Essays on Two-Sided Matching Theory

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This thesis is a collection of three essays in market design concerning designs of matching markets with aggregate constraints, affirmative action schemes, and investigating boundaries of simultaneous efficiency-stability relaxation for one-to-one matching mechanisms.

In Chapter 1, I establish and propose a possible solution for a college housing crisis, a severe ongoing problem taking place in many countries. Every year many colleges provide housing for admitted students. However, there is no college admissions process that considers applicants' housing needs, which often results in college housing shortages. In this chapter, I formally introduce housing quotas to the college admissions problem and solve it for centralized admissions with common dormitories. The proposed setting is inspired by college admissions where applicants apply directly to college departments, and colleges are endowed with common residence halls. Such setting has many real-life applications: hospital/residents matching in Japan (Kamada and Kojima, 2011, 2012, 2015), college admissions with scholarships in Hungary (Biró, 2012), etc.

A simple example shows that there may not be a stable allocation for the proposed setting. Therefore, I construct two mechanisms that always produce some weakened versions of a stable matching: a Take-House-from-Applicant-stable and incentive compatible cumulative offer mechanism that respects improvements, and a Not-Compromised-Request-from-One-Agent-stable (stronger version of stability) cutoff minimising mechanism. Finally, I propose an integer programming solution for detecting a blocking-undominated Not-Compromised-Request-from-One-Agent-stable matching. Building on these results, I argue that presented

procedures could serve as a helpful tool for solving the college housing crisis.

In Chapter 2, I propose a number of solutions to resource allocation problems in an affirmative action agenda. Quotas are introduced as a way to promote members of minority groups. In addition, reserves may overlap: any candidate can belong to many minority groups, or, in other words, have more than one trait. Moreover, once selected, each candidate fills one reserve position for each of her traits, rather than just one position for one of her traits. This makes the entire decision process more transparent for applicants and allows them to potentially utilize all their traits. I extend the approach of Sönmez and Yenmez (2019a) who proposed a paired-admissions choice correspondence that works under no more than two traits. In turn, I allow for any number of traits focusing on extracting the best possible agents, such that the chosen set is *non-wasteful*, *the most diverse*, and *eliminates collective justified envy*. Two new, *lower*- and *upper-dominant* choice rules and a class of *sum-minimizing* choice correspondences are introduced and characterized.

In Chapter 3, I implement optimization techniques for detecting the efficient trade off between ex-post Pareto efficiency (for one side of a two-sided matching market) and ex-ante stability for small one-to-one matching markets. Neat example (Roth, 1982) proves that there is no matching mechanism that achieves both efficiency (for one side of the one-to-one matching market) and stability. As representative mechanisms I choose deferred-acceptance for stability, and top trading cycles for Pareto efficiency (both of them are strategy-proof for one side of the market). I compare performances of a randomized matching mechanism that simultaneously relaxes efficiency and stability, and a convex combination of two representative mechanisms. Results show that the constructed mechanism significantly improves efficiency and stability in comparison to mentioned convex combination of the benchmark mechanisms.

Contents

List of Tables

List of Figures

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Chapter 1

College Admissions with Housing Quotas

1.1 Introduction

The COVID-19 pandemic has affected higher education processes in numerous ways. Among others, college education and admissions procedures became fully online in many countries for almost two years. And, after colleges reopened their doors to (newly accepted) students, it turned out that there were often far more admitted applicants with housing needs than available residence options.

> *"Students at colleges from California to Florida were denied on-campus housing last fall and found themselves sitting out the year at home or living in motel rooms or vehicles as surging rents and decades of failing to build sufficient student housing came to a head.*
>
> *For some colleges, the housing crunch was related to increased demand by students who had been stuck at home during the pandemic. For others, including many in California, the shortage reflects a deeper conflict between the colleges and homeowners who don't want new housing built for students who they say increase congestion and noise".*

ABC News, April 26th, 2022

Furthermore, the student housing crisis was present in the country (especially in California) prior to the pandemic, so it got much worse after the virus receded. Besides the United States, various press articles indicate that students from the United Kingdom, the Netherlands, France, Hungary, Ireland and many other countries face similar problems. It is often noted that the ongoing housing crisis, the lack of on-campus housing, and the steady increase in the number of applicants will continuously exacerbate the student accommodation crisis in these countries if nothing is done to address it.[1]

In this paper, I propose a possible solution to this problem by modifying the college admission procedure. In particular, I model a college admissions market that takes into account applicants preferences over housing. The classic college admissions problem was introduced in the seminal work of Gale and Shapley (1962).[2] It is a two-sided many-to-one matching market with a finite set of *applicants* on the one side, and a finite set of *colleges* on the other. Each college possesses a positive quota – a number of applicants that this college can potentially admit. Also, each agent has *strict preferences* over the opposite side of the market together with a *stay alone* option, thus, an applicant may have a set of unacceptable colleges and vice versa.

A solution to the college admissions problem is an allocation of applicants to colleges with the following desirable features:

- each applicant is matched to at most one college, while each college admits applicants within its quota (*feasibility*);
- no agent is matched to an unacceptable partner (*individual rationality*);

[1] See, for instance, the following articles from *The Guardian* (https://www.theguardian.com/education/2022/dec/26/uk-student-housing-reaching-crisis-point-as-bad-as-1970s-charity-warns) and *Los Angeles Times* (https://www.latimes.com/california/story/2022-09-26/college-housing-shortage-pushes-students-into-crisis-as-most-uc-classes-start-up).

[2] See Roth and Sotomayor (1990), and Roth (1984, 1991) for a survey and applications.

- there does not exist a (*blocking*) pair of an applicant and a college, such that they strictly prefer each other to their current matches.

Individual rationality (IR) together with the absence of blocking pairs constitute *stability*. Indeed, if an allocation is stable, then there is no agent or student-college pair that can perform a profitable deviation. Gale and Shapley (1962) construct a student-proposing deferred acceptance mechanism (SDA) and prove that it produces a stable and feasible solution for any college admissions problem. Moreover, they show that, under the resulting allocation, every applicant is at least as well off as he would be under any other feasible and stable solution (*student-optimality*).

The absence of blocking pairs can be separated into two conditions (Balinski and Sönmez, 1999): *fairness* and *non-wastefulness*. Namely, a feasible allocation or a *matching* is stable if and only if it is IR, and both

- *fair*: there is no blocking pair such that this college prefers this applicant to some other admitted applicant;[3] and

- *non-wasteful*: there is no blocking pair such that this college's quota is not exhausted.

Thus, fairness is guided by the absence of justified envy by any non-admitted applicant of an admitted one, whereas non-wastefulness takes care of the maximum utilization of quotas.[4]

In this paper, I consider a novel setting by introducing housing quotas to the college admissions problem. A distinctive feature of my model is that now each college has a common housing quota and can contain several departments, each with its own capacity. So, each applicant, first, is applying directly to a department, and, second, is free to signal her decision on whether or not it is feasible for her to study at this department without a place in a college dorm.

[3]This property is also called *elimination of justified envy*.

[4]Indeed, as shown in Appendix 1.7, after college admissions in Russia in 2021, many applicants and colleges were complaining about the unfairness and wastefulness of the final allocation.

Furthermore, only applicants are considered as strategic agents in my model, while colleges and departments are objects: all quotas and department priorities are publicly known prior to admissions.[5]

1.1.1 Motivating Example

Consider the following admissions market. There are three applicants a_1, a_2, and a_3 and three departments d_1, d_2, and d_3, such that d_1 and d_2 are contained in the college c_1 and d_3 is alone in its college c_2. Each department has a unit capacity and strictly prefers a_1 to a_2, and a_2 to a_3. Also, each applicant strictly prefers d_1 to d_2, and d_2 to d_3. It is easy to verify that there is only one stable matching: $\mu = \{(a_1, d_1), (a_2, d_2), (a_3, d_3)\}$, thus, any stable college admissions process should yield exactly this matching.

After the admissions process is over and the unique stable matching μ is finalized, each applicant has an opportunity to apply for college housing. Each college has a unit capacity dorm. It turns out that applicants a_1 and a_2 cannot afford off-campus housing, while a_3 does not need to live on campus regardless of the department. Thus, only applicants a_1 and a_2 apply for one housing slot at college c_1, and a_1 gets the room (because all departments prefer a_1 to a_2) while a_2 drops out of the college and returns home. As a result, the final matching is: $\{(a_1, d_1, 1), (a_3, d_3, 0)\}$, where a digit 0 or 1 indicates whether an applicant receives a housing place at the corresponding college.

Obviously, the obtained matching is not stable under the classic admissions model. Moreover, it is also not stable under the setting with housing constraints, because a_2 and d_3 strictly prefer each other to their current matches. So, if someone took into account housing constraints during the admissions process, then he should have obtained the following unique stable solution: $\{(a_1, d_1, 1), (a_2, d_3, 1), (a_3, d_2, 0)\}$.

[5] Such models are sometimes called *student placement* (Balinski and Sönmez, 1999), or *school choice* (Abdulkadiroglu and Sönmez, 2003).

1.1.2 Contribution

The proposed model of college admissions with housing quotas (CAH) is a generalization of the model of matching with distributional constraints studied by Kamada and Kojima (2011, 2012, 2015), which was inspired by the Japanese medical residency matching market. The authors build a hospital-resident matching model with regions (REG), in which each hospital has a quota, and the set of all hospitals is partitioned into regions, each having its own quota. The proposed CAH model can be reduced to the REG if we assume that no student wants to study in any department without getting a place in a college dormitory. The authors show the nonexistence of a strongly stable matching and propose a weaker concept of a *weakly stable* matching that always exists.[6,7]

A weakly stable matching under the REG model tolerates only the following blocking pairs of a doctor and a hospital:

- first, this hospital prefers any admitted doctor to this doctor;
- second, this doctor is matched to a hospital from the same region as this hospital;
- third, the corresponding regional quota is exhausted.

In other words, weak stability prohibits transferring only if the transfer is within one region, the corresponding region is completely filled, and this transfer is not caused by a justified envy.[8]

In turn, Aziz et al. (2021) generalize the REG model of Kamada and Kojima (2015) by constructing the summer internship problem with budgets (SIP). They assume that, first, regions may overlap, and, second, each region has a budget to

[6]Under strong stability Kamada and Kojima (2015) allow for a blocking pair, such that its implementation would violate common quotas. In this paper such notion is named *stability*.

[7]Che and Tercieux (2019b) and Che et al. (2019) study stability under large finite economies with preferences that exhibit substitutability.

[8]Kamada and Kojima (2018) observe that *"moving a doctor from one hospital to another involves administrative tasks on the part of relevant regions . . . hence disallowing only those blocking pairs that Pareto-improve the relevant regions is, in our view, the most plausible notion in our environment"*.

spend on admitted applicants, such that each admitted applicant receives exactly one unit of funding. They adapt the notion of weak stability in their setting and propose a stronger concept of *cutoff stability.* Given a set of cutoffs (one for each department), an allocation is constructed as follows: each applicant is choosing the best department, such that she exceeds the corresponding cutoff score. The authors note that a matching is fair if and only if it is induced by a set of cutoffs. A matching is cutoff stable if it is induced by some set of cutoff scores and, after decreasing any of these cutoffs by one, the resulting allocation is no longer feasible. The authors prove the existence of a cutoff stable matching and show that cutoff stability implies weak stability under SIP.

In addition, Aziz et al. (2021) propose an algorithm that always finds some cutoff stable solution. However, a simple example from their paper shows that this algorithm can find a not (strongly) stable but a cutoff stable matching even if a (strongly) stable matching exists under a given SIP market.

The rest of the paper is structured as follows. Section 1.1.3 discusses related literature. Section 1.2 formally introduces the CAH model and relevant definitions. I consider a two-sided many-to-one admissions problem with applicants and departments partitioned into colleges. Departments are non-strategic agents, equipped with exogenously determined capacities and strict priority rankings over applicants and a stay alone option. In turn, each applicant has a strict preference ordering over corresponding contracts and a stay alone option, where a contract is a triple of an applicant, a department and an indicator of the presence or absence of a housing slot (a *bed*). Thus, one applicant can have at most two available contracts for a given department: with and without a bed. The total amount of housing slots or a *housing quota* is exogenously determined for each college.

An *allocation* under a CAH problem is a set of contracts. The presence of a contract with an applicant, a department and the presence (absence) of a housing slot in an allocation implies that this applicant is admitted to this department and will (not) be given a bed in a college dorm. A *matching* is an allocation, such that, first, each applicant has at most one contract, second, departments quotas are respected, and, third, colleges housing quota constraints are not violated.

A matching is stable if it is IR, fair and non-wasteful. In terms of fairness I consider four types of blocking contracts, depending on the existence of a chosen contract with or without a housing that is *blocked by* a not chosen contract with or without a housing: *(no housing-by-no housing)-, (housing-by-no housing)-, (no housing-by-housing)-*, and *(housing-by-housing)*-blocking. In turn, non-wastefulness takes care of another two types: *(∅-by-no housing)-*, and *(∅-by-housing)*-blocking contracts, depending on the existence of an empty department seat that is blocked by a not chosen contract with or without a housing.[9]

This paper studies the college admissions model with housing quotas under two types of housing constraints. Section 1.3 solves the previously unsolved CAH problem under *single-department* housing constraints, i.e. when each department has its own housing quota. This assumption eliminates any kind of complementarity across contracts and allows for the construction of a unique individually rational and stable department choice rule. After embedding this choice rule into the cumulative offer process of Hatfield and Milgrom (2005) I obtain the unique student-optimal stable mechanism: student-proposing deferred acceptance with housing quotas (SDAH). This mechanism never penalizes an applicant for a better performance during exams (respects improvements). Moreover, SDAH induces a truthful preferences submission as a weakly dominant strategy for any applicant; in other words, SDAH is strategy-proof (Theorem 1).

Section 1.4 studies the general CAH problem. A simple Example 1 shows that there may not be a stable matching. Thus, I first adapt SDAH to the general problem, obtain a strategy-proof and respecting improvements mechanism SDAH(G), and analyze its stability features. I show that the matching induced by SDAH(G) will not be changed after the admissions if we oblige applicants who

[9]For instance, under the matching $\{(a_1, d_1, 1), (a_3, d_3, 0)\}$ from the motivating example (Section 1.1.1) there are two blocking contracts: (no housing-by-housing)-blocking $(a_2, d_3, 1)$, and (∅-by-no housing)-blocking $(a_3, d_2, 0)$.

wish to transfer to look for a needed bed only among ones that has been already distributed. In other words, we ban colleges with unfilled beds from participating in the secondary market. I call such weakened version of stability a *Take-House-from-Applicant*-stability (THfA).

Second, I generalize the concept of weak stability from Kamada and Kojima (2015) and propose a stronger concept of *Not-Compromised-Request-from-One-Agent*-stability (NC-RfOA). Third, I construct a NC-RfOA-stable mechanism (CUT). Then, I introduce a new solution concept of *sub-market stability* based on the idea of minimizing the dissatisfaction of high-performing applicants caused by their blocking contracts. Unlike NC-RfOA-stability, a sub-market stable matching is stable if there is at least one stable matching. Theorem 2 proves that sub-market stability is stronger than NC-RfOA-stability, and that the five stability concepts do not coincide (stability and sub-market, NC-RfOA-, weak and THfA- stabilities).

Unfortunately, under the general CAH model there is no college choice rule that could be used in a cumulative offer process that will always produce a weakly stable solution (Proposition 12). Thus, Section 1.5 is dedicated to constructing an integer programming mechanism (SM-IP) that always finds a sub-market stable matching (Theorem 3).

Section 1.6 discusses the results, concludes and provides avenues for future research. Appendix 1.7 presents the college admissions procedure currently used in Russia (RCA), shows that the resulting allocations in 2021 and 2022 were very unstable, adapts the sequential IDAM+DA mechanism of Bó and Hakimov (2021) to single-department CAH setting and shows that a straightforward type-strategy profile is the only ex-post equilibrium of the induced game (Proposition 15).[10] Thus, a policymaker could also use this sequential mechanism under single-department housing constraints in order to get a student-optimal stable final allocation. Appendix 1.8 presents all the omitted proofs.

[10]Dynamic sequential and decentralized college admissions mechanisms are also studied in Alcalde and Romero-Medina (2000, 2005), Che and Koh (2016), Andersson et al. (2018), Balakin (2021), Bonkoungou (2021), Dur et al. (2021), and many others.

1.1.3 Related Literature

The fundamental research of Gale and Shapley (1962) has given rise to an extremely rich literature on market design. In particular, real-life school and college admissions procedures were analyzed and modified (not only theoretically) for various cities and countries, including Boston (Abdulkadiroglu and Sönmez, 2003; Abdulkadiroglu et al., 2005b; Ergin and Sönmez, 2006; Pathak and Sönmez, 2008), New York (Abdulkadiroglu et al., 2005a), Chicago (Pathak and Sönmez, 2013), the United Kingdom (Pathak and Sönmez, 2013), Turkey (Balinski and Sönmez, 1999; Yuret and Dogan, 2011), Brazil (Aygün and Bó, 2021; Bó and Hakimov, 2021), Germany (Westkamp, 2013), Hungary (Biró, 2008, 2012), Chile (Correa et al., 2022), China (Pu, 2021), and Taiwan (Dur et al., 2022). In this paper, I also briefly analyse currently used and upcoming Russian college admissions procedures.

The cornerstone of the general CAH model is the presence of aggregate housing constraints imposed on colleges. Besides Kamada and Kojima (2011, 2012, 2015) and Aziz et al. (2021), some recent studies also share similar types of regional constraints. Biró et al. (2010) and Goto et al. (2014) assume that each region has its own ranked master list of applicants. Goto et al. (2017) and Kamada and Kojima (2017) study the college admissions problem with common quotas under the *heredity* property. Namely, a matching problem satisfies the heredity property if the feasibility of a matching is monotone in the number of applicants matched. Kamada and Kojima (2018) investigate a general model with regional constraints under a hierarchical regional structure, where an applicant has at most one contract for a department. In contrast, the CAH model does not impose college master lists, does not satisfy the *heredity* property, and under CAH an applicant may have more that one contract for a department. In addition, Hafalir et al. (2022) deal with a setting where each school district is endowed with a well-defined choice rule, which is not possible under the general CAH model (Propositions 6 and 12).[11]

The main concern of this article is a fair distribution of available student housing. Abdulkadiroglu and Sönmez (1999) and Sönmez and Ünver (2008) develop and explore the well-behaved *You Request My House-I Get Your Turn* mechanism for allocating dormitory rooms to students on college campuses.[12] However, this procedure is assumed to start after the admissions process, and may thus leave some admitted students unmatched. Such outcome may either force these students to drop out of the college, or make their lives much more challenging financially and mentally.

1.2 Model

There are a finite set of **applicants** A with a typical element a, and a finite set of **departments** D with a typical element d. Each applicant a has a set of **attainable departments** $D_a \subseteq D$, which is the set of all departments that an applicant a is allowed to apply to.[13] Each department d has a strictly positive **department quota** $q_d > 0$.

Also, there is a finite set of **colleges** C with a typical element c, such that each college contains at least one department, and any department is contained in exactly one college. So, C is a partition of D without an empty set.[14] Each college c has a positive **housing quota** $q_c^H \geq 0$, which is weakly less than the sum of all its departments quotas: $q_c^H \leq \sum_{d \in c} q_d$. Denote by $c(d)$ a college that contains a department d.

A **contract** is a triple $x = (a, d, i) \in A \times D \times \{0, 1\} \equiv \mathcal{X}$, where $i \in \{0, 1\}$ indicates whether an applicant a gets a bed at a college $c(d)$; and x_A, x_D, x_I are, correspondingly, an applicant, a department, and a housing indicator in x. For

[11]See Aziz et al. (2022) for a complete survey of matching models with distributional constraints.

[12]Their setting is a generalization of *housing markets* model by Shapley and Scarf (1974), and house allocation problems by Hylland and Zeckhauser (1979).

[13]For instance, an applicant who decides not to take a chemistry exam will not be able to apply to any chemistry department.

[14]$\cup_{c \in C} = D$, $c_1 \cap c_2 = \{\}$ for any $\{c_1, c_2\} \subseteq C$, and $c \neq \{\}$ for any $c \in C$.

any set of contracts $X \subseteq \mathcal{X}$ and an applicant $a \in A$, we denote by $X_a \subseteq X$ the set of all contracts from X that contain a: $X_a = \{x \in X \mid x_A = a\}$. By analogy, we denote $X_d = \{x \in X \mid x_D = d\}$ for any department d, and $X_i = \{x \in X \mid x_I = i\}$ for any indicator $i \in \{0, 1\}$.

Each applicant a has a strict **preference ordering** P_a over a corresponding set of contracts $\{a\} \times D_a \times \{0, 1\}$ and a *stay alone* option $\varnothing$, where $x \, P_a \, x'$ means that a contract x is strictly better for a than x', and $\varnothing P_a x$ indicates that a contract x is unacceptable for a.[15] Also, each department d has a strict **priority ranking** P_d over A and a *stay alone* option $\varnothing$, such that an applicant a is acceptable if and only if d is attainable for her: $a \, P_d \, \varnothing$ if and only if $d \in D_a$.

A **college admissions market with housing quotas** (CAH) is a tuple $\langle A, D, C, (P_a, D_a)_{a \in A}, (P_d, q_d)_{d \in D}, (q_c^H)_{c \in C} \rangle$.

Definition 1. (Matching). A set of contracts $\mu \subseteq \mathcal{X}$ is a *matching* if the following holds:

- any applicant has at most one contract: $|\mu_a| \leq 1$ for any $a \in A$;
- any department has no more contracts than its quota: $|\mu_d| \leq q_d$ for any $d \in D$;
- any college gives no more dormitory places than its housing quota: $\sum_{d \in c} |(\mu_d)_1| \leq q_c^H$ for any $c \in C$.

So, any matching respects all three types of quota constraints: of applicants, of departments, and of colleges. An applicant a without a contract in μ, $|\mu_a| = 0$, is said to be unmatched under μ, so, $\mu_a = \{\} = \varnothing$.

I assume that applicants only care about their own matches, so that their preferences over matchings coincide with their preferences over their contracts in these matchings.

[15]Note that a collection $\{D_a\}_{a \in A}$ is public information, so any submitted preference P'_a (truthful or not) must report as acceptable only contracts with departments from D_a: if $x_D \notin D_a$, then $\varnothing \, P'_a \, x$ for a submitted P'_a.

Definition 2. (Individual rationality). A matching μ is *individually rational* (IR) if no applicant gets an unacceptable contract: $\mu_a \, P_a \, \varnothing$ for any $a \in A$ with $|\mu_a| = 1$.

In other words, a matching is IR if all applicants are either alone or are matched with acceptable contracts. Note that, in our setting, if a matching is IR, then any department will enroll only acceptable applicants.

Now we present six possible types of blocking contracts: four for fairness and two for non-wastefulness.

Definition 3. ((i'-by-i)-blocking contract). A contract $x = (a, d, i) \notin \mu$ is *(i'-by-i)-blocking* for a matching μ if, first, an applicant $x_A = a$ strictly prefers x to her current assignment μ_a, $x \; P_a \; \mu_a$; and, second, there exists a contract $x' \in \mu$, such that $x'_D = d$ and $x'_I = i'$, and d strictly prefers x_A to x'_A, $x_A \; P_d \; x'_A$, and $(\mu \cup x) \backslash \{x', \mu_a\}$ is a matching.[16]

For clarity we will use the following names for each of the four types of (i'-by-i)-blocking contracts:

1. *(0-by-0)-blocking contract x for μ:* this means that a chosen under μ contract without a housing option is blocked by a not chosen under μ contract x that also does not contain a housing option. Thus, we also call x a *(no housing-by-no housing)-blocking*, or *(NH-by-NH)-blocking* for short;

2. *(0-by-1)-blocking contract x for μ:* this means that a chosen under μ contract without a housing option is blocked by a not chosen under μ contract x with a housing option. Thus, we also call x a *(no housing-by-housing)-blocking*, or *(NH-by-H)-blocking* for short;

3. *(1-by-0)-blocking contract x for μ:* this means that a chosen under μ contract with a housing option is blocked by a not chosen under μ contract x that does not contain a housing option. Thus, we also call x a *(housing-by-no housing)-blocking*, or *(H-by-NH)-blocking* for short;

[16]Note that μ_a is an empty set $\varnothing$, if a is unmatched under μ.

4. *(1-by-1)-blocking contract x for μ:* this means that a chosen under μ contract with a housing option is blocked by a not chosen under μ contract x that also contains a housing option. Thus, we also call x a *(housing-by-housing)-blocking*, or *(H-by-H)-blocking* for short.

Definition 4. (Fairness). A matching μ is *fair* if there is no (i'-by-i)-blocking contract for it for any $i, i' \in \{0, 1\}$.

In simple terms, there is no pair of applicants a and a', such that a justifiably envies a'.

Definition 5. (($\varnothing$-by-i)-blocking contract). A contract $x = (a, d, i) \notin \mu$ is *($\varnothing$-by-i)-blocking* for a matching μ if, first, the quota of a corresponding department $x_D = d$ is not completely filled, $|\mu_d| < q_d$; second, an applicant $x_A = a$ strictly prefers x to her current assignment μ_a, $x P_a \mu_a$; and, third, $(\mu \cup x) \backslash \mu_a$ is a matching.

Again, for clarity we will use the following names for each of two types of ($\varnothing$-by-i)-blocking contracts:

1. *($\varnothing$-by-0)-blocking contact x for μ:* this means that an empty place is blocked by a not chosen under μ contract x without a housing option. Thus, we also call x a *($\varnothing$-by-no housing)-blocking*, or *($\varnothing$-by-NH)-blocking* for short;

2. *($\varnothing$-by-1)-blocking contact x for μ:* this means that an empty place is blocked by a not chosen under μ contract x with a housing option. Thus, we also call x a *($\varnothing$-by-housing)-blocking*, or *($\varnothing$-by-H)-blocking* for short.

Definition 6. (Non-wastefulness). A matching μ is *non-wasteful* if there is no ($\varnothing$-by-i)-blocking contract for it for any $i \in \{0, 1\}$.

In other words, there is no such pair of an applicant a and a department d, that a has a justified claim for an empty seat at d.

We also say that a contract x is *blocking* for a matching μ, if it is either (i'-by-i)-blocking or ($\varnothing$-by-i)-blocking for μ for any $i, i' \in \{0, 1\}$. Note that not all of the above blocking types are mutually exclusive (for example, there may exist a blocking contract that is both (NH-by-H)- and ($\varnothing$-by-H)-blocking).

Definition 7. (Stability). A matching μ is *stable* if it is individually rational, fair, and non-wasteful.

Under a stable matching no applicant is willing and has a right to change her current assignment, because satisfying any blocking contract will not produce a matching.[17]

Denote by R_a weak preferences of an applicant a induced by P_a.

Definition 8. (Pareto domination). A matching μ *dominates* another matching μ' if for any applicant a: $\mu \, R_a \, \mu'$, and for some applicant a': $\mu \, P_{a'} \, \mu'$.

Recall that all departments' priority rankings are exogenous. So, only applicants are considered to be strategic agents in this model. An applicant strategizes over her submitted strict preferences over acceptable contracts. A *preference profile* P is a list of all applicants' submitted preferences over (acceptable) contracts (e.g., $P = \{P_a\}_{a \in A}$, if everyone reveals their true preferences).[18]

Definition 9. (Matching mechanism). A (direct) *matching mechanism* φ is a mapping from the set of all possible preference profiles to the set of all matchings.

A matching mechanism is called IR, fair, non-wasteful, or stable if the resulting matching is such for any preference profile.

Definition 10. (Strategy-proofness). A (direct) matching mechanism φ is *strategy-proof* if for any applicant a, for any possible submission of her strict preferences over contracts P'_a, and for any possible submitted preferences of all other applicants P'_{-a}:

$$\varphi(P_a, P'_{-a}) \, R_a \, \varphi(P'_a, P'_{-a}).$$

[17]Thus, our stability notion is an analog of a *strong stability* introduced by Kamada and Kojima (2017).

[18]Recall that an applicant a cannot submit any not attainable department $d \notin D_a$ as acceptable.

So, a mechanism is strategy-proof if for any applicant truthful preference revelation is always in his best interests.

Now I introduce a notion of a *choice rule* for a college.

Definition 11. (Choice rule). A *choice rule* for a college c, Ch_c, is a mapping that for any set of contracts $X \subseteq A \times c \times \{0,1\}$, where each applicant has at most one contract, produces a subset $Ch_c(X) \subseteq X$ that is a matching.

So, a choice rule makes sure that none of the quota constraints of a college and all its departments is violated.

Definition 12. (Choice rule: fairness, IR, non-wastefulness, stability). A choice rule Ch_c is *fair/IR/non-wasteful/stable* if for any set of contracts $X \subseteq A \times c \times \{0,1\}$, where each applicant has at most one contract, it produces a subset $Ch_c(X) \subseteq X$ that is a fair/IR/non-wasteful/stable matching for the sub-market that considers only college c.

1.3 Single-Department Housing Constraints

This section solves the case where $|C| = |D|$, so each college contains exactly one department. As a result, we can focus only on departments: a department d housing quota is denoted by $q_d^H = q_c^H (\leq q_d)$, where $d \in c$.[19]

For the case of single-department housing constraints we will use department choice rules, where $Ch_d = Ch_{c(d)}$.

Now I present the unique stable department choice rule Ch_d^* for a college admissions problem with single-department housing constraints.

Take a department d with its strict priority ranking P_d, quota q_d and housing quota q_d^H. Take a proposed set of contracts $X \subseteq A \times \{d\} \times \{0,1\}$, where each applicant has at most one contract. The department choice rule Ch_d^* works as follows.

[19] A real-life setting is the following: each college divides all its housing quota among all its departments before a college admissions procedure takes place.

1. Start with an empty set of contracts $X^* = \{\}$ and take a set of all acceptable contracts from the proposed set X: $X' = \{x \in X \mid x\, P_{x_A}\, \varnothing\}$.

2. Take the best contract x for d from X'. If $X^* \cup x$ is feasible, then add x to X^*, otherwise, do nothing. Take the next best contract for d from X' and do the same. After either considering all acceptable contracts or exhausting a department quota set $Ch_d^*(X) = X^*$.

In other words, this choice rule takes care of, first, respecting all quota constraints; second, the maximum possible utilization of a department quota; third, choosing the best possible contracts for the department.[20]

Note that in the classical college admissions setting with no housing constraints the constructed choice rule Ch_d^* will choose the best $\min\{q_d, |X|\}$ contracts from $X \subseteq A \times \{d\}$.

Proposition 1. *A department choice rule is stable if and only if it is Ch_d^*.*

As a result, if stability is what we desire, then Ch_d^* is the only department choice rule to use. In order for our choice rule to be successfully used in a cumulative offer process we also need to impose the following properties of a college choice rule.

Definition 13. (Law of aggregate demand (Hatfield and Milgrom, 2005)). A choice rule Ch_c satisfies the *law of aggregate demand* if for any set $X \subseteq A \times c \times \{0, 1\}$, where each applicant has at most one contract, and for any contract $x \in X$ we have

$$|Ch_c(X \backslash \{x\})| \leq |Ch_c(X)|.$$

In other words, the size of the choice set $Ch_c(X)$ never shrinks if we include a new contract in the proposed set X.

Proposition 2. *Ch_d^* satisfies the law of aggregate demand.*

[20]Note that Ch_d^* is not isomorphic to a branch *bid-for-your-career* choice rule from Sönmez (2013).

Definition 14. (Substitutes (Hatfield and Kojima, 2010)). A choice rule Ch_c satisfies *substitutes* if for any set of contracts $X \subseteq A \times c \times \{0, 1\}$, where each applicant has at most one contract, and for any contracts $x \in X$ and $z \in A \times c \times \{0, 1\}$, where $x_A \neq z_A$, and z_A does not have a contract in X, the following holds:

$$\text{if } x \notin Ch_c(X), \text{ then } x \notin Ch_c(X \cup \{z\}).$$

In other words, if a college c rejects a contract from the set X, then this contract would still be rejected from the set X together with a new contract from a new applicant. So, we do not allow two contracts to be complements for a college.

