathbb{R}$ calculus, define different bicomplex gradient operators with respect to each conjugate, and prove a Leibniz rule for these gradients. In Section 5, we introduce and study the multivariate bicomplex calculus and related bicomplex gradient operators in the case of several variables. Finally, we apply the techniques developed in the previous sections in order to derive two different version of bicomplex least mean square (BLMS) algorithms in Section 6.

2 | THE COMPLEX CASE

This section is an expository reminder of several concepts from complex analysis, including polyanalytic function theory, $\mathbb{C}-\mathbb{R}$ calculus (or Wirtinger calculus), concluding with a complex LMS algorithm as an application. We use the regular setting of a complex-valued function $f : \Omega \subset \mathbb{C} \longrightarrow \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is a domain. We consider f to be real analytic with respect to its real and imaginary parts, therefore the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are well-defined by

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right); \quad z = x + \mathrm{i}y.$$

Real differentiable functions in the kernel of $\frac{\partial}{\partial \overline{z}}$ are called *complex analytic, or, equivalently holomorphic* on their domains. For easier readability, we would like to recall that Cauchy's theorem states that these types of functions are locally convergent power series around any point in their domains.

2.1 | $\mathbb{C}-\mathbb{R}$ calculus in the complex case

In the papers [11, 21], an application of $\mathbb{C}-\mathbb{R}$ calculus to complex neural networks is discussed and we elaborate on some of the finer points of the theory in this subsection.

We observe that if a function $f(z)$ is real valued, we obtain

$$\overline{\left(\frac{\partial}{\partial z} f(z)\right)} = \frac{\partial}{\partial \overline{z}} f(z),$$

and it is clear that if f belongs to the kernel of $\frac{\partial}{\partial \overline{z}}$ it will be in the kernel of $\frac{\partial}{\partial z}$ and vice versa.

We now introduce the following class of functions which can be considered the basis to study the $\mathbb{C}-\mathbb{R}$ calculus. We remind the reader that $\mathbb{C}-\mathbb{R}$ stands for *complex-real or Wirtinger calculus* in this paper (see [11, 21]).

Definition 2.1. Let $\Omega \subset \mathbb{C}$ be a complex domain, symmetric with respect to the real axis. We say that a function $f : \Omega \longrightarrow \mathbb{C}$ is $\mathbb{C}-\mathbb{R}$ **analytic (or $\mathbb{C}-\mathbb{R}$ regular) on** Ω if there exists a complex analytic function of two complex variables $g : \Omega \times \Omega \longrightarrow \mathbb{C}, (z_1, z_2) \mapsto g(z_1, z_2)$ such that

i) $f(z) = g(z, \overline{z}), \quad \forall z \in \Omega,$
ii) $\frac{\partial f}{\partial z} = \left(\frac{\partial g}{\partial z_1}\right)_{z_1 = z,\, z_2 = \overline{z}},$
iii) $\frac{\partial f}{\partial \overline{z}} = \left(\frac{\partial g}{\partial z_2}\right)_{z_1 = z,\, z_2 = \overline{z}}.$

Remark 2.2. If g is entire we have

$$g(z_1, z_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{k,j} z_1^j z_2^k, \qquad \text{yielding} \qquad f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{k,j} z^j \overline{z}^k,$$

and the following forms for the derivatives follows:

$$\frac{\partial f}{\partial z} = \sum_{j=1}^{\infty} \left(\sum_{k=0}^{\infty} a_{k,j} \overline{z}^k \right) j\, z^{j-1}, \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}} = \sum_{k=1}^{\infty} \left(\sum_{j=0}^{\infty} a_{k,j} z^k \right) k\, \overline{z}^{k-1}.$$

Remark 2.3. In a natural way, any complex analytic function will be $\mathbb{C}-\mathbb{R}$ regular.

Here are other examples of $\mathbb{C}-\mathbb{R}$ regular functions which are not complex analytic.

Example 2.4. The function $f_1(z) = \mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ is a real-valued $\mathbb{C}-\mathbb{R}$ analytic function with $g_1(z_1, z_2) = \frac{z_1+z_2}{2}$. Also, the function $f_2(z) = z^2\overline{z}$ is a complex-valued $\mathbb{C}-\mathbb{R}$ analytic function with $g_2(z_1, z_2) = z_1^2 z_2$. Note that neither f_1 nor f_2 are complex analytic.

Other interesting examples include $|z|^2$ and $\ln(z\bar{z})$.

Remark 2.5. The following formulas hold in the complex case:

$$\frac{\partial |z|^{2k}}{\partial \overline{z}} = kz|z|^{2k-2}; \qquad \frac{\partial |z|^{2k+1}}{\partial \overline{z}} = \frac{2k+1}{2} z|z|^{2k-1}, \qquad k \geq 0.$$

In classic complex analysis, we see that any function $f : \mathbb{C} \longrightarrow \mathbb{C}$ can be seen as a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$ as $f(z) = f(x + iy) = f(x, y)$. For improved clarity, here we re-formulate some results of Brandwood from his work [11].

Theorem 2.6. *Let f be $\mathbb{C}-\mathbb{R}$ regular on $\mathbb{C}$, i.e., as in Definition 2.1, there exists a complex analytic function of two complex variables $g : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}, (z_1, z_2) \mapsto g(z_1, z_2)$ such that*

$$f(z) = f(x, y) = g(z, \overline{z}); \quad z = x + iy.$$

Then, the differential operators follow the expected rules:

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(x, y),$$
$$\frac{\partial g}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(x, y),$$

where $\frac{\partial g}{\partial z} := \left(\frac{\partial g}{\partial z_1}\right)_{z_1=z,\, z_2=\overline{z}}, \qquad \frac{\partial g}{\partial \overline{z}} := \left(\frac{\partial g}{\partial z_2}\right)_{z_1=z,\, z_2=\overline{z}}.$

Definition 2.7. A real-valued real-differentiable function $f : \Omega \longrightarrow \mathbb{R}$, defined on an open domain $\Omega \subset \mathbb{C}$, has a stationary point $(x_0, y_0) \in \Omega$ if

$$\frac{\partial}{\partial x} f(x_0, y_0) = \frac{\partial}{\partial y} f(x_0, y_0) = 0.$$

It is easy to see that for such real-valued function we have:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Leftrightarrow \frac{\partial f}{\partial z} = 0 \Leftrightarrow \frac{\partial f}{\partial \overline{z}} = 0,$$

at the point $z_0 = x_0 + iy_0$, which leads to the following result in [11].

Lemma 2.8. *Let $f : \mathbb{C} \longrightarrow \mathbb{R}$ be a $\mathbb{C}-\mathbb{R}$ regular, real-valued function of a complex variable z (i.e., $f(z) = g(z, \overline{z})$ with $g : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{R}$, analytic with respect to each variable). Then, f has a stationary point if and only if*

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

at this given point, where this is defined as in Definition 2.1. In a similar way, the condition below is also a necessary and sufficient condition for f to have a stationary point

$$\frac{\partial f}{\partial z} = 0,$$

at this given point.

Remark 2.9. We note that in Theorem 2.6 and Lemma 2.8 we can replace $\mathbb{C}\times\mathbb{C}$ by $\Omega\times\Omega$ for a domain symmetric with respect to the real line.

In what follows, we will present an overview of applications of $\mathbb{C}-\mathbb{R}$ Calculus to the study of polyanalytic functions in Section 2.2 and to the complex LMS algorithm in Section 2.3.

2.2 | Classical polyanalytic functions

Under the conditions above, we have:

Definition 2.10. Let $\Omega \subset \mathbb{C}$ be a complex domain and f be a $C^\infty(\Omega)$-function (i.e., smooth in its real components). If f is in the kernel of a power $n \geq 1$ of the classical Cauchy–Riemann operator $\frac{\partial}{\partial \overline{z}}$, that is,

$$\frac{\partial^n}{\partial \overline{z}^n} f(z) = 0, \quad \forall z \in \Omega,$$

then f is called a polyanalytic function of order n on Ω. The space of all complex polyanalytic function of order n on Ω is denoted by $H_n(\Omega)$.

An interesting fact regarding these functions is that any polyanalytic function of order n can be decomposed in terms of n analytic functions so that we have a decomposition of the following form (see [9, 10]):

$$f(z) = \sum_{k=0}^{n-1} \overline{z}^k f_k(z), \tag{2.1}$$

for which all f_k are complex analytic functions on Ω. Again, in the case $n = 1$ one recovers the classical case of complex analytic functions via Cauchy's theorem, namely complex analytic functions are polyanalytic of order 1.

In particular, for any $n \geq 1$, expanding each analytic component using the series expansion theorem leads to an expression of the form

$$f(z) = \sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \overline{z}^k z^j a_{k,j}, \tag{2.2}$$

where $(a_{k,j})$ are complex coefficients.

In this paper, we are also interested in the case where the expansion (2.2) is *of infinite order* (this case was also considered in [4]), we now write a new definition:

Definition 2.11. A function of the form

$$f(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \overline{z}^k z^j a_{k,j}, \tag{2.3}$$

where the coefficients $a_{k,j}$ are non-zero for an infinite number of indices k, will be called a polyanalytic function of infinite order.

We note that such functions were discussed in [9, 10] as well, where they were mentioned as *conjugate analytic functions*.

Proposition 2.12. *Any polyanalytic function (of finite or infinite order) on a domain Ω symmetric with respect to the $x-$axis is $\mathbb{C} - \mathbb{R}$ analytic. The converse is also true.*

Proof. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a polyanalytic function of order $n = 1, 2, \ldots$. We know by poly-decomposition that there exists unique analytic functions $f_0, \ldots, f_{n-1}$ such that

$$f(z) = \sum_{k=0}^{n-1} \overline{z}^k f_k(z); \quad \forall z \in \mathbb{C}.$$

Then, we consider the analytic function of two complex variables defined by

$$g(z_1, z_2) = \sum_{k=0}^{n-1} z_2^k f_k(z_1), \quad \forall (z_1, z_2) \in \mathbb{C}^2.$$

It is clear that $g(z,\overline{z}) = f(z)$ and then we can easily check that f is $\mathbb{C}-\mathbb{R}$ analytic.

The infinite order case is treated similarly and we leave the proof to the reader.

The converse easily follows from the definition. □

Example 2.13. We consider the function $f(z) = e^{|z|^2}$. It is clear that f is $\mathbb{C}-\mathbb{R}$ analytic with

$$g(z,w) := \sum_{n=0}^{\infty} \frac{z^n w^n}{n!} = e^{zw}; \quad \forall z,w \in \mathbb{C}.$$

However, f is not polyanalytic of a finite order.

For $n = 1, 2, \ldots$, we recall that polyanalytic Fock spaces of order n can be defined as follows:

$$\mathcal{F}_n(\mathbb{C}) := \left\{ g \in H_n(\mathbb{C}), \quad \frac{1}{\pi}\int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} dA(z) < \infty \right\}.$$

The reproducing kernel associated with the space $\mathcal{F}_n(\mathbb{C})$ is given by

$$K_n(z,w) = e^{z\overline{w}} \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \binom{n}{k+1} |z-w|^{2k}, \tag{2.4}$$

for every $z, w \in \mathbb{C}$.

The poly-Bergman space $A_n^2(B(0,1))$ of polyanalytic functions of order n in the complex unit disc $B(0,1)$ is given by

$$A_n^2(B(0,1)) = \{f \in H_n(B(0,1)); \int_{B(0,1)} |f(z)|^2 \, dA(z) < \infty\}.$$

We note that $A_n^2(B(0,1))$ is a reproducing kernel Hilbert space, whose reproducing kernel is given by

$$B_n(z,w) = \frac{n}{\pi(1-\overline{w}z)^{2n}} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n+k}{n} |1-\overline{w}z|^{2(n-1-k)} |z-w|^{2k}, \tag{2.5}$$

for every $z, w \in B(0,1)$.

Important examples of polyanalytic functions were introduced by Itô in [17]; these are called *complex Hermite polynomials* and they are used in the study of stochastic processes. We write about these functions as well as a corresponding bicomplex function theory in a recent paper which complements this work [7].

2.3 | The complex gradient operator and the complex LMS algorithm

We now move to the case of several complex variables and use all the $\mathbb{C}-\mathbb{R}$ calculus techniques described above. We define the complex gradient operators with respect to the variables $\mathbf{z}$ and $\overline{\mathbf{z}}$.

Definition 2.14. Let $\Omega_k \subset \mathbb{C}$, $1 \leq k \leq n$ be n complex domains, each symmetric with respect to the real axis. We say that a function $f : \Pi_{k=1}^n \Omega_k \longrightarrow \mathbb{C}$ is $\mathbb{C}-\mathbb{R}$ analytic (or $\mathbb{C}-\mathbb{R}$ regular) on $\Omega = \Pi_{k=1}^n \Omega_k$ if there exists a complex analytic function of $2n$ complex variables $g : \Pi_{k=1}^n \Omega_k^2 \longrightarrow \mathbb{C}, (z_1, w_1, z_2, w_2, \ldots, z_n, w_n) \mapsto g(z_1, w_1, z_2, w_2, \ldots, z_n, w_n)$ such that

i) $f(z_1, z_2, \ldots, z_n) = g(z_1, \overline{z_1}, z_2, \overline{z_2}, \ldots, z_n, \overline{z_n}), \quad \forall (z_1, z_2, \ldots, z_n) \in \Pi_{k=1}^n \Omega_k$,

ii) $\frac{\partial f}{\partial z_l} = \left(\frac{\partial g}{\partial z_l}\right)_{z_l = z_l,\, w_l = \overline{z_l}}$, for any $1 \leq l \leq n$

iii) $\frac{\partial f}{\partial \overline{z_l}} = \left(\frac{\partial g}{\partial w_l}\right)_{z_l = z_l,\, w_l = \overline{z_l}}$, for any $1 \leq l \leq n$.

With this in mind we can now define the multi-variate gradient operators and we have:

Definition 2.15. Let $\mathbf{z} = (z_1, \dots, z_n)^T \in \mathbb{C}^n$, with $z_k = x_k + iy_k$ for any $k = 1, \dots, n$. Then, the complex gradient operator with respect to $\mathbf{z}$ is defined to be

$$\nabla_{\mathbf{z}} := (\partial_{z_1}, \dots, \partial_{z_n})^T, \tag{2.6}$$

where ∂_{z_l} are the complex derivatives with respect to the variable z_l and $l = 1, \dots, n$.

In a similar way, the gradient operator with respect to the conjugate is defined by

$$\nabla_{\overline{\mathbf{z}}} := (\partial_{\overline{z_1}}, \dots, \partial_{\overline{z_n}})^T. \tag{2.7}$$

In the same way as in the single variable case, one defines the notion of stationary points and we have the following characterization statement:

Lemma 2.16. *Let $f : \mathbb{C}^n \longrightarrow \mathbb{R}$ be a real-valued $\mathbb{C} - \mathbb{R}$ analytic function of a complex vector variable $\mathbf{z} = (z_1, z_2, \dots, z_n)$ (i.e., $f(\mathbf{z}) = g(z_1, \overline{z_1}, z_2, \overline{z_2}, \dots, z_n, \overline{z_n})$ with $g : \mathbb{C}^{2n} \longrightarrow \mathbb{R}$ is an entire function of $2n$ complex vector variables which is analytic with respect to each component). Then, f has a stationary point if and only if*

$$\nabla_{\mathbf{z}} f = 0.$$

In a similar way, the condition below is also a necessary and sufficient condition for f to have a stationary point

$$\nabla_{\overline{\mathbf{z}}} f = 0.$$

Proof. The proof follows the single variable case, Theorem 2.8, and, since $z_k = x_k + \mathrm{i}y_k$, we have:

$$f(\mathbf{z}) = f(x_1, y_1, \dots, x_k, y_k, \dots, x_n, y_n).$$

□

Here, we summarize the properties of the gradient operators $\nabla_{\mathbf{z}}$ and $\nabla_{\overline{\mathbf{z}}}$, we leave the proof for the reader.

Proposition 2.17. *Let $\mathbf{a} = (a_1, \dots, a_n)^T \in \mathbb{C}^n$ and R be in $\mathbb{C}^{n \times n}$ respectively, then the following properties hold true:*

(1) $\nabla_{\mathbf{z}}(\mathbf{a}^T \mathbf{z}) = \nabla_{\mathbf{z}}(\mathbf{z}^T \mathbf{a}) = \mathbf{a}$
(2) $\nabla_{\mathbf{z}}(\overline{\mathbf{z}}^T \mathbf{a}) = \nabla_{\mathbf{z}}(\mathbf{a}^T \overline{\mathbf{z}}) = \mathbf{0}$
(3) $\nabla_{\mathbf{z}}(\overline{\mathbf{z}}^T R\mathbf{z}) = R\,\overline{\mathbf{z}}$
(4) $\nabla_{\mathbf{z}}(\overline{\mathbf{z}}^T \overline{\mathbf{z}}) = \mathbf{0}$
(5) $\nabla_{\mathbf{z}}(\mathbf{z}^T \mathbf{z}) = 2\mathbf{z}$
(6) $\nabla_{\overline{\mathbf{z}}}(\mathbf{a}^T \mathbf{z}) = \nabla_{\overline{\mathbf{z}}}(\mathbf{z}^T \mathbf{a}) = \mathbf{0}$
(7) $\nabla_{\overline{\mathbf{z}}}(\overline{\mathbf{z}}^T \mathbf{a}) = \nabla_{\overline{\mathbf{z}}}(\mathbf{a}^T \overline{\mathbf{z}}) = \mathbf{a}$
(8) $\nabla_{\overline{\mathbf{z}}}(\overline{\mathbf{z}}^T R\mathbf{z}) = R\,\mathbf{z}$.

The complex LMS algorithm is an important application of the complex $\mathbb{C} - \mathbb{R}$-calculus as follows. The authors of [11, 37] extend to the complex case the well-known real least mean square (LMS) algorithm which was originally introduced in the real case by Widrow and Hof,f see [36]. In fact, at discrete integer time t, let us consider the output given by

$$s_t = \mathbf{z}_t^T \mathbf{w}_t = \mathbf{w}_t^T \mathbf{z}_t,$$

with $\mathbf{z}_t = (z_{t,1}, \dots, z_{t,n})^T$ and $\mathbf{w}_t = (w_{t,1}, \dots, w_{t,n})^T$. Thus, we have

$$s_t = \sum_{l=1}^{n} z_{t,l} w_{t,l}.$$

We denote by d_t the desired response then the complex signal error is given by

$$e_t = d_t - s_t = d_t - \mathbf{w}_t^T \mathbf{z}_t.$$

Let us consider the complex LMS rule considered in Brandwood

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \mu \nabla_{\overline{\mathbf{w}_t}}(e_t \overline{e_t}), \tag{2.8}$$

where $\mu > 0$ and

$$\nabla_{\overline{\mathbf{w}_t}} := \left(\partial_{\overline{w_{t,1}}}, \dots, \partial_{\overline{w_{t,n}}}\right)^T.$$

Theorem 2.18. *It holds that*

$$\mathbf{w}_{t+1} = \mathbf{w}_t + \mu e_t \overline{\mathbf{z}_t}. \tag{2.9}$$

Proof. As in, [36], we have

$$\nabla_{\overline{w_t}}(e_t \overline{e_t}) = e_t \nabla_{\overline{w_t}}(\overline{e_t}) + (\nabla_{\overline{w_t}} e_t)\overline{e_t}, \tag{2.10}$$

and the proof follows. □

Remark 2.19. The learning rule given by formula (2.9) is the complex LMS algorithm proposed by Widrow and his collaborators in [11, 37].

In Section 4, we will use similar arguments of complex gradient operators and apply them to the study of bicomplex gradient ones. In particular, we write two LMS algorithms in the bicomplex settings.

3 | INTRODUCTION TO BICOMPLEX NUMBERS

The algebra of bicomplex numbers was first introduced by Segre in [30]. During the past decades, a few isolated works analyzed either the properties of bicomplex numbers, or the properties of holomorphic functions defined on bicomplex numbers, and, without pretense of completeness, we direct the attention of the reader first to the book of Price [27], where a full foundation of the theory of multicomplex numbers was given, then to some of the works describing some analytic properties of functions in the field [8, 12, 13, 33]. Applications of bicomplex (and other hypercomplex) numbers can be also found in the works of Alfsmann et al. [1, 2].

We now introduce, in the same fashion as in [12, 22, 27], the key definitions and results for the case of bicomplex holomorphic functions of bicomplex variables. The algebra of bicomplex numbers is generated by two commuting imaginary units $\mathbf{i}$ and $\mathbf{j}$ and we will denote the bicomplex space by $\mathbb{BC}$. The product of the two commuting units $\mathbf{i}$ and $\mathbf{j}$ is denoted by $\mathbf{k} := \mathbf{ij}$ and we note that $\mathbf{k}$ is a hyperbolic unit, that is, it is a unit which squares to 1. Because of these various units in $\mathbb{BC}$, there are several different conjugations that can be defined naturally. We will make use of appropriate conjugations in this paper, and we refer the reader to [22, 33] for more information on bicomplex and multicomplex analysis.

3.1 | Properties of the bicomplex algebra

The bicomplex space, $\mathbb{BC}$, is not a division algebra, and it has two distinguished zero divisors, $\mathbf{e}_1$ and $\mathbf{e}_2$, which are idempotent, linearly independent over the reals, and mutually annihilating with respect to the bicomplex multiplication:

$$\mathbf{e}_1 := \frac{1+\mathbf{k}}{2}, \qquad \mathbf{e}_2 := \frac{1-\mathbf{k}}{2},$$
$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \qquad \mathbf{e}_1^2 = \mathbf{e}_1, \qquad \mathbf{e}_2^2 = \mathbf{e}_2,$$
$$\mathbf{e}_1 + \mathbf{e}_2 = 1, \qquad \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{k}.$$

Just like $\{1, \mathbf{j}\}$, these form a basis of the complex algebra $\mathbb{BC}$, which is called the *idempotent basis*. If we define the following complex variables in $\mathbb{C}(\mathbf{i})$:

$$\beta_1 := z_1 - \mathbf{i}z_2, \qquad \beta_2 := z_1 + \mathbf{i}z_2,$$

the $\mathbb{C}(\mathbf{i})$–*idempotent representation* for $Z = z_1 + \mathbf{j}z_2$ is given by

$$Z = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2.$$

The $\mathbb{C}(\mathbf{i})$–idempotent is the only representation for which multiplication can be taken component-wise, as shown in the next lemma.

Remark 3.1. The addition and multiplication of bicomplex numbers can be realized component-wise in the idempotent representation above. Specifically, if $Z = a_1\,\mathbf{e}_2 + a_2\,\mathbf{e}_2$ and $W = b_1\,\mathbf{e}_1 + b_2\,\mathbf{e}_2$ are two bicomplex numbers, where $a_1, a_2, b_1, b_2 \in \mathbb{C}(\mathbf{i})$, then

$$Z + W = (a_1 + b_1)\,\mathbf{e}_1 + (a_2 + b_2)\,\mathbf{e}_2,$$
$$Z \cdot W = (a_1 b_1)\,\mathbf{e}_1 + (a_2 b_2)\,\mathbf{e}_2,$$
$$Z^n = a_1^n\,\mathbf{e}_1 + a_2^n\,\mathbf{e}_2.$$

Moreover, the inverse of a bicomplex number $Z = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ is defined when $a_1 \cdot a_2 \neq 0$ and is given by

$$Z^{-1} = a_1^{-1}\mathbf{e}_1 + a_2^{-1}\,\mathbf{e}_2,$$

where a_1^{-1} and a_2^{-1} are the complex multiplicative inverses of a_1 and a_2, respectively.