Proposition 3. *Ch^*_d satisfies substitutes.*

Definition 15. (Irrelevance of rejected contracts (Aygün and Sönmez, 2013)). A choice rule Ch_c satisfies the *irrelevance of rejected contracts* (IRC) if for any set of contracts $X \subseteq A \times c \times \{0, 1\}$, where each applicant has at most one contract, and for any contract $x \in (A \times c \times \{0, 1\}) \backslash X$, where x_A does not have a contract in X, the following holds:

$$\text{if } x \notin Ch_c(X \cup \{x\}), \text{ then } Ch_c(X) = Ch_c(X \cup \{x\}).$$

In other words, the removal of a rejected contract has no effect on the choice set.

Proposition 4. *Ch^*_d satisfies the irrelevance of rejected contracts.*

Proof. As noted in Aygün and Sönmez (2013), the law of aggregate demand together with substitutes imply IRC. Thus, Propositions 2 and 3 give us the result. $\square$

Thus, Theorem 1 from Hatfield and Kojima (2010) implies that for any college admissions problem with single-department housing constraints there exists at least one stable matching.

1.3.1 Student-Optimal Stable Mechanism

Gale and Shapley (1962) introduced the student-proposing deferred acceptance mechanism (SDA) for a classical college admissions problem. This mechanism is stable (Gale and Shapley, 1962), strategy-proof (Roth, 1982; Dubins and Freedman, 1981) and dominates any other fair mechanism (Balinski and Sönmez, 1999; Abdulkadiroglu and Sönmez, 2003).

Hatfield and Milgrom (2005) introduced a generalized version of SDA for an environment with contracts. For the setting with single-department housing quotas we obtain the following student-proposing DA mechanism with housing quotas (SDAH).

- *Step 1:* Each applicant a proposes with her best acceptable contract from $\{a\} \times D_a \times \{0,1\}$.

  Denote by $X_{d,t}$ the set of all contracts proposed to d during the *Step* t together with all contracts tentatively accepted by d during the *Step* $(t-1)$, if $t > 1$. Each department d tentatively accepts $Ch^*_d(X_{d,1})$. All remaining contracts are rejected.

- *Step t:* Each applicant a who was rejected in the previous step proposes her next best acceptable contract from $\{a\} \times D_a \times \{0,1\}$ (if it exists).

  Each department d tentatively accepts $Ch^*_d(X_{d,t})$. All remaining contracts are rejected.

The algorithm stops if there are no rejected contracts: all tentative acceptances become final assignments.

Theorem 1. *SDAH is stable and strategy-proof. Moreover, the resulting matching dominates any other stable matching.*

Proof. This is a direct implication of Propositions 2, 3 and 4 above combined with Theorems 5 and 7 from Hatfield and Kojima (2010) and Theorem 2 from Hirata and Kasuya (2014). □

Moreover, we do not want our mechanism to punish an applicant with a strictly worse assignment if she got better exam scores. In other words, we want the mechanism to *respect improvements* (Balinski and Sönmez, 1999). Consider two sets of all department priorities over applicants and a stay alone option: $Pr = \{P_d\}_{d \in D}$ and $Pr' = \{P'_d\}_{d \in D}$. We say that Pr is an *unambiguous improvement for an applicant a over Pr'* if the following holds. First, Pr and Pr' induce the same set of all department priorities if we exclude an applicant a. Second, for any department d, such that a is acceptable, the number of applicants who are better for d than a is weakly smaller under Pr than under Pr', and for some department d' – it is strictly smaller.

Definition 16. (Respecting improvements). A mechanism *respects improvements* if an applicant a never gets a strictly worse assignment as a result of an unambiguous improvement for a.

Proposition 5. *SDAH respects improvements.*

This result implies that applicants will indeed try their best during exams in order to be higher in departments rankings during the college admissions process SDAH.

1.4 General Problem

In this section I consider a general model, where a college may contain more than one department.

First, I show that there could be a college admissions market without a stable matching, which was not the case for the single-department constraints. Consider the following adaptation of Example 1 from Kamada and Kojima (2017).

Example 1. Take the following CAH with two applicants and one college containing two departments. Quotas are $q_{d_1} = q_{d_2} = q_c^H = 1$. Preferences are:

a_1	a_2	d_1	d_2
$(d_1, 1)$	$(d_2, 1)$	a_2	a_1
$(d_2, 1)$	$(d_1, 1)$	a_1	a_2

Table 1.1: Market without a stable matching

There are five possible matchings: $\mu_0 = \{\}$, $\mu_1 = \{(a_1, d_1, 1)\}$, $\mu_2 = \{(a_2, d_2, 1)\}$, $\mu_3 = \{(a_1, d_2, 1)\}$, and $\mu_4 = \{(a_2, d_1, 1)\}$. The following is true: μ_0 and μ_1 have a (H-by-H)-blocking contract $(a_2, d_1, 1)$; μ_2 has a (H-by-H)-blocking contract $(a_1, d_2, 1)$; μ_3 has a ($\varnothing$-by-H)-blocking contract $(a_1, d_1, 1)$; and μ_4 has a ($\varnothing$-by-H)-blocking contract $(a_2, d_2, 1)$.

Thus, all five matchings are unstable: μ_1 and μ_2 are not fair; μ_3 and μ_4 are wasteful; and μ_0 is both.

Moreover, due to the following, we cannot use a cumulative offer mechanism to get a stable matching even if one exists.

Proposition 6. *There is no stable college choice rule that satisfies substitutes.*

So, we need to relax the notion of stability so there will always be such a matching.

1.4.1 Take-House-from-Applicant Stability

Suppose that a not stable matching μ is chosen. Once an applicant a with a blocking contract (a, d, i) decides to go for it, the following *sequence of events should happen*:

1. a gives up on μ_a: all resources (bed (if any) and a department seat) from μ_a are now returned back to a corresponding college;

2. a approaches a college of interest $c(d)$ and,

    - if $i = 0$, requests a needed place at d either from $c(d)$, or from an admitted to d lower ranked applicant;

- if $i = 1$, requests both a needed place at d and a bed from an admitted to d lower ranked applicant.

If at some stage of this sequence of events an applicant a is stuck and, as a result, is not admitted with (d, i), then we tolerate such blocking contract (a, d, i) under *Take-House-from-Applicant* protocol.

Definition 17. A matching is *THfA-stable* if it is individually rational, and any blocking contract is tolerated under THfA protocol.

Corollary 1. *A blocking contract is not tolerated under THfA protocol if and only if it requires an additional college resource (a bed).*

As a result, under THfA-stability, the only ($\varnothing$-by-i)-blocking contracts that we can allow are ($\varnothing$-by-H)-blocking, and the only (i'-by-i)-blocking contracts that we can allow are (NH-by-H)-blocking. So, we do not tolerate a justified envy of an applicant a towards another applicant a' if and only if a requires more resources than a'. The next section shows that this is guaranteed for a matching if and only if it is induced by a set of cutoffs.

1.4.2 Introducing Cutoffs

Fix a CAH. Take any department d with its strict priority ranking P_d over all applicants and a stay alone option. Given all applicants preferences over contracts $\{P_a\}_{a \in A}$, d's induced strict priority ranking over acceptable contracts is:[21]

- for any $a \in A$: if $(a, d, i)\ P_a\ \varnothing$, then $(a, d, i)\ P_d\ \varnothing$;
- for any $a \in A$: if $\varnothing\ P_a\ (a, d, i)$, then $\varnothing\ P_d\ (a, d, i)$;[22]
- for any pair $a, a' \in A$: if $(a, d, i)\ P_a\ \varnothing$, $(a', d, i')\ P_{a'}\ \varnothing$, and $a\ P_d\ a'$, then $(a, d, i)\ P_d\ (a', d, i')$;

[21] I abuse notation and use P_d also for strict priority ranking of a department d over its contracts.

[22] Thus, a contract (a, d, i) is acceptable for a if and only if it is acceptable for d.

- for any $a \in A$: if $(a, d, i)\ P_a\ (a, d, i')\ P_a\ \varnothing$, then $(a, d, i)\ P_d\ (a, d, i')$.

Thus, what we are doing here is aligning each department's priorities with applicants preferences over contracts for this department.

Example 2. Consider the following market with three applicants and two departments. Preferences are:

a_1	a_2	a_3	d_1	d_2
$(d_1, 0)$	$(d_2, 1)$	$(d_2, 0)$	a_1	a_2
$(d_2, 1)$	$(d_1, 1)$		a_3	a_3
$(d_2, 0)$	$(d_2, 0)$		a_2	a_1
	$(d_1, 0)$			

Table 1.2: Initial preferences of applicants and departments

So, the departments' implied preferences over acceptable contracts are:

d_1	d_2
$(a_1, 0)$	$(a_2, 1)$
$(a_2, 1)$	$(a_2, 0)$
$(a_2, 0)$	$(a_3, 0)$
	$(a_1, 1)$
	$(a_1, 0)$

Table 1.3: Alighned department preferences over contracts

Now, construct all departments rankings over contracts $\{P_d\}_{d \in D}$. Assign two integer numbers to any contract $x \in \mathcal{X}$. Its rank v_x in department's ranking, or a department x_D utility of x is[23]

$$v_x = \begin{cases} 1 + |\{y \in (A \times \{x_D\} \times \{0, 1\}) \mid x\ P_{x_D}\ y\ P_{x_D}\ \varnothing\}|, \text{ if } x\ P_{x_D}\ \varnothing, \\ -1, \text{ otherwise.} \end{cases}$$

[23] The same role can be played by the total exam scores of an applicant x_A for the department x_D.

And its score u_x in applicant's ordering, or an applicant x_A utility of x is

$$u_x = \begin{cases} 1 + |\{y \in (\{x_A\} \times D \times \{0,1\}) \mid x \, P_{x_A} \, y \, P_{x_A} \, \varnothing\}|, \text{ if } x \, P_{x_A} \, \varnothing, \\ -1, \text{ otherwise.} \end{cases}$$

By construction, the following holds. First, $v_x, u_x \geq 1$ for any acceptable contract x, and $v_x = u_x = -1$ for any unacceptable one. Second, for any acceptable contracts x, y with $x_A = y_A$: $u_x > u_y$ if and only if $x \, P_{x_A} \, y$; and for any acceptable contracts x, y with $x_D = y_D$: $v_x > v_y$ if and only if $x \, P_{x_D} \, y$.

Also assign two integer cutoffs to each department d: a department cutoff t_d and a housing cutoff t_d^H, such that $0 < t_d \leq t_d^H$. Denote by T a collection of all department cutoffs $T = \{t_d\}_{d \in D}$, and by T^H a collection of all housing cutoffs $T^H = \{t_d^H\}_{d \in D}$.

Given all cutoffs T and T^H, there can always can be constructed a unique allocation including the following contracts: for any applicant a take the best possible contract (with the highest score) x, such that its rank is weakly higher than the corresponding cutoff, $v_x \geq (1 - x_I) \cdot t_{x_D} + x_I \cdot t_{x_D}^H$. However, not all allocations are matchings, since some violate quota constraints. Thus, not any set of cutoffs $\mathcal{T} = \{T, T^H\}$ induces a matching. Denote by $Al(\mathcal{T})$ an allocation induced by $\mathcal{T}$.

Proposition 7. *A matching does not have (NH-by-NH)-, (H-by-H)-, or (H-by-NH)-blocking contracts if and only if it is induced by a set of cutoffs.*[24]

1.4.3 SDAH under General Problem

Consider the following SDAH under the general problem mechanism, denoted by SDAH(G) for short.

1. For each college c distribute all its housing quota q_c^H among its departments: $\sum_{d \in c} q_d^H = q_c^H$ and $0 \leq q_d^H \leq q_d$ for all $d \in c$.

[24]This is an adaptation of a well known result (see, e.g. Proposition 1 in Aziz et al. (2021), or Lemma 3 in Fleiner and Jankó (2014)).

2. Perform SDAH on the obtained CAH with single-department housing constraints and get the resulting matching $\mu^{SDAH(G)}$.

What can we say about this mechanism? Theorem 1 and Proposition 5 give us the following.

Corollary 2. *SDAH(G) is strategy-proof and respects improvements.*

Also, we can say the following about its stability.

Proposition 8. *SDAH(G) is THfA-stable.*

Propositions 7 and 8 imply that a resulting matching of SDAH(G) is induced by a set of cutoffs, such that for any department d: $t_d = 1$ if q_d is not completely filled.

So, a resulting matching of SDAH(G) can have either (NH-by-H)-, or (∅-by-H)-blocking contracts. Moreover, there can be a (∅-by-H)-blocking contract x, such that the corresponding applicant x_A is not admitted anywhere. In other words, there can be a not chosen applicant who justifiably claims an empty seat.

Example 3. Consider the following CAH with four applicants and four departments in two colleges: $c_1 = \{d_1, d_2\}$ and $c_2 = \{d_3, d_4\}$. Department quotas are: $q_{d_1} = q_{d_2} = q_{d_4} = 1$, $q_{d_3} = 2$. College housing quotas are $q^H_{c_1} = q^H_{c_2} = 1$. Preferences are:

a_1	a_2	a_3	a_4	$\{d_1$	$d_2\}$	$\{d_3$	$d_4\}$
$(d_1, 1)$	$(d_2, 1)$	$(d_3, 1)$	$(d_4, 0)$	a_2	a_1	a_1	a_4
$(d_2, 1)$	$(d_1, 1)$			a_1	a_2	a_3	
$(d_3, 0)$							

Table 1.4: CAH with not admitted applicant with justified claim under SDAH(G)

During SDAH(G) first college gives its bed to the department d_1, and second college – to the department d_4. As a result, we got the following matching: $\mu = \{(a_1, d_3, 0), (a_2, d_1, 1), (a_4, d_4, 0)\}$. This matching has one blocking contract: a (∅-by-H)-blocking $(a_3, d_3, 1)$, and moreover, a_3 is not admitted anywhere.

In the next sections I show that there always exists a matching only with (NH-by-H)- or ($\varnothing$-by-H)-blocking contracts, such that there is no not admitted applicant who justifiably claims an empty seat. Moreover, under such matching any applicant with ($\varnothing$-by-H)-blocking contract ends up in the college that contains her department of interest.

1.4.4 Weak Stability

In this section I adapt the notion of weak stability from Kamada and Kojima (2017) to my setting.

Definition 18. (Weak stability). A matching μ is *weakly stable* if a blocking contract x is either ($\varnothing$-by-H)-blocking or (NH-by-H)-blocking, and if it is ($\varnothing$-by-H)-blocking, then the housing quota $q^H_{c(x_D)}$ is exhausted.

Note that weak stability of μ implies that for a ($\varnothing$-by-H)-blocking contract $(a, d, 1)$ there exists another contract $(a, d', 1) \in \mu$, where d' and d are from the same college. So, if an applicant justifiably claims an empty place at a college, then she is already admitted to another department from this college.[25]

Definition 19. (Unconstrained set of cutoffs). We call a set of cutoffs $\mathcal{T}$ *unconstrained* if the following holds. First, $Al(\mathcal{T})$ is a matching. Second, for any department d, such that q_d is not exhausted we have:

- if $q^H_{c(d)}$ is not exhausted, then $t_d = t^H_d = 1$;
- if $q^H_{c(d)}$ is exhausted, then $t^H_d \geq t_d = 1$.

In other words, an unconstrained set of cutoffs makes admissible any acceptable contract with $i = 0$ for a department if its quota is not completely filled, and makes admissible all acceptable contracts for a department if the housing quota of its college is also not completely filled.

[25]This may be useful if a college has disjoint subsets of departments with similar programs: if each such subset receives its own housing quota prior to the admissions process, then for each applicant who justifiably claims an empty slot at some department from this college, it will be true that he has already been admitted to a similar department from the same college.

Proposition 9. *A matching is weakly stable if and only if it is induced by an unconstrained set of cutoffs.*

Note that a matching produced by a SDAH(G) is not necessarily a weakly stable matching and vice versa.[26] However, any weakly stable matching is THfA-stable.

1.4.5 Not-Compromised-Request-from-One-Agent Stability

In this section I construct a stronger notion than THfA-stability that still gives us existence, and argue why the *"compromised (∅-by-H)-blocking contract"* building block of this notion is essential.

Definition 20. (Compromised (∅-by-H)-blocking contract). We call a (∅-by-H)-blocking under μ contract $(a, d, 1)$ a *compromised blocking contract* if there exists another applicant a', s.t. first, a' has an acceptable contract $(a', d, 1)$ that he prefers to $\mu_{a'}$, second, d prefers a' to a, and third, $(a', d, 1)$ is not (∅-by-H)-blocking.

To get the intuition behind this notion we should look at the dynamics of satisfying a (∅-by-H)-blocking contract. Recall that once an applicant a has decided to go for a (∅-by-H)-blocking contact $(a, d, 1)$ under a matching μ, she first returns her current bed and a department seat (if any) to a corresponding college $c((\mu_a)_D)$.

Once it is done we look at the other applicant a', who previously had a not chosen and not (∅-by-H)-blocking contract $(a', d, 1)$ that he prefers to $\mu_{a'}$. The presence of such contract implies that the housing quota of $c(d)$ should be exhausted. At the same time, a contract $(a, d, 1)$ is (∅-by-H)-blocking under μ.

[26] Example 3 shows that a matching produced by SDAH(G) can be not weakly stable. On the other hand, consider the following CAH with two applicants and two departments in one college. Quotas are $q_{d_1} = q_{d_2} = q_c^H = 1$. Preferences are: $(d_1, 1)\, P_{a_1}\, (d_1, 0)$, $(d_2, 1)\, P_{a_2}\, (d_2, 0)$ (so, for d_1 only a_1 is acceptable, and for d_2 only a_2 is acceptable). Matching $\{(a_1, d_1, 0), (a_2, d_2, 0)\}$ is weakly stable by Proposition 9, but obviously cannot be a resulting matching of SDAH(G), since there are only two of them: $\{(a_1, d_1, 1), (a_2, d_2, 0)\}$ and $\{(a_1, d_1, 0), (a_2, d_2, 1)\}$.

Combining these two facts, we infer that under μ, first, there is an empty seat at the department d, and second, applicant a indeed had a bed and returned it to the college $c(d)$.

Thus, now a college $c(d)$ has both a seat at d and a bed to give to a more preferred applicant a' instead of a. As a result, applicant a will not be able to get the desired contract $(a, d, 1)$, because it turns to be compromised by another contract $(a', d, 1)$.

As an illustration, recall the simple market from Example 1 that does not have a stable matching. Take the following matching $\mu_4 = \{(a_2, d_1, 1)\}$. There is exactly one blocking contract under μ_4: a ($\varnothing$-by-H)-blocking $\{(a_2, d_2, 1)\}$. However, this contract turns out to be compromised by $\{(a_1, d_2, 1)\}$, thus, applicant a_2 will not go for her blocking contract under μ_4.

Now, we turn to a new protocol. Suppose that a not stable matching μ is chosen. Once an applicant a with a blocking contract (a, d, i) decides to go for it, the following *sequence of events should happen*:

1. a gives up on μ_a: all resources (bed (if any) and a department seat) from μ_a are now returned back to a corresponding college;

2. a approaches a college of interest $c(d)$ and,

    - if $i = 0$, requests a needed place at d either from $c(d)$, or from an admitted to d lower ranked applicant;

    - if $i = 1$, requests both a needed place at d and a bed either from an admitted to d lower ranked applicant, *or directly from college* $c(d)$*, if* (a, d, i) *is not compromised.*

If at some stage of this sequence of events an applicant a is stuck and, as a result, is not admitted with (d, i), then we tolerate such blocking contract (a, d, i) under *Not-Compromised-Request-from-One-Agent* protocol.

Definition 21. A matching is *NC-RfOA-stable* if it is individually rational, and any blocking contract is tolerated under NC-RfOA protocol.

As a result, under NC-RfOA-stability, in order to be tolerated a blocking contract should be either only (NH-by-H)-blocking, or only compromised (∅-by-H)-blocking, or only both.

Corollary 3. *NC-RfOA-stability implies THfA-stability.*

Note that under NC-RfOA protocol we still require applicants to ask for resources only from one agent, however, unlike under THfA protocol, now they can request beds directly from colleges.

1.4.6 NC-RfOA-Stable Mechanism

Given a set of department cutoffs T denote by T_{-d} a set of cutoffs where a cutoff t_d is decreased by one if $t_d > 1$, while all the others are kept the same: $T_{-d} = \{t_1, \ldots, \max\{1, (t_d - 1)\}, \ldots, t_{|D|}\}$. By analogy, denote by T^H_{-d} a set of housing cutoffs where a cutoff t^H_d is decreased by one if $t^H_d > 1$, while all the others are kept the same: $T^H_{-d} = \{t^H_1, \ldots, \max\{1, (t^H_d - 1)\}, \ldots, t^H_{|D|}\}$. Also, given a set of cutoffs $\mathcal{T} = \{T, T^H\}$ denote $\mathcal{T}_{-d} = \{T_{-d}, T^H\}$; and $\mathcal{T}_{-d,H} = \{\min\{T, T^H_{-d}\}, T^H_{-d}\}$, where $\min\{T, T^H_{-d}\}$ is taking the minimum of t_d and $\max\{1, (t^H_d - 1)\}$, while keeping all the others department cutoffs the same.

Definition 22. (Minimal set of cutoffs for a matching). A set of cutoffs $\mathcal{T}$ that induces a matching μ is called *minimal for* μ if μ cannot be induced by any of the following sets of cutoffs that differ from $\mathcal{T}$: $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for any $d \in D$.

So, given a matching we can search for the minimal set of cutoffs that induces it. Moreover, there will always be a unique such set of cutoffs.

Proposition 10. *For any matching induced by some set of cutoffs there exists a unique minimal set of cutoffs.*

Now we establish the following equivalence.

Proposition 11. *A matching μ induced by the minimal set of cutoffs $\mathcal{T}$ is NC-RfOA-stable if and only if there does not exist a different matching induced by a set of cutoffs $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for some $d \in D$.*

Note that for any set of cutoffs $\mathcal{T}$ it is straightforward to check whether $Al(\mathcal{T})$ is a matching. All we have to do is to make sure that all quota constraints are respected. Now consider the following NC-RfOA-stable mechanism CUT.

1. Take the set of cutoffs $\mathcal{T}$, such that all cutoffs are maximal: $t_d = t_d^H = |X_d| + 1$ for all d, where X is the set of all acceptable contracts. Thus, $Al(\mathcal{T})$ is an empty matching. Also, randomly pick a permutation π on the set $\{1, 2, \ldots, |D|\}$ that induces the following department order: $d_{\pi(1)}, d_{\pi(2)}, \ldots,$ $d_{\pi(|D|)}$. If at some point all cutoffs are equal to one, terminate the following procedure.

2. Take the first department $d_{\pi(1)}$ and try to decrease its housing cutoff: check whether $Al\left(\mathcal{T}_{-d_{\pi(1)},H}\right)$ is feasible. If it is, then update the cutoffs $\mathcal{T} = \mathcal{T}_{-d_{\pi(1)},H}$. Take the next department and do the same step (go to $d_{\pi(1)}$ if before we considered $d_{\pi(|D|)}$).

3. Suppose that at the beginning of some step when we should take some department $d_{\pi(j)}$ there have been exactly $|D|$ steps without cutoff updating. Then, take the department $d_{\pi(j)}$ and try to decrease its department cutoff: check whether $Al\left(\mathcal{T}_{-d_{\pi(j)}}\right)$ is feasible. If it is, then update the cutoffs $\mathcal{T} = \mathcal{T}_{-d_{\pi(j)}}$, and continue trying to decrease the housing cutoffs of the next departments as described above. Otherwise, take the next department and do the same step: again try to decrease its department cutoff (go to $d_{\pi(1)}$ if before we considered $d_{\pi(|D|)}$).

4. If, at the beginning of some step there have been exactly $2|D|$ steps without cutoff updating, then terminate the procedure. The resulting matching is $\mu^{CUT} = Al(\mathcal{T})$.

By construction, CUT always produces some NC-RfOA-stable matching. Moreover, CUT always tries to pick as many high ranked contracts as possible.

Corollary 4. *For any CAH there always exist a NC-RfOA-stable matching.*

However, due to the following, we cannot use a cumulative offer mechanism to get a weakly stable matching (and, as a result, a NC-RfOA-stable matching) even though one always exists.

Proposition 12. *There is no weakly stable college choice rule that satisfies substitutes.*

Proof. The proof is equivalent to the proof of Proposition 6 after changing "stability" to "weak stability". □

Also note that we can choose any matching μ as a starting point of CUT. Moreover, if by decreasing any of the cutoffs of μ the resulting allocation appears to be a different matching μ', then μ' dominates μ. In particular, the following holds.

Corollary 5. *For any not NC-RfOA-stable matching μ there exists a NC-RfOA-stable matching μ' that dominates μ.*

However, Theorem 5 from Aziz et al. (2021) implies the following drawbacks of CUT. First, it is not strategy-proof (for the applicants); second, CUT does not always find a stable matching whenever one exists; third, changing the permutation π can change the final allocation; and, fourth, there exists a NC-RfOA-stable matching that cannot be produced as a result of CUT. In order to fix the latter problem, in the next section I introduce a stronger notion of stability, that always produce a stable matching if one exists, and gives a NC-RfOA-stable matching otherwise.

1.4.7 Sub-Market Stability

Denote by R_d weak preferences of a department d induced by P_d.

Definition 23. (Blocking-domination). Consider two matchings μ and μ'. For each department d take the best blocking contract x_d under μ and the best blocking contract x'_d under μ' (if any, otherwise, set $x_d = \varnothing$ or $x'_d = \varnothing$). We say that μ

blocking-dominates μ' if the following holds: $x'_d \, R_d \, x_d$ for all $d \in D$, and $x'_{d'} \, P_{d'} \, x_{d'}$ for some $d' \in D$.

Suppose that a matching μ blocking-dominates another matching. For each of these two matchings and for each department take the highest ranked applicant who is disappointed by having a blocking contract with this department. Under a blocking dominating matching μ each such applicant will have lower rank at the corresponding department, that is she got lower test scores for this department prior to the admissions.

We consider a blocking-undominated subset of the set of all weakly stable matchings. So, we minimize the amount of justified envy and claims for applicants with higher test scores, while keeping the final matching weakly stable.

Definition 24. (Sub-market stability). A matching μ is *sub-market stable* if it is weakly stable, and there is no other weakly stable matching that blocking-dominates μ.

Why is it called sub-market stability? Given an initial CAH market, we can construct a *trimmed sub-market* in the following way.

Definition 25. (Trimmed sub-market). Fix a CAH Δ. A CAH Δ' differing only in applicants preferences over contracts $\{P'_a\}_{a \in A}$ is a *trimmed sub-market of* Δ, denoted by $\Delta' \in Tr(\Delta)$, if the following holds for any $a \in A$ and $x, x' \in (\{a\} \times D \times \{0, 1\}) \cup \{\varnothing\}$: $x \, P_a \, x'$ and $x' \, P'_a \, x$ only if

- first, $x_I = 1$;
- second, $\varnothing \, P'_a \, x$; and
- third, there is no contract $y \in A \times \{x_D\} \times \{1\}$, such that $y \, P'_{y_A} \, \varnothing$ and $x_A \, P_{x_D} \, y_A$.

So, all acceptable contracts with $i = 0$ stay so under a trimmed sub-market, while some acceptable contracts with $i = 1$ from the bottom of departments rankings may become unacceptable. Note that any CAH Δ is a trimmed sub-market of itself: $\Delta \in Tr(\Delta)$ for any Δ.

Example 4. Consider a CAH market Δ with four applicants, two departments, and one college $c = \{d_1, d_2\}$. Preferences of applicants over contracts and departments over applicants imply the following preferences of departments over acceptable contracts:

d_1	d_2
$(a_1, 1)$	$(a_2, 0)$
$(a_2, 0)$	$(a_4, 1)$
$(a_3, 1)$	$(a_1, 1)$
$(a_3, 0)$	$(a_1, 0)$
$(a_4, 1)$	$(a_3, 1)$

Table 1.5: Department preferences over contracts

If, for instance, applicants a_4 and a_3 decide to make contracts $(a_4, d_1, 1)$ and $(a_3, d_2, 1)$ unacceptable, then we will obtain a trimmed sub-market $\Delta' \in Tr(\Delta)$. If, in addition, applicant a_3 decides to make a contract $(a_3, d_1, 1)$ unacceptable, then we will obtain a trimmed sub-market $\Delta'' \in Tr(\Delta)$ (and also $\Delta'' \in Tr(\Delta')$). If, in addition, applicant a_4 decides to make a contract $(a_4, d_2, 1)$ unacceptable, then we will obtain a market Δ''', which is not a trimmed sub-market, $\Delta''' \notin Tr(\Delta)$, because $(a_4, d_2, 1)\ P_{d_2}\ (a_1, d_2, 1)$ and $(a_1, d_2, 1)$ is acceptable under Δ'''.

If, starting from Δ again, applicants a_1, a_3 and a_4 decide to make contracts $(a_4, d_1, 1)$, $(a_3, d_2, 1)$, and $(a_1, d_2, 1)$ unacceptable, then we will obtain a trimmed sub-market $\Delta'''' \in Tr(\Delta)$, which is also a trimmed sub-market of Δ', but not of Δ''.

As we can see from this example, there may be many trimmed sub-markets and not for any pair of them can we say which one is *more trimmed*. Also, in order to get a trimmed sub-market we can simply set a trim threshold $\bar{t}_d^H \geq 1$ for each department d, such that any contract $x = (a, d, 1)$ with $v_x < \bar{t}_d^H$ becomes unacceptable for any $a \in A$.

We do not want to exclude more contracts than necessary, so, we want to consider one of the largest trimmed sub-markets (in terms of the inclusion of sets of acceptable contracts), such that there exists a stable matching that is weakly stable under initial CAH.

Definition 26. (Maximal trimmed sub-market). Fix a CAH Δ. A trimmed sub-market $\Delta' \in Tr(\Delta)$ is called *maximal* if, first, there exists a matching that is stable under Δ' and weakly stable under Δ, and, second, there does not exist another trimmed sub-market $\Delta'' \in Tr(\Delta)$, such that $\Delta' \in Tr(\Delta'')$, and there exists a matching that is stable under Δ'' and weakly stable under Δ.

By Definition 24, if a matching μ is sub-market stable, then the set of contracts that should be excluded for μ to be stable is minimal (in terms of inclusion), so the following holds.

Proposition 13. *A matching is sub-market stable if and only if it is stable under some maximal trimmed sub-market.*

The main result is the following.

Theorem 2. *Stability implies sub-market stability, which implies NC-RfOA-stability, which implies weak stability, which implies THfA-stability. Moreover, all these notions do not coincide.*

As a result, the proposed notion of sub-market stability considers an always non-empty blocking-undominated subset of NC-RfOA-stable matchings.

Corollary 6. *For any NC-RfOA-stable but not sub-market stable matching μ^{CT}, there exists a sub-market stable matching μ^{SM} that blocking-dominates μ^{CT}.*

Moreover, once a maximal trimmed sub-market is fixed, we want to pick an undominated stable matching.

Definition 27. (Sub-market undominated stability). A matching is *sub-market undominated stable* if it is an undominated stable matching under some maximal trimmed sub-market.

By definition, a sub-market undominated stable matching is sub-market stable. Note that if there is at least one stable matching under a given CAH, then a sub-market undominated stable matching is an undominated stable matching.

1.5 Integer Programming Solution

In this section I develop an integer programming (IP) solution for finding a sub-market undominated stable matching (recall that there can be multiple undominated stable matchings under a maximal trimmed sub-market).[27]

Fix a CAH Δ. Recall that each contract $x \in \mathcal{X}$ has its applicant utility u_x and its department utility v_x. Also each department d has two integer positive cutoff scores: a department cutoff t_d and a housing cutoff t_d^H, such that $1 \leq t_d \leq t_d^H$. A contract x can be chosen only if $v_x \geq t_{x_D}$. A contract x with $x_I = 1$ can be chosen only if $v_x \geq t_{x_D}^H$.