One can see this also by computing directly which product on the bicomplex numbers of the form

$$x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4, \qquad x_1, x_2, x_3, x_4 \in \mathbb{R}$$

is component wise, and one finds that the only one with this property is given by the mapping:

$$x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \mapsto ((x_1 + x_4) + \mathbf{i}(x_2 - x_3), (x_1 - x_4) + \mathbf{i}(x_2 + x_3)), \tag{3.11}$$

which corresponds exactly with the idempotent decomposition

$$Z = z_1 + \mathbf{j}z_2 = (z_1 - \mathbf{i}z_2)\mathbf{e}_1 + (z_1 + \mathbf{i}z_2)\mathbf{e}_2,$$

where $z_1 = x_1 + \mathbf{i}x_2$ and $z_2 = x_3 + \mathbf{i}x_4$.

Remark 3.2. These split the bicomplex space in $\mathbb{BC} = \mathbb{C}\mathbf{e}_1 \bigoplus \mathbb{C}\mathbf{e}_2$, as:

$$Z = z_1 + \mathbf{j}z_2 = (z_1 - \mathbf{i}z_2)\mathbf{e}_1 + (z_1 + \mathbf{i}z_2)\mathbf{e}_2 = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2. \tag{3.12}$$

Simple algebra yields:

$$\begin{aligned} z_1 &= \frac{\lambda_1 + \lambda_2}{2}, \\ z_2 &= \frac{\mathbf{i}(\lambda_1 - \lambda_2)}{2}. \end{aligned} \tag{3.13}$$

Because of these various units in $\mathbb{BC}$, there are several different conjugations that can be defined naturally and we will now define the conjugates in the bicomplex setting, as in [12, 22]

Definition 3.3. For any $Z \in \mathbb{BC}$ we have the following three conjugates:

$$\overline{Z} = \overline{z_1} + \mathbf{j}\overline{z_2},$$

$$Z^\dagger = z_1 - \mathbf{j}z_2,$$

$$Z^* = \overline{Z^\dagger} = \overline{z_1} - \mathbf{j}\overline{z_2}.$$

Remark 3.4. Moreover, following Definition 3.3, if we write $Z = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ in the idempotent representation,we have

$$Z = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2,$$

$$Z^* = \overline{\lambda_1}\mathbf{e}_1 + \overline{\lambda_2}\mathbf{e}_2,$$

$$Z^\dagger = \lambda_2\mathbf{e}_1 + \lambda_1\mathbf{e}_2,$$

$$\overline{Z} = \overline{\lambda_2}\mathbf{e}_1 + \overline{\lambda_1}\mathbf{e}_2.$$

We refer the reader to [22] for more details.

3.2 | Hyperbolic subalgebra and the hyperbolic-valued modulus

A special subalgebra of $\mathbb{BC}$ is the set of hyperbolic numbers, denoted by $\mathbb{D}$. This algebra and the analysis of hyperbolic numbers have been studied, for example, in [8, 22, 32] and we summarize below only the notions relevant for our results. A *hyperbolic number* can be defined independently of $\mathbb{BC}$, by $\mathfrak{z} = x + \mathbf{k}y$, with $x, y, \in \mathbb{R}$, $\mathbf{k} \notin \mathbb{R}$, $\mathbf{k}^2 = 1$, and we denote by $\mathbb{D}$ the algebra of hyperbolic numbers with the usual component–wise addition and multiplication. The hyperbolic *conjugate* of $\mathfrak{z}$ is defined by $\mathfrak{z}^\diamond := x - \mathbf{k}y$, and note that:

$$\mathfrak{z} \cdot \mathfrak{z}^\diamond = x^2 - y^2 \in \mathbb{R}, \tag{3.14}$$

which yields the notion of the square of the *modulus* of a hyperbolic number $\mathfrak{z}$, defined by $|\mathfrak{z}|^2_{\mathbb{D}} := \mathfrak{z} \cdot \mathfrak{z}^\diamond$.

Remark 3.5. It is worth noting that both $\overline{Z}$ and $Z^\dagger$ reduce to $\mathfrak{z}^\diamond$ when $Z = \mathfrak{z}$. In particular, $\mathbf{e}_2 = \mathbf{e}_1^\diamond = \mathbf{e}_1^* = \mathbf{e}_1^\dagger$.

Similar to the bicomplex case, hyperbolic numbers have a unique idempotent representation with real coefficients:

$$\mathfrak{z} = s\mathbf{e}_1 + t\mathbf{e}_2, \tag{3.15}$$

where, just as in the bicomplex case, $\mathbf{e}_1 = \frac{1}{2}(1+\mathbf{k})$, $\mathbf{e}_2 = \frac{1}{2}(1-\mathbf{k})$, and $s := x+y$ and $t := x-y$. Note that $\mathbf{e}_1^\diamond = \mathbf{e}_2$ if we consider $\mathbb{D}$ as a subset of $\mathbb{BC}$, as briefly explained in the remark above. We also observe that

$$|\mathfrak{z}|_{\mathbb{D}}^2 = x^2 - y^2 = (x+y)(x-y) = st.$$

The hyperbolic algebra $\mathbb{D}$ is a subalgebra of the bicomplex numbers $\mathbb{BC}$ (see [22] for details). Actually $\mathbb{BC}$ is the algebraic closure of $\mathbb{D}$, and it can also be seen as the complexification of $\mathbb{D}$ by using either of the imaginary unit $\mathbf{i}$ or the unit $\mathbf{j}$.

Definition 3.6. Define the set $\mathbb{D}^+$ of non-negative hyperbolic numbers by:

$$\mathbb{D}^+ = \left\{x + \mathbf{k}y \,\middle|\, x^2 - y^2 \geq 0, x \geq 0\right\} = \left\{x + \mathbf{k}y \,\middle|\, x \geq 0, |y| \leq x\right\}$$
$$= \{s\mathbf{e}_1 + t\mathbf{e}_2 \,|\, s, t \geq 0\}.$$

Remark 3.7. As studied extensively in [8], one can define a partial order relation defined on $\mathbb{D}$ by:

$$\mathfrak{z}_1 \preceq \mathfrak{z}_2 \qquad \text{if and only if} \qquad \mathfrak{z}_2 - \mathfrak{z}_1 \in \mathbb{D}^+, \tag{3.16}$$

and we will use this partial order to study the *hyperbolic-valued* norm, which was first introduced and studied in [8].

The Euclidean norm $\|Z\|$ on $\mathbb{BC}$, when it is seen as $\mathbb{C}^2(\mathbf{i})$, $\mathbb{C}^2(\mathbf{j})$ or $\mathbb{R}^4$ is:

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\operatorname{Re}\left(|Z|_{\mathbf{k}}^2\right)} = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2}.$$

As studied in detail in [22], in idempotent coordinates $Z = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$, the Euclidean norm becomes:

$$\|Z\| = \frac{1}{\sqrt{2}}\sqrt{|\lambda_1|^2 + |\lambda_2|^2}. \tag{3.17}$$

It is easy to prove that

$$\|Z \cdot W\| \leq \sqrt{2}(\|Z\| \cdot \|W\|), \tag{3.18}$$

and we note that this inequality is sharp since if $Z = W = \mathbf{e}_1$, one has:

$$\|\mathbf{e}_1 \cdot \mathbf{e}_1\| = \|\mathbf{e}_1\| = \frac{1}{\sqrt{2}} = \sqrt{2}\,\|\mathbf{e}_1\| \cdot \|\mathbf{e}_1\|,$$

and similarly for $\mathbf{e}_2$.

Definition 3.8. One can define a hyperbolic-valued norm for $Z = z_1 + \mathbf{j}z_2 = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ by:

$$\|Z\|_{\mathbb{D}_+} := |\lambda_1|\mathbf{e}_1 + |\lambda_1|\mathbf{e}_2 \in \mathbb{D}^+.$$

It is shown in [8] that this definition obeys the corresponding properties of a norm, that is, $\|Z\|_{\mathbb{D}_+} = 0$ if and only if $Z = 0$, it is multiplicative, and it respects the triangle inequality with respect to the order introduced above.

3.3 | Hyperbolic-valued modulus of vectors in $\mathbb{BC}$

The previous norm can be generalized to the space of $\mathbb{BC}$ vectors, that is, elements of $\mathbb{BC}^n$, and we will also define an inner product on the space of vectors in $\mathbb{BC}$. Let $\langle X, Y\rangle$ be the usual Hermitian inner product on $\mathbb{C}^n$, then we have the following:

Definition 3.9. For any $\mathsf{X}, \mathsf{Y} \in \mathbb{BC}^n$, we have the following $\mathbb{D}$-valued inner product

$$\langle \mathsf{X}, \mathsf{Y}\rangle_{\mathbb{D}} = \langle X_1, Y_1\rangle \mathbf{e}_1 + \langle X_2, Y_2\rangle \mathbf{e}_2, \tag{3.19}$$

where $\mathsf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2$ and $\mathsf{Y} = Y_1\mathbf{e}_1 + Y_2\mathbf{e}_2$, and $X_l, Y_l \in \mathbb{C}^n$ for $l = 1, 2$.

This inner product yields the hyperbolic-valued modulus of a vector $\mathsf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2$ as:

$$\|\mathsf{X}\|_{\mathbb{D}_+} = \|X_1\|\mathbf{e}_1 + \|X_2\|\mathbf{e}_2.$$

3.3.1 | Finsler-type norm

Another real-valued norm that can be useful in this setting is the one found by multiplying all conjugates of a bicomplex number. This norm has been first considered in [34] and let us recall that it is a fourth-order Finsler-type norm defined by

$$|Z|^4_{\mathcal{F}} := Z\overline{Z}Z^*Z^\dagger.$$

For $Z = z_1 + \mathbf{j}z_2 = \varphi_1 + \mathbf{i}\varphi_2$, with $z_1, z_2 \in \mathbb{C}_{\mathbf{i}}$ and $\varphi_1, \varphi_2 \in \mathbb{C}_{\mathbf{j}}$, recall that:

$$\overline{Z} := \overline{z_1} + \mathbf{j}\overline{z_2} = \varphi_1 - \mathbf{i}\varphi_2, \qquad Z^\dagger := z_1 - \mathbf{j}z_2 = \overline{\varphi_1} - \mathbf{i}\overline{\varphi_2},$$

$$Z^* := \left(\overline{Z}\right)^\dagger = \overline{\left(Z^\dagger\right)} = \overline{z_1} - \mathbf{j}\overline{z_2} = \overline{\varphi_1} + \mathbf{i}\overline{\varphi_2} = \mathfrak{z}_1 - \mathbf{i}\,\mathfrak{z}_2 = \mathfrak{w}_1 - \mathbf{j}\,\mathfrak{w}_2$$

so

$$|Z|^4_{\mathcal{F}} = Z\overline{Z}Z^*Z^\dagger = (z_1 + \mathbf{j}z_2)(\overline{z_1} + \mathbf{j}\overline{z_2})(\overline{z_1} - \mathbf{j}\overline{z_2})(z_1 - \mathbf{j}z_2) = (z_1^2 + z_2^2)(\overline{z_1}^2 + \overline{z_2}^2).$$

The corresponding (square of the) bicomplex moduli are defined as:

$$|Z|^2_{\mathbf{i}} := Z \cdot Z^\dagger = z_1^2 + z_2^2 \in \mathbb{C}_{\mathbf{i}}$$

$$|Z|^2_{\mathbf{j}} := Z \cdot \overline{Z} = \varphi_1^2 + \varphi_2^2 \in \mathbb{C}_{\mathbf{j}}$$

$$|Z|^2_{\mathbf{k}} := Z \cdot Z^* = \mathfrak{z}_1^2 + \mathfrak{z}_2^2 = \mathfrak{w}_1^2 + \mathfrak{w}_2^2 \in \mathbb{D}.$$

so

$$|Z|^4_{\mathcal{F}} = |Z|^2_{\mathbf{i}} \cdot |\overline{Z}|^2_{\mathbf{i}} = \left||Z|^2_{\mathbf{j}}\right|^2_{\mathbf{i}} = ||Z|^2_{\mathbf{i}}|^2_{\mathbf{j}} = ||Z|^2_{\mathbf{k}}|^2_{\mathbf{i}}.$$

In this case, we will obtain a modulus of order 4 as $\mathcal{F}(Z)$ is a real number. This number is positive when Z is not a divisor of 0 and it is equal to 0 when $Z \in \mathfrak{S}$, that is, it is a complex multiple of $\mathbf{e}$ or $\mathbf{e}^\dagger$.

From Remark 3.4, we have:

$$\overline{Z} = \overline{\lambda_2}\mathbf{e}_1 + \overline{\lambda_1}\mathbf{e}_2; \quad Z^* = \overline{\lambda_1}\mathbf{e}_1 + \overline{\lambda_2}\mathbf{e}_2; \quad Z^\dagger = \lambda_2\mathbf{e}_1 + \lambda_1\mathbf{e}_2,$$

which brings the following form for the three bicomplex moduli of order 2 and the modulus of order 4 to:

Lemma 3.10. *In the idempotent representation, we have:*

$$|Z|^2_{\mathbf{i}} = \lambda_1\lambda_2; \quad |Z|^2_{\mathbf{j}} = \lambda_1\overline{\lambda_2}\mathbf{e}_1 + \overline{\lambda_1}\lambda_2\mathbf{e}_2; \quad |Z|^2_{\mathbf{k}} = |\lambda_1|^2\mathbf{e}_1 + |\lambda_2|^2\mathbf{e}_2; \quad \mathcal{F}(Z) = |\lambda_1|^2 \cdot |\lambda_2|^2.$$

For proofs and more details about these moduli, one can see [29, 34].

4 | $\mathbb{BC} - \mathbb{R}$ CALCULUS AND BICOMPLEX GRADIENT OPERATORS

In this section, we extend the study of $\mathbb{C} - \mathbb{R}$ (or Wirtinger) calculus to the bicomplex setting, we define the bicomplex real ($\mathbb{BC} - \mathbb{R}$) calculus, and investigate the notion of bicomplex gradient operators. Applications of $\mathbb{BC} - \mathbb{R}$ calculus are found in Section 5.

4.1 | Bicomplex differential operators

We first define the bicomplex differential operators arising from the rich underlying structure of the space.

In the case of the bicomplex variable, $Z = z_1 + \mathbf{j}z_2$, we have

$$z_1 = \frac{Z + Z^\dagger}{2}, \quad z_2 = \frac{\mathbf{j}}{2}(Z^\dagger - Z),$$

and

$$\overline{z_1} = \frac{\overline{Z} + Z^*}{2}, \quad \overline{z_2} = \frac{\mathbf{j}}{2}(Z^* - \overline{Z}).$$

On the other hand, for $z_1 = x_1 + \mathbf{i}x_2$ and $z_2 = x_3 + \mathbf{i}x_4$ we obtain

$$x_1 = \frac{Z + Z^\dagger + Z^* + \overline{Z}}{4}, \quad x_2 = \frac{\mathbf{i}}{4}(Z^* + \overline{Z} - Z - Z^\dagger),$$

and

$$x_3 = \frac{\mathbf{j}}{4}(Z^* + Z^\dagger - \overline{Z} - Z), \quad x_4 = \frac{\mathbf{k}}{4}(Z + Z^* - \overline{Z} - Z^\dagger).$$

The bicomplex differential operators with respect to the various bicomplex conjugates (see [8, 12]) are

$$\partial_Z := \partial_{z_1} - \mathbf{j}\partial_{z_2} = \partial_{x_1} - \mathbf{i}\partial x_2 - \mathbf{j}\partial_{x_3} + \mathbf{k}\partial x_4 = \partial_{\lambda_1}\mathbf{e}_1 + \partial_{\lambda_2}\mathbf{e}_2\,,$$

$$\partial_{\overline{Z}} := \partial_{\overline{z_1}} - \mathbf{j}\partial_{\overline{z_2}} = \partial_{x_1} + \mathbf{i}\partial x_2 - \mathbf{j}\partial_{x_3} - \mathbf{k}\partial x_4 = \partial_{\overline{\lambda_2}}\mathbf{e}_1 + \partial_{\overline{\lambda_1}}\mathbf{e}_2\,,$$

$$\partial_{Z^*} := \partial_{\overline{z_1}} + \mathbf{j}\partial_{\overline{z_2}} = \partial_{x_1} + \mathbf{i}\partial x_2 + \mathbf{j}\partial_{x_3} + \mathbf{k}\partial x_4 = \partial_{\overline{\lambda_1}}\mathbf{e}_1 + \partial_{\overline{\lambda_2}}\mathbf{e}_2\,,$$

$$\partial_{Z^\dagger} := \partial_{z_1} + \mathbf{j}\partial_{z_2} = \partial_{x_1} - \mathbf{i}\partial x_2 + \mathbf{j}\partial_{x_3} - \mathbf{k}\partial x_4 = \partial_{\lambda_2}\mathbf{e}_1 + \partial_{\lambda_1}\mathbf{e}_2\,.$$

We recall the definition of bicomplex analyticity, which is a more recent concept in hypercomplex analysis, therefore worth re-writing here.

Definition 4.1. Let Ω be a domain in $\mathbb{BC}$, we say that a function $F : \Omega \longrightarrow \mathbb{BC}$ is bicomplex holomorphic if and only if F is in the kernel of the last three differential operators described above, that is, $\partial_{\overline{Z}}, \partial_{Z^*}, \partial_{Z^\dagger}$.

We invite the reader to the following references [8, 12, 22, 27] for more details. Following these works, we can see that a bicomplex holomorphic function will admit a convergent power series representation at each point in Ω. We point out that in Section 7.6 from [22], we can see that a function is bicomplex analytic if and only if it can be split in the idempotent representation in two functions each depending on a single idempotent variable.

As in [12], the concept of bicomplex holomorphy can be extended to several bicomplex variables and we have:

Definition 4.2. Let Ω be a domain in $\mathbb{BC}^n$, we say that a function $F : \Omega \longrightarrow \mathbb{BC}$ is a bicomplex holomorphic function of several bicomplex variables if and only if F is in the kernel of the family of the $3n$ differential operators (three for each variable) $\partial_{\overline{Z_l}}, \partial_{Z_l^*}, \partial_{Z_l^\dagger}$, for any $1 \leq l \leq n$, where the n bicomplex vector variable is denoted by $Z = (Z_1, \ldots, Z_n)$.

We will use this last definition throughout our paper and for more details on the theory of functions of several bicomplex variables we refer the reader to [12] .

4.2 | Notions of $\mathbb{BC} - \mathbb{R}$ calculus

In this subsection, we define the counterpart of $\mathbb{C} - \mathbb{R}$-calculus (or Wirtinger) in the bicomplex space. We start by introducing the notion of $\mathbb{BC} - \mathbb{R}$ analytic functions and discuss some of their main properties. In what follows, we use the notion of open bicomplex domains, namely domains of the form $\Omega_1 \mathbf{e}_1 + \Omega_2 \mathbf{e}_2$, where Ω_1, Ω_2 are complex domains.

Definition 4.3. Let $\Omega \subset \mathbb{BC}$ be a bicomplex domain, symmetric with respect to all conjugations, that is, if $Z \in \Omega$ then $\overline{Z}, Z^*, Z^\dagger \in \Omega$. We say that a function $f : \Omega \longrightarrow \mathbb{BC}$ is $\mathbb{BC} - \mathbb{R}$ analytic (or $\mathbb{BC} - \mathbb{R}$ regular) on Ω if and only if there exists a function

$$g : \Omega^4 \longrightarrow \mathbb{BC}, (Z_1, Z_2, Z_3, Z_4) \mapsto g(Z_1, Z_2, Z_3, Z_4),$$

which is $\mathbb{BC}$-analytic in each of the four bicomplex variables such that

i) $f(Z) = g(Z, \overline{Z}, Z^*, Z^\dagger), \quad \forall Z \in \Omega,$

ii) $\frac{\partial f}{\partial Z} = \left(\frac{\partial g}{\partial Z_1}\right)_{Z_1=Z, Z_2=\overline{Z}, Z_3=Z^*\, Z_4=Z^\dagger},$

iii) $\frac{\partial f}{\partial \overline{Z}} = \left(\frac{\partial g}{\partial Z_2}\right)_{Z_1=Z, Z_2=\overline{Z}, Z_3=Z^*\, Z_4=Z^\dagger},$

iv) $\frac{\partial f}{\partial Z^*} = \left(\frac{\partial g}{\partial Z_3}\right)_{Z_1=Z, Z_2=\overline{Z}, Z_3=Z^*\, Z_4=Z^\dagger},$

v) $\frac{\partial f}{\partial Z^\dagger} = \left(\frac{\partial g}{\partial Z_4}\right)_{Z_1=Z, Z_2=\overline{Z}, Z_3=Z^*\, Z_4=Z^\dagger}.$

Remark 4.4. Just as the complex–real calculus generalizes notions of analytic complex functions, the bicomplex–real calculus generalizes notions of bicomplex analyticity and it is easy to see that a bicomplex holomorphic function will be $\mathbb{BC} - \mathbb{R}$ analytic. This type of analyticity generalizes this concept of holomorphy.

Here are some examples of $\mathbb{BC} - \mathbb{R}$ regular functions which are not bicomplex holomorphic.

Example 4.5. The Finsler-type norm defined by

$$f(Z) = |Z|^4_{\mathcal{F}} := Z\bar{Z}Z^*Z^\dagger$$

is an entire $\mathbb{BC} - \mathbb{R}$ function on $\mathbb{BC}$, with

$$g(Z_1, Z_2, Z_3, Z_4) = Z_1 Z_2 Z_3 Z_4.$$

We have

$$\frac{\partial f}{\partial Z} = \bar{Z}Z^*Z^\dagger, \quad \frac{\partial f}{\partial \overline{Z}} = ZZ^*Z^\dagger,$$

and

$$\frac{\partial f}{\partial Z^*} = Z\bar{Z}Z^\dagger, \quad \frac{\partial f}{\partial Z^\dagger} = Z\bar{Z}Z^*.$$

Proposition 4.6. *Let $f, h : \mathbb{BC} \longrightarrow \mathbb{BC}$ be two $\mathbb{BC} - \mathbb{R}$ analytic functions and $\lambda \in \mathbb{BC}$. Then, the sum $f + h$ and multiplication $f\lambda$ are also $\mathbb{BC} - \mathbb{R}$ analytic.*

Proof. Follows standard arguments. □

Remark 4.7. For $\Omega = \mathbb{BC}$ in Definition 4.3, we call such functions $\mathbb{BC} - \mathbb{R}$ *entire* and the set of all $\mathbb{BC} - \mathbb{R}$ entire functions is a vector space over $\mathbb{BC}$ which is denoted by $\mathcal{H}_{CR}(\mathbb{BC})$.

We note that bicomplex polyanalytic functions of finite order were considered in [14] using finite sums involving different bicomplex conjugates.