1.5.1 Undominated Stable Matching

Suppose that there is at least one stable matching under Δ. Now, we construct all necessary constraints that should be satisfied for a stable matching μ. Let ξ_x be a binary indicator of whether a contract x is chosen under μ:

$$\xi_x \in \{0, 1\},\ \forall x \in \mathcal{X}. \tag{1.1}$$

Choosing a Matching

All quota constraints should be satisfied:

[27] Agoston et al. (2016, 2022) develop (Mixed) IP solutions for various types of many-to-one matching problems.

- no applicant may be assigned to more than one department:

$$\sum_{d \in D} \sum_{i \in \{0,1\}} \xi_{(a,d,i)} \leq 1, \, \forall a \in A; \tag{1.2}$$

- all department quotas are satisfied:

$$\sum_{a \in A} \sum_{i \in \{0,1\}} \xi_{(a,d,i)} \leq q_d, \, \forall d \in D; \tag{1.3}$$

- no colleges housing quotas are violated:

$$\sum_{d \in c} \sum_{a \in A} \xi_{(a,d,1)} \leq q_c^H, \, \forall c \in C. \tag{1.4}$$

Individual Rationality

No unacceptable contract is chosen.

$$\sum_{x \in \mathcal{X} \,:\, u_x < 0} \xi_x \leq 0 \tag{1.5}$$

Setting Up the Cutoffs

All cutoffs are strictly positive, and a housing cutoff of each department is weakly greater than its department cutoff.

$$1 \leq t_d, \, \forall d \in D \tag{1.6}$$

$$t_d \leq t_d^H, \, \forall d \in D \tag{1.7}$$

There is a trimmed bound for any cutoff.

$$t_d^H \leq |\mathcal{X}| + 1, \, \forall d \in D \tag{1.8}$$

If a contract x is chosen, then $v_x \geq t_{x_D}$, and $v_x \geq t_{x_D}^H$, if $x_I = 1$.

$$(1 - x_I) \cdot t_{x_D} + x_I \cdot t^H_{x_D} \leq \xi_x \cdot v_x + (1 - \xi_x) \cdot (|\mathcal{X}| + 1), \; \forall x \in \mathcal{X} \tag{1.9}$$

Minimal Department Cutoff if Department Quota is not Filled

If a quota of a department is not completely filled, then this department should have the minimal department cutoff.

$$f_d \in \{0, 1\}, \; \forall d \in D \tag{1.10}$$

$$f_d \cdot q_d \leq \sum_{(a,i) \in A \times \{0,1\}} \xi_{(a,d,i)}, \; \forall d \in D \tag{1.11}$$

$$t_d \leq (1 - f_d) + f_d \cdot (|\mathcal{X}| + 1), \; \forall d \in D \tag{1.12}$$

If $f_d = 0$, then $t_d = 1$, so all acceptable contracts with $i = 0$ may be chosen. If $f_d = 1$, then the department quota q_d is exhausted, and t_d may be greater than one.

Minimal Housing Cutoff if Housing Quota is not Filled

If a housing quota of a college is not completely filled, then the housing cutoff of each department from this college should be equal to its department cutoff.

$$f_c \in \{0, 1\}, \; \forall c \in C \tag{1.13}$$

$$f_c \cdot q^H_c \leq \sum_{d \in c} \sum_{a \in A} \xi_{(a,d,1)}, \; \forall c \in C \tag{1.14}$$

$$t^H_d \leq \left(1 - f_{c(d)}\right) \cdot t_d + f_{c(d)} \cdot (|\mathcal{X}| + 1), \; \forall d \in D \tag{1.15}$$

If $f_{c(d)} = 0$, then $t^H_d = t_d$, so d has a common cutoff for any acceptable contract. If $f_{c(d)} = 1$, then the housing quota q^H_c is exhausted, and t^H_d may be greater than

t_d.

Fairness and Non-Wastefulness

If an applicant a is not admitted with a contract x or better, then $v_x < t^H_{x_D}$, and $v_x < t_{x_D}$, if $x_I = 0$.

$$v_x + 1 \leq (1 - x_I) \cdot t_{x_D} + x_I \cdot t^H_{x_D} + \left(\sum_{(d,i)\,:\,u_{(x_A,d,i)} \geq u_x} \xi_{(x_A,d,i)} \right) \cdot (|\mathcal{X}| + 1), \forall x \in \mathcal{X} \quad (1.16)$$

Note that, first, an applicant x_A is not admitted with a contract x or better if and only if $\sum_{(d,i)\,:\,u_{(x_A,d,i)} \geq u_x} \xi_{(x_A,d,i)} = 0$, and, second, this sum may be equal to either zero or one. So, if this sum is equal to one, then (1.16) always holds; else, if this sum is equal to zero, then $v_x < t^H_{x_D}$, and $v_x < t_{x_D}$, if $x_I = 0$.

Now we show that there are no ($\varnothing$-by-NH)-, (NH-by-NH)-, (H-by-NH)-, or (H-by-H)-blocking contracts for a matching μ that satisfies (1.1)-(1.16).

- Suppose that $x = (a, d, 0)$ is **($\varnothing$-by-NH)-blocking** for μ. Thus, by Definition 5, department quota q_d is not filled. Thus, by (1.11), $f_d = 0$. Thus, by (1.12) the department cutoff is minimal: $t_d = 1$.

  Also, since x is ($\varnothing$-by-NH)-blocking, then, by (1.16), $v_x < t_d$.

  Combining both results above we get that $v_x < t_d = 1$, which is impossible. Contradiction.

- Suppose that $x = (a, d, 0)$ is **(NH-by-NH)-blocking** for μ. Thus, by (1.16), $v_x < t_d$. So, there cannot be a chosen contract $x' = (a', d, 0) \in \mu$, such that, first, $a \; P_a \; a'$, which implies that $v_{x'} < v_x$, and, second, $v_{x'} \geq t_d$. Contradiction.

- Suppose that $x = (a, d, 0)$ is **(H-by-NH)-blocking** for μ. Thus, by (1.16), $v_x < t_d \leq t^H_d$. So, there cannot be a chosen contract $x' = (a', d, 1) \in \mu$, such that, first, $a \; P_a \; a'$, which implies that $v_{x'} < v_x$, and, second, $v_{x'} \geq t_d^H$. Contradiction.

- Suppose that $x = (a, d, 1)$ is **(H-by-H)-blocking** for μ. Thus, by (1.16), $v_x < t_d^H$. So, there cannot be a chosen contract $x' = (a', d, 1) \in \mu$, such that, first, $a \; P_a \; a'$, which implies that $v_{x'} < v_x$, and, second, $v_{x'} \geq t_d^H$. Contradiction.

What about (NH-by-H)-, and ($\varnothing$-by-H)-blocking contracts?

- Suppose that $x = (a, d, 1)$ is **(NH-by-H)-blocking** for μ. Thus, by (1.16), $v_x < t_d^H$. We have two cases.
  - If housing quota $q_{c(d)}^H$ is not filled, then, by (1.14), $f_c = 0$. Thus, by (1.15), $t_d^H = t_d$. So, there cannot be a chosen contract $x' = (a', d, 0) \in \mu$, such that, first, $a \; P_a \; a'$, which implies that $v_{x'} < v_x$, and, second, $v_{x'} \geq t_d$. Contradiction.
  - If housing quota $q_{c(d)}^H$ is exhausted, it should be that $t_d^H > t_d$ (otherwise, we get a contradiction as above). Also, there should be a chosen contract $x' = (a', d, 0) \in \mu$, such that $a \; P_a \; a'$, which implies that $t_d \leq v_{x'} < v_x < t_d^H$. By Definition 3, an allocation $(\mu \cup x) \backslash \{x', \mu_a\}$ should be a matching. Thus, for (1.4) to hold, an applicant a should have a chosen contract $x'' = (a, d'', 1) = \mu_a$, such that departments d'' and d belong to the same college: $c((\mu_a)_D) = c(d)$.

- Suppose that $x = (a, d, 1)$ is **($\varnothing$-by-H)-blocking** for μ. Thus, by (1.16), $v_x < t_d^H$. Also, by Definition 5, department quota q_d is not filled. Thus, by (1.11), $f_d = 0$. Thus, by (1.12) the department cutoff is minimal: $t_d = 1$. We have two cases.
  - If housing quota $q_{c(d)}^H$ is not filled, then, by (1.14), $f_c = 0$. Thus, by (1.15), $t_d^H = t_d = 1$. So, v_x should be smaller than $t_d^H = 1$, which is impossible. Contradiction.

– If housing quota $q^H_{c(d)}$ is exhausted, it should be that $t^H_d > t_d = 1$, so $1 = t_d \leq v_x < t^H_d$ (otherwise, we get a contradiction as above). By Definition 5, an allocation $(\mu \cup x) \backslash \mu_a$ should be a matching. Thus, for (1.4) to hold, an applicant a should have a chosen contract $x'' = (a, d'', 1) = \mu_a$, such that departments d'' and d belong to the same college: $(\mu_a)_I = 1$ and $c((\mu_a)_D) = c(d)$.

As a result, in order to completely eliminate (NH-by-H)-, and (∅-by-H)-blocking contracts the following constraint should be imposed:

$$x_I \cdot f_{c(x_D)} \cdot \left(1 - \sum_{(d,i)\,:\,u_{(x_A,d,i)} \geq u_x} \xi_{(x_A,d,i)}\right) \cdot \left(\sum_{d \in c(x_D)} \xi_{(x_A,d,1)}\right) \leq 0,\ \forall x \in \mathcal{X} \quad (1.17)$$

It implies that there should not be a contract x, such that, first, $x_I = 1$, second, housing quota $q^H_{c(x_D)}$ is exhausted and $t_d < t^H_d$, third, an applicant x_A is not admitted with a contract x or better, and, fourth, $(\mu_{x_A})_I = 1$ and $c((\mu_{x_A})_D) = c(d)$.

It is straightforward to check that any stable matching satisfies (1.1)–(1.17).

Picking the Best Contracts for Applicants

In order to get an undominated matching we need to pick the best possible contract for every applicant. Thus, the following IP optimization problem produces an undominated stable matching if it exists:

$$\begin{aligned} &\max_{\substack{\text{binary:}\{\xi_x\}_{x\in\mathcal{X}},\{f_d\}_{d\in D},\{f_c\}_{c\in C} \\ \text{integer:}\{t_d, t^H_d\}_{d\in D}}} \sum_{x\in\mathcal{X}} u_x \cdot \xi_x, \\ &\text{s.t. } (1.1)\text{–}(1.17). \end{aligned} \quad (1.18)$$

1.5.2 Maximal Trimmed Sub-Market

If there is no stable matching, then we need to find a maximal trimmed sub-market. So, for each department d we add an integer lower bound $\bar{t}_d^H \geq 1$ for a dormitory cutoff t_d^H, which will play the role of a threshold separating acceptable and unacceptable contracts with $i = 1$: a contract $x = (a, d, 1)$ becomes unacceptable under obtained trimmed sub-market if $v_x < \bar{t}_d^H$. In order to find a maximal trimmed sub-market, we will try to make all these bounds as low as possible.

Thus, we need to introduce integer lower bounds with the following constraints:

$$1 \leq \bar{t}_d^H, \ \forall d \in D \tag{1.19}$$

$$\bar{t}_d^H \leq t_d^H, \ \forall d \in D \tag{1.20}$$

There always exists a matching without (∅-by-NH)-, (NH-by-NH)-, (H-by-H)-, and (H-by-NH)-blocking contracts – a weakly stable matching. As a result, we need to take care about (NH-by-H)- and (∅-by-H)-blocking contracts, which absence was controlled before by constraints (1.15) and (1.17).

Thus, first, we need to make sure that, if for some department d the corresponding housing quota $q_{c(d)}^H$ is not completely filled, then $t_d^H = \max\{t_d, \bar{t}_d^H\}$. So, instead of (1.15) we impose the following constraints.

$$f_d^H \in \{0, 1\}, \ \forall d \in D \tag{1.21}$$

$$0 \leq \left(\bar{t}_d^H - t_d\right) \cdot f_d^H, \ \forall d \in D \tag{1.22}$$

$$t_d^H \leq \left(1 - f_{c(d)}\right) \cdot \left(f_d^H \cdot \bar{t}_d^H + \left(1 - f_d^H\right) \cdot t_d\right) + f_{c(d)} \cdot (|\mathcal{X}| + 1), \ \forall d \in D \tag{1.23}$$

Second, in order for the matching to be sub-market stable it should be induced by an unconstrained set of cutoffs. Thus, by Definition 19 we need to make sure for each department d that if both quotas q_d and $q^H_{c(d)}$ are not exhausted, then t^H_d is still one, thus we need the lower bound $\bar{t}^H_d = 1$.

$$(1 - f_d) \cdot \big(1 - f_{c(d)}\big) \cdot \bar{t}^H_d \leq 1, \; \forall d \in D \tag{1.24}$$

Third, a (NH-by-H)- or (∅-by-H)-blocking contract x should no longer be considered if $v_x < \bar{t}^H_d$. So, instead of (1.17) we impose the following constraint.

$$x_I \cdot f_{c(x_D)} \cdot \left(1 - \sum_{(d,i)\,:\,u_{(x_A,d,i)} \geq u_x} \xi_{(x_A,d,i)}\right) \cdot \left(\sum_{d \in c(x_D)} \xi_{(x_A,d,1)}\right) \cdot \left(v_x - f^H_d \cdot \left(\bar{t}^H_d - 1\right)\right) \leq 0, \; \forall x \in \mathcal{X}, \text{ s.t. } x_D \in D \tag{1.25}$$

Note that for any $d \in D$ constraints (1.23) and (1.25) reduce to (1.15) and (1.17) respectively if $f^H_d = 0$. So, new constraints (1.19)–(1.25) play a new role only if $f^H_d = 1$, which, by (1.22), may happen only if $\bar{t}^H_d \geq t_d$.

In order to find a maximal trimmed sub-market, we need to minimize the sum of all lower bounds. So, an optimization problem is:

$$\min_{\substack{\text{binary:}\{\xi_x\}_{x \in \mathcal{X}}, \{f_d, f^H_d\}_{d \in D}, \{f_c\}_{c \in C} \\ \text{integer:}\{t_d, t^H_d, \bar{t}^H_d\}_{d \in D}}} \sum_{d \in D} \bar{t}^H_d, \quad \text{s.t. } (1.1)\text{–}(1.14), (1.16), (1.19)\text{–}(1.25). \tag{1.26}$$

1.5.3 Sub-Market Stable Mechanism

Fix a CAH Δ. Consider the following sub-market-IP (SM-IP) mechanism.

1. Solve the problem (1.26) under Δ and find all minimal lower bounds $\{\bar{t}^H_d\}_{d \in D}$.

2. Take a trimmed sub-market $\Delta' \in Tr(\Delta)$, where any contract $(a, d, 1) \in \mathcal{X}$

is unacceptable under Δ' if $v_{(a,d,1)} < \bar{t}_d^H$.

3. Solve the problem (1.18) under Δ' and find $\{\xi_x\}_{x \in \mathcal{X}}$ (as a starting point one can pick a stable under Δ' matching obtained during the first step).

4. The resulting matching is $\mu^{SM-IP} = \{x \in \mathcal{X} \mid \xi_x = 1\}$.

Theorem 3. *An SM-IP mechanism produces a sub-market undominated stable matching. Moreover, it produces an undominated stable matching if one exists.*

Proof. By construction of the problem (1.26) a trimmed sub-market obtained during step 2 of SM-IP is a maximal trimmed sub-market. Also, by construction of the problem (1.18) a resulting matching obtained during step 4 of SM-IP is an undominated stable matching under this maximal trimmed sub-market. Thus, it is a sub-market undominated stable matching.

By Definition 13, a sub-market undominated stable matching is an undominated stable matching if one exists. □

1.6 Discussion and Concluding Remarks

In this paper I introduced a brand-new setting of the two-sided many-to-one matching problem with contracts and aggregate constraints, which I refer to as the *college admissions with housing quotas* (CAH). This model allows applicants to report their housing preferences for any department, and imposes housing quotas on colleges, each containing one or several departments. The CAH is inspired by college admissions, where applicants are applying to specific departments and colleges have their common dorms. I investigate a centralized admissions process under two types of constraints. For the case of *single-department housing constraints*, i.e. when each department possesses its own housing quota, I construct the student-optimal stable and strategy-proof SDAH mechanism that respects improvements and produces the unique solution for any given market. This mechanism can be used not only in countries with centralized admissions (Germany), but also in countries with fully decentralized admissions, even if applicants are applying directly to colleges, as in the United States. For instance, the severe housing shortage at the University of California-Berkeley can be overcome by adopting the SDAH procedure.[28,29]

In contrast, under the general problem, there may not be a stable matching. Thus, I first analyze a version of the SDAH mechanism under general problem (SDAH(G)) and find that it is strategy-proof, respects improvements and tolerates only the following blocking contracts: when a contract with a housing option blocks either some contract without such option ((NH-by-H)-blocking) or some unfilled slot ((∅-by-H)-blocking). In order for the former type of blocking to be allowed we can demand the following: if an applicant has a justified envy towards someone, then she should not require strictly more resources. As for the latter type of blocking, note that in order for an applicant to get an empty slot with housing she should get some unfilled housing slot from a college. In general, to resolve such kinds of justified envy or justified claim the department must apply directly to its college and request additional housing. Thus, in order to disable such blocking contracts, the authorities may simply forbid colleges to participate in the secondary market: after admissions, applicants will be able to exchange only with their possessions (a place at a specific department, and housing at the corresponding college (if any)). Such approach yields the first relaxation of stability: *Take-House-from-Applicant-stability.*

Moreover, it turns out that we can always restrict the (∅-by-H) type of blocking. In order to do that, I adapt the *weak stability* (Kamada and Kojima, 2017) and propose a stronger *Not-Compromised-Request-from-One-Agent-stability*, and prove their existence with a constructed NC-RfOA-stable mechanism CUT. Proposition 12 shows that there is no version of the cumulative offer process of Hatfield

[28] On June 23, 2022, the Mercury News reported: *"Happy to be admitted, incoming Cal students worry about future cramped quarters, living in cars or enduring long commutes that might exclude them from campus life".*

[29] The SDAH will for sure terminate in two steps, because any applicant has at most two acceptable contracts for UC-Berkeley.

and Milgrom (2005) that always finds some weakly stable matching.

No weakly stable matching has a ($\varnothing$-by-H)-blocking contract, such that the corresponding applicant is not already admitted to a department from the same college. That is, there may not be an applicant who claims an empty slot at a college while being assigned to some place outside of this college. Thus, weak stability eliminates such kinds of inter-college mobility.

Furthermore, no NC-RfOA-stable matching has a ($\varnothing$-by-H)-blocking contract, such that there is a second contract with housing for a corresponding department, such that, first, it is better for this department than the blocking contract and, second, an applicant from the second contract would prefer to leave his match and choose this contract, but by doing so would violate feasibility. In other words, under a NC-RfOA-stable matching, a department cannot admit an applicant who claims an empty slot and needs housing, and feasibility will hold after admitting her only if it cannot admit someone better than her who would also like to be admitted with housing. We can justify this in the following way. Suppose that each time an applicant with housing is transferred from one department to the other, she should be evicted, because she cannot possess a housing slot at a college if she is not currently admitted to some department at that college. Once she is evicted, she vacates the housing slot of the college, which automatically turns the contract of the better applicant into a ($\varnothing$-by-H)-blocking contract. Thus, she cannot be allowed to transfer.

Then, I construct a novel concept of *sub-market stability* based on the idea of minimizing the department ranks of blocking contracts for a weakly stable matching. Recall that departments' priorities are induced by applicants test scores: the better an applicant performed during exams, the higher she will be in the corresponding departments rankings. Thus, the lower the rank an applicant has, the less effort she has made in order to get into this department. In addition, each blocking contract the applicant increases her frustration with the final matching. Under the sub-market stability, I try to maximally eliminate frustration for applicants with high test scores, while keeping the matching weakly stable. It is proven that the sub-market stability is stronger than the NC-RfOA-stability and, moreover, there always exists a sub-market stable solution (not necessarily unique). Furthermore, I find an integer programming based mechanism SM-IP that always yields a sub-market stable matching. This solution can be used by policymakers under a flexible distribution of college housing seats in order to maximally eliminate disappointment of more motivated applicants with the final allocation.

Of course, the proposed setting does not just apply to the matching of applicants to departments. For instance, it also models the Japanese hospital-resident matching market (Kamada and Kojima, 2015), the summer intern research programs in Australia without overlapping Aziz et al. (2021), and college admissions with scholarships in Hungary Biró (2012). In general it can be applied to any two-sided many-to-one matching market with a strict merit system, disjoint regions, and regionally disjoint sets of scarce indivisible goods to be distributed. Thus, the mechanisms introduced in this paper can be of great use for policymakers.

Besides, it will be interesting to explore more general settings with three and more possible contracts for the same applicant-department pair (e.g. with different levels of funding), or – with overlapping regions (e.g. in Turkey some colleges have both: their own dorms and access to city dorms, which are available to all colleges nearby). Furthermore, it is worth scrutinizing the existence of a non-IP-based sub-market stable mechanism.

1.7 Appendix: Russian College Admissions

In this section I describe the *Russian college admissions procedure* (RCA) that was firstly introduced during the admissions campaign in 2021. I formalize it in the form of a dynamic game and show that any equilibrium outcome is stable and can be induced by some *passive* strategy profile. Unfortunately, it turns out that the most natural *straightforward* passive strategy profile is not a Nash equilibrium of this game. This result may be one of the reasons for very unfair outcomes that took place after college admissions campaigns in 2021 and 2022.

Since any applicant in Russia must be admitted to a particular department, and many colleges have their own common dormitories, instead of the malfunctioning RCA procedure, I propose to adopt one of the mechanisms developed in this paper: SDAH(G), CUT, or SM-IP. In addition, Appendix 1.7.7 proposes an iterative version of SDAH that operates under single-department constraints and yields the student-optimal stable matching as a result of the unique ex-post equilibrium.

1.7.1 Description of the RCA Procedure

Before the RCA procedure starts the following pieces of the economy are known (at least by the authorities) and fixed: the set of all applicants A; the set of all departments D (together with the set of all colleges C); the collection of all potentially acceptable departments for each applicant $\{D_a\}_{a \in A}$;[30] all departments priority rankings over applicants $\{P_d\}_{d \in D}$;[31] all departments quotas $\{q_d\}_{d \in D}$. The current RCA procedure does not consider the dormitory aspect, so for our model this implies that $q_c^H = 0$ for any college $c \in C$, and $x_I = 0$ for any contract.[32] Thus, before introducing dormitory constraints in RCA we will assume that only contracts from $A \times D \times \{0\}$ may be considered by any applicant (or department), and will drop zero from it.

Also, the following time points are announced: the deadline of the whole procedure $T > 1$, and deadlines of each department $\{T_d\}_{d \in D}$, where $1 < T_d \leq T$ for any $d \in D$. Finally, the shortlist constraint is announced: the maximal amount of departments $k \geq 1$ that each applicant a may submit in her unranked list $\widetilde{D_a} \subseteq D_a$, $|\widetilde{D_a}| \leq k$ during the initial stage of the procedure; so, only departments from this list may be considered by a afterwards during RCA.[33]

[30]Each applicant has already taken all her exams.

[31]Before the procedure each department announces its rules for constructing its ranking (based on exam results). By combining these rules with exam results we obtain the ranking itself.

[32]As discussed below, the presence of the shortlist constraint allows colleges to deal with the dormitory places distribution after the enrollment (by giving up fairness of the final allocation).

[33]The actual constraint is even more restrictive: any applicant a may choose no more than 5

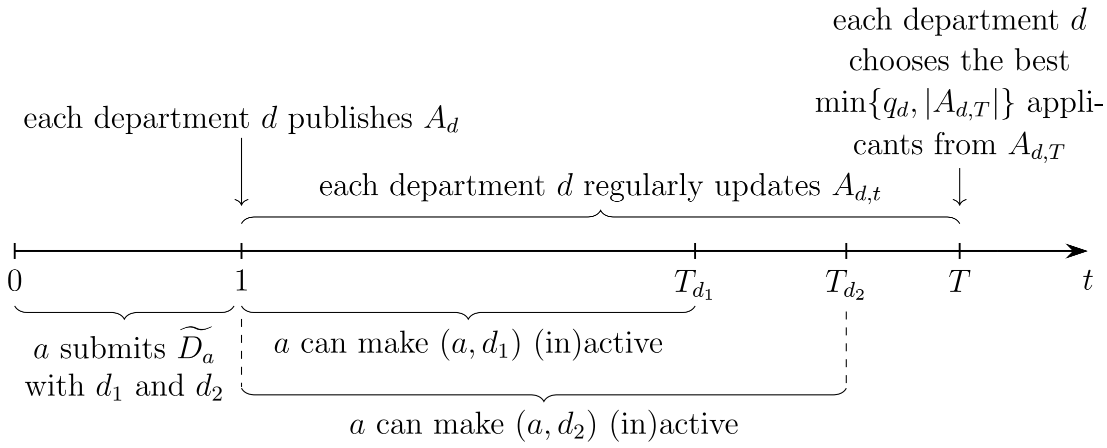

Figure 1.1: Sketch of a timeline of the RCA procedure for an applicant a.

To sum up, prior to the procedure, the following tuple is known (at least by the authorities) and fixed $\langle A, D, \{D_a\}_{a \in A}, \{q_d, P_d, T_d\}_{d \in D}, T, k\rangle$.

Now, the RCA procedure starts. During the initial stage (taking place between time points $t = 0$ and $t = 1$), each applicant a submits her unranked list $\widetilde{D_a}$ with no more than k departments to the central authorities. At $t = 1$, each department d publishes its list of applicants $A_d = \{a \mid d \in \widetilde{D_a}\}$ and ranks it according to P_d.

Take any applicant a. She can only consider contracts from $\{a\} \times \widetilde{D_a}$. Before $t = 1$ all these contracts are *inactive*. For any $d \in \widetilde{D_a}$ during the time period $[1, T_d)$ she can make a contract (a, d) *active* or *inactive*, so that at any moment of time $t \in [1, T)$ she has no more than one active contract from $\{a\} \times \widetilde{D_a}$.

Throughout the period $[1, T]$, each department d is required to regularly publish its list $A_{d,t} \subseteq A_d$ with all applicants from A_d that have an active contract with d at the current time point t.[34]

At the deadline $t = T$ each department d accepts the best $\min\{q_d, |A_{d,T}|\}$ applicants from $A_{d,T}$ (with active contracts with d). All other applicants are rejected. The matching is finalized. The sketch of the RCA timeline for an applicant is depicted in Figure 1.1.

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colleges and include no more than 10 departments from each of these colleges in $\widetilde{D_a}$.

[34]For 2021 the rule was: at least 6 times a day. For 2022, things were different: the list only needs to be updated when it changes.

1.7.2 Debates on 2021 Russian College Admissions

The newly implemented Russian college admissions procedure ended on August 11th, 2021. A week later, on August 18th, *Izvestia* newspaper published an article named "Why applicants with a high United State Exams (USE) score could not get into universities".[35] This article contains numerous testimonies from disgruntled applicants, and, furthermore, names the 2021 admissions procedure the *"August 11th lottery"*. At the same time, it pointed out that *"rectors of Russian universities recognized the admission campaign as successful"*.

On August 26th, *Kommersant* newspaper published an article named "Universities are looking for applicants" that starts with the following:

> *A number of universities, including leading ones, could not recruit the required number of first-year students. …The Minister of Education and Science [Falkov] believes that "disappointment and difficulties" are inevitable for any mechanism of enrollment.*

Besides unfairness and wastefulness, another problem with the final allocation was that some of the best departments experienced a sharp drop in cutoff scores. For example, the cutoff for the Department of Mechanics of the Faculty of Mechanics and Mathematics of the Moscow State University (MSU) dropped from 347 to 270 (out of 410).[36] This is by far the lowest cutoff over the last 12 years, even though normally cutoffs have shown an upward trend over the years. This is how this situation was described in the official group of the Faculty of Mechanics and Mathematics in the social network VK:[37]

> *In fact, everyone, who wanted to, was enrolled. Let's just say it's a disaster. Hell froze over.*

[35] USE reform is discussed in Ampilogov et al. (2013) and Francesconi et al. (2019).
[36] MSU was ranked 33rd best in the world in physics according to the QS World University Rankings by Subject 2021, and the top-1 in Russia according to the QS World University Rankings.
[37] Russian substitute for Facebook.

Despite all the difficulties, some colleges have greatly simplified the life of applicants due to the competent actions of admission committees. A perfect example is my alma mater, the NRU "Higher School of Economics" (HSE). Shortly after the publication of the ranked lists of students, each department also published two lists of applicants, and was updating them throughout the whole process of admission:

- the "Green list" with names of applicants who will be admitted for sure if they want to;
- the "Yellow list" with names of applicants, such that *"HSE cannot yet guarantee admission, but assesses their chances of admission as significant"*.[38]

Obviously, this approach required an increased workload on the members of admission committees. For instance, they were directly calling and asking applicants about the likelihood of applying with an active contract to a department under question. Furthermore, as soon as an applicant appeared on one of these lists, she received an email about it from the admissions office. Needless to say, such strategy resulted in no significant drop in cutoffs, and made life much easier for applicants. Unfortunately, I am not aware of any other college that applied a similar strategy.

However, despite all the problems, on August 27th the Russian news agency *TASS* reported the following:

> *In general, the 2021 admission campaign can already be considered successful, Falkov noted. "The final results of this year's admission campaign - it was recognized as successful, according to the absolute majority of both applicants and rectors", the minister said.*

As for the applicants themselves, shortly after August 11th they created a Change.org petition to the Ministry of Science and Higher Education of the Rus-

[38]This is an analog of a waiting list.

sian Federation named “Make the college admissions procedure fair and transparent - it’s not difficult”.[39] In the text of this petition, the authors provide an impressive list of media that have covered the problem, and offer the following solution: student-proposing deferred acceptance (SDA).

1.7.3 Adding Housing Quotas

To simplify the analysis and focus on the main issue of the RCA procedure we drop the shortlist constraint (set $k = \infty$), so there will be no need for the initial stage: for any applicant a we will set $\widetilde{D_a} = D_a$.

One way to justify it is the following. Some papers (Haeringer and Klijn, 2009; Calsamiglia et al., 2010; Beyhaghi et al., 2017) show that imposing a shortlist restriction leads to failure of the strategy-proofness of SDA due to precautionary behavior of applicants. Using theoretical and experimental approaches the authors find that applicants tend to apply mostly to departments where they are most likely to get accepted, together with a few top-quality departments, and a few “safe” low-quality departments.

So, it seems reasonable to assume that in real-life applications of the RCA procedure (which also has a student-proposing essence) applicants also demonstrate such precautionary behavior.[40] A shortlist constraint motivates an applicant to mainly apply to departments, that will most likely admit her and that are the most convenient ones. Such precautions may result in an unfair final outcome, because, for instance, many strong applicants simply do not apply to top departments.

As a result, a lot of applicants apply to departments in their hometowns, where they do not need college housing. The absence of an over demand for housing places motivates departments not to take it into account during the admission process, and distribute these places afterwards across already admitted students.

[39]Over 26 thousand people have signed it as of today. Unfortunately, this does not seem much. The problem, I suspect, is in the nature of this issue: almost all applicants face the college admissions process only once in a lifetime. So, they do not have much motivation to fix it, because it will not affect their own assignment.

[40]Indeed, many applicants testimonies on television and in the press support this assumption.

However, in general there is a need to respect housing constraints, as was mentioned by a member of the admission committee of Bauman MSTU during an interview on *Russia 1* TV channel:

> *"Suppose that a centralized mechanism assigned you to a department where you need housing, but you are not good enough to get it. There is no way for you to change this assignment, so you will not study anywhere".*[41]

Thus, to compensate for the lack of a shortlist constraint, we will analyze the RCA procedure in the more general context of single-department housing constraints. So, now each department has its own housing quota $q_d^H \leq q_d$.

1.7.4 RCA Dynamic Game

This section presents a discrete-time dynamic game induced by the RCA procedure without a shortlist constraint.

There are $T \geq 1$ periods in the Russian college admissions game (RCG). Before the game starts the following tuple is known for everyone $\langle A, D, \{D_a\}_{a \in A}, \{q_d, q_d^H, T_d\}_{d \in D}, T \rangle$, where for each $d \in D$ we have integer $1 \leq T_d \leq T$. Also, each department d publishes its list of applicants $A_d = \{a \mid d \in D_a\}$ and ranks it according to its preferences P_d. So, during the game each applicant $a \in A$ can consider contracts only from the set $\{a\} \times D_a \times \{0, 1\}$. A department d is called *available* at the (end of a) period t if $t < T_d$.

The T periods of RCG are designed as follows.

- *Period 1:* All contracts are inactive. Each applicant $a \in A$ can choose one contract from the set $\{a\} \times D_a \times \{0, 1\}$ and make it active. Each applicant makes a decision. Denote by X_1 the set of all contracts that are now active. Each available department d publishes all active contracts with it: $X_{d,1} =$

[41] This was an argument against using a centralized mechanism for college admissions in Russia.

$(X_1)_d = \{x \in X_1 \mid x_D = d\}$.[42]

...

- *Period t:* Take an applicant a. If she does not have an active contract, $|(X_{(t-1)})_a| = 0$, then she can make a contract $x \in \{a\} \times D_a \times \{0,1\}$ active only if $t \leq T_{x_D}$.

  If she has an active contract $x' \in (X_{(t-1)})_a$, then she can make it inactive only if $t \leq T_{x'_D}$. If she does so, then she can make a contract $x \in \{a\} \times D_a \times \{0,1\}$ active only if $t \leq T_{x_D}$.

  Each applicant makes a decision. Denote by X_t the set of all contracts that are now active. Each available department d publishes all active contracts with it: $X_{d,t} = (X_t)_d = \{x \in X_t \mid x_D = d\}$.[43]

...

- *Period T:* Take an applicant a. If she does not have an active contract, $|(X_{(T-1)})_a| = 0$, then she can make a contract $x \in \{a\} \times D_a \times \{0,1\}$ active only if $T_{x_D} = T$.

  If she has an active contract $x' \in (X_{(t-1)})_a$, then she can make it inactive only if $T_{x'_D} = T$. If she does so, then she can make a contract $x \in \{a\} \times D_a \times \{0,1\}$ active only if $T_{x_D} = T$.

  Each applicant makes a decision. Denote by X_T the set of contracts that are now active. Each department d chooses $Ch^*_d(X_{d,T})$, assuming that all active contracts are acceptable ($x \in X_{d,T}$ implies $x \, P_{x_D} \, \varnothing$), where $X_{d,T} = (X_T)_d = \{x \in X_T \mid x_D = d\}$. All other contracts are rejected. The assignment is finalized.

The RCG timeline for an applicant is depicted in Figure 1.2.

[42] So, now each applicant knows a list $X_{d,1}$ ranked according to P_d for each available department d.

[43] Only departments with deadlines at t are not announcing their $X_{d,t}$. This aspect guarantees that any Nash equilibrium of this game yields a stable outcome.

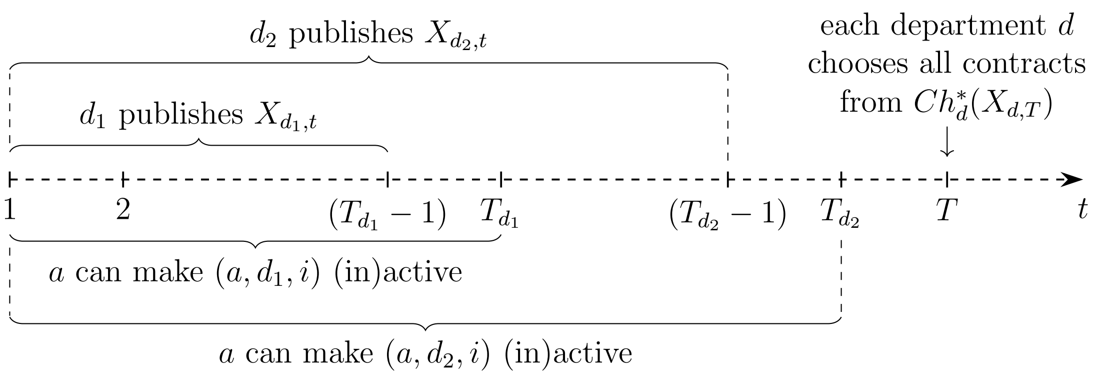


Figure 1.2: Sketch of RCG timeline for an applicant a.

Take any $t \in \{2, ..., T\}$, a *history* at the period t is $\eta_t = \cup_{d\in D}(\cup_{\tau<\min\{T_d,t\}}\{X_{d,\tau}\})$ – a set with all ranked sets of active contracts of all departments from the first period up to the deadline of a given department. Any applicant knows η_t at the beginning of a period t. Set $\eta_1 = \{\}$.

Also, at the beginning of a period t an applicant a knows whether she has an active contract x with $x_A = a$ and $x_D = d$, such that $T_{x_D} < t$. For any $t \in \{2, ..., T\}$ the set of all such active contracts is $X_{a,t} = \cup_{d:T_d<t}\{x \mid x_A = a, x \in X_{d,T_d}\}$.[44] This information is unknown to any other applicant $a' \neq a$ up to the end of RCG. Set $X_{a,1} = \{\}$ for any $a \in A$.

A *strategy* of an applicant a under RCG is a mapping σ_a that for any fixed $t \in \{1, 2, ..., T\}$, for each pair of a history η_t and a set $X_{a,t}$ gives some $x \in (\{a\} \times D_a \times \{0, 1\}) \cup \{\varnothing\}$, such that $x \in X_{a,t}$, if $|X_{a,t}| = 1$, because it is not allowed to change an active contract for a department with passed deadline; and either $x_D \in \{d \mid T_d \geq t\}$ or $x = \varnothing$, otherwise. If $\sigma_a(\eta_t, X_{a,t})$ is a contract, then it is the only active contract for a at the end of the period t, if, otherwise, $\sigma_a(\eta_t, X_{a,t}) = \varnothing$, then no contract is active for a at the end of the period t. Note that $\sigma_a(\eta_t, \{x\}) = x$ always for any $x \in \{a\} \times D_a \times \{0, 1\}$.

For the further analysis I focus on strategies, such that an applicant may decide to change her active contract during some period t (if she can) only if she understands that she would not have been admitted anywhere if $(t - 1)$ was the

[44]Note that $|X_{a,t}| \leq 1$ for any a and t, because an applicant cannot have two or more active contracts at the same time.

last period. In other words, an applicant is always trying to stay with her current choice. We call such strategy of an applicant a *passive* strategy.

Formally, a strategy σ_a is *passive* if the following holds for any $t \in \{2, ..., T\}$. If there is an active contract x with $x_A = a$ at the beginning of a period t, $x \in X_{t-1}$, then $\sigma_a(\eta_t, \{\}) \neq x$ only if $x \notin Ch^*_{x_D}(X_{x_D,t-1})$ assuming that all active contracts are acceptable. A *strategy profile* is a collection of all applicants strategies, $\{\sigma_a\}_{a \in A}$. We call a strategy profile *passive* if it contains only passive strategies.

The following result justifies our focus on passive strategy profiles.

Theorem 4. *Fix a college admissions market with single-department housing quotas. Let $\mathcal{X}^{NE}$, $\mathcal{X}^{SPNE}$, $\mathcal{X}^{SPNEp}$, and $\mathcal{X}^{Stable}$ be, respectively, the set of Nash equilibrium outcomes of RCG, the set of Subgame-Perfect Nash equilibrium outcomes of RCG, the set of passive Subgame-Perfect Nash equilibrium outcomes of RCG, and the set of all stable matchings under the true preferences. Then, $\mathcal{X}^{NE} = \mathcal{X}^{SPNE} = \mathcal{X}^{SPNEp} = \mathcal{X}^{Stable}$.*

Proof. We prove this theorem with the following lemmas.

Lemma 1. *All matchings induced at NE of RCG are stable.*

Lemma 2. *Pick an applicant a. Suppose that all other applicants have some passive strategies. If there exists a contract (a, d, i), such that $(a, d, i) \notin Ch^*_d(X_{d,t} \cup (a, d, i))$ for some $t < T$, then $(a, d, i) \notin Ch^*_d(X_{d,t+\tau} \cup (a, d, i))$ for any $\tau \in \{1, 2, \dots, T - t\}$ regardless of a's strategy.*

Lemma 3. *For each stable matching μ there exists a passive SPNE of RCG inducing μ.*

Using Lemmas 1 and 3, we get that every NE produces a stable outcome, and that every stable outcome can be produced by some passive (SP)NE. This implies the result: $\mathcal{X}^{NE} = \mathcal{X}^{SPNE} = \mathcal{X}^{SPNEp} = \mathcal{X}^{Stable}$.

This concludes the proof of the theorem. □

So, Theorem 4 implies that a passive behavior of applicants may result in any stable outcome in equilibrium.[45] Lemma 2 also implies the following nice property of the RCG.

Corollary 7. *Pick an applicant a. Suppose that all other applicants have some passive strategies. If during some period $t < T$ of RCG the tentative matching (one that would have been obtained if t was the deadline) $\mu(t)$ turned out to be stable, then a weakly dominant strategy in the following sub-game for a that is matched under $\mu(t)$ is to keep the same active contract until the deadline T. So, all following tentative matchings and the resulting matching will be stable and equal to $\mu(t)$.*

In other words, once a stable matching is formed it will not be changed under a common belief that everyone else is using a passive strategy. So, it would be perfect to have a natural passive strategy for everyone that would result in an equilibrium with the student-optimal stable outcome.

1.7.5 Straightforward Strategy

Recall the SDAH mechanism and note that RCG is also based on a student-proposing idea. Thus, in order to obtain the student-optimal stable outcome every applicant need to use the following passive strategy.

A passive strategy σ_a for an applicant a is *straightforward* if the following holds. First, for the first period: $\sigma_a(\eta_1, X_{a,1}) = x$, where x is the best acceptable contract from $\{a\} \times D_a \times \{0, 1\}$. Second, take any period $t > 1$ with a history η_t, if a has an active contract for some department d with $T_d \geq t$, $(a, d, i) \in X_{d,t-1}$, such that $(a, d, i) \notin Ch^*_d(X_{d,t-1})$, then $\sigma_a(\eta_t, \{\}) = (a, d', i')$ is the best acceptable contract from $\{a\} \times D_a \times \{0, 1\}$, such that $(a, d', i') \in Ch^*_{d'}(X_{d',t-1} \cup (a, d', i'))$ and $T_{d'} \geq t$, or, if there is no such contract, then $\sigma_a(\eta_t, \{\}) = \varnothing$.

In other words, during the first period an applicant *activates* her best acceptable contract, and, if at any of the following periods she is out of the quota in

[45] This is a generalization of Proposition 3 from Bó and Hakimov (2021).

the current department (from her active contract) and can still make this contract inactive, then she *activates* the next best possible acceptable contract that would have been chosen (if any, otherwise, she only makes the current contract inactive).

It is easy to see that for a given market under RCG a straightforward strategy profile will generate the student-optimal stable final matching only if all deadlines are far enough: all steps of the SDAH mechanism should take place.[46]

In the Russian admissions market all deadlines of RCG should be determined before applicants take their exams. Unfortunately, the following is true.

Proposition 14. *Fix the set of departments D, such that $|D| \geq 3$. For any set of deadlines $\{T\} \cup \{T_d\}_{d \in D}$ there exists a CAH with single-department housing constraints with $|A| \geq 3$, such that a straightforward strategy profile is not a NE of RCG.*

Thus, a natural straightforward behavior of applicants is not a NE of RCG under any fixed set of all deadlines. The following example illustrates how one manipulation under a straightforward strategy profile may lead to all three drawbacks of RCA that occurred in 2021: unfairness, wastefulness and low cutoffs.

Example 5. Consider a CAH with single-department housing constraints with four departments and four applicants. All deadlines and quotas are the same: $T_d = T \geq 6$, $q_d = 1$, $q_d^H = 0$ for any d. Preferences are:

a_1	a_2	a_3	a_4	d_1	d_2	d_3	d_4
d_1	d_1	d_3	d_2	a_3	a_1	a_2	a_3
d_3	d_3	d_1	d_4	a_2	a_2	a_1	a_4
d_2	d_2	d_2		a_1	a_3	a_3	
		d_4			a_4		

Table 1.6: Preferences of applicants and departments for RCA2021

[46]By Hirata and Kasuya (2014), this result holds even if we allow applicants to hesitate to choose a new active contract under a straightforward strategy and just wait for some finite amount of periods after falling out of the quota.

Under a straightforward strategy profile there will be the following sets of active contracts for each department during periods:

Period	d_1	d_2	d_3	d_4
1	a_1, a_2	a_4	a_3	-
2	a_2	a_4	a_1, a_3	-
3	a_2, a_3	a_4	a_1	-
4	a_3	a_4	a_1, a_2	-
5	a_3	a_1, a_4	a_2	-
6	a_3	a_1	a_2	a_4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	a_3	a_1	a_2	a_4

Table 1.7: Periods of RCA2021 under straightforward strategy profile

Now consider the following manipulation of a_1:

Period	d_1	d_2	d_3	d_4
1	a_2	a_4	a_3	-
2	a_2	a_4	a_3	-
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T-1$	a_2	a_4	a_3	-
T	a_2	a_4	a_1, a_3	-

Table 1.8: Periods of RCA2021 under successful manipulation of a_1

The final matching now is $\{(a_2, d_1), (a_4, d_2), (a_1, d_3)\}$. It is not fair, because (a_3, d_1) is a blocking pair. It is wasteful, because (a_3, d_4) is a blocking pair (d_4 has an empty seat that is acceptable for an unassigned applicant a_3). As for the drop in cutoffs, note that under the student-optimal matching a department d_2 has only the best applicant a_1, while now it has only the worst applicant a_4.

1.7.6 Discussion of Admissions Procedures in 2022 and 2023

Now we turn to details of the most recent Russian admissions mechanisms. First, this section talks about the 2022 Russian college admissions process and discusses undesirable features of the final allocation. Second, it introduces the upcoming Russian college admissions procedure that will take place in 2023, which differs from the original 2021 RCA and 2022 mechanisms, and briefly explains why the resulting matching will suffer from the same problems as in the previous two years.

Russian College Admissions 2022

The Russian college admissions procedure in 2022 differed from the RCA in two aspects. First, colleges were given an opportunity to combine their departments with common rankings into disjoint bundles that were playing the role of single departments. For instance, before, in 2021 an applicant Alex was able to apply separately to the Department of Computer Science and the Department of Applied Mathematics at the State College, but in 2022 these two departments were put by the State College into one High-Tech Bundle, so Alex could have only applied to this bundle, not to the two separate departments. And, if in the final matching Alex was admitted to this High-Tech Bundle, then the State College decided which of the two departments to place Alex in. This innovation obviously deprived applicants of control over the final result even further.

Second, it became much harder to switch active contracts, because the online platform was working poorly and the only option to make a new contract active without this platform was to physically deliver your diploma to a corresponding college, which was pricey and impossible to do reasonably fast. This aspect greatly reduced applicants' mobility across cities during the 2022 admissions.

As a result, all the shortcomings of the RCA procedure only intensified in 2022. The final matching was once again very unfair, wasteful, and resulted in even lower cutoffs than in 2021 for many departments (including leading ones, even in NRU HSE this time) across the country.

Upcoming Russian College Admissions 2023

On August 26, 2022, the Ministry of Science and Higher Education of the Russian Federation introduced nineteen pages of amendments to the college admissions law. First, they imposed a common deadline for all departments. As a result, any applicant will be able to reapply to any college from her shortlist until the very end of the procedure. Second, the shortlist constraint was tightened: any applicant will be allowed to apply to at most five colleges with at most five departments in each (not ten departments as before).

However, the biggest alteration was the following. The RCA23 admissions procedure will again have two stages, as before. During the first one each applicant will submit a shortlist, as before. But together with shortlists all applicants will need to report their strict rankings across chosen departments in each chosen college. For instance, an applicant a will submit: "State College 1: Department of Math P_a Department of Physics P_a Department of Computer Science; State College 2: Department of Applied Math P_a Department of Math P_a Department of Computer Science", where P_a is now a strict partial ordering on the set of contracts included in a's shortlist that can compare only contracts with departments from the same college.

After all applicants submit their shortlists with strict partial orderings, the second stage will start. At the beginning, each department will publish a ranked list of all applicants that included this department in their shortlists. During the second stage, until the common deadline each applicant will be able to have at most one active college (not one active department as before, during RCA) from her shortlist at a time, with an option to change her active college as many times as she wants. Also, each department will continuously update its published ranked list by highlighting all applicants who currently consider this college as active. Finally, at the common deadline each college will perform SDA on all its departments and all agents that still consider this college as active. The SDA choices of all colleges will constitute the final matching.[47]

Why could the resulting matching be unfair and wasteful again? Let me make the following simplifying assumptions. Assume that there will not be any shortlist constraint, so each applicant will be able to choose all acceptable departments during the first stage. Also assume that each time a department updates its ranked list, the corresponding college also publishes the current cutoff scores for all its departments obtained from he SDA procedure on all departments from this college and all agents that currently consider this college as active. This will help an applicant to understand at any given moment of the second stage whether she is tentatively admitted or not to a specific department. Otherwise, it will not be possible for an applicant, because she will not know other applicants' preferences. Finally, also assume that every applicant will submit truthful strict partial preferences during the first stage.

If all the above assumptions hold, then for any applicant the following will also hold. At any moment of the second stage, she knows for sure which college to make active now in order to be tentatively accepted by the currently best possible department (the best department, such that this applicant passes its current cutoff score), conditional on the fact that no-one else changes their active contracts at this moment. Thus, the RCA23 procedure can now be modeled again with the discrete-time dynamic game RCG with a common deadline. As a result, Example 5 implies that there can be a successful manipulation under the straightforward behavior of all other applicants, such that the resulting matching is unfair, wasteful and suffers from a sharp drop in cutoffs.

To sum up, unfortunately, the introduction of strict partial rankings to the procedure does not solve the general underlying issue of the RCA process: the most natural straightforward behavior of applicants does not constitute a Nash

[47]The choice of SDA has not been specified in law yet. So, in general, each college will have the right to use any choice rule. I picked SDA as the unique student-optimal stable choice rule.

equilibrium of the induced dynamic game.

1.7.7 Iterative Mechanism for RCA

Experimental studies suggest that, under large markets, iterative versions of strategy-proof direct mechanisms lead to a significant increase in truth-telling behaviour of agents in real life (Bó and Hakimov, 2020; Chen and Kesten, 2019; Hakimov and Kesten, 2018). In light of these studies, a natural way to enforce a truth-telling behaviour would be to make a centralized mechanism sequential: each step of an algorithm becomes a wave of an iterative procedure. In this section I propose an iterative version of the SDAH introduced above.[48]

What is the main concern about such design? As we can see, the total number of steps of SDAH that lead to a final student-optimal matching depends on the market. As the result, an iterative implementation of this mechanism may take too much time. Usually a policymaker needs to schedule the whole admissions procedure in advance (where the matching stage is just one of many); hence, there should be an option to impose a trimmed bound on the duration of this mechanism.

The way to solve this without losing the properties of an iterative SDAH is to allow it to be sequential for some fixed amount of waves, and then switch to a centralized version for the final waves that will take almost no time to simulate. However, this approach clearly has the following drawback. Consider a period right before the switch to a centralized version: each unmatched applicant now needs to submit a strict ranking list among all acceptable contracts with departments that have not rejected her yet. As noted above, in real-life applications this may lead to misrepresentation even for strategy-proof mechanisms.

To partially deal with this I suggest the following: before each wave ask all applicants who are still in the mechanism to submit such rankings. The stated motivation is that an applicant's list will only be used by the mechanism if, for

[48] Another problem may arise from the fact that some applicants do not have full information on their own preferences (Grenet et al., 2022).

whatever reason, she is not directly participating in the wave (e.g. due to personal reasons, a poor internet connection, or a website error). An applicant will be able to change her *tentative ranking* at the beginning of any wave. So, this feature will motivate applicants to think over and over again about their true rankings throughout the whole iterative part of the mechanism, and, as a result, the submitted rankings right before the switch will be more elaborate.

Now I present the iterative SDAH mechanism (ISDAH).[49] Initially all applicants are *active*. Suppose that a policymaker wants to spend no more than T periods on this procedure.

- *Wave 1:* Each department d publishes its ranking over all contracts from $(\cup_{a\in A}(\{a\} \times D_a \times \{0,1\}))_d$.

  Each applicant a is asked to submit a strict ranking P'_a among any subset of $\{a\} \times D_a \times \{0,1\}$. This will be considered as her tentative ranking over all her acceptable contracts during this wave.

  Each applicant a is asked to apply with one contract from $\{a\}\times D_a\times\{0,1\}$.[50] If the deadline of the wave has passed and applicant a has not applied anywhere, then she automatically applies with the best contract according to the previously submitted P'_a.

  Denote by $X_{d,t}$ the set of all contracts proposed to d during the wave t together with all contracts tentatively accepted by d during the wave $(t-1)$, if $t > 1$. After the deadline of the first wave each department d tentatively chooses contracts $Ch^*_d(X_{d,1})$.

...

- *Wave $t < T$:* The previous wave choices are announced: for each department d, each applicant with a contract from $Ch^*_d(X_{d,t-1})$ is tentatively accepted by d, and all other contracts are rejected. These decisions are added to all

[49]ISDAH is an adaptation of the IDAM+DA mechanism from Bó and Hakimov (2021).

[50]Note that she can even apply with a contract not listed in her tentative ranking P'_a.

departments' published rankings, and the sizes of all available quotas are updated.

If there are no rejected contracts, then end the procedure: tentative acceptances become final matches. Otherwise, continue.

Each active applicant is given a chance to change her tentative ranking P'_a among any set of contracts from $\{a\} \times D_a \times \{0, 1\}$ that are not tentatively accepted, and have not been rejected yet.

Each active rejected applicant a is asked to apply with some not previously rejected contract from $\{a\} \times D_a \times \{0, 1\}$, or become inactive. If the deadline of the wave has passed and an active rejected applicant a has not applied anywhere, then she automatically applies to the best department according to her current P'_a.[51] An active rejected applicant without a new application after the deadline becomes inactive (and will be alone in the final matching).

After the deadline each department updates its published ranking by excluding all inactive applicants. Then, each department d tentatively chooses contracts $Ch^*_d(X_{d,t})$.

...

- *Wave T:* The previous wave choices are announced: for each department d, each applicant with a contract from $Ch^*_d(X_{d,T-1})$ is tentatively accepted by d, and all other contracts are rejected. These decisions are added to all departments' published rankings, and the sizes of all available quotas are updated.

  If there are no rejected contracts, then end the procedure: tentative acceptances become final matches. Otherwise, continue.

  Each active applicant is given a chance to change her tentative ranking P'_a among any set of contracts from $\{a\} \times D_a \times \{0, 1\}$ that are not tentatively accepted, and have not been rejected yet.

[51] A tentatively accepted applicant cannot reapply during this wave.

The SDAH mechanism is performed starting from the current tentative matching using current preferences P'_a of active applicants. The resulting matching of this SDAH is the final matching of the whole procedure.

A *history* $\tilde{\eta}_t$ known by each applicant at the beginning of a period t is $\tilde{\eta}_t = \cup_{d \in D}(\cup_{\tau<t}\{X_{d,\tau}\})$ for $t \in \{2, \ldots, T\}$, and for $t = 1$: $\tilde{\eta}_1 = \{\}$. A *strategy* $\tilde{\sigma}_a$ of an active applicant a is a mapping, such that for any history $\tilde{\eta}_t$ with $t < T$ where a does not have a tentatively accepted contract at the beginning of a period t it either gives an contract from $\{a\} \times D_a \times \{0, 1\}$ that has not been rejected yet (if there is at least one such contract) or makes a inactive; and for $t = T$ gives a ranked list of some not yet rejected acceptable contracts from $\{a\} \times D_a \times \{0, 1\}$.

A *type-strategy* Σ_a of a is a function that maps any preferences P'_a over $(\{a\} \times D_a \times \{0, 1\}) \cup \{\varnothing\}$ to a strategy $\tilde{\sigma}_a$.

A strategy $\tilde{\sigma}_a$ is *straightforward with respect to preferences* P'_a if for any history $\tilde{\eta}_t$ with $t < T$ where active a does not have a tentatively accepted contract at the beginning of a period t it gives the best acceptable (according to P'_a) contract with a that has not been rejected yet, if there is at least one such contract, and makes a inactive, otherwise; and for $t = T$, if a is active, it gives preferences P'_a over the not yet rejected contracts acceptable for a (according to P'_a).

A type-strategy Σ_a of a is *straightforward* if, for every preference P'_a a resulting strategy $\Sigma_a(P'_a)$ is straightforward with respect to P'_a.

Definition 28. (Ex-post equilibrium). A type-strategy profile $\{\Sigma_a\}_{a \in A}$ is an *ex-post equilibrium* of ISDAH if for any applicant a, for any possible strategy $\tilde{\sigma}_a$ of a, and for any possible preferences of all other applicants P'_{-a}: a resulting matching from a strategy profile $\big(\Sigma_a(P_a), (\Sigma_{-a}(P'_{-a})\big)$ is weakly better for a then a resulting matching from a strategy profile $\big(\tilde{\sigma}_a, (\Sigma_{-a}(P'_{-a})\big)$.

In other words, a type-strategy profile is an ex-post equilibrium if, for any applicant, given her true preferences, the strategy yielded by her type-strategy is a best-response regardless of the preferences of all other applicants.

In contrast with Proposition 14 the following holds for ISDAH.

Proposition 15. *A straightforward type-strategy profile is an ex-post equilibrium of ISDAH for any $T \geq 1$.*

Moreover, if we assume that any active applicant with all her acceptable contracts being rejected always chooses to become inactive instead of applying with some unacceptable contract, then a straightforward type-strategy profile becomes the only ex-post equilibrium of ISDAH for any $T \geq 1$.

Proof. This result follows directly from Propositions 2 and 3 above, and Theorems 1 and 2 in Bó and Hakimov (2021). □

Proposition 15 holds because under a straightforward type-strategy profile any manipulation of an applicant is outcome-equivalent to a corresponding misrepresentation in the direct SDAH mechanism, which cannot be profitable due to Theorem 1.

As a result, if we consider a market with single-department housing constraints, the ISDAH mechanism should be chosen by a policymaker if he wants to obtain the most efficient (for applicants) stable matching, and believes that applicants do not have enough experience to compile their true strict preferences before the procedure starts. However, if he is sure that there is no problem for applicants to come up with their strict preferences, then the direct SDAH mechanism should be chosen.

1.8 Appendix: Omitted Proofs

Proposition 1. *A department choice rule is stable if and only if it is Ch_d^*.*

Proof. Take a set of contracts $X \subseteq A \times \{d\} \times \{0, 1\}$, where each applicant has at most one contract.

($\Leftarrow$) Ch_d^* is IR by construction: it considers only acceptable contracts $X' = \{x \in X \mid x \, P_{x_A} \, \varnothing\}$.

Ch_d^* is non-wasteful by construction: if there is an empty seat, $|Ch_d^*(X)| < q_d$, then there is no applicant with a contract in $X \backslash Ch_d^*(X)$, who can take this seat.

By construction of Ch_d^* the following is true.

- *Property 1:* if a contract $(a, d, 1)$ is chosen, then any contract $(a', d, i) \in X$, such that $a' \, P_d \, a$, is also chosen (for any $i \in \{0, 1\}$).

- *Property 2:* if a contract $(a, d, 0)$ is chosen, then any contract $(a', d, 0) \in X$, such that $a' \, P_d \, a$, is also chosen.

Ch_d^* is fair. Suppose, by contrary, that there is an applicant a without a contract in $Ch_d^*(X)$, who justifiably envies some applicant a' with a contract (a', d, i') in $Ch_d^*(X)$. Then, first, it should be that $a \, P_d \, a'$, and, second, there should not be a chosen contract $(a, d, i) \in X$, such that the set of contracts $Ch_d^*(X) \backslash \{(a', d, i')\} \cup \{(a, d, i)\}$ satisfies all quotas of d. There can be four cases:

1. $i = i' = 1$ is not possible by property 1;

2. $i = i' = 0$ is not possible by property 2;

3. $i = 0$, $i' = 1$ is not possible by property 1;

4. $i = 1$, $i' = 0$ is not possible, because, by construction of Ch_d^*, this implies that the whole housing quota q_d^H is filled, so a cannot justifiably envy a' by Definitions 11 and 12.

($\Rightarrow$) We need to prove that if for some department choice rule Ch_d there is a contract $x \in X$, such that either $x \notin Ch_d^*(X)$ and $x \in Ch_d(X)$, or $x \in Ch_d^*(X)$ and $x \notin Ch_d(X)$, then Ch_d is either not IR, or not non-wasteful, or not fair. So, we have two cases.

1. There exists a contract x, such that $x \notin Ch_d^*(X)$ and $x \in Ch_d(X)$. We have two cases.

   (a) If x is unacceptable, $\varnothing \, P_{x_A} \, x$, then Ch_d is not IR.

   (b) If x is acceptable, $x \, P_{x_A} \, \varnothing$, then we have two cases

i. If $x_I = 1$, then for each contract $x' \in Ch_d^*(X)$ the following holds: if $x'_I = 1$, then $x'_A \, P_d \, x_A$. Moreover $Ch_d^*(X)$ contains exactly q_d^H contracts with $i = 1$. This implies that there exists at least one contract x', such that, first, $x'_I = 1$, second, $x' \in Ch_d^*(X)$, third, $x' \notin Ch_d(X)$, and, fourth, $x'_A \, P_d \, x_A$. Thus, Ch_d is not fair.

ii. If $x_I = 0$, then for each contract $x' \in Ch_d^*(X)$ the following holds: $x'_A \, P_d \, x_A$. Moreover $Ch_d^*(X)$ contains exactly q_d contracts. This implies that there exists at least one contract x', such that, first, $x' \in Ch_d^*(X)$, second, $x' \notin Ch_d(X)$, and, third, $x'_A \, P_d \, x_A$. Thus, Ch_d is not fair.

2. There exists a contract x, such that $x \in Ch_d^*(X)$ and $x \notin Ch_d(X)$. Then, x is acceptable, $x \, P_{x_A} \, \varnothing$. We have two cases.

   (a) If $x_I = 1$, then

   i. if there exists a contract $x' \in Ch_d(X)$ with $x'_I = 1$, such that $x_A \, P_d \, x'_A$, then Ch_d is not fair;

   ii. if all contracts in $Ch_d(X)$ with $i = 1$ are better than x, then

   A. if there exists a contract $x' \in Ch_d(X)$ with $x'_I = 0$, such that $x_A \, P_d \, x'_A$, then Ch_d is not fair;

   B. if all contracts in $Ch_d(X)$ are better than x, then Ch_d is wasteful.

   (b) If $x_I = 0$, then

   i. if there exists a contract $x' \in Ch_d(X)$, such that $x_A \, P_d \, x'_A$, then Ch_d is not fair;

   ii. if all contracts in $Ch_d(X)$ are better than x, then Ch_d is wasteful.

This concludes the proof. □

Proposition 2. *Ch_d^* satisfies the law of aggregate demand.*

Proof. If $\varnothing \, P_d \, x_A$ then $Ch_d^*(X \backslash \{x\}) = Ch_d^*(X)$, because Ch_d^* is individually rational, which implies $|Ch_d^*(X \backslash \{x\})| = |Ch_d^*(X)|$. So, now we need to consider only contracts with acceptable applicants.