Definition 4.8. A bicomplex-valued function $f : \mathbb{BC} \longrightarrow \mathbb{BC}$, $f \in C^\infty(\mathbb{BC})$ (i.e. smooth in each of its real components), which satisfies

$$\frac{\partial^l f}{\partial \overline{Z}^l} = \frac{\partial^k f}{\partial (Z^*)^k} = \frac{\partial^q f}{\partial (Z^\dagger)^q} = 0,$$

where l, k, and q are strictly positive integers, is called a bicomplex polyanalytic function of finite multi-order (l, k, q).

Remark 4.9. In the case $l = k = q = 1$, one obtains the special case of bicomplex holomorphic functions, which become bicomplex polyanalytic functions of multi-order (1,1,1).

For example, in [14], Proposition 3.8 shows that a function is as in Definition 4.8 if and only if

$$g(Z) := \sum_{m=0}^{l-1} \sum_{n=0}^{k-1} \sum_{p=0}^{q-1} g_{m,n,p}(Z) \overline{Z}^m (Z^*)^n (Z^\dagger)^p,$$

where $g_{m,n,p}$ are bicomplex holomorphic functions.

We introduce the class of global bicomplex polyanalytic functions of infinite order as follows.

Definition 4.10. A bicomplex-valued function $f : \mathbb{BC} \longrightarrow \mathbb{BC}$ is called a global bicomplex polyanalytic of infinite order if it can be represented as a power series with respect to the variables $Z, \overline{Z}, Z^*, Z^\dagger$ so that we have

$$f(Z) = \sum_{m,n,p,q=0}^{\infty} a_{m,n,p,q} Z^m \overline{Z}^n (Z^*)^p (Z^\dagger)^q; \tag{4.20}$$

where $(a_{m,n,p,q})_{m,n,p,q \geq 0}$ are suitable bicomplex coefficients, non-zero for an infinite number of indices n, p, or q.

Theorem 4.11. *A bicomplex-valued function $f : \mathbb{BC} \longrightarrow \mathbb{BC}$ is global polyanalytic of finite or infinite order if and only if it is a $\mathbb{BC} - \mathbb{R}$ analytic function.*

Proof. From the definition of $\mathbb{BC} - \mathbb{R}$ analyticity, since g is analytic in the four variables, we have:

$$g(Z_1, Z_2, Z_3, Z_4) := \sum_{m,n,p,q=0}^{\infty} a_{m,n,p,q} Z_1^m Z_2^n Z_3^p Z_4^q.$$

□

These global polyanalytic types of functions can take many forms, depending on whether they are in the kernels of the operators $\frac{\partial}{\partial \overline{Z}}, \frac{\partial}{\partial Z^*}, \frac{\partial^q f}{\partial Z^\dagger}$ or their powers, yielding power series in terms of specific conjugates, or exhibiting finite-order polyanalytic-type behavior, respectively.

When we consider the extension of the complex Laplacian to the bicomplex setting, there exists more than one form, depending on the choice of bicomplex conjugate. For example, one can define three of these Laplacians as follows.

Definition 4.12. We define the Laplacians with respect to **i**, **j** and **k** as follows

$$\Delta_{\mathbf{i}} = \frac{\partial^2}{\partial Z \partial \overline{Z}}; \tag{4.21}$$

$$\Delta_{\mathbf{j}} := \frac{\partial^2}{\partial Z \partial Z^{\dagger}}; \tag{4.22}$$

$$\Delta_{\mathbf{k}} = \frac{\partial^2}{\partial Z \partial Z^{*}}. \tag{4.23}$$

The following example relates to the Laplacian given by Z^* and we have a result concerning the logarithm of a bicomplex function $f(Z)$, denoted by $\mathsf{Ln}(f(Z))$, which extends a complex result proven in Exercise 4.2.23 from [3], also found in [15], to the bicomplex setting:

Proposition 4.13. *Let h be a $\mathbb{BC}$ holomorphic function, we consider the function defined by*

$$f(Z) = ||h(Z)||_{\mathbf{k}}^2 g(Z).$$

We denote by $\Delta_{\mathbf{k}}$ the bicomplex Lalplacian given by

$$\Delta_{\mathbf{k}} := \frac{\partial^2}{\partial Z \partial Z^{*}}.$$

Then, it holds that

$$\Delta_{\mathbf{k}} \mathsf{Ln} f(Z) = \Delta_{\mathbf{k}} \mathsf{Ln} g(Z). \tag{4.24}$$

Proof. We observe that for $Z = z_1 + z_2\mathbf{j} = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ we have

$$h(Z) = h_1(Z)\mathbf{e}_1 + h_2(Z)\mathbf{e}_2, \quad g(Z) = g_1(\lambda_1)\mathbf{e}_1 + g_2(\lambda_2)\mathbf{e}_2,$$

and

$$\Delta_{\mathbf{k}} = \Delta_{\lambda_1}\mathbf{e}_1 + \Delta_{\lambda_2}\mathbf{e}_2.$$

Thus, using the hyperbolic norm expression we have

$$||h(Z)||_{\mathbf{k}}^2 = |h_1(Z)|^2\mathbf{e}_1 + |h_2(Z)|^2\mathbf{e}_2.$$

Hence, applying the bicomplex multiplication rules we get

$$\begin{aligned} f(Z) &= ||h(Z)||_{\mathbf{k}}^2 g(Z) \\ &= |h_1(\lambda_1)|^2 g_1(\lambda_1)\mathbf{e}_1 + |h_2(\lambda_2)|^2 g_2(\lambda_2)\mathbf{e}_1 \end{aligned}$$

It follows that

$$\Delta_{\mathbf{k}} \mathsf{Ln} f(Z) = \Delta_{\lambda_1} \log(|h_1(\lambda_1)|^2 g_1(\lambda_1))\mathbf{e}_1 + \Delta_{\lambda_2} \log(|h_2(\lambda_2)|^2 g_2(\lambda_2))\mathbf{e}_2 \tag{4.25}$$

Hence, applying the complex result we obtain

$$\begin{aligned} \Delta_{\mathbf{k}} \mathsf{Ln} f(Z) &= \Delta_{\lambda_1} \log(g_1(\lambda_1))\mathbf{e}_1 + \Delta_{\lambda_2} \log(g_2(\lambda_2))\mathbf{e}_2 \\ &= \Delta_{\mathbf{k}} \mathsf{Ln} g(Z). \end{aligned}$$

□

4.3 | A type of bicomplex Fock space

A useful type of polyanalyticity is the one defined by the $*$-conjugation and we have:

Definition 4.14. If f is in the kernel of $\frac{\partial}{\partial \overline{Z}}, \frac{\partial}{\partial Z^{\dagger}}$ and in the kernel of a power $n \geq 1$ of the $*$ operator $\frac{\partial}{\partial Z^{*}}$, that is,

$$\frac{\partial^n}{\partial {Z^*}^n} f(Z) = 0, \quad \forall Z \in \Omega \subset \mathbb{BC},$$

where $\Omega = \Omega_1 \mathbf{e}_1 + \Omega_2 \mathbf{e}_2$ is a product (or split) domain in $\mathbb{BC}$, then f is called a bicomplex $*$ polyanalytic function of order n on Ω.

The space of all bicomplex $*$ polyanalytic function of order n is denoted by $H_n^{split}(\Omega)$.

Just as in the case of complex polyanalytic function of order n can be decomposed in terms of n analytic functions so that we have a decomposition of the following form:

Theorem 4.15. *Let f be a bicomplex $*$ polyanalytic function of order n on $\Omega \subset \mathbb{BC}$ a split domain, then we have:*

$$f(Z) = \sum_{l=0}^{n-1} {Z^*}^l f_l(Z), \tag{4.26}$$

for which all f_l are bicomplex analytic functions on Ω. In particular, expanding each analytic component using the series expansion theorem leads to an expression of this form.

We can extend the definition of a polyanalytic Fock space in Section 2, to the bicomplex case as follows.

For $n = 1, 2, \ldots$ we recall that polyanalytic Fock spaces of order n can be defined as follows:

$$\mathcal{F}_n(\mathbb{BC}) := \left\{ g \in H_n^{split}(\mathbb{BC}), \quad g(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2) = g_1(\lambda_1)\mathbf{e}_1 + g_2(\lambda_2)\mathbf{e}_2,\ g_1, g_2 \in \mathcal{F}_n(\mathbb{C}) \right\},$$

and we have:

Proposition 4.16. *The reproducing kernel associated to the space $\mathcal{F}_n(\mathbb{BC})$ is given by*

$$\mathsf{K}_n(Z, W) = e^{ZW^*} \sum_{l=0}^{n-1} \frac{(-1)^l}{l!} \binom{n}{l+1} |Z - W|_{\mathbf{k}}^{2l}, \tag{4.27}$$

for every $Z, W \in \mathbb{BC}$.

Proof. We will show that for $Z = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2$ and $W = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2$ we have

$$\mathsf{K}_n(Z, W) = K_n(\lambda_1, \mu_1)\mathbf{e}_1 + K_n(\lambda_2, \mu_2)\mathbf{e}_2, \tag{4.28}$$

where $K_n(\lambda_1, \mu_1)$ and $K_n(\lambda_2, \mu_2)$ are given in (2.4). We have:

$$\mathsf{K}_n(Z, W) = e^{ZW^*} \sum_{l=0}^{n-1} \frac{(-1)^l}{l!} \binom{n}{l+1} |Z - W|_{\mathbf{k}}^{2l}$$

In the idempotent representation

$$e^{ZW^*} = e^{(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2)(\overline{\mu_1} \mathbf{e}_1 + \overline{\mu_2} \mathbf{e}_2)} = e^{\lambda_1 \overline{\mu_1}} \mathbf{e}_1 + e^{\lambda_2 \overline{\mu_2}} \mathbf{e}_2$$

and

$$|Z - W|_{\mathbf{k}}^2 = (Z - W)(Z^* - W^*)$$

$$= ((\lambda_1 - \mu_1)\mathbf{e}_1 + (\lambda_2 - \mu_2)\mathbf{e}_2)(\overline{(\lambda_1 - \mu_1)}\mathbf{e}_1 + \overline{(\lambda_2 - \mu_2)}\mathbf{e}_2)$$

which yields:

$$|Z - W|^2_{\mathbf{k}} = |\lambda_1 - \mu_1|^2\mathbf{e}_1 + |\lambda_2 - \mu_2|^2\mathbf{e}_2,$$

and, using Equation (2.4), the result follows. □

Another special case follows:

Definition 4.17. The bicomplex $*$-poly-Bergman space $\mathcal{A}_n(\mathbb{K})$ of polyanalytic functions of order n in the bicomplex product-type unit ball denoted here by $\mathbb{K} = B(0,1)\mathbf{e}_1 + B(0,1)\mathbf{e}_2$ is given by

$$\mathcal{A}_n(\mathbb{K}) := \left\{ g \in H_n^{split}(\mathbb{K}), \quad g(\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2) = g_1(\lambda_1)\mathbf{e}_1 + g_2(\lambda_2)\mathbf{e}_2,\ g_1, g_2 \in A^2_n(B(0,1)) \right\},$$

where

Remark 4.18. We note also that $\mathcal{A}_n(\mathbb{K})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$\mathsf{B}_n(Z,W) = \frac{n}{\pi(1 - W^*Z)^{2n}} \sum_{\ell=0}^{n-1} (-1)^{\ell} \binom{n}{\ell+1}\binom{n+\ell}{n} |1 - W^*Z|^{2(n-1-\ell)}_{\mathbf{k}} |Z - W|^{2\ell}_{\mathbf{k}} \tag{4.29}$$

for every $Z, W \in \mathbb{K}$.