If $|Ch_d^*(X)| = q_d$, then $|Ch_d^*(X \backslash \{x\})| \leq |Ch_d^*(X)|$, because Ch_d^* cannot pick more than q_d applicants. So, we assume that $|Ch_d^*(X)| < q_d$.

This implies that any contract $x \in X$ with $x_A \, P_d \, \varnothing$ and $x_I = 0$ should be in $Ch_d^*(X)$, because, otherwise, Ch_d^* will be wasteful. Moreover, either all housing quota q_d^H of d is filled, or any contract $x \in X$ with $x_A \, P_d \, \varnothing$ and $x_I = 1$ is in $Ch_d^*(X)$, because, otherwise, Ch_d^* again will be wasteful.

So, we have two cases.

1. If all housing quota q_d^H of d is filled we again have two cases:
   (a) if we exclude a contract x with $x_A \, P_d \varnothing$ and $x_I = 0$, then $Ch_d^*(X \backslash \{x\}) = Ch_d^*(X) \backslash \{x\}$, because we cannot pick a new contract with a housing place; thus, $|Ch_d^*(X \backslash \{x\})| < |Ch_d^*(X)|$;
   (b) if we exclude a contract x with $x_A \, P_d \, \varnothing$ and $x_I = 1$, then either $Ch_d^*(X \backslash \{x\}) = Ch_d^*(X)$ if $x \notin Ch_d^*(X)$, because all housing quota is still filled with better contracts, or $Ch_d^*(X \backslash \{x\})$ includes at most one new contract with housing place and acceptable applicant that comes right next to x in department's d ranking over X (if such contract exists), thus $|Ch_d^*(X \backslash \{x\})| \leq |Ch_d^*(X)|$.
2. If any contract $x \in X$ with $x_A \, P_d \, \varnothing$ and $x_I = 1$ is in $Ch_d^*(X)$, then all contracts with acceptable applicants from X (there are $< q_d$ of them) are chosen by Ch_d^*. Hence, by excluding any contract with an acceptable applicant we will make a chosen set even smaller: $Ch_d^*(X \backslash \{x\}) = Ch_d^*(X) \backslash \{x\}$ for any contract x with $x_A \, P_d \, \varnothing$. Thus, $|Ch_d^*(X \backslash \{x\})| < |Ch_d^*(X)|$.

This concludes the proof. □

Proposition 3. *Ch_d^* satisfies substitutes.*

Proof. If $\varnothing \, P_d \, x_A$ then x will not be chosen regardless of any other factors. So, we consider x to have an acceptable applicant. Also, a chosen set will not change by adding a contract with an unacceptable applicant, thus we also consider z_A to be acceptable.

Denote by x^l and x^{lh} respectively the worst contract and the worst contract with a housing place ($x_I^{lh} = 1$) for d from $Ch_d^*(X)$.[52]

If $x_I = 0$ (and $x \notin Ch_d^*(X)$), then $|Ch_d^*(X)| = q_d$. We have two cases.

1. If $x_A^l \, P_d \, z_A$, then $x \notin Ch_d^*(X) = Ch_d^*(X \cup \{z\})$.

2. If $z_A \, P_d \, x_A^l$, then we have the following. We know that $x_A^l \, P_d \, x_A$, because x is not chosen. We have the following cases.

   (a) If $z_I = 0$ (and $z_A \, P_d \, x_A^l$), then $Ch_d^*(X \cup \{z\}) = (Ch_d^*(X) \backslash \{x^l\}) \cup \{z\}$, so x is not chosen.

   (b) If $z_I = 1$ and $x_A^{lh} P_d z_A$ (and $z_A P_d x_A^l$), then $x \notin Ch_d^*(X) = Ch_d^*(X \cup \{z\})$, so x is not chosen.

   (c) If $z_I = 1$ and $z_A \, P_d \, x_A^{lh}$, then $Ch_d^*(X \cup \{z\}) = (Ch_d^*(X) \backslash \{x^{lh}\}) \cup \{z\}$, so x is not chosen.

If $x_I = 1$ (and $x \notin Ch_d^*(X)$), then $x_A^{lh} \, P_d \, x_A$, since, otherwise, Ch_d^* is not stable. We have two cases.

1. If $x_A^l \, P_d \, x_A$, then above presented arguments imply that there is no contract z, such that $x \in Ch_d^*(X \cup \{z\})$.

2. If $x_A^{lh} \, P_d \, x_A \, P_d \, x_A^l$, then housing quota q_d^H is completely filled under $Ch_d^*(X)$. If $x_A^l \, P_d \, z_A$, then $x \notin Ch_d^*(X) = Ch_d^*(X \cup \{z\})$, otherwise, we have the following cases.

   (a) If $z_I = 0$ (and $z_A \, P_d \, x_A^l$), then $Ch_d^*(X \cup \{z\}) = (Ch_d^*(X) \backslash \{x^l\}) \cup \{z\}$, so x is not chosen.

[52] We have $x^l = x^{lh}$.

(b) If $z_I = 1$ and $x_A^{lh} P_d z_A$ (and $z_A P_d x_A^l$), then $x \notin Ch_d^*(X) = Ch_d^*(X \cup \{z\})$, so x is not chosen.

(c) If $z_I = 1$ and $z_A \; P_d \; x_A^{lh}$, then $Ch_d^*(X \cup \{z\}) = (Ch_d^*(X) \backslash \{x^{lh}\}) \cup \{z\}$, so x is not chosen.

This concludes the proof. □

Proposition 5. *SDAH respects improvements.*

Proof. Take an applicant a and two sets of department priorities Pr and Pr', such that Pr is an unambiguous improvement for a over Pr'. We will look at the final matchings of SDAH under each of these sets. By Theorem 2 from Hirata and Kasuya (2014) and Propositions 3 and 4 the outcome of SDAH will stay the same if we first run it on all applicants but a until it stops, and only then include a into the market.

SDAH under Pr: Suppose, we obtained a matching μ after running SDAH on all applicants but a. Now an applicant a proposes with his best acceptable contract x_1. This may induce a chain of rejections that, eventually, may end up in rejecting x_1. If it happens, then a proposes the next best acceptable contract x_2. This goes on until either all acceptable contracts of a are rejected (in this case outcome of SDAH under Pr' cannot be worse for a), or a gets accepted with some kth best acceptable contract x_k. So, I suppose that a gets some contract x_k.

SDAH under Pr': After running SDAH on all applicants but a we again obtain a matching μ. Now an applicant a proposes with his best acceptable contract x_1. Since a is weakly worse for $(x_1)_D$ than under Pr, then either x_1 will be rejected right away, or her proposal will cause exactly the same chain of rejections as above. This goes on until a is about to propose her kth best acceptable contract x_k. Thus, an outcome of SDAH under Pr' is weakly worse than under an unambiguous improvement Pr. □

Proposition 6. *There is no stable college choice rule that satisfies substitutes.*

Proof. Take a college c with two departments d_1 and d_2. Quotas are $q_{d_1} = q_{d_2} = q_c^H = 1$. Consider a set of acceptable contracts $X = \{(a_1, d_1, 1), (a_2, d_2, 1)\}$, thus for d_1 only a_1 is acceptable, and for d_2 only a_2 is acceptable. A stable choice rule should choose one of two matchings: $\mu_1 = \{(a_1, d_1, 1)\}$ or $\mu_2 = \{(a_2, d_2, 1)$.

Suppose that $Ch_c(X) = \mu_1$. Thus, a contract $(a_2, d_2, 1)$ is not chosen. Take an applicant a_3, such that $a_3 \, P_{d_1} \, a_1$. Consider the following super-set of X: $X' = X \cup (a_3, d_1, 0)$. Now, a stable choice rule should choose the following matching: $\mu' = \{(a_3, d_1, 0), (a_2, d_2, 1)\} = Ch_c(X')$. A contract $(a_2, d_2, 1)$ is chosen, thus Ch_c does not satisfy substitutes.

Otherwise, suppose that $Ch_c(X) = \mu_2$. Thus, a contract $(a_1, d_1, 1)$ is not chosen. Take an applicant a_4, such that $a_4 \, P_{d_2} \, a_2$. Consider the following super-set of X: $X'' = X \cup (a_4, d_2, 0)$. Now, a stable choice rule should choose the following matching: $\mu'' = \{(a_1, d_1, 1), (a_4, d_2, 0)\} = Ch_c(X'')$. A contract $(a_1, d_1, 1)$ is chosen, thus Ch_c does not satisfy substitutes. □

Proposition 7. *A matching does not have a (NH-by-NH)-, (H-by-H)-, and (H-by-NH)-blocking contracts if and only if it is induced by a set of cutoffs.*

Proof. ($\Leftarrow$) Take a matching μ induced by a set of cutoffs $\mathcal{T}$. We need to consider three types of blocking contracts.

1. (NH-by-NH)-blocking: Suppose that there is a (NH-by-NH)-blocking contract $(a, d, 0)$. This implies that there is an admitted contract $(a', d, 0) \in \mu$, such that $a \, P_d \, a'$, which implies that $v_{(a,d,0)} > v_{(a',d,0)} \geq t_d$. Thus, applicant a has a better contract in μ than $(a, d, 0)$ (she could have chosen it otherwise). As a result, $(a, d, 0)$ cannot be blocking. Contradiction.

2. (H-by-NH)-blocking: Suppose that there is a (H-by-NH)-blocking contract $(a, d, 0)$. This implies that there is an admitted contract $(a', d, 1) \in \mu$, such that $a \, P_d \, a'$, which implies that $v_{(a,d,0)} > v_{(a',d,1)} \geq t_d^H \geq t_d$. Thus, applicant a has a better contract in μ than $(a, d, 0)$ (she could have chosen it otherwise). As a result, $(a, d, 0)$ cannot be blocking. Contradiction.

3. <u>(H-by-H)-blocking:</u> Suppose that there is a (H-by-H)-blocking contract $(a, d, 1)$. This implies that there is an admitted contract $(a', d, 1) \in \mu$, such that $a \; P_d \; a'$, which implies that $v_{(a,d,1)} > v_{(a',d,1)} \geq t_d^H$. Thus, applicant a has a better contract in μ than $(a, d, 1)$ (she could have chosen it otherwise). As a result, $(a, d, 1)$ cannot be blocking. Contradiction.

($\Rightarrow$) Suppose that a matching μ does not have a (NH-by-NH)-, (H-by-H)-, and (H-by-NH)-blocking contracts. For any department d set a department cutoff t_d to be equal to the rank of the worst admitted contract (if it exists, otherwise set $t_d = |X| + 1$, so no contract can pass), and set a housing cutoff t_d^H to be equal to the rank of the worst admitted contract with $i = 1$ (if it exists, otherwise set $t_d^H = |X| + 1$, so no contract with $i = 1$ can pass). By construction, the obtained set of cutoffs induces μ. $\square$

Proposition 8. *There are no (NH-by-NH)-, (H-by-H)-, (H-by-NH)-, or (∅-by-NH)-blocking contracts for a resulting matching of SDAH(M).*

Proof. Let μ be a resulting matching of SDAH(M). Since μ is stable under some CAH with single-department housing constraints, then there cannot be a blocking contract with $i = 0$. Thus, there are no (NH-by-NH)-, (H-by-NH)-, or (∅-by-NH)-blocking contracts for a matching μ. Moreover, there cannot be a chosen contract with housing that is blocked by another contract with housing. Hence, there are no (H-by-H)-blocking contracts either. $\square$

Proposition 9. *A matching is weakly stable if and only if it is induced by an unconstrained set of cutoffs.*

Proof. ($\Leftarrow$) Take a matching μ induced by an unconstrained set of cutoffs $\mathcal{T}$. Proposition 7 implies that there are no (NH-by-NH)-, (H-by-H)-, or (H-by-NH)-blocking contracts for it. Also, we can have (NH-by-H)-blocking contracts. So, we need to consider two types of blocking contracts.

1. <u>(∅-by-NH)-blocking:</u> Suppose that there is a (∅-by-NH)-blocking contract $(a, d, 0)$. This implies that a quota q_d is not completely filled, which in turn

implies that $t_d = 1$. Thus, applicant a has a better contract in μ than $(a, d, 0)$ (she could have chosen it otherwise). As a result, $(a, d, 0)$ cannot be blocking. Contradiction.

2. (∅-by-H)-blocking: Suppose that there is a (∅-by-H)-blocking contract $(a, d,$ $1)$, where $d \in c$. This implies that a quota q_d is not completely filled, which in turn implies that $t_d = 1$. We have two cases.

   - If q_c^H is not completely filled, then $t_d^H = t_d = 1$. Thus, applicant a has a better contract in μ than $(a, d, 1)$ (she could have chosen it otherwise). As a result, $(a, d, 1)$ cannot be blocking. Contradiction.
   - If q_c^H is exhausted, then $(a, d, 1)$ is acceptable for a weak stability.

(⇒) Take a weakly stable matching μ. By Definition 18 there are no (NH-by-NH)-, (H-by-NH)-, and (H-by-H)-blocking contracts. Thus, by Proposition 7 μ is induced by a set of cutoffs. Also, by Definition 18 there are no (∅-by-NH)-blocking contracts, hence, for any department d with unfilled quota q_d the department cutoff t_d can be minimal: $t_d = 1$.

Now consider a department d, such that both q_d and $q_{c(d)}^H$ are not exhausted. Again, we can set $t_d = 1$. Furthermore, by Definition 5 any blocking contract is (∅-by-H)-blocking. Thus, by Definition 18 there are no blocking contracts, because $q_{c(d)}^H$ is not exhausted. As a result, we can also set $t_d^H = 1$. This implies that μ is indeed induced by an unconstrained set of cutoffs. □

Proposition 10. *For any matching induced by some set of cutoffs there exists a unique minimal set of cutoffs.*

Proof. Take any matching μ induced by some set of cutoffs. We will construct a unique minimal set of cutoffs for μ. Suppose that X is the set of all acceptable contracts.

Take any department d. Find the set $X_d' \subseteq X_d$ of all acceptable contracts for d, such that for any $x \in X_d'$ the following holds: $x \; P_{x_A} \; \mu_{x_A}$. If the set is empty, then set $t_d = t_d^H = 1$, otherwise, continue.

Take the best contract x for d from this set X'_d. If $x_I = 0$, then set $t_d = t_d^H = v_x + 1$. If, otherwise, $x_I = 1$, then set $t_d^H = v_x + 1$, and find the best contract x' for d from X'_d with $x'_I = 0$. If there is no such x', then set $t_d = 1$, otherwise, set $t_d = v_{x'} + 1$.

By construction, first, $1 \leq t_d \leq t_d^H$ for all $d \in D$, second, each applicant a will get exactly μ_a, and, third, the induced allocation will change by decreasing any of the cutoffs. □

Proposition 11. *A matching μ induced by the minimal set of cutoffs $\mathcal{T}$ is NC-RfOA-stable if and only if there does not exist a different matching induced by a set of cutoffs $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for some $d \in D$.*

Proof. ($\Rightarrow$) Take a NC-RfOA-stable matching μ that is induced by the minimal set of cutoffs $\mathcal{T}$. Also take any department d. We know that there are no blocking contracts with d and $i = 0$. This implies that $Al(\mathcal{T}_{-d})$ is not feasible. If, in addition, there are no blocking contracts with d and $i = 1$, then $Al(\mathcal{T}_{-d,H})$ is also not feasible.

Not suppose that there is a blocking contract with d and $i = 1$, but there are no ($\varnothing$-by-H)-blocking contracts with d, thus q_d is exhausted. Take the best for d (NH-by-H)-blocking contract $x = (a, d, 1)$. By construction of the minimal set of cutoffs, t_d^H should be strictly greater then v_x. If $t_d^H > v_x + 1$, then $Al(\mathcal{T}_{-d,H})$ is not feasible, since a contract with d and rank $v_d = t_d^H - 1$ is not (NH-by-H)-blocking. Otherwise, if $t_d^H = v_x + 1$, then $Al(\mathcal{T}_{-d,H})$ is not feasible, since q_d is exhausted, so we should exclude the worst accepted contract with $i = 0$, but we cannot increase t_d.

Finally, suppose that there is a ($\varnothing$-by-H)-blocking contract, thus q_d is not exhausted. Take the best for d ($\varnothing$-by-H)-blocking contract $x = (a, d, 1)$. Since mu is NC-RfOA-stable, then there exists an applicant $a' \neq a$ with an acceptable contract $x' = (a', d, 1)$, such that $a' P_d a$, and $x' P_{a'} \mu_{a'}$, but $(\mu \cup x') \backslash \mu_{a'}$ is not feasible. Take the highest ranked such contract x'' for d, such that $x'' = (a'', d, 1)\ P_{a''}\ \mu_{a''}$.

From above we conclude that $v_{x''} \geq v_{x'} > v_x$. By construction of the minimal set of cutoffs, $t_d^H = r_{x''} + 1$. We have two cases.

- If x'' is not (NH-by-H)-blocking, then $Al(\mathcal{T}_{-d,H}) = \mu \cup x''$ is not feasible, since $(\mu \cup x'')\backslash\mu_a$ is not feasible.

- If x'' is only (NH-by-H)-blocking, then $Al(\mathcal{T}_{-d,H}) = \mu \cup x''$ is not feasible, since x'' is not ($\varnothing$-by-H)-blocking.

As a result, there does not exist a different matching induced by a set of cutoffs $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for some $d \in D$.

($\Leftarrow$) Take a matching μ induced by minimal set of cutoffs $\mathcal{T}$, such that there does not exist a different matching induced by a set of cutoffs $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for some $d \in D$. By Proposition 7 there are no (NH-by-NH)-, (H-by-H)-, or (H-by-NH)-blocking contracts. Since we cannot decrease any department cutoff from T, then there is no ($\varnothing$-by-NH)-blocking contract.

Now suppose that there is a ($\varnothing$-by-H)-blocking contract $x = (a, d, 1)$, but there does not exist an applicant $a' \neq a$ with an acceptable contract $x' = (a', d, 1)$, such that $a' \, P_d \, a$, and $x' \, P_{a'} \, \mu_{a'}$, but $(\mu \cup x')\backslash\mu_{a'}$ is not feasible. Thus, q_d is not exhausted. Thus, $t_d \leq v_x$, which implies that $t_d^H = v_x + 1$. However, $(\mu \cup x)\backslash\mu_a$ is feasible, since x is ($\varnothing$-by-H)-blocking. So, t_d^H is not minimal, contradiction.

As a result, μ is NC-RfOA-stable. $\square$

Theorem 2. *Stability implies sub-market stability, which implies NC-RfOA-stability, which implies weak stability, which implies THfA-stability. Moreover, all these notions do not coincide.*

Proof. Fix a CAH Δ.

(Stability $\Rightarrow$ sub-market stability). If a matching μ is stable, then it is stable under $\Delta \in Tr(\Delta)$. Thus, Δ is a maximal trimmed sub-market of Δ. As a result, μ is sub-market stable.

(Sub-market stability $\Rightarrow$ NC-RfOA-stability). Take a sub-market stable matching μ, that is stable under some maximal trimmed sub-market $\Delta' \in Tr(\Delta)$. By

Definition 24 μ is weakly stable, thus, by Proposition 9 it is induced by an unconstrained set of cutoffs. Thus, the minimal set of cutoffs $\mathcal{T}$ that induces μ (under Δ) is also unconstrained. Next, we show that there is no different from μ matching that can be induced by any of the following sets of cutoffs: $\mathcal{T}_{-d}$ or $\mathcal{T}_{-d,H}$ for any $d \in D$. This will imply that μ is NC-RfOA-stable.

Suppose, by contrary, that $Al(\mathcal{T}_{-d}) \neq \mu$ is a matching for some d. This may happen only if, first, there exists an acceptable under Δ contract $x = (a, d, 0)$ with rank $v_x = t_d - 1$, and, second, department quota q_d is not exhausted under μ, because we need $Al(\mathcal{T}_{-d}) = (\mu \cup x)\backslash\mu_a$ to be a matching. But because $\mathcal{T}$ is unconstrained, then we should have $t_d = 1$, which implies $\mathcal{T}_{-d} = \mathcal{T}$. Thus, $Al(\mathcal{T}_{-d}) = \mu$. Contradiction.

Suppose now, by contrary, that $Al(\mathcal{T}_{-d,H}) \neq \mu$ is a matching for some d. This may happen only if there exists an acceptable under Δ contract $x = (a, d, i)$ with rank $v_x = t_d^H - 1$, such that $Al(\mathcal{T}_{-d,H}) = (\mu \cup x)\backslash\mu_a$. If $i = 0$, then $Al(\mathcal{T}_{-d}) = (\mu \cup x)\backslash\mu_a \neq \mu$, which cannot be true from above. Thus, $i = 1$. We have two cases.

- If a department quota q_d is exhausted, then for $Al(\mathcal{T}_{-d,H}) = (\mu \cup x)\backslash\mu_a$ to be a matching we should have $(\mu_a)_D = d$ and $(\mu_a)_I = 0$. Thus $Al(\mathcal{T}_{-d,H}) = (\mu \cup x)\backslash\mu_a$ is stable under Δ'', where x is now acceptable. Thus, first, $\Delta' \in Tr(\Delta'')$, and, second, $Al(\mathcal{T}_{-d,H})$ is stable under Δ'' and is induced by an unconstrained set of cutoffs under Δ. As a result, μ is not sub-market stable. Contradiction.

- If a department quota q_d is not exhausted. We have two cases.

  - If x is acceptable under Δ', then x is a blocking contract for μ under Δ'. As a result, μ is not sub-market stable. Contradiction.

  - If x is not acceptable under Δ', then x is ($\varnothing$-by-H)-blocking under Δ, then Δ' is not a maximal trimmed sub-market, because if we make x acceptable we will obtain another trimmed sub-market Δ'', such that, first, $\Delta' \in Tr(\Delta'')$, and, second, the following matching μ' is stable under Δ'' and weakly stable under Δ: $\mu' = ((\mu \cup x)\backslash \mu_a) \cup x'$, where x' is the highest ranked not chosen contact with $x'_I = 0$ and $x'_D = (\mu_a)_D$ (if it exists). It is so, because, first, $c((\mu_a)_D) = c(x_D)$, and, second, $(\mu_a)_I = 1$. As a result, μ is not sub-market stable. Contradiction.

Thus, μ is NC-RfOA-stable.

(NC-RfOA-stability $\Rightarrow$ weak stability). Holds by Corollary 3.

(Weak stability $\Rightarrow$ THfA-stability). Holds by corresponding definitions.

Example 3 shows that THfA- and weak stabilities do not coincide.

The next example shows that stability, sub-market stability, NC-RfOA-stability, and weak stability also do not coincide.

Example 6. Consider a CAH market Δ with five applicants, five departments and two colleges: $c_1 = \{d_1, d_2, d_3\}$ and $c_2 = \{d_4, d_5\}$. Quotas are $q_{d_j} = 1$ for all $j \in \{1, \ldots, 5\}$, $q^H_{c_1} = 2$, and $q^H_{c_2} = 1$. Preferences of applicants and departments over all acceptable contracts are:

a_1	a_2	a_3	a_4	a_5	$\{d_1$	d_2	$d_3\}$	$\{d_4$	$d_5\}$
$(d_1, 1)$	$(d_2, 1)$	$(d_3, 1)$	$(d_4, 1)$	$(d_5, 1)$	$(a_2, 1)$	$(a_1, 1)$	$(a_1, 1)$	$(a_5, 1)$	$(a_4, 1)$
$(d_2, 1)$	$(d_1, 1)$		$(d_5, 1)$	$(d_4, 1)$	$(a_1, 1)$	$(a_2, 1)$	$(a_3, 1)$	$(a_4, 1)$	$(a_5, 1)$
$(d_3, 1)$									

Since a sub-market of Δ that contains only applicants a_4 and a_5, and departments d_4 and d_5 from college c_2 is exactly a CAH market from Example 1, then there is no stable matching for this market.

Consider a trimmed sub-market Δ' of Δ, where only a contract $(a_4, d_4, 1)$ is no longer acceptable. The following matching is stable under Δ' and weakly stable under Δ: $\mu^{SM} = \{(a_1, d_1, 1), (a_2, d_2, 1), (a_4, d_5, 1)\}$. Thus, Δ' is a maximal trimmed sub-market, and μ^{SM} is sub-market stable.

Consider the following set of cutoffs $\mathcal{T}$: $t_{d_j} = 1$ for all $j \in \{1, \ldots, 5\}$, $t^H_{d_1} = 3$, $t^H_{d_2} = 2$, $t^H_{d_3} = 1$, $t^H_{d_4} = 3$, $t^H_{d_5} = 2$. It induces the following matching: $Al(\mathcal{T}) =$

$\{(a_1, d_2, 1), (a_3, d_3, 1), (a_4, d_5, 1)\} = \mu^{CT}$. It is easy to check that, first, $\mathcal{T}$ is minimal for μ^{CT}, and, second, μ^{CT} is NC-RfOA-stable.

Suppose that μ^{CT} is also sub-market stable under some maximal trimmed sub-market Δ''. A contract $(a_4, d_4, 1)$ is blocking for μ^{CT} under Δ, as a result, $(a_4, d_4, 1)$ should be unacceptable under Δ''. As a result, $\Delta'' \in Tr(\Delta')$, so it should be equal to Δ' in order to be a maximal trimmed sub-market. But, μ^{CT} is not stable under Δ': $(a_1, d_1, 1)$ is a ($\varnothing$-by-H)-blocking contract. Thus, μ^{CT} is NC-RfOA-stable, but not sub-market stable.

Consider the following matching: $\mu^W = \{(a_1, d_3, 1), (a_2, d_1, 1), (a_4, d_5, 1)\}$. The minimal set of cutoffs $\mathcal{T}'$ that induces μ^W is: $t_{d_j} = 1$ for all $j \in \{1, \ldots, 5\}$, $t^H_{d_1} = 2$, $t^H_{d_2} = 3$, $t^H_{d_3} = 2$, $t^H_{d_4} = 3$, $t^H_{d_5} = 2$. Obviously, it is unconstrained, thus, by Proposition 9 μ^W is weakly stable.

The matching μ^W is not NC-RfOA-stable, since $\mathcal{T}'_{-d_2,H}$ also induces a matching: $\mu = \{(a_1, d_2, 1), (a_2, d_1, 1), (a_4, d_5, 1)\} \neq \mu^W$. Thus, μ^W is weakly stable, but not NC-RfOA-stable.

This concludes the proof of the theorem. □

Lemma 1. *All matchings induced at NE of RCG are stable.*

Proof. Suppose that the final matching is not stable. Thus, there is an applicant a, such that her assignment after the deadline T is strictly worse than some contract (a, d, i), such that $(a, d, i) \in Ch^*_d(X_{d,T_d} \cup (a, d, i))$. This implies that (a, d, i) was not active at T_d.

What should be a deviating strategy σ'_a of a in order to get (a, d, i)? She should have $\sigma'_a(\eta_t, \{\}) = \varnothing$ for any $t = T_{d'} < T_d$ for some $d' \in D$; $\sigma'_a(\eta_t, \{\}) = (a, d, i)$ for $t = T_d$; and we have two cases:

1. if either there is some contract $(a, d'', i) \in X_T$, such that $T_{d''} \geq T_d$, or $(X_T)_a = \{\}$, then $\sigma'_a(\eta_t, \{\}) = \sigma_a(\eta_t, \{\})$ for $t < T_d$ and $t \neq T_{d'}$ for some $d' \in D$;

2. if there is some contract $(a, d'', i) \in X_T$, such that $T_{d''} < T_d$, then $\sigma'_a(\eta_t, \{\}) = \sigma_a(\eta_t, \{\})$ for $t < T_{d''}$ and $t \neq T_{d'}$ for some $d' \in D$, and $\sigma'_a(\eta_t, \{\}) = \varnothing$ for $T_{d''} \leq t < T_d$.

Such strategy will not affect a history η_t up to T_d, thus (a, d, i) will be chosen by d at T. So, current strategy profile is not a NE. □

Lemma 2. *Pick an applicant a. Suppose that all other applicants have some passive strategies. If there exists a contract (a, d, i), such that $(a, d, i) \notin Ch^*_d(X_{d,t} \cup (a, d, i))$ for some $t < T$, then $(a, d, i) \notin Ch^*_d(X_{d,t+\tau} \cup (a, d, i))$ for any $\tau \in \{1, 2, \ldots, T - t\}$ regardless of a's strategy.*

Proof. Suppose that there exists a contract (a, d, i), such that $(a, d, i) \notin Ch^*_d(X_{d,t} \cup (a, d, i))$ for some $t < T$. This implies $Ch^*_d(X_{d,t} \cup (a, d, i)) = Ch^*_d(X_{d,t})$, because Ch^*_d satisfies IRC.

Consider the next period $(t+1)$. Since all agents with contracts from $Ch^*_d(X_{d,t})$ use a passive strategy (a is not necessarily among them), then $Ch^*_d(X_{d,t}) \subseteq X_{d,t+1}$. From IRC we have $Ch^*_d(X_{d,t}) = Ch^*_d(Ch^*_d(X_{d,t})) = Ch^*_d(X_{d,t} \cup (a, d, i)) = Ch^*_d(Ch^*_d(X_{d,t}) \cup (a, d, i))$. Hence, $(a, d, i) \notin Ch^*_d(Ch^*_d(X_{d,t}) \cup (a, d, i))$.

Now take any contract $(a', d, \cdot) \in X_{d,t+1} \backslash Ch^*_d(X_{d,t})$. From substitutes we get that $(a, d, i) \notin Ch^*_d(Ch^*_d(X_{d,t}) \cup (a', d, \cdot) \cup (a, d, i))$. Now take another contract $(a'', d, \cdot) \in X_{d,t+1} \backslash Ch^*_d(X_{d,t})$. From substitutes we get that $(a, d, i) \notin Ch^*_d(Ch^*_d(X_{d,t}) \cup (a', d, \cdot) \cup (a'', d, \cdot) \cup (a, d, i))$. ...Now take the last contract $(a^{'\cdots'}, d, \cdot) \in X_{d,t+1} \backslash Ch^*_d(X_{d,t})$. From substitutes we get that $(a, d, i) \notin Ch^*_d(Ch^*_d(X_{d,t}) \cup (a', d, \cdot) \cup (a'', d, \cdot) \cup \cdots \cup (a^{'\cdots'}, d, \cdot) \cup (a, d, i)) = Ch^*_d(X_{d,t+1} \cup (a, d, i))$.

Using the same reasoning we get that the contract $(a, d, i) \notin Ch^*_d(X_{d,t+\tau} \cup (a, d, i))$ for any $\tau \in \{1, 2, \ldots, T - t\}$. □

Lemma 3. *For each stable matching μ there exists a passive SPNE of RCG inducing μ.*

Proof. Take a stable matching μ. Consider a passive strategy for an applicant a, such that the following holds. If a has some contract $x = (a, d, i)$ in μ, then at the first period she makes x active. If, otherwise, a is alone under μ, then she always chooses $\varnothing$ (no active contract).

Now take a strategy profile, such that all applicants use such strategies. By construction it will produce μ as an outcome.