So, as in the case of the bicomplex poly-Fock space we can show that for $Z = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ and $W = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ we have

$$\mathsf{B}_n(Z,W) = B_n(\lambda_1, \mu_1)\mathbf{e}_1 + B_n(\lambda_2, \mu_2)\mathbf{e}_2, \tag{4.30}$$

where $B_n(\lambda_1, \mu_1)$ and $B_n(\lambda_2, \mu_2)$ are given in Equation (2.5).

Remark 4.19. In the case $n = 1$, we recover the Bergman kernel of the bicomplex Bergman space, found in Corollary 6.2 of [26]. In this case, the functions will be bicomplex analytic.

In our work [7], we discuss important families of $\mathbb{BC} - \mathbb{R}$ analytic functions such as bicomplex Hermite polynomials and we show that they generate spaces of bicomplex functions with certain properties.

5 | MULTIVARIATE BICOMPLEX–REAL ($\mathbb{BC} - \mathbb{R}$) CALCULUS

In this section, we will consider spaces of vectors of bicomplex components and the gradient operators corresponding to different types of bicomplex conjugations.

5.1 | Bicomplex gradient operators

We first define the notion of multi-variate $\mathbb{BC} - \mathbb{R}$ analytic functions.

Definition 5.1. Let $\Omega_k \subset \mathbb{BC}$, $1 \leq k \leq n$ be n bicomplex domains, each symmetric with respect to all conjugations. We say that a function $f : \Pi_{k=1}^n \Omega_k \longrightarrow \mathbb{BC}$ is $\mathbb{BC} - \mathbb{R}$ analytic (or $\mathbb{BC} - \mathbb{R}$ regular) on $\Omega = \Pi_{k=1}^n \Omega_k$ if there exists a bicomplex analytic function of $4n$ bicomplex variables $g : \Pi_{k=1}^n \Omega_k^4 \longrightarrow \mathbb{BC}, (Z_1, W_1, X_1, Y_1, \dots, Z_n, W_n, X_n, Y_n) \mapsto g(Z_1, W_1, X_1, Y_1, \dots, Z_n, W_n, X_n, Y_n)$ such that

i) $f(Z_1, \dots, Z_n,) = g(Z_1, \overline{Z_1}, Z_1^*, Z_1^\dagger, \dots, Z_n, \overline{Z_n}, Z_n^*, Z_n^\dagger)$,
$\forall (Z_1, \dots, Z_n,) \in \Pi_{k=1}^n \Omega_k$,

ii) $\frac{\partial f}{\partial Z_l} = \left(\frac{\partial g}{\partial Z_l}\right)_{Z_l=Z_l, W_l=\overline{Z_l} X_l=Z_l^*, Y_l=Z_l^\dagger}$, for any $1 \le l \le n$,

iii) $\frac{\partial f}{\partial \overline{Z_l}} = \left(\frac{\partial g}{\partial W_l}\right)_{Z_l=Z_l, W_l=\overline{Z_l} X_l=Z_l^*, Y_l=Z_l^\dagger}$, for any $1 \le l \le n$,

iv) $\frac{\partial f}{\partial Z_l^*} = \left(\frac{\partial g}{\partial X_l}\right)_{Z_l=Z_l, W_l=\overline{Z_l} X_l=Z_l^*, Y_l=Z_l^\dagger}$, for any $1 \le l \le n$,

v) $\frac{\partial f}{\partial Z_l^\dagger} = \left(\frac{\partial g}{\partial Y_l}\right)_{Z_l=Z_l, W_l=\overline{Z_l} X_l=Z_l^*, Y_l=Z_l^\dagger}$, for any $1 \le l \le n$.

We can now define the four bicomplex gradient operators and prove some of their properties.

Definition 5.2 (Bicomplex gradient operator). Let $\mathsf{Z} = \mathbf{z_1} + \mathbf{j}\mathbf{z_2} = \Lambda_1\mathbf{e}_1 + \Lambda_2\mathbf{e}_2 \in \mathbb{BC}^n$, where $\mathsf{Z} = (Z_1, \dots, Z_n)^T$, with complex vector components $\mathbf{z}_1 = (z_{11}, \dots, z_{1n})^T$, $\mathbf{z}_2 = (z_{21}, \dots, z_{2n})^T$ and $\Lambda_1 = (\lambda_{11}, \dots, \lambda_{1n})^T$, $\Lambda_2 = (\lambda_{21}, \dots, \lambda_{2n})^T$ belong to $\mathbb{C}^n$. Then, we define the various bicomplex gradient operators with respect to the variables $\mathsf{Z}, \overline{\mathsf{Z}}$, and Z^* by:

i) Bicomplex gradient operator:

$$\nabla_{\mathsf{Z}} := \nabla_{\mathbf{z_1}} - \mathbf{j}\nabla_{\mathbf{z_2}} = \nabla_{\Lambda_1}\mathbf{e}_1 + \nabla_{\Lambda_2}\mathbf{e}_2;$$

ii) Bicomplex gradient-bar operator:

$$\nabla_{\overline{\mathsf{Z}}} := \nabla_{\overline{\mathbf{z_1}}} - \mathbf{j}\nabla_{\overline{\mathbf{z_2}}} = \nabla_{\overline{\Lambda_2}}\mathbf{e}_1 + \nabla_{\overline{\Lambda_1}}\mathbf{e}_2;$$

iii) Bicomplex gradient-$*$ operator:

$$\nabla_{\mathsf{Z}^*} := \nabla_{\overline{\mathbf{z_1}}} + \mathbf{j}\nabla_{\overline{\mathbf{z_2}}} = \nabla_{\overline{\Lambda_1}}\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}\mathbf{e}_2;$$

iv) Bicomplex gradient-$\dagger$ operator:

$$\nabla_{\mathsf{Z}^\dagger} := \nabla_{\mathbf{z_1}} + \mathbf{j}\nabla_{\mathbf{z_2}} = \nabla_{\Lambda_2}\mathbf{e}_1 + \nabla_{\Lambda_1}\mathbf{e}_2.$$

In order to study the LMS algorithm in this setting, we will use the bicomplex gradient operator ∇_{Z^*} and $\nabla_{\overline{\mathsf{Z}}}$. In particular, we prove various interesting properties for these differential operators:

Theorem 5.3 (Bicomplex-gradient Leibniz rules). *Let f and g be two bicomplex-valued BC-R analytic functions. Then, it holds that*

$$\nabla_{\mathsf{Z}}(fg) = f\nabla_{\mathsf{Z}}(g) + \nabla_{\mathsf{Z}}(f)g; \tag{5.31}$$

$$\nabla_{\overline{\mathsf{Z}}}(fg) = f\nabla_{\overline{\mathsf{Z}}}(g) + \nabla_{\overline{\mathsf{Z}}}(f)g; \tag{5.32}$$

$$\nabla_{\mathsf{Z}^*}(fg) = f\nabla_{\mathsf{Z}^*}(g) + \nabla_{\mathsf{Z}^*}(f)g; \tag{5.33}$$

and

$$\nabla_{\mathsf{Z}^\dagger}(fg) = f\nabla_{\mathsf{Z}^\dagger}(g) + \nabla_{\mathsf{Z}^\dagger}(f)g. \tag{5.34}$$

Proof. We will prove formula (5.33), with respect to the bicomplex gradient-$*$ operator. Indeed, we write $Z = \Lambda_1\mathbf{e}_1 + \Lambda_2\mathbf{e}_2$, $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ and $g = g_1\mathbf{e}_1 + g_2\mathbf{e}_2$. Then,

$$fg = f_1g_1\mathbf{e}_1 + f_2g_2\mathbf{e}_2.$$

So, using the gradient operator representation in terms of the bicomplex variables (Λ_1, Λ_2) we have

$$\begin{aligned}\nabla_{Z^*}(fg) &= (\nabla_{\overline{\Lambda_1}}\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}\mathbf{e}_2)(fg)\\ &= \nabla_{\overline{\Lambda_1}}(fg)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(fg)\mathbf{e}_2\end{aligned}$$

However, we note that

$$\nabla_{\overline{\Lambda_1}}(fg)\mathbf{e}_1 = \nabla_{\overline{\Lambda_1}}(f_1g_1)\mathbf{e}_1^2 + \nabla_{\overline{\Lambda_1}}(f_2g_2)\mathbf{e}_2 \cdot \mathbf{e}_1.$$

Thus, using the fact that $\mathbf{e}_1^2 = \mathbf{e}_1$ and $\mathbf{e}_2 \cdot \mathbf{e}_1 = 0$ we obtain

$$\nabla_{\overline{\Lambda_1}}(fg)\mathbf{e}_1 = \nabla_{\overline{\Lambda_1}}(f_1g_1)\mathbf{e}_1.$$

In a similar way, we have also

$$\nabla_{\overline{\Lambda_2}}(fg)\mathbf{e}_2 = \nabla_{\overline{\Lambda_2}}(f_2g_2)\mathbf{e}_2.$$

Moreover, by the classical complex case we already know that

$$\nabla_{\overline{\Lambda_1}}(f_1g_1)\mathbf{e}_2 = (f_1\nabla_{\overline{\Lambda_1}}(g_1) + \nabla_{\overline{\Lambda_1}}(f_1)g_1)\mathbf{e}_1,$$

and

$$\nabla_{\overline{\Lambda_2}}(f_2g_2)\mathbf{e}_2 = (f_2\nabla_{\overline{\Lambda_2}}(g_2) + \nabla_{\overline{\Lambda_2}}(f_2)g_2)\mathbf{e}_2.$$

In particular, this leads to the following:

$$\begin{aligned}\nabla_{Z^*}(fg) &= \nabla_{\overline{\Lambda_1}}(f_1g_1)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(f_2g_2)\mathbf{e}_2\\ &= (f_1\nabla_{\overline{\Lambda_1}}(g_1) + \nabla_{\overline{\Lambda_1}}(f_1)g_1)\mathbf{e}_1 + (f_2\nabla_{\overline{\Lambda_2}}(g_2) + \nabla_{\overline{\Lambda_2}}(f_2)g_2)\mathbf{e}_2\\ &= \left(f_1\nabla_{\overline{\Lambda_1}}(g_1)\mathbf{e}_1 + f_2\nabla_{\overline{\Lambda_2}}(g_2)\mathbf{e}_2\right) + \left(\nabla_{\overline{\Lambda_1}}(f_1)g_1\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(f_2)g_2\mathbf{e}_2\right)\\ &= (f_1\mathbf{e}_1 + f_2\mathbf{e}_2)\left(\nabla_{\overline{\Lambda_1}}(g_1)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(g_2)\mathbf{e}_2\right) + \left(\nabla_{\overline{\Lambda_1}}(f_1)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(f_2)\mathbf{e}_2\right)(g_1\mathbf{e}_1 + g_2\mathbf{e}_2)\end{aligned}$$

However, using the properties of $\mathbf{e}_1$ and $\mathbf{e}_2$ we can easily check the following computations:

$$\nabla_{Z^*}(g) = \nabla_{\overline{\Lambda_1}}(g_1)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(g_2)\mathbf{e}_2;$$

and

$$\nabla_{Z^*}(f) = \nabla_{\overline{\Lambda_1}}(f_1)\mathbf{e}_1 + \nabla_{\overline{\Lambda_2}}(f_2)\mathbf{e}_2.$$

Hence, it follows that

$$\nabla_{Z^*}(fg) = f\nabla_{Z^*}(g) + \nabla_{Z^*}(f)g.$$

This ends the proof. □

Remark 5.4. The computations for expressions (5.31) and (5.32) in the previous theorem follow similar arguments used in the proof of Equation (5.33).