Why is it SPNE? Take any applicant a. Note that for any contract $x' = (a, d', i')$ which is better than her outcome under μ we have $x' \notin Ch^*_{d'}(X_{d',1} \cup x')$, because μ is stable. From Lemma 2 we get that $x' \notin Ch^*_{d'}(X_{d',t} \cup x')$ for any $t \in \{1, 2, \ldots, T\}$ regardless of a's strategy. So, there is no profitable manipulation for a. □

Proposition 14. *Fix the set of departments D, such that $|D| \geq 3$. For any set of deadlines $\{T\} \cup \{T_d\}_{d \in D}$ there exists a CAH with single-department housing constraints with $|A| \geq 3$, such that a straightforward strategy profile is not a NE of RCG.*

Proof. Consider a set of CAH markets with single-department housing constraints, such that $\{a_1, a_2, a_3\} \subseteq A$; $\{d_1, d_2, d_3\} \subseteq D$; $q_d = 1$ and $q^H_d = 0$ for any $d \in \{d_1, d_2, d_3\}$ (so, we can drop i from contracts); $D_a = \{d_1, d_2, d_3\}$ for any $a \in \{a_1, a_2, a_3\}$; for any $d \in \{d_1, d_2, d_3\}$ and $a \in A \backslash \{a_1, a_2, a_3\}$ we have $d \notin D_a$; and the preferences are ($\alpha, \beta, \gamma \in \{1, 2, 3\}$ and are different):

a_1	a_2	a_3	d_α	d_β	d_γ
d_α	d_α	d_γ	a_3	a_1	a_2
d_γ	d_γ	d_α	a_2	a_2	a_1
d_β	d_β	d_β	a_1	a_3	a_3

If $T_{d_i} \geq 5$ for any $i \in \{1, 2, 3\}$, then the straightforward strategy profile will produce the student-optimal stable matching for the sub-market with these three applicants and three departments. There will be the following sets of active contracts for each department during periods:

Period	d_α	d_β	d_γ
1	a_1, a_2	-	a_3
2	a_2	-	a_1, a_3
3	a_2, a_3	-	a_1
4	a_3	-	a_1, a_2
5	a_3	a_1	a_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	a_3	a_1	a_2

Suppose that a deadline of some department is < 5. Without loss of generality assume that $T_{d_1} < 5$. Choose a CAH where $\beta = 1$. Now the final matching under a straightforward strategy profile is not stable (some applicant is always unmatched). By Lemma 1, a straightforward strategy profile is not a NE.

Now suppose that all three deadlines are ≥ 5. Without loss of generality assume that $5 \leq T_{d_3} \leq T_{d_2} \leq T_{d_1}$. Choose a CAH where $\alpha = 1$, $\beta = 2$, $\gamma = 3$. Consider a manipulation of a_1 that results into the following sets of active contracts for each department during periods:

Period	d_1	d_2	d_3
1	a_2	-	a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$T_{d_3} - 1$	a_2	-	a_3
T_{d_3}	a_2	-	a_1, a_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

So, a_1 is assigned to d_3 after the manipulation. Note that a straightforward strategy of a_1 would have produced d_2 as her partner. Thus, the manipulation was successful: $d_3 \, P_{a_1} \, d_2$. $\square$

Chapter 2

Affirmative Action with Overlapping Reserves: Choice Rules under the One-to-All Approach

2.1 Introduction

Besides efficiency, fairness is currently one of the main desirable characteristics of any resource allocation mechanism. Specifically, policymakers want to ensure that the resulting matching faithfully reflects the inherent diversity among all candidates in terms of gender, race, socioeconomic status, sexual orientation, etc. To meet this requirement, some changes should be made to the existing allocation algorithms. Strictly speaking, there are two broadly used sets of approaches to increasing the diversity of the final allocation:

1. *priorities*: any individual from an underrepresented group gets boosted up in the common merit list (that is based on the underlying scores, e.g., exam scores)[1];

[1]Priorities in context of matching markets are discussed in Celebi and Flynn (2020b) and Echenique and Yenmez (2015).

2. *quotas*: each minority group gets a number of reserved slots to be filled exclusively by agents from that group.

Celebi and Flynn (2020a) study the trade-off between the two classes of approaches and conclude that they are identical if an authority is certain of the true distribution of types of the applicants; however, if this distribution is unknown to the very risk-averse authorities, then the introduction of quotas becomes more optimal. In this paper, the affirmative action policy is implemented solely through the availability of quotas.

In general, affirmative action policies aim to help distribute any indivisible goods to candidates from under-represented minority groups: vaccines among patients (Pathak et al., 2020b), school places among pupils, or working places among potential employees[2] (Kojima, 2012; Abdulkadiroglu, 2005; Dogan, 2016; Kitahara and Okumura, 2021), immigration visas among candidates (Pathak et al., 2020a), etc. These and similar policies are widely used and studied in many countries: US (Dur et al., 2018, 2020), Germany (Westkamp, 2013), Brazil (Aygün and Bó, 2020), India (Aygün and Turhan, 2017, 2020; Sönmez and Yenmez, 2020, 2019b), Chile (Correa et al., 2019; Sönmez and Yenmez, 2019a), etc.

Since reserves may *overlap* in any such allocation problem: one candidate may belong to more than one minority group (have more than one trait), a mechanism designer has two main approaches to construct a choice rule that will extract the best subset from the initial set of candidates:

- *one-to-one reserve matching*: each agent may fill only one reserved position for only one of her traits;
- *one-to-all reserve matching*: each agent may fill only one reserved position for each of her traits.

As a motivating example, consider the recruitment process with one employer (e.g., a government agency) and a group of job seekers. Each applicant may belong

[2]Student assignment and school choice problems were introduced by Balinski and Sönmez (1999) and Abdulkadiroglu and Sönmez (2003) respectively.

to none, one, or many categories (traits) that are considered in the framework of an affirmative action policy: women, African American, LGBTQ+, etc. Suppose that each category has a predefined number of reserved positions to be filled and that an employer wants to hire a certain number of employees in total. Which convention should a mechanism designer use?

Although the *one-to-one* approach is more widely studied due to its elegance,[3] the *one-to-all* reserve matching rule could be easier to comprehend for potential candidates. In the context of justified envy, under the one-to-all convention any unpicked candidate may easily verify that she is a better fit than a worse candidate that was picked. However, this will be much harder under the one-to-one approach, since in this case the unpicked candidate may need to reassign the reserved positions to some picked candidates in some complicated way in order to understand whether she can justifiably envy someone. Additionally, she will need to know the sets of all traits for each selected candidate, while under the one-to-all convention all she needs is set of traits of the picked candidate she envies and the information on the filled reserved positions.

Moreover, because an employer wants to hire the best workers to fill the reserved positions as much as possible, he will be much more satisfied with the one-to-all approach, as the restrictions are less stringent for an employer if any candidate can fill a reserved position for each of their traits.

In addition, in some applications, the allocation rule "resembles" the one-to-all convention. For example, in elections (at various levels) there is some evidence that white male groups support minority female candidates as they meet both quotas and thus open more positions for white men. Hughes (2011) points out that counting for both being a woman and a minority benefits the minority women more than using just one of these traits. To verify this, consider the following simple example.

Suppose that you want to hire three out of five candidates, so at least one of

[3] From Sönmez and Yenmez (2019a): "...if there is any flexibility to select one of the conventions we believe the case for the one-to-one reserve matching is much stronger."

them is a woman and one is from a minority group. So we have two possible traits. Suppose also that according to the merit list, the best and second best candidates have no traits, the third best candidate is a woman, the fourth best candidate is from minorities, and the fifth best candidate is a minority woman. If the minority woman can fill only one reserved seat, as under the one-to-one convention, she will not be hired: the first, third and fourth agents will be hired under the one-to-one convention. Thus, the second best agent has no chance. However, if we move to a one-to-all approach, where a minority woman can fill both reserved positions (one for each characteristic), then it makes sense to hire the first, second, and fifth best agents. Thus, the second best candidate is chosen with a one-to-all approach. Therefore, a one-to-all approach not only favors more depressed candidates, but also helps an employer hire more candidates from the top of the merit list.

In their paper, Sönmez and Yenmez (2019a) propose a choice rule under the one-to-all approach that deals with no more than two traits. This choice rule produces all possible solutions that are non-wasteful, eliminate justified envy, and fill as much of the reserved positions as possible. Unfortunately, due to naturally arising complementarities between candidates, this choice rule is constructed using a brute force case-by-case analysis. The complexity of this paired-admissions choice correspondence means that creating an analog for more than two traits will be nearly impossible. However, the number of possible traits in the real world is definitely greater than two.

For instance, on Tuesday, September 8, 2020, the Academy of Motion Picture Arts and Sciences released a new affirmative action rule[4] to qualify for the Best Picture category, starting at the 96th Oscars in 2024. More than two traits that in the future can be considered as completely independent with independent reserve policies are mentioned there: women; racial or ethnic group; LGBTQ+; people with cognitive or physical disabilities, or who are deaf or hard of hearing.

[4]The statement is available at `https://www.oscars.org/news/academy-establishes-representation-and-inclusion-standards-oscarsr-eligibility`.

Moreover, on December 1, 2020 Nasdaq filed a proposal[5] with the US Securities and Exchange Commission to adopt new listing rules related to board diversity and disclosure. The goal is to increase representation of women, underrepresented minorities, and the LGBTQ+ community on company boards. Again, more than two traits are considered.

To cope with such matching problems one will need choice rules that can operate on more than two traits under the one-to-all reserve convention. In this paper I propose three solutions to the problem of choosing the best subset of candidates, given all the constraints mentioned: two choice rules, each producing a unique outcome, and one choice correspondence that may give a class of final outcomes. Each of these three mechanisms produces solutions that utilize all quotas (total quota and traits reserved positions) as much as possible, and are incomparable in terms of containing the best possible agents from the common merit list. The *lower-dominant* choice rule picks the subset of applicants, where an applicant with the lowest rank is the best possible one, given the constraints. In other words, this rule tries to avoid picking low ranked applicants. On the other hand, the *upper-dominant* choice rule picks as many high ranked individuals as possible at expense of hiring very low ranked ones. In turn, the *sum-minimizing* choice correspondence chooses subsets of applicants that maximize the total *significance* of their members. In order to do that we need to provide it with quantitative significance level of each applicant in advance. Depending on the agenda of the employer he can choose any of the proposed solutions. It is worth noting that all candidates are considered acceptable in this model.

The rest of the paper is structured as follows. Section 2.2 sets up a general model and states the main features of choice rules. Section 2.3 introduces a lower-dominant strict ordering relation on the power set of all applicants and constructs a corresponding choice rule. Section 2.4 introduces an upper-dominant strict ordering relation on the power set of all applicants and constructs a corresponding

[5]The proposal can be found at `https://www.nasdaq.com/press-release/nasdaq-to-advance-diversity-through-new-proposed-listing-requirements-2020-12-01`.

choice rule together with a well-behaved choice correspondence. Section 2.5 explains why it is impossible to obtain a stable allocation with any meaningful rule under a one-to-all approach. Section 2.6 presents a real-life application of this research with three traits. Section 2.7 concludes.

2.2 Model

In this section, I set up a general model to study choice rules under a given reservation market and introduce a set of desired requirements that our choice rules should satisfy. I also construct an extensive illustrative example from a galaxy far, far away that will last throughout the study. Furthermore, the main goal of this study is explicitly formulated in the second half of this section.

2.2.1 Primitives

The following elements constitute the basis of our model. There exists a set of potential applicants $\mathcal{I}$ and a set of traits $\mathcal{T}$. Each trait stands for a specific sub-group, that could be, for instance, a kind of minority: race, gender, sexual orientation, etc. Each applicant may possess none, one, or many of those traits. In other words, they could be a member of none, one, or multiple sub-groups. In order to formalize this, I introduce a correspondence $\tau : \mathcal{I} \twoheadrightarrow \mathcal{T}$ that returns a subset of traits for each given applicant. There also exists a strict priority ordering π over all applicants, a so-called *merit list*. So, for any two agents $i, j \in \mathcal{I}$: either i is strictly better than j, denoted as $i \ \pi \ j$, or j is strictly better than i, denoted as $j \ \pi \ i$.

The principal wants to pick exactly q applicants, so q is the total amount of seats (quota). Moreover, for any trait $t \in \mathcal{T}$ there is a number of reserved positions for each trait $\{r_t\}_{t \in \mathcal{T}}$, such that the total number of reserved positions does not exceed quota, $\sum_{t \in \mathcal{T}} r_t \leq q$.[6]

[6] This assumption is restrictive but quite natural for this setting.

Finally, a **reservation market** is a tuple $\langle \mathcal{I}, \mathcal{T}, \tau, \pi, q, \{r_t\}_{t\in\mathcal{T}}\rangle$ and a **choice rule** is a function that for any initial subset of applicants $I \subseteq \mathcal{I}$ picks a final subset $C(I) \subseteq I$, under a given reservation market.[7]

Now, fix a reservation market $\langle \mathcal{I}, \mathcal{T}, \tau, \pi, q, \{r_t\}_{t\in\mathcal{T}}\rangle$. Define a function $\rho : 2^{\mathcal{I}} \to \mathbb{Z}^{|\mathcal{T}|}$ that for a given subset of agents $I \in \mathcal{I}$ produces a number for each trait t: $\rho_t(I) = (-r_t) + |\{i \in I | t \in \tau(i)\}|$ and that describes how unfilled or overfilled are the reserved positions for this trait t. We will say that trait t's reserved positions are **exactly filled** under set of agents $I \subseteq \mathcal{I}$ if $\rho_t(I) \geq 0$.[8]

We will say that a set $I \subseteq \mathcal{I}$ is **weakly more diverse** than a set $J \subseteq \mathcal{I}$, if for all $t \in \mathcal{T} : \min\{\rho_t(I), 0\} \geq \min\{\rho_t(J), 0\}$. A set is just **more diverse** if at least one inequality is strict. Hence, a binary relation that is "weakly more diverse" is a non-strict partial order (reflexive, transitive, and antisymmetric), while a binary relation that is "more diverse" is a strict partial order (irreflexive, transitive, and antisymmetric). Furthermore, in our model we have soft upper bounds.

Of course, we want our choice set to be as diverse as possible until each trait's reserved positions are exactly filled.

Definition 29. A **choice rule** C **is the most diverse** if for any $I \subseteq \mathcal{I}$ a chosen set $C(I)$ is weakly more diverse than any other subset of I.

We also want to exhaust all the quota, q, if we can.

Definition 30. A **choice rule** C **is non-wasteful** if for every $I \subseteq \mathcal{I}$,

$$|C(I)| = \min\{|I|, q\}.$$

Consider the following illustrative example from a galaxy far, far away.

[7] Throughout the paper I assume that there is a need to pick a subset, $|I| > q$, and that all applicants from I are acceptable.

[8] From now on we will assume that there always exists at least one subset of I with q elements that exactly fills all the traits' reserved positions. Otherwise, we could have just redefined the numbers of reserved positions for each trait $t \in \mathcal{T}$ to be $r_t = \min\left\{r_t, \sum_{i\in I} \mathbb{I}\{t \in \tau(i)\}\right\}$. This guarantees the existence of such subset due to the initial constraint $\sum_{t\in\mathcal{T}} r_t \leq q$.

Illustrative Example. There are 8 potential applicants, $|\mathcal{I}| = 8$, for 4 Jedi Master positions in the Jedi High Council, $q = 4$.

The Jedi Grand Temple's leading HR cares about 3 traits, $|\mathcal{T}| = 3$:

- women (applicant's name is in italics: *Shaak*), with 2 reserved positions, $r_w = 2$;
- rare species (applicant's name is in bold: **Yoda**), with 1 reserved position, $r_{rs} = 1$;
- natives of Outer Rim worlds (applicant's name is underlined: Dooku), with 1 reserved position, $r_{or} = 1$.

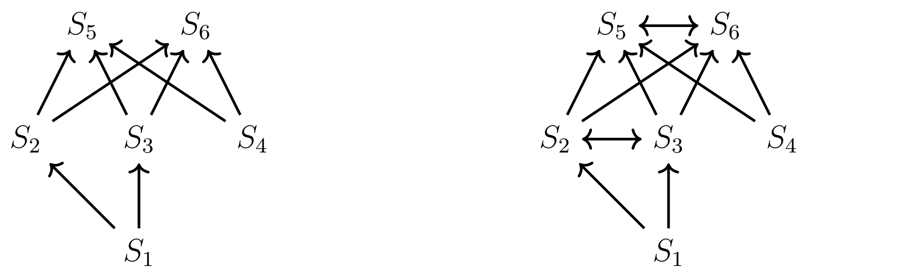
Suppose that all 8 applicants did apply, $I = \mathcal{I}$. After analyzing the applicants' portfolios the following merit list, π, was constructed (from the best applicant to the worst):

> Obi-Wan, Mace, Anakin, *Jocasta*, **Grogu**, ***Yaddle***, *Ahsoka*, ***Luminara***.

As a simple exercise, let us compare the following six subsets through binary relations "weakly more diverse," and "more diverse":

- $S_1 = \{$Obi-Wan, Mace, Anakin$\}$, $\rho(S_1) = (\rho_w(S_1), \rho_{rs}(S_1), \rho_{or}(S_1)) = (-2, -1, 0)$;
- $S_2 = \{$Obi-Wan, Mace, Anakin, *Jocasta*$\}$, $\rho(S_2) = (-1, -1, 0)$;
- $S_3 = \{$Obi-Wan, Mace, Anakin, *Ahsoka*$\}$, $\rho(S_3) = (-1, -1, 1)$;
- $S_4 = \{$Obi-Wan, Mace, *Jocasta*, **Grogu**$\}$, $\rho(S_4) = (-1, 0, -1)$;
- $S_5 = \{$Anakin, *Jocasta*, ***Luminara***$\}$, $\rho(S_5) = (0, 0, 1)$;
- $S_6 = \{$*Jocasta*, **Grogu**, ***Yaddle***, *Ahsoka*$\}$, $\rho(S_6) = (1, 1, 0)$.

We get the following Hasse diagrams (an arrow from S_i to S_j means that S_i is less diverse than S_j).



(a) "More diverse" relation (b) "Weakly more diverse" relation

Figure 2.1: Hasse diagrams

As we can observe, subset S_1 is less diverse than subsets S_2 and S_3, which in turn are less diverse than subsets S_5 and S_6. Also, subset S_4 is less diverse than subsets S_5 and S_6, but cannot be compared to subsets S_1, S_2, or S_3.

Now, consider the following choice set: $C_1 = \{$Obi-Wan, *Jocasta*, **Grogu**, *Ahsoka*$\}$. Under this set the total quota is met, $|C_1| = 4 = q$, and reserved positions for all 3 traits are exactly filled, $\rho(C_1) = (0, 0, 0)$. Hence, the set C_1 can be the outcome of the most diverse and non-wasteful choice rule.

Another very important feature that a choice rule should satisfy is the elimination of justified envy. Otherwise this choice rule will produce additional costs due to court cases.

Definition 31. A **choice rule** C **eliminates justified envy** if for any $I \subseteq \mathcal{I}$ there are no such agents $i \in C(I)$ and $i' \in I \backslash C(I)$ that $i' \, \pi \, i$ and set $(C(I) \backslash \{i\}) \cup \{i'\}$ is weakly more diverse than set $C(I)$.

In other words, an agent i' who is not chosen may envy a chosen agent i only if i' has a higher ranking in the merit list, and i' is no worse than i in exactly filling all reserved positions under the choice rule C.

Illustrative Example (Continued). Let us look again at the current choice set, $C_1 = \{$Obi-Wan, *Jocasta*, **Grogu**, *Ahsoka*$\}$.

Can it be chosen by the most diverse and non-wasteful choice rule that eliminates justified envy? Yes! Since neither Mace nor Anakin can justifiably envy *Jocasta*, **Grogu**, or *Ahsoka*; and **Yaddle** also cannot justifiably envy *Ahsoka*.

However, there is another subset of four applicants that is unambiguously better than C_1: $C_2 = \{$Obi-Wan, Anakin, *Jocasta*, ***Yaddle***$\}$, $\rho(C_2) = (0,0,0)$. This is so, since Anakin π **Grogu** and ***Yaddle*** π *Ahsoka*.

This implies that if a subset C_1 was chosen, then a group of two Jedi, Anakin and ***Yaddle***, could have sued the Jedi Grand Temple in the Supreme Court of the Galactic Alliance. Since they justifiably envy the group of two chosen Jedi, **Grogu**, and *Ahsoka*, even though they could not have filed two lawsuits separately.

Thus, not only can separate not chosen individuals envy chosen individuals, but groups of not chosen individuals can as well. But how can we compare these groups?

For any set $I \subseteq \mathcal{I}$ denote by I^k its k-th best element according to π. We will say that a set $I \subseteq \mathcal{I}$ **dominates** a set $J \subseteq \mathcal{I}$ if $|I| = |J|$ and for any possible k: either $I^k = J^k$ or $I^k \ \pi \ J^k$, and for some $k : I^k \ \pi \ J^k$.[9]

Illustrative Example (Continued). As we can see, the set C_1 dominates C_2, since the set $\{$Anakin, ***Yaddle***$\} \subset C_1$ dominates the set $\{$**Grogu**, *Ahsoka*$\} \subset C_2$.

So, we also do not want our choice rule to produce civil court cases.

Definition 32. A **choice rule** C **eliminates collective justified envy** if for any $I \subseteq \mathcal{I}$ there are no such groups of agents $S \subseteq C(I)$ and $S' \subseteq I \backslash C(I)$ that S' dominates S and set $(C(I) \backslash S) \cup S'$ is weakly more diverse than set $C(I)$.

First, note that both groups of agents from the Definition 32, S and S' should have the same cardinality, $|S| = |S'|$. Second, elimination of collective justified envy naturally implies elimination of justified envy.

2.2.2 What Are We Looking for?

Given all the features defined above, it is natural to formulate the following problem: *find all the most diverse and non-wasteful choice rules that eliminate collective justified envy.*

[9]A similar logic is used in definition of Stochastic Dominance.

Definition 33. A **choice rule** C **is undominated** if it is the most diverse, and for any $I \subseteq \mathcal{I}$ there is no subset of I that dominates $C(I)$ and is weakly more diverse than $C(I)$.

The following theorem implies that it is equivalent to focus our attention on the undominated and non-wasteful choice rules.

Proposition 16. *A choice rule is the most diverse and eliminates collective justified envy if, and only if, it is undominated.*

Proof. Necessity. Suppose that a choice rule C is the most diverse and eliminates collective justified envy. Suppose, to the contrary, that C is not undominated. Hence, under some set of candidates, I, there is a set $C^* \subseteq I$ that dominates $C(I)$, and is weakly more diverse than $C(I)$ (hence, $|C^*| = |C(I)|$). Now we construct the following two sets: $S' = C^* \backslash C(I)$ and $S = C(I) \backslash C^*$.

Obviously, $S \subseteq C(I)$ and $S' \subseteq I \backslash C(I)$ and S' dominates S. Moreover, set $C^* = (C(I) \backslash S) \cup S'$ is weakly more diverse than set $C(I)$. This contradicts choice C eliminating collective justified envy. Hence, C is undominated.

Sufficiency. Suppose that a choice rule C is undominated. Hence, by the Definition 33, C is the most diverse.

Now suppose, to the contrary, that this C does not eliminate collective justified envy. Hence, under some set of candidates, I, there are such groups of agents $S \subseteq C(I)$ and $S' \subseteq I \backslash C(I)$ that S' dominates S and set $(C(I) \backslash S) \cup S'$ is weakly more diverse than set $C(I)$. This implies that the obtained set $(C(I) \backslash S) \cup S'$ dominates the initial choice set $C(I)$ and is weakly more diverse than the initial choice set $C(I)$. This contradicts choice C being undominated. Hence, C eliminates collective justified envy. □

The following example illustrates why just elimination of justified envy is not enough for the "if and only if" statement in the Proposition 16.

Example 7.

- There are 5 agents in the set I with merit list: $i_1 \; \pi \; i_2 \; \pi \; i_3 \; \pi \; i_4 \; \pi \; i_5$.
- There are 3 traits, each with 1 reserved position.
- Quota is $q = 3$, so $\sum_{t \in \mathcal{T}} r_t = 3 \leq q$.
- Sets of traits for each agent are:

i_1	i_2	i_3	i_4	i_5
-	t_1	t_3	t_2	t_1
	-	-	t_3	t_2
			-	-

Table 2.1: Sets of agents' traits for Example 7

Note that $C(I) = \{i_1, i_3, i_5\}$, $|C(I)| = 3 = q$, exactly fills all reserved positions, $\rho(C(I)) = (0, 0, 0)$. Moreover, neither agent i_3 nor agent i_5 can be justifiably envied by non chosen agent i_2, and not chosen agent i_4 cannot justifiably envy agent i_5. Hence, C may indeed be the most diverse choice rule that eliminates justified envy.

However, there is an obvious better choice, $S = \{i_1, i_2, i_4\}$, $|S| = 3 = q$, that exactly fills all reserved positions, $\rho(S) = (0, 0, 0)$, and dominates $C(I)$. So, C is not undominated.

Note also that in this example, under the choice rule C, a set $\{i_2, i_4\}$ collective justifiably envies a set $\{i_3, i_5\}$.

So, under three and more traits elimination of justified envy is not enough for elimination of collective justified envy. But what about under two traits?

Proposition 17. *Under two traits or less, the elimination of justified envy is equivalent to the elimination of collective justified envy.*

Proof. If the number of possible traits is zero or one then this statement is trivial.

Suppose that there are two possible traits. We need to prove that for any initial set of candidates I there is no choice set $C \subseteq I$, such that the following holds:

- there is no pair of agents $i \in C$ and $i' \in I \backslash C$ such that i' justifiably envies i;
- there are two sets $S \subseteq C$ and $S' \subseteq I \backslash C$ such that $|S| = |S'| \geq 2$ and S' collective justifiably envies S.

Suppose, to the contrary, that for some initial set I there is such a set $C \subseteq I$. Now we calculate how many reserved positions only agents from the set S fill if we pick all agents from the set C, $\min\{\rho(C), 0\} - \min\{\rho(C \backslash S), 0\} = (u_1, u_2)$. Also, since S' collective justifiably envies S, and thus all agents from S' should fill no less than (u_1, u_2) reserved seats, $|\{i' \in S' | t_1 \in \tau(i')\}| \geq u_1$ and $|\{i' \in S' | t_2 \in \tau(i')\}| \geq u_2$.

What can we say about these non-negative integer numbers u_1 and u_2?

- $\underline{u_1 + u_2 \geq |S| = |S'| \geq 2}$: First, note that *each member of the set S should fill at least one reserved position out of these $u_1 + u_2$.* Otherwise, if there is some agent $i \in S$ that does not help C to exactly fill reserved positions (even if $\tau(i) \neq \emptyset$), $\min\{\rho(C \backslash \{i\}), 0\} = \min\{\rho(C), 0\}$. Hence the best agent i' from the set S' justifiably envies i: first, since the set S' dominates the set S, then $i' \; \pi \; i$, second, $(C \backslash \{i\}) \cup \{i'\}$ is weakly more diverse than C. So, the sum of u_1 and u_2 should be weakly greater than $|S| = |S'| \geq 2$.
- $\underline{u_1 + u_2 \leq |S| = |S'|}$: First, note that since S' dominates S and they are mutually exclusive, for any agent $i' \in S'$ there is a worse agent $i \in S$. Hence, due to elimination of justified envy there cannot be an agent $i' \in S'$ with both traits, $\tau(i') = \{t_1, t_2\}$, since otherwise, such an agent $i' \in S'$ will justifiably envy a worse agent $i \in S$. Thus, *each member of the set S' may have at most one trait.* So, together all agents from S' would not be able to fill more that $|S'|$ reserved positions in total. This implies than $u_1 + u_2 \leq |S'|$.

Combining the above yields the following three results:

- $u_1 + u_2 = |S| = |S'| \geq 2$;

- *under the chosen set C, each member of the set S should fill exactly one reserved position*;

- *each member of the sets S' should have exactly one trait.*

Now take the best agent from S', i'. WLOG suppose that she has the first trait, $\tau(i') = \{t_1\}$. Hence, there should be an agent $i \in S$ that fills exactly one reserved position for the first trait under the chosen set C. Since i' is the best agent in S' and S' dominates S, $i' \, \pi \, i$. This means that $i' \in I \backslash C$ justifiably envies $i \in C$. This is contradiction. $\square$

Using Proposition 17 together with Proposition 16, we may conclude that the paired-admissions choice correspondence constructed by Sönmez and Yenmez (2019a) contains all non-wasteful and undominated choice rules under no more than two traits.

So, the problem that we would like to solve now is: *find all non-wasteful and undominated choice rules.* However, it is easy to notice that, given a set of candidates I, there can be many undominated and the most diverse subsets with q elements, since domination plus the most diversity create only a partial ordering on the set of all subsets of I with q elements (I assume that $q < |I|$). For this reason, the next sections will focus on more restrictive versions of domination that will produce a strict ordering: lower-domination and upper-domination.

2.3 Lower-Dominant Choice Rule

What undominated and non-wasteful subset of I should we pick if we want to avoid choosing weak candidates as much as possible? The answer is: a lower-dominant non-wasteful subset, as defined below.

For any set $I \subseteq \mathcal{I}$ denote by I_k its k-th worst element according to π (the k-th element from the worst one, I_1, if all elements are ordered according to π). Pick any two subsets I and J of $\mathcal{I}$, such that $|I| = |J|$. We will say that I **lower-dominates** J if there is a natural number k, such that for any natural $l < k$:

$I_l = J_l$, and $I_k \; \pi \; J_k$.[10] It is easy to see that lower-domination relation creates a strict ordering on the set of all subsets of I with q elements.

Definition 34. A **choice rule** C **satisfies lower-domination** if it is the most diverse, and there is no subset of I that lower-dominates $C(I)$ and is weakly more diverse than $C(I)$.

Note that domination implies lower-domination, and if I lower-dominates J, then J does not dominate I. So, if a choice rule C satisfies lower-domination, then it is undominated.

Now we define our choice rule C^{LD}, where all considered individuals are from some $I \subseteq \mathcal{I}$:

- **Step 1:** Create a set A with one element - the best individual with at least one trait, name her i_1. If there is no such individual, go to the last step.

- **Step** k**:** Choose the best individual with at least one trait that was not considered before, name her i_k. If there is no such individual, go to the trimming step. If a set $A \cup \{i_k\}$ is not more diverse than a set A, then go to the next step $(k+1)$; if it is more diverse than A, update a set $A = A \cup \{i_k\}$ and go to the next step $(k+1)$.

- **Trimming step:** Go from the worst agent in A to the best and proceed as follows: take a current agent, i, and check whether A is more diverse than $A \backslash \{i\}$. If it is, then go to the next agent. If it is not, then exclude this agent i: $A = A \backslash \{i\}$, and go to the next agent.

- **Last step:** If the previous step was the Trimming step, then, if $|A| < \min\{|I|, q\}$, we can hire $(\min\{|I|, q\} - |A|)$ more of the best possible individuals not from A (but from I). Add them to A. Finally, set $C^{LD}(I) = A$.[11]

[10] I use a $\min\max$ logic here.

[11] Note that if we exclude the trimming step and replace "is more diverse than" with "increasing reserve utilization," we will obtain the exact the envelope choice rule $C^{\boxtimes}$ from Sönmez and Yenmez (2019a) for the one-to-one matching convention.

Now I present the main theorem characterizing the introduced choice function C^{LD}.

Theorem 5. *A choice rule C is non-wasteful and satisfies lower-domination if, and only if, it is C^{LD}.*

Proof. The choice rule C^{LD} is non-wasteful by construction: during the last step, we make sure that exactly $\min\{|I|, q\}$ agents are hired.

The choice rule C^{LD} satisfies lower-domination. First, note that by construction of C^{LD} the following is true: if $i \in C^{LD}(I)$ and she was hired before the last step, then $C^{LD}(I)$ is more diverse than $C^{LD}(I)\backslash\{i\}$. So, each agent hired before the last step is crucial for exactly filling the reserved positions r_t for at least one trait t. Moreover, we know that $\sum_{t\in\mathcal{T}} r_t \leq q$, hence for any $I \subseteq \mathcal{I}$ we will need no more than q agents from I in order to create a set C' (which is not unique), such that there will be no other subset of I that is more diverse than C'. By construction, the choice rule C^{LD} picks this set plus agents hired during the last step. Hence, $C^{LD}(I)$ is weakly more diverse than any other subset of I.

Now suppose that there is a subset $S \subseteq I$, different from $C^{LD}(I)$, that is weakly more diverse than $C^{LD}(I)$ and that has the same cardinality, $|S| = |C^{LD}(I)|$. From above we know that S cannot be more diverse than $C^{LD}(I)$, only weakly more diverse.