We formulate a hypothesis proposal for the bicomplex stationary point conditions:

Lemma 5.5. *Let $f : \mathbb{BC}^n \longrightarrow \mathbb{R}$ be a real-valued function of a bicomplex variable Z such that $f(Z) = f(Z_1, \dots, Z_n) = g(Z_1, \overline{Z_1}, Z_1^*, Z_1^\dagger, \dots, Z_n, \overline{Z_n}, Z_n^*, Z_n^\dagger)$ with $g : \mathbb{BC}^{4n} \longrightarrow \mathbb{R}$ is a function of four variables which is analytic with respect to each variable (i.e. f is $\mathbb{BC} - \mathbb{R}$ entire on $\mathbb{BC}^n$). Then, f has a stationary point if and only if one of the following equivalent conditions is satisfied:*

i) $\nabla_Z g = 0$;
ii) $\nabla_{\overline{Z}} g = 0$;
iii) $\nabla_{Z^*} g = 0$;
iv) $\nabla_{Z^\dagger} g = 0$.

Proof. We write $f(Z) = f_1(Z) + \mathbf{i} f_2(Z) + \mathbf{j} f_3(Z) + \mathbf{k} f_4(Z)$. Then, since $f = g$ is real valued, we have $f = f_1$ and it is clear that the conditions above are all equivalent in this case. Moreover, f has a stationary point if and only one of these conditions hold. In fact, since f is real valued and using the bicomplex gradient operators definition we have $\nabla_Z f = \nabla_Z f_1 = 0$ if and only if $\nabla_{\mathbf{z_1}} f_1 = \nabla_{\mathbf{z_2}} f_1 = 0$ and then by the Brandwood complex result this is equivalent to $\nabla_{\overline{\mathbf{z_1}}} f_1 = \nabla_{\overline{\mathbf{z_2}}} f_1 = 0$. Hence, in order to have a stationary point it is enough to have $\nabla_Z g = 0$ or $\nabla_{\overline{Z}} g = 0$ or $\nabla_{Z^*} g = 0$ or $\nabla_{Z^\dagger} g = 0$. □

Now, we mention various properties of the three bicomplex gradient operators. For the sake of brevity, we leave their proofs to the reader:

Proposition 5.6. *Let Z, a be column vectors in $\mathbb{BC}^n$ and R a matrix in $\mathbb{BC}^{n \times n}$, then the following properties hold true:*

(1) $\nabla_{\overline{Z}}(a^T Z) = 0$
(2) $\nabla_{\overline{Z}}(\overline{Z}^T a) = a$
(3) $\nabla_{\overline{Z}}(\overline{Z}^T R Z) = RZ.$
(4) $\nabla_{Z^*}(a^T Z) = 0$
(5) $\nabla_{Z^*}((Z^*)^T a) = a$
(6) $\nabla_{Z^*}((Z)^{*T} R Z) = RZ.$

6 | BICOMPLEX LMS ALGORITHMS

In this section, we apply the previous results in order to study two bicomplex extensions of the LMS algorithm. For more details on these algorithms, we refer to [6].

Let Y_ℓ be the actual bicomplex output and D_ℓ denote the desired output at time ℓ. We note that the actual output is defined by

$$Y_\ell = X_\ell^T W_\ell = W_\ell^T X_\ell = \sum_{k=1}^n X_{\ell,k} W_{\ell,k};$$

where the bicomplex vectors X_ℓ and W_ℓ in $\mathbb{BC}^n$ are given respectively by $X_\ell = (X_{\ell,1}, \dots, X_{\ell,n})$ and $W_\ell = (W_{\ell,1}, \dots, W_{\ell,n})$. Finally, we introduce the bicomplex-valued error signal at time ℓ which is defined to be

$$E_\ell := D_\ell - Y_\ell.$$

In the next subsections, we will introduce and prove two bicomplex extensions of the well-known LMS algorithm and some related results.

6.1 First bicomplex least mean square ($\mathbb{BC}-\mathbb{R}$ LMS) algorithm

We propose here the first bicomplex LMS algorithm which can be introduced thanks to the use of the bicomplex $*$ conjugate instead of the complex conjugate as in the classical paper of Widrow et al. This allows us to extend the LMS algorithm from complex to the bicomplex setting. Another extension will be presented in the next section.

Definition 6.1 (The first $\mathbb{BC}-\mathbb{R}$ LMS learning rule). We define the first bicomplex LMS algorithm (BLMS-**I**) by the following learning rule:

$$\mathsf{W}_{\ell+1}=\mathsf{W}_{\ell}-\mu\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}\mathsf{E}_{\ell}^{*}); \tag{6.35}$$

where E_{ℓ} is the bicomplex signal error, $*$ is the bicomplex conjugate and $\mu>0$ is a real constant.

In the next result, we prove that the first bicomplex LMS algorithm can be derived by applying the bicomplex gradient operator $\nabla_{\mathsf{Z}^{*}}$:

Theorem 6.2 (First BLMS algorithm). *The learning rule at time ℓ of the BLMS-**I** algorithm has the following explicit expression:*

$$\mathsf{W}_{\ell+1}=\mathsf{W}_{\ell}+2\mu\mathsf{E}_{\ell}\mathsf{X}_{\ell}^{*}. \tag{6.36}$$

Proof. We write the first bicomplex LMS algorithm at time ℓ as follows:

$$\mathsf{W}_{\ell+1}=\mathsf{W}_{\ell}-\mu\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}\mathsf{E}_{\ell}^{*}).$$

We have also

$$\mathsf{E}_{\ell}\mathsf{E}_{\ell}^{*}=\mathsf{D}_{\ell}\mathsf{D}_{\ell}^{*}-\mathsf{D}_{\ell}\mathsf{Y}_{\ell}^{*}-\mathsf{Y}_{\ell}\mathsf{D}_{\ell}^{*}+\mathsf{Y}_{\ell}\mathsf{Y}_{\ell}^{*}.$$

Then, starting from Equation (6.35) it is clear that we only have to compute the quantity $\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}\mathsf{E}_{\ell}^{*})$. We apply Theorem 5.3 and get

$$\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}\mathsf{E}_{\ell}^{*})=\mathsf{E}_{\ell}\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}^{*})+\mathsf{E}_{\ell}^{*}\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}).$$

First, we note that the error E_{ℓ} is given by

$$\mathsf{E}_{\ell}:=\mathsf{D}_{\ell}-\mathsf{Y}_{\ell}=\mathsf{D}_{\ell}-\sum_{k=1}^{n}\mathsf{X}_{\ell,k}\mathsf{W}_{\ell,k}$$

So, using the properties of the operator $\nabla_{\mathsf{Z}^{*}}$ we have

$$\begin{aligned}\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell})&=-\sum_{k=1}^{n}\mathsf{X}_{\ell,k}\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{W}_{\ell,k})\\&=0.\end{aligned}$$

On the other hand, since

$$\mathsf{E}_{\ell}^{*}=\mathsf{D}_{\ell}^{*}-\mathsf{Y}_{\ell}^{*}=\mathsf{D}_{\ell}^{*}-\sum_{k=1}^{n}\mathsf{X}_{\ell,k}^{*}\mathsf{W}_{\ell,k}^{*};$$

we have the following computations:

$$\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{E}_{\ell}^{*})=-\sum_{k=1}^{n}\mathsf{X}_{\ell,k}^{*}\nabla_{\mathsf{W}_{\ell}^{*}}(\mathsf{W}_{\ell,k}^{*}).$$

We observe the following fact:

$$\partial_{W^*_{\ell,k}}(W^*_{\ell,k}) = 2, \quad \partial_{W^*_{\ell,s}}(W^*_{\ell,k}) = 0; \qquad \forall k, s = 1, \dots, n, s \neq k.$$

In particular, we note that for every $k = 1, \dots, n$ we have

$$X^*_{\ell,k} \nabla_{W^*_\ell}(W^*_{\ell,k}) = \begin{pmatrix} X^*_{\ell,k}\partial_{W^*_{\ell,1}}(W^*_{\ell,k}) \\ \vdots \\ X^*_{\ell,k}\partial_{W^*_{\ell,k}}(W^*_{\ell,k}) \\ X^*_{\ell,k}\partial_{W^*_{\ell,k+1}}(W^*_{\ell,k}) \\ \vdots \\ X^*_{\ell,k}\partial_{W^*_{\ell,n}}(W^*_{\ell,k}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 2X^*_{\ell,k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, we obtain

$$\sum_{k=1}^{n} X^*_{\ell,k} \nabla_{W^*_\ell}(W^*_{\ell,k}) = 2 \begin{pmatrix} X^*_{\ell,1} \\ \vdots \\ X^*_{\ell,k} \\ \vdots \\ X^*_{\ell,n} \end{pmatrix} = 2X^*_\ell. \tag{6.37}$$

Thus, it follows that at time ℓ we have

$$\begin{aligned} \nabla_{W^*_\ell}(E^*_\ell) &= -\sum_{k=1}^{n} X^*_{\ell,k} \nabla_{W^*_\ell}(W^*_{\ell,k}) \\ &= -2X^*_\ell. \end{aligned}$$

Hence, to sum up we have

$$\nabla_{W^*_\ell}(E_\ell) = 0;$$

and

$$\nabla_{W^*_\ell}(E^*_\ell) = -2X^*_\ell.$$

Finally, we obtain

$$\nabla_{W^*_\ell}(E_\ell E^*_\ell) = E_\ell \nabla_{W^*_\ell}(E^*_\ell) = -2E_\ell X^*_\ell;$$

which yields at time ℓ to the first bicomplex LMS learning rule given by

$$W_{\ell+1} = W_\ell + 2\mu E_\ell X^*_\ell.$$

□

In the next results, we will express the learning rule for the first BLMS algorithm in terms of two classical complex LMS algorithms using two bicomplex decompositions:

Theorem 6.3. *Let us consider the decomposition of the bicomplex weights, error and input which are given respectively by*

$$W_\ell = w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2, \quad E_\ell = e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2, \quad \textit{and } X_\ell = x_{\ell,1}\mathbf{e}_1 + x_{\ell,2}\mathbf{e}_2.$$

Then, at time ℓ the learning rule of the first bicomplex LMS algorithm can be expressed in terms of two complex LMS algorithms as follows:

$$\begin{aligned}\mathsf{W}_{\ell+1} &= (w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,1}})\mathbf{e}_1 + (w_{\ell,2} + 2\mu e_{\ell,2}\overline{x_{\ell,2}})\mathbf{e}_2\\ &= w_{\ell+1,1}\mathbf{e}_1 + w_{\ell+1,2}\mathbf{e}_2.\end{aligned}$$

Proof. We use the expression of the first bicomplex LMS algorithm obtained in Theorem 6.2 and get

$$\begin{aligned}\mathsf{W}_{\ell+1} &= \mathsf{W}_\ell + 2\mu \mathsf{E}_\ell \mathsf{X}_\ell^*\\ &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2)(x_{\ell,1}\mathbf{e}_1 + x_{\ell,2}\mathbf{e}_2)^*\\ &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2)(\overline{x_{\ell,1}}\mathbf{e}_1 + \overline{x_{\ell,2}}\mathbf{e}_2)\end{aligned}$$

So, applying the product formula and properties of the $*$-bicomplex conjugate we have

$$\begin{aligned}\mathsf{W}_{\ell+1} &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\overline{x_{\ell,1}}\mathbf{e}_1 + e_{\ell,2}\overline{x_{\ell,2}}\mathbf{e}_2)\\ &= (w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,1}})\mathbf{e}_1 + (w_{\ell,2} + 2\mu e_{\ell,2}\overline{x_{\ell,2}})\mathbf{e}_2\\ &= w_{\ell+1,1}\mathbf{e}_1 + w_{\ell+1,2}\mathbf{e}_2.\end{aligned}$$

Finally, the first BLMS algorithm can be represented by the following two complex LMS algorithms given by

$$w_{\ell+1,1} = w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,1}};$$

and

$$w_{\ell+1,2} = w_{\ell,1} + 2\mu e_{\ell,2}\overline{x_{\ell,2}}.$$

□

Proposition 6.4. *Let us consider the decomposition of the bicomplex weights, error and input at time ℓ given by*

$$\mathsf{W}_\ell = W_{\ell,1} + W_{\ell,2}\mathbf{j}, \mathsf{E}_\ell = E_{\ell,1} + E_{\ell,2}\mathbf{j}, \quad \text{and } \mathsf{X}_\ell = X_{\ell,1} + X_{\ell,2}\mathbf{j}.$$

Then, the learning rule of the first bicomplex LMS algorithm can be expressed in terms of two complex LMS algorithms as follows:

$$\mathsf{W}_{\ell+1} = \left(W_{\ell,1} + 2\mu(E_{\ell,1}\overline{X_{\ell,1}} + \mathsf{E}_{\ell,2}\overline{\mathsf{X}_{\ell,2}})\right) + \mathbf{j}\left(W_{\ell,2} + 2\mu(E_{\ell,2}\overline{X_{\ell,1}} - \mathsf{E}_{\ell,1}\overline{\mathsf{X}_{\ell,2}})\right). \tag{6.38}$$

Proof. We apply the first BLMS algorithm obtained in Theorem 6.2 and the bicomplex decomposition to get

$$\begin{aligned}\mathsf{W}_{\ell+1} &= \mathsf{W}_\ell + 2\mu \mathsf{E}_\ell \mathsf{X}_\ell^*\\ &= \left(W_{\ell,1} + W_{\ell,2}\mathbf{j}\right) + 2\mu(E_{\ell,1} + E_{\ell,2}\mathbf{j})(\overline{X_{\ell,1}} - \overline{X_{\ell,2}}\mathbf{j})\\ &= W_{\ell,1} + W_{\ell,2}\mathbf{j} + 2\mu(E_{\ell,1}\overline{\mathsf{X}_{\ell,1}} - E_{\ell,1}\overline{\mathsf{X}_{\ell,2}}\mathbf{j} + E_{\ell,2}\overline{X_{\ell,1}}\mathbf{j} + \mathsf{E}_{\ell,2}\overline{X_{\ell,2}})\\ &= \left(W_{\ell,1} + 2\mu(E_{\ell,1}\overline{X_{\ell,1}} + \mathsf{E}_{\ell,2}\overline{\mathsf{X}_{\ell,2}})\right) + \mathbf{j}\left(W_{\ell,2} + 2\mu(E_{\ell,2}\overline{X_{\ell,1}} - \mathsf{E}_{\ell,1}\overline{\mathsf{X}_{\ell,2}})\right).\end{aligned}$$

This ends the proof. □

6.2 | The second bicomplex least mean square ($\mathbb{BC}-\mathbb{R}$ LMS) algorithm

This part is reproduced by analogy with the previous subsection by taking the *bar*-bicomplex conjugate in the learning rule rather than the $*$-conjugate. Some proofs are based on similar arguments used for the first BLMS, so we omit to give all the details. This allows to obtain a second extension of the complex LMS algorithm invented by Widrow et al. to the bicomplex setting.

Definition 6.5 (The second $\mathbb{BC}-\mathbb{R}$ LMS learning rule). We define the second bicomplex LMS algorithm (BLMS-**II**) by the following learning rule:

$$\mathsf{W}_{\ell+1} = \mathsf{W}_\ell - \mu\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell\overline{\mathsf{E}_\ell}); \tag{6.39}$$

where E_ℓ is the error, $-$ is the bicomplex *bar*-conjugate, and $\mu > 0$ is a real constant.

In the next result, we prove that the second bicomplex LMS algorithm introduced in Definition 6.5 can be obtained by applying the bicomplex gradient operator $\nabla_{\overline{\mathsf{Z}}}$ as follows:

Theorem 6.6 (The second BLMS algorithm). *The learning rule at time ℓ of the BLMS-**II** algorithm has the following explicit expression:*

$$\mathsf{W}_{\ell+1} = \mathsf{W}_\ell + 2\mu\mathsf{E}_\ell\overline{\mathsf{X}_\ell}. \tag{6.40}$$

Proof. We have

$$\mathsf{E}_\ell\overline{\mathsf{E}_\ell} = \mathsf{D}_\ell\overline{\mathsf{D}_\ell} - \mathsf{D}_\ell\overline{\mathsf{Y}_\ell} - \mathsf{Y}_\ell\overline{\mathsf{D}_\ell} + \mathsf{Y}_\ell\overline{\mathsf{Y}_\ell}. \tag{6.41}$$

Starting from Equation (6.39) it is clear that we only have to compute the quantity $\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell\overline{\mathsf{E}_\ell})$. To this end, we apply Theorem 5.3 and get

$$\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell\overline{\mathsf{E}_\ell}) = \mathsf{E}_\ell\nabla_{\overline{\mathsf{W}_\ell}}(\overline{\mathsf{E}_\ell}) + \overline{\mathsf{E}_\ell}\nabla_{\mathsf{W}_\ell^*}(\mathsf{E}_\ell).$$

First, we note that the error E_ℓ is given by

$$\mathsf{E}_\ell := \mathsf{D}_\ell - \mathsf{Y}_\ell = \mathsf{D}_\ell - \sum_{k=1}^{n}\mathsf{X}_{\ell,k}\mathsf{W}_{\ell,k}$$

So, using the properties of the operator $\nabla_{\overline{\mathsf{Z}}}$ we have

$$\begin{aligned}\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell) &= -\sum_{k=1}^{n}\mathsf{X}_{\ell,k}\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{W}_{\ell,k})\\ &= 0.\end{aligned}$$

On the other hand, since

$$\overline{\mathsf{E}_\ell} = \overline{\mathsf{D}_\ell} - \overline{\mathsf{Y}_\ell} = \overline{\mathsf{D}_\ell} - \sum_{k=1}^{n}\overline{\mathsf{X}_{\ell,k}}\,\overline{\mathsf{W}_{\ell,k}},$$

and

$$\partial_{\overline{\mathsf{W}_{\ell,k}}}(\overline{\mathsf{W}_{\ell,k}}) = 2, \quad \partial_{\overline{\mathsf{W}_{\ell,s}}}(\overline{\mathsf{W}_{\ell,k}}) = 0 \qquad \forall k, s = 1, \dots, n, s \neq k\,,$$

we will see that the argument follows and the learning algorithm has the required form.

We follow a similar reasoning as for the first BLMS algorithm (now with respect to the bicomplex *bar*-gradient operator) to obtain

$$\begin{aligned}\nabla_{\overline{\mathsf{W}_\ell}}(\overline{\mathsf{E}_\ell}) &= -\sum_{k=1}^{n} \overline{\mathsf{X}_{\ell,k}} \nabla_{\overline{\mathsf{W}_\ell}}(\overline{\mathsf{W}_{\ell,k}}) \\ &= -2\overline{\mathsf{X}_\ell}.\end{aligned}$$

Hence, to sum up we have

$$\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell) = 0;$$

and

$$\nabla_{\overline{\mathsf{W}_\ell}}(\overline{\mathsf{E}_\ell}) = -2\overline{\mathsf{X}_\ell}.$$

Finally, we obtain

$$\nabla_{\overline{\mathsf{W}_\ell}}(\mathsf{E}_\ell \overline{\mathsf{E}_\ell}) = \mathsf{E}_\ell \nabla_{\overline{\mathsf{W}_\ell}} \overline{\mathsf{E}_\ell} = -2\mathsf{E}_\ell \overline{\mathsf{X}_\ell};$$

which yields to the second BLMS learning rule given by

$$\mathsf{W}_{\ell+1} = \mathsf{W}_\ell + 2\mu \mathsf{E}_\ell \overline{\mathsf{X}_\ell}.$$

□

In the next results, we express the learning rule for the second bicomplex LMS algorithm in terms of two complex LMS algorithms based on the two bicomplex decompositions

Theorem 6.7. *Let us consider the decomposition of the bicopmplex weights, error and input which are given respectively by*

$$\mathsf{W}_\ell = w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2, \mathsf{E}_\ell = e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2, \quad \textit{and } \mathsf{X}_\ell = x_{\ell,1}\mathbf{e}_1 + x_{\ell,2}\mathbf{e}_2.$$

Then, at time ℓ the learning rule of the second BLMS algorithm can be expressed in terms of two complex LMS algorithms as follows:

$$\begin{aligned}\mathsf{W}_{\ell+1} &= (w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,2}})\mathbf{e}_1 + (w_{\ell,2} + 2\mu e_{\ell,2}\overline{x_{\ell,1}})\mathbf{e}_2 \\ &= w_{\ell+1,1}\mathbf{e}_1 + w_{\ell+1,2}\mathbf{e}_2.\end{aligned}$$

Proof. We use the expression of the second bicomplex LMS algorithm obtained in Theorem 6.6 and get

$$\begin{aligned}\mathsf{W}_{\ell+1} &= \mathsf{W}_\ell + 2\mu \mathsf{E}_\ell \overline{\mathsf{X}_\ell} \\ &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2)\overline{(x_{\ell,1}\mathbf{e}_1 + x_{\ell,2}\mathbf{e}_2)} \\ &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\mathbf{e}_1 + e_{\ell,2}\mathbf{e}_2)(\overline{x_{\ell,1}}\mathbf{e}_2 + \overline{x_{\ell,2}}\mathbf{e}_1)\end{aligned}$$

So, applying the product rule and properties of the *bar*-bicomplex conjugate, such as $\overline{\mathbf{e}_1} = \mathbf{e}_2$ and $\overline{\mathbf{e}_2} = \mathbf{e}_1$, we obtain

$$\begin{aligned}\mathsf{W}_{\ell+1} &= (w_{\ell,1}\mathbf{e}_1 + w_{\ell,2}\mathbf{e}_2) + 2\mu(e_{\ell,1}\overline{x_{\ell,2}}\mathbf{e}_1 + e_{\ell,2}\overline{x_{\ell,1}}\mathbf{e}_2) \\ &= (w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,2}})\mathbf{e}_1 + (w_{\ell,2} + 2\mu e_{\ell,2}\overline{x_{\ell,1}})\mathbf{e}_2 \\ &= w_{\ell+1,1}\mathbf{e}_1 + w_{\ell+1,2}\mathbf{e}_2.\end{aligned}$$

Finally, the second BLMS algorithm can be represented by the following two complex LMS algorithms given by

$$w_{\ell+1,1} = w_{\ell,1} + 2\mu e_{\ell,1}\overline{x_{\ell,2}};$$

and

$$w_{\ell+1,2} = w_{\ell,2} + 2\mu e_{\ell,2}\overline{x_{\ell,1}}.$$

□

Proposition 6.8. *Let us consider the decomposition of the bicopmplex weights, error and input at time ℓ given by*

$$\mathsf{W}_\ell = W_{\ell,1} + W_{\ell,2}\mathbf{j}, \mathsf{E}_\ell = E_{\ell,1} + E_{\ell,2}\mathbf{j}, \quad \text{and } \mathsf{X}_\ell = X_{\ell,1} + X_{\ell,2}\mathbf{j}.$$

Then, the learning rule of the second bicomplex LMS algorithm can be expressed in terms of two complex LMS algorithms as follows:

$$\mathsf{W}_{\ell+1} = \left(W_{\ell,1} + 2\mu(E_{\ell,1}\overline{X_{\ell,1}} - \mathsf{E}_{\ell,2}\overline{\mathsf{X}_{\ell,2}})\right) + \mathbf{j}\left(W_{\ell,2} + 2\mu(E_{\ell,2}\overline{X_{\ell,1}} + \mathsf{E}_{\ell,1}\overline{\mathsf{X}_{\ell,2}})\right). \tag{6.42}$$

Proof. The computations follow similar arguments as in the first BLMS algorithm using the bicomplex *bar*-conjugation instead of the $*$-conjugation. □

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