Now we show that $C^{LD}(I)$ must lower-dominate S. To the contrary, suppose that S lower-dominates $C^{LD}(I)$. Take the lowest possible natural number k, such that $S_k \ \pi \ C^{LD}(I)_k$. Note that for any natural number $l < k$ there should be $S_l = C^{LD}(I)_l$. Finally, by construction of C^{LD} agent $C^{LD}(I)_k$ is crucial for filling the reserved positions r_t exactly for at least one trait t (she is the best possible agent that further fills this trait's reserved positions). Hence, under the set S the number of unfilled reserved positions for this trait t is strictly lower than it is under the set $C^{LD}(I)$, which is less than or equal to zero. This means that $C^{LD}(I)$ is more diverse than S, which is a contradiction.

Finally, note that if a choice rule C is non-wasteful and satisfies lower-domination, then it picks the best – in terms of lower-domination – set out of all sets, for which the following is true: cardinality is $\min\{|I|, q\}$, and there is no subset of I that is more diverse than this set. Since any two different sets with cardinality $\min\{|I|, q\}$ can be compared through lower-domination, the best set is unique. And, by construction, choice rule C^{LD} always picks it. $\square$

In other words, the lower-dominant choice rule tries to fill all reserved positions for each trait exactly and avoid picking bad candidates according to the merit list.[12]

Illustrative Example (Continued). If the Jedi use the lower-dominant choice rule C^{LD}, then four seats of Jedi High Council will be taken by $C^{LD}(I) = C_2 =$ {Obi-Wan, <u>Anakin</u>, *Jocasta*, ***Yaddle***}, $\rho(C^{LD}(I)) = (0, 0, 0)$. So, the worst hired Jedi is the 6th worst applicant (out of 8).

Now we prove that the constructed choice rule cannot cause court cases due to justified envy.

Proposition 18. *A choice rule C^{LD} eliminates justified envy.*

Proof. Direct proof. To the contrary, suppose that there is a pair of agents $i \in C^{LD}(I)$ and $i' \in I \backslash C^{LD}(I)$, such that $i' \ \pi \ i$ and set $(C^{LD}(I) \backslash \{i\}) \cup \{i'\}$ is weakly more diverse than set $C^{LD}(I)$.

Note, however, that, since $i' \ \pi \ i$, set $(C^{LD}(I) \backslash \{i\}) \cup \{i'\}$ lower-dominates set $C^{LD}(I)$. So, set $(C^{LD}(I) \backslash \{i\}) \cup \{i'\}$ both lower-dominates set $C^{LD}(I)$ and is weakly more diverse than set $C^{LD}(I)$. Hence, our choice rule C^{LD} does not satisfy lower-domination. This contradicts Theorem 5.

Alternative proof. From Theorem 5 we know that C^{LD} satisfies lower-domination. Hence, C^{LD} is undominated. Using Proposition 16 we obtain that C^{LD} eliminates collective justified envy. Hence, C^{LD} eliminates justified envy. $\square$

[12] So, the lower-dominant choice rule C^{LD} is a generalization of the extreme paired-admission choice rule $C^{\max\min}$ from Sönmez and Yenmez (2019a).

Through the following example I show that being undominated instead of satisfying lower-domination is not enough for the "if and only if" statement in Theorem 5.

Example 8.

- There are 8 agents in the set I with merit list: $i_1 \, \pi \, i_2 \, \pi \, \ldots \, \pi \, i_7 \, \pi \, i_8$.
- There are 4 traits with the following numbers of reserved positions:
  - $r_{t_1} = r_{t_3} = 2$;
  - $r_{t_2} = r_{t_4} = 1$.
- Quota is $q = 6$, so $\sum_{t \in \mathcal{T}} r_t = 2 \cdot (2 + 1) = 6 \leq q$.
- Sets of traits for each agent are:

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
-	-	-	t_1	t_1	t_3	t_2	t_1
			-	t_2	-	t_3	t_3
				-		-	t_4
							-

Table 2.2: Sets of agents' traits for Example 8

It is easy to verify that $C^{LD}(I) = \{i_1, i_2, i_3, i_5, i_6, i_8\}$, $|C^{LD}(I)| = 6 = q$. Also, $\rho(C^{LD}(I)) = (0, 0, 0, 0)$, so $C^{LD}(I)$ exactly fills all reserved positions.

However, there is another set $S = \{i_1, i_2, i_3, i_4, i_7, i_8\}$, $|S| = 6 = q$. And again $\rho(S) = (0, 0, 0, 0)$, so S also exactly fills all reserved positions. Moreover, $C^{LD}(I)$ and S cannot be compared through domination, but $C^{LD}(I)$ lower-dominates S.

So, is there a way to construct a non-wasteful and undominated choice rule that will pick a subset S from the example above?

2.4 Upper-Dominant Choice Rule

What undominated and non-wasteful subset of I should we pick if we want to choose as many best candidates as possible? The answer is: an upper-dominant non-wasteful subset, as defined below.

Recall that for any set $I \subseteq \mathcal{I}$ we denoted by I^k its k-th best element according to π (the k-th element from the best one, I^1, if all elements are ordered according to π). Pick any two subsets I and J of $\mathcal{I}$, such that $|I| = |J|$. We will say that I **upper-dominates** J if there is a natural number k, such that for any natural $l < k$: $I^l = J^l$, and $I^k \ \pi \ J^k$. It is easy to see that upper-domination relation creates a strict ordering on the set of all subsets of I with q elements.

Definition 35. A **choice rule** C **satisfies upper-domination** if it is the most diverse, and there is no subset of I that upper-dominates $C(I)$ and is weakly more diverse than $C(I)$.

By analogy with lower-domination, note that domination implies upper-domination, and if I upper-dominates J, then J does not dominate I. So, if a choice rule C satisfies upper-domination, then it is undominated.

2.4.1 Minimizing Sum of Weights

Suppose that each agent from I has a weight according to her index in the merit list: the l-th best agent in π has weight l. Another way to choose the best possible agents under the restrictions is the following: try to exactly fill all reserved positions with no more than q agents, such that their sum of weights is minimal under given constraints.

This can be formulated as an integer linear programming problem. So far we have a quota q, number of all agents $n = |I|$ and number of traits $k = |\mathcal{T}|$. Now we construct a k-by-n binary matrix for traits, T, as follows

$$T_{ti} = \begin{cases} 1 & \text{if } t \in \tau(i), \\ 0 & \text{otherwise.} \end{cases}$$

So, we have $T_{ti} = 1$ only if an agent with index i has trait t. Also, we have weights for all agents, which we can combine into a n-by-1 vector $v = v^s \equiv$

$[1, 2, \ldots, n]'$. Now we can construct an optimization problem, where x is a n-by-1 binary vector with $x_l = 1$ only if agent l is chosen, and $x_l = 0$ otherwise:

$$\begin{aligned} & \min_x v' \cdot x \\ \text{s.t.} \quad & T \cdot x \geq r \\ & \textstyle\sum_{i=1}^n x_i = q \end{aligned} \tag{2.1}$$

So, we can define a sum-minimizing choice correspondence $C^{SM}(v)$:

- **Step 1:** Redefine the vector with the numbers of reserved positions as follows: $r = \min\{r, \sum_{i=1}^n T_{\cdot i}\}$ (what if it is impossible to exactly fill the initial numbers of reserved positions?). There is now a sure a solution for (2.1), since $\sum_{t \in \mathcal{T}} r_t \leq q$ by assumption.

- **Step 2:** Construct and solve problem (2.1).[13] Set the collection of solutions to be the outcome of $C^{SM}(v)$.

We call any selection from the sum-minimizing choice correspondence a sum-minimizing choice rule. Through the construction of a sum-minimizing choice rule the following proposition holds.

Proposition 19. *A sum-minimizing choice rule is non-wasteful and undominated.*

Non-wastefulness follows straightly from the second constraint of the problem (2.1). A sum-minimizing choice rule is undominated because there cannot exist any other the most diverse subset of applicants that dominates the sum-minimizing choice rule solution, since its sum of weights will be strictly smaller, which is impossible due to construction of the problem (2.1).

In general, the same will hold under any weighting vector v with weights that are strictly decreasing with the merit list: if $i' \, \pi \, i$, then i' has strictly smaller weight than i, $v_{i'} < v_i$. This gives us a large class of choice correspondences (constructed with different weighting vectors), each containing only non-wasteful and undominated choice rules.

[13]Use an Integer Linear Programming routine, e.g. Branch-and-Bound algorithm realization.

2.4.2 Choice Rule

Now suppose that we want to find out whether there is such binary vector x that satisfies both restrictions from the integer LP problem (2.1) and hires the best l agents from I, $x_1 = x_2 = \cdots = x_l = 1$. In order to do this, we need to modify the weighting vector v in the following way: $v(l) = [1, 2, \ldots, l, \sum_{i=1}^{l} i+(l+1), \sum_{i=1}^{l} i+(l+2), \ldots, \sum_{i=1}^{l} i + n]'$. Now we solve the optimization problem (2.1) with the new vector $v = v(l)$. If there is solution x, such that $x_1 = x_2 = \cdots = x_l = 1$, then the answer is yes. Let us define the following binary function, given q, r, T, for any $l \in \{1, 2, \ldots, n\}$:

$$f(l) = \begin{cases} 1 & \text{if } \exists \text{ solution } x \text{ for (2.1) with } v = v(l), \text{ s.t. } x_1 = x_2 = \cdots = x_l = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can find the maximum number, l_m, of best agents that we can pick without skipping anyone and satisfy both restrictions: $l_m = \max l$, s.t. $f(l) = 1$. Note that l_m may be zero, if there is no solution that picks the best possible agent from I.

Now we have everything we need in order to construct an upper-dominant choice rule C^{UD}:

- **Step 0:** Redefine the vector with the numbers of reserved positions as follows: $r = \min\{r, \sum_{i=1}^{n} T_{\cdot i}\}$ (what if it is impossible to exactly fill the initial numbers of reserved positions?). There is now a sure solution for (2.1), since $\sum_{t \in \mathcal{T}} r_t \leq q$ by assumption.

- **Step 1:** Start with all agents I, quota q, reserved positions vector r, and trait matrix T for I. Find l_m,

  – if $l_m > 0$, set $A = \{I^1, \ldots, I^{l_m}\}$ – hired agents (recall that I^l is the l-th best agent in I), set $C = A$ – set of the best agents we considered hiring so far;

  – if $l_m = 0$, set $C = A = \{\}$.

- **Step k:** Set $I_{rem} = I \backslash C$. Recalculate the remaining quota $q_{rem} = q - |A|$, numbers of reserved positions $r_{rem} = \max\{0, -\rho(A)\}$ and trait matrix T_{rem} for all agents from I_{rem}. If $q_{rem} = 0$ – end the procedure.

  Find l_m under $I_{rem}, q_{rem}, r_{rem}$ and T_{rem},

  – if $l_m > 0$, set $A = A \cup \{I^1_{rem}, \ldots, I^{l_m}_{rem}\}$ – hired agents, set $C = C \cup A$;

  – if $l_m = 0$, set $C = C \cup I^1_{rem}$.

Through the construction of the upper-dominant choice rule, all the previous results hold.

Theorem 6. *A choice rule C is non-wasteful and satisfies upper-domination if, and only if, it is C^{UD}.*

In other words, the upper-dominant choice rule tries to exactly fill reserved positions for each trait and attempts to pick the best candidates according to the merit list.[14] Note that the upper-dominant choice rule would have chosen the set S under Example 8.

Illustrative Example (Continued). If the Jedi use the upper-dominant choice rule C^{UD}, then four Jedi Council seats will be taken by $C^{UD}(I) = \{$Obi-Wan, Mace, *Jocasta*, ***Luminara***$\}$, $\rho(C^{UD}(I)) = (0, 0, 0)$. So, the best two applicants were hired by the Temple.

Proposition 20. *A choice rule C^{UD} eliminates justified envy.*

Proof. Direct proof. To the contrary, suppose that there is a pair of agents $i \in C^{UD}(I)$ and $i' \in I \backslash C^{UD}(I)$, such that $i' \; \pi \; i$ and set $(C^{UD}(I) \backslash \{i\}) \cup \{i'\}$ is weakly more diverse than set $C^{UD}(I)$.

Note, however, that since $i' \; \pi \; i$, set $(C^{UD}(I) \backslash \{i\}) \cup \{i'\}$ upper-dominates set $C^{UD}(I)$. So, set $(C^{UD}(I) \backslash \{i\}) \cup \{i'\}$ both upper-dominates set $C^{UD}(I)$, and

[14]So, the upper-dominant choice rule C^{UD} is a generalization of the extreme paired-admission choice rule $C^{\min\max}$ from Sönmez and Yenmez (2019a).

is weakly more diverse than set $C^{UD}(I)$. Hence, our choice rule C^{UD} does not satisfy lower-domination. This contradicts Theorem 6.

Alternative proof. From Theorem 6 we know that C^{UD} satisfies upper-domination. Hence, C^{UD} is undominated. Using Proposition 16 we obtain that C^{UD} eliminates collective justified envy. Thus, C^{UD} eliminates justified envy. □

The following example further motivates why we need both lower-dominant and upper-dominant choice rules.

Example 9.

- There are 8 agents in the set I with merit list: $i_1 \, \pi \, i_2 \, \pi \, \ldots \, \pi \, i_7 \, \pi \, i_8$.
- There are 4 traits, each with 1 reserved position.
- Quota is $q = 4$, so $\sum_{t \in \mathcal{T}} r_t = 4 \leq q$.
- Sets of traits for each agent are:

i_1	i_2	i_3	i_4	i_5	i_6	i_7	i_8
-	-	-	t_1	t_2	t_3	t_4	t_1
			-	-	-	-	t_2
							t_3
							t_4
							-

Table 2.3: Sets of agents' traits for Example 9

It is easy to verify that $C^{LD}(I) = \{i_4, i_5, i_6, i_7\}$, $|C^{LD}(I)| = 4 = q$. Also, $\rho(C^{LD}(I)) = (0, 0, 0, 0)$, so $C^{LD}(I)$ exactly fills all reserved positions.

However, one can argue that we can pick the three best agents i_1, i_2, i_3 if the worst agent i_8 fills all the reserved positions. This exact issue occurs if we use the upper-dominant choice rule: $C^{UD}(I) = \{i_1, i_2, i_3, i_8\}$, $|C^{UD}(I)| = 4 = q$. Again, $\rho(C^{UD}(I)) = (0, 0, 0, 0)$, so $C^{UD}(I)$ also exactly fills all reserved positions.

Moreover, under the Example 2 from Sönmez and Yenmez (2019a) all three choice rules: C^{LD}, C^{UD} and C^{SM} with the simplest weighting vector v^s produce different solutions (sum-minimizing choice correspondence produces only one solution). This means that, in general, neither C^{LD} nor C^{UD} is included in $C^{SM}(v^s)$.

Proposition 21. *Given a reservation market and a set of all candidates $I \subseteq \mathcal{I}$, the following is true. The set of chosen candidates with no traits after using any non-wasteful and undominated choice rule C is a subset of a set of chosen candidates with no traits after using the upper-dominant choice rule,* $\{i \in C(I), |\tau(i)| = 0\} \subseteq \left\{i \in C^{UD}(I), |\tau(i)| = 0\right\}$.

Proof. Note that, due to the elimination of justified envy, if a k-th best candidate has no traits and she is chosen, then all better candidates are also chosen under any non-wasteful and undominated choice rule.

Now, suppose by contrary, that there are more agents with no traits in $C(I)$ than in $C^{UD}(I)$. From above we can infer that the worst agent with no traits from $C(I)$ is worse than the worst agent with no traits from $C^{UD}(I)$. This implies that $C(I)$ upper-dominates $C^{UD}(I)$. Contradiction. $\square$

So, if one wants to hire additional strong candidates by offering some slots to low-skilled depressed ones, then one should use the upper-dominant choice rule. Otherwise, if one wants to favor depressed candidates (to choose more candidates with traits) by avoiding hiring low-skilled depressed ones, then one should use the lower-dominant choice rule.

Note that, in general, if an agent with no traits is chosen under C^{LD}, then she may not be chosen by some other non-wasteful and undominated choice rule.

Example 10.

- There are 7 agents in the set I with merit list: $i_1 \ \pi \ i_2 \ \pi \ \ldots \ \pi \ i_6 \ \pi \ i_7$.
- There are 4 traits, each with 1 reserved position.
- Quota is $q = 4$, so $\sum_{t \in \mathcal{T}} r_t = 4 \leq q$.
- Sets of traits for each agent are:

i_1	i_2	i_3	i_4	i_5	i_6	i_7
-	-	t_1	t_2	t_1	t_2	t_3
		-	-	t_3	t_4	t_4
				-	-	-

Table 2.4: Sets of agents' traits for Example 10

It is easy to verify that $C^{LD}(I) = \{i_1, i_2, i_5, i_6\}$, $|C^{LD}(I)| = 4 = q$. Also, $\rho(C^{LD}(I)) = (0, 0, 0, 0)$, so $C^{LD}(I)$ exactly fills all reserved positions.

However, there is another non-wasteful and undominated solution $S = \{i_1, i_3, i_4, i_7\}$, $|S| = 4 = q$. Again, $\rho(S) = (0, 0, 0, 0)$, so S also exactly fills all reserved positions. However, an agent with no traits i_2 is not chosen now.

This means that C^{LD} does not always pick the greatest possible number of depressed players (which is the case for depressed players with exactly one trait under no more than two traits).[15]

2.5 No Strategy-Proofness

When taking into account the derived choice rules above, it is natural to ask whether it is natural to ask the following: is it possible to use them in a Deferred Acceptance algorithm for markets with multiple employers. This is possible only if these rules satisfy both the *irrelevance of rejected individuals* and *substitutes* conditions defined below.[16]

Definition 36 (Aygün and Sönmez (2013))**.** A choice rule C satisfies the **irrelevance of rejected individuals** condition, if, for every $I \subseteq \mathcal{I}$, $i \in I \backslash C(I) \implies C(I \backslash \{i\}) = C(I)$.

Proposition 22. *Choice rules C^{LD} and C^{UD} satisfy the irrelevance of rejected individuals condition.*

Proof. To the contrary, suppose that an agent $i \in I$ is not hired, $i \notin C^{LD}(I)$ ($i \notin C^{UD}(I)$), but $C^{LD}(I \backslash \{i\}) \neq C(I)$ ($C^{UD}(I \backslash \{i\}) \neq C(I)$). However, from

[15] Proposition 2, point (2).(a) from Sönmez and Yenmez (2019a).
[16] See Hatfield and Milgrom (2005) and Aygün and Sönmez (2013).

Theorem 5 (Theorem 6) we know that $C^{LD}(I)$ lower-dominates ($C^{UD}(I)$ upper-dominates) any other subset of I with $\min\{|I|, q\}$ elements. This implies that $C^{LD}(I)$ lower-dominates ($C^{UD}(I)$ upper-dominates) any other subset of $I\backslash\{i\}$ with $\min\{|I|-1, q\}$ elements, since $i \notin C^{LD}(I)$ ($i \notin C^{UD}(I)$). Hence, $C^{LD}(I\backslash\{i\}) = C^{LD}(I)$ ($C^{UD}(I\backslash\{i\}) = C^{UD}(I)$). This is a contradiction. □

Definition 37 (Kelso and Crawford (1982))**.** A choice rule C satisfies the **substitutes** condition, if, for every $I \subseteq \mathcal{I}$,

$$i \in C(I) \text{ and } i' \neq i \implies i \in C(I\backslash\{i'\}).$$

Proposition 23. *Any non-wasteful and undominated choice rule C does not satisfy the substitutes condition.*

Proof. The example for the substitutes condition is taken from Sönmez and Yenmez (2019a).

- There are 4 agents in the set I with merit list: $i_1 \ \pi \ i_2 \ \pi \ i_3 \ \pi \ i_4$.
- There are 2 traits, each with 1 reserved position.
- Quota is $q = 2$, so $\sum_{t\in\mathcal{T}} r_t = 2 \leq q$.
- Sets of traits for each agent are:

i_1	i_2	i_3	i_4
-	t_1	t_1	t_2
	-	t_2	-
		-	

Table 2.5: Sets of agents' traits for no substitutes

Our choice rules will yield $C(I) = \{i_1, i_3\}$, $|C(I)| = 2 = q$ and $\rho(C(I)) = (0, 0)$, since the most diverse choice rule could have chosen a subset out of $S_1 = \{i_1, i_3\}$, $S_2 = \{i_2, i_3\}$, $S_3 = \{i_2, i_4\}$, or $S_4 = \{i_3, i_4\}$, and S_1 dominates all the others. So, $i_1 \in C(I)$. However, if we exclude i_3 from I, our choice rule will choose

$C(I\backslash\{i_3\}) = \{i_2, i_4\}$, and again $\rho(C(I\backslash\{i_3\})) = (0,0)$. Therefore, i_1 is not chosen. This example proves that no one-to-all choice function that tries to exactly fill all reserved positions satisfies the substitutes condition. □

Note that no agent i can increase their chances of being hired by not revealing any of her traits.[17] It is also natural to assume that nobody can credibly lie about having any trait that they do not have. This means that everyone will reveal all of their traits. In other words, nobody will strategize over her set of traits.

However, since any non-wasteful and undominated choice rule does not satisfy the substitutes condition, we cannot guarantee strategy-proofness of the agent-proposing Deferred Acceptance algorithm (DAA) (potential employees propose to employers) or the stability of the final allocation. To sum up, any agent will reveal all her traits, but may lie about her preferences over employers.

2.6 Application: The Law of Social Quotas in Brazil

On August 29, 2012, President Dilma Rousseff enacted the *Law of Social Quotas* in Brazil, which requires public colleges to set reserve positions for students. The following three traits were considered in this affirmative action law:

- students with a public school education (trait p),
- students with low income (trait l), and
- students who are black, mixed, or indigenous descent (trait m).

Denote the total quota of a public college as q and a share of minorities in the State population as $s \leq 1/2$. Any public college should reserve at least half of its seats, $\lceil q/2 \rceil$, for graduates of public high schools. Within this half, according to the normative ordinance 18 published by Brazil's Ministry of Education on

[17] Aygün and Bó (2020) state this as a *privilege monotonicity* property of a choice function.

October 11 of the same year, the following quotas were suggested for traits l and m and their combination:

- for students from minorities with a public school education and low income (traits p, l, and m) – $\lceil \frac{q}{4}s \rceil$ seats,
- for not low income students from minorities with a public school education (traits p and m only) – $\lceil \frac{q}{4}s \rceil$ seats,
- for students not from minorities with a public school education and low income (traits p and l only) – $\lceil \frac{q}{4}(1-s) \rceil$ seats, and
- for not low income students not from minorities with a public school education (trait p only) – $\lceil \frac{q}{4}(1-s) \rceil$ seats.

Aygün and Bó (2020) showed that the currently used in Brazil mechanism is not strategy proof, since applicants may trick the system by strategizing over their traits. On the other hand, choice rules C^{LD}, C^{UD} and C^{SM} proposed in this paper eliminate such possibility for an applicant. Besides, these choice rules can be applied to the Brazilian case due to the following reasons. First, college admissions process in Brazil is decentralized, so there is no need for a choice rule that satisfies substitutes condition; second, from above we conclude that Brazil's Ministry of Education is willing to reserve $r_p = \lceil \frac{q}{2} \rceil$ seats for applicants with a trait p, $r_l = \lceil \frac{q}{4}s \rceil + \lceil \frac{q}{4}(1-s) \rceil = \lceil \frac{q}{4} \rceil$ seats for applicants with traits p and l, and $r_m = \lceil \frac{q}{4}s \rceil + \lceil \frac{q}{4}s \rceil = \lceil \frac{q}{2}s \rceil \leq \lceil \frac{q}{4} \rceil$ seats for applicants with traits p and m, hence, our initial constraint on total amount of reserved seats being smaller than quota holds.[18] Moreover, an application of the one-to-all approach will introduce more flexibility and transparency into the mechanism.

So, once we receive the merit list, π, with all $|I| = n$ applicants, and corresponding sets of traits, we should, at first, delete all traits for any applicant without a trait p: if $p \notin \tau(i)$, hence, $\tau(i) \equiv \emptyset$.

[18] Since we do not consider one's traits if she does not have a trait p.

Now we can construct a reservation market. We have a set of all applicants $I \in \mathcal{I}$; a set of all traits $\mathcal{T} = \{p, l, m\}$; correspondence τ (if $p \notin \tau(i)$, and hence $\tau(i) \equiv \emptyset$); a merit list π with all, $|I| = n$, applicants; total quota of a public college, q; reserved positions for each trait: $r_p = \lceil q/2 \rceil$, $r_l = \lceil q/4 \rceil$, $r_m = \lceil q \cdot s\% \rceil$. From above, our initial constraint holds: $r_p + r_l + r_m = \lceil \frac{q}{2} \rceil + \lceil \frac{q}{4} \rceil + \lceil \frac{q}{2} s \rceil \leq q$. Hence, it is possible to apply any of our choice rules, C^{LD}, C^{SM}, or C^{UD} in order to obtain an undominated and non-wasteful solution.

2.7 Final Remarks

This paper develops a framework for quota-based affirmative action allocation markets under the one-to-all matching convention. Depending on the agenda of the policymaker, three different diversity maximizing solutions for a choice problem are proposed. If he would like to avoid picking low-skilled applicants as much as possible, then the *lower-dominant* choice rule is the best solution. If, however, he would like to pick as many high-skilled applicants as possible, then the *upper-dominant* choice rule should be chosen. Finally, if his goal lies somewhere in between, he should choose a corresponding weighting vector and pick a solution from the *sum-minimizing* choice correspondence outcome. Unfortunately, due to complementarities between agents following from naturally imposed non-wastefulness, any meaningful rule under the one-to-all approach cannot be used together with an agent-proposing Deferred Acceptance algorithm.

Therefore, constructed solutions may be successfully applied in presence of one principal and a set of applicants with overlapping traits structure, such as a hiring process, or a college admission procedure, e.g. the one established in Brazil.

Chapter 3

Relaxing Stability and Efficiency in Two-Sided Matching Markets

3.1 Motivation

Consider a two-sided matching market where one side is not strategic. As a major example, I will refer to a school choice problem introduced by Abdulkadiroglu and Sönmez (2003). Students are applying to colleges by submitting their strict preferences over the other side of the market, while merit list for each college is explicitly constructed using only external to this college information (e.g., students' test scores, walk zone, sibling, etc.).

As mentioned by Abdulkadiroglu and Sönmez (2003) there does not exist stable and Pareto efficient for students matching mechanism.[1] Depending on a policy maker's agenda, one of these two desirable properties may be considered as more important. Suppose that each college has only one empty slot to fill, and that a student may have unacceptable colleges.[2]

If stability is chosen, then Gale and Shapley's deferred-acceptance algorithm (DA) (Gale and Shapley, 1962) is a natural choice, since it is stable and strategy-proof (for students), and moreover weakly Pareto dominates any other stable

[1] Due to example by Roth (1982).

[2] She prefers staying alone to being matched with this college.

mechanism. If, conversely, efficiency for students is more important for a policy maker, then Gale's top trading cycles (TTC) (Shapley and Scarf, 1974) is a natural mechanism to use,[3] since it is also strategy proof and, as shown by Abdulkadiroglu et al. (2020), it minimizes justified envy among all Pareto efficient and strategy-proof mechanisms in one-to-one matching.[4]

Therefore, if so happens that one property is preferred over the other, then a policy maker has a definite matching mechanism to use. Though, it remains unclear which mechanism to pick if both properties are important. Naturally, the following questions arises. What is the best matching mechanism that is *approximately* stable and *approximately* efficient?[5] So we allow for simultaneous relaxation of both properties. In the context of randomized matching mechanisms we focus on relaxing ex-ante stability[6] and ex-post Pareto efficiency for students.[7] Note that impossibility of ex-ante stable and ex-post Pareto efficient for students randomised matching mechanism trivially follows from the impossibility of stable and Pareto efficient for students matching mechanism, mentioned above.

Che and Tercieux (2019a) approach this question by considering large markets. They construct asymptotically efficient and asymptotically stable extension of DA. Their findings suggest that commonly used strategy of achieving one property with a minimal sacrifice of the other (as DA or TTC do) may not be the best there is. Lee and Yariv (2018) study a trade-off between utilitarian efficiency and stability in large markets. Their results suggest that using (ordinal) stable matching mechanisms for obtaining average efficiency is justified if markets are balanced and agents' preferences are not severely correlated. Conversely, Heo (2019) achieves stable (fair) and efficient DA and TTC mechanisms by restricting

[3]Cutoff characterization of TTC mechanisms for school choice is introduced by Leshno and Lo (2020).

[4]Abdulkadiroglu and Grigoryan (2020) resolve the trade-off between efficiency and stability under weak priorities for sequential dictatorships and hierarchical exchange rules.

[5]Approximate ex-ante efficiency is discussed by Immorlica et al. (2017).

[6]Given preferences for all students and colleges, in the resulting randomized matching there is probability zero for any pair to become blocking.

[7]Only Pareto efficient for students matchings have positive corresponding probabilities in the resulting distribution over all matchings.

preference profiles set. She shows that any preference profile resulting in stable and efficient TTC matching also produces stable and efficient DA matching. Also, for weak preferences environment Erdil and Ergin (2008) construct the stable improvement cycles mechanism that maximises efficiency. However, it fails to be strategy-proof.

3.2 Approach

Unlike mentioned approaches, I try to construct and investigate behaviour of a matching mechanism that efficiently relaxes efficiency and stability for a small one-to-one matching market with three students and three colleges without restrictions on preference profiles. I construct loss functions for both, ex-ante stability and ex-post efficiency (ex-ante stability violation and ex-post efficiency violation) and minimize a convex combination of these functions on a large randomly sampled set of preference profiles.

For three-by-three matching market each student may have $3 \cdot 3! = 18$ different possibly truncated strict preferences over the set of all colleges.[8] I assume that all students are acceptable for a college, hence each college may have $3! = 6$ different strict preferences over the set of all students. Hence, there are over 1.2 million possible preference profiles for this market.[9] I investigate environments with varying degrees of preference correlation along each side of the market. For each environment I randomly sample 50,000 profiles. Now we turn to the model.

The rest of the Chapter is structured as follows. Section 3.3 formally presents an underlying model and related definitions. In section 3.4 I introduce the well known state of the art mechanisms: DA and TTC. Section 3.5 constructs loss functions for ex-ante stability and ex-post PE, and presents the general optimization problem. Four matching specifications for computer simulations are summarised in Section 3.6. Section 3.7 discusses the simulations results and concludes.

[8] I omit possibility of every college being unacceptable.

[9] Over 34 million if we allow truncated preferences for colleges

3.3 Model

Consider a one-to-one matching market with n students and m colleges. Denote set of students as S and set of colleges as C. Denote as $\mathcal{M}$ the set of all possible matchings, where each matching $\mu \in \mathcal{M}$ consists of either pairs (s, c), or singletons $(s, \perp)$ or $(\perp, c)$, where $s \in S$ and $c \in C$, such that any agent from $S \cup C$ is met exactly once. A singleton with a student $s \in S$ means that this student is alone in the current matching. Same for a college. So, $\perp$ denotes an empty option. For convenience let us denote $\overline{C} = C \cup \{\perp\}$. For a given matching $\mu' \in \mathcal{M}$ we denote a partner of an agent $a \in S \cup C$ as $\mu(a) \in S \cup \overline{C}$.

Each student $s \in S$ has a strict preference order $\succ_s$ over the set $\overline{C}$. If a student $s \in S$ prefers college $c \in C$ to another college $c' \in C$ we write $c \succ_s c'$. If $\perp \succ_s c$, hence college $c \in C$ is unacceptable for a student $s \in S$. Each college $c \in C$ has a strict preference order $\succ_c$ over the set S. If a college $c \in C$ prefers student $s \in S$ to another student $s' \in S$ we write $s \succ_c s'$.

We denote a preference profile (set of preference orders of all agents) as $P = \{\succ_{s_1}, \ldots, \succ_{s_n}, \succ_{c_1}, \ldots, \succ_{c_m}\} \in \mathcal{P}$, where $\mathcal{P}$ is the set of all preference profiles.

3.3.1 Pareto Efficiency and Stability

Matching $\mu \in \mathcal{M}$ is called **Pareto efficient** (for students) (PE), if there does not exist any other matching $\mu' \in \mathcal{M}$, such that some student's $s \in S$ partner is strictly better for her, $\mu'(s) \succ_s \mu(s)$, and any student's $s' \in S$ partner is not worse for her, $\mu(s') \succ_{s'} \mu'(s')$ does not hold for all $s' \in S$. Simply put, efficient matching cannot be weakly improved for all students by some other matching. Denote a set of all PE matchings under a preference profile P as $\mathrm{PE}(\mathcal{M}, P) \subseteq \mathcal{M}$.

Matching $\mu \in \mathcal{M}$ is called **individually rational** (IR), if any agent $a \in S \cup C$ is either alone, $\mu(a) = \perp$, or has an acceptable partner, $\mu(a) \succ_a \perp$. In other words, for any matching there does not exist a student paired with an unacceptable college.

A pair (s, c) of a student $s \in S$ and a college $c \in C$ is called a **blocking pair** for a matching $\mu \in \mathcal{M}$, if both agents prefer each other to their matches, $c \succ_s \mu(s)$ and $s \succ_c \mu(c)$. If such matching is formed, both agents will drop their assigned partners and match with each other. So, a matching $\mu \in \mathcal{M}$ is called **stable** if is IR and there does not exist blocking pairs.

A **matching mechanism** ψ is a function that maps preference profiles $P \in \mathcal{P}$ to matchings $\mu \in \mathcal{M}$, $\psi : \mathcal{P} \mapsto \mathcal{M}$. A matching mechanism ψ is called PE, IR or stable, if for any preference profile $P \in \mathcal{P}$, its image $\psi(P)$ is correspondingly PE, IR or stable.

3.3.2 Randomized matchings

For a given matching market a **randomized matching mechanism** φ is a function that maps preference profiles $P \in \mathcal{P}$ to probability distributions over the set of all matchings $\rho \in \triangle(\mathcal{M})$, $\varphi : \mathcal{P} \mapsto \triangle(\mathcal{M})$.[10]

So, a randomized matching $\rho \in \triangle(\mathcal{M})$ is a vector of probabilities ρ_μ for any matching $\mu \in \mathcal{M}$, such that they are all non-negative, $\rho_\mu \geq 0$ for any $\mu \in \mathcal{M}$, and sum up to one, $\sum_{\mu \in \mathcal{M}} \rho_\mu = 1$. It is easy to show that if there are n students and m colleges and $k = \min\{n, m\}$, we have $M(n, m) = 1 + \sum_{i=0}^{k-1} \binom{n}{k-i} \cdot \binom{m}{k-i} \cdot (k-i)!$ different matchings (including a *singleton matching* with no pairs).[11] Hence, a randomized matching $\rho \in \triangle(\mathcal{M})$ consists of $M(n, m)$ probabilities.

Definition 38. (Ex-post Pareto efficiency). A randomized matching mechanism φ is called *ex-post Pareto efficient* if for any preference profile $P \in \mathcal{P}$ the resulting randomized matching assigns positive probabilities only to PE matchings: $\forall P \in \mathcal{P}$: $\varphi(P)_\mu > 0$ only if $\mu \in \mathrm{PE}(\mathcal{M}, P)$.

In addition, for any randomized matching $\rho \in \triangle(\mathcal{M})$ we can construct an $(n+1)$-by-$(m+1)$ randomized matching matrix $R(\rho)$ with marginal probabilities of matching together any pair of student and college and for any agent to stay alone.

[10] $\triangle(X)$ denotes the probability simplex on the set X.
[11] $\binom{n}{k}$ denotes a binomial coefficient n-choose-k, and $n!$ denotes a factorial of n.

For each pair $(s, c) \in S \times C$ the corresponding marginal probability is calculated as follows: $R(\rho)_{sc} = \sum_{\mu \in \mathcal{M}} \mathbb{I}((s, c) \in \mu) \cdot \rho_\mu \geq 0$.[12] For any student $s \in S$ a marginal probability of staying alone under distribution ρ is $R(\rho)_{s\perp} = 1 - \sum_{c \in C} R(\rho)_{sc} \geq 0$. Analogously, for any college $c \in C$ a marginal probability of staying alone under distribution ρ is $R(\rho)_{\perp c} = 1 - \sum_{s \in S} R(\rho)_{sc} \geq 0$.

Definition 39. (Individual rationality). A randomized matching $\rho \in \triangle(\mathcal{M})$ is called *IR* if there does not exist an agent $a \in S \cup C$ with a positive marginal probability of being matched with an unacceptable partner.

So, if, for instance, a college c is unacceptable for a student s, $\perp \succ_s c$, and the corresponding marginal probability $R(\rho)_{sc}$ is positive, hence our randomized matching ρ is not IR, because it leaves positive probability for not IR matching to be formed.

Definition 40. (Elimination of ex-ante justified envy). A randomized matching $\rho \in \triangle(\mathcal{M})$ *eliminates ex-ante justified envy* if there does not exist a pair of student and college $(s, c) \in S \times C$, such that

- there exists another college or an empty slot $c' \in \overline{C}$, such that the marginal probability for him and student s of being matched $R(\rho)_{sc'}$ is positive, while s strictly prefers c, $c \succ_s c'$;
- there exists another student or an empty slot $s' \in S$, such that the marginal probability for her and college c of being matched $R(\rho)_{s'c}$ is positive, while c strictly prefers s, $s \succ_c s'$.

So, if a randomized matching ρ does not eliminate ex-ante justified envy, it leaves positive probability for a blocking pair (s, c) to arise.

Definition 41. (Ex-ante stability). A randomized matching mechanism φ is called *ex-ante stable* if for any preference profile $P \in \mathcal{P}$ the resulting randomized matching is individually rational and eliminates ex-ante justified envy.

[12] $\mathbb{I}(\cdot)$ represents an indicator function.

3.4 Deferred Acceptance and Top Trading Cycles

In this section I consider two representative mechanisms: *deferred acceptance*, which is stable but not Pareto efficient for students, and *top trading cycles* (TTC), which is Pareto efficient for students but not stable. Both mechanisms are strategy-proof for students.

3.4.1 Deferred Acceptance

As an input of the deferred acceptance matching mechanism we receive a preference profile $P \in \mathcal{P}$ with strict preferences of each agent over the agents on the other side of the market and an empty option. The student-proposing DA is used.

- **Step 1:**
  1. Each student $s \in S$ proposes to her best college.
  2. Each college $c \in C$ accepts his best student among the set of received offers and reject all the others.
- **Step $(\mathbf{k+1})$:**
  1. Each student s rejected at step k proposes to her best college among those who have not yet rejected him.
  2. Each college c accepts his best student among the set of received during this step offers and his best student choice from the previous step (if any) and rejects all the others.
- **Final Step:** If there are no rejected students at the previous step, then include all current pairs of colleges and accepted students into the resulting matching. All other agents are alone in the resulting matching (matched to an empty option $\perp$).

So, output of a DA is a matching $\mu^{DA} \in \mathcal{M}$. Student-proposing DA is stable and strategy-proof for students (Roth and Sotomayor, 1990).

3.4.2 Top Trading Cycles

Again, as input for a top trading cycles matching mechanism we receive a preference profile $P \in \mathcal{P}$ with strict preferences of each agent over the agents on the other side of the market (and an empty option for students).

- **Step 1:** Each agent and an empty option are considered as nodes of a directed graph. Each agent *points* to her best available partner (so far everyone are available). So, there is a directed edge from an agent a_1 to the other agent a_2 if a_2 is the best possible partner for a_1 (nobody points to the empty option during the first step, since everyone has at least one acceptable partner).

  Obtained directed graph has at least one cycle. All present cycles are disjoint, so we can resolve all of them: for any student s in any cycle we match her with a college c she points to.

  All obtained pairs are added to the final matching (so all matched students and colleges become unavailable for the others).

- **Step $(\mathbf{k+1})$:** Directed graph is constructed analogously, but now some agent can point to an empty option $\perp$. If there are no available colleges or no available students – stop the algorithm (all left agents are singletons).

  Otherwise, there should be at least one cycle. Resolve all cycles. Add obtained pairs to the final matching. Also add to the final matching all students who points to an empty option as singletons (they do not have any available acceptable college).

  Make all newly added to the matching agents unavailable for the others.

- **Final Step:** Once there are only students left, or only colleges left, or no one left – make all left agents singletons and stop the algorithm.

3.5 Optimization Problem

In this section I formulate the problem of finding the best randomized two-sided matching mechanism that relaxes ex-ante stability and ex-post efficiency as an optimization problem.

3.5.1 Preference Profiles

Recall that a randomized two-sided matching mechanism is a mapping $\varphi : \mathcal{P} \mapsto \triangle(\mathcal{M})$. So, for each preference profile $P \in \mathcal{P}$ we need to be able to transform ordinal preferences $\succ_a$ of each agent $a \in S \cup C$ to cardinal preferences p_a, that will constitute as input for our minimization problem.

For each student $s \in S$ suppose that his ordinal strict preferences $\succ_s$ over $\overline{C}$ are represented by a $(m+1)$-by-one vector with natural numbers, where for any $i \in \{1, \dots, m\}$ i-th number equals $(m+1)$ minus a position of a college c_i in student's s ranking, and the last $(m+1)$-th number equals $(m+1)$ minus a position of an empty option $\perp$ in student's s ranking. For instance, if there are $m = 3$ colleges, and for some student s we have $c_2 \succ_s c_3 \succ_s \perp \succ_s c_1$, hence $\succ_s = (0, 3, 2, 1)$.

Note that we already can use $\succ_s$ as cardinal preferences, however it is a good practice to center and normalize data. So, given a vector $\succ_s$, we firstly center it by subtracting its last element from all elements and, since the last $(m+1)$-th element is now zero, we cut it off and obtain an m-by-one vector. Going back to the example: from $(0, 3, 2, 1)$ we obtain $(-1, 2, 1)$. And finally, we normalize it by dividing each element by m. So, our resulting vector of cardinal preferences is $p_s = \left(-\frac{1}{3}, \frac{2}{3}, \frac{1}{3}\right)$. Note, that if i-th college c_i is unacceptable for student s, hence $p_s(i) < 0$. Procedure of constructing a cardinal preference vector p_c for any college $c \in C$ is defined analogously.

So, the input vector p for out minimization problem is a $2mn$-by-one vector with all cardinal preference vectors combined: $p = (p_{s_1,c_1}, \dots, p_{s_1,c_m}, p_{s_2,c_1}, \dots, p_{s_2,c_m}, \dots, p_{s_n,c_1}, \dots, p_{s_n,c_m}, p_{c_1,s_1}, \dots, p_{c_1,s_n}, \dots, p_{c_m,s_1}, \dots, p_{c_m,s_n})$.

The output – is a probability distribution over all matchings represented by a $M(n,m)$-by-one vector ρ of probabilities, where each probability is a non-negative number and the sum of all probabilities equals one.

3.5.2 Loss Function

In this section I construct metrics for violation of ex-ante stability (stv) and violation of ex-post Pareto efficiency (pev), which convex combination will constitute a loss function $\mathcal{L}$ that will be minimized on a randomly sampled data.

Ex-ante stability violation

For notational convenience we denote $\varphi_{sc}(p) = R\left(\varphi(p)\right)_{sc}$ as a marginal probability of a match between $s \in S$ and $c \in \overline{C}$. For each pair of student $s \in S$ and college $c \in C$, according to Definition 40, we define the elimination of ex-ante justified envy violation at preference profile $p \in \mathcal{P}$ as in Ravindranath et al. (2021) (assume that $p_{\cdot\perp} = p_{\perp\cdot} = 0$):

$$ev_{sc}(\varphi, p) = \underbrace{\left(\sum_{s' \in S} \varphi_{s'c}(p) \cdot \max\{p_{cs} - p_{cs'}, 0\}\right)}_{\text{college } c \text{ part}} \cdot \underbrace{\left(\sum_{c' \in \overline{C}} \varphi_{sc'}(p) \cdot \max\{p_{sc} - p_{sc'}, 0\}\right)}_{\text{student } s \text{ part}} \tag{3.1}$$

Also, according to Definition 39, we define IR violation at profile p for all agents as in Ravindranath et al. (2021):

$$irv(\varphi, p) = \underbrace{\frac{1}{2m}\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi_{s_ic_j}(p)\cdot\max\{-p_{c_js_i}, 0\}}_{\text{colleges' part}} + \underbrace{\frac{1}{2n}\sum_{i=1}^{n}\sum_{j=1}^{m}\varphi_{s_ic_j}(p)\cdot\max\{-p_{s_ic_j}, 0\}}_{\text{students' part}} \quad (3.2)$$

Finally, according to Definition 41, we define stability violation at profile p as

$$stv(\varphi, p) = \frac{1}{2}\left(\frac{1}{n} + \frac{1}{m}\right)\sum_{s\in S}\sum_{c\in C} ev_{sc}(\varphi, p) + irv(\varphi, p) \quad (3.3)$$

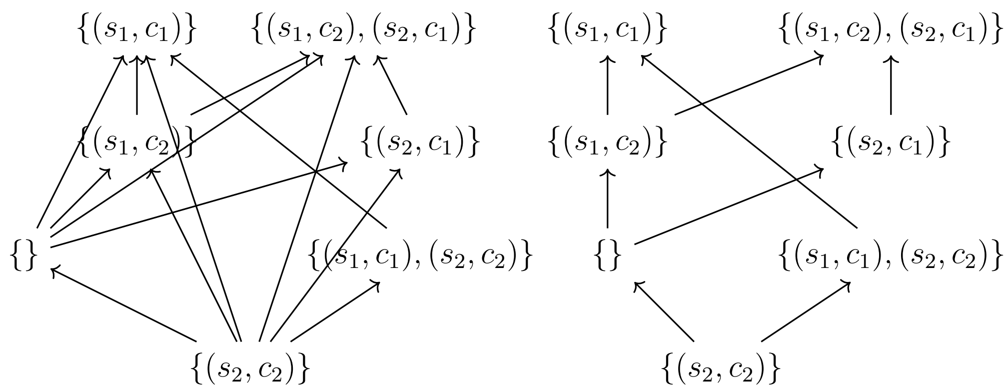
The average stability violation on a data set (set of cardinal preference profiles) is denoted as $stv(\varphi)$. We also define the expected stability violation as $STV(\varphi) = \mathbb{E}_{\mathcal{P}}[stv(\varphi, p)]$.

Proposition 24. *(Ravindranath et al., 2021) A randomized matching mechanism φ is ex-ante stable up to zero-measure events if and only if $STV(\varphi) = 0$.*

Ex-post Pareto efficiency violation

According to Definition 38, a randomized matching $\rho \in \triangle(\mathcal{B})$ is not ex-post PE if there is at least one not PE matching $\mu \in \mathrm{PE}(\mathcal{M}, P)$ with a positive assigned probability, $\rho_\mu > 0$. So, the most simple way to define an ex-post Pareto efficiency violation of a given randomized matching is just to sum up all its elements assigned to not PE matchings. However, not any two not PE matchings are equally not PE. So, probability assigned to a *more not PE* matching should receive a bigger weight during calculation of PE violation. The question is: how to measure a degree of being *not PE* for a matching?

Note that a Pareto ordering relation on the set of all matchings $\mathcal{M}$ is a strict partial order (irreflexive, transitive and asymmetric). Hence, under fixed two-sided n-by-m matching market, for any given preference profile $P \in \mathcal{P}$ we can



(a) Directed acyclic graph. (b) Hasse diagram.

Figure 3.1: Strict partial Pareto ordering representations

construct a directed acyclic graph (DAG) with $M(n, m)$ nodes, where there is a directed edge from a node i to a node j if and only if matching μ_j Pareto dominates matching μ_i.

As a simple example consider two-by-two two-sided matching market and the following preference profile $P \in \mathcal{P}$:

- for s_1: $c_1 \succ_{s_1} c_2 \succ_{s_1} \perp$;
- for s_2: $c_1 \succ_{s_2} \perp \succ_{s_2} c_2$;
- for c_1: $s_1 \succ_{c_1} s_2 \succ_{c_1} \perp$;
- for c_2: $s_2 \succ_{c_2} s_1 \succ_{c_2} \perp$.

There are seven different possible matchings (for each matching only contained in it pairs of agents are written, all other agents are alone): $\{\}$, $\{(s_1, c_1)\}$, $\{(s_1, c_2)\}$, $\{(s_2, c_1)\}$, $\{(s_2, c_2)\}$, $\{(s_1, c_2), (s_2, c_1)\}$, $\{(s_1, c_1), (s_2, c_2)\}$. Now we pairwise Pareto compare all matchings and obtain a DAG on Figure 3.1a (and a corresponding Hasse diagram on Figure 3.1b). We see that there are two PE matchings: $\{(s_1, c_1)\}$ and $\{(s_1, c_2), (s_2, c_1)\}$.

Now I formulate what it means for a matching to be *less PE*.

Definition 42. (Degree of not PE). For a given preference profile P, a matching $\mu \in \mathcal{M}$ has *degree-l of not PE*, if $l \geq 0$ is the length of a minimal path across all maximal paths from this matching to each PE matching in the corresponding DAG.[13] Denoted as $\pi(\mu, P) = \pi(\mu, p) = l$.

To get the intuition let me go back to our example. For each PE matching: path to itself has length zero and path do any other PE matching does not exist, hence our PE matchings $\mu_2 = \{(s_1, c_1)\}$ and $\mu_6 = \{(s_1, c_2), (s_2, c_1)\}$ both have degree-0 of not PE. So, $\pi(\mu_2, P) = \pi(\mu_6, P) = 0$.

For matching $\mu_3 = \{(s_1, c_2)\}$: length of a maximal path to one PE matching μ_2 is 1, length of a maximal path to another one PE matching μ_6 is 1. Hence, matching μ_3 has degree-1 of not PE. Analogously, matchings $\mu_4 = \{(s_2, c_1)\}$ and $\mu_7 = \{(s_1, c_1), (s_2, c_2)\}$ both have degree-1 of not PE. So, $\pi(\mu_3, P) = \pi(\mu_4, P) = \pi(\mu_7, P) = 1$.

For matching $\mu_1 = \{\}$: length of a maximal path to one PE matching μ_2 is 2, length of a maximal path to another one PE matching μ_6 is 2. Hence, matching μ_1 has degree-2 of not PE. So, $\pi(\mu_1, P) = 2$.

For matching $\mu_5 = \{(s_2, c_2)\}$: length of a maximal path to one PE matching μ_2 is 3, path to another one PE matching μ_6 does not exist. Hence, matching μ_5 has degree-3 of not PE. So, $\pi(\mu_5, P) = 3$. Therefore, matching μ_5 is *the farthest* one from Pareto efficiency among all possible matchings. Note that it contains only one pair, where a student is matched with her unacceptable college.

Now, finally, we can construct ex-post Pareto efficiency violation of a randomized matching as a weighted sum of probabilities assigned to not PE matchings ($\varphi_\mu(p)$ denotes probability assigned to a matching μ):

$$pev(\varphi, p) = \sum_{\mu \in \mathcal{M}} \pi(\mu, p) \cdot \varphi_\mu(p) \tag{3.4}$$

The average PE violation on a data set (set of cardinal preference profiles) is denoted as $pev(\varphi)$. We also define the expected PE violation as $PEV(\varphi) = \mathbb{E}_{\mathcal{P}}[pev(\varphi, p)]$.

[13]The maximal path between any two nodes in a DAG can be found using Bellman-Ford algorithm (Bellman, 1989) with inverted edge weights. For an arbitrary graph it is an NP-hard problem.

Proposition 25. *A randomized matching mechanism φ is ex-post PE up to zero-measure events if and only if $PEV(\varphi) = 0$.*

Proof. Since for all preference profiles $p \in \mathcal{P}$: $pev(\varphi, p) \geq 0$, then $PEV(\varphi) = \mathbb{E}_{\mathcal{P}}[pev(\varphi, p)] = 0$ if and only if $pev(\varphi, p) = 0$ for all preference profiles $p \in \mathcal{P}$ except on zero measure events. In turn, if for all preference profiles $pev(\varphi, p) = 0$, hence for each $p \in \mathcal{P}$ probability $\varphi_\mu(p)$ for each not PE matching is zero. By Definition 38, it implies that a randomized matching mechanism φ is ex-post PE. □

Loss function construction

Now we can construct a loss function $\mathcal{L}$ that our mechanism φ will minimize on a subset $\mathcal{P}'$ of the set of all preference profiles $\mathcal{P}$, $\mathcal{P}' \subseteq \mathcal{P}$. We formulate the optimization problem as follows:

$$\min_{\varphi} \lambda \cdot stv(\varphi) + (1 - \lambda) \cdot pev(\varphi), \tag{3.5}$$

where $\lambda \in [0, 1]$ is a fixed number that calibrates how much we want to relax ex-post PE compared to ex-ante stability.

Since minimization can be performed independently for each preference profile, we can reformulate the optimization problem as follows. For each preference profile $p \in \mathcal{P}'$:

$$\begin{aligned} &\min_{x} \lambda \cdot stv(x, p) + (1 - \lambda) \cdot pev(x, p), \text{ s.t.} \\ &\sum_{i=1}^{M(n,m)} x_i = 1 \\ &x_i \geq 0 \text{ for } 1 \leq i \leq M(n, m) \end{aligned} \tag{3.6}$$

So, for each sampled preference profile p the problem (3.6) can be solved by using a constraint nonlinear solver (e.g., *fmincon* function in Matlab).

3.6 Matching Specifications

In this section I present different matching specifications with different pairs of truncation and correlation probabilities, Pr_{tr} and Pr_{corr}. If there is no truncation of preferences, meaning – nobody has unacceptable partners, and no correlation among preferences ($Pr_{tr} = Pr_{corr} = 0$), hence each preference profile for the data set is sampled uniformly at random from the set of all preference profiles without unacceptable partners.

If correlation probability is zero, $Pr_{corr} = 0$, but truncation probability is positive, $Pr_{tr} > 0$, we proceed as follows. We firstly uniformly at random sample each preference profile from the set of all preference profiles without unacceptable partners. Then, for each of n preference orders of students in each preference profile with probability $Pr_{tr} > 0$ we perform truncation: we uniformly at random sample a college, which is not the worst one for the considered student, and make all worse colleges unacceptable for her.

If both probabilities are positive, $Pr_{tr} > 0$ and $Pr_{corr} > 0$, we proceed as follows. First, we sample and truncate preference profiles as described above. Then, for each preference profile we uniformly at random sample and truncate one student preference order and sample one college preference order. Finally, for all students: with correlation probability $Pr_{corr} > 0$ we replace her preferences with a sampled one for students, and for all colleges: with correlation probability $Pr_{corr} > 0$ we replace his preferences with a sampled one for colleges.

I investigate four different specifications for $n = 3$ students and $m = 3$ colleges: $(Pr_{tr}, Pr_{corr}) \in \{(0.2, 0), (0.2, 0.75), (0, 0), (0, 0.75)\}$. For each specification I calculate results for DA, TTC and optimized results for $\lambda \in \{0.1, 0.2, \ldots, 0.9\}$.

Also for each specification I calculate the result of the following mechanism, that is efficient and more stable than TTC. Mechanism $\widetilde{\varphi}$: for a given preference profile p obtain matchings for DA and TTC, $\varphi_{DA}(p)$ and $\varphi_{TTC}(p)$,

- if $\varphi_{DA}(p)$ is efficient, then $\widetilde{\varphi}(p) = \varphi_{DA}(p)$;

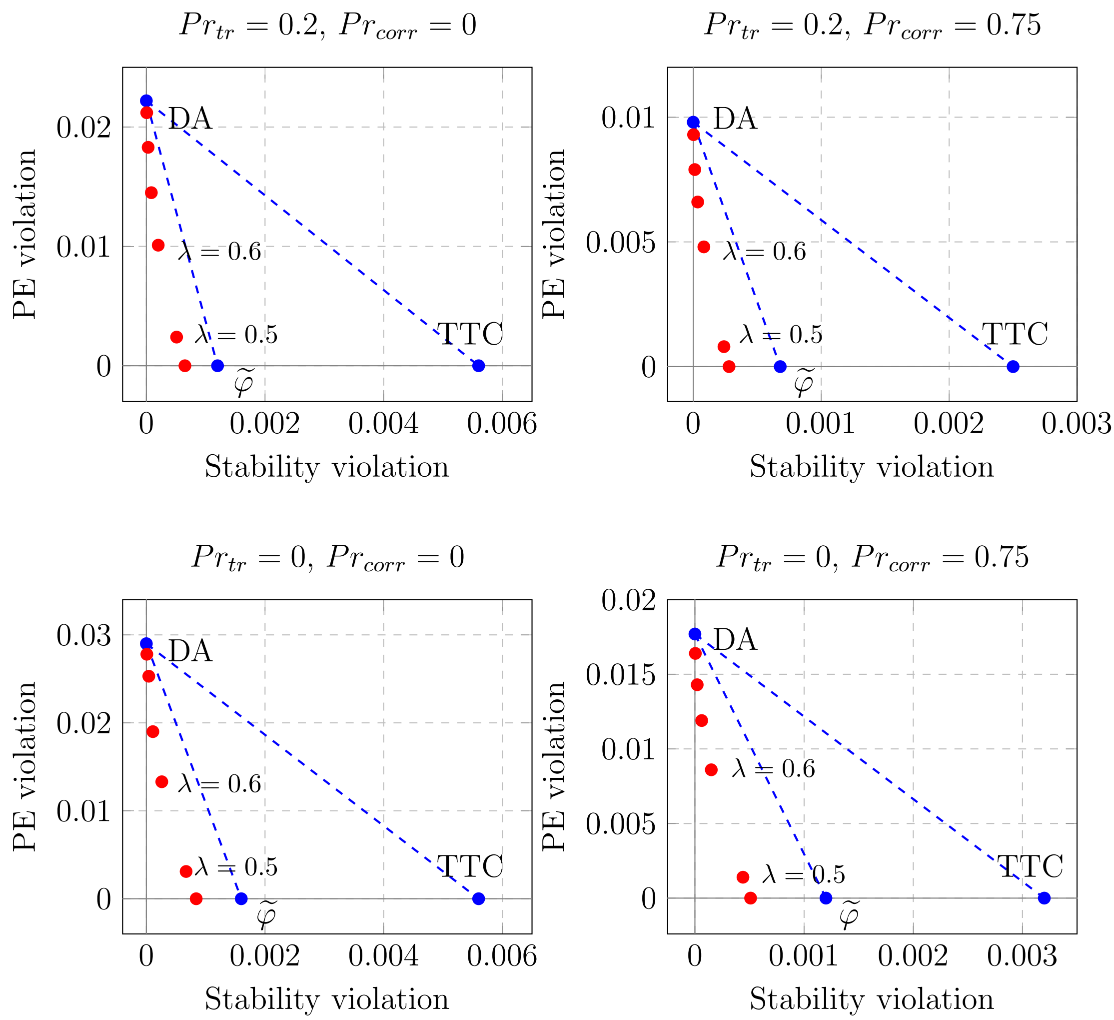


Figure 3.2: Simulation results

- if $\varphi_{DA}(p)$ is not efficient, then $\widetilde{\varphi}(p) = \varphi_{TTC}(p)$.

3.7 Results and Future Research

The obtained simulation results are depicted on Figure 3.2. For each specification we have a plot with *stv* and *pev* on axles with performances of DA, TTC and $\widetilde{\varphi}$, and optimized results for $\lambda \in \{0, 0.1, 0.2, \ldots, 0.9\}$.

As we can see, regardless of the specification, for any weight λ there exists a mechanism that performs a simultaneous relaxation of stability and efficiency much better than the corresponding convex combination of DA and TTC. More-

$Pr_{tr} = 0.2,\ Pr_{corr} = 0$

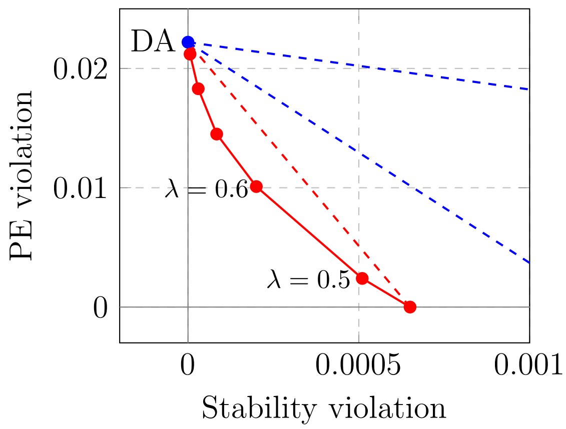


Figure 3.3: Zoomed in results

over it performs it at least twice as good as a convex combination of DA and $\widetilde{\varphi}$. Also, for $\lambda \in \{0, 0.1, 0.2, 0.3, 0.4\}$ the optimized results are always the same: it produces an efficient matching; and only after we make stability at least as important as efficiency, it departs from the x-axis and starts to violate efficiency.[14].

In addition, interestingly, for specifications with positive truncation probability the change in correlation probability from zero to 0.75 shrinks all the resulting violations by the factor of two. While, the same change for specifications with zero truncation probability does so only for stability violations, and PE violations got reduced by the factor of 1.5. In general, we can observe that increase in correlation probability decreases both violations, while increase in truncation probability decreases only PE violation and leaves stability violation almost intact.

To sum up, the main result of this chapter is robust to different specifications: in order to optimally relax efficiency and stability under two-sided matching markets we should definitely look beyond the celebrated benchmark mechanisms. And, while DA results are indeed limits for the optimal mechanism as λ tends to one, the TTC results are more than five times less stable than the limit as λ tends to zero. This implies that absence of strategy-proof requirement has no effect on PE violation, while is crucial for stability violation.[15] Thus, observing zoomed in

[14]During simulations *stv* were multiplied by 20 to be comparable with *pev*

[15]Indeed, DA is constrained efficient Gale and Shapley (1962), while TTC is the most stable

results on Figure 3.3, even if we consider DA and the most efficient mechanism (bottom red dot on the plot) as border mechanisms, their convex combination with a given $\lambda \in (0,1)$ will not be the optimal mechanism that puts a weight of λ on ex-post stability and $(1-\lambda)$ on ex-post PE.

As for the future research, one could further investigate the simple one-to-one markets trying to infer the underlying structure of optimal mechanisms. Also, for a specification with no truncation and no correlation ($Pr_{tr} = Pr_{corr} = 0$) it may be interesting to calculate students' strategy-proofness violation for each λ. If there are m colleges, under a preference profile P, any student s can misreport in $(m!-1)$ ways. We can denote a set of all possible student's preference orders as $\mathcal{P}_s$ (contains $m!$ elements).

Thus, for each student $s \in S$ strategy-proofness violation is (Ravindranath et al., 2021)

$$spv_s(\varphi, p) = \max_{\succ'_s \in \mathcal{P}_s} \left(\max_{c' \in C, c' \succ_s \perp} \sum_{c \in C: c \succ_s c'} \varphi_{sc}(\succ'_s, \succ_{-s}) - \varphi_{sc}(\succ_s, \succ_{-s}) \right) \tag{3.7}$$

So, students strategy-proofness violation at profile p is

$$spv(\varphi, p) = \frac{1}{n} \sum_{s \in S} spv_s(\varphi, p) \tag{3.8}$$

For instance, we can investigate what happens to the results once we decide to control for strategy-proofness. In other words, we can perform the same study in the class of only strategy-proof mechanisms.

out of all efficient and strategy-proof mechanisms in one-to-one matching (Abdulkadiroglu et al., 2020).

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